2018 SAC 3 PREP 3 - SOLUTIONS

Question 1

The gestation period for pandas is approximately normally distributed with a mean of 135 days and standard deviation of 10 days.

a. What percentage of pandas have a gestation period greater than 150 days. Give your answer to the nearest whole percent.

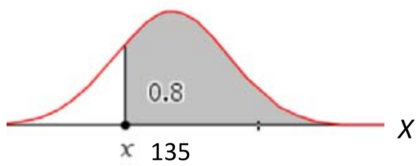
$X \sim N(135, 10^2)$

$\Pr(X > 150) = 0.0668$

Therefore approximately 7% have a gestation period greater than 150 days.

1 mark

b. 80% of pandas have a gestation period greater than *x* days. Find the value of *x* to the nearest whole number. Include a diagram in your solution.



$\Pr(X > x) = 0.8$

$\Pr(X < x) = 0.2$

$x = 127\ days$

3 mark

It is known that 70% of people in a particular city are in favour of the pandas being kept in zoos. A random sample of 500 people was taken from this city and asked whether they were in favour of the pandas being kept in zoos.

c. What is the probability, correct to four decimal places, that more than 360 people in the survey are in favour of the pandas being kept in zoos? (Do not use the normal approximation)

$Y \sim Bi(500, 0.7)$

$\Pr(X > 360) = \Pr(X \geq 361) = 0.1527$

2 marks

d. Find $\Pr(\mu - 2\sigma < X < \mu + 2\sigma)$. Give your answer correct to two decimal places. (Do not use the normal approximation).

$\mu = 500 \times 0.7 = 350$

$\sigma = \sqrt{500 \times 0.7 \times 0.3} = \sqrt{105}$

$Pr(350 - \sqrt{105} < Y < 350 + 2\sqrt{105})$

$= \Pr(329.51 < Y < 370.494)$

$= \Pr(330 \leq X \leq 370)$

$= 0.95$

3 marks

f. The probability of selecting a person in favour of pandas being kept in zoos in another city is 0.45. What is the least number of people who should be asked their opinion, if we want to ensure that the probability of selecting at least one person in favour of pandas being kept in zoos is more than 0.8?

$X \sim Bi(n, 0.45)$

$\Pr(X \geq 1) > 0.8$

$1 - \Pr(X = 0) > 0.8$

$\Pr(X = 0) < 0.2$

$\binom{n}{0}(0.45)^0(0.55)^n < 0.2$

$0.55^n < 0.2$

$n > \frac{log_e(0.2)}{log_e(0.55)}$

$n > 2.69$

$\therefore 3\ people$

2 marks

It is known that 10% of a certain species has a gestation period of more than 200 days and 5% of the species less than 175 days. The gestation period is normally distributed.

i. Find the mean and standard deviation of the gestation period for this species. Give your answer to the nearest whole number.

$\Pr(X_1 > 200) = 0.1$

$\Pr(X_1 < 200) = 0.9$

$\Pr(X_2 < 175) = 0.05$

$\frac{175-\mu}{\sigma} = -1.6448536259066$

$\frac{200-\mu}{\sigma} = 1.2815515665787$

invNorm(0.9,0,1)	1.28155
invNorm(0.05,0,1)	-1.64485
solve$\left(\frac{200-a}{b}=1.2815515665787 \text{ and } \frac{175-a}{b}\right.$ ▸	
	a=189.052 and b=8.5429

Solve simultaneously: $\mu = 189 \; and \; \sigma = 9$

3 marks

Total 13 marks

Question 2

Xavier is a dental inspector. He has just opened a new dental practice and has been informed that in his local area, about 20% of young children have tooth decay. Xavier wants to conduct some of his own research.

a. If Xavier randomly selected a sample of 100 children from his area:

i. How many would be expected to have tooth decay?

Let D be the numbe rof children with tooth decay $D \sim Bi(100, 0.2)$

$E(D) = 100 \times 0.2 = 20$

Find the variance of the number of children with tooth decay.

$var(D) = 100 \times 0.2 \times 0.8 = 16$

ii. What would be the probability, correct to three decimal places, of finding exactly the expected number of children with tooth decay in the sample?

$Pr(D = 20) = 0.099$

1 + 2 + 2 = 5 marks

b. Xavier would like to reduce the number of children with tooth decay.

Find the probability that the number of children in the sample with decay would be no more than 15. Give your answer correct to three decimal places. 1 mark

$Pr(D \leq 15) = 0.129$

c. Xavier examines all the children in the sample and finds 29 of them have tooth decay.

i. Given 20% of students are thought to have tooth decay find $\mu \pm 2\sigma$ and interpret the interval $[\mu - 2\sigma, \mu + 2\sigma]$ Comment on whether Xavier could have cause to doubt the accuracy of information he was given.

$\mu - 2\sigma = 20 - 2 \times 4 = 12$

$\mu + 2\sigma = 20 + 2 \times 4 = 28$ Therefore the interval is [12,28]

29 children having tooth decay is more than 2 standard deviations away from the expected number of children with tooth decay. The chance of more than 28 children having tooth decay is less than approximately 2.5%. Xavier could have cause to doubt the accuracy of the information but should realise that one sample is not enough to be a reliable estimate.

2 marks

Total 13 marks

Question 3

A sample of six house elves is to be drawn from a large population in which 20% of the house elves are free.

a. Find the probability that the sample contains:

i. three free house elves

$E \sim Bi(6, 0.2)$

$\Pr(E = 3) = 0.08192 = \frac{256}{3125}$

ii. less than three free house elves

$\Pr(E < 3) = \Pr(E \leq 2)$

$\Pr(E < 3) = 0.90112 = \frac{2816}{3125}$

2 + 1 = 3 marks

b. Another large population contains a proportion p of free house elves.

i. Write down an expression in terms of p, for P the probability that a sample of six contains one or two free house elves.

$$\mathrm{P} = \binom{6}{1}(p)^1(1-p)^5 + \binom{6}{2}(p)^2(1-p)^4 = 6p(1-p)^5 + 15p^2(1-p)^4$$

ii. Use a suitable method to determine a value of p for which P is greatest. Hence find the maximum value of P.

$Let\ P = 3p(p-1)^4(3\mathrm{p}+2)$

$\frac{dP}{dx} = 6p(p-1)^3(9p^2+2p-1)$

$For\ maximum\ \frac{dP}{dx} = 0$

$0 = 6p(p-1)^3(9p^2+2p-1)$

$p = \frac{-1-\sqrt{10}}{9}\ or\ p = -\frac{-1+\sqrt{10}}{9}\quad or\quad p = 1$

Consider the shape of the graph. We find that P is a maximum when $p = -\frac{-1+\sqrt{10}}{9}$. For this value of p the value of $P\ is\ 0.6534$

2 + 3 = 8 marks

Total 11 marks

Question 5

Suppose that, in flight, plane engines fail with probability q, independently of each other, and that a plane will complete the flight successfully if at least half of its engines are still working. For what values of q is a two-engine plane to be preferred to a four0engine plane?

For a two-engine plane

Let X be the number of engines which will fail.

The plane will successfully complete its journey if 0 or 1 engines fail.

$$\begin{aligned}\Pr(X = 0) + \Pr(X = 1) &= (1-q)^2 + 2q(1-q)\\ &= 1 - 2q + q^2 + 2q - 2q^2\\ &= 1 - q^2\end{aligned}$$

For a four-engine plane:

Let Y be the number of engines which will fail.

The plane will successfully complete its journey if 0, 1 or 2 engines fail.

$$\begin{aligned}\Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) &= (1-q)^4 + 4q(1-q)^3 + 6q^2(1-q)^2\\ &= (1-q)^2[(1-q)^2 + 4q(1-q) + 6q^2]\\ &= (1-q)^2[1 - 2q + q^2 + 4q - 4q^2 + 6q^2]\\ &= (1-q)^2[1 + 2q + 3q^2]\end{aligned}$$

To find when a two-engine plane is to be preferred to a one-engine consider the inequality

$$1 - q^2 > (1-q)^2(1 + 2q + 3q^2)$$

$$(1-q)(1+q) > (1-q)^2(1 + 2q + 3q^2)$$

$$\therefore \quad (1+q) > (1-q)(1 + 2q + 3q^2)$$

$$(1+q) > 1 + 2q + 3q^2 - q - 2q^2 - 3q^3q > q + q^2 - 3q^3$$

$$0 > q^2 - 3q^3$$

$$0 > q^2(1 - 3q)$$

$$\therefore \quad \frac{1}{3} \le q \le 1$$

A two-engine plane it to be preferred to a four-engine plane for $\frac{1}{3} \le q \le 1$