Name: ______________________________

Teacher: ____________________________

School No: _____________

Scotch College

Department of Mathematics

2017

Mathematical Methods (CAS)

<u>Unit 3 SAC 1a</u>

Application Task

CAS calculator allowed

Distribution Date:	Tuesday 23 May 2017 (Day 5)
Due Date:	Monday 5 June 2017 (Day 2)
Total Marks:	100

Instructions

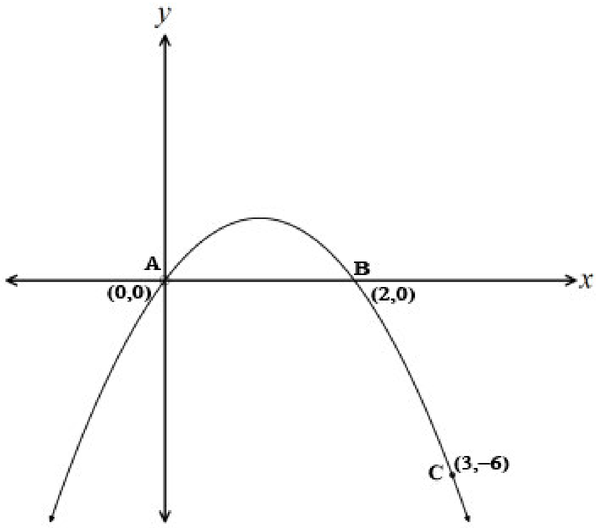
Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this task are **not** drawn to scale.

The graph of the function $f:\mathbb{R}\to\mathbb{R}, f(x)=ax(m-x)$ is shown below, where a and m are real constants.



1 **a** From the information provided on the graph above, find the values of a and m

b **i** Find $f'(x)$ in terms of a and m.

ii Solve $f'(x)=0$ for x in terms of m.

c **i** Use your values of a and m from part **1a** to find an expression for $f'(x)$

ii Solve $f'(x) = 0$ for x.

d **i** State the range of f in terms of a and m.

ii Use your values of a and m from part **1a** to state the range of f.

e Let $g:\mathbb{R}\rightarrow\mathbb{R}$ where $g(x)=2x(4-x)$.

i State the set of values of p for which the graph with equation $y=-g(x)+p$ has real solutions where $p\in\mathbb{R}$.

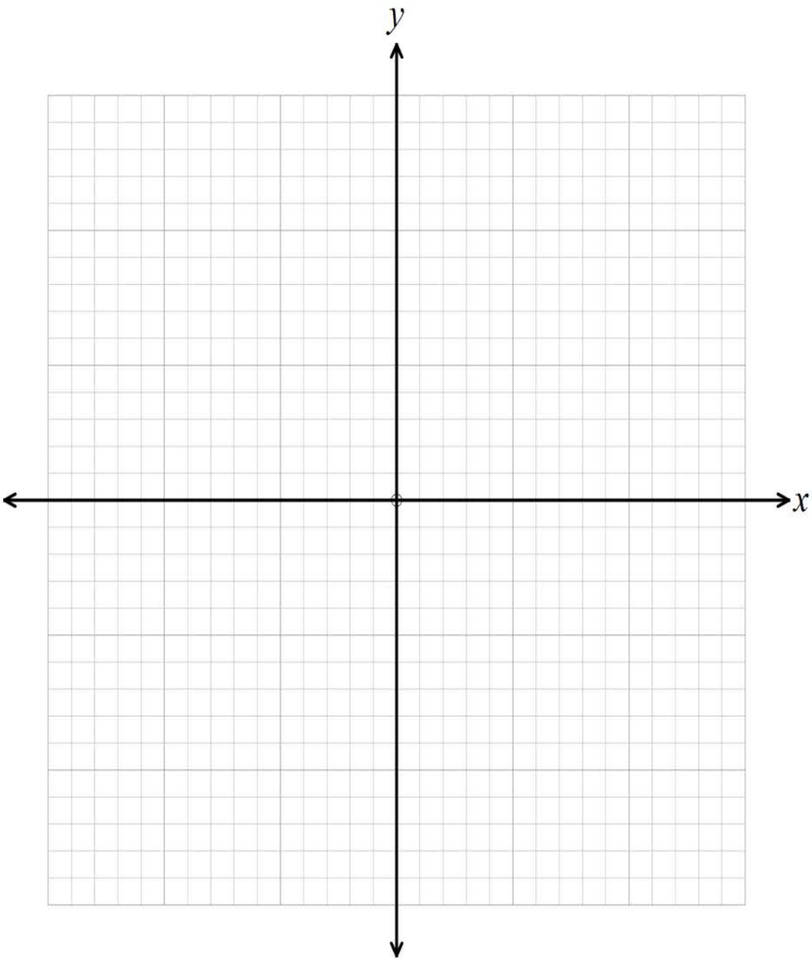
ii State the values of p for which the graph of equation $y=-f(x)+p$ has real solutions in terms of a and m, where $p\in\mathbb{R}$.

f Let $h:[-2,3]\rightarrow\mathbb{R},\ h(x)=ax^m$ and $w:[-2,3]\rightarrow\mathbb{R},\ w(x)=(k-x)^n$, were a, k, m and n are positive integers.

$g(x)=h(x)\times w(x)$

i Define fully $g(x)$

ii For $a = 1,\ k = 2,\ m = 2$ and $n = 1$, sketch the graph of $y = g(x)$ labelling the coordinates of end-points in exact form, axial intercepts and turning points.



g. Let $a=1$ and $k=2$

i By changing the values of m and n as indicated in the table below, describe the nature of the stationary points formed as indicated by the example shown,

$$g(x)=x^m(2-x)^n$$

m	n	Local minimum	Stationary point of inflection	Local maximum
2	1	$x=0$	None	$x=\frac{4}{3}$
1	2			
2	2			
1	3			
3	1			
2	3			
3	3			

ii How do the values of m and n affect the graph of $y=g(x)$?

iii Find the x-coordinates of the stationary points of $g(x)=x^m(2-x)^n$ in terms of m and n where appropriate.

iv Verify your answer to part **g iii.** Using $m = 2$ and $n = 3$

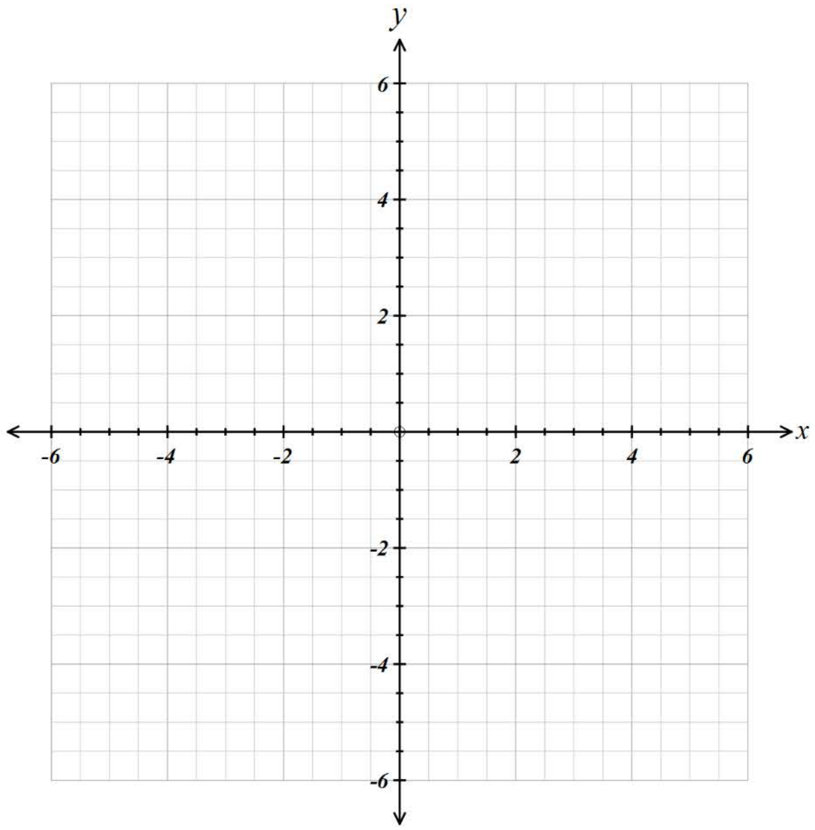
(Marks: 2 + (1 + 1) + (1+ 1) + (2 + 1) + (2 + 2) + (2 + 3) + (3 + 1 + 4 + 1) = 27)

2 **a** Consider the functions:

$p:[-4,4] \to \mathbb{R}$, $p(x) = x^{\frac{1}{3}}$ and $q:\mathbb{R} \to \mathbb{R}$, $q(x) = x^2$

i Fully define the composite function $h(x) = q(p(x))$

ii Sketch the graph of $y = h(x)$, showing axial intercepts.



iii A transformation is described through $X' = TX + B$ where $T = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Find the image of $h(x)$ and sketch on the same set of axes above the graph of the image of $h(x)$ showing axes intercepts and any other significant points.

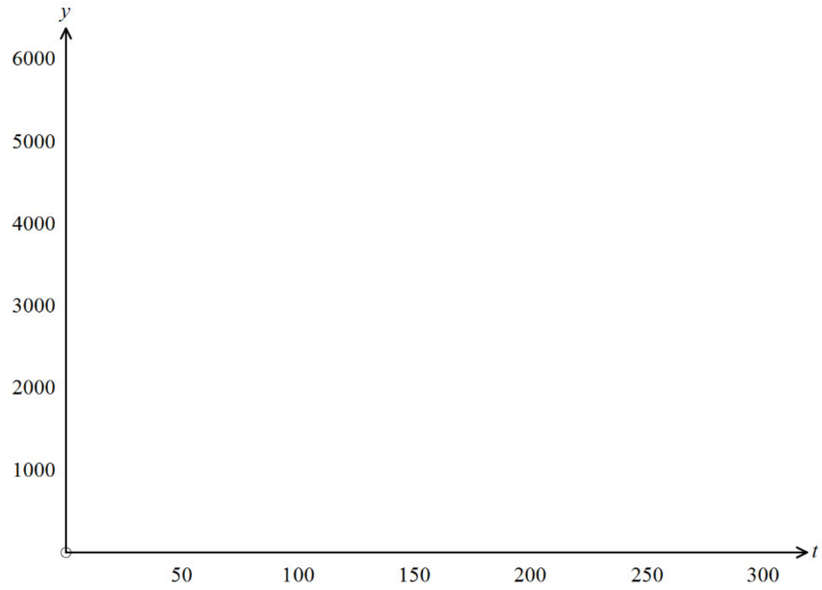
(Marks: 1+ 2 + 6 = 9)

3 Scientists are concerned that a species of birds may be in danger of extinction, as low numbers have been observed in the forests where they live. In an effort to protect the species from extinction, the Department of Conservation and Natural Resources decided to trap 1400 of these birds and move them to a remote island off the Australian coast. It is hoped that the bird population will recover there, safe from the problems on the mainland.

a A mathematical model for the number N_1 of birds on the island *t* years after the initial 1400 are settled there is given by $N_1(t) = 4800 - 3400e^{-kt}$ where $t \geq 0$ and $t = 0$ represents the year, the 1st of January 2017 and *k* is a real constant.

i If the population of these birds increased by 400 in next 5 years, find the value of *k* correct to **two decimal places**.

ii On the set of axes provided, sketch the graph of $y = N_1(t)$ for $0 \le t \le 300$ and label clearly any intercepts with the axes , asymptotes and plot the coordinate point indicating the population of these birds at the 1st of January 2067 (in 50 years).



iii Comment on the population of the birds in the long run.

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b Another species of birds, N_2, was already part of the islands' habitat. The scientists believe that the population of this species will be effected by the introduction of the N_1 birds in this island. The scientists developed a mathematical model for the N_2 birds to be $N_2(t) = 2500e^{-0.03t} + 2500$, where t represents the number of years after 2017.

i What was the initial population of N_2 ?

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ii What will be the population of N_2 in 2067?

iii On the diagram in part **a. ii.** above, sketch the graph of $y = N_2(t)$, for $0 \le t \le 300$ showing any intercepts with the axes, asymptotes and the coordinates of the point when $t = 50$.

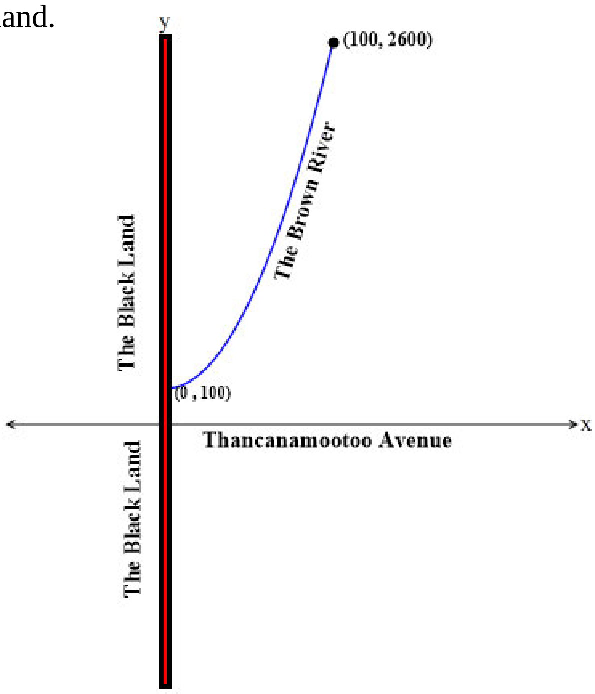
iv In which month of which year would the two species of birds have the same population? How many birds would there be?

v If k is a controlling factor for the growth of the population of N_1, for what value of k (to four decimal places), could the two populations be equal in 50 years' time?

vi If the scientists decided to prolong the time when the two species have equal populations to the 1st of January 2117. Given that $N_1(t) = 4800 - 3400e^{-mt}$ and $N_2(t) = 2500e^{-nt} + 2500$ where m and n are the scientists population controlling factors. Find to four decimal places the values of m and n for the two populations to reach 3500 birds in 2117.

(Marks: (1 + 4 +1) + (1 + 1 + 4 + 2 + 1 + 2) =17)

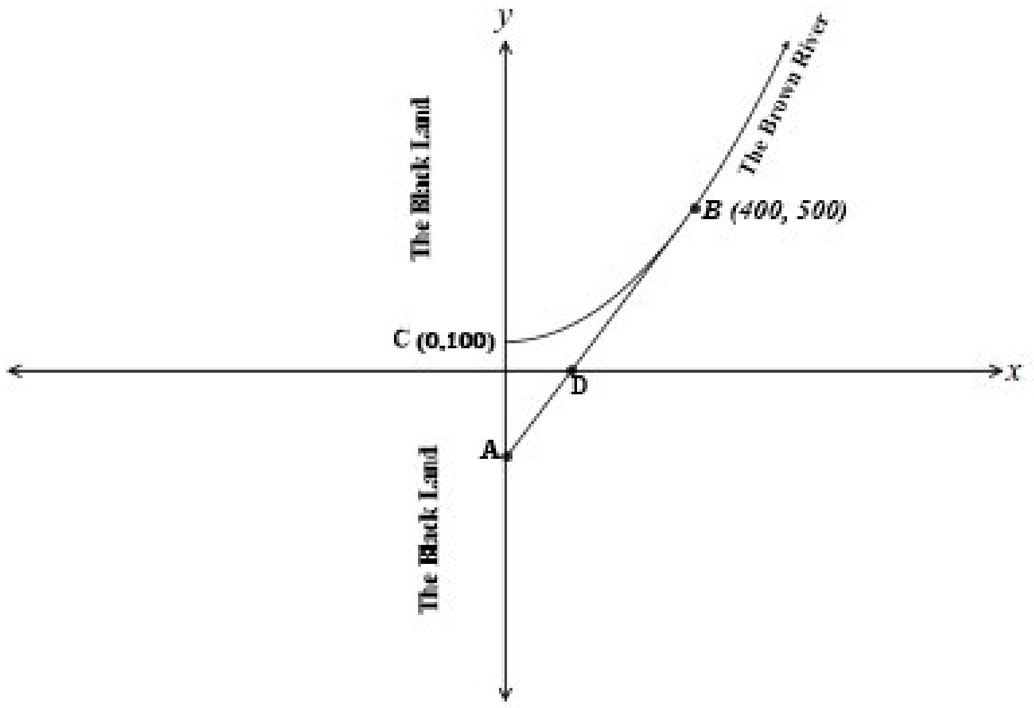
4 In the land of Shtam, the population is ageing. A wealthy benefactor donated a large area of land to build two new age care facilities. The land is situated near the "Brown" river (named after the distinct brown colour of the water). See the diagram below. The *y*-axis is a high wall separating Shtam from 'The Black" land. This dense dark forest is forbidden for the citizens of Shtam to enter, hence a high wall is built to keep people away. The positive *x*-axis is a straight horizontal road going through the benefactor's land.



A town planner was given the job to build both the age care facilities. He is using the wall and the road as axes and established an equation that represents the Brown river as:

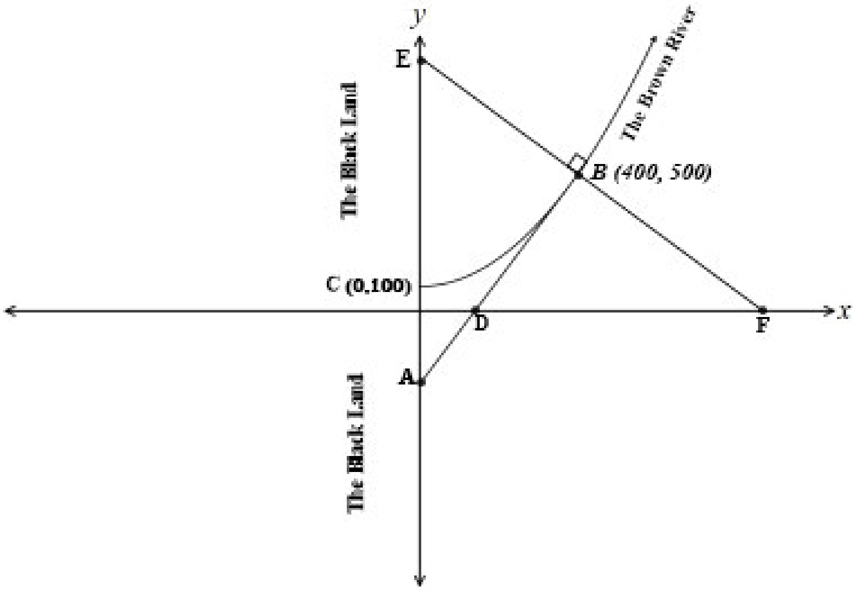
$y = \frac{1}{400}x^2 + 100$, $0 \le x \le 100$. Both horizontal and vertical measurements are in metres and the width of the river is assumed to be negligible.

a Initially the benefactor wanted his residence to be built at point $B(400, 500)$ on the river bank. A fence is to be built forming a tangent at B and ending at the high wall.



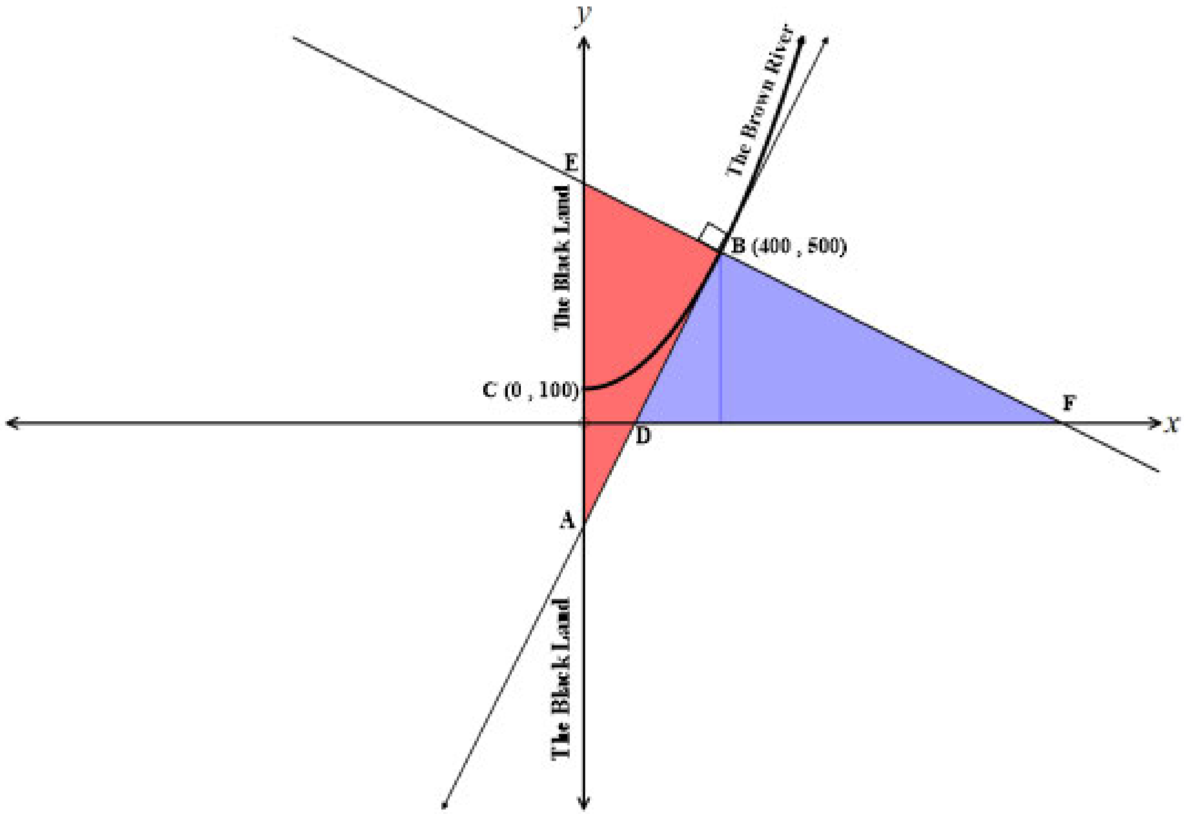
i Find the equation of tangent line and the coordinates of points A and D.

A second fence will be used as part of the side fencing of the two age care facilities. It will be a straight line perpendicular to the tangent and passing through the benefactor's residence at B.



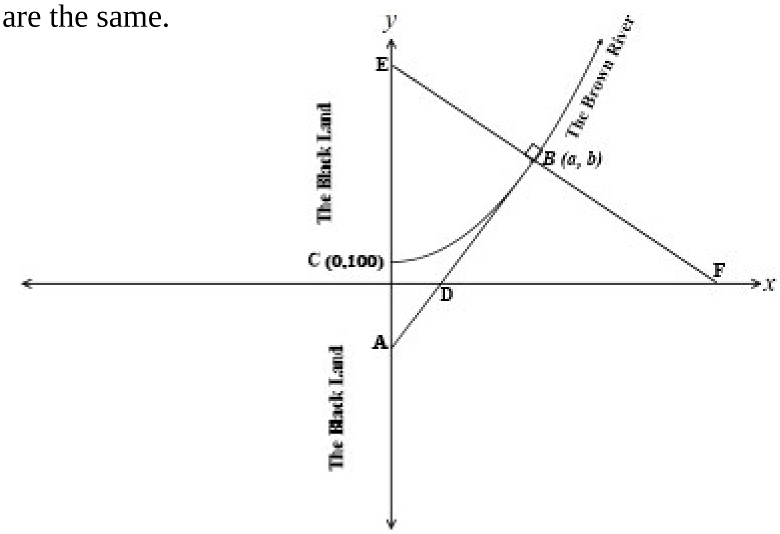
ii Find the equation of this perpendicular line and the coordinates of points E and F as shown in the above diagram.

The fences, the wall and the road form boundaries for two distinct areas represented by Δ ABE and Δ DBF .



iii Find the areas of $\Delta\, ABE$ and $\Delta\, DBF$ in m^2 .

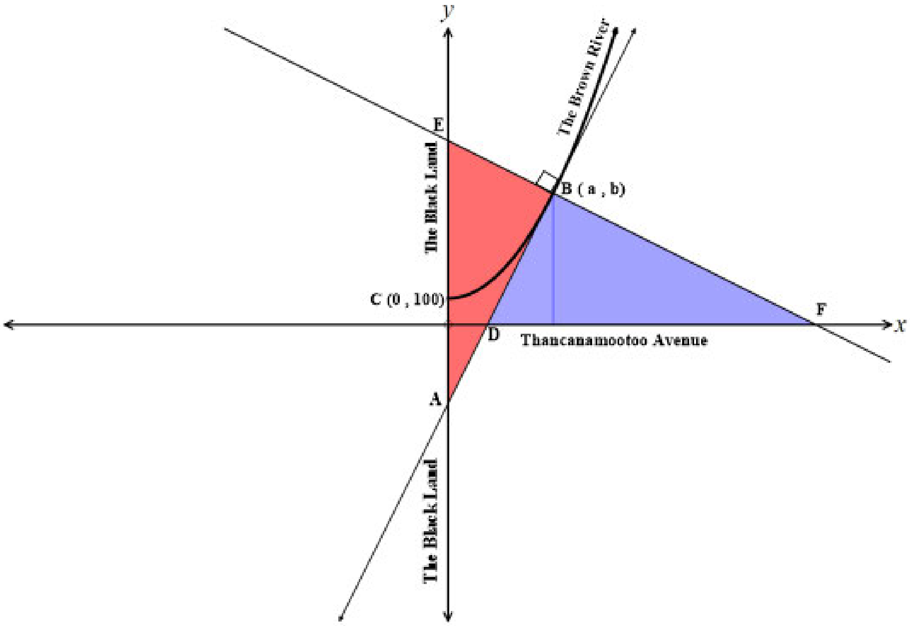
Evidently, the two regions are not of equal proportions. The benefactor's wishes were that the two age care facilities are to be of equal area.
The town planner replied that the position of the benefactor's residence is causing this problem. His task with your assistance is to find B (a, b) so that the areas of the triangular estates are the same.



b Repeat steps **a i , ii, iii** , using point B (a , b).

i Find the tangent at B (a , b) and the coordinate points A and D in terms of a.

ii Find the normal at B (a , b) and the coordinates E and F in terms of a.

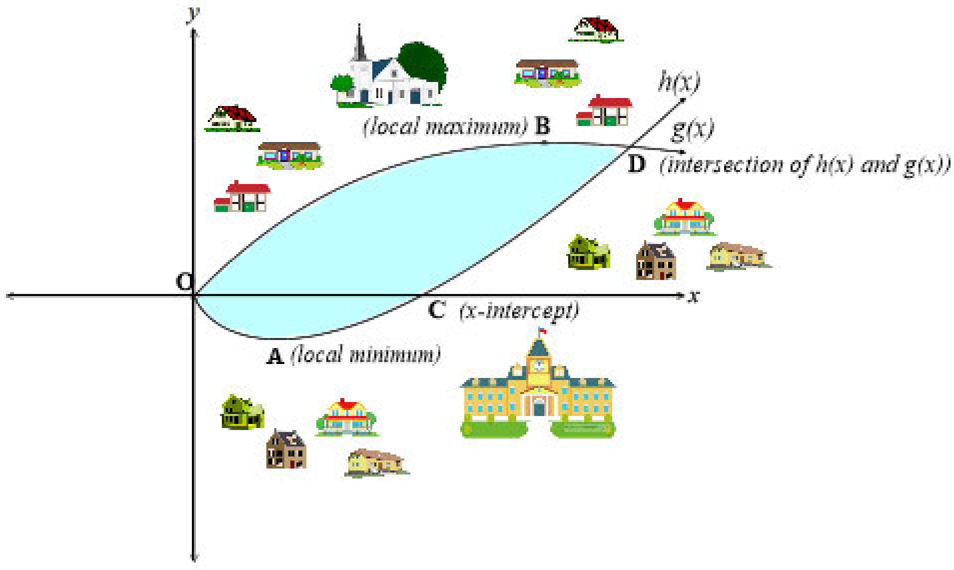


iii Find the areas of $\Delta\, ABE$ and $\Delta\, DBF$ in terms of a and hence find the value of a that will make it possible for both age care facilities to have equal area. State the coordinates of the benefactor's residence and the area that each facility will have.

(Marks: (4 + 4 + 4) + (4 + 4 + 6) = 26)

5 One of the estates in Shtam is built around a picturesque lake as shown in the diagram below. (North is the direction of the positive *y*-axis, East is the direction of the positive *x*-axis). The southern bank of the lake is approximated by the function defined as $h:(0,\infty)\to\mathbb{R},\ \ h(x)=x\log_e(x)$ and the northern bank is approximated by the function $g:(0,\infty)\to\mathbb{R},\ \ g(x)=\dfrac{x^2}{e^x-1}$.

a The measurements of the clipart in this diagram are not to scale. Distances in both *x* and *y* directions are considered as kilometres.



i Find $h'(x)$

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ii Find $g'(x)$

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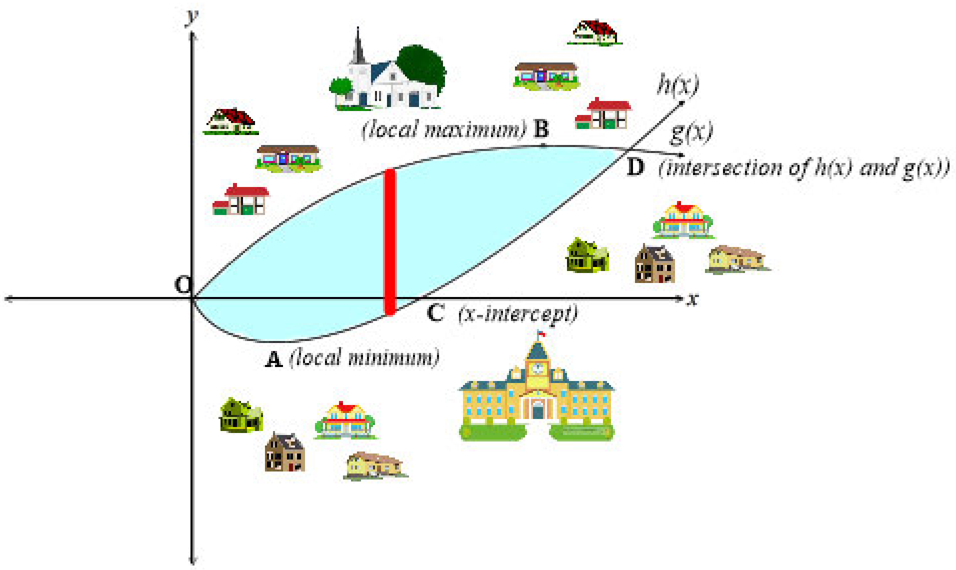
iii Find the exact coordinates of A and the coordinates of B correct to two decimal places.

iv Find the exact coordinates of C, the x-intercept of function h.

v Find the coordinates of D (correct to two decimal places), the point where functions h and g intersect.

b The people of this estate thought that a **bridge** connecting the southern bank to the northern bank would improve transportation, particularly to the local school and the church.

The vertical distances from one side of the lake bank to the other is represented by function f such that $f(x) = g(x) - h(x)$.



i Define $f(x)$

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ii Find the value of x, the location where the bridge will be situated in order to construct the longest possible bridge parallel to the y-axis and calculate the length of this bridge to one decimal place.

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iii The costs for this project was beyond the budget of this community and the engineer recommended a smaller bridge. For which values located on the x-axis could they situate a 0.5 km bridge? Express your answers to one decimal place.

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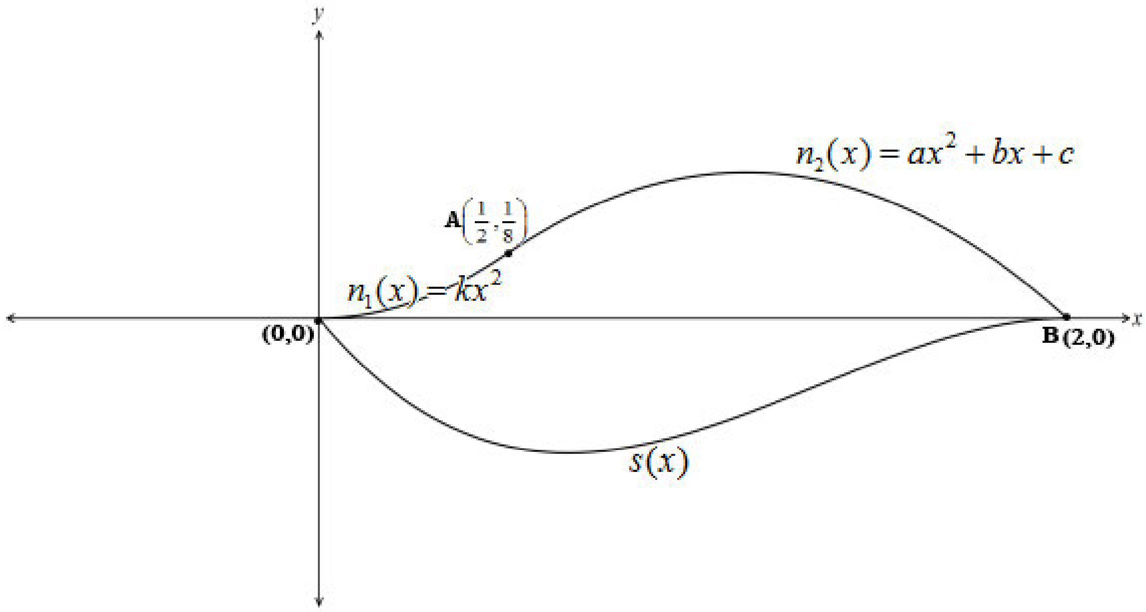
c When plans were submitted to the Shtam Council for the construction of the bridge, the plans did not pass. The functions used to represent the banks of the lake were incorrect. It was recommended that northern bank be approximated by two quadratic piecewise functions **smoothly** connected at a point A $\left(\frac{1}{2}, \frac{1}{8}\right)$

such that, $n(x) = \begin{cases} kx^2 & x \in [0, 0.5] \\ ax^2 + bx + c & x \in (0.5, 2] \end{cases}$

and the southern bank is best approximated by the cubic function

$s(x) = mx(x-2)^2, \quad x \in [0, 2]$, where k, a, b, c and $m \in R$.

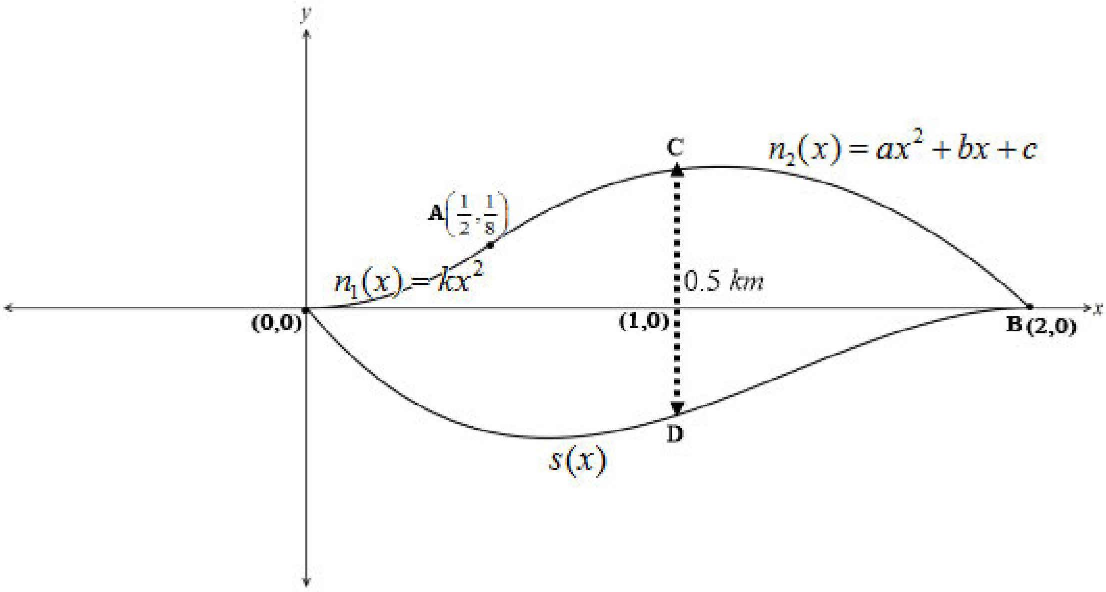


i From the information provided, find the value of k, the dilation factor of the first quadratic function defined from the origin to point A.

ii The second quadratic function from point A $\left(\frac{1}{2},\frac{1}{8}\right)$ to point B $(2,0)$ is represented by $n(x)=ax^2+bx+c$. Given that these quadratic functions are smoothly connected, find the values of $a,b,$ and c.

The southern bank of the lake is represented by the function $s(x) = mx(x-2)^2$, $x \in [0, 2]$.

The engineer had to find the dilation factor *m* of this function. The only information provided was that the bridge had to be 0.5 km long and it had to be located 1 km east of the origin.



iii Find the exact value of *m*.

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(Marks: (1 + 1 + 4 + 2 + 1) + (1 + 1 + 1) + (2 + 4 + 3) = 21)

End of SAC 1a