Name: _____

Teacher: _____

School No: _____

Scotch College

Department of Mathematics

2017

Mathematical Methods (CAS)

Unit 3 SAC 1c

Application Task

CAS calculator and scientific calculator allowed

No notes allowed

Date: Thursday 8 June 2017

Time: 60 minutes

Marks: 34

Instructions

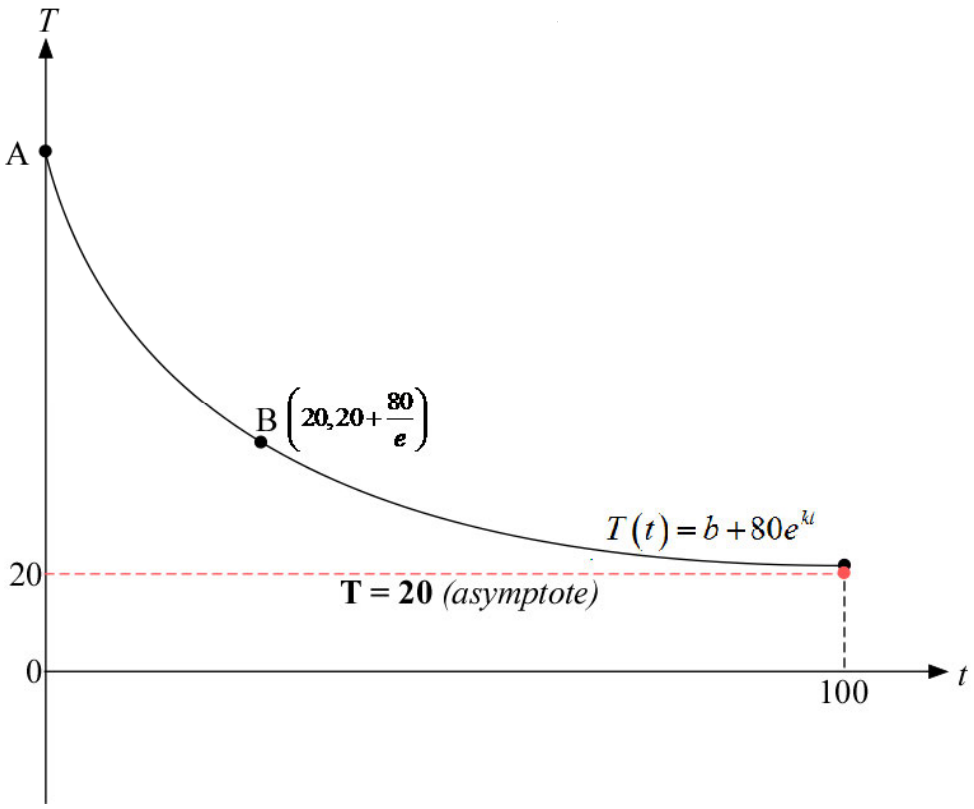
Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this task are **not** drawn to scale.

1 According to Newton's Law of cooling, the temperature $T\ ^oC$ with respect to time t minutes of my soup, is given by $T:[0,100]\to\mathbb{R}, T(t)=b+80e^{kt}$, where k and b are real constants.



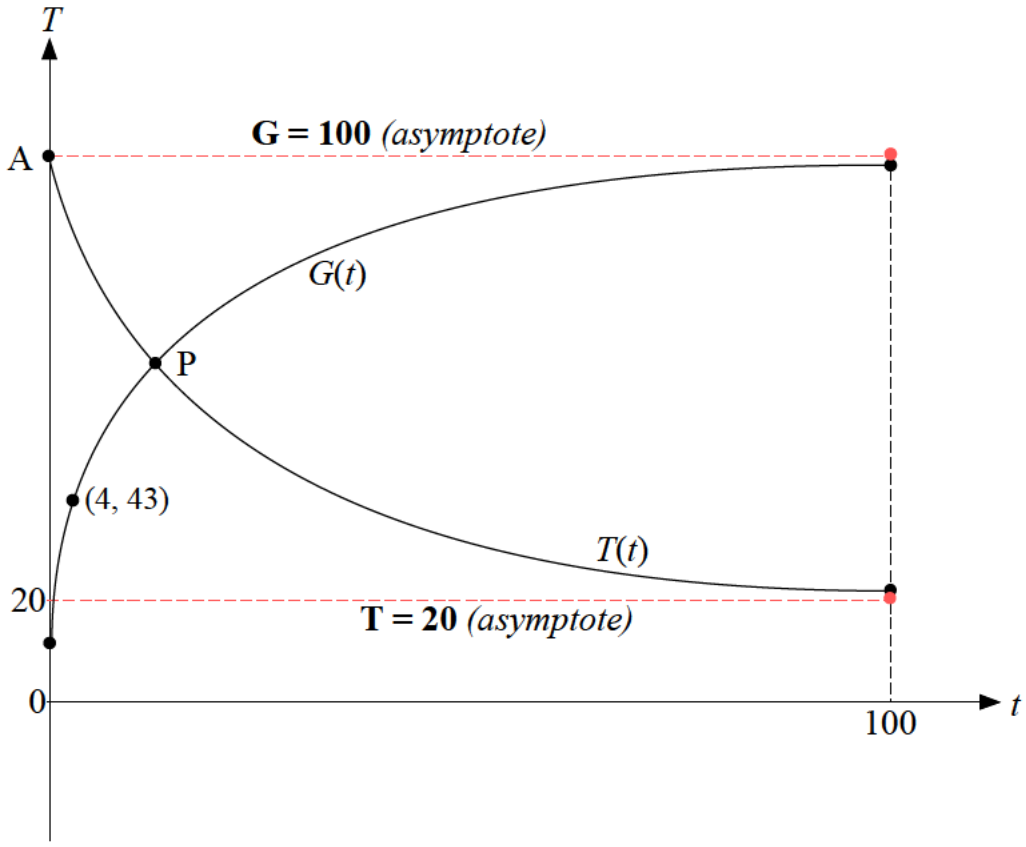
a Find the value of b and then the coordinates of A.

b Given that the curve passes through $\left(20, 20+\frac{80}{e}\right)$, show that $k=-\frac{1}{20}$.

This function is used to model the relationship between temperature and time as a bowl of soup cools. In a separate pan, a special mixture is being heated. The temperature of the special mixture is defined as: $G:[0,100]\to\mathbb{R}, G(t)=100-85e^{-ht}$, where $G\ ^oC$ is the temperature after t minutes.

c Find the initial temperature of G.

d The temperature reached 43^oC in 4 minutes. Find the value of h expressing your answer to **one decimal place**.



The graphs of both functions are shown above.

e Find to **one** decimal place, the time and the temperature when both the soup and the special mixture will reach the same temperature.

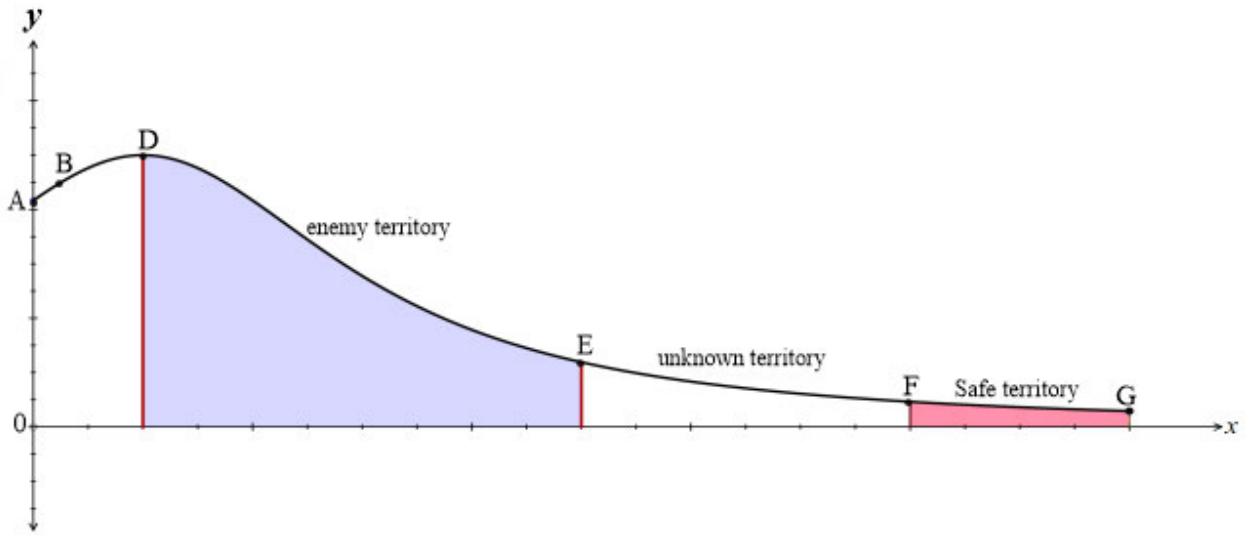
f If the chef states that it is best to prolong the time when both the soup and the special mixture have equal temperature to exactly half an hour. Find to **three** decimal places the values of k and h so that both reach $55^{\circ}C$ at half an hour.

(Marks: 2 + 1+ 1 + 1 + 2 + 2 = 9)

2 Agent Patel is a secret agent on a special mission. He is attempting to escape enemy territory. The cross section of the terrain surrounding him is given by the function

$$h:[0,100] \to \mathbb{R},\ h(x) = \frac{2500}{x^2 - 20x + 600}$$

where h represents the height in metres of the ground above sea level and x represents the horizontal distance in metres from Agent Patel's starting point A. The graph of $y = h(x)$ is shown below.

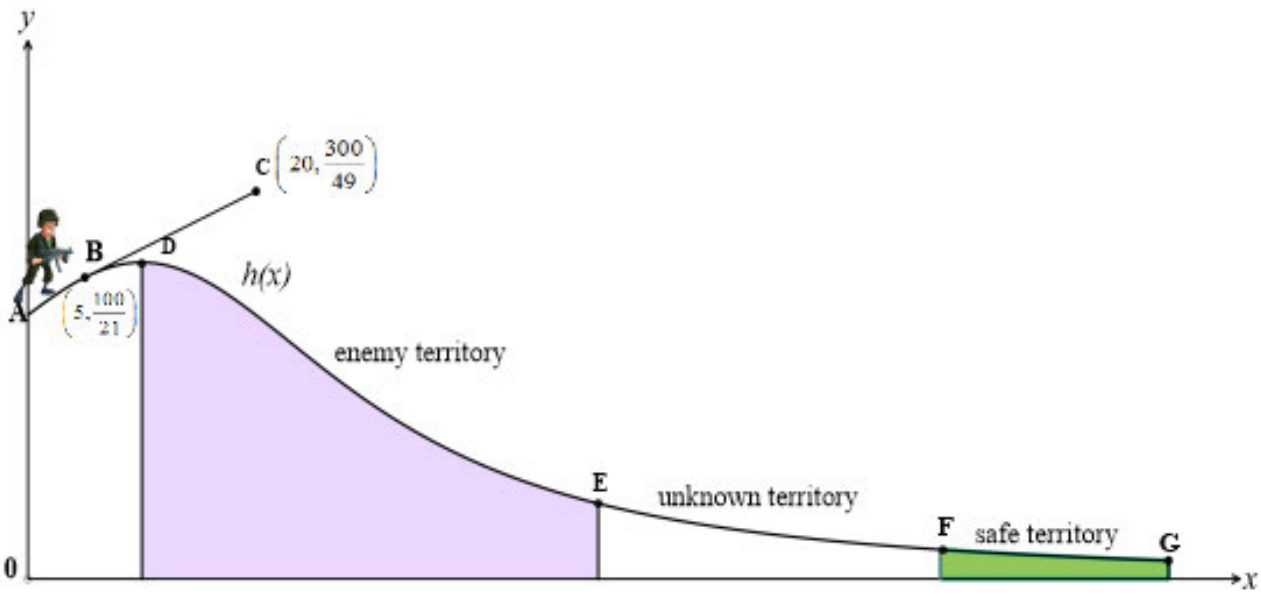


Between points A and D, Agent Patel is hidden behind the top of small hill which shields him from the enemy guard post located at point E. At point F, there are friendly guards and safe territory extends to the right of F. Unknown territory lies between points E and F. Agent Patel is trying to make it to point F or beyond without being caught.

a State the coordinates of point A.

b Agent Patel walked to point B, which is at a **horizontal** distance of 5 metres from his starting point A. Show that the height above sea level at point B is $\frac{100}{21}$.

Agent Patel has hi-tech flying equipment strapped to his body which can propel him into the air. He uses this equipment at point B and once airborne at point B, he follows a straight path, **tangential** to the curve $h(x)$ point B, until he reaches point C. The path from B to C is shown below.



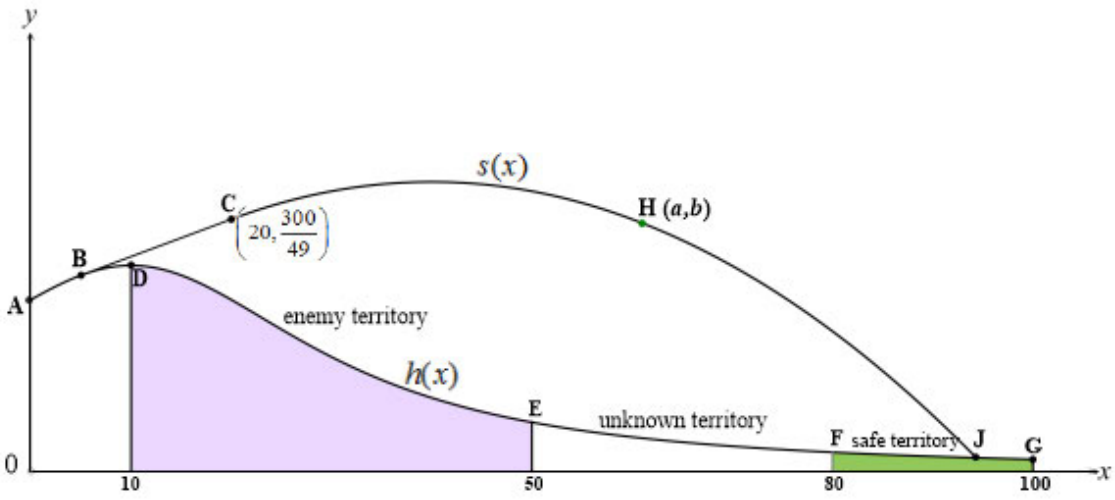
c How far above the ground would Agent Patel be at point C? Express your answer in metres correct to **two** decimal places.

d Find the equation of Agent Patel's linear path.

After Agent Patel reached point C, his flight path has changed. His new path was programmed so that the height above sea level in metres is given by

$$s(x) = -\frac{1}{k}(x-40)^2 + \frac{3100}{441} \text{ , where } k \in \mathbb{R}$$

This parabolic path was smoothly connected to his linear path at the point where his horizontal distance was 20 metres.



e What is the maximum height above sea level that Agent Patel will have whilst on this new flight path?

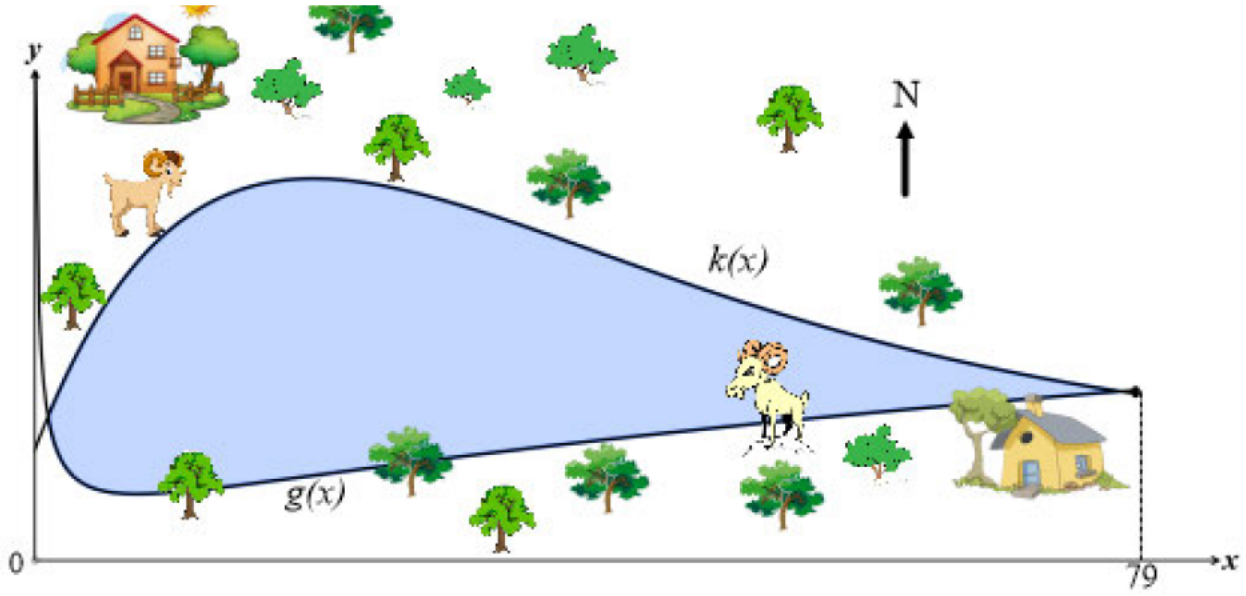
f Find the value(s) of k that will make the curves have a smooth transition from the linear to the parabolic path.

g Find the coordinates of point J where Agent Patel landed correct to **two** decimal places.

h At point H, the agent fired a missile towards the enemy territory, aiming at E. Find the coordinates of H if it is known that the path of the missile was perpendicular to the tangent at H and express your answer to **two** decimal places.

(Marks: 1 +1 + 1 + 2 + 1 + 2 + 1 + 3 =12)

3 Telemachus and Karmenia leave in farms on opposite sides of a lake in Shtam. The northern bank of the lake is defined by function $y = k(x)$ and the southern bank by $y = g(x)$ as shown on the diagram. The value of x represents the horizontal distance in metres and the functions $k(x)$ and $g(x)$ represent the vertical distance in metres.



$$k : [1, 79) \to \mathbb{R}, \quad k(x) = mxe^{-\frac{x}{n}} + 6$$

$$g : [1, 79) \to \mathbb{R}, \quad g(x) = \left(\log_e \left(\frac{x}{n} \right) \right)^2 + \log_e \left(\left(\frac{x}{m} \right)^2 \right) \text{ where } m \text{ and } n \in \mathbb{Z}.$$

a Using Calculus, find in terms of m and n:

i $k'(x)$

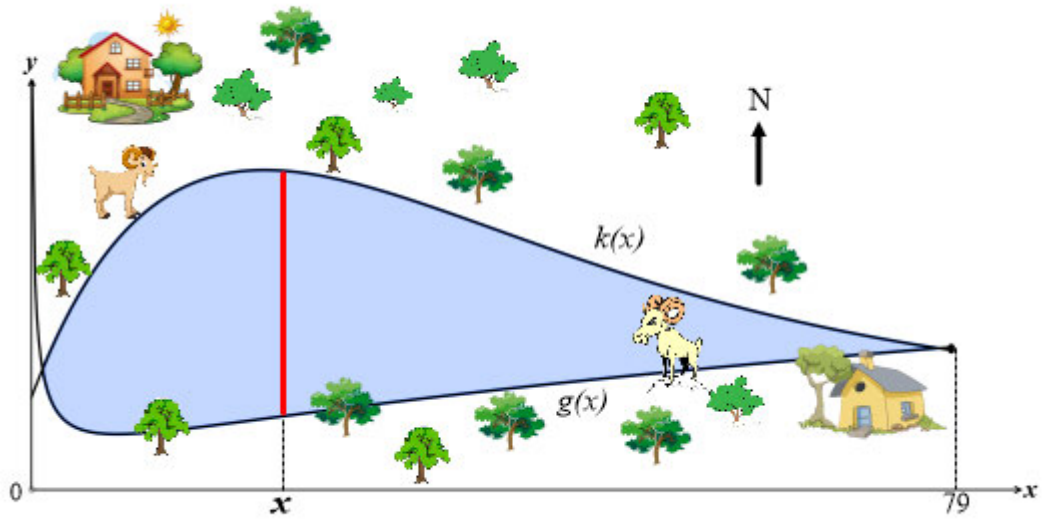
ii $g'(x)$

iii If P $\left(20, \frac{40}{e}+6\right)$ is the most northern point of the lake, find the exact values of m and n.

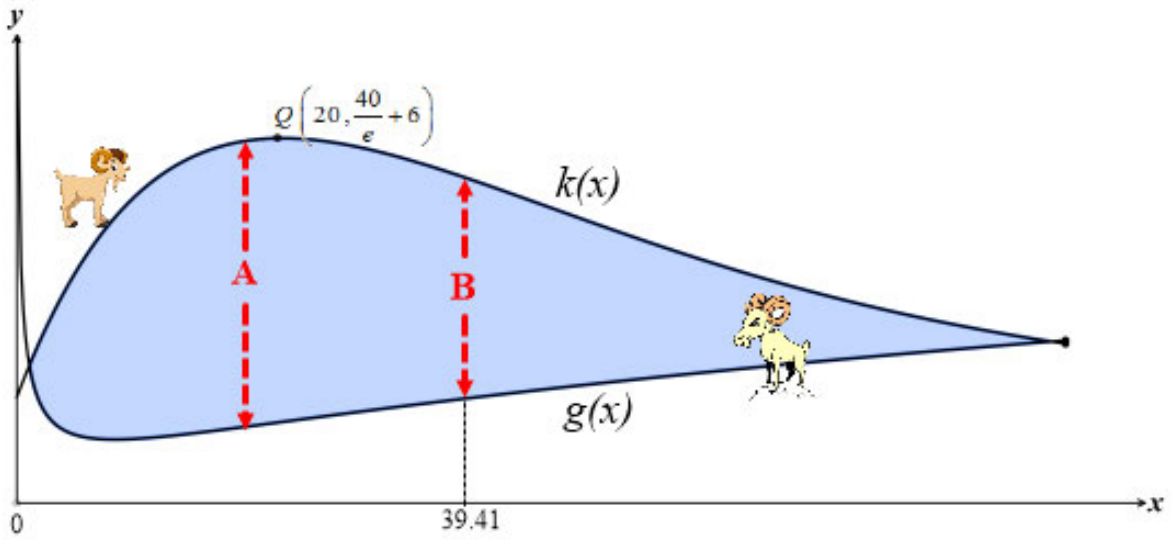
b The vertical distance from one side of the lake bank to the other is represented by w such that $w(x)=k(x)-g(x)$. Using the values of m and n you obtained from **a iii**,

i Define $w(x)$

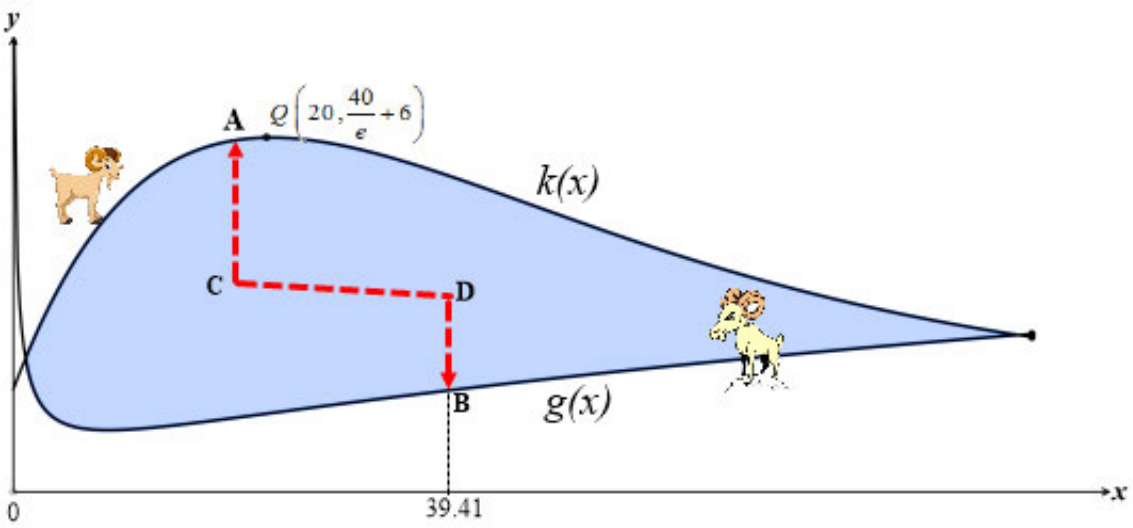
The two friends can't meet because the lake is too wide between the two farms. Gary, the farmer from the North of the lake suggested to Toby the farmer from the South to jointly build the longest possible bridge so that the two friends can meet.



ii Find the value of x where the longest possible bridge A can be built and the length in metres of the bridge. Give your answer to **two** decimal places.



iii Toby had a different opinion. He thinks that the bridge should be built in the middle of the lake, where $x = 39.41$ m. Find the length of bridge B, the bridge that Toby suggested. Give your answer to **two** decimal places.



c Telemachus and Karmenia were not happy with either recommendation. To compromise, the farmers decided to build a bridge half the length of Gary's initial suggestion, from A to C and half the length of Toby's initial suggestion, from B to D and another section joining these midpoints, from C to D.

i Find the coordinates of the mid-points C and D. Give your answer to **two** decimal places.

ii Hence find the length in metres of the section from C to D to **one** decimal place.

(Marks: (2 + 2 + 2) + (1 + 2 + 1) + (2 + 1) = 13)

END OF SAC 1c