Name: ______________________________

Teacher: ____________________________

School No: _____________

Scotch College

Department of Mathematics

2018

Mathematical Methods (CAS)

<u>Unit 3 SAC 1a – Project Component</u>

Date:	**Monday 21 May 2018**
Due date:	**Tuesday 5 June 2018**
Marks:	**65**

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this task are **not** drawn to scale.

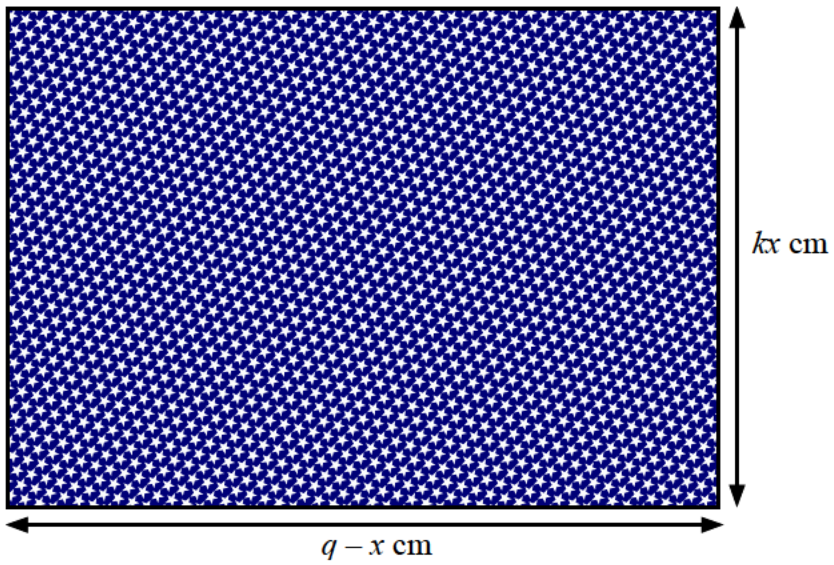
THEME: Maxima and minima

Your project centres around the determination of maximum and minimum points for some given problems or set of functions. You should use mathematics appropriate to the content of Mathematical Methods. Whilst starting points are prescribed, there is room for student initiative, independence and originality is encouraged.

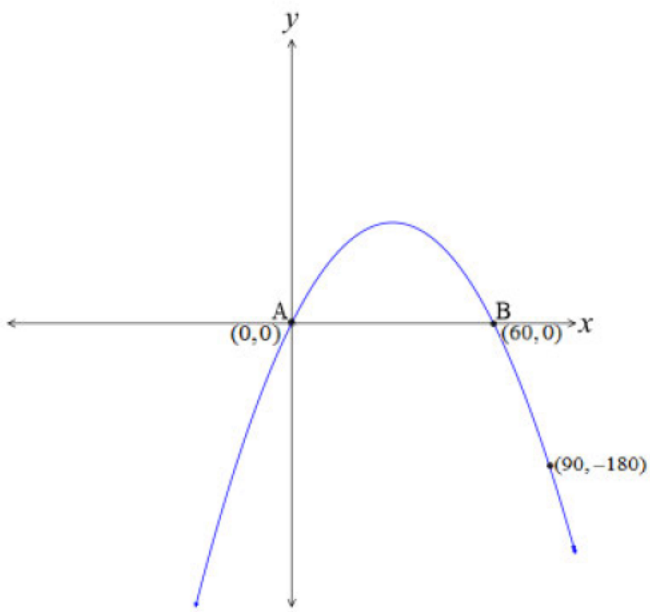
Section A. Wrapping paper

1 In the land of Shtam, everything is calculated accurately and precisely. People on this island try to economise and minimise expenditure and wastage. Their gifts are thoughtfully selected and are placed in appropriate containers and wrapped carefully without any wasted material.

Firstly, consider a particular situation constructing a box in the shape of an open topped rectangular prism from a thick rectangular cardboard sheet of length $q-x$ *cm* and width kx *cm*.



a An example of the graph of the function $f:\mathbb{R}\rightarrow\mathbb{R}, f(x)=kx(q-x)$ is shown below, where k and q are positive, real constants.



i From the information provided on the graph above find the values of k and q, in this case and hence state the equation of $f(x)$.

ii Using differentiation, find the coordinates of the stationary point.

b Now considering the function $A:(a,b) \to \mathbb{R}, A(x) = kx(q-x)$, where $A(x)$ is the rule representing the area of the cardboard sheet.

i State a suitable domain for $A(x)$ in terms of k and q (if appropriate).

ii Find $A'(x)$ in terms of k and q.

iii Solve $A'(x)=0$ in terms of k and q (if appropriate).

iv What are the dimensions of the cardboard sheet for the maximum area?

v State the range of A in terms of k and q.

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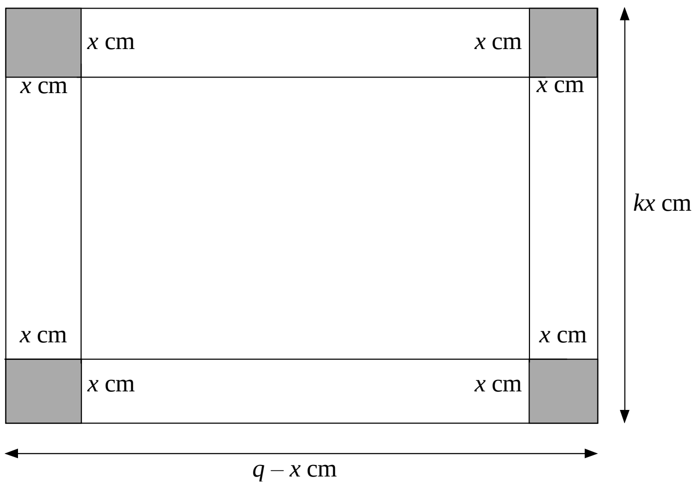
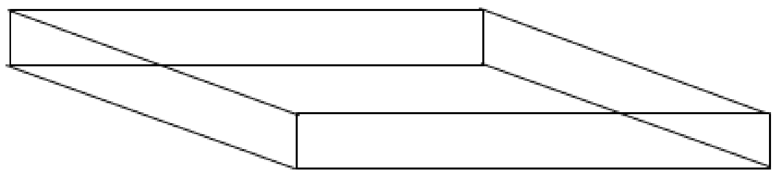


Four squares of side length *x* are cut from each corner of the cardboard and an open box is formed as shown.



i Find the volume of the box in terms of *x*, *k* and *q*.

ii Explain why $k > 2$.

iii Find the possible values of x for a maximum volume in terms of k and q.

iv Sketch the graph of volume against x.

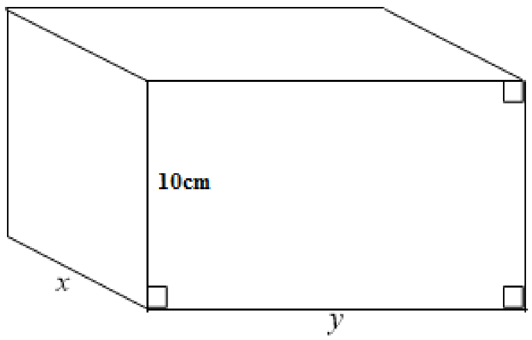
v Find the external surface area of the box in terms of x, k and q.

(Marks: (2 + 2) + (1 + 1 + 1 + 2 + 1) + (1 + 1 + 3 + 2 + 1) = 18)

2 There are many situations in which we need to minimise the surface area of a solid. The area concerned may be the whole or part of a surface. At times we may be concerned with the area of material required to “wrap” a solid. In this problem you will need to assume that the solid under consideration has a **fixed** volume. You will examine the relationships between the volume, total surface area and dimensions of solids to determine the conditions necessary for a solid, or its wrapping, to have a minimum surface area.

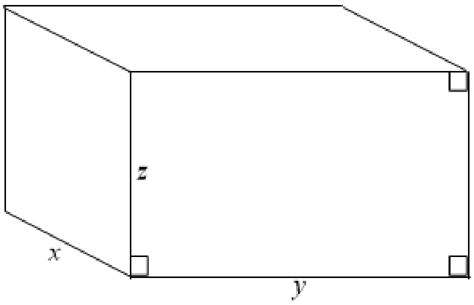
a Consider a closed rectangular box with fixed volume 200cm^3 and dimensions x, y, z.

i To begin with, assume that the value of $z = 10\text{cm}$ is fixed while x and y may vary. Show that the surface area of the box is a minimum when $x = y$.



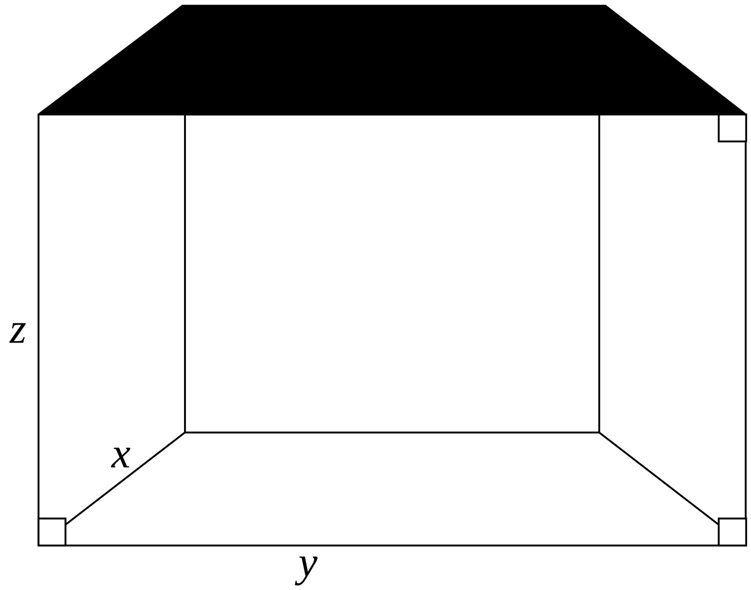
ii How can you be sure your answer gives a minimum?

iii Now show mathematically that if the volume is a fixed pronumeral *V* and one of the dimensions is a fixed pronumeral *z*, then the surface area of the open box is minimum when $x = y$.



iv Now let the value of *z* vary as well, keeping $x = y$. Determine the shape of the open box with the minimum surface area.

b Consider an open box of fixed volume and dimensions x, y, z. The top of the rectangular box being removed so that it has five sides instead of six.



i Consider this rectangular box of fixed volume $200\ \text{cm}^3$ and dimensions x, y, z. Where $z = 10\ cm$ while x and y may vary. What do you notice about the shape of this open box with minimum surface area?

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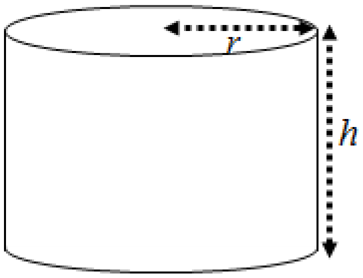
ii Now more generally, if the volume is a fixed pronumeral V, and one of the dimensions is a fixed pronumeral z, then find the shape of this open box for a minimum surface area, showing all working.

c Now consider a closed **cylindrical** box with a fixed volume and dimensions r, h.

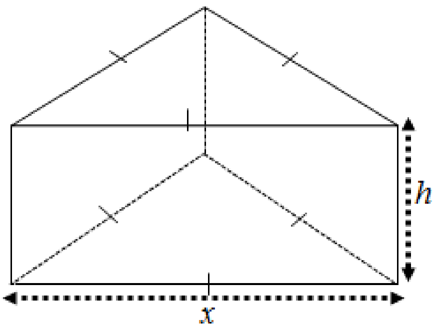
Both r and h vary.



i Find the expression for r and h in terms of V for a minimum surface area, and hence find a relationship between r and h.

ii Now suppose that the top of the cylindrical box is removed. In this case, would there be any difference to the dimensions of the open cylindrical container for a minimum surface area? Support your answer with mathematical working.

d Now consider a closed **triangular** box with an equilateral cross-section.



i Find the relationship between the dimensions of the triangular box for a minimum surface area.

ii Now suppose that the top of the triangular box is removed. In this case, would there be any difference to the dimensions of the open triangular container for a minimum surface area?

(Marks: (2 + 2 + 5 + 4) + (4 + 5) + (4+ 2) + (4 + 3) = 35)

Section B: The Hills

A Mathematics teacher in Shtam asked their students to determine a hybrid function to represent the outline of the two hills in the land. To assist the students they defined the curve of the first hill as the function f and also provided some further information for g , the function for the second hill.

$$h(x)=\begin{cases} f:[0,3]\to\mathbb{R}, & f(x)=x^2e^{3-x} \\ g:(3,m)\to\mathbb{R}, & g(x)=ax^3+bx^2+cx+d \end{cases}$$

The teacher provided some further information in order to assist the students to determine the second function g.

- The two curves are connected **smoothly** at the point $(3,9)$
- The maximum point of g is at $(6,16)$
- Each unit on the graph represents 300 metres.
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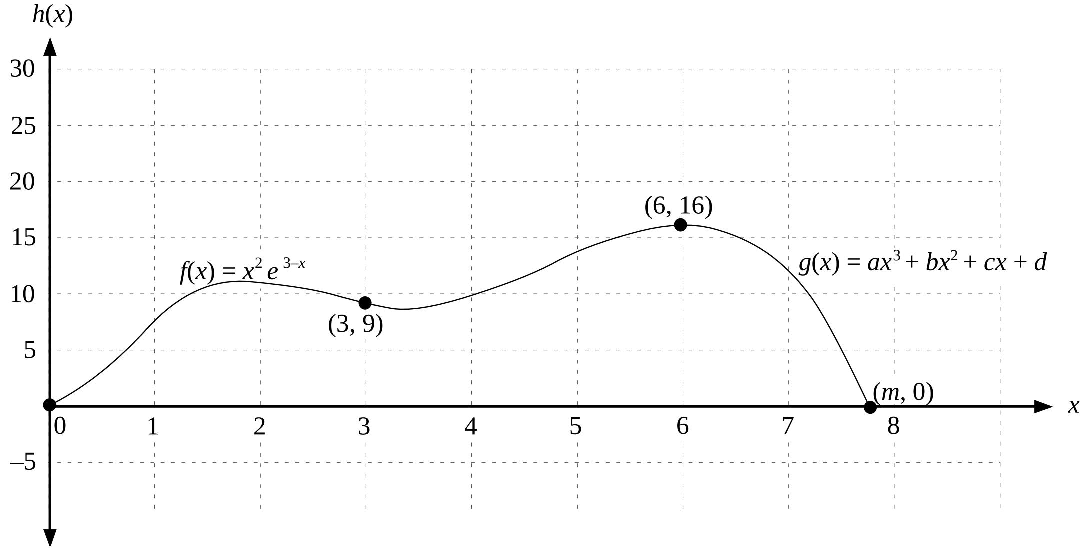

a From the information provided above, find the equation of the function g and the value of m , correct to one decimal place.

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At the point where $x = \frac{11}{2}$, there is a beacon light. It projects a beam towards the first hill.

b Find the equation of a beam of light from $x = \frac{11}{2}$ which is a tangent to the second hill.

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c Find the equation of a beam of light from $x = \frac{11}{2}$ which reaches the maximum point of the first hill.

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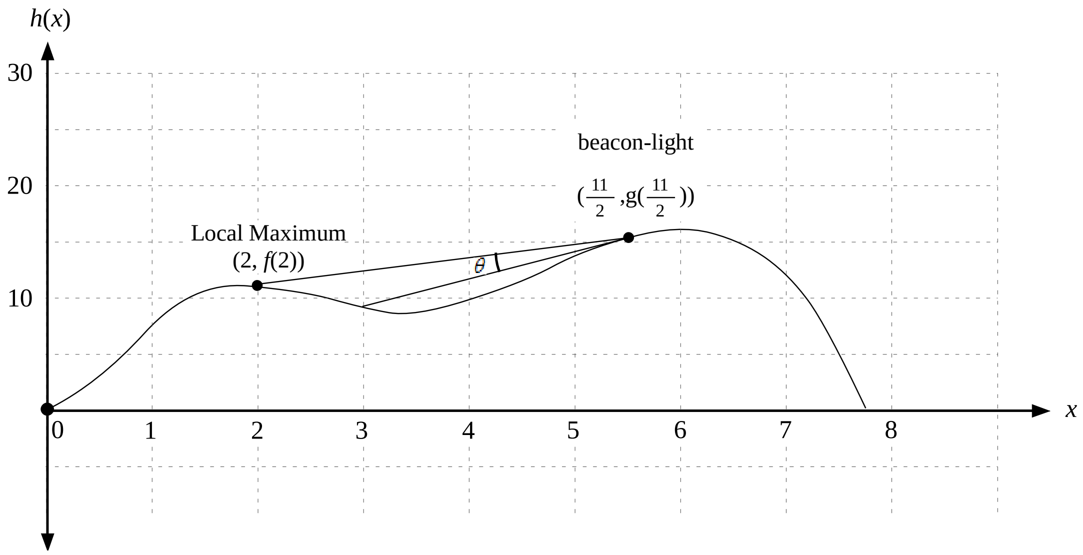

d The first hill is obstructing the projection of the beacon light. This obstruction is measured by angle θ as shown in the above diagram. Find to the nearest degree the value of the angle of obstruction.

(Marks: 3 + 3 + 3 + 3 = 12)

END OF SAC 1a