2021 SAC-1a

Tuesday, May 18, 2021 6:33 PM

Solutions

	Scotch Student ID #			
Circle the relevant digits				
	0	0	0	0
	1	1	1	1
	2	2	2	2
	3	3	3	3
	4	4	4	4
	5	5	5	5
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	7	7	7	7
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	9	9	9	9

Teacher's Name

Scotch College

MATHEMATICAL METHODS

U3-SAC 1a – Application Task: Project

Date of distribution: Wednesday 19th May 2021

Due date: Wednesday 2nd June 2021, prior to SAC 1b

Task Sections	Marks	Your Marks
Extended Response Questions	75	
Total Marks	**75**	

Remote Declaration

I declare that any work I have submitted for this VCE assessment is wholly my own, unless properly referenced or authorised for use by my teacher. I have had no assistance from any person in my home nor have I been assisted by, or given assistance to, a boy in my class or cohort unless specifically permitted to do so by my teacher. I have not used the internet or other sources to assist me in my responses unless specifically permitted by my teacher. I acknowledge my work may be reproduced, communicated, compared and archived for the purposes of detecting plagiarism and collusion.

Signature: ____________________

General Instructions

- Answer all questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this task are not drawn to scale.

Allowed Materials

- A scientific calculator and a CAS calculator.
- Any notes or references.

At the end of the task

- Submit the task to your teacher by the due date.

Question 1 (6 marks)

A pair of equations are defined as follows:

$$(k+1)x - ky = 6$$
$$3x + 2ky + 4 = 0$$

where k is a real constant.

For what values of k do the pair of equations have:

a. a unique solution? 3 marks

$y = \frac{k+1}{k}x - \frac{6}{k}$ | $y = \frac{-3}{2k}x - \frac{2}{k}$

For a unique solution

$\frac{k+1}{k} \neq \frac{-3}{2k}$

$2k^2 + 2k \neq -3k$

$2k^2 + 5k \neq 0$

$k(2k+5) \neq 0$

$\therefore k \in R \setminus \{0, -5/2\}$

b. no solutions? 2 marks

$\frac{k+1}{k} = \frac{-3}{2k}$ and $k = 0$ or $k = -5/2$

c. infinitely many solutions? 1 mark

No values of k will make them identical

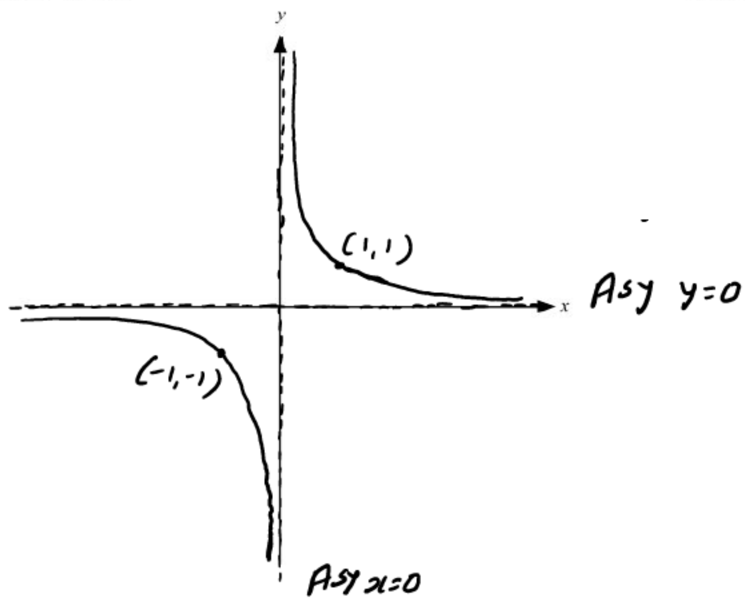
Question 2 (5 marks)

The function f has rule $f(x) = x^{-\frac{1}{3}}$.

a. State the maximal domain for f. 1 mark

$x \in R \setminus \{0\}$

b. Sketch the graph of $y = f(x)$ on the axes below, labelling all intercepts, asymptotes and endpoints, if they exist. 3 marks



c. State the values of x for which the function f is strictly decreasing. 1 mark

$R \setminus \{0\}$

Question 3 (9 marks)

$f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers.

a. The graph of $y = f(x)$ has stationary points at $(-3, 8)$ and $(5, 3)$. Find the rule $f(x)$. 2 marks

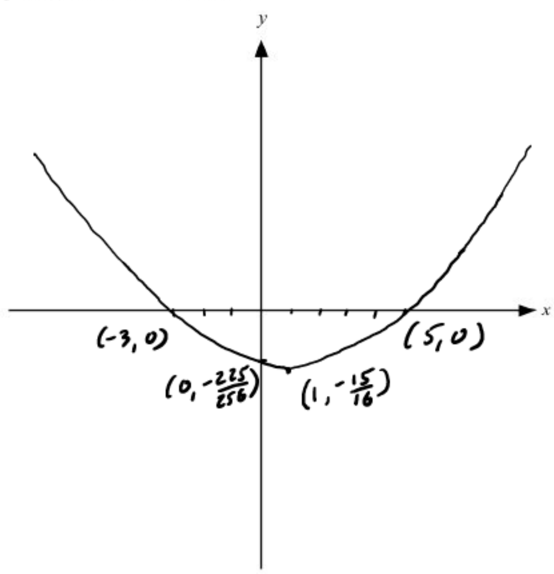
$f(-3) = 8 \quad f'(-3) = 0$

$f(5) = 3 \quad f'(5) = 0$

$a = \frac{5}{256}$, $b = \frac{-15}{256}$, $c = -\frac{225}{256}$, $d = \frac{1643}{256}$

$f(x) = \frac{5}{256}x^3 - \frac{15}{256}x^2 - \frac{225}{256}x + \frac{1643}{256}$

b. Sketch the graph of $y = f'(x)$ on the axes below, labelling all intercepts and the stationary point(s) with their coordinates. 2 marks



c. State the solutions to the equation $f'(x) = 0$. 1 mark

$x = -3$ or $x = 5$ From b

d. Consider another quadratic equation $x^2 + px + q = 0$, where p and q are real numbers and $p > 0$, $q > 0$.

i. Find the value(s) of p, in terms of q, for which this equation has exactly one solution. 1 mark

$\Delta = 0 \quad p^2 - 4q = 0$

$p = \pm 2\sqrt{q}$

$\therefore p = 2\sqrt{q}$ for $p > 0$ & $q > 0$

ii. Find the one solution to the equation in this case, in terms of q. 1 mark

$x^2 + 2\sqrt{q}x + q = 0$

$(x + \sqrt{q})^2 = 0$

$x = -\sqrt{q}$

e. The equation of a curve is $y = x^2 + px + q$, where p and q are real numbers.

i. For what values of p is the x-coordinate of the turning point of the curve positive? 1 mark

$x^2 + px + \left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 + q$

$\left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}$

$\therefore \; -\frac{p}{2} > 0 \qquad \therefore p < 0$

ii. Find the coordinates of the turning point of the curve in terms of p and q. 1 mark

$\left(-\frac{p}{2}, q - \frac{p^2}{4}\right)$ From e-i

Question 4 (13 marks)

Two functions f and g are defined as:

$$f : \mathbb{R} \to \mathbb{R}, f(x) = e^{3x} - 2$$
$$g : \mathbb{R} \to \mathbb{R}, g(x) = e^{x}$$

a. List a sequence of transformations which maps the graph of $y = g(x)$ to the graph of $y = f(x)$. 2 marks

- Dilated by a factor of 1/3 from the y-axis
- then translated by 2-units in the negative direction of y-axis.

b. Find a rule for the inverse, f^{-1}, of f and state its domain. 3 marks

Let $y = f(x)$

$y = e^{3x} - 2$

For inverse swap x and y, $x = e^{3y} - 2$

$x + 2 = e^{3y}$

$3y = \log_e(x+2)$

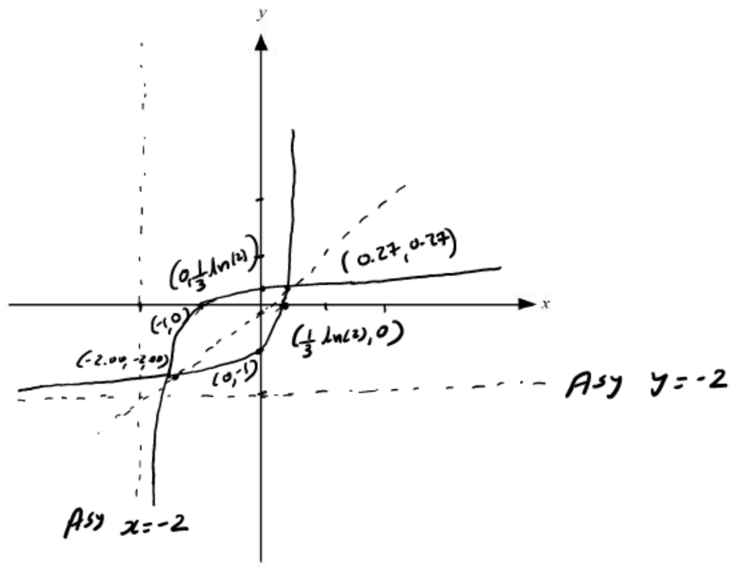
$y = \frac{1}{3}\log_e(x+2)$

$\therefore f^{-1}(x) = \frac{1}{3}\log_e(x+2)$ and $x \in (-2, \infty)$

or

$f^{-1} : (-2, \infty) \to R, f^{-1}(x) = \frac{1}{3}\log_e(x+2)$

c. Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes below, labelling all intercepts with exact coordinates, all asymptotes with their equations and the intersection point(s) between the curves to two decimal places. 4 marks



d. Fully define $h(x) = f\left(f^{-1}(x)\right)$. 2 marks

$h: (-2, \infty) \to R, \; h(x) = x$

e. Show that $f\left(-2f^{-1}(3x)\right) = -\dfrac{18x^2+24x+7}{9x^2+12x+4}$. 2 marks

$f(x) = e^{3x} - 2$

$f^{-1}(x) = \frac{1}{3}\log_e(x+2)$

$f^{-1}(3x) = \frac{1}{3}\log_e(3x+2)$

$-2f^{-1}(3x) = \frac{-2}{3}\log_e(3x+2)$

$f\left(-2f^{-1}(3x)\right) = e^{3\left(\frac{-2}{3}\log_e(3x+2)\right)} - 2$

$= (3x+2)^{-2} - 2$

$= \dfrac{1}{9x^2+12x+4} - 2$

$= \dfrac{1-2(9x^2+12x+4)}{9x^2+12x+4}$

$= -\dfrac{18x^2+24x+7}{9x^2+12x+4}$

Question 5 (17 marks)

Consider the function f such that

$$f : D \to \mathbb{R}, f(x) = \frac{-1}{\sqrt{4 - x^2}}$$

a. Find D, the maximal domain of f. 1 mark

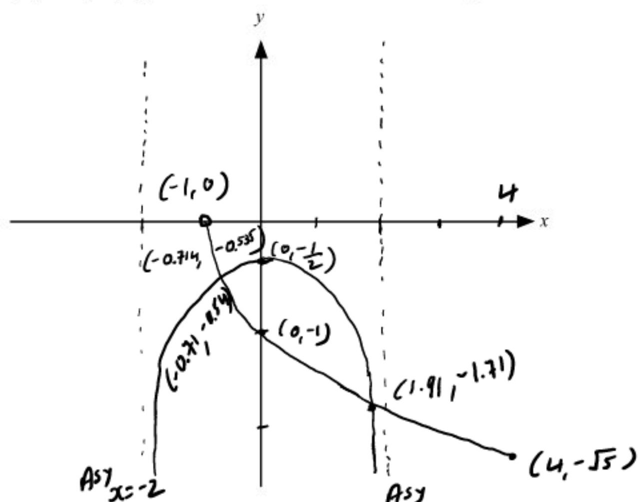
$4 - x^2 > 0$

$\therefore$ Domain $= (-2, 2)$

or

$f : (-2, 2) \to \mathbb{R}, f(x) = \frac{-1}{\sqrt{4-x^2}}$

b. Sketch the graph of $y = f(x)$ over its maximal domain. Label all key features. 3 marks



c. Hence, or otherwise, find the absolute maximum of f. 1 mark

Absolute maximum $= -\frac{1}{2}$

d. Consider the function g such that

$$g : (-1, 4] \to \mathbb{R}, g(x) = -\sqrt{x + 1}$$

On the above axes, sketch $y = g(x)$ including the exact coordinates of all endpoints and intercepts and the coordinates of the point(s) of intersection with $y = f(x)$ correct to two decimal places. 3 marks

e. Show that the function $f(g(x))$ does not exist. 1 mark

Rang $g \notin$ Dom f

$[-\sqrt{5}, 0) \not\subset (-2, 2)$

f. Let $g^*: X \to \mathbb{R}, g^*(x) = -\sqrt{x+1}$ where X is the domain of g^*.

i. Find the maximal domain X such that $f(g^*(x))$ exists. 1 mark

Range of $g^*(x) = (-2, 0)$

$\therefore$ max dom $[-1, 3)$

ii. Find the rule $f(g^*(x))$. 1 mark

$$f(g^*(x)) = \frac{-1}{\sqrt{3-x}}$$

g. Let $p:(0,4)\to\mathbb{R},\ p(x)=\dfrac{2}{\sqrt{x(4-x)}}-1$. The graph of p can be obtained by applying the transformation T to the graph of f where

$$T:\mathbb{R}^2\to\mathbb{R}^2, T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&m\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}q\\n\end{bmatrix}$$

Find the values of m, n and q. 3 marks

$x' = x+q \quad \Rightarrow x = x'-q$

$y' = my+n \quad \Rightarrow y = \dfrac{y'-n}{m}$

$\dfrac{y-n}{m} = \dfrac{-1}{\sqrt{4-(x-2)^2}}$

$y = \dfrac{-m}{\sqrt{4-(x-2)^2}} + n$

$m=-2$

$n=-1$

$q=2$

h. Find the coordinates of the point P on the curve with equation $y=f(x)$ at which the tangent to $f(x)$ is normal to the line $3x+8y=15$, giving your answer correct to two decimal places. 3 marks

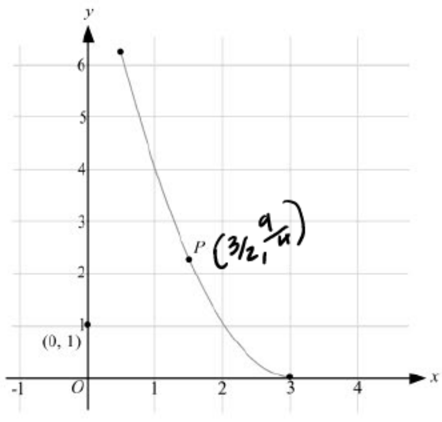
$y = -\dfrac{3}{8}x + \dfrac{15}{8}$

$f'(x) = 8/3$

$(-1.80, -1.14)$

Question 6 (9 marks)

The curve $f(x)=(x-3)^2$ where $x\in\left[\frac{1}{2},3\right]$ is shown below. Point P on $f(x)$ has an x-value of $\frac{3}{2}$.



a. Calculate the distance from $(0,1)$ to Point P. 2 marks

$d=\sqrt{(9/4-1)^2+(3/2-0)^2}$

$=\frac{\sqrt{61}}{4}$

b. Point B lies on $y=f(x)$ and has an x-coordinate of m. State the coordinates of point B in terms of m. 1 mark

$(m,(m-3)^2)$

c. Hence show that the distance $D(m)$ between $(0,1)$ and Point B is given by the function:

$$D(m)=\sqrt{m^2+(m-4)^2(m-2)^2}$$

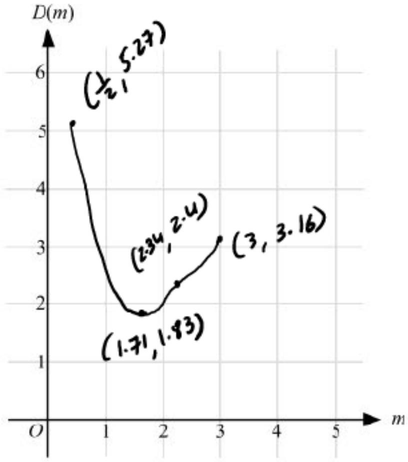
2 marks

$D(m)=\sqrt{((m-3)^2-1)^2+(m-0)^2}$

$=\sqrt{((m-3-1)(m-3+1))^2+m^2}$

$=\sqrt{m^2+(m-4)^2(m-2)^2}$

d. Sketch $D(m) = \sqrt{m^2 + (m-4)^2(m-2)^2}$ for $m \in \left[\frac{1}{2}, 3\right]$ on the axes below. Label the coordinates of all endpoints and stationary points, rounded to two decimal places. 2 marks



e. Find the value(s) of m for which $D(m) = 4$, giving your answer(s) correct to two decimal places. 2 marks

$$4 = \sqrt{m^2 + (m-4)^2(m-2)^2}$$

$$m \approx 0.78$$

Question 7 (16 marks)

A function $N(t)$ measures the intensity level of a treadmill workout t minutes after a workout begins. All workouts begin with the treadmill stationary, at intensity level zero, that is, $N(0) = 0$.

If the treadmill workout reaches a maximum intensity level of 10 (that is, $N(t) = 10$) the treadmill enters the **override** mode. This means it will reduce the intensity level at a constant rate over a 5 minute period until $N(t) = 0$, and the treadmill stops.

$N(t) = \frac{1}{9000}(at-10)^3(b-t)+1$, where $t \geq 0$, $0 \leq N(t) \leq 10$, and a and b are real constants.

a. **i.** Show that $b = 9$. 2 marks

$N(0) = 0 \quad \therefore \; 0 = \frac{1}{9000}(0-10)^3(b-0)+1$

$0 = \frac{-1000b}{9000}+1$

$\frac{b}{9} = 1 \qquad \therefore b = 9$

ii. When $a = 0$, calculate the time taken, in minutes, to achieve the maximum intensity level of 10. 2 marks

$10 = \frac{1}{9000}(0-10)^3(9-t)+1$

$t = 90$ minutes

iii. When $a = 2$, calculate the time taken after the workout has started for the intensity level, $N(t)$, to reduce to zero. State your answer in minutes correct to one decimal place. 2 marks

$\frac{1}{9000}(2t-10)^3(9-t)+1 = 0$

$t \approx 12.1$ minutes

iv. State, in terms of a, the possible value(s) of t for which $(t, N(t))$ is a stationary point of the function $N(t)$. 2 marks

$N'(t) = 0$

$t = \frac{10}{a}$ or $t = \frac{27a+10}{4a}$, $a \neq 0$

v. For what value(s) of a, $a \in [0, 6)$ does $N(t)$ have **no** stationary points? 1 mark

$a \in [0, 0.0124]$

vi. For what value(s) of a, $a \in [0, 6)$ does $N(t)$ have **one** stationary point? 2 marks

$\frac{10}{a} = \frac{27a+10}{4a}$

$\therefore a = \frac{10}{9}$

vii. What is the maximum number of stationary points $N(t)$ can have? 1 mark

2

b. **i.** What is the maximum intensity level achieved when $a = 6$ correct to one decimal place. 1 mark

$N(t) = \frac{1}{9000}(6t-10)^3(9-t)+1$

$N'(t)=0$ $t=1.\dot{6}$ or $t=7.1\dot{6}$

$N(1.\dot{6})=1$ $N(7.1\dot{6})=8.3$ max

ii. When $a = 6$, at what time is this maximum intensity level achieved? Give your answer in minutes, correct to one decimal place. 1 mark

$t=7.2$ mins

c. a is restricted to the possible set of values below:

$$a \in \{1,2,3,4,5,6,7,8,9,10\}.$$

What is the least value of $a \in \{1,2,3,4,5,6,7,8,9,10\}$, which will cause the treadmill to enter the **override** mode, and how many minutes in total will this workout last? State your answer in minutes, correct to one decimal place. 2 marks

$10 = \frac{1}{9000}(at-10)^3(9-t)+1$

$a=7$, $t=5.5+5$

$\therefore$ $t=10.5$ minutes

END OF SAC 1a