	Scotch Student ID #			
Circle the relevant digits				
	0	0	0	0
	1	1	1	1
	2	2	2	2
	3	3	3	3
	4	4	4	4
	5	5	5	5
	6	6	6	6
	7	7	7	7
	8	8	8	8
	9	9	9	9

Teacher's Name

Scotch College

MATHEMATICAL METHODS

WORKED SOLUTIONS

U3-SAC 1a – Application Task: Project

Date of distribution: Monday 23rd May 2022

Due date: Thursday 2nd June 2022, prior to SAC 1b

Task Sections	Marks	Your Marks
Extended Response Questions	60	
Total Marks	**60**	

Remote Declaration
I declare that any work I have submitted for this VCE assessment is wholly my own, unless properly referenced or authorised for use by my teacher. I have had no assistance from any person in my home nor have I been assisted by, or given assistance to, a boy in my class or cohort unless specifically permitted to do so by my teacher. I have not used the internet or other sources to assist me in my responses unless specifically permitted by my teacher. I acknowledge my work may be reproduced, communicated, compared and archived for the purposes of detecting plagiarism and collusion.
Signature: ______________________

General Instructions

- Answer all questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this task are not drawn to scale.

Allowed Materials

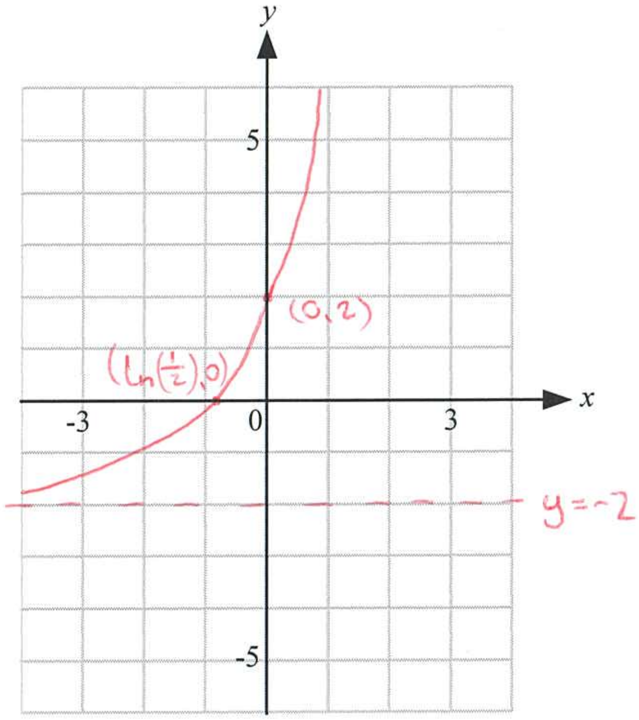
- A scientific calculator and a CAS calculator.
- Any notes or references.

At the end of the task

- Submit the task to your teacher by the due date.

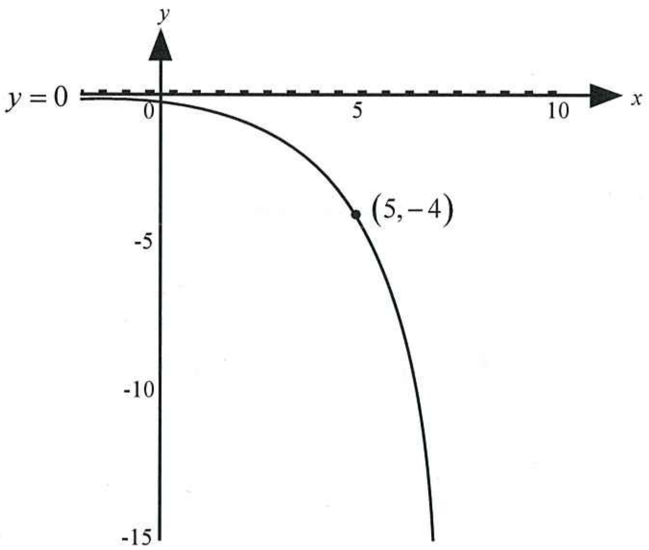
Question 1 (3 marks)

Sketch the graph of $y = 4e^x - 2$. Label all axis intercepts with their co-ordinate points and all asymptotes with their equations.



Question 2 (15 marks)

a. **i.** The graph below has rule $f(x) = -2^{x+b}$, where $b \in \mathbb{R}$. Find the value of b. 1 mark

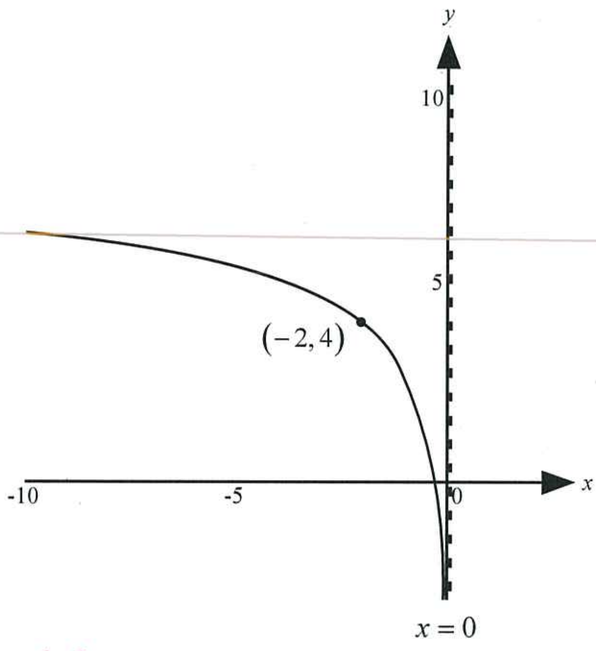


$-4 = -2^{(5+b)}$

$\therefore\ 2 = 5+b$

$b = -3$

ii. The graph below has rule $g(x) = \log_2(-x) + m$, where $m \in \mathbb{R}$. Find the value of m. 1 mark



$4 = \log_2(2) + m$

$m = 3$

iii. Show that $g(x) = f^{-1}(x)$. 1 mark

Let $y = f(x)$. For inverse, swap x and y:

$x = -2^{y-3}$

$-x = 2^{y-3}$

$\log_2(-x) = y - 3$

$\therefore f^{-1}(x) = \log_2(-x) + 3$

$= g(x)$

b. The curve given by rule $h(x) = a \times 2^x - b$ passes through the points $(1,5)$ and $(2,11)$.

i. Find the rule $h(x)$. 2 marks

$5 = 2a - b$ ①

$11 = 4a - b$ ②

② − ① $6 = 2a$

$a = 3$

sub into ① $5 = 6 - b$

$b = 1$

ii. Find the rule $h^{-1}(x)$. 2 marks

Let $y = 3 \times 2^x - 1$

For inverse, swap x and y.

$x = 3 \times 2^y - 1$

$\frac{x+1}{3} = 2^y$

$y = \log_2\left(\frac{x+1}{3}\right)$

$\therefore h^{-1}(x) = \log_2\left(\frac{x+1}{3}\right)$

c. **i.** Fully define p^{-1}, the inverse of $p:[-4,6) \to \mathbb{R}$, $p(x) = 2\log_{10}(6-x)+1$. State the range of p^{-1}. 4 marks

Let $y = 2\log_{10}(6-x) + 1$

For inverse, swap x and y

$x = 2\log_{10}(6-y) + 1$

$\frac{x-1}{2} = \log_{10}(6-y)$

$6 - y = 10^{\left(\frac{x-1}{2}\right)}$

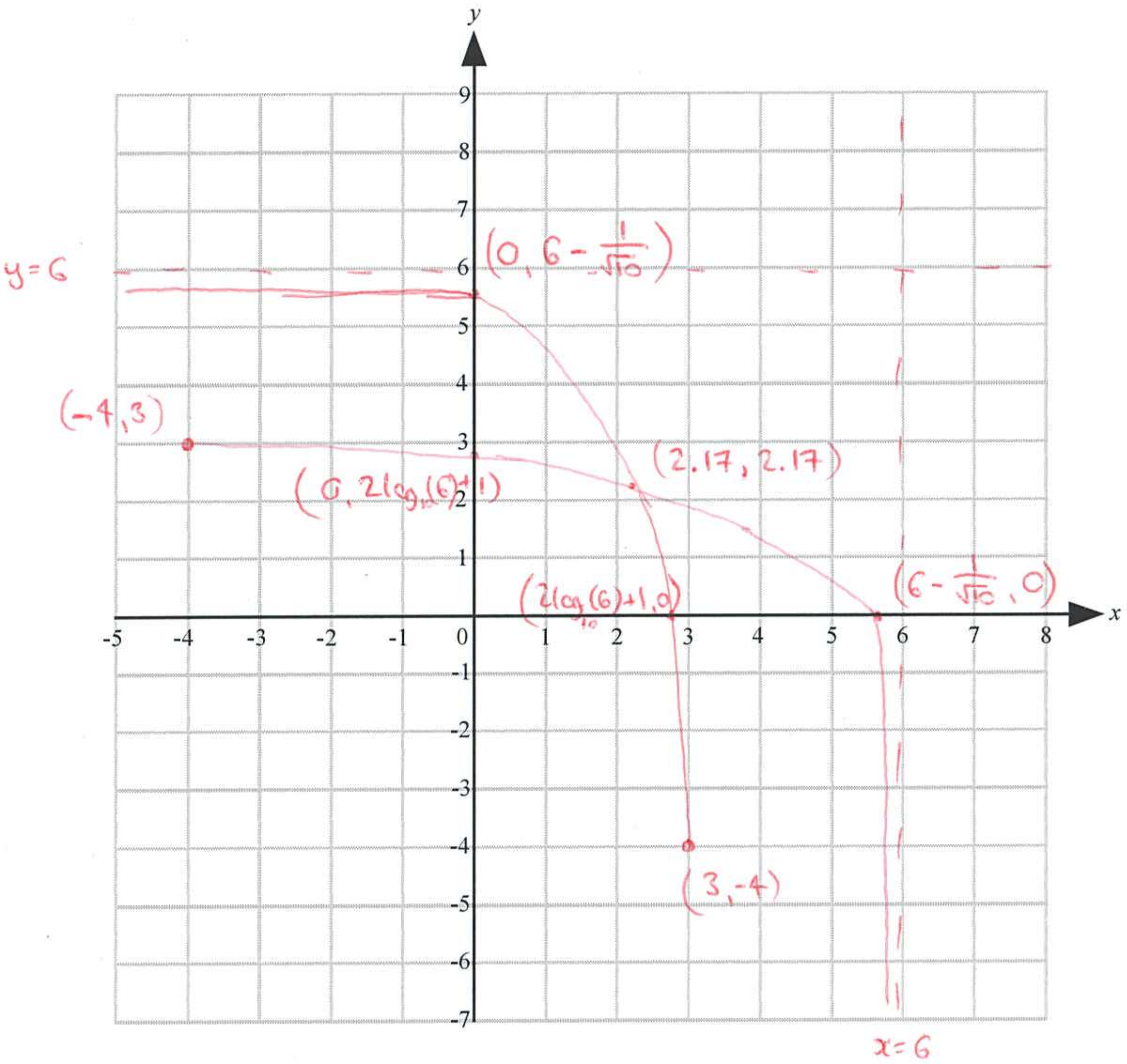
$y = 6 - 10^{\left(\frac{x-1}{2}\right)}$

dom p^{-1} = ran g = $(-\infty, 3]$

$p^{-1}: (-\infty, 3] \to \mathbb{R}$, $p^{-1}(x) = 6 - 10^{\left(\frac{x-1}{2}\right)}$

ran p^{-1} = dom p = $[-4, 6)$

ii. Sketch the graph of $y = p(x)$ and $y = p^{-1}(x)$ on the same set of axes. Label all axis intercepts, endpoints and points of intersection with their co-ordinates. Points of intersection co-ordinates to be given correct to two decimal places. Label all asymptotes with their equations. 4 marks



Question 3 (5 marks)

Consider the function $f(x) = e^{\sqrt{x}}$.

a. Find $f'(x)$. 1 mark

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} e^{\sqrt{x}}$

b. For the graph of $y = f(x)$, find, in terms of a, the equation of the tangent to the graph at the point where $x = a$. 2 marks

$f'(a) = \frac{1}{2\sqrt{a}} e^{\sqrt{a}}$

$\therefore \quad y - e^{\sqrt{a}} = \frac{1}{2\sqrt{a}} e^{\sqrt{a}} (x - a)$

$y = \frac{e^{\sqrt{a}}}{2\sqrt{a}} x + e^{\sqrt{a}} - \frac{a e^{\sqrt{a}}}{2\sqrt{a}}$

c. Find the value of a such that the tangent to the graph of $y = f(x)$ at $x = a$ passes through the origin. 1 mark

$0 = e^{\sqrt{a}} - \frac{a e^{\sqrt{a}}}{2\sqrt{a}}$

$\therefore \quad a = 4$

d. Using the value of a found in **part c**, find the equation of the normal to the graph of $y = f(x)$ at $x = a$. 1 mark

$y = -\frac{4}{e^2} x + \frac{16}{e^2} + e^2$

Question 4 (10 marks)

a. The graphs of $y=-(x-2)^2$ and $y=x^2-k$ are to join smoothly (that is, their gradients are equal at the point of intersection). Find the value of k and the co-ordinates of their point of intersection. 3 marks

① $-(x-2)^2 = x^2-k$

② $-2(x-2) = 2x$

② ⇒ $x=1$

sub into ① $-1 = 1-k$

$k=2$

∴ intersection at $(1,-1)$

b. **i.** The graph of the equation $y=\frac{1}{10}(x-3)(x+2)(x-6)$, where $x\in[-2,8]$, is to join smoothly to the graph of a certain parabola, where $x\in[8,16]$, at the point $(8,10)$. The parabola also passes through the point $(12,26)$. Find the equation of the parabola. 3 marks

$y = ax^2+bx+c$

$dy/dx = 2ax+b$

∴ $10 = 64a+8b+c$

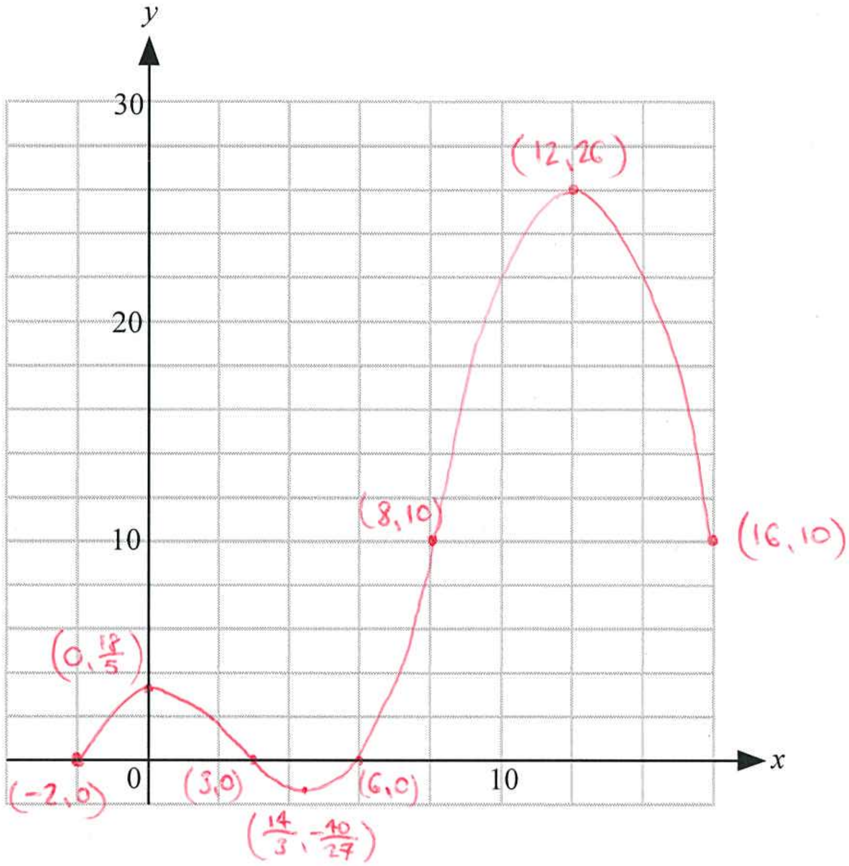
$26 = 144a+12b+c$

$8 = 16a+b$

$a=-1$ $b=24$ $c=-118$

∴ $y=-x^2+24x-118$

ii. Sketch the cubic and parabola from **part i** on the same set of axes below. Label all stationary points, intercepts, points of intersection and endpoints with their co-ordinates. 4 marks



Question 5 (5 marks)

Consider the functions $f:\mathbb{R}\to\mathbb{R}, f(x)=e^{\frac{7}{2}x^2-x-1}$ and $g(x)=\begin{cases} e^{\frac{7}{2}x^2-x-1} & ,-\infty < x \le a \\ \log_e(x-k)+2 & ,a < x < \infty \end{cases}$,

where $a,k\in\mathbb{R}$ and $a>k$.

a. Solve $f'(x)=0$. 2 marks

$f'(x)=(7x-1)e^{\frac{7}{2}x^2-x-1}$

$f'(x)=0$

$x=\frac{1}{7}$

b. Hence state the largest interval over which f is strictly increasing. 1 mark

$[\frac{1}{7},\infty)$

c. Find the values of a and k (both correct to two decimal places) such that $g'(x)$ exists for all $x\in\mathbb{R}$. 2 marks

$e^{\frac{7}{2}a^2-a-1} = \log_e(a-k)+2$

$(7a-1)e^{\frac{7}{2}a^2-a-1} = \frac{1}{a-k}$

$\therefore$ $a=0.66$

$k=0.34$

Question 6 (8 marks)

a. Find the common tangent for the graphs with equation $y = e^x - 1$ and $y = \log_e(x+1)$. 2 marks

$e^x - 1 = \log_e(x+1)$

$\therefore x = 0$

check: $dy/dx = e^x = 1$ at $x=0$ $dy/dx = \frac{1}{x+1} = \frac{1}{1} = 1$ at $x=0$

$\therefore$ $y = x$ is common tangent

b. The graphs with equations $y = e^{x+2} - 1$ and $y = \log_e(x+h) + k$ intersect at a single point and have a common tangent with gradient 1 at that point.

i. Find the values of h and k. 2 marks

$y = e^{x+2} - 1$ is a translation 2 units left from $y = e^x + 1$.

$\therefore$ $h = 3$

$k = 0$

ii. Find the equation of the common tangent. 1 mark

$y = x + 2$

c. The graphs with equations $y = e^{x+p} + q$ and $y = \log_e(x+2) + 2$ intersect at a single point and have a common tangent with gradient 1 at that point.

i. Find the values of p and q. 2 marks

Translating $y = \log_e(x+1)$ to $y = \log_e(x+2) + 2$

is a translation 1 unit left and 2 units up.

Apply transformations to $y = e^x - 1$

$\therefore$ $p = 1$

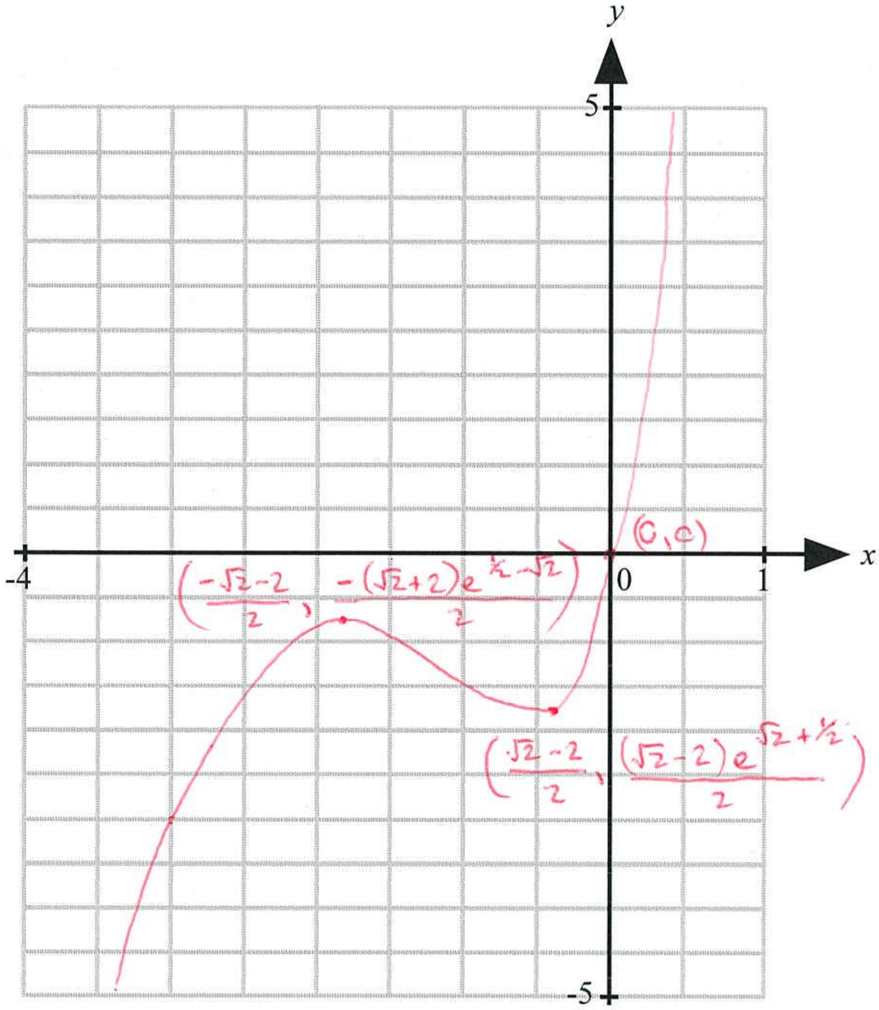
$q = 1$

ii. Find the equation of the common tangent. 1 mark

$y = x + 3$

Question 7 (7 marks)

a. Sketch the graph of $y = xe^{x^2+4x+3}$ on the axes below, labelling all intercepts and stationary points with their co-ordinates. 3 marks



b. Consider the family of graphs represented by equations of the form $y = xe^{ax^2+bx+c}$, where $a \neq 0$.

i. Find the relationship of b in terms of a for the graph of $y = xe^{ax^2+bx+c}$ to have one stationary point. 2 marks

$\frac{dy}{dx} = x(2ax+b)e^{ax^2+bx+c} + e^{ax^2+bx+c}$

$= (2ax^2+bx+1)e^{ax^2+bx+c}$

As $e^{ax^2+bx+c} > 0$, solving $\frac{dy}{dx} = 0$ gives

$2ax^2 + bx + 1 = 0$

$\Delta = b^2 - (2a) \times 4 = 0$

$b = \pm 2\sqrt{2a}$

ii. Given the graph of $y = xe^{ax^2+bx+c}$ has one stationary point at $\left(1, e^{\frac{1}{2}}\right)$, find the values of a, b and c. 2 marks

$e^{\frac{1}{2}} = e^{a+b+c}$

$(2a+b+1) = 0$

$b = 2\sqrt{2a}$

$-2a-1 = 2\sqrt{2a}$

No solutions

$b = -2\sqrt{2a}$

$\therefore a = \frac{1}{2}$

$b = -2$

$c = 2$

Question 8 (7 marks)

The number of virions in a person's body t days after becoming infected is modelled by

$$f:[0,\infty)\to\mathbb{R}, f(t)=3t\times10^8\times e^{\frac{-t^2+3}{3}}$$

a. Calculate the number of virions 1 day after infection. Give your answer correct to three significant figures. 1 mark

5.84×10^8

b. If the number of virions is greater or equal to 5×10^8, a person has a heightened risk of being admitted to hospital.

i. Find how long after becoming infected a person will have a heightened risk of being admitted to hospital. Give your answer in days, correct to three decimal places. 2 marks

$f(t)=5\times10^8$

$\therefore t=0.734$ days

ii. For how long does a person experience the heightened risk of hospital admission. Give your answer in days, correct to two decimal places. 1 mark

1.06 days

c. Find the maximum number of virions predicted by the model. Give your answer correct to three significant figures. 1 mark

$f'(t)=0$

$t=\frac{\sqrt{6}}{2}$

$f\left(\frac{\sqrt{6}}{2}\right)=6.06\times10^8$

A new medication is in trials and the number of virions in a person's body t days after infection, $g(t)$, is linked to the amount of the drug administered, k, where $k \in (0,10]$, such that

$$g:[0,\infty) \to \mathbb{R}, g(t) = kt \times 10^8 \times e^{\frac{-t^2+3}{k}}$$

d. Find the value of k for which the peak number of virions is smallest. 2 marks

$g'(t) = 0$

$t = \frac{\sqrt{k}}{2}$

$h(k) = g\left(\frac{\sqrt{k}}{2}\right)$

solve $h'(k) = 0$

$k = 2$

END OF SAC 1a

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET