Scotch Student ID #				
Circle the relevant digits	0	0	0	0
	1	1	1	1
	2	2	2	2
	3	3	3	3
	4	4	4	4
	5	5	5	5
	6	6	6	6
	7	7	7	7
	8	8	8	8
	9	9	9	9

Teacher's Name

Scotch College

MATHEMATICAL METHODS

Unit 3-SAC 1c – Application Task: Test

WORKED SOLUTIONS

Thursday 2nd June 2022

Reading Time	none
Writing Time	45 minutes

Task Sections	Marks	Your Marks
Extended Response Questions	30	
Total Marks	**30**	

Declaration

I declare that any work I have submitted for this VCE assessment is wholly my own, unless properly referenced or authorised for use by my teacher. I have had no assistance from any person in my home nor have I been assisted by, or given assistance to, a boy in my class or cohort unless specifically permitted to do so by my teacher. I have not used the internet or other sources to assist me in my responses unless specifically permitted by my teacher. I acknowledge my work may be reproduced, communicated, compared and archived for the purposes of detecting plagiarism and collusion.

Signature: ________________________________

General Instructions

- Answer all questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this task are not drawn to scale.

Allowed Materials

- Calculators are allowed
- Notes and/or references are not allowed.

At the end of the task

- Ensure you cease writing upon request.

Electronic Devices

Students are **not** allowed to have a mobile phone, smart watch and/or any other unauthorised electronic device in the SAC, unless it is TURNED OFF and is placed on the front teacher desk.

Question 1 (3 marks)

Consider the function $f:\mathbb{R}\to\mathbb{R}, f(x)=4\times e^{(x-2)}+1$.

a. Find $f^{-1}(x)$. 1 mark

Let $y=f(x)$. Swap x and y for inverse.

$x = 4e^{y-2}+1$

$\frac{x-1}{4} = e^{y-2}$

$y = 2+\log_e\left(\frac{x-1}{4}\right)$

$\therefore f^{-1}(x) = 2+\log_e\left(\frac{x-1}{4}\right)$

b. State the domain and range for f^{-1}. 2 marks

dom f^{-1} = ran f = $(1,\infty)$

ran f^{-1} = dom f = $\mathbb{R}$

Question 2 (4 marks)

$$f(x)=\begin{cases} x^2+4x+1, & x \le a \\ -x^2+2x+k, & x > a \end{cases}$$

The function f is continuous and joins smoothly at $x = a$.

a. Find the values of a and k. 3 marks

$2a+4 = -2a+2$ and $a^2+4a+1 = -a^2+2a+k$

$\therefore a = -\frac{1}{2}$

$\therefore k = \frac{1}{2}$

b. Find the co-ordinates of the point at which the joining occurs. 1 mark

$f(-\frac{1}{2}) = -\frac{3}{4}$

So $(-\frac{1}{2}, -\frac{3}{4})$

Question 3 (5 marks)

a. Find, in terms of a, the equation of the tangent to the graph of $y = 2x^2 - 8x + 11$ at the point where $x = a$. 1 mark

$\frac{dy}{dx} = 4x - 8$

$\therefore \quad y - (2a^2 - 8a + 11) = (4a - 8)(x - a)$

$y = (4a - 8)x - 2a^2 + 11$

b. Find the equation of the normal to the graph of $y = 2x^2 - 8x + 11$ at the point $(3,5)$. 1 mark

at $(3,5)$, gradient of tangent $= 4$

$\therefore$ gradient of normal $= -\frac{1}{4}$

$y = -\frac{1}{4}x + \frac{23}{4}$

c. The tangent to the graph of $y = 2x^2 - 8x + 11$ at the point (b,c) is parallel to the normal to the graph of $y = 2x^2 - 8x + 11$ at the point $(1,5)$. Find the values b and c. 3 marks

Normal at $(1,5)$ has gradient $m = \frac{1}{4}$

$\therefore$ tangent at (b,c) has gradient $m = \frac{1}{4}$

$\therefore \quad 4b - 8 = \frac{1}{4}$

$b = \frac{33}{16}$

$c = \frac{385}{128}$

Question 4 (11 marks)

The concentration (given in mL/L) of a particular medication in a person's bloodstream

t minutes after it is taken is modelled by $f:[0,240]\to\mathbb{R}, f(t)=\dfrac{3^6\log_{10}(t+1)}{(t+1)e^3}$.

a. Calculate the concentration of medicine in the person's blood stream 1 minute after the medicine is taken, giving your answer in mL/L correct to two decimal places. 1 mark

5.46 mL/L

The medicine is effective whilst the concentration in the bloodstream is greater than 1mL/L. The concentration of medicine in the blood stream is considered toxic when greater than 15mL/L.

b. Find the maximum concentration of medicine in the bloodstream in mL/L, and the time at which this occurs in minutes. Give both answers correct to one decimal place. 2 marks

Max = 5.8 mL/L

which occurs at $t = 1.7$ mins.

c. For how long is the medicine effective? Give your answer in minutes correct to two decimal places. 2 marks

$f(t) = 1$

$t = 0.0702\ldots$ and $t = 65.0524\ldots$

$\therefore$ effective for 64.98 mins

In an attempt to increase the effective period of the medicine to be greater than 3 hours, the chemicals in the medicine are changed such that the concentration in the bloodstream t minutes after the medicine is taken is now modelled by $g:[0,240] \to \mathbb{R}, g(t) = \dfrac{4^6 \log_{10}(t+1)}{(t+1)e^4}$.

d. **i.** Does the model predict that the concentration of medicine in the blood stream will become toxic? Justify your answer. 1 mark

Maximum value $\approx$ 12 mL/L, so it does not become toxic.

ii. Does the model predict an effective period of more than 3 hours? Justify your answer. 2 marks

$g(t) = 1$

$t = 0.0321\ldots$ or $t = 165.689\ldots$

$\therefore$ effective period $\approx$ 165.7 mins, so NOT longer than 3 hours.

It is found that by adjusting the amount of Chemical K (k) in the medicine, the concentration in the bloodstream t minutes after the medicine is taken can be modelled by

$h:[0,240] \to \mathbb{R}, h(t) = \dfrac{k^6 \log_{10}(t+1)}{(t+1)e^k}$, where $k \in [2,10]$.

e. For what value of k does the concentration of the medicine in the bloodstream reach its maximum value? 2 marks

$h'(t) = 0$

$t = e - 1$

Solve $\frac{d}{dk}(h(e-1)) = 0$

$k = 6$ as $k \in [2,10]$

f. For what values of k does the concentration of the medicine become toxic? Give your answer correct to two decimal places. 1 mark

$h(e-1) = 15$

$k \in (4.55, 7.72)$

Note: (,) as concentration must be greater than 15.

Question 5 (7 marks)

a. The graphs of the equations $y_1 = 2e^{x-3} - 3$ and $y_2 = e^{2(x-3)} - 2$ intersect at a single point and have a common tangent at that point. Find the equation of the common tangent at that point. 3 marks

$\frac{dy_1}{dx} = \frac{dy_2}{dx}$

$2e^{x-3} = 2e^{2(x-3)}$

$x-3 = 2x-6$

$3 = x$

at $x=3$, $y_1 = -1$ and $m = 2$

$y-(-1) = 2(x-3)$

$y = 2x-7$

b. Consider the graphs of the equations $y_1 = ae^{(x+1)} - k$ and $y_2 = e^{a(x+1)} - p$ where $k, p \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0,1\}$. Given that the graphs intersect at a single point and have a common tangent at that point,

i. find p in terms of a and k so that a common tangent exists 2 marks

$\frac{dy_1}{dx} = \frac{dy_2}{dx}$

$\therefore x = -1$

at $x=-1$, $y_1 = a-k$ and $y_2 = 1-p$

$\therefore a-k = 1-p$

$p = 1+k-a$

ii. find the equation of the common tangent in terms of a and k 1 mark

at $x=-1$, $y_1 = a-k$ and $m = a$

$y-(a-k) = a(x--1)$

$y = ax+2a-k$

iii. given that the graphs only intersect at $(-1,1)$ and have a common tangent with equation $y = 3x + 4$ at this point, find the values of a, k and p. 1 mark

at $x = -1$, $y = 1$ and $m = 3$

$\therefore$ $a = 3$

$k = 2$

$p = 0$

END OF SAC 1c

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$	$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$	$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$	

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET