Scotch Student ID #			
0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9

Circle the relevant digits

Teacher's Name

Scotch College

MATHEMATICAL METHODS

Unit 3-SAC 1b – Application Task: Test

WORKED SOLUTIONS

Thursday 2nd June 2022

Reading Time	none
Writing Time	45 minutes

Task Sections	Marks	Your Marks
Extended Response Questions	30	
Total Marks	**30**	

Declaration

I declare that any work I have submitted for this VCE assessment is wholly my own, unless properly referenced or authorised for use by my teacher. I have had no assistance from any person in my home nor have I been assisted by, or given assistance to, a boy in my class or cohort unless specifically permitted to do so by my teacher. I have not used the internet or other sources to assist me in my responses unless specifically permitted by my teacher. I acknowledge my work may be reproduced, communicated, compared and archived for the purposes of detecting plagiarism and collusion.

Signature: ______________________________

General Instructions

- Answer all questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this task are not drawn to scale.

Allowed Materials

- Calculators are not allowed
- Notes and/or references are not allowed.

At the end of the task

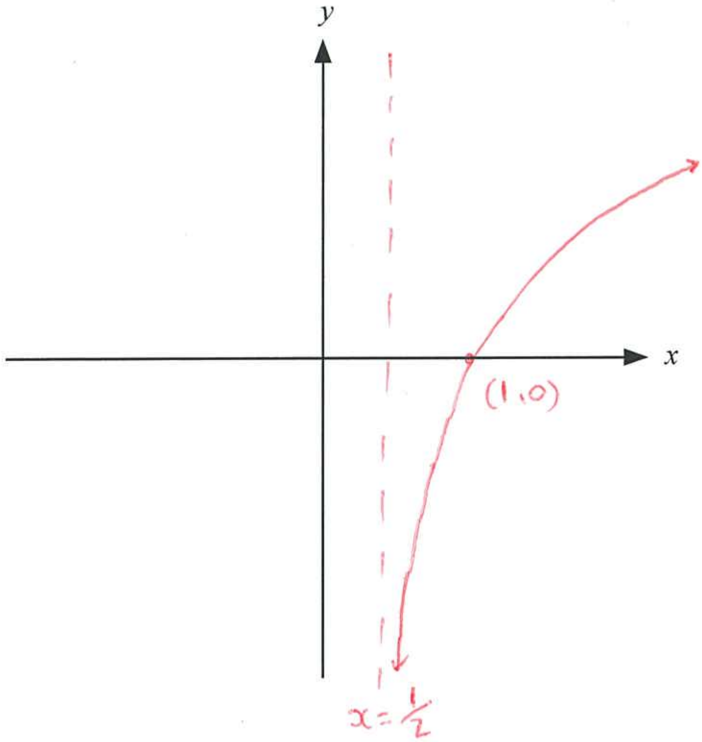
- Ensure you cease writing upon request.

Electronic Devices

Students are **not** allowed to have a mobile phone, smart watch and/or any other unauthorised electronic device in the SAC, unless it is TURNED OFF and is placed on the front teacher desk.

Question 1 (3 marks)

Sketch the graph of $y = \log_e(2x-1)$. Label all axis intercepts with their co-ordinates and all asymptotes with their equations.



Question 2 (3 marks)

Solve each of the equations below.

a. $2 \times 3^x = 5$ 1 mark

$3^x = \frac{5}{2}$

$x = \log_3\left(\frac{5}{2}\right)$

b. $\log_5(2) - \log_5(x) = 3$ 2 marks

$\log_5\left(\frac{5}{x}\right) = 3$

$\frac{5}{x} = 5^3$

$\frac{1}{25} = x$

Question 3 (4 marks)

a. The graph given by the rule $y = a \times 10^x - 2$ passes through the point $(\log_{10}(3), 4)$.

Find the value of a. 2 marks

$4 = a \times 10^{\log_{10}(3)} - 2$

$6 = 3a$

$2 = a$

b. The graph given by the rule $y = 3 \times \log_2(x - b)$ passes through the point $\left(\frac{3}{2}, -3\right)$.

Find the value of b. 2 marks

$-3 = 3 \times \log_2\left(\frac{3}{2} - b\right)$

$-1 = \log_2\left(\frac{3}{2} - b\right)$

$2^{-1} = \frac{3}{2} - b$

$b = 1$

Question 4 (5 marks)

Consider the polynomial $P(x) = 2x^3 + 3x^2 - 12x - 6$.

a. Find $P'(x)$. 1 mark

$P'(x) = 6x^2 + 6x - 12$

b. Hence, find the co-ordinates of all stationary points of the graph of $y = e^{2x^3+3x^2-12x-6}$. 4 marks

$\frac{dy}{dx} = (6x^2 + 6x - 12)\, e^{2x^3 + 3x^2 - 12x - 6}$

$\frac{dy}{dx} = 0$ when $0 = 6x^2 + 6x - 12$

$0 = 6(x^2 + x - 2)$

$0 = 6(x+2)(x-1)$

$x = 1$, $x = -2$

when $x = 1$, $y = e^{-13}$

when $x = -2$, $y = e^{14}$

$\therefore (1, e^{-13})$ and $(-2, e^{14})$

Question 5 (6 marks)

Consider the function $f:(0,\infty)\to\mathbb{R}, f(x)=\log_e(x^2)$.

a. Find the rule of the derivative, $f'(x)$. 1 mark

$f'(x) = \frac{2x}{x^2} = \frac{2}{x}$

b. Show that the tangent to the graph of $y=f(x)$ at the point where $x=a$ is given by

$y=\frac{2x}{a}+2(\log_e(a)-1)$. 2 marks

at $x=a$, $y=\log_e(a^2)$ and $m=\frac{2}{a}$

$\therefore\ y-\log_e(a^2) = \frac{2}{a}(x-a)$

$y = \frac{2}{a}x - 2 + \log_e(a^2)$

$= \frac{2}{a}x - 2 + 2\log_e(a)$

$= \frac{2x}{a} + 2(\log_e(a)-1)$

c. Find the value of a such that the tangent to the graph passes through the origin. 1 mark

$x=0,\ y=0$ $\quad 0 = 2(\log_e(a)-1)$

$1 = \log_e(a)$

$a = e$

d. Using the value of a you found in **part c**, find the equation of the normal to the graph of $y=f(x)$ at the point $x=a$. 2 marks

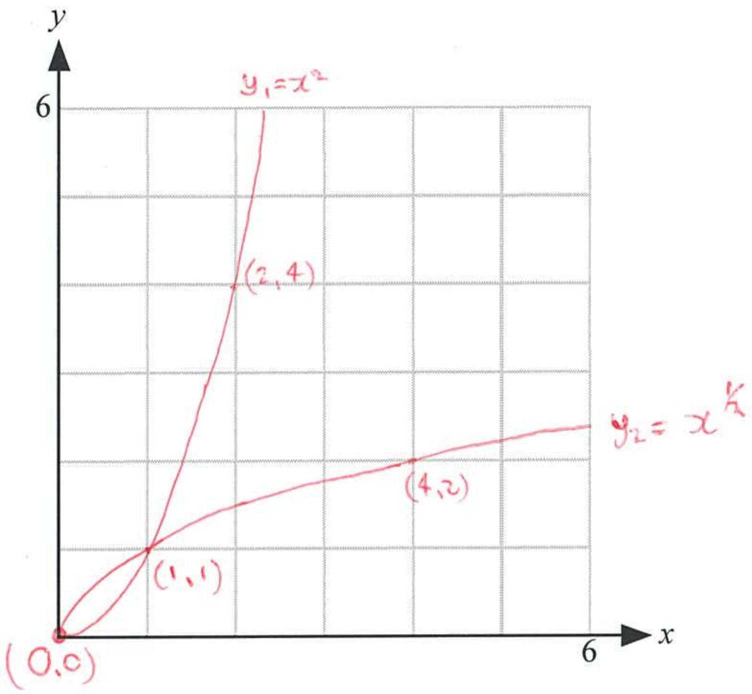
$x=e$, $f(e)=\log_e(e^2)=2$ $\quad f'(e)=2/e$

$\therefore\ y-2 = -\frac{e}{2}(x-e)$

$y = -\frac{e}{2}x + 2 + \frac{e^2}{2}$

Question 6 (9 marks)

a. Sketch the graphs of $y_1 = x^2,\ x \ge 0$ and $y_2 = x^{\frac{1}{2}},\ x \ge 0,$ on the same axes below, labelling any axis intercepts, endpoints and points of intersection with their co-ordinates. 3 marks



b. For each of the graphs in **part a** find the co-ordinates of the point where the gradient of the tangent to the curve is one. 3 marks

$\frac{dy_1}{dx} = 2x$ $\qquad$ $\frac{dy_2}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$1 = 2x$ $\qquad$ $1 = \frac{1}{2\sqrt{x}}$

$x = 1/2$ $\qquad$ $\sqrt{x} = \frac{1}{2}$

so at point $\left(\frac{1}{2}, \frac{1}{4}\right)$ $\qquad$ $x = 1/4$

so point $\left(\frac{1}{4}, \frac{1}{2}\right)$

c. Find the value of k so that the graphs with equations $y = x^2 + k$ and $y = (x-k)^{\frac{1}{2}}$ have a common tangent at a single point of intersection. 1 mark

gradient = 1 at points $\left(\frac{1}{2}, \frac{1}{4}+k\right)$ and $\left(\frac{1}{4}+k, \frac{1}{2}\right)$

$\therefore k = \frac{1}{4}$

d. Let $f : [0, \infty) \to \mathbb{R}, f(x) = x^3 + b$, and

$g : [b, \infty) \to \mathbb{R}, g(x) = (x-b)^{\frac{1}{3}}$.

Find the value of b such that the graphs of $y = f(x)$ and $y = g(x)$ have a common tangent at a single point of intersection. 2 marks

$f'(x) = 3x^2$

$1 = 3x^2$

$\frac{1}{3} = x^2$

$x = \frac{1}{\sqrt{3}}$

gradient = 1 at points $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{27}} + b\right)$ and $\left(\frac{1}{\sqrt{27}} + b, \frac{1}{\sqrt{3}}\right)$

$\therefore \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{27}} + b$

$\frac{3\sqrt{3}}{9} = \frac{\sqrt{3}}{9} + b$

$b = \frac{2\sqrt{3}}{9}$

END OF SAC 1b

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$		$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right)=an(ax+b)^{n-1}$		$\int (ax+b)^n dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c, n\neq -1$	
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$		$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$	
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$		$\int \frac{1}{x}dx=\log_e(x)+c,\ x>0$	
$\frac{d}{dx}\left(\sin(ax)\right)=a\ \cos(ax)$		$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$	
$\frac{d}{dx}\left(\cos(ax)\right)=-a\ \sin(ax)$		$\int \cos(ax)dx=\frac{1}{a}\sin(ax)+c$	
$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A\|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET