SAC 1 Worked Solutions

Scotch Student ID #				
Circle the relevant digits				
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	1	1	1	1
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Teacher's Name

Scotch College

MATHEMATICAL METHODS

U3-SAC 1 – Application Task

MARKSCHEME

2023

Task Sections	Marks	Your Marks
Investigation	25	
Total Marks	**25**	

Declaration

I declare that any work I have submitted for this VCE assessment is wholly my own, unless properly referenced or authorised for use by my teacher. I have had no assistance from any person in my home nor have I been assisted by, or given assistance to, a boy in my class or cohort unless specifically permitted to do so by my teacher. I have not used the internet or other sources to assist me in my responses unless specifically permitted by my teacher. I acknowledge my work may be reproduced, communicated, compared and archived for the purposes of detecting plagiarism and collusion.

Signature: ______________________________

General Instructions

- Answer all questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

Allowed Materials

- A scientific calculator and a CAS calculator.
- Any notes or references.

At the end of the task

- Submit the task to your teacher.

Component I

a. Consider functions of the form $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^{3x}-k$ where k is a real constant.

i. Investigate for which value(s) of k the graph of $y=f(x)$ has a positive y-intercept, a y-intercept of zero and a negative y-intercept respectively.

ii. Investigate for which value(s) of k the graph of $y=f(x)$ has a positive x-intercept, an x-intercept at $x=0$ and a negative x-intercept respectively.

a(i) y-intercept occurs at $y = 1-k$

$\therefore$ positive y-intercept when $k \in (-\infty, 1)$

y-intercept of 0 when $k = 1$

negative y-intercept when $k \in (1, \infty)$

(ii) positive x-intercept when $k \in (1, \infty)$

x-intercept of 0 when $k = 1$

negative x-intercept when $k \in (0, 1)$

b. Consider functions of the form $f : D \to \mathbb{R},\ f(x) = \log_e(3x - h)$, where h is a real constant.

i. Find the maximal domain D of f, giving your answer in terms of h.

ii. Investigate for which value(s) of h the graph of $y = f(x)$ has an x-intercept at point $(2,0)$.

iii. Investigate for which value(s) of h the graph of $y = f(x)$ has a negative x-intercept.

b(i) $3x - h > 0$

$\therefore\ x > \frac{h}{3}$ $\qquad D = \left(\frac{h}{3}, \infty\right)$

(ii) $0 = \log_e(6 - h)$

$h = 5$

(iii) Consider transformations taking $y = \log_e(x)$ to $y = \log_e(3x - h)$

- Translation h units in positive x-direction
- Dilation factor $\frac{1}{3}$ from the y-axis.

$y = \log_e(x)$ has an x-intercept at $(1,0)$, so $y = \log_e(3x - h)$ has a negative x-intercept when $h \in (-\infty, -1)$

Component II

a. Consider functions of the form $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^{ax}-k$ where a and k are real constants and $a>0$.

i. Investigate for which value(s) of k the graph of $y=f(x)$ has a positive y-intercept, a y-intercept of zero and a negative y-intercept respectively.

ii. Investigate for which value(s) of k the graph of $y=f(x)$ has a positive x-intercept, an x-intercept at $x=0$ and a negative x-intercept respectively.

a (i) positive y-intercept when $k \in (-\infty, 1)$

y-intercept of 0 when $k=1$

negative y-intercept when $k \in (1, \infty)$

(ii) positive x-intercept when $k \in (1, \infty)$

x-intercept of 0 when $k=1$

negative ~~x-intercept~~ x-intercept when $k \in (0, 1)$

b. Consider functions of the form $f:\mathbb{R}\to\mathbb{R},\ f(x)=ax^3-4ax+6$, where $a>0$.

i. By considering the coordinates of stationary points, or otherwise, investigate the number of x-intercepts the graph of $y=f(x)$ may have, giving the range of possible a values in each case.

ii. Let $a=2$. Find the value(s) of k for which the graph of $y=f(k-x)$ has zero, one, two and three positive x-intercepts, respectively.

b(i) $f'(x)=0$ at points $\left(-\frac{2\sqrt{3}}{3},\ \frac{16a\sqrt{3}}{9}+6\right)$

and $\left(\frac{2\sqrt{3}}{3},\ 6-\frac{16a\sqrt{3}}{9}\right)$

As $a>0$, $\frac{16a\sqrt{3}}{9}+6>0$.

Also, y-intercept of f occurs at $(0,6)$

$\therefore$ number of x-intercepts is determined by the value of $\left(6-\frac{16a\sqrt{3}}{9}\right)$.

- f has one x-intercept when $6-\frac{16a\sqrt{3}}{9}>0$, so when $a\in\left(0,\frac{9\sqrt{3}}{8}\right)$
- f has two x-intercepts when $6-\frac{16a\sqrt{3}}{9}=0$, so when $a=\frac{9\sqrt{3}}{8}$
- f has three x-intercepts when $6-\frac{16a\sqrt{3}}{9}<0$, so when $a\in\left(\frac{9\sqrt{3}}{8},\infty\right)$

b (ii) $\quad f(x) = 2x^3 - 8x + 6$

$f(k-x)$ has x-intercepts at :

$$x = k + \frac{1-\sqrt{13}}{2}, \quad x = k-1 \quad \text{and} \quad x = k + \frac{1+\sqrt{13}}{2}$$

$\therefore$ • zero positive x-intercepts when $k + \frac{1+\sqrt{13}}{2} \leq 0$, so $k \in \left(-\infty, \frac{-\sqrt{13}-1}{2}\right]$

• 1 positive x-intercept when $k-1 \leq 0 \cap k + \frac{1+\sqrt{13}}{2} > 0$, so $k \in \left(\frac{-\sqrt{13}-1}{2}, 1\right]$

• 2 positive x-intercepts when $k + \frac{1-\sqrt{13}}{2} \leq 0 \cap k-1 > 0$, so $k \in \left(1, \frac{\sqrt{13}-1}{2}\right]$

• 3 positive x-intercept when $k + \frac{1-\sqrt{13}}{2} > 0$, so $k \in \left(\frac{\sqrt{13}-1}{2}, \infty\right)$

Component III

Consider functions of the form $f:\mathbb{R}\to\mathbb{R},\ f(x)=ax^3+bx^2+cx+5,$ where a, b and c are real constants.

The graph of $y=f(x)$ has a turning point at $(2,-2)$.

a. Generate two simultaneous equations and hence express a and b in terms of c.

b. Hence investigate for which value(s) of c all stationary points on the graph of $y=f(x)$ have positive x-coordinates.

(a) $f(2)=-2 \quad \therefore \quad -2 = 8a+4b+2c+5$

$-7 = 8a+4b+2c \qquad ①$

$f'(2)=0 \quad \therefore \quad 0 = 3a(2)^2+2b(2)+c$

$0 = 12a+4b+c \qquad ②$

$7 = 4a-c \qquad ①-②$

$\therefore\ a = \dfrac{7+c}{4}$

$-21 = 24a+12b+6c \qquad ①\times 3$

$0 = 24a+8b+2c \qquad ②\times 2$

$\therefore\ -21 = 4b+4c$

$b = \dfrac{-21-4c}{4}$

(b) $f(x) = \left(\dfrac{7+c}{4}\right)x^3 + \left(\dfrac{-21-4c}{4}\right)x^2 + cx + 5$

$f'(x)=0$

$\therefore\ x=2,\quad x=\dfrac{2c}{3(c+7)}$

solve $\dfrac{2c}{3(c+7)} > 0 \quad \therefore \quad c\in(-\infty,-7)\cup(0,\infty)$

Note: when $c = -7$, $a = 0$ so $f(x) = \frac{7}{4}x^2 - 7x + 5$

One turning point which occurs at $x = 2$.

$\therefore \; c \in (-\infty, -7] \cup (0, \infty)$

END OF SAC 1

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$		$\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$		$\int \cos(ax) dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
trapezium rule approximation	$Area \approx \frac{x_n - x_0}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$		