Scotch Student ID #			
0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9

Circle the relevant digits

Scotch College

Teacher's Name

MATHEMATICAL METHODS

Unit 3-SAC 1c – Application Task: Test

June 2023

SOLUTIONS

Reading Time	none
Writing Time	45 minutes

Task Sections	Marks	Your Marks
Extended Response Questions	30	
Total Marks	**30**	

Declaration

I declare that any work I have submitted for this VCE assessment is wholly my own, unless properly referenced or authorised for use by my teacher. I have had no assistance from any person in my home nor have I been assisted by, or given assistance to, a boy in my class or cohort unless specifically permitted to do so by my teacher. I have not used the internet or other sources to assist me in my responses unless specifically permitted by my teacher. I acknowledge my work may be reproduced, communicated, compared and archived for the purposes of detecting plagiarism and collusion.

Signature: ___________________________

General Instructions

- Answer all questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this task are not drawn to scale.

Allowed Materials

- Calculators are allowed
- Notes and/or references are not allowed.

At the end of the task

- Ensure you cease writing upon request.

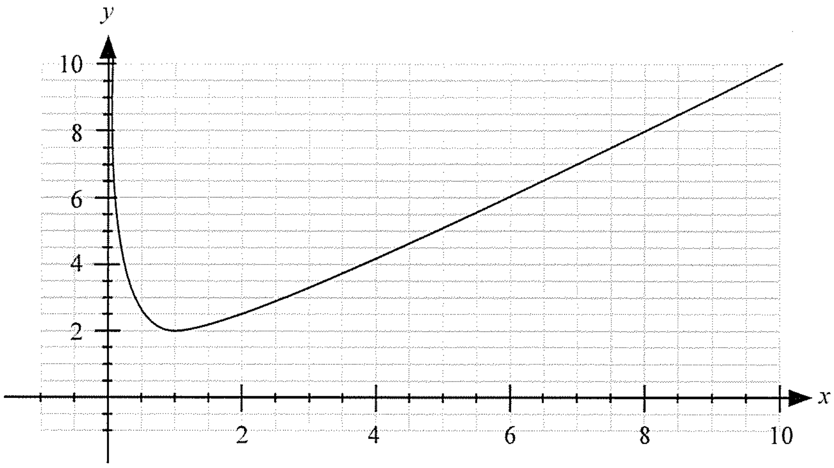
Electronic Devices

Students are **not** allowed to have a mobile phone, smart watch and/or any other unauthorised electronic device in the SAC, unless it is TURNED OFF and is placed on the front teacher desk.

Question 1 (6 marks)

Functions f and g are defined as $f : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \frac{1}{x}$ and $g : [0, \infty) \to \mathbb{R},\ g(x) = x$.

Let $k(x) = f(x) + g(x)$. Part of the graph of $y = k(x)$ is shown below.



a. Show that $k(x) = \frac{x^2+1}{x}$ 1 mark

$k(x) = f(x) + g(x)$

$= \frac{1}{x} + x$

$= \frac{1}{x} + \frac{x^2}{x}$

$= \frac{x^2+1}{x}$ as required.

b. Determine the domain of k. 1 mark

$(0, \infty)$ (or $\mathbb{R}^+$)

c. Show that $k'(x) = \frac{(x-1)(x+1)}{x^2}$ 2 marks

$k'(x) = \frac{2x \cdot x - 1(x^2+1)}{x^2}$

$= \frac{2x^2 - x^2 - 1}{x^2}$

$= \frac{x^2-1}{x^2}$

$= \frac{(x-1)(x+1)}{x^2}$

d. Determine the coordinates of the stationary point on the graph of $y = k(x)$. 1 mark

$(1, 2)$

e. Determine the values of x for which $k(x)$ is strictly increasing. 1 mark

$[1, \infty)$

Question 2 (12 marks)

Consider the function given by $f(x)=3(x-1)^3 e^{(1-x)}+3$.

a. The graph of f has a horizontal asymptote at $y=a$. State the value of a. 1 mark

$a = 3$

b. The graph of f passes through the point $(0,b)$. Find the exact value of b. 1 mark

$b = 3(-1)^3 e^1 + 3$

$b = -3e + 3$

c. Consider the function $g_1 : \mathbb{R} \to \mathbb{R}, g_1(x) = f(2x-h)+k$, where h and k are positive real numbers. List the sequence of transformations which maps the graph of $y = f(x)$ to the graph of $y = g_1(x)$. 3 marks

- dilation of factor $\frac{1}{2}$ from y axis
- translated k units in the positive y direction
- translated $\frac{h}{2}$ units in the positive x direction.

(alternatives possible

$x' = \frac{x+h}{2}$ $\quad y' = y + k$

$= \frac{x}{2} + \frac{h}{2}$)

d. Consider the function $g_2 : \mathbb{R} \to \mathbb{R}, g_2(x) = f(2x - p) + q,$ where p and q are real numbers.

i. Determine the coordinates of the y-intercept and the stationary points of g_2, giving your answers in terms of p and q. 2 marks

y intercept $\left(0, -3(p+1)^3 e^{p+1} + q + 3\right)$

stat pts. $\left(\frac{p+1}{2}, q+3\right)$ and $\left(\frac{p+4}{2}, q+81e^{-3}+3\right)$

ii. Hence or otherwise, find the value(s) of q for which g_2 has exactly one x-intercept. 2 marks

$\{-3-81e^{-3}\} \cup [-3, \infty)$

iii. If $p = q$, find the non zero value(s) of q for which the graph of g_2 cuts the y-axis at $y = \dfrac{1-e^q}{2}$, giving your answer(s) correct to two decimal places. 1 mark

$q = -5.52, -0.25$

e. Let $g_3 : \mathbb{R} \to \mathbb{R},$ where $g_3(x) = (x-a)^3 e^{(1-x)},$ where $a \in \mathbb{R}$.

The coordinates of the stationary points of g_3 are $A(a, 0)$ and $B\left(a+r+s, 27e^{-(a+r)}\right),$ where r and s are positive integers.

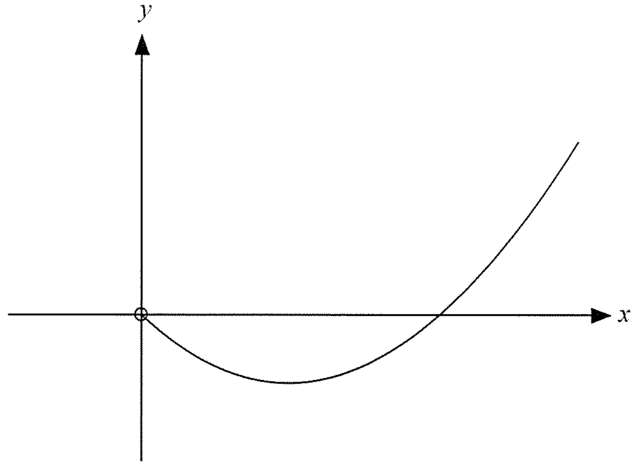
Find the values of r and s. 2 marks

$r = 2$

$s = 1$

Question 3 (4 marks)

Let $f:(0,\infty)\to\mathbb{R}, f(x)=x\log_e(x)-x$. Part of the graph of f is shown below.



a. Find the values of x for which $f(x)>0$. 1 mark

$x \in (e, \infty)$

b. The equation of the tangent to the graph of $y=f(x)$ at the point $(\sqrt{n}, f(\sqrt{n}))$ is $y=\frac{\log_e(n)}{2}x+b$. Find b in terms of n. 1 mark

$b = -\sqrt{n}$

c. Find the value(s) of n for which the tangents to the graph of $y=f(x)$ at the points with coordinates $(n, f(n))$ and $(\frac{1}{n}, f(\frac{1}{n}))$ are perpendicular, giving your answer(s) correct to two decimal places. 2 marks

$m_1 = \log_e(n)$ $\quad$ $m_2 = \log_e(\frac{1}{n})$

$\log_e(n) \times \log_e(\frac{1}{n}) = -1$

$n = 0.37, \quad 2.72$

Question 4 (8 marks)

Functions f and g are defined as $f:(-14,\infty)\rightarrow\mathbb{R}, f(x)=\log_e(\frac{x}{2}+7)$ and $g:(0,\infty)\rightarrow\mathbb{R}, g(x)=\log_e(\frac{x}{2})$.

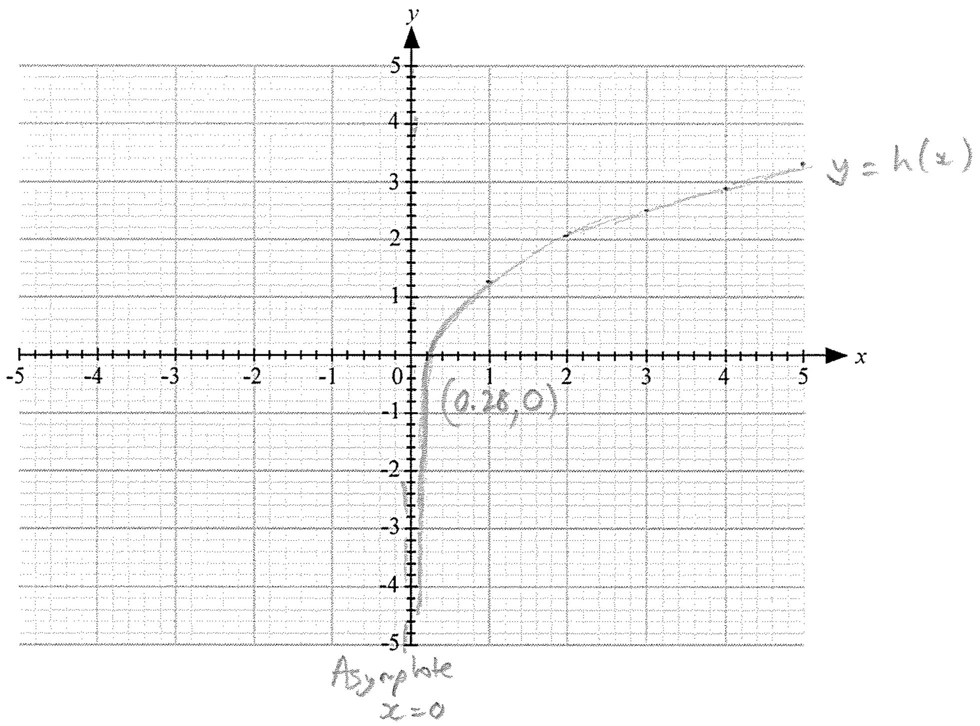
Let $h(x)=f(x)+g(x)$, for $x\in(0,\infty)$.

a. The x-intercept of function h is $x=\sqrt{a}-b$. Find the value of a and b. 1 mark

$a=53$

$b=7.$

b. Sketch the graph $y=h(x)$ on the set of axes below. Label the asymptote with its equation and the axial intercept with its coordinates correct to two decimal places. 2 marks



c. **i.** Find the rule for the inverse function h^{-1}. 1 mark

$h^{-1}(x)=\sqrt{4e^{x}+49}-7$

ii. Find the coordinates of the point(s) of intersection of the graphs of $y = h(x)$ and $y = h^{-1}(x)$, giving your answer(s) correct to two decimal places. 1 mark

(0.42, 0.42) and (2.17, 2.17)

Consider the function $h_2(x) = \log_e\left(\frac{x}{k}+7\right)+\log_e\left(\frac{x}{k}\right)$ where k is a positive real constant.

d. Find the value of x in terms of k such that $h_2'(x) = 1$. 1 mark

$x = \dfrac{\sqrt{49k^2+4} - 7k + 2}{2}$

e. Hence or otherwise, find the value of k so that the graphs of h_2 *and* h_2^{-1} have only one point of intersection. Give your answer correct to two decimal places. 2 marks

$h_2(x) = x$ and $x = \dfrac{\sqrt{49k^2+4} - 7k + 2}{2}$

$k = 2.71$

END OF SAC 1c