Scotch Student ID #			
0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9

Circle the relevant digits

Teacher's Name
Solutions

Scotch College

MATHEMATICAL METHODS

Unit 3-SAC 1b – Application Task: Test

June 2023

Reading Time	none
Writing Time	45 minutes

Task Sections	Marks	Your Marks
Extended Response Questions	30	
Total Marks	**30**	

Declaration
I declare that any work I have submitted for this VCE assessment is wholly my own, unless properly referenced or authorised for use by my teacher. I have had no assistance from any person in my home nor have I been assisted by, or given assistance to, a boy in my class or cohort unless specifically permitted to do so by my teacher. I have not used the internet or other sources to assist me in my responses unless specifically permitted by my teacher. I acknowledge my work may be reproduced, communicated, compared and archived for the purposes of detecting plagiarism and collusion.   *Signature:* ______________________________

General Instructions

- Answer all questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this task are not drawn to scale.

Allowed Materials

- Calculators are not allowed
- Notes and/or references are not allowed.

At the end of the task

- Ensure you cease writing upon request.

Electronic Devices

Students are **not** allowed to have a mobile phone, smart watch and/or any other unauthorised electronic device in the SAC, unless it is TURNED OFF and is placed on the front teacher desk.

Question 1 (4 marks)

a. If $y=\frac{\log_e(3x)}{2x}$, show that $\frac{dy}{dx}=\frac{1-\log_e(3x)}{2x^2}$. 2 marks

Let $u=\log_e(3x)$ $\quad v=2x$

$\frac{du}{dx}=\frac{3}{3x}$ $\quad \frac{dv}{dx}=2$

$=\frac{1}{x}$

$\frac{dy}{dx}=\frac{\frac{du}{dx}v-u\frac{dv}{dx}}{v^2}$

$=\frac{\frac{2x}{x}-2\log_e(3x)}{(2x)^2}$

$=\frac{2-2\log_e(3x)}{4x^2}$

$=\frac{1-\log_e(3x)}{2x^2}$

b. Let $f(x)=x(2x+1)^3$. Find $f'(1)$. 2 marks

Let $u=x$ $\quad v=(2x+1)^3$

$\frac{du}{dx}=1$ $\quad \frac{dv}{dx}=3(2)(2x+1)^2=6(2x+1)^2$

$f'(x)=\frac{du}{dx}v+\frac{dv}{dx}u$

$=(2x+1)^3+6x(2x+1)^2$

$f'(1)=(3)^3+6(3)^2$

$=27+54$

$=81$

Question 2 (7 marks)

Consider the polynomial function p with rule $p(x) = x^3 - 3x^2 - 8x + 10$ over the domain $[m, n)$ where $m, n \in \mathbb{R}$.

a. **i.** Show that $(x-1)$ is a factor of $p(x)$. 1 mark

$p(1) = (1)^3 - 3(1)^2 - 8(1) + 10$

$= 1 - 3 - 8 + 10$

$= 0$ $\therefore$ $(x-1)$ is a factor of $p(x)$

ii. Hence, express $p(x)$ in the form $p(x) = f(x) \times g(x)$, where $f(x) = x - 1$ and $g(x) = x^2 - ax - b$ where $a, b \in \mathbb{Z}$. 1 mark

$x^2 - 2x - 10$

$x - 1 \overline{) x^3 - 3x^2 - 8x + 10}$

$-(x^3 - 1x^2)$

$-2x^2 - 8x$

$-(-2x^2 + 2x)$

$-10x + 10$

$-(-10x + 10)$

0

$p(x) = (x-1)(x^2 - 2x - 10)$

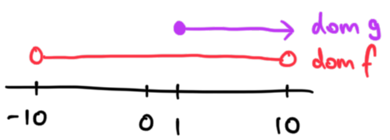
where $f(x) = x - 1$

$g(x) = x^2 - 2x - 10$

b. The domain of f is $(-10, 10)$ and the domain of g is $[1, \infty)$. Using this information, find the domain of p, in the form $[m, n)$. 1 mark

if $p(x) = f(x) \times g(x)$

then dom p = dom $f \cap$ dom g $\therefore$ dom $p = [1, 10)$



c. The function f undergoes the following sequence of transformations to produce the function f_t.

- Dilation of factor 2 from the x-axis
- Reflection in the x-axis
- Translation of 3 units in the positive direction of the x-axis.

Write down the rule for the transformed function f_t. 2 marks

$(x, y) \rightarrow (x, 2y) \rightarrow (x, -2y) \rightarrow (x+3, -2y)$

$x' = x+3$ $\quad y' = -2y$ $\quad y = f(x)$

$x' - 3 = x$ $\quad -\frac{y'}{2} = y$ $\quad -\frac{y'}{2} = f(x'-3)$

$-\frac{y'}{2} = (x'-3) - 1$

$-\frac{y'}{2} = x' - 4$

$y' = -2x' + 8$

d. Find the domain and range for the transformed function f_t. 2 marks

$(x, y) \rightarrow (x+3, -2y)$

dom $f = (-10, 10) \rightarrow$ dom $f_t = (-7, 13)$

ran $f = (-11, 9) \rightarrow$ ran $f_t = (-18, 22)$

Question 3 (4 marks)

Let $f(x) = \sqrt{x-3}$ and $g(x) = x^2 - 13$, which are both defined over their maximal domains.

a. State the maximal domain and range of $g \circ f$. 2 marks

$g \circ f = g[f(x)]$

$\text{dom}(g \circ f) = \text{dom } f = [3, \infty)$

$g \circ f = g[f(x)] = g[\sqrt{x-3}] = (\sqrt{x-3})^2 - 13$

$= x - 3 - 13$

$= x - 16$

For $x \in [3, \infty)$, $\text{ran}(g \circ f) = [-13, \infty)$

b. Find the maximal domain of g such that $f \circ g$ is defined. 2 marks

If $f \circ g$ defined, then $\text{ran } g \subseteq \text{dom } f$

$\text{dom } f = [3, \infty)$ $\therefore$ Solve $g(x) \geq 3$

$\text{ran } g = [-13, \infty)$

$x^2 - 13 \geq 3$

$x^2 - 16 \geq 0$

$(x-4)(x+4) \geq 0$

$x \leq -4$ OR $x \geq 4$

$\therefore$ max domain of g

such that $f \circ g$ is defined

$= (-\infty, -4] \cup [4, \infty)$

Question 4 (4 marks)

A hybrid function h is defined as follows:

$$h(x)=\begin{cases}\log_e(x)+a & \text{for } 1 \le x < e \\ e^{x-(e+b)} & \text{for } e \le x \le 2e\end{cases}$$

where $a, b \in \mathbb{Z}$.

a. Express a in terms of b if h is a continuous function. 2 marks

$\log_e(e) + a = e^{e-(e+b)}$

$1 + a = e^{-b}$

$a = e^{-b} - 1$

b. Assuming h is a continuous function, find the value of b such that h is differentiable for $x \in (1, 2e)$. 2 marks

$\frac{1}{e} = e^{e-(e+b)}$

$\frac{1}{e} = e^{-b}$

$\frac{1}{e} = \frac{1}{e^b} \quad \therefore \quad b = 1$

Question 5 (2 marks)

Let $T_1:\mathbb{R}^2 \to \mathbb{R}^2,\ T_1(x,y)=(x-2,\ 2y)$ and $T_2:\mathbb{R}^2 \to \mathbb{R}^2,\ T_2(x,y)=(3x,-y)$.

Find the image of the curve $y=\frac{1}{x}$, under the transformation $T_1 \circ T_2$

$T_1 \circ T_2 = T_1[T_2(x,y)] = T_1[3x,-y] = (3x-2,-2y)$

$x' = 3x-2 \qquad y' = -2y$

$\frac{x'+2}{3} = x \qquad -\frac{y'}{2} = y$

$y = \frac{1}{x} \rightarrow \quad -\frac{y'}{2} = \frac{1}{\frac{x'+2}{3}}$

$-\frac{y'}{2} = \frac{3}{x'+2}$

$y' = \frac{-6}{x'+2}$

Question 6 (6 marks)

Consider the function $f:(a,\infty)\to\mathbb{R},\ f(x)=-4\log_e\left(\sqrt{2x+3}\right)$, where a is the smallest real number such that f is defined.

a. What is the value of a? 1 mark

$2x+3>0$

$x>-\frac{3}{2} \quad \therefore \quad a=-\frac{3}{2}$

b. The function $f(x)=-4\log_e\left(\sqrt{2x+3}\right)$ can be written as $f(x)=\cancel{b}^{\,k}\log_e(2x+3)$.

Show that $\cancel{b}^{\,k}=-2$. 1 mark

$f(x) = -4\log_e\left((2x+3)^{\frac{1}{2}}\right)$

$= \frac{-4}{2}\log_e(2x+3)$

$= -2\log_e(2x+3) \quad \therefore \quad k=-2$

c. List the sequence of transformations which maps the graph $y=f(x)$ to the graph $y=6\log_e(x-5)$. 2 marks

$y=-2\log_e(2x+3) \longrightarrow y'=6\log_e(x'-5)$

$\frac{-y}{2}=\log_e(2x+3) \qquad \frac{y'}{6}=\log_e(x'-5)$

$x'-5=2x+3$

$x'=2x+8$

$\frac{y'}{6}=\frac{-y}{2}$

$y'=-3y$

- Dilation of factor 2 from the y-axis, followed by a translation of 8 right
- A reflection in the x-axis and a dilation of factor 3 from the x-axis

d. The gradient of the normal to the graph of f at the point $(b, f(b))$ is $\frac{2}{3}$.

Find the value of b. 2 marks

$f(x) = -2\log_e(2x+3)$

$f'(x) = \frac{-2(2)}{2x+3} = \frac{-4}{2x+3}$

Gradient of normal $= \frac{2}{3}$

As $m_T \times m_N = -1 \quad \therefore \; m_T = -\frac{3}{2}$

$\frac{-4}{2x+3} = -\frac{3}{2}$

$-8 = -3(2x+3)$

$-8 = -6x - 9$

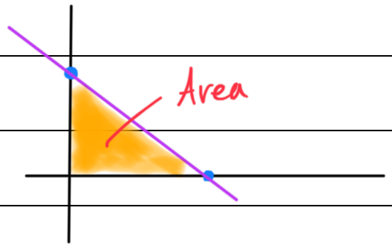
$x = -\frac{1}{6} \qquad \therefore \; b = -\frac{1}{6}$

Question 7 (3 marks)

The function $g(x)=25-x^2$ has a tangent at the point $(p,\ g(p))$ which has the equation $y=-2px+p^2+25,$ where $p>0.$

Find the value of p for which the area enclosed by the tangent at the point $(p,\ g(p))$, the x-axis and the y-axis is a minimum and find this minimum area.

$y=-2px+p^2+25$, as $p>0$



$\underline{y\text{int}}$: $y=p^2+25$

$\underline{x\text{int}}$: $0=-2px+p^2+25$

$x=\dfrac{p^2+25}{2p}$

Let A = Area enclosed

$$A=\frac{1}{2}\left(p^2+25\right)\left(\frac{p^2+25}{2p}\right)$$

$$=\frac{\left(p^2+25\right)^2}{4p}$$

Minimum area where $\dfrac{dA}{dp}=0$

Let $u=\left(p^2+25\right)^2$

$\dfrac{du}{dp}=2(2p)\left(p^2+25\right)$

$=4p\left(p^2+25\right)$

$v=4p$

$\dfrac{dv}{dp}=4$

$$\frac{dA}{dp}=\frac{v\dfrac{du}{dp}-u\dfrac{dv}{dp}}{v^2}$$

$$=\frac{4p\left[4p\left(p^2+25\right)\right]-4\left(p^2+25\right)^2}{(4p)^2}$$

$$=\frac{16p^2\left(p^2+25\right)-4\left(p^2+25\right)^2}{16p^2}$$

$$=\frac{4p^2\left(p^2+25\right)-\left(p^2+25\right)^2}{4p^2}$$

Min area when $\dfrac{dA}{dp}=0$

$0=4p^2\left(p^2+25\right)-\left(p^2+25\right)^2$

$0=\left(p^2+25\right)\left(4p^2-\left(p^2+25\right)\right)$

$0=\left(p^2+25\right)\left(3p^2-25\right)$

$p=\dfrac{5}{\sqrt{3}}$

$$A=\frac{\left(\left(\frac{5}{\sqrt{3}}\right)^2+25\right)^2}{4\left(\frac{5}{\sqrt{3}}\right)}$$

$$=\left(\frac{25}{3}+\frac{75}{3}\right)^2\times\frac{\sqrt{3}}{20}$$

$$=\frac{10000\sqrt{3}}{180}$$

$$=\frac{500\sqrt{3}}{9}\text{ units}^2$$

END OF SAC 1b

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
trapezium rule approximation	$Area \approx \frac{x_n - x_0}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$		