Circle the relevant digits	Scotch Student ID #			
	0	0	0	0
	1	1	1	1
	2	2	2	2
	3	3	3	3
	4	4	4	4
	5	5	5	5
	6	6	6	6
	7	7	7	7
	8	8	8	8
	9	9	9	9

Teacher's Name

Scotch College

MATHEMATICAL METHODS

U4-SAC 2 – Investigation Task

2023

Task Sections	Marks	Your Marks
Investigation	25	
Total Marks	**25**	

Declaration
I declare that any work I have submitted for this VCE assessment is wholly my own, unless properly referenced or authorised for use by my teacher. I have had no assistance from any person in my home nor have I been assisted by, or given assistance to, a boy in my class or cohort unless specifically permitted to do so by my teacher. I have not used the internet or other sources to assist me in my responses unless specifically permitted by my teacher. I acknowledge my work may be reproduced, communicated, compared and archived for the purposes of detecting plagiarism and collusion.   *Signature:* ______________________________________

General Instructions

- Answer all questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

Allowed Materials

- A scientific calculator and a CAS calculator.
- One double-sided A4 page of notes to be handed in at the end of each session.

At the end of the task

- Submit the task to your teacher.

Component I

In Component I, you will investigate *X*, a random variable with a probability distribution given by:

x	0	1	2	3	4
$\Pr(X = x)$	0.05	a	0.23	b	0.09

a. Consider the case when $a = 0.2$

i. Calculate the value of *b*.

ii. Calculate the expected value of *X*.

iii. Calculate the standard deviation of *X*, giving your answer correct to four decimal places.

b. Calculate the range of possible expected values of *X* as *a* and *b* vary.

c. Calculate $\Pr(X \geq 1 | X \leq 3)$. Explain why your answer is unaffected by the values of a and b.

d. **i.** Calculate the expected value of X in terms of a.

ii. Calculate the maximum possible standard deviation of X, giving your answer correct to four decimal places, and state the exact value of a for which this occurs.

iii. Calculate the minimum possible standard deviation of X, giving your answer correct to four decimal places, and state the exact value of a for which this occurs.

Component II

In Component II, you will investigate X, a random variable with probability density function f given by:

$$f(x)=\begin{cases} kx\left(a^2-x^2\right) & 0\le x\le a \\ 0 & \text{elsewhere} \end{cases}$$

where a and k are positive real numbers.

a. Let $a=1$. Find the value of k, the expected value of X and the standard deviation of X.

b. Find the value of k in terms of a for which f is a probability density function and hence express the expected value of X and the standard deviation of X in terms of a.

c. Let $a = 10$. Find the value of t for which $\Pr(t \le X \le t+2)$ is a maximum, giving your answer correct to four decimal places.

Component III

a. **i.** The standard normal distribution for a random variable Z with a mean of 0 and a standard deviation of 1 has a probability density function given by:

$$f\left(x\right)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

Sketch the graph of $f\left(x\right)$ below, labelling the coordinates of all turning points.

ii. A random variable X is normally distributed with a mean of a and a standard deviation of b.

The probability density function for the distribution is given by:

$$g(x) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$$

Sketch the graphs of $g(x)$ for two different sets of a and b values below, labelling the coordinates of all turning points.

b. **i.** List the sequence of transformations which map the graph of $y = f(x)$ to the graph of $y = g(x)$, giving your answers in terms of a and b.

ii. Hence write down the maximum value of g and the value of x for which this occurs, giving your answers in terms of a and b.

c. A student completes three tests, scoring 82%, 76% and 65%. The results on each test are normally distributed with a mean and standard deviation given in the table below.

Test	Student's score	Test mean	Test standard deviation
Test 1	82	76.1	6.1
Test 2	76	70.2	4.7
Test 3	65	60.3	3.9

Let Z represent the random variable with the standard normal distribution.

i. Calculate the student's z-value for each test, and explain what these z-values represent.

ii. Another student's score is picked at random from the list of all test scores. Given the score for this test is less than 71, find the probability the score is from Test 3. Give your answer correct to four decimal places.

END OF SAC 2

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
trapezium rule approximation	$Area \approx \frac{x_n - x_0}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A\|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$
binomial coefficient	$\binom{n}{x} = \dfrac{n!}{x!(n-x)!}$		

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
binomial	$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$	$\mu = np$	$\sigma^2 = np(1-p)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET