XAVIER MATHS

MATHEMATICAL METHODS

Problem Solving Task

SAC 2

2016

Student Name: ________________________________

Teacher's Name: ______________________________

Directions

Reading Time	10 Minutes
Writing Time	120 Minutes

- COMPLETE THIS TASK IN THE SPACES PROVIDED.
- **FOR QUESTIONS WORTH MORE THAN 1 MARK THE METHOD OF SOLUTION MUST BE CLEARLY EVIDENT.**
- YOU ARE PERMITTED TO USE A CAS CALCULATOR AND BOUND REFERENCE BOOK.
- UNITS MUST BE INCLUDED WHERE APPROPRIATE.
- GIVE EXACT ANSWERS UNLESS INSTRUCTED OTHERWISE.
- THIS SAC HAS A TOTAL OF 63 MARKS.
- BEFORE YOU COMMENCE WRITING PLEASE SIGN THE STUDENT DECLARATION ON THE NEXT PAGE.

COMPULSORY STUDENT DECLARATION:

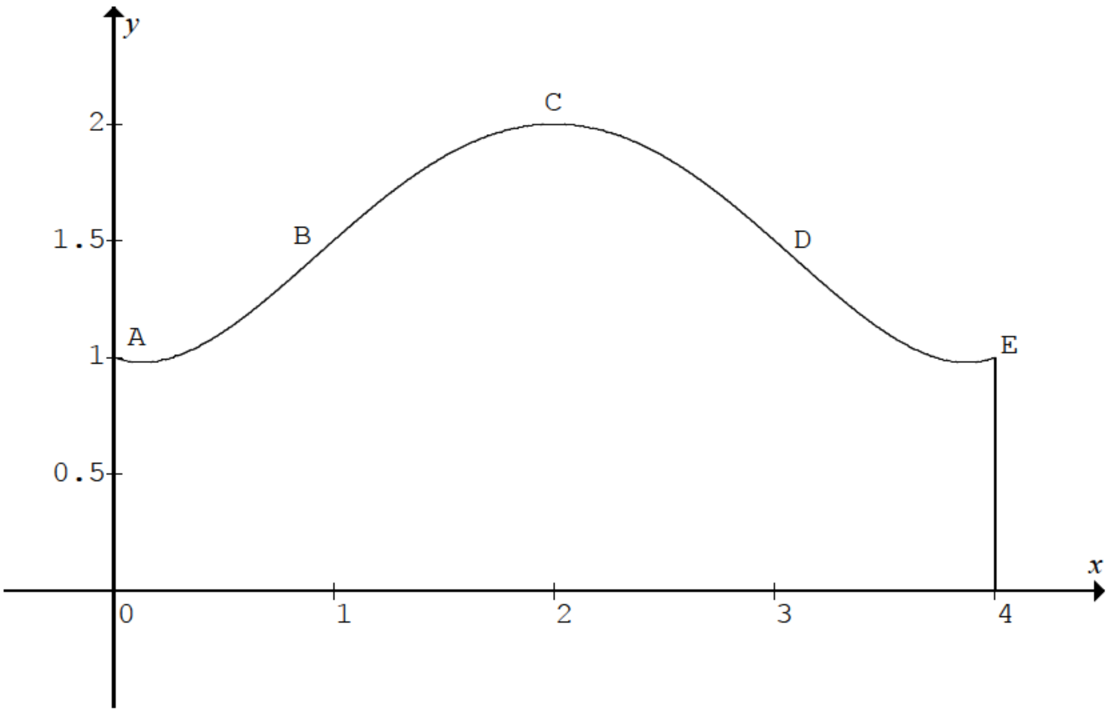
I, ____,
acknowledge that I have read the SAC/examination conditions and understand which items/materials I am permitted to use and have in my possession.
******If you have any doubts as to what is permitted, raise your hand and DO NOT sign this declaration******

Student's signature: ________________________

Teacher's name: ___________________________

Question 1

The design of a section of fencing ABCDE is shown below, along with the x and y axes with dimensions in metres. The fence must pass through the points $A(0,1)$, $B\left(1,\frac{3}{2}\right)$, $C(2,2)$ $D\left(3,\frac{3}{2}\right)$ and $E(4,1)$. The point C is the highest point on the fence, while points A and E are the lowest points on the fence.



a. If the area of the fence is approximated by four equally spaced left rectangles, determine in square metres the area of the fence.

2 mark

b. One design for the fence is a function of the form $g(x) = p + q\cos(nx)$ for $0 \le x \le 4$.

i. Find the values of p, q and n.

3 marks

ii. Using this design, write down a definite integral which gives the average height of the fence.

1 mark

iii. Find the average height of the fence in metres

1 mark

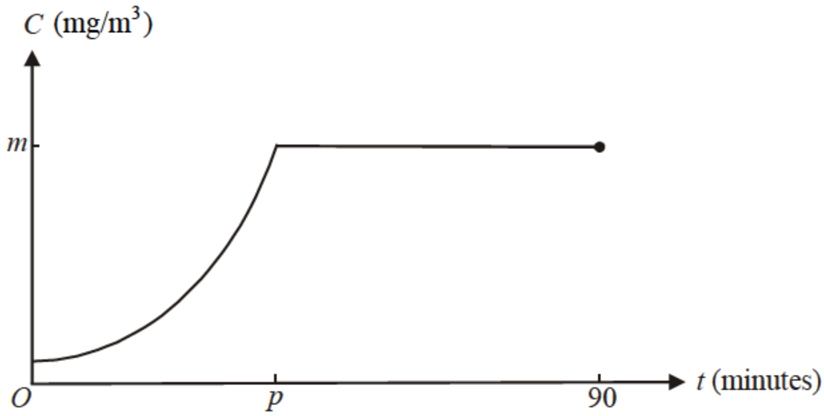
Question 2

Rugged Rob is a spy. He is trapped in a space where poisonous gas is leaking.
The concentration C, in mg/m^3, of the gas t minutes after Rugged Rob became trapped is given by the continuous function

$$C(t) = \begin{cases} \dfrac{500}{100-t}, & 0 \le t \le p \\ m, & p < t \le 90 \end{cases}$$

where m and p are constants.

A graph of the function is shown below.



a. What is the initial concentration of the gas in mg/m^3?

1 mark

b. Find an expression for m in terms of p.

1 mark

c. Find the minimum and maximum values of m.

2 marks

d. Find the function $C'(t)$ for $0 < t < p$.

1 mark

e. If the rate at which the concentration of the gas is increasing was 1 mg/m^3 per minute, find the value of *t*. Express your answer in minutes correct to 2 decimal places.

2 marks

f. If $p = 10$, find the average concentration of the gas between $t = 0$ and $t = p$, correct to 2 decimal places.

2 marks

If the concentration of the gas reaches 6 mg/m^3 then a human cannot survive.

g. Given that Rugged Rob is trapped for 90 minutes in this space, find the possible values of *p* in order for him to survive.

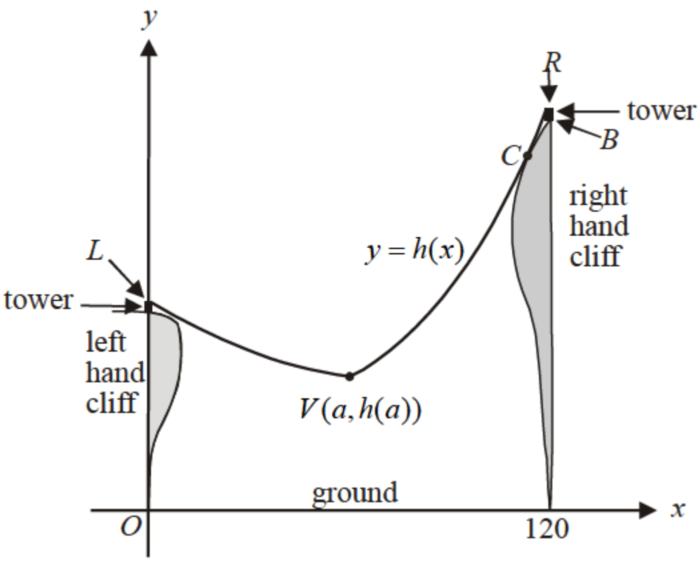
2 marks

Question 3

Flying Frances is a spy. Whilst on a chairlift that spans a valley between two cliffs, her enemies destroyed the control tower causing the chairlift cable to slacken. As a result, the position of the cable above the ground can be described by the **continuous** function

$$h(x)=\begin{cases}\frac{1}{100}(x-50)^2+40, & \text{for } x\in[0,a]\\ \frac{1}{100}(x-30)^2+40, & \text{for } x\in(a,120]\end{cases}$$

where x represents the horizontal distance in metres of the cable from the base of the left hand cliff and y represents the height in metres of the cable above the ground.



The graph of h is shown above.
The cable is attached to the top of 5m high towers on the left and right hand cliffs at points L and R respectively. The base of the tower on the right hand cliff is indicated by point B.
The cable touches the right hand cliff at point C and starts to fray. Frances is left stationary at the point $V(a,h(a))$.

a. Show that $a = 40$.

(2 marks)

b. How high is Frances above the ground?

(1 mark)

The gradient of the cable at point C is 1.48.

c. Find the coordinates of point C giving your answer correct to two decimal places.

(3 marks)

Frances has equipment with her that enables her to lower herself to the ground from her stationary position. Let v be her height, in metres, above the ground t seconds after she starts her descent. The rate at which she descends is given by

$$\frac{dv}{dt} = \frac{-23}{21} e^{\frac{-t}{42}}.$$

d. **i.** Find an expression for v in terms of t.

(3 marks)

ii. Hence find the time that it takes for Frances to reach the ground. Express your answer correct to 3 decimal places.

(1 mark)

At the same instant that Frances starts her descent, an enemy spy who is located at point *B*, starts to abseil down to point *C*. He drops 0.24 metres vertically each second. Once at *C* he takes *n* seconds to cut the cable.

e. **i.** Find an expression for the total time *T*, in terms of *n*, that it takes for him to abseil down from point *B* and cut the cable.

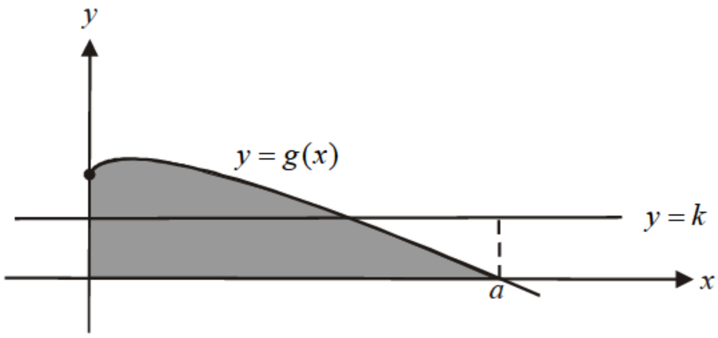
(3 marks)

ii. Assuming Frances will be safe once she reaches the ground, find the values of *n* for which Frances will be safe.

(2 mark)

Question 4

Let $g:[0,\infty) \to R,\ g(x) = \sqrt{x} + 2 - x$. The graphs of $y = g(x)$ and the line with equation $y = k$, where k is a positive constant, are shown below.



The graph of g has an x-intercept at the point $(a,0)$ where $a > 1$. The area enclosed by the graph of $y = g(x)$ and the x and y axes is shaded.

a. Show that $a = 4$.

(3 marks)

b. Find the values of k for which there is one point of intersection with the line $y = k$ and the graph of $y = g(x)$.

(3 marks)

c. If the area enclosed by the lines with equations $y = k$ and $x = a$ and the x and y axes, is equal to the area of the shaded region, then find the value of k.

(2 marks)

d. Find the area of the shaded region.

(1 mark)

The graphs of $y = g(x)$ and $y = k$, intersect at a point where $x = 0.25\left(\sqrt{9-4k}+1\right)^2$.

e. Find the value of k for which the area of the shaded region above the line $y = k$ is equal to the area of the shaded region below the line $y = k$. Express your answer correct to 2 decimal places.

(3 marks)

Question 5

Given the cubic function $f: R \to R$, $f(x) = x^3 + bx^2 + cx + 6$ where *b* and *c* are real constants.

i. Find $f'(x)$ in terms of *b* and *c*. (1 mark)

ii. Find in terms of *b* and *c*, the equation of the tangent $t(x)$ to the cubic at the point *P* where $x = 2$. Give the answer in simplest form. (3 marks)

iii. This tangent $t(x)$ intersects the cubic again at the point $Q(-1, 6)$.
Write down simultaneous equations involving *b* and *c*, and hence show that $b = -3$ and $c = -4$. (3 marks)

iv. Write down a definite integral A_1 not involving b and c, which gives the area between the tangent $t(x)$ and the cubic function.

(2 marks)

v. Find the equation of the tangent $t_1(x)$ to the cubic at the point $Q(-1,6)$. (2 marks)

vi. At point R, this tangent $t_1(x)$ intersects the cubic function again. Find the coordinates of the point R. (1 mark)

vii. Write down a definite integral A_2 not involving b and c, which gives the area between the tangent $t_1(x)$ and the cubic function.

(1 mark)

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viii. Find the value of $\frac{A_2}{A_1}$ (1 mark)

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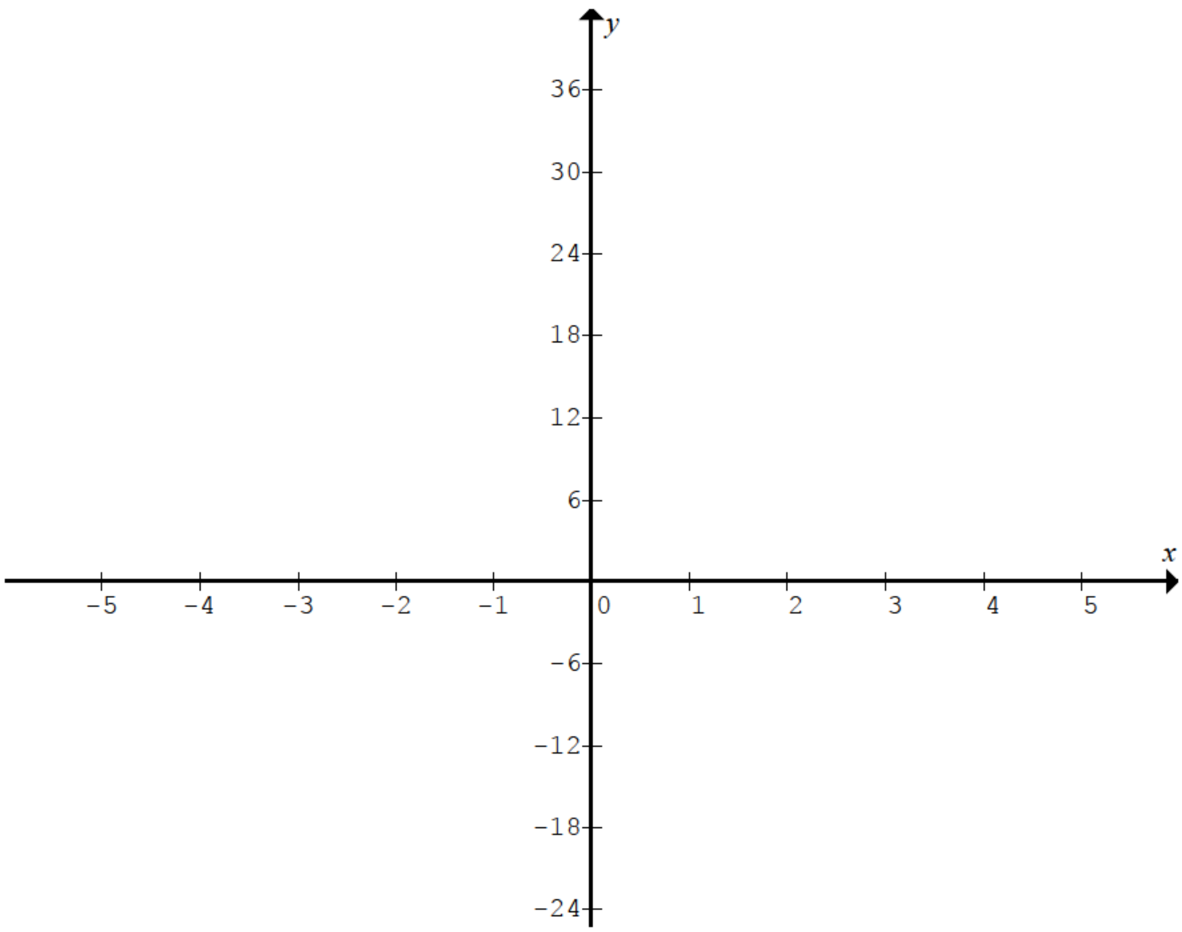
ix. On the axes below, sketch the graphs of $y = f(x)$ and the tangents at P and Q and shade the area A_2. Give the coordinates of the axes intercepts and intersection points. (4 marks)



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