MATHEMATICAL METHODS

Problem Solving Task

SAC 2

Wednesday 9 August 2017

Student Name: ____________________________

Teacher's Name: __________________________

Directions

Reading Time 10 Minutes

Writing Time 120 Minutes

- COMPLETE THIS TASK IN THE SPACES PROVIDED.
- **FOR QUESTIONS WORTH MORE THAN 1 MARK THE METHOD OF SOLUTION MUST BE CLEARLY EVIDENT.**
- YOU ARE PERMITTED TO USE A CAS CALCULATOR AND BOUND PREPARATION BOOKLET.
- UNITS MUST BE INCLUDED WHERE APPROPRIATE.
- GIVE EXACT ANSWERS UNLESS INSTRUCTED OTHERWISE.
- THIS SAC HAS A TOTAL OF 47 MARKS.
- BEFORE YOU COMMENCE WRITING PLEASE SIGN THE STUDENT DECLARATION ON THE NEXT PAGE.

COMPULSORY STUDENT DECLARATION:

I, ____,
acknowledge that I have read the SAC/examination conditions and understand which items/materials I am permitted to use and have in my possession.
****If you have any doubts as to what is permitted, raise your hand and DO NOT sign this declaration****

Student’s signature: ____________________________

Teacher’s name: _______________________________

Teacher’s name:
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Q1.

The Chiko Roll is an Australian savoury snack invented by Frank McEncroe, inspired by the Chinese spring roll and first sold in 1951 as the "Chicken Roll" despite not actually containing chicken. The snack was designed to be easily eaten on the move without a plate or cutlery. It is in the shape of a cylinder, with a pastry outer cover (curved surface and two circular ends) and a filling primarily made of cabbage, barley, carrot, green beans, beef, beef tallow, wheat cereal, celery and onion.

The dimensions of a typical Chiko roll are: height 12cm and diameter 4cm.

a. Show that the volume (quantity of filling) and total surface area (quantity of pastry) of a typical Chiko roll, as exact values, are 48π cm^3 and 56π cm^2 respectively.

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2 Marks

You are going to investigate what the maximum volume of filling would be for Chiko roll with the same surface area

b. Using the total surface area of a typical Chiko roll from part a, find an expression for the height (h) in terms of the radius (r).

2 Marks

c. Find the equation for the Volume in terms of the radius.

2 Marks

d. Find the value of the radius and height for which the maximum volume occurs, giving your answers as exact values.

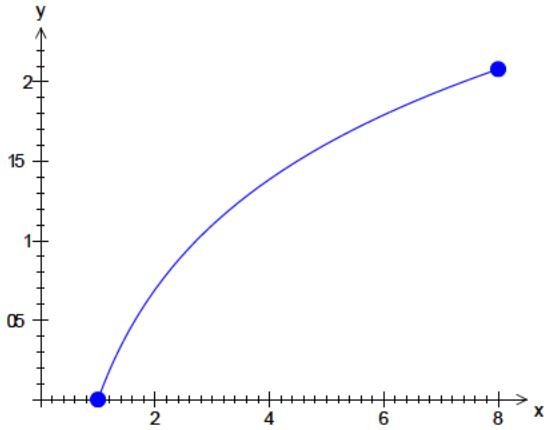
3 Marks

e. Give the maximum volume of filling for a reconfigured Chiko roll, as an exact value.

1 Marks

Q2. Nick is designing the roof for a semi-dome shaped chalet at Mt Baw Baw. He consults his good friend Hannah for advice. The roof needs to span a building that is 7 metres long. Hannah suggests the following model:

$$f:[1,8]\to R, f(x)=\log_e x$$



Nick knows that the engineers will need to find the area under the roof. But he is not sure how he will find the anti-derivative of $\log_e x$.

Hannah suggests the following.

a. Show that if $y = x\log_e x$ then $\frac{dy}{dx} = 1 + \log_e x$

1 Mark

b. Hence show that $\int \log_e x \, dx = x\log_e x - x + c$

2 Marks

c. Using this result, find, as an exact value, $\int_1^8 \log_e x \, dx$, ie. the area under the roof.

2 Marks

d. A chimney will be located on the roof. Eric, the architect, recommends that visually it looks best if placed at the point which is the average height of the roof. That is, the average value. Find the coordinates of this point, giving your answers to two decimal places.

2 Marks

e. Robert, the roof plumber, needs to know the angle of the roof at a point when $y = \frac{1}{2}\log_e 3$. Find the angle in degrees.

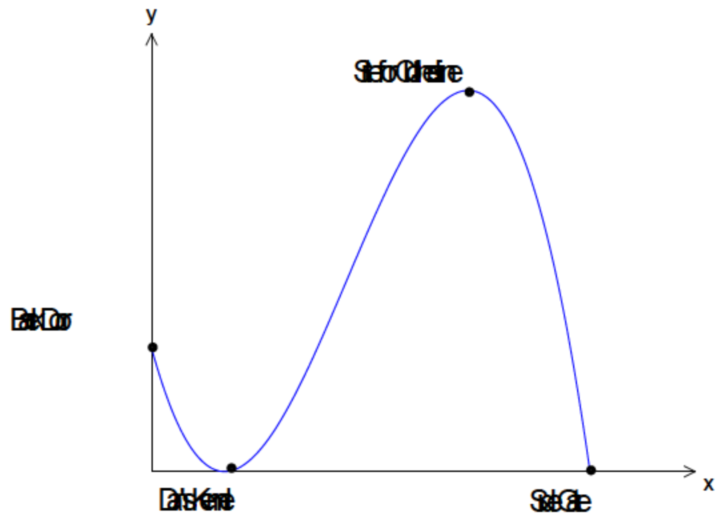
2 Marks

Q3. Milly is landscaping her backyard. For the purposes of planning she creates the following plan on a cartesian plane. Each unit of the scale is a metre.

The back wall of her house is the positive y axis and the boundary fence is the positive x-axis

Her back door is located at (0,3). On the boundary fence, Dan the Dog's kennel is located at (1,0) and the side gate is located at (6,0).

She wants to install an all-weather path joining the points in the shape of a cubic curve, so that one of the turning points occurs at Dan the Dog's kennel as shown in the diagram:



a. If the path has the general equation: $y = a(x-b)^2(x-c)$, using the features of the graph justify that $a = -\frac{1}{2}, b = 1, c = 6$

2 Marks

b. If the clothesline is to be located at the second turning point, find the **exact** co-ordinates of the clothesline.

2 Marks

c. Milly marks out a string line, $f(x)$ between the back door and the point on the path (4,9). Find the equation of this string line.

2 Marks

d. The area between the string line, $f(x)$ and the path will be covered with lawn. Establish a definite integral for the area of lawn that will be sown. Using CAS calculator or otherwise, find the area of lawn

2 Marks

e. Milly is thinking of putting in a retaining wall for a vegetable garden. The retaining wall will be perpendicular to the path at $x=2$ and end at the boundary fence. Find the equation of $g(x)$, representing this retaining wall.

2 Marks

f. Determine if this retaining wall will meet the boundary fence between the kennel and the side gate.

2 Marks

Q4. Bat Mann is attempting to escape at night from a villain without being seen. A grass quadrangle is bordered by four concrete paths along the x-axis, the y-axis and the lines y=46 and x=46.

His starting point is at $O(0,0)$ shown on the diagram below. The space has two floodlit zones, either side of a corridor which is in darkness.

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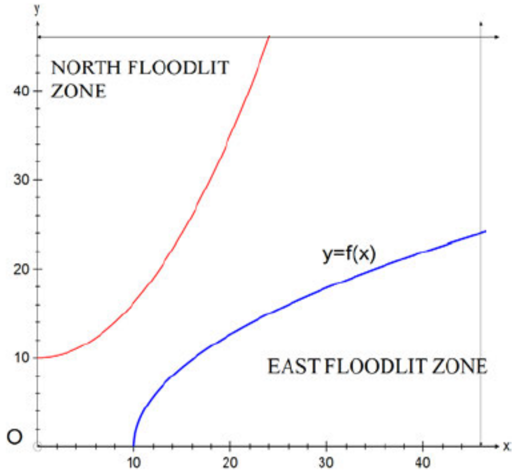


The *x*-axis runs in an east-west direction.
The east floodlit zone is enclosed by the *x*-axis, the concrete path $x = 46$ and the function

$$f:[10,46] \to R,\ f(x) = 4\sqrt{x-10}\ .$$

Each unit of the scale is a metre.

a. Find $\int 4\sqrt{x-10}\ dx$

1 Marks

b. Use your answer to part a. to find the area bound between f(x), the x- axis and x = 46.

2 Marks

The north floodlit zone is enclosed by the *y*-axis, the concrete path $y = 46$ and the graph of the function f^{-1}**;** the inverse function of *f*.

ci. Show that the rule for the inverse function is $f^{-1}(x) = \dfrac{x^2}{16} + 10$

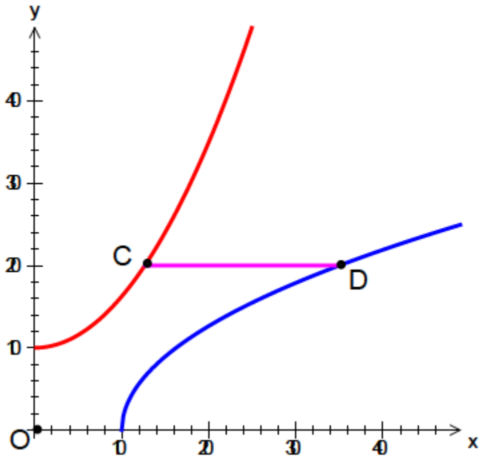
cii. Find the domain for f^{-1}

2+1 Marks

d. Find the derivative of the inverse function

1 Mark

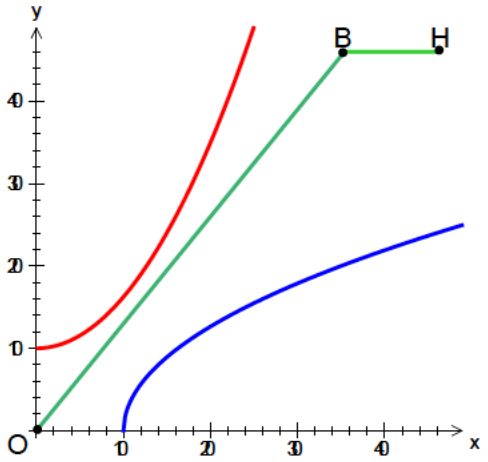
If Bat moves in a straight line from $O(0,0)$ he may pass over a sensor wire that runs in an east-west direction along the line $y=20$ in the area that is **not** floodlit.
The point where he passes over the sensor wire is given by $(s,20)$.



e. Find the co-ordinates of C and D, as exact values.

2 Marks

Bat needs to rendezvous with a helicopter. If he moves in a straight line from his starting point at $O(0,0)$ until he reaches the concrete path at $B(x,46)$ where he will meet with Robyn. From this point they will move due east where he will be able to safely board the helicopter at $H(46,46)$.



On the grass, he can able to move at a speed on 1m/s, on the concrete path he can move at 3m/s

The time *T*, in seconds, taken by Bat to move from *O* to *H* via *B* is given by

$$T = \sqrt{x^2 + 2116} + \frac{46 - x}{3}, \quad x \in [0, 46].$$

fi. Find the value of *x*, correct to 2 decimal places, for which Bat would reach the helicopter, via B, in the minimum time.

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fii. Hence find the minimum value of *T* correct to 2 decimal places.

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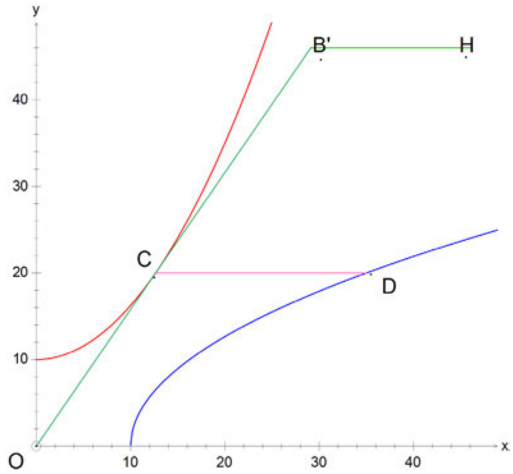
1 + 1 Marks

Bat realises that this pathway takes him into the floodlit zone. To be undetected he must follow the path that is a tangent to f^{-1} where the sensor wire meets the curve at C to the point B'.



gi. Find the equation of the tangent at the point C, proving that is passes through the origin. Use exact values.

gii. Find the co-ordinates of B'. Use exact values.

2+1 Marks

h. Find the time it takes to reach the helicopter via B'. Give answer to two decimal places.

1 Mark

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x}\,dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$		$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$		$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		