Mathematical Methods SAC 2

Modelling/Problem Solving

Wednesday 8th August 2018

10 minutes reading time

120 minutes writing time

- This task is to be completed in one session of duration 130 minutes.
- During this task, you may use your calculator and refer only to one set of bound notes. No other pieces of paper may be used.
- You must work silently and independently for the duration of this task.
- All answers are to be written within this booklet.
- Questions worth more than one mark require some explanation of working or process.
- When drawing graphs, ensure that a pencil is used and that significant features of the graph are clearly indicated in pen.
- **Exact values** are expected throughout, unless otherwise stated and **appropriate units must be included**.
- Your CAS calculator may be collected at the conclusion of the SAC so that all memories can be cleared, and will be returned to you in your first Mathematical Methods class.
- No electronic devices (such as mobile phones) may be brought into the examination room.

Total : 50 marks

Student Name : ________________________________

Teacher Name: ________________________________

COMPULSORY STUDENT DECLARATION

I, (*print your name neatly*) ____
acknowledge that I have read the SAC/examination conditions and understand which items/materials I am permitted to use and have in my possession.

****If you have any doubts as to what is permitted, raise your hand and DO NOT sign this declaration****

Student's Signature: ____

Student's Name: ____

Teacher's Name: ____

The grade awarded to this SAC is subject to statistical moderation

by the VCAA and is likely to change.

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Question 1 – Gary Eastwood's Pools (12 marks)

Gary Eastwood is the director of a swimming pool construction company. He has been asked to upgrade the construction process for Olympic size swimming pools. Gary has come up with two models for the cross section of an Olympic size swimming pool.

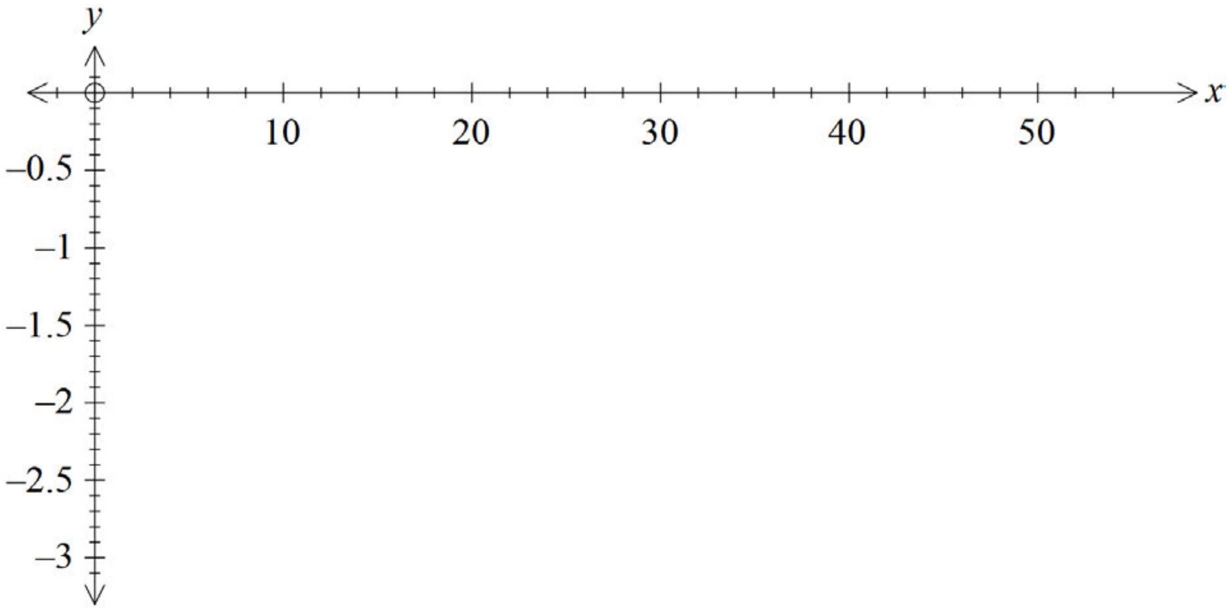
The first model is the function $p:[0,50]\to R$ where $p(x)=\frac{3x^2}{6250}-\frac{6x}{125}-2$

The second model is the function $q:[0,50]\to R$ where $q(x)=\frac{-3\sqrt{2x}}{25}-2$

All measurements are in metres

a. Sketch the graph of $p(x)$ and $q(x)$ on the axes below. Label the endpoints and the point of intersection of the two graphs with their coordinates correct to two decimal places.

(3 marks)



b. i) Write down a definite integral which gives the area bound by $p(x)$ and the x – axis.

(1 mark)

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ii) Hence, by showing your working, find the cross sectional area of the swimming pool defined by the model $p(x)$.

(2 marks)

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iii) If the pool were 25m wide, what would the volume of the swimming pool be?

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c. i) Write down a definite integral which gives the area bound by $q(x)$ and the $x-$axis.

(1 mark)

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ii) Hence, by showing your working, find the cross sectional area of the swimming pool defined by the model $q(x)$.

(3 marks)

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d. Compare the areas of each model.

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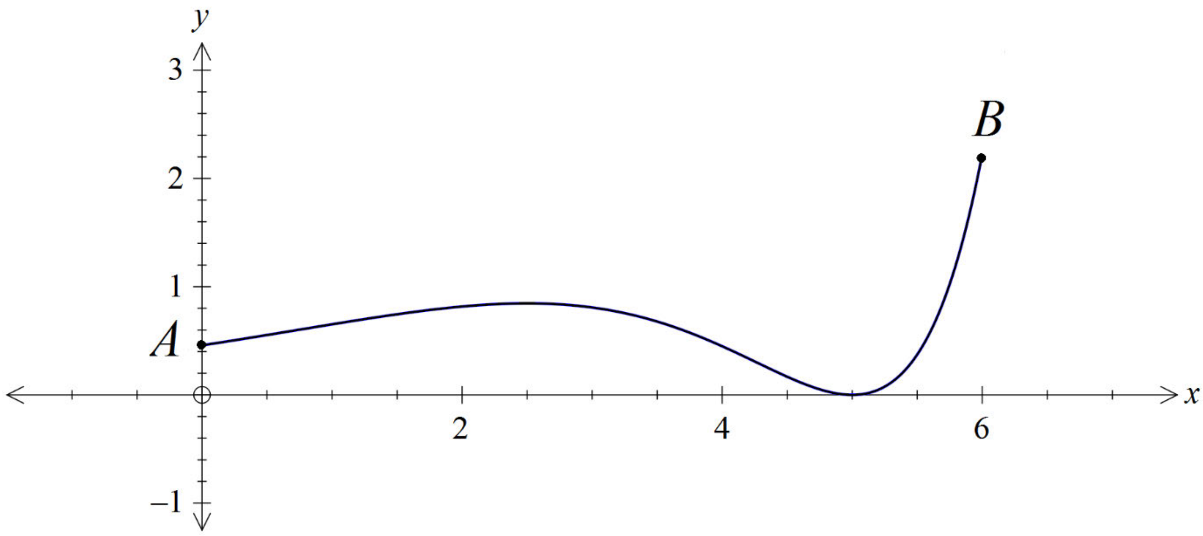
Question 2 – Geraldine Northforest's Request (16 marks)

Gary has been asked by Geraldine Northforest, a particularly prickly customer, to have an above ground pool built, with the cross section having the equation

$$f:[0,6] \to R, \text{where } f(x) = (x-5)^2 e^{\frac{4x}{5}-4}$$

The graph of $f(x)$ is shown below. All measurements are metres.



a. Find the exact coordinates of A and B

(2 marks)

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b. Use calculus (show your working) to find $f'(x)$. Give your answer as the sum of two terms.

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Gary knows he will need to build some steel supports to hold up the right edge of the pool. In addition to a vertical support from (6,0) to B, he also needs to build a steel support which is a tangent to the curve at $\left(\frac{11}{2}, \frac{1}{4}e^{\frac{2}{5}}\right)$.

The steel support can be modelled by the equation $y = ax + b$

c. Show that $a = \frac{6}{5}e^{\frac{2}{5}}$

(2 marks)

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d. Hence, find the exact value of b.

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e. The stationary point(s) of $f(x)$ are of interest to Gary. Find the exact coordinates of the stationary point(s) of $f(x)$

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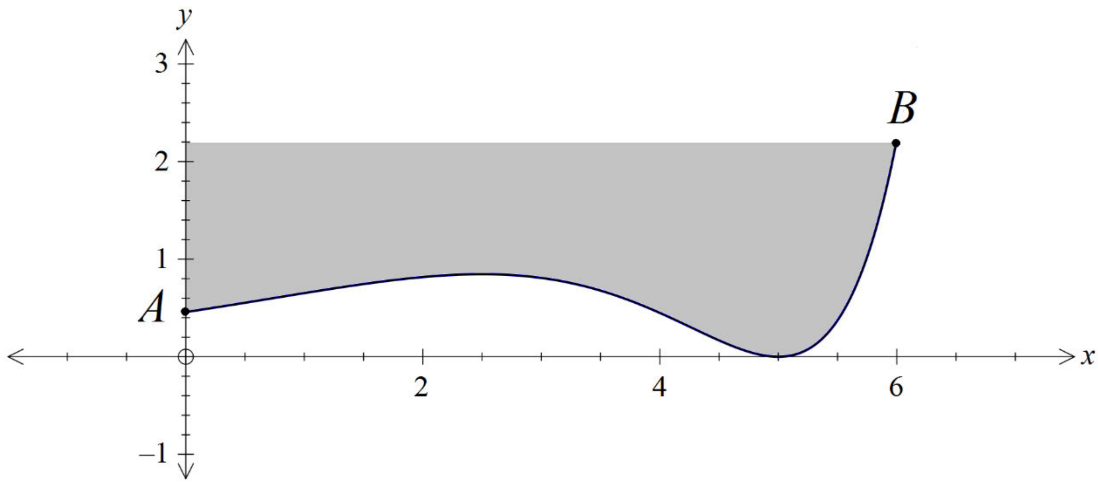
The pool is filled with water to the top as shown.



f. Find the area of the shaded region correct to one decimal place.

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g. How deep is the pool at $x = 2$? Give your answer correct to two decimal places.

(1 mark)

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h. What is the **average** depth of the pool? Give your answer correct to three decimal places

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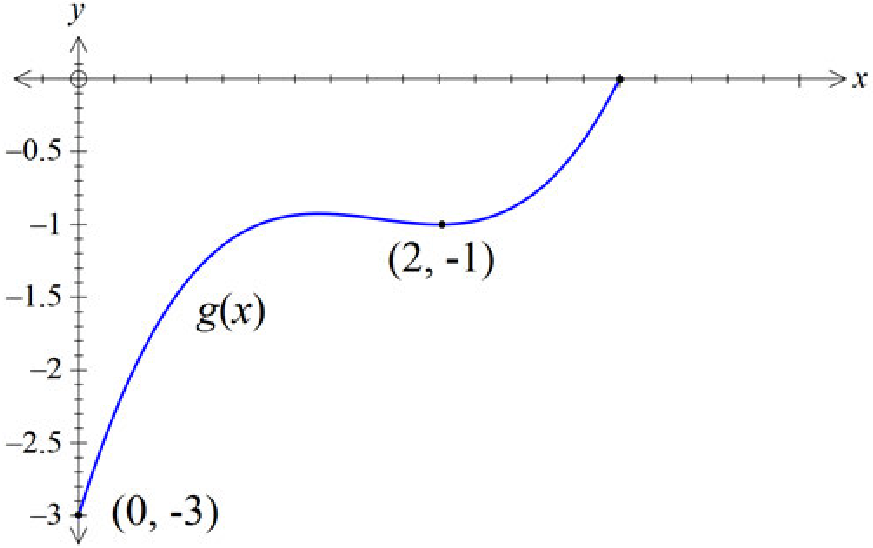
Question 3 – The Envy of Frank Southglade (22 marks)

Frank Southglade is a neighbour of Geraldine's, and once her pool is installed, he becomes envious. He desires his own pool, but he demands it to be specifically built. He wants the cross section of his pool to follow the equation

$$g:[0,3] \to R, \text{where } g(x) = \frac{1}{2}x^3 + bx^2 + cx - 3$$

The graph of g is shown below, where (2, –1) is a local minimum.



a. Find $g'(x)$

(1 mark)

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b. i) Use the fact that (2, –1) is a local minimum to write down two equations which can be solved simultaneously to find b and c

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ii) Hence, show that $b = -\frac{5}{2}$ and $c = 4$

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c. Find the x-coordinate of the other stationary point of $g(x)$ and prove that is a local maximum. Show your working.

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d. Hence write down the exact coordinates of the other stationary point of $g(x)$.

(1 mark)

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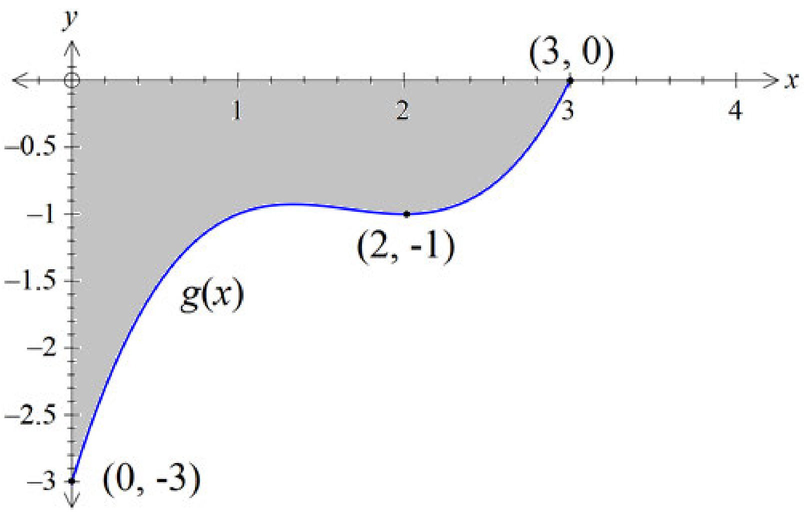
e. Use calculus (i.e show your working) to find the exact cross sectional area of Frank's pool as shown below.



(3 marks)

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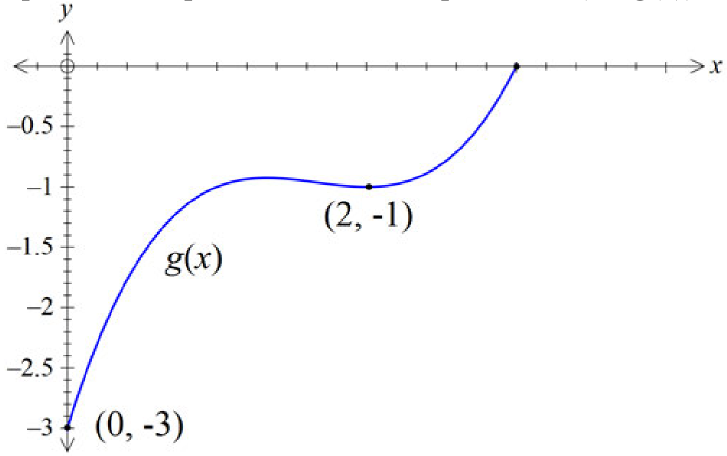
Frank has many young nieces and nephews (but no children of his own), and thinks that the deep end of the pool is too deep. He wants Gary to construct a retractable ramp which can be used when his nieces and nephews come around to block off the deep end of the pool.

The ramp will originate from (0, 0) and reach a point on the pool floor (i.e. $g(x)$) such that it makes a right angle with it (i.e. it is perpendicular to $g(x)$ at the point where it touches it). The ramp is also the shortest possible ramp that can reach the pool floor (i.e. $g(x)$) from (0, 0)



f. i) Find the coordinates on $g(x)$ where the ramp touches. Give your answer correct to 3 decimal places

(3 marks)

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ii) Find the length of the ramp correct to 2 decimal places

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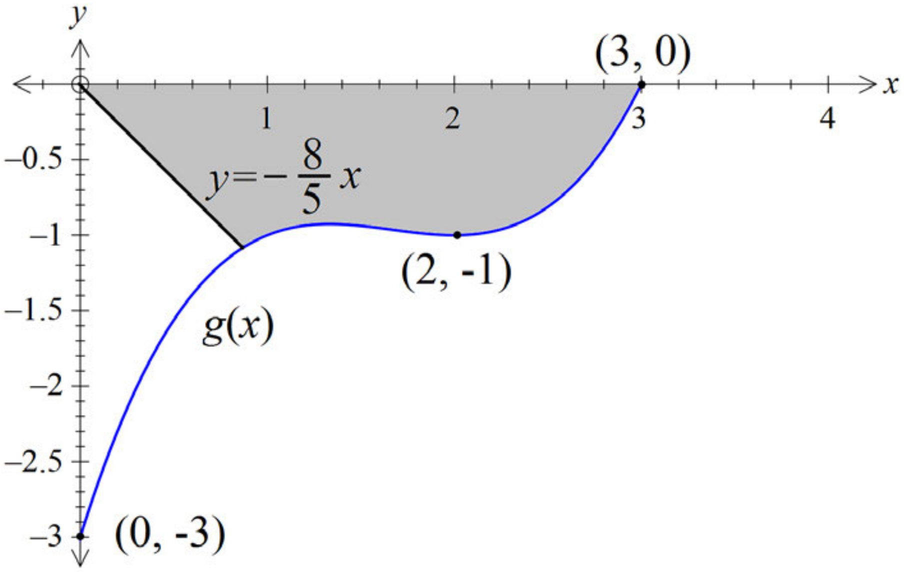
Gary decides that the ramp that Frank has asked for is too difficult to make, and has opted to instead build a ramp which has the equation $y = -\frac{8}{5}x$ as shown below.



g. Find the coordinates of intersection of this ramp and $g(x)$ correct to three decimal places. (1 mark)

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g. Hence or otherwise, find the area of the shaded region above correct to three decimal places.

(2 marks)

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h. What percentage of the area of the entire pool is blocked off by the ramp? Give your percentage rounded to the nearest whole number.

(2 marks)

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END OF SAC