XAVIER MATHS

Mathematical Methods SAC 1 (2019)

APPLICATION TASK

Swinging Boys

Booklet 2 – 27nd May

3:45 – 5:55

10 minutes reading time

120 minutes writing time

This task is to be completed in one session of duration 130 minutes.
During this task, you may use your calculator and refer to your notebook and/or Preparation Booklet. No other pieces of paper may be used.
You must work silently and independently for the duration of this task.
All answers are to be written within this booklet.
When drawing graphs, ensure that a pencil is used and that significant features of the graph are clearly indicated in pen.
Exact values are expected throughout, unless otherwise stated and units must be included.
Your CAS calculator will be cleared at the conclusion of the SAC.
No electronic devices (such as mobile phones or watches) may be brought into the examination room.

Total : 65 marks

Student Name : ______________________________

Teacher Name: ______________________________

The grade awarded to this SAC is subject to statistical moderation by the VCAA and is likely to change.

COMPULSORY STUDENT DECLARATION

I, (*print your name neatly*) ____
acknowledge that I have read the SAC/examination conditions and understand which items/materials I am permitted to use and have in my possession.

****If you have any doubts as to what is permitted, raise your hand and DO NOT sign this declaration****

Student's Signature: ____

Student's Name: ____

Teacher's Name: ____

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

SWINGING BOYS

After completing the Roller Coaster Exercise the previous week along with their other work, the boys, Linus, Ollie and Brody take a break. They go to the Hayes Paddock and play on the swing. Whilst they are there, the boys met two friends Gianni and Alberto.   The robust play of Gianni and Alberto, soon lands one of them in trouble and you are asked to assist with medical procedures and recuperation.

Question One **19 marks**

Gianni and Alberto play on the swing. When Gianni pulls the swing back and lets Alberto go, the swing's displacement can be approximately modelled by a function of the form

$$s = Ae^{-\frac{t}{b}}\cos(ct)$$

where ***t*** measures the time (in *seconds*) from when the swing is released, and ***s*** gives the horizontal displacement of the seat of the swing (in *metres*) from the equilibrium position.
(The equilibrium position is the position at which the swing will eventually come to rest).
A, b and c are *positive* constants.

a Consider firstly the function where $A = b = c = 1$.

Sketch the graph of the function over the first $\frac{7\pi}{2}$ seconds, that is $\left[0 \leq t \leq \frac{7\pi}{2}\right]$.

Give exact values of the ***s*** and ***t*** intercepts and coordinates of endpoints.

(Hint: the first ***t***-intercept is at $t = \frac{\pi}{2}$) **.**

You may need to *exaggerate* the rises and falls of the graph in order to see the pattern clearly.

[3]

b Do you consider that this equation provides a realistic model of the swing's movements?
Give a reason for your answer.

[1]

c Consider your graph in part (*a*).
What is the horizontal distance of the swing from the equilibrium point, in metres, when Gianni lets the swing go?

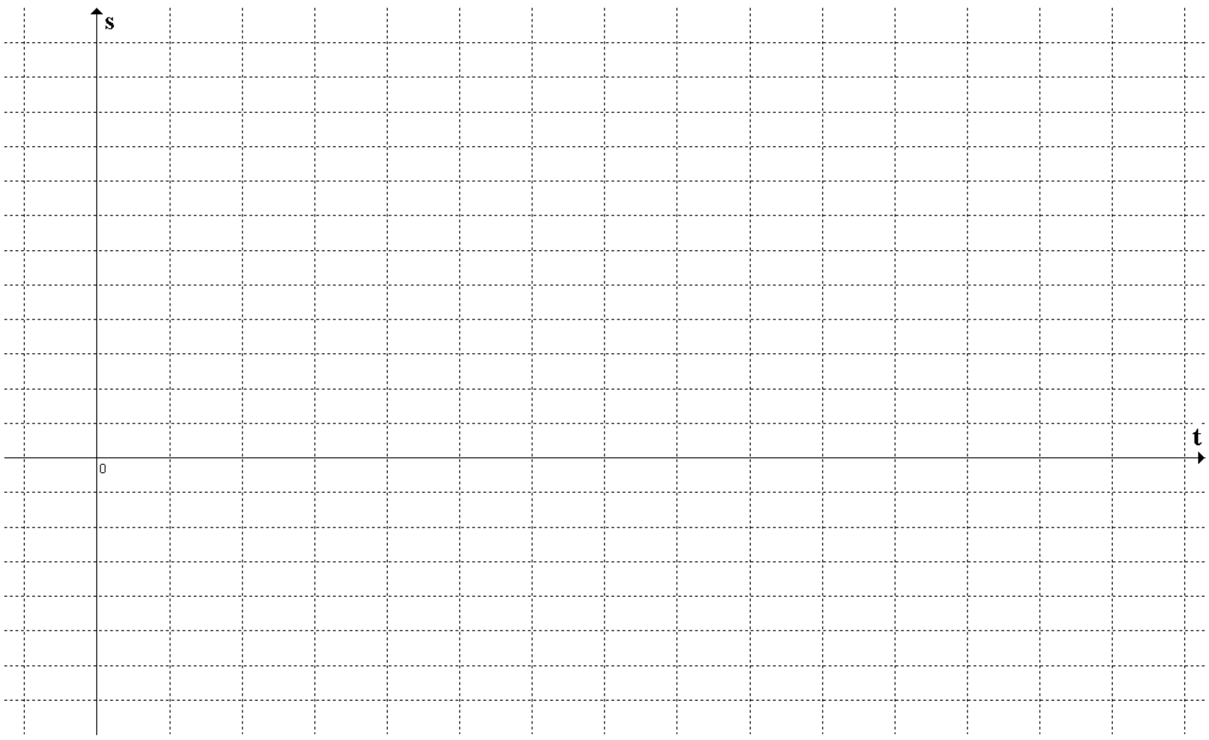
[1]

d Now consider the function where $A = b = 1$ and $c = 2$.

Sketch the graph of the function over the first $\dfrac{7\pi}{4}$ seconds, that is $\left[0 \le t \le \dfrac{7\pi}{4}\right]$.

Give **exact** values of the ***s*** and ***t*** intercepts and coordinates of endpoints.

[3]



Gianni stands and watches Alberto swing back and forth. Gianni is an able mathematician and notices that the graph of $s = e^{-t}\cos(2t)$ appears to be exponentially decreasing.

While Alberto is on the swing, Gianni takes out his *i*Pad and sketches the graph of

$$f : \left[0, \frac{7\pi}{2}\right] \to R, f(t) = e^{-t}.$$

Note the change of the domain.

e Sketch the graphs of both $f(t) = e^{-t}$ and $s = e^{-t}\cos(2t)$ for $t \in \left[0, \frac{7\pi}{2}\right]$ on one pair of axes.

Label the coordinates of the endpoints and any minimum points of both graphs, using exact values.
(*There is no need to label intersection points between the 2 graphs*).

[5]

f Using your graphs, or by other means, in part (*e*) to write down exact values of the coordinates of the first three points of intersection of the two graphs ***f*** and ***s*** for the domain $t \in \left[0, \frac{7\pi}{2}\right]$.

[3]

g Describe the pattern in the ***t***-values of the points of intersection in part (*f*).

[1]

Finally, Gianni pulls Alberto as far back as possible before he lets him go, so that Alberto's path follows the graph of

$$s = 3e^{-\frac{t}{5}}\cos(2t)$$

where ***t*** measures the time in seconds from when the swing is released, and ***s*** gives the horizontal displacement of the seat of the swing (in *metres*) from the equilibrium position (the position at which the swing will eventually come to rest).

h How far horizontally does Gianni pull Alberto back before he lets the swing go?

[1]

Alberto is badly injured when he eventually falls off the swing, at a horizontal distance of 1 metre from the equilibrium point for the *fourth time.*

i For how long does Alberto remain on the swing before falling off?
Give your answer to the correct to two decimal places.

[1]

Question Two **17 marks**

Alberto and Gianni are both rushed by ambulance to hospital, where Alberto learns that his leg is broken, requiring an operation.
Doctors schedule the operation for 10.00 am Monday morning, the day after the accident.

After the operation, Alberto's temperature follows the function with the rule

$$T(t) = 2.5\cos\left(\frac{\pi t}{6}\right) + 38$$

where $\boldsymbol{t}$ is the time, in hours after 10 am Monday, and T°C is Alberto's temperature at time $\boldsymbol{t}$.

a State the amplitude and period of the function with the rule $y = T(t)$.

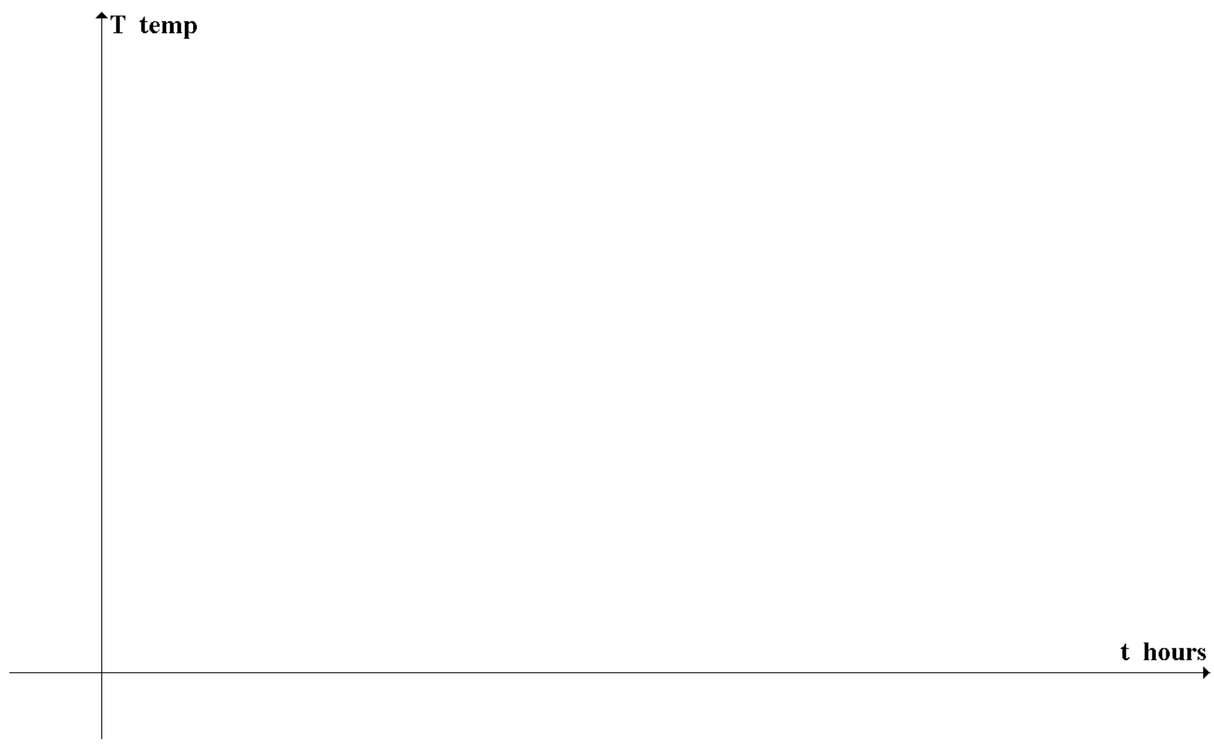
[2]

b Sketch a graph of the function

$$T(t) = 2.5\cos\left(\frac{\pi t}{6}\right) + 38 \qquad \text{for } \ 0 \le t \le 24$$

on the pair of axes.
Label the coordinates of maximum and minimum points, and endpoints, correct to **1 decimal place**.

[4]



c In the first 24 hours after his operation, what is Alberto's highest temperature? Also give the **time(s)** when his temperature reaches this maximum.

[3]

Alberto's temperature rises quite quickly at certain times.

d On your graph in part (*b*), draw a chord which could be used to find the average rate of change in Alberto's temperature between 6 pm and 8 pm on the day of the operation. Clearly label the endpoints, giving values correct to **2 decimal places**.

[2]

e Find the *average rate of change* in Alberto's temperature (in $^\circ C$/hour) from 6 pm till 8 pm on the day of the operation. Answer correct to 2 decimal places.

[2]

f Between what times of day over the first 24-hour period will this average rate of change be repeated?

[2]

g The doctor needs to be called when the *magnitude* of the *instantaneous* rate of change of Alberto's temperature exceeds $1.25\ ^\circ\text{C/hour}$.
This happens twice in the first 24 hours after the operation.
At what times will the doctor need to be called in the first 24 hours? Explain your answer.

[2]

Question Three **17 marks**

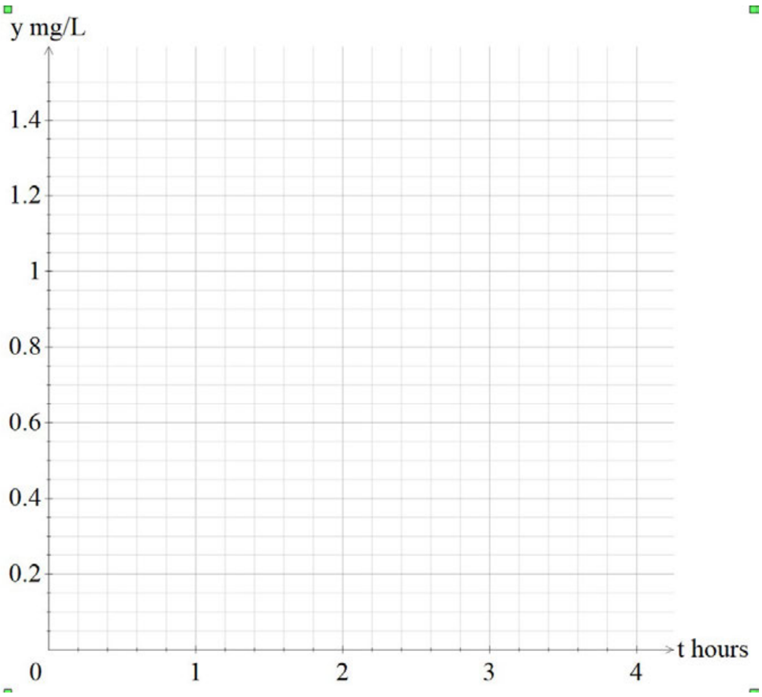
Once Alberto's operation is over, he needs pain relief, administered by his nurses.
The safe medicinal use of a drug requires knowledge of both its rate of absorption by the body (which may depend on how the drug is given) and its rate of clearance (principally through the kidneys). Alberto wants the absorption to be swift, and the clearance (of pain relief) to last a long time. A class of functions which can be used to model this process are functions of the form

$$y = At^b e^{-ct}$$

where ***y*** represents the concentration of the drug in the blood in milligrams per litre (mg/L) and ***t*** represents the time, in hours, that have elapsed since the drug was injected.
A, b and c are arbitrary positive constants.

a Firstly consider the function where $A = b = c = 1$.
Sketch the graph of the concentration, ***y*** **,** over the first 4 hours, clearly labelling the co-ordinates of any maximum and minimum points, and endpoints, with values correct to **2 decimal places.**

[3]

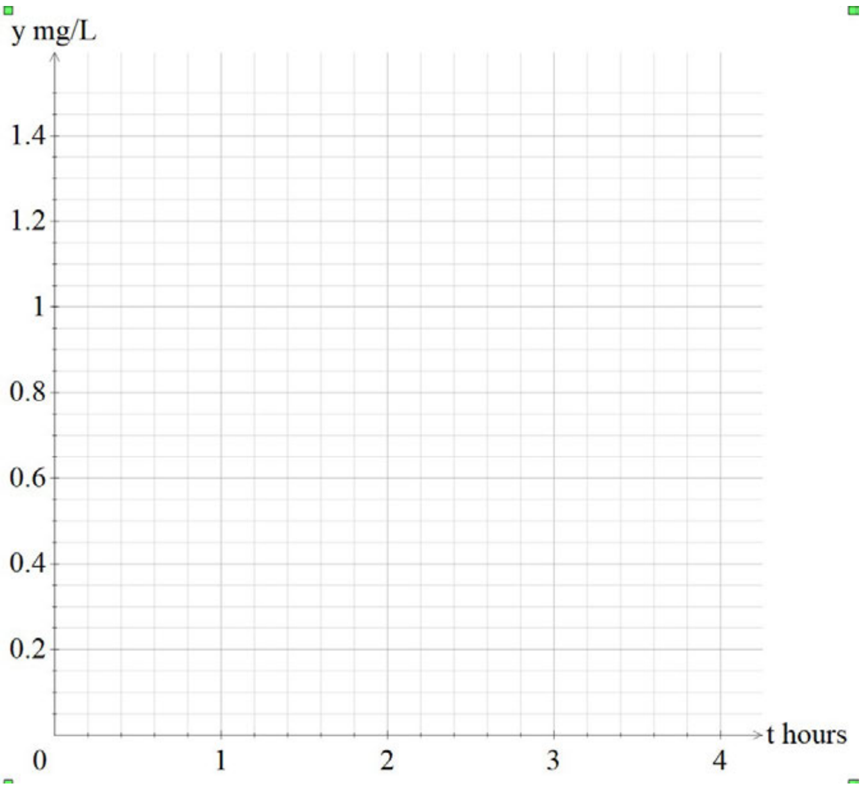


b What percentage of the maximum concentration is lost between the time it reaches its maximum and the end of the 4-hour period? Give your answer correct to one decimal place.

[2]

c The concentration of a second drug follows the equation $y = At^{b}e^{-ct}$ with $A = c = 1$ and $b = 3$. Sketch the graph of this concentration, over the first 4 hours, with co-ordinates of important points correct to **2 decimal places**.

[3]



d Compare the two graphs you drew in parts (*a*) and (*c*).
Which do you consider to be the more appropriate to meet Alberto's needs?
Explain your answer.

[2]

e If the nurses use the method described in part (*c*), how long will Alberto have to wait for maximum pain relief?

[1]

The graphs of parts (*a*) and (*c*) suggest that there is a pattern to the values of ***b*** in the equation

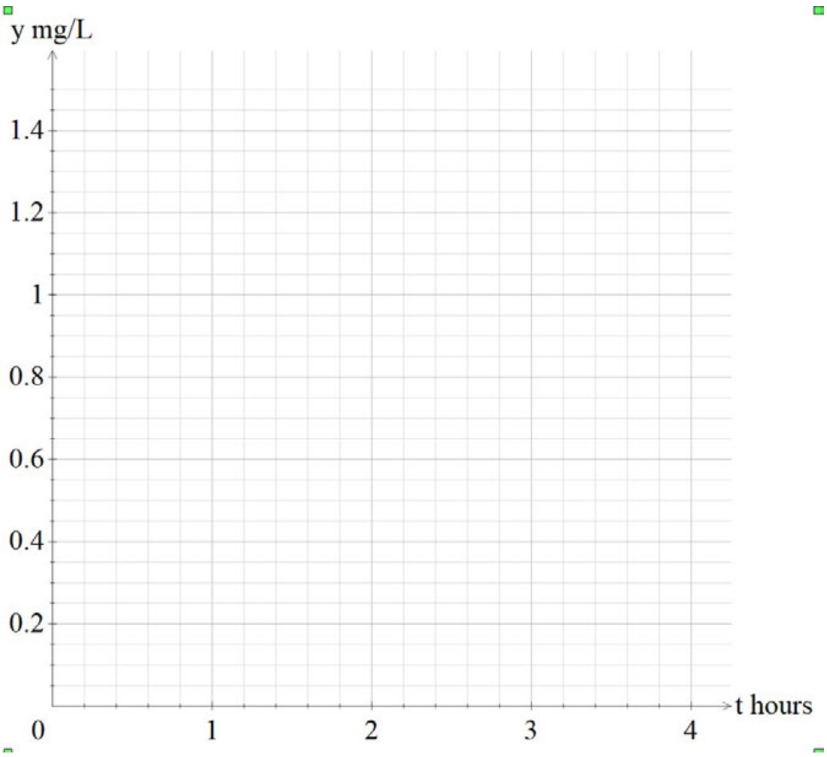
$$y = At^b e^{-ct}.$$

f Whichever drug is used, its concentration must reach its maximum 30 minutes after injection. Find the value of ***b*** in the model of the drug concentration for which this will occur. Let ***c*** = 1.

[1]

g Sketch a graph of the function $y = At^b e^{-ct}$ over 4 hours. Use your value of ***b*** from part (***f***) and the values $A = 0.46633$ and $c = 1$. Give values on your graph to **2 decimal places**.

[3]



h Ideally, the concentration after two hours should be close to half its maximum amount.
Does the model of part (***g***) satisfy this condition? Give a reason.

[2]

Question Four **12 marks**

On the Wednesday following Alberto's operation, an ambulance takes Alberto and Gianni home.
Both boys live on Timbertop Road, a long straight road linking a Cement Works (*C*) and a Pulp Mill (*M*), which are 10 kilometres apart.
Alberto lives 3 kilometres from the Cement Works, Gianni lives 1 kilometre from the Pulp Mill.

Both these industries produce air pollution. The concentration (in parts per million) of the pollution produced by each industry can be modelled by values of y_1 and y_2 in the equations

Cement Works C : $y_1 = \dfrac{p}{x+1}$ and

Pulp Mill M : $y_2 = \dfrac{q}{11-x}$.

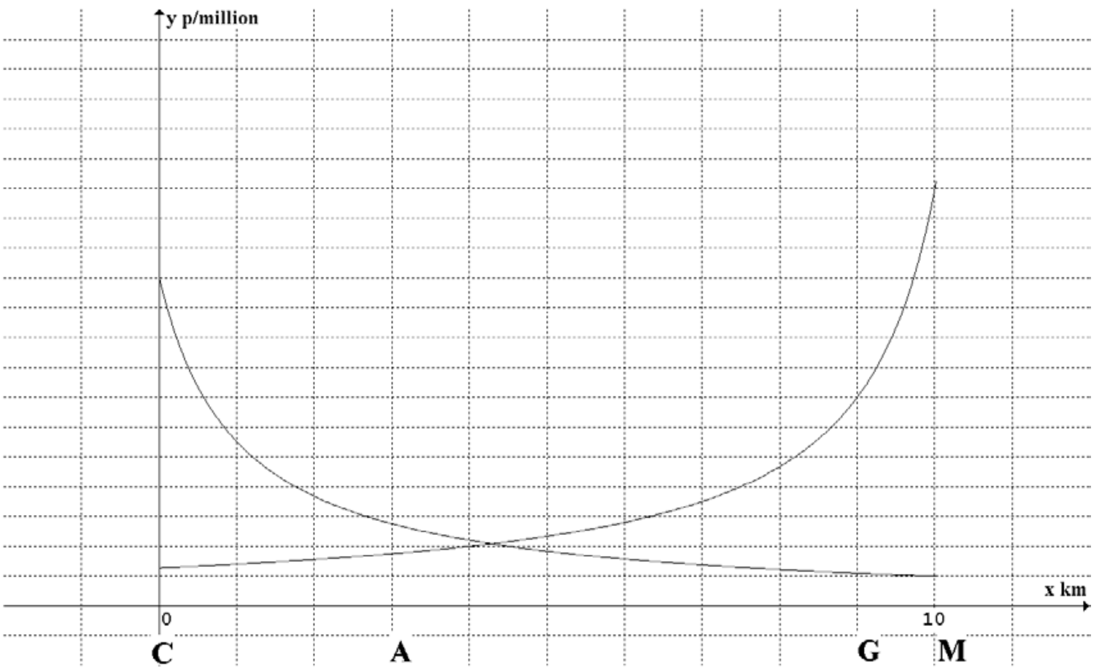
In these formulas, *p* and *q* are constants which vary from day to day, and *x* kilometres is the distance along Timbertop Road from the Cement works C $(0 \le x \le 10)$.

The *total* air pollution is given by $y = y_1 + y_2 = \dfrac{p}{x+1} + \dfrac{q}{11-x}$ ppm.

a On the Wednesday that Alberto and Gianni arrive home, the values of *p* and *q* are such that sections of the graphs of $y_1 = \dfrac{p}{x+1}$ and $y_2 = \dfrac{q}{11-x}$ against *x* are as shown below.

On this set of axes, sketch also the graph of $y = y_1 + y_2$.
Label the endpoints of your graph in terms of *p* and *q*.

[2]



A Alberto's house **G** Gianni's house Key

b The total air pollution can be expressed as $y = \dfrac{mx+n}{(x+1)(11-x)}$ where m and n are constants..

Show, using algebraic methods, that $m = q - p$ and $n = 11p + q$.

[3]

c On the Wednesday that they arrive home, the total air pollution at Alberto's house is 4.5 units, whereas it is 8.1 units at Gianni's home.
Form two equations and use them to show that, on this Wednesday, $p = 11$ and $q = 14$.

[3]

d As the ambulance drives from Albertos' house towards Gianni's home, the ambulance driver notes that the level of air pollution is changing.
Find the *exact* instantaneous rate of change in the total air pollution (with respect to distance x) when the ambulance is exactly halfway between the Cement Works and the Pulp Mill.

Is the total air pollution increasing or decreasing at this point?
Explain your answer.

[2]

e On the same day, the total air pollution reaches a minimum at some point P on Timbertop Road between the Cement Works and the Pulp Mill.

Find the distance of point P from the Pulp Mill M, in kilometres correct to **4 decimal places**.
State also the concentration of air pollution at P on this day, correct to **3 decimal places**.

[2]

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End of SAC