READING START:
WRITING START:
WRITING END:

Mathematical Methods SAC 2

Modelling/Problem Solving

Wednesday 7th August 2019

10 minutes reading time

120 minutes writing time

- This task is to be completed in one session of duration 130 minutes.
- During this task, you may use a CAS calculator and refer to one set of bound notes and SAC preparation material.
- You must work silently and independently for the duration of this task.
- All answers are to be written within this booklet.
- Questions worth more than one mark require some explanation of working or process.
- **Exact values** are expected throughout, unless otherwise stated.
- Your will be required to clear all CAS memory at the conclusion of the SAC
- No electronic devices (such as mobile phones) may be brought into the examination room.
- Questions seeking clarification may occur in reading time only
- There are FOUR questions.

Total : 60 marks

Student Name : ____________________________________

Teacher Name: ____________________________________

The grade awarded to this SAC is subject to statistical moderation by the VCAA and is likely to change.

COMPULSORY STUDENT DECLARATION

I, (*print your name neatly*) ____

acknowledge that I have read the SAC/examination conditions and understand which items/materials I am permitted to use and have in my possession, and will undertake to clear the memory of my CAS calculator at the conclusion of the SAC.

****If you have any doubts as to what is permitted, raise your hand and DO NOT sign this declaration****

Student's Signature: ____

Student's Name: ____

Teacher's Name: ____

Mathematical Methods formulas

Mensuration

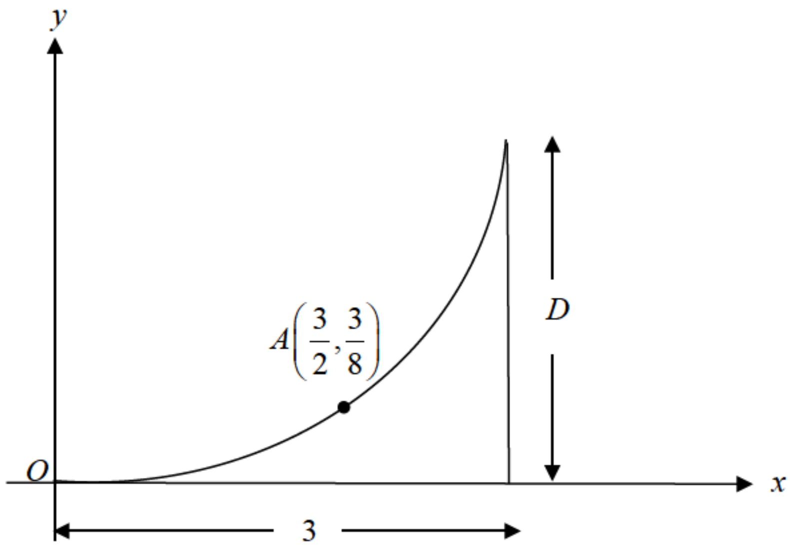
area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$		$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right)=an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c, n\neq -1$	
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$		$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$	
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$		$\int \frac{1}{x}dx=\log_e(x)+c,\ x>0$	
$\frac{d}{dx}\left(\sin(ax)\right)=a\ \cos(ax)$		$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$	
$\frac{d}{dx}\left(\cos(ax)\right)=-a\ \sin(ax)$		$\int \cos(ax)dx=\frac{1}{a}\sin(ax)+c$	
$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$		

Question 1 (14 marks)

At a circus, roller skaters move up and down a ramp. The ramp is designed as a function of x, the horizontal distance of the ramp from the origin O and y represents the height of the ramp above the ground. The lengths are measured in metres. The ramp has a width of 3 metres and a height of D metres. The diagram below show a cross section of the ramp.



The point A, is located horizontally at the midpoint of the ramp and is $\frac{3}{8}$ metres above the ground. For the safety of the performers, the function to represent the ramp must meet certain criteria

CRITERIA 1: At the origin, the gradient of the graph is zero.

CRITERIA 2: At the point A, the tangent to the ramp is inclined at 45^0.

CRITERIA 3: The gradient of the ramp is always increasing.

The designers of the ramp try to model the ramp by various functions, $f(x), g(x)$ and $h(x)$.

a. Let $f(x) = ax^2$ where a is a positive constant. Show that this function does satisfy CRITERIA 1, however it is not possible to satisfy CRITERIA 2.

2 marks

b. Let $g(x) = bx^3 + cx^2$ where b and c are constants.

i. Find the values of b and c if this function satisfies CRITERIA 2.

3 marks

ii. Explain why this function does not satisfy CRITERIA 3.

2 marks

c. Let $h(x) = \frac{x^n}{k}$ where n is a positive integer and k is a positive constant.

i. If this function satisfies CRITERIA 2 show that $k = \frac{27}{2}$ and n = 4.

2 marks

ii. For this function find the value of D.

1 mark

iii. For this function find the angle made with the horizontal, to the nearest degree, of the tangent at the highest point on the ramp.

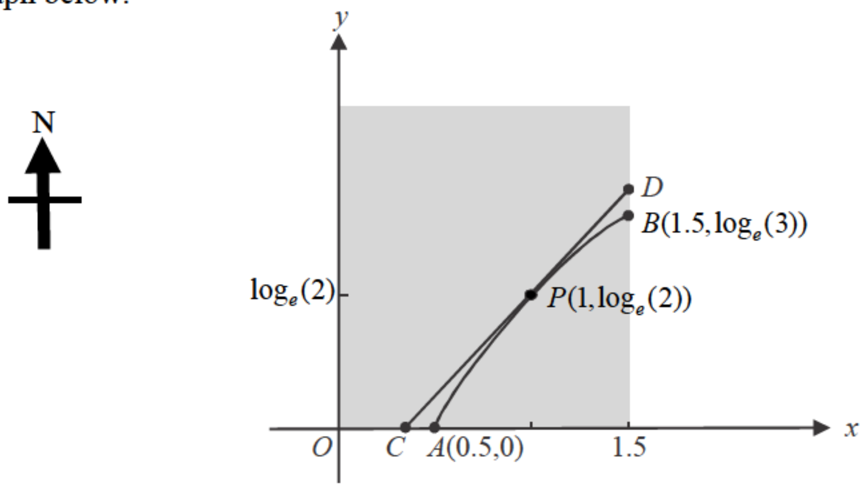
2 marks

iv. For this function find using calculus, the cross-sectional area under the ramp, in square metres.

2 marks

Question 2 (16 marks)

A dry creek bed that follows a curve with equation $y = \log_e(2x)$, between points A and B, is shown on the graph below.



The creek bed runs through a rectangular piece of flat, public land which is shaded in the diagram.
A straight walking track runs between points C and D. The line CD is a tangent to the graph of $y = \log_e(2x)$ at the point $P(1, \log_e(2))$.
Each unit represents a kilometre and the positive y-axis is directly north.

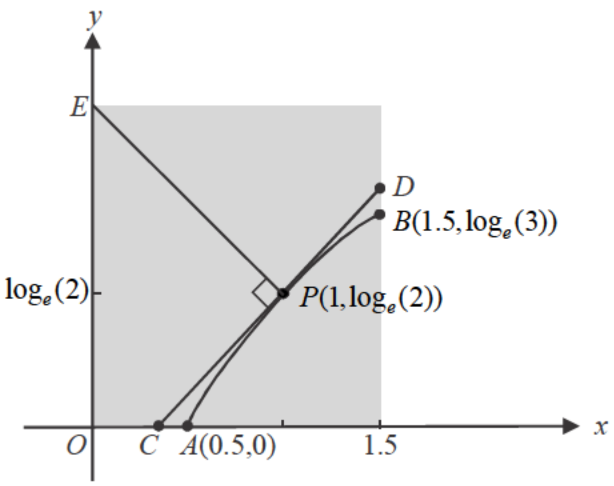
a. **i.** Show that A is the point $(0.5, 0)$ 1 mark

ii. Find $\frac{dy}{dx}$ when $x = 1$. 2 mark

iii. Show that the x-coordinate of point C is $1 - \log_e(2)$. 1 mark

iv. Show that the y-coordinate of point D is $0.5+\log_e(2)$. 1 mark

A second straight track is built. It runs from point E at the top corner of the public land to point P. This second track, EP, runs at right angles to CD.



b. Find the area of this piece of public land. Express your answer as an exact value in square kilometers. 3 marks

In order to create a bike track as well as a pedestrian track, local council plans to build a **straight** track that would run from point A to point B.

c. **i.** Show that the equation of this straight track AB is $y = x\log_e(3) - 0.5\log_e(3)$. 2 marks

ii. Find the maximum distance, in the north direction, between the dry creek bed and this new straight track between A and B. Express your answer in kilometers correct to 2 decimal places. 3 marks

The local council is considering replacing the dry creek bed with a barrel drain. The position of this drain would be slightly different to the current position of the dry creek bed.

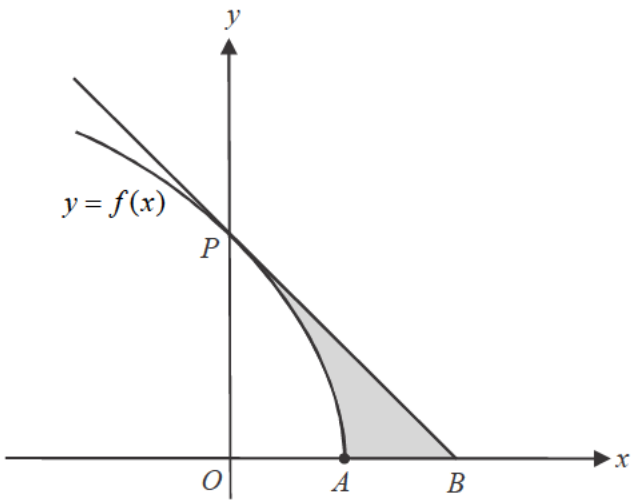
One model for the position of the new drain is obtained by dilating each point on the curve $y = \log_e(2x)$, for $0.5 \le x \le 1.5$ by a factor of n from the x-axis and by a factor of q from the y-axis.

Point $P(1, \log_e(2))$ would become point P' according to this transformation where P' is the midpoint of CD.

d. Find the values of n and q. Express your values correct to two decimal places. 3 marks

Question 3 (13 marks)

The graph of the function $f:(-\infty,1]\to R, f(x)=k\sqrt{1-x}$ where k is a positive integer, is shown below.



Point P is the y-intercept of the graph of f. The tangent to the graph of f at point P has its x-intercept at point B as shown.

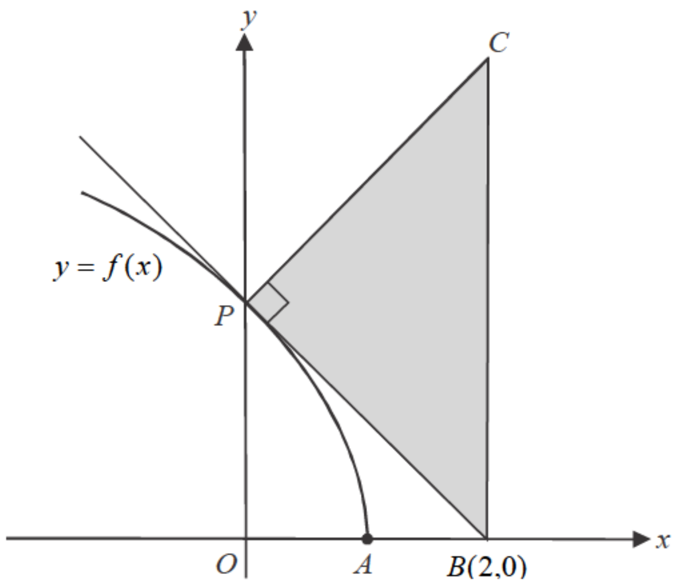
a. Find the equation of the tangent to the graph of f at point P, in terms of k. 2 marks

b. Show that B is the point $(2,0)$. 1 mark

c. The shaded region in the diagram is enclosed by the graph of *f*, the tangent at point *P* and the *x*-axis. Find, in terms of *k*, the area of this shaded region, in square units. 2 marks

d. The average value of function *f* between $x = d, (d < 0)$ and $x = 1$, is *k*.
Find the value of *d*. 2 marks

The normal to the graph of f at point P intersects with the line $x = 2$ at the point C.



ei. Find the co-ordinates of the point C, in terms of k. 2 marks

eii. Hence show that the area of ΔBCP, expressed in square units in terms of k, can be found by $A(k) = k + \frac{4}{k}$.

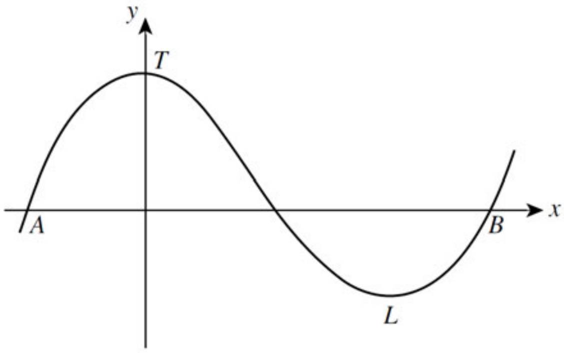
2 marks

f. Find $A'(k)$. 1 mark

g. Find the minimum area, in square units, of ΔBCP. 1 mark

Question 4 (17 marks)

Phil Hotham is a construction engineer working to build a new ski resort at Fool's Creek. Part of the plan of the main ski run on the mountain is shown on the diagram below.



This section of the mountain can be mapped by the curve with equation:

$$f(x) = \frac{x^3}{480} - \frac{7x^2}{48} + 75$$

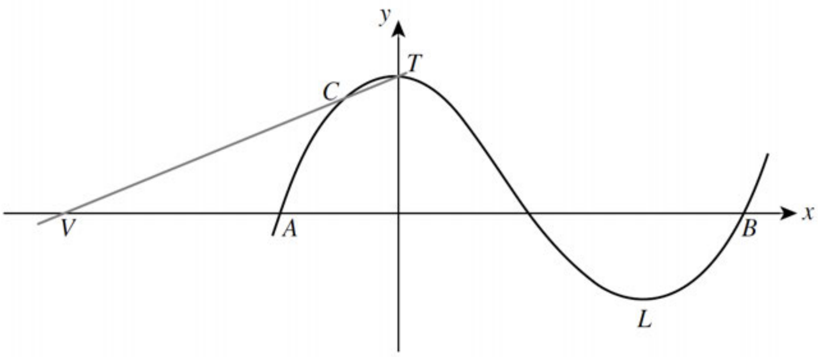
from A to B. The x-axis represents the level at which the village will be built. T is the highest point of the ski run and L is the lowest point. All distances are measured in metres.

a. Find the coordinates of A and B. 2 marks

b. Find the height of the top of the mountain above the village, T, in metres. 1 mark

c. Find the average **gradient** of the ski run from *T* to *L*. 3 marks

Phil is planning a chairlift to take the skiers to the top of the mountain from the village. He plans to build the village at the point *V*, which is 40 m from *A*. The path of the chairlift is shown on the diagram below from *V* to *T*.



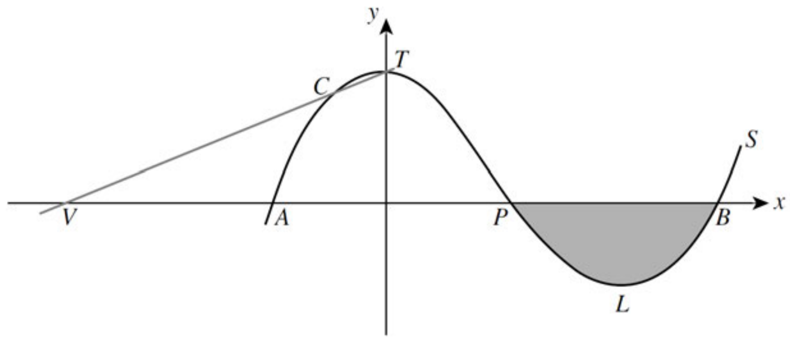
d. Find the equation of the line which models the path of the chairlift. 1 mark

Phil has found that the chairlift will reach the mountainside at the point C before it gets to the top of the mountain. He decides to build a tunnel to complete the journey to the top, T.

e. Find the coordinates of point C, to two decimal places. 2 marks

f. Find the length of the tunnel required to get to the top of the mountain from C to T. State your answer to the nearest centimetre. 2 marks

The developers of Fool's Creek would like to entice visitors to the resort all year round. Phil discovers that when the snow melts, a pond is formed in the valley between P and B, as shown in the diagram below.



g. Find the area of the cross-section of the pond, in square metres. 2 marks

Phil is considering adding a water slide to be built down the other side of the mountain, beginning at point S and entering the pond at point B. Safety regulations require that the average gradient of the slide be no more than **one fifth** of the depth of the water at the deepest part of the pond.

h. Find the coordinates of S, the highest point on the mountain that the water slide could start from and still satisfy the safety regulations. Give your answer to the nearest centimetre. 4 marks

AT THE CONCLUSION OF THE SAC IT IS YOUR REPOSIBILITY TO CLEAR YOUR CAS CALCULATOR'S MEMORY

FAILURE TO DO SO IS A BREACH OF THE SAC CONDITIONS

PROCEDURE

Press	Home on
Select	2: My Documents
Press	Menu
Select	C: Delete All
Choose	OK