XAVIER COLLEGE
SAC / Assessment Conditions

	Date:	Time:
MATHEMATICAL METHODS	**THURSDAY 17TH JUNE**	**3.45 pm – 5.55 pm**

- Listen carefully to the supervisor's instructions.
- Permissible items include: pens, pencils, highlighters, erasers, sharpeners, rulers.
- You are not permitted to use white out (liquid paper).
- You have 10 minutes reading and 2 hrs writing to complete this part.
- Complete this task in the spaces provided.
- Give answers in exact form unless told otherwise.
- You can use your CAS calculator and a bound set of notes to complete this task.
- A number of questions are consequential in nature. You are advised to show all working, even for questions worth one mark. In questions worth more than 1 mark, working is required to gain full marks.
- You must work silently and independently for the duration of the task. Only questions of clarification can be asked of your teacher.
- It is not in your interest to talk about this task with students from other classes.

PLEASE NOTE: ***Students are NOT permitted to have mobile phones or any other unauthorised electronic devices in their possession during a SAC/examination***

COMPULSORY STUDENT DECLARATION

I, (*print your name neatly*) _____
acknowledge that I have read the SAC/examination conditions and understand which items/materials I am permitted to use and have in my possession.

****If you have any doubts as to what is permitted, raise your hand and DO NOT sign this declaration****

Student's Signature: _____

Student's Name: _____

Teacher's Name: _____

XAVIER MATHS

MATHEMATICAL METHODS APPLICATION SAC 2021

PART TWO

51 Marks

Question 1 (14 marks)

Once the drill has reached its destination, the nuclear device is detonated. The blast radius of the explosion is modelled by the exponential function

$$R_1(t) = a \cdot e^{kt} + b, t \geq 0$$

Where R_1 is the blast radius of the explosion in kilometres, t is time in seconds and $a, k, b \in \mathbb{R}$.

a Write down $R_1'(t)$

_____

_____

1 mark

The following information is known about the blast radius.

1. The blast radius is initially zero
2. The blast radius is 6 km after 1 second
3. The blast radius is increasing at $24\log_e(2)$ km/s after 2 seconds.

b **i** Using point 1, show that $b = -a$

_____

_____

1 mark

ii Using point 2, and the fact that $b = -a$ show that $a = \dfrac{6}{e^k - 1}$

_____

_____

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2 marks

iii Using point 3, and the fact that $k = \log_e(2)$, show that $a = 6$.

2 marks

c Hence, write down the rule for $R_1(t)$ in terms of t only.

1 mark

d Find the average rate of change of the blast radius over the first 3 seconds. Give your answer in kilometres per second.

2 marks

e Find the blast radius after 8 seconds.

_____

_____

1 mark

After 8 seconds, the model for the blast radius changes. The new model is given by

$$R_2(t) = p\log_e(t-q), t \geq 8$$

Where R_2 is the blast radius in kilometres, t is time in seconds since the initial detonation and $p, q \in \square$.

f Given that the function modelling the blast radius is smooth and continuous for $t \geq 0$, find the values of p and q correct to two decimal places

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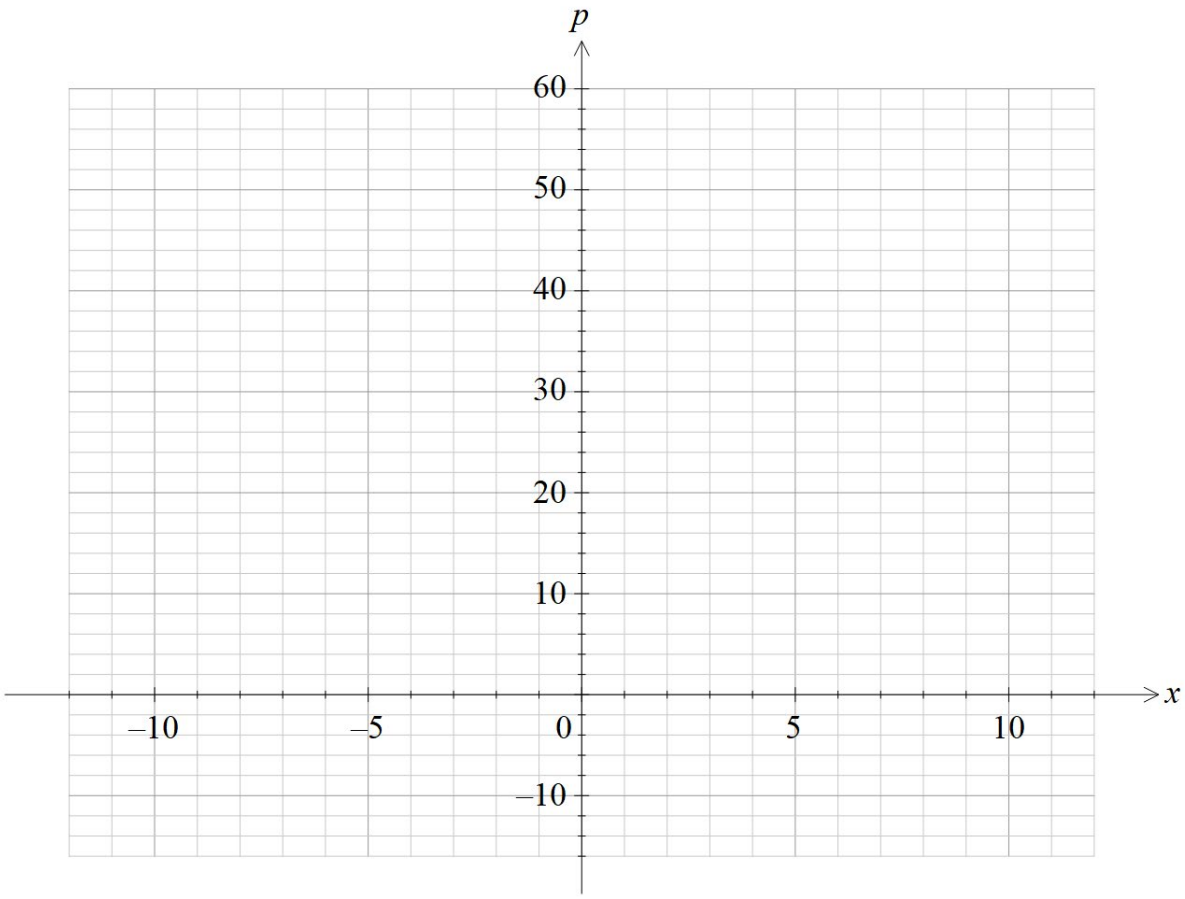
4 marks

Question 2 (13 marks)

a Sketch the function $p:[-10,10] \to R, p(x) = \frac{1}{10}(x-15)^2 - 10$ on the axes below.

Clearly indicate the coordinates of the endpoints and the axis intercepts correct to two decimal places.



3 marks

The astronauts who were on the shuttle need to escape the blast of the nuclear device. One of the astronauts suggests a path described by the function

$$f:[-10,10]\to R, f(x)=\frac{1}{10}(x-h)^2+k,$$

Where $k\in R$ and $h>10$. f is a decreasing function for $x\in[-10,10]$

Another astronaut proposes that the path they could follow could be $f^{-1}(x)$.

b Show that the rule for the inverse function is $f^{-1}(x)=h-\sqrt{10(x-k)}$

2 marks

c Write down the domain of $f^{-1}(x)$ in terms of h and k.

1 mark

d Given that the inverse has an x – intercept at (–1, 0) and the original function has a right endpoint at (10, –12) show that

$$k = -1 - \frac{h^2}{10}, \quad \text{and}$$

$$k = -12 - \frac{1}{10}(10 - h)^2$$

2 marks

e Hence, find the value of h

2 marks

f Hence, write down the value of k

1 mark

g **i** Write down a definite integral which would give the area enclosed by the inverse function, the x-axis and the line $x = -\frac{481}{40}$

_____

_____

1 mark

ii Hence, find the area enclosed by the inverse function, the x-axis and the line $x = -\frac{481}{40}$.

_____

_____

1 mark

Question 3 (10 marks)

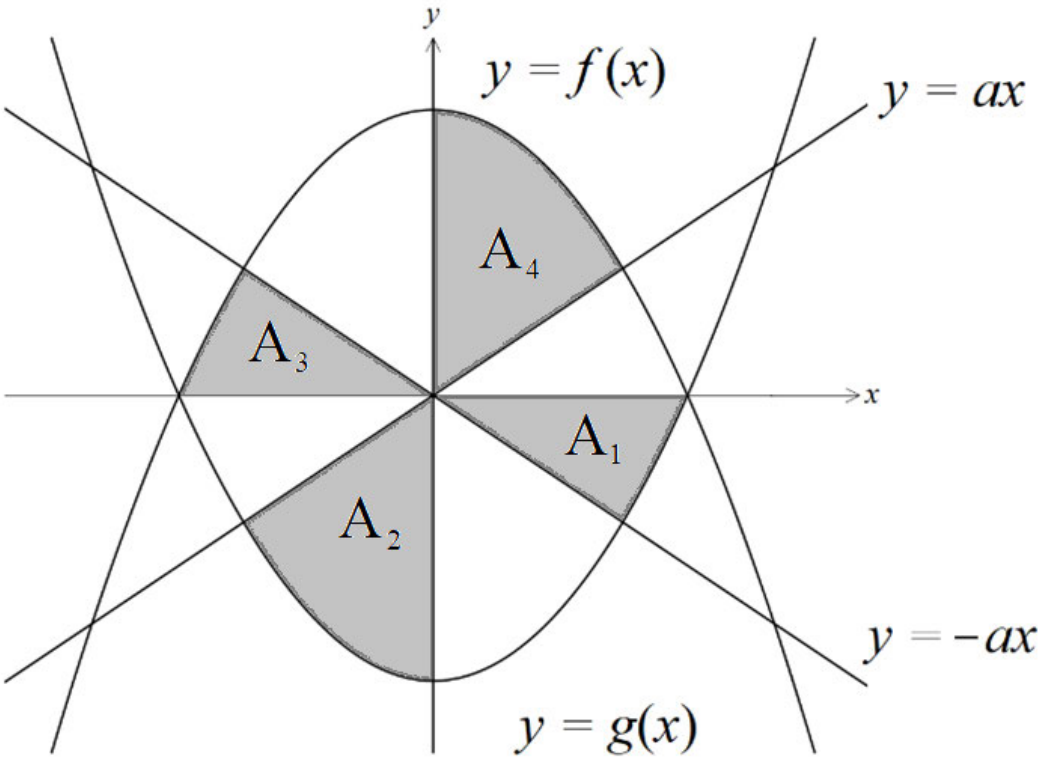
The asteroid is successfully destroyed and the astronauts are safely on their way back to Earth. Designers at NASA are creating a logo to celebrate the mission success.

The logo will be a pinwheel design contained within two symmetrical parabolas, as in the diagram below (which is not to scale). Where

$$f(x) = 9 - x^2,$$

$$g(x) = x^2 - 9, \text{ and}$$

$$a \in R^+$$



a Evaluate $\int_0^3 f(x)dx$

_____

_____

1 mark

b For $a = 2$, Find the total area enclosed by the graph of $g(x)$, the x – axis and the line $y = -ax$. (A_1 in the diagram) Give your answer correct to two decimal places.

_____

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2 marks

c Let $a = \frac{2}{3}$

i Find the x – values of the points of intersection of $g(x)$ and the line $y = -ax$

_____

_____

1 mark

ii Find the x – values of the points of intersection of $f(x)$ and the line $y = ax$

_____

_____

1 mark

iii Hence, or otherwise, find the shaded area of the whole pinwheel logo if $a = \frac{2}{3}$.

2 marks

The line $y = ax$ actually divides the integral found in **part a** into two equal portions. Remember that $a \in R^+$

d If $y = ax$ intersects $f(x)$ at $x = p$, find a in terms of p. (Hint use $\frac{rise}{run}$)

1 mark

e Hence, find the value of p that divides the integral found in **part a** into two equal portions. Give your answer correct to 2 decimal places.

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2 marks

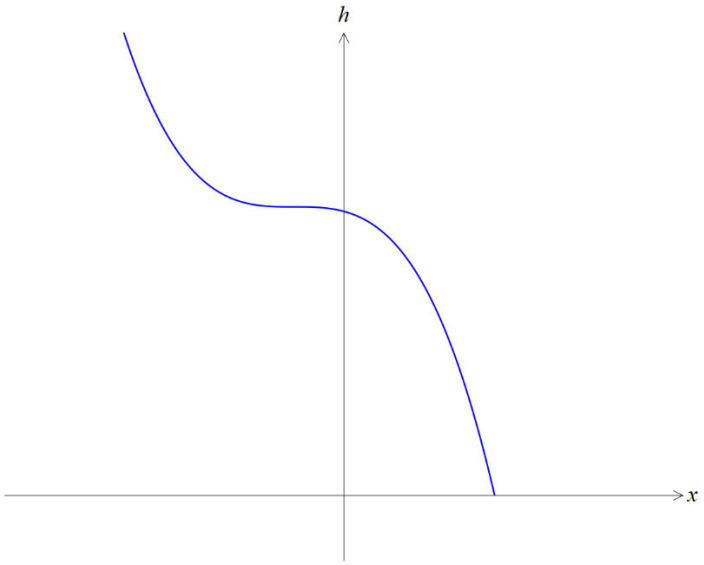
Question 4 (8 marks)

The shuttle containing the astronaut crew is re-entering the Earth's atmosphere and landing back on earth. Relative to an oil rig in the ocean, which is at the origin (0, 0), the path of the shuttle is given by the function

$$h(x) = -\frac{1}{200}(x+5)^3 + 40$$

Where h is the height of the shuttle in kilometres and x is the horizontal position of the shuttle relative to the oil rig at the origin, also in kilometres.

The graph of the path is shown below



The shuttle is planning on landing on a floating landing pad at sea level, i.e. $h = 0$.

a What are the coordinates of the landing pad?

_____

_____

1 mark

b What is the **vertical** distance of the shuttle from the oil rig when the shuttle passes directly above it? Give your answer in kilometres.

_____

_____

1 mark

There is an important piece of monitoring equipment on the oil rig which measures the **direct** distance from the oil rig to the shuttle at all times.

c Find the coordinates of the shuttle when it is at a minimum distance from the oil rig. Give your answer correct to two decimal places.

_____

_____

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_____

_____

3 marks

d Verify that these coordinates give a minimum distance.

2 marks

e Therefore, find the minimum distance of the shuttle from the oil rig on its re-entry to earth. Give your answer in kilometres correct to two decimal places.

_____

_____

1 mark

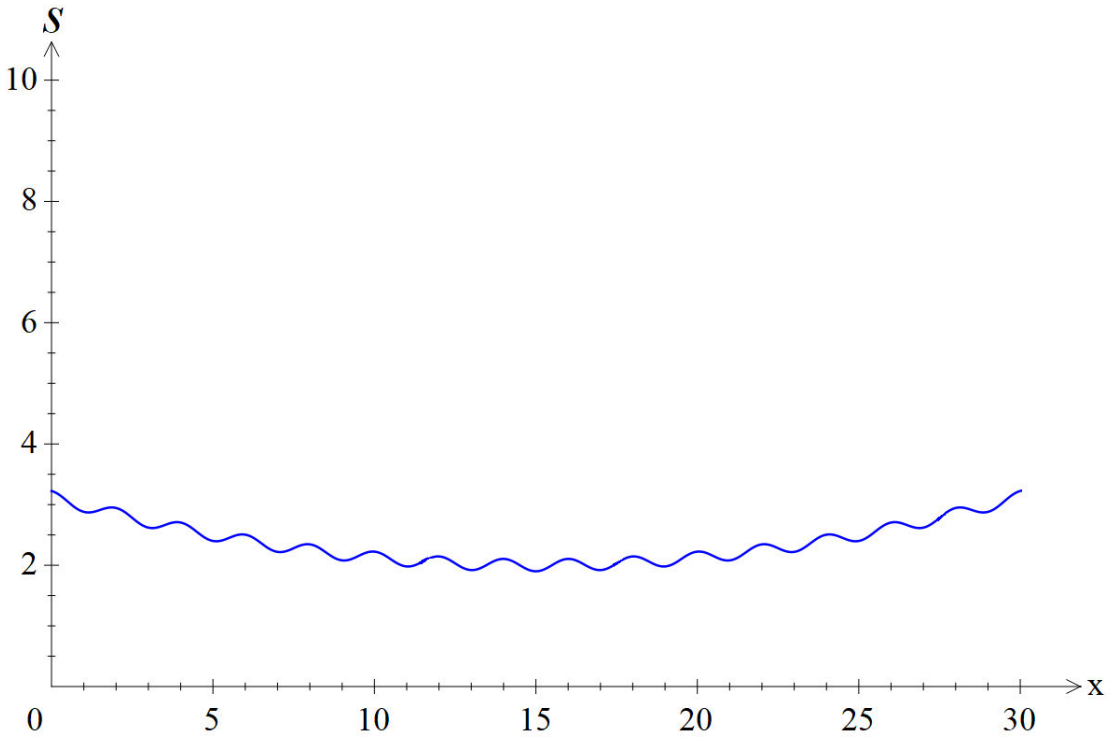
Question 5 (6 marks)

The surface of the landing pad is created in such a way to improve the comfort of the astronauts when the shuttle lands. The cross section follows the rule

$$s(x) = \frac{1}{10}\cos(\pi x) + \frac{1}{200}(x-15)^2 + 2, x \in [0,30].$$

Where s is the height of the surface in metres, and x is in metres.

The graph below shows a close up of the cross section of the landing pad surface.



a Evaluate $\int_0^{30} s(x)dx$

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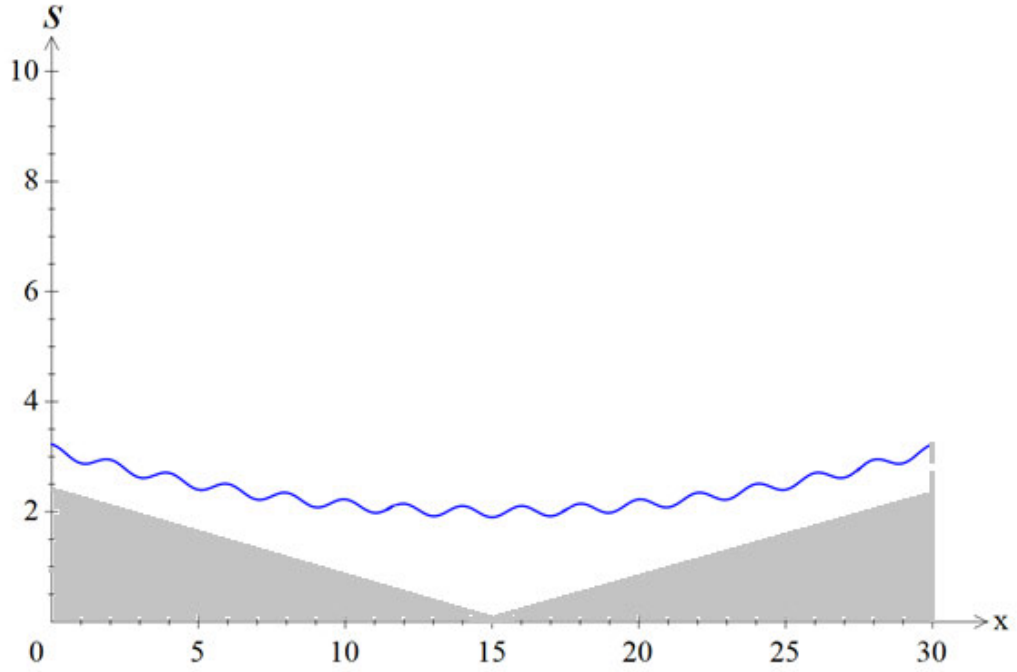
1 mark

b To ensure that the landing pad is strong enough to support the landing of the shuttle, the engineers decide to add in two triangular supports made of reinforced concrete, as shown in the diagram below.



The top of the reinforced sections are straight line graphs with the equations $y = -a(x-15)$ and $y = a(x-15)$ where $a > 0$. Find the value of a if the area remaining between $s(x)$ and the concrete sections is $\frac{1}{2}$ of the original area between $s(x)$ and the x – axis.

2 marks

c What is the average value of the height of the landing pad for $x \in [0, 20]$?

_____

_____

1 mark

d The engineers want to make the landing pad wider, to ensure that it is wide enough for the astronauts to land. To do this, they want to translate the graph of the landing pad to the right by k units, and increase the right endpoint of the domain, also by k units, in order to make the area between $s(x)$ and the x – axis 100m^2. Given that $0 < k < 10$, Find the value of k correct to 2 decimal places.

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2 marks

END OF PART TWO