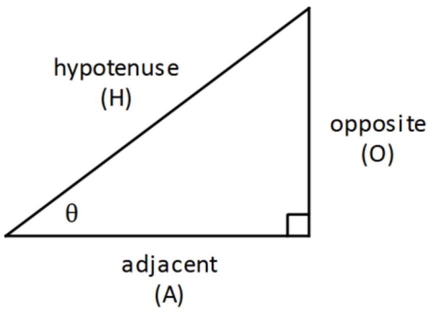
T05 Circular Functions

WS1 – Reviewing trigonometric ratios and Exact values

When we have a right-angled triangle, we can use trigonometric ratios to find side lengths and angles.

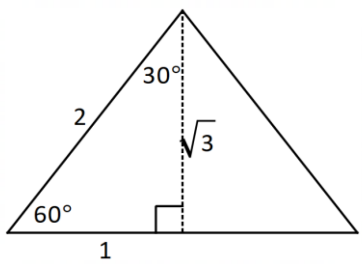


Remember: SOH CAH TOA

$\sin\theta = \frac{O}{H}$ $\cos\theta = \frac{A}{H}$ $\tan\theta = \frac{O}{A}$

Exact values of circular functions

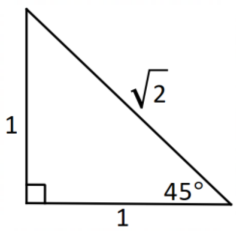
For $30°$ and $60°$: use an equilateral triangle, cut in half, with side lengths of 2 units.



$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

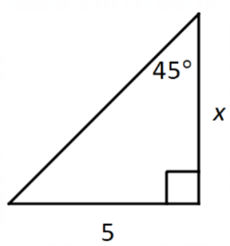
For $45°$: use a right-angled isosceles triangle, with side lengths of 1 unit.



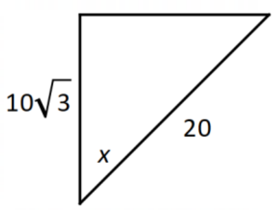
$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\tan 45° = \frac{1}{1} = 1$

Example: Without using CAS, find the value of *x* in the following triangles:

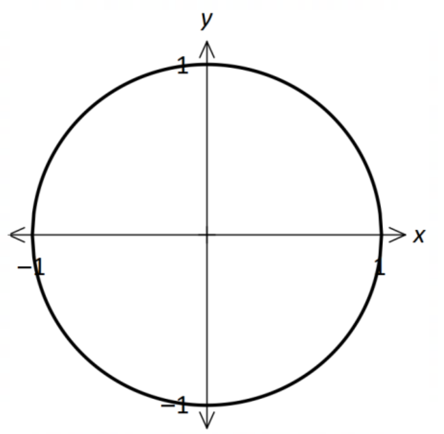
a)



b)



Ex 14A – Measuring angles in degrees and radians



A *unit circle* has a radius of 1 unit.

The circumference of a circle is: $C = 2\pi r$

But if $r = 1$, then $C = 2\pi$

The angle in a circle is $360°$. This means that the circumference and the angle trace the same distance. So:

$2\pi = 360°$ or $\pi = 180°$

To convert from degrees to radians: $\times\dfrac{\pi}{180}$ To convert from radians to degrees: $\times\dfrac{180}{\pi}$

Example: convert between radians and degrees.

a) $45°$

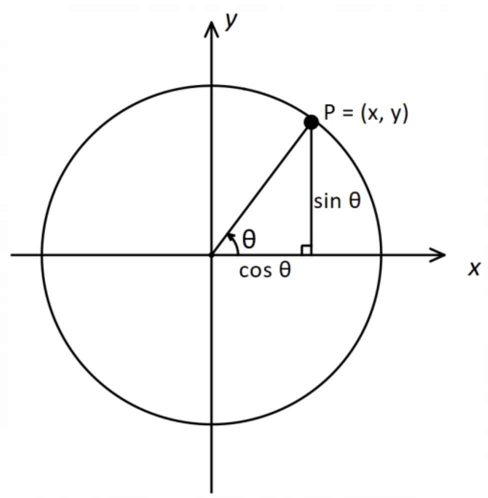
b) $\dfrac{\pi}{6}$

Ex 14B – Defining circular functions: sine and cosine

The unit circle is a circle with radius of 1 and is drawn on the Cartesian plane with its centre at (0, 0).

The value of the co-ordinates x and y can be described in terms of θ:

In general: $x = \cos\theta$
$y = \sin\theta$



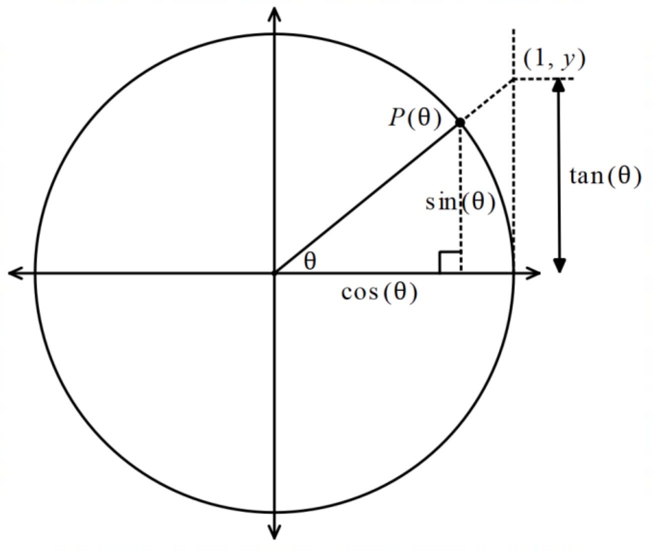
These are known as *circular functions*.

Positive values of θ are represented by an *anticlockwise* rotation from the positive x-axis.

Example: If $\theta = \dfrac{\pi}{3}$, find the co-ordinates of the point P on the unit circle.

Ex 14C – Another circular function: tangent

The tangent function can be defined using similar triangles.



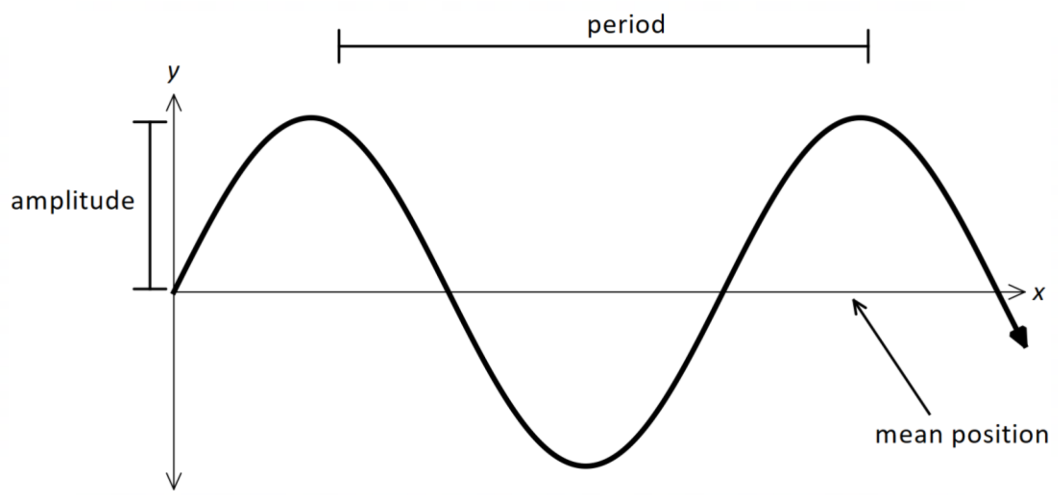
In general, $\tan\theta = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$

Example: If $\theta = 135°$, find $\tan\theta$.

Example: If $\theta = \dfrac{2\pi}{3}$, find $\tan\theta$.

Ex 14F – Graphs of sine and cosine

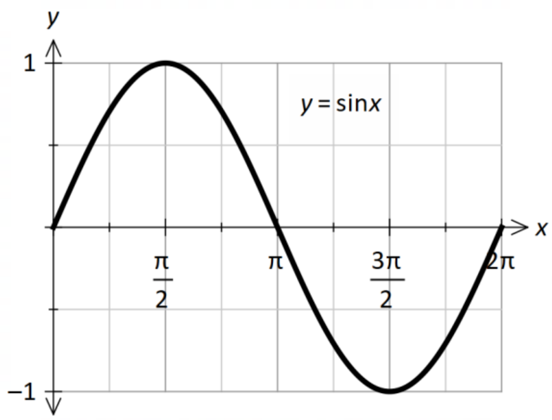
Sine and Cosine graphs are called sinusoidal curves. They have an amplitude of 1 and a period of 2π.

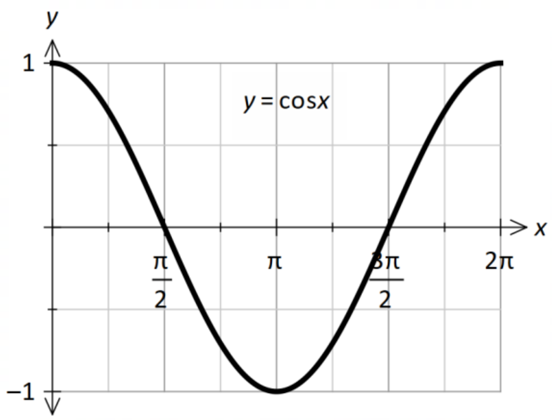


Amplitude: The distance between the mean position and the maximum or minimum value.

Period: The value of θ that it takes to compete one full cycle of the unit circle.

The basic shapes are shown below:





Transformations of the basic graphs

Graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$ can be obtained from $y = \sin(x)$ and $y = \cos(x)$ by applying a dilation factor of a from the x-axis and a factor of $\frac{1}{n}$ from the y-axis. This will have an effect on the amplitude and period of the functions.

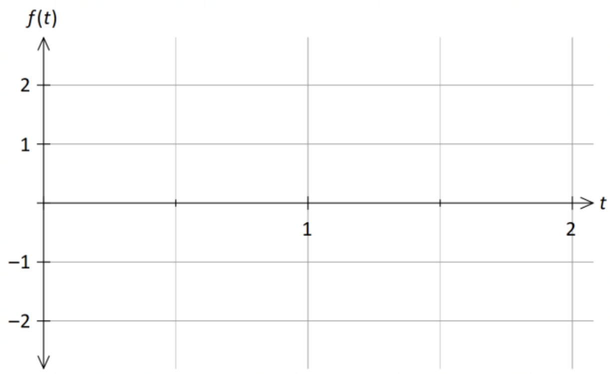
$$\boxed{\begin{array}{l} (x,y) = \left(\frac{1}{n}x, ay\right) \\ x' = \frac{1}{n}x \quad y' = ay \\ x = nx' \quad y = \frac{1}{a}y' \end{array}} \qquad \boxed{\begin{array}{l} y = \sin x \\ \frac{1}{a}y' = \sin(nx') \\ y' = a\sin(nx') \end{array}}$$

In general:

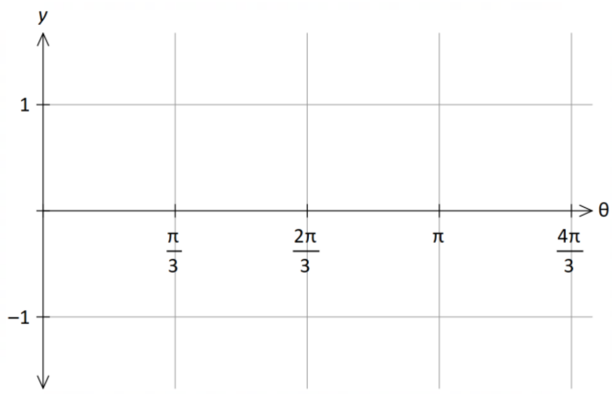
- Amplitude: a
- Period: $\frac{2\pi}{n}$

Example: State the amplitude and the period and hence sketch the following functions for one complete cycle:

a) $f(t) = 2\sin(\pi t)$



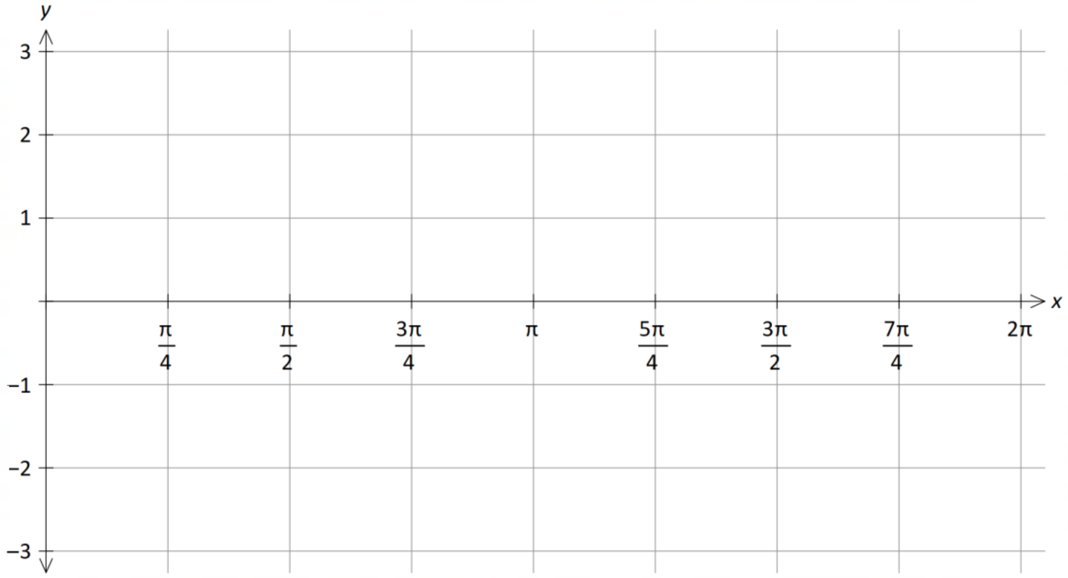
b) $y = -\cos\left(\frac{3\theta}{2}\right)$



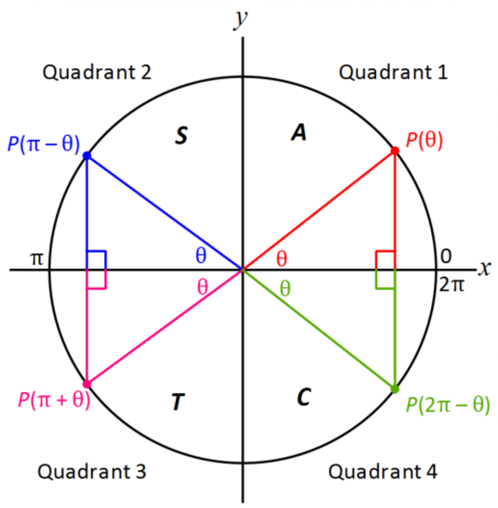
Ex 14H – Sketch graphs of $y = a\sin n(t \pm \varepsilon)$ and $y = a\cos n(t \pm \varepsilon)$

Graphs of these functions have undergone a translation parallel to the x-axis.

Example: Sketch the graph of $y = 3\sin 2\left(x - \frac{\pi}{4}\right)$ for $0 \le x \le 2\pi$.



Ex 14D / 14E – Exact values and Symmetry properties



The Cartesian plane is divided into four equal regions, called quadrants.

Using symmetry between the circular functions, we can extrapolate the value for angles in other quadrants.

Quadrant 1: all are positive
Quadrant 2: only sin positive
Quadrant 3: only tan positive
Quadrant 4: only cos positive

Negative angles are measured from the positive x-axis in a clockwise direction.

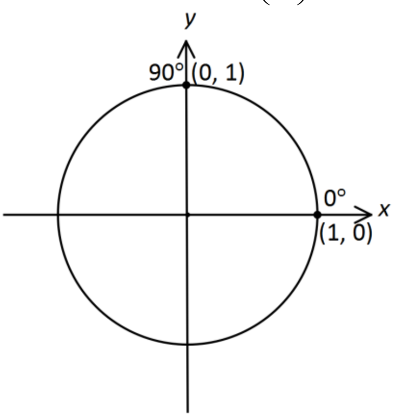
Example: If $\sin x = 0.8$ find:

a) $\sin(\pi - \theta)$ b) $\sin(2\pi - \theta)$

Example: If $\cos x = -\cos\left(\frac{\pi}{6}\right)$ and $\pi < x < \frac{3\pi}{2}$, find the value of *x*.

Exact values of circular functions in radians

For $0°$ (0^c) and $90°$ $\left(\frac{\pi}{2}\right)$: use the co-ordinates in a unit circle.



$\sin 0 = 0$

$\cos 0 = 1$

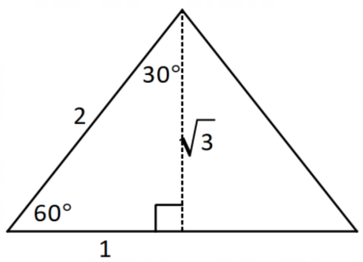
$\tan 0 = \frac{0}{1} = 0$

$\sin\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$

$\tan\frac{\pi}{2} = \frac{1}{0} = undef$

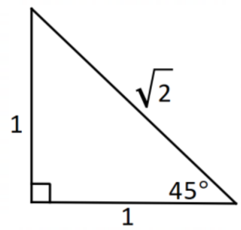
For $30^\circ\ \left(\frac{\pi}{6}\right)$ and $60^\circ\ \left(\frac{\pi}{3}\right)$: use an equilateral triangle, cut in half, with side lengths of 2 units.



$$\sin\frac{\pi}{6}=\frac{1}{2} \qquad \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2} \qquad \tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

$$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3}=\frac{1}{2} \qquad \tan\frac{\pi}{3}=\frac{\sqrt{3}}{1}=\sqrt{3}$$

For $45^\circ\ \left(\frac{\pi}{4}\right)$: use a right-angled isosceles triangle, with side lengths of 1 unit.



$$\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2} \qquad \tan\frac{\pi}{4}=\frac{1}{1}=1$$

Example: Without using CAS, evaluate the following:

a) $\cos\frac{3\pi}{4}$ b) $\tan\frac{11\pi}{2}$ c) $\sin\frac{5\pi}{3}$

<u>Ex 14G – Solutions of trigonometric equations</u>

In practice, it is usually simplest to find the first two solutions using the unit circle, and then find any others in the specified domain by adding multiples of the period to them.

Example:

a) Find all the values of x for which $\cos\theta=-0.58$ and $\theta\in[0,180^\circ]$

Your CAS will always give you the Q1 angle.
i.e. $0^\circ\le\theta\le 90^\circ$

$0\le\theta\le\frac{\pi}{2}$

We call this the Reference Angle (or RA)

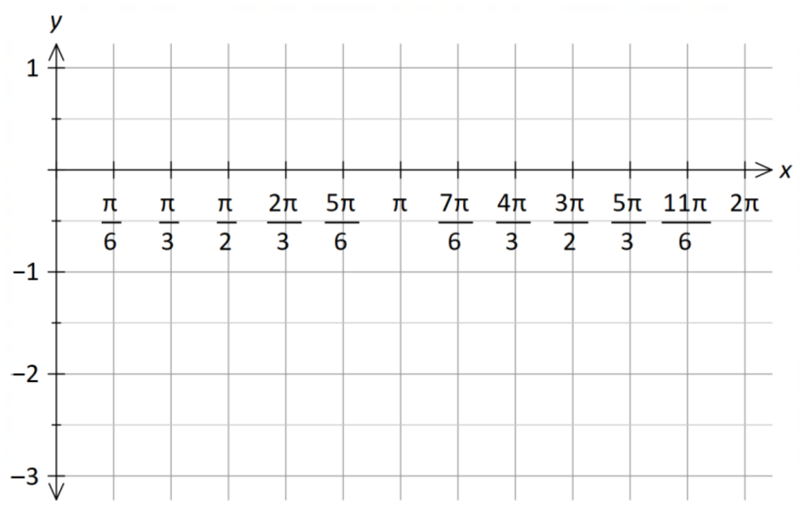
b) Find x when $2\sin x+\sqrt{3}=0$ between $0\le x\le 2\pi$

c) Solve $2\cos 3x=\sqrt{3}$ between $[0,2\pi]$

to find:
$\min = -amp + y$-translation
$\max = +amp + y$-translation

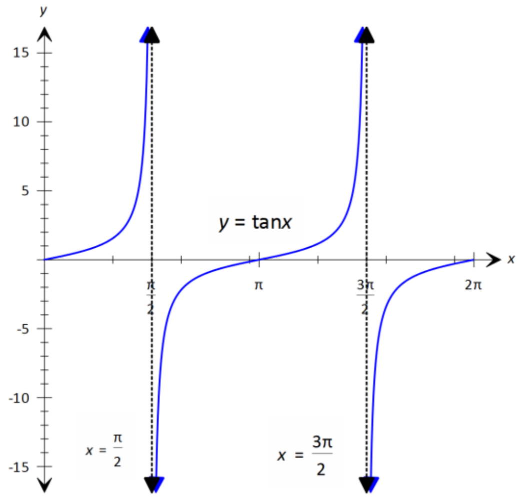
Ex 14I – Sketch graphs of $y=a\sin n(t\pm\varepsilon)\pm b$ and $y=a\cos n(t\pm\varepsilon)\pm b$

Example: Sketch the graph of $y=2\cos(2x)-1$ over $x\in[0,2\pi]$



Ex 14K – The tangent function (worksheet)

The basic shape of a tangent function is shown.



Key features of the basic graph:

Amp:	R	Range:	R
Period:	π	Domain:	R

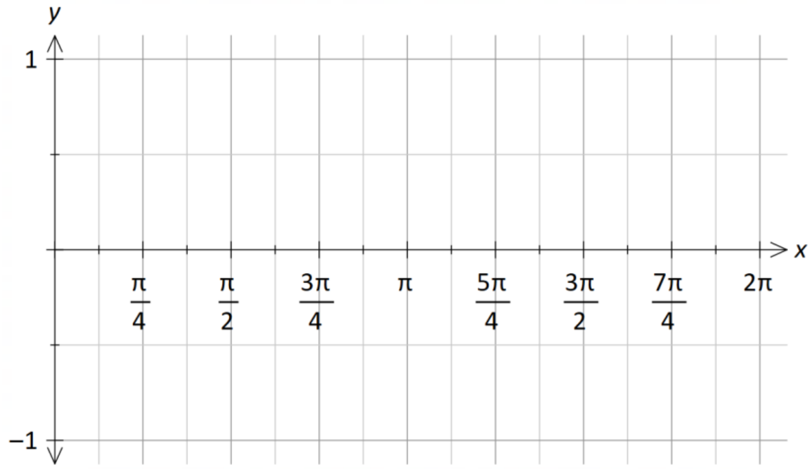
Transformations of the basic graph

Graphs of $y = a\tan(nt)$ can be obtained from $y = \tan(t)$ by applying a dilation factor of a from the t-axis and a factor of $\frac{1}{n}$ from the y-axis.

In general: Period = $\frac{\pi}{n}$

remember:
$\tan\theta = \frac{\sin\theta}{\cos\theta}$

Example: Sketch the graph of $y = \tan 2x$ over $x \in [0, 2\pi]$, showing all x-intercepts and asymptotes

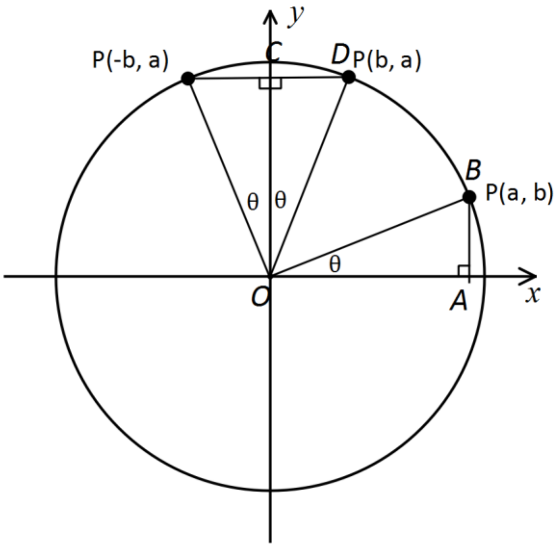


Example: Solve $\tan(2x) = \sqrt{3}$ over $0 \le x \le 2\pi$

Ex 14J – Further symmetry properties and the Pythagorean Identity

Complementary functions

Consider the unit circle shown on the right:



The triangles *OAB* and *ODC* are congruent because they have all corresponding angles equal and the hypotenuse equal (radius = 1).

So all corresponding sides are equal and it follows that:

$\sin(90-\theta)^\circ = a = \cos\theta$
$\cos(90-\theta)^\circ = b = \sin\theta$

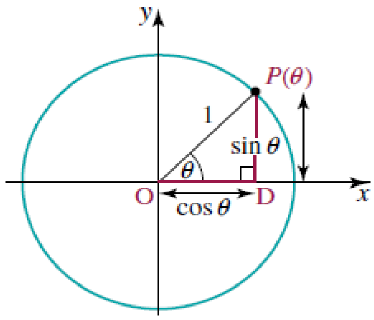
or

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta = x$
$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta = y$

We say that sine and cosine are complementary functions.

The Pythagorean identity

Consider the right-angled triangle in the unit circle shown.



Applying Pythagoras' theorem gives the identity:

$$\sin^2\theta + \cos^2\theta = 1$$

Example: If $\sin\theta = 0.3$ find the values of:

a) $\sin(-\theta)$

b) $\cos\left(\frac{\pi}{2}+\theta\right)$

Example: Given that $\sin x = -\frac{4}{5}$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

Ex 14M – General solution of circular function equations

note: "RA" is the reference or Q1 angle

Cosine:

If $\cos(x)=a$, where $0<a<1$ (i.e. quadrants 1 and 4)

$$x=2n\pi \pm RA,\ n\in Z$$

If $\cos(x)=a$, where $-1<a<0$ (i.e. quadrants 2 and 3)

$$x=(2n+1)\pi \pm RA,\ n\in Z$$

If $\cos(x)=-1$

$x=(2n+1)\pi,\ n\in Z$

If $\cos(x)=0$

$x=(2n+1)\frac{\pi}{2},\ n\in Z$

If $\cos(x)=1$

$x=2n\pi,\ n\in Z$

Sine:

If $\sin(x)=a$, where $0<a<1$ (i.e. quadrants 1 and 2)

$x=2n\pi+RA,\ n\in Z$ or $x=(2n+1)\pi-RA,\ n\in Z$

(or you can use $x=n\pi+(-1)^n RA,\ \ n\in Z$)

If $\sin(x)=a$, where $-1<a<0$ (i.e. quadrants 3 and 4)

$x=2n\pi-RA,\ n\in Z$ or $x=(2n+1)\pi+RA,\ n\in Z$

(or you can use $x=n\pi-(-1)^n RA,\ \ n\in Z$)

If $\sin(x)=-1$

$x=-\frac{\pi}{2}\pm 2n\pi,\ n\in Z$

If $\sin(x)=0$

$x=2n\pi$ or $(2n+1)\pi,\ n\in Z$

If $\sin(x)=1$

$x=\frac{\pi}{2}\pm 2n\pi,\ n\in Z$

Tangent:

If $\tan(x)=a$, where $a\in R^+$ (i.e. quadrants 1 and 3)

$x=2n\pi+RA,\ n\in Z$ or $x=(2n+1)\pi+RA,\ n\in Z$ (alternatively: $x=n\pi+RA,\ n\in Z$)

If $\tan(x)=a$, where $a\in R^{-}$ (i.e. quadrants 2 and 4)

$$x=2n\pi-RA,\ n\in Z \quad \text{or} \quad x=(2n+1)\pi-RA,\ n\in Z \qquad \text{(alternatively: } x=n\pi-RA,\ n\in Z\text{)}$$

If $\tan(x)=0$

$x=n\pi,\ n\in Z$

If $\tan(x)$ is undefined

$x=(2n+1)\frac{\pi}{2},\ n\in Z$

Example 1: Find the general solution to the equation $\cos(2x)=-\frac{1}{2}$.

Example 2: Find the general solution to the equation $\tan(3x)=\frac{1}{\sqrt{3}}$.

Ex 14N – Applications of circular functions

- The general equations: $\begin{array}{l} y = a\sin n(t \pm \varepsilon) \pm b \\ y = a\cos n(t \pm \varepsilon) \pm b \end{array}$ Period: $\dfrac{2\pi}{n}$ Amplitude: a

- To find the maximum value of a function: $\max = amp + y\text{-}trans$
- To find the minimum value of a function: $\min = -amp + y\text{-}trans$

- To find the initial value: let $t = 0$

Example:
E. coli is a type of bacterium. Its concentration, P parts per million (ppm), at a particular beach over a 24-hour period t hours after 6 am, is described by the function: $P = 0.05\sin\left(\dfrac{\pi t}{12}\right) + 0.1$. Find the:

a) Maximum E. coli levels at this beach.

b) Minimum E. coli levels at this beach.

c) What is the level of E. coli at 3 pm, correct to 3 decimal places?

d) How long is the level *above* 0.125 parts per million during the first 12 hours after 6 am?