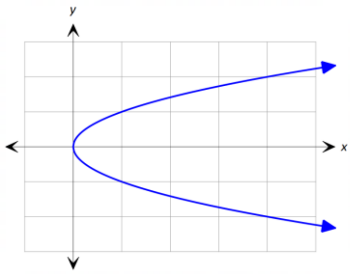
T02: Graphs and Transformations

Ex 4C – The graph of $y^2 = x$



The graph of $y^2 = x$ is a parabola. It can be obtained from the graph of $y = x^2$ by a reflection in the line $y = x$.

The vertex is at $(0,0)$ and the axis of symmetry is the x-axis.

The general equation is given by $(y-k)^2 = a^2(x-h)$, where the value of '*h*' and '*k*' gives the vertex (h,k) and the axis of symmetry is $y = k$.

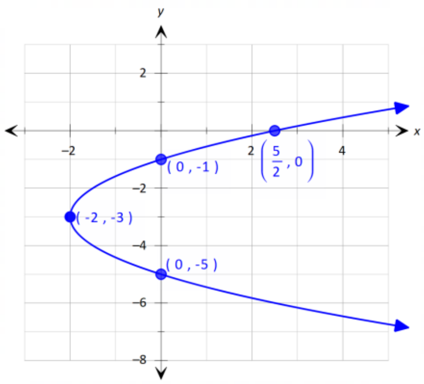
Example: Sketch the graph of $y^2 + 6y - 2x + 5 = 0$.

$y^2 + 6y = 2x - 5$

$(y^2 + 6y + 3^2) - 3^2 = 2x - 5$

$(y+3)^2 = 2x + 4$

$(y+3)^2 = 2(x+2)$

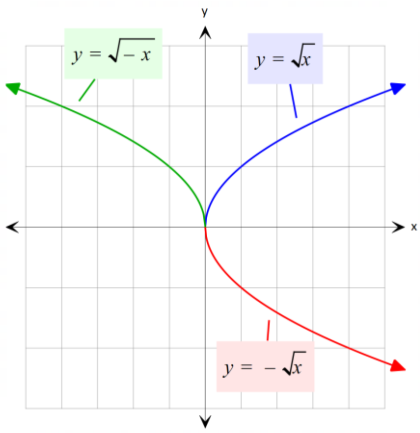


WS1 – Square-root graph $y = \sqrt{x} = x^{\frac{1}{2}}$

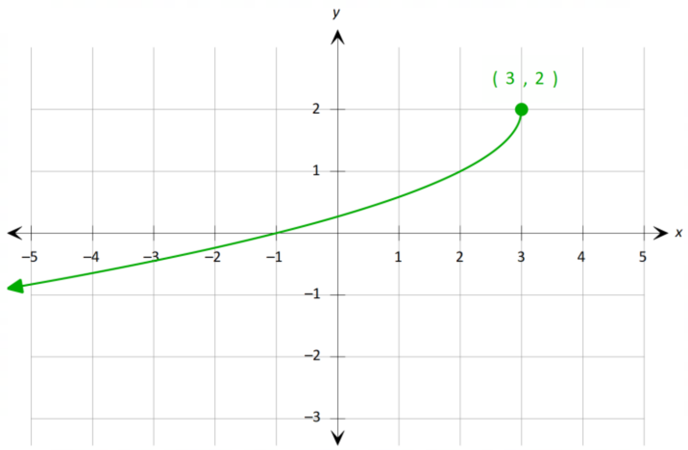
Consider the graphs of $y = \sqrt{x}$, $y = -\sqrt{x}$ and $y = \sqrt{-x}$.

The graph starts at $(0,0)$ and has no asymptotes.

The general equation is given by: $y = a\sqrt{x-h} + k$, where the value of '*h*' and '*k*' gives the starting point (h,k).



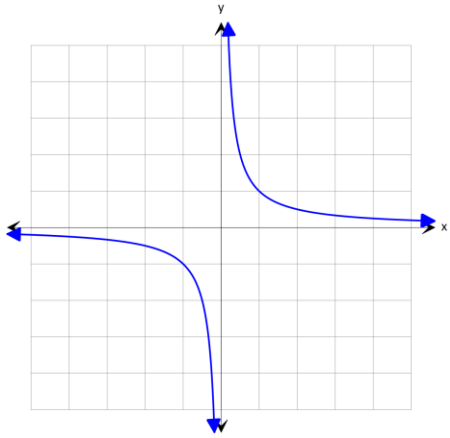
Example: Sketch the graph of $y = -\sqrt{-(x-3)} + 2$. State the domain and range.



domain: $(-\infty, 3]$

range: $(-\infty, 2]$

Ex 4A / WS3 – Rectangular Hyperbolas



A rectangular hyperbola has the equation: $y = \frac{1}{x}$

The graph goes through the points $(1,1)$ and $(-1,-1)$.

By inspection we can see that the graph approaches the *x* and *y* axes but never quite crosses.

For a rectangular hyperbola, we say: as $x \to +\infty, y \to {}^{+}0$

$x \to -\infty, y \to {}^{-}0$

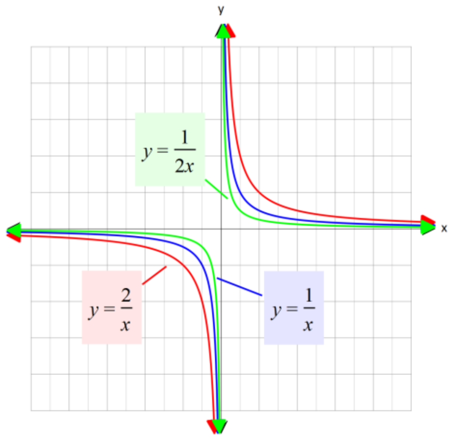
$x \to {}^{+}0, y \to +\infty$

$x \to {}^{-}0, y \to -\infty$

This is explained by the presence of asymptotes. Asymptotes are straight lines that are never crossed. For $y = \frac{1}{x}$ at $x = 0$, *y* is undefined and vice versa.

The general equation of a rectangular asymptote is given by: $y = \frac{a}{x-h} + k$

Consider the graphs of $y = \frac{1}{x}$, $y = \frac{2}{x}$ and $y = \frac{1}{2x}$.

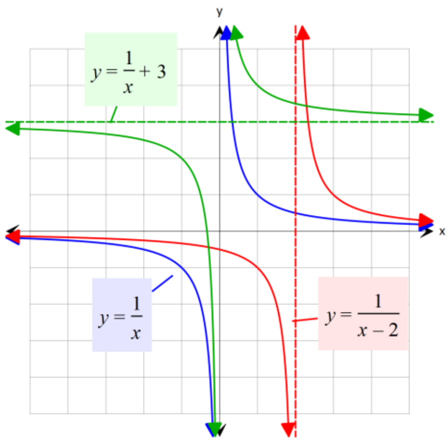


Here we are changing the 'dilation' factor of the equation, given the variable '*a*'.

We say the dilated graph has been 'stretched', or dilated from the *x*-axis.

A reflection can also occur in the *x* or the *y* axes.

Consider the graphs of $y = \frac{1}{x}$, $y = \frac{1}{x-2}$ and $y = \frac{1}{x} + 3$.



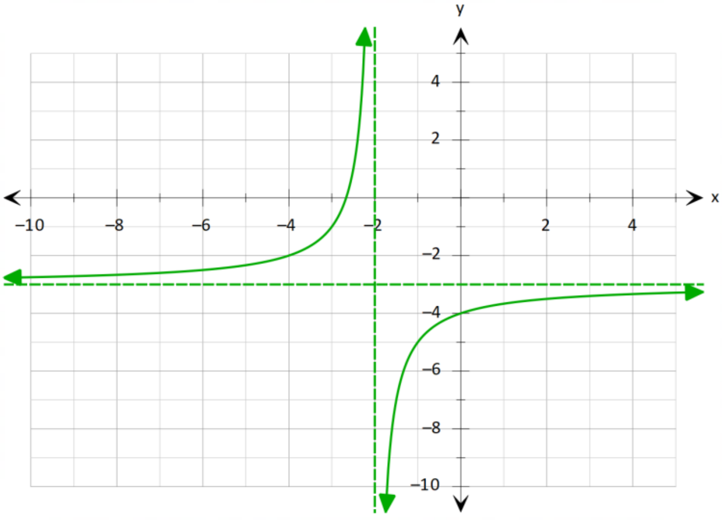
Here we are changing the horizontal and vertical translation, given the variables '*h*' and '*k*' respectively

By changing the value of '*h*' and '*k*', we can change the equation of the vertical and horizontal asymptotes respectively.

Vertical asymptote: $x = h$

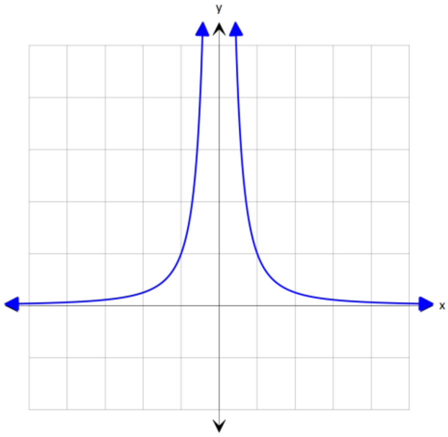
Horizontal asymptote: $y = k$

Example: Sketch the graph of $y = -\dfrac{2}{x+2} - 3$. State the domain and range.



domain: $R \setminus \{-2\}$
range: $R \setminus \{-3\}$

Ex 4B / WS4 – The Truncus



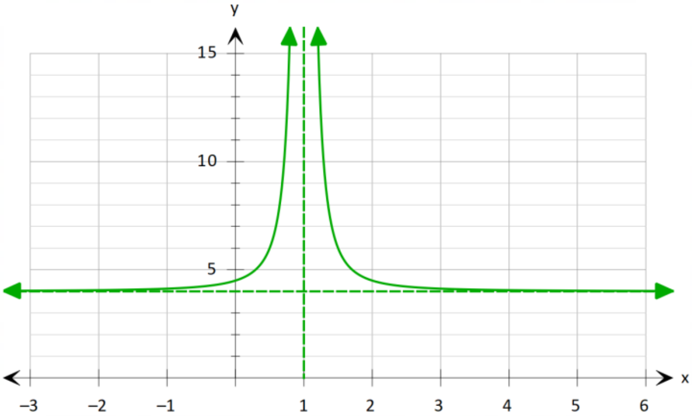
The truncus behaves in a similar fashion to the hyperbola, although has a slightly different shape.

The equation for at truncus is $y = \dfrac{1}{x^2}$

We write the general equation as: $y = \dfrac{a}{(x-h)^2} + k$

The asymptotes are given by the equations: $x = h$ and $y = k$

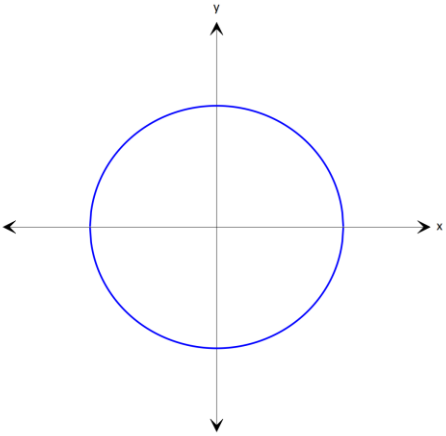
Example: Sketch the graph of $y = \dfrac{1}{2(x-1)^2} + 4$. State the domain and range.



domain: $R \setminus \{1\}$
range: $(4, \infty)$

Ex 4E – Circles

Consider the equation $x^2 + y^2 = 1$



This gives a circle with radius of 1 and origin at $(0,0)$.

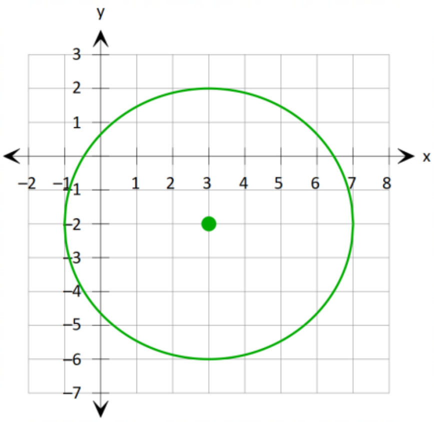
We can transform this graph in much the same way as with the other graphs we have looked at so far.

The general equation of a circle is: $(x-h)^2 + (y-k)^2 = r^2$

- '*h*' and '*k*' give the horizontal and vertical translation of the graph respectively, hence, origin of the graph can be found at (h,k).
- '*r*' changes the size of the circle through changing the length of the radius.

Examples:

1. Sketch the graph of $(x-3)^2 + (y+2)^2 = 4^2$

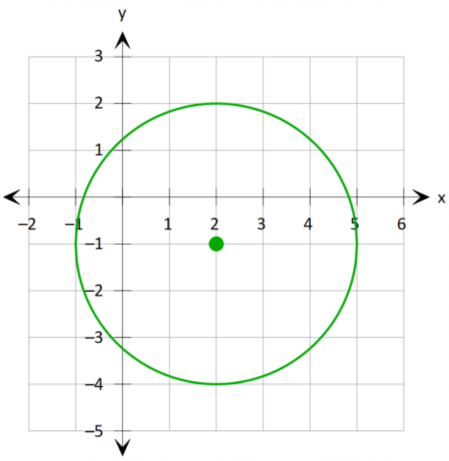


$centre = (3,-2)$
$radius = 4$

$x - \text{int}$
$(x-3)^2 + (0+2)^2 = 16$
$(x-3)^2 + 4 = 16$
$(x-3)^2 = 12$
$x - 3 = \pm\sqrt{12}$
$x = 3 \pm \sqrt{12}$
$x \simeq -0.464 \ \ and \ \ 6.464$

$y - \text{int}$
$(0-3)^2 + (y+2)^2 = 16$
$9 + (y+2)^2 = 16$
$(y+2)^2 = 7$
$y + 2 = \pm\sqrt{7}$
$y = -2 \pm \sqrt{7}$
$y \simeq -4.646 \ \ and \ \ 0.646$

2. Sketch the graph of $x^2 + y^2 - 4x + 2y = 4$



$(x^2 - 4x) + (y^2 + 2y) = 4$
$(x^2 - 4x + 2^2) - 2^2 + (y^2 + 2y + 1^1) - 1^2 = 4$
$(x-2)^2 + (y+1)^2 = 9$
$\therefore centre = (2,-1)$
$radius = 3$

$x - \text{int}$
$(x-2)^2 + (0+1)^2 = 9$
$(x-2)^2 + 1 = 9$
$x = 2 \pm \sqrt{8}$
$x \simeq -0.828 \ \ and \ 4.828$

$y - \text{int}$
$(0-2)^2 + (y+1)^2 = 9$
$4 + (y+1)^2 = 9$
$y = -1 \pm \sqrt{5}$
$y \simeq -3.236 \ \ and \ \ 1.236$

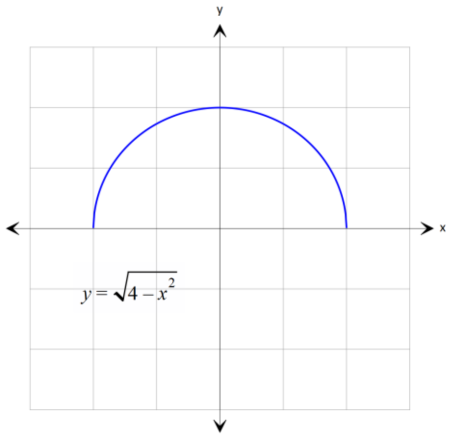
$x^2 + y^2 = r^2$
$y^2 = r^2 - x^2$
$y = \pm\sqrt{r^2 - x^2}$

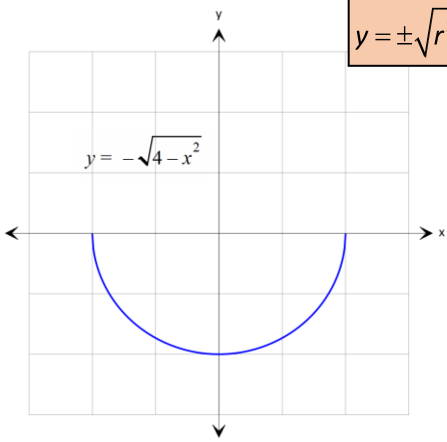
$(y-k)^2 = r^2 - (x-h)^2$
$y - k = \pm\sqrt{r^2 - (x-h)^2}$
$y = \pm\sqrt{r^2 - (x-h)^2} + k$

Semi-circles

By transposing the equation of a circle we can find the equation of a semi-circle.

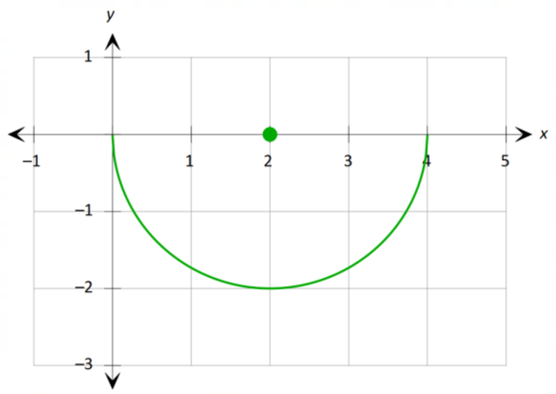
$(x-h)^2 + (y-k)^2 = r^2$





Semi-circles will follow transformations in the same way as circles.

Example: Sketch the graph of $y = -\sqrt{4 - (x-2)^2}$



$x - \text{int}$
$0 = -\sqrt{4 - (x-2)^2}$
$\sqrt{4 - (x-2)^2} = 0$
$4 = 0$
$-(x-2)^2 = -4$
$(x-2)^2 = 4$
$(x-2) = \pm 2$
$x = 2 \pm 2$
$x = 0 \;\; and \;\; 4$

$centre = (2, 0)$
$radius = 2$

Ex 4F – Determining Rules

Example:

The rectangular hyperbola $y = \frac{a}{x} + k$ passes through the points $(1, 8)$ and $(-1, 7)$. Find the values of *a* and *k*.

$(1,8) \rightarrow 8 = \frac{a}{1} + k$
$8 = a + k \quad ...[1]$
$(-1,7) \rightarrow 7 = \frac{a}{-1} + k$
$7 = -a + k \quad ...[2]$

$[1] + [2]$
$15 = 2k$
$k = \frac{15}{2} = 7.5$

sub $k = \frac{15}{2}$ into $[1]$
$a = 8 - \frac{15}{2}$
$a = \frac{1}{2}$

$\therefore y = \frac{1}{2x} + \frac{15}{2}$

Example:

Find the equation of the circle whose centre is at the point $(-2,3)$ and which passes through the point $(-3,3)$.

$(x-h)^2+(y-k)^2=r^2$

$(x+2)^2+(y-3)^2=r^2$

sub in $(-3,3)$ to find r

$(-3+2)^2+(3-3)^2=r^2$

$(-1)^2+(0)^2=r^2$

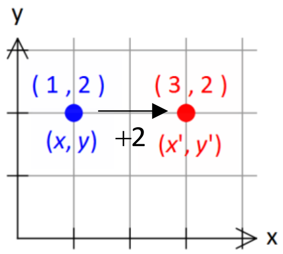
$r=1$

$\therefore (x+2)^2+(y-3)^2=1$

Ex 7A/B/C – Combinations of transformations (image notation)

For all transformations it is usual to let the original point be denoted by (x,y) and the *image* (after translation) by (x',y').

A translation of *h* units in the positive *x* direction and *k* units in the positive *y* direction will translate the point $(x,\ y)$ to $(x+h,\ y+k)$.



The image will therefore have the points:

$x'=x+h$ $\qquad\qquad$ $y'=y+k$

For graph with rule $y=f(x)$, we can find the rule for the translated graph by solving for *x* and *y* and substituting into the rule.

$x=x'-h$ $\qquad\qquad$ $y=y'-k$

In general for the equation: $\quad y=f(x)$

The translated function will be: $y'-k=f(x'-h)$ or $y'=f(x'-h)+k$

Example:

For the graph of $y=\frac{1}{x}$, write the new rule if it undergoes a translation of 2 units in the positive x-direction and 3 units in the negative y-direction.

$(x,y)\to(x+2,y-3)$

$x'=x+2 \qquad y'=y-3$

$x=x'-2 \qquad y=y'+3$

$\therefore y=\frac{1}{x}$

$y'+3=\frac{1}{x'-2}$

$y'=\frac{1}{x'-2}-3$

Dilations and reflections on graphs can be determined in much the same way as translations.

A dilation *from* the x-axis of factor a affects the y-values.

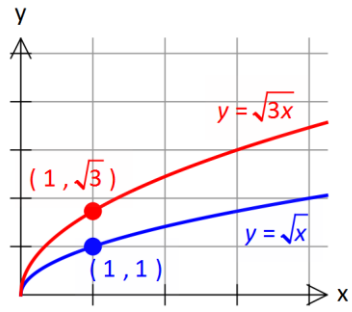
$$(x,\ y) \rightarrow (x,\ ay)$$

$$\therefore y = a \times f(x)$$

A dilation *from* the y-axis of factor b affects the x-values.

$$(x,\ y) \rightarrow (bx, y)$$

$$\therefore y = f\left(\frac{x}{b}\right)$$



A reflection in the x-axis affects the y-values.

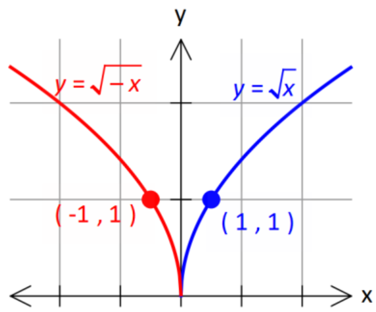
$$(x,\ y) \rightarrow (x,\ -y)$$

$$\therefore y = -f(x)$$

A reflection in the y-axis affects the x-values.

$$(x,\ y) \rightarrow (-x,\ y)$$

$$\therefore y = f(-x)$$



Example: Write down the rule obtained when $y = \frac{1}{x}$ is:

a) Dilated by a factor of 2 from the x axis

$$(x,y) \rightarrow (x,2y)$$
$$x' = x \qquad y' = 2y$$
$$x = x' \qquad y = \frac{1}{2}y'$$

$$y = \frac{1}{x}$$
$$\frac{1}{2}y' = \frac{1}{x'}$$
$$\therefore y' = \frac{2}{x'}$$

b) Reflected about the y-axis, following by a translation of 4 units in the negative x-direction.

$$(x,y) \rightarrow (-x,y) \rightarrow (-x-4,y)$$
$$x' = -x-4 \qquad y' = y$$
$$-x = x'+4 \qquad y = y'$$
$$x = -(x'+4)$$

$$y = \frac{1}{x}$$
$$y' = \frac{1}{-(x'+4)}$$
$$\therefore y' = -\frac{1}{(x'+4)}$$

Ex 7D – Determining Transformations

To determine transformations, we can do mapping in reverse.

Example: Find the sequence of transformations which takes the graph of $y=\sqrt{x}$ to $y=-\sqrt{x-3}+1$

$-(y'-1)=\sqrt{x'-3}$

$x=x'-3 \quad y=-(y'-1)$

$x'=x+3 \quad y'=-y+1$

$(x,y)\rightarrow(x+3,\ -y+1)$

reflection in x-axis
translation 3 units in positive x-dir
and 1 unit in positive y-dir.

check:
if $x=1,\ y=\sqrt{1}=1 \ \therefore (1,1)$
apply transformations to this point
$(1,1)\rightarrow(1,-1)\rightarrow(4,0)$
So if $x=4$
$y=-\sqrt{4-3}+1=0$
which is true!
So this sequence of transformations is correct.

Example: Find the sequence of transformation which takes the graph of $y=(5x-1)^2+6$ to the graph of $y=x^2$.

$y-6=(5x-1)^2$

$x'=5x-1 \quad y'=y-6$

$(x,y)\rightarrow(5x-1,y-6)$

dilation of 5 from the y-axis
translation 1 unit in the negative x-dir
and 6 units in the negative y-dir.

check:
if $x=1,\ y=(5(1)-1)^2+6 \ \therefore (1,22)$
apply transformations to this point
$(1,22)\rightarrow(5,22)\rightarrow(4,16)$
So if $x=4$
$y=(4)^2=16$
which is true!
So this sequence of transformations is correct.

What if we wrote the transformation as:
$(x,y)\rightarrow\left(5\left(x-\frac{1}{5}\right),y-6\right)$
The order of transformations would change:
- translate $\frac{1}{5}$ in negative x-direction and
  6 units in negative y-direction
- dilation of 5 units from the y-axis

check:
if $x=1,\ y=(5(1)-1)^2+6 \ \therefore (1,22)$
apply transformations to this point
$(1,22)\rightarrow\left(\frac{4}{5},16\right)\rightarrow(4,16)$
So if $x=4$
$y=(4)^2=16$
which is true!
So this sequence of transformations is correct.

Ex 7E – Transformations of graphs of functions

Example:

A transformation is defined by the rule $(x,y)\to(-2y-1,3x+1)$. Find the image of the straight line with equation $y=3x-4$ under this transformation.

reflection in the line $y=x$

$x'=-2y-1,\quad y'=3x+1$

$y=\dfrac{-x'-1}{2},\quad x=\dfrac{y'-1}{3}$

the graph of $y=3x-4$ is mapped to:

$$\frac{-x'-1}{2}=3\left(\frac{y'-1}{3}\right)-4$$

$$\frac{-x'-1}{2}=y'-5$$

$$y'=\frac{-x'-1}{2}+5$$

$$y=-\frac{x}{2}+\frac{9}{2}$$

Example:

Find a sequence of transformations that takes the graph of the quadratic function with rule $f_1(x)=x^2+4x+12$ to the graph of the quadratic function with rule $f_2(x)=3x^2+6x+5$.

transformations of quadratics

CTS for each rule:

$f_1(x)=(x+2)^2+8$ and $f_2(x)=3(x+1)^2+2$

To find transformations, write:

$y=(x+2)^2+8$ and $y'=3(x'+1)^2+2$

rearrange:

$y-8=(x+2)^2$ and $\dfrac{y'-2}{3}=(x'+1)^2$

equate and solve for x' and y':

$y-8=\dfrac{y'-2}{3}$ and $x+2=x'+1$

$y'=3y-22$ and $x'=x+1$

$\therefore$ dilation of factor 3 from x-axis, followed by a translation 1 unit in the positive x-direction and 22 units in the negative y-direction.