T05-T06 Mixed Application Questions – Part 2 SOLUTIONS – CAS

Instructions: CAS is permitted.

Question 1

A liquid has been heated using a heating element. The liquid then cools down according to the model $L(t) = 40e^{-kt} + 20$, where $L°C$ is the temperature of the liquid and t is the time in minutes from the time the heating element is first switched off. The value of k is different for different types of liquid.

a. What is the temperature of the liquids when the heating is first switched off?

$L(0) = 40e^{0} + 20$

$= 60°C$

This means substitute ***t=0***

b. If a particular liquid cools to 50^0C after three minutes, find the value of k, to three decimal places, for this liquid.

$40e^{-3k} + 20 = 50$ M1 Use CAS to solve for k

$k = 0.096$

This means substitute ***t=3***

2 marks

solve$\left(40 \cdot e^{-3 \cdot k} + 20 = 50, k\right)$ $k = 0.095894024151$

c. For another liquid the value of k is 0.157. How many minutes will it take for this liquid to cool to half its temperature when the heating element was first switched off? Give your answer to one decimal place.

This means ***half of***

$40e^{-0.157t} + 20 = 30$ M1 Use CAS to solve for t

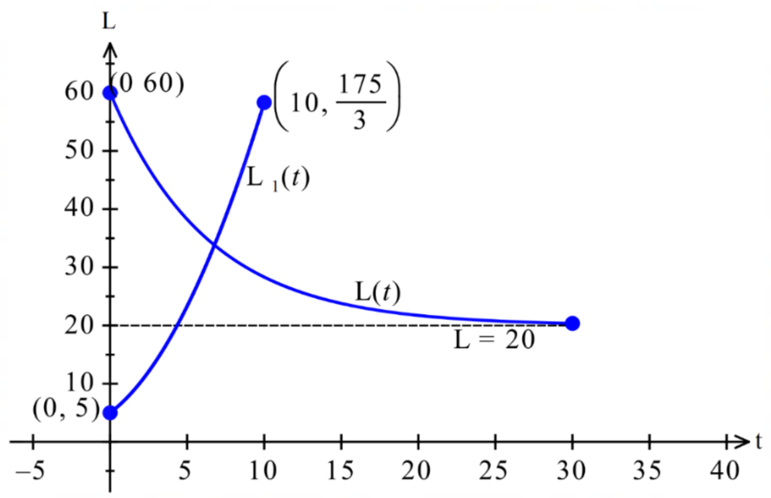
$t = 8.8\ mins$

$l(t) := 40 \cdot e^{-0.157 \cdot t} + 20$ *Done*

solve$(l(t) = 30, t)$ $t = 8.8299003893$

arks

d. Sketch the graph of $y = L(t)$, where $L(t) =$ ... ow, showing any the coordinates of any axis intercepts and giving the equation of any asymptotes.



3 marks

e. Another liquid is heated for 10 minutes. It's temperature, L_1°C , is given by the rule

$L_1 = a(t+b)^2 + 2, 0 \le t \le 10$ where *a* and *b* are **positive** real numbers and *t* is measured in minutes from when the liquid is placed on a heating element.

This means at *t=0*

When it is first placed on the heating element the temperature of this liquid is $5°C$ and after 3 minutes of being heated its temperature is $14°C$

So 2 points on the graph are (0, 5) and (3, 14)

i) **By forming and solving** a set of two simultaneous equations, show that $a = \frac{1}{3}$ and $b = 3$.

Full algebraic working must be shown.

2 marks

(0, 5): $5 = ab^2 + 2$ (3, 14): $14 = a(3+b)^2 + 2$

$3 = ab^2$ (1) $12 = a(3+b)^2$ (2) M1

OR Method 2: $a = \frac{3}{b^2}$ (1) $a = \frac{12}{(3+b)^2}$ (2)

Method 1: (2)÷ (1): $\frac{12}{3} = \frac{a(3+b)^2}{ab^2}$ M1

$4 = \frac{(3+b)^2}{b^2}$

$4b^2 = (3+b)^2$

$4b^2 = 9 + 6b + b^2$

$3b^2 - 6b - 9 = 0$

$b^2 - 2b - 3 = 0$

$(b-3)(b+1) = 0$

$b = 3, b = -1,$ but $b > 0$

$\therefore b = 3$

Equating these expressions for a:

$\frac{3}{b^2} = \frac{12}{(3+b)^2}$

$3(3+b)^2 = 12b^2$

$(3+b)^2 = 4b^2$

$9 + 6b + b^2 = 4b^2$

(then as for method 1)

ii) Sketch the graph of L_1 on the axes given in d) showing the coordinates of any end points.

Note the end points are (0, 5) and (10, 175/3)

2 marks

iii) If the heating element for the liquid in part c) is turned off at the same time as the liquid in part d) is put on the heating element, find the number of minutes correct to one decimal place when the these two liquids will be the same temperature.

1 mark

$\frac{1}{3}(t+3)^2 + 2 = 40e^{-0.157t} + 20$ Solve with CAS or find point of intersection 2 graphs on CAS

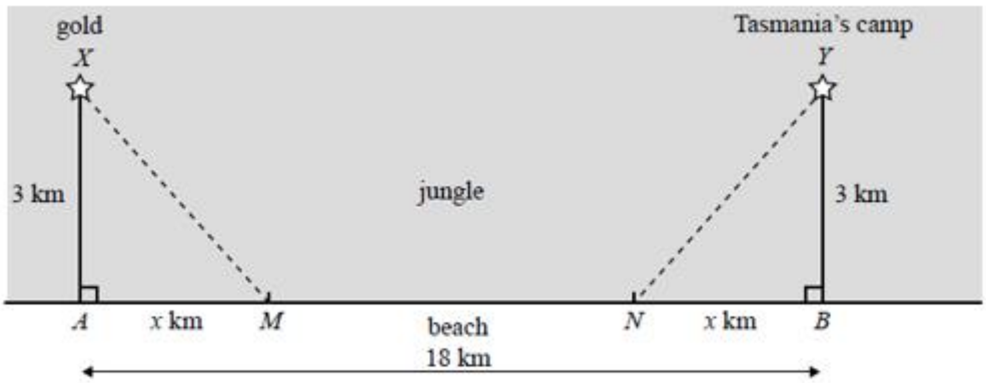
$t = 6.8$mins

Question 2

Tasmania Jones is in the jungle, digging for gold. He finds the gold at X which is $3\,km$ from point A.

Point A is on a straight beach.

Tasmania's camp is at Y which is a distance of $3\,km$ from a point B. Point B is also on the straight beach. $AB = 18km$ and $AM = NB = x\ km$ and $AX = BY = 3\ km$.



While he is digging up the gold, Tasmania is bitten by a snake which injects toxin into his blood. After he is bitten, the concentration of the toxin in his bloodstream increases over time according to the equation

$$y = 50\log_e(1+2t)$$

where y is the concentration, and t is the time in hours after the snake bites him.

The toxin will kill him if its concentration reaches 100.

a. Find the time, to the nearest minute, that Tasmania has to find an antidote (that is, a cure for the toxin).

[2 marks]

$50\log_e(1+2t) = 100 \quad ...[M1]$

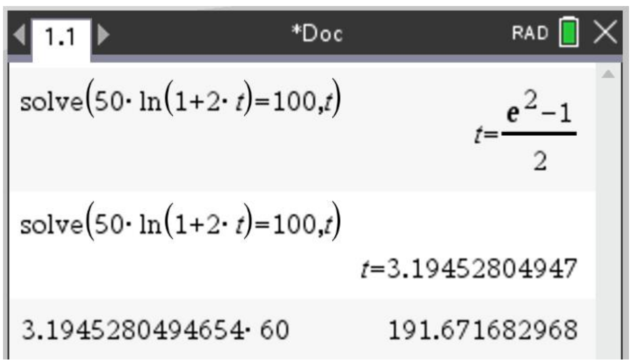
$\log_e(1+2t) = 2$

solve for t

$t = 3.19452....\ hrs \qquad (3.19452.... \times 60 = 191.6716...)$

$= 191.6716...\text{min}$

$= 192\,\text{min}$ (*to the nearest minute*) $\quad ...[A1]$



Tasmania has an antidote to the toxin at his camp. He can run through the jungle at $5km/h$ and he can run along the beach at $13km/h$.

b. Show that he will not get to the antidote in time if he runs directly to his camp through the jungle.

[1 mark]

$$\textit{time required to reach camp is } \frac{18}{5} = 3.6hrs$$

$$3.6 > 3.1945... \quad \therefore \textit{he will not get the antidote in time.} [A1]$$

In order to get the antidote, Tasmania runs through the jungle to M on the beach, runs along the beach to N and then runs through the jungle to the camp at Y. M is $x\ km$ from A and N is $x\ km$ from B (See diagram).

c. Show that the time taken to reach the camp, T hours, is given by $T = 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right)$.

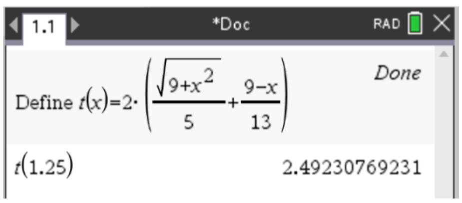
$$\textit{using } s = \frac{d}{t} \therefore t = \frac{d}{s} \quad \textit{and pythagoras' theorem}$$

$$\left.\begin{aligned} t_{XM} &= t_{NY} = \frac{\sqrt{9+x^2}}{5} \\ t_{MN} &= \frac{18-2x}{13} \end{aligned}\right\} \quad ...[A1]$$

$$\left.\begin{aligned} T &= 2\left(\frac{\sqrt{9+x^2}}{5}\right) + \frac{18-2x}{13} \\ &= 2\left(\frac{\sqrt{9+x^2}}{5}\right) + \frac{2(9-x)}{13} \\ &= 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right) \end{aligned}\right\} \quad ...[A1]$$

d. If $x = 1.25$, show that he gets to his camp in time to get the antidote.

[1 mark]



$$T = 2\left(\frac{\sqrt{9+(1.25)^2}}{5} + \frac{9-1.25}{13}\right)$$

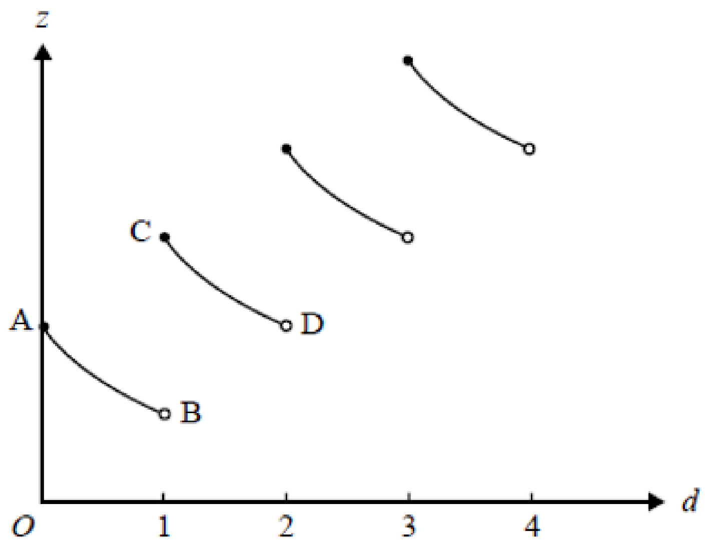
$$= 2.492...$$

$$2.492... < 3.194... \textit{ therefore he gets the antidote in time.} \quad ...[A1]$$

At his camp, Tasmania Jones takes a capsule containing 16 units of antidote to the toxin. After taking the capsule the quantity of antidote in his body decreases over time.

At exactly the same time on successive days, he takes another capsule containing 16 units of antidote and again the quantity of antidote decreases in his body.

The graph of the quantity of antidote z units in his body at time d days after taking the first capsule looks like this (See graph below). Each section of the curve has exactly the same shape as curve AB.



The equation of the curve AB is $z=\dfrac{16}{d+1}$.

e. Write down the coordinates of the points A and C.

$$\begin{aligned}
&\text{Point } A: \quad d=0,\ z=16 \quad (0,16)\\
&\text{Point } B: \quad d=1,\ z=8\\
&\text{Point } C: \quad d=1,\ z=8+16=24 \quad (1,24)\\
&A(0,16) \quad ...[A1]\\
&C(1,24) \quad ...[A1]
\end{aligned}$$

f. Find the equation of the curve CD.

$$\begin{aligned}
&z=\frac{16}{d+1}\\
&\text{curve } AB \text{ is translated } 1 \text{ unit right and } 8 \text{ units up.}\\
&(d,z)\to(d+1,z+8) \quad ...[M1]\\
&d'=d+1 \qquad z'=z+8\\
&d=d'-1 \qquad z=z'-8\\
&\text{image} \quad z'-8=\frac{16}{d'-1+1}\\
&\qquad\qquad z'=\frac{16}{d'}+8\\
&\text{curve } AC = z=\frac{16}{d}+8 \qquad ...[A1]
\end{aligned}$$

Tasmania will no longer be affected by the snake toxin when he first has 50 units of the antidote in his body.

g. Assuming he takes a capsule at the same time each day, on how many days does he need to take a capsule so that he will no longer be affected by the snake toxin?

[1 mark]

Day	z
1	16 to 8
2	24 to 16
3	32 to 24
4	40 to 32
5	48 to 40
6	56 to 48

$6\ days \quad ...[A1]$

Question 3 **NOTE: A continuous function appears as one smooth curve, with no holes or gaps. Continuous graphs are those that can be drawn without lifting a pencil.**

Tasmania Jones is attempting to recover the lost Zambeji diamond. The diamond is buried at a point 4 km into Death Gorge, which is infested with savage insects. In order to recover the diamond, Tasmania will need to run into the gorge, dig up the diamond and return the same way that he came.

The concentration of insects in the gorge is a **continuous** function of time. The concentration *C*, insects per square metre, is given by

$$C(t)=\begin{cases}1000(\cos\left(\dfrac{\pi(t-8)}{2}\right)+2)^2-1000 & 8\le t\le 16\\ m & 0\le t<8 \quad or \quad 16<t\le 24\end{cases}$$

Where *t* is the number of hours after midnight and *m* is a real constant.

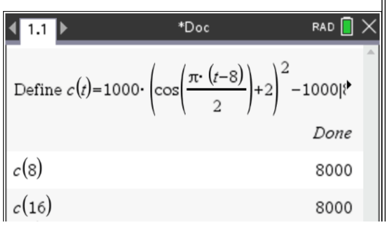
a. What is the value of *m*?

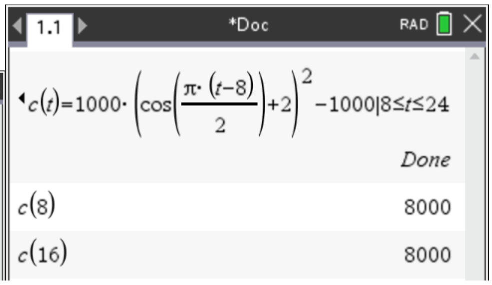
$C(8)=8000$

$C(8)=8000$

$\therefore m=8000 \ \ for \ \ C(t) \ to\ be\ continuous.$

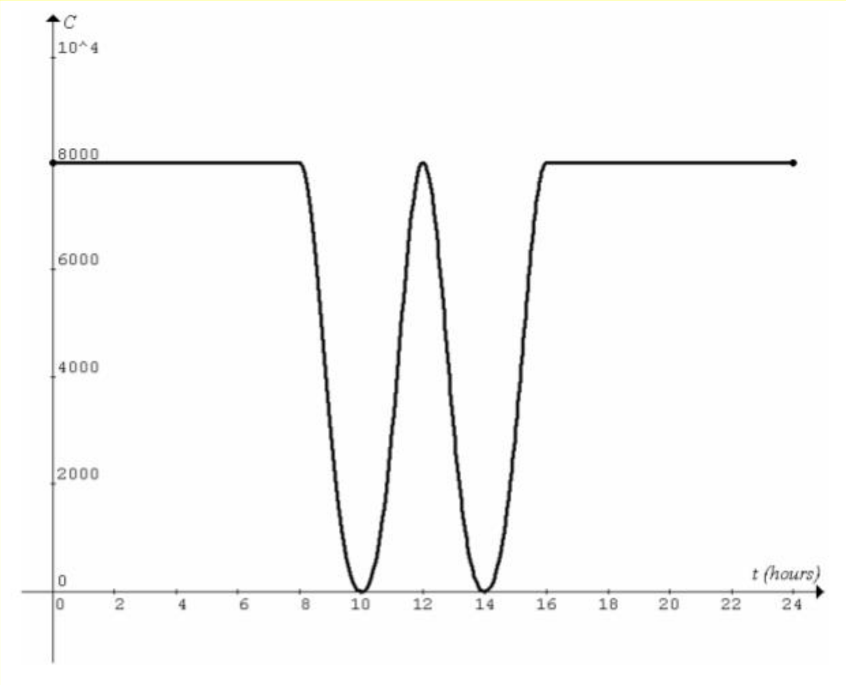
NOTE: Add the domain restriction when defining the equation.





1 mark

b. Sketch the graph of *C* for $0 \le t \le 24$



3 marks

c. What is the minimum concentration of insects and at what value(s) of *t* does that occur?

looking at graph:

$C_{\min} = 0$ *when* $t = 10$ *or* $14.$

2 marks

The insects infesting the gorge are known to be deadly if their concentration is more than 1250 insects per square metre.

d. At what time after midnight does the concentration of insects first stop being deadly?

solve $(c(t) = 1250, t)$

$t = 9\frac{1}{3}$

$\therefore$ *Safety level starts at* 9:20*am*

1 mark

e. During a 24-hour period, what is the total length of time for which the concentration of insects is less than 1250 insects per square metre?

CAS with a restriction

$$solve\left(1000\left(\cos\left(\frac{\pi(t-8)}{2}\right)+2\right)^2-1000=1250,t\right)|8\le t\le 24$$

$$t=\frac{32}{3}-\frac{28}{3}+\frac{44}{3}-\frac{40}{3}$$

$$time=\frac{8}{3}hours$$

$$\therefore 2hours\ and\ 40\min utes$$

This answer can also be obtained by looking at the graph.

2 marks

Due to the uneven surface of the gorge, the time, *T* minutes, that Tasmania will take to run *x* km into the gorge is given by $T(x)=p(q^x-1)$, where *p* and *q* are constants.

f. Tasmania knows that it will take him 5 minutes to run the first kilometre and 12.5 minutes to run the first two kilometres.

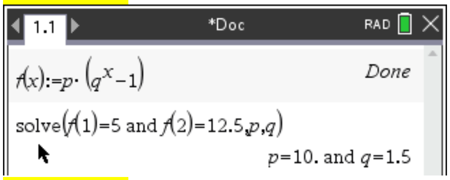
i. Find the values of *p* and *q*.

$T(1)=5$ $T(2)=12.5$

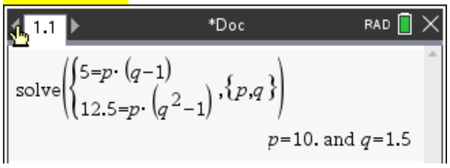
$p=10, q=1.5$

The 2 ways to do it on the CAS should be shown.

Method 1:



Method 2:



$$T(x)=10(1.5^x-1)$$

$$T(4)=40.625\text{ minutes}$$

ii. Find the length of time that Tasmania will take to run the 4 km to reach the buried diamond.

$$T(x)=10(1.5^x-1)$$

$$T(4)=10(1.5^4-1)=40.625$$

$$\therefore 40\min utes\ and\ 37.5\sec onds$$

2 + 1 = 3 marks

g. Tasmania takes 19 minutes to dig up the diamond and he is able to run back through the gorge in half the time it took him to reach the diamond. Show that it is possible for him to recover the diamond successfully and state how much time he has to spare.

$$\text{Required time} = 40.625 + 19 + \frac{1}{2} \times 40.625 = 79.9375 \text{ minutes}$$

$$\text{Available time} = \frac{4}{3} \text{ hours} = 80 \text{ minutes}$$

$$\text{Spare time} = 80 - 79.9375 = 0.0625 \text{ minutes}$$

$$\therefore 3.75 \text{ seconds}$$

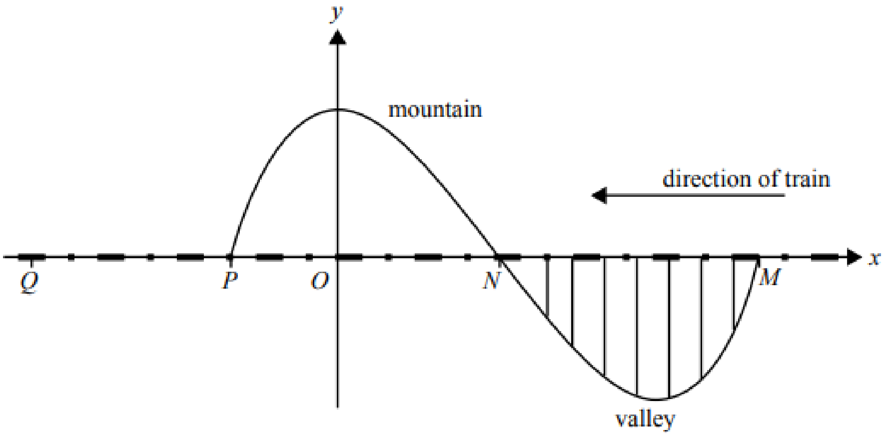
$$79.9375 \text{ mins} < 80 \text{ mins}$$

Can recover diamond

3 marks

See Question 4 on the next page

Question 4



A train is travelling at 120 km/h along a straight level track from *M* towards *Q*.

The train will travel along a section of track *MNPQ.*

Section *MN* passes along a bridge over a valley.

Section *NP* passes through a tunnel in a mountain.

Section *PQ* is 6.2 km long.

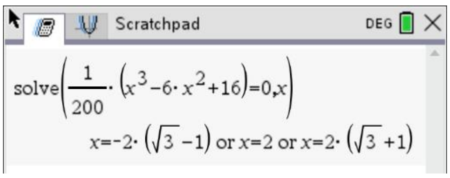
From *M* to *P*, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph of $y = \frac{1}{200}(x^3 - 6x^2 + 16)$. All measurements are in kilometers.

a. Find, giving exact values:

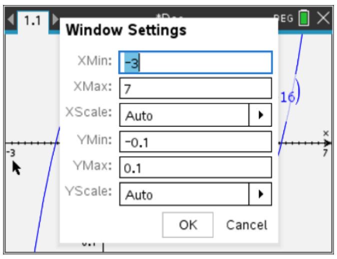
i. the co-ordinates of *M* and *P*

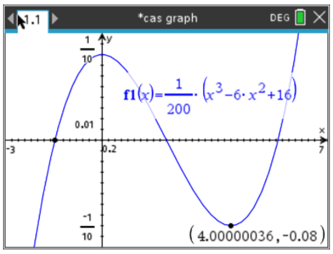
$P\ (2-2\sqrt{3},0)\quad M\ (2+2\sqrt{3},0)\quad$ A1 mark

co-ordinates &exact answers requried, (2,0) is at N



ii. the maximum depth of the valley below the train track





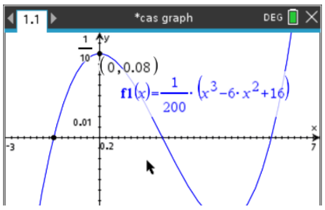
0.08km
80m
one of − A1

unit & postive value requried

1 mark

iii. the maximum height of the mountain above the train track

1 mark



$$\left.\begin{array}{l}0.08\text{km}\\80\text{m}\end{array}\right\}\text{ one of }-\text{A1}$$

unit & postive value requried

iv. the length of the tunnel.

1 mark

$$\left.\begin{array}{l}NP=2-2+2\sqrt{3}\\ \quad =2\sqrt{3}\text{ km}\\ \quad =2000\sqrt{3}\text{ m}\end{array}\right\}\text{A1}-\text{one of}$$

unit & exact answer required

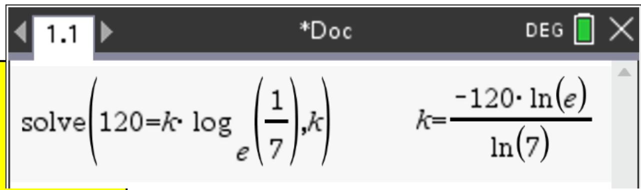
The driver sees a large rock on the track at point *Q*, 6.2 km from *P*. The driver puts on the brakes at the instant that the front of the train comes out of the tunnel at *P*.

From its initial speed of 120 km/h, the train slows down from point *P* so that its speed *v* km/h is

given by
$$y=k\log_e\left(\frac{(d+1)}{7}\right)$$

Where *d* km is the distance of the front of the train from *P* and *k* is a real constant.

b. Find the exact value of *k*.



$PQ=6.2\text{km}$ At P, $d=0$ $v=120$ km/h

$$120=k\log_e\left(\frac{0+1}{7}\right)$$

$$120=k\log_e\left(\frac{1}{7}\right)\quad\text{or from here use CAS in exact mode}$$

$$120=k\log_e 7^{-1}$$

$$120=-k\log_e 7$$

$$k=-\frac{120}{\log_e 7}\quad ....\text{A1}$$

1 mark

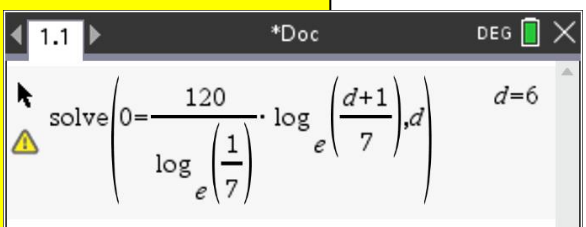
c. Find the exact distance from the front of the train to the large rock when the train finally stops.

1 mark

when $v=0$

$$0=-\frac{120}{\log_e\frac{1}{7}}\log_e\left(\frac{(d+1)}{7}\right)\quad\text{use CAS to solve for } d \text{ - faster or the long way:}$$

$$0=\log_e\left(\frac{(d+1)}{7}\right)\quad\left(Note:-\frac{120}{\log_e\frac{1}{7}}\text{ is a number}\right)$$



$$e^0=\frac{(d+1)}{7}$$

$$1=\frac{(d+1)}{7}$$

$$d=6\text{ km}$$

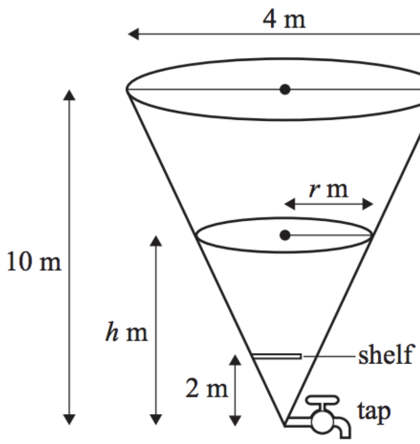
Distance: 6.2 - 6

$=0.2$km or 200m ...A1

Question 5

Tasmania Jones is in the jungle, searching for the Quetzalotl tribe's valuable emerald that has been stolen and hidden by a neighbouring tribe. Tasmania has heard that the emerald has been hidden in a tank shaped like an inverted cone, with a height of 10 metres and a diameter of 4 metres (as shown below).

The emerald is on a shelf. The tank has a poisonous liquid in it.



a. If the depth of the liquid in the tank is *h* metres

i. Find the radius, *r* metres, of the surface of the liquid in terms of *h*.

$$\frac{r}{2} = \frac{h}{10}$$

$$r = \frac{2h}{10}$$

$$r = \frac{h}{5} \quad \ldots [1\ \text{mark}]$$

1 mark

ii. Show that the volume of the liquid in the tank is $\frac{\pi h^3}{75} m^3$

$$V = \frac{1}{3}\pi r^2 h$$

[1 mark – for *all the following* lines of working]

$$V = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h$$

$$V = \frac{1}{3}\pi \times \frac{h^2}{25} \times h$$

$$V = \frac{\pi h^3}{75} m^3$$

1 mark

The tank has a tap at its base that allows the liquid to run out of it. The tank is initially full. When the tap is turned on, the liquid flows out of the tank at such a rate that the depth, *h* metres, of the liquid in the tank is given by

$$h = 10 + \frac{1}{1600}(t^3 - 1200t),$$

Where *t* minutes is the length of time after the tap is turned on until the tank is empty.

b. Show that the tank is empty when *t = 20.*

Tank is empty when *h=0* (sub t=20 into the equation)

1 mark for all of the following lines of working in yellow:

$$\mathrm{h} = 10 + \frac{1}{1600}(20^3 - 1200 \times (20))$$

$$h = 0$$

V=0 *ie.* the tank is empty

1 mark

c. When *t = 5* minutes, find the depth of the liquid in the tank.

Sub *t*=5 into the equation

$$h = 10 + \frac{1}{1600}(5^3 - 1200(5))$$

$h = 6.328125\ \mathrm{m}$ OR $\frac{405}{64}m$ ....[1 mark]

Exact answer required, as above.

1 mark

d. The shelf on which the emerald is placed is 2 metres above the vertex of the cone. From the moment the liquid starts to flow from the tank, find how long, in minutes, it takes until *h = 2*. (Give your answer correct to one decimal place.)

Sub *h*=2 into the equation & solve for t

When $h = 2m \rightarrow 10 + \frac{1}{1600}(t^3 - 1200t) = 2$

$t = -39.1, 12.2, 26.9$ ..[1 mark]

But check which values of *t* are valid

$t \neq -39.1, 26.9$ Since $0 \leq t \leq 20,$

So $t = 12.2\ minutes$ ..[1 mark]

2 marks

e. As soon as the tank is empty, the tap turns itself off and poisonous liquid starts to flow into the tank at a rate of $0.2m^3/minute$. How long, in minutes, after the tank is first empty will the liquid once again reach a depth of 2 metres?

$$V = \frac{\pi h^3}{75} m^3$$

$the\ voume\ when\ when\ the\ depth, h = 2$

$$V = \frac{\pi (2)^3}{75}$$

$V = \frac{8\pi}{75}$ ..[1 mark]

rate = volume (m^3)/ time $(minute)$

$$time = \frac{volume}{rate}$$

$$time = \frac{8\pi}{75} \div 0.2$$

$$time = \frac{8\pi}{75} \times \frac{1}{0.2}$$

$time\ required = \frac{8\pi}{15} minutes$ ...[1 mark] Exact answer required

2 marks

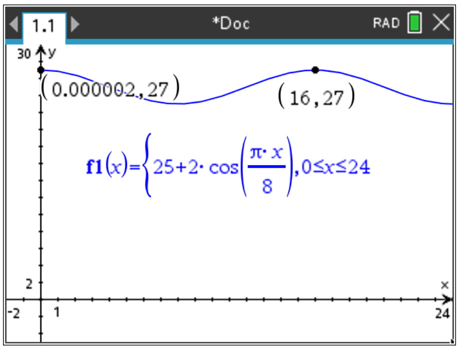
Question 6

Trigg the gardener is working in a temperature controlled greenhouse. During a particular 24-hour time interval, the temperature (T °C) is given by $T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right), 0 \leq t \leq 24$, where t is the time in hours from the beginning of the 24-hour time interval,

a. State maximum temperature in the greenhouse and the values of t when this occurs.

$T\ max = 27\ °\text{C}[\text{A1}]$

$at\ t = 0 \text{ or } 16\ \text{h}[\text{A1}]$



b. State the period of the function T.

$$Period = \frac{2\pi}{\pi/8} = \frac{8}{\pi} \times 2\pi = 16\ \text{h}[\text{A1}]$$

1 mark

c. Find the smallest value of t for which $T = 26$.

$$\text{Solve } 25 + 2\cos\frac{\pi t}{8} = 26 \text{ for } t[\text{M1}]$$

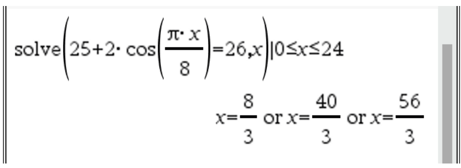
$$t = \frac{8}{3}\ \text{h}\ \ [\text{A1}]$$

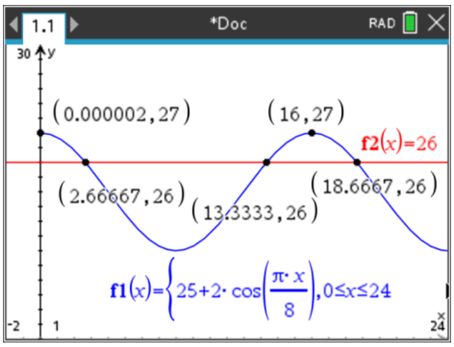
2 marks

d. For how many hours in the 24-hour time interval is $T \geq 26$?

$\text{Solve } T(t) \geq 26 \text{ for } 0 \leq t \leq 24$ [M1]

$t = \frac{8}{3} + \frac{56}{3} - \frac{40}{3} = 8\ \text{h}$ [A1]

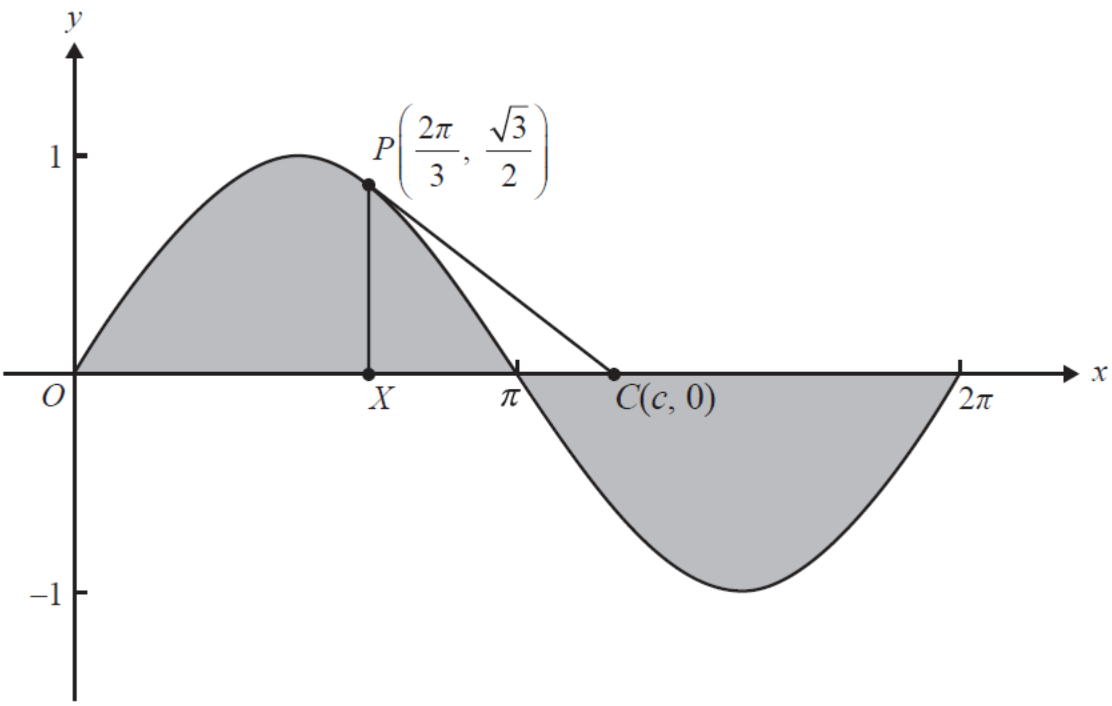




2 marks

Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y = \sin(x)$ for $0 \le x \le 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram below. He includes a garden path, which is shown as line segment PC.

The line through points $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $C(c, 0)$ is a tangent to the graph of $y = \sin(x)$ at point P.



e. If the gradient of PC is $-\frac{1}{2}$ Show that the value of c is $\sqrt{3} + \frac{2\pi}{3}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{\sqrt{3}}{2} - 0}{\frac{2\pi}{3} - c} = -\frac{1}{2}$$

$$\sqrt{3} = c - \frac{2\pi}{3}$$

$$c = \sqrt{3} + \frac{2\pi}{3}$$

1 mark

In further planning for the garden, Trigg uses a transformation of the plane defined as a dilation of factor k from the x-axis and a dilation of factor m from the y-axis, where k and m are positive real numbers.

f. Let X′, P′ and C′ be the image, under this transformation, of the points X, P and C respectively.

i. Find the values of k and m if X′P′ = 10 and X′C′ = 30.

$$XC = \frac{2\pi}{3} + \sqrt{3} - \frac{2\pi}{3} = \sqrt{3} \qquad XP = \frac{\sqrt{3}}{2}$$

$$X'C' = 30 \qquad X'P' = 10$$

$$30 = m\sqrt{3} \qquad 10 = k\frac{\sqrt{3}}{2}$$

$$m = \frac{30}{\sqrt{3}} = 10\sqrt{3} \qquad k = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3}$$

2 marks

ii. Find the coordinates of the point P′.

$$P' = \left(\frac{2\pi}{3} \times 10\sqrt{3}, \frac{\sqrt{3}}{2} \times \frac{20\sqrt{3}}{3}\right)$$

$$P' = \left(\frac{20\pi\sqrt{3}}{3}, 10\right)$$

1 mark