T04 Polynomials

Ex 6A – Polynomial Arithmetic & Binomial expansion

A polynomial is an expression containing only positive integer powers of *x*.

A *polynomial function* can be written in the form $P(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ where *n* is a natural number (or zero) and $a_0,...,a_n$ are coefficients.

The *leading term*, a_nx^n is the term of highest power.

The *degree of a polynomial* is the index *n* of the leading term.

A *monic polynomial* is one whose leading term has a coefficient of 1.

The constant term is the term of power 0 (i.e. a_0x^0).

When performing arithmetic on polynomials, the result is always a polynomial.

Example: If $P(x)=x^2+1$ and $Q(x)=3x^2-2$, find:

a) $P(x)+Q(x)$

$=(x^2+1)+(3x^2-2)$

$=4x^2-1$

b) $P(x)Q(x)$

$=(x^2+1)(3x^2-2)$

$=3x^4-2x^2+3x^2-2$

$=3x^4+x^2-2$

You are familiar with the quadratic perfect squares formula:

- Sum: $(a+b)^2=a^2+2ab+b^2$
- Difference: $(a-b)^2=a^2-2ab+b^2$

An identity can also be found for perfect cubes:

- Sum: $(a+b)^3=a^3+3a^2b+3ab^2+b^3$
- Difference: $(a-b)^3=a^3-3a^2b+3ab^2-b^3$

And also for quartics: $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$

Pascal's Triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Can you see the pattern?

Example: Use the rules to expand:

a) $(x+5)^3$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$(x+5)^3=(x)^3+3(x)^2(5)+3(x)(5)^2+(5)^3$

$=x^3+15x^2+75x+125$

b) $(2x-1)^3$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$

$(2x-1)^3=(2x)^3-3(2x)^2(1)+3(2x)(1)^2-(1)^2$

$=8x^3-12x^2+6x-1$

Equating coefficients

Two polynomials are equal only if their corresponding coefficients are equal.

Example: It is known that $x^3+3x^2+2x+1=x^3+(a-2)x^2+(b-2a)x-2c$. Find the values of *a*, *b* and *c*.

$3=a-2$
$a=5$

$2=b-2a$
$2=b-2(5)$
$b=2+10$
$b=12$

$1=-2c$
$c=-\frac{1}{2}$

Ex 6B – Division of polynomials

When sketching cubics of the form $y=ax^3+bx^2+cx+d$ we begin by finding the *x*-intercepts. All cubics have at least one *x*-intercept, but can have up to 3.

Remember that *x*-intercepts are found by factorising and solving using the null factor law.

The technique used to factorise cubics uses long division.

Recall basic long division: $325\div12$

$$
\begin{array}{rrl}
 & 027 & \leftarrow \text{quotient, Q(x)} \\
\text{divisor} \rightarrow & 12\overline{)325} & \leftarrow \text{dividend} \\
 & -24\downarrow & \\
 & \overline{85} & \\
 & -84 & \\
 & \overline{1} & \leftarrow \text{remainder, R}
\end{array}
$$

The answer is therefore $27\frac{1}{12}$, which we could also write as $12\times27+1$.

A similar technique is used for polynomial division.

Example: Divide the following and write in the form $P(x)=(ax-b)Q(x)+R$.

a) $2x^3+x^2-4x+3,\ \ x-1$

$$
\begin{array}{r}
2x^2+3x-1 \\
x-1\overline{)2x^3+x^2-4x+3} \\
-(2x^3-2x^2) \\
\hline
3x^2-4x+3 \\
-(3x^2-3x) \\
\hline
-x+3 \\
-(-x+1) \\
\hline
2
\end{array}
$$

$\therefore P(x)=(x-1)(2x^2+3x-1)+2$

for polynomial division:
think: "$2x^3\div x=2x^2$"
write this in the quotient
multiply out: $x\times2x^2=2x^3$
$-1\times2x^2=-2x^2$
put brackets, and subtract.
bring down the rest of the function
start again:
think: "$3x^2\div x=3x$"
.... etc

b) $x^3+3x-4,\ \ x+1$

$$\begin{array}{r} x^2-x+4 \\ x+1\overline{)\,x^3+0x^2+3x-4} \\ -\underline{\left(x^3+x^2\right)} \\ -x^2+3x-4 \\ -\underline{\left(x^2-x\right)} \\ 4x-4 \\ -\underline{\left(4x+4\right)} \\ -8 \end{array}$$

$\therefore P(x)=(x+1)\left(x^2-x+4\right)-8$

c) $2x^3-3x^2-29x-30,\ \ 2x+3$

$$\begin{array}{r} x^2-3x-10 \\ 2x+3\overline{)\,2x^3-3x^2-29x-30} \\ -\underline{\left(2x^3+3x^2\right)} \\ -6x^2-29x-30 \\ -\underline{\left(6x^2-9x\right)} \\ -20x-30 \\ -\underline{\left(-20x-30\right)} \\ 0 \end{array}$$

$\therefore P(x)=(2x+3)\left(x^2-3x-10\right)$

An alternative way of dividing polynomials is to *equate coefficients*.

Example: Divide the polynomial x^3+x^2-2x+3 by $x-1$ by equating coefficients.

first write the identity as:
$x^3+x^2-2x+3=(x-1)\left(ax^2+bx+c\right)+r$
expand and simplify the RHS:
$x^3+x^2-2x+3=ax^3+bx^2+cx-ax^2-bx-c+r$
$x^3+x^2-2x+3=ax^3+(b-a)x^2+(c-b)x-c+r$

$\therefore a=1$
$b-a=1$
$\therefore b=2$
$c-b=-2$
$\therefore c=0$

$-c+r=3$
$\therefore r=3$

$\therefore x^3+x^2-2x+3=(x-1)\left(x^2+2x\right)+3$

<u>Ex 6C – Factorisation of Polynomials</u>

<u>The Remainder Theorem</u>

When $P(x)$ is divided by $ax+b$, the remainder will be given by $P\left(-\frac{b}{a}\right)$.

$$ax+b=0$$
$$\therefore x=-\frac{b}{a}$$

For example, x^3-3x^2+4x-1 divided by $x+2$:

Using long division:

$$\begin{array}{r} x^2-5x+14 \\ x+2\overline{)\,x^3-3x^2+4x-1} \\ -\underline{\left(x^3+2x^2\right)} \\ -5x^2+4x-1 \\ -\underline{\left(-5x^2-10x\right)} \\ 14x-1 \\ -\underline{\left(14x+28\right)} \\ -29 \end{array}$$

Using remainder theorem:

$$\begin{aligned} P(-2)&=(-2)^3-3(-2)^2+4(-2)-1 \\ &=-8-12-8-1 \\ &=-29 \end{aligned}$$

Example: Without using division find the remainder when $10x^3-3x^2+4x-1$ is divided by $2x+1$.

$$P\left(-\frac{1}{2}\right)=10\left(-\frac{1}{2}\right)^3-3\left(-\frac{1}{2}\right)^2+4\left(-\frac{1}{2}\right)-1$$
$$=-\frac{10}{8}-\frac{3}{4}-2-1$$
$$=-\frac{8}{4}-3$$
$$=-5$$

Example: If x^3+ax^2+3x-5 has a remainder of -3 when divided by $x-2$, find the value of a.

$$P(2)=(2)^3+a(2)^2+3(2)-5$$
$$-3=8+4a+6-5$$
$$-3=9+4a$$
$$-12=4a$$
$$a=-3$$

<u>The Factor Theorem</u>

The factor theorem states that if $ax+b$ is a factor of $P(x)$ then $P\left(-\frac{b}{a}\right)=0$.

This means that for $ax+b$ to be a factor of $P(x)$ the remainder must be equal to zero.

Example: **Show that** $x+2$ is a factor of $2x^3-3x^2-11x+6$

$$P(-2)=2(-2)^3-3(-2)^2-11(-2)+6$$
$$=-16-12+22+6$$
$$=0$$
$$\therefore(x+2) \text{ is a factor}$$

in "show that" questions you must explicitly show the substitution.

you will need to "guess" the first factor.
start with 1, -1, 2, -2, etc

Example: Factorise the following:

a) x^3+3x^2+3x+1

$$P(-1)=(-1)^3+3(-1)^2+3(-1)+1=0$$
$$\therefore(x+1) \text{ is a factor}$$

$$
\begin{array}{r l}
 & x^2+2x+1 \\
x+1 \enclose{longdiv}{} & x^3+3x^2+3x+1 \\
 & -(x^3+x^2) \\
 & 2x^2+3x+1 \\
 & -(2x^2+2x) \\
 & x+1 \\
 & -(x+1) \\
 & 0
\end{array}
$$

$$\therefore(x+1)(x^2+2x+1)$$
$$=(x+1)(x+1)^2$$
$$=(x+1)^3$$

b) $x^3-21x+20$

$$P(1)=(1)^3-21(1)+20=0$$
$$\therefore(x-1) \text{ is a factor}$$

$$
\begin{array}{r l}
 & x^2+x-20 \\
x-1 \enclose{longdiv}{} & x^3+0x^2-21x+20 \\
 & -(x^3-x^2) \\
 & x^2-21x+20 \\
 & -(x^2-x) \\
 & -20x+20 \\
 & -(-20x+20) \\
 & 0
\end{array}
$$

$$\therefore(x-1)(x^2+x-20)$$
$$=(x-1)(x-4)(x+5)$$

The Rational-root Theorem:

The rational-root theorem can be used when the solution to the polynomials is not an integer.

It states that if we let $P(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ and if $\beta x+\alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

Example: Factorise $P(x)=3x^3+8x^2+2x-5$

using the rational-root theorem:

$$P\left(\frac{-5}{3}\right)=3\left(\frac{-5}{3}\right)^3+8\left(\frac{-5}{3}\right)^2+2\left(\frac{-5}{3}\right)-5=0$$

$\therefore (3x+5)$ is a factor

Dividing gives:

$(3x+5)(x^2+x-1)$

Factorise (in any way)

$$(3x+5)\left(x+\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(x+\frac{1}{2}-\frac{\sqrt{5}}{2}\right)$$

Special cases:

Sum of cubes: $x^3+a^3=(x+a)(x^2-ax+a^2)$

Difference of cubes: $x^3-a^3=(x-a)(x^2+ax+a^2)$

Example: Factorise the following:

a) x^3+64

$$\begin{aligned}x^3+64&=x^3+4^3\\&=(x+4)(x^2-4x+4^2)\\&=(x+4)(x^2-4x+16)\end{aligned}$$

b) $64m^3-27n^3$

$$\begin{aligned}64m^3-27n^3&=(4m)^3-(3n)^3\\&=(4m-3n)\left((4m)^2+(4m)(3n)+(3n)^2\right)\\&=(4m-3n)(16m^2+12mn+9n^2)\end{aligned}$$

Ex 6D – Solving cubic equations

Remember: the Null Factor Law is used to solve equations; if $a\times b=0$, then $a=0$ and $b=0$.

Example: Factorise the following

a) $(x-1)(x+3)(x+7)=0$

$(x-1)(x+3)(x+7)=0$

$x-1=0,\quad x+3=0,\quad x+7=0$

$x=1,\quad x=-3,\quad x=-7$

b) $x^3+2x^2-8x=0$

$x^3+2x^2-8x=0$

$x(x^2+2x-8)=0$

$x(x-4)(x+2)=0$

$x=0,\quad x-4=0,\quad x+2=0$

$x=0,\quad x=4,\quad x=-2$

c) $3x^3-4x^2-13x-6=0$

$P(-1)=3(-1)^3-4(-1)^2-13(-1)-6=0$

$\therefore (x+1)$ is a factor

$$\begin{array}{r} 3x^2-7x-6 \\ x+1\overline{)3x^3-4x^2-13x-6} \\ -(3x^3+3x^2) \\ -7x^2-13x-6 \\ -(-7x^2-7x) \\ -6x-6 \\ -(-6x-6) \\ 0 \end{array}$$

$(x+1)(3x^2-7x-6)=0$

$(x+1)(3x^2-7x-6)=0$

$(x+1)(x-3)(3x+2)=0$

$x+1=0,\quad x-3=0,\quad 3x+2=0$

$x=-1,\quad x=3,\quad x=-\frac{2}{3}$

d) $2x^3+250=0$

$2x^3+250=0$

$2(x^3+125)=0$

$2(x+5)(x^2-5x+25)=0$

$\therefore x+5=0$

$x=-5$

e) $(x^3+4x^2)-(11x+44)=0$

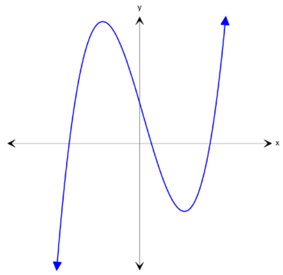
$x^2(x+4)-11(x+4)=0$

$(x+4)(x^2-11)=0$

$x+4=0,\quad x^2-11=0$

$x=-4,\quad x=\pm\sqrt{11}$

Ex 6E – Cubics of the form $f: R \to R, f(x) = a(x-h)^3 + k$

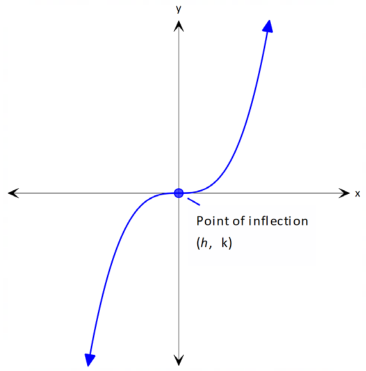


Cubic functions are polynomials of degree 3.

The general form is shown on the right and has the equation

$$y = ax^3 + bx^2 + cx + d.$$

A basic cubic is given by the equation $y = ax^3$



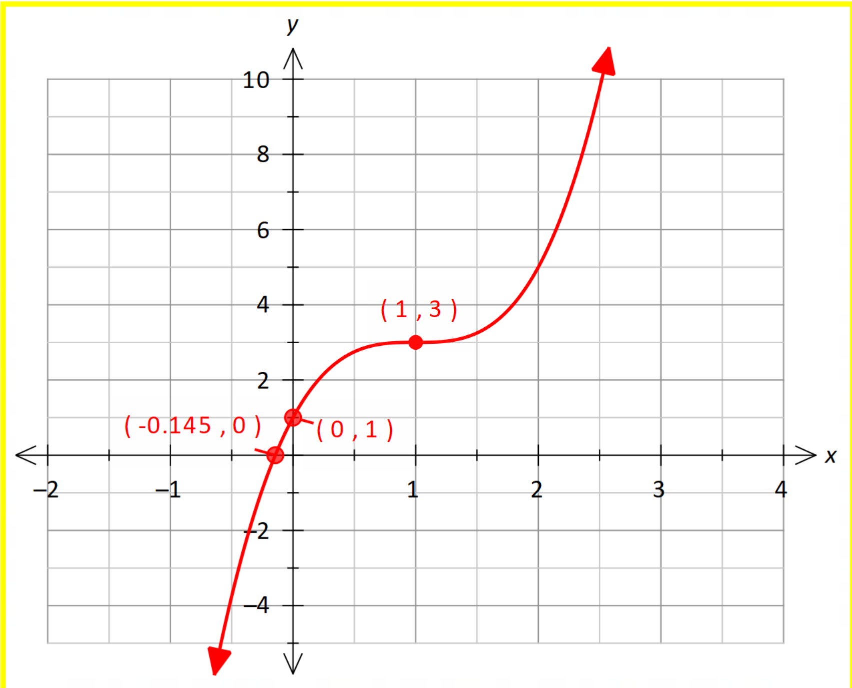
We can transform this graph in the same way we transform quadratics.

$$y = a(x-h)^3 + k$$

This equation shows a horizontal translation of *h* units to the right and a vertical translation *k* units up.

The point of inflection has a gradient of zero and is given by (h,k).

Example: Sketch the graph of $y = 2(x-1)^3 + 3$



Find x and y intercepts in the normal way:
for x-int, let $y = 0$
for y-int, let $x = 0$

Remember that you can cube root negative numbers! $\sqrt[3]{-8} = -2$

x-int, let $y = 0$

$$2(x-1)^3 + 3 = 0$$

$$(x-1)^3 = -\frac{3}{2}$$

$$x - 1 = \sqrt[3]{-\frac{3}{2}}$$

$$x = \sqrt[3]{-\frac{3}{2}} + 1$$

$$\simeq -0.1447142...$$

Inverse functions

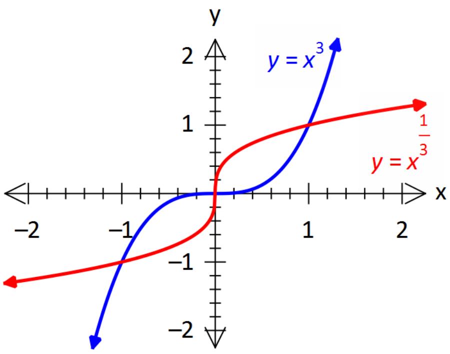
The rule $f: R \rightarrow R, f(x) = x^{\frac{1}{3}}$ is the inverse function of a cubic.

$y = x^3$

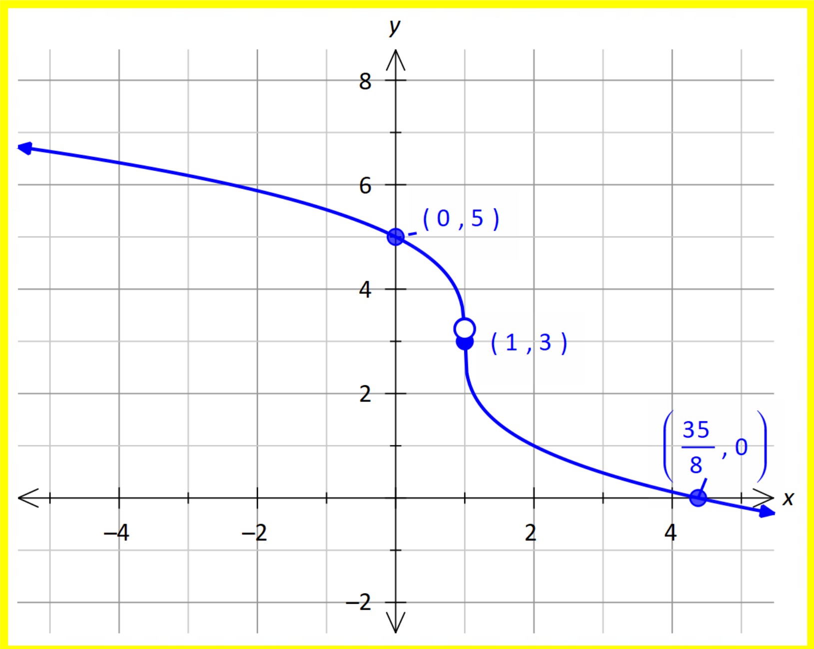
swap $x \Leftrightarrow y$

$x = y^3$

$y = \sqrt[3]{x} = x^{\frac{1}{3}}$



Example: Sketch the graph of $y = -2(x-1)^{\frac{1}{3}} + 3$



x-int, let $y = 0$

$$-2(x-1)^{\frac{1}{3}} + 3 = 0$$

$$(x-1)^{\frac{1}{3}} = \frac{3}{2}$$

$$x - 1 = \left(\frac{3}{2}\right)^3$$

$$x = \frac{27}{8} + 1$$

$$x = \frac{35}{8}$$

$$= 4.375$$

y-int, let $x = 0$

$$y = -2(0-1)^{\frac{1}{3}} + 3$$

$$= -2(-1)^{\frac{1}{3}} + 3$$

$$= -2(-1) + 3$$

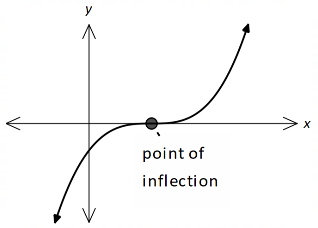
$$= 2 + 3$$

$$= 5$$

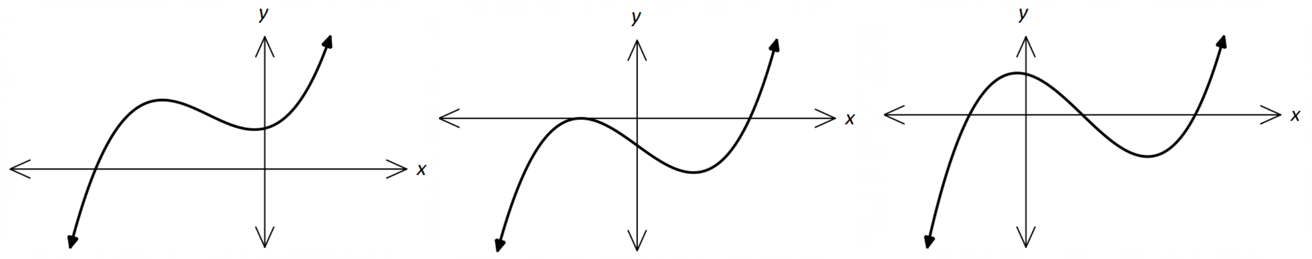
Ex 6F – Graphs of cubic functions

There are two types of cubic function graphs:

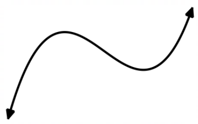
The first type has the rule $y=a(x-h)^3+k$ and looks like $\rightarrow$



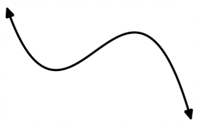
The second type is usually of the general form $y=ax^3+bx^2+cx+d$ or in intercept form $y=a(x-d)(x-e)$ and can have one, two or three x-intercepts:



If the coefficient of the x^3 term is positive, the graph will have the right hand side pointing UP.



If the coefficient of the x^3 term is negative, the graph will have the right hand side pointing DOWN.

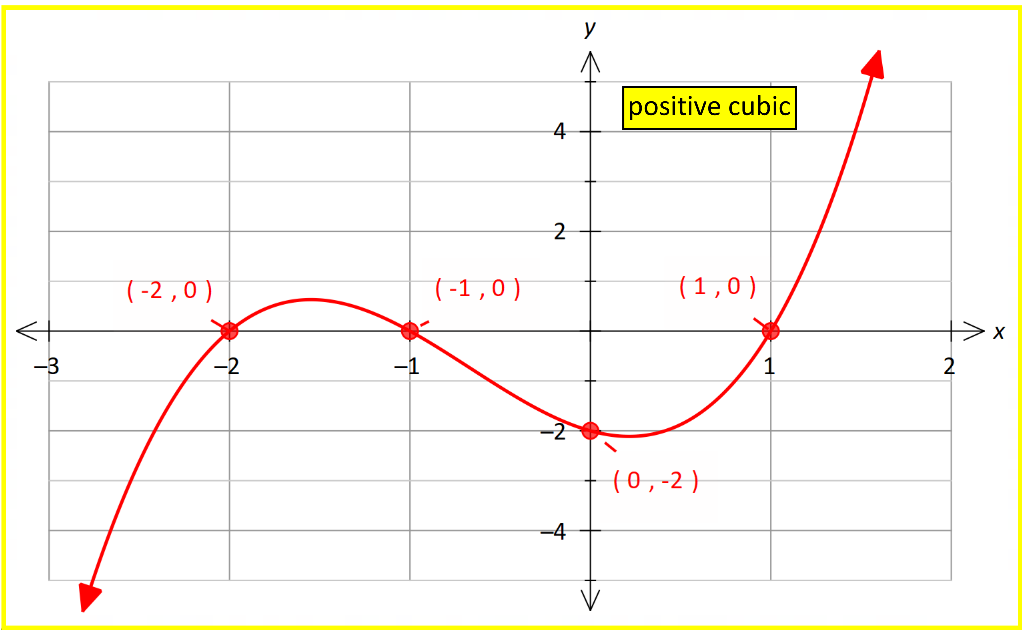


To sketch graphs of cubic functions:

- Determine if the graph is positive or negative
- Find the *x*-intercepts. Determine if any are turning points or inflections.
- Find the *y*-intercept
- Sketch the graph!

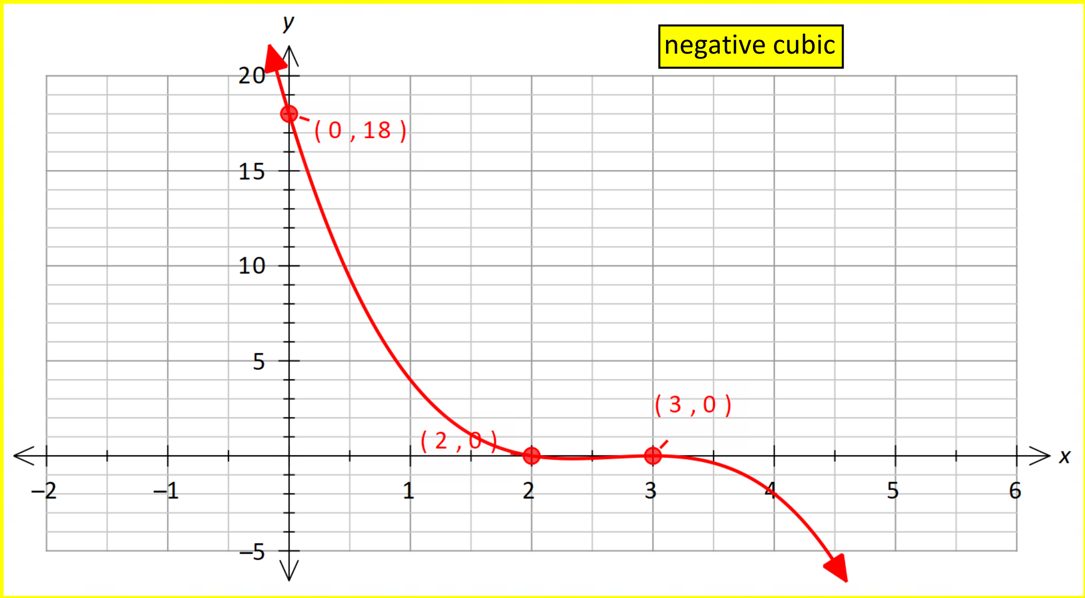
Note that cubic graphs are ***not symmetrical***. Turning points are often not exactly between the x-intercepts.

Example: Sketch the graph of $y=(x-1)(x+1)(x+2)$



note: you do not need to find the TPs right now. Calculus is needed to find them, so for now, just turn the graph at approximately the correct place.

Example: Sketch the graph of $y=(2-x)(x-3)^2$

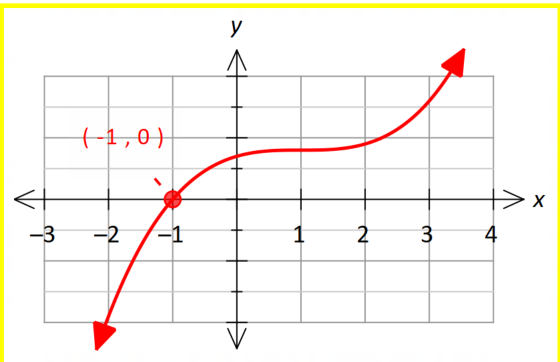


<u>Ex 6G – Solving cubic inequalities</u>

We solve cubic inequations in the same way as we solve quadratic inequations. Often this means we have to consult the graph.

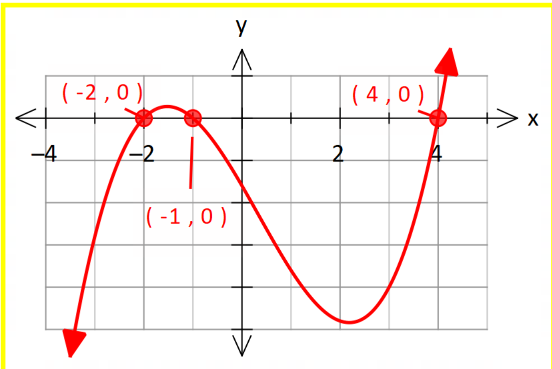
Example: Solve the following for *x*.

a) $(x-1)^3+8\le 0$



we are asked for the values
$y\le 0$
$\therefore$ from the graph, the parts below the x-axis are:
$x\in(-\infty,-1]$

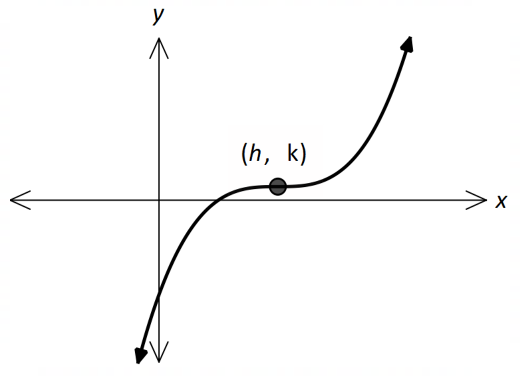
b) $(x+1)(x+2)(x-4)\ge 0$



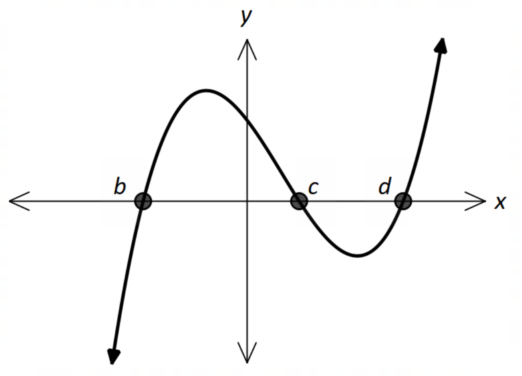
we are asked for the values
$y\ge 0$
$\therefore$ from the graph, the parts above the x-axis are:
$x\in[-2,-1]\cup[4,\infty)$

Ex 6H – Families of cubic polynomial functions

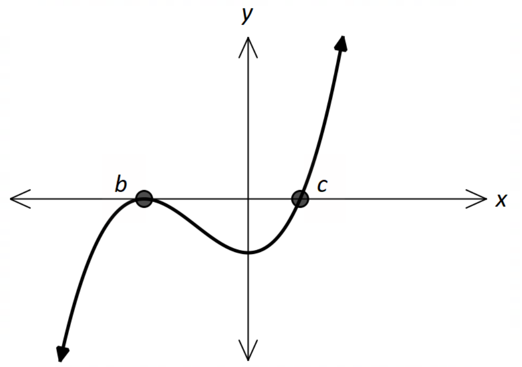
Turning point form: $y = a(x-h)^3 + k$



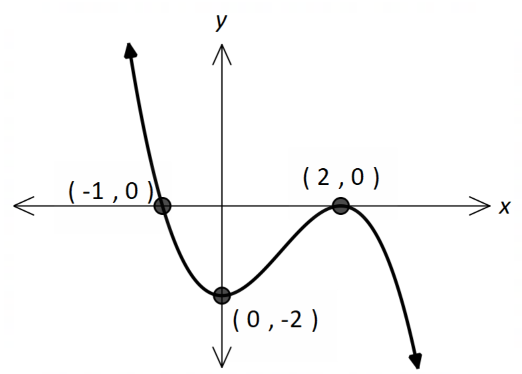
Intercept form: $y = a(x-b)(x-c)(x-d)$



Repeated factor form: $y = a(x-b)^2(x-c)$



Example: Find the equation of the following.



There's a repeated factor

$y = a(x-b)^2(x-c)$

$y = a(x-2)^2(x+1)$

to find a, sub in $(0,-2)$

$-2 = a(0-2)^2(0+1)$

$-2 = 4a$

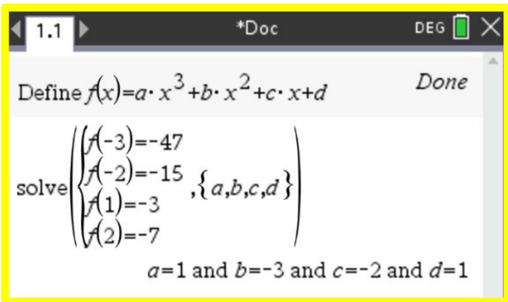
$a = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}(x-2)^2(x+1)$

If given four random points that do not fit into any of the above equations, use simultaneous equations by substituting each point into the equation $y = ax^3 + bx^2 + cx + d$.

Example: Find the equation which passes through the points $(-3,-47),(-2,-15),(1,-3),(2,-7)$.

$y = x^3 - 3x^2 - 2x + 1$



Ex 6I – Quartic and other polynomial functions

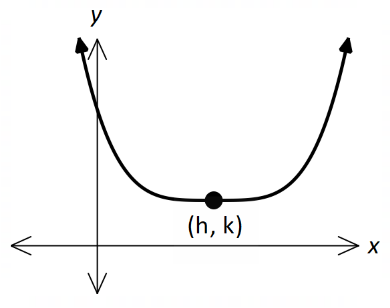
Quartic functions are polynomials of degree 4 and have a general equation of:

$$y = ax^4 + bx^3 + cx^2 + dx + e\,.$$

The basic graph is given by the equation $y = ax^4$ and can undergo transformations:

$$y = a(x-h)^4 + k$$

The turning point is given by (h, k).



quartics have a flatter TP than a quadratic

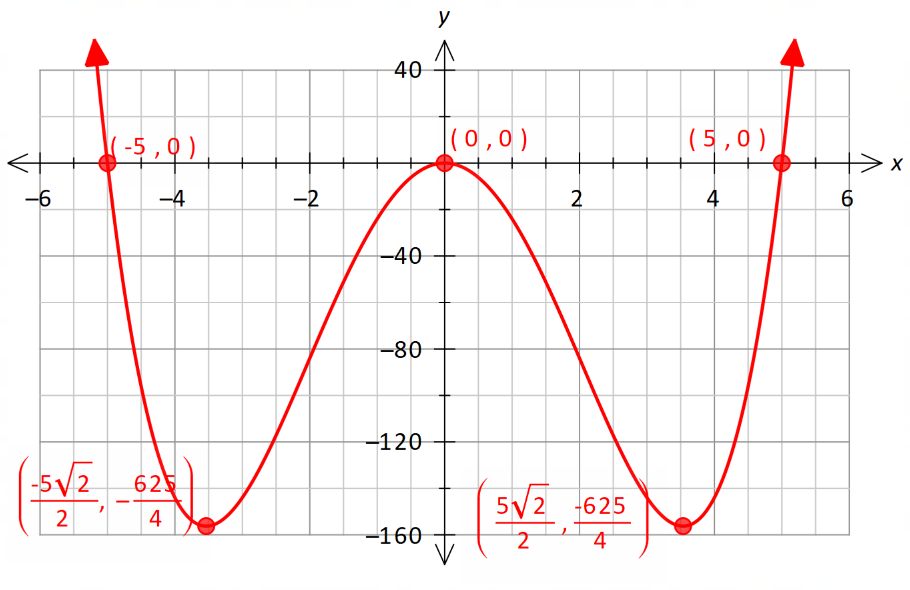
The techniques used for graphing quartic functions are similar to those used for cubics.
A CAS calculator may be used to help sketch, but care must be taken to not miss any TPs or intercepts.

Example: Solve $(x^2-4)(x^2-9)=0$ for x.

$(x^2-4)(x^2-9)=0$
$(x-2)(x+2)(x-3)(x+3)=0$
$x=2,\ \ x=-2,\ \ x=3,\ \ x=-3$

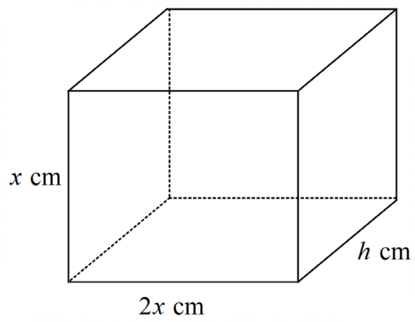
Example: Sketch the graph of $y = x^4 - 25x^2$. Give the coordinates of all axes intercepts and turning points. Use your CAS to find the coordinates of the turning points.

$y = x^4 - 25x^2$
$y = x^2(x^2-25)$
$y = x^2(x-5)(x+5)$
x-int, let $y=0$
$x^2(x-5)(x+5)=0$
$x^2=0,\ \ x-5=0,\ \ x+5=0$
$x=0,\ \ x=5,\ \ x=-5$



Ex 6J – Applications of Polynomial Functions

Example: A piece of wire 400 cm long is used to make the 12 edges of a cuboid, with dimensions shown.



a) Find h in terms of x.

$$4(x)+4(2x)+4(h)=400$$
$$3x+h=100$$
$$h=100-3x$$

b) Find the volume, V cm^3, in terms of x.

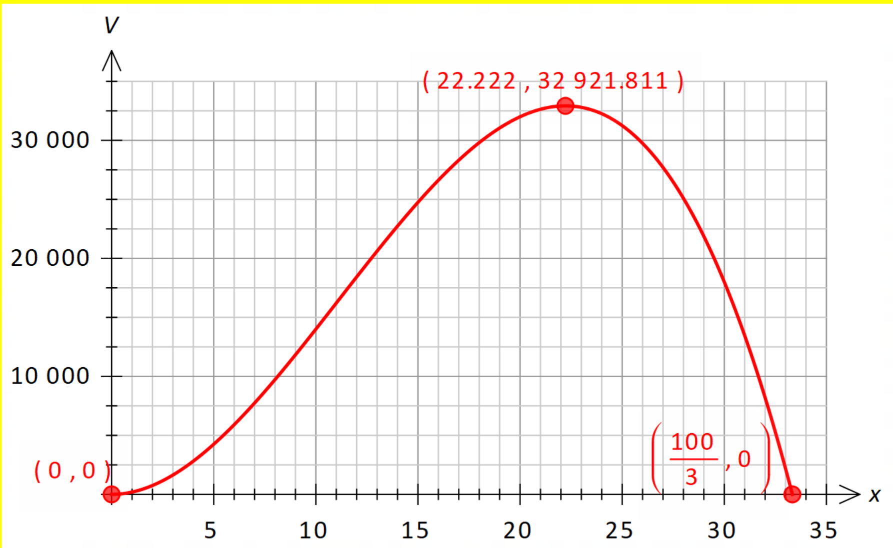
$$V=l\times w\times h$$
$$=(x)(2x)(h)$$
$$=2x^2(100-3x)$$
$$=200x^2-6x^3$$

c) State the possible values of x.

$$x>0 \text{ and } 100-3x>0$$
$$100>3x$$
$$x<\frac{100}{3}$$

$$\therefore \left\{x:\ 0<x<\frac{100}{3}\right\}$$

d) Plot the graph of V vs. x



e) Find the maximum volume to 3 d.p. and the corresponding value of x.

Max $V=32921.811\,cm^3$ when $x=22.222\,cm$