T01 Linear and Quadratic Relations

W4 – Completing the square and graphing quadratics

We can 'complete the square' using our knowledge of perfect squares and difference of perfect squares. This method is useful for easily determining the turning point of a quadratic graph.

Follow these simple steps:

Step 1: Take out any common factors (*Note*: the coefficient of x^2 MUST be 1)
Step 2: Half and square the middle term
Step 3: Include in the equation (**add** and **minus** the term so that we have not changed the equation)
Step 4: The first three terms create a perfect square which we then factorise and simplify
Step 5: Square root and square the final term to create a difference of perfect squares. (*Note*: if the final term is positive we cannot continue further).

Example: Factorise the following by completing the square.

a) x^2+2x-5

$$=\left(x^2+2x+1^2\right)-1^2-5$$
$$=(x+1)^2-6$$
$$=(x+1)^2-\left(\sqrt{6}\right)^2$$
$$=\left(x+1-\sqrt{6}\right)\left(x+1+\sqrt{6}\right)$$

b) $2x^2+12x+4$

$$=2\left[x^2+6x+2\right]$$
$$=2\left[\left(x^2+6x+3^2\right)-3^2+2\right]$$
$$=2\left[(x+3)^2-7\right]$$
$$=2\left(x+3-\sqrt{7}\right)\left(x+3+\sqrt{7}\right)$$

c) $-2x^2+3x+4$

$$=-2\left[x^2-\frac{3}{2}x-2\right]$$
$$=-2\left[\left(x^2-\frac{3}{2}x+\left(\frac{3}{4}\right)^2\right)-\left(\frac{3}{4}\right)^2-2\right]$$
$$=-2\left[\left(x-\frac{3}{4}\right)^2-\frac{41}{16}\right]$$
$$=-2\left(x-\frac{3}{4}-\frac{\sqrt{41}}{4}\right)\left(x-\frac{3}{4}+\frac{\sqrt{41}}{4}\right)$$

The axis of symmetry

For a quadratic function written in the form $y=ax^2+bx+c$, the axis of symmetry of its graph has the equation $x=-\dfrac{b}{2a}$

Example:
Use the axis of symmetry to find the turning point of the graph, and hence express in turning point form:

$y=x^2-4x+3$

x-coord of AoS is:

$$x=-\frac{-4}{2(1)}=2$$

Find the y-coord by subbing $x=2$ into equation

$$y=(2)^2-4(2)+3=-1$$

Therefore, the coords of the TP is $(2,-1)$

TP form: $y=a(x-h)^2+k$

$\therefore y=(x-2)^2-1$

Graphing Quadratics

To sketch the graph of a quadratic function:

- Find the y-intercept
- Find the x-intercept(s)
- Find the equation of the axis of symmetry
- Find the co-ordinates of the turning point

Example: sketch the following parabolas, clearly showing all important points.

a) $y = 2x^2 - 4x$

Find the y-int, let $x = 0$

$\therefore (0,0)$

Find the x-ints using NFL

$2x^2 - 4x = 0$

$2x(x-2) = 0$

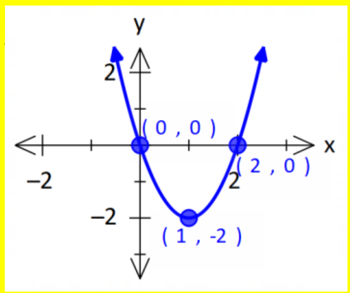
$\therefore (0,0)$ and $(2,0)$

Axis of symmetry $= \dfrac{2+0}{2} = 1$

Find the turning point

when $x = 1, y = 2(1)^2 - 4(1) = -2$

$\therefore$ TP at $(1,-2)$



b) $y = 6 - x - x^2$

Find the y-int, let $x = 0$

$\therefore (0,6)$

Find the x-ints using NFL

$6 - x - x^2 = 0$

$-(x-2)(x+3) = 0$

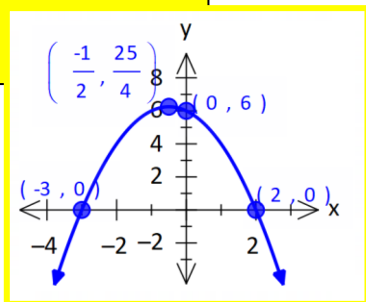
$\therefore (2,0)$ and $(-3,0)$

Axis of symmetry $= \dfrac{2+-3}{2} = -\dfrac{1}{2}$

Find the turning point

when $x = -\dfrac{1}{2}, y = 6 - \left(-\dfrac{1}{2}\right) - \left(-\dfrac{1}{2}\right)^2 = \dfrac{25}{4}$

$\therefore$ TP at $\left(-\dfrac{1}{2}, \dfrac{25}{4}\right)$



Ex 3G – Solving Quadratic Inequalities

To solve a quadratic inequality:

- Factorise and solve the *equation* normally (i.e. let it $= 0$)
- Sketch the graph of the quadratic (you only need the x-intercepts for this)
- Use the graph to determine the set of x-values for which the inequality is satisfied.

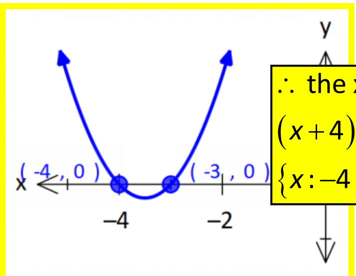
Example: Solve each of the following inequalities

a) $(x+4)(x+3) < 0$

solve $(x+4)(x+3) = 0$

$\therefore x = -4$ and $x = -3$

Sketch the quadratic:



$\therefore$ the x values which satisfy $(x+4)(x+3) < 0$ are:

$\{x : -4 < x < -3\}$

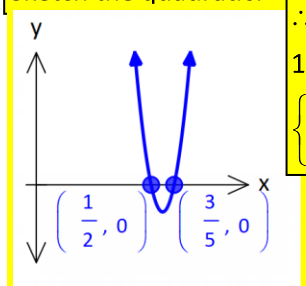
b) $10x^2 - 11x + 3 \geq 0$

solve $10x^2 - 11x + 3 = 0$

$(2x-1)(5x-3) = 0$

$\therefore x = \dfrac{1}{2}$ and $x = \dfrac{3}{5}$

Sketch the quadratic:



$\therefore$ the x values which satisfy $10x^2 - 11x + 3 \geq 0$ are:

$\left\{x : x \leq \dfrac{1}{2} \cup x \geq \dfrac{3}{5}\right\}$

Ex 5A – Set notation and sets of numbers

A set is a collection of 'elements' that have a common theme.

Consider $A=\{1,3,4,5,9\}$. *A* is a **set** with elements 1, 3, 4, 5 and 9.

- If $x=3$ (where is *x* is an arbitrary element), therefore we can say that $x \in A$ (*x* is an element of *A*)
- If $x=2$, $\Rightarrow x \notin A$

Also consider $B=\{1,4,9\}$. *B* contains elements common to *A*.

- We say: $B \subseteq A$ (*B* is a subset of *A*)

Now consider $A=\{1,3,4,5,9\}$, $B=\{1,4,6,9,10\}$ and $C=\{2,6,7,8\}$.

- Elements common to both *A* and *B* are the **intersection** of *A* and *B*.
  - $A \cap B=\{1,4,9\}$
- If there are no common elements, *A* and *C* contain an empty or **null** set.
  - $A \cap B=\varnothing$
- All elements in *A*, *B* and *C* are the **union** of *A*, *B* and *C*.
  - $A \cup B \cup C=\{1,2,3,4,5,6,7,8,9,10\}$
- $A \setminus B=\{x : x \in A, x \notin B\}$. This means that *x* is an element of *A*, but not an element of *B*.
  - $A \setminus B=\{3,5\}$
  - $B \setminus A=\{6,10\}$

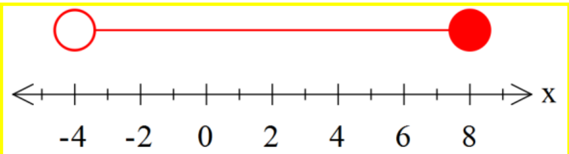
We can illustrate sets intervals on a number line.

Remember: Closed circle → included

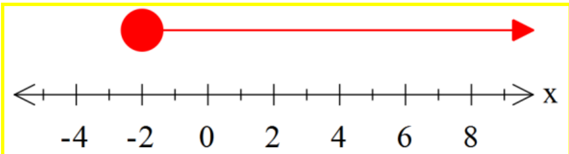
Open circle → not included

Example: Illustrate the following intervals on a number line:

a) $(-4, 8]$



b) $[-2, \infty)$



Ex 5B – Relations, Domain and Range

Ordered pairs (denoted (x, y)) are a pair of elements *x* and *y* (i.e. *x* and *y* co-ordinates). A **relation** is made up of ordered pairs. The set of elements for *x* gives the **domain** of the relation, and the set of elements for *y* gives the **range**.

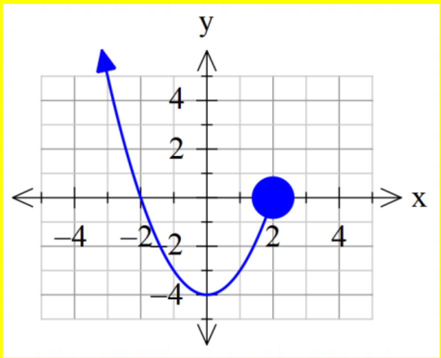
Some relations may be defined by a rule. To fully define a relation, we need to specify both the rule and the domain. $\{(x, y): y = 2x - 3, x \in R\}$

Implied (maximal) domain refers to the domain that is possible for a relation given the restrictions of the equation.

e.g. $y = \frac{1}{x}$

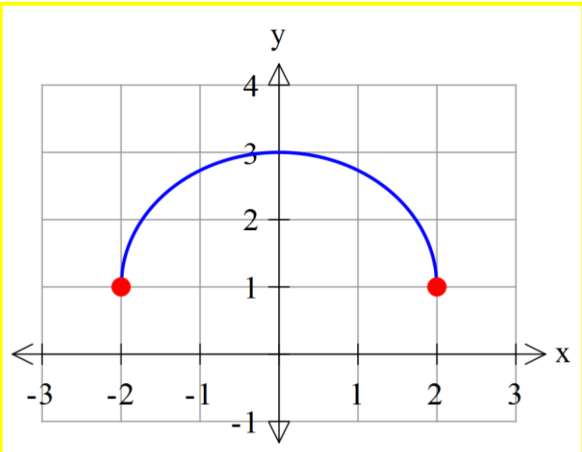
$x \neq 0 \ \therefore x \in R \setminus \{0\}$

Example: Sketch $y = x^2 - 4$ for $x \in (-\infty, 2]$ and state the range of the function.



$range = [-4, \infty)$

Example: Sketch $y = \sqrt{4 - x^2} + 1$ and state the maximal domain and range.



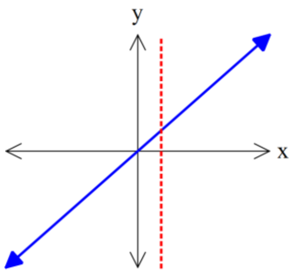
$max\ domain = [-2, 2]$

$range = [1, 3]$

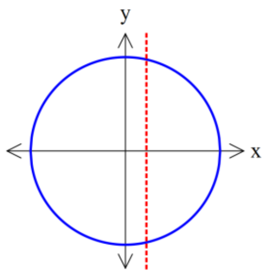
Ex 5C – Functions

A function is a relation that has a unique *y* value of every *x* value in an ordered pair. This is called a one-to-one relation (we will cover this further next exercise).

The **vertical line test** can be used to identify if a relation is a function. If a vertical line cuts the curve only once anywhere it is drawn, then it is an function.



This is a function



This is NOT a function

Function Notation: $f:R\to R, f(x)=f(x)$

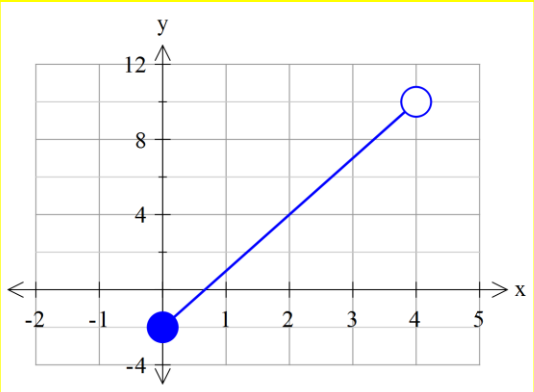
"the function *f* such that the domain maps onto the co-domain, of $f(x)$ equals to the rule"

Example: If $f(x)=x^2+3$, find $f(-1)$ and $f(a^2)$

$$f(-1)=(-1)^2+3 \\ =1+3 \\ =4$$

$$f(a^2)=(a^2)^2+3 \\ =a^4+3$$

Example: Sketch $f:[0,4)\to R, f(x)=3x-2$



For every ordered pair (x,y) of a function, the element *y* is called the image of *x* under *f* or the value of *f* at *x*. The corresponding value of *y* is called the pre-image of *y*.

Example: For $f(x)=2x-4$, find the image of 3 and pre-image of 6.

The image of 3...
$f(3)=2(3)-4$
$=6-4$
$=2$

The pre-image of 6...
$6=2x-4$
$10=2x$
$x=5$

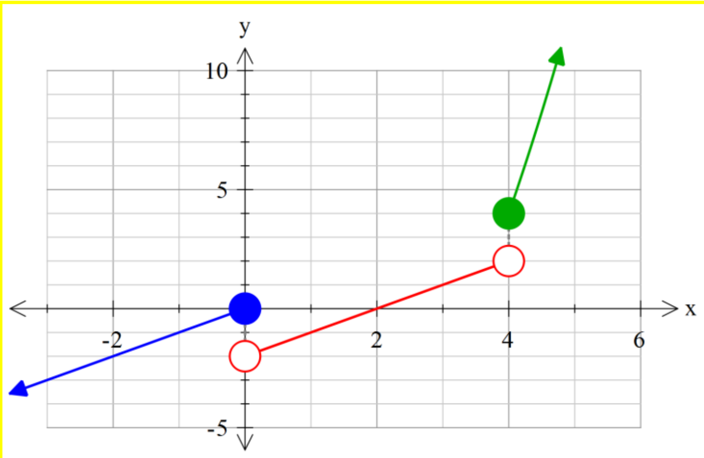
<u>Ex 5E – Piecewise-defined functions</u>

Piecewise-defined (or hybrid) functions have different rules for different subsets of the domain.

Example: Sketch the following function and state its range.

$$f(x)=\begin{cases} x, & (-\infty,0] \\ x-2, & (0,4) \\ x^2-12, & [4,\infty) \end{cases}$$

$$Range:(-\infty,2)\cup[4,\infty)$$
$$R\setminus[2,4)$$



<u>Ex 3H – The Quadratic Formula</u>

The Quadratic Formula can be used to solve for the value of *x* by substituting the corresponding values into the following equation:

For a quadratic in the form: $ax^2+bx+c=0$

then: $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Note: This formula is derived by completing the square of $ax^2+bx+c=0$.

Example: Solve

a) $x^2+3x-4=0$

$$x=\frac{-(3)\pm\sqrt{(3)^2-4(1)(-4)}}{2(1)}$$
$$x=\frac{-3\pm\sqrt{9+16}}{2}$$
$$x=\frac{-3\pm5}{2}$$
$$x=\frac{-3-5}{2} \quad or \quad \frac{-3+5}{2}$$
$$x=-4 \quad or \quad x=1$$

two RATIONAL solutions

b) $-3x^2+8x-2=0$

$$x=\frac{-(8)\pm\sqrt{(8)^2-4(-3)(-2)}}{2(-3)}$$
$$x=\frac{-8\pm\sqrt{64-24}}{-6}$$
$$x=\frac{-8\pm\sqrt{40}}{-6}$$
$$x=\frac{-8\pm2\sqrt{10}}{-6}$$
$$x=\frac{4\pm\sqrt{10}}{3}$$

two IRRATIONAL solutions

Ex 3I – The Discriminant

Under the square root is an expression called the *discriminant*, Δ.
We use the discriminant to determine how many solutions an equation has.

$$\Delta = b^2 - 4ac$$

If $\quad \Delta > 0$ there will be 2 real solutions

$\Delta = 0$ there will be 1 real solution $\left(x = -\dfrac{b}{2a}\right)$

$\Delta < 0$ there will be no real solutions $\left(\text{since } \sqrt{\Delta} \text{ is undefined when } \Delta \text{ is negative}\right)$

Example: Determine the number of solutions for the following.

a) $2x^2 + 5x + 7 = 0$

$\Delta = b^2 - 4ac$
$= (5)^2 - 4(2)(7)$
$= 25 - 56$
$= -31$
$\therefore$ no real solutions

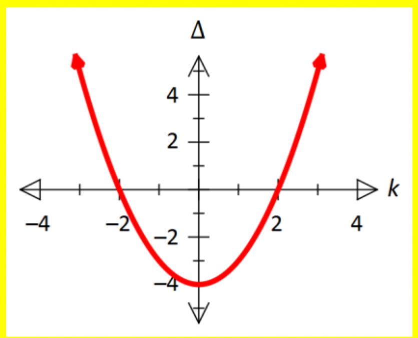
b) $x^2 - 4x + 2 = 0$

$\Delta = b^2 - 4ac$
$= (-4)^2 - 4(1)(2)$
$= 16 - 8$
$= 8$
$\therefore$ 2 real solutions

$x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$
$= \dfrac{-(-4) \pm \sqrt{8}}{2(1)}$
$= \dfrac{4 \pm 2\sqrt{2}}{2}$
$= 2 \pm \sqrt{2}$

Example:
For which values of k does $x^2 + kx + 1 = 0$ have:

$\Delta = b^2 - 4ac$
$= (k)^2 - 4(1)(1)$
$= k^2 - 4$
$= (k-2)(k+2)$



a) no solutions

when $\Delta < 0$
$\therefore -2 < k < 2$

b) one solution

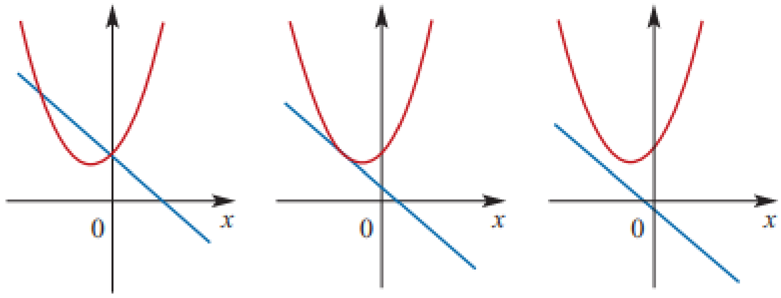
when $\Delta = 0$
$\therefore k = -2$ and $k = 2$

c) two solutions

when $\Delta > 0$
$\therefore k < -2$ and $k > 2$

Ex 3J – Solving simultaneous linear and quadratic equations

Depending on where a linear line cuts (or does not cut) a quadratic, there may be zero, one or two intersection points between a parabola and a straight line.



Two points of intersection   One point of intersection   No point of intersection

The **discriminant** may also be used to determine the number of intersection points.

Remember that if $y = ax^2 + bx + c$, then $\Delta = b^2 - 4ac$, where:

- If $\Delta > 0$, there are two intersection points
- If $\Delta = 0$, there is one intersection point (i.e. a tangent to the curve)
- If $\Delta < 0$, there are no intersection points.

Example: Solve the following pair of equations:

$y = x^2 + x - 5$
$y = -x - 8$

equate and rearrange
$x^2 + x - 5 = -x - 8$
$x^2 + 2x + 3 = 0$

use discriminant to determine the number of solutions:
$\Delta = (2)^2 - 4(1)(3) = -8$
no solutions, so do not proceed.

Example: Solve the following pair of equations:

$y = 5x^2 + 9x$
$y = 12 - 2x$

equate and rearrange
$5x^2 + 9x = 12 - 2x$
$5x^2 + 11x - 12 = 0$

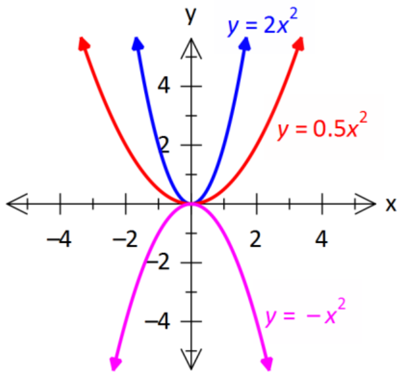
use discriminant to determine the number of solutions:
$\Delta = (11)^2 - 4(5)(-12) = 361$
$\therefore$ there are two solutions

$5x^2 + 11x - 12 = 0$
$(x+3)(5x-4) = 0$
$\therefore x = -3$ and $x = \frac{4}{5}$

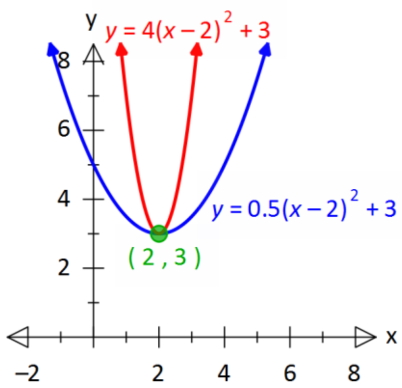
Find y values by substituting back into original equations
$\therefore (-3, 18)$ and $\left(\frac{4}{5}, \frac{52}{5}\right)$

Ex 3K – Families of quadratic polynomial functions

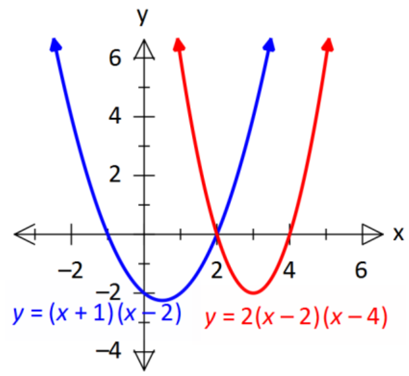
- $y = ax^2$, $a \neq 0$
  - All graphs with vertices at the origin



- $y = a(x-2)^2 + 3$, $a \neq 0$
  - All graphs have a turning point at $(2,3)$
  - The width of the curve changes



- $y = a(x-h)(x-2)$, $a \neq 0$
  - All graphs have one x-intercept at $(2,0)$



Example:

A quadratic rule for a particular parabola is of the form $y = ax^2$. The parabola passes through the point with co-ordinates $(2,8)$. Find the value of a.

$y = ax^2$

sub in $(2,8)$

$8 = a(2)^2$

$8 = 4a$

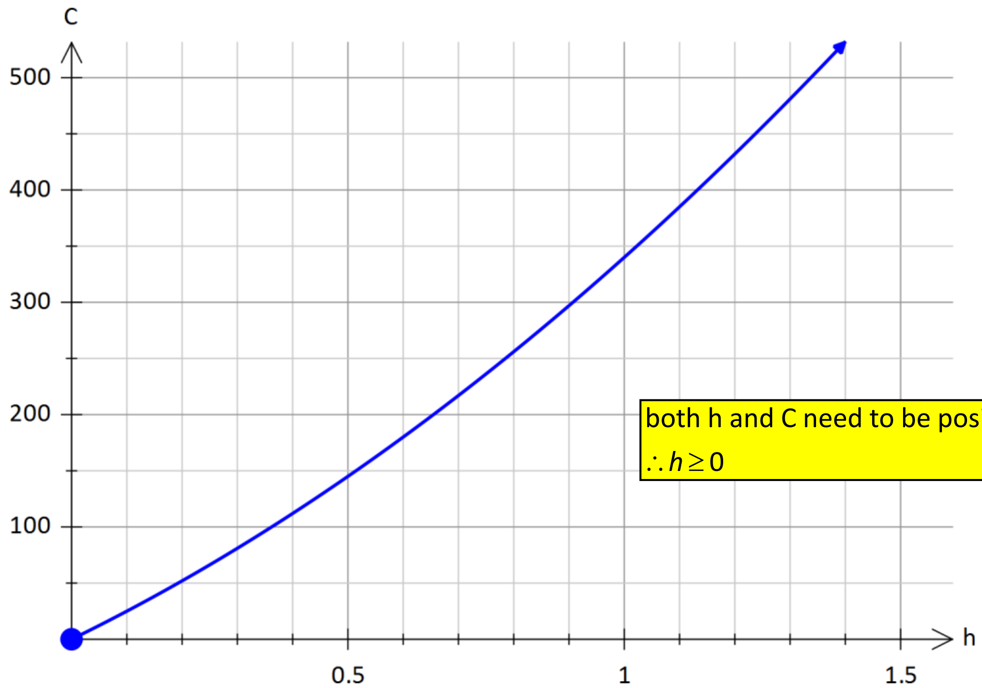
$a = 2$

Ex 3L – Quadratic Models

In order to solve quadratic models we must first manipulate the information given to form a quadratic equation which can then be solved.

Example: (p 134, Q6)
A construction firm has won a contract to build cable-car pylons at various positions on the side of a mountain. Because of the difficulties associated with construction in alpine areas, the construction firm will be paid an extra amount, $\$C$, given by the formula $C = 240h + 100h^2$, where h is the height in km above sea level.

a. Sketch the graph of C as a function of h. Comment on the possible values of h.



both h and C need to be positive
$\therefore h \geq 0$

b. Does C have a maximum value? *No*

c. What is the value of C for a pylon built at an altitude of 2500m?

$2500m = 2.5km$
$C = 240(2.5) + 100(2.5)^2$
$= \$1225$