T04 Polynomials

Ex 6A – Polynomial Arithmetic & Binomial expansion

A polynomial is an expression containing only positive integer powers of *x*.

A *polynomial function* can be written in the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ where *n* is a natural number (or zero) and $a_0, ..., a_n$ are coefficients.

The *leading term*, $a_n x^n$ is the term of highest power.

The *degree of a polynomial* is the index *n* of the leading term.

A *monic polynomial* is one whose leading term has a coefficient of 1.

The constant term is the term of power 0 (i.e. $a_0 x^0$).

When performing arithmetic on polynomials, the result is always a polynomial.

Example: If $P(x) = x^2 + 1$ and $Q(x) = 3x^2 - 2$, find:

a) $P(x) + Q(x)$

b) $P(x)Q(x)$

You are familiar with the quadratic perfect squares formula:

- Sum: $(a+b)^2 = a^2 + 2ab + b^2$
- Difference: $(a-b)^2 = a^2 - 2ab + b^2$

An identity can also be found for perfect cubes:

- Sum: $(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$
- Difference: $(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$

And also for quartics: $(a+b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$

Pascal's Triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Can you see the pattern?

Example: Use the rules to expand:

a) $(x+5)^3$

b) $(2x-1)^3$

Equating coefficients

Two polynomials are equal only if their corresponding coefficients are equal.

Example: It is known that $x^3+3x^2+2x+1=x^3+(a-2)x^2+(b-2a)x-2c$. Find the values of *a*, *b* and *c*.

Ex 6B – Division of polynomials

When sketching cubics of the form $y=ax^3+bx^2+cx+d$ we begin by finding the *x*-intercepts. All cubics have at least one *x*-intercept, but can have up to 3.

Remember that *x*-intercepts are found by factorising and solving using the null factor law.

The technique used to factorise cubics uses long division.

Recall basic long division: $325 \div 12$

$$\begin{array}{rrl} & 027 & \leftarrow \text{quotient, Q(x)} \\ \text{divisor} \rightarrow & 12\overline{)325} & \leftarrow \text{dividend} \\ & -24\downarrow & \\ \hline & 85 & \\ & -84 & \\ \hline & 1 & \leftarrow \text{remainder, R} \end{array}$$

The answer is therefore $27\frac{1}{12}$, which we could also write as $12\times 27+1$.

A similar technique is used for polynomial division.

Example: Divide the following and write in the form $P(x)=(ax-b)Q(x)+R$.

a) $2x^3+x^2-4x+3, \quad x-1$

for polynomial division:
think: "$2x^3 \div x = 2x^2$"
write this in the quotient
multiply out: $x \times 2x^2 = 2x^3$
$-1 \times 2x^2 = -2x^2$
put brackets, and subtract.
bring down the rest of the function
start again:
think: "$3x^2 \div x = 3x$"
.... etc

b) $x^3+3x-4,\ \ x+1$

c) $2x^3-3x^2-29x-30,\ \ 2x+3$

An alternative way of dividing polynomials is to *equate coefficients*.

Example: Divide the polynomial x^3+x^2-2x+3 by $x-1$ by equating coefficients.

<u>Ex 6C – Factorisation of Polynomials</u>

<u>The Remainder Theorem</u>

When $P(x)$ is divided by $ax+b$, the remainder will be given by $P\left(-\frac{b}{a}\right)$.

$$\begin{aligned} &ax+b=0 \\ &\therefore x=-\frac{b}{a} \end{aligned}$$

For example, x^3-3x^2+4x-1 divided by $x+2$:

Using long division:

$$\begin{array}{r} x^2-5x+14 \\ x+2\overline{)\,x^3-3x^2+4x-1} \\ -\underline{(x^3+2x^2)} \\ -5x^2+4x-1 \\ -\underline{(-5x^2-10x)} \\ 14x-1 \\ -\underline{(14x+28)} \\ -29 \end{array}$$

Using remainder theorem:

$$\begin{aligned} P(-2)&=(-2)^3-3(-2)^2+4(-2)-1 \\ &=-8-12-8-1 \\ &=-29 \end{aligned}$$

Example: Without using division find the remainder when $10x^3 - 3x^2 + 4x - 1$ is divided by $2x + 1$.

Example: If $x^3 + ax^2 + 3x - 5$ has a remainder of -3 when divided by $x - 2$, find the value of *a*.

<u>The Factor Theorem</u>

The factor theorem states that if $ax + b$ is a factor of $P(x)$ then $P\left(-\frac{b}{a}\right) = 0$.

This means that for $ax + b$ to be a factor of $P(x)$ the remainder must be equal to zero.

Example: **Show that** $x + 2$ is a factor of $2x^3 - 3x^2 - 11x + 6$

in "show that" questions you must explicitly show the substitution.

you will need to "guess" the first factor.
start with 1, -1, 2, -2, etc

Example: Factorise the following:

a) $x^3 + 3x^2 + 3x + 1$

b) $x^3 - 21x + 20$

The Rational-root Theorem:

The rational-root theorem can be used when the solution to the polynomials is not an integer.

It states that if we let $P(x)=a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ and if $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

Example: Factorise $P(x)=3x^3+8x^2+2x-5$

Special cases:

Sum of cubes: $x^3+a^3=(x+a)(x^2-ax+a^2)$

Difference of cubes: $x^3-a^3=(x-a)(x^2+ax+a^2)$

Example: Factorise the following:

a) x^3+64

b) $64m^3-27n^3$

Ex 6D – Solving cubic equations

Remember: the Null Factor Law is used to solve equations; if $a \times b = 0$, then $a = 0$ and $b = 0$.

Example: Factorise the following

a) $(x-1)(x+3)(x+7)=0$

b) $x^3 + 2x^2 - 8x = 0$

c) $3x^3 - 4x^2 - 13x - 6 = 0$

d) $2x^3 + 250 = 0$

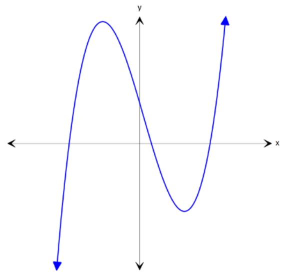
e) $(x^3 + 4x^2) - (11x + 44) = 0$

Ex 6E – Cubics of the form $f: R \to R, f(x) = a(x-h)^3 + k$

Cubic functions are polynomials of degree 3.

The general form is shown on the right and has the equation

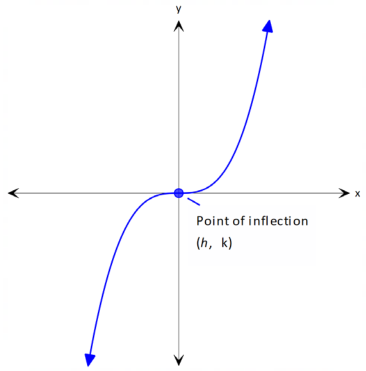
$$y = ax^3 + bx^2 + cx + d\,.$$



A basic cubic is given by the equation $y = ax^3$

We can transform this graph in the same way we transform quadratics.

$$y = a(x-h)^3 + k$$



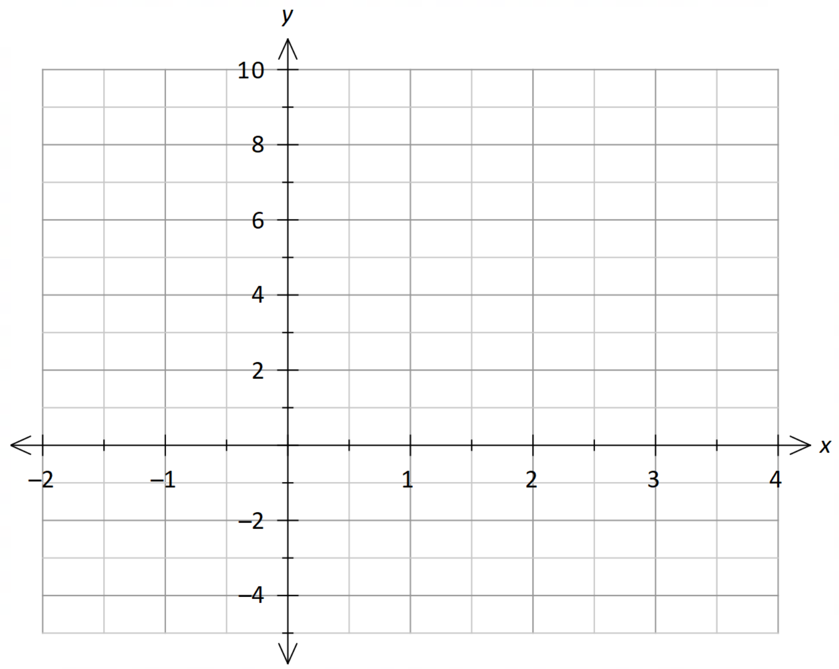
This equation shows a horizontal translation of *h* units to the right and a vertical translation *k* units up.

The point of inflection has a gradient of zero and is given by (h, k).

Example: Sketch the graph of $y = 2(x-1)^3 + 3$

Find x and y intercepts in the normal way:
for x-int, let $y = 0$
for y-int, let $x = 0$

Remember that you can cube root negative numbers! $\sqrt[3]{-8} = -2$



Inverse functions

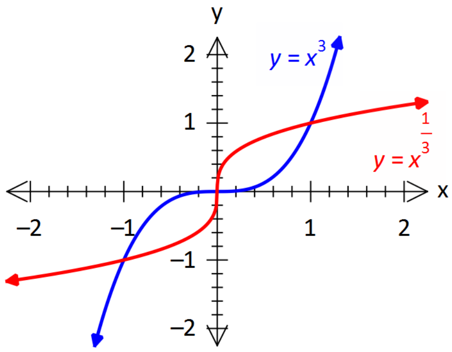
The rule $f: R \to R, f(x) = x^{\frac{1}{3}}$ is the inverse function of a cubic.

$y = x^3$

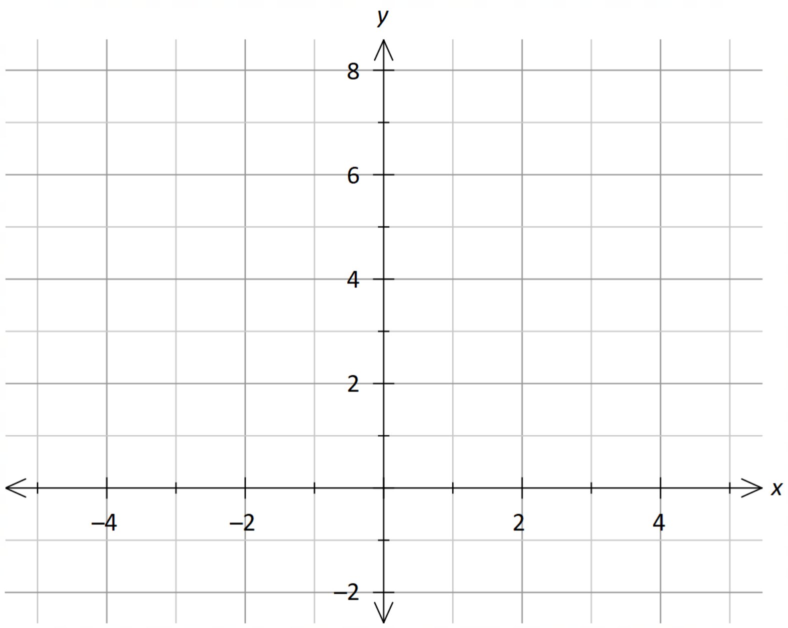
swap $x \Leftrightarrow y$

$x = y^3$

$y = \sqrt[3]{x} = x^{\frac{1}{3}}$



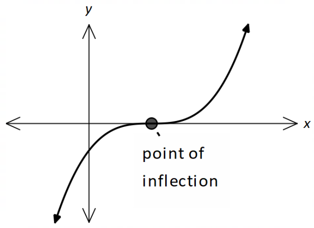
Example: Sketch the graph of $y = -2(x-1)^{\frac{1}{3}} + 3$



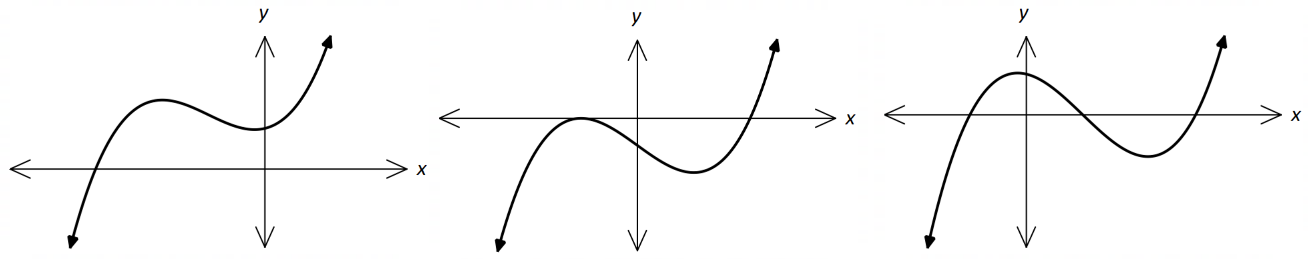
Ex 6F – Graphs of cubic functions

There are two types of cubic function graphs:

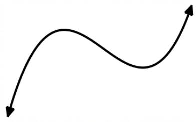
The first type has the rule $y = a(x-h)^3 + k$ and looks like $\rightarrow$



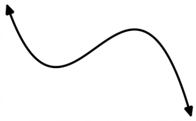
The second type is usually of the general form $y = ax^3 + bx^2 + cx + d$ or in intercept form $y = a(x-d)(x-e)$ and can have one, two or three x-intercepts:



If the coefficient of the x^3 term is positive, the graph will have the right hand side pointing UP.



If the coefficient of the x^3 term is negative, the graph will have the right hand side pointing DOWN.

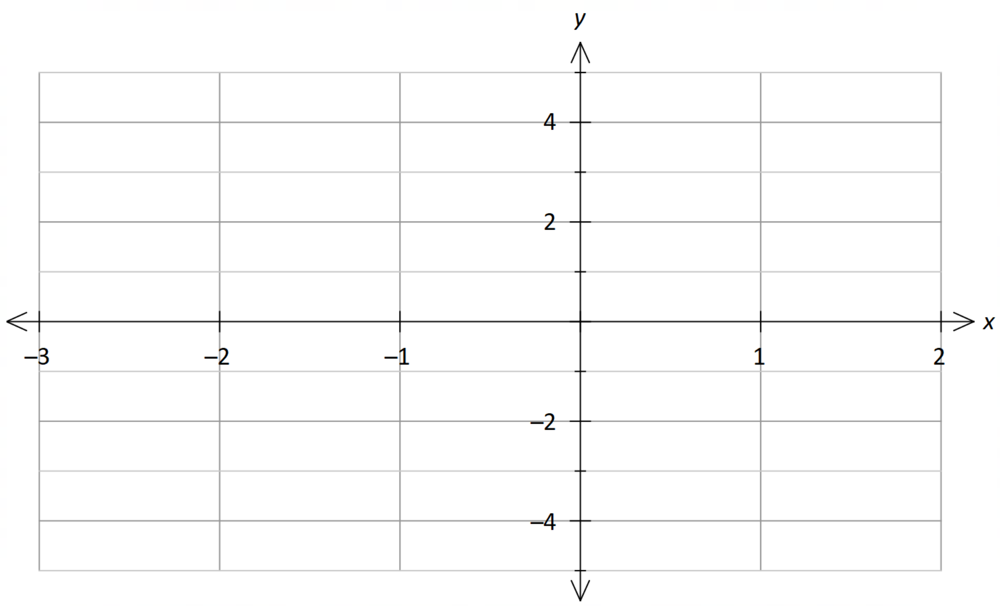


To sketch graphs of cubic functions:

- Determine if the graph is positive or negative
- Find the *x*-intercepts. Determine if any are turning points or inflections.
- Find the *y*-intercept
- Sketch the graph!

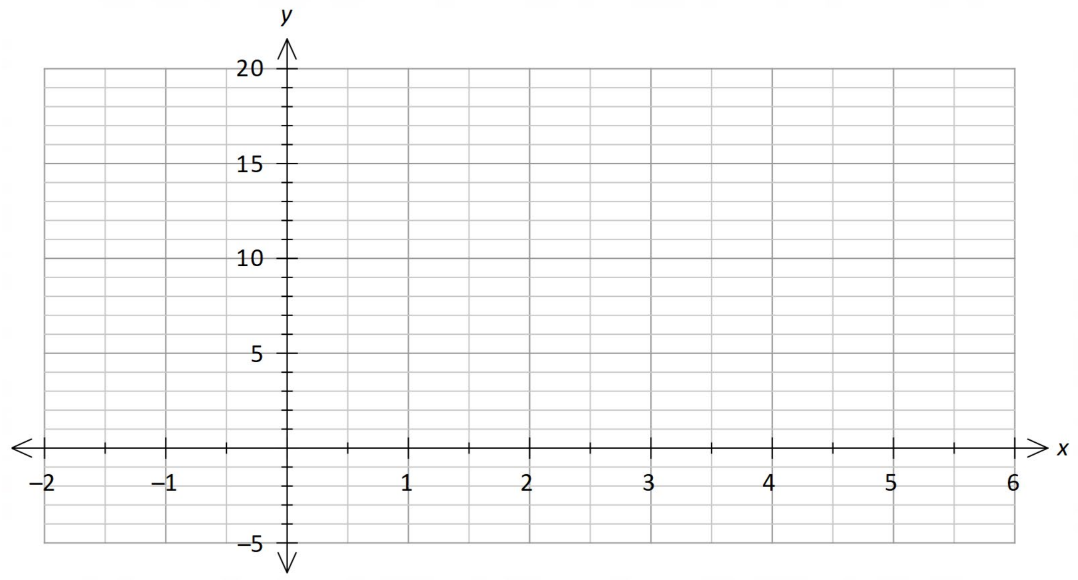
Note that cubic graphs are ***not symmetrical***. Turning points are often not exactly between the x-intercepts.

Example: Sketch the graph of $y = (x-1)(x+1)(x+2)$



note: you do not need to find the TPs right now. Calculus is needed to find them, so for now, just turn the graph at approximately the correct place.

Example: Sketch the graph of $y = (2 - x)(x - 3)^2$

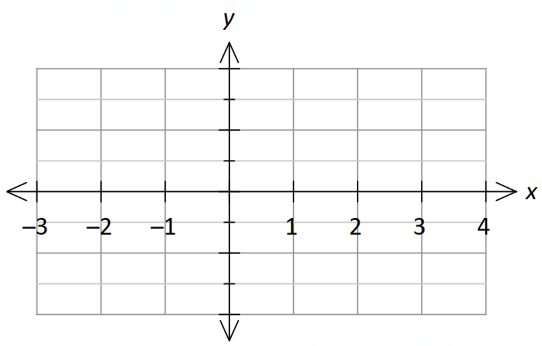


Ex 6G – Solving cubic inequalities

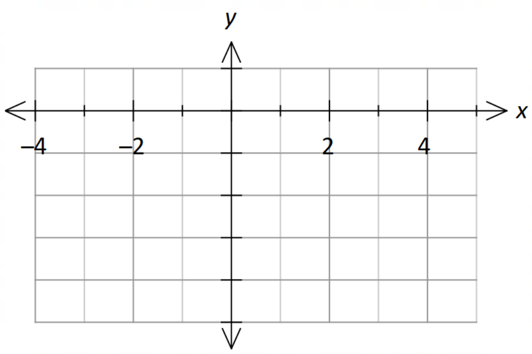
We solve cubic inequations in the same way as we solve quadratic inequations. Often this means we have to consult the graph.

Example: Solve the following for *x*.

a) $(x-1)^3 + 8 \le 0$

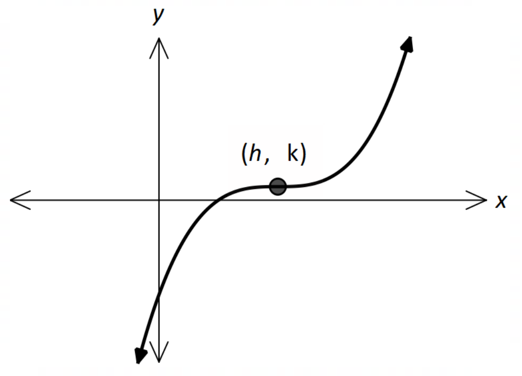


b) $(x+1)(x+2)(x-4) \ge 0$

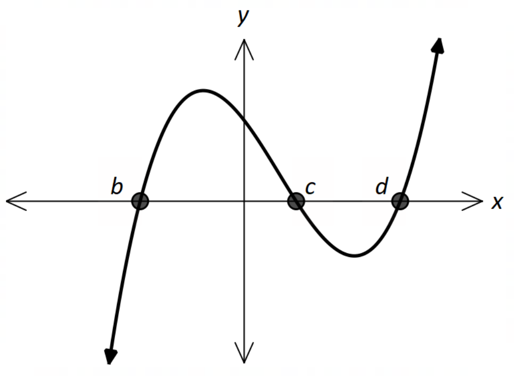


Ex 6H – Families of cubic polynomial functions

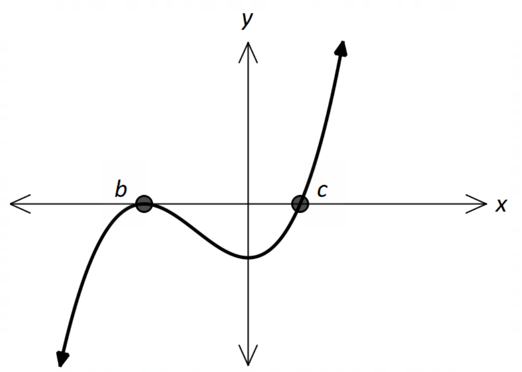
Turning point form: $y=a(x-h)^3+k$



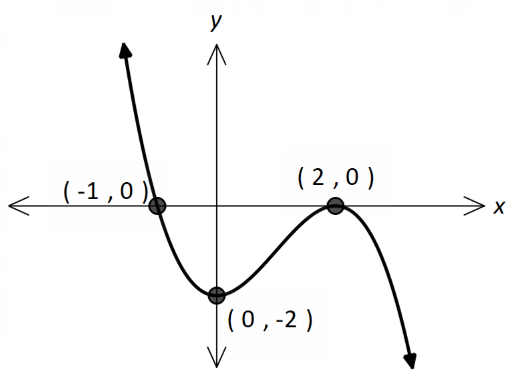
Intercept form: $y=a(x-b)(x-c)(x-d)$



Repeated factor form: $y=a(x-b)^2(x-c)$



Example: Find the equation of the following.



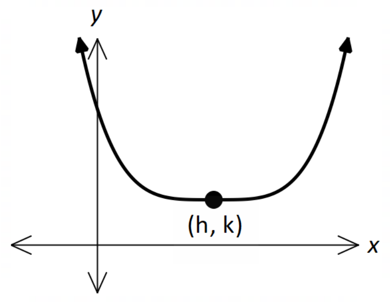
If given four random points that do not fit into any of the above equations, use simultaneous equations by substituting each point into the equation $y=ax^3+bx^2+cx+d$.

Example: Find the equation which passes through the points $(-3,-47),(-2,-15),(1,-3),(2,-7)$.

Ex 6I – Quartic and other polynomial functions

Quartic functions are polynomials of degree 4 and have a general equation of:

$$y = ax^4 + bx^3 + cx^2 + dx + e\,.$$



The basic graph is given by the equation $y = ax^4$ and can undergo transformations:

$$y = a\left(x-h\right)^4 + k$$

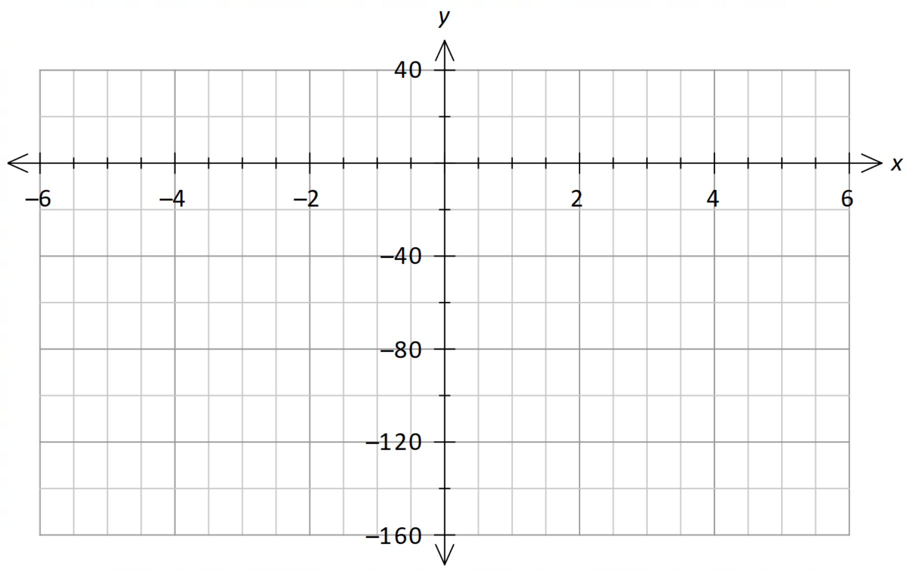
The turning point is given by $\left(h,k\right)$.

quartics have a flatter TP than a quadratic

The techniques used for graphing quartic functions are similar to those used for cubics.
A CAS calculator may be used to help sketch, but care must be taken to not miss any TPs or intercepts.

Example: Solve $\left(x^2-4\right)\left(x^2-9\right)=0$ for *x*.

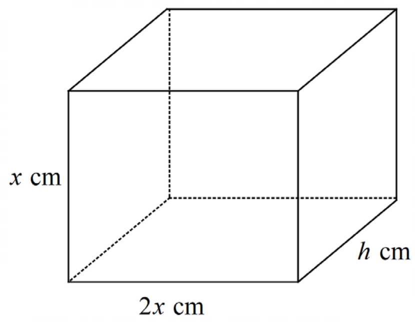
Example: Sketch the graph of $y = x^4 - 25x^2$. Give the coordinates of all axes intercepts and turning points. Use your CAS to find the coordinates of the turning points.



Ex 6J – Applications of Polynomial Functions

Example: A piece of wire 400 cm long is used to make the 12 edges of a cuboid, with dimensions shown.

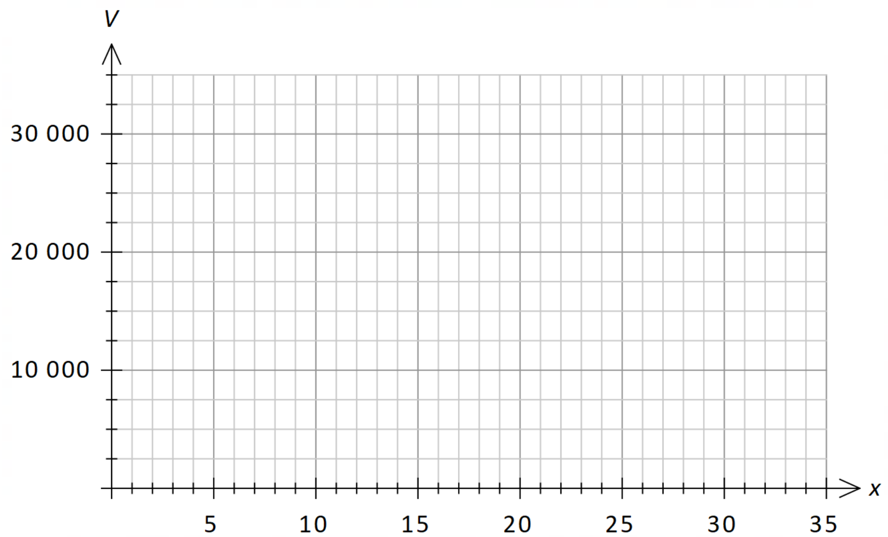
a) Find h in terms of x.



b) Find the volume, V cm^3, in terms of x.

c) State the possible values of x.

d) Plot the graph of V vs. x



e) Find the maximum volume to 3 d.p. and the corresponding value of x.