# SPECIALIST MATHEMATICS

# Unit 3 SAC

## Application task

## The bakery

**Student Name……………………………………………………….**

**Student Number…………………………………………………….**

This task is divided into three parts to be completed over a period of 1–2 weeks in 4–6 hours.
The SAC assesses the 3 outcomes, with the marking allocation as outlined below.

**Marking allocation**

| **Outcomes** | **Marks allocated** |
|---|---|
| Outcome 1 | 15 |
| Outcome 2 | 20 |
| Outcome 3 | 15 |
| **TOTAL MARKS** | **50** |

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## The Bakery

Contents covered: trigonometry functions, complex numbers, differential calculus, integration, and differential equations.

### Background information

Joanne enjoys baking cakes. During the extensive lockdown through the pandemic, she tries different bakery recipes and cake decorations. Joanne would like to document those in preparation for opening a local bakery in the near future. Some Mathematical knowledge could be used to help perfect her recipes.

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**Problem One – Cake decoration**

Joanne would like to begin decorating her favourite cake with fruits. She makes a plain cake with a radius of 10 cm and will decorate its top face. Joanne puts the cake on a complex plane. From the top view, the centre of the cake's top surface is at the origin. All units are measured in centimetres.

**a.** Find the cartesian equation for the outline of the cake's top face.

**b.** The cake's top face can also be expressed using the complex equation

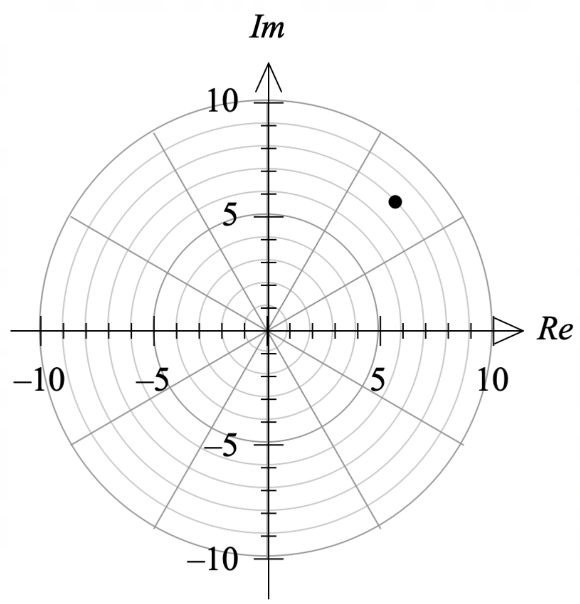
$$|z| = a$$

where $a \in \mathbb{R}$.

**i.** State the value of $a$.

**ii.** Give a geometrical interpretation of the graph of $|z| = a$ in relation to the point that represents $a$.

Joanne starts by putting six strawberries evenly spaced out on the cake. The position of the first strawberry is at $S_1 = 4\sqrt{2} + 4\sqrt{2}i$, plotted on the complex plane below.




**c.** Express $S_1$ in polar form.

**d.** Show that the positions of $S_2$, which represents the strawberry adjacent to $S_1$ in the second quadrant, is $8\text{cis}\left(\frac{7\pi}{12}\right)$.

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**e.** Use an appropriate compound angle formula to express $S_2$ in cartesian form.

**f.** $S_2$ can also be written as $8(-\sin(\alpha) + i \cdot \cos(\alpha))$. Find the value of $\alpha$.

**g.** Plot the positions of all strawberries on the complex plane provided above.

**h.** Write down a possible complex equation, the solution of which would give the position of the six strawberries above.

Joanne then puts one blueberry halfway between every two strawberries. Let $B_1$ be the position of a blueberry and is the midpoint of $S_1$ and $S_2$.

**i.** Express $B_1$ in polar form.

To finish up, Joanne wants to dust some sugar powder on top. She places a cardboard over the cake to cover the majority of it, and only dusts the rest of the cake. The cardboard is placed along a ray given by $\text{Arg}\left(z + 5 + 5\sqrt{3}i\right) = \frac{\pi}{6}$

**j.** Sketch this ray on the complex plane provided above.

**k.** Calculate the area of the cake to be dusted with sugar powder.

Joanne would like to share her wonderful cake with some friends. She wishes to cut the cake into six equal pieces, each containing one strawberry.

**l.** Find the possible cartesian equation of three lines, if cut along, that could divide the whole cake into six equal pieces without cutting through any strawberry.

**m.** Select one of the lines you created in part **l**, which is neither horizontal nor vertical. Its equation can also be expressed using the complex equation

$$|z - 3| = |z - b|$$

where $b \in \mathbb{C}$. Find the value of $b$ for the line selected.

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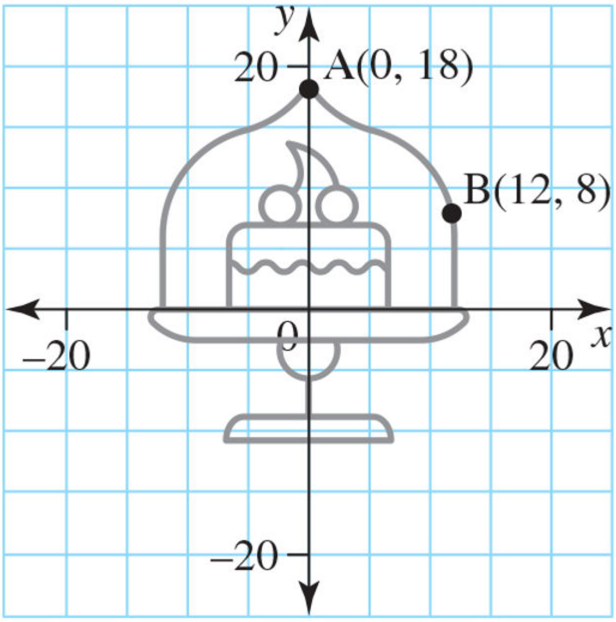
**Problem Two – Cake packaging**

Cherries are in season during summer. After making a beautiful black forest cake, Joanne wants to create a package to protect the cake during transportation. The cake sits on a round cake stand with a base radius of 12 cm that needs to be fully covered. Its highest point is a cherry placed at the centre of the cake, with a height of 14 cm from the base of the cake.

Joanne sketches a diagram of the cake cover she would like to make. When placed on a cartesian plane, the curve in the first quadrant models the outline of the cake cover. It passes point $A(0, 18)$ and $B(12, 8)$ as shown on the diagram below.




The curve between point $A$ and $B$ can be modelled with the function

$$f(x) = a \cdot \arcsin\big(b \cdot (x + c)\big) + d$$

where parameters $a, b, c, d \in \mathbb{R}$.

**a.** The function $f(x)$ joins with a vertical line at point $B$ to form the cake cover. Write down the equation of this line.

**b.** Find the values of $a, b, c$ and $d$.

**c.** Explain why the graph of $y = f(x)$ and the line found in part **a** join smoothly.

**d.** Find another possible function that could approximately model the curve between point $A$ and point $B$.

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The cake cover is formed by rotating the function $y = f(x)$ and the function found in part **a** around the $y$-axis. Assume the thickness of the cover itself is negligible.

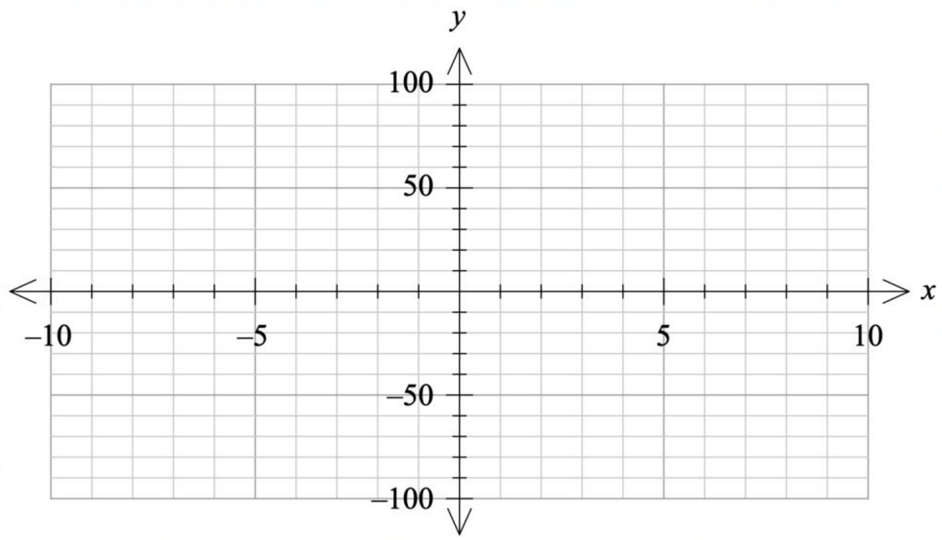
**e.** Find the volume inside the cake cover. Show algebraic working out.

Joanne further investigates a rational function with equation

$$r(x) = \frac{x^3 + 100}{x}$$

**f.** State the equation of asymptotes for the graph of $y = r(x)$

**g.** Sketch the graph of $y = r(x)$ on the axis provided below.




A sequence of transformations listed below maps the curve with equation $y = r(x)$ to the curve with equation $y = p(x)$:

- Dilation by a factor of $k$ from the $y$-axis;
- Translation of $m$ units right and $n$ units up.

The graph of the function $y = p(x)$ could model the outline of the cake cover.

**h.** Investigate the effects of $k$, $m$ and $n$ on the function $r(x)$. Find a possible set of appropriate values for these three parameters.

Joanne finds out the cake cover of this shape takes up too much space during transportation. She now looks for an alternative cake cover.

**i.** Create a function that could represent the outline of a cake cover, when rotated about the $y$-axis, which forms a smaller volume inside. Make sure your design is

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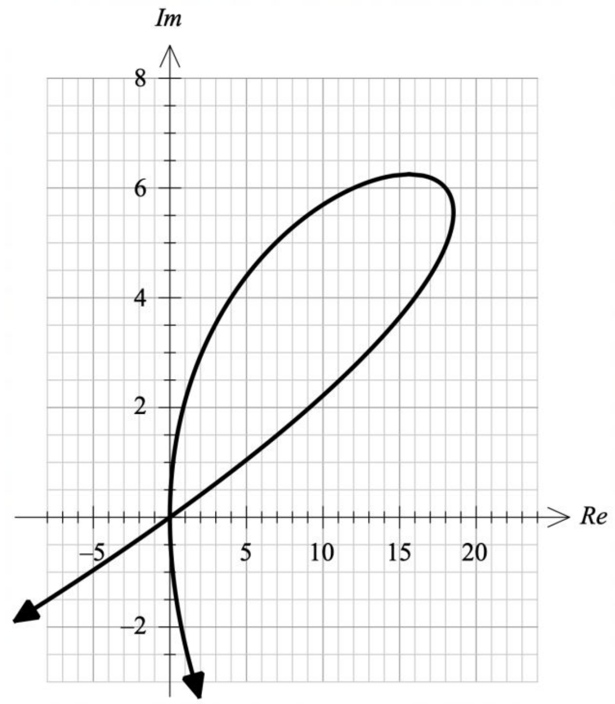
large enough to cover this cake. Show appropriate calculations to support your design.

Joanne wants to make a small logo for her homemade cake to stick on the cake cover. She designed a curve using the function

$$x^2 + y^3 = 5xy$$

A diagram of this logo is shown below.




Joanne wishes to cut off this logo from paper, along the tangents of the curve that are parallel to the $x$-axis and $y$-axis in the first quadrant.

**j.** Find the equation of the tangent lines Joanne will cut along.

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**Problem Three – Making bread**

As a creative backer, Joanne never stops exploring new recipes. She plans to try a special bread recipe. Some Mathematical calculations would help her to determine the optimal cooking time.

A frozen dough needs to be thawed before use. Joanne has a freezer at a temperature of $-18$°C. She takes a dough out of the freezer and places it in a room with constant temperature of 25°C. The rate of change of the dough's temperature is directly proportional to the difference between the temperature of the dough and the surrounding. Let $t$ be the time in minutes the dough is thawed, and $T$°C be the temperature of the dough at time $t$.

**a.** Show that, the rate of change of the dough's temperature is

$$\frac{dT}{dt} = k(T - 25)$$

**b.** Show that, $T = Ae^{kt} + 25$ where $A, k \in \mathbb{R}$ satisfied the differential equation in part **a.**

It takes 2 hours for the dough to reach a temperature of 0°C.

**c.** Use integration to find the value of $k$ for thawing this dough.

**d.** Find how long it takes for the dough to reach 10°C. Leave your answer to the nearest minute.

After mixing the dough with some yeast, Joanne left the dough to rise. The volume of the dough, $V$ cm$^3$, after $s$ hours of raising, can be modelled by the differential equation

$$\frac{dV}{dt} = \frac{1200V - V^2}{400}$$

Initially, the size of the dough is 500 cm$^3$.

**e.** Set up a definite integral which can be used to express $t$ in terms of $V$.

**f.** Use partial fractions to integrate the expression above. Express $V$ in terms of $t$.

**g.** Find the time it takes for the volume of the dough to be doubled. Leave your answer correct to the nearest minute.

**h.** What does the model predict the eventual volume of dough will be?

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**i.** Find the expression for $\frac{d^2V}{dt^2}$ in terms of $V$.

**j.** Hence, or otherwise, find when the volume of the dough increases the fastest.

After setting the dough to rise at room temperature, Joanne puts it into a preheated oven set at a constant temperature $T_O$°C. 10 minutes later, the temperature of the dough is raised to 60°C; 25 minutes later, the temperature of the dough is raised to 100°C. The temperature of the bread at time $r$ hours put into the oven is $T_B$°C.

**k.** Set up a differential equation to model the rate of change of the dough's temperature inside the oven.

**l.** Find the temperature of the oven. Leave your answer correct to the nearest whole number.

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