SPECIALIST MATHEMATICS

Unit 4 SAC 1

Modelling task

THE AMUSEMENT PARK

Student Name……………………………………………………..

Student Number…………………………………………………..

This task is divided into three parts to be completed over a period of 1 week in 2–3 hours. The SAC assesses the 3 outcomes, with the marking allocation as outlined below.

Marking allocation

Outcomes	**Marks allocated**
Outcome 1	8
Outcome 2	10
Outcome 3	7
TOTAL MARKS	**25**

Amusement Park

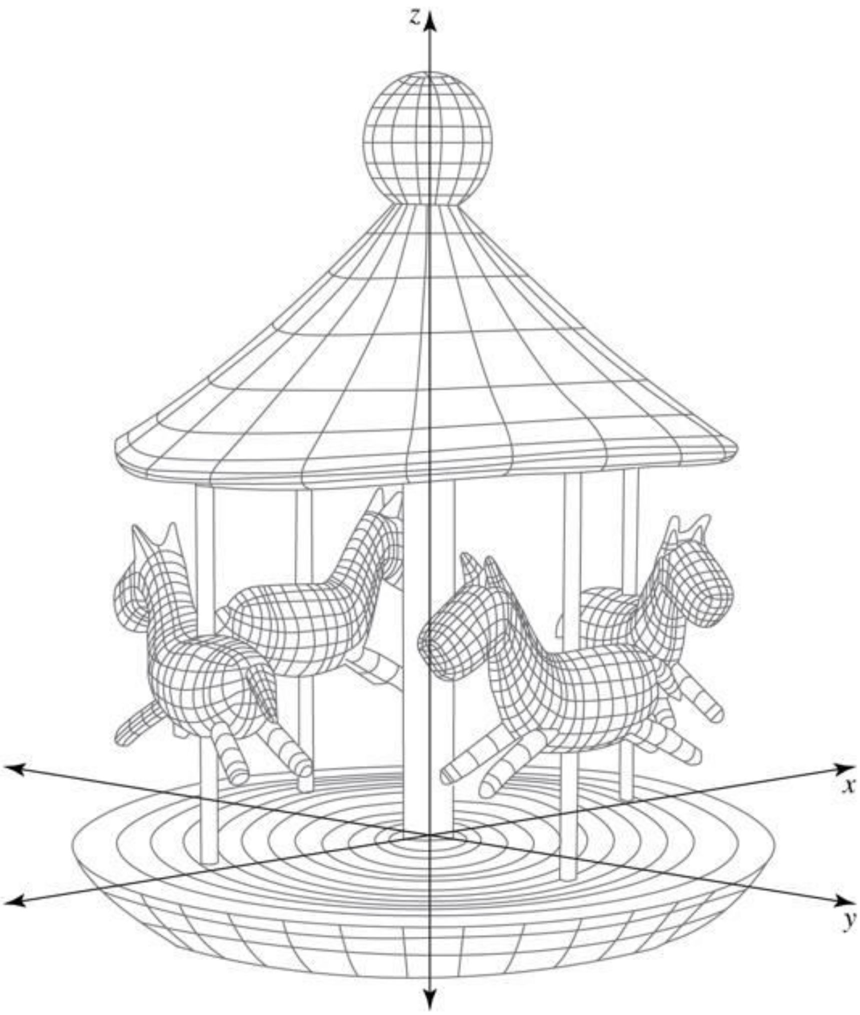
Contents covered: kinematics and vector calculus.

Andrew and his sister Karen went to an amusement park during the school holidays. They had a great day exploring different attractions in the park.

Throughout the SAC, assume air resistance is negligible, and the acceleration due to gravity is given by $g = 9.8\ \text{m/s}^2$.

Problem One

Karen enjoys riding the small merry-go-round at the park. The merry-go-round has four fixed horse seats evenly spread out on the floor. The floor rotates on a horizontal plane formed by x-axis and y-axis, around the vertical pole labelled as z-axis at the centre of the ride, as illustrated on the diagram below. Define $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ to be the unit vector in the positive direction along the x-axis, y-axis, and z-axis respectively.



Note: Not drawn to scale.

The floor of this ride is 50 cm above the ground, rotating at a constant rate. The height of each horse seat is fixed. Let the centre of the horse seat represent the position of its rider. Karen's position can be described by the equation

$$\boldsymbol{C}(t) = 5\sin(nt)\,\boldsymbol{i} + 5\cos(nt)\,\boldsymbol{j} + 1\boldsymbol{k}$$

relative to the base of the central pole, where t is the time in seconds since the start of the ride, and n is a constant relating to the rate that the floor rotates.

a. How high is Karen's horse seat above the ground?

b. How long does it take for the merry-go-round to complete one round if $n = \frac{\pi}{4}$?

c. One ride of this merry-go-round takes 3 minutes. Karen returned to her starting point exactly 6 times during this ride. Find the exact value of n.

d. Derive the Cartesian equation for the path of Karen's ride on the horizontal plane of the floor.

e. Write down another possible vector function, that could give the same Cartesian equation for Karen's ride as part **d**. State the domain of your vector equation.

Andrew, who rides on a horse seat that is directly opposite Karen, is 4 meters away from the central pole.

f. Derive the Cartesian equation of Andrew's path on the horizontal plane.

To increase the excitement, the theme park manager turns on the extra movement on Andrew's horse seats. At the start of the ride, the horse seat is at the lowest height of 1 meter above the ride's floor. This seat now moves up and down vertically and can reach a maximum height of 2.4 meters.

g. Write down a possible vector equation $\boldsymbol{A}(t)$ that describes Andrew's position.

For all the remaining questions, use $n = \frac{\pi}{8}$.
Karen feels very excited and waves her hands during the ride. She accidentally dropped the lollipop from her hand at $t = 4$.

h. Show that, the velocity vector of the lollipop at the moment it is dropped is $-\frac{5\pi}{8}\boldsymbol{j}$.

i. Describe the motion of the lollipop after it left Karen's hand. Discuss the motion in $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ directions respectively.

j. Show that, the vertical velocity of the lollipop is $(-9.8t)\boldsymbol{k}$.

k. When does the lollipop hit ground?

l. At what angle, correct to the nearest tenth of a degree, does the lollipop hit the ground?

m. What distance does the lollipop travel from the time it's dropped to the time it hits the ground? Give your answer correct to the three decimal places.

Problem Two

After enjoying the ride at the merry-go-round, Andrew and Karen went to have fun at the water park area. They hired two water blasters to play with.

Karen holds her water blaster at an angle of 30° from the horizontal direction, at a height of 1.2 meters above the ground. She blasts water out at a speed of 10 meters per second.

a. What's the horizontal velocity of the water when it's just shot from the water blaster?

b. What's the vertical velocity of the water when it's just shot from the water blaster?

c. Find the time taken for the water to hit the ground. Leave your answer correct to two decimal places.

d. Find the horizontal distance water travelled when it hits the ground. Leave your answer correct to one decimal place.

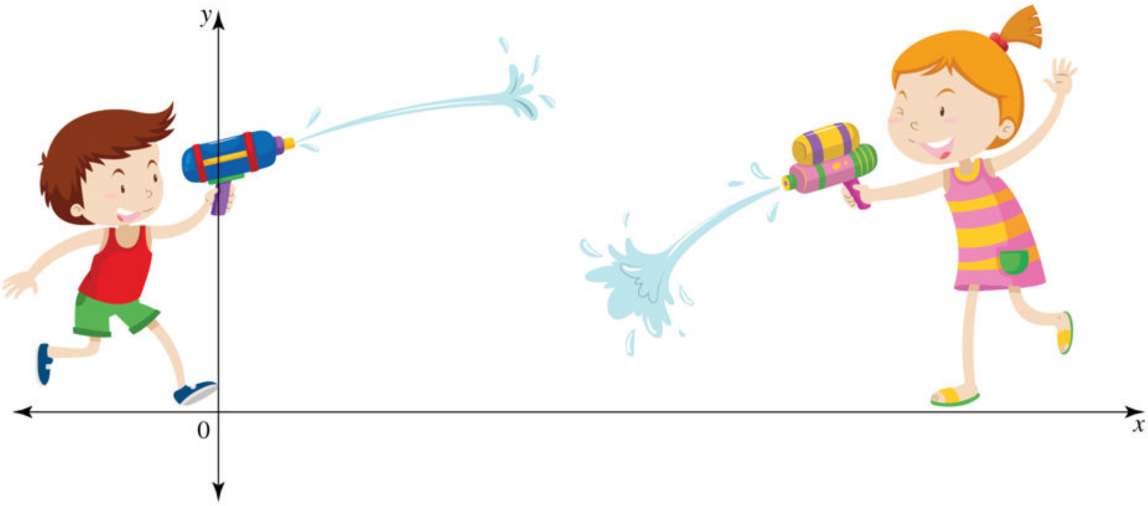
Andrew, who is 12 meters away from Karen, holds his water blaster at an angle θ radians from the horizontal direction, at a height of 1 meter above the ground. He shoots water out at a speed of U meters per second, at the same time as Karen blasts her water. Let T represents the time in seconds from the moment water blasts from both siblings.

e. Write down the horizontal displacement of the water in terms of T, U and θ.
f. Write down the vertical displacement of the water in terms of T, U and θ.

g. Find an expression, in terms of U and θ, for the range reached by the Andrew's water blaster when water returns to its initial height.

h. Show that, the maximum possible range found in part **g** is achieved when $\theta = \frac{\pi}{4}$

A cartesian plane is constructed to illustrate the siblings' relative position shown in the figure below. Let Andrew's left foot represent the origin and assume his hand holding the water blaster is directly above this foot.



i. Derive the cartesian equation of Andrew's water blast path.

j. Hence, or otherwise, derive the cartesian equation of Karen's water blast path.

Andrew wishes that the water he blasts could meet the water from Karen's blaster in the air, halfway between them.

k. Show that, the point of intersection between two water traces is at the point $\left(6, 2\sqrt{3} - \frac{144}{125}\right)$.

l. When $U = 10$, find the values of θ for this to happen. Leave your answer to the nearest degree.

m. Find the relationship between U and θ for their water to meet halfway in the air.

n. Find the minimum initial speed of Andrew's water blast that allows it to meet Karen's water halfway in the air. Leave your answer correct to 2 decimal places.

A sudden strong wind blows in the water park that affects Andrew and Karen's water blast. The air resistance is no longer negligible. The siblings now blast water vertically upwards to compete for the maximum height reached by the water.
When Andrew blasts water, it is subject to an acceleration of a m/s^2, given by

$$a = -9.8 - 0.1v^2$$

where v is the velocity of water in m/s. He blasts water out at a speed of 10 m/s.

o. Find the displacement of Andrew's water in terms of v.

When Karen blasts water, it is subject to an acceleration of a m/s^2, given by

$$a = -6\sqrt{x}$$

where x is the displacement of water in meters. She blasts water out at a speed of 8 m/s.

p. Find the displacement of Karen's water in terms of v.

q. Which sibling reaches the highest water blast? Use Mathematical calculations to justify your answer.