Chapter 1 – Displaying and describing data distributions

Solutions to Exercise 1A

1

a see definition page 2

b see definition page 3

2 see diagram page 4

3 see diagram page 4

4

a a measured quantity ⇒ numerical

b a counted quantity⇒ numerical

c Bank account numbers are used only to name or identify a person's account. They have no other numerical properties ⇒ categorical

d In this instance, a person's height is not measured (in cm, say), but recorded as falling into one of three categories (short, average, tall) ⇒ categorical

e a measured quantity ⇒ numerical

f a counted quantity ⇒ numerical

g eye colour, recorded as (brown, blue, green), is a quality *not* a quantity ⇒ categorical

h Postcodes are used only to name or identify a geographic area. They have no other numerical properties ⇒ categorical

5

a Is a quality or characteristic that can only be used to name or identify but *not* order. ⇒ nominal

b Is a quality or characteristic that can only be used to name or identify but *not* order. ⇒ nominal

c Is a quality or characteristic that can be used to order. ⇒ ordinal

d Is a quality or characteristic that can be used to order ⇒ ordinal

e Is a quality or characteristic that can be used to order. ⇒ ordinal

f Is a quality or characteristic that can only be used to name or identify but *not* order. ⇒ nominal

6

a a counted quantity ⇒ discrete

b If cost is charged to the nearest cent, eg. \$23.67, \$23.68, \$23.69, etc., as is everyday practice, then cost is best regarded as discrete. ⇒ discrete

c a measured quantity ⇒ continuous

d a measured quantity ⇒ continuous

e a counted quantity ⇒ discrete

f a measured quantity ⇒ continuous

Solutions to Exercise 1B

1

a the most frequently occurring value

b **i** B occurs 5 times and is the most frequent value, so it is the mode.

ii size 8 occurs 5 times and is the most frequent value, so it is the mode.

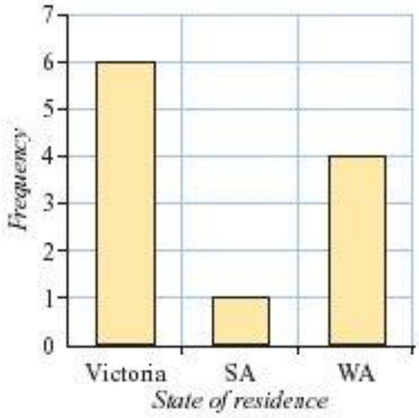
2

a *State of residence* is a quality or characteristic of a person. ⇒ categorical.

b The table is constructed by counting the number of responses for each category and then calculating percentages of the whole.

State of residence	*Frequency*	
	Count	*Percent*
Victoria	6	54.5
SA	1	9.1
WA	4	36.4
Total	11	100.0

c The bar chart is constructed by counting the number of responses for each category and then drawing the bar for that response to the appropriate height.



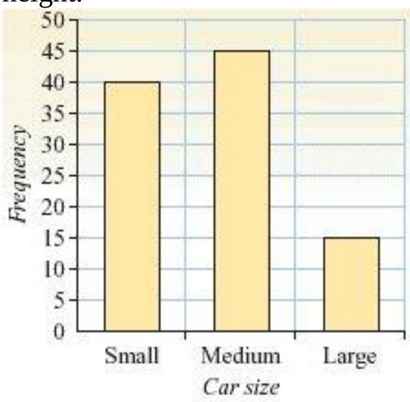
3

a *Car size* recorded as 'small', 'medium' or 'large' is a *quality* or *characteristic* of a car. ⇒ categorical.

b The table is constructed by counting the number of responses for each category and then calculating percentages of the whole.

Car size	*Frequency*	
	Count	*Percent*
Small	8	40
Medium	9	45
Large	3	15
Total	20	100

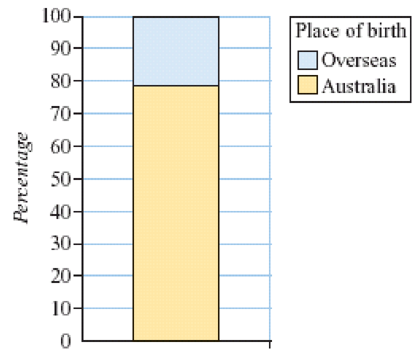
c The bar chart is constructed by counting the number of responses for each category and then drawing the bar for that response to the appropriate height.



4

a *Place of birth* is a characteristic of a person that can be used to name or identify where a person was born. However, it has no ordering properties. ⇒ nominal

b The segmented bar chart is constructed by reading percentages of each response from the table and then drawing the bar for that response to the appropriate height.



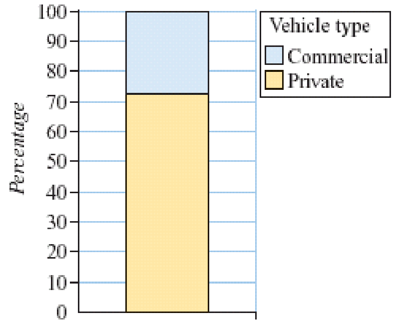
5

a *Type of car*, recorded as ‘private’ or ‘commercial’, can be used to identify a car. However, this information cannot be used to order cars in any meaningful way. ⇒ nominal

b Sum the two count values to get the total count, which represents 100%. Divide each of the count values by the total count and multiply by 100 to get each of the two percentages.

Type of vehicle	*Frequency*	
	Count	*Percent*
Private	132 736	73
Commercial	49 109	27
Total	181 845	100

c The segmented bar chart is constructed by reading percentages of each response from the completed table and then drawing the bar for that response to the appropriate height.



6

a The total number of schools sums to 20 and the missing percentage must be 55, since the 3 percentages must sum to 100.

b Reading off the table, 5 schools

c Summing the number column: 20 schools

d As calculated in **a**, 55%

e Taking information from the table: ‘Report: **20** schools were classified according to school type. The majority of these schools, **55%**, were found to be **Government** schools. Of the remaining schools, **25%** were **Independent** while **20%** were **Catholic** schools.’

7

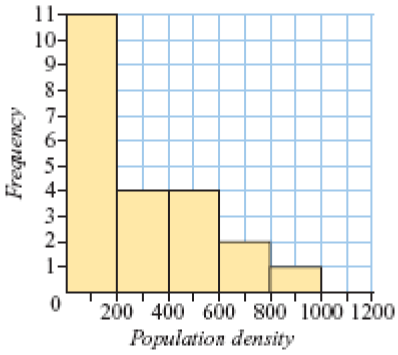
a 7, 45.5, 100.0.
The count for ‘rarely’ can be calculated by subtracting the counts for ‘regularly’ and ‘sometimes’ from the total. The total percentage must be 100% and the ‘sometimes’ percentage will be 100% minus the percentages for ‘regularly’ and ‘rarely’.

b Taking the frequencies and percentages from the **completed** table: ‘Report: When 22 students were asked the question, “How often do you play sport?”, the dominant response was ‘Sometimes’, given by **45.5%** of the students. Of the remaining students, **31.8%** of the students responded that they played sport ‘Rarely’ while **22.7%** said that they played sport ‘Regularly’.’

8 Taking the frequencies and percentages from the table, we can report that:
‘The eye colours of 11 children were recorded. The majority, 54.5%, had brown eyes. Of the remaining children, 27.3% had blue eyes and 18.2% had hazel eyes.’

Solutions to Exercise 1C

1 Draw in the axes and scales, labelling both. Draw in the histogram bars to the appropriate height as given in the table.



2

a **i** Reading from the graph: 17%
ii Reading from the graph: 13%
iii Reading from the graph: 46%
iv Summing the first three columns: 33%

b **i** Multiplying the percentage as read from the graph by 30 and dividing by 100% = 6
ii Multiplying the percentage as read from the graph by 30 and dividing by 100% = 4

c 15–19 is the highest bar and is therefore the mode in this case.

3

a Read the frequencies from the histogram and sum: 3 + 2 + 3 + 3 + 2 + 3 + 4 + 5 +1 = 21

b **i** Read the appropriate frequencies from the histogram and sum: 3 + 2 + 3 + 4 + 1 = 13
ii Read the appropriate frequencies from the histogram and sum: 3 + 2 + 3 = 8
iii Read the appropriate frequencies from the histogram and sum: 3 + 2 = 5
iv None of these cricketers had a batting average between 40 and 50.

c **i** $\frac{1}{21} \times 100 = 4.8\%$

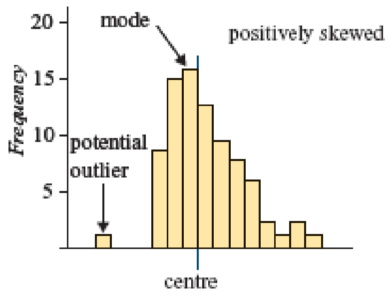
ii $\frac{12}{21} \times 100 = 57.1\%$

4 See calculator instructions page 15 (TI) or 17 (CASIO).

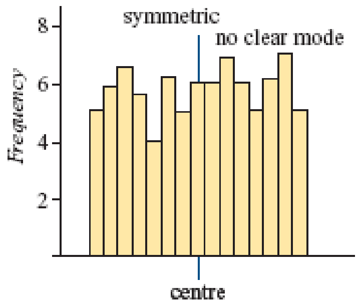
5 See calculator instructions page 15 (TI) or 17 (CASIO).

6

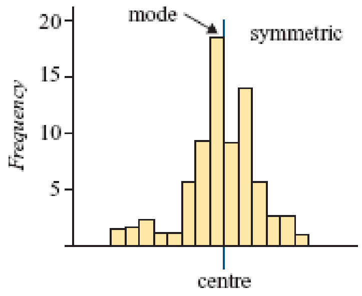
a The mode is the highest bar. There is one potential outlier and the histogram tails off to the right so is positively skewed.



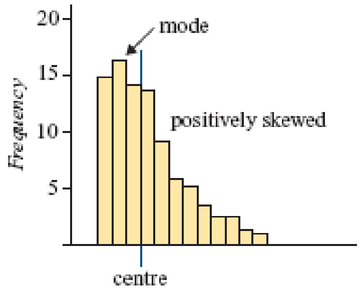
b The mode is not clear since two bars appear to be the same size. There are no outliers. The histogram is approximately symmetrically spread around its centre.



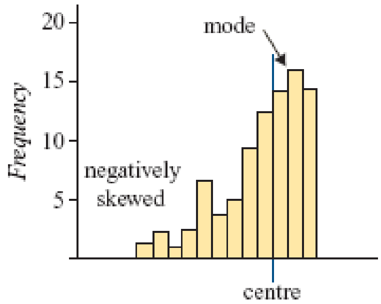
c The mode is the highest bar. There are no outliers and the histogram appears to be approximately symmetrically spread around its centre.



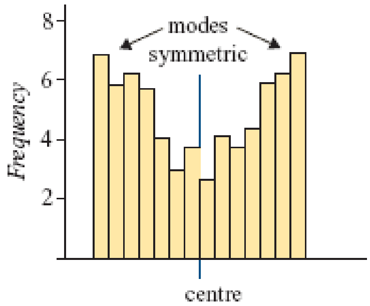
d The mode is the highest bar. There are no outliers and the histogram tails off to the right so is positively skewed.



e The mode is the highest bar. There are no potential outliers and the histogram tails off to the left so is negatively skewed



f There are two highest bars and thus two modes. There are no outliers and the histogram approximately symmetrically spread around its centre.



7

a All the distributions appear approximately symmetric around their respective centres.

b There are no clear outliers in any of the distributions.

c In A the central mark lies in the interval 8–10, in B it lies in the interval 24–26 and in C it lies in the interval 40–42. Note that the central interval in A and B is also the mode for those distributions.

d The spread is lowest in B since the range is only 8, compared to 14 for A and 18 for C.

e The spread was greatest in C, with a range of 18.

8 Reading the information from the histogram: 'Report: For the **28** students, the distribution of pulse rates is **approximately symmetric** with an outlier. The centre of the distribution lies in the interval **75–80** beats per minute and the spread of the distribution is **55** beats per minute. The outlier lies in the interval **110–115** beats per minute.'

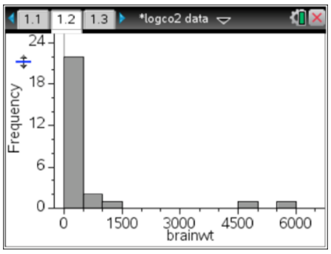
9 Suggest for worked solutions:
Shape of distribution is positively skewed.
Centre of distribution here is the median, corresponding to the data point $\frac{42+1}{2} = 21.5$, so median is between 65 and 70.
Spread here is the largest value – smallest value, i.e. 95 – 55 = 40.

Solutions to Exercise 1D

1 **a** to **h**
To calculate the log of a number:
i Type **log(number),** eg. log(2.5), and press **ENTER** (or **EXE**).
ii Round the answer to one decimal place:
eg. log(2.5) = 0.3979 … = 0.4 (to 1 d.p.)

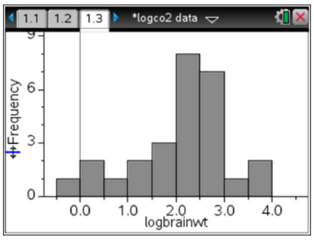
2 **a** to **d**
To convert the log of a number to its number, the log of a number:
i Type**10^(number),** eg. 10^(-2.5), and press **ENTER** (or **EXE**).
ii Round the answer to two significant figures as required:
eg. log(-2.5) = 0.00316…
= 0.0032 (to 2 sig.figs.)

3 **a** To construct a histogram for the raw data, see instructions on page 15(TI) or page 17(CASIO).



The histogram tails off to the right indicating positive skew. The are also two isolated bars well to the right of the main body of data which appear to be outliers.

b To construct a histogram with a log scale, see instructions on page 31 (TI) or page 32(CASIO).



The histogram is reasonably evenly spread around its central region indicating that it is approximately symmetric.

4

a log(0.4) = –0.397… = –0.4 to 1 d.p.
b log(5712) = 3.756… = 3.76 to 3 sig. figs.
c 10^2 = 100g
d 10^{-1} = 0.1g
e i Over 1000g is greater than 3 on the log scale. Reading from the histogram, there are 5 (=3 + 2) animals with brain weights over 1000g.
ii Between 1 and 100 g on the log scale is between 0 and 2. Reading from the histogram, there are 12 (= 4 + 8) animals with brain weights in this range.
i Over 1g is greater than 0 on the log scale. Reading from the histogram, there are 24 (= 4 + 8 +7 + 3 + 2) animals with brain weights over 1g

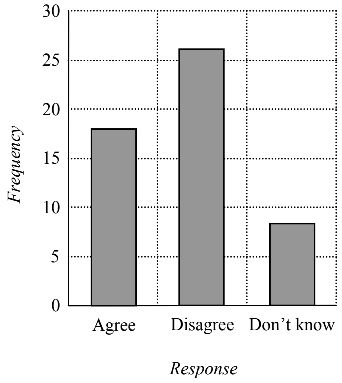
Chapter Review: Extended-response questions

1 a Reading from the histogram, 19% of students prefer to watch television.

b Reading from the histogram, approximately 13% prefer movies and approximately 10% prefer reading, a total of 23%.

c Sport is the preferred leisure activity. Approximately 29% prefer sport.
29% of 121= 35.09
$\therefore$ Rounding to the nearest whose number, conclude that 35 students prefer sport.

2 a

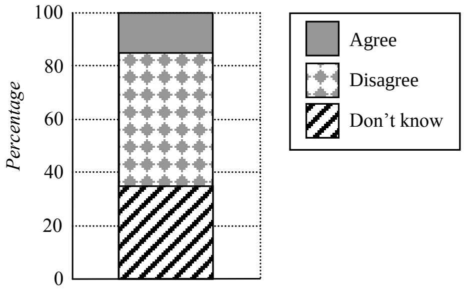


b Agree: $\frac{18}{52}=\frac{9}{26}\times 100\%= 34.6\%$ (to one decimal place)

Disagree: $\frac{26}{52}=\frac{50}{100}\times 100\%= 50\%$

Don't know: $\frac{8}{52}=\frac{2}{13}\times 100\% = 15.4\%$ (to one decimal place)

c



d Using the percentages calculated in **b**, conclude that: 50% of the 52 students disagree. Of the remaining students, 34.6% agreed while 15.4% said that they didn't know.

3 a i Number of students surveyed
$= 3 + 5 + 5 + 9 + 6 + 8 + 8 + 2 + 2 + 2 = 50$

ii 5 students spent from \$100 to \$105 per month.

b The modal interval is \$105–\$110. This is the location of the tallest bar in the histogram.

c Number of students who spent \$110 or more
$= 6 + 8 + 8 + 2 + 2 + 2 = 28$

d $3 + 5 = 8$ students spent \$100 or less per month.
$\frac{8}{50}=\frac{16}{100}\times 100\% = 16\%$
16% of students spent \$100 or less per month.

e i From the histogram, the distribution is approximately symmetric.

ii 22 students spent less than \$110 and 22 students spent more than \$115. Therefore \$110–\$115, with 6 students, is the interval containing the centre of the distribution.

iii The range $= \$140 - \$90 = \$50$

4 Inspection of the histogram shows that it is negatively skewed with two outliers. The range is 55 seconds. Noting that there are 17 data values either side of the median, tallying the frequencies from left to right locates the median in the interval 25 and 30 seconds. The two outliers lie somewhere between 50 and 55. This information can then be put together into a report as follows.

Report
For the 34 cars, the distribution of

waiting times is negatively skewed with an outlier. The distribution is centred between 25 and 30 seconds and has a spread of 55 seconds. There is an outlier located in the interval 50–55 seconds.

Chapter Review:
Solutions to Multiple–choice questions

1 Number of cars owned is a numerical variable but car size is categorical. $\Rightarrow$ **D**

2 Head diameter is a continuous numerical variable but sex is a nominal variable. $\Rightarrow$ **E**

3 The brown hair segment is approximately 36 percentage points in size. Since there are 200 students, this equates to $0.36 \times 200 = 72$ students with brown hair. $\Rightarrow$ **D**

4 The widest segment represents the students with brown hair, so brown hair is the most common hair colour $\Rightarrow$ **C**

5 Reading from the 'No' row of the 'Female' column, the number of females is 111 $\Rightarrow$ **C**

6 Reading from the 'Yes' row of the 'Male' column, the number of males on a diet is 31 $\Rightarrow$ **A**

7 Reading from the 'Total' row of the 'Female' column, the number of females not on a diet is 66. The percentage of females not on a diet is

$\frac{66}{111} \times 100 = 59.45...\%$ $\Rightarrow$ **C**

8 Reading from the 'Yes' row of the table, 31 males were on a diet. The percentage of people on a diet who were male is $\frac{31}{76} \times 100 = 40.78...\%$

$\Rightarrow$ **B**

9 The total number of students is given by the sum of the frequencies: $3 + 4 + 6 + 3 + 2 + 1 + 1 = 20$ $\Rightarrow$ **C**

10 $3 + 4 + 6 + 3 = 16$ $\Rightarrow$ **D**

11 The main body of data is approximately symmetric and there appears to an outlier. $\Rightarrow$ **D**

12 There are 20 scores in total. The bottom 10 scores are below the median and top 10 scores are above the median. Thus the median value lies in the interval 10–12 $\Rightarrow$ **B**

13 For a histogram, the spread of the data values can be estimated from the maximum range which is 22 $\Rightarrow$ **E**

14 $100 = 10^2$ so $\log(100) = 2$ $\Rightarrow$ **C**

15 The number whose log is 2.314 is $10^{2.314} = 206$ to the nearest whole number. $\Rightarrow$ **C**

16 $\log(16.8) = 1.22\ldots$ so on the log scale, Australia lies in the interval 1 – < 1.5 $\Rightarrow$ **D**

17 $\log(10) = 1$ so the percentage of countries with CO_2 emissions under 10 tonnes is approximately $100 - (11+1) = 88\%$ $\Rightarrow$ **E**

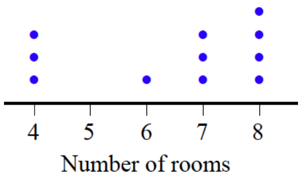
Chapter 2 – Summarising numerical data

Solutions to Exercise 2A

1

a The variable *number of rooms* is discrete: the data values are obtained by counting.

b See Example 1 instructions on constructing a dot plot.



2

a The variable *number of children* is discrete: the data values are obtained by counting.

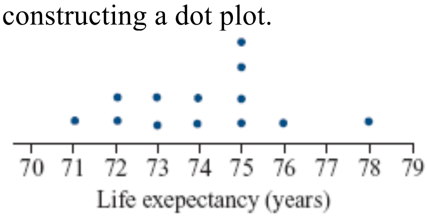
b See Example 1 instructions on constructing a dot plot.

c 4 children: the mode is the most frequently or commonly occurring value.

3

a The variable *life expectancy* is continuous: it can take any value between 70 and 79, but has been rounded to the nearest whole number.

b See Example 1 instructions on constructing a dot plot.



c 75 years: the mode is the most frequently or commonly value

4

a The variable *urbanisation* is continuous: it is a percentage that can potentially take any value between 0 and 100, but has been rounded to the nearest percent.

b See Example 2 instructions on constructing a stem plot.

key: 1|6 = 16

0	3 3 6 9 9
1	2 2 6 7
2	0 2 2 5 7 8 9
3	1 5
4	
5	4 6
6	
7	
8	
9	9 9
10	0

c The distribution is multimodal; six values 3, 99, 12 and 99.

5

a The variable *wrist circumference* is continuous because data values can take any value 16.5 and 19.9 cm but have been rounded to one decimal place.

b See page 45 for information on stem plots with split stems on page 45 .

i

16|5 =16.5

16	579
17	0 1 2 3 6 6 7
18	2 4 5
19	3 9

ii

16|5 =16.5

16	5 7 9
17	0 1 2 3
17	6 6 7
18	2 4
18	5
19	3
19	9

Solutions to Exercise 2B

1

a The range is the spread of all values and is the difference between the largest and smallest value in any given data set.

b For an odd number of data values, the median value is the middle value in the ordered data set. If the data set has an even number of values, there is no single middle data value. To find the median, take the average of the two middle values.

c Quartiles are the three numbers that divide a distribution into four equal parts. Note that Q_2 is the same as the median, while Q_1 is the median of the lower 50% of values and Q_3 is the median of the upper 50% of values.

d The interquartile range is the spread of the middle 50% of values and is the difference between Q_3 and Q_1.

2

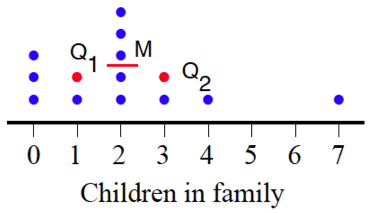
a After ordering the 6 values, the two middle values are 4 and 6,
median = (4 +6)/2 =5
To check that the median splits the data set into two equal parts, write out the date values in order, locate the median (see red line below)and count the number of data values each side of the median.
1 3 4 | 6 8 9

b After ordering the 5 values,
middle value = median = 12
To check that the median splits the data set into two equal parts, write out the date values in order, locate the median (see red data value below)and count the number of data values each side of the median.
9 10 12 14 20

3 After ordering the 9 values,
middle value = median = $850

430 500 650 750 850 1200 1790 2950 3500

4
Use the dot plot to locate the median and the quartiles and use these values to answer the questions.



a From the plot, $M = 2$

b From the plot, $Q_1 = 1$ & $Q_3 = 3$.

c $IQR = Q_3 - Q_1 = 3 - 1 = 2$

d Range = max. – min. = 7 – 0 =7

e

0 0 0 1 1 2 2 □2 2 2 3 3 4 7
Q_1 M Q_3

5
Use the dot plot to locate the median and the quartiles and use these values to answer the questions.

a Positively skewed with some outliers: The data rapidly tails off to the right and there are three values (3, 4 and 6) that appear to be well separated from the main body of the data.

b From the plot, $M = 0$.

c From the plot, $Q_1 = 0$ & $Q_3 = 1$
IQR = $Q_3 - Q_1 = 1 - 0 = 1$
Range = max. – min. = 6 – 0 = 6

6 Locate the required values on the stem plot. See Example 7 if you need help.

a Of the 14 values, the two middle values are 10 and 12;
average = median = 11.

b Middle value of the lower 50% (7 lowest values) = $Q_1 = 10$;
Middle value of the upper 50% (7 highest values) = $Q_3 = 15$.

c $IQR = Q_3 - Q_1 = 5$

R = highest value – lowest value = 18

To check your answers, write the data values in a line, check that $Q_1 = 10$, $Q_3 = 15$ and M is in the middle of 10 and 12.

7

a the data values are relatively evenly spread around the centre of the stem plot, so the distribution is approximately symmetric. No individual data values stand out from the main body of the data so there are no outliers.

b Of the 20 values, the two middle values are 25 and 27;
average = median = 26.

c Two middle values of lower 50% (10 lowest values) are 16 and 19,
average = Q_1 = 17.5;
two middle values of upper 50% (10 highest values) are 30 and 31,
average = Q_3 = 30.5.

d $IQR = Q_3 - Q_1 = 13$

R = highest value – lowest value = 29

8

a Of the 23 values,
middle value = median = 20

b Middle value of the lower 50% (11 lowest values) = Q_1 = 8;
middle value of the upper 50% (11 highest values) = Q_3 = 26.

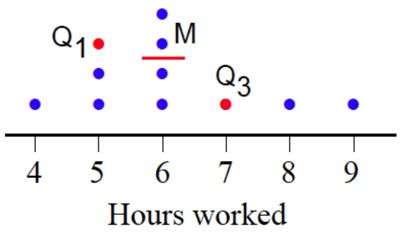
c $IQR = Q_3 - Q_1 = 18$

R = highest value – lowest value = 54

Solutions to Exercise 2C

1

Locate Q_1, M and Q_3 on the dot plot.



Use these and the minimum and maximum values (read from the dot plot) to form the five-number summary

Five number summary:
4, 5, 6, 7, 9

2

Locate Q_1, M and Q_3 on the stem plot.

key: 13 |6 =136

13	6	7				
14	3	6	8	8	9	
15	2	5	8	8	8	
16	4	5	5	6	7	9
17	8	8	9			
18	2	9				

Use these and the minimum and maximum values (read from the stem plot) to form the five-number summary

Five number summary:
136, 148, 158, 169, 189

3

a, b See worked example 8 (page 54) for method.

4 & 5

Follow the calculator instructions on page 56 (TI) or page 57 (CASIO) to construct these box plots.

6

a **i** Median is middle vertical of coloured box = 10
ii Q_1 is left edge of the box = 5, Q_3 is right edge of the box = 21
iii $IQR = Q_3 - Q_1 = 16$
iv Minimum is left end of horizontal line = 0, maximum is right end of horizontal line = 45
v No outliers

b **i** Median is middle vertical of coloured box = 27
ii Q_1 is left edge of coloured box = 12, Q_3 is right edge of coloured box = 42
iii $IQR = Q_3 - Q_1 = 30$
iv Minimum is left end of horizontal line = 5, maximum is right end of horizontal line = 50
v No outliers

c **i** Median is middle vertical of coloured box = 38
ii Q_1 is left edge of coloured box = 32, Q_3 is right edge of coloured box = 42
iii $IQR = Q_3 - Q_I = 10$
iv Minimum is leftmost outlier = 5, maximum is right end of horizontal line = 50
v Outlier is 5

d **i** Median is middle vertical of coloured box = 16
ii Q_1 is left edge of coloured box = 14, Q_3 is right edge of coloured box = 21
iii $IQR = Q_3 - Q_1 = 7$
iv Minimum is leftmost outlier = 1.5, maximum is rightmost outlier = 50
v Outliers are 1.5, 3, 36, 40, 50

7

a **i** Upper fence is right edge of coloured box + 1.5 times the IQR
$= 40 + 1.5 \times 10 = 55$
ii Lower fence is left edge of coloured box – 1.5 times the IQR
$= 30 - 1.5 \times 10 = 15$

b **i** Upper fence is right edge of coloured box + 1.5 times the IQR
$= 70 + 1.5 \times 20 = 100$
ii Lower fence is left edge of coloured box – 1.5 times the IQR
$= 50 - 1.5 \times 20 = 20$

8

a Lower fence is left edge of coloured box – 1.5 times the IQR
$= 39 - 1.5 \times 6 = 30$

b no-31 would still lie inside the lower fence

9

a 39 is the first quartile; 25% of values are less that the first quartile

b 45 is the third quartile; 75% of values are less than the first quartile

c 45 is the third quartile; 25% of values are greater than the first quartile

d 39 is the first quartile and 45 is the third quartile; 50% of values are between that the first and third quartiles

e 5 is less than the lowest data value and 45 is the third quartile; 75% of values are greater than 5 and less than the third quartile.

10

a 59 is the third quartile; 25% of values are above the third quartile

b 49.5 is the first quartile; 25% of values are below the first quartile

c 50% of values are between the first and third quartiles

d 57 is the median; 25% of values are between the median and the third quartile

e 75% of values are lower than the third quartile

f 70 is greater than any data value is the first quartile; 50% of values are between the median and any value greater than 67

Solutions to Exercise 2D

Boxplot 1 matches histogram **B** because the histogram symmetric and has approximately equal length quartiles.

Box plot 2 is negatively skewed with a possible outlier at the upper end. Histogram **D** is negatively skewed distribution, with a high–value outlier.

Box plot 3 matches histogram **C** because the central body of data is approximately symmetric and there is both a high–value and low–value outlier.

Box plot 4 matches Histogram **A** because it is symmetric and the upper and lower quartiles are more spread out compared to the two middle quartiles.

Solutions to Exercise 2E

1

a Because median value is well towards the right hand side of the box and the left whisker is longer than the right whisker we can say that the distribution is negatively skewed. The dot on the right indicates there is one outlier.
Reading values from the boxplot, we can say:
'The distribution is negatively skewed with an outlier. The distribution is centred at 39, the median value. The spread of the distribution, as measured by the *IQR,* is 10 and, as measured by the range, 45. There is an outlier at 5.'

b Because the median value of 16 is well towards to the left hand side of the box and the right hand whisker is longer than the left hand whisker we can say that the distribution is positively skewed. The five dots indicate that there are 5 outliers.
Reading values from the boxplot, we can say:
'The distribution is positively skewed with outliers. The distribution is centred at 16, the median value. The spread of the distribution, as measured by the *IQR,* is 6 and, as measured by the range, 35. The outliers are at 5, 8, 36, and 40.'

c Because the median value of 41.5 is well towards right hand side of the box and the left hand whisker is longer than the right hand whisker we can say that the distribution is negatively skewed. There are no dots, so there are no outliers.
Reading values from the boxplot, we can say:
'The distribution is negatively skewed with no outliers. The distribution is centred at 41.5, the median value. The spread of the distribution, as measured by the *IQR,* is 15 and, as measured by the range, 47.'

d Because the median value of 41 is approximately in the middle of the box and whiskers are similar in length, we can say that distribution is approximately symmetric. The four dots indicate that there are 4 outliers.
Reading values from the boxplot, we can say:
'The distribution is approximately symmetric and has 4 outliers. The distribution is centred at 41, the median value. The spread of the distribution, as measured by the *IQR,* is 7 and, as measured by the range, 36. The outliers are at 10, 15, 20 and 25.'

2

a The distributions of pulse rates are approximately symmetric for both men and women. There are no outliers. The median pulse rate for females (M = 76 beats/minute) is greater than for males (M = 73 beats/minute). The *IQR* is also greater for females (*IQR* =14 beats/minute) than males (*IQR* = 8 beats/minute). The range of pulse rates is also greater for females (R =30 beats/minute) than males (R =19 beats/minute).

b Comparing the two graphs and noting that the female distribution is much broader, we can say:
'For this group of males and females, the females on average had higher and more variable pulse rates.'

3

a The median battery lifetime for Brand A (M =34 hours) is greater than for Brand B (M =28 hours).
The IQR is for Brand A (*IQR*=14 hours) is approximately the same as for Brand B (*IQR* = 14 hours). The range of lifetimes for Brand A (R = 26 hours) is also less for Brand B (R = 39 hours).

b Comparing the two box plots and noting that the brand B distribution is much broader, we can say:
'On average, Brand A batteries have longer and less variable lifetimes.'

Solutions to Exercise 2F-1

1

a By definition, the median divides a distribution in half whereas the mean only does so for a symmetric distribution.

b The median and mean are the same only for a symmetric distribution.

c The median is the middle value when the data values are put in order, so isn't affected by the numerical values of outliers. The mean takes all values into account and thus is affected.

d The median would be more appropriate as there are likely to be a few very high salaries that would skew the mean value positively.

2

a n = number of values = 4;

$\sum x$ = sum of values = 12;

$$\bar{x} = \frac{\sum x}{n} = 3$$

b n = number of values = 5;

$\sum x$ = sum of values = 104;

$$\bar{x} = \frac{\sum x}{n} = 20.8$$

c n = number of values = 7;

$\sum x$ = sum of values = 21;

$$\bar{x} = \frac{\sum x}{n} = 3$$

3

a After ordering and summing the 11 values, mean = $\frac{33}{11}$ = 3;

median = 6^{th} value = 3;

mode = most common value = 2

b After ordering and summing the 12 values, mean = $\frac{60}{12}$ = 5;

median = average of two middle values (5 and 5) = 5;

mode = most common value = 5

4 Challenge question, no solutions.

5

a After ordering and summing the 8 values,

i mean = $\frac{288.8}{8}$ = 36.1;

ii median = average of two middle values (36.0 and 36.0) = 36.0

b Because the mean and median are very close to each other, the distribution can be assumed to be close to symmetric.

6

a After ordering and summing the 7 values,

i mean = $\frac{\$25.55}{7}$ = \$3.65;

ii median = middle value = \$1.70

b In this case, the median is a much better marker of a typical amount spent. This is because the mean has been positively skewed by the large positive outlier of \$16.55.

7

a Mean shouldn't be used, due to the distribution being strongly negatively skewed.

b No reason not to use the mean, the distribution is approximately symmetric.

c Mean shouldn't be used, due to outliers.

d Mean shouldn't be used, due to the distribution being very positively skewed.

e Mean shouldn't be used, due to the presence of outliers and the distribution being positively skewed.

f No reason not to use the mean.

8

a Since the distribution is approximately symmetric, either could be used, the distribution is approximately symmetric..

b mean = 82.8 (sum the data values and divide by 23);

median = middle value = 83

(the stem plot orders the data values so you can easily locate the middle value by inspection).

Solutions to Exercise 2F-2

1

a The *IQR*, by definition, will always incorporate 50% of the scores, specifically the middle 50% of scores.

b Since range = highest score – lowest score, the range only uses the smallest and largest scores.

c The standard deviation is the average amount by which the scores differ from the mean.

2 As all of the data values have the same, 7.1, the mean is 7.1 As data values do not vary, the standard deviation is 0.

3 It doesn't make sense to calculate a mean and standard deviation for *sex* (**b**), *post code* (**d**) and *weight* (underweight, etc.) (**f**), since all three are categorical variables and cannot be used to perform meaningful numerical calculations.

4

a Use you calculator. For instructions, see page 71 (TI) or 72 (CASIO).

b When the median and mean are similar in value, the distribution can be assumed to be close to symmetric.

5

a Use you calculator. For instructions, see page 71 (TI) or 72 (CASIO).

6

a Use you calculator. For instructions, see page 71 (TI) or 72 (CASIO).

Solutions to Exercise 2G

1

a 68% of values lie within 1 standard deviation of the mean.
134 + 20 = 154, 134 – 20 = 114,
$\Rightarrow$ 68% of values lie between 114 and 154.

b 95% of values lie within 2 standard deviations of the mean.
134 + 40 = 174, 134 – 40 = 94,
$\Rightarrow$ 95% of values lie between 94 and 174.

c 99.7% of values lie within 3 standard deviations of the mean.
134 + 60 = 194, 134 – 60 = 74,
$\Rightarrow$ 99.7% of values lie between 74 and 194.

d 16% of values are greater than 1 standard deviation above the mean.
134 + 20 = 154,
$\Rightarrow$ 16% of values are above 154.

e 2.5% of values are less than 2 standard deviations below the mean.
134 – 40 = 94,
$\Rightarrow$ 2.5% of values are below 94.

f 0.15% of values are less than 3 standard deviations below the mean.
134 – 60 = 74,
$\Rightarrow$ 2.5% of values are below 74.

g 50% of values are greater than the mean.
$\Rightarrow$ 50% of values are greater than 134.

2

a 1.68 and 2.08 are both 1 standard deviation from the mean.
$\Rightarrow$ 68% of values are between 1.68 and 2.08.

b 1.28 and 2.48 are both 3 standard deviations from the mean.
$\Rightarrow$ 99.7% of values are between 1.28 and 2.48.

c 2.08 is 1 standard deviation above the mean.
$\Rightarrow$ 16% of values are above 2.08.

d 2.28 is 2 standard deviations above the mean.
$\Rightarrow$ 2.5% of values are above 2.28.

e 1.28 is 3 standard deviations below the mean.
$\Rightarrow$ 0.15% of values are below 1.28.

f 1.88 is the mean.
$\Rightarrow$ 50% of values are above 1.88.

3

a **i** 11 is one standard deviation below the mean.
$\Rightarrow$ 84% of values are more than 11.
ii 14 is the mean.
$\Rightarrow$ 50% of values are less than 14.
iii 20 is two standard deviations above the mean.
50% of values are greater than the 14 (the mean)
2.5% of values are greater than20 (2 SDs above the mean)
$\Rightarrow$ 50–2.5 = 47.5 % of values are between 14 and 20.

b 8 is 2 standard deviations below the mean.
$\Rightarrow$ 2.5% of values are below 8.
$2.5 \times \frac{1000}{100} = 25$ walkers would be expected to complete the circuit in less than 8 minutes if 1000 walkers attempted it.

4

a **i** 155 and 185 are both 3 standard deviations from the mean.
$\Rightarrow$ 99.7% of values are between 155 and 185.

ii 180 is 2 standard deviation above the mean.
$\Rightarrow$ 2.5% of values are above 180.

iii 160 is 2 standard deviations below the mean.
$\Rightarrow$ 2.5% of values are below 160.
175 is 1 standard deviations above the mean.
$\Rightarrow$ 16% of values are above 175.
$\Rightarrow$ 100–2.5 –16 = 81.5% of values are between 160 and 175

b 175 is 1 standard deviation above the mean.
$\Rightarrow$ 16% of values are above 175.
$16 \times \frac{5000}{100} = 800$

5

a **i** 66 is the mean.
$\Rightarrow$ 50% of values are below 66.

ii 70 is 1 standard deviation above the mean.
$\Rightarrow$ 16% of values are above 70.
66 is the mean.
$\Rightarrow$ 50% of values are above 66
$\Rightarrow$ 50–16 = 34 % of values are between 160 and 175.

iii 62 is 1 standard deviation below the mean.
$\Rightarrow$ 16% of values are below 62.
74 is 2 standard deviations above the mean.
$\Rightarrow$ 2.5% of values are above 74.
$\Rightarrow$ 100–16 –2.5 = 81.5% of values are between 62 and 74

b 54 and 78 are both 3 standard deviations from the mean.
$\Rightarrow$ 99.7% of values are between 54 and 78.
$99.7 \times \frac{2000}{100} = 1994$

Solutions to Exercise 2H

1

a $z = \frac{120 - 100}{20} = 1$

b $z = \frac{140 - 100}{20} = 2$

c $z = \frac{80 - 100}{20} = -1$

d $z = \frac{100 - 100}{20} = 0$

e $z = \frac{40 - 100}{20} = -3$

f $z = \frac{110 - 100}{20} = 0.5$

2

a $x = 100 + 1 \times 20 = 120$

b $x = 100 + 0.8 \times 20 = 116$

c $x = 100 + 2.1 \times 20 = 142$

d $x = 100 + 0 \times 20 = 100$

e $x = 100 + (-1.4) \times 20$
$= 100 - 28 = 72$

f $x = 100 + (-2.5) \times 20$
$= 100 - 50 = 50$

3

a *English*: $z = \frac{69 - 60}{4} = 2.25$

Biology: $z = \frac{75 - 60}{5} = 3$

Chemistry: $z = \frac{55 - 55}{6} = 0$

Further Maths: $z = \frac{55 - 44}{10} = 1.1$

Psychology: $z = \frac{73 - 82}{4} = -2.25$

b *English*: A z–score of 2.25 is at least 2 standard deviations above the mean, so the student was within the top 2.5% of scores for English.

Biology: A z–score of 3 is 3 standard deviations above the mean, so the student was within the top 0.15% of scores for Biology.

Chemistry: A z–score of 0 is the mean, so the student was exactly average for Chemistry.

Further Maths: A z–score of 1.1 is at least 1 standard deviation above the mean, so the student was within the top 16% of scores for Further Maths.

Psychology: A z–score of –2.25 is at least 2 standard deviations below the mean, so the student was within the bottom 2.5% of scores for Psychology.

4

a $z = \frac{56 - 54}{10} = 0.2$

b $x = 54 + (-0.75) \times 10 = 46.5$

c A weight of 74 kg is 2 standard deviations above the mean so its standardised score is $z = 2$. Thus, using the 68-95-99.7% rule, 2.5% of these girls have weights greater than 74kg.

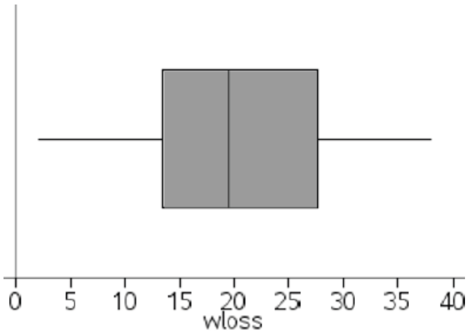
d A weight of 54 kg is the mean weight of these girls so its standardised score is $z = 0$. A weight of 64 kg is one standard deviation above the mean so its standardised score is $z = 1$. Thus, using the 68-95-99.7% rule, 34% of these girls have weights between 54 and 74kg.

e A standardised weight of $z = -1$ is one standard deviation below the mean. Thus, using the 68-95-99.7% rule, 16% of these girls have standardise weight less than –1.

f A standardised weight of $z = -2$ is two standard deviation below the mean. Thus, using the 68-95-99.7% rule, 97.5% of these girls have standardise weight more than –1.

Chapter Review: Extended-response questions

1 **a** Enter the data into your CAS calculator and follow the instructions for constructing a box plot on page 55 (TI) and page 57 (CASIO).

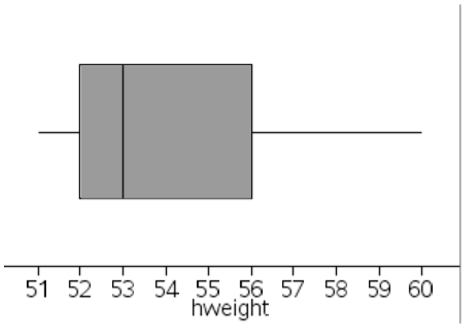


b From the box plot in **a**, $Q_1 = 13.5$, the median is 19.5 and $Q_3 = 27.5$.

c The middle 50% of people who exercised had weight losses between 13.5 kg and 27.5 kg. Twenty-five per cent of people lost less than 13.5 kg.

d The distribution is approximately symmetric with no outliers. The distribution is centred at 19.5 kg, the median value. The spread of the distribution, as measured by the IQR, is 14 kg and, as measured by the range, 36 kg.

2 **a** Enter the data into your CAS calculator and follow the instructions for constructing a box plot on page 55 (TI) and page 57 (CASIO)..



b

$$\begin{aligned} \text{Minimum} &= 51 \\ Q_1 &= 52 \\ \text{Median} &= 53 \\ Q_3 &= 56 \\ \text{Maximum} &= 60 \end{aligned}$$

c

$$\begin{aligned} \text{The interquartile range} &= Q_3 - Q_1 \\ &= 56 - 52 \\ &= 4 \end{aligned}$$

d The distribution is positively skewed with no outliers. The distribution is centred at 53 kg, the median value. The spread of the distribution, as measured by the IQR, is 4 kg and, as measured by the range, 9 kg.

3 **a** The five-number summary for the predicted and actual are estimated to be as follows:

Predicted	Actual
Minimum ≈ 20	Minimum ≈ 15
$Q_1 \approx 30$	$Q_1 \approx 27$
Median ≈ 38	Median ≈ 35
$Q_3 \approx 40$	$Q_3 \approx 40$
Maximum ≈ 58	Maximum ≈ 50

The teacher tended to **overestimate** her students' marks.

b

$$\begin{aligned} \text{Range of predicted marks} &\approx 48 - 20 \\ &\approx 28 \\ \text{Range of actual marks} &\approx 50 - 15 \\ &\approx 35 \\ \text{IQR of predicted marks} &= Q_3 - Q_1 \\ &\approx 40 - 30 \\ &\approx 10 \\ \text{IQR of actual marks} &= Q_3 - Q_1 \\ &\approx 40 - 27 \\ &\approx 13 \end{aligned}$$

The predicted marks have a smaller range and a smaller IQR, so the teacher's marks are **less variable** than the actual marks.

c The distribution of the predicted marks is negatively skewed with no outliers. The distribution of actual marks is approximately symmetric with no outliers. The median of the predicted marks (38) is greater than the median of the actual marks (35). The IQR of the predicted marks (10) is less than the IQR of the actual marks (13). The range of the predicted marks (28) is also less than the range of the actual marks (35).

d The teacher's predicted marks are, on average, higher than the students' actual marks and less variable.

4 **a** Median household income was greater in Suburb A.
Median Suburb A = 40 000
Median Suburb B = Median Suburb C = 22 000

b Suburb C had the most variable income.
Suburb C has the greatest range and the greatest IQR.

c The outliers represent families with exceptionally high incomes for the suburb.

d Statement **i** is true as Q_1 of Suburb A is approximately 26 000 which is greater than the median of Suburb B of approximately 22 000.
Statement **ii** is true as the median of Suburb A is approximately 40 000.
Statement **iii** is not true as the distribution is positively skewed.
Statement **iv** is true as the distribution is positively skewed with outliers.

5 **a** Median age of mothers ≈ 40; median age of fathers ≈ 45
$\therefore$ true

b Q_3 for fathers ≈ 48
$\therefore$ true

c Q_3 for mothers $\approx$ 46; median age of fathers $\approx$ 45
$\therefore$ false

d 50% of the mothers were clearly younger than 41, so fewer were aged between 42 and 48.
$\therefore$ false

e 25% of the fathers were aged 48 (Q_3) years or older, so fewer were aged 50 years or older
$\therefore$ false

Chapter Review: Multiple–choice questions

1 The dot plot tails off to the right (positively skewed) and there is one possible outlier.

⇒ **E**

2 The dot plot automatically orders the data so the median is the middle value in the plot and can be determined by inspection. There are 20 data points, so the median lies between the 10th and 11th value so the median is $M = 1$. Check by counting the number of data points either side of the median. They should be equal.

⇒ **B**

3 The first quartile is the median of the first half of the data points. There are 10 data points in the first half of the data points, so, so the median lies between the 5th and 6th value so the median is $Q_1 = 0$. Check by counting the number of data points either side of Q_1 They should be equal.

⇒ **A**

4 The third quartile is the median of the second half of the data points. There are 10 data points in the second half of the data points, so, so the median lies between the 15th and 16th value so the median is $Q_2 = 2$. Check by counting the number of data points either side of Q_2 They should be equal.

⇒ **C**

5 range = highest value – lowest value
= 6 – 0 = 6

⇒ **E**

6 Minimum value = 22,
Q_1 = 3rd value = 23,
median = average of two middle values = 24.5,
Q_3 = 8th value = 27,
maximum value = 29

⇒ **B**

7 The main body of the data in the stem plot is roughly evenly spread around its centre, so the distribution is approximately symmetric. While there appears to be an outlier at 60, it is inside the lower fence which is at 63.25 (=38.5 + 1.5 ×16.5).s

⇒ **A**

8 The stem plot automatically orders the data so the median is the middle value in the plot and can be determined by inspection. There are 25 data points, so the median 13th value so the median is $M = 28$. Check by counting the number of data points either side of the median. They should be equal.

⇒ **C**

9 $IQR = Q_3 - Q_1$
Locate the first and third quartiles by inspection to find.
$IQR = 38.5 - 22 = 16.5$

⇒ **B**

10 Reading from the midline in box plot A, the median is $M = 53$.

⇒ **B**

11 The length of the box in box plot B is 9 (= 75– 66), so the $IQR = 9$.

⇒ **A**

12 range = highest value – lowest value
In box plot C, the highest value is 80 (the end of the right hand whisker) and the lowest value is the outlier at 49, so range = 80 – 49 = 31

⇒ **D**

13 In box plot A, the mid line is towards the left-hand side of the box and the left hand whisker is much shorter than the right hand whisker, so the distribution is positively skewed.

⇒ **D**

14 In box plot B, the mid line is centred in the box and the left hand whisker and the right hand whiskers are approximately equal in length, so the distribution is approximately symmetric.

⇒ **A**

15 In box plot D, the mid line is just off centre the box and the left hand whisker and the right hand whiskers are approximately equal in length, so the main body of the data is approximately symmetric. There are also 4 outliers as indicated by the shaded circles. distribution is approximately symmetric with outliers.

⇒ **B**

16 In box plot D, the third quartile is 53, so 25% of the data values are great than 35.

$\Rightarrow$ **B**

17 $IQR = Q_3 - Q_1$

$= 65 - 60 = 5;$

$60 - 1.5 \times 5 = 52.5$ = lower fence,

$65 + 1.5 \times 5 = 72.5$ = upper fence

$\Rightarrow$ **A**

18 Enter the data into your CAS calculator and calculate the mean and the standard deviation.

$\Rightarrow$ **D**

19 Since phone numbers are categorical variables and not numerical, calculating the mean and standard deviation a set of phone numbers will not give anything meaningful. $\Rightarrow$ **B**

20 The mean is generally only an appropriate measure of centre when the distribution is symmetric and there are no outliers. The median is a much more robust measure of centre and would be preferred when the data is clearly skewed and/or there are outliers.

$\Rightarrow$ **D**

21 $z = \dfrac{50 - 55}{2.5}$

$= \dfrac{-5}{2.5} = -2$

$\Rightarrow$ **B**

22 Because the trips are normally distributed, 50% of values lie above and below the mean. The mean is 78 mins.

=> **C**

23 The standard deviation is 4 mins, therefore 74 to 82 mins is 1 standard deviation above and below the mean. From the 68 – 95 – 99.7% rule, 68% of values lie within 1 standard deviation of the mean.

=> **D**

24 82 mins is 1 standard deviation above the mean. From the 68 – 95 – 99.7% rule it is seen that 84% of values are less than 82 mins.

=> **E**

25 70 mins is 2 standard deviations below the means.
From the 68 – 95 – 97.5% rule it is seen that 97.5% of trips take more than 70 mins.
82 mins is 1 standard deviation above the mean.
From the 68 – 95 – 97.5% rule it is seen that 16% of trips take more than 82 mins.
As a result, 97.5 – 16 = 81.5% **of** trips take between 70 and 82 minutes. As there are 200 trips in total, the number taking between 70 and 82 minutes is 81.5% of 200 = 163

=> **E**

26 $z = \dfrac{71-78}{4} = -1.75$

=> **A**

27

actual time

= mean + z×standardised time

$= 78 + (-0.25) \times 4 = 77$

=> **A**

28

A standardised time of 2.1 is more than 2 standard deviations above the mean so it is very much above average.

=> **E**

29 As every garden stake is reduced by the same amount, the mean e value will decrease by this same amount, 5 cm, so $\bar{x} = 180.5 - 5 = 175.5$ cm.
Reducing the length of each stake by the same amount does not change the variability in length so the standard deviation stays the same, $s = 2.9$ cm

=> **C**

Chapter 3 – Investigating associations between two variables

Solutions to Exercise 3A

1

a If we plan to predict (or explain) a fish's toxicity from its colour, the variable *colour* is explanatory variable (EV) and *toxicity* is the response variable (RV).

b It makes more sense to explain a person's *weight loss* in terms of a change in the *type of diet* they follow rather than the other way around. Thus *type of diet* is the explanatory variable and *weight loss* is the response variable.

c It makes more sense to explain the change in a second hand car's *price* in terms of its *age* rather than the other way around. Thus *age* is the explanatory variable and *price* is the response variable.

d By suggesting that the cost of heating a house depends on the type of fuel used designates type of *fuel* as the explanatory variable. *Cost* is then the response variable.

e It makes more sense to explain the change in a house's *price* in terms of its *location* rather than the other way around. Thus *location* is the explanatory variable and *price* is the response variable.

2

a It makes more sense to explain a change in a person's *exercise level* in terms of their *age* rather than the other way around. Thus *age* is the explanatory variable.

b It makes more sense to explain a change in a person's *salary level* in terms of their *years of education* rather than the other way around. Thus *years of education* is the explanatory variable.

c It makes more sense to explain the change in a person's *comfort level* in terms of a change in *temperature* rather than the other way around. Thus *temperature* is the explanatory variable.

d It makes more sense to explain the change in the *incidence of hay fever* in terms of a change in the *time of year* rather than the other way around. Thus *time of year* is the explanatory variable.

e It makes more sense to explain the difference in people's *musical taste* in terms of their *age group* rather than the other way around.
Thus *age group* is the explanatory variable.

f It makes more sense to explain the differences in the *AFL team supported* by people in terms of the differences in their *state of residence* rather than the other way around.
Thus *state of residence* is the explanatory variable.

Solutions to Exercise 3B

1

a It is more likely that *enrolment status* determines *drinking behaviour* rather than the other way around, making *enrolment status* the explanatory variable.

b No: since the ‘yes’ and ‘no’ percentages were very similar for both full-time and part-time students, we can make the following statement: ‘The percentage of full-time and part-time students who drank alcohol is similar, 80.5% to 81.8%. This indicates that drinking behaviour is not associated with enrolment status.’

2

a As *handedness* and *sex* are both natural attributes that cannot be manipulated in any way, the choice of the response variable is arbitrary. However, the way the table has been set up with *handedness* (left, right) defining the rows of the table, handedness is assumed to be the response variable (see page 94).

b Sum the columns to find the total number males (22 + 222 = 242) and females (16 + 147 = 163) and use these values to convert the table to percentages by making the following computations:

male left: $\frac{22}{244} \times 100 = 9.0\%$

male right: $\frac{222}{244} \times 100 = 91.0\%$

female left: $\frac{16}{163} \times 100 = 9.8\%$

female right: $\frac{147}{163} \times 100 = 90.2\%$

c No: since the percentage of left handed males (9.0%) and females (9.8%) is similar, there is no evidence of an association between *handedness* and *sex*.

3

a It is possible that the *sex* of a person determines the how often they *exercised* but not the other way around, so *sex* is the explanatory variable.

b *Exercised* is an ordinal variable because its categories ‘rarely’, ‘sometimes’ and ‘regularly’ can be used to order the individuals involved in terms of how often they exercised.

c The percentage of females who exercised sometimes is given by the percentage in the ‘female’ column and the ‘sometimes’ row (54.9%).

d Yes: for example, since a significantly greater percentage of males (18.6%) exercised regularly compared to females (5.9%), we can conclude that the variable *exercised* and *sex* are associated.

4

a The percentage of widowed people who found life dull is given by the percentage in the ‘widowed’ column and the ‘dull’ row (11.9%).

b The percentage of people who were never married and found life exciting is given by the percentage in the ‘never’ column and the ‘exciting’ row (52.3%)

c The way the table is set up with marital status defining the columns, marital status can be assumed to be the explanatory variable.

d *Attitude to life* is as an ordinal variable because its categories ‘exciting’, ‘pretty routine’ and ‘dull’ can be used to order the individuals involved in terms of their satisfaction with life.

e Yes: the are many ways this question can be answered. Several possibilities are given in the answers. In essence, it is sufficient to find two percentages in the same row (defined by the RV) but different columns (defined by the EV) that differ significantly to infer an association. For example, focusing on the ‘dull’ row, the fact that the percentage of ‘married’ people’ who found life ‘dull’ (3.7%) is very much less than the percentage of ‘widowed’ people who find life dull (11.9%) is sufficient to infer an association between *attitude to life* and *marital status*.

Solutions to Exercise 3C

1

a Battery *lifetime* is measured in hours, so it is a numerical variable.
Battery *price* is classified as 'high', 'medium, and 'low' so it is a categorical variable.

b Yes: for example, focusing on the median lifetimes of the batteries we see that as the price increases from 'low', 'medium' and 'high' the median life time also increases.
Alternatively, the price increases from 'low', 'medium' and 'high', the variability in lifetimes (as given by the IQR) increases.

2 Yes: for example, the median number of days spent away from home is quite different for Japanese tourists (M = 17 nights, obtained by locating the 'middle' dot in the plot) and Australian tourists (M = 7) indicating that the number of days sent away from home is associated with country of origin.

3 Yes: the median age of female patients (M = 34 years – obtained by locating the middle value in the stem plot) is quite different from the median age of male patients (M = 25.5 years) indicating that the age of patients admitted to this hospital is associated with the patients' sex.

Solutions to Exercise 3D

1

a Plotting the variable number of seats on the horizontal axis of the scatterplot implies that in this investigation, *number of seats* is the explanatory variable.

b *Airspeed* values are recorded in km/h so the variable is numerical.

c 8. Count the number of data points (dots).

d Around 800 km/h. Locate the data point that represents the aircraft that can seat 300 passengers and read off its airspeed.

2–5 Enter data into your calculator and follow the instructions on page 104 to generate the required plots.

Solutions to Exercise 3E

1 If necessary, see the 'Warning!' box on page 112 of the textbook for a list of the assumptions**.**

2

a As a general rule no association would be expected.

b A positive association would be expected, as the higher the level of education, the higher the level of skills usually required in high paying occupations.

c A positive association would be expected; tax paid should increase with salary.

d A positive association would be expected, the more frustrated people become with a difficult situation, the more they are likely they are to exhibit aggressive behaviour.

e A negative association would be expected. In a typical Australian city, population density tends because individual dwellings tend to occupy a greater area of land.

f A negative association would be expected, the more time spent using social media reduces the time available for studying.

2

a

- **i** The dots drift upwards to the right, so it is a positive association.
- **ii** The dots drift downwards to the right, so it is a negative association.
- **iii** The dots drift upwards to the right, so it is a positive association.
- **iv** There is no clear pattern in the cloud of dots that make up the scatterplot indicating there is no association.

b To help you answer these questions, refer to the scatterplots on pages 110–111 and the table on page 111.

i The correlation coefficient is approximately +0.7, which indicates a moderate positive association.

ii The correlation coefficient is approximately –0.4, which indicates a weak negative association.

iii The correlation coefficient is approximately +0.9, which indicates a strong positive association.

iv There is no association, so the correlation coefficient will be close to 0 (in correlation terms between –0.24 and +0.24).

Solutions to Exercise 3F

1

a Scatterplot A shows a strong, positive, non-linear association with no outliers.
Scatterplot B shows a strong, negative, linear association with one outlier.
Scatterplot C shows a weak (to moderate), negative, linear association with no outliers.

b It wouldn't be appropriate to use the correlation coefficient for scatterplot A (as the association is non-linear) or for scatterplot B (as the association has an outlier).

2 (Optional) Using the information given in the table, we can draw the following table:

x	$x-\bar{x}$	y	$y-\bar{y}$	$(x-\bar{x})\times(y-\bar{y})$
2	–2	1	–4	8
3	–1	6	1	–1
6	2	5	0	0
3	–1	4	–1	1
6	2	9	4	8
Sum	**0**		**0**	**16**

$$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{(n-1)s_x s_y}$$

$$= \frac{16}{4 \times 1.871 \times 2.915}$$

$$= 0.73$$

3 a, b & c

Enter data into your calculator and follow the instructions on page 114 (TI) or page 115 (CASIO) to calculate the correlation coefficient.

Solutions to Exercise 3G

1

a Coefficient of determination:
$r^2 = (0.675)^2 = 0.456 = 45.6\%$

b Coefficient of determination:
$r^2 = (0.345)^2 = 0.119 = 11.9\%$

c Coefficient of determination:
$r^2 = (-0.567)^2 = 0.321 = 32.1\%$

d Coefficient of determination:
$r^2 = (-0.673)^2 = 0.453 = 45.3\%$

e Coefficient of determination:
$r^2 = (0.124)^2 = 0.015 = 1.5\%$

2

a $r^2 = 0.8215$,
$r = \sqrt{0.8215} = 0.906$ (from the scatterplot, the association is positive)

b $r^2 = 0.1243$,
$r = \sqrt{0.1243} = -0.353$ (from the scatterplot, the association is negative)

3 To help you answer questions **3a** to **e**, remember that:
the coefficient of determination (as a percentage) tells us the variation in the response variable (RV) explained by the variation in the explanatory variable (EV).

a The coefficient of determination = $r^2 = (-0.611)^2 = 0.373$ or 37.3%, which can be interpreted as '37.3% of the variation observed in hearing test scores (RV) can be explained by the variation in age (EV)'.

b The coefficient of determination = $r^2 = (0.716)^2 = 0.513$ or 51.3%, which can be interpreted as '51.3% of the variation observed in mortality rates can be explained by the variation in smoking rates'.

c The coefficient of determination = $r^2 = (-0.807)^2 = 0.651$ or 65.1%, which can be interpreted as '65.1% of the variation observed in life expectancy can can be explained by the variation in birth rates'.

d The coefficient of determination = $r^2 = (0.818)^2 = 0.669$ or 66.9%, which can be interpreted as '66.9% of the variation observed in daily maximum temperature can be explained by the variation in daily minimum temperatures'.

e The coefficient of determination = $r^2 = (0.8782)^2 = 0.771$ or 77.1%, which can be interpreted as '77.1% of the variation in the runs scored by a batsman can be explained by the variation in the number of balls they face.'

Solutions to Exercise 3H

1 to 4 The solutions are integrated into the answers for these questions

Solutions to Exercise 3I

To help you answer these questions, consult the table on page 125.

1

a Two categorical variables $\Rightarrow$ segmented bar chart

b Two numerical variables $\Rightarrow$ scatterplot

c Numerical response variable (hours spent at the beach) and categorical explanatory variable (state of residence) with two or more sub categories $\Rightarrow$ parallel box plots

d Two numerical variables $\Rightarrow$ scatterplot

e Two numerical variables $\Rightarrow$ scatterplot

f Two categorical variables $\Rightarrow$ segmented bar chart

g Two categorical variables $\Rightarrow$ segmented bar chart

h Numerical response variable (cigarettes smoked per day) and categorical explanatory variable (sex) $\Rightarrow$ parallel box plots or a back-to-back stem plot because the variable sex only has two categories

2 The response variable is numerical and the explanatory variable is categorical but with only two categories so the correct response is **E** (back-to-back stem plot)

Chapter Review: Extended-response questions

1 **a** *number of accidents* and *age* are both categorical variables.

b From their location in the table, we see that *age* is assumed to be the explanatory variable and *number of accidents* the response variable.

c Reading from the ‘more than one accident’ row and the ‘age < 30’ column we see that 470 drivers under the age of 30 had more than one accident.

d

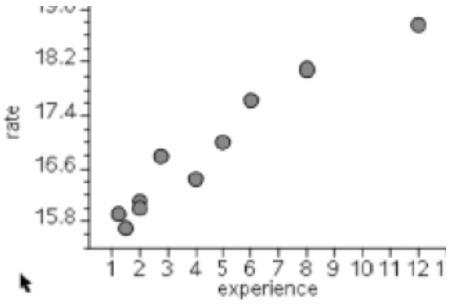
Number of accidents	*Age* < 30	*Age* ≥ 30
At most one accident	$\frac{130}{600} \times \frac{100}{1}\% \approx$ 21.7%	$\frac{170}{400} \times \frac{100}{1}\% =$ 42.5%
More than one accident	$\frac{470}{600} \times \frac{100}{1}\% \approx$ 78.3%	$\frac{230}{400} \times \frac{100}{1}\% =$ 57.5%
Total	100%	100%

e The statement ‘*Younger drivers (age < 30) are more likely than older drivers (age ≥ 30) to have had more than one accident*’ is correct. Of drivers aged less than 30, 78.3% had more than one accident compared to only 57.5% of the older drivers.

2 **a** *Age at marriage* (in years) is numerical and *sex* is categorical.

b The parallel box plots do support the contention that the age a person marries depends on their sex. The median age at marriage of men and women differs, indicating that *age at marriage* is associated with to *sex*.

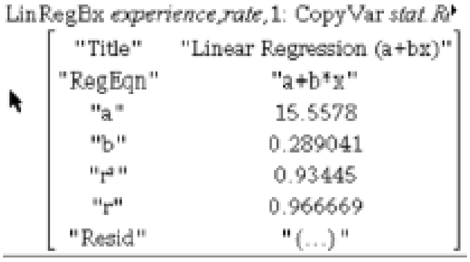
3 **a**



Enter the data into your calculator and follow the instructions given on pages 104 and 105. The vertical axis has been used for *rate*, as *rate* is the response variable.

b Because the scatterplot points tend to drift up the plot from left to right closely following a linear pattern, we can conclude that there is a strong positive linear association between pay rate and experience, that is, people with more experience are generally being paid a higher starting pay rate. There are no apparent outliers.

c Follow the instruction on pages 114 (TI) and 115 (CASIO) to find that $r \approx 0.967$
Note, there is no need to re-enter the data as it will be still be stored in your calculator from when you constructed the scatterplot.



d Coefficient of determination is $r^2 = 0.934$. This can be read directly from your calculator screen when you calculate r . From this value, it can be concluded that, 93.45% of variation in pay *rate* is explained by the variation in *experience*.
Note: The coefficient of determination (as a percentage) tells us the variation in the response variable (RV) explained by the variation in the explanatory variable (EV).

Chapter Review: Multiple–choice questions

1 Plays sport and sex are both categorical variables. $\Rightarrow$ **A**

2 'Females–No' = 175 – 79
= 96 $\Rightarrow$ **D**

3 Percentage 'Males–No'
$= \frac{34}{102} \times 100\% = 33.3\%$ $\Rightarrow$ **B**

4 When the table is percentaged, the percentage of males who play sport is much higher than the percentage of females who play sport indicating an association between playing sport and sex. $\Rightarrow$ **D**

5 Battery life is a numerical variable but brand is a categorical variable. $\Rightarrow$ **C**

6 The first statement supports the contention because the difference in median battery life shows one brand to be superior.
The second and third statements support the contention because they both show one brand to be more reliable than the other.
Therefore, all three statements support the contention. $\Rightarrow$ **E**

7 Weight at age 21 and weight at birth are both measured in kg so they are numerical variables. $\Rightarrow$ **D**

8 The dots drift upwards towards the right of the graph and appear to be aggregated fairly closely around an imagined straight line fitted to the data. From this we conclude that there is a strong positive linear association. $\Rightarrow$ **E**

9 If $r = –0.9$, this means the relationship is a strongly negative linear one. This means that as drug dosage increases, response time tends to decrease. $\Rightarrow$ **C**

10 Enter the data into your calculator and follow the instructions on page 114 (TI) or Page 115 (CASIO) to get:
$r = 0.7863$ (to 4 decimal places)
$\Rightarrow$ **C**

11 Coefficient of determination = r^2
$= (–0.7685)^2 = 0.5906$ $\Rightarrow$ **D**

12 Coefficient of determination = r^2
$= (0.765)^2 = 0.585 = 58.5\%$.
Since heart weight is the response variable, this result tells us that, for these mice, '58.5% of the variation in heart weight can be explained by the variation in body weights'. $\Rightarrow$ **A**

13 This association tells us that heavier mice tend to have heavier hearts (D) and lighter mice tend to have lighter hearts, but no more. Statement such as 'increasing the body weights of mice will increase their heart' (E) might also seem to be a reasonable option. However it is causal in nature and not a justifiable conclusion to draw when all we know is that heart weight and body weight are strongly correlated.
$\Rightarrow$ **E**

14 Since the response variable (*weight*) is numerical and the explanatory variable (*level of nutrition*) is categorical with more than two categories, we would use parallel box plots. $\Rightarrow$ **B**

15 Since the two variables are both categorical, we would use a segmented bar chart. $\Rightarrow$ **C**

16 This association tells us that people on higher salaries tend to recycle more garbage but we cannot assume that it is the higher salary alone that encourages a higher level of recycling of garbage. It may be that those on higher salaries have more education and thus are more aware of the environmental impact of recycling. $\Rightarrow$ **E**

17 As there is no logical reason to expect an association between marriage rate and falling out of a fishing boat and drowning, we conclude that the association is just coincidence. $\Rightarrow$ **C**

Chapter 4 – Regression: fitting lines to data

Solutions to Exercise 4A

1 A residual is the vertical difference between a value on a plot and the regression line drawn to fit the plot.

2 To find the least squares regression line, we must minimise the sum of the squares of the residual values. $\Rightarrow$ **C**

3 See page 134.

Solutions to Exercise 4B

1

a As stated in the question, *traffic volume* is to be used to predict *pollution level* so *traffic volume* is the predictor or explanatory variable (EV) and *pollution level* is the response variable (RV).

b $b = r \times \frac{s_x}{s_y}$

$= 0.94 \times \quad = 49.2 \ldots$

$a = \bar{y} - b\bar{x}$

$= 230.7 - 49.2 \times 11.38 = -329.2 \ldots$

Therefore,

pollution level = *a* + *b* × *traffic volume*

= –330 + 49 × *traffic volume* (2 sig. figs.)

2

a As stated in the question, *birth rate* is to be used to predict *life expectancy*, so *birth rate* is the predictor or explanatory variable (EV) and *life expectancy* is the response variable (RV).

b $b = r \times \frac{s_x}{s_y}$

$= -0.810 \times \frac{9.99}{5.41} = -1.4957\ldots$

$a = \bar{y} - b\bar{x}$

$= 55.1 + 1.4957 \times 34.8 = 107.15\ldots$

Therefore,

life expectancy = *a* + *b* × *x*

= 110 – 1.5 × *birth rate* (2 sig. figs.)

3

a As stated in the question, *age* is to be used to predict *distance travelled*, so *age* is the predictor or explanatory variable (EV) and *distance travelled* is the response variable (RV).

b $b = r \times \frac{s_x}{s_y}$

$= 0.947 \times \frac{42.61}{3.64} = 11.09 \ldots$

$a = \bar{y} - b\bar{x}$

$= 78.04 - 11.09 \times 5.63 = 15.6 \ldots$

Therefore,

distance travelled = *a* + *b* × *x*

= 16 + 11 × *age* (2 sig. figs.)

4

a If the slope is negative, the correlation coefficient must also be negative. This follows from the rule: $b = r \times \frac{s_y}{s_x}$

b If the correlation coefficient is zero, this means the least squares regression line will be horizontal and thus will have a slope of zero. This also follows from the rule: $b = r \times \frac{s_y}{s_x}$

c The correlation coefficient being zero means the line will be horizontal and thus will have a constant *y*-value for its entire length. This *y*-value will thus be the average of all the *y*-values and will be the mean *y*-value, which is $\bar{y}$. This follows from the rule: $a = \bar{y} - b\bar{x}$ when *b* = 0.

5-6 Enter the data into your calculator and follow the instructions on page 136 (TI) or page 137 (CASIO).

7 **a** answer given in the question

b *runs* = –2.6 + 0.73 × *balls faced*

8 **a** RV is *number of TVs*

b answers given in question

c *number of TVs* = 61.2 +0.930 × *number of cars*

Solutions to Exercise 4C

1 read values from the graph
intercept ≈ 80
slope = $\frac{46-80}{8-0} = 4.25$ or 4.3 rounded to 1 d.p.

2

a Because of the way in which the regression equation is written:
hand span = 2.9 + 0.33 × *height*
the EV can be assumed to be *height*.

b Reading values from the equation of the least squares line:
slope = 0.33
intercept= 2.9

c Substitute *height* = 160 into the equation of the regression line and evaluate:
hand span = 2.9 + 0.33 × 160
= 55.7 cm

d error of prediction = residual
residual = actual – predicted
= 58.5 – 55.7 = 2.8 cm

3

a Because of the way in which the regression equation is written:
fuel consumption = –0.1 + 0.01 × *weight*
the RV can be assumed to be *fuel consumption*.

b Reading values from the equation of the least squares line:
slope = 0. 01
intercept = –0.1

c Substitute *weight* = 980 into the equation of the regression line and evaluate:
fuel consumption = –0.1 + 0.01 × 980
= 9.7 litres/100 km

d error of prediction = residual
residual = actual – predicted
= 8.9 – 9.7 = –0.8 litres/100 km

4

a The equation of the least squares line is:
energy content = 27.8 +14.7 × *fat content*
Reading values from this equation:
slope = 14.7
intercept= 27.8

b 14.7: 'the energy content for every increase of 1 g of fat' is given by the slope of the least squares line

c $r = \sqrt{r^2} = \sqrt{0.7569}$ = 0.87

d In general terms, the coefficient of determination (r^2) gives the proportion (or percentage) of the variation in a response variable that can be explained by the explanatory variable. Thus, since 0.7569 × 100% = 75.7%, we can say that 75.7% of the variation in energy content can be explained by the variation in fat content.

e **i** Substitute *fat content* = 132 into the equation of the regression line and evaluate:
energy content = 27.8 + 14.7 × 8
= 145.4 calories

ii error of prediction = residual
residual = actual – predicted
= 132 – 145.4 = –13.4 calories

5

a The equation of the least squares line is:
success rate = 98.5 –0.278 × *distance*
Reading values from this equation:
slope = –0.278
The slope gives the average change in the RV for each one unit change in the EV or, in the context of this question, ‘On average, the probability of success decreases by 27.8% for each extra metre the golfer is from the hole.’
Note: 27.8% per metre is equivalent to 0.278% per cm

b Substitute *distance* = 90 into the equation of the regression line and evaluate:
success rate = 98.5 – 0.278 × 90
= 98.5 – 25.02 = 73.5%

c 0 = 98.5 – 0.278 × *distance*
or
98.5 = 0.278 × *distance*
distance = 354.3 cm = 3.54 m

d $r = \pm\sqrt{0.497} = \pm 0.705$ to 3 d.p.
However, the association is negative because the slope of the regression line is negative, so $r = –0.705$.

e The coefficient of determination:
$r^2 = 0.497$.
Thus, since 0.497 × 100% = 49.7%, we can say that 49.7% of the variation in *success rate* is explained by the variation in distance of the golfer from the hole.

6 A and C: Solutions/explanations are part of the the answers.

7

a Appropriate: the scatterplot shows that there is a clear linear association.

b Coefficient of determination =
$r^2 = (0.967)^2 = 0.9351 \ldots$

c In general terms, the coefficient of determination (r^2) gives the proportion (or percentage) of the variation in a response variable that can be explained by the explanatory variable.
Since 0.9351… × 100% ≈ 93.5%, we can say that 93.5% of the variation in a person’s *pay rate* is explained by the variation in their years of work *experience.*

d Replacing *x* and *y* by *experience* and *pay rate* respectively in the equation
$y = 8.56 + 0.289 \times x$
gives
pay rate = 8.56 + 0.289 × *experience*

e The *y*-intercept is by definition the *pay rate* when *experience* = 0, and thus is the pay for an employee who has just begun work.

f the slope gives the average change in the RV for each one unit change in the EV or, in the context of this question: On average, the *pay rate* increases $0.29 per hour for each additional year of experience.

g **i** *pay rate* = 8.56 + 0.289 × *experience*
= 8.56 + 0.289 × 8 = 8.56 + 2.312
= $10.87

ii residual = actual – predicted
= 11.20 – 10.87 = $0.33

h The absence of a clear pattern in the residual plot supports the assumption of linearity, and this is what is observed in the residual plot.

8

a $r = \pm\sqrt{0.370} = \pm 0.608$ to 3 d.p.
However, the association is negative because the slope of the regression line is negative, so $r = –0.608$.

b In general terms, the coefficient of determination (r^2) gives the proportion (or percentage) of the variation in a response variable that can be explained by the explanatory variable.
Since 0.370 × 100% = 37.0%, we can say that 37.0% of the variation in the *hearing test scores* can be explained by the variation in *age*.

c Replacing *x* and *y* by *age* and hearing test *score* into the equation:
$y = 4.9 – 0.043 \times x$
gives:
score = 4.9 – 0.043 × *age*

d The slope of the line is – 0.043.
Interpretation: The slope gives the average change in the RV for each on-unit change in the EV or, in the context of this question:
On average, hearing test scores decrease by 0.043 for each for each year increase in age.

e **i** *hearing test score* =4.9 – 0.043 × *Age*
= 4.9 – 0.043 × 20
= 4.04
ii residual = actual – predicted
= 2.0 – 4.04 = –2.04

f **i** The plot point at 35 years is a vertical distance of 0.3 above the regression line, so the residual is 0.3.
ii The plot point at 55 years is a vertical distance of 0.4 below the regression line, so the residual is –0.4.

g The absence of a clear pattern in the residual plot supports the assumption of linearity which is observed in the residual plot.

9 **A: negative** – From the scatterplot, we can see that the association is negative since the points in the scatterplot drift downwards towards the right of the plot.
B: *drug dose* – This is the explanatory variable.
C: –0.9492 As read from the regression results screen.
D: 55.9 As read from the regression results screen.
E: –9.3 As read from the regression results screen.
F: decreases The slope of the regression line is negative.
H: 9.3 The slope of the regression line is –9.306
I: 55.9 The *y*-intercept is 55.89
J: 90.1– Since $r^2 = 0.901$
K: *response time* and ***drug dose*** Drug dosage determines response time since drug dosage is the explanatory variable.
L: clear pattern – The residual graph can be seen to show a parabolic–like pattern.

10 Use the report in question 9 as a model.

Solutions to Exercise 4D

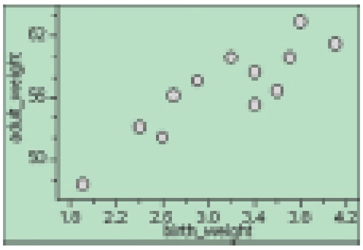
1 & 2 Enter the data into your calculator and follow the instruction on page 153 (TI) and 154 (CASIO).

3

a The aim is to predict an adult's weight (*adult weight*) from their *birth rate*, which means that
RV: *adult weight*
EV: *birth rate*.

To provide the answers necessary to answer questions parts **b** to **i**, enter the data into your calculator and follow the instruction on page 153 (TI) and 154 (CASIO).

b



c **i** The points in the scatterplot tend to drift upwards to the right of the plot following a roughly a linear pattern with a relatively small amount of scatter.
This is evidence of a strong positive, linear association. There are no apparent outliers.
ii Comparing the scatter with the standard plots on page 11 suggest an *r* of around 0.9

d Noting that the explanatory variable is *birth weight* and using the regression results generated by your calculator, the following the regression equation can be written down:
adult weight = 38.4 + 5.86 × *birth weight* (3 sig. figs.)

Using the regression results generated by you calculator you will find that the coefficient of determination:
$r^2 = 0.7653$
So $r = \sqrt{7653} = 0.875$

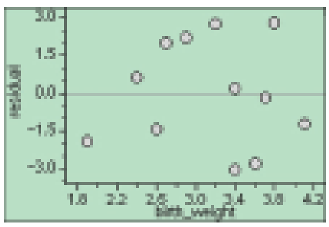
e Since 0.7653… × 100% ≈ 76.5%, we can say, using the standard interpretation, that 76.5% of the variation in *adult weight* is can be explained by the variation in birth weight.

f The slope of the regression line is 5.86 … ≈ 5.9. Using the standard interpretation, this tells us that, on average, adult weight increases by 5.9 kg for each additional kg of birth weight.

g **i** *adult weight* = 38.4 + 5.86 × *birth weight*
= 38.4 + 5.86 × 3.0
= 56.0 kg (to 1 d.p).
ii *adult weight* = 38.4 + 5.86 × *birth weight*
= 38.4 + 5.9 × 2.5 = 38.4 + 14.8
= 53.1 kg (to 1 d.p).
iii *adult weight* = 38.4 + 5.86 × *birth weight*
= 38.4 + 5.86 × 3.9
= 61. 3 kg (to 1 d.p.)

h This contention is supported by the data, since the data suggests that 76.5% of the variation in a person's adult weight explained by the variation in their birth weight. Only 23.5% is explained by other factors

i The absence of a clear pattern in a residual plot supports the assumption of linearity. This is observed in the residual plot below.



Chapter Review: Extended-response questions

1 The answers to these questions all relate to the interpretation and use of the regression equation:
hours of sunshine
$= 2850 - 6.88 \times$ *days of rain*

a The EV is ***days of rain*.**

b The slope is **–6.88** and the intercept is **2850**.

c Substitute *days of rain* = 120 in to the equation and evaluate.
hours of sunshine
$= 2850 - 6.88 \times 120 \approx$ **2024**

d The slope of the regression line is –6.88 so we can say that the hours of sunshine per year will **decrease** by **6.88** hours for each additional day of rain.

e $r^2 = 0.484$
$r = \pm\sqrt{0.4838}$
$= \pm 0.696$ (to 3 sig. figs.)
Because the slope of the line is negative r is negative, so
$\boldsymbol{r \approx -0.696}$

f $r^2 = 0.484$ or 48.4%
From this it follows that:
48.4% of the variation in sunshine hours can be explained by the variation in days of rain.

g **i** For 142 days of rain:
hours of sunshine =
$2850 - 6.88 \times 142 \approx$ **1873**

ii The residual value for this city
= data value – predicted value
= 1390 – 1873 = –480
The residual value for this city is **–483** hours.

h Making predictions within the range of data used to determine the regression equation is called **interpolation**.

2

a In this situation, the RV is *cost*.

b Enter the data into your calculator and follow instructions on page 136 (TI) or page 137 (CASIO) and use the values of the statistics generated to arrive at the equation:
cost = 81.5 +2.10 × *number of meals*

c **i** *cost* = 81.50 + 2.10 × 48
= \$182.30
As we are making a prediction within the range of data, we are interpolating.

ii *cost* = 81.50 + 2.10 × 21
= \$125.60
As we are making a prediction outside the range of data, we are extrapolating.

d **i** \$81.50: The intercept of the regression line gives the predicted *cost* when the *number of meals* = 0, this represents the fixed costs of preparing meals.

ii \$2.10: The slope of the regression line predicts the average cost of preparing each meal after the fixed costs have been accounted for.

e The coefficient of determination
$= r^2 = 0.978^2 \approx 0.956$ or 95.6%
This indicates that 95.6% of the variation in the *cost* of preparing meals is explained by the variation in the *number of meals* produced.

3 The answers to these questions all relate to the interpretation and use of the regression equation:

female income
= 13 000 + 0.35 × *male income*

a From the equation it can be seen that the EV is *male income*.

b From the equation it can be seen, that for each dollar increase in male salaries, females salaries increased by \$0.35.
Thus for each \$100 increases in male salaries, female salaries increased by 1000 × \$0.35 =\$350

c **i** *female income*
= 13 000 + 0.35 × 15 000
= \$18 250

ii Unreliable because we are making a prediction well outside the data.

4

a From the information given in the question, height is to be predicted from femur length. Thus *femur length* is the EV and *height* is the RV.

b Use the following rules to determine the slope and intercept of the least squares line.

$$b = r\frac{s_y}{s_x} \quad a = \bar{y} - b\bar{x}$$

$$b = r\frac{s_y}{s_x}$$

$$= 0.9939\left(\frac{10.086}{1.873}\right) = 5.352\ldots$$

$$a = \bar{y} - b\bar{x}$$

$$= 166.092 - 5.352 \times 24.246 = 36.32\ldots$$

Thus, correct to 3 sig. figs., the equation of the least squares line is:
height = 36.3 +5.35 × *femur length*

c Given the slope = 5.35, it can be deduced that, on average, *height* increases by 5.35 cm for each cm increases in *femur length*.

d Given $r = 0.9939$,
$r^2 = 0.9939^2 \approx 0.988$ or 98.8%
From this it can be deduced that 98.8% of the variation in *height* can be explained by the variation in *femur length*.

5

a From the information given in the question, height is to be predicted from age. Thus *age* is the EV and *height* is the RV.

b Enter the data into your calculator and follow instructions on page 136 (TI) or page 137 (CASIO) and use the values of the statistics generated to arrive at the equation:
height = 76.64 + 6.366 × *age*

c *height* = 76.64 + 6.366×1 ≈ 83 cm
As we are making a prediction outside the range of the data it is an extrapolation.

d Given the slope ≈ 6.4, it can be deduced that, on average, *height* increases by 6.4 cm for each extra year in *age*.

e Given $r = 0.9973$,
$r^2 = 0.9973^2 \approx 0.995$ or 99.5%
From this it can be deduced that 99.5% of the variation in *height* can be explained by the variation in *age*.

f **i** When the age is 10,
height = 76.64 + 6.366 × 10
= 140.3 cm (to1 d.p.)
The height of a 10-year-old child is predicted to be 140.3 cm.

ii residual = actual – predicted
= 139.6 – 140.3 = –0.7 cm

g **i** use your calculator to generate the residual plot shown
ii There is a clear pattern in the residual plot which does not support the assumption of linearity.

Chapter Review: Multiple–choice questions

1 To use a straight line to model an association work, it is assumed that the variables being modelled are linearly related. ⇒ **C**

2 The constant value = –1.2 = *y*-intercept; the gradient value = 0.52 = the slope.
⇒ **D**

3 $r = \pm\sqrt{0.25} = \pm 0.5$
However, since the slope of the regression line is negative, *r* must be negative. Thus, *r* = –0.5. ⇒ **A**

4 $y = 8 - 9x = 8 - 9 \times 5$
$= 8 - 45 = -37$ ⇒ **B**

5 Enter the data into your calculator and follow instructions on page 136 (TI) or page 137 (CASIO).
⇒ **B**

6 Enter the data into your calculator and follow instructions on page 136 (TI) or page 137 (CASIO).
⇒ **D**

7 Substituting the given values into the equation, we get:
$$b = \frac{0.733 \times 3.391}{1.871}$$
$= 1.33$ ⇒ **C**

8 residual = actual – predicted
–5.4 = actual value – 78.6
actual value = 78.6 – 5.4
= 73.2 ⇒ **A**

9 The *y*-intercept, the constant term in the equation, is approximately 9. The slope $\equiv \frac{0-9}{10} = -0.9$ so that the equation is y ≡ 9 –0.9 *x* which is closest to option A: y = 8.7– 0.9*x*.
⇒ **A**

10 The intercept cannot be determined directly from the graph because the *y*-axis in this graph is located at *x* = 20 not *x* = 0, so proceed as follows.
The line passes through the points (20,2) and (25,6).so that the slope
$$\frac{6-2}{25-20} = \frac{4}{5} = 0.8$$
Our equation is now $y = a + 0.8x$.
Substituting in the point (20,2), we get:
$2 = a + 0.8 \times 20$
$0 = a + 16$
$a = -16$
Of the available responses, this is closest to $y = -14 + 0.8x$. ⇒ **A**

11 All of the statements except for D are true. Reading from the equation of the least squares line, the intercept is –96, not 96, so option D is false. ⇒ **D**

12 The slope the regression line is 0.95 . This tells us that, on average, *weight* increases by 0.95 kg for reach 1 cm increase in *height*.
⇒ **E**

13 If *r* = 0.79, then $r^2 = 0.6241$.
Since 0.6241 × 100% = 62%, we can say that 62% of the variation in *weight* is explained by the variation in *height*.
⇒ **A**

14 *weight* = –96 + 0.95 × height
= –96 + 0.95 × 179
= –96 + 170.05 = 74
residual = actual– predicted
= 82 – 74 = 8 kg.
⇒ **C**

15 $r = \pm\sqrt{0.5} = \pm 0.7$
The slope of the graph is negative, so *r* = –0.7 ⇒ **A**

Chapter 5 – Data transformation

Solutions to Exercise 5A

1

a Matching the shape of the scatterplot with the shapes shown on the circle of transformations, we see that the following transformations have the potential to linearise the plot:

$\log(x)$, $\dfrac{1}{x}$, $\log(y)$ and $\dfrac{1}{y}$

b The circle of transformations only applies to plots with a consistently increasing or decreasing trend.
Since the plot has a decreasing trend followed by an increasing a trend, the circle of transformations does not apply.

c Matching the shape of the scatterplot with the shapes shown on the circle of transformations, we see that the following transformations have the potential to linearise the plot:

$\log(y)$, $\dfrac{1}{y}$ and x^2

d Matching the shape of the scatterplot with the shapes shown on the circle of transformations, we see that the following transformations have the potential to linearise the plot:
y^2 and x^2.

Solutions to Exercise 5B

1

a $y = 7 + 8x^2$
When $x = 1.25$,
$y = 7 + 8 \times 1.25^2 = 19.5$

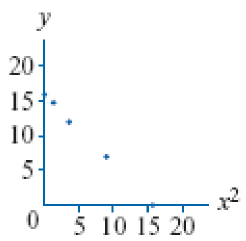
b $y = 7 + 3x^2$
When $x = 1.25$,
$y = 7 + 3 \times 1.25^2 = 11.7$ to 1 d.p.

c $y = 24.56 - 0.457x^2$
When $x = 1.23$,
$y = 24.56 - 0.457 \times 1.23^2$
$= 23.8$ to 1 d.p.

d $y = -4.75 + 5.95x^2$
When $= 4.7$,
$y = -4.75 + 5.95 \times 4.7^2$
$= 126.7$ to 1 d.p.

2

a Follow the calculator instructions on pages 196 (TI) and 170 (CASIO), enter the raw data into your calculator, apply an x^2 transformation and generate a scatterplot of y vs x^2 to show that the x^2 transformation has linearised the data.



b Fit a least squares line to the transformed to obtain the equation:

$y = 16 - x^2$

c Substitute $x = -2$ into the equation and evaluate.

$y = 16 - x^2$

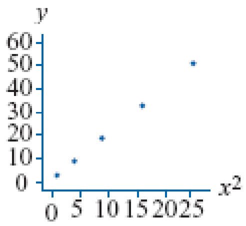
when $x = -2$:

$y = 16 - (-2)^2$

$y = 12$

3

a Follow the calculator instructions on pages 196 (TI) and 170 (CASIO), enter the raw data into your calculator, apply an x^2 transformation and generate a scatterplot of y vs x^2 to show that the x^2 transformation has linearised the data.



b Fit a least squares line to the transformed to obtain the equation:
$y = 1 + 2x^2$

c Substitute $x = 6$ into the equation and evaluate.

$y = 1 + 2x^2$

when $x = 6$:

$y = 1 + 2(6)^2$

$y = 73$

4

a $y^2 = 16 + 4x$
When $x = 1.57$,
$y^2 = 16 + 4 \times 1.57$
or
$y = \pm\sqrt{(16 + 4 \times 1.57)}$
$y = \pm 4.7$ to 1 d.p.

b $y^2 = 1.7 - 3.4x$
When $x = 0.03$,
$y^2 = 1.7 - 3.4 \times 0.03$
or
$y = \pm\sqrt{(1.7 - 3.4 \times 0.030)}$
$= \pm 1.3$ to 1 d.p.

c $y^2 = 16 + 2x$
When $x = 10$,
$y = \pm\sqrt{(16 + 2 \times 10)} = \pm 6$
However, the question specifies that we only want the positive solution ($y > 0$), so:
$y = 6$

d $y^2 = 58 + 2x$
When $x = 3$,
$y = \pm\sqrt{(58 + 2 \times 3)} = \pm 8$
However, the question specifies that we only want the negative solution ($y < 0$), so:
$y = -8$

5

a Follow the calculator instructions on pages 169 (TI) and 170 (CASIO), enter the raw data into your calculator but, this time, and apply a y^2 transformation and generate a scatterplot of y^2 vs x to show that the y^2 transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:
$y^2 = 1.5 + 3.1x$ to 2 sig. figs.

c Substitute $x = 9$ into the equation and evaluate.
$y^2 = 1.5 + 3.1 \times 9$
or
$y = \pm\sqrt{29.4} = \pm 5.42 \ldots$
so, because y can only be positive in this situation (see the original scatterplot)
$y = 5.4$ to 1 d.p.

6

Follow the calculator instructions on pages 169 (TI) and 170 (CASIO), enter the raw data into your calculator but, this time, and apply a squared transformation to the variable *diameter* and generate a scatterplot of *number* vs *diameter*^{2} to show that the *diameter*^{2} transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:
$number = 4 \times diameter^2$ to 1 sig. fig.

b Substitute *diameter* = 1.3 into the equation and evaluate.
$number = 4 \times 1.3^2 = 6.76$
$= 7$ to the nearest person

7

Follow the calculator instructions on pages 169 (TI) and 170 (CASIO), enter the raw data into your calculator but, this time, and apply a squared transformation to the variable *time* and generate a scatterplot of *time*^{2} vs *amount* to show that the *time*^{2} transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:
$time^2 = 18 - 9.3 \times amount$ to 2 sig. figs.

b Substitute $x = 0.4$ into the equation and evaluate.
$time^2 = 18 - 9.3 \times 0.4 = 14.28$
or
$time = 3.8$ to 1 d.p.
(*time* must be positive)

Solutions to Exercise 5C

1

a $y = 5.5 + 3.1 \log 2.3$
$= 6.6$ to 1 d.p.

b $y = 0.34 + 5.2 \log 1.4$
$= 1.1$ to 1 d.p.

c $y = -8.5 + 4.12 \log 20$

$= -3.1$ to 1 d.p.

d $y = 196.1 - 23.2 \log 303$
$= 138.5$ to 1 d.p.

2

a Follow the calculator instructions on pages 176 (TI) and 177 (CASIO), enter the raw data into your calculator, apply an log x transformation and generate a scatterplot of y vs log x to show that the log x transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:
$y = 1 + 3 \log x$

c Substitute $x = 100$ into the equation and evaluate.
$y = 1 + 3\log x$
when $x = 100$
$y = 1 + 3\log 100$
$y = 7$

3

a Follow the calculator instructions on pages 176 (TI) and 177 (CASIO), enter the raw data into your calculator, apply an log x transformation and generate a scatterplot of y vs log x to show that the log x transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:
$y = 20 - \log x$

c Substitute $x = 1000$ into the equation and evaluate.
$y = 20 - 5 \log x$
when $x = 1000$
$y = 20 - 5 \log 1000$
$y = 5$

4

a $\log y = 2$
$\Rightarrow y = 10^2 = 100$

b $\log y = 2.34$
$\Rightarrow y = 10^{2.34} = 218.8$ to 1 d.p.

c $\log y = 3.5 + 2x$
When $x = 1.25$,
$\log y = 3.5 + 2 \times 1.25 = 6$
$\Rightarrow y = 10^6 = 1\,000\,000$

d $\log y = -0.5 + 0.024x$
When $x = 17.3$,
$\log y = -0.5 + 0.024 \times 17.3 = -0.0848\ldots$
$\Rightarrow y \approx 10^{-0.0848} = 0.8$ to 1 d.p.

5

a Follow the calculator instructions on pages 176 (TI) and 177 (CASIO), enter the raw data into your calculator but, this time, and apply a log y transformation and generate a scatterplot of log y vs x to show that the log y transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:
$\log y = 1 + 2x$

c Substitute $x = 0.6$ into the equation and evaluate.
$\log y = 1 + 2x$
when $x = 0.6$
$\log y = 2.2$
or
$y = 10^{2.2} = 158.5$ to 1 d.p.

6

Follow the calculator instructions on pages 176 (TI) and 177 (CASIO), enter the raw data into your calculator, apply an log transformation to the variable *time* and generate a scatterplot of *level* vs log *time* to show that the log transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation: *level* = 1.8 + 2.6 log (*time*) (2 sig. figs.)

b Substitute *time* = 2.5 into the equation and evaluate.

level = 1.8 + 2.6 × log 2.5
= 2.8 to 1 d.p.

7

Follow the calculator instructions on pages 176 (TI) and 177 (CASIO), enter the raw data into your calculator but, this time, and apply a log transformation to the variable *number* and generate a scatterplot of log *number* vs *month* to show that the log *number* transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation: log (*number*) = 1.314 + 0.08301 × *month* (4 sig. figs.)

b Substitute *month* =10 into the equation and evaluate.

log (*number*) = 1.314 + 0.08301 × 10
or
log (*number*) = 2.1441…
or
number = $10^{2.1441}$ = 139.34… = 139 to the nearest whole number

Solutions to Exercise 5D

1

a $y = 6 + \frac{22}{x}$

When $x = 3$,

$y = 6 + \frac{22}{3} = 13.3$ to 1 d.p.

b $y = 4.9 - \frac{2.3}{x}$

When $x = 1.1$,

$y = 4.9 - \frac{2.3}{1.1} = 2.8$ to 1 d.p.

c $y = 8.97 - \frac{7.95}{x}$

When $x = 1.97$,

$y = 8.97 - \frac{7.95}{1.97}$

$= 4.9$ to 1 d.p.

d $y = 102.6 + \frac{223.5}{x}$

When $x = 1.08$,

$y = 102.6 + \frac{223.5}{1.08} = 309.5$ to 1 d.p.

2

a Following the calculator instructions on pages 183 (TI) and 184 (CASIO), enter the raw data into your calculator, apply a reciprocal transformation and generate a scatterplot of y vs $1/x$ to show that the reciprocal ($1/x$) transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:

$y = \frac{120}{x}$

c Substitute $x = 5$ into the equation and evaluate.

$y = \frac{120}{x}$

when $x = 5$:

$y = 24$

3

a $\frac{1}{y} = 3x$

When x = 2,

$\frac{1}{y} = 3 \times 2 = 6$

so

$y = \frac{1}{6} = 0.17$ to 2 d.p.

b $\frac{1}{y} = 6 + 2x$

When x = 4,

$\frac{1}{y} = 6 + 2 \times 4 = 14$

so

$y = \frac{1}{14} = 0.07$ to 2 d.p.

c $\frac{1}{y} = -4.5 + 2.4x$

When x = 4.5,

$\frac{1}{y} = -4.5 + 2.4 \times 4.5 = 6.3$

so

$y = \frac{1}{6.3} = 0.16$ to 2 d.p.

d $\frac{1}{y} = 14.7 + 0.23x$

When x = 4.5,

$\frac{1}{y} = 14.7 + 0.23 \times 4.5 = 15.735$

so

$y = \frac{1}{15.735} = 0.62$ to 2 d.p.

4

a Follow the calculator instructions on pages 183 (TI) and 184 (CASIO), enter the raw data into your calculator but, this time, and apply a *reciprocal* transformation and generate a scatterplot of 1/*y* vs *x* to show that the reciprocal (1/ *y*) transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:

$$\frac{1}{y} = x$$

c Substitute *x* = 0.25 into the equation and evaluate.

$$\frac{1}{y} = x$$

when $x = 2.5$

$y = 4$

5

Follow the calculator instructions on pages 196 (TI) and 170 (CASIO), enter the raw data into your calculator, apply a reciprocal transformation to the variable fuel *consumption* and generate a scatterplot of *horsepower* vs 1/ *consumption* to show that the log transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:

$$horsepower = 22.1 + \frac{690}{consumption}$$

(3 sig. figs.)

b Substitute *time* = 2.5 into the equation and evaluate.

$$horsepower = 22.1 + \frac{690}{9} = 99$$

to the nearest whole number.

6

Follow the calculator instructions on pages 196 (TI) and 170 (CASIO), enter the raw data into your calculator but, this time, and apply a reciprocal transformation to the variable *errors* and generate a scatterplot of 1/*errors* vs *times* to show that the reciprocal transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:

$$\frac{1}{errors} = 0.050 \times times$$ (2 sig. figs.)

b Substitute *times* = 6 into the equation and evaluate.

$$\frac{1}{errors} = 0.050 \times 6$$

or

errors = 3.33… = 3

to the nearest whole number

Chapter Review: Extended-response questions

1

a Following the instructions on pages 176 (TI) and 177 (CASIO), enter the data into your calculator, perform a log x transformation and fit a least squares line to the transformed data to obtain the equation:

average age = 2.39 + 5.89 log (*income*)

b *average age*

= 2.39 + 5.89 log (*income*)

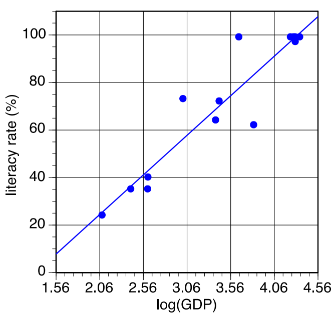
= 2.39 + 5.89 × log(20 000)

= 2.39 + 5.89 × log(20 000)

= 27.72 … = 27.7 to 1 d.p.

2

a Following the instructions on pages 176 (TI) and 177 (CASIO), enter the data into your calculator, perform a log x transformation and use the transformed data to construct a scatterplot with log(*GDP*) as the explanatory variable as shown below.



b Using the transformed data, fit a least squares line to the transformed data to obtain the equation:

literacy rate = –44.2 + 33.3log(*GDP*)

c Reading from the equation:

intercept = –44.2 slope = 33.3

d When *GDP* = $10 000,

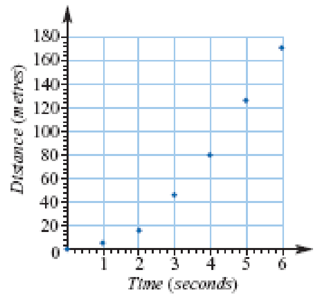
literacy rate

= –44.2 + 33.3 × log(10 000)

= 89%

3

a Following the instructions on pages 169 (TI) and 170 (CASIO), enter the data into your calculator, and construct a scatterplot with *distance* as the response variable as shown below.



Inspection of the scatterplot shows that the association beteween *distance* and *time* is strong, postive and non- linear.

b Perform an x^2 transformation to form a new variable *time*^{2} and complete the table.

c Using the transformed data, fit a least squares line to the transformed data to obtain the equation:

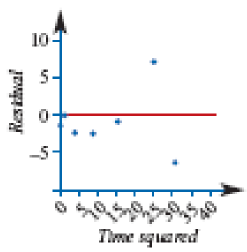
distance = 0.45 + 4.8 × *time*^{2}

c When *time* = 7 seconds,

distance = 0.45 + 4.8 ×7^2

= 235.65 ≈ 236 metres

f Follow the calculator instructions on page 169 (TI) or 170(CASIO) to construct a residual plot with *time*^{2} as the EV as shown below.



There is no clear pattern in the residual plot; it is essentially a random array of points, indicating that the assumption of linearity is justified.

Chapter Review: Multiple–choice questions

1 A square transformation has an expanding or stretching-out effect upon the high values in a data distribution. ⇒ **A**

2 A log transformation has a compressing effect upon the high values in a data distribution. ⇒ **D**

3 The scatterplot can be linearised by stretching out the high end of the *y* axis scale. A y^2 transformation can be used for this purpose. ⇒ **B**.

4 The scatterplot can be linearised by compressing the high end of the *y* axis scale (log *y* or 1/*y*) or compressing the high end of the *x* axis scale (log *x* or 1/*x*). ⇒ **A**

5 The scatterplot can be linearised by stretching out the high end of the *y* axis scale. A y^2 transformation can be used for this purpose.⇒ **B**

6 Following the instructions on pages 176 (TI) and 177 (CASIO), enter the data into your calculator, perform a log *x* transformation and fit a least squares line to the transformed data to obtain the equation, *y* = 7.04 + 3.86 log *x*.⇒ **E**

7 $width^2 = 1.8 + 0.8 \times area$
When *area* = 120,
$width^2 = 1.8 + 0.8 \times 120 = 97.8$
so
$width = \sqrt{97.8} = 9.9$ to 1 d.p. ⇒ **B**

8 For an x^2 transformation, we can write the general equation $y = a + b \times x^2$.
y is the response variable = *weight*,
a is the *y*-intercept = 10,
b is the slope of the regression line = 5,
x is the explanatory variable = *width*.
Substituting all these values in gives us $weight = 10 + 5 \times width^2$.⇒ **D**

9 $mark = 20 + 40 \times \log(hours)$
$= 20 + 40 \times \log(20)$
$= 72.04 \approx 72$ ⇒ **D**

10 $1/y = 0.14 + 0.045x$
$= 0.14 + 0.045 \times 6$
$= 0.41$
or $y = 1/0.41 = 2.439\ldots \approx 2.4$
⇒ **D**

Chapter 6 – Investigating and modelling time series

Solutions to Exercise 6A

1 **Graph A:** There is a general tendency for the points to decrease in value as we go from left to right indicating a **decreasing trend.** The remaining variation appears to be no more than **irregular (random) fluctuations**.

Graph B: There is no general tendency for the points to decrease or increase in value as we go from left to right so there is **no** trend. The sole source of variation appears to be **irregular (random) fluctuations**.

Graph C: There is a general tendency for the points to increase in value as we go from left to right indicating a **increasing trend.** One data point does not appear to follow this general trend indicating the presence of a possible **outlier**. The remaining variation appears to be no more than **irregular (random) fluctuations**.

2 **Graph A:** There is a general tendency for the points to decrease in value as we go from left to right indicating a **decreasing trend.** There also appears to be **seasonality** as there are regularly spaced troughs and peaks with intervals of around one 1 year. The remaining variation appears to be no more than **irregular (random) fluctuations**.

Graph B: There is no apparent trend. However, there appears to be **cyclical variation**, since the troughs and the peaks are regularly spaced but at intervals of greater than 1 year. The remaining variation appears to be no more than **irregular (random) fluctuations**.

Graph C: There is a general tendency for the points to increase in value as we go from left to right indicating a **increasing trend.** There also appears to be **seasonality** as the troughs and peaks are roughly equally spaced with intervals of less than 1 year. The remaining variation appears to be no more than **irregular (random) fluctuations**.

3 **Graph A:** The plot shows a period of **decreasing trend** followed by a period of **increasing trend** which suggests there has been a **structural change**. The remaining variation appears to be no more than **irregular (random) fluctuations**.

Graph B: The values in the plot do not change in any way over time. Other than that, the plot has no defining features.

Graph C: There is no apparent trend but there appears to be **seasonality** as the troughs and peaks are roughly equally spaced with intervals of around one 1 year. The remaining variation appears to be no more than **irregular (random) fluctuations**.

4 There is no clear increasing or decreasing trend, but instead the graph oscillates. The fact that peaks appear to occur every March with troughs every June, means the graph exhibits seasonal variation. The remaining variation appears to be no more than irregular (random) fluctuations.

5

a The plot for the men (red line) shows a general decrease in smoking rates for the period 1945 until 1992 which indicates the presence of a decreasing trend.
The plot for females (blue line) shows a general but slow increase in smoking rates for the period 1945 until 1975. This was followed by a general decrease in female smoking rates from 1975 to 1992 that that paralleled the decrease for men.

b Since the two graphs have become closer and closer together between 1945 and 1992, we can say that the difference in smoking rates has decreased over that period.

6 The plot shows the presence of significant structural change as different sections of the graph show abrupt changes in trends. From 1920 to 1930 there was a strongly increasing trend, followed a by a period of very weak or no trend from 1930 to 1940. From 1940 to 1945 there was strongly decreasing trend which changed abruptly to a strongly increasing trend until 1960. This was followed by a strongly decreasing trend until 1985. There is no cyclic or seasonal variation evident.

Considering the graph, along with a knowledge of history, we can say: 'The number of whales caught increased rapidly between 1920 and 1930 but levelled off during the 1930s, which was the time of the Depression. In the period 1940–1945, which was the time of the Second World War, there was a rapid decrease in the number of whales caught and numbers fell to below the 1920 catch. In the period 1945–1965 the numbers increased again but then fell again until 1985 when numbers were back to around the 1920 level. (This was a time when people were becoming more environmentally aware and realised that whales were becoming an endangered species.)'

7 Follow the calculator instructions given on page 200 (TI) or page 201 (CASIO).

8 Follow the calculator instructions given on page 200 (TI) or page 201 (CASIO).

9 Follow the calculator instructions given on page 200 (TI) or page 201 (CASIO).

10

a Follow the calculator instructions given on page 200 (TI) or page 201 (CASIO).

b Noting the flat trend of the red line but the increasing trend of the blue line in the plot given in the answer, we can say: 'The number of male school teachers has remained relatively constant over the years 1993–2001, whereas the number of female school teachers has increased over this time.'

Solutions to Exercise 6B

1 (33.5+ 21.6 +18.1)/3 =24.4 **C**

2 (21.6 +18.1+16.2 +17.9+26.4)/5
= 20.04 **A**

3 (28.9 + 33.5 + 21.6 +18.1
+16.2 +17.9 + 26.4)/7 = 23.22… **B**

4 M_1 = (28.9 + 33.5)/2 =31.2
M_2 = (33.5 + 21.6)/2 =27.55

Two-mean centred = (31.2 + 27.55)/2
= 29.375 **D**

5 M_1 = (21.6 + 18.1 + 16.2 + 17.9))/4
= 18.45
M_2 = (18.1 + 16.2 + 17.9 + 26.4)/4
= 19.65

Four-mean centred = (18.45 + 19.65)/2
= 19.05 **A**

6

a This is the 3-mean of the *y* values for t = 3, 4, 5, which are 5, 3, 1 respectively. Since $\frac{5+3+1}{3}$ = 3, the 3-mean for t = 4 is 3.

b This is the 3-mean of the *y* values for t = 5, 6, 7, which are 1, 0, 2 respectively. Since $\frac{1+0+2}{3}$ = 1, the 3-mean for t = 6 is 1.

c This is the 3-mean of the *y* values for t = 1, 2, 3, which are 5, 2, 5 respectively. Since $\frac{5+2+5}{3}$ = 4, the 3-mean for t = 2 is 4.

d This is the 5-mean of the *y* values for t = 1, 2, 3, 4, 5, which are 5, 2, 5, 3, 1 respectively. Since $\frac{5+2+5+3+1}{5}$ = 3.2, the 5-mean for t = 3 is 3.2.

e This is the 5-mean of the *y* values for t = 5, 6, 7, 8, 9 which are 1, 0, 2, 3, 0 respectively. Since $\frac{1+0+2+3+0}{5}$ = 1.2, the 5-mean for t = 7 is 1.2.

f This is the 5-mean of the *y* values for t = 2, 3, 4, 5, 6 which are 2, 5, 3, 1, 0 respectively. Since $\frac{2+5+3+1+0}{5}$ = 2.2, the 5-mean for t = 4 is 2.2.

g The 2-mean of t = 2, 3 is $\frac{2+5}{2}$ = 3.5.
The 2-mean of t = 3, 4 is $\frac{5+3}{2}$ = 4.
The centred 2-mean of the two 2-means is $\frac{3.5+4}{2} = 3.75$.

h The 2-mean of t = 2, 3 is $\frac{2+3}{2}$ = 2.5.
The 2-mean of t = 3, 0 is $\frac{3+0}{2} = 1.5$.
The centred 2-mean of the two 2-means is $\frac{2.5+1.5}{2} = 2$.

i The 4-mean of t = 1, 2, 3, 4 is $\frac{5+2+5+3}{4} = 3.75$.
The 4-mean of t = 2, 3, 4, 5 is $\frac{2+5+3+1}{4} = 2.75$
The centred 2-mean of the two 4-means is $\frac{3.75+2.75}{2} = 3.25$.

j The 4-mean of t = 4, 5, 6, 7 is $\frac{3+1+0+2}{4} = 1.5$.
The 4-mean of t = 5, 6, 7, 8 is $\frac{1+0+2+3}{4} = 1.5$.
The centred 2-mean of the two 4-means is $\frac{1.5+1.5}{2} = 1.5$.

Note: For questions 7 to 11, copies of the blank tables are obtainable via the skill sheet icon in the Interactive Textbook.

7 For the 3-mean of a particular *t*-value, add the *y*-value of that *t*-value, the *y*-value of the *t*-value one to the left and the *y*-value of the *t*-value one to the right. Divide the sum by 3 to get the 3-mean for that *t*-value.

For the 5-mean of a particular *t*-value, add the *y*-value of that *t*-value, the two *y*-values of the two *t*-values immediately to the left and the two *y*-values of the two *t*-values immediately to the right. Divide the sum by 5 to get the 5-mean for that *t*-value.
See answer for the completed table.

8

a Follow the calculator instructions given on page 200 (TI) or page 201 (CASIO).

b For the 3-mean of a particular *Day*-value, add the *Temperature*-value of that *Day*-value, the *Temperature*-value of the *Day*-value one to the left and the *Temperature*-value of the *Day*-value one to the right. Divide the sum by 3 to get the 3-mean for that *Day*-value.

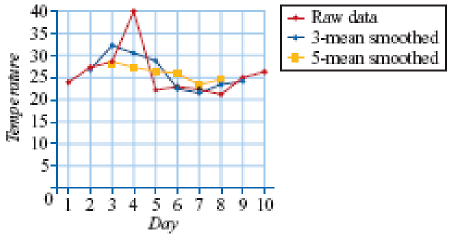
For the 5-mean of a particular *Day*-value, add the *Temperature*-value of that *Day*-value, the two *Temperature*-values of the two *Day*-values immediately to the left and the two *Temperature*-values of the two *Day*-values immediately to the right. Divide the sum by 5 to get the 5-mean for that *Day*-value.
See answer for the completed table.

c Follow the calculator instructions given on page 200 (TI) or page 201 (CASIO). Once constructed, you are expected to comment on the effect of smoothing on the time series plots.

The following comments are based on the plot below.

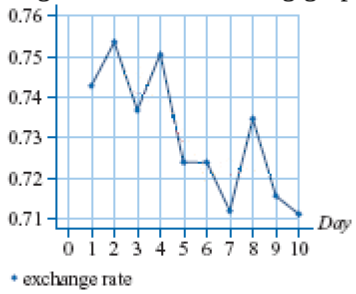
Noting that the smoothed graphs show a much smaller temperature variation from day to day, we can say:

'The smoothed plot show that the 'average' maximum temperature changes relatively slowly over the 10-day period (the 5-day average varies by only 5 degrees) when compared to the daily maximum, which can vary quite widely (for example, nearly 20 degrees between the 4th and 5th day) over the same period of time.'



9

a Plotting the values on a graph with *Day* as the *x*-axis and *Exchange rate* as the *y*-axis gives us the following graph:



b For the 3-mean of a particular *Day*-value, add the *Exchange rate*-value of that *Day*-value, the *Exchange rate*-value of the *Day*-value one to the left and the *Exchange rate*-value of the *Day*-value one to the right. Divide the sum by 3 to get the 3-mean for that *Day*-value.

For the 5-mean of a particular *Day*-value, add the *Exchange rate*-value of that *Day*-value, the two *Exchange rate*-values of the two *Day*-values immediately to the left and the two *Exchange rate*-values of the two *Day*-values immediately to the right. Divide the sum by 5 to get the 5-mean for that *Day*-value.

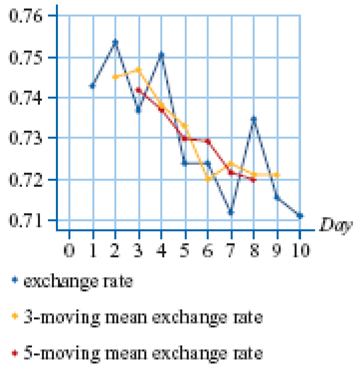
See answer for the completed table

c Follow the calculator instructions given on page 200 (TI) or page 201 (CASIO). Once constructed, you are expected to comment on the effect of smoothing on the time series plots.

The following comments are based on the plot below.

Noting that the downwards trend is more obvious in the smoothed graphs than the original plot, we can say:

‘The exchange rate is dropping steadily over the 10-day period. This is most obvious from the smoothed plots, particularly the 5-moving mean plot.’



10 Calculating the 2-moving means by taking the mean of each pair of two adjacent months and calculating the centred means by taking the 2-moving means of the original 2-moving means gives us the following table:

Month	***Number of births***	**2-moving mean**	**Centred means**
January	10		–
		$\frac{10+12}{2}=11$	
February	12		$\frac{11+9}{2}=10$
		$\frac{12+6}{2}=9$	
March	6		$\frac{9+5.5}{2}=7.25$
		$\frac{6+5}{2}=5.5$	
April	5		$\frac{5.5+13.5}{2}=9.5$
		$\frac{5+22}{2}=13.5$	
May	22		$\frac{13.5+20}{2}=16.75$
		$\frac{22+18}{2}=20$	
June	18		$\frac{20+15.5}{2}=17.75$
		$\frac{18+13}{2}=15.5$	
July	13		$\frac{15.5+10}{2}=12.75$
		$\frac{13+7}{2}=10$	
August	7		$\frac{10+8}{2}=9$
		$\frac{7+9}{2}=8$	
September	9		$\frac{8+9.5}{2}=8.75$
		$\frac{9+10}{2}=9.5$	
October	10		$\frac{9.5+9}{2}=9.25$
		$\frac{10+8}{2}=9$	
November	8		$\frac{9+11.5}{2}=10.25$
		$\frac{8+15}{2}=11.5$	
December	15		–

11 Calculating the 4-moving means by taking the mean of set of four adjacent months and calculating the centred means by taking the 2-moving means of the original 4-moving means gives us the following table:

Month	***Internet usage***	**4-moving mean**	**Centred means**
April	21	–	
May	40	–	
		$\frac{21+40+52+42}{4}=38.75$	
June	52		$\frac{38.75+48}{2}=43.375$
		$\frac{40+52+42+58}{4}=48$	
July	42		$\frac{48+57.75}{2}=52.875$
		$\frac{52+42+58+79}{4}=57.75$	
August	58		$\frac{57.75+65}{2}=61.375$
		$\frac{42+58+79+81}{4}=65$	
September	79		$\frac{65+68}{2}=66.5$
		$\frac{58+79+81+54}{4}=68$	
October	81		$\frac{68+66}{2}=67$
		$\frac{79+81+54+50}{4}=66$	
November	54	–	
December	50	–	

Solutions to Exercise 6C

Note: Copies of all of the graphs in the exercise set can be accessed through the skill sheet icon in the Interactive textbook.

1

a – d

To locate the median of a set of points graphically:

1. locate the median horizontally and mark with a vertical dotted line
2. locate the median vertically and mark with a horizontal line.
3. the median of the points is then given by the point of intersection of these two lines.

See page 212 for a step-by-step worked example.

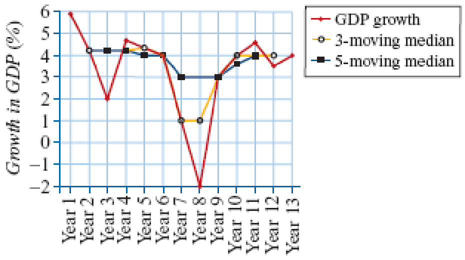
2 – 4

See Example 5 for a step-by-step worked example of three median smoothing.
See Example 6 for a step-by-step worked example of five median smoothing.

5

a See Example 5 for a step-by-step worked example of three median smoothing.
See Example 6 for a step-by-step worked example of five median smoothing.

b The effect of median smoothing is to smooth out local irregular fluctuations in the plot to hopefully reveal any underlying feature. In this case smoothing revealed a dip in GDP growth between years 6 and 10.



Solutions to Exercise 6D

1 We know that the average seasonal index must be 1 by definition. Since there are 12 seasons (months) in this case, the total sum of seasonal indices must be 12.
Since 12 – 1.2 – 1.3 – 1.0 – 1.0 – 0.9 – 0.8 – 0.7 – 0.9 – 1.0 – 1.1 = 1.0, the seasonal index for December is 1.0. **C**

2 deseasonalised value

$$= \frac{\text{actual value}}{\text{seasonal index}}$$

deseasonalised sales

$$= \frac{8.6}{1.1} = 7.818...$$ **B**

3 deseasonalised value

$$= \frac{\text{actual value}}{\text{seasonal index}}$$

deseasonalised sales

$$= \frac{6.0}{0.9} = 6.666...$$ **E**

4 actual value = deseasonalised value × seasonal index
actual sales = 5.6 × 0.7 = 3.92 **B**

5 actual value = deseasonalised value × seasonal index
actual sales = 6.9 × 1.0 = 6.9 **C**

6 Seasonal index of the average month is 1.0.
The seasonal index for February is 1.3 which is 30% greater than the average month **D**

7 Seasonal index of the average month is 1.0.
The seasonal index for September is 0.9 which is 10% less than the average month **B**

8 deseasonalised value

$$= \frac{\text{actual value}}{\text{seasonal index}}$$

The seasonal index for January is 1.2, so

$$\text{deseasonalised value} = \frac{\text{actual value}}{1.2}$$

$$= 0.83 \times \text{actual value}$$

This means that to de-seasonalise the sales for January, you would have to **reduce** the actual sales figures by around 17%. **C**

9 We know that the average seasonal index must be 1 by definition. Since there are 4 quarters in this case, the total sum of seasonal indices must be 4.
Since 4 – 0.8 – 0.7 – 1.3 = 1.2, the seasonal index for Q_4 must be 1.2. **D**

10 deseasonalised value

$$= \frac{\text{actual value}}{\text{seasonal index}}$$

deseasonalised sales

$$= \frac{1060}{0.7} = 1514.285...$$ **D**

11 deseasonalised value

$$= \frac{\text{actual value}}{\text{seasonal index}}$$

deseasonalised sales

$$= \frac{1868}{1.3} = 1436.923...$$ **A**

12 actual value = deseasonalised value × seasonal index
actual sales = 1256 × 0.8 = 1004.8 **B**

13

	Summer	**Autumn**	**Winter**	**Spring**
Year 1	56	125	126	96
Deseasonalised	$\frac{56}{0.5}=112$	$\frac{125}{1}=125$	$\frac{126}{1.3}=96.9...$	$\frac{96}{1.2}=80$
Seasonal index	0.5	1	1.3	4.0 – 0.5 – 1.0 – 1.3 = 1.2

14
a, **c**

	Quarter 1	**Quarter 2**	**Quarter 3**	**Quarter 4**
Year 1	198	145	86	168
Deseasonalised	$\frac{198}{1.3}=152.30...$	$\frac{145}{1.02}=142.15...$	$\frac{86}{0.58}=148.27...$	$\frac{168}{1.10}=152.72...$
Seasonal index	1.30	4– 1.30 – 0.58 –1.10 = 1.02	0.58	1.10

b The seasonal index of the average month is 1.0
The seasonal index for Quarter 1 is 1.3 which is 30% greater than the average quarter
The number of waiters employed in Quarter 1 is 30% greater than the average quarter.

15 Seasonal average = $\frac{60 + 56 + 75 + 78}{4} = 67.25$

Q1	**Q2**	**Q3**	**Q4**
$\frac{60}{67.25}=0.89$	$\frac{56}{67.25}=0.83$	$\frac{75}{67.25}=1.12$	$\frac{78}{67.25}=1.16$

16 Seasonal average = $\frac{12+13+14+17+18+15+9+10+8+11+15+20}{12}=13.5$. The indices in the table below are calculated by dividing the *Sales* figure for each month by 13.5 and rounding to 2 decimal places.

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Sales	12	13	14	17	18	15	9	10	8	11	15	20
Index	0.89	0.96	1.04	1.26	1.33	1.11	0.67	0.74	0.59	0.81	1.11	1.48

17 Seasonal average = $\frac{22+19+25+23+20+18+20+15+14+11+23+30}{12}=20$.
The indices in the table below are calculated by dividing the *Sales* figure for each month by 20 and rounding to two decimal places.

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Sales	22	19	25	23	20	18	20	15	14	11	23	30
Index	1.10	0.95	1.25	1.15	1.00	0.90	1.00	0.75	0.70	0.55	1.15	1.50

Solutions to Exercise 6E

1

a From the plot we can see a steady increasing trend in the data plots, meaning the number of Australian university students has increased steadily over the time period.

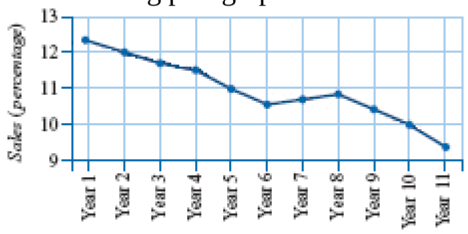
b Enter the data into your calculator and follow the instructions on page 136 (TI) or 137 (CASIO) to fit a straight line to data with *year* as the EV.
number (in thousands)
$= 520 + 10.1 \times year$ (to 3 sig. figs).

slope = 10.1: this means that, on average, the number of university students in Australia increased by 10 100 each year.

c If 1992 is Year 1, then 2020 is Year 29.
number (000s) $= 520 + 10.1 \times year$
$= 520 + 10.1 \times 29$
$= 812\,900 = 813\,000$ (to 3 sig figs)

2

a Plotting the points on a graph gives us the following plot graph:

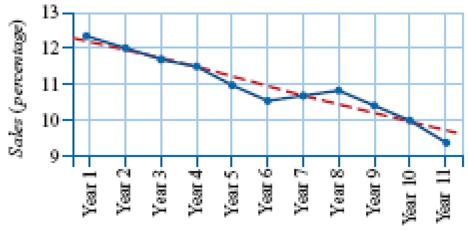


You can do this on your calculator if you wish.

b From the plot, a generally decreasing trend in the percentage of sales made in department stores is evident.

c Enter the data into your calculator and follow the instructions on page 136 (TI) or 137 (CASIO) to fit a straight line to data with *year* as the EV.
sales (%) $= 12.5 - 0.258 \times year$
(to 3 sig figs)
slope = –0.258: this means that, on average, the percentage of sales made in department stores decreased by approximately 0.3% per year.

d Drawing in the trend line onto the plot graph gives us the following graph:



You can do this on your calculator if you wish.

e Substituting *year* = 15 into the equation we get:
sales $= 12.5 - 0.258 \times 15 \approx 8.6\%$

3

a Enter the data into your calculator and follow the instructions on page 136 (TI) or 137 (CASIO) to fit a straight line to data with *year* as the EV.
age $= 27.2 + 0.199 \times year$ (to 3 sig figs)

slope = 0.199: this means that, on average, the age of mothers having their first child in Australia increased by around 0.2 years each year.

b If 1989 is Year 1, then 2018 will be year 30. Substituting *year* = 30 into the equation we get:
age $= 27.20 + 0.199 \times 30 = 33.2$
(to 3 sig figs)

This prediction is likely to be unreliable as we are extrapolating well beyond the data.

4

a Enter the deseasonalised data into your calculator and follow the instructions on page 136 (TI) or 137 (CASIO) to fit a straight line to data with *quarter number* as the EV.
deseasonalised number
$= 50.9 + 1.59 \times quarter\ number$
(to 3 sig. figs).

b The fourth quarter year 4 is quarter number 16. Substituting *quarter number* = 16 into the equation we get:
deseasonalised number
= 50.9 + 1.59 × 16 = 76.34
reseasonalising this value using the seasonal index for quarter 4 we get
actual number = 76.34 × 1.18 = 90.08 or 90 to the nearest whole number.

5

a Replacing 1992 = 1, 1993 = 2, etc., enter the data into your calculator and follow the instructions on page 136 (TI) or 137 (CASIO) to fit a straight line to data with *year* number as the EV and *duration* as the RV.
duration = 10.4 + 0.135 × *year*
(to 3 sig figs)

slope = 0.135: this means that, on average, the median duration between marriage and divorce increased by increased by 0.135 years each year

b If 1992 is Year 1, then 2020 will be year 29. Substituting *year* = 29 into the equation we get:
duration = 10.4 + 0.135 × 29
= 14.315 = 14.3 (to 3 sig figs)

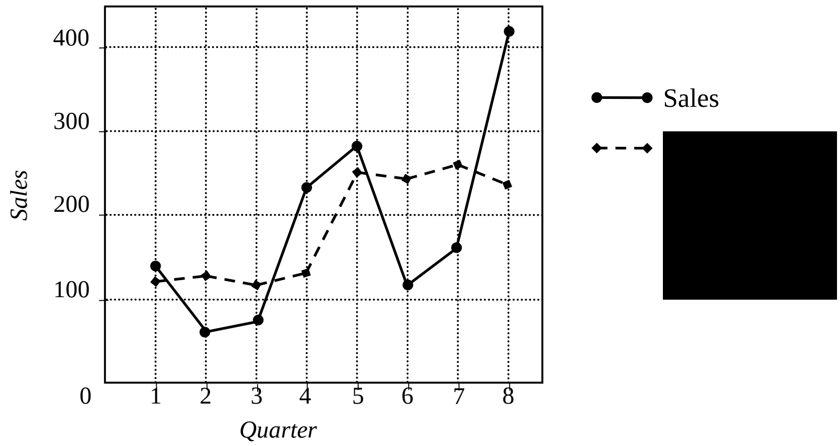
This prediction is likely to be unreliable as we are extrapolating well beyond the data.

6

a Deseasonalised sales and rounded to the nearest whole number.

	Quarter 1	*Quarter 2*	*Quarter 3*	*Quarter 4*
Year 1	$\frac{138}{1.13}=122$	$\frac{60}{0.47}=128$	$\frac{73}{0.62}=`118$	$\frac{230}{1.77}=130$
Year 2	$\frac{283}{1.13}=250$	$\frac{115}{0.47}=245$	$\frac{163}{0.62}=263$	$\frac{417}{1.77}=236$

b Enter the data into your calculator and follow the instructions on page 200 (TI) or 201(CASIO) and construct a time series plot of the actual data and the deseasonalised data as shown below.



Both plots show a general increase in sales over time indicating a positive trend.
The deseasonalised plot suggests that the trend is roughly linear.

c Follow the instructions on page 136 (TI) or 137 (CASIO) to fit a straight line to the deaseasonalised data with *quarter* as the EV.

$$\textit{deseasonalised sales} = 80.8 + 23.5 \times \textit{quarter} \qquad \text{(to 3 sig. figs.)}$$

d For the first quarter of Year 4, *quarter* = 13

$$\therefore \quad \textit{deseasonalised sales} = 80.8 + 23.5 \times 13$$
$$= 386 \quad \text{to the nearest whole number}$$
$$\textit{sales} = \textit{deseasonalised sales} \times \textit{seasonal index}$$
$$= 386 \times 1.13$$
$$= 436 \text{ to the nearest whole number}$$

The forecast for sales in the first quarter of Year 4 is 436.
Note: These answers are slightly different from the answers given in the text as they have been generated using rounded values for the deseasonalised data.

Chapter Review: Extended-response questions

1

a Enter the and construct a time series plot following the calculator instructions given on page 200 (TI) or page 201 (CASIO). The plot should match the plot given in the answer.

b The plot shows that there has been a general increase in *GDP* with time. This indicates a positive trend.

c Follow the instructions on page 136 (TI) or 137 (CASIO) to fit a straight line to data with *time* as the EV to arrive at the equation:

GDP = 20 400 + 507 × *time*

(to 3 sig. figs.)

d error of prediction (= residual) = actual *GDP* – predicted *GDP*

In 2007 (*time* = 27), the actual *GDP* = $34 900.

To find the predicted *GDP* substitute *time* = 27 in the regression equation.

predicted *GDP* = 20 400 + 507 × 27
= $34 089

Thus,

error of prediction = 34 900 – 34 089
= $811

2

a Reading from the graph, the month of the highest rainfall is November.

b Smooth the plot graphically using three median smoothing following the process illustrated in Example 5. See the answer for the solution.

c See the answer for the solution.

3

a see answer for solution

b Enter the and construct a time series plot following the calculator instructions given on page 200 (TI) or page 201 (CASIO). The plot should match the red line in the answer graph.

c Insert the smoothed value for 1992 (11%) into and plot on the same graph the the 3-mean smoothed data.

Note there are no values for 1987 and 1997 when 3-mean smoothing is used. This is indicated by shaded cells in the table.

The plot should match the blue line in the answer graph.

Chapter Review: Multiple–choice questions

1 Since the fluctuations in the graph occur within a time period of a year or less, i.e. the troughs are every half a year, the time series plot is best described as seasonal. **C**

2 $\frac{2.4+3.4+4.4}{3}=3.366...$ **A**

3 $\frac{2.4+3.4+4.4+2.7+5.1}{5}=3.6$ **B**

4 $M_1=\frac{2.7+5.1}{2}=3.9$

$M_2=\frac{5.1+3.7}{2}=4.4$

Two-mean centred = $\frac{3.9+4.4}{2}=4.15$. **E**

5 $M_1=\frac{3.4+4.4+2.7+5.1}{4}=3.9$.

$M_2=\frac{4.4+2.7+5.1+3.7}{4}=3.975$

Four-mean centred =

$\frac{3.9+3.975}{2}=3.9375$ **C**

6 Graphically locate the median of the five points centred on March and read value off the vertical axis (M≈63) . See Example 6 for the process. **B**

7 Graphically locate the median of the five points centred on September and read value off the vertical axis (M≈69) See Example 6 for the process. **E**

8 We know that the average seasonal index must be 1 by definition. Since there are 12 seasons (months) in this case, the total sum of seasonal indices must be 12. Since 12 – 1.0 – 1.1 – 0.9 – 1.0 – 1.0 – 1.2 – 1.1 – 1.1 – 1.1 – 1.0 – 0.7 = 0.8, the seasonal index for Feb is 0.8. **C**

9 To deseasonalise a data value, we divide the data value by its seasonal index: $\frac{423}{1.8}$ = 240 **B**

10 actual value = deseasonalised value × seasonal index

actual sales Winter = 380 × 0.3 = 114 **A**

11 The seasonal index of the average month is 1.0

The seasonal index for spring is 1.5 which is 50% greater than the average month. **E**

12 The seasonal index for Autumn is 0.4, so; deseasonlised value

$=\frac{\text{actual value}}{0.4}$

$=2.50\times\text{actual value}$

This means that to deasonalise the sales for Autumn, the sales figure will need to be 250% of the original – i.e. a 150% increase. **E**

13 Seasonal average

$=\frac{1048+677+593+998}{4}=829$.

Seasonal index for Autumn is the value for Autumn divided by the seasonal average = $\frac{677}{829}$ = 0.82. **D**

14 If 1995 is Year 1, then 2004 will be Year 10.

Substituting *Year* = 10 into *age* = 27.1 + 0.236 × *year*,

we get *Age* = 27.1 + 0.236 × 10

= 29.46 **C**

15 slope = 0.236: this means that, on average, the age of marriage for males increased by 3 months each year Note: Since there are 12 months in a year.

0.236 years = 12 × 0.236 = 2.8

≈ 3 months. **A**

16 Enter the deseasonalised data into your calculator and follow the instructions on page 136 (TI) or 137 (CASIO) to fit a straight line to data with *day* as the EV. In this equation, Sunday = day 1, Monday day 2, etc.

price = 84.3 + 1.25 × *day*. **A**

17 (88.3 + 85.4 + 86.7 + 88.5 + 90.1 + 91.7 + 94.6)/7 = 89.32 … **D**

18 actual value = deseasonalised value × seasonal index

actual price = 90.1 × 0.7= 63.07 **A**

Chapter 7 – Revision: Data analysis

Solutions to Exercise 7A

1 The variables *sex* and *transport* are both nominal variables. The data values they generate that can be used to group individuals into separate categories (eg female, male) but there is no underlying order to these categories. ⇒ **A**

2 The numerical variables are *time* and *number*, so there are two numerical variables. ⇒ **C**

3 Reading the table we see that there are two males (*M*) in the sample, but only one uses public transport (3). ⇒ **B**

4 Since the first quartile starts to the right of the value 30, we can safely say that less than a quarter of all observations are less than 30. ⇒ **B**

5 Since the histogram tails off slowly to the left from the peak it is negatively skewed. ⇒ **B**

6 Negatively skewed distribution has a box plot with the median to the right-hand side of the box, a short right-hand whisker and a long left-hand whisker.
⇒ **D**

7 The histogram tells us that 12 people scored less than 30.
We know that 63 students sat the test.
Since $\frac{12}{63} \times 100\% = 19.0\%$, we conclude that 19% of students failed the test. ⇒ **B**

8 The histogram tells us that 4 students scored between 30 and 35, 7 students between 35 and 40, and 9 students between 40 and 45, so that 20 (= 4 + 7 + 9) students have scores between 30 and 45.
⇒ **E**

9 For grouped data, the mode (or modal interval) is the most commonly occurring group of values. In a histogram this group of values (50–55) is identified by the highest bar.
⇒ **E**

10 Since there are 63 values, the median value is the 32nd value.
Counting up from the left-hand side of the histogram, the 32nd value (the median) is in the 40–45 interval.
⇒ **D**

11 $IQR = Q_3 - Q_1 = 60 - 50 = 10$.
Lower limit for outliers (upper fence)
$= Q_1 - 1.5 \times IQR$
$= 50 - 1.5 \times 10 = 35$.
Upper limit for outliers (lower fence)
$= Q_3 + 1.5 \times IQR$
$= 60 + 1.5 \times 10 = 75$.
Thus, outliers are defined as values lower than 35 or higher than 75. ⇒ **A**

12 Reading from the histogram, around 36% (≈ 21% + 15%) of the ovens have temperatures between 179°C and 181°C.
36% of 300 = 0.36 × 300 = 108.
⇒ **D** .

13 Reading from the box plot, the length of the box (the *IQR*) is around 181.7 – 179 = 2.7. ⇒ **D**

14 From the 68-95-99.7% rule, the range is approximately 6 standard deviations (SD), so, estimating the range from the box plot we have 6 × SD ≈ 13, so SD ≈ 13/6 = 2.16… ≈ 2°C ⇒ **B**

15 From the 68-95-99.7% rule, 95% of ants have lengths between:
mean – 2 *SD* and *mean* + 2 *SD*
or
$4.8 - 2 \times 1.2 = 2.4$ and $4.8 + 2 \times 1.2 = 7.2$

⇒ **B**

16 $actual\ length = mean + z \times SD$
$= 4.8 + (-0.5) \times 4.2$

⇒ **C**

17 3.6 is one SD below the mean.
As a consequence of the 68-95-99.7% rule, 16% of ants have lengths less than 3.6 mm.

⇒ **C**

18 6 is one SD above the mean.
As a consequence of the 68-95-99.7% rule, 16% of ants have lengths more than 6, (100 –16)% = 84% of ants have lengths less than 6 mm.

⇒ **E**

19 3.6 is one SD below the mean.
7.2 is two SDs above the mean.
As a consequence of the 68-95-99.7% rule,

- 84% of ants have lengths more than 3.6 mm
- 2.5% of ants have lengths above 7.2 mm,

so 84–2.5 = 81.5% of ants have lengths between 3.6 and 7.2 mm.

⇒ **D**

20 2.4 is two SDs below the mean.
4.8 is the mean.
As a consequence of 68-95-99.7% rule, 95% of ants have lengths within two standard deviations of the mean so half of these, 47.5%, have lengths between 2.4 and 4.8 mm.
$47.5% of 1000 = 475 ants.

⇒ **C**

21 Standardised score (z) = $\frac{\text{score - mean}}{\text{SD}}$

$$= \frac{45 - 65}{10} = -2$$

⇒ **B**

22 $actual\ score = mean + z \times SD$
$= 65 + 1.2 \times 10 = 77$

⇒ **D**

23 81 for Biology is equivalent to a standardised score

$$z = \frac{(81 - 54)}{15} = 1.8$$

To obtain the same standardised score in Legal Studies, Sashi's actual score would need to be:
$actual\ score = mean + z \times SD$
$= 78 + 1.8 \times 5$
$= 87$

⇒ **D**

Solutions to Exercise 7B

1 For this data, an association between *location of internet use* and *age group* can be deduced if the percentage of '15–19 year olds' using the internet at any one of the three location differs significantly from the percentage of' 20–24 year olds' using the internet at the **same** location. ⇒ **D**

2 From the information given in the question it can be concluded that that,

- all of the box plots are approximately symmetric, so the shape of the distribution does not change: A is true.
- the median pay rates increase from 1980 to 2000: B is true.
- the IQR (as indicated by the box widths increase from 1980 to 2000): C is true.
- because *pay rate* can change with *year*, but not the other way around, *pay rate* is the RV:C is true.
- the median *pay rate* increases between 1980 and 1990 which is consistent with *pay rate* and *year* being associated. E in **not** true.

⇒ **E**

3 The points in the scatterplot drift down to the right and can be imaged to closely follow a straight line with minimal scatter around the line. Of the options given, $r = –0.9$ is best. ⇒ **E**

4 The points in the scatterplot drift down to the right and can be imaged to closely follow a straight line with minimal scatter around the line. The association is best described as strong, negative and linear. ⇒ **B**

5 The correlation coefficient, can be both positive and negative because associations can be both positive and negative. The coefficient of determination equals r^2 so it is positive. The remaining options are statistics that involve measuring the *amount of spread* of a data set in some way so must be positive. ⇒ **D**

6 Moving the outlier at (7, 25) to the point (7, 5) reduces the total scatter in the plot which would not change the direction of the association but would strengthen the correlation. Thus *r* value would remain negative and would be closer to –1. ⇒ **E**

7 Consider each of the statements in turn.

A. the variables *computer ownership* and *car ownership* measured in number /1000 people are both numerical variables: A is true

B. the correlation coefficient = r^2 = $0.92^2 \approx 0.85$ or 85%: B is true.

C. there is no clear causal relationship between *computer ownership* and *car ownership* so either could be the EV: C is true.

D. a correlation $r = 0.92$ indicates a positive association which is consistent with *computer ownership* increasing with *car ownership*: D is true

E. *computer ownership* decreasing with *car ownership* indicates a negative association: E is **not** true

⇒ **E**

8 All we are able to infer from the information given is that there is a positive correlation between reading scores and height, meaning taller students tend to read better and vice versa. ⇒ **D**

9 A back–to–back stem plot is useful for displaying associations between a numerical variable and a categorical variable with **two** categories. *Height* in centimetres versus *sex* (male, female) fulfils these requirements. ⇒ **D**

10 Since both variables are categorical, an appropriately percentaged table is the most appropriate choice. ⇒ **A**

11 Since a numerical variable is being plotted against a categorical variable with **more than two** categories, parallel box plots would be the most appropriate choice. ⇒ **D**

Solutions to Exercise 7C

1 $b = \frac{rs_y}{s_x} = \frac{0.675 \times 4.983}{2.567} = 1.3 \Rightarrow$ **C**

2 The equation of the least squares line is: *number* = 230 4.3 × *temperature*
From this equation we see that the slope is –4.3 ⇒ **A**

To provide the answers necessary to answer questions 3 & 4, enter the data into your calculator and follow the instruction on page 153 (TI) and 154 (CASIO) to fit a least squares line to the data and generate the value of r^2.

3 Coefficient of determination:
$r^2 = 0.89$ ⇒ **D**

4 Given:
number of errors
= 8.8 – 0.12 × *study time*,
When *study time* = 35,
number of errors = 8.8 – 0.12 × 35
= 4.6 ⇒**B**

5 Given: $r^2 = 0.8198$
$r = \pm\sqrt{r^2} = \pm\sqrt{0.8198} = \pm 0.905...$
As the **slope** of the least squares line is **negative**, r is negative, so $r \approx -0.91$
⇒ **A**

6 residual value = actual – predicted
actual value = 6
predicted value = 8.8 – 0.12 × 10
= 7.6
residual value = 6 – 7.6 = –1.6
⇒**B**

7 Option **E** is clearly **not** true, the *EV* is *study time* not *number of errors*.
All of the other options can be shown to be true. ⇒ **E**

8 From the given regression equation, the slope = –0.12.
This means that, on average, the number of errors made:
'decreases by 0.12 for each extra minute spent studying' ⇒ **B**

9 Since $r^2 = 0.8198$ or 81.98 % ≈ 82%, we can say that 82% of the variation in the *number of errors* made can be explained by the variation in *study time*. ⇒ **D**

10 From the graph, the *y*-intercept can be seen to be approximately 210. The slope of the graph is negative and the line drops around 11 cm of rainfall for every 1 degree increase in temperature.
Hence, *average rainfall* = 210 – 11 × *temperature range*. ⇒ **A**

11 From the given correlation coefficients,

- option A is **not** true: $(-0.357)^2 \approx$ 12.7%
- Option B is **not** true: the correlation between population density and distance from the centre of the city is negative.
- Option C is **not** true: the correlation coefficient is positive
- Option D is **not** true: the correlation coefficient is negative, so the slope of the regression line is negative
- Option E is **true**: a correlation coefficient of $r = -0.563$ indicates a stronger association than a correlation coefficient of $r = 0.357$

⇒ **E**

12 Following the calculator instructions on pages 196 (TI) and 170 (CASIO), enter the raw data into your calculator, apply a $\frac{1}{y}$ transformation to obtain the required equation:
$\frac{1}{y} = 0.068 + 0.16x$ ⇒ **A**

13 *population* = 58 170 + 43.17 × $year^2$
When *year* = 10,
population = 58 170 + 43.17 × 10^2
= 62 487 ⇒ **E**

14 $weight^2 = 52 + 0.78 \times area$

When $area = 8.8$,

$weight^2 = 52 + 0.78 \times 8.8$

$weight = \sqrt{58.86} \ldots \approx 7.7$

$\Rightarrow$ **C**

15 $\log number = 1.31 + 0.083 \times month$

When $month = 6$,

$\log number = 1.31 + 0.083 \times 6 = 1.808$

or $number = 10^{1.808} = 64.2 \ldots \approx 64$

$\Rightarrow$ **D**

Solutions to Exercise 7D

1 There is a clear upwards trend but no variation that can be said to be clearly cyclical or seasonal. $\Rightarrow$ **A**

2 This question is concerned with the **difference** in share price between the two companies. This distance is represented by the distance between the two time series lines which clearly **decreases** with time. $\Rightarrow$ **A**

3 $\frac{4+5+4}{3} = 4.33...$ $\Rightarrow$ **A**

4 $\frac{4+4+8+6+91}{5} = 6.2$ $\Rightarrow$ **B**

5 $\text{M1} = \frac{8+6}{2} = 7$

$\text{M2} = \frac{6+9}{2} = 7.5$

two mean centred = $\frac{7+7.5}{2} = 7.25$

$\Rightarrow$**E**

6 $\text{M1} = \frac{4+5+4+4}{4} = 4.25$

$\text{M2} = \frac{5+4+4+8}{4} = 5.25$

four mean centred =

$\frac{4.25+5.25}{2} = 4.75$ $\Rightarrow$**B**

7 $\frac{18}{0.6} = 30$ $\Rightarrow$**E**

8 seasonal average

$= \frac{(21+36+49+23)}{4}$

= 33.5

$\text{SI} = \frac{28}{33.5} = 0.835...$ $\Rightarrow$**B**

9 deseasonalised sales = $\frac{800}{0.8} = 1000$

$\Rightarrow$ **D**

10 SI = 4 –1.1 –0.9–0.8 = 1.2 $\Rightarrow$ **E**

11 actual sales = 91 564 × 1.45

= $132 767.8 $\Rightarrow$**E**

12 the seasonal index of an average quarter is 1.0 the seasonal index for quarter 3 is 0.85 which is 15% less than the average quarterly $\Rightarrow$**B**

13 deseasonalised value = actual value/seasonal index

The SI for summer is 0.8, so deseasonalised value = actual value/0.8

≈ 1.25 ×actual value

This means that deseasonalise the sales for summer, you would have to **increase** the actual sales figures by around 25% $\Rightarrow$**E**

14 $\text{M1} = \frac{2016000+3900000}{2} = 2\,958\,000$

$\text{M1} = \frac{3900000+4830000}{2} = 4\,365\,000$

two mean centred =

$\frac{295800+4365000}{2} = 3\,661\,500$ $\Rightarrow$**D**

15 Graphically smooth the five points centred on the three-median smoothed number of calls for month 9 to obtain the answer: 362

Following this procedure, the three-median smoothed number of calls for month 9 is found to be close to 362

See page 213 for a step-by-step worked example using graphical three-median smoothing. $\Rightarrow$ **B**

16 Graphically smooth the five points centred on the five-median smoothed number of calls for month 10 to obtain the answer: 375

See page 213 for a step-by-step worked example using graphical five-median smoothing. $\Rightarrow$ **D**

17 The line goes from (0,20) to (10,4), which is a *y*-intercept of 20 and a slope of $\frac{4-20}{10} = \; = -1.6$.

Hence, $y = 20 - 1.6t$. $\Rightarrow$ **A**

Chapter Review: 7E Extended-response questions

1 **a**

i Complete the table by counting separately, the number of girls who walked (10), sat or stood (9), and ran (9).

ii $\frac{9}{28}\times 100 = 32.1\%$ to 1 d.p.

b Nominal: categories cannot be used to order, only to identify (name) the type of activity a girl engages in

2 **a** See answer: the solution is integrated with the answer

b Ordinal: the categories can be used to order the girls by year level.

3

a The **mode** is the most frequently occurring value: 78; **range** = maximum –minimum = 79 –70 = 9

b To be an outlier, 70 must below the lower fence.
The lower fence is located at the point $Q_1 - 1.5 \times IQR$
Determine Q_1 and Q_3 by inspection:
$Q_1 = 75$ and $Q_3 = 78$
Determine the IQR:
$IQR = Q_3 - Q_1 = 78 - 75 = 3$
Determine the lower fence:
$75 - 1.5 \times 3 = 70.5$
Conclude that 70 is an outlier because 70 < 70.5

4 **a**

i 25.0 years, which can be read directly from the stem plot.

ii Median age = 28.2 which can be determined by finding the middle value in the stem plot.

b $IQR = Q_3 - Q_1$
$= 30.9 - 29.9 = 1$ year

c To be an outlier, 26.0 must below the lower fence.
The lower fence is located at the point $Q_1 - 1.5 \times IQR$
Determine the lower fence:
$29.9 - 1.5 \times 1 = 28.4$
Conclude that 26.0 is an outlier because 26.0 < 28.4

5

a 42.1 + 23.4 = 65.5%: read values from table (column 1, row 3)

b Yes: see answer for an explanation

c *year of marriage* because it is the variable used to make the prediction.

6

a The middle 50% lie between Q_1 and Q_3, i.e. 124 and 148 read from the box plot.

b Median arm span (along with other quantities) increases with year level

c Outliers will be lower than $Q_1 - 1.5 \times IQR$ or greater than $Q_1 - 1.5 \times IQR$.
From the box plot for year 10 girls, $Q_1 = 160$, $Q_3 = 170$
$IQR = 170 - 160 = 10$
Lower fence $= Q_1 - 1.5 \times IQR$
$= 160 - 1.5 \times 10$
$= 145$
Her real arm span is still below 145 cm so it is still an outlier.

7

a Use the equation to locate to widely spaced points on the line.

1. minimum temperature = 0, maximum temperature =13 + 0.67 × 0 = 13
2. minimum temperature = 20, maximum temperature =13 + 0.67 × 20 = 26.4

Draw a line on the graph passing through the two points (0, 13) and (0, 26.4)

b intercept = 13: this means that when the minimum temperature is 0°C, the predicted maximum temperature is 13°C

c given that *r* =0.630 and a roughly linear scatterplot, conclude that there is a moderate, positive, linear association.

d slope = 0.67: this means that, on average, the maximum temperature increases by 0.67°C for each 1°C increase in the minimum temperature.

e coefficient of determination = 0.630^2 = 0.3969 or around 40%: this means that 40% of the variation in the maximum temperature can be explained by the variation in the minimum temperature.

f residual = actual – predicted
= 12.2 – (13 + 0.67 × 11.1)
= –8 °C to the nearest degree

8 **a** Follow the calculator instructions on pages 196 (TI) and 170 (CASIO), enter the raw data into your calculator, apply an $\log_{10} x$ transformation and fit a least squares line to the transformed data; log (*area*) is the EV.
The resulting equation should be:
population = 7.7 + 7.7 $\log_{10}$ (*area*) correct to 1 d.p.

b *population* = 7.7 + 7.7 $\log_{10}$ (90)
= 22.747 … thousand people
= 23 000 to the nearest 1000 people

9 **a** 1964: read from the graph

b *mean surface temperature* = –12.36 + 0.013 × *year*

i When *year* = 2010
mean surface temperature = –12.36 + 0.013 × 2010
= 13.77 °C to 2 d.p.

ii residual = actual – predicted
predicted
= –12.36 + 0.013× 2000 =13.64
residual = 13.55 – 13.64
= –0.09 °C to the nearest degree

iii +0.013: the slope gives the average change in temperature per year

c There is a general increase in temperature with increasing time, indicating a positive trend.
The temperature in 1998 seems to very much higher than in the nearby years suggesting that it is an outlier.

Chapter 8 – Modelling growth and decay using recursion

Solutions to Exercise 8A

1

a 2
2 + 6 = 8
8 + 6 = 14
14 + 6 = 20
20 + 6 = 26

b 5
5 – 3 = 2
2 – 3 = –1
–1 – 3 = –4
–4 – 3 = –7

c 1
1 × 4 = 4
4 × 4 = 16
16 × 4 = 64
64 × 4 = 256

d 10
10 ÷ 2 = 5
5 ÷ 2 = 2.5
2.5 ÷ 2 = 1.25
1.25 ÷ 2 = 0.625

e 6
6 × 2 + 2 = 14
14 × 2 + 2 = 30
30 × 2 + 2 = 62
62 × 2 + 2 = 126

f 12
12 × 0.5 + 3 = 9
9 × 0.5 + 3 = 7.5
7.5 × 0.5 + 3 = 6.75
6.75 × 0.5 + 3 = 6.375

2

a

4	4
4+2	6
6+2	8
8+2	10
10+2	12

b

24	24
24–4	20
20–4	16
16–4	12
12–4	8

c

2	2
2·3	6
6·3	18
18·3	54
54·3	162

d

50	50
50/5	10
10/5	2
2/5	0.4
0.4/5	0.08

e

5	5
5·2+3	13
13·2+3	29
29·2+3	61
61·2+3	125

f

18	18
18·0.8+2	16.4
16·4.0.8+2	15.12
15·12.0.8+2	14.096
14·096.0.8+2	13.2768

Solutions to Exercise 8B

1

a $W_0 = 2$
$W_1 = 2 + 3 = 5$
$W_2 = 5 + 3 = 8$
$W_3 = 8 + 3 = 11$
$W_4 = 11 + 3 = 14$

b $D_0 = 50$
$D_1 = 50 - 5 = 45$
$D_2 = 45 - 5 = 40$
$D_3 = 40 - 5 = 35$
$D_4 = 35 - 5 = 30$

c $M_0 = 1$
$M_1 = 3 \times 1 = 3$
$M_2 = 3 \times 3 = 9$
$M_3 = 3 \times 9 = 27$
$M_4 = 3 \times 27 = 81$

d $L_0 = 3$
$L_1 = -2 \times 3 = -6$
$L_2 = -2 \times -6 = 12$
$L_3 = -2 \times 12 = -24$
$L_4 = -2 \times -24 = 48$

e $K_0 = 5$
$K_1 = 2 \times 3 - 1 = 9$
$K_2 = 2 \times 9 - 1 = 17$
$K_3 = 2 \times 17 - 1 = 33$
$K_4 = 2 \times 33 - 1 = 65$

f $F_0 = 2$
$F_1 = 2 \times 2 + 3 = 7$
$F_2 = 2 \times 7 + 3 = 17$
$F_3 = 2 \times 17 + 3 = 37$
$F_4 = 2 \times 37 + 3 = 77$

g $S_0 = -2$
$S_1 = 3 \times -2 + 5 = -1$
$S_2 = 3 \times -1 + 5 = 2$
$S_3 = 3 \times 2 + 5 = 11$
$S_4 = 3 \times 11 + 5 = 38$

h $V_0 = -10$
$V_1 = -3 \times -10 + 5 = 35$
$V_2 = -3 \times 35 + 5 = -100$
$V_3 = -3 \times -100 + 5 = 305$
$V_4 = -3 \times 305 + 5 = -910$

2

a

12	12
12·6−15	57
57·6−15	327
327·6−15	1947
1947·6−15	11667

b

20	20
20·3+25	85
85·3+25	280
280·3+25	865
865·3+25	2620

c

2	2
2·4+3	11
11·4+3	47
47·4+3	191
191·4+3	767

d

64	64
64·0.25−1	15
15·0.25−1	2.75
2.75·0.25−1	−0.3125
−0.3125·0.25−1	−1.078125

e

48000	48000
48000−3000	45000
45000−3000	42000
42000−3000	39000
39000−3000	36000

f

25000	25000
25000·0.9−550	21950
21950·0.9−550	19205
19205·0.9−550	16734.5
16734.5·0.9−550	14511.05

3

150	150
150·0.6−5	85
85·0.6−5	46
46·0.6−5	22.6
22·6.0.6−5	8.56
8.56·0.6−5	0.136
0.136·0.6−5	−4.9184

Count the number of positive terms in the sequence generated using the calculator. There are 6 positive numbers in this sequence.

4

30	30
30·0.8+2	26
26·0.8+2	22.8
22.8·0.8+2	20.24
20.24·0.8+2	18.192
18.192·0.8+2	16.5536
16.5536·0.8+2	15.24288

Although these numbers are getting smaller, they will always be greater than zero. Exploring on the calculator further, the values will get closer and closer to zero. None of them will be negative.

Solutions to Exercise 8C

1

a $V_0 = 2000$
$V_1 = 2000 + 76 = 2076$
$V_2 = 2076 + 76 = 2152$
$V_3 = 2152 + 76 = 2228$

The value of the investment after 1 year is \$2076. The value of the investment after 2 years is \$2152 and the value of the investment after 3 years is \$2228.

b Use the calculator to count the number of years until a value greater than \$3000 is reached.

2000	2000
2000+76	2076
2076+76	2152
2152+76	2228
2228+76	2304
2304+76	2380
2380+76	2456

2456+76	2532
2532+76	2608
2608+76	2684
2684+76	2760
2760+76	2836
2836+76	2912
2912+76	2988

2988+76	3064

It takes 14 years for the value of the investment to be more than \$3000.

c Interest = 6% of \$1500

$$= \frac{6}{100} \times \$1500$$

$= \$90$

$V_0 = 1500,\ V_{n+1} = V_n + 90$

2

a $V_0 = 7000$
$V_1 = 7000 + 518 = 7518$
$V_2 = 7518 + 518 = 8036$
$V_3 = 8036 + 518 = 8554$

The value of the loan after 1 year is \$7518. The value of the loan after 2 years is \$8036 and the value of the loan after 3 years is \$8554.

b Use the calculator to count the number of years until a value greater than \$10 000 is reached.

7000	7000
7000+518	7518
7518+518	8036
8036+518	8554
8554+518	9072
9072+518	9590
9590+518	10108

It takes 6 years for the value of the loan to be more than \$10 000.

c Interest = 8.2% of \$12 000

$$= \frac{8.2}{100} \times \$12\ 000$$

$= \$984$

$V_0 = 12\ 000,\ V_{n+1} = V_n + 984$

3

a **i** The principal of the investment is V_0. The principal is \$15 000.

ii The interest is the amount added in the recurrence relation. The interest is \$525.

iii $\text{interest rate} = \dfrac{525}{15\,000} \times 100\%$

$= 3.5\%$ per annum

b Double the principal is \$30 000.

15000	15000
15000+525	15525
15525+525	16050
16050+525	16575
16575+525	17100
17100+525	17625
17625+525	18150

⋮	
29175+525	29700
29700+525	30225

It takes 29 years for the investment amount to double.

4

a $V_0 = 2550$
$V_1 = 2550 - 400 = 2100$
$V_2 = 2100 - 400 = 1700$
$V_3 = 1700 - 400 = 1300$

The value of the computer after 1 year is \$2100. The value of the computer after 2 years is \$1700 and the value of the computer after 3 years is \$1300.

b Use the calculator to count the number of years until a value less than \$1000 is reached.

2500	2500
2500-400	2100
2100-400	1700
1700-400	1300
1300-400	900

It takes 4 years for the value of the computer to be less than \$1000.

c $V_0 = 1800,\ V_{n+1} = V_n - 350$

5

a $V_0 = 23\ 000$
$V_1 = 23\ 000 - 805 = 22\ 195$
$V_2 = 22\ 195 - 805 = 21\ 390$
$V_3 = 21\ 390 - 805 = 20\ 585$

The value of the car after 1 year is \$222 195. The value of the car after 2 years is \$21 390 and the value of the car after 3 years is \$20 585.

b Use the calculator to count the number of years until a value less than \$10 000 is reached.

23000	23000
23000-805	22195
22195-805	21390
21390-805	20585
20585-805	19780
19780-805	18975
18975-805	18170

18170-805	17365
17365-805	16560
16560-805	15755
1755-805	14950
14950-805	14145
14145-805	13340
13340-805	12535

12535-805	11730
11730-805	10925
10925-805	10120
10120-805	9315

It takes 17 years for the value of the computer to be less than \$10 000.

c $V_0 = 37\ 000,\ V_{n+1} = V_n - 700$

d Interest $= 4.5\%$ of \$12 000

$= \dfrac{4.5}{100} \times \$12\ 000$

$= \$540$

$V_0 = 12\ 000,\ V_{n+1} = V_n - 540$

6

a **i** The purchase price of the television is V_0. The purchase price of the television is $1500.

ii The depreciation is the amount subtracted in the recurrence relation.
The depreciation is $102.

iii $\text{interest rate} = \frac{102}{1500} \times 100\%$
$= 6.8\%$ per annum

b Use the calculator to find the value of the television after 8 years.

1500	1500
1500-102	1398
1398-102	1296
1296-102	1194
1194-102	1092
1092-102	990
990-102	888

888-102	786
786-102	684

The selling price of the television is $684.

7

a

450	450
450-0.05	449.95
449.94-0.05	449.90
⋮	

⋮	
449.15-0.05	449.1
449.1-0.05	449.05
449.05-0.05	449

b $V_0 = 300,\ V_{n+1} = V_n - 0.08$

8

a $V_0 = 48\ 000$
$V_1 = 48\ 000 - 200 = 47\ 800$
$V_2 = 47\ 800 - 200 = 47\ 600$
$V_3 = 47\ 600 - 200 = 47\ 400$

The value of the van after 1000 km is $47 800. The value of the van after 2000 km is $47 600 and the value of the computer after 3000 km is $47 400.

b 15 000 km is 15 lots of 1000 km.
Press ENTER or EXE 15 times.

48000	48000
48000-200	47800
47800-200	47600
47600-200	47400
47400-200	47200
47200-200	47000
47000-200	46800

46800-200	46600
46600-200	46400
46400-200	46200
46200-200	46000
4600-200	45800
45800-200	45600
45600-200	45400

45400-200	45200
45200-200	45000

c Continue using the calculator until the value is $43 000

⋮	
43400-200	43200
43200-200	43000

This requires another 10 presses of ENTER or EXE for a total of 25. This is 25 lots of 1000 km or a total of 25 000 km

Solutions to Exercise 8D

1

a $A_n = 4 + 2n$
$A_{20} = 4 + 2\times20$
$A_{20} = 44$

b $A_n = 10 - 3n$
$A_{20} = 10 - 3\times20$
$A_{20} = -50$

c $A_n = 5 + 8n$
$A_{20} = 5 + 8\times20$
$A_{20} = 165$

d $A_n = 300 - 18n$
$A_{20} = 300 - 18\times20$
$A_{20} = -60$

2

a The principal is $8000.

b The interest charged each year is $512.

c **i** $V_{12} = 8000 + 512\times12$
$V_{12} = 14\,144$

ii $V_n = 16\,000$
$16\,000 = 8000 + 512n$
$512n = 8000$
$n = \frac{8000}{512}$
$n = 16$ years (rounded up)

d $V_{15} = 8000 + 512\times15$
$V_{15} = 15\,680$

3

a The principal is $2000.

b The interest charged each year is $70.

c **i** $V_6 = 2000 + 70\times6$
$V_6 = 2420$

ii $V_n = 4000$
$4000 = 2000 + 70n$
$70n = 2000$
$n = \frac{2000}{70}$
$n = 29$ years (rounded up)

d $V_{10} = 2000 + 70\times10$
$V_{10} = 2700$

4

a Annual interest = 5.4% of $5000
$= \frac{5.4}{100} \times \$5000$
= $270

b **i** $V_0 = 5000,\ V_{n+1} = V_n + 270$

ii $V_n = 5000 + 270n$

c $V_9 = 5000 + 270\times9$
$V_9 = 7430$

After 9 years, Webster will owe the bank $7430.

5

a Annual interest = 7.2% of $12 000
$= \frac{7.2}{100} \times \$12\,000$
= $864

b **i** $V_0 = 12\,000,\ V_{n+1} = V_n + 864$

ii $V_n = 12\,000 + 864n$

c $V_9 = 12\,000 + 864\times9$
$V_9 = 19\,776$

After 9 years, Anthony will owe the bank $19 776.

6

a The purchase price of the sewing machine is $1700.

b The sewing machine in depreciated by $212.50 each year.

c $V_4 = 1700 - 212.5\times4$
$V_4 = 850$

After 4 years, the sewing machine has value $850.

d $V_8 = 1700 - 212.5\times8$
$V_8 = 0$

After 8 years, the sewing machine will have value $0.

7

a The purchase price of the harvester is \$65 000.

b The harvester is depreciated by \$3250 each year.

c % depreciation $= \dfrac{3250}{65\,000} \times 100\%$

$= 5\%$ per annum.

d $V_7 = 65\,000 - 3250 \times 7$

$V_7 = 42\,250$

After 7 years, the value of the harvester is \$42 250.

e $V_n = 29\,250$

$29\,250 = 65\,000 - 3250n$

$3250n = 65\,000 - 29\,250$

$3250n = 35\,750$

$n = 35\,750 / 3250$

$n = 11$

It takes 11 years for the harvester to reach a value of \$29 250.

8

a depreciation $= 22.5\%$ of \$5600

$= \dfrac{22.5}{100} \times 5600$

$= \$1260$

The annual depreciation is \$1260.

b **i** $V_0 = 5600,\ V_{n+1} = V_n - 1260$

ii $V_n = 5600 - 1260n$

c **i** $V_3 = 5600 - 1260 \times 3$

$V_3 = 1820$

After 3 years, the value of the computer is \$1820.

ii If the computer is worth nothing,

$V_n = 0$

$0 = 5600 - 1260 \times n$

$1260 \times n = 5600$

$n = \dfrac{5600}{1260}$

$n = 5$

After 5 years, the computer will be worth nothing.

9

a depreciation $= 17.5\%$ of \$7000

$= \dfrac{17.5}{100} \times 7000$

$= \$1225$

The annual depreciation is \$1225.

b **i** $V_0 = 7000,\ V_{n+1} = V_n - 1225$

ii $V_n = 7000 - 1225n$

c **i** $V_2 = 7000 - 1225 \times 2$

$V_2 = 4550$

After 2 years, the value of the machine is \$4550.

ii The machine is written off when its value is \$875.

$V_n = 875$

$875 = 7000 - 1225 \times n$

$1225n = 7000 - 875$

$1225n = 6125$

$n = \dfrac{6125}{1225}$

$n = 5$

The machine will be used for 5 years.

10

a The purchase price of the taxi is \$29 000.

b The value of the taxi is depreciated by \$0.25 (25 cents) per kilometre.

c $V_{20000} = 29\,000 - 0.25 \times 20\,000$

$V_{20000} = \$24\,000$

d If the taxi is valued at \$5000,

$V_5 = 5000$

$5000 = 29\,000 - 0.25 \times n$

$0.25 \times n = 29\,000 - 5000$

$0.25 \times n = 24\,000$

$n = 24\,000 / 0.25$

$n = 96\,000$

The taxi is valued at \$5000 after it has travelled 96 000 kilometres.

11

a total depreciation = \$35 400 – \$25 000
= \$9700

b depreciation per km $= \dfrac{9700}{25\,000}$
= 0.388

The car is depreciated at the 38.8 cents per kilometre.

c $V_n = 35\,400 - 0.388\text{n}$

d If the car has value \$6688,
$V_n = 6688$
$6688 = 35\,400 - 0.388n$
$0.388n = 35\,400 - 6688$
$0.388n = 28\,712$
$n = 28\,712 / 0.388$
$n = 74\,000$

When the car has value \$6688 it has travelled 74 000 kilometres.

e If the car has value \$0,
$V_n = 0$
$0 = 35\,400 - 0.388n$
$0.388n = 35\,400$
$n = 35\,400 / 0.388$
$n = 91\,237.1134$

The car is expected to drive 91 237.1134 km before it has value zero
To the nearest kilometre, it will have a value of zero after 91 238 km.

12

a **i** depreciation = \$110 000 – \$2500
= \$107 500

unit cost $= \dfrac{107\,500}{4\,000\,000}$
= 0.026875 dollars per page
(2.6875 cents per page)

ii $V_n = 110\,000 - 0.026875 \times n$
$V_{1\,500\,000} = 110\,000 - 0.026875 \times 1\,500\,000$
$V_{1\,500\,000} = 69\,687.50$

After printing 1.5 million pages, the printer has value \$69 687.50

iii depreciation for 750 000 pages
= 750 000 × 0.026875
= \$20 156.25

The annual depreciation of the printer is \$20 156.25

b $V_n = 110\,000 - 20\,156.25 \times n$ where n is the number of years.
$V_5 = 110\,000 - 20\,156.25 \times 5$
$V_5 = 9218.75$

After 5 years, the printer has value \$9218.75

c If the value of the machine is \$70 000,
$V_n = 70\,000$
$70\,000 = 110\,000 - 0.026875 \times n$
$0.026875 \times n = 110\,000 - 70\,000$
$0.026875 \times n = 40\,000$
$n = 40\,000 / 0.026875$
$n = 1\,488\,373$

By the time the printer has value \$70 000, it has printed 1 488 373 pages.

Solutions to Exercise 8E

1

a $V_1 = 1.042 \times 6000 = 6252$
$V_2 = 1.042 \times 6252 = 6514.58$
$V_3 = 1.042 \times 6514.58 = 6788.20$

After 1 year, the value of the investment is \$6252, after 2 years the value of the investment is \$6514.58 and after 3 years, the value of the investment is \$6788.20

b Use the calculator, counting the number of times ENTER or EXE is pressed for the value to exceed 8000.

6000	6000
6000·1.042	6252
6252·1.042	6514.58
6514.58·1.042	6788.20
6788.20·1.042	7073.30
7073.30·1.042	7370.38
7370.38·1.042	7679.94

7679.94·1.042	8002.49

Note: The values in the calculator screens above have been rounded to 2 decimal places.

It takes 7 years for the value of the investment to first exceed \$8000.

c $V_0 = 5000,\ V_{n+1} = 1.068\ V_n$

2

a $V_1 = 1.063 \times 20\,000 = 21\,260$
$V_2 = 1.063 \times 21\,260 = 22\,599.38$
$V_3 = 1.063 \times 22\,599.38 = 24\,023.14$

After 1 year, the value of the loan is \$21 260, after 2 years the value of the loan is \$22 599.38 and after 3 years, the value of the loan is \$24 023.14

b Use the calculator, counting the number of times ENTER or EXE is pressed for the value to exceed 30 000.

20000	20000
20000·1.063	21260
21260·1.063	22599.38
22599.38·1.063	24023.14
24023.14·1.063	25536.60
25536.60·1.063	27145.40
27145.40·1.063	28855.57

28855.57·1.063	30673.47

Note: The values in the calculator screens above have been rounded to 2 decimal places.

It takes 7 years for the value of the loan to first exceed \$30 000.

c $V_0 = 18\,000,\ V_{n+1} = 1.094\ V_n$

3

a Principal invested = V_0 = 7600
annual interest rate is 6% so monthly
interest rate = $\frac{6}{12}$% = 0.5%

$R = 1 + \frac{0.5}{100} = 1.005$

$V_0 = 7600,\ V_{n+1} = 1.005\ V_n$

b Use the calculator and perform 5 steps of the recurrence relation.

7600	7600
7600·1.005	7638
7638·1.005	7676.19
7676.19·1.005	7714.57
7714.57·1.005	7753.14
7753.14·1.005	7791.91

Note: The values in the calculator screens above have been rounded to 2 decimal places.

Wayne’s investment is worth $7791.91 after 5 months.

4

a Principal borrowed = V_0 = 3500
annual interest rate is 8% so quarterly
interest rate = $\frac{8}{4}$% = 2%

$R = 1 + \frac{2}{100} = 1.02$

$V_0 = 3500,\ V_{n+1} = 1.02\ V_n$

b 1 year is 4 quarters.
Use the calculator and perform 4 steps of the recurrence relation.

3500	3500
3500·1.02	3570
3570·1.02	3641.4
3641.4·1.02	3714.228
3714.228·1.02	3788.51256

If Jessica paid back everything she owes after 1 year, she would need $3788.51

5

a Purchase price = V_0 = 9800
annual interest rate is 3.5%

$R = 1 - \frac{3.5}{100} = 0.965$

$V_0 = 9800,\ V_{n+1} = 0.965\ V_n$

b Use the calculator and perform 5 steps of the recurrence relation.

9800	9800
9800·0.965	9457
9457·0.965	9126.01
9126.01·0.965	8806.59
8806.59·0.965	8498.36
8498.36·0.965	8200.92

Note: The values in the calculator screens above have been rounded to the nearest cent.

c The motor cycle is valued at $8200.92 after 5 years.

d depreciation in 3rd year = $V_2 - V_3$
= $9126.01 – $8806.59
= $319.41

6

a Purchase price = V_0 = 18 000
annual interest rate is 4.5%

$R = 1 - \frac{4.5}{100} = 0.955$

$V_0 = 18\,000,\ V_{n+1} = 0.955\ V_n$

b Use the calculator and perform 5 steps of the recurrence relation.

18000	18000
18000·0.955	17190
17190·0.955	16416.45
16416.45·0.955	15677.71
15677.71·0.955	14972.21
14972.21·0.955	14298.46

Note: The values in the calculator screens above have been rounded to the nearest cent.

c The furniture is valued at $15 677.71 after 3 years.

d total depreciation = $V_0 - V_5$
= $18 000 – $14 298.46
= $3701.54

Solutions to Exercise 8F

1

a $V_n = 2^n \times 6$
$V_4 = 2^4 \times 6$
$V_4 = 96$

b $V_n = 3^n \times 10$
$V_4 = 3^4 \times 10$
$V_4 = 810$

c $V_n = 0.5^n \times 1$
$V_4 = 0.5^4 \times 1$
$V_4 = 0.0625$

d $V_n = 0.25^n \times 80$
$V_4 = 0.25^4 \times 80$
$V_4 = 0.3125$

2

a **i** $V_0 = 3000$ so the amount invested is $3000.

ii $1.1 = 1 + \dfrac{r}{100}$

$1.1 - 1 = \dfrac{r}{100}$

$0.1 = \dfrac{r}{100}$

$r = 0.1 \times 100$
$r = 10$
The annual interest rate for this investment is 10%.

b $V_n = 1.1^n \times 3000$

c $V_5 = 1.1^5 \times 3000$
$V_5 = 4831.53$

d V_5 is the value of the investment after 5 years.

3

a **i** $V_0 = 2000$ so the amount borrowed is $2000.

ii $1.06 = 1 + \dfrac{r}{100}$

$1.06 - 1 = \dfrac{r}{100}$

$0.06 = \dfrac{r}{100}$

$r = 0.06 \times 100$
$r = 6$
The annual interest rate for this loan is 6%.

b $V_n = 1.06^n \times 2000$

c $V_5 = 1.06^4 \times 2000$
$V_5 = 2524.95$

d $V_6 = 1.06^6 \times 2000$
$V_6 = 2837.04$

Interest paid $= V_6 - V_0$
$= \$2837.04 - \$2000$
$= \$837.04$

4

a $V_0 = 8000$
$r = 12.5$
$R = 1 + \frac{12.5}{100}$
$R = 1.125$

$V_n = (1.125)^n \times 8000$

b $V_3 = (1.125)^3 \times 8000$
$V_3 = 11\ 390.63$

After 3 years, the value of the investment is $11 390.63

c interest earned $= V_3 - V_0$
$= 11\ 390.63 - 8000$
$= 3390.63$

After 3 years, the interest earned is $3390.63

d interest earned in 3rd year $= V_3 - V_2$

$V_2 = (1.125)^2 \times 8000$
$V_2 = 10\ 125$

interest 3rd year $= 11\ 390.63 - 10\ 125$
$= 1265.63$

The interest earned in 3rd year is $1265.63

5

a annual interest rate = 6%
monthly interest rate $= \frac{6}{12}\%$
$= 0.5\%$

$R = 1 + \frac{0.5}{100}$
$= 1.005$

$V_0 = 3300$

$V_n = (1.005)^n \times 3300$

b $V_{10} = (1.005)^{10} \times 3300$
$= 3468.76$

After 10 months, the value of the loan is $3468.76

c interest charged $= V_{10} - V_0$
$= 3468.76 - 3300$
$= 168.76$

Over 10 months, the interest charged is $168.76

6

a **i** The purchase price is V_0.
The purchase price of the stereo system is $1200.

ii $R = 0.88$
$1 - \frac{r}{100} = 0.88$
$1 - 0.88 = \frac{r}{100}$
$0.12 = \frac{r}{100}$
$r = 0.12 \times 100$
$r = 12$

The stereo system is deprecating at the rate of 12% per annum.

b $V_n = (0.88)^n \times 1200$

c $V_7 = (0.88)^7 \times 1200$
$= 490.41$

After 7 years, the stereo system has value $490.41

7

a Purchase price = $38 500, so
$V_0 = 38\ 500$

$r = 9.5$, so
$R = 1 - \frac{9.5}{100}$
$= 0.905$

$V_n = (0.905)^n \times 38\ 500$

b $V_5 = (0.905)^5 \times 38\ 500$
$= 23\ 372.42$

After 5 years, the car has value $23 372.42

c total depreciation over 5 years

$= V_0 - V_5$

$= 38\,500 - 23\,372.42$

$= 15\,127.58$

Over 5 years, the total depreciation of the car is $15 127.58

8 $V_0 = 3500$

$r = 6.75$ so,

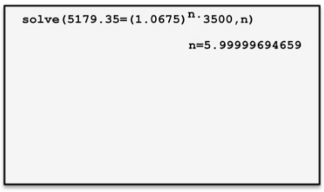
$R = 1 + \frac{6.75}{100}$

$= 1.0675$

$V_n = (1.0675)^n \times 3500$

If the value of the investment is now $5179.35,

$5179.35 = (1.0675)^n \times 3500$



Sarah investment her money for 6 years.

9 $V_0 = 200$

$V_n = 20\,000$

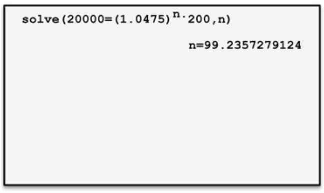
$r = 4.75$ so,

$R = 1 + \frac{4.75}{100}$

$= 1.0475$

$V_n = (1.0475)^n \times 200$

$20\,000 = (1.0475)^n \times 200$



Since the value of the investment must exceed $20 000, the value of n must be rounded up to 100.

It would take $200 100 years to exceed $20 000 if it was invested at 4.75% per annum compound interest.

10 $V_0 = 1000$

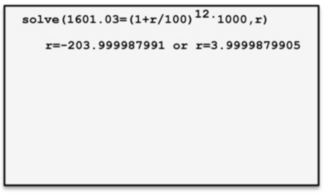
$n = 12$

$V_{12} = 1601.03$

$r = ?$

$V_n = (1 + \frac{r}{100})^n \times V_0$

$1601.03 = (1 + \frac{r}{100})^{12} \times 1000$



Ignore the negative value for r.
The investment has an annual interest rate of 4%.

11 $V_0 = 8000$

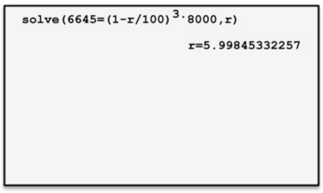
$n = 3$

$V_3 = 6645$

$r = ?$

$V_n = (1 - \frac{r}{100})^n \times V_0$

$6645 = (1 - \frac{r}{100})^3 \times 8000$



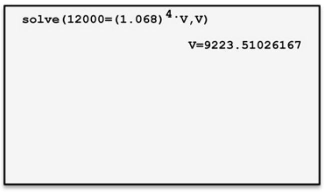
The depreciation rate is 6% per annum.

12 $V_0 = ?$
$n = 4$
$V_n = 12\ 000$

$r = 6.8$ so,

$$R = 1 + \frac{6.8}{100}$$
$$= 1.068$$

$$12\ 000 = (1.068)^4 \times V_0$$



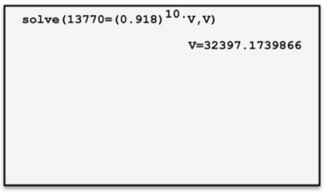
$9223.51 must be deposited.

13 $V_0 = ?$
$n = 10$
$V_{10} = 13\ 770$

$r = 8.2$ so,

$$R = 1 - \frac{8.2}{100}$$
$$= 0.918$$

$$13\ 770 = (0.918)^{10} \times V_0$$



The initial cost of the machine was
$32 397.17

Solutions to Exercise 8G

1

a 4.95% per annum $= \frac{4.95}{12}\%$ monthly

$= 0.4125\ \%$ monthly

b 8.3% per annum $= \frac{8.3}{4}\%$ quarterly

$= 2.08\%$ quarterly

c 6.2% per annum $= \frac{6.2}{26}\%$ fortnightly

$= 0.238\%$ fortnightly

d 7.4% per annum $= \frac{7.4}{52}\%$ weekly

$= 0.142\%$ weekly

e 12.7% per annum $= \frac{12.7}{365}\%$ daily

$= 0.0348\%$ daily

2

a 0.54% monthly $= 0.54 \times 12\%$ annually

$= 6.48\%$ annually

b 1.45% quarterly $= 1.45 \times 4\%$ annually

$= 5.8\%$ annually

c 0.57% fortnightly $= 0.57 \times 26\%$ annually

$= 14.82\%$ annually

d 0.19% weekly $= 0.19 \times 52\%$ annually

$= 9.88\%$ annually

e 0.022% daily $= 0.022 \times 365\%$ annually

$= 8.03\%$ annually

3

a

```
eff(6.2,12)      6.3792531
          OR
convEff(12,6.2) 6.3792531
```

The effective rate is 6.38% per annum.

b

```
eff(8.4,365)       8.761838
          OR
convEff(365,8.4)   8.761838
```

The effective rate is 8.76% per annum.

c

```
eff(4.8,52)        4.9147426
          OR
convEff(52,4.8)    4.9147426
```

The effective rate is 4.91% per annum.

d

```
eff(12.5,4)        13.098239
          OR
convEff(4,12.5)    13.098239
```

The effective rate is 13.10% per annum.

e

```
eff(7.5,2)         7.640625
          OR
convEff(2,7.5)     7.640625
```

The effective rate is 7.64% per annum.

4

a More frequent compounds means more interest is earned or charged. Since Brenda is investing money, it is better for her to earn more interest.

b $r = 4.60$
There are 4 quarters in a year so $n = 4$

$$r_{\text{effective}} = ((1+\frac{4.6}{4\times100})^4 - 1)\times 100$$
$$= 4.679960\ldots$$

or

```
eff(4.6,4)          4.679960
          OR
convEff(4,4.6)      4.679960
```

The effective interest rate for the loan is 4.68% per annum (2 d.p).

c $r = 4.60$
There are 12 months in a year so $n = 12$

$$r_{\text{effective}} = ((1+\frac{4.6}{12\times100})^{12} - 1)\times 100$$
$$= 4.69823\ldots$$

or

```
eff(4.6,12)          4.69823
          OR
convEff(12,4.6)      4.69823
```

The effective interest rate for the loan is 4.70% per annum (2 d.p).

d The effective interest rate for monthly compounds is higher than for quarterly compounds. Brenda will earn more interest in one year with monthly compounds, which are more frequent than quarterly compounds.

5

a Less frequent compounds means more interest is earned or charged. Since Stella is borrowing money, it is better for her to pay less interest.

b $r = 7.94$
There are 26 fortnights in a year so $n = 26$

$$r_{\text{effective}} = ((1+\frac{7.94}{26\times100})^{26} - 1)\times 100$$
$$= 8.2506\ldots$$

or

```
eff(7.94,26)          8.2506
          OR
convEff(26,7.94)      8.2506
```

The effective interest rate for the loan is 8.25% per annum (2 d.p).

c $r = 7.94$
There are 12 months in a year so $n = 12$

$$r_{\text{effective}} = ((1+\frac{7.94}{12\times100})^{12} - 1)\times 100$$
$$= 8.2354\ldots$$

or

```
eff(7.94,12)          8.2354
          OR
convEff(12,7.94)      8.2354
```

The effective interest rate for the loan is 8.24% per annum (2 d.p).

d The effective interest rate for monthly compounds is lower than for fortnightly compounds. Stella will pay less interest in one year with monthly compounds, which are less frequent than fortnightly compounds.

6

a <u>Option A</u>:

$r = 8.3$
$n = 12$

$$r_{\text{effective}} = ((1+\frac{8.3}{12\times100})^{12} - 1) \times 100$$
$$= 8.62314\ldots$$

or

```
eff(8.3,12)          8.62314
            OR
convEff(12,8.3)      8.62314
```

The effective interest rate for the loan is 8.62% per annum (2 d.p).

<u>Option B:</u>

$r = 7.8$
$n = 52$

$$r_{\text{effective}} = ((1+\frac{7.8}{52\times100})^{52} - 1) \times 100$$
$$= 8.10594\ldots$$

or

```
eff(7.8,52)          8.10594
            OR
convEff(52,7.8)      8.10594
```

The effective interest rate for the loan is 8.11% per annum (2 d.p).

b <u>Option A:</u>

$$\text{interest} = 8.62\% \text{ of } \$35\,000$$
$$= \frac{8.62}{100} \times 35\,000$$
$$= \$3017$$

<u>Option B:</u>

$$\text{interest} = 8.11\% \text{ of } \$35\,000$$
$$= \frac{8.11}{100} \times 35\,000$$
$$= \$2838.50$$

c Luke should choose loan option B because he will pay less interest with this loan. This loan has a lower effective interest rate.

7

a <u>Option A:</u>

$r = 5.3$
$n = 12$

$$r_{\text{effective}} = ((1+\frac{5.3}{12\times100})^{12} - 1) \times 100$$
$$= 5.43066\ldots$$

or

```
eff(5.3,12)          5.43066
            OR
convEff(12,5.3)      5.43066
```

The effective interest rate for the investment is 5.43% per annum (2 d.p).

<u>Option B:</u>

$r = 5.5$
$n = 4$

$$r_{\text{effective}} = ((1+\frac{5.5}{4\times100})^{4} - 1) \times 100$$
$$= 5.61448\ldots$$

or

```
eff(5.4,4)          5.510337
            OR
convEff(4,5.4)      5.510337
```

The effective interest rate for the investment is 5.51% per annum (2 d.p).

b <u>Option A:</u>

interest $= 5.43066\%$ of \$140 000

$= \dfrac{5.43066}{100} \times 140\ 000$

$= \$7602.92$

<u>Option B:</u>

interest $= 5.61448\%$ of \$140 000

$= \dfrac{5.61488}{100} \times 140\ 000$

$= \$7860.27$

c Sharon should choose investment option B because she will earn more interest with this investment. This investment has a higher effective interest rate.

Chapter 9 – Modelling and analysing reducing-balance loans and annuities

Solutions to Exercise 9A

1

a $T_0 = 50,\ T_{n+1} = 2T_n - 10$

$T_0 = 50$
$T_1 = 2 \times 50 - 10 = 90$
$T_2 = 2 \times 90 - 10 = 170$
$T_3 = 2 \times 170 - 10 = 330$
$T_4 = 2 \times 330 - 10 = 650$
$T_5 = 2 \times 650 - 10 = 1290$

Or

50	50
50·2-10	90
90·2-10	170
170·2-10	330
330·2-10	650
650·2-10	1290

b $Z_0 = 128,\ Z_{n+1} = 0.5Z_n + 32$

$Z_0 = 128$
$Z_1 = 0.5 \times 128 + 32 = 96$
$Z_2 = 0.5 \times 96 + 32 = 80$
$Z_3 = 0.5 \times 80 + 32 = 72$
$Z_4 = 0.5 \times 72 + 32 = 68$
$Z_5 = 0.5 \times 68 + 32 = 66$

Or

128	128
128·0.5+32	96
96·0.5+32	80
80·0.5+32	72
72·0.5+32	68
68·0.5+32	66

c $P_0 = 1000,\ P_{n+1} = 1.02P_n - 20$

$P_0 = 1000$
$P_1 = 1.02 \times 1000 - 20 = 1000$
$P_2 = 1.02 \times 1000 - 20 = 1000$
$P_3 = 1.02 \times 1000 - 20 = 1000$
$P_4 = 1.02 \times 1000 - 20 = 1000$
$P_5 = 1.02 \times 1000 - 20 = 1000$

Or

1000	1000
1000·1.02-20	1000
1000·1.02-20	1000
1000·1.02-20	1000
1000·1.02-20	1000
1000·1.02-20	1000

2

a

150000	150000
150000·0.986+1000	148900
148900·0.986+1000	147815.4
147815.4·0.986+1000	146745.9844
146745.9844·0.986+1000	145691.540644
145691.540644·0.986+1000	144651.859

After 5 days there will be 144 651.859 litres of water in the swimming pool.

b

.	
.	
.	
131535.7072·0.986+1000	130694.2073
130694.2073·0.986+1000	129864.4884

It will take 21 days for there to be less than 130 000 litres of water in the swimming pool.

3

a The initial number of kangaroos is K_0.
$K_0 = 3000$

There were initially 3000 kangaroos in the population.

b

```
3000                    3000
3000·1.02-200           2860
2860·1.02-200         2717.2
2717.2·1.02-200     2571.544
```

Apply the recurrence relation rule and observe the effect.

The population of kangaroos is declining.

c From the recurrence relation rule . . .

$R = 1.02$

$1 + \frac{r}{100} = 1.02$

$\frac{r}{100} = 1.02 - 1$

$\frac{r}{100} = 0.02$

$r = 0.02 \times 100$

$r = 2$

The kangaroo birth rate is growing at the rate of 2% per year.

d The additional number of kangaroos that leave the population is the number subtracted in the recurrence relation.

200 additional kangaroos leave the population each year.

e

```
3000                    3000
3000·1.02-200           2860
2860·1.02-200         2717.2
2717.2·1.02-200     2571.544
2571.544·1.02-200
                  2422.97488
2422.97488·1.02-200
                 2271.434378
```

After 5 years, the expected number of kangaroos in the population is 2271, correct to the nearest whole number.

Solutions to Exercise 9B

1 $V_0 = 2500, \quad V_{n+1} = 1.08V_n - 626.00$

```
2500                        2500
2500·1.08-626               2074
2074·1.08-626            1613.92
1613.92·1.08-626 1117.0336
1117.0336·1.08-626
                        580.396288
580.396288·1.08-626
                        0.82799104
```

a The balance of the loan after three payments have been made is $1117.03, correct to the nearest cent.

b The loan is not fully repaid after 5 payments of $626. The last line of the calculator screen is 0.8279… which means the last payment must be increased by $0.83 to fully repay the loan.

The last payment must be $626.83 to fully repay the loan after 5 payments.

2 $V_0 = 5000, \quad V_{n+1} = 1.01V_n - 862.70$

```
5000                        5000
5000·1.01-862.70          4187.3
4187.3·1.01-862.70      3366.473
3366.473·1.01-862.70
                      2537.43773
2537.43773·1.01-862.70
                      1700.112107
```

```
1700.112107·1.01-862.70
                      854.4132284
854.4132284·1.01-862.70
                     0.2573606567
```

a After two payments, the balance of the loan is $3366.47, correct to the nearest cent.

b The final payment must be increased by $0.26 so will be $862.96

3

a The principal amount borrowed is $2000. So, $V_0 = 2000$.

Interest rate is 6% p.a compounding monthly, so

$r = \dfrac{6}{12} = 0.5\%$ per month

$R = 1 + \dfrac{0.5}{100}$ 0.5/100

$R = 1.005$

Monthly payments of $339 are made, so $D = 339$

$V_0 = 2000, \quad V_{n+1} = 1.005\, V_n - 339$

b

```
2000                        2000
2000·1.005-339              1671
1671·1.005-339          1340.355
1340.355·1.005-339   1008.056775
1008.056775·1.005-339
                     674.0970589
```

After 4 months, the balance of the loan is $674.10, correct to the nearest cent.

c

```
2000                        2000
2000·1.005-339              1671
1671·1.005-339          1340.355
1340.355·1.005-339   1008.056775
1008.056775·1.005-339
                     674.0970589
674.0970589·1.005-339
                     338.4675442
```

```
338.4675442·1.005-339
                      1.15988189
```

In order to ensure that the loan is fully repaid after 6 months, the final payment must be:

$339 + $1.16 = $340.16

4

a The principal amount borrowed is \$10 000. So, $B_0 = 10\ 000$.

Interest rate is 12% p.a compounding quarterly, so

$r = \frac{12}{4} = 3\%$ per month

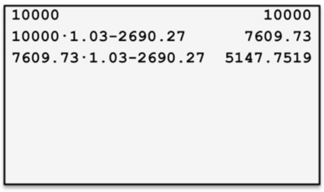
$R = 1 + \frac{3}{100}$

$R = 1.03$

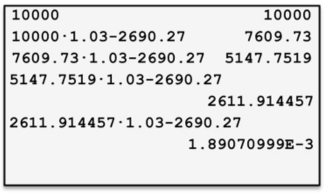
Monthly payments of \$2690.27 are made, so $D = 2690.27$

$B_0 = 10\ 000$,

$B_{n+1} = 1.03\ B_n - 2690.27$

b 

After 2 payment have been made, the balance of the loan is \$5147.75, correct to the nearest cent.

c 

After 4 payments, the balance of the loan is $1.8907\ldots \times 10^{-3}$, or \$0.00189

This balance, when rounded to the nearest cent, is zero, so 4 payments will fully repay the loan.

5

a The principal amount borrowed is \$3500. So, $V_0 = 3500$.

Interest rate is 4.8% p.a compounding monthly, so

$r = \frac{4.8}{12} = 0.4\%$ per month

$R = 1 + \frac{0.4}{100}$

$R = 1.004$

Monthly payments of \$280 are made, so $D = 280$

$V_0 = 3500$, $V_{n+1} = 1.004\ V_n - 280$

b The principal amount borrowed is \$150 000. So, $V_0 = 150\ 000$.

Interest rate is 3.64% p.a compounding fortnightly, so

$r = \frac{3.64}{26} = 0.14\%$ per fortnight

$R = 1 + \frac{0.14}{100}$

$R = 1.0014$

Fortnightly payments of \$650 are made, so $D = 650$

$V_0 = 150\ 000$,

$V_{n+1} = 1.0014\ V_n - 650$

6

a Annual interest rate is 12%

Monthly interest rate = $\frac{12}{12}\% = 1\%$.

The monthly interest rate is 1%.

b Balance after payment 1 is \$1675

Interest paid = 1% of \$1675

$= \frac{1}{100} \times \$1675$

$= \$16.75$

The interest paid when payment 2 is made is \$16.75

c Payment 4 is made up of interest and principal reduction:

$345.00 = $10.15 + principal reduction
principal reduction = $345.00 – $10.15
= $334.85

The principal repaid from payment 4 is $334.85

d Balance after payment 5
= balance after payment 4
– principal reduction
= $680.37 – $338.20
= $342.17

The balance of the loan after payment 5 has been made is $342.17

e The total cost of repaying the loan
= sum of all the payments
= 5 × $345 + 1 × $345.60
= $2070.60

f The total interest paid
= total cost of repayment
– principal
= $2070.60 – $2000
= $70.60

or

total interest
= sum of interest paid
= $20.00 + $16.25 + $13.47
+ $10.15 + $6.80 + $3.42
= $70.59

Note: the difference in the two answers is due to rounding errors in the amortisation table.

7

a **i** The principal value is the balance of the loan for payment number 0.

The principal of the loan is $4000.

ii The quarterly payment is $557.85

iii In the row for payment 1, the value in the interest column is 100. Interest paid from payment 1 is $100.

iv In the row for payment 2, the value in the principal reduction column is 469.30. The principal reduction from payment 2 is $469.30

v After payment 3, the balance of the loan is $2591.83

vi The balance of the loan after 8 quarters is 0.17 so $0.17 must be added to the final payment.

The final payment must be $558.02 in order to fully repay the loan after 8 quarters.

b **i** The interest component of payment 1 is $100. This is calculated on the principal of the loan.

quarterly interest rate
$= \frac{100}{4000} \times 100\%$
= 2.5% per quarter

ii The annual interest rate
= 4 × quarterly interest rate
= 4 × 2.5%
= 10% per annum

c A = interest paid with payment 4
= payment – principal reduction
= 557.85 – 493.05
= 64.80

B = Principal reduction of payment 5
= payment 5 – interest
= 557.85 – 52.47
= 505.38

C = balance after payment 6
= balance after payment 5 – principal reduction of payment 6
= 1593.39 – 518.02
= 1075.37

D = interest paid with payment 7
= 2.5% of balance after payment 6
$= \frac{2.5}{100} \times C$
= 0.025 × 1075.37
= 26.88

E = Principal reduction of payment 7
= payment 7
– interest of payment 7
= 557.85 – D
= 557.85 – 26.88
= 530.97

Note: Because the values in an amortisation table are rounded to the nearest cent, there can be small differences in the values depending on the calculation method used.

Solutions to Exercise 9C

Note: in the financial solver screens below, the shaded box is the quantity that is found by pressing ENTER or tapping Solve.

1

a

N:	6
I%:	4.5
PV:	8000
Pmt or PMT:	-350
FV:	-6061.909746
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Balance after 6 months is $6061.91

b

N:	12
I%:	7.8
PV:	25000
Pmt or PMT:	-1200
FV:	-12095.12642
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Balance after 1 year is $12 095.13

c

N:	5x4
I%:	8.3
PV:	240000
Pmt or PMT:	-7900
FV:	-168519.4015
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Balance after 5 years is $168 519.40

d

N:	2x4
I%:	6.9
PV:	75000
Pmt or PMT:	-4800
FV:	-45196.7775
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Balance after 2 years is $45 196.78

d

N:	12
I%:	4.6
PV:	50000
Pmt or PMT:	-350
FV:	-33735.98685
Pp/y or P/Y:	52
Cp/Y or C/Y:	52

Balance after 1 year is $33 735.99

2

a

N:	30
I%:	6.8
PV:	17000
Pmt or PMT:	-617.7975955
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Monthly payments of $617.80 are required to fully repay this loan in 30 months.

b

N:	2x12
I%:	4.2
PV:	9500
Pmt or PMT:	-413.3829547
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Monthly payments of $413.38 are required to fully repay this loan in 2 years.

c

N:	6
I%:	9.6
PV:	2800
Pmt or PMT:	-506.6411379
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Quarterly payments of $506.64 are required to fully repay this loan in 6 quarters.

d

N:	15x4
I%:	8.6
PV:	140000
Pmt or PMT:	-4175.10529
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Quarterly payments of $4175.11 are required to fully repay this loan in 15 years.

e

N:	25x26
I%:	5.2
PV:	250000
Pmt or PMT:	-687.6499306
FV:	0
Pp/y or P/Y:	26
Cp/Y or C/Y:	26

Fortnightly payments of $687.65 are required to fully repay this loan in 25 years.

3

a

N:	30x12
I%:	11
PV:	90000
Pmt or PMT:	-857.091056
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The monthly payment is $857.09

b Number of payments = 30 × 12 = 360
Total cost of paying off loan
= number of payments × $857.09
= 360 × $857.09
= $308 552.40

Correct to the nearest dollar, the total cost of paying off the loan is $308 552.

c Interest paid = total cost – principal
= $308 552.40 - $90 000
= $218 552.40

Interest paid on this loan is $218 550 (to nearest 10 dollars)

4

a

N:	12x12
I%:	10.25
PV:	240000
Pmt or PMT:	-2200
FV:	-197793.8516
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After 12 years, there is still $197 793.85 still owing on this loan.

b **i**

N:	12x12
I%:	10.25
PV:	240000
Pmt or PMT:	-2902.956296
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

If the loan is fully repaid in 12 years, the monthly payment must be $2902.96, correct to the nearest cent.

ii Total repaid
= number payments
× payment amount
= 144 × $2902.96
= $418 026.24

The total amount repaid on this loan is $418 030 (to the nearest 10 dollars)

iii Total interest
= total repaid – principal
= $418 026.24 - $240 000
= $178 026.24

The total amount of interest paid on this loan is $178 030 (to the nearest 10 dollars)

5

a

N:	54.99160484
I%:	9.5
PV:	20000
Pmt or PMT:	-450
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The number of monthly payments required to pay out this loan is 55, correct to the nearest month.

b Total paid
= number of payments
× payment amount
= 55 × $450
= $24 750

Amount of interest
= Total paid – Principal
= $24 750 – $20 000
= $4750

6

a

N:	43.05464293
I%:	9.5
PV:	20000
Pmt or PMT:	-550
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Correct to the nearest month, it takes Joan 43 months to fully repay the loan.

b Total paid
= number of payments
× payment amount
= 43 × $550
= $23 650

Amount of interest
= Total paid – Principal
= $23 650 – $20 000
= $3650

Interest saved
= interest from Q5b – $3650
= $4750 – $3650
= $1100

Joan would save $1100 in interest by paying the higher monthly payment.

7

a

N:	20x12
I%:	10.5
PV:	35000
Pmt or PMT:	-349.4329604
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The payments for this loan are $349.43, rounded to the nearest cent.

b Total paid
= number of payments
× payment amount
= 240 × $349.43
= $83 863.20

Amount of interest
= Total paid – Principal
= $83 863.20 – $35 000
= $48 863.20

Interest paid is $48 900 (to nearest 100 dollars)

c Change the payment value to $349.43

N:	4x12
I%:	10.5
PV:	35000
Pmt or PMT:	-349.43
FV:	-32437.90263
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After 4 years, there is still $32 437.90 owing on this loan.

d 4 years has already passed. If the loan is repaid in 20 years total, there are still 16 years of the loan left.

Take value of loan after 4 years as the new principal value. Use the value that has been rounded to the nearest cent.

Change PV to $32437.90

Change the interest rate to 13.75%

N:	16x12
I%:	13.75
PV:	32437.90
Pmt or PMT:	-418.6566546
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The new payment required is $418.66

e Amount paid with no interest change
= 20 × 12 × $349.43
= $83 863.20

Amount paid with interest change
= 4 × 12 × $349.43 + 16
× 12 × 418.66
= $97 155.36

Extra paid = $97 155.36 – $83 863.20
= $13 292.16

The extra amount to be repaid is $13 300 (to the nearest 100 dollars)

The calculations in this solution have used payments and balances rounded to the nearest cent because in practice it is not possible to pay fractions of cents.

8

a

N:	7x12
I%:	7.5
PV:	150000
Pmt or PMT:	-1100
FV:	-132119.8209
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After 7 years, there is still $132 119.82 owing on the loan.

b Change interest rate to 8.5
There are 18 years left on the loan
The new principal is $132 119.82

N:	18x12
I%:	8.5
PV:	132119.82
Pmt or PMT:	-1196.2887
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The new monthly payment required is $1196.29

9

a Use a financial solver to find the principal amount borrowed.

N:	30x12
I%:	8.5
PV:	179994.2425
Pmt or PMT:	-1384
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Selling price = deposit + principal
= $50 000 + $179 994.24
= $229 994.24

b total paid on loan = 30 × 12 × $1384
= $498 240

Interest = total paid – principal
= $498 240 – $179 994.24
= $318 245.76

Interest to be paid is $318 246 (to the nearest dollar)

c

N:	6x12
I%:	8.5
PV:	179994.24
Pmt or PMT:	-1384
FV:	-169798.79566
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After 6 years, the couple still owe $169 798.80

d New principal value = $169 798.80
New interest rate = 8.5% + 0.9% = 9.4%
Time left on loan = 24 years

N:	24x12
I%:	9.4
PV:	169798.80
Pmt or PMT:	-1384
FV:	-111567.6961
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After the original 30 year time period, the couple still owe $111 567.70

e Change the future value to zero and calculate the new payment.

N:	24x12
I%:	9.4
PV:	169798.80
Pmt or PMT:	-1487.287542
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The new payment amount is $1487.29

Solutions to Exercise 9D

Note: in the financial solver screens below, the shaded box is the quantity that is found by pressing ENTER or tapping Solve.

1

N:	1
I%:	7.15
PV:	100000
Pmt or PMT:	-595.8333333
FV:	-100000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The monthly payment on this interest-only loan is \$595.83

2

N:	1
I%:	8.15
PV:	50000
Pmt or PMT:	-339.5833333
FV:	-50000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Jamie will need to pay \$339.58 each month.

3 Jackson will pay 12 payments on this interest-only loan before he sells his painting.

N:	1
I%:	9.25
PV:	30000
Pmt or PMT:	-231.25
FV:	-30000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Each payment will be \$231.25

Total paid in interest = 12 × \$231.25
= \$2775.00

Total cost of the painting
= \$30 000 + \$2775.00
= \$32 775.00

Jackson will need to sell the painting for \$32 775 in order to lose no money.

Solutions to Exercise 9E

Note: in the financial solver screens below, the shaded box is the quantity that is found by pressing ENTER or tapping Solve.

1

a

```
5000                      5000
5000·1.01-1030            4020
4020·1.01-1030          3030.2
3030.2·1.01-1030      2030.502
```

After 3 payments, the balance of the annuity is \$2030.50

b

```
5000                      5000
5000·1.01-1030            4020
4020·1.01-1030          3030.2
3030.2·1.01-1030      2030.502
2030.502·1.01-1030
                    1020.80702
```

```
1020.80702·1.01-1030
                     1.0150902
```

After 5 payments, the balance of the annuity is \$1.02 so the annuity is not fully paid out after 5 payments.

The last payment = \$1030.00 + \$1.02
= \$1031.02

2

a

```
6000                      6000
6000·1.005-1518           4512
4512·1.005-1518        3016.56
```

After 2 payments, the balance of the annuity is \$3016.56

b

```
6000                      6000
6000·1.005-1518           4512
4512·1.005-1518        3016.56
3016.56·1.005-1518
                     1513.6428
1513.6428·1.005-1518
                      3.211014
```

After 4 payments, the balance of the annuity is \$3.21 so the annuity is not fully paid out after 4 payments.

The last payment = \$1518.00 + \$3.21
= \$1521.21

3

a Helen invests \$40 000 so $V_0 = 40\ 000$

Interest rate = 6% per annum

$= \frac{6}{4}$% per quarter

= 1.5% per quarter

$R = 1 + \frac{1.5}{100}$

$= 1.015$

Helen receives payments of \$10 380 each quarter, so
$D = 10\ 380$

$V_0 = 40\ 000,\ V_{n+1} = 1.015\ V_n - 10\ 380$

b 6 months is equivalent to 2 quarters.

```
40000                    40000
40000·1.015-10380        30220
30220·1.015-10380      20293.3
```

After 6 months (2 quarters) the balance of the annuity is \$20 293.30

4

a monthly interest rate

$= \frac{\text{annual interest rate}}{12}$

$= \frac{3}{12}$% per month

= 0.25% per month

b From the row for payment 1, the interest earned is \$15.

c From the row for payment 3, the principal is reduced by \$495.47

d From the row for payment 5, the balance of the annuity is \$3522.64

e A = 0.25% or previous balance

$= \frac{0.25}{100} \times 3522.64$

= 8.81 (nearest cent)

B = payment 6 – interest earned

= 508.00 – A

= 508.00 – 8.81

= 499.19

C = balance 5 – B

= 3522.64 – 499.19

= 3023.45

f The last payment = \$508.00 + \$1.97

= \$509.97

g The total return from the annuity

= sum of all the payments received

= 11 × \$508.00 + 1 × \$509.97

= \$6097.97

h Total interest earned

= total return – principal

= \$6097.97 – \$6000

= \$97.97

5 Remember to make PV negative because this value is given to the bank. Remember to make Pmt or PMT positive because this amount is given from the bank.

N:	58.33701358
I%:	6.25
PV:	-64000
Pmt or PMT:	1275
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Leigh's investment will last 58 months, correct to the nearest month.

6

N:	20.65241295
I%:	7.25
PV:	-85500
Pmt or PMT:	5000
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Raj's annuity will last for 21 quarters, correct to the nearest quarter.

7

N:	10x12
I%:	7.5
PV:	-40000
Pmt or PMT:	474.8070765
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Note: the payment value is positive because this amount is given from the bank to Stephanie.

Stephanie will receive \$474.81 each month from her annuity.

8

a

N:	15x12
I%:	6.4
PV:	-80000
Pmt or PMT:	692.4955241
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Helen can withdraw \$692.50 from her annuity each month.

b Make sure that the payment is retained in the financial solve. Just change the value of N.

N:	2x12
I%:	6.4
PV:	-80000
Pmt or PMT:	692.4955241
FV:	73213.06535
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

c Transfer the future value after 2 years to be the new principal value.

Change the interest rate to 6.2%.

Make the future value zero.

N:	153.333373
I%:	6.2
PV:	-73213.06535
Pmt or PMT:	692.4955241
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Helen's investment will last a further 153 months.

d After 2 years, there are still 13 years to go.

N:	13x12
I%:	6.2
PV:	-73213.06535
Pmt or PMT:	684.7323509
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Payments of $684.73 will ensure Helen's investment lasts for 15 years in total.

9

a

N:	10x12
I%:	6.2
PV:	-150000
Pmt or PMT:	0
FV:	278394.4862
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

There is $278 394.49 in the account after 10 years (to the nearest cent)

b FV from Q9a becomes PV.

N:	12x12
I%:	6.573132304
PV:	-278394.4862
Pmt or PMT:	2800
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Sameep will need an annual interest rate of 6.57% (2 d.p) in order to exhaust his investment after 12 additional years.

Solutions to Exercise 9F

Note: in the financial solver screens below, the shaded box is the quantity that is found by pressing ENTER or tapping Solve.

1

a $\$2500 = 2.5\%$ of investment

$\$2500 = \frac{2.5}{100} \times \text{investment}$

$\$2500 \times 100 = 2.5 \times \text{investment}$

$\text{investment} = \$\frac{2500 \times 100}{2.5}$

investment = \$100 000

Geoff should invest \$100 000 to provide for his perpetuity donation to the RSPCA.

b $\text{interest rate} = \$\frac{2500}{80\,000} \times 100\%$

$= 3.125\%$

The minimum interest rate Geoff needs is 3.125% per annum.

2

a $\$500 = 2.7\%$ of investment

$\$500 = \frac{2.7}{100} \times \text{investment}$

$\$500 \times 100 = 2.7 \times \text{investment}$

$\text{investment} = \$\frac{500 \times 100}{2.7}$

investment = \$18 518.52

Barbara should invest \$18 519 (nearest dollar) to provide the scholarship.

b $\text{interest rate} = \frac{\$500}{\$12\,000} \times 100\%$

$= 4.166\ldots\%$

The minimum interest rate Barbara needs is 4.2% per annum (2 d.p)

3 $\$5500 = 2.75\%$ of investment

$\$5500 = \frac{2.75}{100} \times \text{investment}$

$\$5500 \times 100 = 2.75 \times \text{investment}$

$\text{investment} = \$\frac{5500 \times 100}{2.75}$

investment = \$200 000

Kathy should invest \$200 000 to provide for her donation.

4

N:	1
I%:	5.75
PV:	-1000000
Pmt or PMT:	4791.666667
FV:	1000000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Craig will receive a monthly payment of \$4791.67 from his annuity investment.

5

a Remember to change the compounds per year to 4 (quarterly)

N:	1
I%:	6.1
PV:	-642000
Pmt or PMT:	9790.5
FV:	642000
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Suzie receives a quarterly payment of \$9790.50 from her investment.

b Since Suzie is withdrawing only the interest that is earned from her investment, the full invested amount will remain in the account in perpetuity.

After 5 quarterly payments, Suzie has \$642 000 in the perpetuity.

c The amount invested in a perpetuity will remain constant, provided the interest rate and remains the same.

After 10 quarterly payments, Suzie has \$642 000 in the perpetuity.

Solutions to Exercise 9G

Note: in the financial solver screens below, the shaded box is the quantity that is found by pressing ENTER or tapping Solve.

1

a

```
2000                       2000
2000·1.08+1000             3160
3160·1.08+1000           4412.8
4412.8·1.08+1000       5765.824
5765.824·1.08+1000
                     7227.08992
```

```
7227.08992·1.08+1000
                    8805.257114
```

The balance of the investment after 5 years is $8805.26

b The principal value is V_0.
The principal of this investment is $2000.

c The amount added to the investment each year is $1000.

d The value of R from the recurrence relation is

$R = 1.08$

$= 1 + \frac{r}{100}$

$1 + \frac{r}{100} = 1.08$

$\frac{r}{100} = 0.08$

$r = 8\%$ per annum

2

a

```
20000                      20000
20000·1.025+2000           22500
22500·1.025+2000         25062.5
25062.5·1.025+2000
                      27689.0625
```

The balance of the investment after 3 quarters is $27 689.06

b The principal value is V_0.
The principal of this investment is $20 000.

c The amount added to the investment each year is $2000.

d The value of R from the recurrence relation is

$R = 1.025$

$= 1 + \frac{r}{100}$

$1 + \frac{r}{100} = 1.025$

$\frac{r}{100} = 0.025$

$r = 2.5\%$ per quarter

$r = 4 \times 2.5\%$ per annum

$= 10\%$ per annum

3 Sarah invests $1500, so $V_0 = 1500$.

Sarah adds $40 to the investment each month so $D = 40$.

The annual interest rate is 9%.
The monthly interest rate

$= \frac{9}{12}\%$ per month

$= 0.75\%$ per month

$R = 1 + \frac{r}{100}$

$= 1 + \frac{0.75}{100}$

$= 1.0075$

$V_0 = 1500,\ V_{n+1} = 1.0075\ V_n + 40$

4

a **i** The original amount invested is the balance of the annuity after 0 payments.

The original amount invested is $5000.

ii The amount added to the principal each month is the payment made.

Amount added to principal is $100.

iii Using the row for payment 1, the interest earned is \$50.

iv monthly interest rate

$= \frac{50}{5000} \times 100\%$

= 1% per month

v annual interest rate

= monthly interest rate × 12

= 12 × 1%

= 12% per annum

vi Using the row for payment 2, the principal increase is \$151.50

vii Using the row for payment 8, the value of the investment after 8 months is \$6242.85

b **i** A = 1% of the previous balance

$= \frac{1}{100} \times 5301.50$

= 53.015

= 53.02 (2 d.p)

B = Payment made + interest earned

= 100 + 53.02

= 153.02

C = previous balance + principal increase

= 5301.50 + 153.02

= 5454.52

ii total interest

= balance after 8 months

– principal

– payments made

= 6242.85 – 5000 – 8 × 100

= 442.85

or

total interest

= sum of interest column

= 50 + 51.50 + 53.02 + 54.55

+ 56.09 + 57.65 + 59.23 + 60.82

= 442.86

note: The difference of 1 cent between these two methods is due to the rounded values in the amortisation table.

5

N:	10
I%:	5.7
PV:	-12000
Pmt or PMT:	-250
FV:	15136.4594
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After 10 months, the value of Lee's investment is \$15 136.46

6

N:	8.093680127
I%:	5.2
PV:	-10000
Pmt or PMT:	-200
FV:	12000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The number of months it takes the investment to have value \$12 000 is slightly more than 8 months. This means we have to round UP to 9 months.

It will take 9 months for the investment to have a value of at least \$12 000

7

a 5 years = 5 × 12 = 60 months

N:	5x12
I%:	6.15
PV:	-25000
Pmt or PMT:	-120
FV:	42378.58889
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

If Bree makes regular deposits of \$120 per month, she will have \$42 378.59 after 5 years.

b

N:	5x12
I%:	6.15
PV:	-25000
Pmt or PMT:	120
FV:	25569.07218
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

If Bree makes regular withdrawals of $120 per month, she will have $25 569.07 after 5 years.

8 Jarrod did not make an initial deposit, so PV = 0.

a

N:	10x12
I%:	6
PV:	0
Pmt or PMT:	-500
FV:	81939.6734
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After 10 years, Jarrod will have $81 939.67 in his account.

b At the end of the first 10 years, the FV becomes the PV.

The Pmt or PMT value must be positive now because the bank is giving this money to Jarrod.

N:	10x12
I%:	6
PV:	-81939.6734
Pmt or PMT:	500
FV:	67141.10077
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After another 10 years, Jarrod has $67 141.10 in his account

Solutions to Review: Extended-response questions

Note: in the financial solver screens below, the shaded box is the quantity that is found by pressing ENTER or tapping Solve.

1

a Initially, there were 840 fish in the pond, so $F_0 = 840$.

The population grows at the rate of 6% every month, so $r = 6$ and

$R = 1 + \dfrac{6}{100}$

$R = 1.06$

Every month, 80 fish are removed from the pond so

$D = 80$

D is subtracted in the recurrence relation.

$F_0 = 840$, $F_{n+1} = 1.06\, F_n - 80$

b

```
840                              840
840·1.06-80                    810.4
810.4·1.06-80                779.024
779.024·1.06-80            745.76544
745.76544·1.06-80        710.5113664
710.5113664·1.06-80
                         673.1420484
```

After 5 months, there are 673 (to the nearest whole fish) in the pond.

c

```
673.1420484·1.06-80
                         633.5305713
633.5305713·1.06-80
                         591.5424056
          .
          .
          .
          .
```

```
          .
          .
          .
          .
80.09316889·1.06-80
                         4.898759022
4.898759022·1.06-80
                        -74.80731544
```

After 18 months there will be no fish in the pond.

2

a Barry is considering borrowing $250 000, so $V_0 = 250\,000$.

The interest rate of the loan is 4.9% compounding monthly, so

$r = 4.9/12$ and

$R = 1 + 4.9/12/100$

Barry would make monthly payments of $1800 so

$D = 1800$

D is subtracted in the recurrence relation.

$V_0 = 250\,000$,

$V_{n+1} = (1+4.9/12/100)\, V_n - 1800$

b

```
250000                             250000
250000·(1+4.9/12/100)-1800
                                249220.8333
249220.8333·(1+4.9/12/100)-1800
                                248438.4851
          .
          .
          .
```

```
          .
          .
          .
242063.5913·(1+4.9/12/100)-1800
                                241252.0176
241252.0176·(1+4.9/12/100)-1800
                                  240437.13
```

After 12 months, Barry would owe $240 437.13

c It would take 58 months to reduce the loan to below $200 000

d **i** interest on first month

$= \dfrac{4.9}{12 \times 100} \times \$250\ 000$

$= \$1020.83$

ii In the first year, Barry is only paying interest, so

Total interest $= 12 \times \$1020.83$

$= \$12\ 249.96$

iii After the 12th interest only payment, Barry would still owe the original principal of $250 000.

3

a Interest rate is 6.25% per annum, compounding monthly so,

interest received

$= \dfrac{6.25}{12 \times 100} \times \$150\ 000$

$= \$781.25$

Samantha would receive $781.25 each month.

b

N:	1x12
I%:	6.25
PV:	-150000
Pmt or PMT:	1000
FV:	147298.4838
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Samantha would have $147 298.48 left after 12 months.

c

N:	37.27814231
I%:	6.25
PV:	-150000
Pmt or PMT:	2000
FV:	100000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

It takes 37.278… month for the annuity to have value $100 000 so the value of the investment would drop below $100 000 after 38 months.

d **i**

N:	41.8304015
I%:	6.25
PV:	-150000
Pmt or PMT:	4000
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The investment would last for 42 monthly payments.

ii

N:	42
I%:	6.25
PV:	-150000
Pmt or PMT:	4000
FV:	-676.9316138
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

note: The future value is negative, which means Samantha will need to effectively pay this money back to the bank.

Samantha's final payment is $676.93 less than normal.

Final payment $= \$4000 - \$676.93$

$= \$3323.07$

4

a

N:	5x4
I%:	11
PV:	10000
Pmt or PMT:	-656.7173061
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

The quarterly payment is $656.72

b total cost = 20 payments of $656.72

$= 20 \times \$656.72$

$= \$13\ 134.40$

The total cost is $13 134 to the nearest dollar

c interest paid = total cost – principal

$= \$13\ 134.40 - \$10\ 000$

$= \$3134.40$

The total interest paid is $3134 to the nearest dollar

5

a

N:	10x4
I%:	12.75
PV:	65000
Pmt or PMT:	-2580.208848
FV:	-25000
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Note: The future value is negative, because this is money that must be paid back to the bank in the future.

The quarterly payment on this loan is $2580.21

b

N:	20x4
I%:	12.75
PV:	65000
Pmt or PMT:	-2184.632753
FV:	-25000
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

The quarterly payment on this loan is $2184.63

c

N:	10x4
I%:	12.75
PV:	65000
Pmt or PMT:	-2897.917502
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

The quarterly payment on this loan is $2897.92

d

N:	20x4
I%:	12.75
PV:	65000
Pmt or PMT:	-2255.106349
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

The quarterly payment on this loan is $2255.11

6

a

N:	39.42661088
I%:	10.8
PV:	24800
Pmt or PMT:	-750
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The number of months (N) is 39.4266… which means slightly more than 39 payments of $750 are required. In a practical sense, 40 payments are required, with the last one being less than all the others

The least number of payments required will be 40.

b Find the future value after 40 payments.

N:	40
I%:	10.8
PV:	24800
Pmt or PMT:	-750
FV:	429.2197799
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Note: The future value is positive, which means this is money the bank must pay back to the Andersons.

The bank must pay back $429.22, so
final payment = $750 – $429.22
= $320.78

c $750 is one third of $2250 so each quarter, the family is paying the same amount. However, more regular compounds mean the balance is reduced more often, which means less interest is paid in the long run. So, the Andersons made the right decision.

7

a interest rate = $\frac{9.6}{4}$ % per quarter

= 2.4% per quarter

interest paid with first payment
= 2.4% of principal
= $\frac{2.4}{100} \times \$100\,000$
= $2400

payment amount can be found using a finance solver

N:	25x4
I%:	9.6
PV:	100000
Pmt or PMT:	-2647.03847
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

principal reduction
= payment – interest
= $2647.04 – $2400
= $247.04

b

N:	10x4
I%:	9.6
PV:	100000
Pmt or PMT:	-2647.03847
FV:	-83713.47545
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

After 10 years, the amount owing on the loan is $83 713.48

8

a

N:	10x12
I%:	4.5
PV:	-750000
Pmt or PMT:	0
FV:	1175244.582
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After 10 years, Helene will have $1 175 244.58

b

N:	290.8221344
I%:	3.5
PV:	-1175244.582
Pmt or PMT:	6000
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Note: Use the copy and paste feature of your calculator to transcribe the FV from the previous calculation to the PV for this one.

Helene's annuity investment would last for 290.8 months, or 291 months with a smaller than usual payment in the last month.

c interest rate = 3.5% per annum

= $\frac{3.5}{12}$ % per month

= 0.29166…% per month

perpetuity payment
= $\frac{0.29166...}{100} \times \$1\,175\,244.58$
= $3427.80

or

N:	1
I%:	3.5
PV:	-1175244.582
Pmt or PMT:	3427.796698
FV:	1175244.582
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Helene would receive a monthly payment of $3427.80 from her perpetuity investment.

9

a **i**

N:	20x12
I%:	6.25
PV:	250000
Pmt or PMT:	-1827.320506
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The monthly payment on this loan is $1827.32

ii Total interest
= total paid – principal
= 240 × $1827.320506 – $250 000
= $438 556.92 – $250 000
= $188 556.92

The total interest that will be paid is $188 557 to the nearest dollar

b

N:	60
I%:	6.25
PV:	250000
Pmt or PMT:	-1827.320506
FV:	-213117.7716
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After 60 payments, the value of the loan will be $213 118, correct to the nearest dollar.

c 9 years = 9 × 12 payments
= 108 payments

With a new principal of $100 000, and new payments of $1250 each month,

N:	103.7565931
I%:	6.25
PV:	100000
Pmt or PMT:	-1250
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

it would take a further 103.756… payments, or a further 104 payments with a smaller than usual last payment.

total number of payments
= 108 + 104
= 212

Solutions to Review: Multiple-choice questions

Note: in the financial solver screens below, the shaded box is the quantity that is found by pressing ENTER or tapping Solve.

1 There is an 8% increase in wombat numbers each year, so

$R = 1 + \frac{8}{100}$

$R = 1.08$

There are 60 wombats that die each year, so
$D = 60$ and this is subtracted in the recurrence relation

There are currently 490 wombats, so
$V_0 = 490$

$V_0 = 490,\ V_{n+1} = 1.08\ V_n - 60$ **C**

2 From the recurrence relation,
$R = 1.007$ (monthly)
$1 + r/100 = 1.007$
$r/100 = 0.007$
$r = 0.7$

monthly interest rate = 0.7%
annual interest rate = 12 × 0.7%
= 8.4% **D**

3 Amount invested is $18 000, so
$V_0 = 18\,000$

Interest rate is 6.8% per annum, with yearly compounds, so
$R = 1 + 6.8/100$
$R = 1.068$

$2500 is added to the investment each year so
$D = 2500$ and this is added in the recurrence relation.

$V_0 = 18\,000, \quad V_{n+1} = 1.068\ V_n + 2500$ **C**

4

N:	5
I%:	6.4
PV:	28000
Pmt or PMT:	-1200
FV:	-22690.33146
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

B

5 First, calculate how many months it takes to pay back the loan.

N:	24.9845157
I%:	6.4
PV:	28000
Pmt or PMT:	-1200
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Find the balance after 25 months:

N:	25
I%:	6.4
PV:	28000
Pmt or PMT:	-1200
FV:	18.53254474
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The future value is positive, which means the bank must give back $18.53

Final payment = $1200 – $18.53
= $1181.47 **D**

6 payment = interest charged

$$= \frac{5.9}{12}\% \times \$175\ 000$$

$$= \frac{5.9}{12 \times 100} \times \$175\ 000$$

$$= \$860.42$$

or

N:	1
I%:	5.9
PV:	175000
Pmt or PMT:	-860.4166667
FV:	175000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

C

7 interest from perpetuity = \$400
3.4% of investment = \$400
3.4/100 × investment = \$400

$$\text{investment} = \$\frac{400 \times 100}{3.4}$$

investment = \$11 764.70588 **D**

8

N:	12
I%:	10
PV:	6000
Pmt or PMT:	-584.9227619
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

A

9

N:	12
I%:	7
PV:	-35300
Pmt or PMT:	220
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

A

10 First, calculate the monthly payments:

N:	5x12
I%:	12
PV:	12000
Pmt or PMT:	-266.9333722
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

then calculate future value after 2 years:

N:	2x12
I%:	12
PV:	12000
Pmt or PMT:	-266.9333722
FV:	-8036.697849
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

C

11 The principal is the balance after payment 0. **C**

12 The periodic payment is the value in the payment amount column **D**

13 Using row 5 of the table, the reduction in value is \$243.86 **C**

14 Using row 6 of the table, the interest is \$155.16

interest as percentage of payment

$$= \frac{155.16}{400} \times 100\%$$

$$= 38.79\%$$

B

Chapter 10 – Revision: Recursion and financial modelling

Solutions to Exercise 10A: Multiple-choice questions

Note: in the financial solver screens below, the shaded box is the quantity that is found by pressing ENTER or tapping Solve.

1 Test each sequence by applying the rule "multiply by 3 and subtract 2":

A: $5 \times 3 - 2 = 13$, not 14
B: $4 \times 3 - 2 = 10$, correct
$10 \times 3 - 2 = 28$, correct
$28 \times 3 - 2 = 82$, correct
$82 \times 3 - 2 = 244$, correct
C: $10 \times 3 - 2 = 28$, not 17
D: $2 \times 3 - 2 = 4$, not 8
E: $3 \times 3 - 2 = 7$, not 11

Answer: B

2 Starting value is 6, so $V_0 = 6$
Multiply by 0.5 and add 4 means
$V_{n+1} = 0.5 \times V_n + 4$

$V_0 = 6, \quad V_{n+1} = 0.5V_n + 4$

Answer: D

3 $V_0 = 50, \; V_{n+1} = 0.84V_n + 15$

```
50                          50
50·0.84+15                  57
57·0.84+15               62.88
62.88·0.84+15          67.8192
67.8192.88·0.84+15
                      71.968128
```

```
71.968128.88·0.84+15
                    75.45322752
```

Answer: C

4 $A_0 = 270, \; A_{n+1} = 0.92A_n - 8$

Use calculator recursion and notice that for a large number of iterations (large value of n), the value of A_n approaches –99.99999

Or

0.92 x –100 = –92
–92 – 8 = –100

If the value of A_n was exactly –100, it would stay as –100. But A_n will never be exactly –100, but will always be greater than it.

Answer: C

5 Geometric growth occurs with a multiplying factor greater than 1.

A: multiplying factor of 0.93 which is less than 1.
B: multiplying factor of 1.05 which is greater than 1.
C: multiplying factor of 0.94 which is less than 1.
D: no multiplying factor so not geometric growth.
E: multiplying factor of 1.04 but there is also an element of decay (–2).

Answer: B

6 The recurrence relation
$V_0 = 20\ 000$, $V_{n+1} = 1.045V_n + 200$

shows geometric growth plus an element of linear growth

Growth rate
$r = (1.045 - 1)\times 100\%$
$r = 4.5\%$

periodic addition = 200

A: has annual payment, there are additions, not payments
B: correct
C: depreciation is decay, not growth
D: interest rate is not 5.4%
E: interest rate is not 5.4%, there are additions, not payments

Answer: B

7 $R = 1.0017$ (fortnightly)

$1 + r/100 = 1.0017$
$r = (1.0017 - 1)\times 100\%$
$r = 0.17\%$ per fortnight
$r = 0.17 \times 26\%$ per annum

annual interest rate = 4.42%

Answer: C

8 $r = 4.5\%$
$R = 1 - 4.5/100$ for depreciation
$R = 0.955$

Purchase price
$V_0 = \$24\ 990$

Rule: $V_n = R^n \times V_0$
$V_n = (0.955)^n \times 24\ 990$

Answer: D

9 depreciation = 15% of \$5000
= 15/100 × \$5000
= \$750

Let V_n be the value of the machine after n years.

$V_n = 5000 - n \times 750$

If the value is \$1500,

$1500 = 5000 - n \times 750$
$750n = 5000 - 1500$
$750n = 3500$
$n = 3500 / 750$
$n = 4.666\ldots$

The number of years is closest to 5.

Answer: D

10 Flat rate depreciation means the graph will be linear (straight line).
The purchase price is \$20 000 so the graph passes through point (0,20 000)
The value after 5 years is \$4000 so the graph passes through point (5,4000)

Answer: A

11 $r = 6\%$ per annum
$r = 6/12\ \%$ per month
$r = 0.5\%$
$R = 1 + 0.5/100 = 1.005$

initial investment = \$3000

n years = $12\times n$ months

Value after n years = $1.005^{12n} \times 3000$

Answer: B

12

N:	2x12
I%:	6
PV:	-5000
Pmt or PMT:	-500
FV:	18351.7765
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Answer: C

13 Value of investment after 4 years:

N:	4x4
I%:	5.9
PV:	-6350
Pmt or PMT:	0
FV:	8026.360928
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Value of investment after 5 years:

N:	5x4
I%:	5.9
PV:	-6350
Pmt or PMT:	0
FV:	8510.497042
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Interest in 5th year
= value after 5 years – value after 4 years
= \$8510.497042 – \$8026.360928
= \$484.136114

Answer: C

14 The value of the investment is growing, so the multiplying factor must be greater than 1.

Discount answer B and D

The rule for the investment is of the form
$V_n = R^n \times V_0$

Discount answer C and E

Answer: A

15 R=1+6.5/100 = 1.065

V_0 = 15 000, $V_{n+1} = 1.065V_n$

```
15000                    15000
15000·1.065              15975
15975·1.065          17013.375
17013.375·1.065
                   18119.24438
```

Answer: B

16

N:	15x12
I%:	6.5
PV:	40000
Pmt or PMT:	-348.4429461
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Answer: B

17 Total depreciation
= purchase price – depreciated value
= \$6500 – \$2000
= \$4500

Depreciation per year
= \$4500 / 5 years
= \$900 per year

Answer: B

18 Find value of loan after 10 years:

N:	10x12
I%:	5.6
PV:	80000
Pmt or PMT:	-555
FV:	-50866.10895
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

After 10 years, Peter still owes approx. $51000, so answer A is incorrect.

Find the balance after 5 years:

N:	5x12
I%:	5.6
PV:	80000
Pmt or PMT:	-555
FV:	-67454.55437
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Balance after 5 years is not $40 000, so answer B is incorrect.

Reducing balance loans have interest calculated on the remaining balance each month. As the balance decreases each month, so will the amount of interest paid. Answer C is incorrect.

Monthly payments:

N:	239.8710942
I%:	5.6
PV:	80000
Pmt or PMT:	-555
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

repay the loan after 240 payments, or 240/12 = 20 years.

Weekly payments and weekly compounds:

N:	982.5028011
I%:	5.6
PV:	80000
Pmt or PMT:	-132
FV:	0
Pp/y or P/Y:	52
Cp/Y or C/Y:	52

repay the loan after 983 payments or 983/52 = 18.9 years

This is less time than monthly payments so answer D is correct.

Answer: D

19 Find the interest earned in the first 6 months:

N:	6
I%:	6.1
PV:	-15000
Pmt or PMT:	0
FV:	15463.35362
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Interest earned
= \$15 463.35 – \$15 000
= \$463.35

For the next six months:
FV becomes PV and the rate changes.

N:	6
I%:	6.25
PV:	-15463.35362
Pmt or PMT:	0
FV:	15952.91934
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Interest earned
=\$15 952.92 – \$15 463.35
= \$489.57

Total interest = \$463.35 + \$489.57
= \$952.92

Answer: D

Solutions to Exercise 10B: Extended-response questions

Note: in the financial solver screens below, the shaded box is the quantity that is found by pressing ENTER or tapping Solve.

1

a Purchase price is the value of the bike at year 0. From the graph, the purchase price of the bike was \$8000.

b From the graph, the value of the bike after 2 years was \$5000.

Amount of depreciation = \$8000 – \$5000
= \$3000

annual depreciation = \$3000 / 2
= \$1500

c i Purchase price was \$8000, so
$V_0 = 8000$
Annual depreciation was \$1500, so
$D = 1500$

$V_0 = 8000, \quad V_{n+1} = V_n - 1500$

ii $V_0 = 8000$
$V_1 = 8000 - 1500 = 6500$
$V_2 = 6500 - 1500 = 5000$
$V_3 = 5000 - 1500 = 3500$
$V_4 = 3500 - 1500 = 2000$
$V_5 = 2000 - 1500 = 500$

Hugo's bike will be valued at \$500 after 5 years.

d Value after 2 years is \$5000.
Depreciation over 2 years is \$3000

number of km = \$3000 / \$0.25
= 12 000

The bike travelled 12 000 km over 2 years.

e i Initial investment is \$5000, so
$V_0 = 5000$

$r = 4.8\%$ per annum
$r = 4.8/12 = 0.4\%$ per month

$R = 1 + 0.4/100 = 1.004$

Additional payment is \$200 each month, so $D = 200$

$V_0 = 5000, \quad V_{n+1} = 1.004V_n + 200$

ii Doubling means the value will be at least \$10 000:

```
5000                          5000
5000·1.004+200                5220
5220·1.004+200             5440.88
5440.88·1.004+200
                         5662.64352
        .
        .
        .
```

```
        .
        .
        .
9571.2819·1.004+200
                        9809.567027
9809.567027·1.004+200
                         10048.8053
```

iii Find the balance of the loan after 1 year:

N:	1x12
I%:	4.8
PV:	-5000
Pmt or PMT:	-200
FV:	7698.861414
Ppl/y or P/Y:	12
Cp/Y or C/Y:	12

Interest
= value after 1 year – principal – additional payments
= \$7698.86 – \$5000 – 12 × \$200
= \$298.86

Hugo will earn \$298.86 interest after 1 year.

2

a interest rate per quarter
= 6.4/4
= 1.6% per quarter

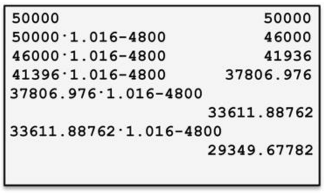
b **i** Principal borrowed is $50 000, so
$V_0 = 50\ 000$

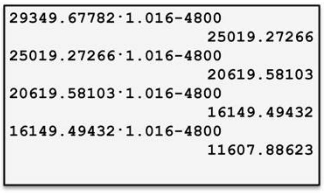
interest rate = 1.6%, so
$R = 1 + 1.6/100$
$R = 1.016$

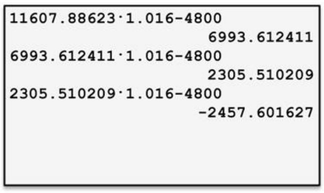
Quarterly payments are $4800, so
$D = 4800$

$V_0 = 50\ 000$,
$V_{n+1} = 1.016V_n - 4800$

ii







Term of loan
= 12 quarters
= 12/4 years
= 3 years.

It will take Daniel 3 years to repay his loan.

3

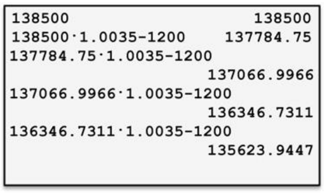
a Principal of loan is $138 500, so
$V_0 = 138\ 500$

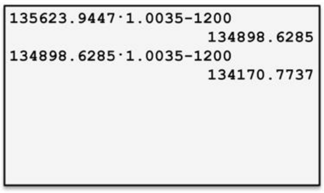
interest rate = 4.2% per annum, so
$R = 1 + 4.2/12/100$
$R = 1.0035$

Monthly payments are $1200, so
$D = 1200$

$V_0 = 138\ 500$, $V_{n+1} = 1.0035V_n - 1200$

b





The sequence of numbers representing the value of Leanne's loan is:

138 500, 137 784.75, 137 067.00,
136 346.73, 135 623.94, 134 898.63,
134 170.77, …

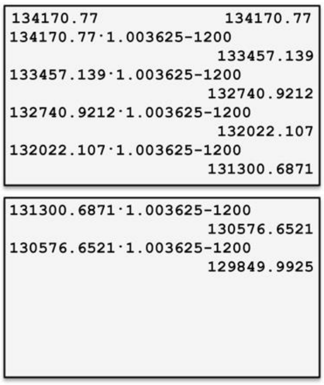
c Principal of loan is \$134 170.77, so
$V_0 = 134\ 170.77$

interest rate = 4.35% per annum, so
$R = 1 + 4.35/12/100$
$R = 1.003625$

Monthly payments are \$1200, so
$D = 1200$

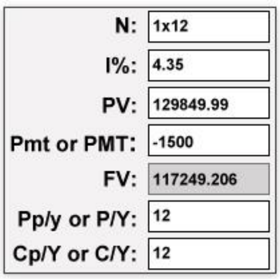
$V_0 = 134\ 170.77$,
$V_{n+1} = 1.003625V_n - 1200$

d



After a further 6 months, Leanne owes \$129 849.99 on her loan.

e



After a further 12 months, Leanne will owe \$117 249.21

4

a The car depreciates at 25% per year, reducing-balance depreciation.

$R = 1 - 25/100$
$R = 0.75$

Let V_n be the value of the car after n years

$V_n = 0.75^n \times 40\ 000$

i after 1 year:
$V_1 = 0.75 \times 40\ 000$
$V_1 = 30\ 000$

After 1 year, the car has value \$30 000.

ii after 3 years
$V_3 = 0.75^3 \times 40\ 000$
$V_3 = 16\ 875$

After 3 years, the car has value \$16 875

b The computer depreciates at \$8000 per year, flat-rate depreciation.

Let C_n be the value of the computer after n years.

$C_n = 40\ 000 - n \times 8000$

After 2 years
$C_2 = 40\ 000 - 2 \times 8000$
$C_2 = 24\ 000$

After 2 years, the computer has value \$24 000

c Use a table to compare the depreciation:

	Car V_n	**Computer C_n**
n	$V_n = 0.75^n \times 40\,000$	$C_n = 40\,000 - n \times 8000$
0	40 000	40 000
1	30 000	32 000
2	22 500	24 000
3	16 875	16 000

The depreciated value of the car is first more than that of the computer after 3 years.

d percentage depreciation
= \$8000 / \$40 000 × 100%
= 20%

Annual flat rate depreciation of the computer equipment is 20%.

5

a i annual interest rate = 7.9%
quarterly interest rate = 7.9/4%
= 1.975 %

First interest payment
= 1.975% of \$25 000
= 1.975 / 100 × \$25 000
= \$493.75

\$493.75 of Lucy's first payment is interest.

ii Determine the balance after 4 years:

N:	4x4
I%:	7.9
PV:	25000
Pmt or PMT:	-1600
FV:	-4420.270689
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

After 4 years, Lucy still owes money (\$4420.27) so no, these payments will not allow Lucy to repay the loan within 4 years.

b i

N:	20.00372067
I%:	7.9
PV:	25000
Pmt or PMT:	-1600
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

Lucy will need to make 21 payments to fully repay the loan. This will take 21 quarters.

ii

N:	17.00685248
I%:	7.9
PV:	25000
Pmt or PMT:	-1745
FV:	0
Pp/y or P/Y:	4
Cp/Y or C/Y:	4

If Lucy increased her payments to \$1745 each quarter, it would take her 18 quarters, which is a reduction of 3 quarters compared to quarterly payments of \$1525.

6

a amount paid to superannuation
= 21% or Roslyn’s monthly salary
= 21/100 × ($54 200/12)
= $948.50

b Roslyn begins her investment with no principal at all, just regular payments of $948.50

Roslyn will turn 60 after 23 years.

N:	23x12
I%:	4.2
PV:	0
Pmt or PMT:	-948.50
FV:	439829.2563
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

At her 60th birthday Roslyn will have $439 829.26 available in her superannuation.

c

N:	1
I%:	4.25
PV:	-439829.2563
Pmt or PMT:	-1557.728658
FV:	439829.2563
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

or

monthly payment
= 4.25/12 % of $439829.2563
= 4.25/12/100 × $439829.2563
= $1557.728…

Roslyn will receive a monthly payment of $1557.73

7

a

N:	5x12
I%:	5.5
PV:	-825000
Pmt or PMT:	-1500
FV:	1188776.847
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Shelly will have $1 188 776.85 in the account when she retires.

b

N:	237.1399317
I%:	5.75
PV:	-1188776.847
Pmt or PMT:	8400
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Shelly’s annuity investment will last for 238 months.

c

N:	1
I%:	5.75
PV:	-1188776.847
Pmt or PMT:	5696.222392
FV:	1188776.847
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

The payment from a perpetuity would be $5696.22 each month.

8

a *Note: Make sure the payments per year and compounds per year are both 1. This annuity investment pays annual interest with an annual prize.*

N:	2
I%:	8.25
PV:	-100000
Pmt or PMT:	10000
FV:	96355.625
Pp/y or P/Y:	1
Cp/Y or C/Y:	1

After the first two scholarships are awarded, there is $96 355.625 left in the account.

b

N:	10
I%:	8.25
PV:	-100000
Pmt or PMT:	10000
FV:	74345.55335
Pp/y or P/Y:	1
Cp/Y or C/Y:	1

After 10 years of awarding scholarships, Robyn has $74 345.55 left in her account.

c

N:	1
I%:	8.25
PV:	-100000
Pmt or PMT:	8250
FV:	100000
Pp/y or P/Y:	1
Cp/Y or C/Y:	1

or

scholarship value = interest earned
= 8.25% of $100 000
= 8.25/100 × $100 000
= $8250

If the scholarship is awarded forever, the value of the scholarship is $8250.

d value of scholarship
= 8.25% of investment

so,

8.25% of investment = $10 000
8.25 / 100 × investment = $10 000
investment = $10 000 × 100 / 8.25
investment = $121 212.12

Robyn will need to invest $121 212 in order to pay the $10 000 scholarship in perpetuity.

Chapter 11 – Matrices I

Solutions to Exercise 11A

1

a A 3 × 5 matrix has three rows and 5 columns, so each row has 5 elements. Thus a 3 × 5 matrix has 3 × 5 = 15 elements.

2 **A** $m \times n$ matrix has mn elements. From this it follows matrices of the orders 12 × 1, 12 × 1, 6 × 2, 2 × 6, 3 × 4, 4 × 3 all have 12 elements.

3 To find the transpose of a matrix, interchange its rows and columns. Thus,

a $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

b $\begin{bmatrix} 3 \\ 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \end{bmatrix}$

c $\begin{bmatrix} 9 & 1 & 0 & 7 \\ 8 & 9 & 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 9 & 8 \\ 1 & 9 \\ 0 & 1 \\ 7 & 5 \end{bmatrix}$

4

a Square matrices have the same number of rows as columns. The square matrices are C and E (they have dimensions 2×2 and 3×3 respectively).

b Matrix B has 3 rows.

c The row matrix is A (it contains a single row of elements only).

d The column matrix is B (it contains a single column of elements only).

e Matrix D has 4 rows and 2 columns.

f The order of matrix E is 3×3. It has 3 rows and 3 columns.

g The order of matrix A is 1×5. It has 1 row and 5 columns.

h The order of matrix B is 3×1. It has 3 rows and 1 column.

i The order of matrix D is 4×2. It has 4 rows and 2 columns.

j There are 9 elements in matrix E.

k There are 5 elements in matrix A.

l $a_{14} = 0$: a_{14} is the element in row 1 and column 4

m $b_{31} = 1$: b_{31} is the element in row 3 and column 1

n $c_{11} = 0$.

o $d_{41} = 4$.

p $e_{22} = -1$.

q $d_{32} = 3$.

r $b_{11} = 3$.

s $c_{12} = 1$.

5

a $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

In an upper triangular matrix, all of the elements below the leading diagonal are 0 giving the upper elements a triangular appearance.

b none

An identity matrix is a square diagonal matrix in which all of the diagonal elements are 1 and the remaining elements are zero. None of the matrices given satisfy this criterion.

c $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

In a diagonal matrix all of the non-diagonal elements are zero.

d $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

In a symmetric matrix, the elements in the leading diagonal are a mirror image of the elements above the diagonal.

6 Set up a blank 3×2 matrix and call it *B*.

$$B = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

The rule $b_{ij} = i \times j$ tells us to find the value each element, multiply its row number by its column number.

$$B = \begin{bmatrix} 1\times1=1 & 1\times2=2 \\ 2\times1=2 & 2\times2=4 \\ 3\times1=3 & 3\times2=6 \end{bmatrix}$$

7 Set up a blank 4×1 column matrix and call it *C*.

$$C = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

The rule $c_{ij} = i+2j$ tells us to find the value each element, add twice its column number to its row number.

$$C = \begin{bmatrix} 1+2\times1=3 \\ 2+2\times1=4 \\ 3+2\times1=5 \\ 4+2\times1=6 \end{bmatrix}$$

8

a Follow the instructions on page 371 (TI) or page 371 (CAS) to enter the matrix and determine the transpose.

$$B^T = \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}$$

b As for **a** above.

$$C^T = \begin{bmatrix} 4 & -2 \\ -4 & 6 \end{bmatrix}$$

c As for **a** above.

$$E^T = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

d As for **a** above.

$$F^T = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

Solutions to Exercise 11B

1

a See Example 6 for an illustrative example .
Construct the matrix by enclosing the numbers in the table in square brackets to form the square matrix.

$$\begin{bmatrix} 4 & 2 & 1 \\ 6 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

Order: 3×3: the matrix has 3 rows and 3 columns.

b Construct the matrix by enclosing the numbers in row B of the table in square brackets to form the row matrix.

$$\begin{bmatrix} 6 & 2 & 3 \end{bmatrix}$$

Order: 1×3: the matrix has 1 row and 3 columns.

c Construct the matrix by enclosing the numbers in the 'Computers' column in square brackets to form the column matrix.

$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Order: 3 ×1: the matrix has 3 rows and 1 column.

The sum of the elements will represent the total number of computers owned by members of the three households.

2

a See Example 6 for an illustrative example.
Construct the matrix by enclosing the numbers in the table in square brackets to form the rectangular matrix.

$$\begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix}$$

Order: 2×3: 2 rows and 3 columns

b Row matrix:

$$\begin{bmatrix} 24 & 32 & 11 \end{bmatrix}$$

Order = 1×3.

c Column matrix:

$$\begin{bmatrix} 24 \\ 32 \end{bmatrix}$$

Order: 2×1.
The sum of the elements will represent the total number of small cars sold by both dealers.

3 See Example 7 for an illustrative example.
The 2×8 matrix representing the paired digits listed one under the other is as follows:

$$\begin{bmatrix} 3 & 5 & 8 & 7 & 0 & 2 & 3 & 6 \\ 4 & 2 & 2 & 9 & 0 & 0 & 0 & 9 \end{bmatrix}$$

4 See Example 8 for an illustrative example.
The matrices for the given graphs are given below:

a

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

b

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

c

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

5 See Example 8 for an illustrative example.
With Town 1 as the first row and column, Town 2 as the second and Town 3 as the third, the following matrix can be created:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$$

6 See Example 9 for an illustrative example.

a $f_{3\,4} = 1$, so girls 3 and 4 are friends.

b $f_{2\,5} = 0$, so girls 2 and 5 are **not** friends.

c The sum of row 3 elements will tell us the total number of friends for girl 3: three friends.

d The girl with the least friends is girl 1, with only 1 friend. The girl with the most friends is girl 3, with 3 friends.

7

a **i** the '1' in column B, row L of matrix E is generated by the 'eat' arrow from 'birds' to 'lizards' which tells us that 'birds eat lizards'.

ii in the matrix, the columns represent the 'eaters' and the rows represent the 'eaters'. The row of zeros indicates that 'nothing eats birds'.

b Complete the matrix column-by-column as set out below.

Insect (*I*) column:
From the arrows in the diagram we see that insects do not eat any of the other creatures shown. Thus, the last blank element in this column should be a '0'.

Bird (*B*) column:
From the arrows in the diagram, it can be seen that birds eat insects (I), lizards (L) and frogs (F). Thus, the last blank element should be a '1' because birds eat frogs.

Lizard (*L*) column:
From the arrows in the diagram, lizards (*L*) eat insects (*I*) and frogs (*F*) so the elements in row *I* and *F* should be '1's. The remaining elements are '0's.

Frog (*F*) column:
From the arrows in the diagram, frogs (*F*) eat insects (*I*) only so the element in row *I* should be a '1'. The remaining elements are '0's.

The final matrix should be as follows:

$$z = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{matrix} I & B & L & F \\ \end{matrix}$$

$$z = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} I \\ B \\ L \\ F \end{matrix}$$

Solutions to Exercise 11C

1

a The only matrices to be equal are *C* and *F* They are the same order and contain identical elements in identical positions.

b *A* and *B* have the same order as each other (1×2); *C* and *F* are of the same order (2×3); *D* and *E* are of the same order (2×2).

c Matrices that can be added and subtracted must be of the same order as each other. Thus, the 3 pairs of matrices – *A* and *B*, *C* and *F*, and *D* and *E* – can all be added to and subtracted from each other.

d **i** $A + B = [3 + 1 = 4 \quad 3 + 1 = 4]$

ii $D + E = \begin{bmatrix} 0 + 1 = 1 & 1 + 0 = 1 \\ -1 + 2 = 1 & 2 + -1 = 1 \end{bmatrix}$

iii $C - F = \begin{bmatrix} 0 - 0 = 0 & 1 - 1 = 0 & 4 - 4 = 0 \\ 3 - 3 = 0 & 2 - 2 = 0 & 1 - 1 = 0 \end{bmatrix}$

iv $A - B = [1 - 3 = -2 \quad 3 - 1 = 2]$

v $E - D = \begin{bmatrix} 1 - 0 = 1 & 0 - 1 = -1 \\ 2 - -1 = 3 & -1 - 2 = -3 \end{bmatrix}$

vi $3B = [3 \times 3 = 9 \quad 3 \times 1 = 3]$

vii $4F = \begin{bmatrix} 4 \times 0 = 0 & 4 \times 1 = 4 & 4 \times 4 = 16 \\ 4 \times 3 = 12 & 4 \times 2 = 8 & 4 \times 1 = 4 \end{bmatrix}$

viii $3C + F = 4F$ (since $C = F$)

$$= \begin{bmatrix} 0 & 4 & 16 \\ 12 & 8 & 4 \end{bmatrix}$$

ix $4A - 2B = [4 \times 1 - 2 \times 3 = -2 \quad 4 \times 3 - 2 \times 1 = 10]$

x *E* + *F* is not possible. The matrices are not of the same order, so this addition cannot be performed and thus is undefined.

2 These example are designed to be done quickly by hand.

a $\begin{bmatrix} 1+4=5 & 2+3=5 \\ 4+1=5 & 3+2=5 \end{bmatrix}$

b $\begin{bmatrix} 1-4=-3 & 2-3=-1 \\ 4-1=3 & 3-2=1 \end{bmatrix}$

c $\begin{bmatrix} 1+8=9 & 2+6=8 \\ 4+2=6 & 3+4=7 \end{bmatrix}$

d $\begin{bmatrix} 1-1=0 & 1-1=0 \end{bmatrix}$

e $\begin{bmatrix} 0+1=1 \\ 1+0=1 \end{bmatrix}$

f $\begin{bmatrix} 0+2=2 \\ 3+0=3 \end{bmatrix}$

g Matrices are not of the same order so this addition cannot be performed and thus is undefined.

h $\begin{bmatrix} 0-2=-2 \\ 3-0=3 \end{bmatrix}$

j Matrices are not of the same order so this addition cannot be performed and thus is undefined.

3 Following the instructions given on page 381 (TI) or page 382 (CASIO), enter the matrices involved into your calculator, and perform the required computations to obtain the answers.

4

a $A = \begin{bmatrix} 2.4 \\ 3.5 \\ 1.6 \end{bmatrix}$

$B = \begin{bmatrix} 2.8 \\ 3.4 \\ 1.8 \end{bmatrix}$

$C = \begin{bmatrix} 2.5 \\ 2.6 \\ 1.7 \end{bmatrix}$

$$D = \begin{bmatrix} 3.4 \\ 4.1 \\ 2.1 \end{bmatrix}$$

b The sum of the 4 matrices represents the total sale of CDs in a year for each of the three stores.

$$\begin{bmatrix} 2.4 + 2.8 + 1.5 + 3.4 = 11.1 \\ 3.5 + 3.4 + 2.6 + 4.1 = 13.6 \\ 1.6 + 1.8 + 1.7 + 2.1 = 7.2 \end{bmatrix}$$

5

a $A = \begin{bmatrix} 16 & 104 & 86 \\ 75 & 34 & 94 \end{bmatrix}$

$B = \begin{bmatrix} 24 & 124 & 100 \\ 70 & 41 & 96 \end{bmatrix}$

b $C = A + B = \begin{bmatrix} 16 + 24 = 40 & 104 + 124 = 228 & 86 + 100 = 186 \\ 75 + 70 = 145 & 34 + 41 = 75 & 94 + 96 = 190 \end{bmatrix}$

Matrix *C* represents the total numbers of males and females enrolled in all the programs for the two years of data given.

c $D = B - A = \begin{bmatrix} 24 - 16 = 8 & 124 - 104 = 20 & 100 - 86 = 14 \\ 70 - 75 = -5 & 41 - 34 = 7 & 96 - 94 = 2 \end{bmatrix}$

Matrix *D* represents the increase in the number of males and females in all the programs from 2005 to 2006. The negative element in the matrix represents a decrease in numbers in that category from 2005 to 2006.

d $E = 2B = \begin{bmatrix} 2 \times 24 = 48 & 2 \times 124 = 248 & 2 \times 100 = 200 \\ 2 \times 70 = 140 & 2 \times 41 = 82 & 2 \times 96 = 192 \end{bmatrix}$

Solutions to Exercise 11D

1

a **i** *AB*-yes: (1×2) (2×1) **ii** *BA*-yes: (2×1) (1×2) **iii** *AC*-no: (1×1) (1×3)
iv *CE*-yes: (1×3) (3×1) **v** *EC*-yes: (3×1) (1×3) **vi** *EA*-yes (3×1) (1×2)
vii *DB*-yes: (2×2) (2×1) **viii** *CD*-no: (1×3) (2×2)

b **i** $AB = [1 \times 3 + 3 \times 1 = 6]$

ii $CE = [1 \times 2 + 0 \times 1 + -1 \times 0 = 2]$

iii $DB = \begin{bmatrix} 0 \times 3 + 1 \times 1 = 1 \\ -1 \times 3 + 2 \times 1 = -1 \end{bmatrix}$

iv $AD = \begin{bmatrix} 1\times 0+(-1)\times 3=-3 & 1\times 1+3\times 2=7 \end{bmatrix}$

c Following the instructions given on page 391 (TI and CASIO), enter the matrices involved into your calculator, and perform the required computations to obtain the answers.

2

a $[0\times 1+2\times 0=1]=[0]$

b $[1\times 1+0\times 2=1]=[1]$

c $[2\times 1+0\times 3+1\times 1=1]=[3]$

d $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\times\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 1\times 1+2\times 0=1 \\ 3\times 1+4\times 0=3 \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

e $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\times\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 1\times 4+1\times 1=5 & 1\times 3+1\times 2=5 \\ 0\times 4+1\times 1=1 & 0\times 3+1\times 2=2 \end{bmatrix}=\begin{bmatrix} 5 & 5 \\ 1 & 2 \end{bmatrix}$

f $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\times\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 1\times 2+0\times 0+1\times 1=3 \\ 0\times 2+1\times 0+0\times 1=0 \\ 1\times 2+1\times 0+0\times 1=2 \end{bmatrix}=\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$

3 Following the instructions given on page 391 (TI and CASIO), enter the matrices involved into your calculator, and perform the required computations to obtain the answers.

4

a To sum the rows of a 5×5 matrix, *post* multiply by the 1×5 summing matrix (a column matrix of 1s).

$$\begin{bmatrix} 2 & 4 & 1 & 7 & 8 \\ 1 & 9 & 0 & 0 & 2 \\ 3 & 4 & 3 & 3 & 5 \\ 2 & 1 & 1 & 1 & 7 \\ 5 & 3 & 6 & 7 & 9 \end{bmatrix}\times\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 2+4+1+7+8 \\ 1+9+0+0+2 \\ 3+4+3+3+5 \\ 2+1+1+1+7 \\ 5+3+6+7+9 \end{bmatrix}=\begin{bmatrix} 22 \\ 12 \\ 18 \\ 12 \\ 30 \end{bmatrix}.$$

b To sum the rows of a 3×5 matrix, *pre* multiply by the 1×3 summing matrix (a row matrix of 1s).

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 5 & 1 & 2 & 1 \\ 0 & 3 & 4 & 5 & 1 \\ 4 & 2 & 1 & 7 & 9 \end{bmatrix}$$

$$=\begin{bmatrix} 4+0+4 & 5+3+2 & 1+4+1 & 2+5+7 & 1+!+9 \end{bmatrix}.$$

$$=\begin{bmatrix} 8 & 10 & 6 & 14 & 11 \end{bmatrix}$$

5

$RP =$

$$\begin{bmatrix} 4\times 2+1\times 1+0\times 0=9 \\ 3\times 2+1\times 1+1\times 0=7 \\ 3\times 2+0\times 1+2\times 0=6 \\ 1\times 2+2\times 1+2\times 0=4 \\ 1\times 2+1\times 1+3\times 0=3 \\ 0\times 2+1\times 1+4\times 0=1 \end{bmatrix}$$

6 $TE =$

$$\begin{bmatrix} 10\times 25+20\times 40+30\times 65=3000 \\ 15\times 25+20\times 40+25\times 65=2800 \\ 20\times 25+20\times 40+20\times 65=2600 \\ 30\times 25+20\times 40+10\times 65=2200 \end{bmatrix}$$

7

a Q has 2 rows and 3 columns, so the order of Q is 2×3.

$M = QP$

b **i**

$$=\begin{bmatrix} 2500 & 3400 & 1890 \\ 1765 & 4588 & 2456 \end{bmatrix}\begin{bmatrix} 14.50 \\ 21.60 \\ 19.20 \end{bmatrix}$$

$$=\begin{bmatrix} 145978.00 \\ 171848.50 \end{bmatrix}$$

ii The total revenue at each of the two shopping centres from selling A, B and C.

c PQ is not defined as the no of columns of P(1) is **not equal** to the number of rows of Q(2).

8 **a** To sum the rows of a 3×2 matrix, post multiply by the 2×1 column matrix $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b To sum the columns of a 3×2 matrix, pre multiply by the 1×3 row matrix $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

9 To sum the columns of a 3×3 matrix, pre multiply by the 1×3 row matrix $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 2 \\ 1 & 7 & 3 \\ 8 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 9+1\times 1+1\times 8 & 1\times 0+1\times 7+1\times 3 & 1\times 2+1\times 3+1\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 10 & 9 \end{bmatrix}$$

10

$$\begin{bmatrix} 1 & 3 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1\times x+3\times y \\ 4\times x-4\times y \end{bmatrix}$$

$$= \begin{bmatrix} x+3y \\ 4x-4y \end{bmatrix} = \begin{bmatrix} 16 \\ 5 \end{bmatrix}$$

so

$x + 3y = 16$
$4x - 4y = 5$

11

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2x+y+3z \\ 3x+2y-z \\ 2x+0y+3z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

so

$2x + y + 3z = 2$
$3x + y - z = 4$
$2x + 3z = 3$

Solutions to Exercise 11E

1, 2 & 3 Enter the relevant matrix or matrices into your calculator, use worked Example 22 as a model for performing these computations.

Solutions to Exercise 11F

1 B only: a permutation matrix is a square binary matrix with only one '1' per row.

2

a $$PX = \begin{bmatrix} S & H & U & T \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} H & U & T & S \end{bmatrix}.$$

b $$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I\text{: the identity matrix}$$

so

$P^4X = I^4X = X$

3

a Follow the instructions in Example 24 (page 400) to construct the following 3× 3 communication matrix.

$$C = \begin{array}{c} \begin{matrix} M & F & L \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} M \\ F \\ L \end{matrix}$$

b Follow the instructions in Example 24 (page 400) to construct the following 3× 3 communication matrix.
Use your calculator to square the communication matrix to obtain

$$C^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

c To find the total number of ways Mei can communicate with Freya, calculate

$$T = C + C^2$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{array}{c} \begin{matrix} M & F & L \end{matrix} \\ \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \end{array} \begin{matrix} M \\ F \\ L \end{matrix}$$

From the matrix, we can see that there are 2 ways that Mei can send a message to Freya (Row M, Row F).

4

a In the communication matrix, a zero indicates there is no *direct* communicate link. For example, from the diagram we see that there is no way for a person in tower 1 to directly communicate with a person in tower 4 and this is indicated in the matrix by placing a '0' in row T1, column T4.

b The '1' in row T1, column T2, and row T3, column T2, indicate that a person in towers 1 and 3 can both communicate directly with a person in tower 2.

c From the diagram, it can be seen that the missing element in

- column T2 is zero, because there is no direct communication link between tower T2 and tower T4.
- column T3 is a '1' because there is a direct communication link between tower T2 and tower T3.

d The C^2 matrix gives the number of ways a person can communicate with a person in another tower via a third tower. For example, the '1' in row T1, column T3 indicates that there is one two-step communication link between tower T1 and tower T3. From this diagram, we see that this two-step communication link is T1→T2→T3.

e 6 (=1 + 2 + 2 +1): A redundant communication link is one in which a person eventually end up receiving the message they sent. The communication links in the leading diagonal of the matrix are all redundant communication links. For example the 1 in row T1, column 1, represents the redundant communication link: T1→T2→T1.

f The matrix $T = C + C^2$ shows the total number of one and two-step links between pairs of tower.

g Tower 1 and tower 4 (T1→T2→T3 →T4): for a person to communicate with a person in tower 4, they need to first communicate with a person in tower 2 who will need to pass the message onto person in tower 3. This person can then pass the message onto the person in tower 4 tower link and vice versa.

5 Follow the instructions given in Example 24 (page 400) to arrive at the matrix but note that there are both uni-directional and bi-directional communication links. Because of this, it is very important to be very clear in defining whether the source of the communication is representing by the rows or the columns. In the solution below, the rows represent the source of the communication. For example, the '1' in row A, column E, indicates that a message can be sent *from* A *to* E.

$$\textit{From}\;\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\overset{\textstyle \textit{To}}{\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}}$$

6

a Follow the instructions given in Example 24 (page 400) to arrive at the matrix. Use the row labels to indicate the winners and the column labels to represent the losers as in the answer shown below.

$$\textit{winners}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

To find the total one-step dominances for each team, sum the rows: *A*(2), *B*(1), *C*(0), *D*(3)
Ranking the players according to one-step dominances: *D*, *A*, *B*, *C*

b To find the total two-step dominances for each team, square the one-step dominance matrix and sum the rows.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

To rank the teams by one and two-step dominances, add the one-step and two-step dominance matrices and sum the rows.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{array}{c} 3 \\ 1 \\ 0 \\ 6 \end{array}$$

Ranking the teams using both one and two step dominances we have: *D*(6), *A*(3), B(1), *C*(0) (the same ranking as before).

7

a To rank the players according to one-step dominances, find the total one-step dominances for each of the players by summing the rows and rank the players from highest to lowest according to their total one-step dominances.

$$\text{winners}\quad \begin{array}{c} \\ \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \text{losers} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} = D$$

To find the total one-step dominances for each player, sum the rows: *A*(4), *B*(1), *C*(0), *D*(2), *E*(2)
Ranking the players according to one-step dominances: *A; D* & *E* equal; *B; C*

b To find the total two-step dominances for each player, square the one-step dominance matrix and sum the rows.

$$D^2 = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix} \begin{matrix} 5 \\ 0 \\ 0 \\ 1 \\ 3 \end{matrix}$$

c To rank the players by one and two-step dominances, add the one-step and two-step dominance matrices and sum the rows.

$$T = D + D^2 = \begin{bmatrix} 0 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{bmatrix} \begin{matrix} 9 \\ 1 \\ 0 \\ 3 \\ 5 \end{matrix}$$

Ranking the players using both one and two step dominances we have: *A*(9), *E*(5), D(3), B(1), *C*(0).

Chapter Review: Extended-response questions

1 **a**

- Construct a square matrix with as many rows as there are towns and number both the rows and columns.

$$\begin{matrix} 1 & 2 & 3 \end{matrix}$$
$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

- Count the number of roads connecting each pair of two towns and insert this number into the appropriate cell of the matrix to obtain the matrix below.

For example, for town 1, there are no roads connecting town 1 to itself so insert a 0 in row 1 column 1, and so on to obtain.

$$\begin{matrix} 1 & 2 & 3 \end{matrix}$$
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

b Follow the procedure for **1a** but start with a matrix with 4 rows and 4 columns because there are 4 towns.

$$\begin{matrix} 1 & 2 & 3 & 4 \end{matrix}$$
$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

c Follow the procedure for **1a**, but start with a matrix with 2 rows and 2 columns because there are two towns. As there are no roads connecting the two towns, it will be a matrix of zeros.

2 **a** $C = \begin{bmatrix} 30.45 \\ 2.54 \end{bmatrix}$

C is a 2 × 1 matrix; 2 rows and 1 column.

b $J = [\,5 \quad 4\,]$

J is a 1 × 2 matrix; 1 row and 2 columns.

c *JC* is defined, as the number of columns in *J* equals the number of rows in *C*.

d $JC = [\,5 \quad 4\,]\begin{bmatrix} 30.45 \\ 2.54 \end{bmatrix} = [162.41]$

Jodie's height is 162.41 cm.

e $HC = \begin{bmatrix} 5 & 8 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 30.45 \\ 2.54 \end{bmatrix} = \begin{bmatrix} 172.57 \\ 185.24 \end{bmatrix}$

The heights of the two people are 172.57 cm and 185.24 cm.

3 **a** Bookshop 1 carries 456 non-fiction paperbacks.

b $A = \begin{bmatrix} 334 & 876 \\ 213 & 456 \end{bmatrix}$

Order of A is 2×2.

c $B = \begin{bmatrix} 354 & 987 \\ 314 & 586 \end{bmatrix}$

d $C = A + B$

$$= \begin{bmatrix} 334 + 354 & 876 + 987 \\ 213 + 314 & 456 + 586 \end{bmatrix} = \begin{bmatrix} 688 & 1863 \\ 527 & 1042 \end{bmatrix}$$

C represents the total number of each type of book which is in stock at either bookshop. (Assume no common titles.)

e **i** $E = \begin{bmatrix} 45.00 \\ 18.50 \end{bmatrix}$

The order of E is 2×1, two rows, 1 coulmn.

ii $AE = \begin{bmatrix} 334 & 876 \\ 213 & 456 \end{bmatrix} \begin{bmatrix} 45.00 \\ 18.50 \end{bmatrix} = \begin{bmatrix} 31\,236 \\ 18\,021 \end{bmatrix}$

iii The product represents the total value of fiction and non-fiction books in Bookshop 1. The fiction books are worth \$31 236. The non-fiction books are worth \$18 021.

f $2A = 2\begin{bmatrix} 334 & 876 \\ 213 & 456 \end{bmatrix} = \begin{bmatrix} 668 & 1752 \\ 426 & 912 \end{bmatrix}$

4 **a** P has 1 row and 5 columns; so it has the order 1×5 .

b **i**

$$R = NP$$

$$= \begin{bmatrix} 460 \\ 360 \end{bmatrix} \begin{bmatrix} 0.05 & 0.125 & 0.175 & 0.45 & 0.20 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 57.5 & 80.5 & 207 & 92 \\ 18 & 45 & 63 & 162 & 72 \end{bmatrix}$$

ii R_{24} represents the number of Chemistry students that are expected to be awarded a D.

c **i**

$$F = \begin{bmatrix} \text{Bio Fee} & \text{Chem Fee} \\ 110 & 95 \end{bmatrix}$$

ii

$$L = FN$$

$$= \begin{bmatrix} 110 & 95 \end{bmatrix} \begin{bmatrix} 460 \\ 360 \end{bmatrix}$$

$$= \begin{bmatrix} 110 \times 460 + 95 \times 360 \end{bmatrix}$$

$$= \begin{bmatrix} 84\,800 \end{bmatrix}$$

The total fees paid is \$84 800.

5 **a** $N = \begin{bmatrix} 4 & 8 & 2 \end{bmatrix}$: List the number of successful shots: 4, 8 & 2, in a 1×3 matrix called *N*.

b $P = N \times G = \begin{bmatrix} 4 & 8 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \times 1 + 8 \times 2 + 2 \times 3 \end{bmatrix} = \begin{bmatrix} 26 \end{bmatrix}$

c *P* gives the total number of points scored by Oscar. This can be seen from the way the matrix product has been evaluated above.

Chapter Review: Multiple–choice questions

1 W is the row matrix as it contains a single row. $\Rightarrow$ **C**

2 Square matrices have the same number of rows as columns. U and Y are both 2×2 square matrices. $\Rightarrow$ **D**

3 The order of matrix X is 2 rows by 3 columns = 2×3. $\Rightarrow$ **B**

4 U and Y can be added as they are both 2×2 matrices. $\Rightarrow$ **D**

5 XY is undefined as X is a 2 × 3 matrix and Y is a 2 × 2 matrix. $\Rightarrow$ **D**

6 $-2Y = \begin{bmatrix} -2 \times 0 = 0 & -2 \times 1 = -2 \\ -2 \times -1 = 2 & -2 \times 2 = -4 \end{bmatrix}$ $\Rightarrow$ **A**

7 X is a 2×3 matrix and Z is a 3 × 1 matrix so XZ is 2×1 matrix. $\Rightarrow$ **B**

8 To obtain U^T from U, interchange the rows and columns in U.
If $U = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ then $U^T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow$ **C**

9 $a_{2\,3} = 3$ $\Rightarrow$ **D**

10 $\begin{bmatrix} 4 - -1 = 5 & 0 - 0 = 0 \\ -2 - 1 = -3 & 2 - 1 = 1 \end{bmatrix}$ $\Rightarrow$ **E**

11 $[1 \times 3 + 2 \times 2 + 3 \times 1 = 10]$ $\Rightarrow$ **A**

12 $\begin{bmatrix} 1 \times 2 + 2 \times 1 = 4 \\ 3 \times 2 + 4 \times 1 = 10 \end{bmatrix}$ $\Rightarrow$ **D**

13 X is 3×2, Y is 2×3 and Z is 2×2
XY is defined as columns of X match rows of Y.
YX is defined as columns of Y match rows of X.
XZ is defined as columns of X match rows of Z, this result in a 3×2 matrix, therefore $XZ - 2X$ is defined.
YX results in a matrix of order 2×2, this has the same order as matrix Z, therefore YX + 2Z is defined.
$XY - YX$: XY is a 3×3 matrix, YX is a 2×2 matrix, so $XY - YX$ is not defined.
$\Rightarrow$ **E**

14 The mean of 3, 5, 2, 4 is given by $\frac{3+5+2+4}{4}$ which gives a mean of $\frac{14}{4}$.
The matrix that displays the mean must have an order of 1×1. The only option which gives a 1×1 matrix is D. $\Rightarrow$ **D**

15

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow$ **C**

16 Multiplying the matrices through, we get the equations $3x - 2y = 1$ and $x + 4y = 2$. $\Rightarrow$ **B**

17 Use the matrix to eliminate the incorrect responses.

A: *A* and *D* from the matrix we see that a car can travel from *A* to *D* (there is a '1' in column *A* and row *D*) but not from *D* to *A* (there is a '0' in column *D* and row *A*), so this cannot be the solution.

B: *B* and *C* from the matrix we see that a car can travel from *B* to *C* (there is a '1' in column *B* and row *C*) but not from *C* to *B* (there is a '0' in column *C* and row *B*) , so this cannot be the solution.

C: *C* and *D* from the matrix we see that a car cannot travel from *C* to *D* (there is a '0' in column *C* and row *D*) so this cannot be the solution.

D: *D* and *E* from the matrix we see that a car can travel from *D* to *E* (there is a '1' in column *D* and row *E*) but not from *D* to *E* (there is a '0' in column *D* and row *E*), so this cannot be the solution.

E: *C* and *E* from the matrix we see that a car can travel from *C* to *E* (there is a '1' in column *C* and row *E*). It can also travel from *E* to *C* (there is a '1' in column *E* and row *C*), so this is the solution.

$\Rightarrow$ **E**

Chapter 12 – Matrices II

Solutions to Exercise 12A

1

a **i** 2×2 identity matrix: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ii 3×3 identity matrix: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii 4×4 identity matrix: $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b $AI = \begin{bmatrix} 1 \times 1 + 2 \times 0 = 1 & 1 \times 0 + 2 \times 1 = 2 \\ 0 \times 1 + 3 \times 0 = 0 & 0 \times 0 + 3 \times 1 = 3 \end{bmatrix} = A$

$IA = \begin{bmatrix} 1 \times 1 + 0 \times 0 = 1 & 1 \times 2 + 0 \times 3 = 2 \\ 0 \times 1 + 1 \times 0 = 0 & 0 \times 2 + 1 \times 3 = 3 \end{bmatrix} = A$

Thus, we can see that $AI = IA = A$.

c $CI = \begin{bmatrix} 1 \times 1 + 2 \times 0 + 0 \times 0 = 1 & 1 \times 0 + 2 \times 1 + 0 \times 0 = 2 & 1 \times 0 + 2 \times 0 + 0 \times 1 = 0 \\ 3 \times 1 + 1 \times 0 + 0 \times 0 = 3 & 3 \times 0 + 1 \times 1 + 0 \times 0 = 1 & 3 \times 0 + 1 \times 0 + 0 \times 1 = 0 \\ 0 \times 1 + 1 \times 0 + 2 \times 0 = 0 & 0 \times 0 + 1 \times 1 + 2 \times 0 = 1 & 0 \times 0 + 1 \times 0 + 2 \times 1 = 2 \end{bmatrix} = C$

$IC = \begin{bmatrix} 1 \times 1 + 0 \times 3 + 0 \times 0 = 1 & 1 \times 2 + 0 \times 1 + 0 \times 1 = 2 & 1 \times 0 + 0 \times 0 + 0 \times 2 = 0 \\ 0 \times 1 + 1 \times 3 + 0 \times 0 = 3 & 0 \times 2 + 1 \times 1 + 0 \times 1 = 1 & 0 \times 0 + 1 \times 0 + 0 \times 2 = 0 \\ 0 \times 1 + 0 \times 3 + 1 \times 0 = 0 & 0 \times 2 + 0 \times 1 + 1 \times 1 = 1 & 0 \times 0 + 0 \times 0 + 1 \times 2 = 2 \end{bmatrix} = C$

Thus, we can see that $CI = IC = C$.

2 Multiplying all the pairs of matrices results in the 2×2 identity matrix:

eg

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 2+1\times -1=1 & 1\times 2+2\times -1=0 \\ 1\times 2+2\times -1=0 & 1\times -1+2\times -1=1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

3

a det(A) = 1×3 – 0×2 = 3

b det(B) = 0×4 – 1×3 = –3

c det(C) = 1×4 – 2×2 = 0

d det(D) = –1×4 – 2×2 = –8

4 Follow the instructions on page 419 (TI) or 420 (CASIO) to generate the following inverse matrices using your calculator.

a $A^{-1} = \begin{bmatrix} \frac{10}{11} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

b $B^{-1} = \begin{bmatrix} \frac{20}{9} & \frac{1}{18} \\ -\frac{50}{9} & \frac{1}{9} \end{bmatrix}$

c D^{-1} does not exist, since det(D) = 0.

d $E^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Solutions to Exercise 12B

1 The given sets of simultaneous equations can be written as per below:

a $\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

b $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

c

$$\begin{bmatrix} 5 & 0 & -2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

2 If the equations are inconsistent (graphs parallel with no point of intersection) or dependent (graphs that coincide), the graphs of will not intersect, or intersect at a single point, so there is no unique solution.

3 The matrix equation of the form $AX = C$ will not have a unique solution if $\det(A) = 0$.

4 The matrix equation of the form $AX = C$ will not have a unique solution if $\det(A) = 0$.
Matrix equation **b** does not have a unique solution because $\det A = 4\times1 - 2\times2 = 0$.

5

a If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$
$\det A = 1\times3 - 1\times2 = 1 \neq 0$ so there is a unique solution.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$x = 5$, $y = -3$

b If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$
$\det = 4\times4 - 8\times2 = 0$.
$\Rightarrow$ no unique solution

c If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$
$\det = 1\times1 - -2\times-1 = -1 \neq 0$ so there is a unique solution.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$x = -3$, $y = -5$

6

a If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$
det A = 1 so there is a unique solution.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

$x = 33$, $y = 18$

b If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$
det A = $\frac{1}{2}$ so there is a unique solution.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$x = -5$, $y = 3$

c If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$
$\det = 0 \Rightarrow$ no solution

d If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$
det = $\frac{1}{7}$ so there is a unique solution.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ \frac{4}{7} & \frac{1}{7} & -\frac{6}{7} \\ \frac{1}{7} & \frac{2}{7} & -\frac{5}{7} \end{bmatrix} \times \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}$$

$x = -2$, $y = 2.6$, $z = 0.14$ (to 2 sig. figs.)

7

The graphs of $2x + 2y = 10$ and $2x + y = 5$ are identical so there is **no** unique solution. Divide the first equation through by 2 to see.

The equations:

$x = 0$ and $x + y = 6$

clearly has a unique solution:

$x = 0$ and $y = 6$

See by substituting $x = 0$ in the second equation

The graphs of $x + y = 3$ and $x + y = 3$ are identical so there is **no** unique solution.

The graphs of $4x + y = 5$ and $2x + y = 16$ are parallel lines so there is **no** unique solution

The equations:

$x = 0$ and $y = 8$

clearly has a unique solution:

$x = 0$ and $y = 8$

Thus only two of the sets of equations have unique solutions. $\Rightarrow$ B

8 Multiplying out the matrices we will find:

$$\begin{bmatrix} x - 2y \\ x + 3z \\ 2y - z \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$$

so

$x - 2y = 4$

$x + 3y = 11$

$2y - z = -5$

so, $x + 3z = 11$ is one of the equations

$\Rightarrow$ D

9 If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$

Write the equations in matrix form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -1 & 2 \\ -2 & 0 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}$$

$\Rightarrow$ A

10 If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{5}{3} \\ 2 & -\frac{5}{3} \end{bmatrix} \times \begin{bmatrix} 150 \\ 150 \end{bmatrix}$$

$x = 25, y = 50$

11 If $AX = C$ the $X = A^{-1}C$ if $\det A \neq 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.0075 & 0.01 & -0.015 \\ -0.0235 & -0.04 & 0.055 \\ 0.004 & 0.02 & -0.02 \end{bmatrix} \times \begin{bmatrix} 175\,000 \\ 149\,000 \\ 183\,500 \end{bmatrix}$$

$x = 50, y = 20, z = 10$

Solutions to Exercise 12C-1

1 **a**

$$\begin{array}{cc} & \textit{from} \\ & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.40 & 0.55 \\ 0.60 & 0.45 \end{bmatrix} \textit{to} \end{array}$$

b

$$\begin{array}{cc} & \textit{from} \\ & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{bmatrix} 0.70 & 0.25 \\ 0.30 & 0.75 \end{bmatrix} \textit{to} \end{array}$$

c

$$\begin{array}{cc} & \textit{from} \\ & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.6 & 0.15 & 0.22 \\ 0.1 & 0.7 & 0.23 \\ 0.3 & 0.15 & 0.55 \end{bmatrix} \textit{to} \end{array}$$

d

$$\begin{array}{cc} & \textit{from} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.45 & 0.35 & 0.15 \\ 0.25 & 0.45 & 0.20 \\ 0.30 & 0.20 & 0.65 \end{bmatrix} \textit{to} \end{array}$$

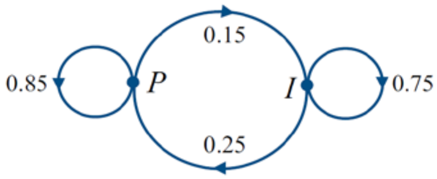
Solutions to Chapter 12C-2

1

a

$$\begin{array}{c} \text{this time} \\ \begin{array}{cc} P & I \end{array} \end{array}$$

$$T = \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix} \begin{array}{c} P \\ I \end{array} \quad \text{next time}$$



b 85% of 80 = 0.85 × 80 = 68

c 25% of 60 = 0.25 × 60 = 15

d 0.15 × 120 + 0.75 × 40 = 48

2

Reading from table:

a

i 10%

ii 80%

iii 10%

b

i 80% of 850 = 0.80 × 850 = 680

ii 10% of 850 = 0.10 × 850 = 85

c

i 100% of 1150 = 1 × 1150 = 1150

ii 0% of 1150 = 0 × 1150 = 0

ii 0% of 1150 = 0 × 1150 = 0

d All (100%) of the sea birds that nest at site A this year will nest at site A next year.

3

a

i 91% of 84 000 = 76 440

ii 9% of 84 000 = 7560

b

i 22% of 25 000 = 5500

ii 22% of 5500 = 1210

ii 22% of 1210 = 266

4 **Option A:** not true-there is nothing in the transition matrix that indicates that equal numbers of children under take each of the activities in the first week. This information would be found in the initial state matrix which is not given.

Option B: true- the percentage of children who do not change activities is given by the elements in the leading diagonal of the matrix. As these elements are all 0.50 or greater, option B is true.

Option C: not true- for all of the children to chose the same activity in the long term, one of the diagonal elements would need to be 1.

Option D: not true- here is nothing in the transition matrix that indicates that which activity was the most popular in the first week or any week. This information would be found in the initial state matrix which is not given.

Option A: not true-there is nothing in the transition matrix that indicates that 50% of the children do swimming each week. The 0.50 the swimming column tells us that 50% of children who do swimming this week will do swimming the next week and this is not the same thing as 50% of children do swimming each week.

⇒ B

5 Wendy lunches with Craig on Monday (today).

The transition matrix tells us she will lunch with Edgar Tuesday (tomorrow).

If Wendy lunches with Edgar on Tuesday, the transition matrix tells us she will lunch with Daniel on Wednesday.

If Wendy lunches with Daniel on Wednesday, the transition matrix tells us she will lunch with Betty on Thursday.

If Wendy lunches with Betty on Thursday, the transition matrix tells us she will lunch with Angela on Friday.

Thus, for the next four days (Tuesday to Friday) , Angela will lunch with Edgar, Daniel, Betty and Angela in that order.

⇒ E

Solutions to Chapter 12C-3

1

a **i** $S_1 = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 130 \\ 170 \end{bmatrix}$

ii $S_2 = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \times \begin{bmatrix} 130 \\ 170 \end{bmatrix} = \begin{bmatrix} 151 \\ 149 \end{bmatrix}$

iii $S_3 = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \times \begin{bmatrix} 151 \\ 149 \end{bmatrix} = \begin{bmatrix} 165.7 \\ 134.3 \end{bmatrix}$

b $T^5 = \begin{bmatrix} 0.72269 & 0.55462 \\ 0.27731 & 0.44538 \end{bmatrix}$

c **i** $S_2 = \begin{bmatrix} 0.83 & 0.34 \\ 0.17 & 0.66 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 151 \\ 149 \end{bmatrix}$

ii $S_3 = \begin{bmatrix} 0.781 & 0.438 \\ 0.219 & 0.562 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 165.7 \\ 134.3 \end{bmatrix}$

iii $S_7 = \begin{bmatrix} 0.6941 & 0.6118 \\ 0.3059 & 0.3882 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 191.76 \\ 108.24 \end{bmatrix}$

d

$S_{10} = T^{10} S_0 = \begin{bmatrix} 197.175 \\ 102.825 \end{bmatrix}$

$S_{15} = T^{15} S_0 = \begin{bmatrix} 199.525 \\ 100.475 \end{bmatrix}$

$S_{21} = T^{21} S_0 = \begin{bmatrix} 199.92 \\ 100.06 \end{bmatrix}$

$S_{22} = T^{22} S_0 = \begin{bmatrix} 199.96 \\ 100.04 \end{bmatrix}$

As n increases $S_n \to \begin{bmatrix} 200 \\ 100 \end{bmatrix}$

2

a **i** $S_1 = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 180 \\ 130 \\ 290 \end{bmatrix}$

ii $S_2 = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 180 \\ 130 \\ 290 \end{bmatrix} = \begin{bmatrix} 207 \\ 136 \\ 257 \end{bmatrix}$

iii $S_3 = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 207 \\ 136 \\ 257 \end{bmatrix} = \begin{bmatrix} 225 \\ 132.1 \\ 242.9 \end{bmatrix}$

b **i** $S_2 = \begin{bmatrix} 0.58 & 0.37 & 0.25 \\ 0.19 & 0.24 & 0.23 \\ 0.23 & 0.39 & 0.52 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 207 \\ 136 \\ 257 \end{bmatrix}$

ii $S_3 = \begin{bmatrix} 0.505 & 0.394 & 0.319 \\ 0.204 & 0.215 & 0.229 \\ 0.291 & 0.391 & 0.452 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 225 \\ 132.1 \\ 242.9 \end{bmatrix}$

iii $S_7 = \begin{bmatrix} 0.4210 & 0.4099 & 0.4027 \\ 0.2145 & 0.2159 & 0.2169 \\ 0.3645 & 0.3742 & 0.3805 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 244.9 \\ 129.7 \\ 225.4 \end{bmatrix}$

c

$$S_{10} = T^{10}S_0 = \begin{bmatrix} 246.67 \\ 129.46 \\ 223.86 \end{bmatrix}$$

$$S_{15} = T^{15}S_0 = \begin{bmatrix} 247.04 \\ 129.42 \\ 223.55 \end{bmatrix}$$

$$S_{17} = T^{17}S_0 = \begin{bmatrix} 247.05 \\ 129.41 \\ 223.54 \end{bmatrix}$$

$$S_{18} = T^{18}S_0 = \begin{bmatrix} 247.06 \\ 129.41 \\ 223.53 \end{bmatrix}$$

As n increases $S_n \to \begin{bmatrix} 247.1 \\ 129.4 \\ 223.5 \end{bmatrix}$

3

a Let J be the first row and column and P be the second row and column.

$T = \begin{bmatrix} 0.80 & 0.25 \\ 0.20 & 0.75 \end{bmatrix}$

b Let the first row be J and the second row be P.

$S_0 = \begin{bmatrix} 400 \\ 400 \end{bmatrix}$

c $S_1 = T \times S_0$

$$S_1 = \begin{bmatrix} 0.80 & 0.25 \\ 0.20 & 0.75 \end{bmatrix} \times \begin{bmatrix} 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 420 \\ 380 \end{bmatrix}$$

Thus 420 people are expected to go to Jill's next week and 380 to Pete's.

d $S_5 = T^5 \times S_0$

$$\begin{bmatrix} 0.5779 & 0.5276 \\ 0.4221 & 0.4724 \end{bmatrix} \times \begin{bmatrix} 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 442.2 \\ 357.8 \end{bmatrix}$$

Thus 442 people are expected to go to Jill's after 5 weeks and 358 to Pete's.

e **As n increases we find that** $S_n \to \begin{bmatrix} 444.4 \\ 355.6 \end{bmatrix}$

Thus, in the long term, 444 people are expected to go to Jill's each week and 356 to Pete's.

4

a Let H be the first row and column and U be the second row and column.

$$T = \begin{bmatrix} 0.90 & 0.60 \\ 0.10 & 0.40 \end{bmatrix}$$

b Let the first row be J and the second row be P.

$$S_0 = \begin{bmatrix} 1500 \\ 500 \end{bmatrix}$$

c $S_1 = T \times S_0$

$$S_1 = TS_0$$
$$= \begin{bmatrix} 0.90 & 0.60 \\ 0.10 & 0.40 \end{bmatrix} \begin{bmatrix} 1500 \\ 500 \end{bmatrix}$$
$$= \begin{bmatrix} 1650 \\ 350 \end{bmatrix}$$

The next day, are expected 1650 to be happy and 350 sad.

d

$$S_4 = T^4 S_0$$
$$= \begin{bmatrix} 0.90 & 0.60 \\ 0.10 & 0.40 \end{bmatrix}^4 \begin{bmatrix} 1500 \\ 500 \end{bmatrix}$$
$$= \begin{bmatrix} 1712.55 \\ 287.45 \end{bmatrix}$$

After 4 days, 1713 happy and 287 sad.

e **As n increases we find that** $S_n \to \begin{bmatrix} 1714.3 \\ 285.7 \end{bmatrix}$

In the long term, 1714 are expected to be happy and 286 sad.

5

a

$$S_0 = \begin{bmatrix} 1200 \\ 600 \\ 200 \end{bmatrix} \begin{matrix} happy \\ neither \\ sad \end{matrix}$$

b

$$S_1 = TS_0$$
$$= \begin{bmatrix} 0.80 & 0.40 & 0.35 \\ 0.15 & 0.30 & 0.40 \\ 0.05 & 0.30 & 0.25 \end{bmatrix} \begin{bmatrix} 1200 \\ 600 \\ 200 \end{bmatrix}$$
$$= \begin{bmatrix} 1270 \\ 440 \\ 290 \end{bmatrix}$$

1270 people are expected to be happy

c

$$S_5 = T^5 S_0$$
$$= \begin{bmatrix} 0.80 & 0.40 & 0.35 \\ 0.15 & 0.30 & 0.40 \\ 0.05 & 0.30 & 0.25 \end{bmatrix}^5 \begin{bmatrix} 1200 \\ 600 \\ 200 \end{bmatrix}$$
$$= \begin{bmatrix} 1310.33 \\ 429.82 \\ 259.85 \end{bmatrix}$$

1310 people are expected to be happy

d

As n increases we find that $S_n \rightarrow \begin{bmatrix} 1311.7 \\ 429.1 \\ 259.1 \end{bmatrix}$

In the long term, 1312 are expected to be happy, 429 neither happy nor sad and 260 sad.

Solutions to Chapter 12C-4

1 a i $S_1 = TS_0 = \begin{bmatrix} 80 \\ 120 \end{bmatrix}$

ii $S_2 = TS_1 = \begin{bmatrix} 72 \\ 128 \end{bmatrix}$

$S_3 = TS_2 = \begin{bmatrix} 68.8 \\ 131.2 \end{bmatrix}$

b i

$$S_1 = TS_0 + R = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 90 \\ 125 \end{bmatrix}$$

ii

$$S_2 = TS_1 + R = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 90 \\ 125 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 79 \\ 136 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 89 \\ 141 \end{bmatrix}$$

c i

$$S_1 = TS_0 - B = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

ii

$$S_2 = TS_1 - B = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

2
a

i

$$S_2 = TS_1 = \begin{bmatrix} 11\,500 \\ 8\,500 \\ 10\,000 \end{bmatrix}$$

ii

$$S_3 = TS_2 = \begin{bmatrix} 12\,800 \\ 7\,300 \\ 9\,850 \end{bmatrix}$$

7300 birds at site B

b *A:* 3000 *B:* 0 *C:* 0
Explanation included in answer.

c i

$$S_2 = TS_1 + N = \begin{bmatrix} 9\,500 \\ 9\,500 \\ 11\,000 \end{bmatrix}$$

ii

$$S_3 = TS_2 + N = \begin{bmatrix} 9000 \\ 9150 \\ 11850 \end{bmatrix}$$

iii $$S_4 = TS_3 + N = \begin{bmatrix} 8507.5 \\ 8912.5 \\ 12580 \end{bmatrix}$$

Chapter Review: Extended-response questions

1

a **i** For the second game, the attendance matrix, A_2, is:

$$\begin{bmatrix} 1.2 & -0.3 \\ 0.2 & 0.7 \end{bmatrix} \times A_1$$

$$= \begin{bmatrix} 1.2 & -0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 2\,000 \\ 1\,000 \end{bmatrix}$$

$$= \begin{bmatrix} 2\,100 \\ 1\,100 \end{bmatrix}$$

ii 2100 + 1100 = 3200

b **i** For game 10, the attendance matrix is

$$A_{10} = G^9 A_1$$

$$= \begin{bmatrix} 1.2 & -0.3 \\ 0.2 & 0.7 \end{bmatrix}^9 \begin{bmatrix} 2000 \\ 1000 \end{bmatrix}$$

$$= \begin{bmatrix} 2612.58 \\ 1612.58 \end{bmatrix},$$

or rounding up,

$$= \begin{bmatrix} 2613 \\ 1613 \end{bmatrix}$$

c Compare

$$\begin{bmatrix} 1.2 & -0.3 \\ 0.7 & 0.7 \end{bmatrix}^{80} \begin{bmatrix} 2000 \\ 1000 \end{bmatrix} \text{and}$$

$$\begin{bmatrix} 1.2 & -0.3 \\ 0.7 & 0.7 \end{bmatrix}^{81} \begin{bmatrix} 2000 \\ 1000 \end{bmatrix} \text{which equal}$$

$$\begin{bmatrix} 2999.78 \\ 1999.78 \end{bmatrix} \text{and} \begin{bmatrix} 2999.80 \\ 1999.80 \end{bmatrix} \text{respectively.}$$

These results round up to $\begin{bmatrix} 3000 \\ 2000 \end{bmatrix}$.

The steady state has been achieved.

For the Dinosaurs, the attendance slowly rises from 2000 to 3000 and then remains steady at this number.

d In this new situation,

$$S_1 = \begin{bmatrix} 2000 \\ 1800 \end{bmatrix} \begin{matrix} D \\ S \end{matrix}$$

so

$$A_{80} = \begin{bmatrix} 1.2 & -0.3 \\ 0.2 & 0.7 \end{bmatrix}^{80} \begin{bmatrix} 2000 \\ 1800 \end{bmatrix}$$

$$\approx \begin{bmatrix} 600 \\ 400 \end{bmatrix}$$

$$A_{81} = \begin{bmatrix} 1.2 & -0.3 \\ 0.2 & 0.7 \end{bmatrix}^{81} \begin{bmatrix} 2000 \\ 1800 \end{bmatrix}$$

$$\approx \begin{bmatrix} 600 \\ 400 \end{bmatrix}$$

The steady state has been achieved.

For the Dinosaurs, the attendance figures decrease from 2000 to 600 and then stay at this figure.

Chapter Review: Multiple-choice questions

1 V cannot be raised to a power as it is not a square matrix. $\Rightarrow$ **B**

2 det(U) = 2×1 – 1×0 = 2 $\Rightarrow$ **D**

3 det(Y) = 1×4 – 2×2 = 0.
Thus, the inverse of Y is undefined.
$\Rightarrow$ **E**

4 det(U) = 2

$$U^{-1} = \begin{bmatrix} \frac{1}{2} = 0.5 & \frac{0}{2} = 0 \\ \frac{-1}{2} = -0.5 & \frac{2}{2} = 1 \end{bmatrix} \Rightarrow \textbf{A}$$

5 UW: U & W are both 2×2 matrices
$\Rightarrow$ **D**

6 none: to be a lower triangular matrix, all elements above the leading diagonal must be zero. None of the matrices satisfy this condition. $\Rightarrow$ **A**

7 Z is a permutation matrix. It is the only binary matrix with only one '1' per row and column. $\Rightarrow$ **D**

8 From the transition diagram:

$$T = \begin{bmatrix} 0.75 & 0.05 \\ 0.25 & 0.95 \end{bmatrix} \Rightarrow \textbf{B}$$

9 From the transition diagram:

$$T = \begin{bmatrix} 0.75 & 0.05 & 0.30 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \Rightarrow \textbf{A}$$

10 det(II) = 2×4 – 4×2 = 0. Thus, system II does not have a unique solution. Systems I and III do have unique solutions. $\Rightarrow$ **D**

11 In matrix form, the equations can be written as:

$$\begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Rightarrow \textbf{D}$$

12 $$S_1 = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 160 \\ 140 \end{bmatrix}$$
$\Rightarrow$ **C**

13 $$T^2 = \begin{bmatrix} 0.56 & 0.55 \\ 0.44 & 0.45 \end{bmatrix} \Rightarrow \textbf{B}$$

14 $$S_3 = \begin{bmatrix} 0.556 & 0.555 \\ 0.444 & 0.445 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$$
$\Rightarrow$ **B**

15 $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$: Obtain by evaluating T_nS_0 for increasing large values of n until there is little or no change in the state matrix (the steady or equilibrium state).
$\Rightarrow$ **C**

16

$$L_1 = TS_0 + B$$
$$= \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix} + \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$
$$= \begin{bmatrix} 170 \\ 160 \end{bmatrix} \Rightarrow \textbf{C}$$

17

$$P_1 = TS_0 - 2B$$
$$= \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix} - 2\begin{bmatrix} 10 \\ 20 \end{bmatrix}$$
$$= \begin{bmatrix} 140 \\ 100 \end{bmatrix} \Rightarrow \textbf{A}$$

18

$$S_2 = GS_1 = \begin{bmatrix} -10 \\ 25 \end{bmatrix} \Rightarrow \textbf{B}$$

19

$$S_5 = TS_4$$
$$T^{-1}S_5 = T^{-1}TS_4$$
$$T^{-1}S_5 = S_4$$
$$S_4 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}^{-1} \begin{bmatrix} 22 \\ 18 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} \Rightarrow \textbf{B}$$

20 Eventually all of the birds will settle at location B because the '1' in row B column B, tells us that once a bird settles at location B it will never leave.

$\Rightarrow$ B

21 From the transition matrix, the number of people who plan to change who they vote for after 1 week is:

$0.25 \times 5692 + 0.24 \times 3450 = 2251$

$\Rightarrow$ C

22

$$S_{10} = T S_0$$

$$= \begin{bmatrix} 0.75 & 0.24 \\ 0.25 & 0.76 \end{bmatrix} \begin{bmatrix} 5692 \\ 3450 \end{bmatrix}$$

$$= \begin{bmatrix} 4479.15... \\ 4662.84... \end{bmatrix}$$

$$\approx \begin{bmatrix} 4479 \\ 4663 \end{bmatrix} \begin{matrix} Rob \\ Anna \end{matrix}$$

$\therefore$ Anna wins by: 4663– 4479 =182 votes

$\Rightarrow$ E

Chapter 13 – Revision: Matrices and applications

Solutions to Exercise 13A: Multiple-choice questions

1 Z is the column matrix (3×1). $\Rightarrow$ **E**

2 U is the square matrix: has 3 rows and 3 columns. $\Rightarrow$ **A**

3 Y has 3 rows and 2 columns, so its order is 3 × 2. $\Rightarrow$ **C**

4 None of the matrices are of the same order, which means none can be added to each other. $\Rightarrow$ **E**

5 Of the given matrix products, only WZ is defined, since W has 2 columns and Z has 2 rows. $\Rightarrow$ **E**

6 $-4 \times V =$
$[-4 \times 4 = -16 \quad -4 \times 1 = -4 \quad -4 \times 0 = 0]$
$\Rightarrow$ **C**

7 U has 3 rows and Y has 2 columns, so the order of UY will be (3 × 2). $\Rightarrow$ **D**

8 Transposing a matrix involves interchanging its row with its columns. Thus, when transposed, a 3 × 2 matrix becomes a 2 × 3 matrix. $\Rightarrow$ **B**

9 A binary matrix contains only 0s and 1s. $\Rightarrow$ **C**

10 $a_{3\,2}$ is the element in the 3rd row and the second column, so $a_{3\,2} = -4$ $\Rightarrow$ **A**

11 Multiply out the matrices (by hand, is quicker) to obtain the matrix [S T O P]
$\Rightarrow$ **A**

12 Multiply out the matrices (by hand, is quicker) to obtain the matrix:
$[\,1 \times 1 + 0 \times 0 + 1 \times 0 + 0 \times 1 + 0 \times 1\,]$
$=[1]$
$\Rightarrow$ **B**

13 R is a 3 × 3 matrix so that $R^2 = R \times R$ is a 3 × 3 matrix.
$\Rightarrow$ **B**

14 Matrix A is a 3 × 2 matrix.
Matrix B is a 4 × 3 matrix.
Thus, BA is a 4 × 2 matrix.
$\Rightarrow$ **D**

15 Both matrices have inverses, so they must be square.
Thus $p = q$ and $q = r$.
Both products XY^{-1} and $X^{-1}Y$ are defined so the matrices X and Y must be of the same order.
Thus $p = q = r$.
$\Rightarrow$ **C**

16 From the diagram, the matrix can be seen to be:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \mathbf{C}$$

17 Multiplying out the matrices gives us $2x = 1$ as the first equation and $x + 3y = 4$ as the second. $\Rightarrow$ **B**

18 W cannot be raised to a power as it is not a square matrix. $\Rightarrow$ **C**

19 $\det(X) = 0.75 \times 0.5 - 0.25 \times 0.5$
$= 0.25$ $\Rightarrow$ **C**

20 $\det(Y) = 1{\times}4 - 2{\times}2 = 0$, so Y^{-1} is not defined.
$\Rightarrow$ **E**

21 $YW+WX$ is not defined because WX is not defined. $\Rightarrow$ **E**

22 $(U–W)^2$ is not defined because $U–W$ is not defined because U and W have different orders. $\Rightarrow$ **E**

23 Consider options one by one.
D is not true.
If $AB = BA = I$, then:
$B = A^{-1}$ and $A = B^{-1}$
However for A to be equal B, A^{-1} must equal A and this is not necessarily true.
$\Rightarrow$ **D**

24 $S_1 = TS_0 = \begin{bmatrix} 90 \\ 110 \end{bmatrix}$

$\Rightarrow$ **A**

25 $S_1 = T^5 S_0 = \begin{bmatrix} 93.1 \\ 106.9 \end{bmatrix}$

$\Rightarrow$ **B**

26

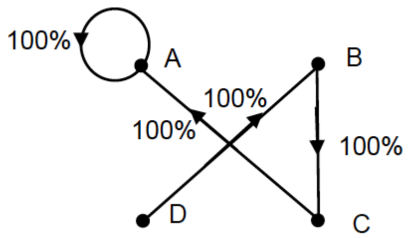
$S_{\text{steady state}} = T^n S_0$ for n large

Using successively large values of n, :

$S_{\text{steady state}} \approx \begin{bmatrix} 94.1 \\ 105.9 \end{bmatrix}$

$\Rightarrow$ **C**

27 The easiest way to answer this question is to construct a diagram to represent the transition matrix as shown below.



If Kerry chooses D as his first answer, the sequence of answers he will give to the first six questions is: *DBCAAA*
$\Rightarrow$ **B**

28 20% of 2800 = 560 $\Rightarrow$ **A**

29 Birds that nest at sites B and D have a 100% probability of nesting there the following year. Therefore no birds will change nests once they are nesting at these locations. Birds that nest at C or A have a probability greater than 0 to nest at B or D the following year. $\Rightarrow$ **D**

30 An equal but unknown number of birds nested at each of the four sites A, B, C and D in 2007. Let x be this number. 6000 birds nested at site B in 2008 Then, using either the transition matrix or the diagram,

$$6000 = x + 0.15\,x + 0.35x$$

So $6000 = 1.5x$ or $x = 4000$ and the total number of birds nesting at all four sites in 2007 was $4 \times 4000 = 16\,000$ birds

$\Rightarrow$ **D**

31 Let D = the dominance matrix.

Let T = the matrix showing the sum of the one and two-step dominances.

$T = D + D^2$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} = \begin{array}{c} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{row sum} \\ 3 \\ 8 \\ 2 \\ 6 \\ 7 \end{array}$$

From the row sums shown on the right hand side of the matrix we can rank the players in order of the sum of their one and two-step dominances as follows:
Ben (8), Elise (7), Donna (6), Alex (3), Cindy (2). $\Rightarrow$ **A**

32 Post multiplying M by S will sum the rows in matrix M to give the sum of marks for each class. Dividing the by 15 will then give the class average.

The answer is: $\frac{1}{15} M \times S$

$\Rightarrow$ **D**

Solutions to Exercise 13B: Extended-response questions

1 a

$$C = \begin{bmatrix} 1.316 \\ 1.818 \\ 0.167 \end{bmatrix} \begin{matrix} \text{US dollar rate} \\ \text{Euro rate} \\ \text{HK dollar rate} \end{matrix}$$

Order of C is 3×1 (3 rows, 1 column).

b $H = [\ 102 \quad 262 \quad 516\ ]$

Order of H is 1×3.

(1 rows, 3 columns)

c The matrix product HC is defined as H has order 1×3 and C has order 3×1.

The order of HC is 1×1.

d i

$$HC = [\ 102 \quad 262 \quad 516\ ] \begin{bmatrix} 1.316 \\ 1.818 \\ 0.167 \end{bmatrix}$$

$= [\ 102 \times 1.316 + 262 \times 1.818 + 516 \times 0.167\ ]$

$= 696.72$

ii This is the total amount converted to Australian dollars.

Solution integrated with answer.

e

$$MC = \begin{bmatrix} 125 & 216 & 54 \\ 0 & 34 & 453 \\ 0 & 356 & 0 \end{bmatrix} \begin{bmatrix} 1.316 \\ 1.818 \\ 0.167 \end{bmatrix}$$

$$= \begin{bmatrix} 566.21 \\ 137.46 \\ 647.21 \end{bmatrix}$$

Person 1 receives \$566.21, person 2 receives \$137.46 and person 3 receives \$647.21.

2 a $x + 2y = -4$

$3x - 2y = 12$

$$\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

b

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}, A^{-1} = \begin{bmatrix} 0.25 & 0.25 \\ 0.375 & -0.125 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, C = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

c

$$X = A^{-1}C = \begin{bmatrix} 0.25 & 0.25 \\ 0.375 & -0.125 \end{bmatrix} \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$\Rightarrow x = 2, y = -3$

d i Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}$, $\det(A) = 1 \times 12 - 3 \times 4 = 0$

The equations are

$$x + 3y = 6 \quad \boxed{1}$$

and

$$4x + 12y = 3 \quad \boxed{2}$$

The second equation simplifies to

$$x + 3y = \frac{3}{4} \quad \boxed{2'}$$

The equations $\boxed{1}$ and $\boxed{2}$ are inconsistent. They correspond to distinct parallel lines.

ii Let $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$, $\det(A) = 2 \times 9 - 3 \times 6 = 0$

The equations are

$$2x + 3y = 6 \quad \boxed{1}$$

and

$$6x + 9y = 18 \quad \boxed{2}$$

By dividing through by 3, the second equation simplifies to:

$$2x + 3y = 6 \quad \boxed{2'}$$

The equations $\boxed{1}$ and $\boxed{2'}$ are identical, therefore the equations $\boxed{1}$ and $\boxed{2}$ are dependent.

iii Let $A = \begin{bmatrix} -2 & 10 \\ 1 & -5 \end{bmatrix}$, $\det(A) = (-2) \times (-5) - 10 \times 1 = 0$

The equations are

$$-2x + 10y = 12 \quad \boxed{1}$$

and

$$x - 5y = 10 \quad \boxed{2}$$

By dividing through by –2, the first equation simplifies to

$$x - 5y = -6 \quad \boxed{2'}$$

The equations $\boxed{1}$ and $\boxed{2}$ are inconsistent. They correspond to distinct parallel lines.

3 a

$$\begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

b

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$, then (using your calculator), $A^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0.4 \\ 0.2 & 0.2 & -0.6 \\ 0.8 & -0.2 & 0.6 \end{bmatrix}$

Letting $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ we have:

$AX = C$ or $X = A^{-1}C$

c To find X, enter the matrices A and C into your CAS calculator and evaluate $A^{-1}C$ as shown below.

$$\begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \to a \qquad \begin{bmatrix} 0. & 1. & 1. \\ 3. & 1. & -1. \\ 1. & -1. & 0. \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} \to c \qquad \begin{bmatrix} 2. \\ 2. \\ 5. \end{bmatrix}$$

$$a^{-1} \cdot c \qquad \begin{bmatrix} 2.8 \\ -2.2 \\ 4.2 \end{bmatrix}$$

The solution is $x = 2.8$, $y = -2.2$ and $z = 4.2$

4 a

$$\begin{array}{ccc} & & \textit{From} \\ & & \begin{array}{cc} \textit{Blue} & \textit{Green} \end{array} \\ \textit{To} & \begin{array}{c} \textit{Blue} \\ \textit{Green} \end{array} & \begin{bmatrix} 0.67 & 0.28 \\ 0.33 & 0.72 \end{bmatrix} \end{array}$$

b

$$S_0 = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix} \begin{array}{c} \textit{Blue} \\ \textit{Green} \end{array}$$

c

$$\begin{bmatrix} 0.67 & 0.28 \\ 0.33 & 0.72 \end{bmatrix} \begin{bmatrix} 4000 \\ 6000 \end{bmatrix} = \begin{bmatrix} 4360 \\ 5640 \end{bmatrix}$$

Tomorrow, we expect 4360 fish in Lake Blue and 5640 fish in Lake Green.

d

$$\begin{bmatrix} 0.67 & 0.28 \\ 0.33 & 0.72 \end{bmatrix}^3 \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$$

$$= \begin{bmatrix} 4555.156 \\ 5444.844 \end{bmatrix} \approx \begin{bmatrix} 4555 \\ 5445 \end{bmatrix} \begin{array}{c} \textit{Blue} \\ \textit{Green} \end{array} \quad \text{(to the nearest whole number)}$$

In three days, we expect 4555 fish in Lake Blue and 5445 fish in Lake Green.

e Estimate the steady state solution by trying successively large values of n until there is little change between successive state matrices: $n = 30$ has been used here.

$$\begin{bmatrix} 0.67 & 0.28 \\ 0.33 & 0.72 \end{bmatrix}^{30} \begin{bmatrix} 4000 \\ 6000 \end{bmatrix} \approx \begin{bmatrix} 4590 \\ 5410 \end{bmatrix} \begin{array}{c} \textit{blue} \\ \textit{green} \end{array} \quad \text{(to the nearest whole number)}$$

In the long term, we expect 4590 fish in Lake Blue and 5410 fish in Lake Green.

5

$$T = \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

a $S_{n+1} = TS_n$

i $S_1 = TS_0$

$$= \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$
$$= \begin{bmatrix} 126 \\ 174 \end{bmatrix}$$

ii $S_2 = TS_1$
$$= \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix} \begin{bmatrix} 126 \\ 174 \end{bmatrix}$$
$$= \begin{bmatrix} 143.16 \\ 156.84 \end{bmatrix} \text{ (to 2 d.p.)}$$

iii $S_3 = TS_2$
$$= \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix} \begin{bmatrix} 143.16 \\ 156.84 \end{bmatrix}$$
$$= \begin{bmatrix} 154.49 \\ 145.51 \end{bmatrix} \text{ (to 2 d.p.)}$$

b
$$T^6 = \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix}^6$$
$$= \begin{bmatrix} 0.62 & 0.54 \\ 0.38 & 0.46 \end{bmatrix} \text{ (to 2 d.p.)}$$

c $S_n = T^n S_0$

i $S_2 = T^2 S_0$
$$= \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix}^2 \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$
$$= \begin{bmatrix} 143.16 \\ 156.84 \end{bmatrix} \text{ (to 2 d.p.)}$$

ii $S_3 = T^3 S_0$
$$= \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix}^3 \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$
$$= \begin{bmatrix} 154.49 \\ 145.51 \end{bmatrix} \text{ (to 2 d.p.)}$$

iii $S_5 = T^5 S_0$
$$= \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix}^5 \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$
$$= \begin{bmatrix} 166.9 \\ 133.1 \end{bmatrix} \text{ (to 1 d.p.)}$$

d $S_n = T^n S_0$
$S_{10} = T^{10} S_0$
$$= \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix}^{10} \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$
$$\therefore S_{10} = \begin{bmatrix} 175.271 \\ 124.729 \end{bmatrix} \text{ (to 3 d.p.)}$$

$S_{15} = T^{15}S_0$

$= \begin{bmatrix} 176.320 \\ 123.680 \end{bmatrix}$ (to 3 d.p.)

$S_{20} = T^{20}S_0$

$= \begin{bmatrix} 176.471 \\ 123.529 \end{bmatrix}$ (to 3 d.p.)

$S_{21} = T^{21}S_0$

$= \begin{bmatrix} 176.458 \\ 123.542 \end{bmatrix}$ (to 3 d.p.)

The steady state solution is close to $\begin{bmatrix} 176.5 \\ 123.5 \end{bmatrix}$ (to 1 d.p.)

6

a $\begin{bmatrix} 1.2 & 20.1 & 4.2 \\ 6.7 & 0.4 & 0.6 \end{bmatrix}$

b **i** $AB = \begin{bmatrix} 2 \times 531 + 2 \times 41 + 1 \times 534 + 1 \times 212 \end{bmatrix}$

$= \begin{bmatrix} 1890 \end{bmatrix}$

ii $\underset{(4\times1)}{B} \times \underset{(1\times4)}{A} = \underset{(4\times4)}{BA}$

iii The total energy content of all these four foods in the same sandwich.

c $\begin{bmatrix} 1.2 & 6.7 & 10.7 & 0 \\ 20.1 & 0.4 & 3.5 & 12.5 \\ 4.2 & 0.6 & 4.6 & 0.1 \\ 531 & 41 & 534 & 212 \end{bmatrix} \begin{bmatrix} b \\ m \\ p \\ h \end{bmatrix} = \begin{bmatrix} 53 \\ 101.5 \\ 28.5 \\ 3568 \end{bmatrix}$

Using a CAS calculator:

$b = 4$, $m = 4$, $p = 2$, $h = 1$

7

a Sum the numbers in the initial state matrix

$400 + 200 + 100 = 700$

b Reading from column E and row J of the transition matrix: 0.5 or 50%

c **i** $S_1 = TS_0 = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 160 \\ 280 \\ 180 \\ 80 \end{bmatrix} \begin{matrix} E \\ J \\ A \\ D \end{matrix}$

ii Reading from the row J of the state matrix: 280

iii $S_4 = T^4 S_0 = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix}^4 \begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 10.24 \\ 56.32 \\ 312.96 \\ 320.48 \end{bmatrix} \begin{matrix} E \\ J \\ A \\ D \end{matrix}$

From row J of the state matrix: 56 to the nearest whole number

iv Evaluate successive state matrices using the rule $S_n = T^n S_0$ until the state matrix first shows that the number of eggs is first less than 1. This will occur for $n = 7$

$S_7 = \begin{bmatrix} 0.655 \\ 6.06 \\ 196 \\ 497. \end{bmatrix} \begin{matrix} E \\ J \\ A \\ D \end{matrix}$

v $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 700 \end{bmatrix} \begin{matrix} E \\ J \\ A \\ D \end{matrix}$

All insects will be dead.

d **i** $S_1 = TS_0 + BS_0 = \begin{bmatrix} 160 \\ 280 \\ 180 \\ 80 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 190 \\ 280 \\ 180 \\ 80 \end{bmatrix}$

ii $S_2 = TS_1 + BS_1$

Number of live eggs $= 0.4 \times 190 + 0.3 \times 180 = 130$

or determine S_2 and read the value from the matrix.

8

a **i** $S_2 = TS_1 = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \begin{bmatrix} 540 \\ 36 \end{bmatrix} = \begin{bmatrix} 493.2 \\ 82.8 \end{bmatrix} \approx \begin{bmatrix} 493 \\ 83 \end{bmatrix} \begin{matrix} \textit{attend} \\ \textit{not attend} \end{matrix}$

ii $S_5 = T^4 S_1 = \begin{bmatrix} 0.9 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \begin{bmatrix} 540 \\ 36 \end{bmatrix} = \begin{bmatrix} 421.46 \\ 154.54 \end{bmatrix} \approx \begin{bmatrix} 421 \\ 155 \end{bmatrix} \begin{matrix} \textit{attend} \\ \textit{not attend} \end{matrix}$

The number of students History attending the fifth lecture is 421.

b $S_n = T^{n-1} S_1$

c $S_n = T^{n-1} S_1$

$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}^{8-1} \begin{bmatrix} 540 \\ 36 \end{bmatrix} \approx \begin{bmatrix} 397 \\ 179 \end{bmatrix} \begin{matrix} \textit{attend} \\ \textit{not attend} \end{matrix}$

Thus attendance first drops below 400 in lecture 8.

d Using a large value for n

$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}^{100-1} \begin{bmatrix} 540 \\ 36 \end{bmatrix} \approx \begin{bmatrix} 384 \\ 192 \end{bmatrix} \begin{matrix} \textit{attend} \\ \textit{not attend} \end{matrix}$

384 students are predicted to attend lectures in the long-term.

9

$$O_{2009} = AS_{2008} + B$$

a

$$= \begin{bmatrix} 0.75 & 0 \\ 0 & 0.68 \end{bmatrix} \begin{bmatrix} 456 \\ 350 \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 360 \\ 250 \end{bmatrix}$$

b

$$O_{2009} = CO_{2008} - D = \begin{bmatrix} 360 \\ 250 \end{bmatrix}$$

$$O_{2010} = CO_{2009} - D = \begin{bmatrix} 248 \\ 162 \end{bmatrix}$$

The bookshop manager should order 248 mathematics textbooks.

Chapter 14 – Graphs, networks and trees: travelling and connecting problems

Solutions to Exercise 14A

1

a **i** There are three edges connected to town A, so $\deg(A) = 3$.

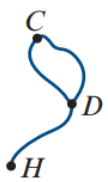
ii There are two edges connected to town B, so $\deg(B) = 2$.

iii There is one edge connected to town H, so $\deg(H) = 1$.

b sum of degrees

$$\begin{aligned} &= \deg(A) + \deg(B) \\ &\quad + \deg(C) + \deg(D) + \deg(H) \\ &= 3 + 2 + 4 + 4 + 1 \\ &= 14 \end{aligned}$$

c A possible subgraph that contains only towns H, D and C is

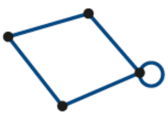


Note: Some subgraphs are possible where vertices are isolated. These subgraphs have not been shown.

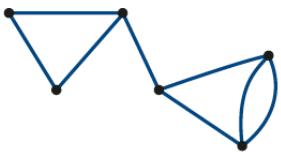
2 *Note: only one alternative has been shown for the answers to the following questions. Others are possible.*

a

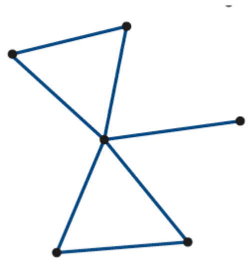
b



c



d



3

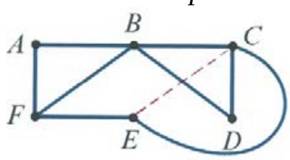
a The graphs in **ii**, **iii** and **iv** have all connections between vertices the same, but the graph in **i** does not. For example, they have two edges between A and C but the graph in i does not.
Graph **i** is not isomorphic to the others.

b The graphs in **i**, **iii** and **iv** have all connections between vertices the same, but the graph in **ii** does not. For example, they do not have an edge between A and C but the graph in **ii** does.
Graph **ii** is not isomorphic to the others.

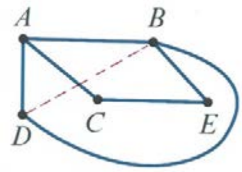
c The graphs in **i**, **iii** and **iv** have all connections between vertices the same, but the graph in **ii** does not. For example, they do not have an edge between E and C but the graph in **ii** does.
Graph **ii** is not isomorphic to the others.

4 *Note: In the graphs for this question, dotted edges show the edges that are repositioned in order to demonstrate the planar nature of the graphs. There are other solutions possible.*

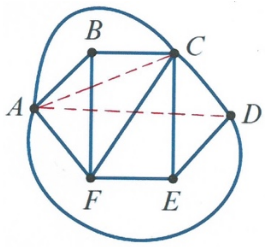
a



b



c



d This graph is non-planar and cannot be redrawn.

5

a **i** There are six faces, so $f = 6$.
There are eight vertices, so $v = 8$.
There are twelve edges, $e = 12$.

ii $v - e + f = 8 - 12 + 6$
$= 2$
Euler's rule is verified.

b **i** There are eight faces, so $f = 8$.
There are six vertices, so $v = 6$.
There are twelve edges, $e = 12$.

ii $v - e + f = 6 - 12 + 8$
$= 2$
Euler's rule is verified.

c **i** There are seven faces, so $f = 7$.
There are seven vertices, so $v = 7$.
There are twelve edges, $e = 12$.

ii $v - e + f = 7 - 12 + 7$
$= 2$
Euler's rule is verified.

6

a $v - e + f = 2$
$8 - 10 + f = 2$
$f = 2 - 8 + 10$
$f = 4$

b $v - e + f = 2$
$v - 14 + 4 = 2$
$v = 2 - 4 + 14$
$v = 12$

c $v - e + f = 2$
$5 - 14 + f = 2$
$f = 2 - 5 + 14$
$f = 11$

d $v - e + f = 2$
$10 - e + 11 = 2$
$-e = 2 - 10 - 11$
$-e = -19$
$e = 19$

Solutions to Exercise 14B

1

a

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

c

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

d

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

e

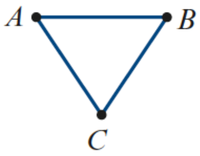
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

f

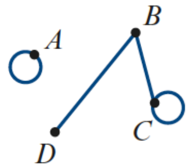
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

2

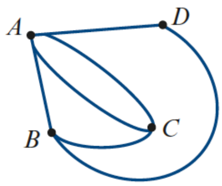
a



b



c



3 The zero in row *C*, column *A* show that vertex *C* is not connected to vertex *A*. There is a zero in row *C*, column *B* and row *C*, column *C* as well. This means that *C* is not connected to any other vertex, so it is isolated.

4 If every vertex has a loop, there will be a '1' in every position along the diagonal, that is in position (*A*,*A*), (*B*,*B*), …

5 The graph

- has no loops, so the diagonal will be all zeros
- has no duplicate edges, so there will only be '0' or '1'
- is complete, so every vertex is connected to every other vertex. Every position in the matrix will be a '1', except for the diagonal.

The adjacency matrix for the graph is

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Solutions to Exercise 14C

Note: There are multiple possible answers to the questions in this exercise.

1

a This walk starts and ends at different vertices so it is not a cycle, nor circuit. The walk does not repeat edges, nor vertices, so it is a **path**.

b This walk starts and ends at different vertices so is not a cycle nor circuit. The walk has a repeated vertex, but not a repeated edge so the walk is a **trail**.

c This walk starts and ends at different vertices so it is not a cycle, nor circuit. The walk does not repeat edges, nor vertices, so it is a **path**.

d This walk starts and ends at the same vertex, so it could be a circuit or a cycle. however, there is a repeated vertex and a repeated edge, so it will be neither. It cannot be a trail or a path because of the repeated edge and vertex, so this walk is only a **walk**.

e This walk starts and ends at different vertices, so it is not a cycle, nor circuit. The walk has a repeated vertex but not a repeated edge, so this walk is a **trail**.

f This walk starts and ends at different vertices, so it is not a cycle, nor circuit. The walk has not repeated edge and no repeated vertex, so the walk is a **path**.

2

a This walk starts and ends at different vertices, so it is not a cycle, nor circuit. The walk has a repeated edge and vertex, so it is a **walk** only.

b This walk starts and ends at the same vertex, so it could be a cycle or a circuit. There is no repeated edge, nor vertex, so the walk is a **cycle**.

c This walk starts and ends at different vertices, so it is not a cycle, nor circuit, There are no repeated edges, nor vertices, so the walk is a **path**.

d This walk starts and ends at different vertices, so it is not a cycle, nor circuit. There are repeated edges and vertices, so the walk is a **walk** only.

e This walk starts and ends at different vertices, so it is not a cycle, nor circuit. There are no repeated edges, nor vertices, so the walk is a **path**.

f This walk starts and ends at different vertices, so it is not a cycle, nor circuit. There are repeated edges and vertices, so the walk is a **walk** only.

3

a **i** This graph has two odd-degree vertices (A and E) and so it will have an eulerian trail.

ii One possible eulerian trail for this graph is $A–B–E–D–B–C–D–A–E$. There are other trails possible.

b **i** This graph has all vertices of odd degree. Neither an eulerian trail, nor eulerian circuit, are possible.

c **i** This graph has two odd-degree vertices (A and F) and so it will have an eulerian trail.

ii One possible eulerian trail for this graph is $A–C–E–C–B–D–E–F$. There are other trails possible

d **i** This graph has all vertices of even degree. An eulerian circuit is possible.

ii One possible eulerian circuit for this graph is $A–B–C–D–E–C–A$. There are other circuits possible

e **i** This graph has all vertices of even degree. An eulerian circuit is possible.

ii One possible eulerian circuit for this graph is $F–E–A–B–E–D–C–B–D–F$. There are other circuits possible

4

a A hamiltonian cycle for this graph is *A–B–C–F–I–H–E–G–D–A*.

b A hamiltonian cycle for this graph is *A–B–C–D–E–F–A*.

c A hamiltonian cycle for this graph is *A–B–D–C–E–A*.

5 *F–A–B–C–D–E–H–G*.

6

a There are five edges connected to vertex *W* so deg(*W*) = 5.

b **i** The salesman will visit each location (vertex) only once and will not return to his starting point. This is an example of a hamiltonian path

ii One possible order is *E–W–D–C–B–A–F*.

c **i** An eulerian circuit is not possible because there are two vertices that have an odd degree. Eulerian circuits are only possible if the graph has all vertices with an even degree.

ii An eulerian trail is possible because there are exactly two vertices that have an odd degree, *W* and *C*. These vertices will be the starting and ending vertices of an eulerian trail. If *W* is the start, the eulerian trail will end at vertex *C*.

7

a

Vertex	Degree
A	4
B	2
C	5
D	2
E	4
F	4
G	3
***SUM*:**	24

b **i** To walk the minimum distance, would mean walking along each path only once. Walking along each of the paths only once would be an eulerian circuit or trail. An eulerian circuit is not possible because there are two odd-degree vertices (*C* and *G*).
Walking an eulerian trail therefore, can start at either *C* or *G*.

ii An eulerian trail would follow each of the paths only once, so the total distance Jamie will walk is equal to the total length of paths that can be walked.

Path	length
A–B	200
A–C	400
A–F	200
A–G	200
B–C	250
C–D	150
C–E	350
C–F	150
D–E	100
E–F	250
E–G	300
F–G	250
TOTAL:	2800

Jamie would walk a total of 2800 m

c Michelle should walk the cycle *F–G–A–B–C–D–E–F*.

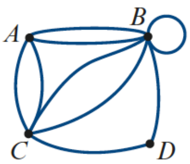
8

a By inspection, there are 7 different trails from town *A* to town *D*. Where there are different routes between two towns, the route is shown by a subscript.

$A–C_1–D$
$A–C_2–D$
$A–B_1–D$
$A–B_2–D$
$A–B_1–C–D$
$A–C_1–B_1–D$
$A–C_1–B_2–D$

b A vehicle can travel between town *A* and town *B* in two ways, without visiting any other town.

c



d An eulerian circuit is not possible through this network because there are some odd-degree vertices (*B* and *C*).

Solutions to Exercise 14D

1

a The edge showing a weight of 12 is between town *D* and town *E*.

b *C* to *D* via *B*, means *C* to *B* (8 minutes) followed by *B* to *D* (9 minutes) for a total of 9 + 8 = 17 minutes.

c *D* to *E* direct is 12 minutes
D to *E* via *B* is 9 + 11 = 20 minutes.

By driving direct, the motorist will save 20 – 12 = 8 minutes.

d The options for travelling from *A* to *E* and visiting all towns exactly once are:

A–C–D–B–E for a total time of 45 minutes.

A–B–C–D–E for a total time of 36 minutes.

A–C–B–D–E for a total time of 44 minutes.

The shortest time is 36 minutes.

2 By inspection, the shortest path from *A* to *E* will be *A–C–D–E* for a length of 11.

3

a The path *A–B–E–H–I* has length:
5 + 9 + 12 + 8
= 34 kilometres

b The circuit *F–E–D–H–E–A–C–F* has length:
6 + 6 + 10 + 12 + 8 + 4 + 10
= 56 kilometres

c The shortest cycle starting and ending at *E* is via *B* and A, *E–B–A–E* for a distance of:
9 + 5 + 8
= 22 kilometres

d Shortest path from *A* to *I* (by inspection) is either *A–C–F–G–I* for a distance of
4 + 10 + 4 + 8
= 26 kilometres

or *A–E–F–G–I* for a distance of
8 + 6 + 4 + 8
= 26 kilometres

4

a The starting vertex for the problem is the first row vertex. The starting vertex is *A*.

b A cross in the table indicates that the vertex for that column is not directly connected to the vertex in the row.
Vertices *D* and *E* are not directly connected to vertex *A*.

c The next row vertex is the column vertex that has the smallest, unboxed number in the row.
The smallest number in row *A* is 2, from column *C* so *C* is the next row vertex.

d The numbers in any row give the distance between the row vertex and the column vertex.
The length of edge *A–F* is 6.

5

	Q	*R*	*S*	*T*	*U*
P	3	1	×	4	×
R	5	1	6	4	3

6

a Vertex *V* was chosen because column *V* had the smallest number in the row, which was 5. Column *P* also has a 5, so vertex *P* could have been chosen instead.

b The × indicates that a direct connection does not exist. There is an × in row *V* and column *U*, so there is definitely no direct connection between *V* and *U*.

c There is a '6' in row *V* and column *W*. This includes the "5" from the box value of V so the distance between *V* and *W* is 6 – 5 = 1.

7

a The length of the shortest path from *A* to *C* is the last box number in column *C*, or 10.

b The length of the shortest path from *A* to *E* is the last box number in column *E*, or 16.

c Using Dijkstra's algorithm to determine the shortest path, the shortest path is *A*–*B*–*C* –*E*.

8

a The length of the shortest path from *A* to *G* is the last box number in column *G*, or 7.

b The shortest path from *A* to *G* is found from the table of calculations:

	B	C	D	E	F	G	H	I
A	2	1	×	5	×	×	×	×
C	2	1	×	5	4	×	×	×
B	2	1	6	5	4	×	×	×
F	2	1	6	5	4	×	8	×
E	2	1	6	5	4	7	7	×
D	2	1	6	5	4	7	7	×
G	2	1	6	5	4	7	7	8
H	2	1	6	5	4	7	7	8

The shortest path from *A* to *G* is *A*–*E*–*G*.

c The length of the shortest path from *A* to *I* is the last box number in column *I*, or 8.

d The shortest path from *A* to *I* is found from the table of calculations:

	B	C	D	E	F	G	H	I
A	2	1	×	5	×	×	×	×
C	2	1	×	5	4	×	×	×
B	2	1	6	5	4	×	×	×
F	2	1	6	5	4	×	8	×
E	2	1	6	5	4	7	7	×
D	2	1	6	5	4	7	7	×
G	2	1	6	5	4	7	7	8
H	2	1	6	5	4	7	7	8

The shortest path from *A* to *I* is *A*–*E*–*G*–*I*.

9

a

	A	B	C	D	F
S	3	2	×	×	×
B	3	2	×	9	×
A	3	2	9	9	×
C	3	2	9	9	13
D	3	2	9	9	12

The shortest path from *S* to *F* is *S*–*B*–*D*–*F* with length 12.

b

	A	B	C	D	E
S	1	5	×	×	×
A	1	5	4	10	×
C	1	5	4	8	×
B	1	5	4	8	11
D	1	5	4	8	10

The shortest path from *S* to *F* is *S*–*A*–*C*–*D*–*F* with length 10.

c

	A	B	C	D	E	F
S	7	5	8	×	×	×
B	7	5	8	9	12	×
A	7	5	8	9	12	×
C	7	5	8	9	10	×
D	7	5	8	9	10	15
E	7	5	8	9	10	15

The shortest path from *S* to *F* is *S*–*B*–*D*–*F* with length 15.

d

	A	*B*	*C*	*D*	*E*	*G*	*H*	*F*
S	3	×	4	×	×	×	×	×
A	3	5	4	×	10	×	×	×
C	3	5	4	12	10	×	×	×
B	3	5	4	12	10	×	14	×
E	3	5	4	12	10	15	14	×
D	3	5	4	12	10	15	14	×
H	3	5	4	12	10	15	14	20
G	3	5	4	12	10	15	14	19

The shortest path from *S* to *F* is *S*–*A*–*E*–*G*–*F* with length 19.

10

	P	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*B*
A	11	×	8	×	×	×	×	×	×
R	11	10	8	×	16	11	×	×	×
Q	11	10	8	×	×	11	×	×	×
P	11	10	8	26	14	11	×	×	×
U	11	10	8	26	14	11	×	23	×
T	11	10	8	26	14	11	×	17	×
W	11	10	8	26	14	11	26	17	19

The length of the shortest path between town *A* and *B* is 19 kilometres.

11

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*W*
Q	26	×	7	×	×	×	×	×	×	×	×	×	×	×
C	26	13	7	16	×	×	×	×	×	×	×	×	×	×
B	22	13	7	16	21	×	×	×	×	×	×	×	×	×
D	22	13	7	16	21	29	×	×	×	×	×	×	×	×
E	22	13	7	16	21	28	37	67	×	×	×	×	×	×
A	22	13	7	16	21	28	37	67	47	×	×	×	×	×
F	22	13	7	16	21	28	32	67	47	×	×	×	38	×
G	22	13	7	16	21	28	32	67	47	×	×	39	38	×
M	22	13	7	16	21	28	32	67	47	×	×	39	38	52
L	22	13	7	16	21	28	32	64	47	×	52	39	38	46

a The length of the shortest path from *Q* to *W* is 46 kilometres.

b The shortest path from *Q* to *W* is *Q*–*C*–*B*–*E*–*F*–*G*–*L*–*W*.

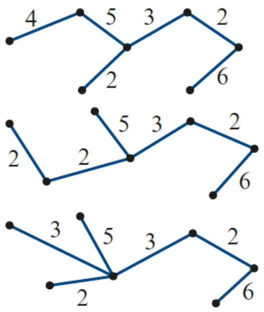
Solutions to Exercise 14E

1

a There are 7 vertices, so the spanning tree will have 7 – 1 = 6 edges.

The network has 12 edges, so 12 – 6 = 6 edges must be removed.

b Note: other trees are possible as answers to this question.



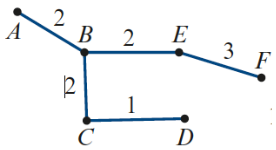
c The first graph has weight:
4 + 5 + 2 + 3 + 2 + 6 = 22

The second graph has weight:
2 + 2 + 5 + 3 + 2 + 6 = 20

The third graph has weight:
3 + 2 + 5 + 3 + 2 + 6 = 21

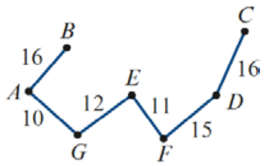
2

a



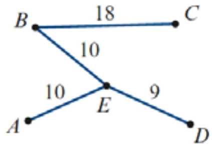
Total weight = 2 + 2 + 2 + 1 + 3
= 10

b



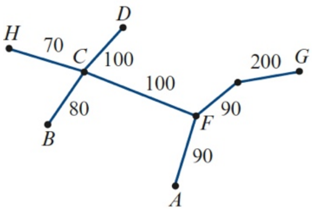
Total weight
= 16 + 10 + 12 + 11 + 15 + 16
= 80

c



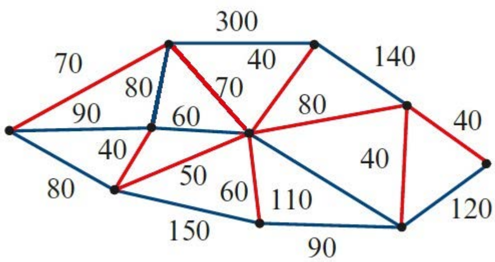
Total weight = 18 + 10 + 10 + 9
= 47

d



Total weight
= 70 + 80 + 100 + 100 + 90 + 90 + 200
= 730

3 The shortest length of pipe required to connect all water storages will be the weight of the minimum spanning tree for the network.

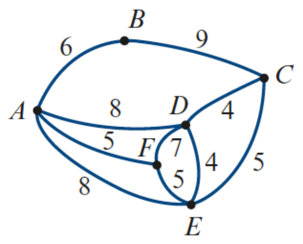


The weight of the minimum spanning tree, shown in red above, is
weight
= 70 + 70 + 40 + 50 + 40 + 60 + 80 + 40
+ 40
= 490

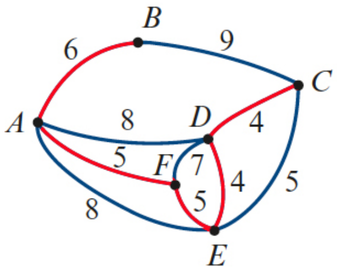
Solutions to Review: Extended-response questions

1

a **i** The only edge missing from the graph is the direct connection between vertex *E* and vertex *C*. There is only one direct connection between these vertices, of length 5, so this must be added to the graph.



ii The cable should be laid along the minimum spanning tree for the graph. The minimum spanning tree is shown in red below:



The weight of the minimum spanning tree
= 4 + 5 + 4 + 5 + 6
= 24

The minimum length of cable required is 24 kilometres.

iii There is:

- one connection between *D* and *C*
- no loop at *D*
- one connection between *D* and *E*
- one connection between *D* and *F*
- one connection between *E* and *C*
- no loop at *E*
- one connection between *E* and *F*
- no connection between *F* and *C*
- no loop at *F*

The matrix is

	A	*B*	*C*	*D*	*E*	*F*
A	0	1	0	1	1	1
B	1	0	1	0	0	0
C	0	1	0	1	1	0
D	1	0	1	0	1	1
E	1	0	1	1	0	1
F	1	0	0	1	1	0

b **i** The route
A – *B* – *A* – *F* – *E* – *D* – *C* – *E* – *F* – *A*

has distance

6 + 6 + 5 + (3+2) + 4 + 4 + 5 + (2+3) + 5
= 45 kilometres

ii This route is not a hamiltonian cycle because some of the vertices are visited more than once, namely *A*, *F* and *E*.

iii There are many answers, but one possible route is:
A – *B* – *C* – *D* – *F* – *E* – *A*

iv The distance travelled will vary depending on the answer for part iii. The route in part **iii** above has distance

6 + 9 + 4 + 7 + 5 + 8
= 39 kilometres

c Starting at *A* and returning to *A* by travelling each track once is an example of an eulerian circuit. This can only occur if all vertices are of even degree.

At the moment, vertex *C* and *F* both have odd degrees, so joining them by a new path will make them both have an even degree, making an eulerian circuit possible.

2

a By inspection, the shortest path from *Amity* to *Bevin* is via checkpoint *R*, taking the shorter of the two possible routes from checkpoint *R* to *Bevin*.

The length of this path = 7 + 4 = 11 km

b It is possible to travel the network on every road exactly once because there are two vertices (*Amity* and checkpoint *V*) that are of an odd degree. All the others have an even degree.

c If the road begins at *Amity*, the race must finish at the other odd-degree vertex, checkpoint *V*.

d The competitor should not travel to checkpoint *U* next. This will mean that all the roads out of *Bevin* in the direction of *Carter* have already been travelled, and so one of them will need to be travelled again to return the competitor to the unvisited checkpoint *R*.

e

	S	***T***	***U***	***V***	***W***	***Carter***
Bevin	3	2	6	×	×	×
T	3	2	5	9	×	×
S	3	2	5	9	11	11
U	3	2	5	8	11	10
V	3	2	5	8	11	10

The shortest path from *Bevin* to *Carter* is *Bevin – T – U – Carter*

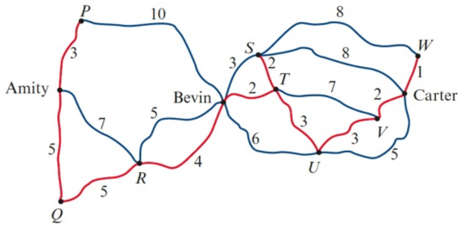
Note: There is an alternative route for the same length: Bevin – T – U – V – Carter

f The shortest distance from *Bevin* to *Carter* is 10 kilometers (answer to part e).

The shortest distance from *Amity* to *Bevin* is 11 km (answer to part a).

The shortest distance from *Amity* to *Carter* is 10 + 11 = 21 kilometres.

g The minimum spanning tree, determined by Primm's algorithm, is shown in red on the network below.



Solutions to Review: Multiple-choice questions

1 Seven vertices can be connected with six edges, one less than the number of vertices. **C**

2 **A** This graph has a cycle so is not a tree.
B This graph has a cycle so is not a tree.
C This graph is a spanning tree.
D This graph has a cycle so is not a tree.
E This graph is a tree but does not include the vertex 2 so it is not a spanning tree. **C**

3 *P* has degree 2
Q has degree 5
R has degree 3
S has degree 4
T has degree 4
U has degree 2 **A**

4 $v = 15$ and $f = 12$
$v - e + f = 2$
$15 - e + 12 = 2$
$-e = 2 - 15 - 12$
$-e = -25$
$e = 25$ **D**

5 An eulerian circuit will exist if all of the vertices have an even degree.

A has two odd-degree vertices.
B has all even-degree vertices.
C has all even-degree vertices.
D has all even-degree vertices.
E has all even-degree vertices. **A**

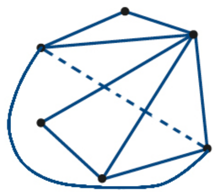
6 $v = 8$ and $e = 13$
$v - e + f = 2$
$8 - 13 + f = 2$
$f = 2 - 8 + 13$
$f = 7$ **C**

7 Hamiltonian cycle starts and ends at the same vertex, so it cannot be option E.

Hamiltonian cycles pass through every vertex only once, so it cannot be option A (visits *E* multiple times) or option C (visits *A* multiple times),

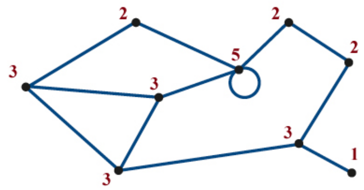
Hamiltonian cycles pass through every vertex in the graph, so it cannot be option D which does not visit vertex *F*. **B**

8 The graph is planar and must be redrawn without edges crossing before counting the faces:



There are five regions defined by the graph in planar form. **B**

9 The graph is drawn below, with the degrees of each vertex written beside them.

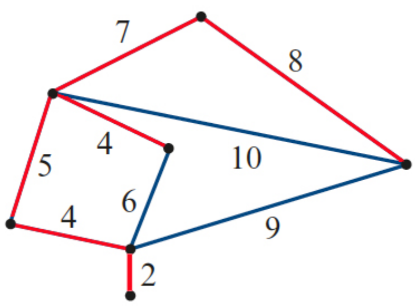


The sum of the degrees is
$3 + 2 + 3 + 3 + 5 + 2 + 2 + 3 + 1$
$= 24$ **E**

10 An eulerian trail exists if there are exactly two odd-degree vertices in a graph. The graph currently has four odd-degree vertices, that is *A*, *E*, *C*, *D*.

Joining two of these by an edge would make their degree even. **B**

11 The minimum spanning tree is shown in red in the diagram below:



The length of the minimum spanning tree
$= 2 + 4 + 5 + 4 + 7 + 8 = 30$ **A**

12 Eulerian circuit will be possible if all of the vertices have an even degree, so it could be option **A** or **B**.

By inspection, a hamiltonian cycle is possible only in option **A**. **A**

13 A complete graph has every vertex connected to every other vertex, so

A – the paper boy would NOT be covering the minimum distance by travelling between every pair of houses

B – the amount of cabling would NOT be the minimum required if it connected every house to every other house

C – All teams (vertices) playing every other team would mean each team or vertex is connected to every other team in a complete graph

D – Six tasks between start and finish would be more linear, one after the other

E – Allocation of assignment to one student means the vertex would be connected to one other vertex. **C**

14 In the given graph,
- *A* is directly connected to *B* in two ways (could be only option B)
- *A* is directly connected to *C* in one way
- *A* is directly connected to *D* in one way
- *B* is directly connected to *C* in one way
- *B* is directly connected to *D* in one way
- *C* is directly connected to *D* in two ways

B

15 An eulerian circuit exists if all the vertices have an even degree.

Option **A** has two odd-degree vertices.
Option **B** has all even-degree vertices.
Option **C** has two odd-degree vertices.
Option **D** has two odd-degree vertices.
Option **E** has two odd-degree vertices. **B**

Chapter 15 – Flow, matching and scheduling problems

Solutions to Exercise 15A

1

a The capacity of $C1$
= 6 + 8
= 14

b The capacity of $C2$
= 3 + 5 + 4
= 12

Note: the edge from E to C is not counted as the flow along this edge is from the sink side to the source side of the cut.

c The capacity of $C3$
= 8 + 10 + 3
= 21

2

a The capacity of $C1$
= 4 + 8
= 12

b The capacity of $C2$
= 5 + 3 + 8
= 16

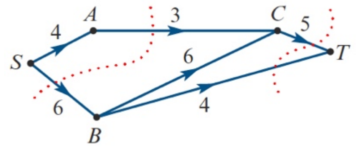
Note: there are two edges where the flow is from the sink side to the source side of the cut. These edges have capacity 4 and 3, and both are not counted in the calculation.

c The capacity of $C3$
= 5 +7 + 4
= 16

Note: there is one edge where the flow is from the sink side to the source side of the cut. This edge has capacity 2 and this is not counted in the calculation.

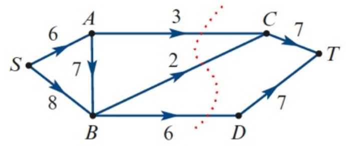
3 *Note: The minimum capacity cut only is shown for each of the graphs in this question. Where there are two equal minimum capacity cuts, both are shown.*

a



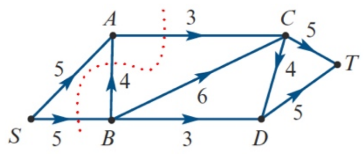
Maximum flow = 3 + 6 = 9
or
Maximum flow = 5 + 4 = 9

b



Maximum flow = 3 + 2 + 6 = 11

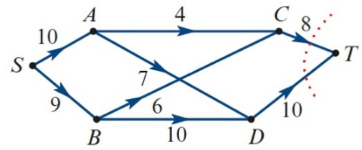
c



Maximum flow = 3 + 5 = 8

Note: The flow from B to A is from the sink side to the source side of the cut and is not counted in the calculation.

d



Maximum flow = 8 + 10 = 18

4

a The capacity of cut A
= 3 + 7 + 4
= 14

The capacity of cut B
= 3 + 7 + 8 + 1 + 4
= 23

The capacity of cut *C*
= 0 + 3 + 4 + 1 + 4
= 12

Note: the edge with capacity 8 is not counted as the flow along this edge is from the sink side to the source side of the cut.

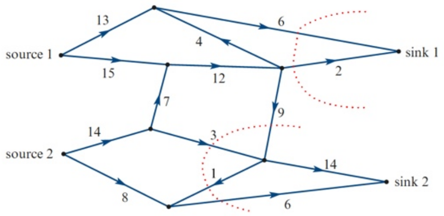
The capacity of cut *D*
= 4 + 8 + 4
16

The capacity of cut *E* cannot be determined. It is not a valid cut.

b Cut *E* is not valid because it does not completely block the flow from Arlie (source) to Bowen (sink).

c The maximum number of seats from Arlie to Bowen is the capacity of the minimum cut. The minimum cut is Cut *C*, with capacity 12, so there is a maximum of 12 seats from Arlie to Bowen.

5 *Note: The minimum capacity cut for each sink is shown on the graph below.*



The maximum flow to sink 1
= 6 + 2
= 8

The maximum flow to sink 2
= 9 + 3 + 6
=18

Note: the edge with capacity 1 is not counted as the flow along this edge is from the sink side to the source side of the cut.

Solutions to Exercise 15B

1

a A bipartite graph can be used to display this information because there are two distinct groups of objects that are, in some way, connected to each other.

One group is the people and the other group is the flavours of ice-cream. Each of the people are connected to one or more ice-cream flavours.

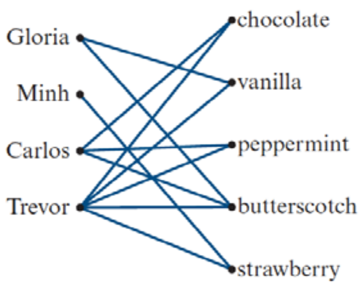
b List the people on the left and the flavours on the right with a vertex for each.

Gloria likes vanilla and butterscotch, so join Gloria's vertex to each of the vertices for these flavours.

Minh only likes strawberry, so join Minh's vertex to the vertex for strawberry.

Similarly, join Carlos' vertex to the vertex for chocolate, peppermint and butterscotch. Join Trevor's vertex to the vertex for every flavour.

The completed bipartite graph is below:



c Trevor is connected to all five ice-cream flavours. The degree of this vertex is 5.

2

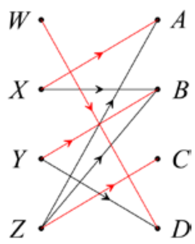
a

	A	B	C	D	
W	110	95	140	80	–80
X	105	82	145	80	–80
Y	125	78	140	75	–75
Z	115	90	135	85	–85

	A	B	C	D
W	30	15	60	0
X	25	2	65	0
Y	50	3	65	0
Z	30	5	50	0
	–25	–2	–50	

	A	B	C	D
W	5	13	10	0
X	0	0	15	0
Y	25	(1)	15	0
Z	5	3	0	0

	A	B	C	D
W	4	12	9	0
X	0	0	15	1
Y	24	0	14	0
Z	0	0	0	1



W must be allocated to *D*, so *Y* cannot.
Y must then be allocated to *B*, so *X* cannot.
X must be allocated to *A*
Z must be allocated to *C*

Allocation: *W* – *D*, *Y* – *B*, *X* – *A*, *Z* – *C*

b

	A	*B*	*C*	*D*	
W	2	4	3	5	−2
X	3	5	3	4	−3
Y	2	3	4	2	−2
Z	2	4	2	3	−2

	A	*B*	*C*	*D*
W	0	2	1	3
X	0	2	0	1
Y	0	1	2	0
Z	0	2	0	1
		−1		

	A	*B*	*C*	*D*
W	0	(1)	1	3
X	0	1	0	1
Y	0	0	2	0
Z	0	1	0	1

	A	*B*	*C*	*D*
W	0	0	1	2
X	0	0	0	0
Y	1	0	3	0
Z	0	0	0	0

Multiple allocations are possible, but all will have the same minimum cost.

One possible allocation is: *W* to *A* (2), *X* to *B* (5), *Y* to *D* (2) and *Z* to *C* (2)

Minimum cost = 2 + 5 + 2 + 2 = 11

3

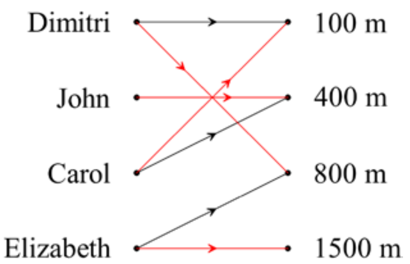
Student	*100 m*	*400 m*	*800 m*	*1500 m*	
Dimitri	11	62	144	379	−11
John	13	60	146	359	−13
Carol	12	61	149	369	−12
Elizabeth	13	63	142	349	−13

Student	*100 m*	*400 m*	*800 m*	*1500 m*
Dimitri	0	51	133	368
John	0	47	133	346
Carol	0	49	137	357
Elizabeth	0	50	129	336
		−47	−129	−336

Student	*100 m*	*400 m*	*800 m*	*1500 m*
Dimitri	0	4	4	32
John	0	0	4	10
Carol	0	(2)	8	21
Elizabeth	0	3	0	0

Student	*100 m*	*400 m*	*800 m*	*1500 m*
Dimitri	0	2	(2)	30
John	2	0	4	10
Carol	0	0	6	19
Elizabeth	2	3	0	0

Student	*100 m*	*400 m*	*800 m*	*1500 m*
Dimitri	0	2	0	28
John	2	0	2	8
Carol	0	0	4	17
Elizabeth	4	5	0	0



John must be allocated to 400 m, so Carol cannot.
If Carol cannot be allocated to 400 m, she must be allocated to 100 m, so Dimitri cannot.
If Dimitri cannot be allocated to 100m, he must be allocated to 800 m, leaving 1500 m for Elizabeth

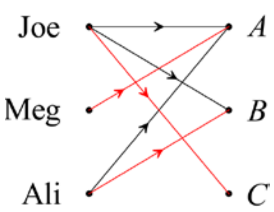
The “best” student allocation is:
Dimitri – 800 m, John – 400 m,
Carol – 100 m, Elizabeth – 1500 m

4

Student	Job A	Job B	Job C	
Joe	20	20	36	–20
Meg	16	20	44	–16
Ali	26	26	44	–26

Student	Job A	Job B	Job C
Joe	0	0	16
Meg	0	4	28
Ali	0	0	18
			–16

Student	Job A	Job B	Job C
Joe	0	0	0
Meg	0	4	12
Ali	0	0	2



Meg must be allocated to job *A*, so Joe and Ali cannot.
If Ali cannot be allocated to job *A*, he must be allocated to job *B*, so Joe cannot.
If Joe cannot be allocated to job *A*, nor job *B*, he must be allocated to job *C*.

The allocation that minimises the time taken to complete the jobs is:

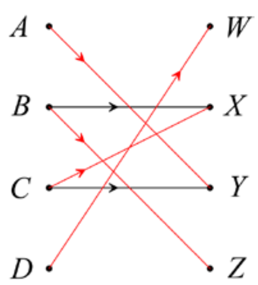
Meg – *A*, Ali – *B*, Joe – *C*

5

Operator	Machine *W*	*X*	*Y*	*Z*	
A	38	35	26	54	–26
B	32	29	32	26	–26
C	44	26	23	35	–23
D	20	26	32	29	–20

Operator	Machine *W*	*X*	*Y*	*Z*
A	12	9	0	28
B	6	(3)	6	0
C	21	3	0	12
D	0	6	12	9

Operator	Machine *W*	*X*	*Y*	*Z*
A	9	6	0	28
B	3	0	6	0
C	18	0	0	12
D	0	6	15	12



A must be allocated to *Y*, so *C* cannot.
If *C* cannot be allocated to *Y*, *C* must be allocated to *X* and so *B* cannot.
If *B* cannot be allocated to *X*, *B* must be allocated to *Z*.
D must be allocated to *W*.

The allocation of machinists to machines that minimises the total cost is:

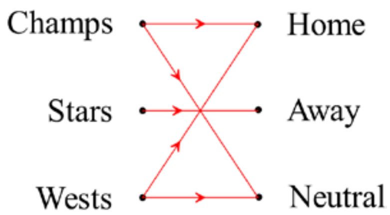
A – *Y*, *B* – *Z*, *C* – *X*, *D* – *W*

6

Team	Home	Away	Neutral	
Champs	10	9	8	–8
Stars	7	4	5	–4
Wests	8	7	6	–6

Team	Home	Away	Neutral
Champs	2	1	0
Stars	3	0	2
Wests	2	1	0
	–2		

Team	Home	Away	Neutral
Champs	0	1	0
Stars	1	0	2
Wests	0	1	0



Stars must play at the away ground. Both Champs and Wests can play at either Home or Neutral, so there are two possible allocations:

Champs – Home (10), Stars – Away (4) and Wests – Neutral (6) for a total of $20 000.

or

Champs – Neutral (8), Stars – Away (4) and Wests – Home (8) for a total of $20 000.

7

Service vehicle	Motorist				
	Jess	Mark	Raj	Karla	
A	18	15	15	16	–15
B	7	17	11	13	–7
C	25	19	18	21	–18
D	9	22	19	23	–9

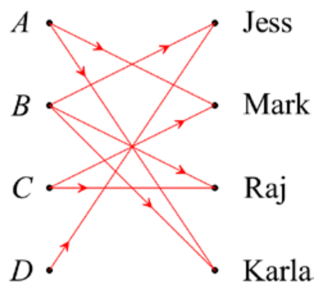
Service vehicle	Motorist			
	Jess	Mark	Raj	Karla
A	3	0	0	1
B	0	10	4	6
C	7	1	0	3
D	0	13	10	14

Service vehicle	Motorist			
	Jess	Mark	Raj	Karla
A	3	0	0	1
B	0	10	4	6
C	7	1	0	3
D	0	13	10	14
				–1

Service vehicle	Motorist			
	Jess	Mark	Raj	Karla
A	3	0	0	0
B	0	10	(4)	5
C	7	1	0	2
D	0	13	10	13

Service vehicle	Motorist			
	Jess	Mark	Raj	Karla
A	7	0	0	0
B	0	6	0	(1)
C	11	1	0	2
D	0	9	6	9

Service vehicle	Motorist			
	Jess	Mark	Raj	Karla
A	8	0	1	0
B	0	5	0	0
C	11	0	0	1
D	0	8	6	8



A must go to Mark or Karla. *B* can go to Jess, Raj or Karla. *C* must go to Mark or Raj. *D* must go to Jess.

The allocations of vehicle to motorist that minimise the total distance travelled are: *A* – Karla (16), *B* – Raj (11), *C*– Mark (19) and *D* – Jess (9) OR *A* – Mark (15), *B* – Karla (13), *C*– Raj (18) and *D* – Jess (9), both for a total of 55 km.

Solutions to Exercise 15C

1

a

Activity	Immediate predecessors
A	–
B	–
C	A
D	A
E	B, C
F	D
G	E

b

Activity	Immediate predecessors
P	–
Q	P
R	P
S	Q
T	Q
U	S, V
V	R
W	R
X	T, U

c

Activity	Immediate predecessors
J	–
K	–
L	J
M	N
N	K
O	K
P	N
Q	L, M
R	P
S	O, R
T	Q

d

Activity	Immediate predecessors
A	–
B	–
C	A
D	A
E	D, B
F	C, E
G	D, B
H	B

e

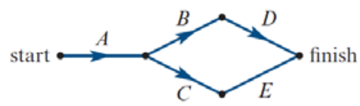
Activity	Immediate predecessors
P	–
Q	–
R	P
S	P
T	Q
U	R
V	S
W	S, T
X	U
Y	W
Z	V, X, Y

f

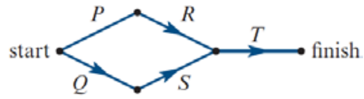
Activity	Immediate predecessors
A	–
B	A
C	A
D	A
E	B
F	C, D
G	D
H	E, F, G
I	G
J	I
K	H

2

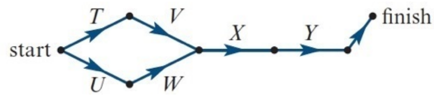
a



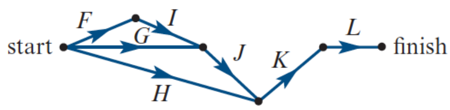
b



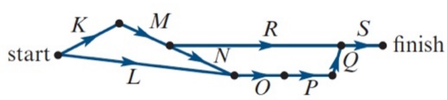
c



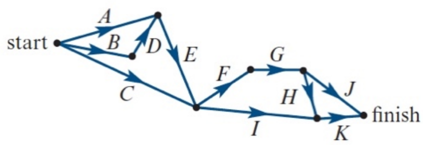
d



e

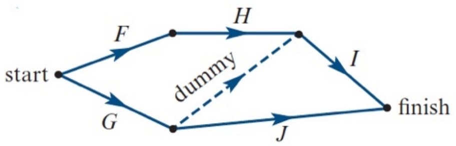


f

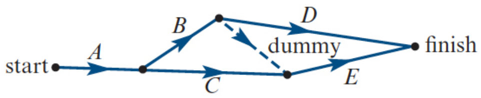


3

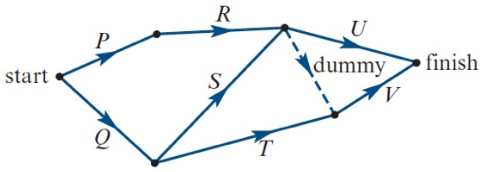
a



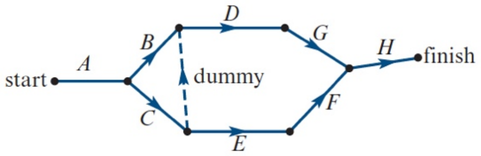
b



c



d



4

a "Remove broken component" is activity *C*.
Look at activity *C* in the activity network.
Activity *C* follows immediately from activity *A*.
Activity *A* is an immediate predecessor of activity *C*.

"Remove panel" is an immediate predecessor of "Remove broken component".

b "Install new component" is activity *F*.
Look at activity *F* in the activity network.
Activity *F* follows immediately from activity *B* and the dummy that follows activity *D*.
Activities *B* and *D* are immediate predecessors of activity *F*.

"Order component" and "Pound out dent" are immediate predecessors of "Install new component".

Solutions to Exercise 15D

1

a Use forward scanning,
$p = 8 + 4$
$p = 12$

b Use forward scanning,
$w = 4 + 6$
$w = 10$

c Using forward scanning,
$m + 4 = 12$
$m = 12 – 4$
$m = 8$

Using backward scanning,
$n = 12 – 4$
$n = 8$

d Using forward scanning,
$c = 6 + 5$
$c = 11$

Using backward scanning,
$a = 15 – 5$
$a = 10$

Using backward scanning,
$b – 3 = 15$
$b = 15 + 3$
$b = 18$

e Using forward scanning,
$f = 3 + 6$
$f = 9$

Using forward scanning, g is the largest of:
$g = 5 + 7$ or $g = f + 0$
$g = 12$ or $g = 9$
So, $g = 12$

f Using forward scanning,
$4 + q = 12$
$q = 12 – 4$
$q = 8$

Using backward scanning,
$9 – p = 4$
$p = 9 – 4$
$p = 5$

Using backward scanning,
$12 – r = 9$
$12 – 9 = r$
$r = 3$

Using forward scanning,
$n + r = 12$
$n + 3 = 12$
$n = 12 – 3$
$n = 9$

2

a Using forward scanning,
6 + duration of A = 9
duration of A = 9 – 6
duration of A = 3

b The critical path follows activities that have no float time. The two numbers in the boxes at the start of these activities will be the same.

The critical path is: $A – C$

c Float time = LST – EST
= 11 – 6
= 5

d LST for D is the second number in the boxes at the start of activity D.

LST for activity D = 13

e Using backward scanning,
15 – duration of D = 13
Duration of D = 15 – 13
Duration of D = 2

3

a Using forward scanning,
0 + duration of B = 12
Duration of B = 12

b LST for E is the second number in the boxes at the start of activity E.

LST for E = 10

c EST for E is the first number in the boxes at the start of activity E.

EST for $E = 9$

d Float time for E = LST – EST
= 10 – 9
= 1

e Using forward scanning,
0 + duration A = 3
duration A = 3

f Using forward scanning,
0 + duration D = 9
duration D = 9

4
a The critical path follows activities to boxes where the EST and LST are the same.

The critical path is: $D – E – F$

b Non-critical activities are A, B, C

Float A = LST (B) – duration of A
= 4 – 3
= 1

Float B = LST (B) – EST (B)
= 4 – 3
= 1

Float C = LST (F) – duration of C
= 22 – 7
= 15

5
a The critical path follows activities to boxes that have EST and LST the same.

The critical path is: $B – E – F – H – J$

b Non-critical activities are A, C, D, G, I

Float A = LST(D) – duration of A
= 11 – 10
= 1

Float C = LST(J) – duration of C
= 17 – 3
= 14

Float D = LST (D) – EST (D)
= 11 – 10
= 1

Float G = LST(I) – EST(G) – duration G
= 15 – 13 – 1
= 1

Float I = LST(J) – EST(I) – duration I
= 17 – 14 – 2
= 1

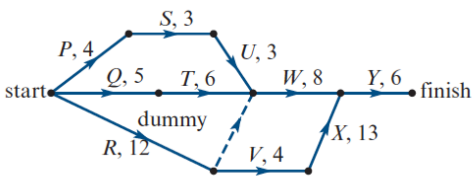
6
a Use information in the activity network.

Activity	Duration (weeks)	Immediate predecessors
A	3	–
B	6	–
C	6	*A*, *B*
D	5	*B*
E	7	*C*, *D*
F	1	*D*
G	3	*E*
H	3	*F*
I	2	*B*

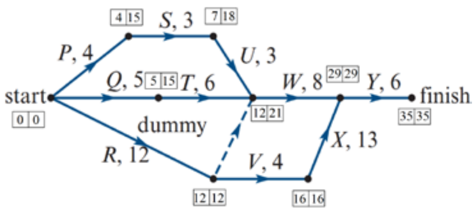
b The critical path follows activities to boxes where the EST and LST are the same.

The critical path is: $B – C – E – G$

7
a



b



Note:
LST (P) = LST (S) – duration P
= 15 – 4
= 11

LST (Q) = LST (T) – duration Q
= 15 – 5
= 10

A table of EST and LST for each activity is shown below:

Activity	***EST***	***LST***
P	0	11
Q	0	10
R	0	0
S	4	15
T	5	15
U	7	18
V	12	12
W	12	21
X	16	16
Y	29	29

c The critical path follows activities to boxes where the EST and LST are the same.

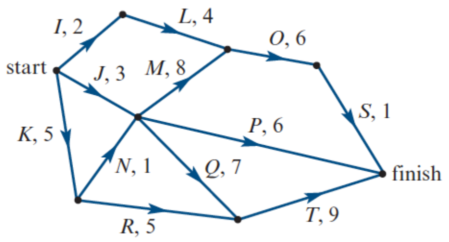
The critical path is: $R – V – X – Y$

d The minimum time to complete the project is the value in the boxes at the finish.

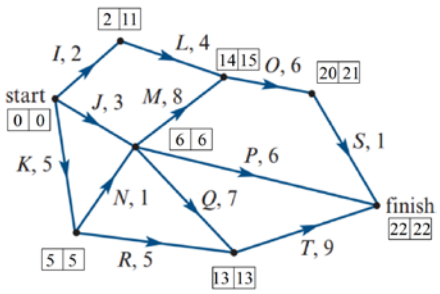
Minimum project completion time is 35 weeks.

8

a



b



Note:
LST (I) = LST (L) – duration I
= 11 – 2
= 9

LST (J) = LST (M, P or Q) – duration J
= 6 – 3
= 3

LST (M) = LST (O) – duration M
= 15 – 8
= 7

LST (P) = LST (*finish*) – duration P
= 22 – 6
= 16

LST (Q) = LST (T) – duration Q
= 13 – 7
= 6

LST (R) = LST (T) – duration R
= 13 – 5
= 8

A table of EST and LST for each activity is shown below:

Activity	***EST***	***LST***
I	0	9
J	0	3
K	0	0
L	2	11
M	6	7
N	5	5
O	14	15
P	6	16
Q	6	6
R	5	8
S	20	21
T	13	13

c The critical path follows activities to boxes where the EST and LST are the same.

The critical path is: $K – N – Q – T$

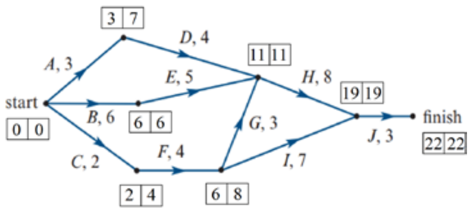
d The minimum time to complete the project is the value in the boxes at the finish.

Minimum project completion time is 22 weeks.

Solutions to Exercise 15E

1

a



The critical path follows activities to boxes where the EST and LST are the same.

The critical path is: $B - E - H - J$

b Total minimum completion time for A and D is 3 + 4 = 7 hours.

Total minimum completion time for activities C, F and G is 2 + 4 + 3 = 9 hours

Reduction in completion time for activity E must not cause the total completion time for activities B and E to drop below 9 hours.

Completion time for activity E cannot be lower than 3.

The maximum reduction in completion time for activity E is 2 hours.

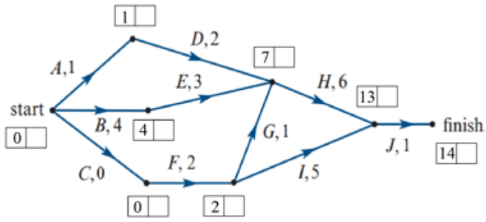
c Total minimum completion time for activities C, F and I is 2 + 4 + 7 = 13 hours.

Activity H can be reduced in duration to ensure duration of activity H + 11 is not lower than 13

Minimum duration of activity H is 2.

Maximum reduction in completion time for activity H is 6 hours.

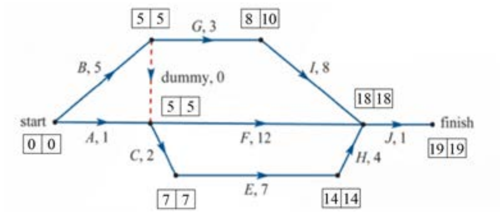
d Reduce times and create a new activity network:



New minimum completion time for the project is 14 hours.

2

a



The critical path follows activities to boxes where the EST and LST are the same.

The critical path is: $B - C - E - H - J$

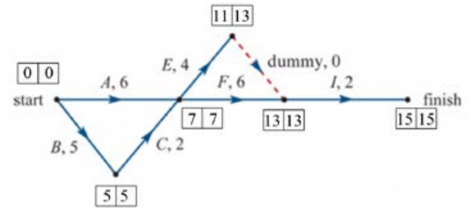
b No time will be saved as activity F is not on the critical path. Activity F already has float time so reducing its duration will have no effect on the overall completion time of the project.

c In order to reduce the overall completion time, activities on the critical path should be considered for duration reductions.

Activity B can be reduced by 4 hours without affecting the critical path.
Activities C, E and H currently have a total completion time of 2 + 7 + 4 = 13 hours.
They could have a total completion time no less than that of activity F, which is 12 hours.
Each activity could therefore be reduced by a maximum of 1 hour.
The maximum saving in time is therefore 4 hours, by reducing the duration of activity B.

3

a



The shortest time for completion of this project is 15 hours.

b The critical path follows activities to boxes where the EST and LST are the same.

The critical path is: $B-C-F-I$.

c Activity *A* is not on the critical path. It already has slack time and reducing the completion time of this activity will have no effect on the overall completion time of the project.

d The duration of activity *B* can be reduced so that the total completion time of activity *B* and *C* combined is not less than 6 hours (the duration of activity *A*).

Activity *B* can be reduced by a maximum of 1 hour only.

The duration of activity *F* can be reduced so that the total completion time of activity *F* is no less than 4 hours (the duration of activity *E* and the dummy).

Activity *F* can be reduced by a maximum of 2 hours only.

Total cost = 1 × \$100 + 2 × \$50
= \$200

Solutions to Review: Extended-response questions

1 Use the Hungarian algorithm to determine the allocation for minimum cost.

Supervisor	*B*	*C*	*D*	*E*	
Ann	25	30	15	35	–15
Bianca	22	34	20	45	–20
Con	32	20	33	35	–20
David	40	30	28	26	–26

Supervisor	*B*	*C*	*D*	*E*
Ann	10	15	0	20
Bianca	2	14	0	25
Con	12	0	13	15
David	14	4	2	0
	–2			

Supervisor	*B*	*C*	*D*	*E*
Ann	8	15	0	20
Bianca	0	14	0	25
Con	10	0	13	15
David	12	4	2	0

Ann must be allocated to *D* so Bianca cannot, so Bianca must be allocated to *B*.

Con must be allocated to *C* and David must be allocated to *E*.

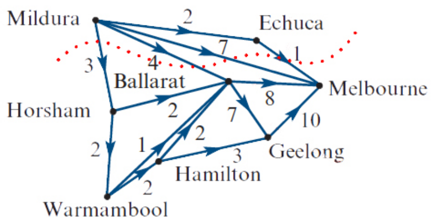
The allocation of supervisors to examination venues that minimises the total travel time is
Ann – *D*, Bianca – *B*,
Con – *C*, David – *E*.

2

a All edges that this cut crosses flow from the source side of the cut to the sink side of the cut, so all of them are used in the capacity calculation.

Cut capacity = 1 + 7 + 8 + 10
= 26

b The minimum capacity cut for this network is shown below:



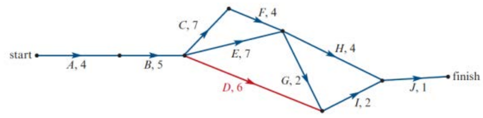
The maximum flow of passengers from Mildura to Melbourne is the capacity of the minimum capacity cut.

Maximum flow = 1 + 7 + 4 + 3
= 15

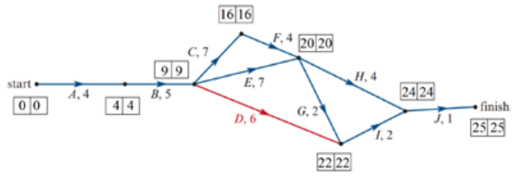
3

a Activity *D* has immediate predecessor activity *B* so it must follow from activity *B*.

Activity *D* is an immediate predecessor of activity *I* so activity *D* must extend from the end of activity *B* to the start of activity *I*, as shown in the diagram below.



b The completed critical path analysis is shown below:



The critical path follows activities to boxes where the EST and LST are the same. This activity network has two possible critical paths:

A – B – C – F – G – I – J
or
A – B – C – F – H – J

4 Use the Hungarian algorithm to determine the allocation for minimum time.

Swimmer	*Backstroke*	*Breaststroke*	*Butterfly*	*Freestyle*	
Rob	76	78	70	62	–62
Joel	74	80	66	62	–62
Henk	72	76	68	58	–58
Sav	78	80	66	60	–60

Swimmer	*Backstroke*	*Breaststroke*	*Butterfly*	*Freestyle*
Rob	14	16	8	0
Joel	12	18	4	0
Henk	14	18	10	0
Sav	18	20	6	0
	–12	–16	–4	

Swimmer	*Backstroke*	*Breaststroke*	*Butterfly*	*Freestyle*
Rob	2	0	4	0
Joel	0	2	0	0
Henk	2	②	6	0
Sav	6	4	2	0

Swimmer	*Backstroke*	*Breaststroke*	*Butterfly*	*Freestyle*
Rob	2	0	4	2
Joel	0	2	0	2
Henk	0	0	4	0
Sav	4	2	0	0

Rob must be allocated to Breaststroke, so Henk cannot.

There are two different allocations possible now:

1. If Joel is allocated to Backstroke, Henk cannot, so Henk must be allocated to Freestyle and Sav to Butterfly.

2. If Joel is allocated to Butterfly, Sav cannot, so Sav must be allocated to Freestyle and Henk to Backstroke.

Allocation 1:
Rob – Breaststroke, Joel – Backstroke, Henk – Freestyle, Sav – Butterfly

Allocation 2:
Rob – Breaststroke, Joel – Butterfly, Henk – Backstroke, Sav – Freestyle

Total time for both allocations is 276.

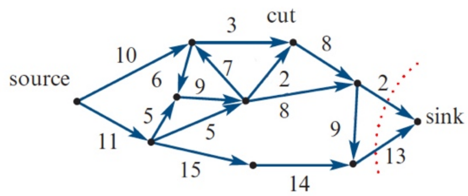
5

a The edge with capacity 9 flows from the sink side of the cut to the source side of the cut, so will not be counted in the capacity calculation.

Cut capacity = 3 + 2 + 8 + 13
= 26

b The maximum flow through this network is equal to the minimum cut capacity.

The cut with minimum capacity is shown below:



Cut capacity = 2 + 13
= 15

Solutions to Review: Multiple-choice questions

1 Identify the shortest path using inspection, or Dijkstra's algorithm

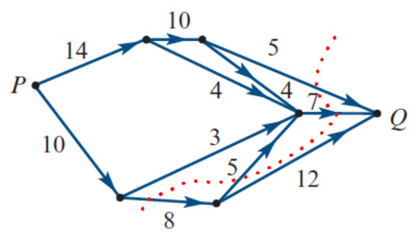
	B	*C*	*D*	*E*	*F*	*G*	*H*	*Z*
A	10	12	×	×	×	×	×	×
B	10	12	17	19	15	×	×	×
C	10	12	17	19	15	22	×	×
F	10	12	17	19	15	18	22	26
D	10	12	17	19	15	18	22	26
G	10	12	17	19	15	18	22	26
E	10	12	17	19	15	18	22	26
H	10	12	17	19	15	18	22	26

D

2 Capacity of cut = 3 + 4 + 3
= 10

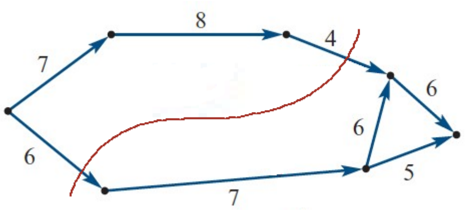
Note: one of the edges with capacity of 4 is not counted as it flows from the sink side of the cut to the source side of the cut. **D**

3 The minimum cut is shown below:



maximum flow = capacity of cut
= 5 + 7 + 8
= 20 **A**

4 The minimum cut is shown below:



maximum flow = capacity of cut
= 6 + 4
= 10 **A**

5 **A**: Travis – basketball and volleyball
Miriam – basketball, athletics, tennis
Swimming is missing so **A** is false.

B: Miriam played 3 sports
Fulvia played 2 sports
total: 5 sports

Andrew played 2 sports
Travis played 2 sports
total: 4 sports

Miriam and Fulvia played more sports than Andrew and Travis so **B** is false

C: Kieran played 3 sports
Miriam played 3 sports so **C** is true.

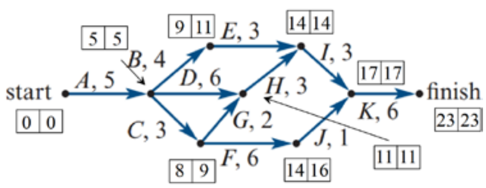
D: Kieran and Travis played basketball, volleyball, swimming, athletics and tennis (5 different sports).

Miriam and Fulvia played swimming, athletics, basketball and tennis (4 different sports).

Travis and Kieran played more sports than Miriam and Fulvia so **D** is false.

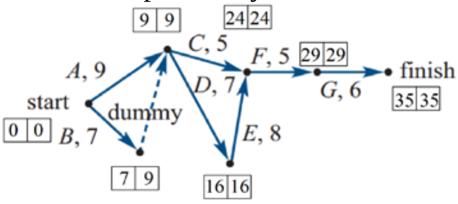
E: Andrew played the same number of sports as Travis and Fulvia so he did not play fewer sports than all others. **E** is false. **C**

6 The critical path analysis is shown below:



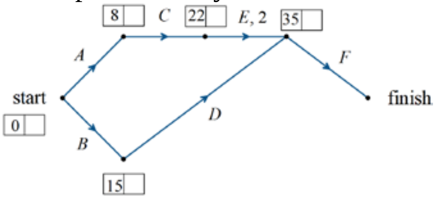
The critical path is: *A* – *D* – *H* – *I* – *K* **B**

7 The critical path analysis is shown below:



The earliest start time for activity *F* is 24. **E**

8 The information in the table has been used to construct the following incomplete activity network.



The duration of activity A
= EST (*C*) – EST (*A*)
= 8

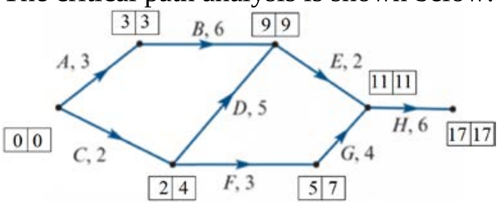
The duration of activity C
= EST (*E*) – EST (*C*)
= 22 – 8
= 14

Total time taken for activities *A*, *C* and *E*
= 8 + 14 + 2
= 24

EST (*F*) is 35 so additional time available
= 35 – 24
= 11 hours

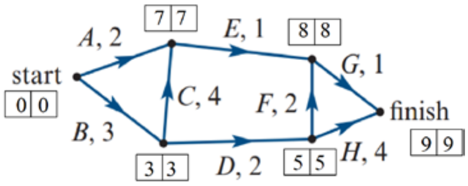
Activity *C* could be increased by 11 hours without affecting the overall project completion time. **D**

9 The critical path analysis is shown below:



The completion time of the project is 17 weeks. **D**

10 The critical path analysis is shown below:



The EST for activity *G* is 8. **E**

Chapter 16 – Revision: Networks and decision mathematics

Solutions to Exercise 16A: Multiple-choice questions

1 The vertices, edges and connections in a subgraph must exist in the original graph.

A has an edge from star to hexagon that does not exist in the original.
B has an edge from the triangle to another shape that does not exist in the original.
C has edges, vertices and connections that all exist in the original.
D has an edge from circle to star that does not exist in the original.
E has an edge from the star to hexagon that does not exist in the original.

Answer: **C**

2 It is possible to travel from *F* and back again without passing through another vertex, so there is a loop at *F*. Answer is either **A**, **B** or **C**.

There are two ways to travel from *G* directly to *F*, so the answer is either **A** or **C**.

There are two ways to travel from *G* directly to *H* so the answer is **A**.

Answer: **A**

3 Apply the Hungarian algorithm.

Name	*A*	*B*	*C*	*D*	*E*	
Francis	12	15	99	10	14	–10
David	10	9	10	7	12	–7
Herman	99	10	11	6	12	–6
Indira	8	8	12	9	99	–8
Natalie	8	99	9	8	11	–8

Name	*A*	*B*	*C*	*D*	*E*
Francis	2	5	89	0	4
David	3	2	3	0	5
Herman	93	4	5	0	6
Indira	0	0	4	1	91
Natalie	0	91	1	0	3
			–1		–3

Name	*A*	*B*	*C*	*D*	*E*
Francis	2	5	88	0	1
David	3	2	2	0	2
Herman	93	4	4	0	3
Indira	0	0	3	1	88
Natalie	0	91	0	0	0

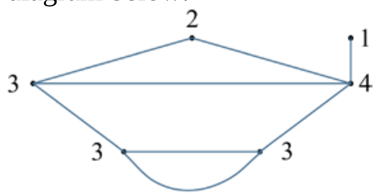
Name	*A*	*B*	*C*	*D*	*E*
Francis	1	4	87	0	0
David	2	1	1	0	1
Herman	92	3	3	0	2
Indira	0	0	3	2	88
Natalie	0	91	0	1	0

Name	*A*	*B*	*C*	*D*	*E*
Francis	0	3	86	0	0
David	1	0	0	0	1
Herman	91	2	2	0	2
Indira	0	0	3	3	89
Natalie	0	91	0	2	1

Francis is the only person who can be allocated task *E*.

Answer: **E**

4 The degree of each vertex is shown in the diagram below:



Sum of degrees
$= 3 + 2 + 4 + 1 + 3 + 3$
$= 16$

Answer: **E**

5 An eulerian trail will be possible if there are exactly two vertices of odd-degree with all others even-degree.
S, *Z*, *U* and *W* are all odd-degree vertices, so joining two of these will cause those joined to be even-degree, leaving only two odd-degree vertices.

Answer: **B**

6 Let n be the number of vertices.

$v = n$, $f = n$ since the number of vertices and faces is equal.
$e = 20$

$v - e + f \quad = 2$
$n - 20 + n \quad = 2$
$2n = 2 + 20$
$2n = 22$
$n \; = 11$

The number of vertices and faces is 11.

Answer: **C**

7 Connecting water pipes with the shortest length of pipe involves connecting each point for water in a minimal spanning tree.

Answer: **B**

8 Analyse each option separately:

A: This statement is true.
B: Activities or tasks in a project can happen simultaneously with tasks on the critical path. The critical path activities do not have to be completed before any other activities can start.
C: Tasks not on the critical path have slack time, which means decreasing their completion time will have no effect on the length of the project.
D: The critical path can be a single activity.
E: Some projects can have multiple critical paths.

Answer: **A**

9 The cut passes across edges with capacity a, b, c, d, e. Edge d flows from the sink side of the cut to the source side of the cut, so is not counted in the calculation of the cut capacity.

Cut capacity = $a + b + c + e$

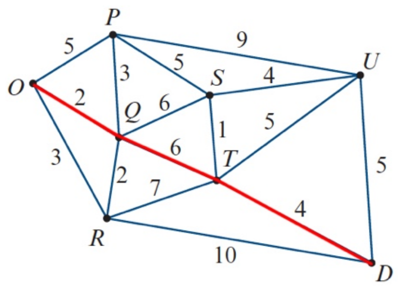
Answer: **C**

10 The shortest path from O to D can be determined using Dijkstra's algorithm, or by inspection.

Using Dijkstra's algorithm:

	P	***Q***	***R***	***S***	***T***	***U***	***D***
O	5	$\boxed{2}$	3	×	×	×	×
Q	5	$\boxed{2}$	$\boxed{3}$	8	8	×	×
R	$\boxed{5}$	$\boxed{2}$	$\boxed{3}$	8	8	×	13
P	$\boxed{5}$	$\boxed{2}$	$\boxed{3}$	$\boxed{8}$	8	14	13
S	$\boxed{5}$	$\boxed{2}$	$\boxed{3}$	$\boxed{8}$	$\boxed{8}$	12	13
T	$\boxed{5}$	$\boxed{2}$	$\boxed{3}$	$\boxed{8}$	$\boxed{8}$	12	$\boxed{12}$

The shortest path is shown in red in the diagram below:



The length of the shortest path
= 2 + 6 + 4
= 12

Answer: **B**

11 Ann has visited two resorts. Matt has visited one. Tom has visited two. Maria has visited three.

A: Ann and Maria have visited 5 in total. Matt and Tom have visited 3 in total. Ann and Maria have visited more than Matt and Tom, not fewer.
B: Matt and Tom have visited 3, not 4.
C: Maria has visited more, not fewer, than anyone else.
D: Neither Ann nor Maria have visited Mt Hutt.
E: Ann and Tom have visited 3 of the 5 resorts, Matt and Maria have visited 4 of them. This option is true.

Answer: **E**

12 In this pipeline, each town needs to be connected to any other town. This can be represented by a minimum spanning tree.

Answer: **C**

13 Consider each option separately.

A: French has two translators, Greek has one. True statement.
B: Sally can translate Spanish, Turkish and French. Kate can translate Italian and Greek. Five languages in total. True statement.
C: John can translate Spanish, Italian and Turkish. Greg can translate French and Turkish. Four different languages. True statement.
D: Kate and John can translate 4 different languages. Sally and Greg can translate 3 different languages. True statement.
E: Sally and John can translate 4 different languages. Kate and Greg can translate 5 different languages. False statement.

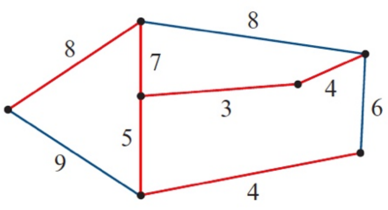
Answer: **E**

14 *O* can donate to any type so vertex *O* must be connected to *O*, *A*, *B* and *AB*. The answer must be either option **D** or **E**.

Each type can donate to its own type, so *O* must be connected to *O*, *A* to *A*, *B* to *B* and *AB* to *AB*. Option **E** has *AB* to *B* – however there is no such information in the question to support this. Thus **D** relays all the information given.

Answer: **D**

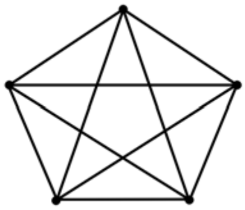
15 The minimum spanning tree is shown in red in the diagram below.



The weight of the minimum spanning tree
$= 8 + 7 + 5 + 3 + 4 + 4$
$= 31$

Answer: **D**

16 A complete graph with five vertices is shown below.



The number of edges in this graph is 10.

Answer: **C**

17 An eulerian circuit exists if all vertices have an even degree. In the graph, vertices *A* and *D* have an odd degree, so joining them with an edge will make them both have an even degree.

Answer: **C**

18 Consider all options separately.

A: *A* and *B* can occur simultaneously. False statement.
B: A does not lead into *F* so *A* does not need to be completed before *F* can start. False statement.
C: *E* and *F* can be completed independently. False statement.
D: *E* and *F* must both be completed before *H* can begin, but they do not necessarily have to finish at the same time. False statement.
E: *E* follows after *A* in the network so this is a true statement.

Answer: **E**

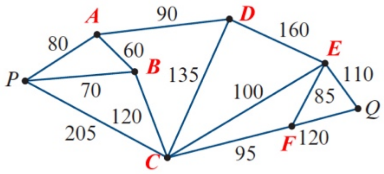
19 $e = 12, f = 4$
$v - e + f = 2$
$v - 12 + 4 = 2$
$v = 10$

Answer: **B**

20 There is one isolated vertex in the graph, so there will be one row and one column consisting of all zeros.
There is one loop, so there will be one “1” along the diagonal.

Answer: **B**

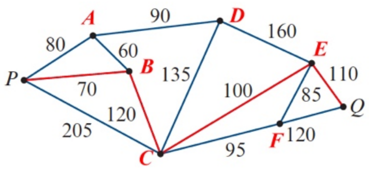
21 The shortest path, or path of minimum cost in this example, from P to Q can be determined using Dijsktra's algorithm, or by inspection. To use Dijkstra's algorithm, the vertices must be labelled with names, as shown below.



Using Dijkstra's algorithm:

	A	B	C	D	E	F	Q
P	80	70	205	×	×	×	×
B	80	70	190	×	×	×	×
Q	80	70	190	170	×	×	×
D	80	70	190	170	330	×	×
C	80	70	190	170	290	285	×
F	80	70	190	170	290	285	405
E	80	70	190	170	290	285	400

The shortest path is shown in red in the diagram below.



Minimum cost = 70 + 120 + 100 + 110
= 400

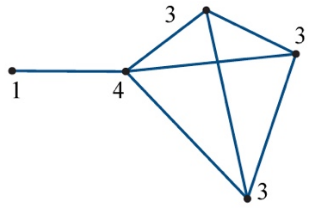
Answer: **A**

22 The capacity of the cut
= 8 + 2 + 3
= 13

Note: one of the edges with weight 3 passes from the sink side of the cut to the source side of the cut, so is not counted in the cut capacity calculation.

Answer: **D**

23 The degree of each of the vertices are shown on the graph below.



The sum of the degrees
= 1 + 4 + 3 + 3 + 3
= 14

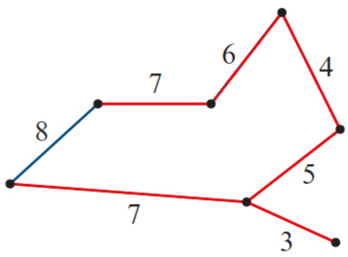
Answer: **C**

24 $v = 9$, $e = 20$

$v - e + f = 2$
$9 - 20 + f = 2$
$f = 2 - 9 + 20$
$f = 13$

Answer: **B**

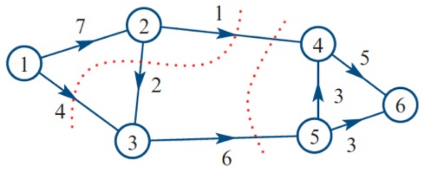
25 The minimum spanning tree is marked in red on the diagram below.



Weight of minimum spanning tree
= 7 + 6 + 4 + 5 + 3 + 7
= 32

Answer: **C**

26 There are two minimum cuts (same capacity) in this diagram. Both are shown below.



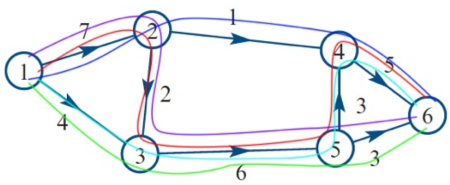
Maximum flow
= 4 + 2 + 1
= 7

or

Maximum flow
= 1 + 6
= 7

Answer: **C**

27 The diagram below shows all the possible walks from vertex 1 to 6.



There are five different walks.

Answer: **E**

28 There are three even-degreed vertices and two odd-degreed vertices.

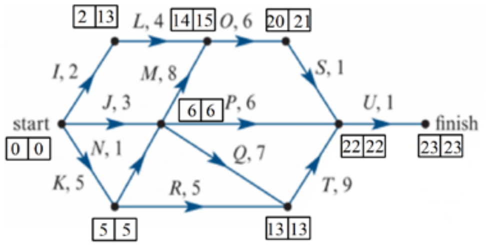
If we add an extra edge, it can either join:

- an even vertex to an even vertex, making them both odd. Result: 1 even, 4 odd. (Option C)
- an even vertex to an odd vertex, swapping the nature of them both. Result: three even, two odd. (Option D)
- an odd vertex to an odd vertex, making them both even. Result: 5 even, none odd. (Option A)
- adding a loop to either an odd or even vertex keeps it either odd or even because a loop adds 2 to the degree.

It will be impossible to make one of the odd vertices become even by adding an edge.

Answer: **E**

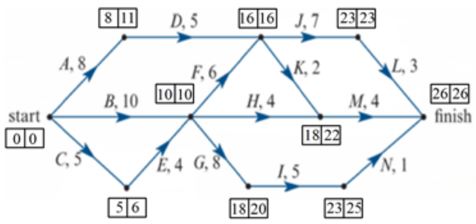
29 The critical path analysis is shown on the diagram below.



Critical path is $K - N - Q - T - U$

Answer: **D**

30 The critical path analysis is shown on the diagram below.



The project completion time is 26 days.

Answer: **D**

Solutions to Exercise 16B: Extended-response questions

1 The least time to complete the project is 30 hrs.

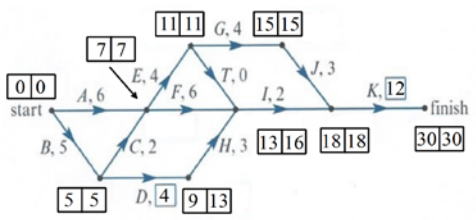
Activity *K* has an EST of 18 hrs, so

duration *K* = 30 – 18
= 12

Activity *D* has an EST of 5.
Activity *H* has an EST of 9.

duration *D* = EST *H* – EST *D*
= 9 – 5
= 4

The critical path analysis can now be completed.



LST *A* = LST *E* (or *F*) – duration *A*
= 7 – 6
= 1

LST *F* = LST *I* – duration *F*
= 16 – 6
= 10

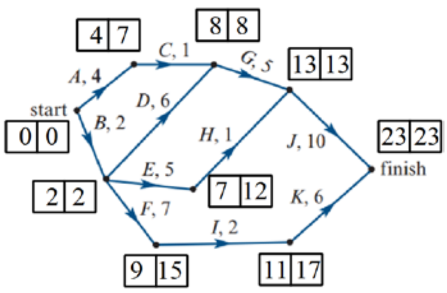
a The completed table is

Activity	Completion time (hours)	Earliest starting time (hours)	Latest starting time (hours)
A	6	0	**1**
B	5	0	0
C	2	5	5
D	**4**	5	9
E	4	7	7
F	6	7	**10**
G	4	11	11
H	3	9	13
I	2	13	16
J	3	15	15
K	**12**	18	18

b The critical path is *B* – *C* – *E* – *G* – *J* – *K*

2

a The activity network for this project is shown below, including the critical path analysis.



b From the activity network:
EST *A* = EST *B* = 0
EST *C* = 4
EST *D* = EST *E* = EST *F* = 2
EST *G* = 8
EST *H* = 7
EST *I* = 9
EST *J* = 13
EST *K* = 11

c The project is estimated to take 23 days.

d From the activity network:
LST *K* = 17
LST *J* = 13
LST *I* = 15
LST *H* = 12
LST *G* = 8
LST *C* = 7
LST *B* = 0

and by calculation:
LST *F* = LST *I* – duration *F*
= 15 – 7
= 8

LST *E* = LST *H* – duration *E*
= 12 – 5
= 7

LST *D* = LST *G* – duration *D*
= 8 – 6
= 2

LST *A* = LST *C* – duration *A*
= 7 – 4
= 3

e The completed critical path is
B – *D* – *G* – *J*

f Activity *K* is not on the critical path.
Slack time for *K* = LST *K* – EST *K*
= 17 – 11
= 6

Activity *K* can be delayed by 6 days without disrupting the completion time of the project.

3

a Activity *E* follows directly from activity *A*, so activity *A* is an immediate predecessor of activity *E*.

The longest path to activity *I* follows *A* – *E* – *G* for a time of
3 + 2 + 3 = 8 hours.

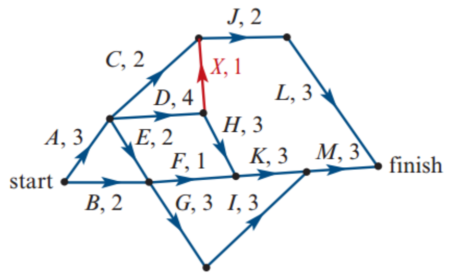
The longest path to activity *M* follows *A* – *D* – *H* – *K* for a time of
3 + 4 + 3 + 3 = 13 hours

Activity	*Immediate predecessor(s)*	*EST*
A	–	0
B	–	0
C	*A*	3
D	*A*	3
E	***A***	3
F	*B*, *E*	5
G	*B*, *E*	5
H	*D*	7
I	*G*	***8***
J	*C*, *X*	8
K	*F*, *H*	10
L	*J*	10
M	*I*, *K*	***13***
X	*D*	7

b From the table, activity *D* is the only immediate predecessor of activity *X*.
From the table, activity *X* is an immediate predecessor of activity *J* only.

Activity *X* follows the end activity *D* to the start of activity *J*.

EST *X* = EST *J* – duration of *X*
7 = 8 – duration of *X*
duration of *X* = 1 hour



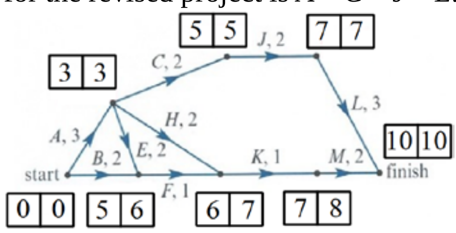
c From the activity network, the durations of the activities on the critical path (*A*, *D*, *H*, *K*, *M*) can be added together.

i Duration = 3 + 4 + 3 + 3 + 3
= 16 hours

ii The critical path is the activities that, if delayed by any amount, would result in the overall completion of the entire project being delayed.

d **i** For the new network, the longest time path from the start to the beginning of activity *K* is *A* – *E* – *F* for a duration of 3 + 2 + 1 = 6 hours.
EST *K* = 6 hours

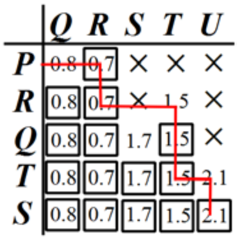
ii By inspection, or using the critical path analysis below the critical path for the revised project is *A* – *C* – *J* – *L*.



iii LST *M* = Earliest completion time of project – duration *M*
= 10 – 2
= 8

4

a **i** By inspection, or using Dijkstra's algorithm below the shortest route from *P* to *U* has length 2.1 km.

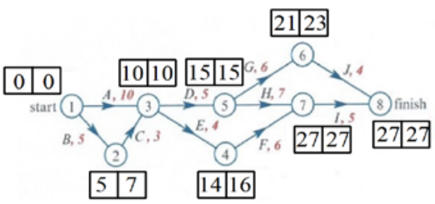


ii The path required is a hamiltonian path. There are multiple answers:
P – *Q* – *R* – *T* – *S* – *U*,
P – *R* – *Q* – *S* – *T* – *U*,
P – *R* – *Q* – *T* – *S* – *U*,
P – *R* – *T* – *Q* – *S* – *U*.

b **i** There are two eulerian trails through this network:
R – *Q* – *P* – *R* – ***T* – *Q* – *S*** – *T* – *U* – *S*
or
R – *Q* – *P* – *R* – ***T* – *S* – *Q*** – *T* – *U* – *S*

ii By travelling an eulerian path, the technician would travel along each of the roads only once. This would minimise the total distance travelled and would avoid having to travel down a street that has already been checked.

5 The activity network containing activity durations and the completed critical path analysis is shown below.



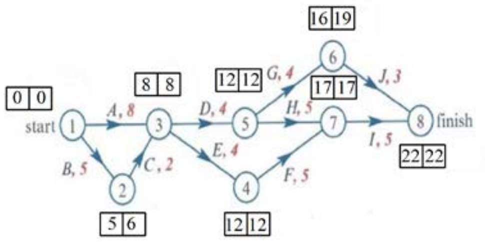
a EST *F* = 14
LST *G* = LST *J* – duration *G*
= 23 – 6
= 17

b **i** The critical path can be identified from the diagram above.
A – *D* – *H* – *I*

ii Length of critical path = 27 days

This is the same as the overall minimum time to complete the project.

c Using the crashed time rather than the normal completion time (from the table), a new activity network can be drawn. This is shown below with the completed critical path analysis.



i LST *B* = LST *C* – duration *B*
= 6 – 5
= 1

EST *J* = 16
LST *J* = 19

The completed table is below.

Activity	*Earliest start time (days)*	*Latest start time (days)*
A	0	0
B	0	**1**
C	5	6
D	8	8
E	8	8
F	12	12
G	12	15
H	12	12
I	17	17
J	**16**	**19**

ii From the new critical path analysis, the shortest time that the project can now be completed is 22 days.

iii Activities *D*, *H* and *F* were reduced in time. Note that by doing so, activities *E* and *F* are now critical.

iv Activity A was reduced by 2 days for a cost of $2 \times \$400 = \$800$.
Activity D was reduced by 1 day for a cost of \$600.
Activity H was reduced by 2 days for a cost of $2 \times \$300 = \$600$.
Activity F was reduced by 1 day for a cost of \$500.

Total cost
$= \$800 + \$600 + \$600 + \$500$
$= \$2500$

6

a A planar graph can be drawn so that none of the edges in that graph intersect, except at the vertices. The edges in a planar graph can be drawn so that they do not cross over.

b $v = 7$ (one for each town)
$e = 11$ (one for each road)
$f = 6$ (one for each subregion)

$v - e + f = 7 - 11 + 6$
$= 2$

Euler's rule has been verified.

c The inspector starts in town B, which has degree three (odd) in the graph.
The inspector must travel every road to inspect it. To do this in the least total distance, she would need to travel each road only once, that is, follow and eulerian trail.
This is possible because there are exactly two odd-degree vertices in the graph, B (where she starts) and C (where she must end).
The inspector will end up in town C.

d The distance travelled will be the sum of all the edge weights
distance $= 18 + 19 + 32 + 29 + 20 + 25 + 28 + 21 + 56 + 16 + 33$
$= 297$ km

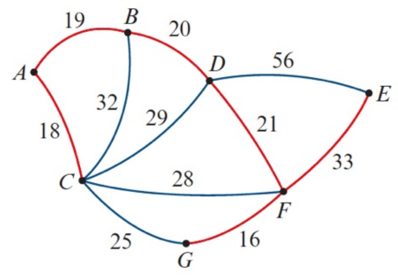
e The route travelled is not unique, because there are two different eulerian trails through this graph.
One is: $B - A - C - B - D - E - F - D - C - F - G - C$
Another is: $B - C - A - B - D - E - F - D - C - F - G - C$

f The shortest path from E to A can be found using inspection, or Dijkstra's algorithm as shown below.

	A	B	C	D	F	G
E	×	×	×	56	33	×
F	×	×	61	54	33	49
G	×	×	61	54	33	49
D	×	74	61	54	33	49
C	79	74	61	54	33	49
B	79	74	61	54	33	49

The shortest path from E to A has length 79 km (along the route $E - F - C - A$).

g The minimal length of cable required will form a minimal spanning tree. This is shown in red in the diagram below.



The minimal spanning tree has weight
$= 18 + 19 + 20 + 21 + 33 + 16$
$= 127$

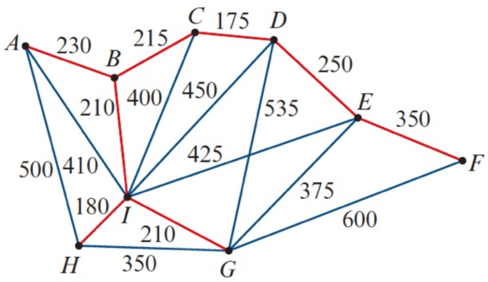
The minimal length of cable required is 127 km.

h A hamiltonian cycle would be
$C - A - B - D - E - F - G - C$
(or the reverse).

The total distance travelled on this cycle
$= 18 + 19 + 20 + 56 + 33 + 16 + 25$
$= 187$ km

7

a The computer network required will be a minimum spanning tree for the graph. The minimum spanning tree is shown in red in the graph below.

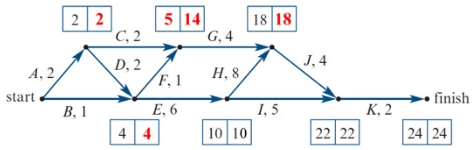


b The minimum length of cable required

= 180 + 210 + 210 + 230 + 215 + 175 + 250 + 350

= 1820 m

8 The completed critical path analysis for the network is shown in red in the diagram below.



a The EST for *G* is 5 hours.

b The shortest time required to complete the project is 24 hours.

c Float *I* = LST *I* – EST *I*

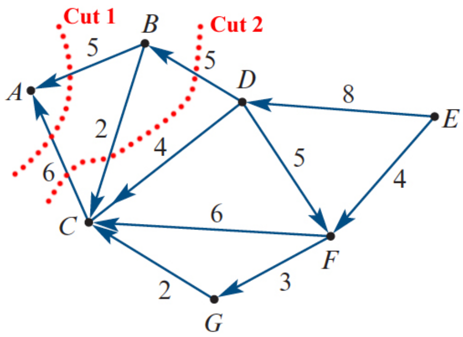
= LST *K* – duration *I* – EST *I*

= 22 – 5 – 10

= 7 hours

9 The maximum flow from *E* to *A* is the minimum cut capacity.
Note: Take care with this question because the source is on the right of the diagram and the sink is on the left.

There are two minimum cuts for this network and both are shown in the diagram below.



Cut 1 capacity = 6 + 5 = 11
Cut 2 capacity = 5 + 6 = 11
Note: the edge from B to C is not counted in the calculation of the cut capacity for Cut 2 because the flow is from the sink side of the cut (left) to the source side of the cut (right).

The maximum flow from *E* to *A*

= 11 megalitres per day.

10

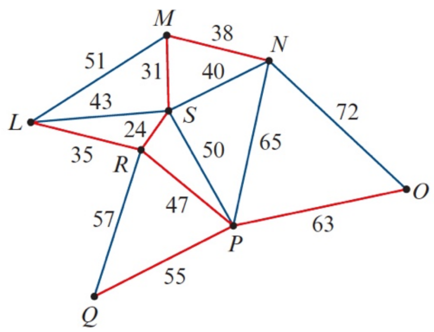
a The shortest path from *S* to *O* is best found using inspection in this question, or Dijkstra's algorithm as shown below.

	L	M	N	O	P	Q	R
S	43	31	40	×	50	×	24
R	43	31	40	×	50	81	24
M	43	31	40	×	50	81	24
N	43	31	40	112	50	81	24
L	43	31	40	112	50	81	24
P	43	31	40	112	50	81	24
Q	43	31	40	112	50	81	24

The shortest path from *S* to *O* has length 112 km (along the route *S* – *N* – *O*).

b i The network within a graph that links all the vertices to give the shortest overall length is called the minimum spanning tree.

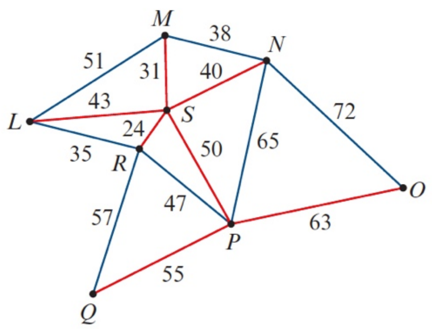
ii The minimum spanning tree for this network is shown in red in the diagram below.



iii The minimum length of pipe required
= total weight of minimum spanning tree
= 35 + 24 + 47 + 55 + 63 + 31 + 38
= 293 kilometres

c In the new tree, *S* will need to be connected directly to *R*, *L*, *M*, *N* and *P*. *O* and *Q* will be connected with the shortest possible pipe length.

The new tree is shown in red in the diagram below.



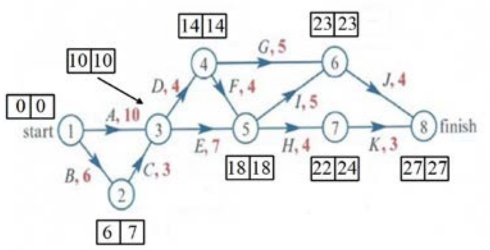
The minimum length of pipe required
= 43 + 24 + 31 + 40 + 50 + 55 + 63
= 306 km

11

a From the activity network activities *A* and *C* lead directly to the start of activity *E* and so are immediate predecessors. Although it is not an *immediate* predecessor, activity *B* must be finished before *E* can start (because *C* cannot begin until it is).

Activities *A*, *B* and *C* must be finished before activity *E* can start.

b The activity network with durations and completed critical path analysis is shown in the diagram below.



LST *B* = LST *C* – duration *B*
= 7 – 6
= 1

EST *E* = 10 (from diagram)

LST *I* = LST *J* – duration *I*
= 23 – 5
= 18

The completed table is below.

Task	*EST*	*LST*
A	0	0
B	0	**1**
C	6	7
D	10	10
E	**10**	11
F	14	14
G	14	18
H	18	20
I	18	**18**
J	23	23
K	22	24

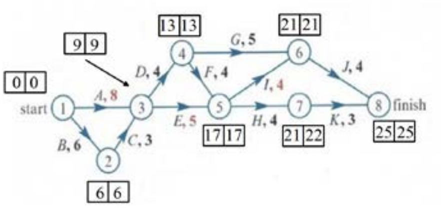
c **i** Using the critical path analysis above, or the tasks for which EST and LST are the same in the table above, the critical path can be determined as

A – *D* – *F* – *I* – *J*

ii The length of the critical path is the sum of the individual durations
= 10 + 4 + 4 + 5 + 4
= 27 months

or it can just be read from the critical path analysis (last box).

d The new durations and critical path analysis after reduction in duration of given activities is shown below.



i The new critical path, from the diagram above, is

$B - C - D - F - I - J$

ii The time taken to complete the project with the new durations is 25 months (from the diagram above).

12

a The completion of the Hungarian algorithm is shown below.

	Tractor based at				
Assigned to	*Location 1*	*Location 2*	*Location 3*	*Location 4*	
Site 1	1130	830	2010	1140	–830
Site 2	1020	1100	690	850	–690
Site 3	2010	1320	1150	1410	–1150
Site 4	960	1210	2100	1530	–960

	Tractor based at			
Assigned to	*Location 1*	*Location 2*	*Location 3*	*Location 4*
Site 1	300	0	1180	310
Site 2	330	410	0	160
Site 3	860	170	0	260
Site 4	0	250	1140	570
				–160

	Tractor based at			
Assigned to	*Location 1*	*Location 2*	*Location 3*	*Location 4*
Site 1	300	0	1180	150
Site 2	330	410	0	0
Site 3	860	170	0	100
Site 4	0	250	1140	410

Tractor at location 1 is assigned to site 4.
Tractor at location 2 is assigned to site 1.
Tractor at location 4 is assigned to site 2, so tractor at location 3 cannot.
Tractor at location 3 is assigned to site 3.

The completed table is shown below.

Tractor at	*Assign to*
Location 1	Site 4
Location 2	**Site 1**
Location 3	**Site 3**
Location 4	**Site 2**

b Use the cost information in the original table to determine the minimum cost of allocating tractors in locations to sites.

The costs corresponding to the allocations are shown in circled in red in the table below.

	Tractor based at			
Assigned to	*Location 1*	*Location 2*	*Location 3*	*Location 4*
Site 1	1130	830	2010	1140
Site 2	1020	1100	690	850
Site 3	2010	1320	1150	1410
Site 4	960	1210	2100	1530

Total cost = \$960 + \$830 + \$1150 + \$850
= \$3790

13

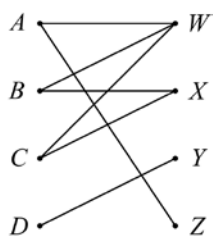
a The completion of the Hungarian algorithm is shown below.

Camp site	*W*	*X*	*Y*	*Z*	
A	30	70	60	20	–20
B	40	30	50	80	–30
C	50	40	60	50	–40
D	60	70	30	70	–30

Camp site	*W*	*X*	*Y*	*Z*
A	10	50	40	0
B	10	0	20	50
C	10	0	20	10
D	30	40	0	40
	–10			

Camp site	*W*	*X*	*Y*	*Z*
A	0	50	40	0
B	0	0	20	50
C	0	0	20	10
D	20	40	0	40

Draw a bipartite graph to see possible allocations.



D must be allocated to *Y*.
A must be allocated to *Z* so cannot be allocated to *W*.
B and *C* can be allocated to the same two residents so there are two different allocations possible.

Allocate:
$D - Y$, $A - Z$, $B - W$, $C - X$
or
$D - Y$, $A - Z$, $B - X$, $C - W$

b The cost will be the same regardless of which of the two allocations are used to calculate it.

Using the first allocation

Camp site	*W*	*X*	*Y*	*Z*
A	30	70	60	(20)
B	(40)	30	50	80
C	50	(40)	60	50
D	60	70	(30)	70

Cost = \$20 + \$40 + \$40 + \$30
= \$130