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CAMBRIDGE
UNIVERSITY PRESS

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.edu.au
Information on this title: www.cambridge.org/9781107567573

First printed 2016

Reprinted 2016

Cover designed by Loupe Design
Typeset by Diacritech
Printed in China by C & C Offset Printing Co. Ltd.

A Cataloguing-in-Publication entry is available from the catalogue of the National Library of Australia at www.nla.gov.au

ISBN 978-1-107-56757-3 Paperback

Additional resources for this publication at www.cambridge.edu.au/seniormaths

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ISBN 978-1-107-56757-3

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Cambridge University Press

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ISBN 978-1-107-56757-3

ISBN 978-1-107-56757-3

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ISBN 978-1-107-56757-3

ISBN 978-1-107-56757-3

Acknowledgements

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ISBN 978-1-107-56757-3

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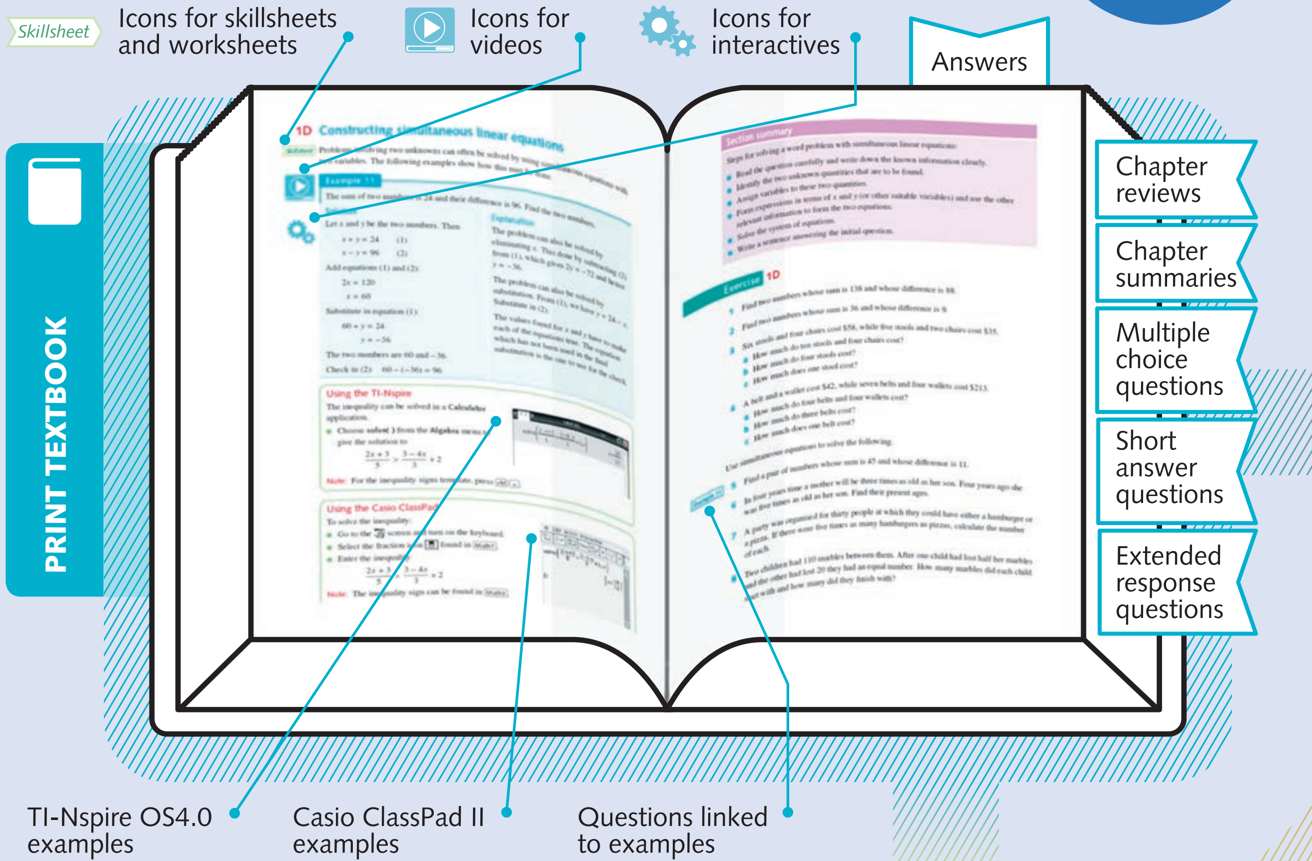
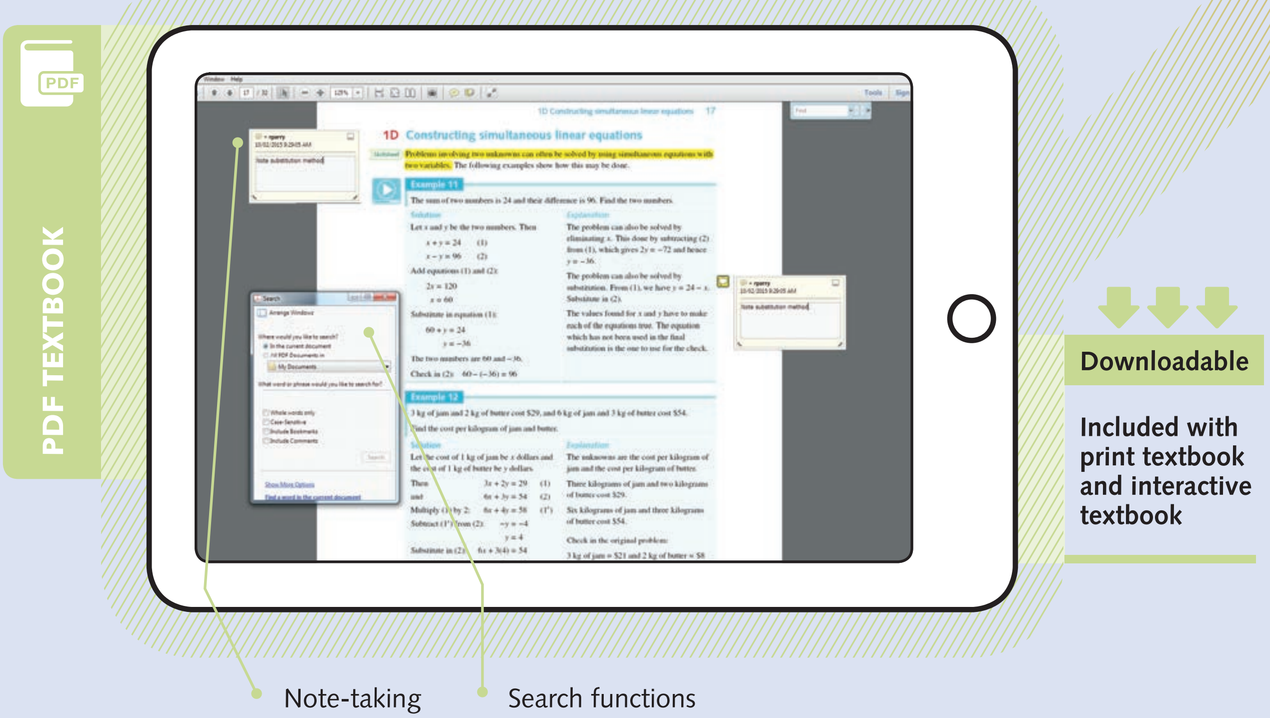
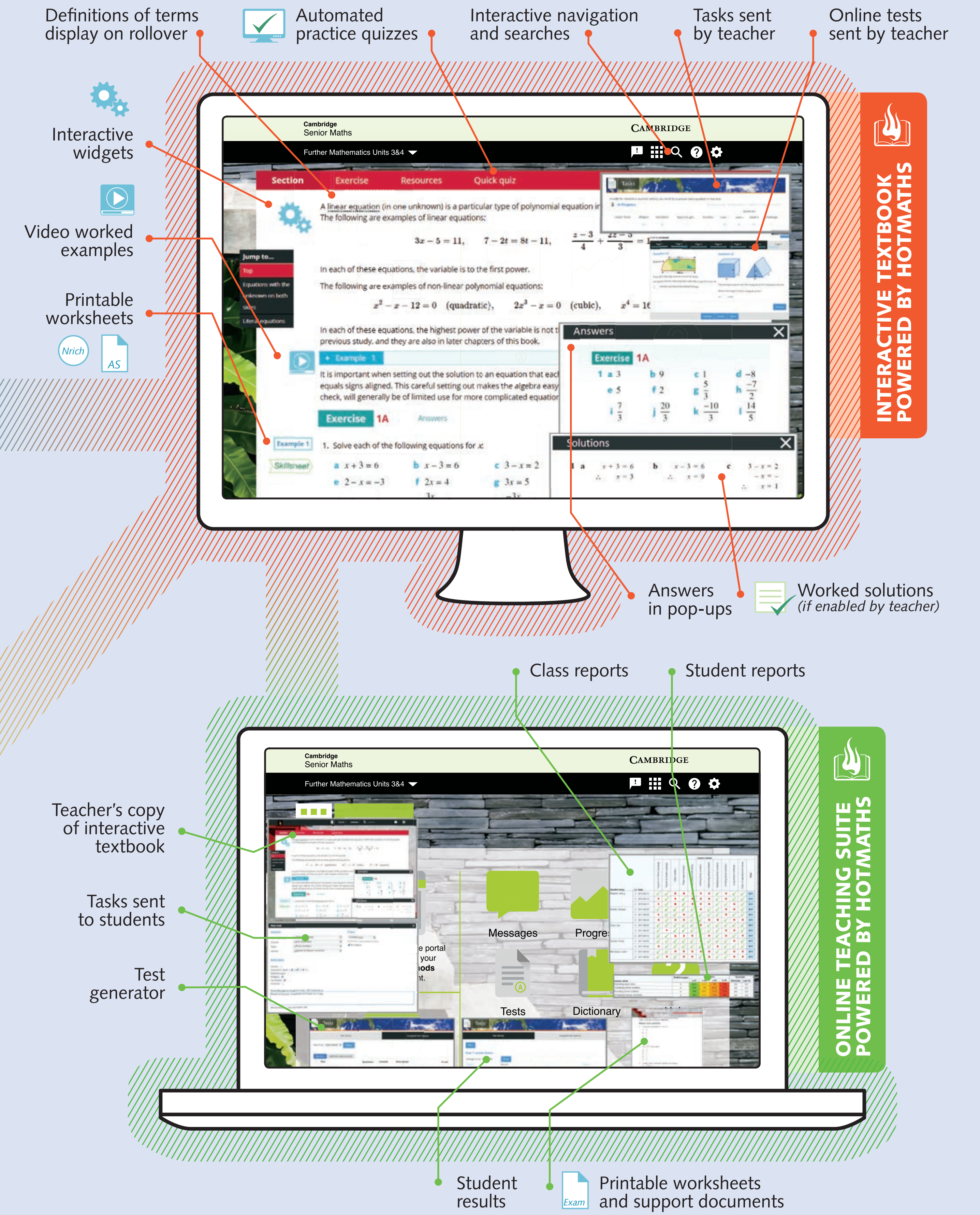




ISBN 978-1-107-56757-3



ISBN 978-1-107-56757-3

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Chapter 1

Core: Data analysis

1

Displaying and describing data distributions

ISBN 978-1-107-56757-3

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1A Classifying data

▶ Data and variables

Skillsheet

Statistics is a science concerned with understanding the world through data.

Some data

The data in the table below were collected from a group of university students.

Height (cm)	*Weight (kg)*	*Age (years)*	*Sex (M male, F female)*	*Fitness level (1 high, 2 medium, 3 low)*	*Pulse rate (beats/min)*
173	57	18	M	2	86
179	58	19	M	2	82
167	62	18	M	1	96
195	84	18	F	1	71
173	64	18	M	3	90
184	74	22	F	3	78
175	60	19	F	3	88
140	50	34	M	3	70

WWW

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Variables

In a dataset, we call the qualities or quantities about which we record information ***variables***.

An important first step in analysing any set of data is to identify the variables involved, their units of measurement (where appropriate) and the values they take.

In this dataset above, there are six variables:

- *height* (in centimetres)
- *weight* (in kilograms)
- *age* (in years)
- *sex* (M = male, F = female)
- *fitness level* (1 = high, 2 = medium, 3 = low)
- *pulse rate* (beats/minute).

Types of variables

Variables come in two general types, *categorical* and *numerical*:

- *Categorical variables*

Categorical variables represent *characteristics* or *qualities* of people or things – for example, a person's eye colour, sex, or fitness level.

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Data generated by a categorical variable can be used to organise individuals into one of several groups or categories that characterise this quality or attribute.
For example, an 'F' in the *Sex* column indicates that the student is a female, while a '3' in the *Fitness level* column indicates that their fitness level is low.

Categorical variables come in two types: *nominal* and *ordinal*.

- *Nominal variables*

  Nominal variables have data values that can be used to both group individuals according to a particular characteristic.

  The variable *sex* is an example of a *nominal* variable.

  The data values for the variable *sex*, for example M or F, can be used to group students according to their sex. It is called a nominal variable because the data values name the group to which the students belong, in this case, the group called 'males' or the group called 'females'.

- *Ordinal variables*

  Ordinal variables have data values that can be used to both group and order individuals according to a particular characteristic.

  The variable *fitness level* is an example of an *ordinal* variable. The data generated by this variable contains two pieces of information. First, each data value can be used to group the students by fitness level. Second, it allows us to logically order these groups according to their fitness level – in this case, as 'low', 'medium' or 'high'.

- *Numerical variables*

  Numerical variables are used to represent *quantities*, things that we can count or measure.

  For example, a '179' in the *Height* column indicates that the person is 179 cm tall, while an '82' in the *Pulse rate* column indicates that they have a pulse rate of 82 beats/minute.
  Numerical variables come in two types: *discrete* and *continuous*.

  - *Discrete variables*

    Discrete variables represent quantities that are *counted*.

    The number of mobile phones in a house is an example. Counting leads to discrete data values such as 0, 1, 2, 3, . . . There can be nothing in between.

ISBN 978-1-107-56757-3

Cambridge University Press

As a guide, *discrete variables* arise when we ask the question 'How many?'

- *Continuous variables*

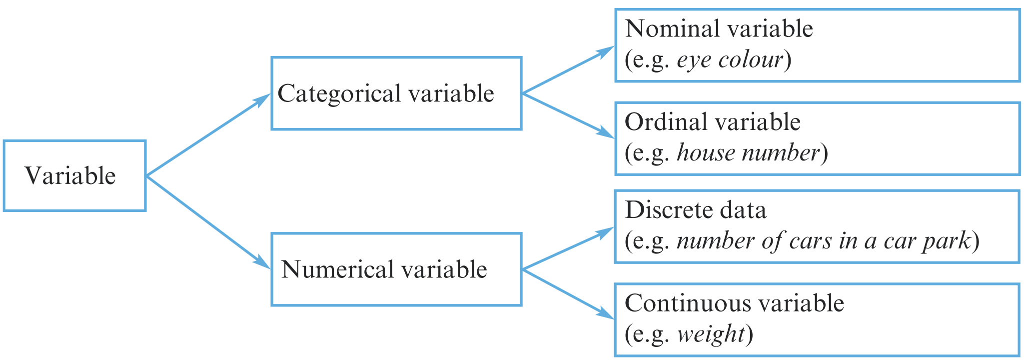
Continuous variables represent quantities that are *measured* rather than counted.

Thus, even though we might record a person's height as 179 cm, their height could be any value between 178.5 and 179.4 cm. We have just rounded to 179 cm for convenience, or to match the accuracy of the measuring device.

As a guide, continuous variables arise when we ask the question 'How much?'

Comparing numerical and categorical variables

The interrelationship between categorical (nominal and ordinal) and numerical variables (discrete and continuous) is displayed in the diagram below.



Numerical or categorical?

Deciding whether data are numerical of categorical is not an entirely trivial exercise. Two things that can help your decision-making are:

1. Numerical data can always be used to perform arithmetic computations. This is not the case with categorical data. For example, it makes sense to calculate the average weight of a group of individuals, but not the average house number in a street. This is a good test to apply when in doubt.
2. It is not the variable name alone that determines whether data are numerical or categorical; it is also the way the data are recorded. For example, if the data for variable *weight* are recorded in kilograms, they are numerical. However, if the data are recorded as 'underweight', 'normal weight', 'overweight', they are categorical.

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Exercise 1A

Basic ideas

1 **a** What is a categorical variable? Give an example.

b What is a numerical variable? Give an example.

2 There are two types of categorical variables. Name them and give an example of each.

3 There are two types of numerical variables. Name them and give an example of each.

Types of variables: categorical or numerical

4 Classify each of the following variables (in *italics*) as categorical or numerical when recording information about:

a *time* (in minutes) spent exercising each day

b *number* of frogs in a pond

c *bank account numbers*

d *height* (short, average, tall)

e *time* spent playing computer games (hours)

f *number of people* in a bus

g *eye colour* (brown, blue, green)

h *post code*.

Categorical variables: nominal or ordinal

5 Classify the categorical variables identified below (in italics) as nominal or ordinal.

a The *colour* of a pencil

b The different *types of animals* in a zoo

c The *floor levels* in a building (0, 1, 2, 3 . . .)

d The *speed* of a car (on or below the speed limit, above the speed limit)

e *Shoe size* (6, 8, 10, . . .)

f Family names

Numerical variables: discrete or continuous

6 Classify the numerical variables identified below (in italics) as discrete or continuous.

a The *number of pages* in a book

b The *cost* (in dollars) to fill the tank of a car with petrol

c The *volume* of petrol (in litres) used to fill the tank of a car

d The *speed* of a car in km/h

e The *number* of people at a football match

f The air *temperature* in degrees Celsius

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1B Displaying and describing the distributions of categorical variables

The frequency table

With a large number of data values, it is difficult to identify any patterns or trends in the raw data.

For example, the set of categorical data opposite, listing the sex (M = male, F = female) of 60 individuals, is hard to make sense of in its raw form.

To help make sense of the data, we first need to organise them into a more manageable form.

F	F	M	F	F	F	M	F	M	M	M	F
M	F	F	F	M	M	M	F	M	F	M	F
M	M	M	F	M	F	M	F	M	F	F	F
M	F	M	F	M	F	M	F	M	M	M	F
M	F	F	F	F	F	F	M	M	F	M	F
F	F	M	F	M	M	M	F	M	F	M	M

The statistical tool we use for this purpose is the **frequency table**.

The frequency table

A frequency table is a listing of the values a variable takes in a dataset, along with how often (frequently) each value occurs.

Frequency can be recorded as a:

- number: the number of times a value occurs, or
- percentage: the percentage of times a value occurs (**percentage frequency**):

$$\text{per cent} = \frac{\text{count}}{\text{total count}} \times 100\%$$

Skillsheet

Example 1 Frequency table for a categorical variable

The sex of 11 preschool children is as shown (F = female, M = male):

F M M F F M F F F M M

Construct a frequency table (including percentage frequencies) to display the data.

Solution

1. Set up a table as shown. The variable *sex* has two categories: 'Male' and 'Female'.
2. Count up the number of females (6) and males (5). Record this in the '*Number*' column.
3. Add the counts to find the total count, 11 (6 + 5). Record this in the '*Number*' column opposite '*Total*'.

Sex	Frequency	
	Number	Percentage
Female	6	54.5
Male	5	45.5
Total	11	100.0

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4 Convert the frequencies into percentage frequencies. Record these in the '*Percentage*' column. For example:

$$\text{percentage of females} = \frac{6}{11} \times 100\%$$
$$= 54.5\%$$

5 Finally, total the percentages and record.

Note: There are two things to note in constructing the frequency table in Example 1.

1 The variable *sex* is nominal, so in setting up this frequency table the order in which we have listed the categories 'Female' and 'Male' is quite arbitrary. However, if the variable was ordinal, say *year level,* with possible values 'Year 10', 'Year 11' and 'Year 12', it would make sense to group the data values in that order.

2 The *Total* should always equal the total number of observations – in this case, 11. The percentages should add to 100%. However, if percentages are rounded to one decimal place a total of 99.9 or 100.1 is sometimes obtained. This is due to rounding error. Totalling the count and percentages helps check on your tallying and percentaging.

How has forming a frequency table helped?

The process of forming a frequency table for a categorical variable:

- displays the data in a *compact* form
- tells us something about the way the data values are *distributed* (the pattern of the data).

The bar chart

Once categorical data have been organised into a frequency table, it is common practice to display the information graphically to help identify any features that stand out in the data.

The statistical graph we use for this purpose is the **bar chart**.

The bar chart represents the key information in a frequency table as a picture. The bar chart is specifically designed to display categorical data.

In a bar chart:

- frequency (or percentage frequency) is shown on the vertical axis
- the variable being displayed is plotted on the horizontal axis
- the height of the bar (column) gives the frequency (count or percentage)
- the bars are drawn with gaps to show that each value is a separate category
- there is one bar for each category.

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Example 2 Constructing a bar chart from a frequency table

The climate type of 23 countries is classified as 'cold', 'mild' or 'hot'. The results are summarised in the table opposite.

Construct a frequency bar chart to display this information.

Climate type	*Frequency*	
	Number	*Percentage*
Cold	3	13.0
Mild	14	60.9
Hot	6	26.1
Total	23	100.0

Solution

a The data enable us to both group the countries by *climate type* and put these groups in some sort of natural order according to the 'warmth' of the different climate types. The variable is ordinal.

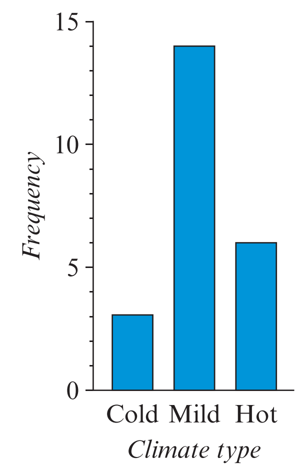
a Ordinal

b **1** Label the horizontal axis with the variable name, 'Climate type'. Mark the scale off into three equal intervals and label them 'Cold', 'Mild' and 'Hot'.

2 Label the vertical axis 'Frequency'. Scale allowing for the maximum frequency, 14. Fifteen would be appropriate. Mark the scale off in fives.

3 For each climate type, draw a bar. There are gaps between the bars to show that the categories are separate. The height of the bar is made equal to the frequency (given in the 'Number' column).

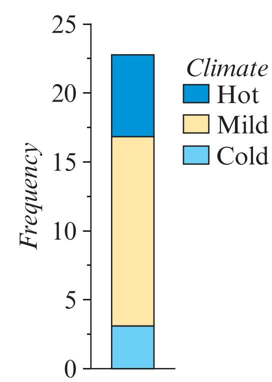
b



Stacked or segmented bar charts

A variation on the standard bar chart is the segmented or stacked bar chart. It is a compact display that is particularly useful when comparing two or more categorical variables.

In a **segmented bar chart**, the bars are stacked one on top of another to give a single bar with several parts or segments. The lengths of the segments are determined by the frequencies. The height of the bar gives the *total* frequency. A legend is required to identify which segment represents which category (see opposite). The segmented bar chart opposite was formed from the climate data used in Example 2. In a *percentage segmented bar chart*, the lengths of each segment in the bar are determined by the percentages. When this is done, the height of the bar is 100.



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Example 3 **Constructing a percentage segmented bar chart from a frequency table**

The climate type of 23 countries is classified as 'cold', 'mild' or 'hot'.

Construct a percentage frequency segmented bar chart to display this information.

	Frequency	
Climate type	*Number*	*Percentage*
Cold	3	13.0
Mild	14	60.9
Hot	6	26.1
Total	23	100.0

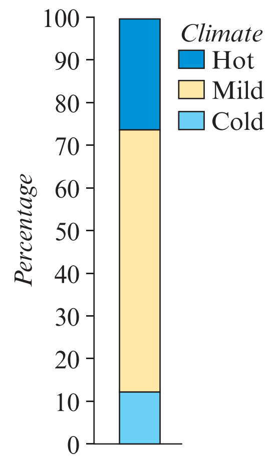
Solution

1 In a segmented bar chart, the horizontal axis has no label.

2 Label the vertical axis 'Percentage'. Scale allowing for the maximum of 100 (%), Mark the scale in tens.

3 Draw a single bar of height 100. Divide the bar into three by inserting dividing lines at 13% and 76.9% (13 + 60.9%).

4 The bottom segment represents the countries with a cold climate. The middle segment represents the countries with a mild climate. The top segment represents the countries with a mild climate. Shade (or colour) the segments differently.



5 Insert a legend to identify each shaded segments by climate type.

The mode

One of the features of a dataset that is quickly revealed with a frequency table or a bar chart is the **mode** or **modal category**.

The *mode* is the most frequently occurring value or category.

In a bar chart, the mode is given by the category with the tallest bar or longest segment. For the bar charts above, the modal category is clearly 'mild'. That is, for the countries considered, the most frequently occurring climate type is 'mild'.

Modes are particularly important in 'popularity' polls. For example, in answering questions such as 'Which is the most watched TV station between 6:00 p.m and 8:00 p.m.?' or 'When is the time a supermarket is in peak demand: morning, afternoon or night?'

Note, however, that the mode is only of real interest when a single category stands out from the others.

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Answering statistical questions involving categorical variables

A **statistical question** is a question that depends on data for its answer.

Statistical questions that are of most interest when working with a single categorical variable are of these forms:

- Is there a **dominant category** into which a significant percentage of individuals fall or are the individuals relatively evenly spread across all of the categories? For example, are the shoppers in a department store predominantly male or female, or are there roughly equal numbers of males and females?
- How many and/or what percentage of individuals fall into each category? For example, what percentage of visitors to a national park are 'day-trippers' and what percentage of visitors are staying overnight?

A short written report is the standard way to answer these questions.

The following guidelines are designed to help you to produce such a report.

Some guidelines for writing a report describing the distribution of a categorical variable

- Briefly summarise the context in which the data were collected including the number of individuals involved in the study.
- If there is a clear modal category, ensure that it is mentioned.
- Include frequencies or percentages in the report. Percentages are preferred.
- If there are a lot of categories, it is not necessary to mention every category, but the modal category should always be mentioned.

Example 4 Describing the distribution of a categorical variable in its context

In an investigation of the variation of climate type across countries, the climate types of 23 countries were classified as 'cold', 'mild' or 'hot'. The data are displayed in a frequency table to show the percentages.

Use the information in the frequency table to write a concise report on the distribution of climate types across these 23 countries.

	Frequency	
Climate type	*Number*	*%*
Cold	3	13.0
Mild	14	60.9
Hot	6	26.1
Total	23	100.0

Solution

Report

The climate types of 23 countries were classified as being, 'cold', 'mild' or 'hot'. The majority of the countries, 60.9%, were found to have a mild climate. Of the remaining countries, 26.1% were found to have a hot climate, while 13.0% were found to have a cold climate.

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Exercise 1B

Constructing frequency tables from raw data

1 **a** In a frequency table, what is the mode?

b Identify the mode in the following datasets.

i *Grades*: A A C B A B B B B D C

ii *Shoe size*: 8 9 9 10 8 8 7 9 8 10 12 8 10

2 The following data identify the *state of residence* of a group of people, where 1 = Victoria, 2 = South Australia and 3 = Western Australia.

2 1 1 1 3 1 3 1 1 3 3

a Is the variable *state of residence*, categorical or numerical?

b Form a frequency table (with both numbers and percentages) to show the distribution of *state of residence* for this group of people. Use the table in Example 1 as a model.

c Construct a bar chart using Example 2 as a model.

3 The *size* (S = small, M = medium, L = large) of 20 cars was recorded as follows.

S S L M M M L S S M

M S L S M M M S S M

a Is the variable *size* in this context numerical or categorical?

b Form a frequency table (with both numbers and percentages) to show the distribution of size for these cars. Use the table in Example 1 as a model.

c Construct a bar chart using Example 2 as a model.

Constructing a percentage segmented bar chart from a frequency table

4 The table shows the frequency distribution of the place of birth for 500 Australians.

a Is *place of birth* an ordinal or a nominal variable?

b Display the data in the form of a percentage segmented bar chart.

Place of birth	*Percentage*
Australia	78.3
Overseas	21.8
Total	100.1

5 The table records the number of new cars sold in Australia during the first quarter of 1 year, categorised by *type of vehicle* (private, commercial).

a Is *type of vehicle* an ordinal or a nominal variable?

	Frequency	
Type of vehicle	*Number*	*Percentage*
Private	132 736	
Commercial	49 109	
Total		

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b Copy and complete the table giving the percentages correct to the nearest whole number.

c Display the data in the form of a percentage segmented bar chart.

Analysing frequency tables and writing reports

6 The table shows the frequency distribution of *school type* for a number of schools. The table is incomplete.

School type	*Frequency*	
	Number	*Percentage*
Catholic	4	20
Government	11	
Independent	5	25
Total		100

a Write down the information missing from the table.

b How many schools are categorised as 'independent'?

c How many schools are there in total?

d What percentage of schools are categorised as 'government'?

e Use the information in the frequency table to complete the following report describing the distribution of school type for these schools.

Report

____ schools were classified according to school type. The majority of these schools, ____ %, were found to be ____. Of the remaining schools, ____ were ____ while 20% were ____.

7 Twenty-two students were asked the question, 'How often do you play sport?', with the possible responses: 'regularly', 'sometimes' or 'rarely'. The distribution of responses is summarised in the frequency table.

Plays sport	*Frequency*	
	Number	*Percentage*
Regularly	5	22.7
Sometimes	10	
Rarely		31.8
Total	22	

a Write down the information missing from the table.

b Use the information in the frequency table to complete the report below describing the distribution of student responses to the question, 'How often do you play sport?'

Report

When ____ students were asked the question, 'How often do you play sport', the dominant response was 'Sometimes', given by ____ % of the students. Of the remaining students, ____ % of the students responded that they played sport ____ while ____ % said that they played sport ____.

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8 The table shows the frequency distribution of the eye colour of 11 preschool children.

Use the information in the table to write a brief report describing the frequency distribution of eye colour.

Eye colour	*Frequency*	
	Number	*Percentage*
Brown	6	54.5
Hazel	2	18.2
Blue	3	27.3
Total	11	100.0

1C Displaying and describing the distributions of numerical variables

The grouped frequency distribution

When looking at ways of organising and displaying numerical data, we are faced with the problem of how to deal with continuous variables that can take a large range of values – for example, age (0–100+). Listing all possible ages would be tedious and produce a large and unwieldy frequency table or graphical display.

To solve this problem, we *group* the data into a small number of convenient intervals. We then organise the data into a frequency table using these data intervals. We call this sort of table a *grouped frequency table*.

Example 5 Constructing a grouped frequency table

The data below give the average hours worked per week in 23 countries.

35.0	48.0	45.0	43.0	38.2	50.0	39.8	40.7	40.0	50.0	35.4	38.8
40.2	45.0	45.0	40.0	43.0	48.8	43.3	53.1	35.6	44.1	34.8	

Form a grouped frequency table with five intervals.

Solution

1 Set up a table as shown. Use five intervals: 30.0–34.9, 35.0–39.9, . . . , 50.0–54.9.

2 List these intervals, in ascending order, under ‘Average hours worked’.

3 Count the number of countries whose average working hours fall into each of the intervals.
Record these values in the ‘Number’ column.

Average hours worked	Frequency	
	Number	Percentage
30.0–34.9	1	4.3
35.0–39.9	6	26.1
40.0–44.9	8	34.8
45.0–49.9	5	21.7
50.0–54.9	3	13.0
Total	23	99.9

ISBN 978-1-107-56757-3

4 Convert the counts into percentages and record in the 'Percentage' column.

5 Total the number and percentage columns, which may not total 100% because of rounding.

Notes:

1 The intervals in this example are of width five. For example, the interval 35.0–39.9 is an interval of width 5.0 because it contains all values from 34.9500 to 39.9499.

2 The intervals are deliberately constructed so that they do not overlap.

3 There are no hard and fast rules for the number of intervals we use when grouping data but, usually, between five and fifteen intervals are used. Usually, the smaller the number of data values, the smaller the number of intervals. Here we have chosen to use five intervals.

How has forming a frequency table helped?

The process of forming a frequency table for a numerical variable:

- *orders* the data displays the data in a *compact* form
- tells us how the data values are *distributed* across the categories
- helps us identify the *mode* (the most frequently occurring value or interval).

The histogram and its construction

The **histogram** is a graphical display of the information in the grouped frequency table.

Constructing a histogram from a frequency table

In a frequency histogram:

- frequency (count or per cent) is shown on the vertical axis
- the values of the variable being displayed are plotted on the horizontal axis
- each bar in a histogram corresponds to a data interval
- the height of the bar gives the frequency (or the percentage frequency).

Example 6 Constructing a histogram from a frequency table

Construct a histogram for the frequency table opposite.

Average hours worked	*Frequency*
30.0–34.9	1
35.0–39.9	6
40.0–44.9	8
45.0–49.9	5
50.0–54.9	3
Total	23

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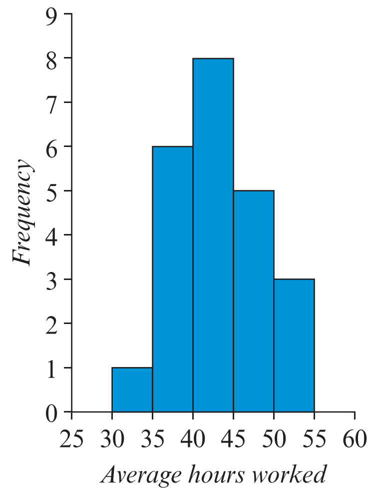
Solution

1. Label the horizontal axis with the variable name, 'Average hours worked'. Mark the scale using the start of each interval: 30, 35, . . .
2. Label the vertical axis 'Frequency'. Scale allowing for the maximum frequency, 8.
3. Finally, for each interval draw a bar, making the height equal to the frequency.



Constructing a histogram from raw data

It is relatively quick to construct a histogram from a frequency table. However, if you have only raw data (as you mostly do), it is a very slow process because you have to construct the frequency table first. Fortunately, a CAS calculator will do this for you.

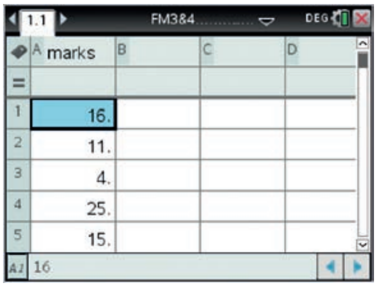
How to construct a histogram using the TI-Nspire CAS

Display the following set of 27 marks in the form of a histogram.

16	11	4	25	15	7	14	13	14	12	15	13	16	14
15	12	18	22	17	18	23	15	13	17	18	22	23	

Steps

1. Start a new document by pressing ctrl + N (or on>**New Document**. If prompted to save an existing document, move cursor to **No** and press enter.
2. Select **Add Lists & Spreadsheet**.
   Enter the data into a list named *marks*.
   - **a** Move the cursor to the name space of column A and type in *marks* as the list name. Press enter.
   - **b** Move the cursor down to row 1, type in the first data value and press enter. Continue until all the data have been entered. Press enter after each entry.



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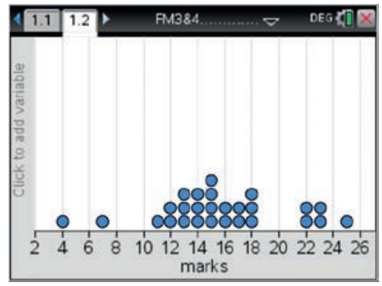
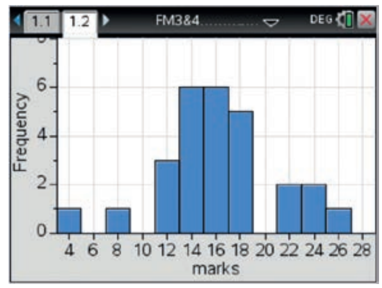
3 Statistical graphing is done through the **Data & Statistics** application. Press ctrl + I (or ctrl doc▾) and select **Add Data & Statistics**.



a Press tab enter (or click on the **Click to add variable box** on the *x*-axis) to show the list of variables. Select *marks*.
Press enter to paste *marks* to that axis.

b A dot plot is displayed as the default. To change the plot to a histogram, press menu>**Plot Type>Histogram**. Your screen should now look like that shown opposite. This histogram has a column (or bin) width of 2 and a starting point of 3.

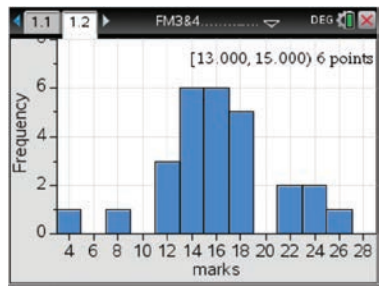


4 Data analysis

a Move the cursor over any column; a ☝ will appear and the column data will be displayed as shown opposite.

b To view other column data values, move the cursor to another column.

Note: If you click on a column, it will be selected.



Hint: If you accidentally move a column or data point, ctrl + esc enter will undo the move.

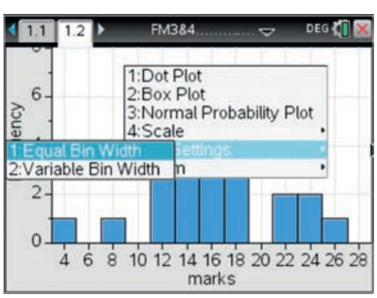
5 Change the histogram column (bin) width to **4** and the starting point to **2**.

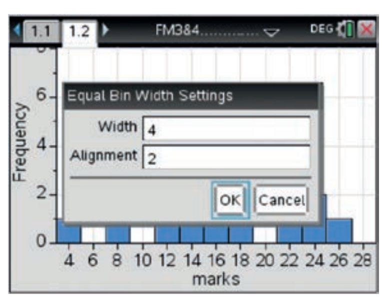
a Press ctrl + menu to get the contextual menu as shown (below left).

Hint: Pressing ctrl + menu enter with the cursor on the histogram gives you a contextual menu that relates only to histograms. You can access the commands through menu>**Plot Properties**.

b Select **Bin Settings**>**Equal Bin Width**.

c In the settings menu (below right) change the Width to 4 and the Starting Point (Alignment) to 2 as shown. Press enter.





d A new histogram is displayed with column width of 4 and a starting point of 2 but it no longer fits the window (below left). To solve this problem, press

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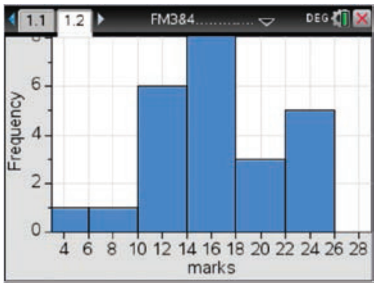
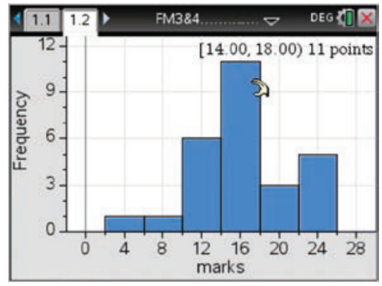
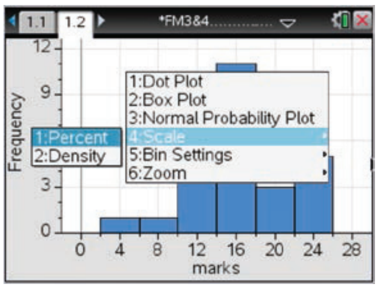
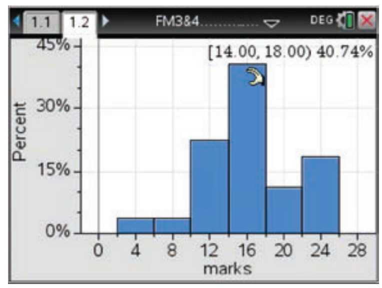
ctrl + menu >**Zoom>Zoom-Data** and enter to obtain the histogram as shown below right.





6 To change the frequency axis to a percentage axis, press ctrl + enter >**Scale>Percent** and then press enter.





How to construct a histogram using the ClassPad

Display the following set of 27 marks in the form of a histogram.

16	11	4	25	15	7	14	13	14	12	15	13	16	14
15	12	18	22	17	18	23	15	13	17	18	22	23	

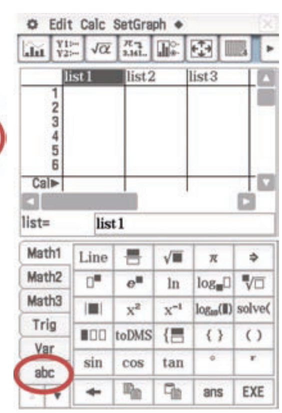
Steps

1 From the application menu screen, locate the built-in **Statistics** application.

Tap Statistics to open.

Tapping Menu from the icon panel (just below the touch screen) will display the application menu if it is not already visible.

2 Enter the data into a list named **marks**.



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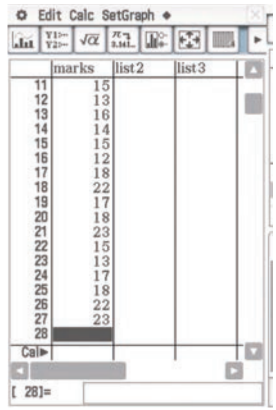
To name the list:

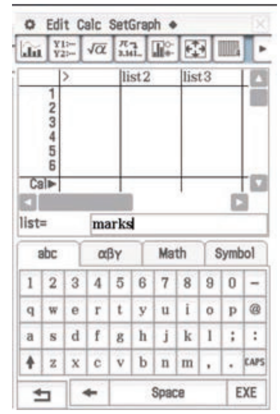
a Highlight the heading of the first list by tapping it.

b Press Keyboard on the front of the calculator and tap the abc tab.

c To enter the data, type the word **marks** and press EXE. Tap ↩ and Keyboard to return to the list screen.

d Type in each data value and press EXE or ▼ (which is found on the cursor button on the front of the calculator) to move down to the next cell.





The screen should look like the one shown above right.

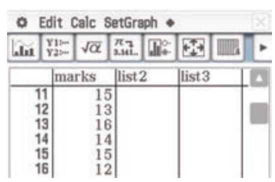
3 Set up the calculator to plot a statistical graph.

a Tap the StatGraph icon from the toolbar. This opens the **Set StatGraphs** dialog box.

b Complete the dialog box as given below.

- **Draw:** select **On**.
- **Type:** select **Histogram** (▼).
- **XList**: select **main\marks** (▼).
- **Freq**: leave as **1**.

c Tap **Set** to confirm your selections.



Note: To make sure only this graph is drawn, select **SetGraph** from the menu bar at the top and confirm that there is a tick only beside **StatGraph1** and no others.

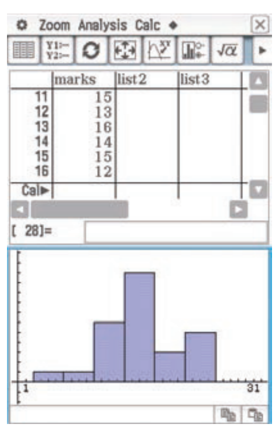
4 To plot the graph:

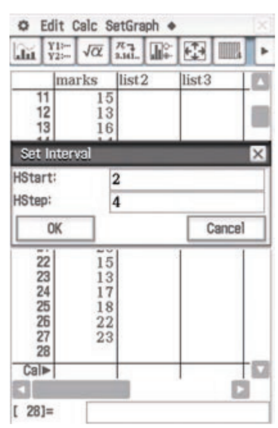
a Tap the graph icon in the toolbar.

b Complete the **Set Interval** dialog box as follows.

- **HStart:** type **2** (i.e. the starting point of the first interval)
- **HStep:** type **4** (i.e. the interval width).

Tap OK to display histogram.





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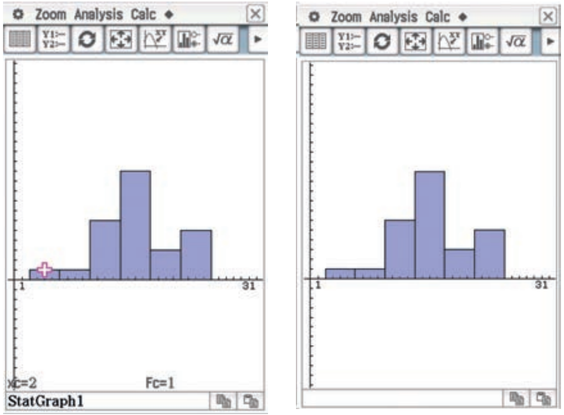
Note: The screen is split into two halves, with the graph displayed in the bottom half, as shown above. Tapping Resize from the icon panel allows the graph to fill the entire screen. Tap Resize again to return to half-screen size.

5 Tapping from the toolbar places a marker (+) at the top of the first column of the histogram (see opposite) and tells us that:

a the first interval begins at 2 **(x_c = 2)**

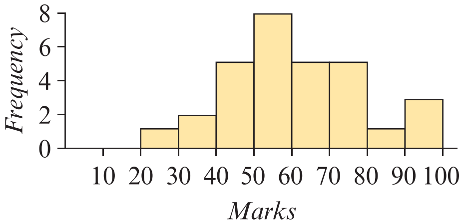
b for this interval, the frequency is 1 **(F_c = 1)**.

To find the frequencies and starting points of the other intervals, use the cursor key arrow (▶) to move from interval to interval.



What to look for in a histogram

A histogram provides a graphical display of a data distribution. For example, the histogram opposite displays the distribution of test marks for a group of 32 students.



The purpose of constructing a histogram is to help understand the *key features* of the data distribution. These features are its:

- **shape** and **outliers**
- **centre**
- **spread**.

Shape and outliers

How are the data distributed? Is the histogram peaked? That is, do some data values tend to occur much more frequently than others, or is the histogram relatively flat, showing that all values in the distribution occur with approximately the same frequency?

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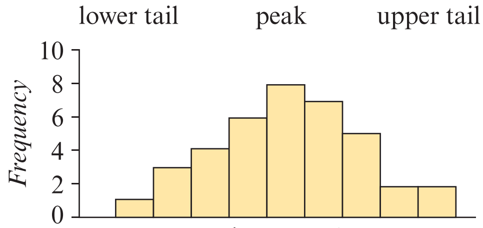
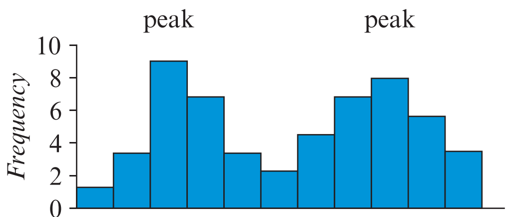
Symmetric distributions

If a histogram is single-peaked, does the histogram region tail off evenly on either side of the peak? If so, the distribution is said to be *symmetric* (see histogram 1).



Histogram 1



Histogram 2

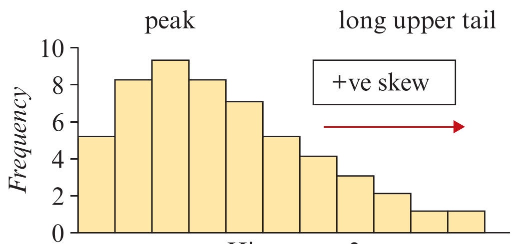
A single-peaked **symmetric distribution** is characteristic of the data that derive from measuring variables such as intelligence test scores, weights of oranges, or any other data for which the values vary evenly around some central value.

The double-peaked distribution (histogram 2) is symmetric about the dip between the two peaks. A histogram that has two distinct peaks indicates a *bimodal* (two modes) distribution.

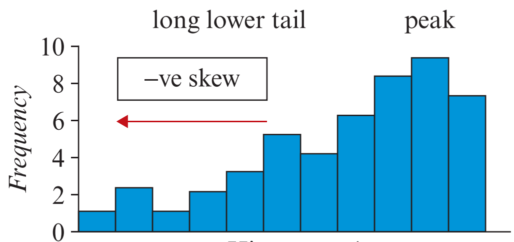
A bimodal distribution often indicates that the data have come from two different populations. For example, if we were studying the distance the discus is thrown by Olympic-level discus throwers, we would expect a bimodal distribution if both male and female throwers were included in the study.

Skewed distributions

Sometimes a histogram tails off primarily in one direction. If a histogram tails off to the right, we say that it is *positively* skewed (histogram 3). The distribution of salaries of workers in a large organisation tends to be positively skewed. Most workers earn a similar salary with some variation above or below this amount, but a few earn more and even fewer, such as the senior manager, earn even more. The distribution of house prices also tends to be positively skewed.



Histogram 3



Histogram 4

If a histogram tails off to the left, we say that it is *negatively* skewed (histogram 4). The distribution of age at death tends to be negatively skewed. Most people die in old age, a few in middle age and fewer still in childhood.

Outliers

Outliers are any data values that stand out from the main body of data. These are data values that are atypically high or low. See, for example, histogram 5, which shows an outlier. In this case it is a data value that is atypically low compared to the rest of the data values.

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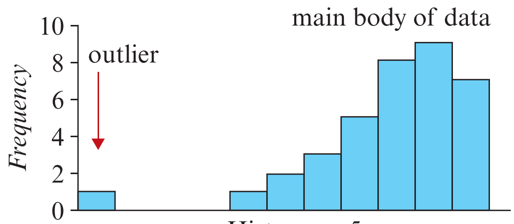
Sports data often contain outliers. For example, the heights of the players in a football side vary, but do so within a limited range. One exception is the 'knock' ruckman, who may be exceptionally tall and well outside the normal range of variation.

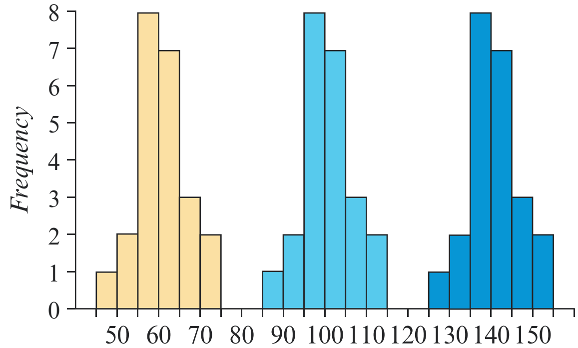


Histogram 5

In statistical terms, the exceptionally tall ruckman is an outlier, because his height does not fit in the range of heights that might be regarded as typical for the team. Outliers can also indicate errors made collecting or processing data – for example, a person's age recorded as 365.

Centre

Histograms 6 to 8 display the distribution of test scores for three different classes taking the same subject. They are identical in shape, but differ in where they are located along the axis. In statistical terms we say that the distributions are 'centred' at different points along the axis. But what do we mean by the **centre of a distribution**?



Histograms 6 to 8

This is an issue we will return to in more detail in the next chapter. For the present we will take the centre to be the *middle* of the distribution. You might know of this point as the **median**.

The middle of a symmetric distribution is reasonably easy to locate by eye. Looking at histograms 6 to 8, it would be reasonable to say that the centre or middle of each distribution lies roughly halfway between the extremes; half the observations would lie above this point and half below. Thus we might *estimate* that histogram 6 (yellow) is centred at about 60, histogram 7 (light blue) at about 100, and histogram 8 (dark blue) at about 140.

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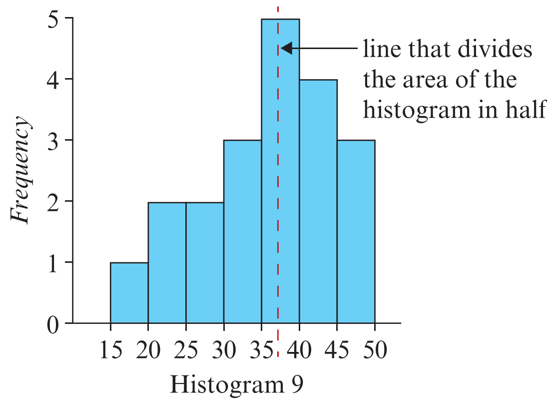
For skewed distributions, it is more difficult to estimate the middle of a distribution by eye. The middle is not halfway between the extremes because, in a skewed distribution, the scores tend to bunch up at one end.

However, if we imagine a cardboard cut-out of the histogram, the midpoint lies on the line that divides the histogram into two equal areas (Histogram 9).

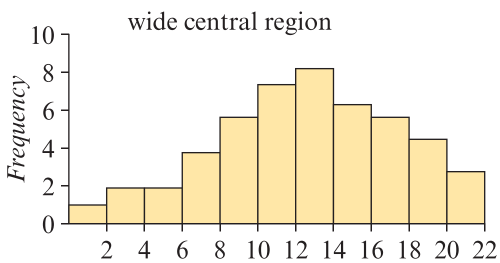
Using this method, we would estimate the centre of the distribution to lie somewhere between 35 and 40, but closer to 35, so we might opt for 37. However, remember that this is only an estimate.



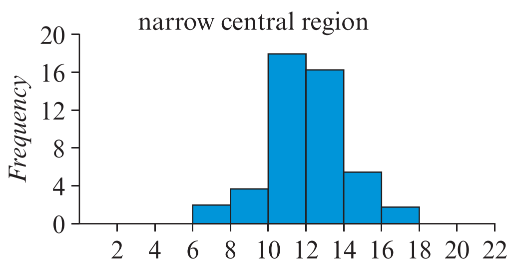
Histogram 9

Spread

If the histogram is single-peaked, is it narrow? This would indicate that most of the data values in the distribution are tightly clustered in a small region. Or is the peak broad? This would indicate that the data values are more widely spread out. Histograms 10 and 11 are both single-peaked. Histogram 10 has a broad peak, indicating that the data values are not very tightly clustered about the centre of the distribution. In contrast, histogram 11 has a narrow peak, indicating that the data values are tightly clustered around the centre of the distribution.



Histogram 10



Histogram 11

But what do we mean by the *spread* of a distribution? We will return to this in more detail later. For a histogram we will take it to be the maximum **range** of the distribution.

> **Range**
>
> Range = largest value − smallest value

For example, histogram 10 has a spread (maximum range) of 22 (22 – 0) units. This is considerably greater than the spread of histogram 11 which has a spread of 12 (18 – 6) units.

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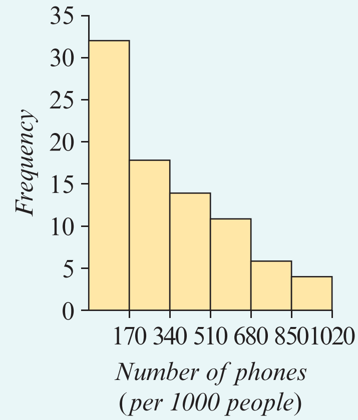
Example 7 **Describing a histogram in terms of shape, centre and spread**

The histogram opposite shows the distribution of the number of phones per 1000 people in 85 countries.

a Describe its shape and note outliers (if any).

b Locate the centre of the distribution.

c Estimate the spread of the distribution.



Solution

a Shape and outliers

The distribution is positively skewed.
There are no outliers.

b *Centre:* Count up the frequencies from either end to find the middle interval.

The distribution is centred between 170 and 340 phones per 1000 people.

c *Spread:* Use the maximum range to estimate the spread.

Spread = 1020 – 0
= 1020 phones/1000 people

Using a histogram to describe the distribution of a numerical variable in the context of its data

If you were using the histogram above to describe the distribution in a form suitable for a statistical report, you might write as follows.

Report

For these 85 countries, the distribution of the number of phones per 1000 people is positively skewed. The centre of the distribution lies between 170 and 340 phones/1000 people. The spread of the distribution is 1020 phones/1000 people. There are no outliers.

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Exercise 1C

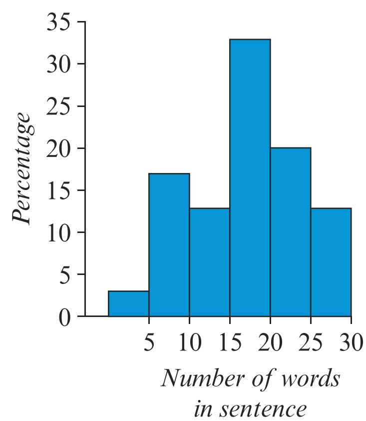
Constructing a histogram from a frequency table

1 Construct a histogram to display the information in the frequency table opposite. Use the histogram in Example 6 as a model. Label axes and mark scales.

Population density	*Frequency*
0–199	11
200–399	4
400–599	4
600–799	2
800–999	1
Total	22

Reading information from a histogram

2 The histogram opposite displays the distribution of the number of words in 30 randomly selected sentences.



a What percentage of these sentences contained:

- **i** 5–9 words?
- **ii** 25–29 words?
- **iii** 10–19 words?
- **iv** fewer than 15 words?

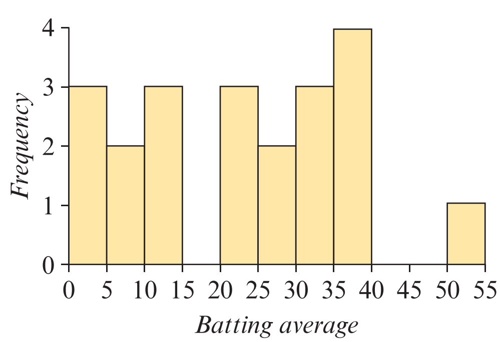
Write answers correct to the nearest per cent.

b How many of these sentences contained:

- **i** 20–24 words?
- **ii** more than 25 words?

c What is the modal interval?

3 The histogram opposite displays the distribution of the average batting averages of cricketers playing for a district team.



a How many players have their averages recorded in this histogram?

b How many of these cricketers had a batting average:

- **i** 20 or more?
- **ii** less than 15?
- **iii** at least 20 but less than 30?
- **iv** of 45?

c What percentage of these cricketers had a batting average:

- **i** 50 or more?
- **ii** at least 20 but less than 40?

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Constructing a histogram from raw data using a CAS calculator

4 The pulse rates of 23 students are given below.

86	82	96	71	90	78	68	71	68	88	76	74
70	78	69	77	64	80	83	78	88	70	86	

a Use a graphics calculator to construct a histogram so that the first column starts at 63 and the column width is two.

b **i** What is the starting point of the third column?

ii What is the 'count' for the third column? What are the *actual* data values?

c Redraw the histogram so that the column width is five and the first column starts at 60.

d For this histogram, what is the count in the interval '65 to <70'?

5 The numbers of children in the families of 25 VCE students are listed below.

1	6	2	5	5	3	4	1	2	7	3	4	5
3	1	3	2	1	4	4	3	9	4	3	3	

a Use a graphics calculator to construct a histogram so that the column width is one and the first column starts at 0.5.

b What is the starting point for the fourth column and what is the count?

c Redraw the histogram so that the column width is two and the first column starts at 0.

d **i** What is the count in the interval from 6 to less than 8?

ii What actual data value(s) does this interval include?

Determining the shape, centre and spread from a histogram

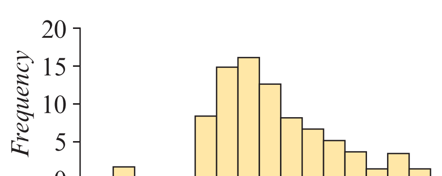
6 Identify each of the following histograms as approximately symmetric, positively skewed or negatively skewed, and mark the following.

i The mode (if there is a clear mode)

ii Any potential outliers

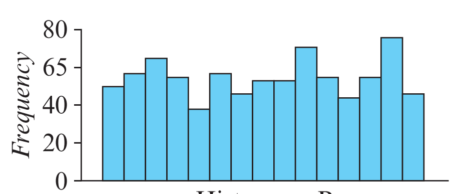
iii The approximate location of the centre

a



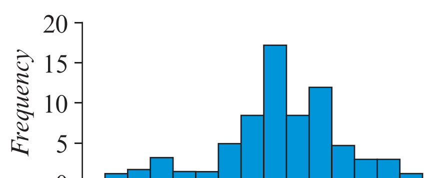
Histogram A

b



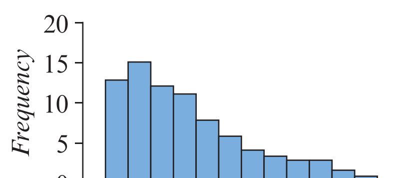
Histogram B

c



Histogram C

d



Histogram D

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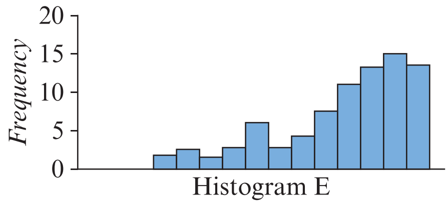
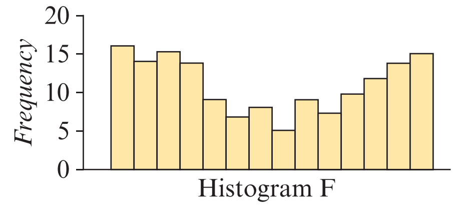
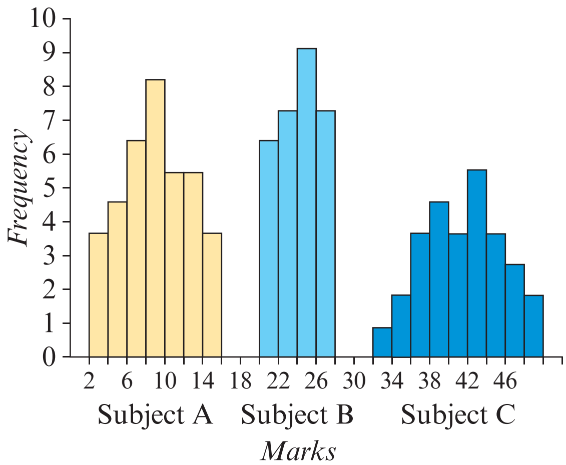
e 

f 

7 These three histograms show the marks obtained by a group of students in three subjects.



a Are each of the distributions approximately symmetric or skewed?

b Are there any clear outliers?

c Determine the interval containing the central mark for each of the three subjects.

d In which subject was the spread of marks the least?

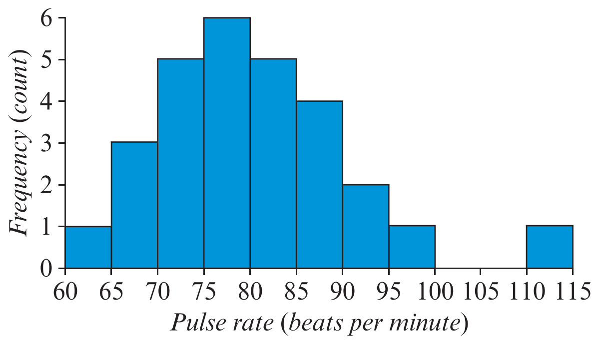
Use the maximum range to estimate the spread.

e In which subject did the marks vary most? Use the range to estimate the spread.

Describing a histogram in the context of its data

8 The histogram opposite shows the distribution of pulse rate for 28 students.

Use the histogram to complete the report below describing the distribution of pulse rate in terms of shape, centre, spread and outliers (if any).



Report

For the ☐ students, the distribution of pulse rates is ☐ with an outlier. The centre of the distribution lies between ☐ beats per minute and the spread of the distribution is ☐ beats per minute. The outlier lies in somewhere between ☐ beats per minute.

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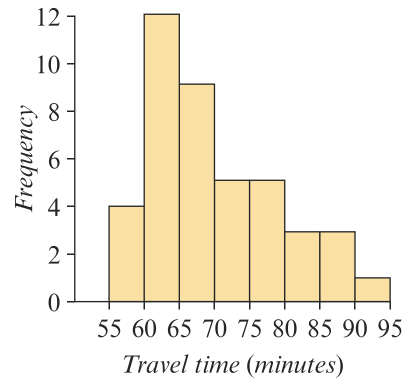
9 The histogram opposite shows the distribution of travel times (in minutes) for 42 journeys from an outer suburban station to the city.

Use the histogram to write a brief report describing the distribution of travel times in terms of shape, centre, spread and outliers (if any).



1D Using a log scale to display data

Many numerical variables that we deal with in statistics have values that range over several orders of magnitude. For example, the population of countries range from a few thousand to hundreds of thousands, to millions, to hundreds of millions to just over 1 billion. Constructing a histogram that effectively locates every country on the plot is impossible.

One way to solve this problem is to use a scale that spreads out the countries with small populations and 'pulls in' the countries with huge populations.

A scale that will do this is called a logarithmic scale (or, more commonly, a **log scale**). However, before you learn to apply log scales, you will have to learn something about logarithms.

A brief introduction to logarithms to the base 10 and their interpretation

Consider the numbers:

0.01, 0.1, 1, 10, 100, 1000, 10 000, 100 000, 1 000 000

Such numbers can be written more compactly as:

10^{-2}, 10^{-1}, 10^0, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6

In fact, if we make it clear we are only talking about powers of 10, we can merely write down the powers:

−2, −1, 0, 1, 2, 3, 4, 5, 6

These powers are called the **logarithms** of the numbers or 'logs' for short.

When we use logarithms to write numbers as powers of 10, we say we are working with logarithms to the *base 10*. We can indicate this by writing $\log_{10}$.

Note: We could also use logarithms to write numbers as powers of two, for example, $8 = 2^3$, or powers of 5 – for example, $625 = 5^4$. In these cases we would be working with logarithms to the base 2 and 5 respectively. Only base 10 logarithms are required for this course.

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Properties of logs to the base 10

1. If a number is greater than one, its log to the base 10 is greater than zero.
2. If a number is greater than zero but less than one, its log to the base 10 is negative.
3. If the number is zero, then its log is undefined.

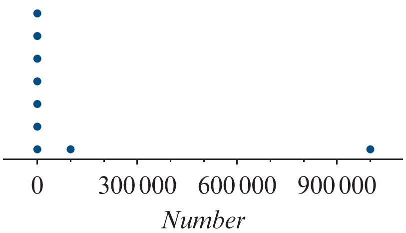
Why use logs?

The set of numbers

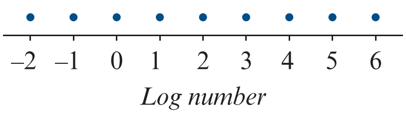
0.01, 0.1, 1, 10, 100, 1000, 10 000, 100 000, 1 000 000

ranges from 0.01 to 1 million.

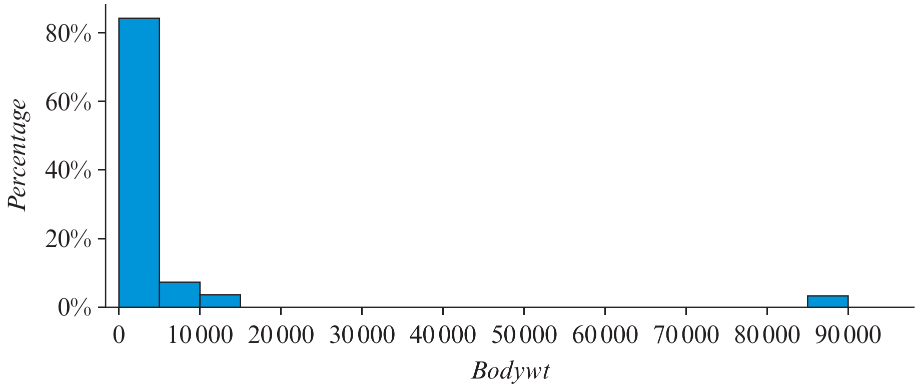
Thus, if we wanted to plot these numbers on a scale, the first seven numbers would cluster together at one end of the scale, while the eighth (1 million) would be located at the far end of the scale.



By contrast, if we plot the logs of these numbers, they are evenly spread along the scale. We use this idea to display a set of data whose values range over several orders of magnitude. Rather than plot the data values themselves, we plot the logs of their data values.



For example, the histogram below displays the body weights (in kg) of a number of animal species. Because the animals represented in this dataset have weights ranging from around 1 kg to 90 tonnes (a dinosaur), most of the data are bunched up at one end of the scale and much detail is missing. The distribution of weights is highly positively skewed, with an outlier.



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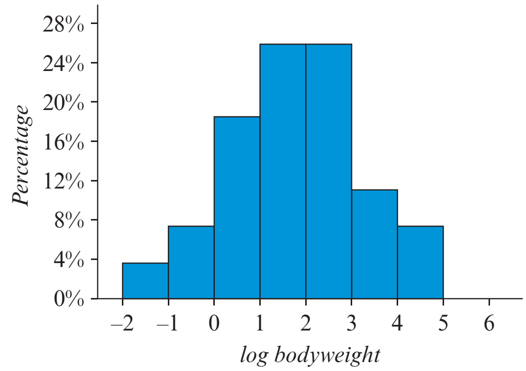
However, when a log scale is used, their weights are much more evenly spread along the scale. The distribution is now approximately symmetric, with no outliers, and the histogram is considerably more informative.



We can now see that the percentage of animals with weights between 10 and 100 kg is similar to the percentage of animals with weights between 100 and 1000 kg.

Note: In drawing this conclusion, you need to remember that log 10 = 1, log 100 = 2, and so on.

▶ Working with logs

To construct and interpret a log data plot, like the one above, you need to be able to:

1. Work out the log for any number. So far we have only done this for numbers such as 10, 100, 100 that are exact powers of 10; for example, $100 = 10^2$, so log 100 = 2.
2. Work backwards from a log to the number it represents. This is easy to do in your head for logs that are exact powers of 10 – for example, if the log of a number is 3 then the number is $10^3 = 1000$. But it is not a sensible approach for numbers that are not exact powers of 10.

Your CAS calculator is the key to completing both of these tasks in practice.

Skillsheet

Example 8 Using a CAS calculator to find logs

a Find the log of 45, correct to two significant figures.

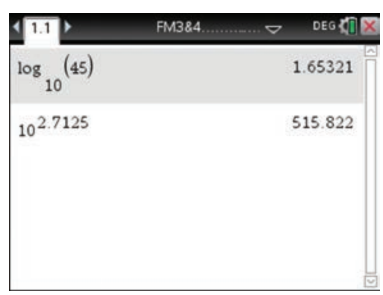
b Find the number whose log is 2.7125, correct to the nearest whole number.

Solution

a Open a calculator screen, type log (45) and press enter. Write down the answer correct to two significant figures.

b If the log of a number is 2.7125, then the number is $10^{2.7125}$.

Enter the expression $10^{2.7125}$ and press enter. Write down the answer correct to the nearest whole number.



a log 45 = 1.65 . . .
= 1.7 (to 2 sig. figs)

b $10^{2.7} = 515.82\ldots$
= 520 (to 2 sig. figs)

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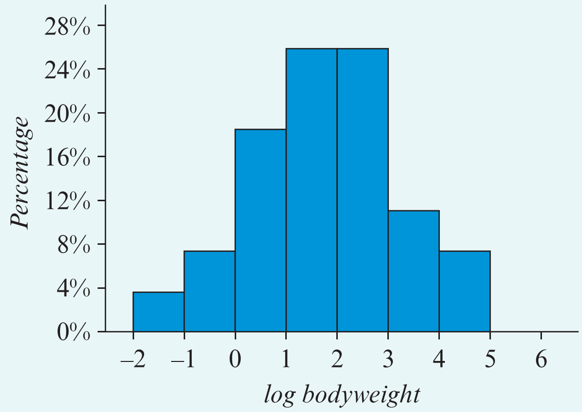
▶ Analysing data displays with a log scale

Now that you know how to work out the log of any number and convert logs back to numbers, you can analyse a data plot using a log scale.

Example 9 Interpreting a histogram with a log scale

The histogram shows the distribution of the weights of 27 animal species plotted on a log scale.



a What body weight (in kg) is represented by the number 4 on the log scale?

b How many of these animals have body weights more than 10 000 kg?

c The weight of a cat is 3.3 kg. Use your calculator to determine the log of its weight correct to two significant figures.

d Determine the weight (in kg) whose log weight is 3.4 (the elephant). Write your answer correct to the nearest whole number.

Solution

a If the log of a number is 4 then the number is $10^4 = 10\,000$.

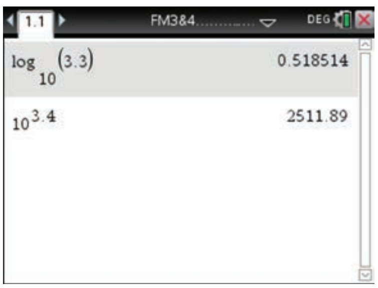
a $10^4 = 10\,000$ kg

b On the log scale, 10 000 is shown as 4.

Thus, the number of animals with a weight greater than 10 000 kg, corresponds to the number of animals with a log weight of greater than 4.

This can be determined from the histogram which shows there are two animals with log weights greater than 4.

b Two animals



c The weight of a cat is 3.3 kg. Use your calculator to find log 3.3. Write the answer correct to two significant figures.

c Cat: $\log 3.3 = 0.518\ldots$

$= 0.52$ kg (to 2 sig. figs)

d The log weight of an elephant is 3.4. Determine its weight in kg by using your calculator to evaluate $10^{3.4}$. Write the answer correct to the nearest whole number.

d Elephant: $10^{3.4} = 2511.88\ldots$

$= 2512$ kg

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▶ Constructing a histogram with a log scale

The task of constructing a histogram is also a CAS calculator task.

Using a TI-Nspire CAS to construct a histogram with a log scale

The weights of 27 animal species (in kg) are recorded below.

1.4	470	36	28	1.0	12 000	2600	190	520
10	3.3	530	210	62	6700	9400	6.8	35
0.12	0.023	2.5	56	100	52	87 000	0.12	190

Construct a histogram to display the distribution:

a of the body weights of these 27 animals and describe its shape

b of the log body weights of these animals and describe its shape.

Steps

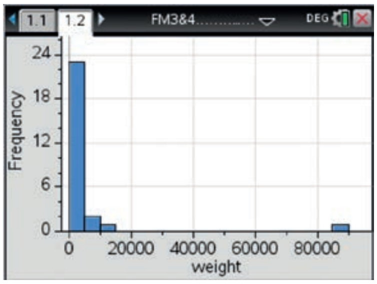
1 a Start a new document by pressing ctrl + N.

b Select **Add Lists & Spreadsheet**.
Enter the data into a column named 'weight'.

2 a Press ctrl + I and select **Add Data & Statistics**.
Click on the **Click to add variable** on the x-axis and select the variable 'weight'. A dot plot is displayed.

b Plot a histogram using menu**>Plot Type>Histogram**.

c Describe the shape of the distribution.

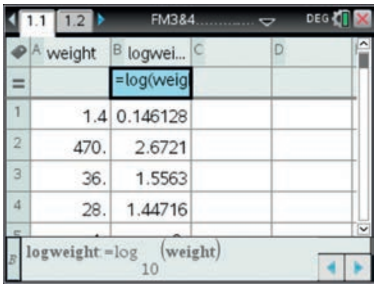


Shape: positively skewed with outliers

3 a Return to the **Lists & Spreadsheet** screen.

b Name another column 'logweight'.

c Move the cursor to the grey cell below the 'logweight' heading. Type in = **log(weight)**.
Press enter to calculate the values of logweight.



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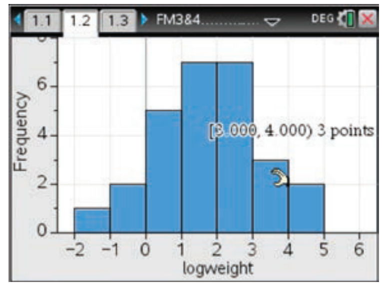
4 **a** Plot a histogram using a log scale. That is, plot the variable 'logweight'.

Note: Use menu>**Plot Properties**>**Histogram Properties**>**Bin Settings**>**Equal Bin Width** and set the column width (bin) to 1 and alignment (start point) to −2 and use menu>**Window/Zoom**>**Zoom-Data** to rescale.



b Describe the shape of the distribution.

Shape: approximately symmetric

Using a ClassPad to construct a histogram with a log scale

The weights of 27 animal species (in kg) are recorded below.

1.4	470	36	28	1.0	12 000	2600	190	520
10	3.3	530	210	62	6700	9400	6.8	35
0.12	0.023	2.5	56	100	52	87 000	0.12	190

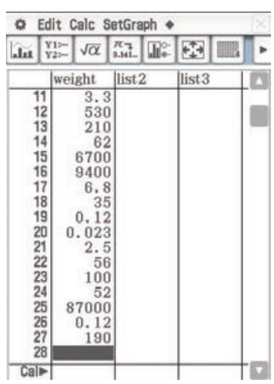
Construct a histogram to display the distribution:

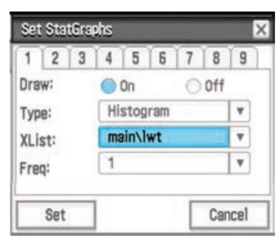
a of the body weights of these 27 animals and describe its shape

b of the log body weights of these animals and describe its shape.

Steps

1 In the statistics application Statistics enter the data into a column named 'weight' as shown.



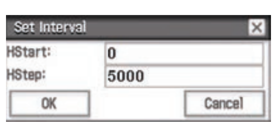


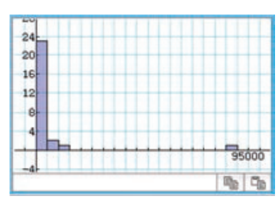
2 Plot a histogram of the data.

a Tap the icon from the toolbar.

b Complete the dialog box.

- **Draw:** select **On**.
- **Type:** select **Histogram** (▼)
- **XList:** select **main\weight**(▼).
- **Freq:** leave as **1**.





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Tap **Set** to confirm your selections.

c Tap in the toolbar.

d Complete the **Set Interval** dialog box as follows:

HStart: 0

HStep: 5000

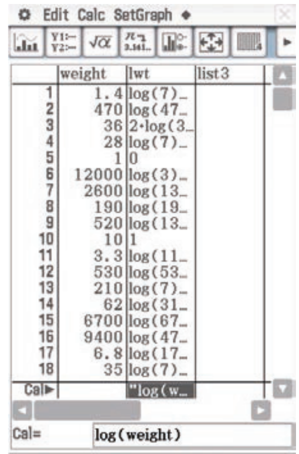
Describe the shape of the distribution. *Shape: positively skewed with outliers*

3 **a** Return to the data entry screen.

b Name another column '1wt', short for log(weight).

c Tap in the calculation cell at the bottom of this column.

Type log(weight) and tap EXE.



4 Plot a histogram to display the distribution of weights on a log scale. That is, plot the variable 1wt.

a Tap from the toolbar.

b Complete the dialog box.

- **Draw**: select **On**.
- **Type**: select **Histogram** (▼).
- **XList**: select **main\1wt** (▼).
- **Freq**: leave as **1**.

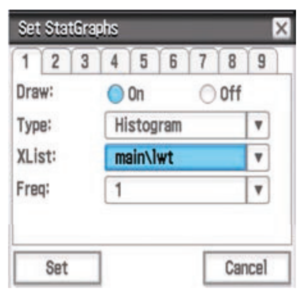
Tap **Set** to confirm your selections.

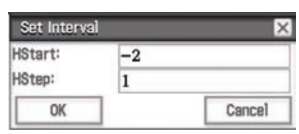
c Tap in the toolbar.

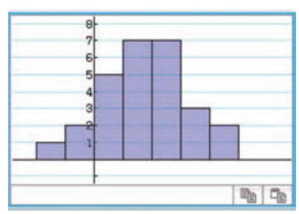
d Complete the **Set Interval** dialog box as follows:

- **HStart**: **type -2**
- **HStep**: type **1**

Tap **OK** to display histogram.







Describe the shape of the distribution. *Shape: approximately symmetric*

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Exercise 1D

Determining logs from numbers

1 Using a CAS calculator, find the logs of the following numbers correct to one decimal place.

a 2.5 **b** 25 **c** 250 **d** 2500
e 0.5 **f** 0.05 **g** 0.005 **h** 0.0005

Determining numbers from logs

2 Find the numbers whose logs are:

a −2.5 **b** −1.5 **c** −0.5 **d** 0

Write your decimal answers correct to two significant figures.

Constructing a histogram with a log scale

3 The brain weights of the same 27 animal species (in g) are recorded below.

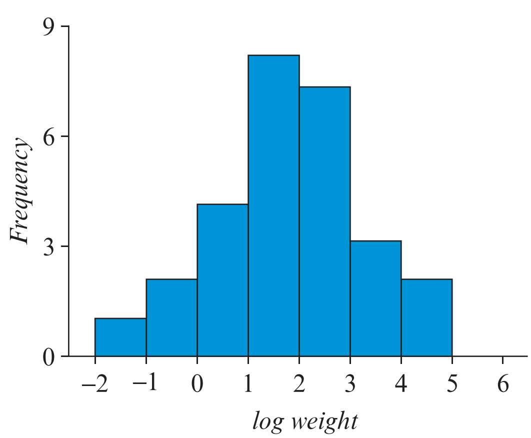
465	423	120	115	5.50	50.0	4600	419	655
115	25.6	680	406	1320	5712	70.0	179	56.0
1.00	0.40	12.1	175	157	440	155	3.00	180

a Construct a histogram to display the distribution of brain weights and comment on its shape.

b Construct a histogram to display the log of the brain weights and note the shape of the distribution.

Interpreting a histogram with a log scale

4 The histogram opposite shows the distribution of brain weights (in g) of 27 animal species plotted on a log scale.



a The brain weight (in g) of a mouse is 0.4 g. What value would be plotted on the log scale?

b The brain weight (in g) of an African elephant is 5712 g. What is the log of this brain weight (to two significant figures)?

c What brain weight (in g) is represented by the number 2 on the log scale?

d What brain weight (in g) is represented by the number –1 on the log scale?

e Use the histogram to determine the number of these animals with brain weights:

i over 1000 g **ii** between 1 and 100 g **iii** over 1 g.

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Review

Key ideas and chapter summary

Univariate data	**Univariate data** are generated when each observation involves recording information about a single variable, for example a dataset containing the heights of the children in a preschool.
Types of variables	Variables can be classified as *numerical* or *categorical*.
Categorical variables	**Categorical variables** are used to represent characteristics of individuals. Categorical variables come in two types: nominal and ordinal. **Nominal variables** generate data values that can only be used by name, e.g. eye colour. **Ordinal variables** generate data values that can be used to both name and order, e.g. house number.
Numerical variables	**Numerical variables** are used to represent quantities. Numerical variables come in two types: discrete and categorical. *Discrete variables* represent quantities – e.g. the number of cars in a car park. *Continuous variables* represent quantities that are measured rather than counted – for example, weights in kg.
Frequency table	A **frequency table** lists the values a variable takes, along with how often (frequently) each value occurs. *Frequency* can be recorded as: ▪ the number of times a value occurs – e.g. the number of females in the dataset is 32 ▪ the percentage of times a value occurs – e.g. the percentage of females in the dataset is 45.5%.
Bar chart	**Bar charts** are used to display frequency distribution of categorical data.
Describing distributions of categorical variables	For a small number of categories, the distribution of a categorical variable is described in terms of the **dominant category** (if any), the *order* of occurrence of each category, and its *relative importance*.
Mode, modal category	The **mode** (or modal interval) is the value of a variable (or the interval of values) that occurs most frequently.
Histogram	A **histogram** is used to display the frequency distribution of a numerical variable. It is suitable for medium- to large-sized datasets.
Describing the distribution of a numerical variable	The distribution of a numerical variable can be described in terms of: ▪ shape: symmetric or skewed (positive or negative) ▪ outliers: values that appear to stand out ▪ centre: the midpoint of the distribution (median) ▪ spread: one measure is the range of values covered (range = largest value – smallest value).
Log scales	**Log scales** can be used to transform a skewed histogram to symmetry.

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Review

Skills check

Having completed this chapter, you should be able to:

- differentiate between categorical data and numerical data
- differentiate between nominal and ordinal categorical data
- differentiate between discrete and continuous numerical data
- interpret the information contained in a frequency table
- identify and interpret the mode
- construct a bar chart, segmented bar chart or histogram from a frequency table
- read and interpret a histogram with a log scale.

Multiple-choice questions

The following information relates to Questions 1 and 2.

A survey collected information about the number of cars owned by a family and the car size (small, medium, large).

1 The variables *number of cars* owned and *car size* (small, medium, large) are:

A both categorical variables
B both numerical variables
C a categorical and a numerical variable respectively
D a numerical and a categorical variable respectively
E a nominal and a discrete variable respectively

2 The variables *head diameter* (in cm) and *sex* (male, female) are:

A both categorical variables
B both numerical variables
C an ordinal and a nominal variable respectively
D a discrete and a nominal variable respectively
E a continuous and a nominal variable respectively

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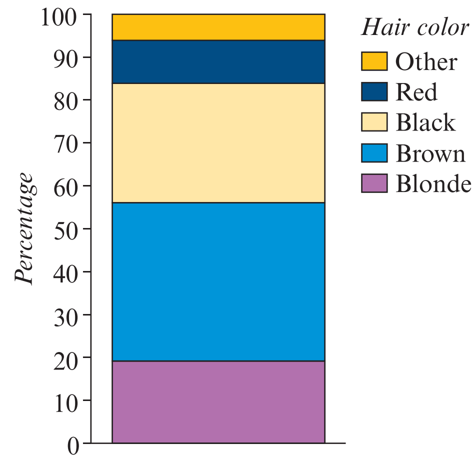
The following information relates to Questions 3 and 4.

The percentage segmented bar chart shows the distribution of hair colour for 200 students.



3 The number of students with brown hair is closest to:

A 4 **B** 34 **C** 57
D 70 **E** 114

4 The most common hair colour is:

A black **B** blonde
C brown **D** red

Questions 5 to 8 relate to the two-way frequency table below.

A group of 189 healthy middle-aged adults were asked whether or not they were currently on a diet. Their responses by sex are summarised in the two-way frequency table below.

	Sex		
Diet	*Male*	*Female*	*Total*
Yes	31	45	76
No	47	66	113
Total	78	111	189

5 The total number of females in the group is:

A 76 **B** 78 **C** 111
D 113 **E** 189

6 The number of males who said they were on a diet is:

A 31 **B** 45 **C** 47 **D** 66 **E** 78

7 The percentage of females *not* on a diet is closest to:

A 39.7% **B** 41.5% **C** 59.5% **D** 60.3% **E** 66.0%

8 The percentage of people on a diet who were male is:

A 39.7% **B** 40.8% **C** 41.5% **D** 58.4% **E** 76.0%

Questions 9 to 13 relate to the histogram shown below.

The histogram opposite displays the test scores of a class of students.

9 The number of students is:

A 6 **B** 18 **C** 20
D 21 **E** 22

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Review

10 The number of students in the class who obtained a test score less than 14 is:

A 4 **B** 10 **C** 14 **D** 16 **E** 28

11 The histogram is best described as:

A negatively skewed
B negatively skewed with an outlier
C approximately symmetric
D approximately symmetric with outliers
E positively skewed

12 The centre of the distribution lies in the interval:

A 8–10 **B** 10–12 **C** 12–14 **D** 14–16 **E** 18–20

13 The spread of the students' marks is closest to:

A 8 **B** 10 **C** 12 **D** 20 **E** 22

14 $\log_{10} 100$ equals:

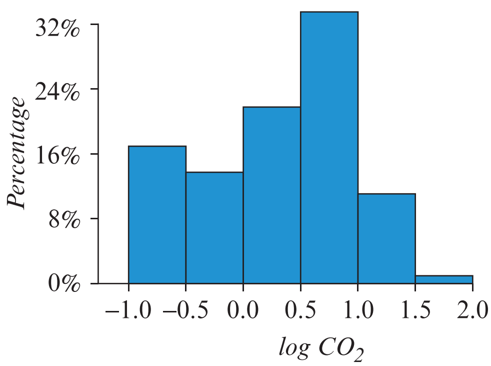
A 0 **B** 1 **C** 2 **D** 3 **E** 100

15 Find the number whose log is 2.314; give the answer to the nearest whole number.

A 2 **B** 21 **C** 206 **D** 231 **E** 20606

The following information relates to Questions 16 and 17.

The percentage histogram opposite displays the distribution of the log of the annual per capita CO_2 emissions (in tonnes) for 192 countries in 2011.



16 Australia's per capita CO_2 emissions in 2011 were 16.8 tonnes. In which column of the histogram would Australia be located?

A –0.5 to <0 **B** 0 to <0.5 **C** 0.5 to <1 **D** 1 to <1.5 **E** 1 to <1.5

17 The percentage of countries with per capita CO_2 emissions of under 10 tonnes is closest to:

A 14% **B** 17% **C** 31% **D** 69% **E** 88%

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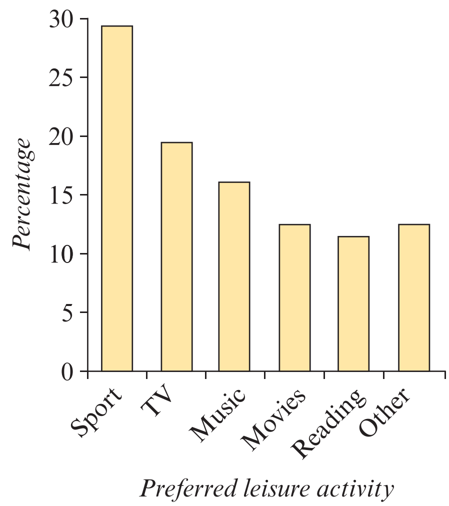
Extended-response questions

1 One hundred and twenty-one students were asked to identify their preferred leisure activity. The results of the survey are displayed in a bar chart.



a What percentage of students nominated watching TV as their preferred leisure activity?

b What percentage of students in total nominated either going to the movies or reading as their preferred leisure activity?

c What is the most popular leisure activity for these students? How many rated this activity as their preferred activity?

2 A group of 52 teenagers were asked, 'Do you agree that the use of marijuana should be legalised?' Their responses are summarised in the table.

a Construct a properly labelled and scaled frequency bar chart for the data.

b Complete the table by calculating the percentages, to one decimal place.

	Frequency	
Legalise	*Number*	*Percentage*
Agree	18	
Disagree	26	
Don't know	8	
Total	52	

c Use the percentages to construct a percentage segmented bar chart for the data.

d Use the frequency table to help you complete the following report.

Report

In response to the question, 'Do you agree that the use of marijuana should be legalised?', 50% of the 52 students ______. Of the remaining students, ______ % agreed, while ______ % said that they ______.

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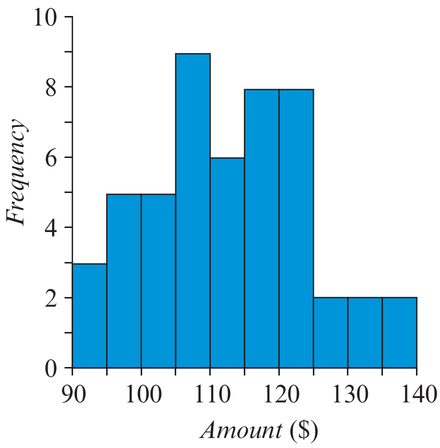
Review

3 Students were asked how much they spent on entertainment each month. The results are displayed in the histogram. Use the histogram to answer the following questions.



a How many students:

i were surveyed?

ii spent \$100–105 per month?

b What is the mode?

c How many students spent \$110 or more per month?

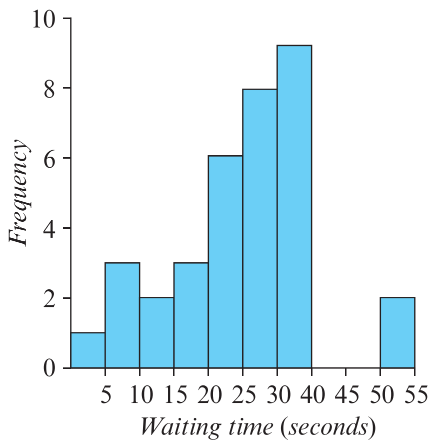
d What percentage spent less than \$100 per month?

e **i** Name the shape of the distribution displayed by the histogram.

ii Locate the interval containing the centre of the distribution.

iii Determine the spread of the distribution using the range.

4 The distribution of the waiting times of 34 cars stopped by a traffic light is shown in the histogram. Use the histogram to write a report on the distribution of waiting times in terms of shape, centre, spread and outliers.



ISBN 978-1-107-56757-3

Core: Data analysis

Chapter 2

2 Summarising numerical data

ISBN 978-1-107-56757-3

2A Dot points and stem plots

Even when we have constructed a frequency table, or a histogram to display a set of numerical data, we are still left with a large amount of information to digest. The way we overcome this problem is to summarise the information and describe the essential features of the distribution with just a few numbers. We call these numbers **summary statistics**.

The two most commonly used types of summary statistics may be classified as:

1 measures of centre (at which point is the distribution centred?)
2 measures of spread (how are the scores in the distribution spread out?).

We encountered the concept of centre and spread in Chapter 1 when we used histograms to describe the distribution of numerical variables. In this chapter, we will look at some more precise ways to define and quantify (give values to) these concepts. To do this, we first need to reintroduce you to the dot plot and the stem plot.

The dot plot

The simplest way to display numerical data is to form a **dot plot**. A dot plot is particularly suitable for displaying discrete numerical data and provides a very quick way to order and display a small dataset.

A dot plot consists of a number line with each data point marked by a dot. When several data points have the same value, the points are stacked on top of each other.

Example 1 Constructing a dot plot

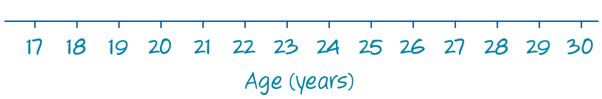
The ages (in years) of the 13 members of a cricket team are:

22 19 18 19 23 25 22 29 18 22 23 24 22

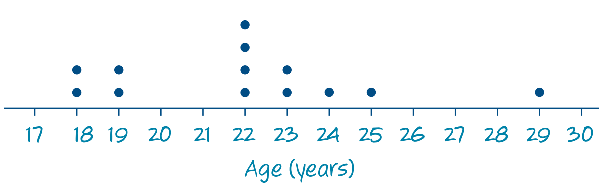
Construct a dot plot.

Solution

1 Draw a number line, scaled to include all data values. Label the line with the variable being displayed.



2 Plot each data value by marking a dot above the corresponding value on the number line.



While some CAS calculators will construct a stem plot, they were designed to be a quick and easy way of ordering and displaying a small dataset by hand.

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▶ The stem plot

The **stem-and-leaf plot**, or **stem plot** for short, is another quick and easy way to display numerical data. Stem plots work well for both discrete and continuous data. They are particularly useful for displaying small- to medium-sized sets of data (up to about 50 data values). Like the dot plot, they are designed to be a pen and paper technique.

In a stem plot, each data value is separated into two parts: its leading digits, which make up the 'stem' of the number, and its last digit, which is called the 'leaf'.

For example, in the stem-and-leaf plot opposite, the data values 21 and 34 are displayed as follows:

	Stem	Leaf
21 is displayed as	2	1
34 is displayed as	3	4

A term 'plot' usually contains a key to show how the numbers in the plot should be interpreted.

Key: 1|2 = 12

0	8
1	2 4 9 9
2	1 1 1 1 2 2 2 3 6 6 9 9
3	2 4
4	4
5	9

Example 2 Constructing a stem plot

University participation rates (%) in 23 countries are given below.

26	3	12	20	36	1	25	26	13	9	26	27
15	21	7	8	22	3	37	17	55	30	1	

Display the data in the form of a stem plot.

Solution

1 The dataset has values in the units, tens, twenties, thirties, forties and fifties. Thus, appropriate stems are 0, 1, 2, 3, 4, and 5. Write these in ascending order, followed by a vertical line.

0	
1	
2	
3	
4	
5	

2 Now attach the leaves. The first data value is '26'. The stem is '2' and the leaf is '6'. Opposite the 2 in the stem, write down the number 6, as shown.

0	
1	
2	6
3	
4	
5	

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3 The second data value is '3' or '03'. The stem is '0' and the leaf is '3'. Opposite the 0 in the stem, write the number 3, as shown.

0	3
1	
2	6
3	
4	
5	

4 Continue systematically working through the data, following the same procedure until all points have been plotted. You will then have the stem plot, as shown.

0	3 1 9 1 7 8 3
1	2 3 5 7
2	6 0 5 6 6 7 1 2
3	6 0 7
4	
5	5

5 To complete the task, write the leaves on each stem in ascending order, then add the variable name and a key.

Rate (%) Key: 1|2 = 12

0	1 1 3 3 7 8 9
1	2 3 5 7
2	0 1 2 5 6 6 6 7
3	0 6 7
4	
5	5

Stem plots with split stems

In some instances, using the simple process outlined above produces a stem plot that is too cramped to give an good overall picture of the variation in the data. This happens when the data values all have the same one or two first digits.

For example, consider the marks obtained by 17 VCE students on a statistics test.

2 12 13 9 18 17 7 16 12 10 16 14 11 15 16 15 17

If we use the process described in Example 2 to form a stem plot, we end up with a 'bunched-up' plot like the one on the right.

0	2 7 9
1	0 1 2 2 3 4 5 5 6 6
	6 7 7 8

We can solve this problem by 'splitting' the stems.

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Generally the stem is split into halves or fifths as shown below.

Key: 1|6 = 16

0	2 7 9
1	0 1 2 2 3 4 5 5 6 6 6 7 7 8

Single stem

Key: 1|6 = 16

0	2
0	7 9
1	0 1 2 2 3 4
1	5 5 6 6 6 7 7 8

Stem split into halves

Key: 1|6 = 16

0	
0	2
0	
0	7
1	0 1
1	2 2 3
1	4 5 5
1	6 6 6 7 7
1	8

Stem split into fifths

The advantage of using a split stem is that it sometimes shows hidden features in the data. For example, the last stem plot reveals that the data are negatively skewed with an outlier. The outlier was not apparent in the original plot.

Exercise 2A

Constructing a dot plot

Note: These exercises are designed to be done by hand.

1 The following data gives the number of rooms in 11 houses.

4 6 7 7 8 4 4 8 8 7 8

a Is the variable *number of rooms* discrete or continuous?

b Construct a dot plot.

2 The following data give the number of children in the families of 14 VCE students:

1 6 2 5 5 3 4 4 2 7 3 4 3 4

a Is the variable *number of children*, discrete or continuous?

b Construct a dot plot.

c Write down the value of the mode. What does the mode represent in the context of the data?

3 The following data give the average life expectancies in years of 13 countries.

76 75 74 74 73 73 75 71 72 75 75 78 72

a Is the variable *life expectancy* discrete or continuous?

b Construct a dot plot.

c Write down the value of the mode. What does the mode represent in the context of the data?

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Constructing a stem plot

4 The data below give the urbanisation rates (%) in 23 countries.

54 99 22 20 31 3 22 9 25 3 56 12
16 9 29 6 28 100 17 99 35 27 12

a Is the variable *urbanisation rate* discrete or continuous?

b Construct a stem plot with an appropriate key.

c Write down the value of the mode. What does the mode represent in the context of the data?

Constructing and stem plot with split stems

5 The data below give the wrist circumference (in cm) of 15 men.

16.9 17.3 19.3 18.5 18.2 18.4 19.9 16.7
17.7 16.5 17.0 17.2 17.6 17.1 17.6

a Is the variable *wrist circumference* discrete or continuous?

b Construct a stem plot for wrist circumference using:

i stems: 16, 17, 18, 19

ii these stems split into halves, that is: 16, 16, 17, 17, . . .

2B The median, range and interquartile range (*IQR*)

The most versatile statistical tools for numerically describing the centre and spread of a distribution are:

- the **median** (the middle value) as its measure of **centre**, and two measures of spread:
- the **range** (the maximum spread of the data values), and
- the **interquartile range** (the spread of the middle half of data values).

While these statistical values could be estimated only approximately from a histogram, they can be determined exactly when we use either a dot or stem plot.

Determining the median

We begin by revisiting the rule for locating the mean of a dataset.

The median

The median is the *middle* value in an ordered dataset.

For n data values the median is located at the $\left(\frac{n+1}{2}\right)$th position.

When:

- n is odd, the median will be the middle data value
- n is even, the median will be the average of the two middle data values.

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Example 3 **Finding the median value in a dataset**

Order each of the following datasets, locate the median, and then write down its value.

a 2 9 1 8 3 5 3 8 1 **b** 10 1 3 4 8 6 10 1 2 9

Solution

a For an odd number of data values, the median will be the *middle* data value.

1 Write down the dataset in order.

1 1 2 3 3 5 8 8 9

1 1 2 3 3 5 8 8 9

2 Locate the middle data value by eye or use the rule.

Median is the $\left(\frac{9+1}{2}\right)$th or fifth value.

3 Write down the median.

Median = 3

b For an even number of data values, the median will be the *average of the two middle data values*.

1 Write down the dataset in order.

1 1 2 3 4 6 8 9 10 10

1 1 2 3 4 6 8 9 10 10

2 Locate the two middle data values and find their average or use the rule.

Median is the $\left(\frac{10+1}{2}\right)$th or 5.5th value.

Write down the median.

Median = $\left(\frac{4+6}{2}\right) = 5$

Note: You should always check that you are correct by counting the number of data values each side of the median. They should be equal.

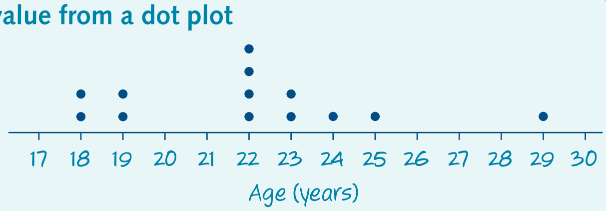
Using a dot plot to help locate medians

The process of calculating a median, as outlined above, is very simple in theory but can be time-consuming in practice. This is because you have to spend most of your time ordering the dataset. For a very large dataset this is a calculator task.

However, even for a reasonably a large dataset, locating a median in a dot or stem plot requires no more than counting because the data are already ordered for you.

Example 4 **Finding the median value from a dot plot**

The dot plot opposite displays the age distribution (in years) of the 13 members of a local cricket team.



Determine the median age of these cricketers and mark its location on the dot plot.

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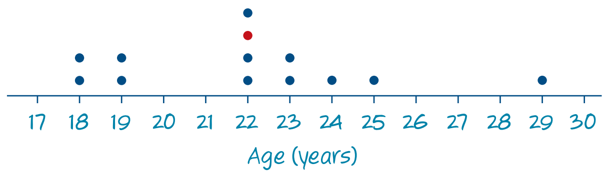
Solution

The median value is the middle data value in the dot plot.

1 Locate the middle data value (or use the rule) and identify it on the dot plot.



2 Write down its value.

Median = 22 years

Example 5 Finding the median value from a stem plot

The stem plot opposite displays the maximum temperature (in °C) for 12 days in January.

Determine the median maximum temperature for these 12 days.

Key: 0|8 = 8°C

1	8 9 9
2	0 2 5 7 8 9 9
3	1 3

Solution

For an even number of data values, as in this example, the median will be the *average of the two middle* data values.

1 Locate the two middle data values in the dot plot by eye (or use the rule) and identify them on the plot.

Key: 0|8 = 8°C

1	8 9 9
2	0 2 5 7 8 9 9
3	1 3

2 Determine the median by finding the average of these two data values.

$$M = \frac{25 + 27}{2} = 26°C$$

Having found the median value in a dot plot or stem plot, we now look at ways of doing the same with the first measure of spread, the range.

▶ The range

The range

The range, R, is the simplest measure of spread of a distribution. It is the difference between the largest and smallest values in the dataset.

R = largest data value − smallest data value

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Example 6 **Finding the range from a stem plot**

The stem plot opposite displays the maximum temperature (in °C) for 12 days in January.

Determine the temperature range over these 12 days.

Key: 0|8 = 8°C

1	8 9 9
2	0 2 5 7 8 9 9
3	1 3

Solution

1 Identify the lowest and highest values in the stem plot and write them down.

Key: 0|8 = 8°C

1	8 9 9
2	0 2 5 7 8 9 9
3	1 3

2 Substitute into the rule for the range and evaluate.

Lowest = 18, highest = 33,
range = 33 − 18 = 15°C

A problem with the range as a measure of spread

The range was the measure of spread we used in Chapter 1 to describe the spread of a histogram. This is because it was easy to estimate.

But because the range depends only on the two extreme values in the data, it is not always an informative measure of spread. For example, one or other of these two values might be an outlier. Furthermore, any data with the same highest and lowest values will have the same range, irrespective of the way in which the data are spread out in between.

A more refined measure of spread that overcomes these limitations of the range is the interquartile range (*IQR*).

The interquartile range (*IQR*)

Quartiles

To understand the *IQR*, you first need to know about quartiles.

Just as the median is the point that divides a distribution in half, **quartiles** are the points that divide a distribution into *quarters*. We use the symbols Q_1, Q_2 and Q_3 to represent the quartiles. Note that the second quartile, Q_2, is the median.

The interquartile range

The interquartile range (*IQR*) is defined as the spread of the middle 50%[1] of data values, so that:

$$IQR = Q_3 - Q_1$$

[1] A practical problem arises when calculating quartiles if the median corresponds to an actual data value. This happens when the number of data values is odd. The question is what to do with the median value when calculating quartiles. The usual strategy is to omit it; this means that there will always be slightly less than 50% of the data values in each 'half' of the distribution. This is the approach we will take. It is also the approach used by CAS calculators.

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To calculate the IQR, it is necessary to first calculate the quartiles Q_1 and Q_3.

In principle, this is straightforward, because:

- Q_1 is the midpoint of the lower half of the data values
- Q_3 is the midpoint of the upper half of the data values.

Again, because the data displayed in a dot or stem plot are automatically ordered, determining quartiles involves little more than counting.

Example 7 Finding quartiles from an ordered stem plot

Use the stem plot to determine the quartiles Q_1 and Q_3, the IQR and the range, R, for life expectancies.

The median life expectancy is $M = 73$.

Stem: 5|2 = 52 years

5	2
5	5 6
6	4
6	6 6 7 9
7	1 2 2 3 3 4 4 4 4
7	5 5 6 6 7 7

Solution

1 Mark the median value, 73, on the stem plot.

2 To find the quartiles, *the median value is excluded*. This leaves 11 values below the median and 11 values above the median. Then:

- Q_1 = midpoint of the bottom 11 data values
- Q_3 = midpoint of the top 11 data values.

Mark Q_1 and Q_3 on the stem plot.
Write these values down.

Stem: 5|2 = 52 years

5	2
5	5 6
6	4
6	6 6 7 9
7	1 2 2 ~~3~~ 3 4 4 4 4
7	5 5 6 6 7 7

Q_1

median

Q_3

3 Determine the IQR using $IQR = Q_3 - Q_1$.

$Q_1 = 66, Q_3 = 75$

$\therefore IQR = Q_3 - Q_1 = 75 - 66$
$= 9$

4 Determine the range using

R = largest data value − smallest data value

$R = 77 - 52 = 25$

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To check that these quartiles are correct, write the data values in order, and mark the median and the quartiles. If correct, the median divides the dataset up into four equal groups.

$$\underbrace{52\ 55\ 56\ 64\ 66}_{5\text{ values}}\ \overset{Q_1}{\mathbf{66}}\ \underbrace{67\ 69\ 71\ 72\ 72}_{5\text{ values}}\ \overset{Q_2\,(=M)}{\mathbf{73}}\ \underbrace{73\ 74\ 74\ 74\ 74}_{5\text{ values}}\ \overset{Q_3}{\mathbf{75}}\ \underbrace{75\ 76\ 76\ 77\ 77}_{5\text{ values}}$$

Why is the *IQR* a more useful measure of spread than the range?

The *IQR* is a measure of spread of a distribution that includes the middle 50% of observations. Since the upper 25% and lower 25% of observations are discarded, the interquartile range is generally not affected by the presence of outliers.

Exercise 2B

Note: The computational exercises in this problem set are designed to build a sound understanding of the concepts of the median, quartile, the *IQR* and range and how these concepts fit together. All of this is lost if you use a calculator to automate the process at this stage. To this end, all of the following exercises have been designed to be done by hand.

Basic ideas

1 Write down in a few words the meaning of the following terms.

a range **b** median **c** quartile **d** interquartile range.

Determining the median from raw data

2 Locate the medians of the following datasets. For each set of data, check that the median divides the ordered dataset into two *equal* groups.

a 4 9 3 1 8 6 **b** 10 9 12 20 14

3 The prices of nine second-hand mountain bikes advertised for sale were as follows.

$650 $3500 $750 $500 $1790 $1200 $2950 $430 $850

What is the median price of these bikes? Check that an equal number of bikes have prices above and below the median.

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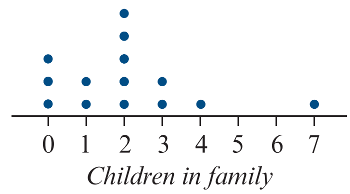
Determining the median, quartiles, *IQR* and range from a dot plot

4 The stem plot shows the distribution of the number of children in each of 14 families.



a Determine the median, M.

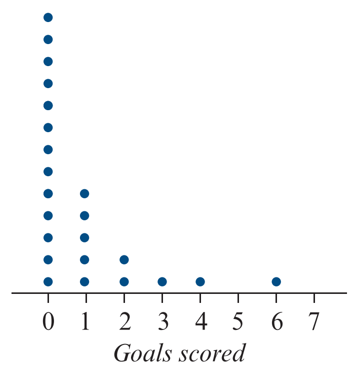
b Determine the quartiles Q_1 and Q_3.

c Calculate the IQR.

d Calculate the range, R.

e By writing the data values in a line, check that the quartiles and the median have divided the dataset up into four equal groups.

5 The stem plot displays the number of goals scored in 23 games.



a Describe the shape of the distribution and note outliers (if any).

b Determine the median, M.

c Calculate the IQR and the range, R.

Determining the median, quartiles, IQR and range from a stem plot

6 The stem plot displays the infant mortality rates (deaths per 1000 live births) in 14 countries.

a Determine the median, M.

b Determine the quartiles Q_1 and Q_3.

c Calculate the IQR and the range, R.

Key: 0|7 = 7

0	
0	7 7 9
1	0 0 0 0 2 2 4
1	5
2	0 1
2	5
3	

7 The stem plot displays the test scores for 20 students.

a Describe the shape of the distribution.

b Determine the median, M.

c Determine the quartiles, Q_1 and Q_3.

d Calculate the IQR and the range, R.

Key: 1|0 = 10

1	0 2
1	5 5 6 9
2	3 3 4
2	5 7 9 9 9
3	0 1 2 4
3	5 9

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8 The stem plot displays the university participation rates (%) in 23 countries.

a Determine the median, M.

b Determine the quartiles Q_1 and Q_3.

c Calculate the IQR and the range, R.

Key: 0|1 = 1

0	1 1 3 3 7 8 9
1	2 3 5 7
2	0 1 2 5 6 6 6 7
3	0 6 7
4	
5	5
6	

2C The five-number summary and the box plot

The five-number summary

Knowing the median and quartiles tells us quite a lot about the centre and spread of the distribution. If we also knew something about the tails (ends) we would have a good picture of the whole distribution. This can be achieved by recording the smallest and largest values of the dataset. Putting all this information together gives the **five-number summary**.

Five-number summary

A listing of the median, M, the quartiles Q_1 and Q_3, and the smallest and largest data values of a distribution, written in the order

$$\text{minimum}, \ Q_1, \ M, \ Q_3, \ \text{maximum}$$

is known as a *five-number summary*.

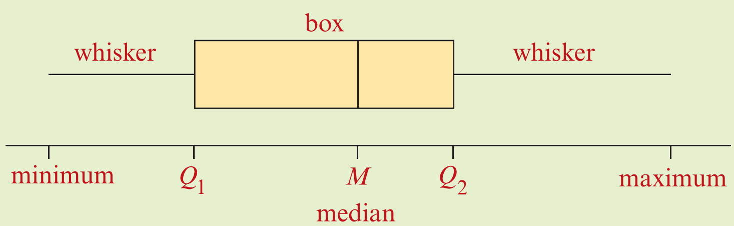
The five-number summary is the starting point for constructing one of the most useful graphical tools in data analysis, the box plot.

The box plot

The **box plot** (or box-and-whisker plot) is a graphical display of a five-number summary. The essential features of a box plot are summarised below.

The box plot

A box plot is a graphical display of a five-number summary.



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In a box plot:

- a box extends from Q_1 to Q_3, locating the middle 50% of the data values
- the median is shown by a vertical line drawn within the box
- lines (called whiskers) are extended out from the lower and upper ends of the box to the smallest and largest data values of the dataset respectively.

Example 8 Constructing a box plot from a five-number summary

The stem plot shows the distribution of life expectancies (in years) in 23 countries.

The five-number summary for these data is:

minimum	52
first quartile (Q_1)	66
median (M)	73
third quartile (Q_3)	75
maximum	77

Key: 5|2 = 52 years

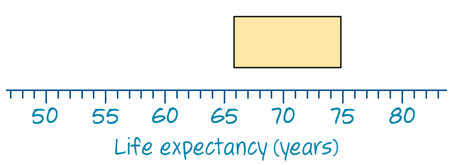
5 | 2 ← minimum
5 | 5 6
6 | 4
Q_1
6 | 6 6 7 9 median
7 | 1 2 2 3 3 4 4 4 4
Q_3
7 | 5 5 6 6 7 7 ← maximum

Use the five-number summary to construct a box plot.

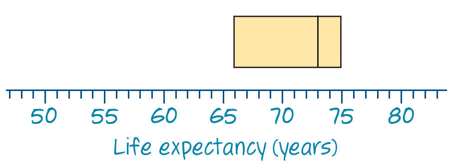
Solution

1 Draw a labelled and scaled number line that covers the full range of values.

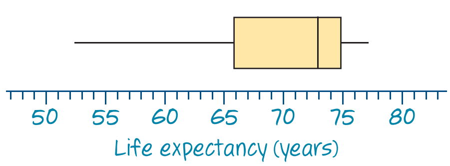
2 Draw a box starting at $Q_1 = 66$ and ending at $Q_3 = 75$.



3 Mark the median value with a vertical line segment at $M = 73$.



4 Draw the whiskers: lines joining the midpoint of the ends of the box to the minimum and maximum values, 52 and 77.



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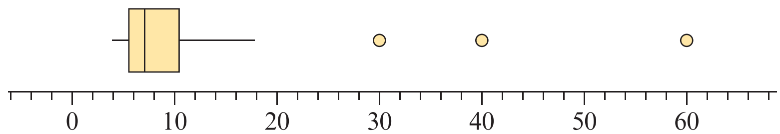
Box plots with outliers

Box plots can also be used to identify and display outliers.

An example of a box plot with outliers is shown opposite.



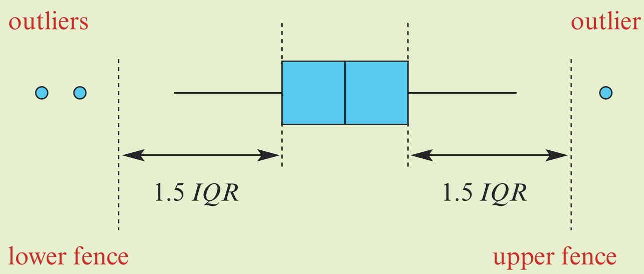
Three of the data values 30, 40, and 60 are possible outliers.

To display outliers on a box plot, we must first determine the location of what we call the **upper** and **lower fences**. These are imaginary lines drawn one and a half interquartile ranges (or box widths) above and below the box ends, as shown in the diagram below. Data values outside these fences are then classified as possible outliers and plotted separately.

Using a box plot to display outliers

In a box plot, possible outliers are defined as being those values that are:

- greater than $Q_3 + 1.5 \times IQR$ (upper fence)
- less than $Q_1 + 1.5 \times IQR$ (lower fence).



When drawing a box plot, any observation identified as an outlier is shown by a dot. The whiskers end at the smallest and largest values that are not classified as outliers.

Two new things to note in a box plot with outliers are that:

1. any points more than 1.5 IQRs away from the end of the box are classified as possible outliers (possible, in that it may be that they are just part of a distribution with a very long tail and we do not have enough data to pick up other values in the tail)
2. the whiskers end at the highest and lowest data values that are less than 1.5 IQRs from the ends of the box.

While we have used a five-number summary as the starting point for our introduction to box plots, in practice the starting point for constructing a box plot is raw data. Constructing a box plot from raw data is a task for your CAS calculator.

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How to construct a box plot with outliers using the TI-Nspire CAS

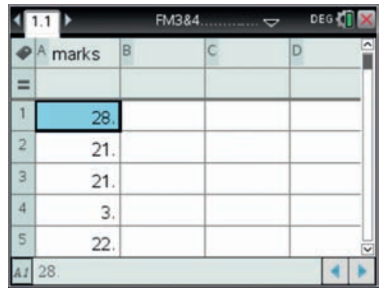
Display the following set of 19 marks in the form of a box plot with outliers.

28	21	21	3	22	31	35	26	27	33
43	31	30	34	48	36	35	23	24	

Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**. Enter the data into a list called ***marks*** as shown.

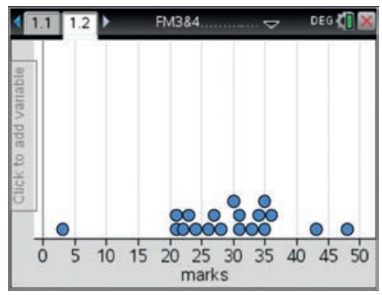


3 Statistical graphing is done through the **Data & Statistics** application.

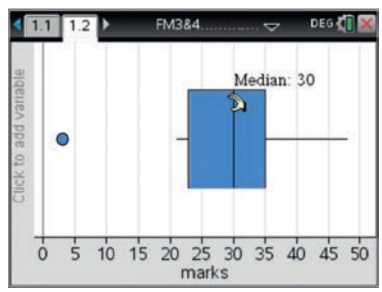
Press ctrl + I and select **Add Data & Statistics**.

Note: A random display of dots will appear – this indicates that list data are available for plotting. Such a dot is not a statistical plot.

a Click on the **Click to add variable** on the x-axis and select the variable ***marks***. A dot plot is displayed by default as shown opposite.



b To change the plot to a box plot press menu>**Plot Type**>**Box Plot**. Your screen should now look like that shown opposite.



4 Data analysis

Key values can be read from the box plot by moving the cursor over the plot or using menu>**Analyze>Graph Trace**.

Starting at the far left of the plot, we see that the:

- minimum value is 3 (an outlier)
- first quartile is 23 (**Q_1 = 23**)
- median is 30 (**Median = 30**)
- third quartile is 35 (**Q_3 = 35**)
- maximum value is 48.

How to construct a box plot with outliers using the ClassPad

Display the following set of 19 marks in the form of a box plot with outliers.

28	21	21	3	22	31	35	26	27	33
43	31	30	34	48	36	35	23	24	

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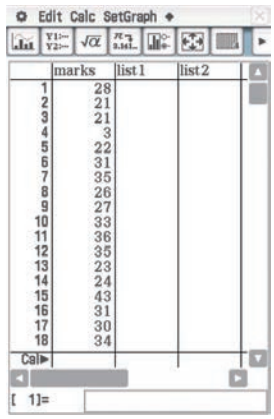
Steps

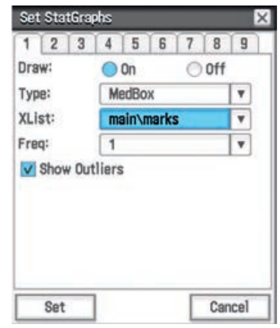
1 Open the **Statistics** application (Statistics) and enter the data into the column labelled **marks**.



2 Open the **Set StatGraphs** dialog box by tapping the icon in the toolbar. Complete the dialog box as shown below.

- **Draw**: select **On**.
- **Type**: select **MedBox** (▼).
- **XList**: select **main\marks** (▼).
- **Freq**: leave as **1**.

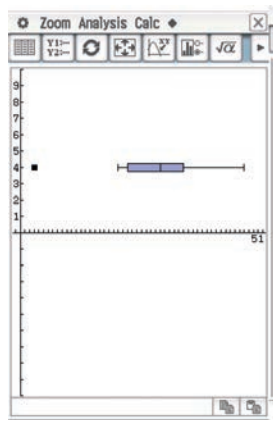
Tap the **Show Outliers** box to tick (☑).

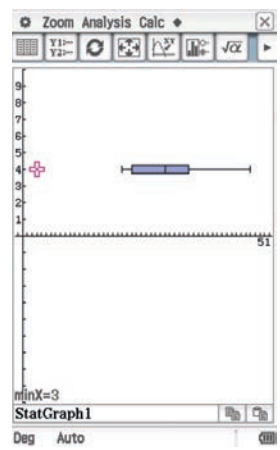


3 Tap **Set** to confirm your selections and plot the box plot b tapping the icon. The graph is drawn in an automatically scaled window, as shown.

4 Tap the Resize icon at the bottom of the screen for a full-screen graph.

Note: If you have more than one graph on your screen, tap the data screen, select **StatGraph** and turn off any unwanted graphs.

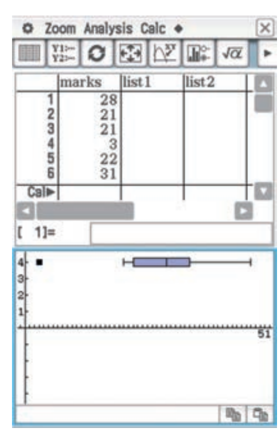




5 Tap the icon to read key values. This places a marker on the box plot (+), as shown. Use the horizontal cursor arrows (◀) and (▶) to move from point to point on the box plot.

We see that the:

- minimum value is 3 (**minX** = **3**; an outlier)
- first quartile is 23 (**Q_1** = **23**)
- median is 30 (**Med** = **30**)
- third quartile is 35 (**Q_3** = **35**)
- maximum value is 48 (**maxX** = **48**).



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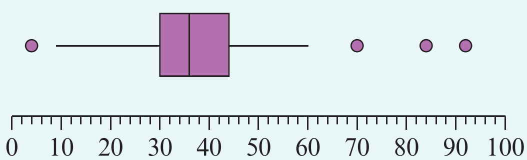
Interpreting box plots

Constructing a box plot is not an end in itself. The prime reason to construct box plots is to help us answer statistical questions. To do this, you need to know how to read values from a box plot and use them to determine statistics such as the median, the interquartile range and the range. We also use box plots to identify possible outliers.

Skillsheet

Example 9 Reading values from a box plot

For the box plot shown, write down the values of:



a the median

b the quartiles Q_1 and Q_3

c the interquartile range (IQR)

d the minimum and maximum values

e the values of any possible outliers

f the smallest value in the upper end of the dataset that will be classified as an outlier

g the largest value in the lower end of the dataset that will be classified as an outlier.

Solution

a The median (the vertical line in the box) — $M = 36$

b Quartiles Q_1 and Q_3 (end points of box) — $Q_1 = 30$, $Q_3 = 44$

c Interquartile range ($IQR = Q_3 - Q_1$) — $IQR = Q_3 - Q_1 = 44 - 30 = 14$

d Minimum and maximum values (extremes) — Min $= 4$, Max $= 92$

e The values of the possible outliers (dots) — 4, 70, 84 and 92 are possible outliers

f Upper fence (given by $Q_3 + 1.5 \times IQR$)

$$\text{Upper fence} = Q_3 + 1.5 \times IQR$$
$$= 44 + 1.5 \times 14 = 65$$

Any value above 65 is an outlier.

g Lower fence (given by $Q_1 - 1.5 \times IQR$)

$$\text{Lower fence} = Q_3 - 1.5 \times IQR$$
$$= 30 - 1.5 \times 14 = 9$$

Any value below 9 is an outlier.

Once we know the location of the quartiles, we can use the box plot to estimate percentages.

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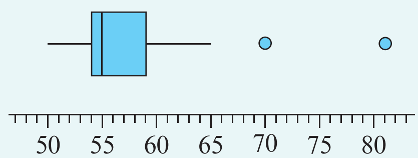
Example 10 Estimating percentages from a box plot

For the box plot shown, estimate the percentage of values:



a less than 54 **b** less than 55
c less than 59 **d** greater than 59
e between 54 and 59 **f** between 54 and 86.

Solution

a 54 is the first quartile (Q_1); 25% of values are less than Q_1. — **a** 25%

b 55 is the median or second quartile (Q_2); 50% of values are less than Q_1. — **b** 50%

c 59 is the third quartile (Q_3); 75% of values are less than Q_1. — **c** 75%

d 75% of values are less than 59 and 25% are greater than 59. — **d** 25%

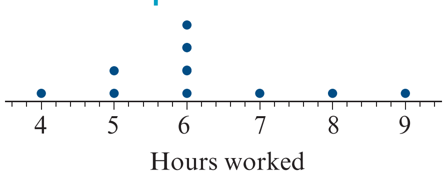
e As 75% of values are less than 59 and 25% are less than 54, 50% of values are between 54 and 59. — **e** 50%

f As 100% of values are less than 86 and 25% of values are less than 54, 75% of values are between 54 and 86. — **f** 75%

Exercise 2C

Constructing a five-number summary for a dot or stem plot

1 Construct a five-number summary for the dot plot opposite.



2 Construct a five-number summary for the stem plot opposite.

Key: 13|6 = 136

13	6 7
14	3 6 8 8 9
15	2 5 8 8 8
16	4 5 5 6 7 9
17	8 8 9
18	2 9

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Constructing a box plot from a five-number summary

3 Use the following five-number summaries to construct box plots.

a Min = 1, $Q_1 = 4$, $M = 8$, $Q_3 = 13.5$, max = 24

b Min = 136, $Q_1 = 148$, $M = 158$, $Q_3 = 169$, max = 189

Constructing a box plot from raw data using a CAS calculator

4 University participation rates (%) in 21 countries are listed below.

3	3	7	8	9	12	13	15	17	20	21
22	25	26	26	26	27	30	36	37	55	

a Use a graphics calculator to construct a box plot with outliers for the data. Name the variable *unirate*.

b Use the box plot to construct a five-number summary. Identify the outliers.

5 The reaction times (in milliseconds) of 18 people are listed below.

38 36 35 35 43 46 42 64 40 48 35 34 40 44 30 25 39 31

a Use a graphics calculator to construct a box plot with outliers for the data. Name the variable *rtime*.

b Use the box plot to construct a five-number summary. Identify the outlier.

Reading values from a box plot

6 For each of the box plots below, estimate the values of:

i the median, M

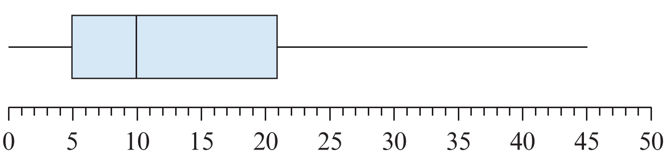
ii the quartiles Q_1 and Q_3

iii the interquartile range, IQR

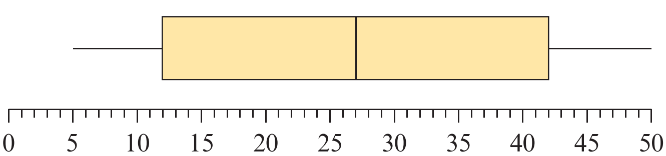
iv the minimum and maximum values

v the values of possible outliers.

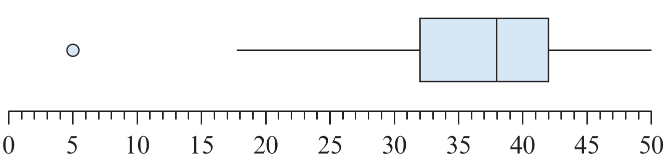
a



b



c



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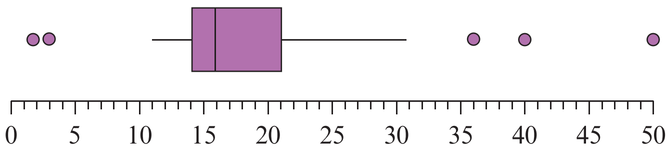
d

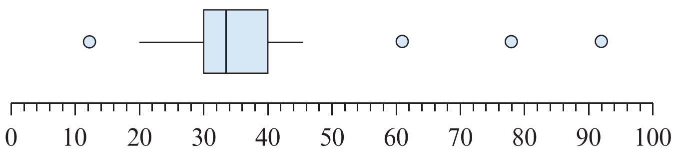


Locating the upper and lower fences in a box plot

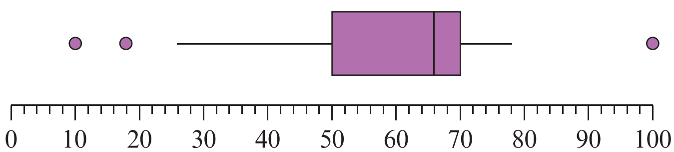
7 For the box plots below, determine the location of:

i the upper fence **ii** the lower fence.

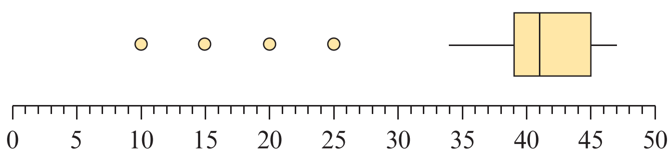
a



b



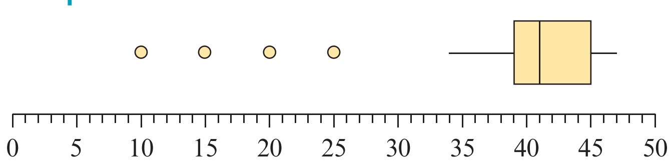
8 **a** Determine the lower fence for the box plot opposite.



b When the data were originally entered, a value of 31 was incorrectly entered as 35. Would the 31 be shown as an outlier when the error is corrected? Explain your answer.

Reading percentages from a box plot

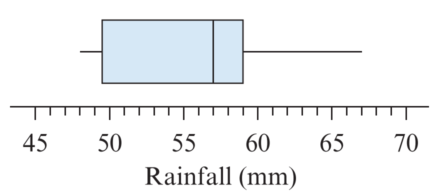
9 Use the box plot opposite to estimate the percentage of values that are:



a less than 38 **b** less than 45 **c** greater than 45

d between 39 and 45 **e** between 5 and 45.

10 The box plot displays the monthly rainfall (in mm) for 12 months.
Use the box plot to estimate the percentage of months in which the monthly rainfall was:



a greater than 59 mm **b** less than 49.5 mm **c** between 49.5 and 59 mm

d between 57 and 59 mm **e** less than 59 mm **f** between 57 and 70 mm.

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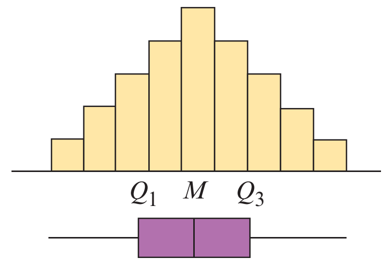
2D Relating a box plot to shape

An almost infinite variety of quantities can be analysed statistically. However, the types of distributions that arise in practice fall into a relatively small number of characteristic forms or shapes, namely symmetric or non-symmetric, and, if single-peaked, positively or negatively skewed. Not surprisingly, each of these shapes has quite distinct box plots.

A symmetric distribution

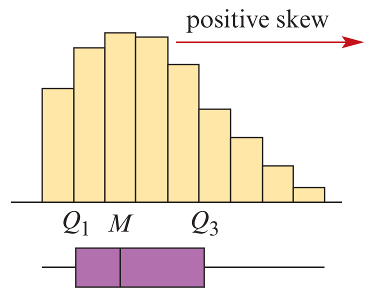
A *symmetric distribution* tends to be centred on its median and have values evenly spread around the median. As a result, its box plot will also be symmetric, its median is close to the middle of the box and its whiskers are approximately equal in length.



Positively skewed distributions

Positively skewed distributions are characterised by a cluster of data values around the *median at the left-hand end* of the distribution with a *gradual tailing off to the right*.

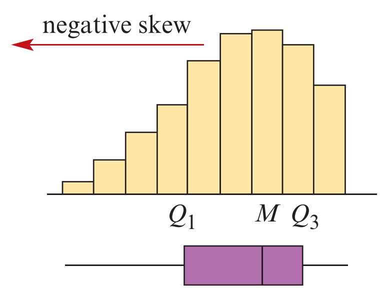
As a result, the box plot of a *positively skewed* distribution will have its *median* off-centre and to the *left-hand side* of its box. The *left-hand whisker* will be *short*, while the *right-hand whisker* will be **long**, reflecting the gradual tailing off of data values to the right.



Negatively skewed distributions

Negatively skewed distributions are characterised by a clustering of data values around the median at the right-hand end of the distribution, with a *gradual tailing off of data values to the left*.

As a result, the box plot of a *negatively skewed* distribution has the *median* off-centre and in the *right-hand side* of its box. The *right-hand whisker* will be *short*, while the *left-hand whisker* will be *long*, reflecting the gradual tailing off of data values to the left.



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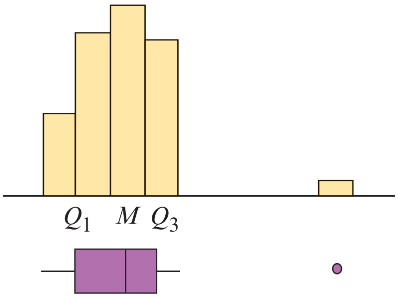
Distributions with outliers

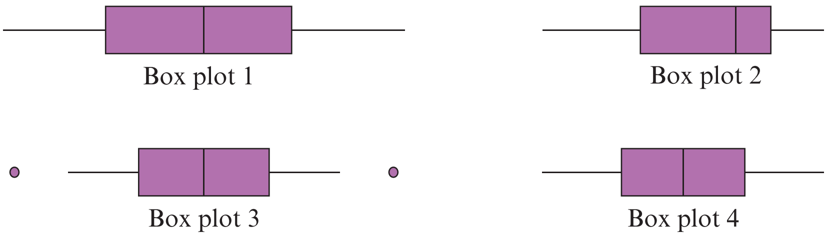
Distributions with outliers are characterised by large *gaps* between the main body and data values in the tails. The histogram opposite displays a distribution with an outlier. In the corresponding box plot, the *box and whiskers* represent the *main body of data* and the *dot*, separated by a gap from the box and whiskers, an *outlier*.

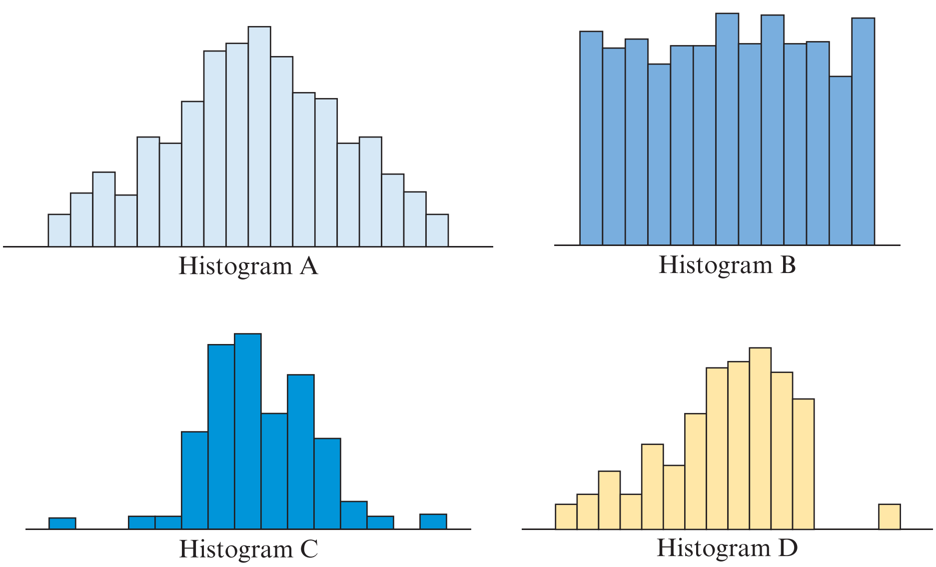


Exercise 2D

Matching histograms to box plots

Match these box plots with their histograms.





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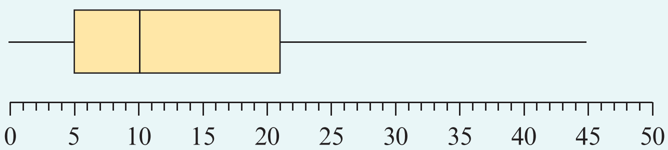
2E Using box plots to describe and compare distributions

Because of the wealth of information contained in a box plot, it is an extremely powerful tool for describing a distribution in terms of shape,[2] centre and spread. They can also be used to help us identify outliers.

Example 11[3] Using a box plot to describe a distribution without outliers

Describe the distribution represented by the box plot in terms of shape, centre and spread. Give appropriate values.

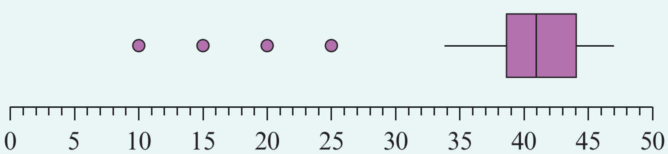


Solution

The distribution is positively skewed with no outliers. The distribution is centred at 10, the median value. The spread of the distribution, as measured by the IQR, is 16 and, as measured by the range, 45.

Example 12 Using a box plot to describe a distribution with outliers

Describe the distributions represented by the box plot in terms of shape and outliers, centre and spread. Give appropriate values.



Solution

The distribution is symmetric but with outliers. The distribution is centred at 41, the median value. The spread of the distribution, as measured by the IQR, is 6 and, as measured by the range, 37. There are four outliers: 10, 15, 20 and 25.

---

2 Beware: identifying the shape of a distribution from of a box plot constructed with only a small amount of data (less than 20–30 data values) can be extremely problematic. This is because the box in the box plot can hide critical detail around the peak of the distribution. If a box plot looks uninterpretable in terms of shape, use a dot plot to check.

3 While not specifically stated, for all of the examples in this book, it can be assumed that the datasets are large enough to allow for valid conclusions to be drawn about the shape of a distribution from its box plot.

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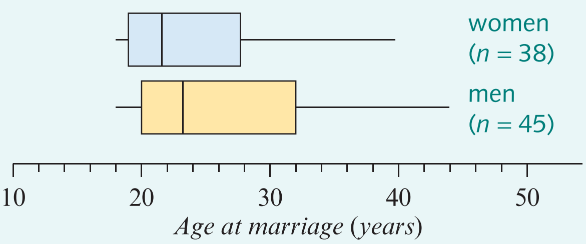
Example 13 **Using a box plot to compare distributions**

The parallel box plots show the distribution of ages of 45 men and 38 women when first married.



a Compare the two distributions in terms of shape (including outliers, if any), centre and spread. Give appropriate values at a level of accuracy that can be read from the plot.

b Comment on how the age of the men when first married compares to that of women.

Solution

a The distributions of age at marriage are positively skewed for both men and women. There are no outliers. The median age at marriage is higher for men ($M = 23$ years) than women ($M = 21$ years). The IQR is also greater for men ($IQR = 12$ years) than women ($IQR = 9$ years). The range of age at marriage is also greater for men ($R = 26$ years) than women ($R = 22$ years).

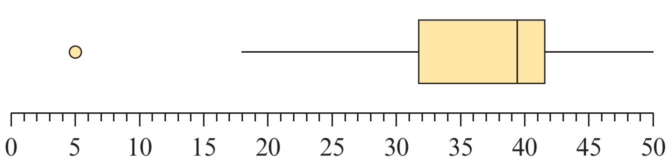
b For this group of men and women, the men, on average, married later and the age at which they first married was more variable.

Exercise 2E

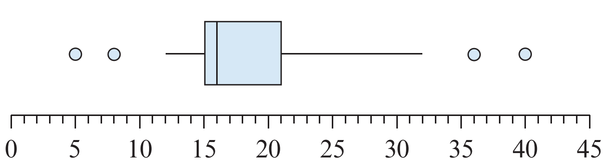
Using box plots to describe distributions in terms of shape, centre and spread

1 Describe the distributions represented by the following box plots in terms of shape, centre, spread and outliers (if any). Give appropriate values.

a



b



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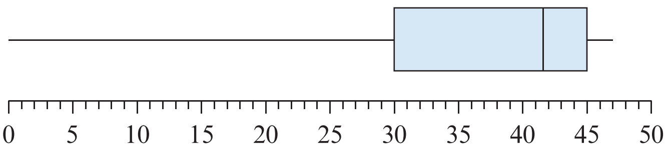
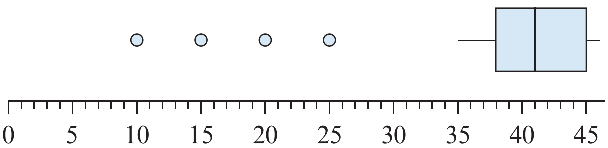
c

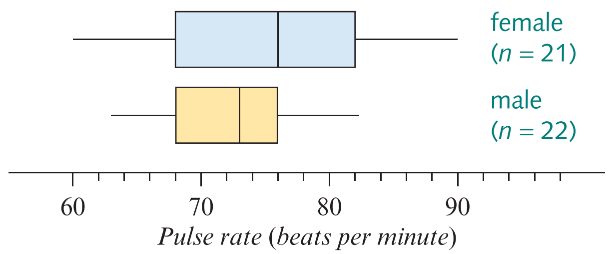


d



Using box plots to compare groups in terms of shape, centre and spread

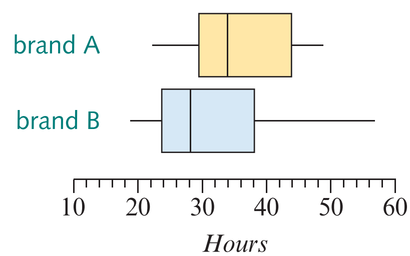
2 The parallel box plots show the distribution of pulse rates of 21 females and 22 males.



a Compare the two distributions in terms of shape, centre and spread. Give appropriate values at a level of accuracy that can be read from the plot.

b Comment on how the pulse rates of females compare to the pulse rates of males for the data.

3 The lifetimes of two different brands of batteries were measured and the results displayed in the form of parallel box plots.



a Compare the two distributions in terms of shape (including outliers), centre and spread. Give appropriate values at a level of accuracy that can be read from the plot.

b Comment on how the lifetime of brand A compares to the lifetime of brand B batteries for the data.

2F Describing the centre and spread of symmetric distributions

So far, we have looked at methods for describing the centre and spread for distributions of any shape. We used the median, *IQR* and range for this purpose. In this section, we will look alternative measures of centre (the mean) and spread (the standard deviation) that are only useful when working with symmetric distributions without outliers. While this may seem unnecessarily restrictive, these two measures have the advantage of being able to fully describe the centre and spread of a symmetric distribution with only two numbers.

ISBN 978-1-107-56757-3

The mean

The **mean** of a set of data is what most people call the 'average'. The mean of a set of data is given by:

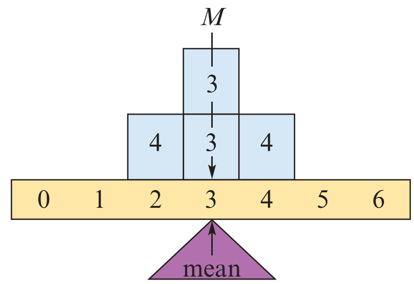
$$\text{mean} = \frac{\text{sum of data values}}{\text{total number of data values}}$$

For example, consider the set of data:

2 3 3 4

The mean of this set of data is given by:

$$\text{mean} = \frac{2+3+3+4}{4} = \frac{12}{4} = 3$$



From a pictorial point of view, the mean is the *balance point* of a distribution (see above).

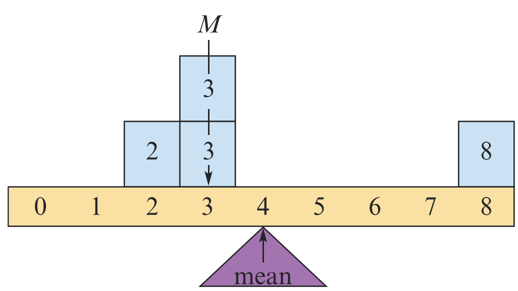
Note that in this case, the mean and the median coincide; the balance point of the distribution is also the point that splits the distribution in half; that is, there are two data points to the left of the mean and two to the right. This is a general characteristic of *symmetric* distributions.

However, consider the dataset:

2 3 3 8

The median remains at $M = 3$, but:

$$\text{mean} = \frac{2+3+3+8}{4} = \frac{16}{4} = 4$$



Note that the mean is affected by changing the largest data value but that the median is not.

Some notation

Because the rule for the mean is relatively simple, it is easy to write in words. However, later you will meet other rules for calculating statistical quantities that are extremely complicated and hard to write out in words.

To overcome this problem, we introduce a shorthand notation that enables complex statistical formulas to be written out in a compact form. In this notation, we use:

- the Greek capital letter sigma, $\sum$, as a shorthand way of writing 'sum of'
- a lower case x to represent a data value
- a lower case x with a bar, $\bar{x}$ (pronounced '*x bar*'), to represent the mean of the data values
- an n to represent the total number of data values.

The rule for calculating the mean then becomes: $\bar{x} = \dfrac{\sum x}{n}$

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Example 14 **Calculating the mean from the formula**

The following is a set of reaction times (in milliseconds): 38 36 35 43 46 64 48 25

Write down the values of the following, correct to one decimal place.

a n **b** $\sum x$ **c** $\bar{x}$

Solution

a n is the number of data values. $n = 8$

b $\sum x$ is the sum of the data values.

$$\sum x = 38 + 36 + 35 + 43 + 46 + 64 + 48 + 25$$
$$= 335$$

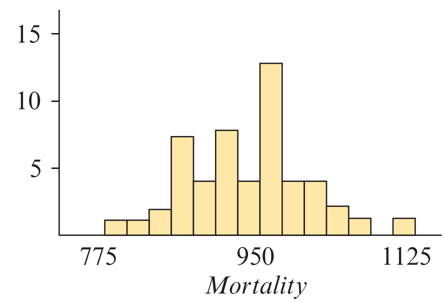
c $\bar{x}$ is the mean. It is defined by $\bar{x} = \dfrac{\sum x}{n}$.

$$\bar{x} = \frac{\sum x}{n} = \frac{335}{8} = 41.9$$

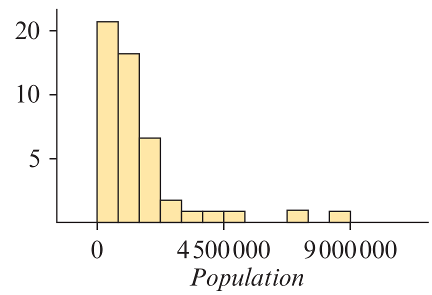
The relationship between the mean and the median

Whereas the *median* lies at the *midpoint* of a distribution, the *mean* is the *balance point* of the distribution. For approximately symmetric distributions, both the median and mean will be approximately equal in value.

An example of a *symmetric distribution* is the distribution of mortality rates for 60 US cities shown opposite. Calculations reveal that the mean mortality rate for the cities is 940 per 100 000 while the median mortality rate is 944 per 100 000 people. As expected, the mean and median are approximately equal in value.



An example of a highly *skewed distribution* is the population distribution of different cities, shown opposite. This distribution is clearly positively skewed with two outliers. The mean population is 1.4 million, while the median population is 0.9 million. They are quite different in value. The mean has been pulled away from the body of the data by the extreme values in the tail and no longer represents the typical city.



When to use the median rather than the mean

Because the value of the *median* is relatively unaffected by the presence of extreme values in a distribution, it is said to be a *resistant* statistic. For this reason, the median is frequently used as a measure of centre when the distribution is known to be clearly *skewed* and/or likely to contain *outliers*.

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For example, median house prices are used to compare housing prices between capital cities in Australia because the distribution of house prices tends to be positively skewed. There are always a small number of very expensive houses sold for much higher prices than the rest of houses sold.

However, if a distribution is symmetric, there will be little difference in the value of the mean and median and we can use either. In such circumstances, the mean is often preferred because:

- it is more familiar to most people
- more can be done with it theoretically, particularly in the area of statistical inference (which you will learn about if you are doing Mathematics Methods).

Choosing between the mean and the median

The *mean* and the *median* are both measures of the *centre* of a distribution. If the distribution is:

- *symmetric* and there are no outliers, either the *mean* or the *median* can be used to indicate the centre of the distribution
- clearly *skewed* and/or there are *outliers*, it is more appropriate to use the *median* to indicate the *centre* of the distribution.

Exercise 2F-1

Basic ideas

1 **a** Which statistic, the median or the mean, always divides a distribution in half?

b In what shaped data distributions do the mean and median have the same value?

c Which is most affected by outliers, the mean or the median?

d Which would be the most appropriate measure of the typical salary of adult workers in Australia, the mean salary or the median salary? Why?

Determining the mean, median and mode from raw data

2 For each of the following datasets, write down the value of n, the value of Σx and hence evaluate $\bar{x}$.

a 2 5 2 3 **b** 12 15 20 32 25 **c** 2 1 3 2 5 3 5

3 Calculate the mean and locate the median and modal value(s) of the following scores.

a 1 3 2 1 2 6 4 5 4 3 2

b 3 12 5 4 3 2 6 5 4 5 5 6

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A challenge

4 Find a set of five whole numbers for which:

a the mean = 4, the median = 3 and the mode = 3

There are several answers – can you find them all?

b the mean = the median = the mode = the range = a single digit number.

Choosing between the mean and the median

5 The temperature of a hospital patient (in degrees Celsius) taken at 6-hourly intervals over 2 days was as follows.

35.6 36.5 37.2 35.5 36.0 36.5 35.5 36.0

a Calculate the patient's mean and median temperature over the 2-day period.

b What do these values tell you about the distribution of the patient's temperature?

6 The amounts (in dollars) spent by seven customers at a corner store were:

0.90 0.80 2.15 16.55 1.70 0.80 2.65

a Calculate the mean and median amount spent by the customers.

b Does the mean or the median give the best indication of the typical amount spent by customers? Explain your answer.

7 For which of the following variables might you question using the mean as a measure of the centre of the distribution? Justify your selection.

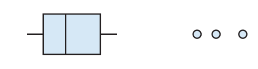
a

Life expectancy (Asia)

b

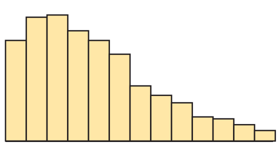
Life expectancy (Europe)

c



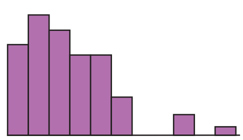
Fuel consumption (cars)

d



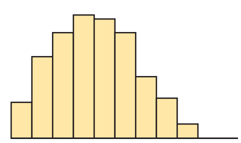
Age distribution in a country

e



Urban car accident rates

f



Blood cholesterol levels

8 The stem plot shows the distribution of weights (in kg) of 23 footballers.

a Name the shape of the distribution. Which measure of centre, the mean or the median, do you think would best indicate the typical weight of these footballers?

b Determine both the mean and median to check your prediction.

Weight (kg)

6	9
7	0 2
7	6 6 7 8
8	0 0 1 2 3 3 4
8	5 5 5 6 9
9	1 2
9	8
10	3

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The standard deviation

To measure the spread of a data distribution around the *median* (M) we use the **interquartile range** (IQR). To measure the spread of a data distribution about the *mean* ($\bar{x}$) we use the **standard deviation** (s).

The standard deviation

The formula for the standard deviation, s, is: $s = \sqrt{\dfrac{\sum(x - \bar{x})^2}{n - 1}}$

Although not easy to see from the formula, the standard deviation is an average of the squared deviations of each data value from the mean. We work with the squared deviations because the sum of the deviations around the mean (the balance point) will always be zero.

Calculating the standard deviation

Normally, you will use your calculator to determine the value of a standard deviation. Instructions for the TI-Nspire or ClassPad follow.

How to calculate the mean and standard deviation using the TI-Nspire CAS

The following are the heights (in cm) of a group of women.

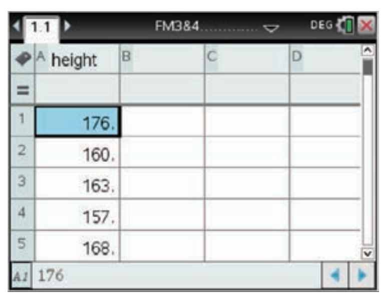
176 160 163 157 168 172 173 169

Determine the mean and standard deviation of the women's heights. Give your answers correct to two decimal places.

Steps

1 Start a new document by pressing ctrl + N.

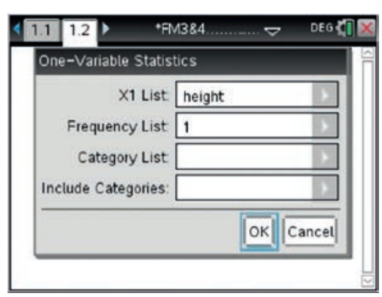
2 Select **Add Lists & Spreadsheet**.
Enter the data into a list named ***height***, as shown.



3 Statistical calculations can be done in either the **Lists & Spreadsheet** application or the **Calculator** application (used here).
Press ctrl + I and select **Add Calculator**.

a Press menu>**Statistics>Stat Calculations>One-Variable Statistics**. Press enter to accept the **Num of Lists** as 1.

b **i** Press enter to generate the pop-up screen, as shown opposite.



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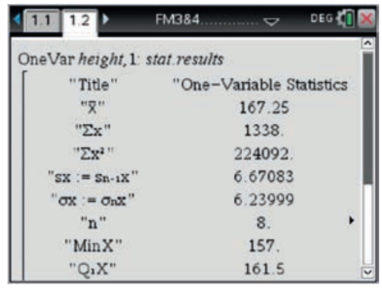
ii To complete this screen, use the ▶ arrow and [enter] to paste in the list name *height*. Pressing [enter] exits this screen and generates the results screen shown opposite.



4 Write down the answers to the required degree of accuracy (i.e. two decimal places).

The mean height of the women is $\bar{x} = 167.25$ cm and the standard deviation is $s = 6.67$ cm.

Notes:

1 The sample standard deviation is **sx**.

2 Use the ▲ ▼ arrows to scroll through the results screen to obtain values for additional statistical values.

How to calculate the mean and standard deviation using the ClassPad

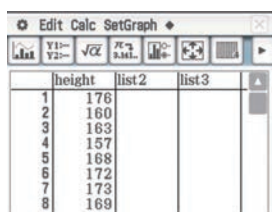
The following are all heights (in cm) of a group of women.

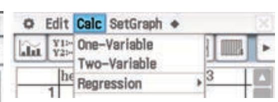
176 160 163 157 168 172 173 169

Determine the mean and standard deviation of the women's heights correct to two decimal places.

Steps

1 Open the **Statistics** application [Statistics] and enter the data into the column labelled **height**.



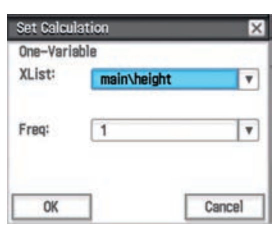


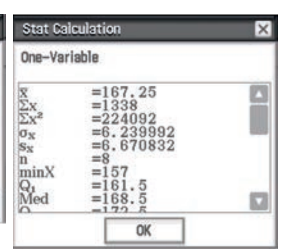
2 To calculate the mean and standard deviation, select **Calc** from the menu *One-Variable* from the drop-down menu to open the **Set Calculation** dialog box shown below.

3 Complete the dialog box as shown.

- **XList**: select **main\height** (▼).
- **Freq**: leave as **1**.

4 Tap **OK** to confirm your selections and calculate the required statistics, as shown.





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5 Write down the answers to two decimal places.

The mean height of the women is $\bar{x} = 167.25$ cm.

The standard deviation is $s_x = 6.67$ cm.

Notes:

1 The value of the standard deviation is given by s_x.

2 Use the side-bar arrows to scroll through the results screen to obtain values for additional statistical values (i.e. median, Q_3 and the maximum value) if required.

Exercise 2F-2

Basic ideas

1 Which measure of spread:

a *always* incorporates 50% of the scores?

b uses only the smallest and largest scores in the distribution?

c gives the average variation around the mean?

2 *Without* using the statistical capabilities of your calculator, write down the mean and standard deviation of the following six data values: 7.1 7.1 7.1 7.1 7.1 7.1

3 For which of the following variables does it *not* make sense to calculate a mean or standard deviation?

a Speed (in km/h) **b** Sex **c** Age (in years)

d Post code **e** Neck circumference (in cm)

f Weight (underweight, normal, overweight)

Calculating the mean and standard deviation using a CAS calculator

Skillsheet

4 A sample of 10 students were given a general knowledge test with the following results.

20 20 19 21 21 18 20 22 23 17

a Calculate the mean and standard deviation of the test scores, correct to one decimal place.

b The median test score is 20, which is similar in value to the mean. What does this tell you about the distribution of test scores?

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5 Calculate the mean and standard deviation for the variables in the table.
Give answers to the nearest whole number for cars and TVs, and one decimal place for alcohol consumption.

Number of TVs/ 1000	*Number of cars/ 1000*	*Alcohol consumption (litres)*
378	417	17.6
404	286	12.5
471	435	16.0
354	370	24.1
539	217	9.9
381	357	9.5
624	550	14.6

6 The table below lists the pulse rates of 23 females and 23 males.

Pulse rate (beats per minute)	
Females	*Males*
65 73 74 81 59 64 76 83 95 70 73 79	80 73 73 78 75 65 69 70 70 78 58 77
64 77 80 82 77 87 66 89 68 78 74	64 76 67 69 72 71 68 72 67 77 73

Calculate the mean and standard deviation for the male and female pulse rates (to one decimal place).

2G The normal distribution and the 68–95–99.7% rule

We know that the interquartile range is the spread of the middle 50% of the dataset. Can we find some similar way in which to interpret the standard deviation?

It turns out we can, but we need to restrict ourselves to symmetric distributions that have an approximate *bell shape*. Again, while this may sound very restrictive, many of the data distributions we work with in statistics (but not all) can be well approximated by this type of distribution. In fact, it is so common that it is called the **normal distribution**.

▶ The normal distribution

Many datasets that arise in practice are roughly symmetrical and have approximate bell shapes, as shown in the four examples below.

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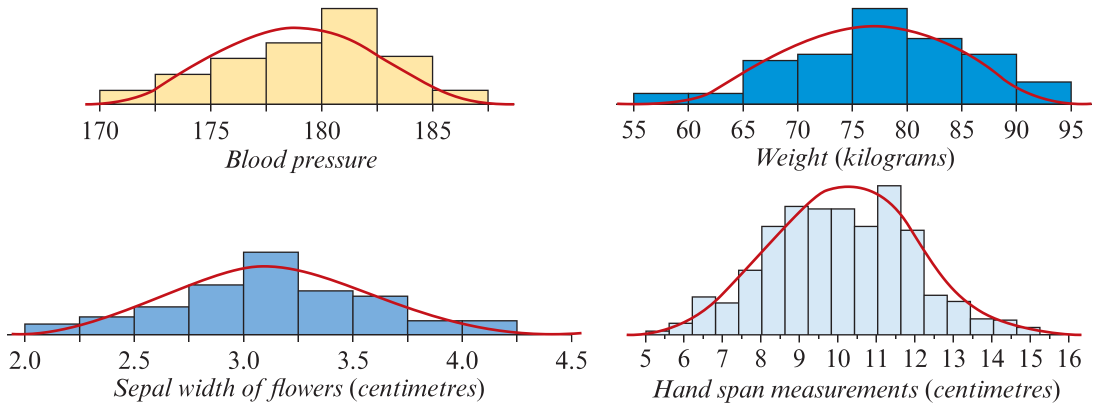


Data distributions that are bell-shaped can be modelled by a *normal* distribution.

The 68–95–99.7% rule

In normal distributions, the percentage of observations that lie within a certain number of standard deviations of the mean can always be determined. In particular, we are interested in the percentage of observations that lie within one, two or three standard deviations of the mean. This gives rise to what is known as the **68–95–99.7% rule**.

The 68–95–99.7% rule

For a *normal* distribution, approximately:

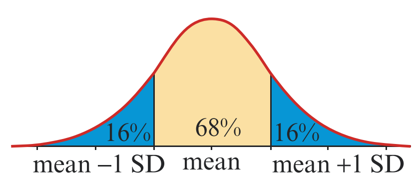
- *68%* of the observations lie within *one* standard deviation of the mean
- *95%* of the observations lie within *two* standard deviations of the mean
- *99.7%* of the observations lie within *three* standard deviations of the mean.

To give you an understanding of what this rule means in practice, it is helpful to view this rule graphically.

The 68–95–99.7% rule in graphical form

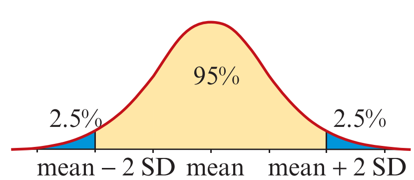
If a data distribution is approximately normal, then:

- around *68%* of the data values will lie within *one standard deviation (SD)* of the mean.



  This also means that *32%* of values lie outside this region. As the distribution is symmetric, we can also say that around 16% of values lie in each of the tails (shaded blue, above).

- around *95%* of the data values will lie within *two standard deviations* of the mean.



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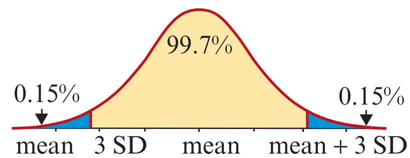
This also means that 5% of values lie outside this region. As the distribution is symmetric, we can also say that around 2.5% of values lie in each of the tails (shaded blue, above).

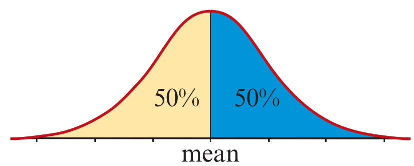
- around *99.7%* of the data values will lie within *three standard deviations* of the mean.



This also means that 0.3% of values lie outside this region. As the distribution is symmetric, we can also say that around 0.15% of values lie in each of the tails (shaded blue, above).

Finally, because the *normal distribution* is *symmetric*, the mean and the median coincide so that:

- *50%* of the data values will lie *above* the mean and *50%* of values will lie *below* the mean.



Example 15 Applying the 68–95–99.7% rule

The distribution of delivery times for pizzas made by House of Pizza is approximately normal, with a mean of 25 minutes and a standard deviation of 5 minutes.

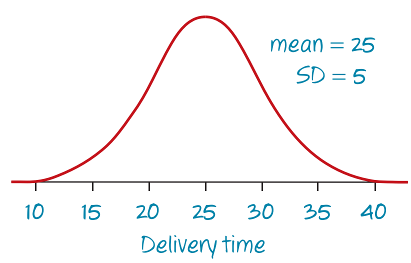
a What percentage of pizzas have delivery times of between 15 and 35 minutes?

b What percentage of pizzas have delivery times of greater than 30 minutes?

c In 1 month, House of Pizza delivers 2000 pizzas. How many of these pizzas are delivered in less than 10 minutes?

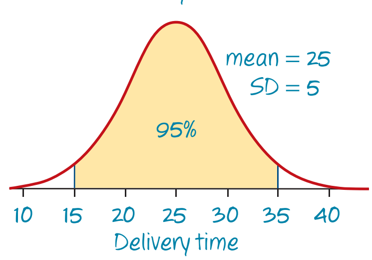
Solution

a **1** Sketch, scale and label a normal distribution curve with a mean of 25 and a standard deviation of 5.



2 Shade the region under the normal curve representing delivery times of between 15 and 35 minutes.

3 Note that delivery times of between 15 and 35 minutes lie within *two* standard deviations of the mean. ($15 = 25 - \mathbf{2} \times 5$ and $35 = 25 + \mathbf{2} \times 5$)



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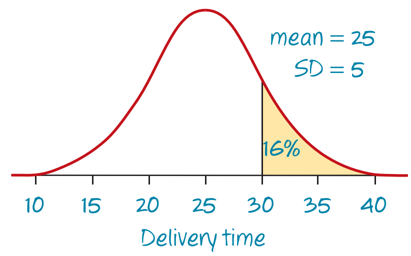
4 95% of values are within two standard deviations of the mean. Use this information to write your answer.	95% of pizzas will have delivery times of between 15 and 35 minutes.
b 1 As before, draw, scale and label a normal distribution curve with a mean of 25 and a standard deviation of 5. Shade the region under the normal curve representing delivery times of greater than 30 minutes.	
2 Delivery times of greater than 30 minutes are more than *one* standard deviation above the mean. (30 = 25 + **1** × 5)	
3 16% of values are more than one standard deviation above the mean. Write your answer.	16% of pizzas will have delivery times of greater than 30 minutes.
c 1 Write down the number of pizzas delivered.	Number = 2000
2 Delivery times of less than 10 minutes are more than *three* standard deviations below the mean. (10 = 25 − **3** × 5).	Percentage delivered in less than 10 minutes = 0.15%
3 0.15% of values are more than *three* standard deviations below the mean. Record this.	
4 Therefore, the number of pizzas delivered in less than 10 minutes is 0.15% of 2000.	Number of pizzas delivered in less than 10 minutes = 0.15% of 2000 $= \frac{0.15}{100} \times 2000 = 3$

Exercise 2G

Routine applications of the 68–95–99.7% rule

Skillsheet

1 The blood pressure readings for executives are approximately normally distributed with a mean systolic blood pressure of 134 and a standard deviation of 20. Given this information it can be concluded that:

a about 68% of the executives have blood pressures between ______ and ______

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b about 95% of the executives have blood pressures between   and
c about 99.7% of the executives have blood pressures between   and
d about 16% of the executives have blood pressures above
e about 2.5% of the executives have blood pressures below
f about 0.15% of the executives have blood pressures below
g about 50% of the executives have blood pressures above   .

2 The average weight of a bag of 10 blood plums picked at U-Pick Orchard is normally distributed with a mean of 1.88 kg and a standard deviation of 0.2 kg.
Given this information the percentage of the bags of 10 plums that weigh:

a between 1.68 and 2.08 kg is approximately   %
b between 1.28 and 2.48 kg is approximately   %
c more than 2.08 kg is approximately   %
d more than 2.28 kg is approximately   %
e less than 1.28 kg is approximately   %
f more than 1.88 kg is approximately   %.

Further applications of the 68–95–99.7% rule

3 The distribution of times taken for walkers to complete a circuit in a park is normal, with a mean time of 14 minutes and a standard deviation of 3 minutes.

a What percentage of walkers complete the circuit in:

i more than 11 minutes? **ii** less than 14 minutes?
iii between 14 and 20 minutes?

b In a week, 1000 walkers complete the circuit. How many will take less than 8 minutes?

4 The distribution of heights of 19-year-old women is approximately normal, with a mean of 170 cm and a standard deviation of 5 cm.

a What percentage of these women have heights:

i between 155 and 185 cm? **ii** greater than 180 cm?
iii between 160 and 175 cm?

b In a sample of 5000 of these women, how many have heights greater than 175 cm?

5 The distribution of resting pulse rates of 20-year-old men is approximately normal, with a mean of 66 beats/minute and a standard deviation of 4 beats/minute.

a What percentage of these men have pulse rates of:

i higher than 66? **ii** between 66 and 70? **iii** between 62 and 74?

b In a sample of 2000 of these men, how many have pulse rates between 54 and 78 beats/minute?

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2H Standard scores

The 68–95–99.7% rule makes the standard deviation a natural measuring stick for normally distributed data.

For example, a person who obtained a score of 112 on an IQ test with a mean of 100 and a standard deviation of 15 has an IQ score less than one standard deviation from the mean. Her score is typical of the group as a whole, as it lies well within the middle 68% of scores. In contrast, a person who scores 133 stands out; her score is more than two standard deviations from the mean and this puts her in the top 2.5%.

Because of the additional insight provided by relating the standard deviations to percentages, it is common to transform data into a new set of units that show the number of standard deviations a data value lies from the mean of the distribution. This is called *standardising* and these transformed data values are called **standardised** or ***z*-scores**.

Calculating standardised (z) scores

To obtain a standard score for an actual score, subtract the mean from the score and then divide the result by the standard deviation. That is:

$$\text{standard score} = \frac{\text{actual score} - \text{mean}}{\text{standard deviation}} \quad \text{or} \quad z = \frac{x - \bar{x}}{s}$$

Let us check to see that the formula works.

We already know that an IQ score of 115 is one standard deviation above the mean, so it should have a standard or z-score of 1. Substituting into the formula above we find, as we had predicted, that:

$$z = \frac{115 - 100}{15} = \frac{15}{15} = 1$$

Standard scores can be both positive and negative:

- a *positive* z-score indicates that the actual score it represents lies *above* the mean
- a *zero* standardised score indicates that the actual score is *equal* to the mean
- a *negative* z-score indicates that the actual score lies *below* the mean.

Example 16 Calculating standard scores

The heights of a group of young women have a mean of $\bar{x} = 160$ cm and a standard deviation of $s = 8$ cm. Determine the standard or z-scores of a woman who is:

a 172 cm tall **b** 150 cm tall **c** 160 cm tall.

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Solution

1 Write down the data value (x), the mean ($\bar{x}$) and the standard deviation (s).

2 Substitute the values into the formula $z = \dfrac{x - \bar{x}}{s}$ and evaluate.

a $x = 172,\ \bar{x} = 160,\ s = 8$

$$z = \frac{x - \bar{x}}{s} = \frac{172 - 160}{8} = \frac{12}{8} = 1.5$$

b $x = 150,\ \bar{x} = 160,\ s = 8$

$$z = \frac{x - \bar{x}}{s} = \frac{150 - 160}{8} = -\frac{10}{8} = -1.125$$

c $x = 160,\ \bar{x} = 160,\ s = 8$

$$z = \frac{x - \bar{x}}{s} = \frac{160 - 160}{8} = \frac{0}{8} = 0$$

Using standard scores to compare performance

Standard scores are also useful for comparing groups that have different means and/or standard deviations. For example, consider a student who obtained a mark of 75 in Psychology and a mark of 70 in Statistics. In which subject did she do better?

Calculating standard scores

We could take the marks at face value and say that she did better in Psychology because she got a higher mark in that subject. The assumption that underlies such a comparison is that the marks for both subjects have the same distribution with the same mean and standard deviation. However, in this case the two subjects have very different means and standard deviations, as shown in the table above.

Subject	*Mark*	*Mean*	*Standard Deviation*
Psychology	75	65	10
Statistics	70	60	5

If we assume that the *marks* are *normally distributed*, then *standardisation* and the *68–95–99.7% rule* give us a way of resolving this issue.

Let us standardise the marks.

Psychology: standardised mark $z = \dfrac{75 - 65}{10} = 1$

Statistics: standardised mark $z = \dfrac{70 - 60}{5} = 2$

What do we see? The student obtained a higher score for Psychology than for Statistics. However, relative to her classmates she did better in Statistics.

- Her mark of 70 in Statistics is equivalent to a z-score of 2. This means that her mark was two standard deviations above the mean, placing her in the top 2.5% of students.
- Her mark of 75 for Psychology is equivalent to a z-score of 1. This means that her mark was only one standard deviation above the mean, placing her in the top 16% of students. This is a good performance, but not as good as for statistics.

ISBN 978-1-107-56757-3

Example 17 Applying standard scores

Another student studying the same two subjects obtained a mark of 55 for both Psychology and Statistics. Does this mean that she performed equally well in both subjects? Use standardised marks to help you arrive at your conclusion.

Solution

1 Write down her mark (x), the mean ($\bar{x}$) and the standard deviation (s) for each subject and compute a standardised score for both subjects.
Evaluate and compare.

Psychology: $x = 55$, $\bar{x} = 65$, $s = 10$

$$z = \frac{x - \bar{x}}{s} = \frac{55 - 65}{10} = \frac{-10}{10} = -1$$

Statistics: $x = 55$, $\bar{x} = 60$, $s = 5$

$$z = \frac{x - \bar{x}}{s} = \frac{55 - 60}{5} = \frac{-5}{5} = -1$$

2 Write down your conclusion.

Yes, her standardised score, $z = -1$, was the same for both subjects. In both subjects she finished in the bottom 16%.

Converting standard scores into actual scores

Having learnt how to calculate standard scores, you also need to be able to convert a standardised score back into an actual score. The rule for converting a standardised score into an actual score is given below.

Converting standardised scores into actual scores

By making the actual score the subject of the rule for calculating standard scores, we arrive at:

$$\text{actual score} = \text{mean} + \text{standard score} \times \text{standard deviation} \quad \text{or} \quad x = \bar{x} + z \times s$$

Example 18 Converting standard scores into actual scores

A class test (out of 50) has a mean mark of $\bar{x} = 34$ and a standard deviation of $s = 4$. Joe's standardised test mark was $z = -1.5$. What was Joe's actual mark?

Solution

1 Write down mean ($\bar{x}$), the standard deviation (s) and Joe's standardised score (z).

$\bar{x} = 34$, $s = 4$, $z = -1.5$

2 Write down the rule for calculating the actual score and substitute these values into the rule.

$$x = \bar{x} + z \times s$$
$$= 34 + (-1.5) \times 4 = 28$$

Joe's actual mark was 28.

ISBN 978-1-107-56757-3

Exercise 2H

Calculating standard scores

1 A set of scores has a mean of 100 and a standard deviation of 20. Standardise the following scores.

a 120 **b** 140 **c** 80 **d** 100 **e** 40 **f** 110

Calculating actual scores from standardised scores

2 A set of scores has a mean of 100 and a standard deviation of 20. Calculate the actual score if the standardised score was:

a 1 **b** 0.8 **c** 2.1 **d** 0 **e** −1.4 **f** −2.5

Applications

3 The table below contains the scores a student obtained in a practice test for each of his VCE subjects. Also shown is the mean and standard deviation for each subject.

Subject	*Mark*	*Mean*	*Standard deviation*
English	69	60	4
Biology	75	60	5
Chemistry	55	55	6
Further Maths	55	44	10
Psychology	73	82	4

a Calculate his standard score for each subject.

b Use the standard score to rate his performance in each subject, assuming a normal distribution of marks and using the 68–95–99.7% rule.

4 The body weights of a large group of 14-year-old girls have a mean of 54 kg and a standard deviation of 10.0 kg.

a Kate weighs 56 kg. Determine her standardised weight.

b Lani has a standardised weight of –0.75. Determine her actual weight.

Assuming the girls' weights are approximately normally distributed with a mean of 54 kg and a standard deviation of 10 kg, determine the:

c percentage of these girls who weigh more than 74 kg

d percentage of these girls who weigh between 54 and 64 kg

e percentage of these girls who have standardised weights less than –1

f percentage of these girls who have standardised weights greater than –2.

2I Populations and samples

This material is available in the Interactive Textbook.

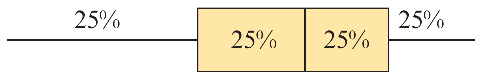
ISBN 978-1-107-56757-3

Key ideas and chapter summary

Summary statistics — **Summary statistics** are used to give numerical values to special features of a data distribution, such as centre and spread.

Median — The **median** is a summary statistic that can be used to locate the **centre** of a distribution. It is the midpoint of a distribution dividing an ordered dataset into two equal parts.

Quartiles — **Quartiles** are summary statistics that divide an ordered dataset into four equal groups.



- The first quartile, Q_1, marks off the first 25% of values.
- The second quartile, or median M, marks off the first 50% of values.
- The third quartile, Q_3, marks off the first 75% of values.

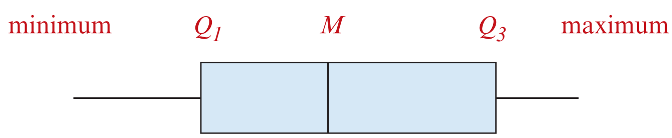
The interquartile range — The **interquartile range** is defined as $IQR = Q_3 - Q_1$. The IQR gives the *spread* of the middle 50% of data values.

Five-number summary — A listing of the median, M, the quartiles Q_1 and Q_3, and the smallest and largest data values of a distribution, written in the order

$$\text{minimum},\ Q_1,\ M,\ Q_3,\ \text{maximum}$$

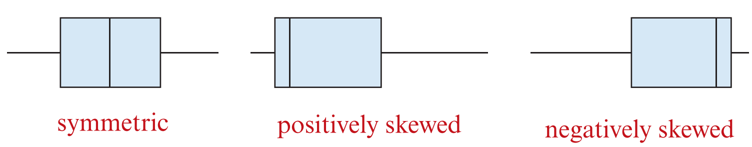
is known as a **five-number summary**.

Box plots — A standard **box plot** is a graphical representation of a five-number summary.



Interpreting box plots — Box plots are powerful tools for picturing and comparing datasets as they give both a visual view and a numerical summary of a distribution.

- *Shape:* symmetric or skewed (positive or negative)



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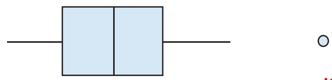
Review

- *Outliers:* values that appear to stand out from the data



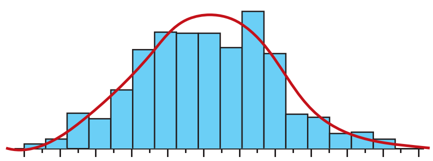
- *Centre:* the midpoint of the distribution (the median)
- *Spread:* the *IQR* and the range of values covered

Outliers In a box plot, outliers are defined as being those values that are:

- greater than $Q_3 + 1.5 \times IQR$ (upper fence)
- less than $Q_1 - 1.5 \times IQR$ (lower fence).

Mean and the standard deviation The **mean**, the balance point of a data distribution, is a summary statistic that can be used to locate the centre of a symmetric distribution. The **standard deviation** is a summary statistic that measures the spread of the data values around the mean.

The normal distribution Data distributions that have a bell shape can be modelled by a **normal distribution.**



The 68–95–99.7% rule The **68–95–99.7%** rule says that for a normal distribution:

- 68% of values are within one standard deviation of the mean
- 95% of values are within two standard deviations of the mean
- 99.7% of values are within three standard deviations of the mean.

Standardised scores The value of the standard score gives the distance and direction of a data value from the mean in terms of standard deviations.
The rule for calculating a **standardised score** is:

$$\text{standardised score} = \frac{\text{actual score} - \text{mean}}{\text{standard deviation}}$$

Skills check

Having completed this chapter you should be able to:

- locate the median and the quartiles of a dataset and hence calculate the *IQR*
- produce a five-number summary from a set of data

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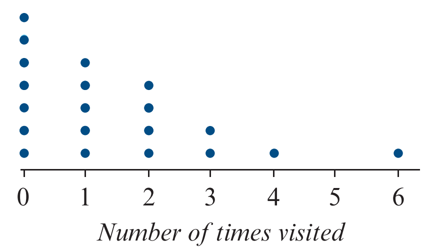
- construct a box plot from a stem plot
- construct a box plot from raw data using a graphics calculator
- use a box plot to identify key features of a dataset, such as shape (including outliers if any), centre and spread
- calculate the mean and standard deviation of a dataset
- understand the difference between the mean and the median as measures of centre and be able to identify situations where it is more appropriate to use the median
- know and be able to apply the 68–95–99.9% rule for bell-shaped distributions
- calculate standard or z-scores and use them to compare performance.
- calculate actual scores from standard or z-scores.

Multiple-choice questions

The following information relates to Questions 1 to 5.

The dot plot opposite displays the *number of times* 20 shoppers visited their supermarket in a week.



1 The shape of the data distribution displayed by this dot plot is best described as:

A symmetric **B** negatively skewed
C negatively skewed with an outlier **D** negatively skewed
E positively skewed with an outlier

2 The median for this data distribution is:

A 0 **B** 1 **C** 2 **D** 3 **E** 6

3 The first quartile (Q_1) for this data distribution is:

A 0 **B** 1 **C** 2 **D** 3 **E** 6

4 The third quartile (Q_3) for this data distribution is:

A 0 **B** 1 **C** 2 **D** 3 **E** 6

5 The range for this data distribution is:

A 0 **B** 1 **C** 2 **D** 3 **E** 6

6 The following is an *ordered* set of 10 daily maximum temperatures (in degrees Celsius):

22 22 23 24 24 25 26 27 28 29

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The five-number summary for these temperatures is:

A 22, 23, 24, 27, 29 **B** 22, 23, 24.5, 27, 29 **C** 22, 24, 24.5, 27, 29

D 22, 23, 24.5, 27.5, 29 **E** 22, 24, 24.5, 27, 29

The following information relates to Questions 7 to 9.

The stem plot opposite displays the distribution of the marks obtained by 25 students.

Stem: 1|5 means 15 marks

Stem	Leaf
0	2
1	5 9 9 9
2	0 4 4 5 5 8 8 8
3	0 3 5 5 6 8 9
4	1 2 3 5
5	
6	0

7 The shape of the data distribution displayed by this dot plot is best described as:

A symmetric **B** symmetric with an outlier

C negatively skewed with an outlier **D** negatively skewed

E positively skewed with an outlier

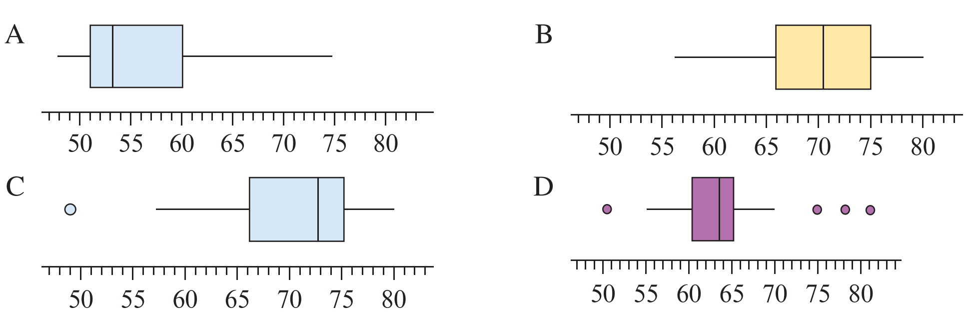
8 The median mark was:

A 20 **B** 27 **C** 28 **D** 29 **E** 30

9 The interquartile quartile range (IQR) for the life expectancies (in years) of these 23 countries is:

A 12 **B** 16.5 **C** 20 **D** 30.5 **E** 31.5

The following information relates to Questions 10 to 17.



10 The median of box plot A is closest to:

A 5 **B** 53 **C** 54.5 **D** 55 **E** 60

11 The IQR of box plot B is closest to:

A 10 **B** 20 **C** 25 **D** 65 **E** 75

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12 The range of box plot C is closest to:

A 4 **B** 13 **C** 20 **D** 30 **E** 80

13 The description that best matches box plot A is:

A symmetric
B symmetric with outliers
C negatively skewed
D positively skewed
E positively skewed with outliers

14 The description that best matches box plot B is:

A symmetric
B negatively skewed with an outlier
C negatively skewed
D positively skewed
E positively skewed with outliers

15 The description that best matches box plot D is:

A symmetric
B symmetric with outliers
C negatively skewed
D positively skewed
E positively skewed with outliers

16 For the data represented by box plot D, the percentage of data values greater than 65 is:

A 2.5% **B** 25% **C** 50% **D** 75% **E** 100%

17 To be an outlier in box plot D, a score must be:

A either less than 52.5 or greater than 72.5
B greater than 72.5
C either less than 55 or greater than 70
D greater than 70
E less than 55

18 The mean ($\bar{x}$) and standard deviation (s) for the following set of test marks

1 1 10 15 16 25 8 10 12

are closest to:

A 7.1, 10.9 **B** 7.5, 10.9 **C** 10.9, 7.1 **D** 10.9, 7.5 **E** 10.8, 7.5

19 It would *not* be appropriate to determine the mean and standard deviation of a group of people's:

A ages **B** phone numbers **C** heights **D** weights **E** family sizes

20 The median is a more appropriate measure of the centre of a distribution than the mean when the distribution is:

A symmetric **B** symmetric with no outliers **C** bell-shaped
D clearly skewed and/or there are outliers **E** normal

ISBN 978-1-107-56757-3

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Review

21 A student's mark on a test is 50. The mean mark for their class is 55 and the standard deviation is 2.5. Their standard score is:

A −2.5 **B** −2.0 **C** 0 **D** 2 **E** 2.5

The following information relates to Questions 22 to 28.

Each week, a bus company makes 200 trips between two large country towns. The time taken to make a trip between the two towns is approximately normally distributed with a mean of 78 minutes and a standard deviation of 4 minutes.

22 The percentage of trips each week that take 78 minutes or more is:

A 16% **B** 34% **C** 50% **D** 68% **E** 84%

23 The percentage of trips each week that take between 74 and 82 minutes is:

A 16% **B** 34% **C** 50% **D** 68% **E** 95%

24 The percentage of trips each week that take less than 82 minutes is:

A 5% **B** 16% **C** 68% **D** 71.5% **E** 84%

25 The number of trips each week that take between 70 and 82 minutes is:

A 4 **B** 32 **C** 68 **D** 127 **E** 163

26 A trip that takes 71 minutes has a standardised time (z-score) of:

A −1.75 **B** −1.5 **C** −1.25 **D** 1.5 **E** 1.75

27 A standardised time for a trip is $z = -0.25$. The actual time (in minutes) is:

A 77 **B** 77.25 **C** 77.75 **D** 78.25 **E** 79

28 The time of a bus trip whose standardised time is $z = 2.1$ is:

A very much below average **B** just below average **C** around average
D just above average **E** very much above average

29 The mean length of 10 garden stakes is $\bar{x} = 180.5$ cm. The standard deviation of the lengths is $s = 2.9$ cm. If the length of each garden stake is then reduced by exactly 5 cm, the mean and standard deviation of the lengths of the stakes will be:

A 175.5 cm and 2.4 cm **B** 180.5 cm and 2.4 cm **C** 175.5 cm and 2.9 cm
D 175.5 cm and 3.4 cm **E** 185.5 cm and 2.9 cm

ISBN 978-1-107-56757-3

Review

Extended-response questions

1 A group of 16 obese people attempted to lose weight by joining a regular exercise group. The following weight losses, in kilograms, were recorded.

26 14 7 38 23 21 17 4 18 34 24 29 2 13 33 15

a Use a CAS calculator to construct a box plot for the data. Name the variable *wloss*.

b Use the box plot to locate the median and the quartiles Q_1 and Q_3.

c Complete the following statements:

'The middle 50% of the people who exercised had weight losses between ☐ and ☐ kilograms.'

'Twenty-five per cent of people lost less than ☐ kilograms.'

d Use the box plot to describe the distributions of weight loss in terms of shape, centre, spread and outliers (if any). Give appropriate values.

2 The weights (in kg) carried by the horses in a handicap race are given below.

60 57 57 55 54 53 53 53 52 52 51.5 51

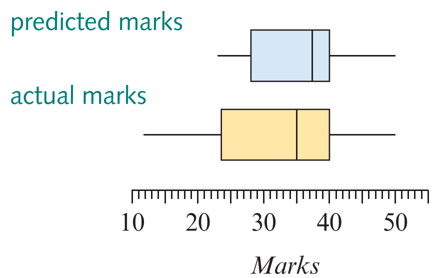
a Use a CAS calculator to construct a box plot. Name the variable *hweight*.

b Complete a five-number summary for the weights carried by the horses.

c What is the interquartile range?

d Use the box plot to describe the distributions of weight carried by the horses in terms of shape, centre, spread and outliers (if any). Give appropriate values.

3 To find out how well she could estimate her students' marks on a test, a statistics teacher set a test and then, before marking the test, predicted the marks she thought her students would get. After marking the test, she produced a parallel box plot to enable her to compare the two sets of marks. The box plots are shown. The test was marked out of 50.



a Overall, did the teacher tend to overestimate or underestimate her students' marks? Explain.

b Were the teacher's marks more or less variable than the actual marks? Explain.

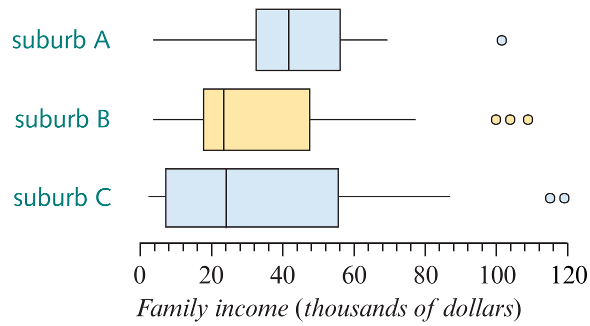
c Compare the two distributions in terms of shape, centre and spread. Give appropriate values at a level of accuracy that can be read from the plot.

d Comment on how the predicted marks of the teacher compared to the students' actual marks.

ISBN 978-1-107-56757-3

Review

4 A random sample of 250 families from three different suburbs was used in a study to try to identify factors that influenced a family's decision about taking out private health insurance. One variable investigated was family income. The information gathered on family incomes is presented opposite in the form of parallel box plots.



a In which suburb was the median household income the greatest?

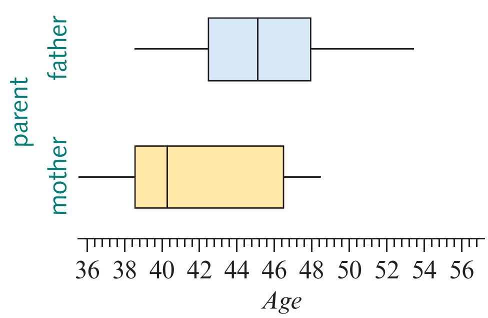
b In which suburb were family incomes most variable?

c What do the outliers represent?

d Which of the following statements are true or appear likely to be true?

- **i** 'At least 75% of the families in suburb A have an income that exceeds the median family income in suburb B.'
- **ii** 'More than 75% of the families in suburb A have incomes under $60 000.'
- **iii** 'The distribution of family incomes in suburb C is approximately symmetric'
- **iv** 'The *mean* family income in suburb B is greater than the *median* family income in suburb B.'

5 The parallel box plots opposite display the distribution of age (in years) of the mothers and fathers of 26 students. Label each of the following statements as true or false.



a The median age of the mothers is lower than the median age of the fathers.

b Approximately 75% of the fathers were 48 years old or younger.

c At least 75% of the mothers were younger than the median age of the fathers.

d Approximately 50% of the mothers were aged between 42 and 48 years.

e More than 25% of the fathers were aged 50 years or older.

ISBN 978-1-107-56757-3

3

Core: Data analysis

Chapter 3

Investigating associations between two variables

ISBN 978-1-107-56757-3

3A Response and explanatory variables

So far you have learned how to display, describe and compare the distributions of single variables. In the process you learned how to use data to answer questions like ‘What is the favourite colour of prep-grade students?’ or ‘How do the weights of net caught tuna fish vary?’ In each case we concentrated on investigating the statistical variables individually.

However, questions like ‘Does a person’s attitude to gun control depend on their sex?’, ‘Does the time it takes students to get to school depend on their mode of transport?’ or ‘Can we predict a student’s test score from time (in hours) spent studying for the test?’ cannot be answered by considering variables separately. All of these questions relate to situations where the two variables are linked in some way (associated) so that they vary together. The data generated in these circumstances is called **bivariate data**.

Analysing associated variables requires a new set of statistical tools. Developing and applying these tools is the subject of the next four chapters.

▶ Identifying response and explanatory variables

When investigating associations between variables, it is helpful to think of one of the variables as the **explanatory variable**. The other variable is then called the **response variable**. We use the explanatory variable to explain changes that might be observed in the response variable.

For example, the question, ‘Does a person’s attitude to gun control depend on their sex?’, suggests that knowing a person’s sex might be useful in explaining their attitude to gun control. In this situation *sex* is the explanatory variable and *attitude to gun control* is the response variable.

It is important to be able to identify the explanatory and response variables before you explore the association between them. Consider the following examples.

Example 1 Identifying the response and explanatory variables

We wish to investigate the question, ‘Does the time it takes a student to get to school depend on their mode of transport?’ The variables here are *time* and *mode of transport*. Which is the response variable (RV) and which is the explanatory variable (EV)?

Solution

In asking the question in this way we are suggesting that a student’s *mode of transport* might explain the differences we observe in the time it takes students to get to school.

EV: mode of transport
RV: time

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Example 2 Identifying the response and explanatory variables

Can we predict people's height (in cm) from their wrist measurement? The variables in this investigation are *height* and *wrist measurement*. Which is the response variable (RV) and which is the explanatory variable (EV)?

Solution

Since we wish to predict height from wrist circumference, we are using *wrist measurement* as the predictor or explanatory variable. *Height* is then the response variable.

EV: wrist measurement
RV: height

It is important to note that, in Example 2, we could have asked the question the other way around; that is, 'Can we predict people's wrist measurement from their height?' In that case *height* would be the explanatory variable, and *wrist measurement* would be the response variable. The way we ask our statistical question is an important factor when there is no obvious explanatory variable.

Response and explanatory variables

When investigating the association between two variables the *explanatory variable (EV)* is the variable we expect to *explain* or *predict* the value of the *response variable (RV)*.

Note: The explanatory variable is sometimes called the independent variable (IV) and the response variable the dependent variable (DV).

Exercise 3A

Identifying explanatory and response variables

For each of the following situations identify the explanatory variable (EV) and response variable (RV). In each situation the variable names are *italicised.*

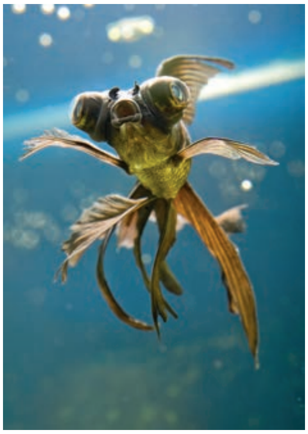
1 **a** We wish to investigate whether a fish's *toxicity* can be predicted from its *colour*. We want to be able to predict *toxicity* from *colour*.

b The relationship between *weight loss* and *type of diet* is to be investigated.

c We wish to investigate the relationship between a used car's *age* and its *price*.

d It is suggested that the *cost* of heating in a house depends on the type of *fuel* used.

e The relationship between the *price* of a house and its *location* is to be investigated.



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2 The following pairs of variables are related. Which is likely to be the explanatory variable? The variable names are italicised.

- **a** *exercise level* and *age*
- **b** *years of education* and *salary level*
- **c** *comfort level* and *temperature*
- **d** *time of year* and *incidence of hay fever*
- **e** *age group* and *musical taste*
- **f** *AFL team supported* and *state of residence*

3B Investigating associations between categorical variables

If two variables are related or linked in some way, we say they are associated. The statistical tool used to investigate associations between two categorical variables is a **two-way frequency table**.

▶ Using a two-way frequency table to investigate an association

Analysing a situation where two categorical variables are associated

Table 3.1 was constructed from data collected to answer the question: 'Is a person's attitude to gun control associated with their sex?'

In this investigation, *attitude* to gun control is the response variable and *sex* is the explanatory variable.

Note: When constructing two-way tables the RV defines the rows, and the EV defines the columns.

Table 3.1

	Sex	
Attitude	*Male*	*Female*
For	32	30
Against	26	12
Total	58	42

RV (rows), EV (columns)

From the table we see that more males than females favoured gun control. However, this doesn't tell us very much, because there were more males in the sample.

To solve this problem, we turn our table entries into percentages by calculating column percentages. See Table 3.2.

Table 3.2

	Sex (%)	
Attitude	*Male*	*Female*
For	55.2	71.4
Against	44.8	28.6
Total	100.0	100.0

If there is *no association* between *attitude* to gun control and *sex*, approximately *equal percentages* of males and females would be 'for' gun control.

This is not the case, so we conclude that *attitude* to gun control and *sex* are *associated*.

To complete our analysis we might write a brief report of our finding.

Report

From Table 3.2 we see that a much higher percentage of females were for gun control than males, 71.4% to 55.2%, indicating that attitude to gun control is associated with sex.

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Example 3 Identifying and describing associations between two categorical variables from a two-way table

A survey was conducted with 100 people.

As part of this survey, people were asked whether or not they supported banning mobile phones in cinemas. The results are summarised in the table.

Ban mobile phones	*Sex*	
	Male	*Female*
Yes	87.9%	65.8%
No	12.1%	34.2%
Total	100.0%	100.0%

Is there is an association between support for banning mobile phones in cinemas and the sex of the respondent? Write a brief response quoting appropriate percentages.

Solution

A large difference in the percentages of males and females supporting the banning of mobile phones indicates an association.

Yes; the percentage of males in support of banning mobile phones in cinemas (87.9%) was much higher than for females (65.8%).

Note: Finding a single row in the two-way frequency distribution in which percentages are clearly different is sufficient to identify an association between the variables.

In the following example, there is *no* association between the variables.

Example 4 Identifying and describing associations between two categorical variables from a two-way table (no association)

In the same survey people were asked whether or not they supported Sunday racing. The results are summarised in the table.

	Sex	
Sunday racing	*Male*	*Female*
For	55.6%	54.5%
Against	44.4%	45.5%
Total	100.0%	100.0%

Is there an association between support for Sunday racing and the sex of the respondent? Write a brief response quoting appropriate percentages.

Solution

The similar percentage of males and females supporting the banning of Sunday racing is consistent with there being no association.

No; the percentage of males (55.6%) supporting Sunday racing is similar to the percentage of females supporting Sunday racing (54.5%).

Note: As a rule of thumb, a difference of at least 5% would be required to classify a difference as significant.

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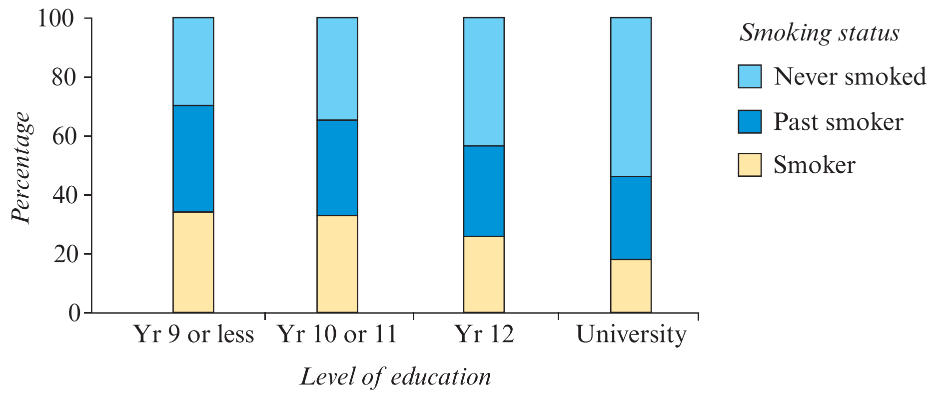
Analysing a situation where two categorical variables have multiple categories

Table 3.3 displays the smoking status of adults (smoker, past smoker, never smoked) by level of education (Year 9 or less, Year 10 or 11, Year 12, university).

Table 3.3

	Education level (%)			
Smoking status	*Year 9 or less*	*Year 10 or 11*	*Year 12*	*University*
Smoker	33.9	31.7	26.5	18.4
Past smoker	36.0	33.8	30.9	28.0
Never smoked	30.0	34.5	42.6	53.6
Total	99.9	100.0	100.0	100.0

The following segmented bar chart has been constructed from this table to help us with the analysis. Each column represents a column from the purple-shaded part of the table.



From the segmented bar chart, it is clear that as education level increases the percentage of smokers decreases, indicating there is an association between smoking and education level. We could report this finding as follows, using the table to determine exact percentages.

Report

From Table 3.3 we see that the percentage of smokers steadily decreases with education level, from 33.9% for Year 9 or below to 18.4% for university. This indicates that smoking is associated with level of education.

Note: A similar conclusion could be drawn by focusing attention on the top segment of each column, which shows that the percentage of non-smokers increases with education level.

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Example 5 Identifying and describing associations between two categorical variables from a two-way table

A survey was conducted with 1000 males under 50 years old. As part of this survey, they were asked to rate their interest in sport as 'high', 'medium', and 'low'. Their age group was also recorded as 'under 18','19–25', '26–35' and '36–50'. The results are displayed in the table.

Interest in sport	*Age group (%)*			
	Under 18 years	*19–25 years*	*26–35 years*	*36–50 years*
High	56.5	50.2	40.7	35.0
Medium	30.1	34.4	36.8	45.8
Low	13.4	13.4	22.5	20.3
Total	100.0	100.0	100.0	100.0

a Which is the explanatory variable, *interest in sport* or *age group*?

b Is there an association between *interest in sport* and *age group*? Write a brief response quoting appropriate percentages.

Solution

a In this situation, *age group* is the obvious EV.

a Age group

b A significant difference in the percentages of the various age groups for any particular level of interest in sport indicates an association. We will choose those with a 'high level' of interest for analysis.

b Yes; the percentage of males with a high level of interest in sport decreases steadily with age group from 56.5% for the 'under 18 years' age group, to 35.0% for the '36–50 years' age group.

Exercise 3B

Using two-way tables to identify associations between two categorical variables

1 A survey was conducted with 242 university students. For this survey, data were collected on the students' *enrolment status* (full-time, part-time) and whether or not each *drinks alcohol* ('Yes' or 'No'). Their responses are summarised in the table opposite.

Drinks alcohol	*Enrolment status (%)*	
	Full-time	*Part-time*
Yes	80.5	81.8
No	19.5	18.2
Total	100.0	100.0

a Which variable is the explanatory variable?

b Is there an association between drinking alcohol and enrolment status? Write a brief report quoting appropriate percentages.

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2 The table opposite was constructed from data collected to see if *handedness* (left, right) was associated with *sex* (male, female).

	Sex	
Handedness	Male	Female
Left	22	16
Right	222	147

a Which variable is the response variable?

b Convert the table to percentages by calculating column percentages.

c Is *handnesss* is associated with *sex*? Write a brief explanation using appropriate percentages.

3 A survey was conducted with 59 male and 51 female university students to determine whether, each day, they exercised, 'regularly', 'sometimes' or 'rarely'. Their responses are summarised in the table.

	Sex (%)	
Exercised	Male	Female
Rarely	28.8	39.2
Sometimes	52.5	54.9
Regularly	18.6	5.9
Total	99.9	100.0

a Which is the explanatory variable?

b Is the variable *exercised* nominal or ordinal?

c What percentage of females exercised sometimes?

d Is there an association between how regularly these students exercised and their sex? Write a brief response quoting appropriate percentages.

4 As part of the General Social Survey conducted in the US, respondents were asked to say whether they found life *exciting*, *pretty routine* or *dull*. Their marital status was also recorded as married, widowed, divorced, separated or never married. The results are organised into a table as shown.

	Marital status (%)				
Attitude to life	*Married*	*Widowed*	*Divorced*	*Separated*	*Never*
Exciting	47.6	33.8	46.7	45.9	52.3
Pretty routine	48.7	54.3	47.6	44.6	44.4
Dull	3.7	11.9	6.7	9.5	3.2
Total	100.1	100.0	100.0	100.0	99.0

a What percentage of widowed people found life 'dull'?

b What percentage of people who were never married found life 'exciting'?

c What is the likely explanatory variable in this investigation?

d Is the variable *attitude to life* nominal or ordinal?

e Does the information you have been given support the contention that a person's attitude to life is related to their marital status? Justify your argument by quoting appropriate percentages.

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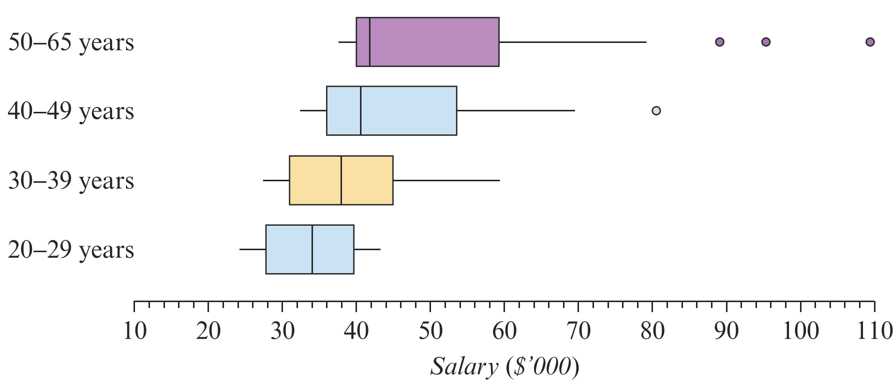
3C Investigating the association between a numerical and a categorical variable

In the previous section, you learned how to identify and describe associations between two categorical variables. In this section, you will learn to identify and describe associations between a numerical variable and a categorical variable. The relationship between the numerical variable *salary* and the categorical variable *age group* is one such example.

The statistical tool most commonly used for investigating associations between a numerical and a categorical variable is the **parallel box plot**. In a parallel box plot, there is one box plot for each category of the categorical variable. Associations can then be identified by comparing the way in which the distribution of the numerical variable changes between categories in terms of shape, centre and spread.

▶ Using parallel box plots to identify and describe associations

The parallel box plots below compare the salary distribution for four different age groups: 20–29 years, 30–39 years, 40–49 years and 50–65 years. Here the numerical variable *salary* is the *response* variable and the categorical variable *age group* is the *explanatory* variable.



If there is *no association* between salary and age group**,** the salary distribution will be similar for all age groups. Equivalently, when working with parallel box plots, each *box plot* will be the *same*. This is not the case here.

There are several ways of identifying and systematically describing the association between salary and age group from this display. This can be done by:

- comparing *medians*

> **Report**
>
> The parallel box plots show that median salaries and age group are associated because median salaries increase with age group. For example, the median salary increased from $34 000 for 20–29 year-olds to $42 000 for 50–65 year-olds.

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- comparing *IQRs* and/or *ranges*

Report

From the parallel box plots we can see that the spread of salaries is associated with age group. For example, the IQR increased from around \$12 000 for 20–29-year-olds to around \$20 000 for 50–65-year-olds.

- comparing *shapes*.

Report

From the parallel box plots we can see that the shape of the distribution of salaries is associated with age group because of the distribution, which is symmetric for 20–29-year-olds, and becomes progressively more positively skewed as age increases. Outliers also begin to appear.

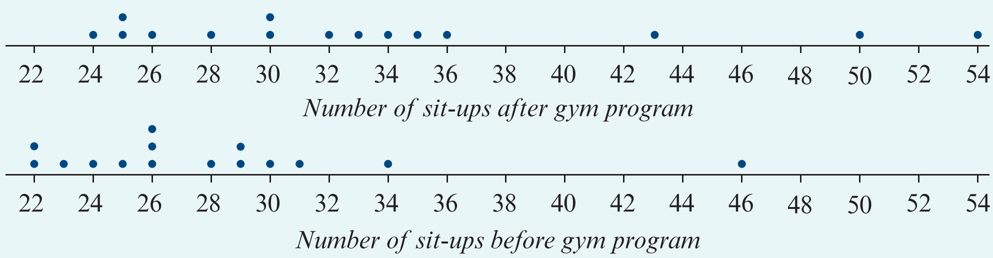
Note: Any one of these reports by themselves can be used to claim that there is an association between salary and age. However, using all three gives a more complete description of this relationship.

Using parallel dot plots and back-to-back stem plots to identify and describe associations

For small datasets, parallel box plots and back-to-back stem plots are ideal displays for identifying and describing associations between a numerical and a categorical variable.

Example 6 Using a parallel dot plot to identify and describe associations

The parallel dot plots below display the distribution of the number of sit-ups performed by 15 people before and after they had completed a gym program.



Do the parallel dot plots support the contention that the number of sit-ups performed is associated with completing the gym program? Write a brief explanation that compares medians.

Solution

1 Locate the median number of sit-ups performed before and after the gym program. You should find them to be $M = 26$ and $M = 32$ sit-ups, respectively.

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2 A significant difference in the median number of sit-ups performed before and after the gym program is sufficient to conclude that the number of sit-ups performed is associated with completing the gym program. Report your conclusion, backed up by a brief explanation.

Yes; the median number of sit-ups performed after attending the gym program ($M = 32$) is considerably higher than the number of sit-ups performed before attending the gym program ($M = 26$). This indicates that the number of sit-ups performed is associated with completing the gym program.

Note: Because it is often difficult to clearly identify the shape of a distribution with a small amount of data, we usually confine ourselves to comparing medians when using dot plots and back-to-back stem plots.

Example 7 Using a back-to back stem plot to identify and describe associations

The back-to-back stem plot below displays the distribution of life expectancy (in years) for 13 countries in 2010 and 1970.

Key: 6|7= 67 years

1970		2010
8	3	
	4	
9 4 2	5	8
9 9 9 8 7 7 1 0	6	9
4	7	1 2 4 4 6 8 9 9
	8	0 0 0

Do the back-to-back stem plots support the contention that life expectancy is increasing over time? Write a brief explanation based on your comparisons of the two medians.

Solution

1 Locate the median life expectancies for 1970 and 2010. You should find them to be 67 and 76 years, respectively.

2 A significant difference in median lifetimes between 1970 and 2010 is sufficient to conclude that life expectancy is increasing over time. Report your conclusion, supported by a brief explanation.

Yes: the median life expectancy in 2010 ($M = 76$ years) is considerably higher than the median life expectancy in 1970 ($M = 67$ years). This indicates that life expectancy is increasing over time.

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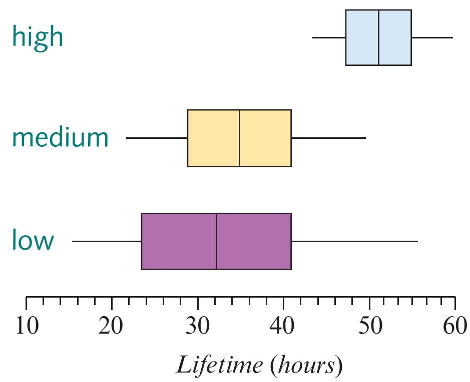
Exercise 3C

Using box plots, dot plots and stem plots to identify and describe associations

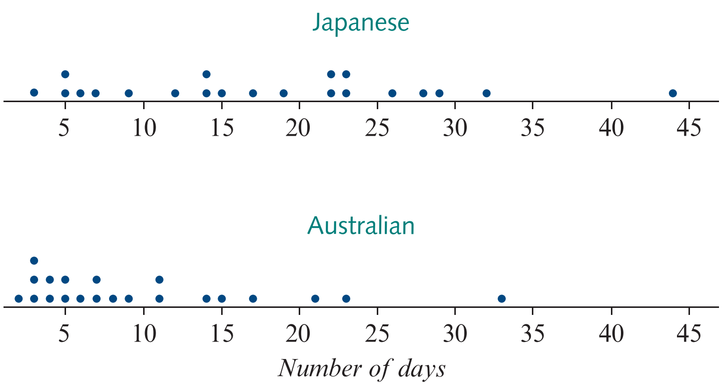
1 The parallel box plots show the distribution of the lifetime (in hours) of three differently priced batteries (low, medium, high).



a The two variables displayed here are battery *lifetime* and battery *price* (low, medium, high). Which is the numerical and which is the categorical variable?

b Do the parallel box plots support the contention that battery lifetime depends on price? Write a brief explanation.

2 The parallel dot plots below compare the number of days spent away from home by Japanese and Australian tourists.



Do the parallel dot plots support the contention that the number of days tourists spent away from home is associated with their country of origin? Write a brief explanation that compares medians.

3 The back-to-back stem plot shown compares the distribution of the average age of male and female patients (in years) admitted to a small hospital during 1 week.
Do the back-to-back plots support the contention that the median age of the patients is associated with their sex? Write a brief explanation that compares medians.

Females		Males
9	0	
5 0	1	3 6
7	2	1 4 5 6 7
7 1	3	4
3 0	4	0 7
0	5	
	6	
9	7	
0|4 = 40 years		4|0 = 40 years

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3D Investigating associations between two numerical variables

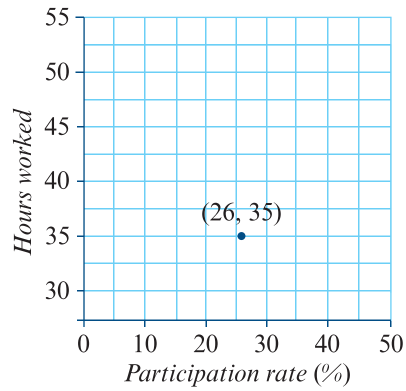
The first step in investigating the association between two numerical variables is to construct a scatterplot. We will illustrate the process by constructing a scatterplot to display average *hours worked* (the RV) against university *participation rate* (the EV) in nine countries. The data are shown below.

Participation rate (%)	26	20	36	1	25	9	30	3	55
Hours worked	35	43	38	50	40	50	40	53	35

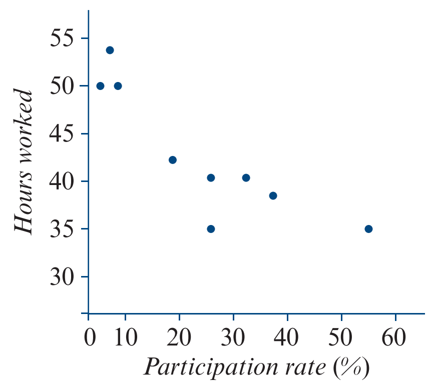
► Constructing a scatterplot

In a **scatterplot**, each point represents a single case; in this instance, a country. The horizontal or x-coordinate of the point represents the university participation rate (the EV). The vertical or y-coordinate represents the average working hours (the RV).

The scatterplot opposite shows the point for a country for which the university participation rate is 26% and average hours worked is 35.



The points for each of the remaining countries are then plotted, as shown opposite.



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Which axis?

When constructing a scatterplot it is conventional to use the *vertical* or *y-axis* for the response variable (*RV*) and the *horizontal* or *x-axis* for the explanatory variable (*EV*).

Note: Following this convention will become very important when we begin fitting lines to scatterplots in the next chapter, so it is a good habit to get into from the start.

How to construct a scatterplot using the TI-Nspire CAS

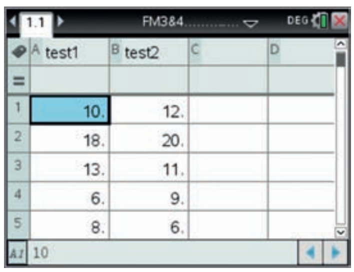
Construct a scatterplot for the set of test scores given below.
Treat *test 1* as the explanatory (i.e. x) variable.

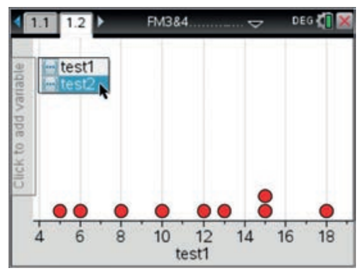
Test 1	10	18	13	6	8	5	12	15	15
Test 2	12	20	11	9	6	6	12	13	17

Steps

1 Start a new document by pressing [ctrl] + [N] [,].

2 Select **Add Lists & Spreadsheet**. Enter the data into lists named *test1* and *test2*.

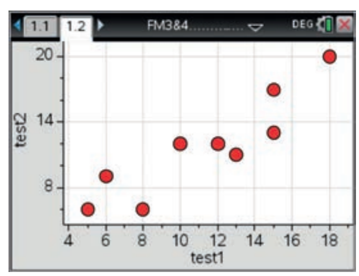




3 Press [ctrl] + [I] and select **Add Data & Statistics**.

4 a Click on **Click to add variable** on the x-axis and select the explanatory variable ***test1***.

b Click on **Click to add variable** on the y-axis and select the response variable ***test2***. A scatterplot is displayed. The plot is scaled automatically.



How to construct a scatterplot using the ClassPad

Construct a scatterplot for the set of test scores given below.
Treat *test 1* as the explanatory (i.e. x) variable.

Test 1	10	18	13	6	8	5	12	15	15
Test 2	12	20	11	9	6	6	12	13	17

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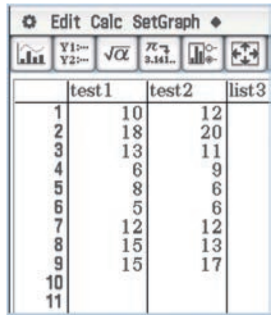
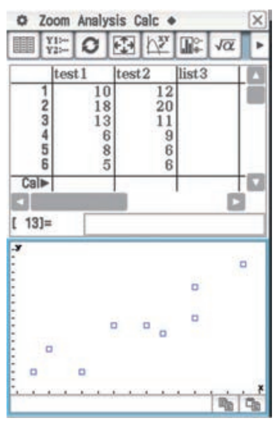
Steps

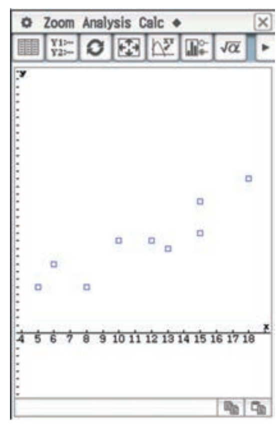


1. Open the **Statistics** application and enter the data into the columns named **test1** and **test2**.
2. Tap [icon] to open the **Set StatGraphs** dialog box and complete as given below.
   - **Draw:** select **On**.
   - **Type:** select **Scatter** ([▼]).
   - **XList:** select **main\test1**([▼]).
   - **YList:** select **main\test2**([▼]).
   - **Freq:** leave as **1**.
   - **Mark:** leave as **square**.

   Tap **Set** to confirm your selections.
3. Tap [icon] in the toolbar at the top of the screen to plot the scatterplot in the bottom half of the screen.
4. To obtain a full-screen plot, tap **Resize** from the icon panel.

Note: If you have more than one graph on your screen, tap the data screen, select StatGraph and turn off any unwanted graphs.

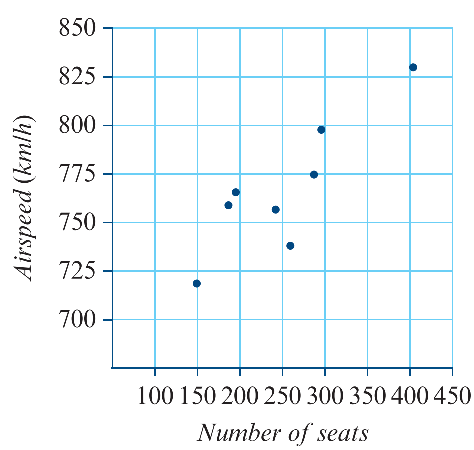




Exercise 3D

The elements of a scatterplot

1. The scatterplot opposite has been constructed to investigate the association between the airspeed (in km/h) of commercial aircraft and the number of passenger seats.
   Use the scatterplot to answer the following questions.
   a Which is the explanatory variable?
   b What type of variable is airspeed?
   c How many aircraft were investigated?
   d What was the airspeed of the aircraft that could seat 300 passengers?



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Using a CAS calculator to construct a scatterplot

2

Minimum temperature (x)	17.7	19.8	23.3	22.4	22.0	22.0
Maximum temperature (y)	29.4	34.0	34.5	35.0	36.9	36.4

The table above shows the maximum and minimum temperatures (in °C) during a hot week in Melbourne. Using a calculator, construct a scatterplot with minimum temperature as the EV. Name the variables *mintemp* and *maxtemp*.

3

Balls faced	29	16	19	62	13	40	16	9	28	26	6
Runs scored	27	8	21	47	3	15	13	2	15	10	2

The table above shows the number of runs scored and the number of balls faced by batsmen in a 1-day international cricket match. Use a calculator to construct an appropriate scatterplot. Remember to identify the EV.

4

Temperature (°C)	0	10	50	75	100	150
Diameter (cm)	2.00	2.02	2.11	2.14	2.21	2.28

The table above shows the changing diameter of a metal ball as it is heated. Use a calculator to construct an appropriate scatterplot. Temperature is the EV.

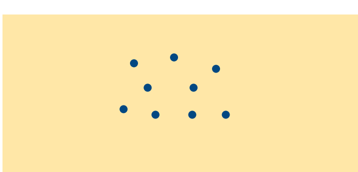
5

Number in theatre	87	102	118	123	135	137
Time (minutes)	0	5	10	15	20	25

The table above shows the number of people in a movie theatre at 5-minute intervals after the advertisements started. Use a calculator to construct an appropriate scatterplot.

3E How to interpret a scatterplot

What features do we look for in a scatterplot that will help us identify and describe any relationships present? First we look to see if there is a *clear pattern* in the scatterplot.



In the example opposite, there is *no clear pattern* in the points. The points are just *scattered randomly* across the plot.

Conclude that there is *no association.*

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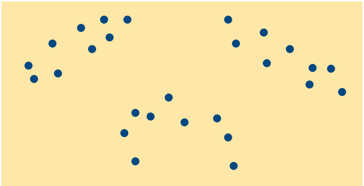


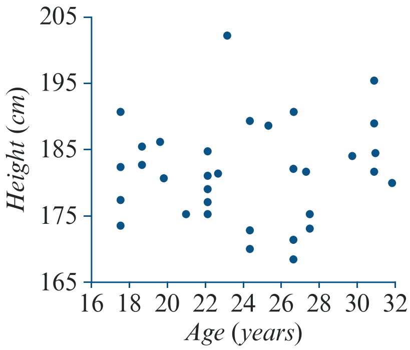
For the three examples opposite, there is a *clear* (but different) *pattern* in each set of points.
Conclude that there is an *association*.

Having found a clear pattern, there are several things we look for in the pattern of points. These are:

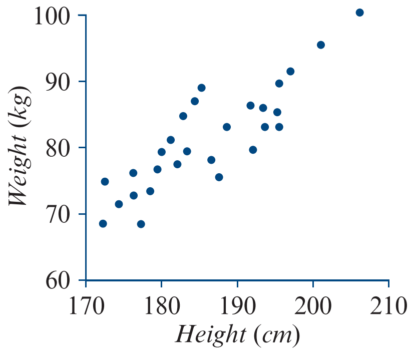
- direction and outliers (if any)
- form
- strength.

Direction and outliers

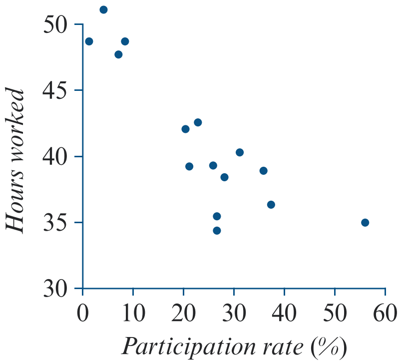
The scatterplot of height against age for a group of footballers (shown opposite) is just a *random* scatter of points. This suggests that there is *no association* between the variables *height* and *age* for this group of footballers. However, there is an *outlier*; the footballer who is 201 cm tall.



In contrast, there is a *clear pattern* in the scatterplot of weight against height for these footballers (shown opposite). The two *variables* are *associated*. Furthermore, the points *drift upwards* as you move across the plot. When this happens, we say that there is a *positive association* between the variables. Tall players tend to be heavy and vice versa. In this scatterplot, there are *no outliers*.



Likewise, the scatterplot of working hours against university participation rates for 15 countries shows a *clear pattern*. The two *variable*s are *associated*. However, in this case the points *drift downwards* as you move across the plot. When this happens, we say that there is a *negative association* between the variables. Countries with high working hours tend to have low university participation rates and vice versa. In this scatterplot, there are *no outliers*.



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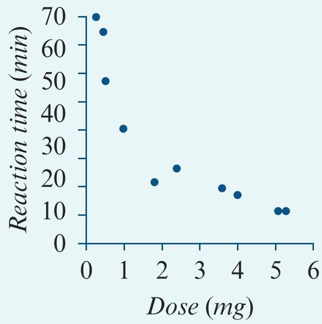
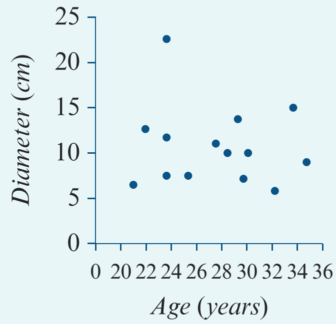
Example 8 Direction of association

Classify each of the following scatterplots as exhibiting positive, negative or no association. Where there is an association, describe the direction of the association in terms of the variables in the scatterplot and what it means in terms of the variables involved.

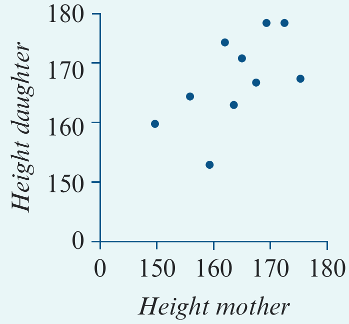
a



b



c



Solution

a There is a *clear pattern* in the scatterplot. The points in the scatterplot drift *downwards* from left to right.

The direction of the association is negative. Reaction times tend to decrease as the drug dose increases.

b There is no pattern in the scatterplot of *diameter* against *age*.

There is no association between diameter and age.

c There is a *clear pattern* in the scatterplot. The points in the scatterplot drift *upwards* from left to right.

The direction of the association is positive. Taller mothers have taller daughters.

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In general terms, we can interpret the *direction of an association* as follows.

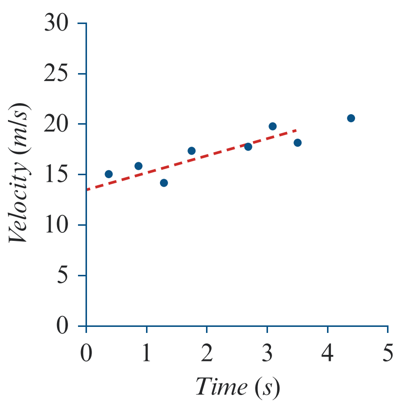
Direction of an association

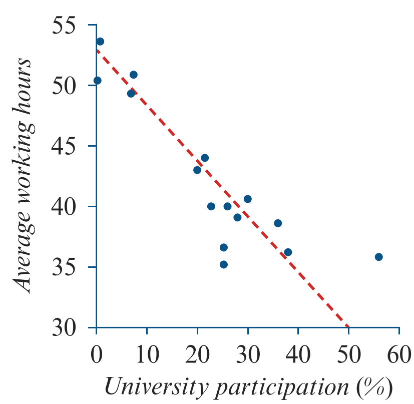
- Two variables have a *positive association* when the value of the response variable tends to increase as the value of the explanatory variable increases.
- Two variables have a *negative association* when the value of response variable tends to decrease as the value of the explanatory variable increases.
- Two variables have *no association* when there is no consistent change in the value of the response variable when the values of the explanatory variable increases.

Form

What we are looking for is whether the pattern in the points has a *linear form*. If the points in a scatterplot appear to be as random fluctuations around a *straight line*, then we say that the scatterplot has a linear form. If the scatterplot has a *linear form*, then we say that the variables are *linearly associated*.

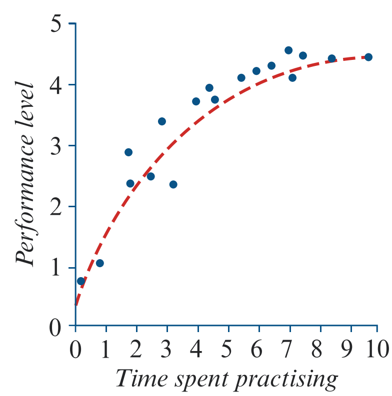
For example, both of the scatterplots below can be described as having a *linear form*; that is, the scatter in the points can be thought of as random fluctuations around a straight line. We can then say that the associations between the variables involved are linear. (The dotted lines have been added to the graphs to make it easier to see the linear form.)





By contrast, consider the scatterplot opposite, plotting performance level against time spent practising a task. There is an association between performance level and time spent practising, but it is clearly non-linear.

This scatterplot shows that while level of performance on a task increases with practice, there comes a time when the performance level will no longer improve substantially with extra practice.



While non-linear relationships exist (and we must always check for their presence by examining the scatterplot), many of the relationships we meet in practice are linear or can

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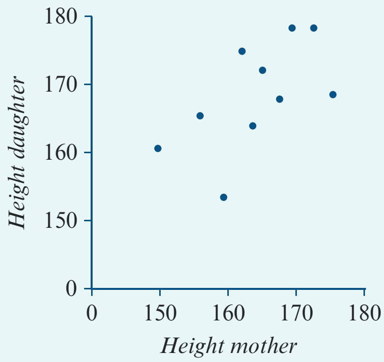
be made linear by transforming the data (a technique you will meet in Chapter 5). For this reason we will restrict ourselves to the analysis of scatterplots with linear forms for now.

Example 9 Form of an association

Classify the *form* of the association in each of scatterplot as linear or non-linear.

a



b

Solution

a There is a *clear pattern*.
The points in the scatterplot can be imagined to be scattered around a *straight line*.

The association is linear.

b There is as a *clear pattern*.
The points in the scatterplot can be imagined to be scattered around a *curved line* rather than a straight line.

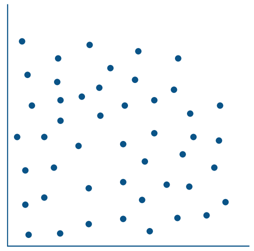
The association is non-linear.

Strength of a linear relationship: the correlation coefficient

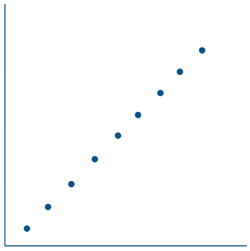
The strength of a linear association is an indication of how closely the points in the scatterplot fit a straight line. If the points in the scatterplot lie exactly on a straight line, we say that there is a perfect linear association. If there is no fit at all we say there is no association. In general, we have an imperfect fit, as seen in all of the scatterplots to date.

To measure the **strength of a linear relationship**, a statistician called Carl Pearson developed a **correlation coefficient**, ***r***, which has the following properties.

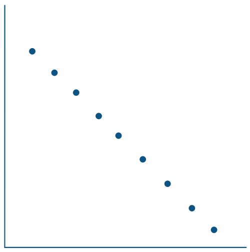
- If there is *no linear* association, $r = 0$.
- If there is a *perfect positive linear* association, $r = +1$.
- If there is a *perfect negative linear* association, $r = -1$.



$r = 0$



$r = +1$



$r = -1$

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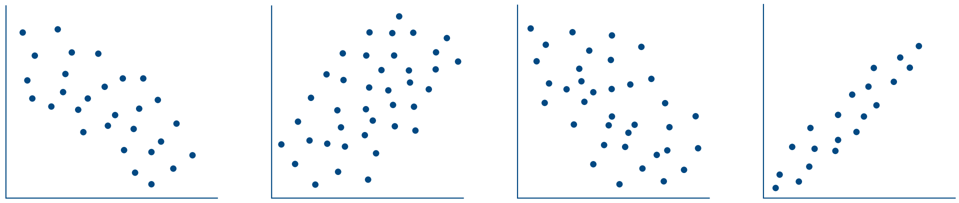
If there is a less than perfect linear association, then the correlation coefficient, r, has a value between -1 and $+1$, or $-1 < r < +1$. The scatterplots below show approximate values of r for linear associations of varying strengths.



$r = -0.7$ $r = +0.5$ $r = -0.3$ $r = +0.9$

At present, these scatterplots with their correlation coefficients should help you get a feel for the association between the correlation coefficient and a scatterplot. In the next chapter, you will learn to calculate its value. At the moment you only have to be able to roughly estimate its value.

Guidelines for classifying the strength of a linear association

Our reason for estimating the value of the correlation coefficient is to give a measure of the strength of the linear relationship. When doing this, it can be useful to classify the strength of the linear relationship as *weak*, *moderate* or *strong*, as shown opposite.

Strong positive association: r between 0.75 and 0.99
Moderate positive association: r between 0.5 and 0.74
Weak positive association: r between 0.25 and 0.49
No association: r between -0.24 and $+0.24$
Weak negative association: r between -0.25 and -0.49
Moderate negative association: r between -0.5 and -0.74
Strong negative association: r between -0.75 and -0.99

For example, the correlation coefficient between scores of a verbal skills test and a mathematical skills test is

$$r_{\text{verbal, mathematical}} = +0.275$$

indicating that there is a *weak* positive linear association.

In contrast, the correlation coefficient between carbon monoxide level and traffic volume is

$$r_{\text{CO level, traffic volume}} = +0.985$$

indicating a *strong* positive linear association between carbon monoxide level and traffic volume.

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Warning!

If you use the value of the *correlation coefficient* as a measure of the strength of an association, you are implicitly assuming that:

1 the variables are *numeric*
2 the association is *linear*
3 there are *no outliers* in the data.

The *correlation coefficient* can give a *misleading* indication of the strength of the linear association if there are outliers present.

Exercise 3E

Basic ideas

1 What three assumptions do you make when using the value of the correlation coefficient as a measure of the strength of an association?

Assessing the direction of an association from the variables

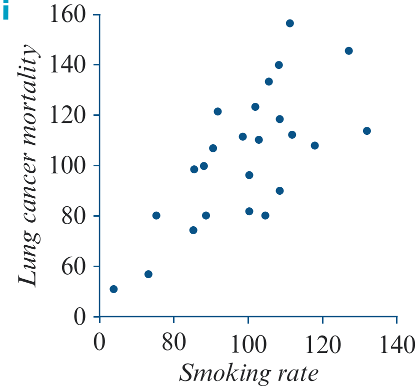
2 For each of the following pairs of variables, indicate whether you expect an association to exist between the variables. If associated, say whether you would expect the variables to be positively or negatively associated.

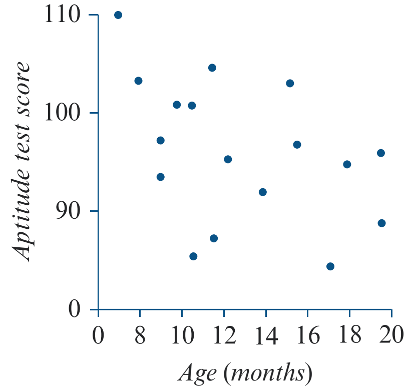
a *intelligence* and *height*
b *level of education* and *salary level*
c *salary* and *tax paid*
d *frustration* and *aggression*
e *population density* and *distance from the city centre*
f *time using social media* and *time spent studying*

Using a scatterplot to assess the direction, form and strength of an association

3 For each of the following scatterplots, state whether the variables appear to be related. If the variables appear to be related:

a state whether the association is positive or negative
b estimate the strength of the association by estimating the value of the correlation coefficient and classifying it as weak, moderate, strong or no association

i


ii


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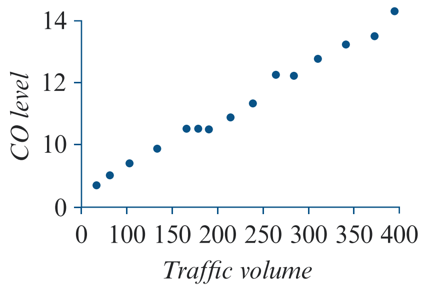
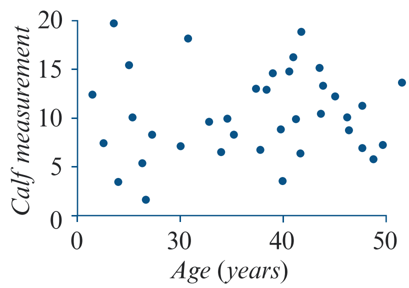
iii



iv



3F Calculating the correlation coefficient

Skillsheet

Pearson's correlation coefficient, r, gives a numerical measure of the degree to which the points in the scatterplot tend to cluster around a straight line.

Formally, if we call the two variables we are working with x and y, and we have n observations, then r is given by:

$$r = \frac{\sum(x - \bar{x})(y - \bar{y})}{(n - 1)s_x s_y}$$

In this formula, $\bar{x}$ and s_x are the mean and standard deviation of the x-values, and $\bar{y}$ and s_y are the mean and standard deviation of the y-values.

▶ Calculating r using the formula (optional)

In practice, you can always use your calculator to determine the value of the correlation coefficient. However, to understand what is involved when you use your calculator, it is best that you know how to calculate the correlation coefficient from the formula first.

How to calculate the correlation coefficient using the formula

Use the formula to calculate the correlation coefficient, r, for the following data.

x	1	3	5	4	7
y	2	5	7	2	9

$\bar{x} = 4,\ s_x = 2.236$
$\bar{y} = 5,\ s_y = 3.082$

Give the answer correct to two decimal places.

Steps

1 Write down the values of the means, standard deviations and n.

$\bar{x} = 4 \quad s_x = 2.236$
$\bar{y} = 5 \quad s_y = 3.082 \quad n = 5$

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2 Set up a table like that shown opposite to calculate $\sum(x-\bar{x})(y-\bar{y})$.

x	$(x-\bar{x})$	y	$(y-\bar{y})$	$(x-\bar{x})\times(y-\bar{y})$
1	−3	2	−3	9
3	−1	5	0	0
5	1	7	2	2
4	0	2	−3	0
7	3	9	4	12
Sum	0		0	23

$\therefore \sum(x-\bar{x})(y-\bar{y}) = 23$

3 Write down the formula for r. Substitute the appropriate values and evaluate, giving the answer correct to two decimal places.

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{(n-1)s_x s_y}$$

$$\therefore r = \frac{23}{(5-1)\times 2.236 \times 3.082}$$

$$= 0.834\ldots = 0.83 \text{ (2 d.p.)}$$

How to calculate the correlation coefficient using the TI-Nspire CAS

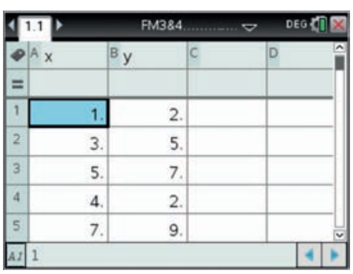
Determine the value of the correlation coefficient, r, for the given data. Give the answer correct to two decimal places.

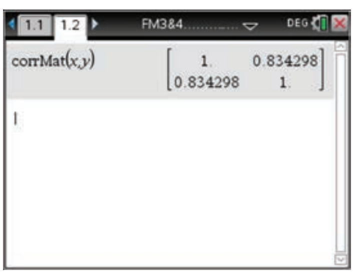
x	1	3	5	4	7
y	2	5	7	2	9

Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet.** Enter the data into lists named x and y.





3 Press ctrl + I and select **Add Calculator**.
Using the **correlation matrix** command: type in **corrmat(*x*, *y*)** and press enter .
Alternatively:

a Press 📖 1 C to access the **Catalog**, scroll down to **corrMat(** and press enter.

b Complete the command by typing in ***x*, *y*** and press enter.
The value of the correlation coefficient is $r = 0.8342\ldots$ or 0.83 (2 d.p.)

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How to calculate the correlation coefficient, *r*, using the ClassPad

The following data show the per capita income (in \$'000) and the per capita carbon dioxide emissions (in tonnes) of 11 countries.

Determine the value of Pearson's correlation coefficient correct to two decimal places.

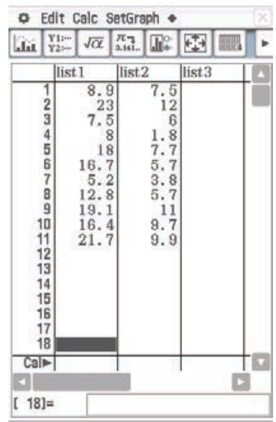
Income (\$'000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 (tonnes)	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

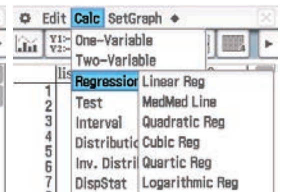
Steps

1 Open the **Statistics** application .

2 Enter the data into the columns:
- **Income** in List1
- **CO2** in List2.

3 Select **Calc>Regression>Linear Reg** from the menu bar.



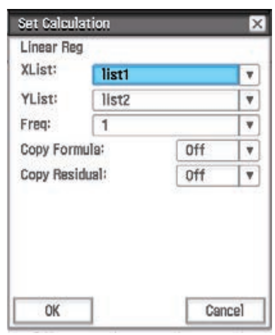


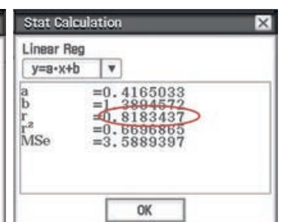
4 Press EXE.

5 Tap **OK** to confirm your selections.

The value of the correlation coefficient is

$r = 0.818\ldots$ or 0.82 (to 2 d.p.).

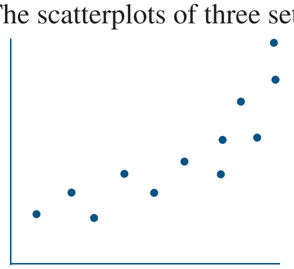




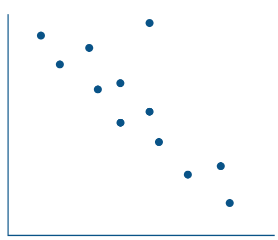
Exercise 3F

Basic ideas

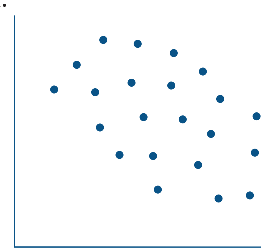
1 The scatterplots of three sets of related variables are shown.



Scatterplot A



Scatterplot B



Scatterplot C

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a For each scatterplot, describe the association in terms of strength, direction, form and outliers (if any).

b For which of these scatterplots would it be inappropriate to use the correlation coefficient, r, to give a measure of the strength of the association between the variables? Give reasons.

Calculating *r* using the formula (optional)

2 Use the formula to calculate the correlation coefficient, r, correct to two decimal places.

x	2	3	6	3	6
y	1	6	5	4	9

$\bar{x} = 4, \; s_x = 1.871$

$\bar{y} = 5, \; s_y = 2.915$

Calculating *r* using a CAS calculator

3 **a** The table below shows the maximum and minimum temperatures during a heat-wave. The *maximum* and *minimum* temperature each day are linearly associated. Use your calculator to show that $r = 0.818$, correct to three decimal places.

Day	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday
Maximum (°*C*)	29.4	34.0	34.5	35.0	36.9	36.4
Minimum (°*C*)	17.7	19.8	23.3	22.4	22.0	22.0

b This table shows the number of runs scored and balls faced by batsmen in a cricket match. *Runs scored* and *balls faced* are linearly associated. Use your calculator to show that $r = 0.8782$, correct to four decimal places.

Batsman	1	2	3	4	5	6	7	8	9	10	11
Runs scored	27	8	21	47	3	15	13	2	15	10	2
Balls faced	29	16	19	62	13	40	16	9	28	26	6

c This table shows the hours worked and university participation rate (%) in six countries. *Hours worked* and university *participation rate* are linearly associated. Use your calculator to show that $r = -0.6727$, correct to four decimal places.

Country	Australia	Britain	Canada	France	Sweden	US
Hours worked	35.0	43.0	38.2	39.8	35.6	34.8
Participation rate (%)	26	20	36	25	37	55

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3G The coefficient of determination

If two variables are associated, it is possible to estimate the value of one variable from that of the other. For example, people's weights and heights are associated. Thus, given a person's height, we can roughly predict their weight. The degree to which we can make such predictions depends on the value of r. If there is a perfect linear association ($r = 1$) between two variables, we can make an exact prediction.

For example, when you buy cheese by the gram there is an exact association between the weight of the cheese and the amount you pay ($r = 1$). At the other end of the scale, there is no association between an adult's height and their IQ ($r \approx 0$). So knowing an adult's height will not enable you to predict their IQ any better than guessing.

The coefficient of determination

The *degree* to which one variable can be predicted from another linearly related variable is given by a statistic called the **coefficient of determination**.

The coefficient of determination is *calculated* by squaring the correlation coefficient:

$$\text{coefficient of determination} = r^2$$

Calculating the coefficient of determination

Numerically, the coefficient of determination = r^2. Thus, if the correlation between weight and height is $r = 0.8$, then the

$$\text{coefficient of determination} = r^2 = 0.8^2 = 0.64 \quad \text{or} \quad 0.64 \times 100 = 64\%$$

Note: We have converted the coefficient of determination into a percentage (64%) as this is the most useful form when we come to interpreting the coefficient of determination.

Interpreting the coefficient of determination

We now know how to calculate the coefficient of determination, but what does it tell us?

Interpreting the coefficient of determination

The coefficient of determination (as a percentage) tells us the *variation in the response variable* that is *explained* by the *variation in the explanatory variable*.

But what does this mean in practical terms?

Let's take the relationship between weight and height that we just considered. Here the coefficient of determination is 0.64 (or 64%).

The coefficient of determination tells us that 64% of the variation in people's weights is explained by the variation in their height.

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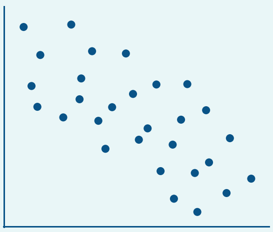
What do we mean by 'explained'?

If we take a group of people, their weights and heights will vary. One explanation is that taller people tend to be heavier and shorter people tend to be lighter. The coefficient of determination tells us that 64% of the variation in people's weights can be explained by the variation in their heights. The rest of the variation (36%) in their weights will be explained by other factors, such as sex, lifestyle, build.

Example 10 Calculating the correlation coefficient from the coefficient of determination

For the relationship described by this scatterplot, the coefficient of determination = 0.5210.

Determine the value of the correlation coefficient, r.



Solution

1 The coefficient of determination = r^2. Use the value of the coefficient of determination to set up an equation for r^2. Solve.

$r^2 = 0.5210$

$\therefore r = \pm\sqrt{0.5210} = \pm 0.7218$

2 There are two solutions, one positive and the other negative. Use the scatterplot to decide which applies.

Scatterplot indicates a negative association.

3 Write down your answer.

$\therefore r = -0.7218$

Example 11 Calculating and interpreting the coefficient of determination

Carbon monoxide (CO) levels in the air and traffic volume are linearly related, with:

$$r_{\text{CO level, traffic volume}} = +0.985$$

Determine the value of the coefficient of determination, write it in percentage terms and interpret. In this relationship, *traffic volume* is the explanatory variable.

Solution

The coefficient of determination is:

$$r^2 = (0.985)^2 = 0.970\ldots \quad \text{or} \quad 0.970 \times 100 = 97.0\%$$

Therefore, 97% of the variation in carbon monoxide levels in the air can be explained by the variation in traffic volume.

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Clearly, traffic volume is a very good predictor of carbon monoxide levels in the air. Thus, knowing the traffic volume enables us to predict carbon monoxide levels with a high degree of accuracy. This contrasts with the next example, which concerns predicting mathematical ability from verbal ability.

Example 12 **Calculating and interpreting the coefficient of determination**

Scores on tests of verbal and mathematical ability are linearly related with:

$$r_{\text{mathematical, verbal}} = +0.275$$

Determine the value of the coefficient of determination, write it in percentage terms, and interpret. In this relationship, *verbal ability* is the explanatory variable.

Solution

The coefficient of determination is:

$r^2 = (0.275)^2 = 0.0756\ldots$ or $0.076 \times 100 = 7.6\%$

Therefore, only 7.6% of the variation observed in scores on the mathematical ability test can be explained by the variation in scores obtained on the verbal ability test.

Clearly, scores on the verbal ability test are not good predictors of the scores on the mathematical ability test; 92.4% of the variation in mathematical ability is explained by other factors.

Exercise 3G

Calculating the coefficient of determination from *r*

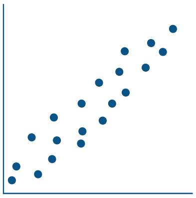
1 For each of the following values of r, calculate the value of the coefficient of determination and convert to a percentage (correct to one decimal place).

a $r = 0.675$ **b** $r = 0.345$ **c** $r = -0.567$ **d** $r = -0.673$ **e** $r = 0.124$

Calculating *r* from the coefficient of determination given a scatterplot

Note: The scatterplots have been included in Questions 2 and 3 to help you decide the sign of r.

2 **a** For the relationship described by the scatterplot shown, the coefficient of determination, $r^2 = 0.8215$.
Determine the value of the correlation coefficient, r (correct to three decimal places).



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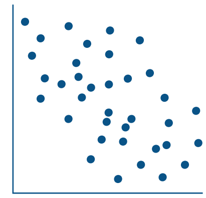
b For the relationship described by the scatterplot shown, the coefficient of determination $r^2 = 0.1243$.
Determine the value of the correlation coefficient, r (correct to three decimal places).



Calculating and interpreting the coefficient of determination

3 For each of the following, determine the value of the coefficient of determination, write it in percentage terms, and interpret.

a Scores on hearing tests and age (EV) are linearly related, with $r_{\text{hearing, age}} = -0.611$.

b Mortality rates and smoking rates (EV) are linearly related, with $r_{\text{mortality, smoking}} = 0.716$.

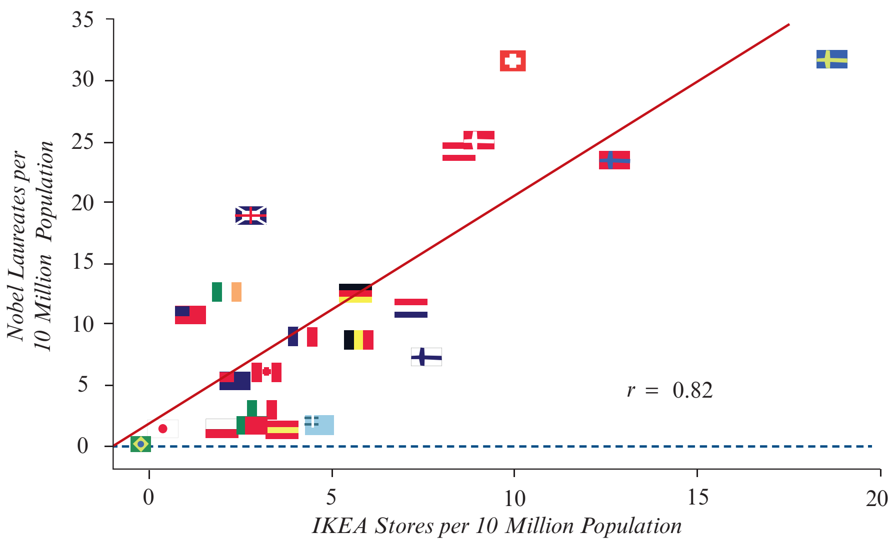
c Life expectancy and birth rates (EV) are linearly related, with $r_{\text{life expectancy, birthrate}} = -0.807$.

d Daily maximum (RV) and minimum temperatures are linearly related, with $r_{\text{max, min}} = 0.818$

e Runs scored (EV) and balls faced by a batsman are linearly related, with $r_{\text{runs, balls}} = 0.8782$

3H Correlation and causality

Recently there has been interest in the strong association between the number of Nobel prizes a country has won and the number of IKEA stores in that country ($r = 0.82$). This strong association is evident in the scatterplot below. Here country flags are used to represent the data points.



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Does this mean that one way to increase the number of Australian Nobel prize winners is to build more IKEA stores?

Almost certainly not, but this association highlights the problem of assuming that a strong *correlation* between two variables indicates the association between them is *causal*.

Correlation does not imply causality

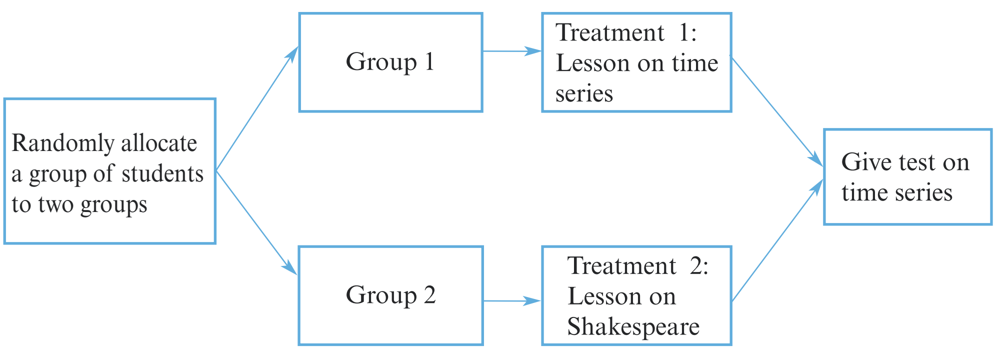
A correlation tells you about the strength of the association between the variables, but no more. It tells you nothing about the source or cause of the association.

Video

To help you with this concept, you should watch the video 'The Question of Causation', which can be accessed through the link below. It is well worth 15 minutes of your time.
http://cambridge.edu.au/redirect/?id=6103

Establishing causality

To establish causality, you need to conduct an *experiment*. In an experiment, the value of the *explanatory variable* is *deliberately manipulated*, while all other possible explanatory variables are kept constant or controlled. A simplified version of an experiment is displayed below.



In this experiment, a class of students is randomly allocated into two groups. Random allocation ensures that both groups are as similar as possible.

Next, group 1 is given a lesson on time series (treatment 1), while group 2 is given a lesson on Shakespeare (treatment 2). Both lessons are given under the same classroom conditions. When both groups are given a test on time series the next day, group 1 does better than group 2.

We then conclude that this was because the students in group 1 were given a lesson on time series.

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Is this conclusion justified?

In this experiment, the students' test score is the response variable and the type of lesson they were given is the explanatory variable. We randomly allocated the students to each group while ensuring that all other possible explanatory variables were controlled by giving the lessons under the same classroom conditions. In these circumstances, the observed difference in the response variable (*test score*) can reasonably be attributed to the explanatory variable (*lesson type*).

Unfortunately, it is extremely difficult to conduct properly controlled experiments, particularly when the people involved are going about their everyday lives.

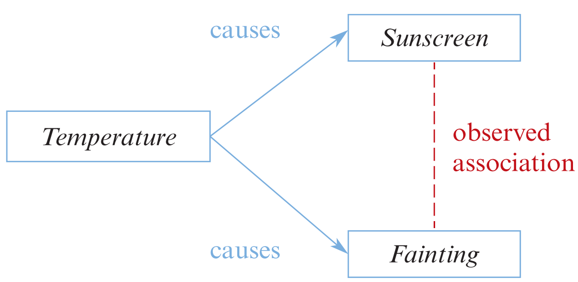
When data are collected through observation rather than experimentation, we must accept that strong association between two variables is insufficient evidence by itself to conclude that an observed change in the response variable has been caused by an observed change in the explanatory variable. It may be, but unless all of the relevant variables are under our control, there will always be alternative non-causal explanations to which we can appeal. We will now consider the various ways this might occur.

▶ Possible non-causal explanations for an association

Common response

Consider the following. There is a strong positive association between the number of people using sunscreen and the number of people fainting. Does this mean that applying sunscreen causes people to faint?

Almost certainly not. On hot and sunny days, more people *apply sunscreen* and more people *faint* due to heat exhaustion. The two variables are associated because they are both strongly associated with a common third variable, *temperature*. This phenomenon is called a *common response*. See the diagram below.



Unfortunately, being able to attribute an association to a single third variable is the exception rather than the rule. More often than not, the situation is more complex.

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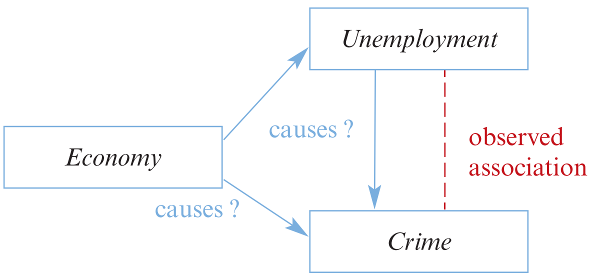
Confounding variables

Statistics show that *crime* rates and *unemployment* rates in a city are strongly correlated. Can you then conclude that a decrease in unemployment will lead to a decrease in crime rates?

It might, but other possible causal explanations could be found. For example, these data were collected during an economic downturn. Perhaps the state of the economy caused the problem. See the diagram below.



In this situation, we have at least two possible causal explanations for the observed association, but we have no way of disentangling their separate effects. When this happens, the effects of the two possible explanatory variables are said to be confounded, because we have no way of knowing which is the actual cause of the association.

Coincidence

It turns out that there is a strong correlation ($r = 0.99$) between the consumption of margarine and the divorce rate in the American state of Maine. Can we conclude that eating margarine causes people in Maine to divorce?

A better explanation is that this association is purely coincidental.

Occasionally, it is almost impossible to identify any feasible confounding variables to explain a particular association. In these cases we often conclude that the association is 'spurious' and it has happened just happened by chance. We call this *coincidence*.

Conclusion

However suggestive a strong association may be, this alone does not provide sufficient evidence for you to conclude that two variables are causally related. Unless the association is totally spurious and devoid of meaning, it will always be possible to find at least one variable 'lurking' in the background that could explain the association.

Association (correlation) and causation

By itself, an observed association between two variables is *never enough* to justify the conclusion that two variables are causally related, no matter how obvious the causal explanation may appear to be.

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Exercise 3H

1 A study of primary school children aged 5 to 11 years finds a strong positive correlation between height and score on a test of mathematics ability. Does this mean that taller people are better at mathematics? What common cause might counter this conclusion?

2 There is a clear positive correlation between the number of churches in a town and the amount of alcohol consumed by its inhabitants. Does this mean that religion is encouraging people to drink? What common cause might counter this conclusion?

3 There is a strong positive correlation between the amount of ice-cream consumed and the number of drownings each day. Does this mean that eating ice-cream at the beach is dangerous? What common cause might explain this association?

4 The number of days a patient stays in hospital is positively correlated with the number of beds in the hospital. Can it be said that bigger hospitals encourage patients to stay longer than necessary just to keep their beds occupied? What common cause might counter this conclusion?

5 Suppose we found a high correlation between smoking rates and heart disease across a group of countries. Can we conclude that smoking causes heart disease? What confounding variable(s) could equally explain this correlation?

6 There is a strong correlation between cheese consumption and the number of people who died after becoming tangled in their bed sheets. What do you think is the most likely explanation for this correlation?

7 There is a strong positive correlation between the number of fire trucks attending a house fire and the amount of damage caused by the fire. Is the amount of damage in a house fire caused by the fire trucks? What common cause might explain this association?

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3I Which graph?

One of the problems when investigating associations is choosing an appropriate graph to display and understand the data you have been given. This decision depends critically on the type of variables involved – that is, whether they are both categorical, both numerical, or one of each.

The following guidelines might help you make your decision. They are guidelines only, because in some instances there may be more than one suitable graph.

Type of variables		
Response variable	*Explanatory variable*	*Graph*
Categorical	Categorical	**Segmented bar chart**, side-by-side (parallel) bar chart
Numerical	Categorical	Parallel box plots, parallel dot plots
Numerical	Categorical (two categories only)	Back-to-back stem plot, parallel dot or box plots
Numerical	Numerical	Scatterplot

Exercise 3I

1 Which graphical display (parallel box plots, parallel dot plots, back-to-back stem plot, a segmented bar chart or a scatterplot) would be appropriate to display the relationships between the following? There may be more than one appropriate graph.

- **a** vegetarian (yes, no) and sex (male, female)
- **b** mark obtained on a statistics test and time spent studying (in hours)
- **c** number of hours spent at the beach each year and state of residence
- **d** number of CDs purchased per year and income (in dollars)
- **e** runs scored in a cricket game and number of ‘overs’ faced
- **f** attitude to compulsory sport in school (agree, disagree, no opinion) and school type (government, independent)
- **g** income level (high, medium, low) and place of residence (urban, rural)
- **h** number of cigarettes smoked per day and sex (male, female)

2 A single back-to-back stem plot would be an appropriate graphical tool to investigate the association between a car’s *speed*, in kilometres per hour, and the

- **A** driver’s *age*, in years
- **B** car’s *colour* (white, red, grey, other)
- **C** car’s *fuel consumption*, in kilometres per litre
- **D** average *distance* travelled, in kilometres
- **E** driver’s *sex* (female, male)

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Review

Key ideas and chapter summary

Two-way frequency tables	**Two-way frequency tables** are used as the starting point for investigating the association between two *categorical variables*.
Identifying associations between two variables	*Associations* between *two categorical variables* are identified by *comparing* appropriate percentages in a two-way frequency table. *Associations* between a *numerical* and a *categorical* variable are identified using parallel dot plots, box plots or a back-to-back stem plot. *Associations* between *two numerical variables* are identified using a scatter plot.
Segmented bar charts	A **segmented bar chart** can be used to graphically display the information contained in a two-way frequency table. It is a useful tool for identifying relationships between two categorical variables.
Parallel box plots	**Parallel box plots** can be used to display, identify and describe the association between a *numerical* and a *categorical* variable.
Scatterplots	A **scatterplot** is used to help identify and describe an association between two *numerical variables*. In a scatterplot, the **response variable** (**RV**) is plotted on the *vertical* axis and the **explanatory variable** (**EV**) is plotted on the *horizontal* axis.
Correlation coefficient, r	The **correlation coefficient**, r, gives a measure of the strength of a linear association.
The coefficient of determination	**Coefficient of determination** = r^2 The coefficient of determination gives the percentage of variation in the RV that can be explained by the variation in the EV.
Correlation and causation	A *correlation* between two variables does not automatically imply that the association is *causal*. Alternative *non-causal explanations* for the association include a *common response* to a common third variable, a *confounded* variable or simply *coincidence*.

Skills check

Having completed this chapter you should be able to:

- interpret the information contained in a two-way frequency table
- identify, where appropriate, the response and explanatory variable in a pair of related variables
- identify associations in tabulated data by forming and comparing appropriate percentages

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- represent a two-way percentaged frequency table by a segmented bar chart and interpret the chart
- choose from among a scatterplot, segmented bar chart and parallel box plots as a means of graphically displaying the association between two variables
- construct a scatterplot
- use a scatterplot to describe an association between two numerical variables in terms of:
  - direction (positive or negative association) and possible outliers
  - form (linear or non-linear)
  - strength (weak, moderate, strong)
- calculate and interpret the correlation coefficient, r
- know the three key assumptions made when using Pearson's correlation coefficient r as a measure of the strength of the linear association between two variables; that is:
  - the variables are numerical
  - the association is linear
  - there are no clear outliers
- calculate and interpret the coefficient of determination
- understand that finding an association between two variables does not automatically indicate a causal association
- identify situations where unjustified statements about causality could be (or have been) made and recognise possible non-causal explanations as examples of a *common response, confounding* or *coincidence*.

Multiple-choice questions

The information in the following frequency table relates to Questions 1 to 4.

	Sex	
Plays sport	*Male*	*Female*
Yes	68	79
No	34	
Total	102	175

1 The variables *plays sport* and *sex* are:

A both categorical variables

B a categorical and a numerical variable, respectively

C a numerical and a categorical variable, respectively

D both numerical variables

E neither numerical nor categorical variables

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Review

2 The number of females who do not play sport is:

A 21 **B** 45 **C** 79 **D** 96 **E** 175

3 The percentage of males who do not play sport is:

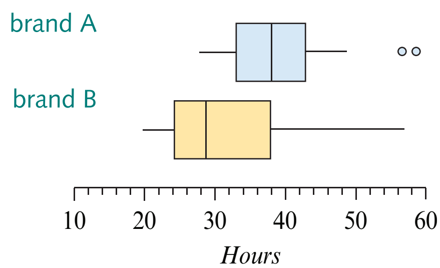
A 19.4% **B** 33.3% **C** 34.0% **D** 66.7% **E** 68.0%

4 The variables *plays sport* and *sex* appear to be associated because:

A more females play sport than males
B fewer males play sport than females
C a higher percentage of females play sport compared to males
D a higher percentage of males play sport compared to females
E both males and females play a lot of sport

The information in the following parallel box plots relates to Questions 5 and 6.

The parallel box plots shown display the distribution of battery life (in hours) for two brands of batteries (brand A and brand B).



5 The variables *battery life* and *brand* are:

A both categorical variables
B a categorical and a numerical variable respectively
C a numerical and a categorical variable respectively
D both numerical variables
E neither a numerical nor a categorical variable

6 Which of the following statements (there may be more than one) support the contention that *battery life* and *brand* are related?

I the median battery life for brand A is clearly higher than for brand B
II battery lives for brand B are more variable than brand A
III the distribution of battery lives for brand A is symmetrical with outliers but positively skewed for brand B

A I only **B** II only **C** III only
D I and II only **E** I, II and III

7 The association between weight at age 21 (in kg) and weight at birth (in kg) is to be investigated. The variables *weight at age 21* and *weight at birth* are:

A both categorical variables
B a categorical and a numerical variable respectively

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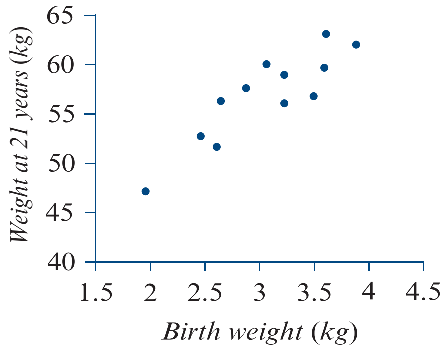
C a numerical and a categorical variable respectively
D both numerical variables
E neither numerical nor categorical variables

8 The scatterplot shows the weights at age 21 and at birth of 12 women. The association is best described as a:



A weak positive linear
B weak negative linear
C moderate positive non-linear
D strong positive non-linear
E strong positive linear

9 The variables *response time* to a drug and *drug dosage* are linearly associated, with $r = -0.9$. From this information, we can conclude that:

A response times are –0.9 times the drug dosage
B response times decrease with decreased drug dosage
C response times decrease with increased drug dosage
D response times increase with increased drug dosage
E response times are 81% of the drug dosage

10 The birth weight and weight at age 21 of eight women are given in the table below.

Birth weight (kg)	1.9	2.4	2.6	2.7	2.9	3.2	3.4	3.6
Weight at 21 (kg)	47.6	53.1	52.2	56.2	57.6	59.9	55.3	56.7

The value of the correlation coefficient is closest to:

A 0.536 **B** 0.6182 **C** 0.7863 **D** 0.8232 **E** 0.8954

11 The value of a correlation coefficient is $r = –0.7685$. The value of the corresponding coefficient of determination is closest to:

A –0.77 **B** –0.59 **C** 0.23 **D** 0.59 **E** 0.77

Use the following information to answer Questions 12 and 13.

The correlation coefficient between heart weight and body weight in a group of mice is $r = 0.765$.

12 Using body weight as the EV, we can conclude that:

A 58.5% of the variation in heart weight is explained by the variation in body weights
B 76.5% of the variation in heart weight is explained by the variation in body weights
C heart weight is 58.5% of body weight
D heart weight is 76.5% of body weight
E 58.5% of the mice had heavy hearts

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13 Given that heart weight and body weight of mice are strongly correlated ($r = 0.765$), we can conclude that:

A increasing the body weights of mice will decrease their heart weights
B increasing the body weights of mice will increase their heart weights
C increasing the body weights of mice will not change their heart weights
D heavier mice tend to have lighter hearts
E heavier mice tend to have heavier hearts

14 We wish to investigate the association between the variables *weight* (in kg) of young children and *level of nutrition* (poor, adequate, good). The most appropriate graphical display would be:

A a histogram **B** parallel box plots **C** a segmented bar chart
D a scatterplot **E** a back-to-back stem plot

15 We wish to investigate the association between the variables *weight* (underweight, normal, overweight) of young children and *level of nutrition* (poor, adequate, good). The most appropriate graphical display would be:

A a histogram **B** parallel box plots **C** a segmented bar chart
D a scatterplot **E** a back-to-back stem plot

16 There is a strong linear positive correlation ($r = 0.85$) between the amount of *garbage recycled* and *salary level.*
From this information, we can conclude that:

A the amount of garbage recycled can be increased by increasing people's salaries
B the amount of garbage recycled can be increased by decreasing people's salaries
C increasing the amount of garbage you recycle will increase your salary
D people on high salaries tend to recycle less garbage
E people on high salaries tend to recycle more garbage

17 There is a strong linear positive correlation ($r = 0.95$) between the marriage rate in Kentucky and the number of people who drown falling out of a fishing boat.
From this information, the most likely conclusion we can draw from this correlation is:

A reducing the number of marriages in Kentucky will decrease the number of people who drown falling out of a fishing boat
B increasing the number of marriages in Kentucky will increase the number of people who drown falling out of a fishing boat
C this correlation is just coincidence, and changing the marriage rate will not affect the number of people drowning in Kentucky in any way
D only married people in Kentucky drown falling out of a fishing boat
E stopping people from going fishing will reduce the marriage rate in Kentucky

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Extended-response questions

1 One thousand drivers who had an accident during the past year were classified according to age and the number of accidents.

Number of accidents	*Age < 30*	*Age ≥ 30*
At most one accident	130	170
More than one accident	470	230
Total	600	400

a What are the variables shown in the table? Are they categorical or numerical?

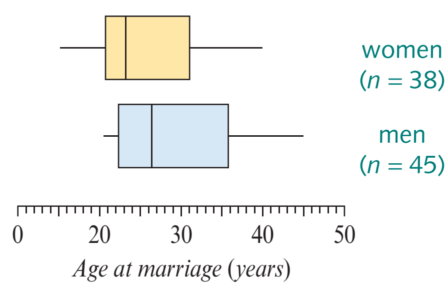
b Determine the response and explanatory variables.

c How many drivers under the age of 30 had more than one accident?

d Convert the table values to percentages by calculating the column percentages.

e Use these percentages to comment on the statement: 'Of drivers who had an accident in the past year, younger drivers (age < 30) are more likely than older drivers (age ≥ 30) to have had more than one accident.'

2 The parallel box plots compare the ages at marriage of 45 men and 38 women.



a The two variables are *age at marriage* and *sex*. Which is numerical and which is categorical?

b Do the parallel box plots support the contention that the age at marriage is associated with their sex? Explain why.

3 The data below give the hourly pay rates (in dollars per hour) of 10 production-line workers along with their years of experience on initial appointment.

Rate ($/h)	15.90	15.70	16.10	16.00	16.79	16.45	17.00	17.65	18.10	18.75
Experience (years)	1.25	1.50	2.00	2.00	2.75	4.00	5.00	6.00	8.00	12.00

a Use a graphics calculator to construct a scatterplot of the data, with the variable *rate* plotted on the vertical axis and the variable *experience* on the horizontal axis. Why has the vertical axis been used for the variable *rate*?

b Comment on direction, outliers, form and strength of any association revealed.

c Determine the value of the correlation coefficient (r) correct to three decimal places.

d Determine the value of the coefficient of determination (r^2) and interpret.

ISBN 978-1-107-56757-3

Chapter 4

4

Core: Data analysis

Regression: fitting lines to data

ISBN 978-1-107-56757-3

Cambridge University Press

4A Least-squares regression line

The process of fitting a straight line to bivariate data is known as **linear regression**.

The aim of linear regression is to model the association between two numerical variables by using a simple mathematical relation, the straight line. Knowing the equation of this line gives us a better understanding of the nature of the association. It also enables us to make predictions from one variable to another – for example, a young boy's adult height from his father's height.

The easiest way to fit a line to bivariate data is to construct a scatterplot and draw the line 'by eye'. We do this by placing a ruler on the scatterplot so that it seems to follow the general trend of the data. You can then use the ruler to draw a straight line. Unfortunately, unless the points are very tightly clustered around a straight line, the results you get by using this method will differ a lot from person to person.

The most common approach to fitting a straight line to data is to use the **least squares method**. This method assumes that the variables are linearly related, and works best when there are no clear outliers in the data.

Some terminology

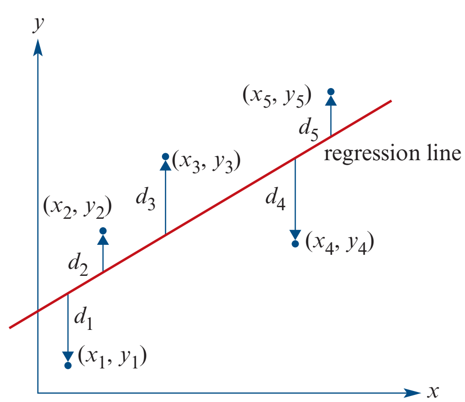
To explain the least squares method, we need to define several terms.

The scatterplot shows five data points, (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) and (x_5, y_5).

A regression line (not necessarily the least squares line) has also been drawn on the scatterplot.

The vertical distances d_1, d_2, d_3, d_4 and d_5 of each of the data points from the regression line are also shown.

These vertical distances, d, are known as **residuals**.



The least squares line

The least squares line is the line where the sum of the squares of the residuals is as small as possible; that is, it minimises:

$$\text{the sum of the squares of the residuals} = d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

Why do we minimise the sum of the *squares* of the residuals and not the sum of the residuals? This is because the sum of the residuals for the least squares line is always zero. The least squares line is like the mean. It balances out the data values on either side of itself. Some residuals are positive and some negative, and in the end they add to zero. Squaring the residuals solves this problem.

ISBN 978-1-107-56757-3

Cambridge University Press

The least squares line

The *least squares line* is the line that *minimises* the *sum of the squares* of the *residuals*.

The *assumptions* for fitting a least squares line to data are the same as for using the correlation coefficient, r. These are that:

- the data is numerical
- the association is linear
- there are no clear outliers.

How do we determine the least squares regression line?

One method is 'trial-and-error'.

We could draw a series of lines, each with a different slope and intercept. For each line, we could then work out the value of each of the residuals, square them, and calculate their sum. The least squares line would be the one that minimises these sums.

To see how this might work, you can simulate the process of fitting a least squares line using the interactive 'Regression line'.

The trial-and-error method does not guarantee that we get the exact solution. Fortunately, the exact solution can be found mathematically, using the techniques of calculus. Although the mathematics is beyond Further Mathematics, we will use the results of this theory summarised below.

The equation of the least squares regression line

The equation of the least squares regression line is given by $y = a + bx$,[*] where:

the **slope** (b) is given by $b = \dfrac{rs_y}{s_x}$

and

the **intercept** (a) is then given by $a = \bar{y} - b\bar{x}$

Here:

- r is the correlation coefficient
- s_x and s_y are the standard deviations of x and y
- $\bar{x}$ and $\bar{y}$ are the mean values of x and y.

* In mathematics you are used to writing the equation of a straight line as $y = mx + c$. However, statisticians write the equation of a straight line as $y = a + bx$. This is because statisticians are in the business of building linear models. Putting the variable term second in the equation allows for additional variable terms to be added; for example, $y = a + bx + cz$. While this sort of model is beyond Further Mathematics, we will continue to use $y = a + bx$ to represent the equation of the regression line because it is common statistical practice.

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Exercise 4A

Basic ideas

1 What is a residual?

2 The least-squares regression line is obtained by:

A minimising the residuals

B minimising the sum of the residuals

C minimising the sum of the squares of the residuals

D minimising the square of the sum of the residuals

E maximising the sum of the squares of the residuals.

3 Write down the three assumptions we make about the association we are modelling when we fit a least squares line to bivariate data.

4B Determining the equation of the least squares line

If we know the values of r, $\bar{x}$ and $\bar{y}$, and s_x and s_y, we can determine the equation of the least squares line by using the formulas from the previous section. If all you have are the actual data values, you will use your CAS calculator to do the computation. Both methods are demonstrated in this section.

Warning!

If you do not correctly decide which is the explanatory variable (the x-variable) and which is the response variable (the y-variable) before you start calculating the equation of the least squares regression line, you may get the wrong answer.

How to determine the equation of a least squares line using the formula

The heights (x) and weights (y) of 11 people have been recorded, and the values of the following statistics determined:

$$\bar{x} = 173.3 \text{ cm} \quad s_x = 7.444 \text{ cm} \quad \bar{y} = 65.45 \text{ cm} \quad s_y = 7.594 \text{ cm} \quad r = 0.8502$$

Use the formula to determine the equation of the least squares regression line that enable weight to be predicted from height. Calculate the slope and intercept correct to two significant figures.

Steps

1 Identify and write down the explanatory variable (EV) and response variable (RV). Label as x and y, respectively.

Note: In saying that we want to predict weight from height, we are implying that height is the EV.

EV: height (x)

RV: weight (y)

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2 Write down the given information.

$\bar{x} = 173.3 \qquad s_x = 7.444$

$\bar{y} = 65.45 \qquad s_y = 7.594$

$r = 0.8502$

3 Calculate the slope.

Slope:

$$b = \frac{rs_y}{s_x} = \frac{0.8502 \times 7.594}{7.444}$$

$= 0.867$ (correct to 3 sig. figs.)

4 Calculate the intercept.

Intercept:

$a = \bar{y} - b\bar{x}$

$= 65.45 - 0.867 \times 173.3$

$= -85$ (correct to 2 sig. figs.)

5 Use the values of the intercept and the slope to write down the least squares regression line using the variable names.

$y = -84.8 + 0.87x$

or

$weight = -85 + 0.87 \times height$

How to determine and graph the equation of a least squares regression line using the TI-Nspire CAS

The following data give the height (in cm) and weight (in kg) of 11 people.

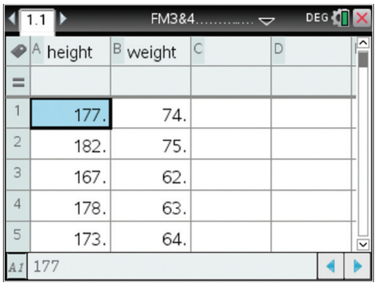
Height (x)	177	182	167	178	173	184	162	169	164	170	180
Weight (y)	74	75	62	63	64	74	57	55	56	68	72

Determine and graph the equation of the least squares regression line that will enable weight to be predicted from height. Write the intercept and slope correct to three significant figures.

Steps

1 Start a new document by pressing [ctrl] + [N].

2 Select **Add Lists & Spreadsheet**. Enter the data into lists named *height* and *weight*, as shown.



3 Identify the explanatory variable (EV) and the response variable (RV).

EV: *height*

RV: *weight*

Note: In saying that we want to predict *weight* from *height*, we are implying that *height* is the EV.

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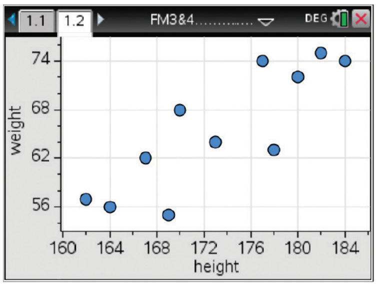
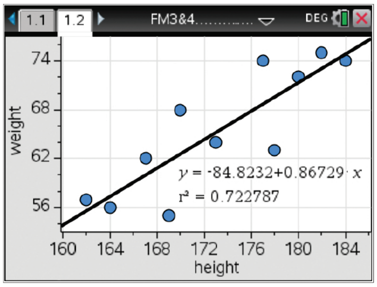
4 Press ctrl + I and select **Add Data & Statistics** and construct a scatterplot with *height* (EV) on the horizontal (or x-) axis and *weight* (RV) on the vertical (or y-) axis.
Press menu **>Settings** and click the **Diagnostics** box. Select **Default** to activate this feature for *all* future documents. This will show the coefficient of determination (r^2) whenever a regression is performed.



5 Press menu >**Analyze**>**Regression**>**Show Linear (a + bx)** to plot the regression line on the scatterplot. Note that, simultaneously, the equation of the regression line is shown on the screen.
The equation of the regression line is:

$$y = -84.8 + 0.867x$$

$$\text{or } weight = -84.8 + 0.867 \times height$$



The coefficient of determination is $r^2 = 0.723$, correct to three significant figures.

How to determine and graph the equation of a least squares regression line using the ClassPad

The following data give the height (in cm) and weight (in kg) of 11 people.

Height (x)	177	182	167	178	173	184	162	169	164	170	180
Weight (y)	74	75	62	63	64	74	57	55	56	68	72

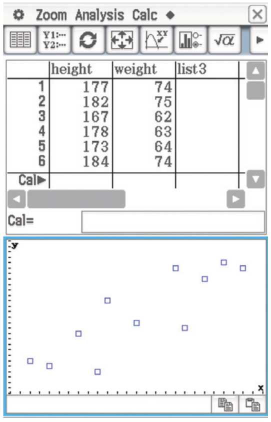
Determine and graph the equation of the least squares regression line that will enable weight to be predicted from height. Write the intercept and slope correct to three significant figures.

Steps

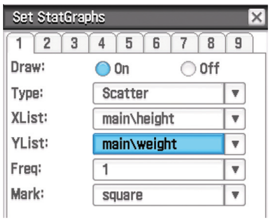
1 Open the **Statistics** application Statistics and enter the data into columns labelled **height** and **weight**.

2 Tap to open the **Set StatGraphs** dialog box and complete as shown.
Tap **Set** to confirm your selections.

3 Tap in the toolbar at the top of the screen to plot the scatterplot in the bottom half of the screen.



	height	weight	list3
1	177	74	
2	182	75	
3	167	62	
4	178	63	
5	173	64	
6	184	74	



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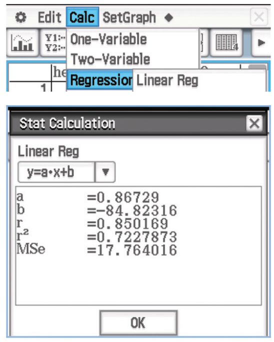
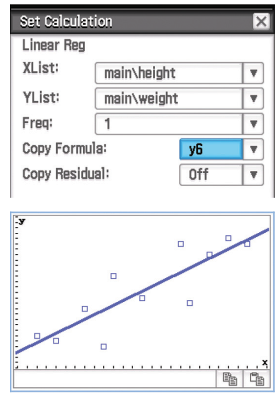
4 To calculate the equation of the least squares regression line:

- Tap **Calc** from the menu bar.
- Tap **Regression** and select **Linear Reg**.
- Complete the **Set Calculations** dialog box as shown.
- Tap **OK** to confirm your selections in the **Set Calculations** dialog box. This also generates the regression results shown opposite.





- Tapping **OK** a second time automatically plots and displays the regression line.

Note: **y6** as the formula destination is an arbitrary choice.

5 Use the values of the slope a and intercept b to write the equation of the least squares line in terms of the variables *weight* and *height*.

Weight $= -84.8 + 0.867 \times$ *height* (to three significant figures)

The coefficient of determination is $r^2 = 0.723$, correct to three significant places.

Exercise 4B

Note: In fitting lines to data, unless otherwise stated, the slope and intercept should be calculated accurate to a given number of significant figures as required in the Further and General Mathematics study designs.

If you feel that you need some help with significant figures see the video on the topic, accessed through the Interactive Textbook.

Using a formula to calculate the equation of a least squares line

1 We wish to find the equation of the least squares regression line that enables *pollution level* beside a freeway to be predicted from *traffic volume*.

a Which is the response variable (RV) and which is the explanatory variable (EV)?

b Use the formula to determine the equation of the least squares regression line that enables the pollution level (y) to be predicted from the traffic volume (x), where:

$r = 0.940 \quad \bar{x} = 11.4 \quad s_x = 1.87$

$\bar{y} = 231 \quad s_y = 97.9$

Write the equation in terms of *pollution level* and *traffic volume* with the y-intercept and slope written correct to two significant figures.

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2 We wish to find the equation of the least squares regression line that enables *life expectancy* in a country to be predicted from *birth rate*.

a Which is the response variable (RV) and which is the explanatory variable (EV)?

b Use the formula to determine the equation of the least squares regression line that enables life expectancy (y) to be predicted from birth rate (x), where:

$$r = -0.810 \quad \bar{x} = 34.8 \quad s_x = 5.41 \quad \bar{y} = 55.1 \quad s_y = 9.99$$

Write the equation in terms of *life expectancy* and *birth rate* with the y-intercept and slope written correct to two significant figures.

3 We wish to find the equation of the least squares regression line that enables *distance* travelled by a car (in 1000s of km) to be predicted from its *age* (in years).

a Which is the response variable (RV) and which is the explanatory variable (EV)?

b Use the formula to determine the equation of the least squares regression line that enables distance travelled (y) by a car to be predicted from its age (x), where:

$$r = 0.947 \quad \bar{x} = 5.63 \quad s_x = 3.64 \quad \bar{y} = 78.0 \quad s_y = 42.6$$

Write the equation in terms of *distance travelled* and *age* with the y-intercept and slope written correct to two significant figures.

Some big ideas

4 The following questions relate to the formulas used to calculate the slope and intercept of the least squares regression line.

a A least squares line is calculated and the slope is found to be negative. What does this tell us about the sign of the correlation coefficient?

b The correlation coefficient is zero. What does this tell us about the slope of the least squares regression line?

c The correlation coefficient is zero. What does this tell us about the intercept of the least squares regression line?

Using a CAS calculator to determine the equation of a least squares line from raw data

5 The table shows the number of sit-ups and push-ups performed by six students.

Sit-ups (x)	52	15	22	42	34	37
Push-ups (y)	37	26	23	51	31	45

Let the number of *sit-ups* be the explanatory (x) variable. Use your calculator to show that the equation of the least squares regression line is:

push-ups = 16.5 + 0.566 × *sit-ups* (correct to three significant figures)

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6 The table shows average hours worked and university participation rates (%) in six countries.

Hours	35.0	43.0	38.2	39.8	35.6	34.8
Rate	26	20	36	25	37	55

Use your calculator to show that the equation of the least squares regression line that enables participation *rates* to be predicted from *hours* worked is:

$$rate = 130 - 2.6 \times hours \text{ (correct to two significant figures)}$$

7 The table shows the number of *runs* scored and *balls* faced by batsmen in a cricket match.

Runs (y)	27	8	21	47	3	15	13	2	15	10	2
Balls faced (x)	29	16	19	62	13	40	16	9	28	26	6

a Use your calculator to show that the equation of the least squares regression line enabling *runs* scored to be predicted from *balls faced* is:

$$y = -2.6 + 0.73x$$

b Rewrite the regression equation in terms of the variables involved.

8 The table below shows the number of TVs and cars owned (per 1000 people) in six countries.

Number of TVs (y)	378	404	471	354	381	624
Number of cars (x)	417	286	435	370	357	550

We wish to predict the *number of TVs* from the *number of cars*.

a Which is the response variable?

b Show that, in terms of x and y, the equation of the regression line is:

$$y = 61.2 + 0.930x \text{ (correct to three significant figures).}$$

c Rewrite the regression equation in terms of the variables involved.

4C Performing a regression analysis

Having learned how to calculate the equation of the least squares regression line, you are well on the way to learning how to perform a full regression analysis. In the process, you will need to use of many of the skills you have so far developed when working with scatterplots and correlation coefficients.

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The elements of a regression analysis

A full regression analysis involves several processes, which include:

- constructing a *scatterplot* to investigate the nature of an association
- calculating the *correlation coefficient* to indicate the strength of the relationship
- determining the equation of the *regression line*
- *interpreting the coefficients* the *y-intercept* (a) and the *slope* (b) of the least squares line $y = a + bx$
- using the *coefficient of determination* to indicate the *predictive power* of the association
- using the *regression line* to make *predictions*
- calculating residuals and using a *residual plot* to test the *assumption of linearity*
- writing a *report* on your findings.

An analysis using some data

We wish to investigate the nature of the association between the price of a second-hand car and its age. The ultimate aim is to find a mathematical model that will enable the price of a second-hand car to be predicted from its age.

To this end, the age (in years) and price (in dollars) of a selection of second-hand cars of the same brand and model have been collected and are recorded in a table (shown).

Age (years)	*Price (dollars)*	*Age (years)*	*Price (dollars)*
1	32 500	4	19 200
1	30 500	5	16 000
2	25 600	5	18 400
3	20 000	6	6 500
3	24 300	7	6 400
3	22 000	7	8 500
4	22 000	8	4 200
4	23 000		

The scatterplot and correlation coefficient

We start our investigation of the association between price and age by constructing a scatterplot and using it to describe the association in terms of strength, direction and form. In this analysis, *age* is the explanatory variable.

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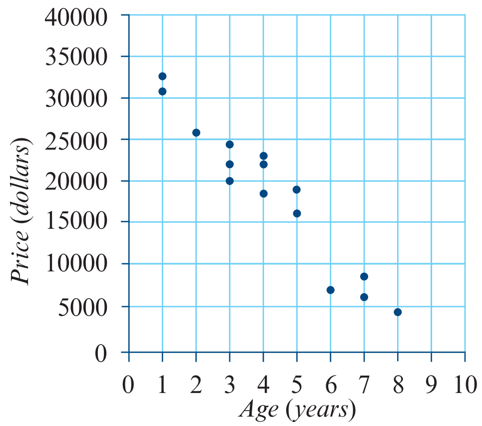
From the scatterplot, we see that there is a *strong negative linear association* between the price of the car and its age. There are no clear outliers. The correlation coefficient is $r = -0.964$.

We would communicate this information in a report as follows.



Report

There is a strong negative linear association between the price of these second-hand cars and their age ($r = -0.964$).

Fitting a least-squares regression line to the data

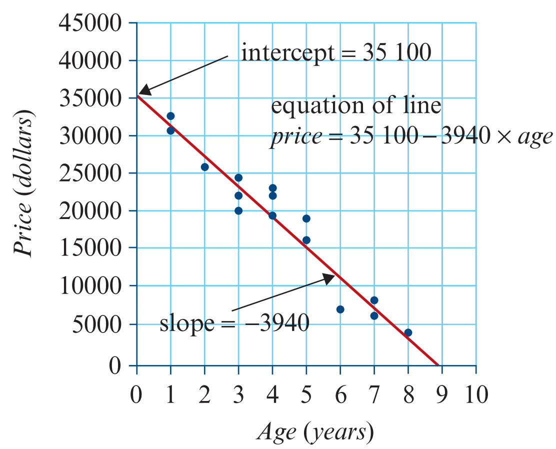
Because the association is linear, it is reasonable to use a least squares regression line to model the association.

Using a calculator, the equation of the least regression line for this data is:

price = 35 100 –3940 × *age*

This line has been plotted on the scatterplot as shown opposite.

We now have a mathematical model to describe, *on average*,[1] how the price of this type of second-hand car changes with time.



Interpreting the slope and the intercept of the regression line

The two key values in our mathematical model are the slope of the line (–3940) and the y-intercept (35 100), and a key step in performing a regression analysis is to know how to interpret these values in terms of the variables *price* and *age*.

[1] We say 'on average' because the line does not pass through every point. Rather, it is a line that is drawn through the average price of this type of car at each age.

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Interpreting the slope and intercept of a regression line

For the regression line $y = a + bx$:

- the slope (b) estimates the average change (increase/decrease) in the *response variable* (y) for each one-unit increase in the *explanatory variable* (x)
- the intercept (a) estimates the average value of the *response variable* (y) when the *explanatory variable* (x) equals 0.

Note: The interpretation of the y-intercept in a data context can be problematic when $x = 0$ is not within the range of observed x-values.

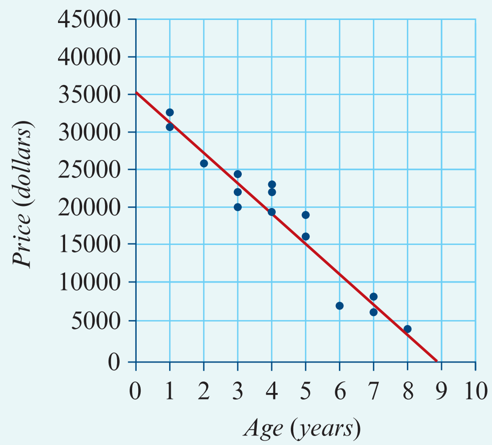
Example 1 Interpreting the slope and intercept of a regression line

The equation of a regression line that enables the *price* of a second-hand car to be predicted from its *age* is:

$$price = 35\,100 - 3940 \times age$$

a Interpret the slope in terms of the variables *price* and *age*.

b Interpret the intercept in terms of the variables *price* and *age*.



Solution

a The *slope* predicts the average change (increase/decrease) in the *price* for each 1-year increase in the *age*. Because the slope is negative, it will be a decrease.

On average, the price of these cars decreases by \$3940 each year.[2]

b The *intercept* predicts the value of the *price* of the car when *age* equals 0; that is, when the car is new.

On average, the price of these cars when new was \$35 100.

[2] A general model for writing about the slope is that, on average, the RV increases/decreases by the value of the 'slope' for each unit increase in the EV.

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Using the regression line to make predictions

Example 2 Using the regression line to make predictions

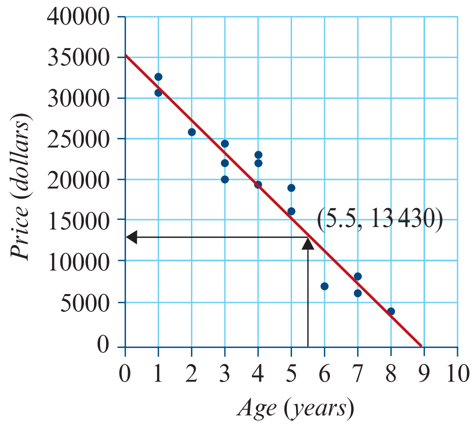
The equation of a regression line that enables the *price* of a second-hand car to be predicted from its *age* is:

$$price = 35\,100 - 3940 \times age$$

Use this equation to predict the price of a car that is 5.5 years old.

Solution

There are two ways this can be done. One is to draw a vertical arrow at *age* = 5.5 up to the graph and then horizontally across to the *price* axis as shown, to get an answer of around \$14 000. A more accurate answer is obtained by substituting *age* = 5.5 into the equation to obtain \$13 430, as shown below.



$$Price = 35\,100 - 3940 \times 5.5$$
$$= \$13\,430$$

Interpolation and extrapolation

When using a regression line to make predictions, we must be aware that, strictly speaking, the equation we have found applies only to the range of data values used to derive the equation.

Predicting *within* the range of data is called **interpolation**.

In general, we can expect a reasonably reliable result when interpolating.

Predicting *outside* the range of data is called **extrapolation**.

With extrapolation, we have no way of knowing whether our prediction is reliable or not.

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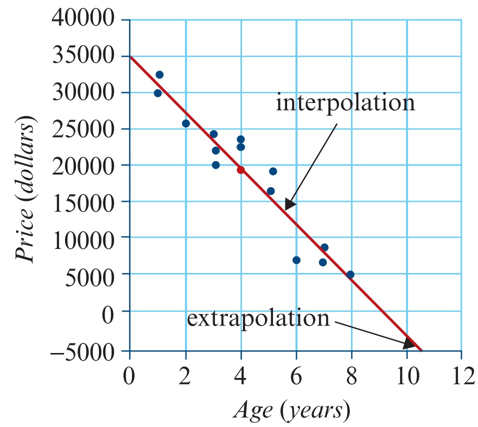
For example, using the regression line to predict the price of a car less than 5.5 years old would be an example of *interpolation*. This is because we are making a prediction *within* the data.

However, using the regression line to predict the price of a 10-year-old car would be an example of *extrapolation*. This is because we are working *outside* the data.

The fact that the regression line would predict a negative value for the car highlights the potential problems of making predictions beyond the data (extrapolating).



The coefficient of determination

The coefficient of determination is a measure of the predictive power of a regression equation. While the association between the price of a second-hand car and its age does not explain all the variation in price, knowing the age of a car does give us some information about its likely price.

For a perfect relationship, the regression line explains 100% of the variation in prices. In this case, with $r = -0.964$ we have the:

$$\text{coefficient of determination} = r^2 = 0.964^2 \approx 0.930 \text{ or } 93.0\%$$

Thus, we can conclude that:

> 93% of the variation in price of the second-hand cars can be explained by the variation in the ages of the cars.[3]

In this case, the regression equation has highly significant (worthwhile) predictive power. As a guide, any relationship with a coefficient of determination greater than 30% can be regarded as having significant predictive power.

Residuals revisited

Calculating residuals

As we saw earlier (Section 4A), the residuals are the vertical distances between the individual data points and the regression line.

[3] A general model for writing about the coefficient of determination is: 'r^2% of the variation in the RV is explained by the variation in the EV'.

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Residuals

residual value = actual data value − predicted value

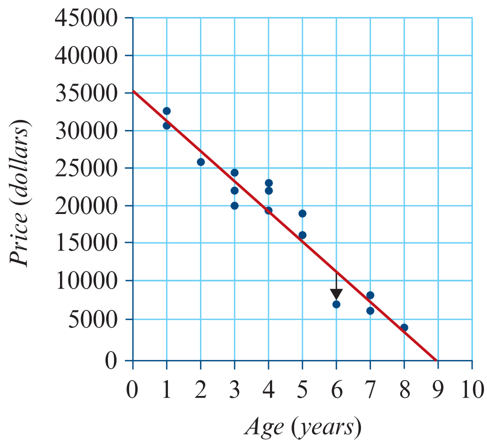
Residuals can be positive, negative or zero.

- Data points *above* the regression line have a *positive residual*.
- Data points *below* the regression line have a *negative residual*.
- Data points *on* the line have a *zero residual*.

We can estimate the value of a residual directly from the scatterplot.

For example, from the scatterplot, we can see that the *residual value* for the 6-year-old car is around –\$5000. That is, the actual price of the car is approximately \$5000 *less* than we would predict.

To determine the value of a residual more precisely, we need to do a calculation.



Example 3 Calculating a residual

The actual price of the 6-year-old car is \$6500. Calculate the residual (error of prediction) when its price is predicted using the regression line: $price = 35\,100 - 3940 \times age$

Solution

1 Write down the actual price. — Actual price: \$6500

2 Determine the predicted price using the regression equation: $price = 35\,100 - 3940 \times age$ — Predicted price $= 35\,100 - 3940 \times 6 = \$11\,460$

3 Determine the residual. — Residual = actual − predicted $= \$6500 - \$11\,460 = -\$4960$

The residual plot: testing the assumption of linearity

A key *assumption* made when calculating a least squares regression line is that the relationship between the variables is *linear*.

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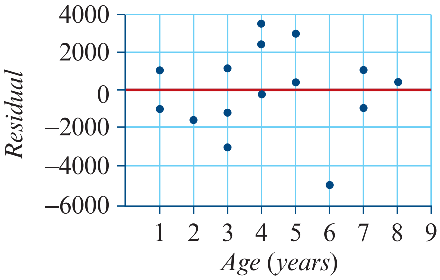
One way of testing this assumption is to plot the regression line on the scatterplot and see how well a straight line fits the data. However, a better way is to use a residual plot, as this plot will show even very small departures from linearity.

A **residual plot** is a plot of the residual value for each data value against the independent variable (in this case, *age*). Because the mean of the residuals is always zero, the horizontal zero line (red) helps us to orient ourselves. This line corresponds to regression in the previous scatterplot.



From the residual plot, we see that there is *no clear pattern*[4] in the residuals. Essentially they are *randomly scattered* around the zero regression line.

Thus, from this residual plot we can report as below.

> **Report**
>
> The lack of a clear pattern in the residual plot confirms the assumption of a linear association between the price of a second-hand car.

Reporting the results of a regression analysis

The final step in a regression analysis is to report your findings. The report below is in a form that is suitable for inclusion in a statistical investigation project.

> **Report**
>
> From the scatterplot we see that there is a strong negative, linear association between the price of a second hand car and its age, $r = -0.964$. There are no obvious outliers.
>
> The equation of the least squares regression line is: price $= 35\,100 - 3940 \times$ age.
>
> The slope of the regression line predicts that, on average, the price of these second-hand cars decreased by \$3940 each year.
>
> The intercept predicts that, on average, the price of these cars when new was \$35 100.
>
> The coefficient of determination indicates that 93% of the variation in the price of these second-hand cars is explained by the variation in their age.
>
> The lack of a clear pattern in the residual plot confirms the assumption of a linear association between the price and the age of these second-hand cars.

[4] From a visual inspection, it is difficult to say with certainty that a residual plot is random. It is easier to see when it is not random, as you will see in the next chapter. For present purposes, it is sufficient to say that a clear lack of pattern in a residual plot indicates randomness.

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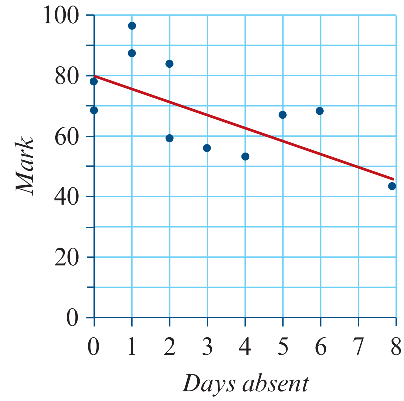
Exercise 4C

Skillsheet

Some basics

1 Use the line on the scatterplot opposite to determine the equation of the regression line in terms of the variables, *mark* and *days absent*. Give the intercept correct to the nearest whole number and the slope correct to one decimal place.



Reading a regression equation, making predictions and calculating residuals

2 The equation of a regression line that enables hand span to be predicted from height is:

$$\textit{hand span} = 2.9 + 0.33 \times \textit{height}$$

Complete the following sentences, by filling in the boxes:

a The explanatory variable is ☐.

b The slope equals ☐ and the intercept equals ☐.

c A person is 160 cm tall. The regression line predicts a hand span of ☐ cm.

d This person has an actual hand span of 58.5 cm.
The error of prediction (residual value) is ☐ cm.

3 For a 100 km trip, the equation of a regression line that enables fuel consumption of a car (in litres) to be predicted from its weight (kg) is:

$$\textit{fuel consumption} = -0.1 + 0.01 \times \textit{weight}$$

Complete the following sentences:

a The response variable is ☐.

b The slope ☐ is and the intercept is ☐.

c A car weighs 980 kg. The regression line predicts a fuel consumption of ☐ litres.

d This car has an actual fuel consumption of 8.9 litres.
The error of prediction (residual value) is ☐ litres.

Interpreting a regression equation and its coefficient of determination

4 In an investigation of the relationship between the food energy content (in calories) and the fat content (in g) in a standard-sized packet of chips, the least squares regression line was found to be:

$$\textit{energy content} = 27.8 + 14.7 \times \textit{fat content} \qquad r^2 = 0.7569$$

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Use this information to complete the following sentences.

a The slope is ____ and the intercept is ____.

b The regression equation predicts that the food energy content in a packet of chips increases by ____ calories for each additional gram of fat it contains.

c $r =$ ____

d ____ % of the variation in food energy content of a packet of chips can be explained by the variation in their ____.

e The fat content of a standard-sized packet of chips is 8 g.

  i The regression equation predicts its food energy content to be ____ calories.

  ii The *actual* energy content of this packet of chips is 132 calories. The error of prediction (residual value) is ____ calories.

5 In an investigation of the relationship between the success rate (%) of sinking a putt and the distance from the hole (in cm) of amateur golfers, the least squares regression line was found to be:

$$\textit{success rate} = 98.5 - 0.278 \times \textit{distance} \qquad r^2 = 0.497$$

a Write down the slope of this regression equation and interpret.

b Use the equation to predict the success rate when a golfer is 90 cm from the hole.

c At what distance (in metres) from the hole does the regression equation predict an amateur golfer to have a 0% success rate of sinking the putt?

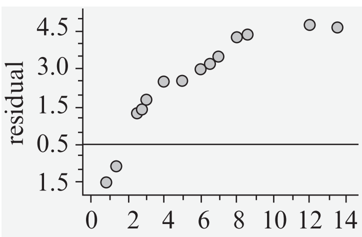
d Calculate the value of r, correct to three decimal places.

e Write down the value of the coefficient in percentage terms and interpret.

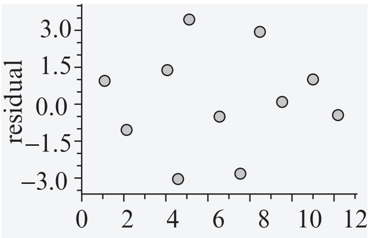
Interpreting residual plots

6 Each of the following residual plots has been constructed after a least squares regression line has been fitted to a scatterplot. Which of the residual plots suggest that the use of a linear model to fit the data was inappropriate? Why?

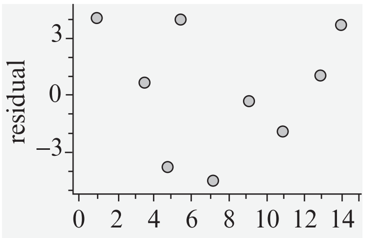
A



B



C



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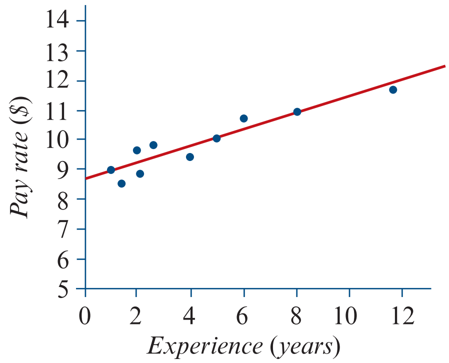
Conducting a regression analysis including the use of residual plots

7 The scatterplot opposite shows the pay rate (dollars per hour) paid by a company to workers with different years of work experience. Using a calculator, the equation of the least squares regression line is found to have the equation:

$y = 8.56 + 0.289x$ with $r = 0.967$



a Is it appropriate to fit a least squares regression line to the data? Why?

b Work out the coefficient of determination.

c What percentage of the variation in a person's pay rate can be explained by the variation in their work experience?

d Write down the equation of the least squares line in terms of the variables *pay rate* and years of *experience*.

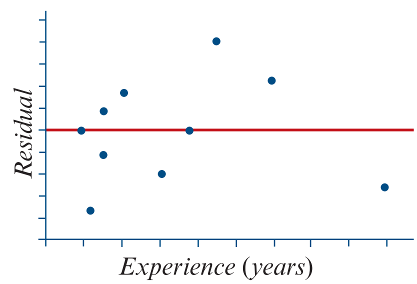
e Interpret the y-intercept in terms of the variables *pay rate* and years of *experience*. What does the y-intercept tell you?

f Interpret the slope in terms of the variables *pay rate* and years of *experience*. What does the slope of the regression line tell you?

g Use the least squares regression equation to:

- **i** predict the hourly wage of a person with 8 years of experience
- **ii** determine the residual value if the person's actual hourly wage is \$11.20 per hour.

h The residual plot for this regression analysis is shown opposite. Does the residual plot support the initial assumption that the relationship between *pay rate* and years of *experience* is linear? Explain your answer.



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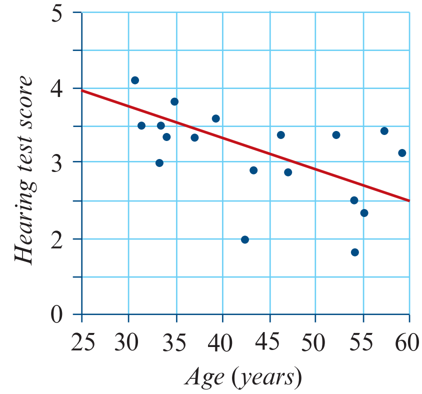
8 The scatterplot opposite shows scores on a hearing test against age. In analysing the data, a statistician produced the following statistics:

- coefficient of determination: $r^2 = 0.370$
- least squares line: $y = 4.9 - 0.043x$



a Determine the value of Pearson's correlation coefficient, r, for the data.

b Interpret the coefficient of determination in terms of the variables *hearing test score* and *age*.

c Write down the equation of the least squares line in terms of the variables *hearing test score* and *age*.

d Write down the slope and interpret.

e Use the least squares regression equation to:

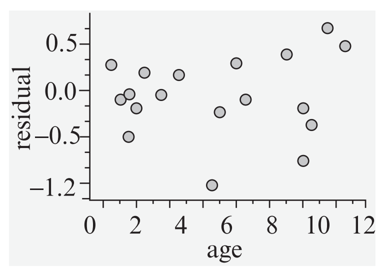
  i predict the hearing test score of a person who is 20 years old

  ii determine the residual value if the person's actual hearing test score is 2.0.

f Use the graph to estimate the value of the residual for the person aged:

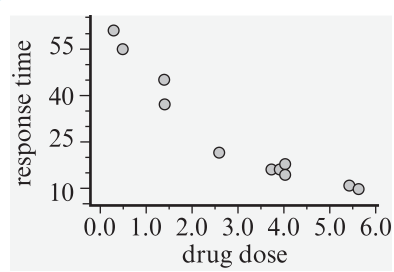
  i 35 years **ii** 55 years.

g The residual plot for this regression analysis is shown opposite.
Does the residual plot support the initial assumption that the relationship between hearing test score and age is essentially linear? Explain your answer.



Writing a statistical report from CAS calculator generated statistics

9 In a study of the effectiveness of a pain relief drug, the response time (in minutes) was measured for different drug doses (in mg). A least squares regression analysis was conducted to enable response time to be predicted from drug dose. The results of the analysis are displayed.



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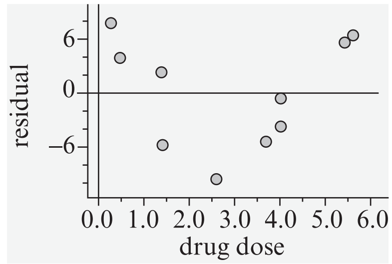
Regression equation: $y = a + bx$

$a = 55.8947$

$b = -9.30612$

$r^2 = 0.901028$

$r = -0.949225$



Use this information to complete the following report. Call the two variables *drug dose* and *response time*. In this analysis *drug dose* is the explanatory variable.

Report

From the scatterplot we see that there is a strong ______ relationship between response time and ______ : r = ______. There are no obvious outliers.

The equation of the least squares regression line is:

response time = ______ + ______ × drug dose

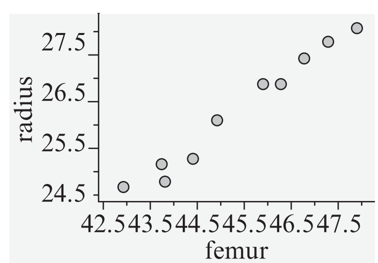
The slope of the regression line predicts that, on average, response time increases/decreases by ______ minutes for a 1-milligram increase in drug dose.

The y-intercept of the regression line predicts that, on average, the response time when no drug is administered is ______ minutes.

The coefficient of determination indicates that, on average, ______ % of the variation in ______ is explained by the variation in ______.

The residual plot shows a ______, calling into question the use of a linear equation to describe the relationship between response time and drug dose.

10 A regression analysis was conducted to investigate the nature of the relationship between femur (thigh bone) length and radius (the short thicker bone in the forearm) length in 18-year-old males. The bone lengths are measured in centimetres. The results of this analysis are reported below. In this investigation, femur length was treated as the independent variable.



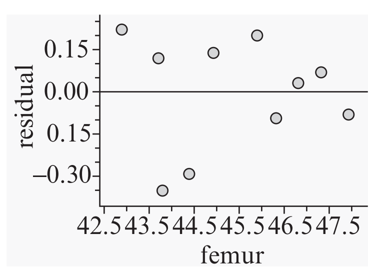
Regression equation $y = a + bx$

$a = -7.24946$

$b = 0.739556$

$r^2 = 0.975291$

$r = 0.927272$



Use the format of the report given in the previous question to summarise findings of this investigation. Call the two variables *femur length* and *radius length*.

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4D Conducting a regression analysis using raw data

In your statistical investigation project you will need to be able to conduct a full regression analysis from raw data. This section is designed to help you with this task.

How to conduct a regression analysis using the TI-Nspire CAS

This analysis is concerned with investigating the association between life expectancy (in years) and birth rate (in births per 1000 people) in 10 countries.

Birth rate	30	38	38	43	34	42	31	32	26	34
Life expectancy (years)	66	54	43	42	49	45	64	61	61	66

Steps

1 Write down the explanatory variable (EV) and response variable (RV). Use the variable names *birth* and *life*.

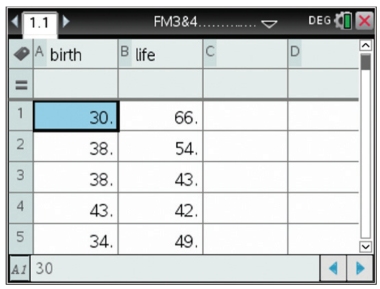
EV: birth

RV: life

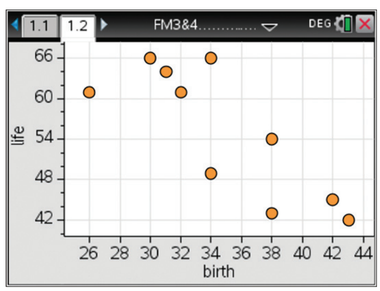
2 Start a new document by pressing ctrl + N.

Select **Add Lists & Spreadsheet**.

Enter the data into the lists named *birth* and *life*, as shown.



3 Construct a scatterplot to investigate the nature of the relationship between life expectancy and birth rate.

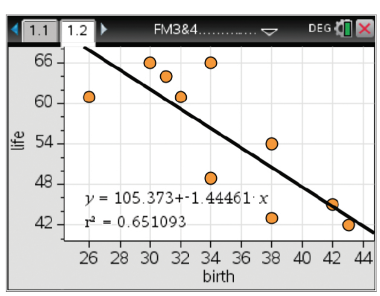


4 Describe the association shown by the scatterplot. Mention direction, form, strength and outliers.

There is a strong, negative, linear relationship between life expectancy and birth rate. There are no obvious outliers.

5 Find and plot the equation of the least squares regression line and r^2 value.

Note: Check if **Diagnostics** is activated using menu > **Settings**.



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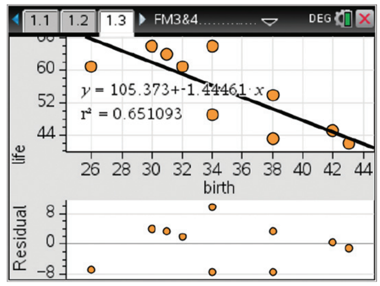
6 Generate a residual plot to test the linearity assumption.

Use ctrl + ◀ (or click on the page tab) to return to the scatterplot.

Press menu >**Analyze>Residuals>Show Residual Plot** to display the residual plot on the same screen.



7 Use the values of the intercept and slope to write the equation of the least squares regression line. Also write the values of r and the coefficient of determination.

Regression equation:

life = 105.4 − 1.445 × birth

Correlation coefficient: $r = 0.8069$

Coefficient of determination: $r^2 = 0.651$

How to conduct a regression analysis using the ClassPad

The data for this analysis are shown below.

Birth rate (per thousand)	30	38	38	43	34	42	31	32	26	34
Life expectancy (years)	66	54	43	42	49	45	64	61	61	66

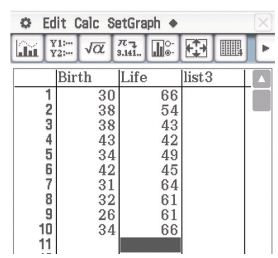
Steps

1 Write down the explanatory variable (EV) and response variable (RV). Use the variable names *birth* and *life*.

EV: birth

RV: life

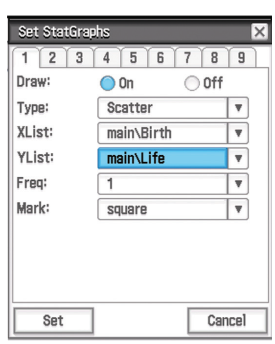
2 Enter the data into lists as shown.

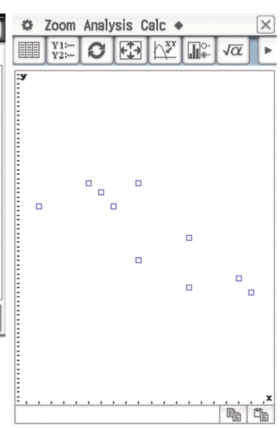


3 Construct a scatterplot to investigate the nature of the relationship between life expectancy and birth rate.

a Tap and complete the **Set Calculations** dialog box as shown.

b Tap to view the scatterplot.





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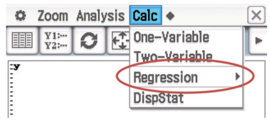
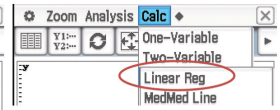
4 Describe the association shown by the scatterplot. Mention direction, form, strength and outliers.

There is a strong negative, linear association between life expectancy and birth rate. There are no obvious outliers.

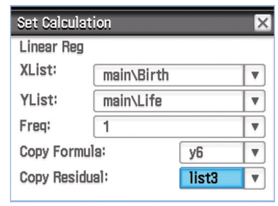
5 Find the equation of the least squares regression line and generate all regression statistics, including residuals.



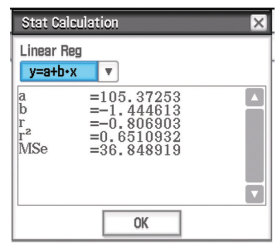


a Tap **Calc** in the toolbar.
Tap **Regression** and select **Linear Reg**.

b Complete the **Set Calculations** dialog box as shown.
Note: **Copy Residual** copies the residuals to **list3**, where they can be used later to create a residual plot.



c Tap **OK** in the Set **Calculation** box to generate the regression results.



d Write down the key results.

Regression equation:

life = 105.4 − 1.445 × birth

Correlation coefficient:

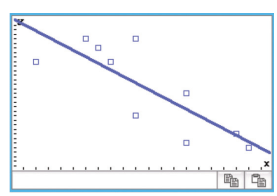
$r = -0.8069$

Coefficient of determination:

$r^2 = 0.651$

6 Tapping **OK** a second time automatically plots and displays the regression line on the scatterplot.

To obtain a full-screen plot, tap **Resize** from the icon panel.



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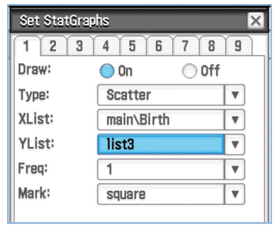
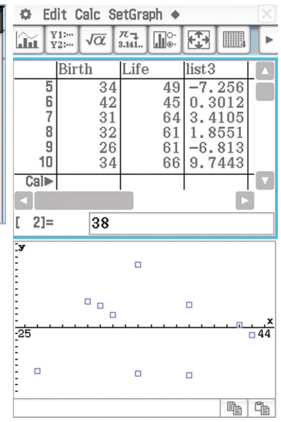
7 Generate a residual plot to test the linearity assumption.
Tap and complete the **Set Calculations** dialog box as shown.
Tap to view the residual plot.





Inspect the plot and write your conclusion.

The random residual plot suggests linearity.

Note: When you performed a regression analysis earlier, the residuals were calculated automatically and stored in **list3**. The residual plot is a scatterplot with **list3** on the vertical axis and **birth** on the horizontal axis.

Exercise 4D

1 The table below shows the scores obtained by nine students on two tests. We want to be able to predict test B scores from test A scores.

Test A score (x)	18	15	9	12	11	19	11	14	16
Test B score (y)	15	17	11	10	13	17	11	15	19

Use your calculator to perform each of the following steps of a regression analysis.

a Construct a scatterplot. Name the variables *test a* and *test b*.

b Determine the equation of the least squares line along with the values of r and r^2.

c Display the regression line on the scatterplot.

d Obtain a residual plot.

2 The table below shows the number of careless errors made on a test by nine students. Also given are their test scores. We want to be able to predict test score from the number of careless errors made.

Test score	18	15	9	12	11	19	11	14	16
Careless errors	0	2	5	6	4	1	8	3	1

Use your calculator to perform each of the following steps of a regression analysis.

a Construct a scatterplot. Name the variables *score* and *errors*.

b Determine the equation of the least squares line along with the values of r and r^2. Write answers correct to three significant figures.

c Display the regression line on the scatterplot.

d Obtain a residual plot.

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3 How well can we predict an adult's weight from their birth weight? The weights of 12 adults were recorded, along with their birth weights. The results are shown.

Birth weight (kg)	1.9	2.4	2.6	2.7	2.9	3.2	3.4	3.4	3.6	3.7	3.8	4.1
Adult weight (kg)	47.6	53.1	52.2	56.2	57.6	59.9	55.3	58.5	56.7	59.9	63.5	61.2

a In this investigation, which would be the RV and which would be the EV?

b Construct a scatterplot.

c Use the scatterplot to:

i comment on the relationship between adult weight and birth weight in terms of direction, outliers, form and strength

ii *estimate* the value of Pearson's correlation coefficient, r.

d Determine the equation of the least squares regression line, the coefficient of determination and the value of Pearson's correlation coefficient, r. Write answers correct to three significant figures.

e Interpret the coefficient of determination in terms of adult weight and birth weight.

f Interpret the slope in terms of adult weight and birth weight.

g Use the regression equation to predict the weight of an adult with a birth weight of:

i 3.0 kg **ii** 2.5 kg **iii** 3.9 kg.

Give answers correct to one decimal place.

h It is generally considered that birth weight is a 'good' predictor of adult weight. Do you think the data support this contention? Explain.

i Construct a residual plot and use it to comment on the appropriateness of assuming that adult weight and birth weight are linearly associated.

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Review

Key ideas and chapter summary

Bivariate data — **Bivariate data** are data in which each observation involves recording information about two variables for the same person or thing. An example would be the heights and weights of the children in a preschool.

Linear regression — The process of fitting a line to data is known as **linear regression**.

Residuals — The vertical distance from a data point to the straight line is called a **residual**: residual value = data value − predicted value.

Least squares method — The **least squares method** is one way of finding the equation of a regression line. It minimises the sum of the squares of the residuals. It works best when there are no outliers.
The equation of the least squares regression line is given by $y = a + bx$, where a represents the *y-intercept* of the line and b the *slope*.

Using the regression line — The regression line $y = a + bx$ enables the value of y to be determined for a given value of x.
For example, the regression line

$$cost = 1.20 + 0.06 \times number\ of\ pages$$

predicts that the cost of a 100-page book is:

$$cost = 1.20 + 0.06 \times 100 = \$7.20$$

Interpolation and extrapolation — Predicting *within* the range of data is called **interpolation**.
Predicting *outside* the range of data is called **extrapolation**.

Slope and intercept — The **slope** of the regression line above predicts that the cost of a textbook increases by 6 cents ($0.06) for each additional page.
The *intercept* of the line predicts that a book with no pages costs $1.20 (this might be the cost of the cover).

Coefficient of determination — The **coefficient of determination** (r^2) gives a measure of the predictive power of a regression line. For example, for the regression line above, the coefficient of determination is 0.81.
From this we conclude that 81% of the variation in the cost of a textbook can be explained by the variation in the number of pages.

Key assumption of regression — Linear regression assumes that the underlying association between the variables is linear.

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Residual plots **Residual plots** can be used to test the linearity assumption by plotting the residuals against the EV.
A residual plot that appears to be a random collection of points clustered around zero supports the linearity assumption.
A residual plot that shows a clear pattern indicates that the association is not linear.

Skills check

Having completed this chapter you should be able to:

- determine the equation of the least squares line using the formulas $b = \dfrac{rs_y}{s_x}$ and $a = \bar{y} - b\bar{x}$
- for raw data, determine the equation of the least squares line using a CAS calculator
- interpret the slope and intercept of a regression line
- interpret the coefficient of determination as part of a regression analysis
- use the regression line for prediction
- calculate residuals
- construct a residual plot using a graphics calculator
- use a residual plot to determine the appropriateness of using the equation of the least squares line to model the association
- present the results of a regression analysis in report form.

Multiple-choice questions

1 When using a least squares line to model a relationship displayed in a scatterplot, one key assumption is that:

A there are two variables **B** the variables are related
C the variables are linearly related **D** $r^2 > 0.5$
E the correlation coefficient is positive

2 In the least squares regression line $y = -1.2 + 0.52x$:

A the y-intercept $= -0.52$ and slope $= -1.2$
B the y-intercept $= 0$ and slope $= -1.2$
C the y-intercept $= 0.52$ and slope $= -1.2$
D the y-intercept $= -1.2$ and slope $= 0.52$
E the y-intercept $= 1.2$ and slope $= -0.52$

3 If the equation of a least squares regression line is $y = 8 - 9x$ and $r^2 = 0.25$:

A $r = -0.5$ **B** $r = -0.25$ **C** $r = -0.0625$ **D** $r = 0.25$ **E** $r = 0.50$

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Review

4 The least squares regression line $y = 8 - 9x$ predicts that, when $x = 5$, the value of y is:

A −45 **B** −37 **C** 37 **D** 45 **E** 53

5 A least squares regression line of the form $y = a + bx$ is fitted to the data set shown.

x	25	15	10	5
y	10	10	15	25

The equation of the line is:

A $y = -0.69 + 24.4x$ **B** $y = 24.4 - 0.69x$ **C** $y = 24.4 + 0.69x$
D $y = 28.7 - x$ **E** $y = 28.7 + x$

6 A least squares regression line of the form $y = a + bx$ is fitted to the data set shown.

y	30	25	15	10
x	40	20	30	10

The equation of the line is:

A $y = 1 + 0.5x$ **B** $y = 0.5 + x$ **C** $y = 0.5 + 7.5x$
D $y = 7.5 + 0.5x$ **E** $y = 30 - 0.5x$

7 Given that $r = 0.733$, $s_x = 1.871$ and $s_y = 3.391$, the slope of the least squares regression line is closest to:

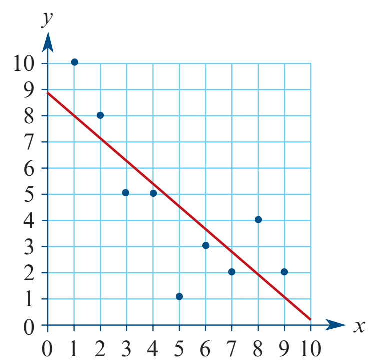
A 0.41 **B** 0.45 **C** 1.33 **D** 1.87 **E** 2.49

8 Using a least squares regression line, the predicted value of a data value is 78.6. The residual value is –5.4. The actual data value is:

A 73.2 **B** 84.0 **C** 88.6 **D** 94.6 **E** 424.4

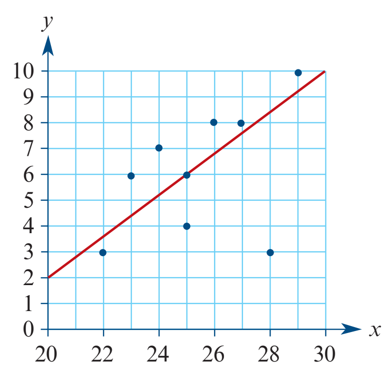
9 The equation of the least squares line plotted on the scatterplot opposite is closest to:

A $y = 8.7 - 0.9x$
B $y = 8.7 + 0.9x$
C $y = 0.9 - 8.7x$
D $y = 0.9 + 8.7x$
E $y = 8.7 - 0.1x$



10 The equation of the regression line plotted on the scatterplot opposite is closest to:

A $y = -14 + 0.8x$
B $y = 0.8 + 14x$
C $y = 2.5 + 0.8x$
D $y = 14 - 0.8x$
E $y = 17 + 1.2x$



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The following information relates to Questions 11 to 14.

Weight (in kg) can be predicted from height (in cm) from the regression line:

$$weight = -96 + 0.95 \times height, \quad \text{with } r = 0.79$$

11 Which of the following statements relating to the regression line is *false*?

A The slope of the regression line is 0.95.
B The independent variable in the regression equation is *height*.
C The least squares line does *not* pass through the origin.
D The intercept is 96.
E The equation predicts that a person who is 180 cm tall will weigh 75 kg.

12 This regression line predicts that, on average, weight:

A decreases by 96 kg for each 1 centimetre increase in height
B increases by 96 kg for each 1 centimetre increase in height
C decreases by 0.79 kg for each 1 centimetre increase in height
D decreases by 0.95 kg for each 1 centimetre increase in height
E increases by 0.95 kg for each 1 centimetre increase in height

13 Noting that the value of the correlation coefficient is $r = 0.79$, we can say that:

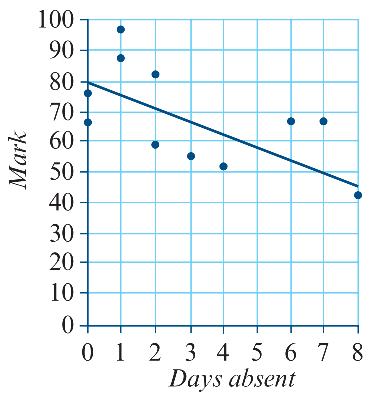
A 62% of the variation in weight can be explained by the variation in height
B 79% of the variation in weight can be explained by the variation in height
C 88% of the variation in weight can be explained by the variation in height
D 79% of the variation in height can be explained by the variation in weight
E 95% of the variation in height can be explained by the variation in weight

14 A person of height 179 cm weighs 82 kg. If the regression equation is used to predict their weight, then the residual will be closest to:

A –8 kg **B** 3 kg **C** 8 kg **D** 9 kg **E** 74 kg

15 The coefficient of determination for the data displayed in the scatterplot opposite is close to $r^2 = 0.5$.
The correlation coefficient is closest to:

A −0.7 **B** −0.25
C 0.25 **D** 0.5
E 0.7



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Review

Extended-response questions

1 In an investigation of the relationship between the hours of sunshine (per year) and days of rain (per year) for 25 cities, the least squares regression line was found to be:

$$\textit{hours of sunshine} = 2850 - 6.88 \times \textit{days of rain}, \text{ with } r^2 = 0.484$$

Use this information to complete the following sentences.

a In this regression equation, the explanatory variable is ☐.

b The slope is ☐ and the intercept is ☐.

c The regression equation predicts that a city that has 120 days of rain per year will have ☐ hours of sunshine per year.

d The slope of the regression line predicts that the hours of sunshine per year will ☐ by ☐ hours for each additional day of rain.

e $r =$ ☐, correct to three significant figures.

f ☐% of the variation in sunshine hours can be explained by the variation in ☐.

g One of the cities used to determine the regression equation had 142 days of rain and 1390 hours of sunshine.

  i The regression equation predicts that it has ☐ hours of sunshine.

  ii The residual value for this city is ☐ hours.

h Using a regression line to make predictions within the range of data used to determine the regression equation is called ☐.

2 The cost of preparing meals in a school canteen is linearly related to the number of meals prepared. To help the caterers predict the costs, data were collected on the cost of preparing meals for different levels of demands. The data are shown below.

Number of meals	30	35	40	45	50	55	60	65	70	75	80
Cost (dollars)	138	154	159	182	198	198	214	208	238	234	244

a Which is the response variable?

b Use your calculator to show that the equation of the least squares line that relates the cost of preparing meals to the number of meals produced is:

$$\textit{cost} = 81.5 + 2.10 \times \textit{number of meals}$$

c Use the equation to predict the cost of producing:

  i 48 meals. In making this prediction are you interpolating or extrapolating?

  ii 21 meals. In making this prediction are you interpolating or extrapolating?

d **i** Write down and interpret the intercept of the regression line.

  ii Write down and interpret the intercept of the regression line.

e If $r = 0.978$, write down the coefficient of determination and interpret.

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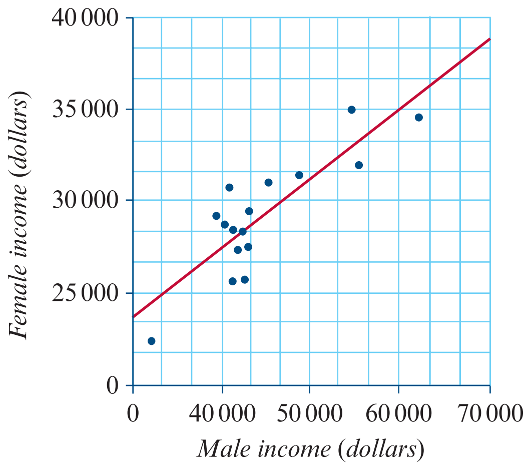
3 In the scatterplot opposite, average annual *female income,* in dollars, is plotted against average annual *male income,* in dollars, for 16 countries. A least squares regression line is fitted to the data.



The equation of the least squares regression line for predicting female income from male income is *female income* = 13 000 + 0.35 × *male income*.

a What is the explanatory variable?

b Complete the following statement by filling in the missing information.
From the least squares regression line equation it can be concluded that, for these countries, on average, female income increases by $ __________ for each $1000 increase in male income.

c **i** Use the least squares regression line equation to predict the average annual female income (in dollars) in a country where the average annual male income is $15 000.

ii The prediction made in **part c i** is not likely to be reliable. Explain why.

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4 We wish to find the equation of the least squares regression line that will enable height (in cm) to be predicted from femur (thigh bone) length (in cm).

a Which is the RV and which is the EV?

b Use the following summary statistics to determine the equation of the least squares regression line that will enable height (y) to be predicted from femur length (x).

$$r = 0.9939 \quad \bar{x} = 24.246 \quad s_x = 1.873 \quad \bar{y} = 166.092 \quad s_y = 10.086$$

Write the equation in terms of height and femur length. Give the slope and intercept accurate to three significant figures.

c Interpret the slope of the regression equation in terms of height and femur length.

d Determine the value of the coefficient of determination and interpret in terms of height and femur length.

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5 The data below show the height (in cm) of a group of 10 children aged 2 to 11 years.

Height (cm)	86.5	95.5	103.0	109.8	116.4	122.4	128.2	133.8	139.6	145.0
Age (years)	2	3	4	5	6	7	8	9	10	11

The task is to determine the equation of a least squares regression line that can be used to predict height from age.

a In this analysis, which would be the RV and which would be the EV?

b Use your calculator to confirm that the equation of the least squares regression line is: $height = 76.64 + 6.366 \times age$ and $r = 0.9973$.

c Use the regression line to predict the height of a 1-year-old child. Give the answer correct to the nearest cm. In making this prediction are you extrapolating or interpolating?

d What is the slope of the regression line and what does it tell you in terms of the variables involved?

e Calculate the value of the coefficient of determination and interpret in terms of the relationship between age and height.

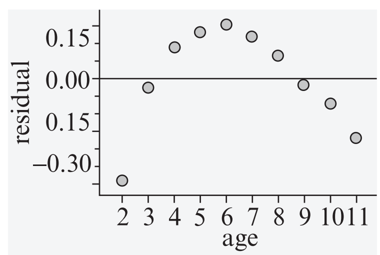
f Use the least squares regression equation to:

i predict the height of the 10-year-old child in this sample

ii determine the residual value for this child.

g **i** Confirm that the residual plot for this analysis is shown opposite.

ii Explain why this residual plot suggests that a linear equation is not the most appropriate model for this relationship.



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5

Core: Data analysis

Chapter 5

Data transformation

ISBN 978-1-107-56757-3

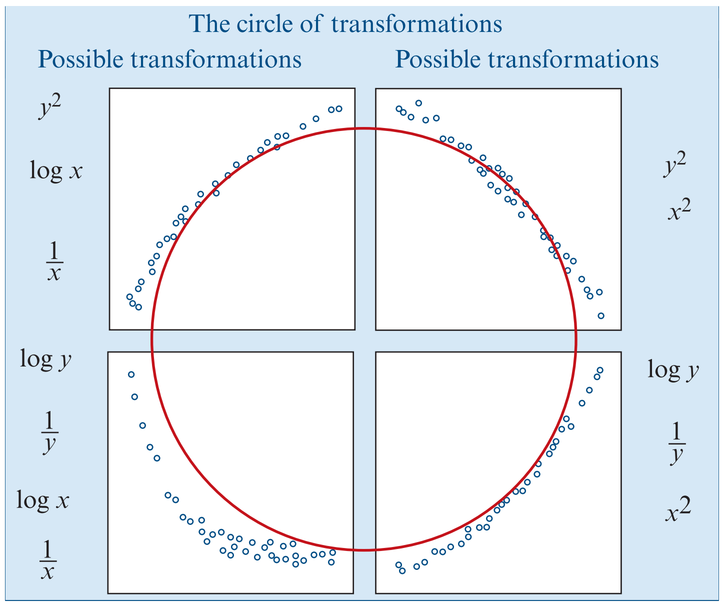
Cambridge University Press

5A Introduction

You first encountered **data transformation** in Chapter 1 where you used a log scale to transform a skewed histogram into a more easily interpreted symmetric histogram. In this chapter, you will learn to use the squared, log and reciprocal transformations to linearise scatterplots, the first step towards solving problems involving non-linear associations.

▶ The circle of transformations

The types of scatterplots that can be transformed by the squared, log or reciprocal transformations can be fitted together into what we call the **circle of transformations**.

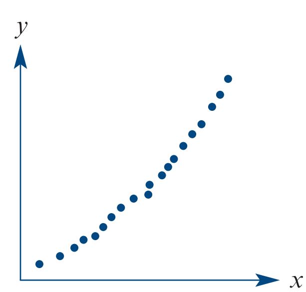


The purpose of the circle of transformations is to guide us in our choice of transformation to linearise a given scatterplot.

There are two things to note when using the circle of transformations:

1. In each case, there is more than one type of transformation that might work.
2. These transformations only apply to scatterplots with a consistently increasing or decreasing trend.

For example, the scatterplot opposite has a consistently increasing trend so the circle of transformations applies.



Comparing the scatterplot to those in the circle of transformations we see that there are three transformations, the x^2, the $1/y$ or the $\log x$, that have the potential to linearise this scatterplot.

At this stage you might find it helpful to use the interactive 'Data transformation' (accessible through the Interactive Textbook) to see how these different transformations can be used to linearise scatterplots.

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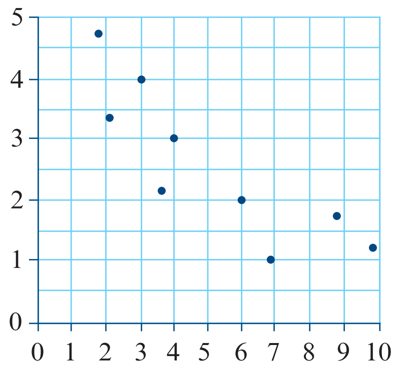
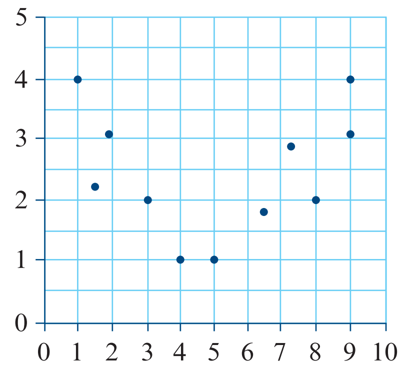
Exercise 5A

1 The scatterplots below are non-linear. For each, identify the transformations x^2, log x, $1/x$, y^2, log y, $1/y$ or none that might be used to linearise the plot.

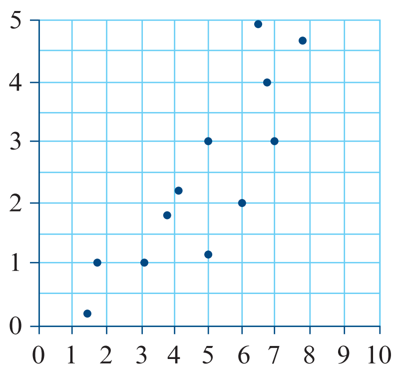
a



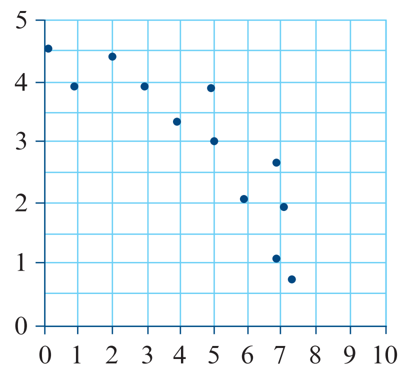
b



c



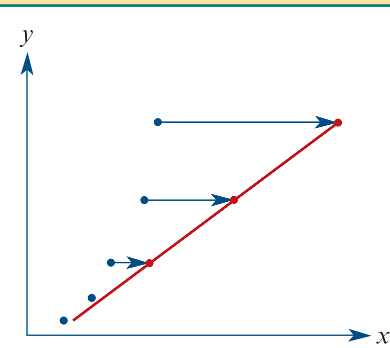
d



5B Using data transformation to linearise a scatterplot

The squared transformation

The **squared transformation** is a *stretching* transformation. It works by *stretching* out the upper end of the scale on either the x- or y-axis. The effect of applying an x^2 transformation to a scatterplot is illustrated graphically below.

Transformation	*Outcome*	*Graph*
x^2	Spreads out the high x-values relative to the lower x-values, leaving the y-values unchanged. This has the effect of straightening out curves like the one shown opposite. The y-squared transformation works in a similar manner but stretches out the scale on the y-axis.	

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The following example shows how the x-squared transformation works in practice.

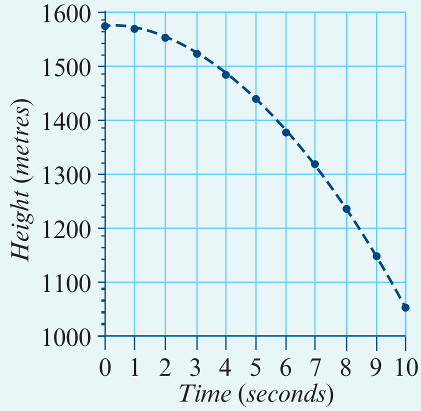
Example 1 Applying the squared transformation

A base jumper leaps from the top of a cliff, 1560 metres above the valley floor. The scatterplot below shows the height (in metres) of the base jumper above the valley floor every second, for the first 10 seconds of the jump. After this time she opened her parachute to bring her safely to the ground.

A scatterplot shows that there is a strong negative association between the *height* of the base jumper above the ground and *time*.

However, the association is clearly non-linear as can be seen from the red dotted line on the scatterplot. Because the association is clearly non-linear, it makes no sense to try to model the association with a straight line.

Before we can fit a least squares line to the data, we need to linearise the scatter plot.



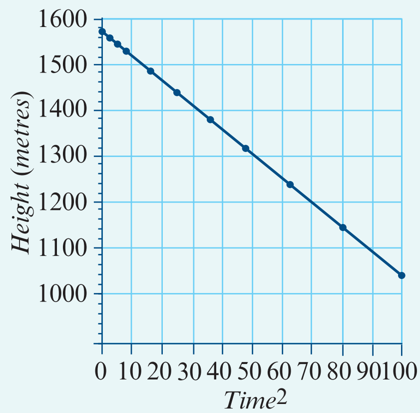
The circle of transformation suggests that we could use either an x^2 or a y^2 to linearise this scatterplot. We will use the x^2 transformation. That involves changing the scale on the *time* axis to *time*^{2}.

When we make this change, we see that the association between *height* and *time*^{2} is linear. See the plot opposite.

Now that we have a linearised scatterplot, we can use a least squares line to model the association between *height* and *time*^{2}.

The equation of this line is:

$$height = 1560 - 4.90 \times time^2$$



Like any regression line, we can use its equation to make predictions.

For example, after 3.4 seconds, we predict that the height of the base jumper is:

$$height = 1560 - 4.90 \times 3.4^2 = 1503 \text{ m (to nearest m)}$$

Performing a data transformation is quite computationally intensive, but your CAS calculator is well suited to the task.

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Using the TI-Nspire CAS to perform a squared transformation

The table shows the height (in m) of a base jumper for the first 10 seconds of her jump.

Time	0	1	2	3	4	5	6	7	8	9	10
Height	1560	1555	1540	1516	1482	1438	1383	1320	1246	1163	1070

a Construct a scatterplot displaying *height* (the RV) against *time* (the EV).

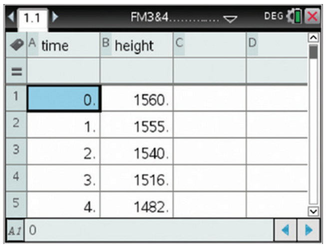
b Linearise the scatterplot and fit a least squares line to the transformed data.

c Use the regression line to predict the height of the base jumper after 3.4 seconds.

Steps

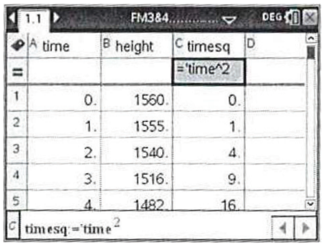
1 Start a new document by pressing ctrl + N .

2 Select **Add Lists & Spreadsheet**.
Enter the data into lists named **time** and **height**, as shown.

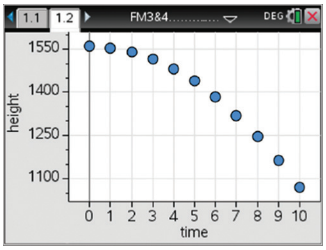


3 Name column C as **timesq** (short for 'time squared').

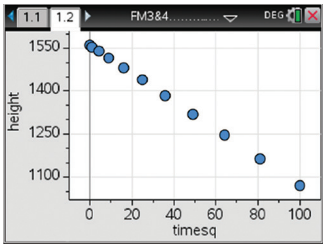
4 Move the cursor to the grey cell below **timesq**. Enter the expression = **time^2** by pressing =, then typing **time^2**. Pressing enter calculates and displays the values of **timesq**.



5 Press ctrl + I and select **Add Data & Statistics**.
Construct a scatterplot of *height* against *time*. Let *time* be the explanatory variable and *height* the response variable. The plot is clearly non-linear.



6 Press ctrl + I and select **Add Data & Statistics**.
Construct a scatterplot of *height* against *time*^{2}.
The plot is now linear.



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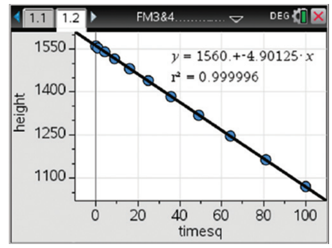
7 Press menu **>Analyze>Regression>Show Linear (a + bx)** to plot the line on the scatterplot with its equation.
Note: The x in the equation on the screen corresponds to the transformed variable $time^2$.



8 Write down the regression equation in terms of the variables *height* and $time^2$.

height = 1560 − 4.90 × $time^2$

9 Substitute 3.4 for *time* in the equation to find the height after 3.4 seconds.

height = 1560 − 4.90 × 3.4^2 = 1503 m

Using the CASIO Classpad to perform a squared transformation

The table shows the height (in m) of a base jumper for the first 10 seconds of her jump.

Time	0	1	2	3	4	5	6	7	8	9	10
Height	1560	1555	1540	1516	1482	1438	1383	1320	1246	1163	1070

a Construct a scatterplot displaying *height* (the RV) against *time* (the EV).

b Linearise the scatterplot and fit a least squares line to the transformed data.

c Use the regression line to predict the height of the base jumper after 3.4 seconds.

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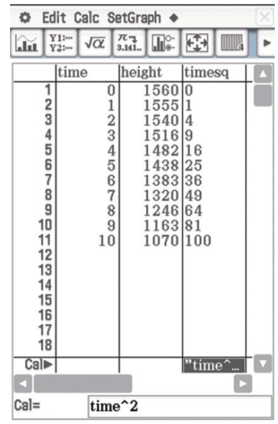
Steps

1 In the Statistics application enter the data into lists named **time** and **height**.

2 Name the third list **timesq** (short for time squared).

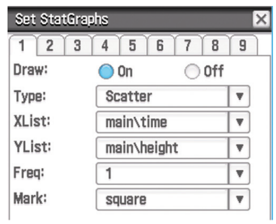
3 Place the cursor in the calculation cell at the bottom of the third column and type **time^2**. This will calculate the values of $time^2$.

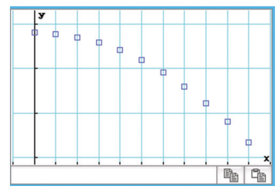
Let *time* be the explanatory variable (x) and *height* the response variable (y).



4 Construct a scatterplot of *height* against *time*.

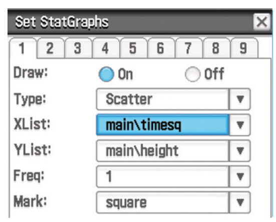
- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot. The plot is clearly non-linear.

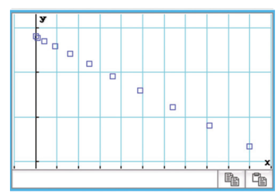




5 Construct a scatterplot of *height* against $time^2$.

- Tap and complete the **Set StatGraphs** box as shown.
- Tap to view the scatterplot. The plot is now clearly linear.



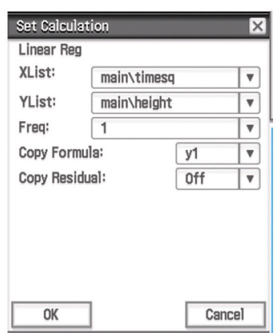


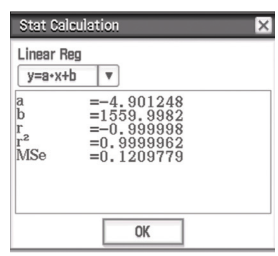
6 Fit a regression line to the transformed data.

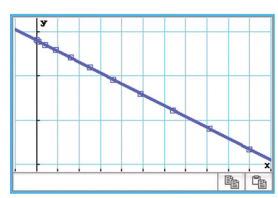
- Go to **Calc, Regression, Linear Reg**.
- Complete the **Set Calculation** dialog box as shown and tap **OK**.

Note: The 'x' in the linear equation corresponds to the transformed variable $time^2$.

- Tap **OK** a second time to plot and display the regression line on the scatterplot.







7 Write down the equation in terms of *height* and $time^2$.

$height = 1560 - 4.90 \times time^2$.

8 Substitute 3.4 for *time* in the equation.

$height = 1560 - 4.90 \times 3.4^2 = 1503$ m

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Exercise 5B

The x-squared transformation: some prerequisite skills

1 Evaluate y in the following expression, correct to one decimal place.

a $y = 7 + 8x^2$ when $x = 1.25$

b $y = 7 + 3x^2$ when $x = 1.25$

c $y = 24.56 - 0.47x^2$ when $x = 1.23$

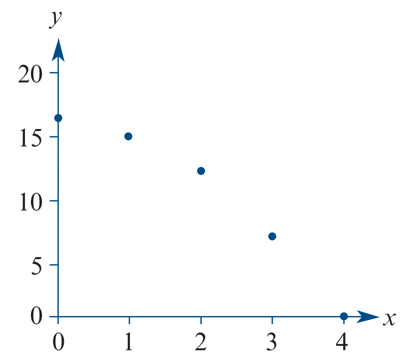
d $y = -4.75 + 5.95x^2$ when $x = 4.7$

The x-squared transformation: calculator exercises

2 The scatterplot opposite was constructed from the data in the table below.

x	0	1	2	3	4
y	16	15	12	7	0

From the scatterplot, it is clear that the association between y and x is non-linear.



a Linearise the scatterplot by applying an x-squared transformation and fit a least squares line to the transformed data.

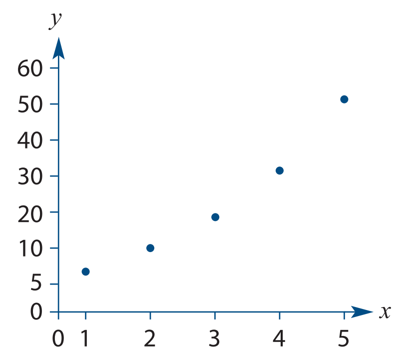
b Give its equation.

c Use the equation to predict the value of y when $x = -2$.

3 The scatterplot opposite was constructed from the data in the table below.

x	1	2	3	4	5
y	3	9	19	33	51

From the scatterplot, the association between y and x is non-linear.



a Linearise the scatterplot by applying an x-squared transformation and fit a least squares line to the transformed data.

b Give its equation.

c Use the equation to predict the value of y when $x = 6$.

The y-squared transformation: some prerequisite skills

4 Evaluate y in the following expression. Give the answers correct to one decimal place.

a $y^2 = 16 + 4x$ when $x = 1.57$

b $y^2 = 1.7 - 3.4x$ when $x = 0.03$

c $y^2 = 16 + 2x$ when $x = 10$ ($y > 0$)

d $y^2 = 58 + 2x$ when $x = 3$ ($y < 0$)

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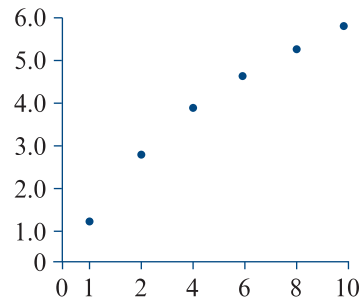
The y-squared transformation: calculator exercises

5 The scatterplot opposite was constructed from the data in the table below.

x	0	2	4	6	8	10
y	1.2	2.8	3.7	4.5	5.1	5.7

From the scatterplot, the association between y and x is non-linear.



a Linearise the scatterplot by applying a y-squared transformation and fit a least squares line to the transformed data.

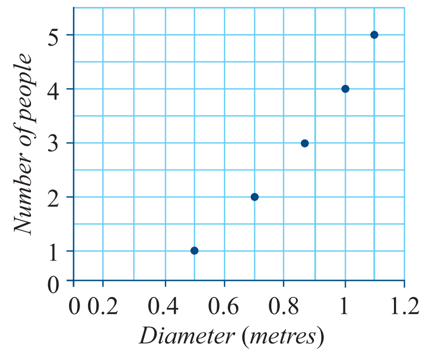
b Give its equation. Write the coefficient, correct to two significant figures.

c Use the equation to predict the value of y when $x = 9$. Give the answer correct to one decimal place.

Applications of the squared transformation

6 The table gives the *diameter* (in m) of five different umbrellas and the *number of people* each umbrella is designed to keep dry. A scatter plot is also shown.

Diameter	*Number*
0.50	1
0.70	2
0.85	3
1.00	4
1.10	5



a Apply the squared transformation to the variable *diameter* and determine the least squares regression line for the transformed data. *Number* is the EV.

Write the slope and intercept of this line, correct to one significant figure, in the spaces provided.

$$number = \boxed{} + \boxed{} \times diameter^2$$

b Use the equation to predict the number of people who can be sheltered by an umbrella of 1.3 m. Give your answer correct to the nearest person.

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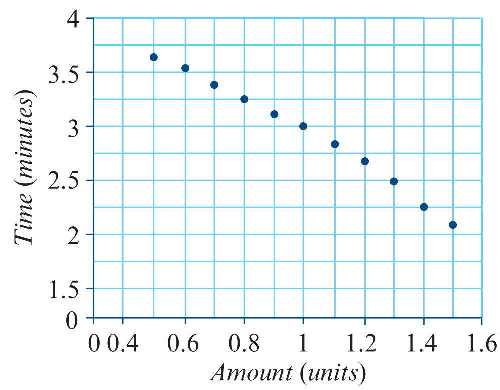
7 The time (in minutes) taken for a local anaesthetic to take effect is associated with to the amount administered (in units). To investigate this association a researcher collected the data.

Amount	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
Time	3.7	3.6	3.4	3.3	3.2	3.0	2.9	2.7	2.5	2.3	2.1

The association between the variables *amount* and *time* is non-linear as can be seen from the scatterplot below. A squared transformation applied to the variable *time* will linearise the scatterplot.



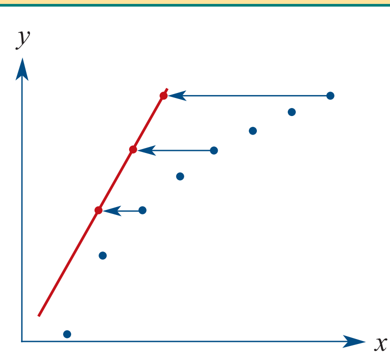
a Apply the squared transformation to the variable *time* and fit a least squares regression line to the transformed data. *Amount* is the EV. Write the equation of this line with the slope and intercept, correct to two significant figures.

b Use the equation to predict the time for the anaesthetic to take effect when the dose is 0.4 units. Give the answer correct to one decimal place.

5C The log transformation

Skillsheet

The **logarithmic transformation** is a *compressing* transformation and the upper end of the scale on either the x- or the y-axis. The effect of applying a log x transformation to a scatterplot is illustrated graphically below.

Transformation	*Outcome*	*Graph*
log x	Compresses the higher x-values relative to the lower x-values, leaving the y-values unchanged. This has the effect of straightening out curves like the one shown. The log y transformation works in similar manner but compressing the scale on the y-axis.	

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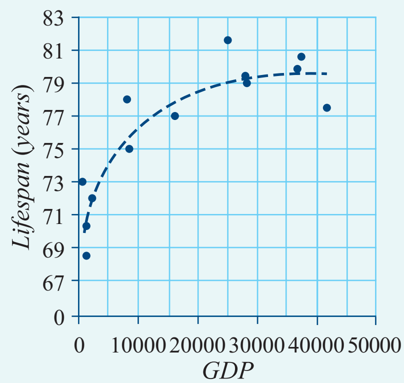
Example 2 Applying the log transformation

The general wealth of a country, often measured by its Gross Domestic Product (*GDP*), is one of several variables associated with *lifespan* in different countries. However, the association it not linear, as can be seen in the scatterplot below which plots *lifespan* (in years) against *GDP* (in dollars) for 13 different countries.

Because the association is non-linear, it makes no sense to try to model the association with a straight line. But before we can fit a least squares regression line to the data, we need to transform the data.

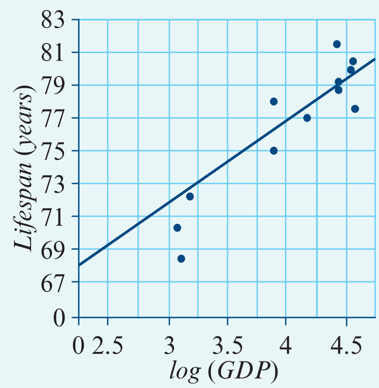


The circle of transformation suggests that we could use the y^2, $\log x$ or $\dfrac{1}{y}$ to linearise the transformation.

We will use the $\log x$ transformation.[1] That is, we change the scale on the *GDP*-axis to log (*GDP*).

When we make this change, we see that the association between the variables *lifespan* and log (*GDP*) is linear. See the plot opposite.

Note: On the plot, when log (*GDP*) = 4, the actual GDP is 10^4 or \$10 000.



We can now fit a least squares line to model the association between the variables *lifespan* and log (*GDP*).

The equation of this line is:

$$lifespan = 54.3 + 5.59 \times \log(GDP)$$

Like any other regression line we can use its equation to make predictions.

For example, for country with a GDP of \$20 000, the lifespan is predicted to be:

$$lifespan = 54.3 + 5.59 \times \log 20\,000 = 78.3 \text{ years (correct to one decimal place)}$$

[1] Following the normal convention, $\log x$ means $\log_{10} x$.

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Using the TI-Nspire CAS to perform a log transformation

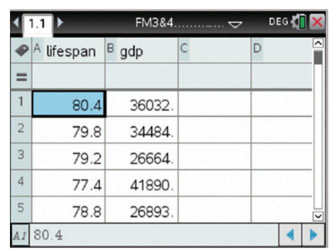
The table shows the *lifespan* (in years) and *GDP* (in dollars) of people in 12 countries. The association is non-linear. Using the log x transformation:

- linearise the data, and fit a regression line to the transformed data (*GDP* is the EV)
- write its equation in terms of the variables *lifespan* and *GDP* correct to three significant figures.
- use the equation of the regression line to predict the lifespan in a country with a GDP of $20 000, correct to one decimal place.

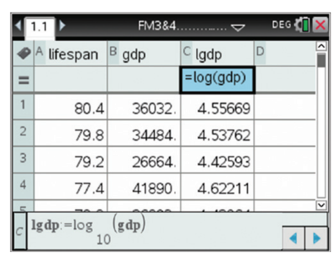
Lifespan	*GDP*
80.4	36 032
79.8	34 484
79.2	26 664
77.4	41 890
78.8	26 893
81.5	25 592
74.9	7 454
72.0	1 713
77.9	7 073
70.3	1 192
73.0	631
68.6	1 302

Steps

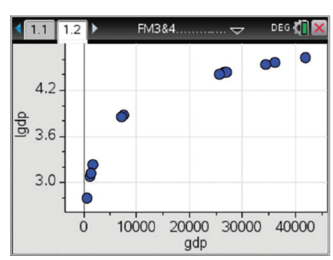
1. Start a new document by pressing ctrl + N .
2. Select **Add Lists & Spreadsheet**.
   Enter the data into lists named **lifespan** and **gdp**.



3. Name column C as **lgdp** (short for log (*GDP*)).
   Now calculate the values of log (*GDP*) and store them in the list named **lgdp**.
4. Move the cursor to the grey cell below the **lgdp** heading.
   We need to enter the expression = **log(gdp)**.
   To do this, press = then type in **log(gdp)**. Pressing enter calculates and displays the values of **lgdp**.



5. Press ctrl + I and select **Add Data & Statistics**.
   Construct a scatterplot of *lifespan* against *GDP*. Let *GDP* be the explanatory variable and *lifespan* the response variable. The plot is clearly non-linear.



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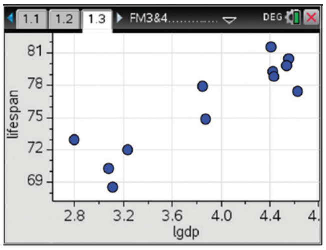
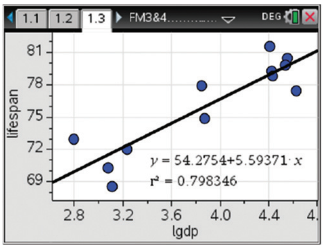
6 Press ctrl + I and select **Add Data & Statistics**.
Construct a scatterplot of *lifespan* against log *GDP*.
The plot is now clearly linear.



7 Press menu **>Analyze>Regression>Show Linear (a + bx)** to plot the line on the scatterplot with its equation.
Note: The x in the equation on the screen corresponds to the transformed variable log (*GDP*).



8 Write the regression equation in terms of the variables *lifespan* and log (*GDP*).

$lifespan = 54.3 - 5.59 \times \log (GDP)$

9 Substitute 20 000 for *GDP* in the equation to find the lifespan of people in a country with GDP of \$20 000.

$lifespan = 54.3 - 5.59 \times \log 20\,000$
$= 78.3$ years

Using the CASIO Classpad to perform a log transformation

The table shows the *lifespan* (in years) and *GDP* (in dollars) of people in 12 countries. The association is non-linear.
Using the log x transformation:

- linearise the data, and fit a regression line to the transformed data (*GDP* is the EV)
- write its equation in terms of the variables *lifespan* and *GDP* correct to three significant figures.
- use the equation to predict the lifespan in a country with a GDP of \$20 000 correct to one decimal place.

Lifespan	*GDP*
80.4	36 032
79.8	34 484
79.2	26 664
77.4	41 890
78.8	26 893
81.5	25 592
74.9	7 454
72.0	1 713
77.9	7 073
70.3	1 192
73.0	631
68.6	1 302

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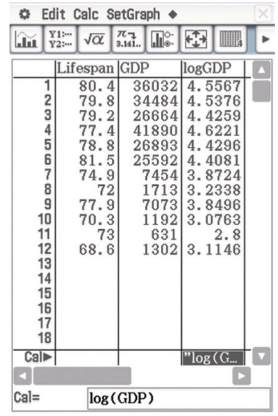
Steps

1 In the Statistics application enter the data into lists named **Lifespan** and **GDP**.

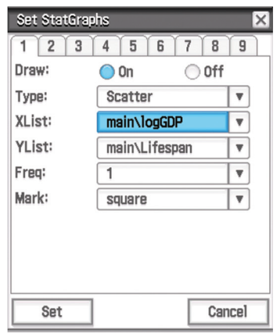
2 Name the third list **logGDP**.

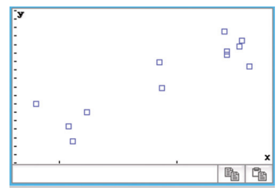
3 Place the cursor in the calculation cell at the bottom of the third column and type **log (GDP)**.
Let *GDP* be the explanatory variable (x) and *lifespan* the response variable (y).



4 Construct a scatterplot of *lifespan* against *log (GDP)*.

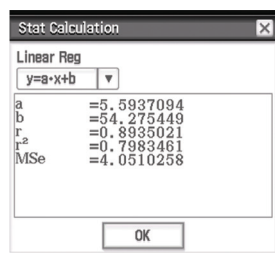
- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot.
- The plot is linear.

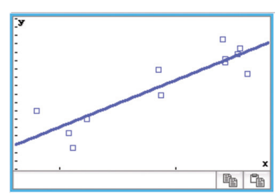




5 To find the least squares regression equation and fit a regression line to the transformed data.

- Go to **Calc, Regression, Linear Reg**.
- Complete the **Set Calculation** dialog box as shown and tap **OK**.
  This generates the regression results.
  Note: The x in the linear equation corresponds to the transformed variable log (GDP).





- Tap **OK** a second time to plot and display the regression line on the scatterplot.

6 Write the equation in terms of *lifespan* and log (GDP).

lifespan $= 54.3 - 5.59 \times \log(GDP)$

7 Substitute 20 000 for GDP in the equation.

lifespan $= 54.3 - 5.59 \times \log 20\,000$
$= 78.3$ years

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Exercise 5C

The log x transformation: some prerequisite skills

1 Evaluate the following expressions correct to one decimal place.

a $y = 5.5 + 3.1 \log 2.3$ **b** $y = 0.34 + 5.2 \log 1.4$

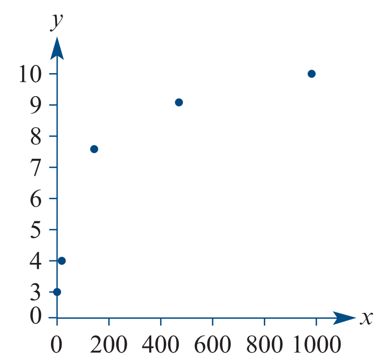
c $y = -8.5 + 4.12 \log 20$ **d** $y = 196.1 - 23.2 \log 303$

The log x transformation: calculator exercise

2 The scatterplot opposite was constructed from the data in the table below.

x	5	10	150	500	1000
y	3.1	4.0	7.5	9.1	10.0

From the scatterplot, it is clear that the association between y and x is non-linear.



a Linearise the scatterplot by applying a log x transformation and fit a least squares line to the transformed data.

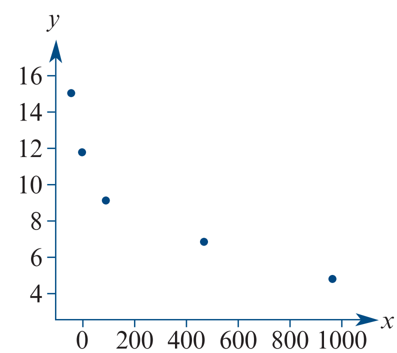
b Write down its equation and the coefficient, correct to one significant figure.

c Use the equation to predict the value of y when $x = 100$.

3 The scatterplot opposite was constructed from the data in the table below.

x	10	44	132	436	981
y	15.0	11.8	9.4	6.8	5.0

From the scatterplot, it is clear that the relationship between y and x is non-linear.



a Linearise the scatterplot by applying a log x transformation and fit a least squares line to the transformed data.

b Write down its equation and coefficient, correct one significant figure.

c Use the equation to predict the value of y when $x = 1000$.

The log y transformation: some prerequisite skills

4 Find the value of y in the following, correct to one decimal place if not exact.

a $\log y = 2$ **b** $\log y = 2.34$

c $\log y = 3.5 + 2x$ where $x = 1.25$ **d** $\log y = -0.5 + 0.024x$ where $x = 17.3$

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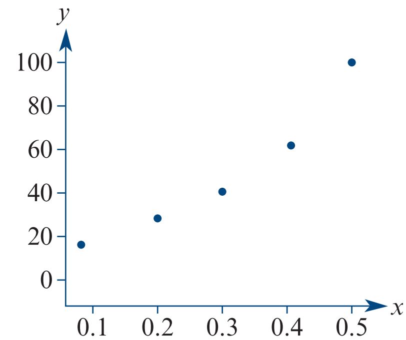
The log y transformation: calculator exercise

5 The scatterplot opposite was constructed from the data in the table below.

x	0.1	0.2	0.3	0.4	0.5
y	15.8	25.1	39.8	63.1	100.0



From the scatterplot, it is clear that the relationship between y and x is non-linear.

a Linearise the scatterplot by applying a log y transformation and fit a least squares line to the transformed data.

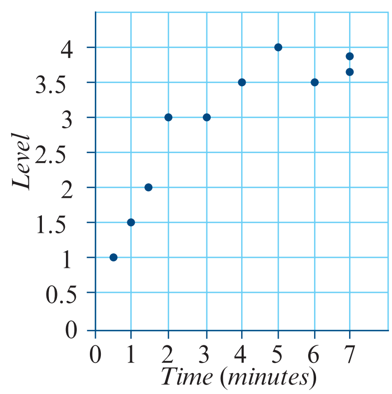
b Write down its equation.

c Use the equation to predict the value of y when $x = 0.6$, correct to one decimal place.

Applications of the log transformation

6 The table below shows the level of performance level achieved by 10 people on completion of a task. Also shown is the time spent (in minutes) practising the task. In this situation, *time* is the EV. The association between the *level* and *time* is non-linear as seen in the scatterplot.

Time	*Level*
0.5	1
1	1.5
1.5	2
2	3
3	3
4	3.5
5	4
6	3.5
7	3.9
7	3.6



A log transformation can be applied to the variable *time* to linearise the scatterplot.

a Apply the log transformation to the variable *time* and fit a least squares line to the transformed data. log (*time*) is the EV.

Write the slope and intercept of this line, correct to two significant figures in the spaces provided.

$$level = \boxed{} + \boxed{} \times \log(time)$$

b Use the equation to predict the level of performance (correct to one decimal place) for a person who spends 2.5 minutes practising the task.

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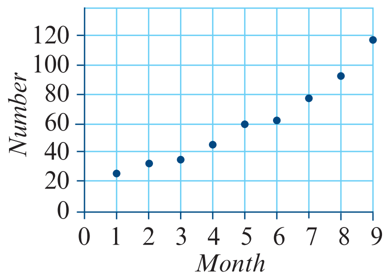
7 The table below shows the number of internet users signing up with a new internet service provider for each of the first nine months of their first year of operation. A scatterplot of the data also shown.

Month	*Number*
1	24
2	32
3	35
4	44
5	60
6	61
7	78
8	92
9	118



The association between *number* and *month* is non-linear.

a Apply the log transformation to the variable *number* and fit a least squares line to the transformed data. *Month* is the EV.
Write the slope and intercept of this line, correct to four significant figures, in the spaces provided.

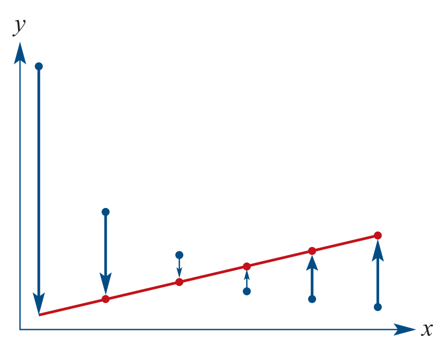
$log\ (number) = \square + \square \times month$

b Use the equation to predict the *number* of internet users after 10 months. Give answer to the nearest whole number.

5D The reciprocal transformation

The **reciprocal transformation** is a stretching transformation that compresses the upper end of the scale on either the x- or y-axis.

The effect of applying a reciprocal y transformation to a scatterplot is illustrated below.

Transformation	*Outcome*	*Graph*
$\frac{1}{y}$	The reciprocal y transformation works by compressing larger values of y relative to lower values of y. This has the effect of straightening out curves like the one shown opposite. The reciprocal x transformation works the same way but in the x-direction.	

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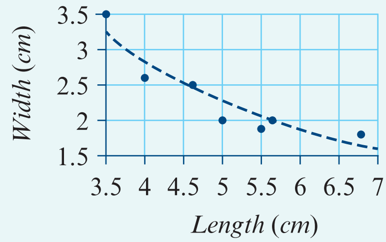
The following example shows how the $1/y$ transformation works in practice.

Example 3 **Applying the reciprocal transformation**

A homeware company makes rectangular sticky labels with a variety of lengths and widths.

The scatterplot opposite displays the *width* (in cm) and *length* (in cm) of eight of their sticky labels.



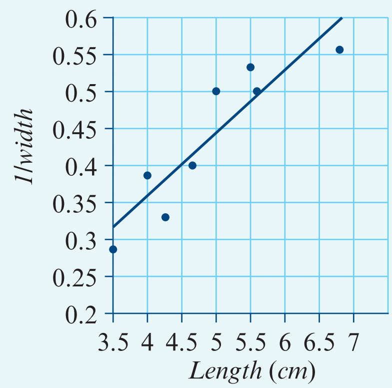
There is a strong negative association between the width of the sticky labels and their lengths, but it is clearly non-linear. Before we can fit a least squares regression line to the data, we need to linearise the scatterplot.

The circle of transformation suggests that we could use the log y, $1/y$, $1/x$ or log x transformation to linearise the scatterplot. We will use the $1/y$ transformation.

That is, we will change the scale on the *width* axis to 1/*width*.

This type of transformation is known as a reciprocal transformation.

When we make this change, we see that the association between 1/*width* and *length* is linear. See the plot opposite.



We can now fit a least squares line to model the association between 1/*width* and *length*.

Note: On the plot opposite, when 1/*width* = 0. 4, the actual width is 1/0.4 = 2.5 cm.

The equation of this line is:

$$1/width = 0.015 + 0.086 \times \ length$$

Like any other regression line we can use its equation to make predictions.

For example, for a sticky label of length 5 cm, we would predict that:

$$1/width = 0.015 + 0.086 \times 5 = 0.445$$

$$\text{or } width = \frac{1}{0.445} = 2.25 \text{ cm (to 2 d.p.)}$$

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Using the TI-Nspire CAS to perform a squared transformation

The table shows the length (in cm) and width (in cm) of eight sizes of sticky labels.

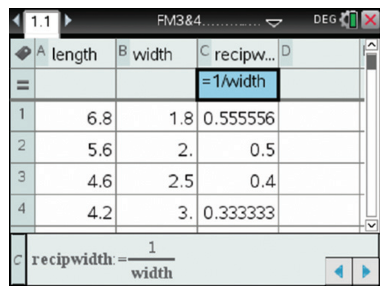
Length	1.8	2.0	2.5	3.0	3.5	2.6	2.0	1.9
Width	6.8	5.6	4.6	4.2	3.5	4.0	5.0	5.5

Using the $1/y$ transformation:

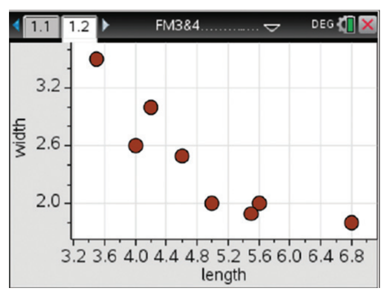
- linearise the data, and fit a regression line to the transformed data (*length* is the EV)
- write its equation in terms of the variables *length* and *width*
- use the equation to predict the width of a sticky label with a length of 5 cm.

Steps

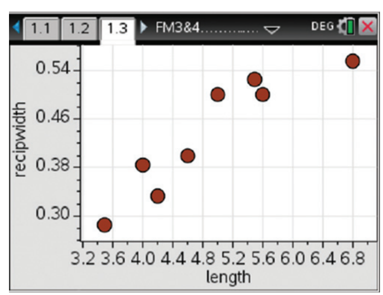
1. Start a new document by pressing ctrl + N.
2. Select **Add Lists & Spreadsheet**.
   Enter the data into lists named **length** and **width**.
3. Name column C as **recipwidth** (short for 1/width).
   Calculate the values of **recipwidth**.
   Move the cursor to the grey cell below the **recipwidth** heading. Type in =**1/width**. Press enter to calculate the values of **recipwidth**.



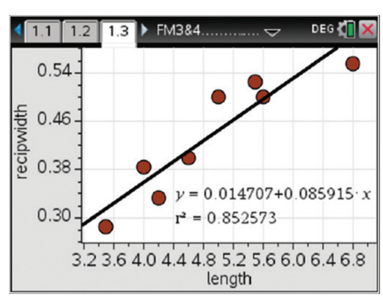
4. Press ctrl + I and select **Add Data & Statistics**.
   Construct a scatterplot of *width* against *length*.
   Let *length* be the explanatory variable and *width* the response variable. The plot is clearly non-linear.



5. Press ctrl + I and select **Add Data & Statistics**.
   Construct a scatterplot of **recipwidth** (1/*width*) against **length**. The plot is now clearly linear.



6. Press menu **>Analyze>Regression>Show Linear (a + bx)** to plot the line on the scatterplot with its equation.
   Note: The y in the equation on the screen corresponds to the transformed variable $1/width$.



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7 Write down the regression equation in terms of the variables *width* and *length*.	1/width = 0.415 + 0.066 × length
8 Substitute 5 cm for *length* in the equation.	1/width = 0.415 + 0.066 × 5 = 0.390 or width = 1/0.390 = 2.56 cm

Using the CASIO Classpad to perform a reciprocal transformation

The table shows the length (in cm) and width (in cm) of eight sizes sticky labels.

Length	1.8	2.0	2.5	3.0	3.5	2.6	2.0	1.9
Width	6.8	5.6	4.6	4.2	3.5	4.0	5.0	5.5

Using the $1/y$ transformation:

- linearise the data, and fit a regression line to the transformed data. *Length* is the RV.
- write its equation in terms of the variables *length* and *width.*
- use the equation to predict the width of a sticky label with length of 5 cm.

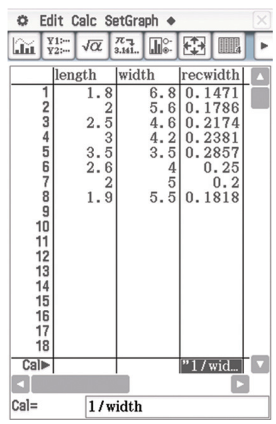
Steps

1 Open the Statistics application and enter the data into lists named **length** and **width**.

2 Name the third list **recwidth** (short for reciprocal width).

3 Place the cursor in the calculation cell at the bottom of the third column and type **1/width**. This will calculate all the reciprocal values of the width.

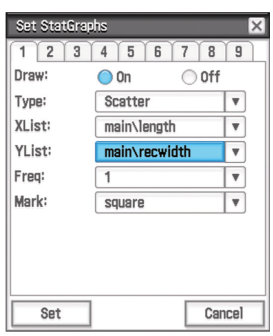
Let *length* be the explanatory variable (x) and *width* the response variable (y).

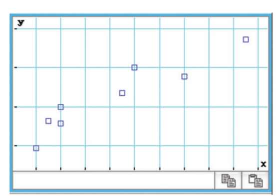


4 Construct a scatterplot of *1/width* against *length*.

- Tap [icon] and complete the **Set StatGraphs** dialog box as shown.
- Tap [icon] to view the scatterplot.

The plot is now clearly linear.





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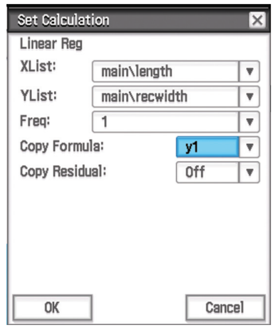
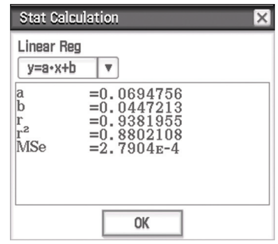
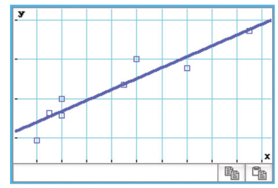
5 Fit a regression line to the transformed data.

- Go to **Calc, Regression, Linear Reg**.
- Complete the **Set Calculation** dialog box as shown and tap **OK**. This generates the regression results.

  Note: The y in the linear equation corresponds to the transformed variable $1/width$; that is $1/y$.







- Tap **OK** a second time to plot and display the line on the scatterplot.

6 Write down the equation in terms of the variables *width* and *length*.

$1/width = 0.045 + 0.069 \times length$

7 Substitute 5 cm for *length* in the equation.

$1/width = 0.045 + 0.069 \times 5 = 0.390$

or $width = 1/0.390 = 2.56$ cm

Exercise 5D

The reciprocal (1/x) transformation: some prerequisite skills

1 Evaluate the following expressions correct to one decimal place.

a $y = 6 + \dfrac{22}{x}$ when $x = 3$

b $y = 4.9 - \dfrac{2.3}{x}$ when $x = 1.1$

c $y = 8.97 - \dfrac{7.95}{x}$ when $x = 1.97$

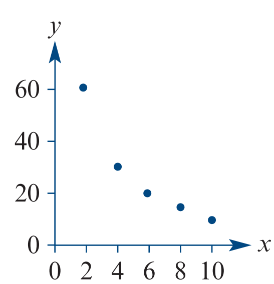
d $y = 102.6 + \dfrac{223.5}{x}$ when $x = 1.08$

The reciprocal (1/x) transformation: calculator exercise

2 The scatterplot opposite was constructed from the data in the table below.

x	2	4	6	8	10
y	60	30	20	15	12

From the scatterplot, it is clear that the association between y and x is non-linear.



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a Linearise the scatterplot by applying a $1/x$ transformation and fit a least squares line to the transformed data.

b Write down its equation.

c Use the equation to predict the value of y when $x = 5$.

The reciprocal (1/y) transformation: some prerequisite skills

3 Evaluate the following expressions correct to two decimal places.

a $\frac{1}{y} = 3x$ when $x = 2$

b $\frac{1}{y} = 6 + 2x$ when $x = 4$

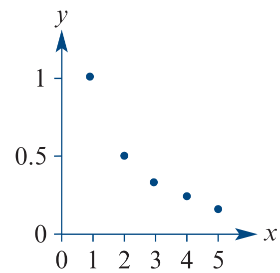
c $\frac{1}{y} = -4.5 + 2.4x$ when $x = 4.5$

d $\frac{1}{y} = 14.7 + 0.23x$ when $x = 4.5$

The reciprocal (1/y) transformation: calculator exercise

4 The scatterplot opposite was constructed from the data in the table below.

x	1	2	3	4	5
y	1	0.5	0.33	0.25	0.20



From the scatterplot, it is clear that the association between y and x is non-linear.

a Linearise the scatterplot by applying a $1/y$ transformation and fit a least squares line to the transformed data.

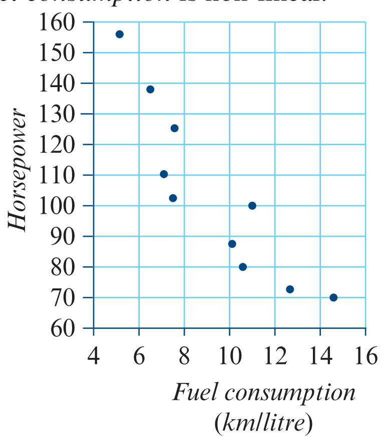
b Write down its equation.

c Use the equation to predict the value of y when $x = 0.25$.

Applications of the reciprocal transformation

5 The table shows the *horsepower* of 10 cars and their *fuel consumption*. The association between *horsepower* and *fuel consumption* is non-linear.

Consumption	*Horsepower*
5.2	155
7.3	125
12.6	75
7.1	110
6.3	138
10.1	88
10.5	80
14.6	70
10.9	100
7.7	103



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a Apply the reciprocal transformation to the variable *time* and fit a least squares line to the transformed data. *Horsepower* is the RV.

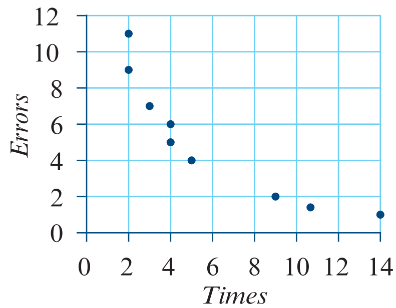
Write the intercept and slope of this line in the provided, correct to three significant figures.

$$horsepower = \boxed{} + \boxed{} \times \frac{1}{consumption}$$

b Use the equation to predict the horsepower of a car with a fuel consumption of 9 km/litre.

6 Ten students were given an opportunity to practise a complex matching task as often as they liked before they were assessed. The number of *times* they practised the task and the number of *errors* they made when assessed are given in the table.

Times	*Errors*
1	14
2	9
2	11
4	5
5	4
6	4
7	3
7	3
9	2



a Apply the reciprocal transformation to the variable *errors* and determine the least squares regression with the number of times the task was practiced as the EV.

Write the intercept and slope of this line in the boxes provided, correct to two significant figures.

$$\frac{1}{errors} = \boxed{} + \boxed{} \times times$$

b Use the equation to predict the number of errors made when the task is practised six times.

Cambridge Senior Maths AC/VCE
Further Mathematics 3&4

ISBN 978-1-107-56757-3

Cambridge University Press

Review

Key ideas and chapter summary

Data transformation	In regression analysis, **data transformation** is used to linearise a scatterplot prior to modelling the association with a least squares line.
Squared transformation	The **squared transformation** *stretches out* the upper end of the scale on an axis.
Logarithmic transformation	The **logarithmic transformation** *compresses* the upper end of the scale on an axis.
Reciprocal transformation	The **reciprocal transformation** *compresses* the upper end of the scale on an axis but to a greater extent than the log transformation.
The circle of transformations	The **circle of transformations** provides guidance in choosing the transformations that can be used to linearise various types of scatterplots. See Section 5A.

Skills check

Having completed this chapter you should be able to:

- use the circle of transformation to identify an appropriate transformation to linearise a scatterplot
- apply a square, log or reciprocal transformation to linearise a scatterplot (to one axis only)
- fit a least squares regression line to a linearised scatterplot, and use its equation to make predictions.

Multiple-choice questions

1 Select the statement that correctly completes the sentence:
'The effect of a squared transformation is to . . . '

A stretch the high values in the data
B maintain the distance between values
C stretch the low values in the data
D compress the high values in the data
E reverse the order of the values in the data

2 Select the statement that correctly completes the sentence:
'The effect of a log transformation is to . . .'

A stretch the high values in the data
B maintain the distance between values
C stretch the low values in the data
D compress the high values in the data
E reverse the order of the values in the data

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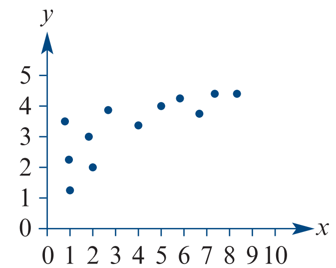
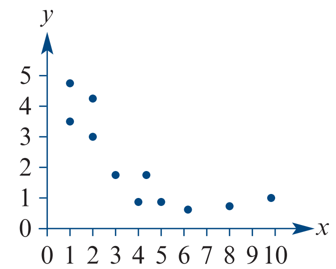
Review

3 The association between two variables y and x, as shown in the scatterplot, is non-linear. In an attempt to transform the relationship to linearity, a student would be advised to:



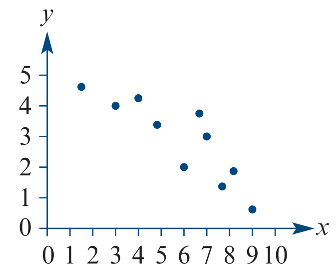
A leave out the first four points
B use a y^2 transformation
C use a $\log y$ transformation
D use a $1/y$ transformation
E use a least squares regression line

4 The association between two variables y and x, as shown in the scatterplot, is non-linear.
Which of the following sets of transformations could possibly linearise this relationship?



A $\log y$, $1/y$, $\log x$, $1/x$ **B** y^2, x^2
C y^2, $\log x$, $1/x$ **D** $\log y$, $1/y$, x^2
E $ax + b$

5 The association between two variables y and x, as shown in the scatterplot, is non-linear.
Which of the following transformations is most likely to linearise the relationship?



A a $1/x$ transformation **B** a y^2 transformation
C a $\log y$ transformation **D** a $1/y$ transformation
E a $\log x$ transformation

6 The following data were collected for two related variables x and y.

x	1	2	3	4	5	6	7	8	9	10	11
y	7	8.6	8.9	8.8	9.9	9.7	10.4	10.5	10.7	11.2	11.1

A scatterplot indicates a non-linear association. The data is linearised using a log x transformation and a least squares line is then fitted. The equation of this line is closest to:

A $y = 7.52 + 0.37 \log x$ **B** $y = 0.37 + 7.52 \log x$
C $y = -1.71 + 0.25 \log x$ **D** $y = 3.86 + 7.04 \log x$
E $y = 7.04 + 3.86 \log x$

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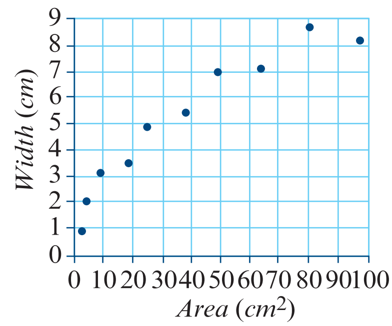
7 The data in the scatterplot opposite shows the width (cm) and the surface area (cm^2) of leaves sampled from 10 different trees. The scatterplot is non-linear.

To linearise the scatterplot, $(Width)^2$ is plotted against area and a least squares regression line is then fitted to the linearised plot.



The equation of this least squares regression line is: $(Width)^2 = 1.8 + 0.8 \times Area$

Using this equation, a leaf with a surface area of 120 cm^2 is predicted to have a width, in cm, closest to:

A 9.2 **B** 9.9 **C** 10.6 **D** 84.6 **E** 97.8

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8 The association between the total *weight* of produce picked from a vegetable garden and its *width* is non-linear. An x^2 transformation is used to linearise the data. When a least squares line is fitted to the data, its y-intercept is 10 and its slope is 5. Assuming that *weight* is the response variable, the equation of this line is:

A $(weight)^2 = 10 + 5 \times \ width$

B $width = 10 + 5 \times \ (weight)^2$

C $width = 5 + 10 \times \ (weight)^2$

D $(weight)^2 = 10 + 5 \times \ (weight)^2$

E $(weight)^2 = 5 + 10 \times \ weight$

9 A model that describes the association between the hours spent studying for an exam and the mark achieved is:

$mark = 20 + 40 \times \log\,(hours)$

From this model, we would predict that a student who studies for 20 hours would score a mark (to the nearest whole number) of:

A 80 **B** 78 **C** 180 **D** 72 **E** 140

10 A $1/y$ transformation is used to linearise a scatterplot.

The equation of a least squares line fitted to this data is:

$1/y = 0.14 + 0.045x$

This regression line predicts that, when $x = 6$, y is closest to:

A 0.17 **B** 0.27 **C** 0.41 **D** 2.4 **E** 3.7

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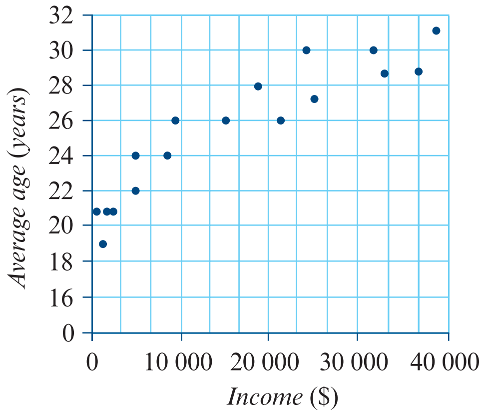
Extended-response questions

1 The average age at first marriage (*average age*) and average yearly income in dollars per person (*income*) was recorded for a group of 17 countries. The results are displayed below. A scatterplot of the data is also shown.

Average age (years)	*Income* (\$)
21	1 750
22	3 200
26	8 600
26	16 000
28	17 000
26	21 000
30	24 500
30	32 000
31	38 500
29	33 000
27	25 500
29	36 000
19	1 300
21	600
24	3 050
24	6 900
21	1 400



The association between *average age* and *income* is non-linear.
A log transformation can be applied to the variable income and used to linearise the data.

a Apply this log transformation to the data and determine the equation of the least squares regression line that allows *average age* to be predicted from log (*income*). Write the coefficients for this equation correct to three significant figures in the spaces provided.

$$\textit{average age} = \boxed{} + \boxed{} \times \log(\textit{income})$$

b Use the equation to predict the average age of women at first marriage in a country with an average income of \$20 000 per person. Write your answer correct to one decimal place.

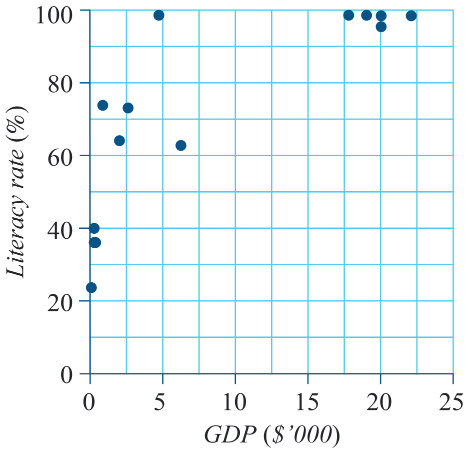
based on VCAA (2010)

ISBN 978-1-107-56757-3

Review

2 The table below shows the percentage of people who can read (*literacy rate*) and the gross domestic product (*GDP*), in dollars, for a selection of 14 countries. A scatterplot of the data is also shown.
The scatterplot can be linearised by using a log x transformation.

GDP	*Literacy rate*
2677	72
260	35
19 904	97
122	24
18 944	99
4 500	99
17 539	99
1 030	73
19 860	99
409	40
406	35
6651	62
22 384	99
2 436	64



a Apply the log transformation to the variable *GDP* and verify that it linearises the data by constructing a scatterplot from the transformed data.

b Fit a least squares line to the transformed data and write down its equation terms of the variable *literacy rate* and log (*GDP*). *Literacy rate* is the RV.

c Give the slope and intercept correct to three significant figures.

d Use the regression line to predict the literacy rate of a country with a GDP of \$10 000 to the nearest per cent.

3 Measurements of the *distance* travelled (metres) and *time* taken (seconds) were made on a falling body. The data are given in the table below.

Time	0	1	2	3	4	5	6
Distance	0	5.2	18.0	42.0	79.0	128.0	168.0
Time^2							

a Construct a scatterplot of the data and comment on its form.

b Determine the values of *time*^{2} and complete the table.

c Construct a scatterplot of *distance* against *time*^{2}.

d Fit a least squares line to the transformed data. *Distance* is the RV.

e Use the regression equation to predict the distance travelled in 7 seconds.

f Obtain a residual plot and comment on the assumption of linearity.

ISBN 978-1-107-56757-3

6

Core: Data analysis

Chapter 6

Investigating and modelling time series

ISBN 978-1-107-56757-3

Cambridge University Press

6A Time series data

Time series data are a special kind of bivariate data, where the explanatory variable is time. An example of time series data is the following table of Australian annual birth rates (average births per female) between 1931 and 2002.

Year	*Birth rate*	*Year*	*Birth rate*	*Year*	*Birth rate*	*Year*	*Birth rate*
1931	1.039	1949	1.382	1967	1.342	1985	0.920
1932	0.967	1950	1.415	1968	1.360	1986	0.894
1933	0.959	1951	1.409	1969	1.360	1987	0.883
1934	0.939	1952	1.468	1970	1.349	1988	0.877
1935	0.941	1953	1.477	1971	1.400	1989	0.882
1936	0.967	1954	1.497	1972	1.296	1990	0.908
1937	0.981	1955	1.532	1973	1.179	1991	0.887
1938	0.976	1956	1.546	1974	1.123	1992	0.906
1939	0.986	1957	1.598	1975	1.049	1993	0.893
1940	1.042	1958	1.603	1976	0.980	1994	0.884
1941	1.094	1959	1.614	1977	0.951	1995	0.875
1942	1.096	1960	1.613	1978	0.930	1996	0.861
1943	1.148	1961	1.668	1979	0.908	1997	0.855
1944	1.179	1962	1.609	1980	0.901	1998	0.848
1945	1.267	1963	1.572	1981	0.924	1999	0.846
1946	1.379	1964	1.480	1982	0.921	2000	0.844
1947	1.416	1965	1.400	1983	0.920	2001	0.833
1948	1.376	1966	1.355	1984	0.883	2002	0.848

This dataset is rather complex, and it is hard to see any patterns just by looking at the data.

However, we can start to make sense of the data by displaying it graphically.

The graph we use for this purpose is called a **time series plot**.

A time series plot is a line graph with time plotted on the horizontal axis. The variable under investigation, the response variable, is plotted on the vertical axis.

The time series plot below has been constructed from the birth rate data tabulated above.

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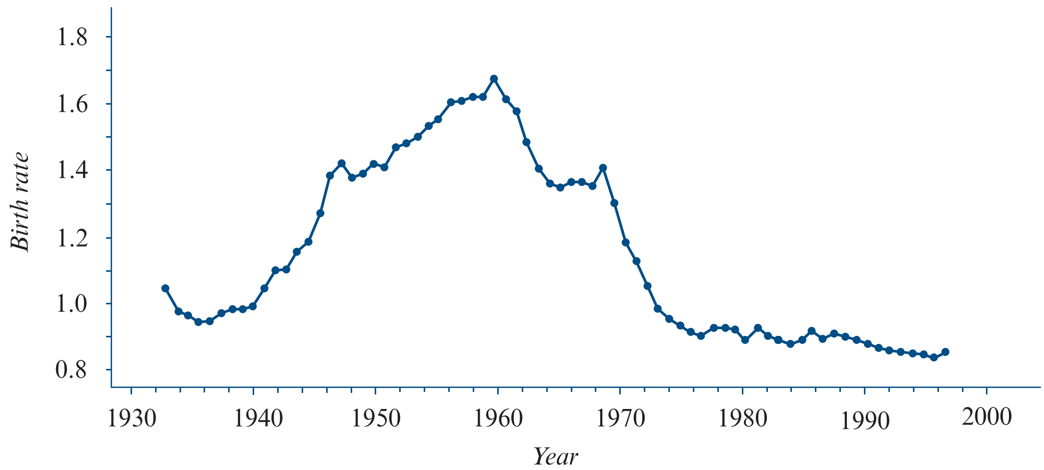
In this time series plot, *birth rate* is the response variable.



Looking for patterns in time series plots

The features we look for in a time series are:

- trend
- cycles
- seasonality
- structural change
- possible outliers
- irregular (random) fluctuations.

One or all of these features can be found in a time series plot.

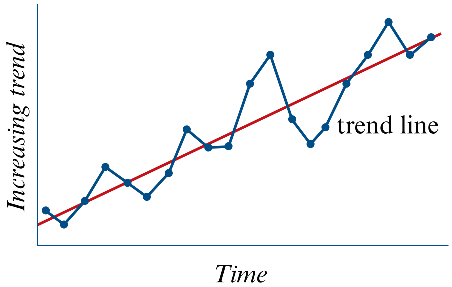
Trend

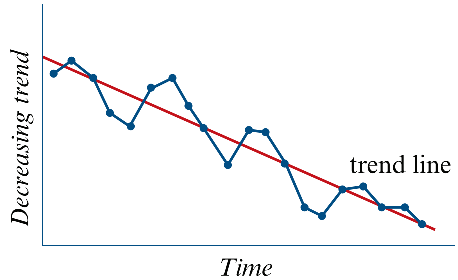
Examining a time series plot we can often see a general upward or downward movement over time. This indicates a long-term change over time that we call a trend.

> **Trend**
>
> The trendency for values in a time series to generally increase or decrease over a significant period of time is called a *trend*.

One way of identifying trends on a time series graph is to draw a line that ignores the fluctuations, but which reflects the overall increasing or decreasing nature of the plot. These lines are called *trend lines*.

Trend lines have been drawn on the time series plots below to indicate an *increasing* trend (line slopes upwards) and a *decreasing* trend (line slopes downwards).





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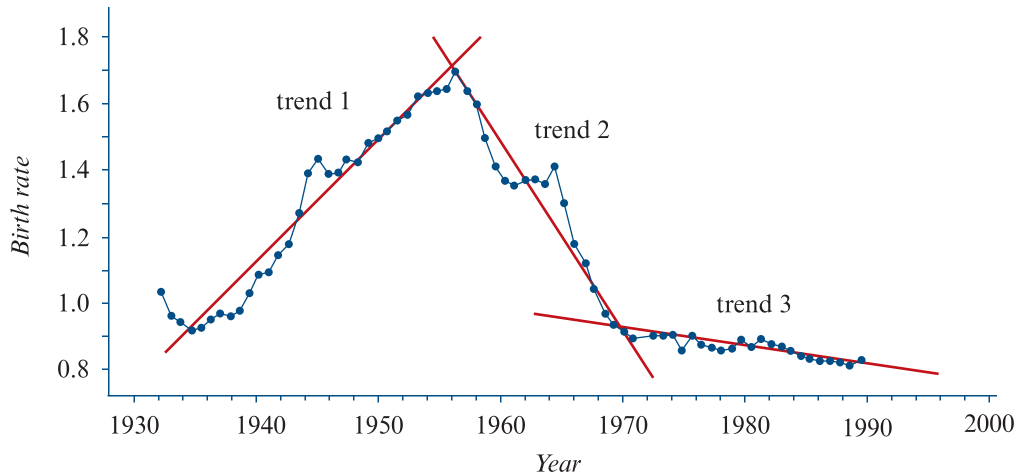
Sometimes, different trends are apparent in a time series for different time periods. For example, in the time series plot of the birth rate data, there are three distinct trends, which can be seen by drawing trend lines on the plot.



Each of these trends can be explained by changing socioeconomic circumstances.

Trend 1: Between 1940 and 1961 the birth rate in Australia grew quite dramatically. Those in the armed services came home from the Second World War, and the economy grew quickly. This rapid increase in the Australian birth rate during this period is known as the 'Baby Boom'.

Trend 2: From about 1962 until 1980 the birth rate declined very rapidly. Birth control methods became more effective, and women started to think more about careers. This period is sometimes referred to as the 'Baby Bust'.

Trend 3: During the 1980s, and up until the early 2000s, the birth rate continued to decline slowly for a complex range of social and economic reasons.

Cycles

Cycles

Cycles are periodic movements in a time series, but over a period greater than 1 year.

Some cycles repeat regularly, and some do not. The following plot shows the sunspot[1] activity for the period 1900 to 2010. The period of this cycle is approximately 11 years.

[1] Sunspots are dark spots visible on the surface of the Sun that come and go over time.

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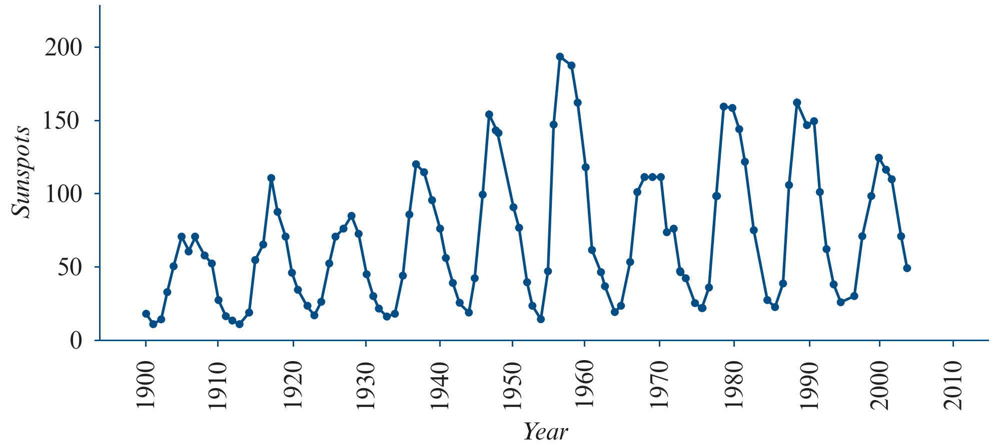


Many business indicators, such as **interest rates** or unemployment figures, also vary in cycles, but their periods are usually less regular. Cycles with calendar-related periods of 1 year or less are of special interest and give rise to what is called 'seasonality'.

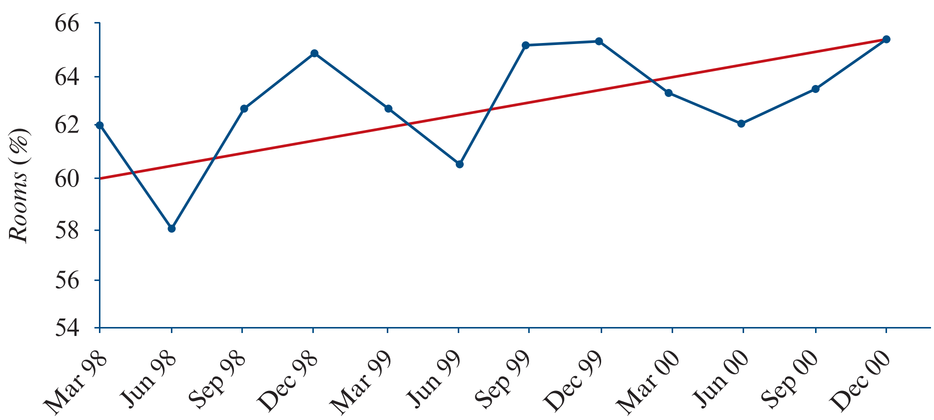
Seasonality

Seasonality

Seasonality is present when there is a periodic movement in a time series that has a calendar-related period – for example a year, a month or a week.

Seasonal movements tend to be more predictable than trends, and occur because of variations in the weather, such as ice-cream sales, or institutional factors, like the increase in the number of unemployed people at the end of the school year.

The plot below shows the total percentage of rooms occupied in hotels, motels and other accommodation in Australia by quarter, over the years 1998–2000.



This time series plot reveals both *seasonality* and *trend* in the demand for accommodation. The *regular peaks and troughs* in the plot that occur at the *same time each year* signal the presence of *seasonality*. In this case, the demand for accommodation is at its lowest in the June quarter and highest in the December quarter.

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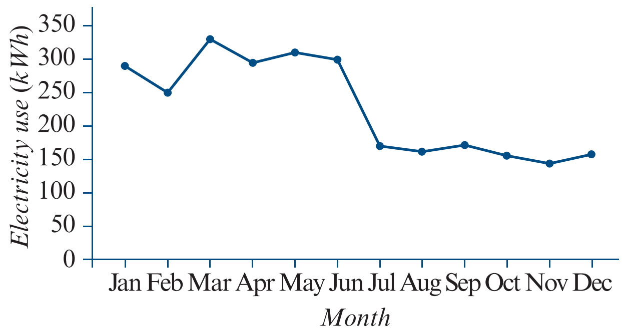
The *upward sloping trend line* signals the presence of a general increasing *trend*. This tells us that, even though demand for accommodation has fluctuated from month to month, demand for hotel and motel accommodation has increased over time.

Structural change

Structural change

Structural change is present when there is a sudden change in the established pattern of a time series plot.

The time series plot below shows the power bill for a rental house (in kWh) for the 12 months of a year.



The plot reveals an abrupt change in power usage in June to July. During this period, monthly power use suddenly decreases from around 300 kWh per month from January to June to around 175k Wh for the rest of the year. This is an example of structural change that can probably be explained by a change in tenants, from a family with two children to a person living alone.

Structural change is also displayed in the birth rate time series plot we saw earlier. This revealed three quite distinct trends during the period 1900–2010. These reflect significant external events (like a war) or changes in social and economic circumstances.

One consequence of structural change is that we can no longer use a single mathematical model to describe the key features of a time series plot.

Outliers

Outliers

Outliers are present when there are individual values that stand out from the general body of data.

The time series plot below shows the daily power bill for a house (in kWh) for a fortnight.

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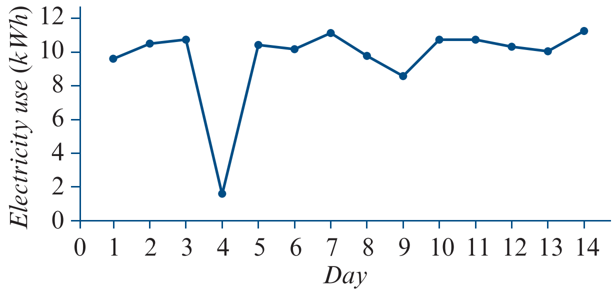

For this household, daily electricity use follows a regular pattern that, although fluctuating, averages about 10 kWh per day. In terms of daily power use, day 4 is a clear outlier, with less than 2 kWh of electricity used. A follow-up investigation found that, on this day, the house was without power for 18 hours due to a storm, so much less power was used than normal.

Irregular (random) fluctuations

Irregular (random) fluctuations

Irregular (random) fluctuations include all the variations in a time series that we cannot reasonably attribute to systematic changes like trend, cycles, seasonality and structural change or an outlier.

There can be many sources of irregular fluctuations, mostly unknown. A general characteristic of these fluctuations is that they are unpredictable.

One of the aims of time series analysis is to develop techniques to identify regular patterns in time series plots that are often obscured by irregular fluctuations. One of these techniques is smoothing, which you will meet in the next section.

Identifying patterns in time series plots

The features we look for in a time series are:

- trend
- cycles
- seasonality
- structural change
- possible outliers
- irregular (random) fluctuations.

Trend is present when there is a *long-term* upward or downward movement in a time series.

Cycles are present when there is a periodic movement in a time series. The period is the time it takes for one complete up and down movement in the time series plot. In practice, this term is reserved for periods greater than 1 year.

Seasonality is present when there is a periodic movement in a time series that has a calendar related period – for example a year, a month or a week.

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Structural change is present when there is a sudden change in the established pattern of a time series plot.

Outliers are present when there are individual values that stand out from the general body of data.

Irregular (random) fluctuations are always present in any real-world time series plot. They include all the variations in a time series that we cannot reasonably attribute to systematic changes like trend, cycles, seasonality and structural change or an outlier.

▶ Constructing time series plots

Most real-world time series data come in the form of large datasets that are best plotted with the aid of a spreadsheet or statistical package. The availability of the data in electronic form via the internet greatly helps this process. However, in this chapter, most of the time series datasets are relatively small and can be readily plotted using a CAS calculator.

How to construct a time series using the TI-Nspire CAS

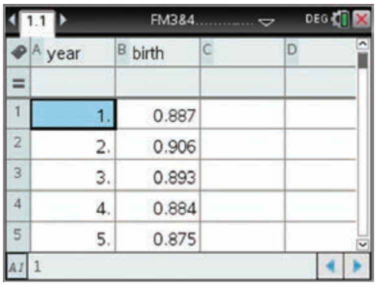
Construct a time series plot for the data presented below. The years have been recoded as 1, 2, . . . , 12, as is common practice.

2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
1	2	3	4	5	6	7	8	9	10	11	12
0.887	0.906	0.893	0.884	0.875	0.861	0.855	0.848	0.846	0.844	0.833	0.848

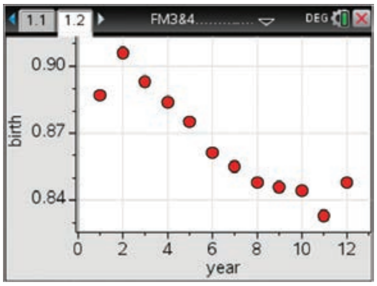
Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet.** Enter the data into lists named *year* and *birth*.



3 Press ctrl + I and select **Add Data & Statistics**. Construct a scatterplot of *birth* against *year*. Let *year* be the explanatory variable and *birth* the response variable.



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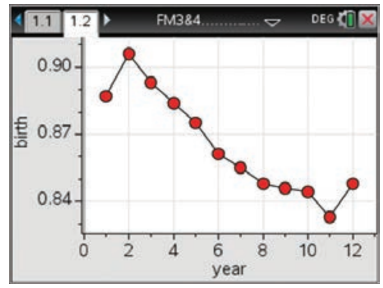
4 To display as a connected time series plot, move the cursor to the main graph area and press ctrl + menu >**Connect Data Points**. Press enter.



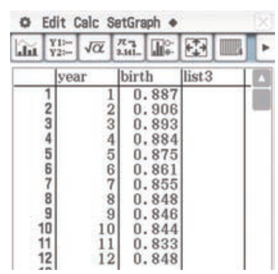
How to construct a time series using the ClassPad

Construct a time series plot for the data presented below. The years have been recoded as 1, 2, . . . , 12, as is common practice.

2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
1	2	3	4	5	6	7	8	9	10	11	12
0.887	0.906	0.893	0.884	0.875	0.861	0.855	0.848	0.846	0.844	0.833	0.848

Steps

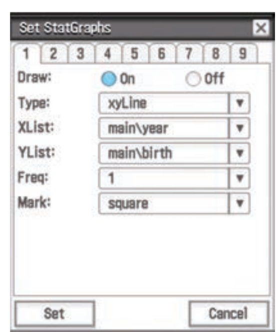
1 Open the **Statistics** application and enter the data into the columns named *year* and *birth*. Your screen should look like the one shown.



2 Tap  to open the **Set StatGraphs** dialog box and complete as follows.

- **Draw:** select **On**.
- **Type:** select **xyLine** (▼).
- **XList:** select **main/year** (▼).
- **YList:** select **main/birth** (▼).
- **Freq:** leave as **1**.
- **Mark:** leave as **square**.

Tap **Set** to confirm your selections.



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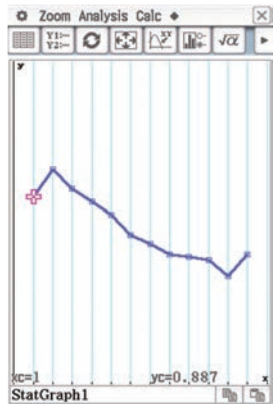
3 Tap in the toolbar at the top of the screen to display the time series plot in the bottom half of the screen.
To obtain a full-screen display, tap Resize from the icon panel.
Tap from the toolbar, and use ▶ and ◀ to move from point to point to read values from the plot.

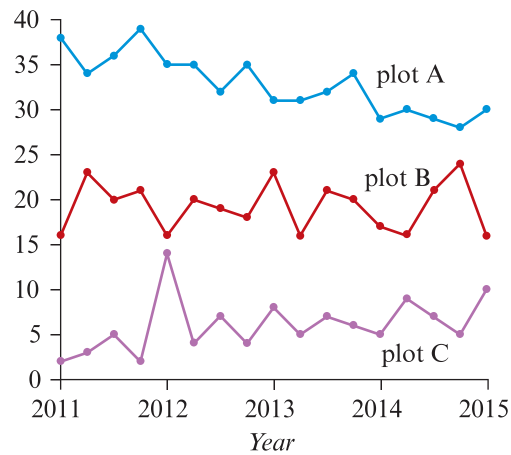


Exercise 6A

Identifying key features in a time series plot

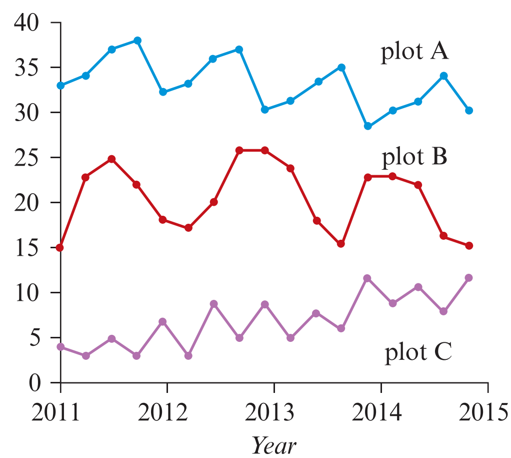
1 Complete the table below by indicating which of the listed features are present in each of the time series plots shown below.

Feature	*Plot* A	B	C
Irregular fluctuations			
Increasing trend			
Decreasing trend			
Cycles			
Outlier			



2 Complete the table below by indicating which of the listed features are present in each of the time series plots shown below.

Feature	*Plot* A	B	C
Irregular fluctuations			
Increasing trend			
Decreasing trend			
Cycles			
Seasonality			



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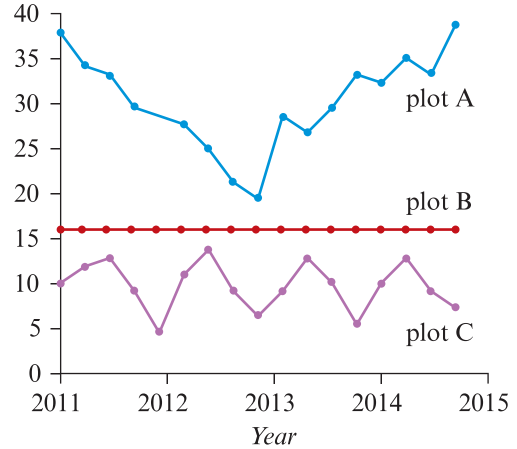
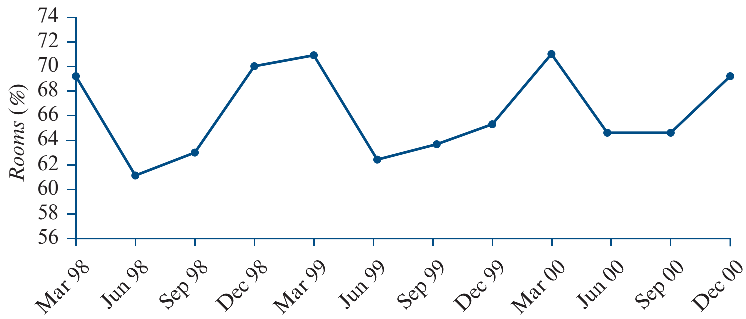
3 Complete the table below by indicating which of the listed features are present in each of the time series plots shown below.

Feature	*Plot* A	B	C
Irregular fluctuations			
Structural change			
Increasing trend			
Decreasing trend			
Seasonality			



Describing time series plots

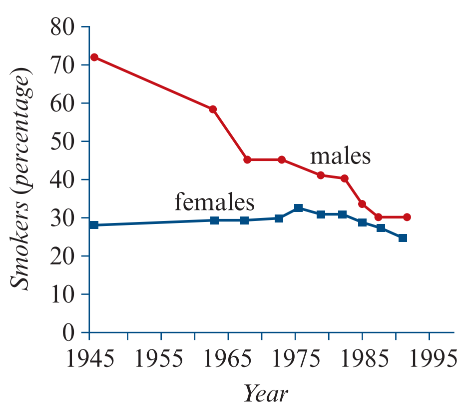
4 The time series plot for the hotel room occupancy rate (%) in Victoria over the period March 1998–December 2000 is shown below. Describe the features of the plot.



5 The time series plot shown shows the smoking rates (%) of Australian males and females over the period 1945–92.

a Describe any trends in the time series plot.

b Did the *difference* in smoking rates increase or decrease over the period 1945–92?



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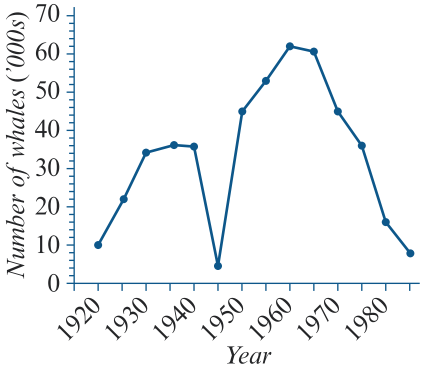
6 The time series plot opposite shows the number of whales caught during the period 1920–85. Describe the features of the plot.

Note: This time series exhibits structural change so cannot be described by a single trend. Here is some relevant historical information:

- The 1930s was the time of the Great Depression.
- 1939–45 was the period of the Second World War.
- 1960–85 was a time when countries began to accept that whales were endangered.



Using a CAS calculator to construct a time series plot

7 Use the data below to construct a time series plot of the Australian birth rate for 1960–70.

Year	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	1970
Rate	1.613	1.668	1.609	1.572	1.480	1.400	1.355	1.342	1.360	1.360	1.349

8 Use the data below to construct a time series plot of the population (in millions) in Australia over the period 1993–2003.

Year	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003
Population	17.8	18.0	18.2	18.4	18.6	18.8	19.0	19.3	19.5	19.8	20.0

9 Use the data below to construct a time series plot for the number of teachers (in thousands) in Australia over the period 1993–2001.

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
Teachers	213	217	218	218	221	223	228	231	239	244	250

10 The table below gives the number of male and female teachers (in thousands) in Australia over the years 1993–2001.

Year	1993	1994	1995	1996	1997	1998	1999	2000	2001
Males ('000s)	77.9	76.6	75.3	75.0	74.9	74.9	76.0	76.6	77.1
Females ('000s)	139.9	141.2	145.5	148.5	152.5	156.0	163.4	167.4	172.5

a Construct a time series plot showing both the male and female teachers on the same graph.

b Describe and comment on any trends you observe.

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6B Smoothing a time series using moving means

A time series plot can incorporate many of the sources of variation previously mentioned: trend, cycles, seasonality, structural change, outliers and irregular fluctuations. One effect of the irregular fluctuations and seasonality can be to obscure an underlying trend. The technique of **smoothing** can sometimes be used to overcome this problem.

Smoothing a time series plot using moving means

This method of smoothing (**moving mean smoothing**) involves replacing individual data points in the time series with their moving means. The simplest method is to smooth over a small number of odd number points – for example, three or five.

The three-moving mean

To use *three-moving mean smoothing*, replace each data value with the mean of that value and the values of its two neighbours, one on each side. That is, if y_1, y_2 and y_3 are sequential data values, then:

$$\text{smoothed } y_2 = \frac{y_1 + y_2 + y_3}{3}$$

The first and last points do not have values on each side, so leave them out.

The five-moving mean

To use *five-moving mean smoothing*, replace each data value with the mean of that value and the two values on each side. That is, if y_1, y_2, y_3, y_4, y_5 are sequential data values, then:

$$\text{smoothed } y_3 = \frac{y_1 + y_2 + y_3 + y_4 + y_5}{5}$$

The first two and last two points do not have two values on each side, so leave them out.

If needed, these definitions can be readily extended for moving means involving 7, 9, 11, . . . points. The larger the number of points we smooth over, the greater the smoothing effect.

Example 1 Three- and five-moving mean smoothing

The table below gives the temperature (°C) recorded at a weather station at 9.00 a.m. each day for a week.

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Temperature	18.1	24.8	26.4	13.9	12.7	14.2	24.9

a Calculate the three-mean smoothed temperature for Tuesday.

b Calculate the five-mean smoothed temperature for Thursday.

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Solution

a **1** Write down the three temperatures centred on Tuesday.

18.1, 24.8, 26.4

2 Find their mean and write down your answer.

$$\text{Mean} = \frac{(18.1 + 24.8 + 26.4)}{3} = 23.1$$

The three-mean smoothed temperature for Tuesday is 23.1°C.

b **1** Write down the five temperatures centred on Thursday.

24.8, 26.4, 13.9, 12.7, 14.2

2 Find their mean and write down your answer.

$$\text{Mean} = \frac{(24.8 + 26.4 + 13.9 + 12.7 + 14.2)}{5} = 18.4$$

The five-mean smoothed temperature for Thursday is 18.4°C.

The next step is to extend these computations to smooth all terms in the time series.

Example 2 Three- and five-moving mean smoothing of a time series

The following table gives the number of births per month over a calendar year in a country hospital. Use the three-moving mean and the five-moving mean methods, correct to one decimal place, to complete the table.

Solution

Complete the calculations as shown below.

Month	*Number of births*	*3-moving mean*	*5-moving mean*
January	10		
February	12	$\frac{10+12+6}{3} = 9.3$	
March	6	$\frac{12+6+5}{3} = 7.7$	$\frac{10+12+6+5+22}{5} = 11.0$
April	5	$\frac{6+5+22}{3} = 11.0$	$\frac{12+6+5+22+18}{5} = 12.6$
May	22	$\frac{5+22+18}{3} = 15.0$	$\frac{6+5+22+18+13}{5} = 12.8$
June	18	$\frac{22+18+13}{3} = 17.7$	$\frac{5+22+18+13+7}{5} = 13.0$
July	13	$\frac{18+13+7}{3} = 12.7$	$\frac{22+18+13+7+9}{5} = 13.8$
August	7	$\frac{13+7+9}{3} = 9.7$	$\frac{18+13+7+9+10}{5} = 11.4$
September	9	$\frac{7+9+10}{3} = 8.7$	$\frac{13+7+9+10+8}{5} = 9.4$
October	10	$\frac{9+10+8}{3} = 9.0$	$\frac{7+9+10+8+15}{5} = 9.8$
November	8	$\frac{10+8+15}{3} = 11.0$	
December	15		

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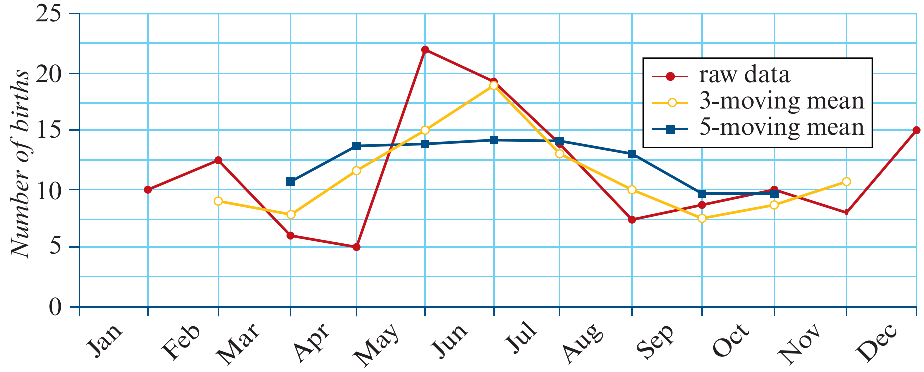
The result of this smoothing can be seen in the plot below, which shows the raw data, the data smoothed with a three-moving means and the data smoothed with a five-moving means.



Note: In the process of smoothing, **data points are lost** at the beginning and end of the time series.

Two observations can be made from this plot:

1 five-mean smoothing is more effective in reducing the irregular fluctuations than three-mean smoothing

2 the five-mean smoothed plot shows that there is no clear trend although the raw data suggest that there might be an increasing trend.

There are many ways of smoothing a time series. Moving means of group size other than three and five are common and often very useful.

However, if we smooth over an even number of points, we run into a problem. The centre of the set of points is not at a time point belonging to the original series. Usually, we solve this problem by using a process called **centring**. Centring involves taking a two-moving mean of the already smoothed values so that they line up with the original time values. It is a two-step process.

Two-mean smoothing with centring

We will illustrate the process by finding the two-moving mean, centred on Tuesday, for the daily temperature data opposite.

Day	*Temperature*
Monday	18.1
Tuesday	24.8
Wednesday	26.4

It is straightforward to calculate a series of two-moving means for this data by calculating the mean for Monday and Tuesday, followed by the mean for Tuesday and Wednesday. However, as we can see in the diagram below, these means do not align with a particular day, but lie between days. We solve this problem by finding the average of these two means. This gives a smoothed value that is now centred on Tuesday.

We call this process two-mean smoothing with centring.

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Day	*Temperature*	*Two-moving means*	*Two-moving mean with centring*
Monday	18.1		
		$\frac{(18.1 + 24.8)}{2} = 21.45$	
Tuesday	24.8		$\frac{(21.45 + 25.6)}{2} = 23.525$
		$\frac{(24.8 + 26.4)}{2} = 25.60$	
Wednesday	26.4		

In practice, we do not have to draw such a diagram to perform these calculations. The purpose of doing so is to show how the centring process works. In practice, calculating two-moving means is a much briefer and routine process as we illustrate in the following example. However, before proceeding, you might find it useful to view the video for this topic.

Example 3 Two-moving mean smoothing with centring

The temperatures (°C) recorded at a weather station at 9 a.m. each day for a week are displayed in the table.

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Temperature	18.1	24.8	26.4	13.9	12.7	14.2	24.9

Calculate the two-mean smoothed temperature for Tuesday with centring.

Solution

1 For two-mean smoothing with centring, write down the **three** data values centred on Tuesday (highlighted in red).

18.1 24.8 26.4

2 Calculate the mean of the first two values (mean 1). Calculate the mean of the second two values (mean 2).

$$\text{Mean 1} = \frac{(18.1 + 24.8)}{2}$$

$$\text{Mean 2} = \frac{(24.8 + 26.4)}{2}$$

3 The centred mean is then the average of mean 1 and mean 2.

$$\text{Centred mean} = \frac{(\text{mean 1} + \text{mean 2})}{2} = \frac{(21.45 + 25.6)}{2} = 23.525$$

4 Write down your answer.

The two-mean smoothed temperature, centred on Tuesday, is 23.5°C (to 1 d.p.).

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The process of four-mean smoothing with centring is the same as two-mean smoothing except that you smooth values in groups of four.

Example 4 **Four-moving mean smoothing with centring**

The table below gives the temperature (°C) recorded at a weather station at 9.00 a.m. each day for a week.

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Temperature	18.1	24.8	26.4	13.9	12.7	14.2	24.9

Calculate the four-mean smoothed temperature with centring for Thursday.

Solution

1 For two-mean smoothing with centring, write down the *five* data values centred on Thursday.

24.8 26.4 13.9 12.7 14.2

2 Calculate the mean of the first four values (mean 1) and the mean of the last four values (mean 2).

$$\text{Mean 1} = \frac{(24.8 + 26.4 + 13.9 + 12.7)}{4} = 19.45$$

$$\text{Mean 2} = \frac{(26.4 + 13.9 + 12.7 + 14.2)}{4} = 16.8$$

3 The centred mean is then the average of mean 1 and mean 2.

$$\text{Centred mean} = \frac{(\text{mean 1} + \text{mean 2})}{2} = \frac{(19.45 + 16.8)}{2} = 18.125$$

4 Write down your answer.

The four-mean smoothed temperature centred on Thursday is 18.1°C (to 1 d.p.).

The next step is to extend these computations to smooth all terms in the time series.

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Exercise 6B

Basic skills

The information below relates to Questions 1 to 5.

The table below gives the temperature (°C) recorded at a weather station at 3.00 p.m. each day for a week.

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Temperature (°C)	28.9	33.5	21.6	18.1	16.2	17.9	26.4

1 The three-mean smoothed temperature for Thursday is closest to:

A 20.0 **B** 23.2 **C** 24.4 **D** 29.4 **E** 31.2

2 The five-mean smoothed temperature for Friday is closest to:

A 20.0 **B** 23.2 **C** 24.4 **D** 29.4 **E** 31.2

3 The seven-mean smoothed temperature for Thursday is closest to:

A 20.0 **B** 23.2 **C** 24.4 **D** 28.0 **E** 31.2

4 The two-mean smoothed temperature with centring for Tuesday is closest to:

A 19.1 **B** 20.0 **C** 24.4 **D** 29.4 **E** 31.2

5 The four-mean smoothed temperature for Friday is closest to:

A 19.1 **B** 20.0 **C** 23.2 **D** 28.0 **E** 31.2

Calculating the smoothed value of individual data points

6

t	1	2	3	4	5	6	7	8	9
y	5	2	5	3	1	0	2	3	0

For the time series data in the table above, find:

a the three-mean smoothed y-value for $t = 4$

b the three-mean smoothed y-value for $t = 6$

c the three-mean smoothed y-value for $t = 2$

d the five-mean smoothed y-value for $t = 3$

e the five-mean smoothed y-value for $t = 7$

f the five-mean smoothed y-value for $t = 4$

g the two-mean smoothed y-value centred at $t = 3$

h the two-mean smoothed y-value centred at $t = 8$

i the four-mean smoothed y-value centred at $t = 3$

j the four-mean smoothed y-value centred at $t = 6$.

Note: Copies of the tables in Questions 7 to 11 can be accessed via the skillsheet icon in the Interactive Textbook.

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Smoothing a table of values

Skillsheet

7 Complete the following table.

t	1	2	3	4	5	6	7	8	9
y	10	12	8	4	12	8	10	18	2
Three-mean smoothed y	–								–
Five-mean smoothed y	–	–						–	–

Smoothing and plotting a time series (three- and five-mean smoothing)

8 The maximum temperature of a city over a period of 10 days is given below.

Day	1	2	3	4	5	6	7	8	9	10
Temperature (°C)	24	27	28	40	22	23	22	21	25	26
Three-moving mean										
Five-moving mean										

a Use a CAS calculator to construct a time series plot of the temperature data.

b Use the three-mean and five-mean smoothing method to complete the table.

c Use a CAS calculator to plot the smoothed temperature data and compare and comment on the plots. This is best done if all plots are on the same graph.

9 The value of the Australian dollar in US dollars (exchange rate) over 10 days is given below.

Day	1	2	3	4	5	6	7	8	9	10
Exchange rate	0.743	0.754	0.737	0.751	0.724	0.724	0.712	0.735	0.716	0.711
Three-moving mean										
Five-moving mean										

a Construct a time series plot of the data. Label and scale the axes.

b Use the three-mean and five-mean smoothing method to complete the table.

c Use a CAS calculator to plot the smoothed exchange rate data and compare the plots and comment on the plots. This is best done if all three plots are on the same graph.

Smoothing a time series (two- and four-mean smoothing)

10 Construct a table with four columns: 'Month', 'Number of births', 'Two-moving mean' and 'Two-moving mean with centring' using the following data.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sep	Oct	Nov	Dec
Number of births	10	12	6	5	22	18	13	7	9	10	8	15

11 Construct a table with four columns: 'Month', 'Internet usage', 'Four-moving mean' and 'Four-moving mean with centring' using the following data.

Month	Apr	May	Jun	July	Aug	Sep	Oct	Nov	Dec
Internet usage	21	40	52	42	58	79	81	54	50

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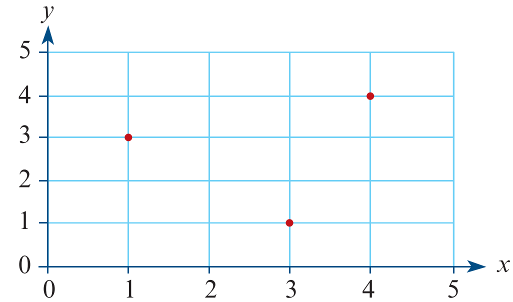
6C Smoothing a time series plot using moving medians

The quickest way to smooth an actual times series plot is to use moving medians because the smoothing can be done directly on the graph without needing to know the exact values of each data point.[2] However, before smoothing a time series plot graphically using moving medians (**moving median smoothing**), you first need to know how to locate medians graphically.

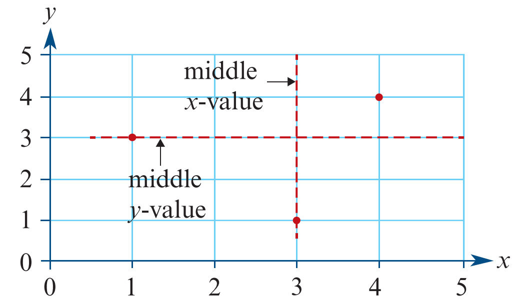
▶ Locating medians graphically

The graph opposite shows three data points plotted on set of coordinate axes. The task is to locate the median of these three points. The median will be a point somewhere on this set of coordinate axes. To locate this point we proceed as follows.



Step 1

Identify the middle data point moving in the x-direction. Draw a vertical line through this value as shown.



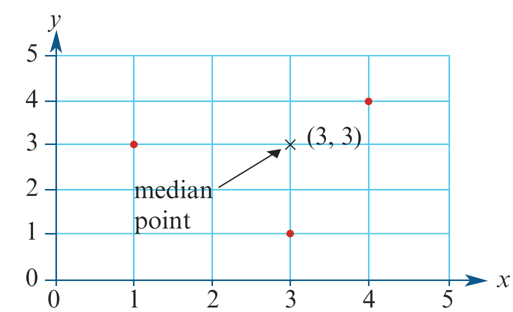
Step 2

Identify the middle data point moving in the y-direction. Draw a horizontal line through this value as shown.

Step 3

The median value is where the two lines intersect – in this case, at the point (3, 3).

Mark this point with a cross (×).



The process of graphically smoothing a time series plot requires no more than repeating the above process for each group of three or five data points in the plot as required. The following worked examples demonstrate the process.

[2] Note that, in this course, median smoothing is restricted to smoothing over an odd number of points, so centring is not required.

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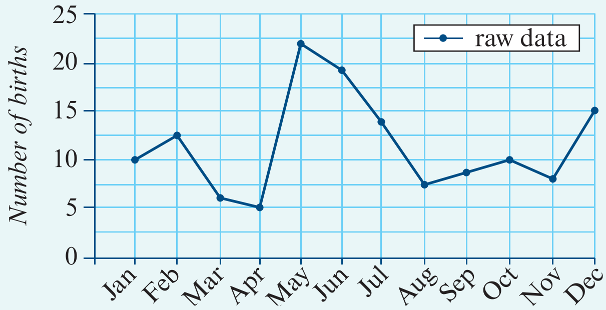
Skillsheet

Example 5 Three-median smoothing using a graphical approach

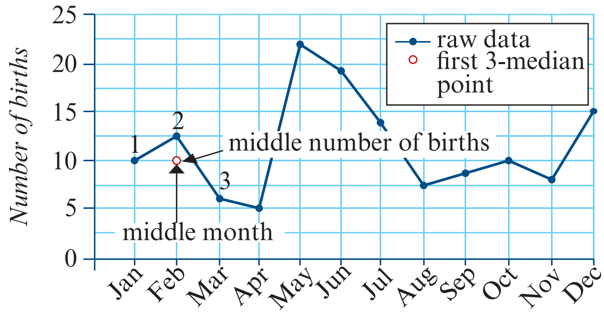
Construct a three-median smoothed plot of the time series plot shown opposite.

Note: When median smoothing graphically you smooth directly onto the time series plot. Copies of all of the plots in this section can be accessed via the skillsheet icon in the Interactive Textbook.

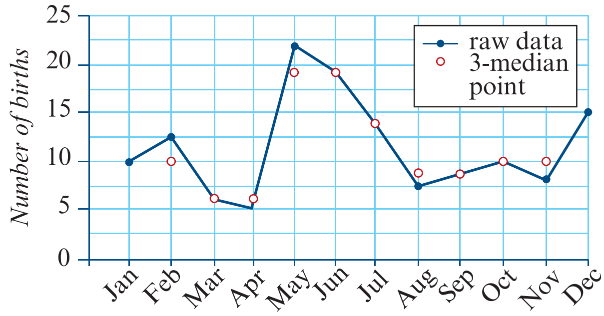


Solution

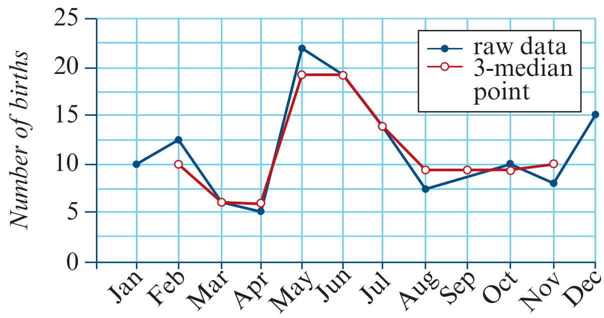
1 Locate on the time series plot the median of the *first* three points (Jan, Feb, Mar).



2 Continue this process by moving onto the next three points to be smoothed (Feb, Mar, Apr).
Mark their medians on the graph, and continue the process until you run out of groups of three.



3 Join the median points with a line segment – see opposite.



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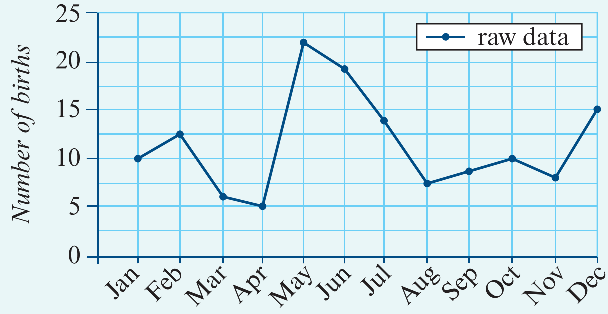
Example 6 Five-median smoothing using a graphical approach

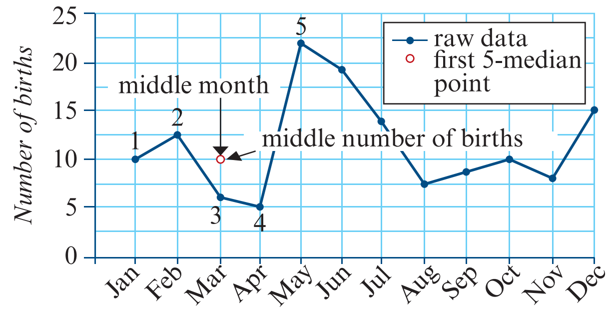
Construct a five-median smoothed plot of the time series plot shown opposite.

Note: The starting point for a median smoothing is a time series plot and you smooth directly onto the plot. Copies of all of the plots in this section can be accessed through the skillsheet icon in the Interactive Textbook.

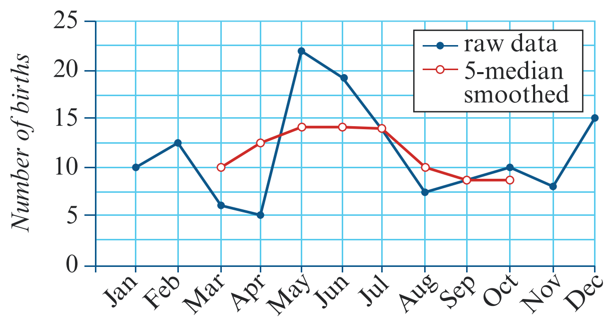


Solution

1 Locate on the time series plot the median of the *first* five points (Jan, Feb, Mar, Apr, May), as shown.



2 Then move onto the next five points to be smoothed (Feb, Mar, Apr, May, Jun). Repeat the process until you run out of groups of five points. The five-median points are then joined up with line segments to give the final smoothed plot, as shown.



Note: The five-median smoothed plot is much smoother than the three-median smoothed plot.

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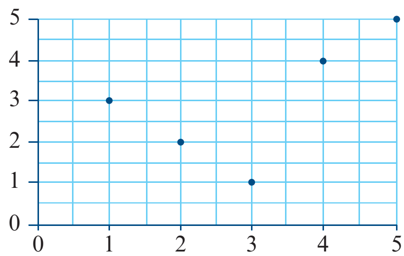
Exercise 6C

Skillsheet Note: Copies of all plots in this section can be accessed through the skillsheet icon in the Interactive Textbook.

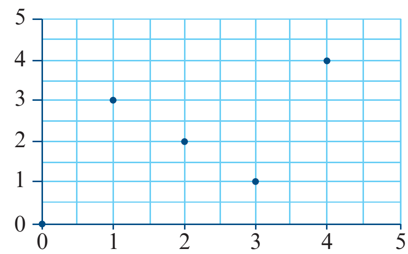
Locating the median of a set of data points graphically

1 Mark the location of the median point for each of the sets of data points below.

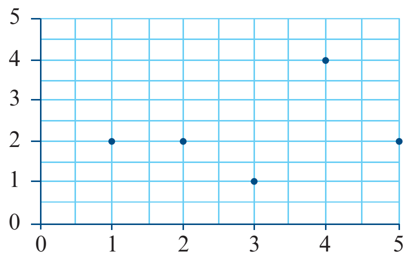
a



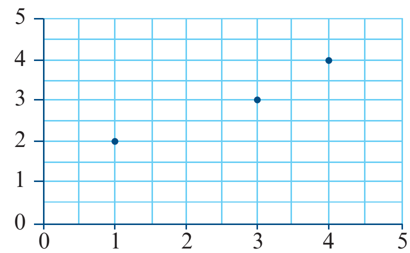
b



c

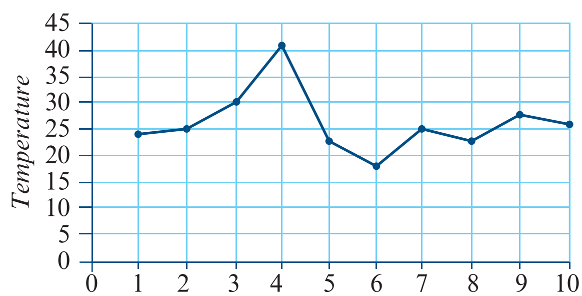


d



Smoothing a time series graphically

2 The time series plot below shows the maximum daily temperatures (in °C) in a city over a period of 10 consecutive days.



Use the graphical approach to determine the smoothed temperature:

a for day 4 using:

i three-median smoothing **ii** five-median smoothing

b for day 8 using:

i three-median smoothing **ii** five-median smoothing.

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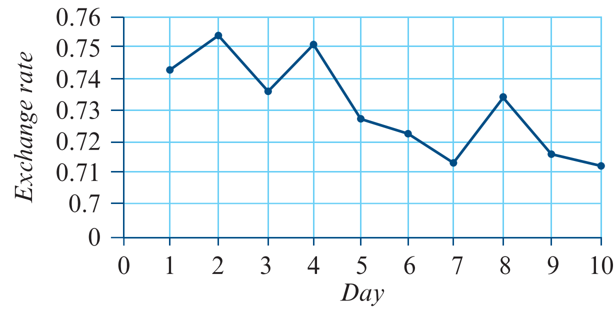
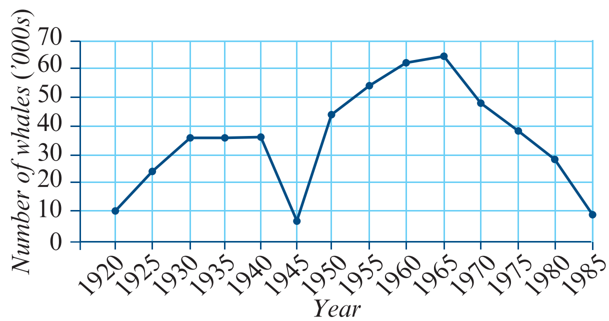
3 The time series plot opposite shows the value of the Australian dollar in US dollars (the exchange rate) over a period of 10 consecutive days in 2009. Use five-median smoothing to graphically smooth the plot and comment on the smoothed plot.



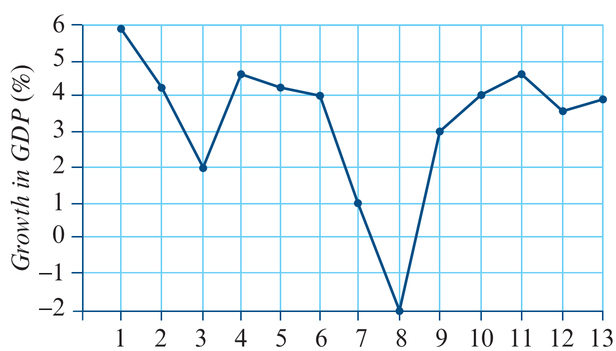
4 Use the graphical approach to smooth the time series plot below using:

a three-median smoothing

b five-median smoothing.



5 The time series plot opposite shows the percentage growth of GDP (gross domestic product) over a 13-year period.



a Smooth the times series graph:

i using three-median smoothing

ii using five-median smoothing.

b What conclusions can be drawn about the variation in GDP growth from these smoothed time series plots?

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6D Seasonal indices

When the data is seasonal, it is often necessary to **deseasonalise** the data before further analysis. To do this we need to calculate seasonal indices.

The concept of a seasonal index

Consider the (hypothetical) monthly seasonal indices for unemployment given in the table.

Jan	*Feb*	*Mar*	*Apr*	*May*	*Jun*	*Jul*	*Aug*	*Sept*	*Oct*	*Nov*	*Dec*	*Total*
1.1	1.2	1.1	1.0	0.95	0.95	0.9	0.9	0.85	0.85	1.1	1.1	12.0

Key fact 1

Seasonal indices are calculated so that their *average* is *1*. This means that the *sum* of the seasonal indices equals the *number of seasons*.

Thus, if the seasons are months, the seasonal indices add to 12. If the seasons are quarters, then the seasonal indices would to 4, and so on.

Key fact 2

Seasonal indices tell us how a particular season (generally a day, month or quarter) compares to the *average season*.

For example:

- seasonal index for unemployment for the month of February is 1.2 or 120%.
  This tells us that February unemployment figures tend to be 20% *higher* than the monthly average. Remember, the average seasonal index is 1 or 100%.
- seasonal index for August is 0.90 or 90%.
  This tells us that the August unemployment figures tend to be only 90% of the monthly average. Alternatively, August unemployment figures are 10% *lower* than the monthly average.

We can use seasonal indices to remove the seasonal component (deseasonalise) from a time series, or to put it back in (**reseasonalise**).

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Using seasonal indices to deseasonalise or reseasonalise a time series

To calculate deseasonalised figures, each entry is divided by its seasonal index as follows.

Deseasonalising data

Time series data are deseasonalised using the relationship:

$$\text{deseasonalised figure} = \frac{\text{actual figure}}{\text{seasonal index}}$$

Example 7 Using a seasonal index to deseasonalise data

The seasonal index (SI) for cold drink sales for summer is SI = 1.33.

Last summer a beach kiosk's actual cold drink sales totalled \$15 653.

What were the *deseasonalised* sales?

Solution

Use the rule

$$\text{deseasonalised sales} = \frac{\text{actual sales}}{\text{seasonal index}}$$

with actual sales = \$15 653
and SI = 1.33.

$$\text{Deseasonalised sales} = \frac{15\,653}{1.33} = 11\,769.17$$

The deseasonalised sales for summer were \$11 769.17.

The rule for determining deseasonalised data values can also be used to reseasonalise data – that is, convert a deseasonalised value into an actual data value.

Reseasonalising data

Time series data are reseasonalised using the rule:

$$\text{actual figure} = \text{deseasonalised figure} \times \text{seasonal index}$$

Example 8 Using a seasonal index to reseasonalise data

The seasonal index for cold drink sales for spring is SI = 0.85.

Last spring a beach kiosk's deseasonalised cold drink sales totalled \$10 870.

What were the *actual* sales?

Solution

Use the rule
actual sales = deseasonalised sales × seasonal index
with deseasonalised sales = \$10 870 and SI = 0.85.

$$\text{Actual sales} = 10\,870 \times 0.85 = 9239.50$$

The actual sales for spring were \$9239.50.

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Example 9 Deseasonalising a time series

The quarterly sales figures for Mikki's shop over a 3-year period are given below.

Year	*Summer*	*Autumn*	*Winter*	*Spring*
1	920	1085	1241	446
2	1035	1180	1356	541
3	1299	1324	1450	659

Use the seasonal indices shown to deseasonalise these sales figures. Write answers correct to the nearest whole number.

Summer	*Autumn*	*Winter*	*Spring*
1.03	1.15	1.30	0.52

Solution

1 To deseasonalise each sales figure in the table, divide by the appropriate seasonal index.
For example, for summer, divide the figures in the 'Summer' column by 1.03.
Round results to the nearest whole number.

$$\frac{920}{1.03} = 893$$

$$\frac{1035}{1.03} = 1005$$

$$\frac{1299}{1.03} = 1261$$

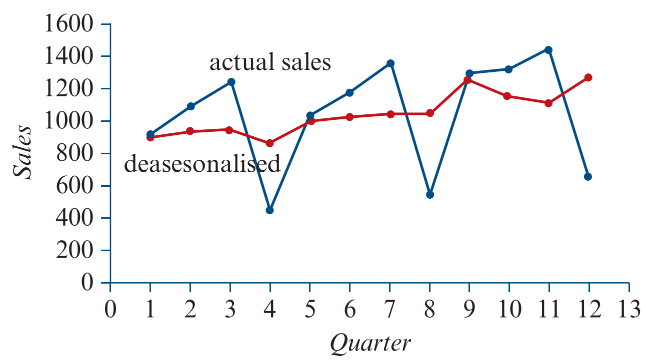
2 Repeat for the other seasons.

Deseasonalised sales figures

Year	Summer	Autumn	Winter	Spring
1	893	943	955	858
2	1005	1026	1043	1040
3	1261	1151	1115	1267

▶ Comparing a plot of the raw data with the deseasonalised data

The plot below shows the time series deseasonalised sales.



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Two things to be noticed are that deseasonalising has:

- removed the seasonality from the time series plot
- revealed a clear underlying trend in the data.

It is common to deseasonalise time series data before you fit a trend line.

Calculating seasonal indices

To complete this section, you will now learn to calculate a seasonal index. We will start by using only 1 year's data to illustrate the basic ideas and then move onto a more realistic example where several years' data are involved.

Example 10 Calculating seasonal indices (1 year's data)

Mikki runs a shop and she wishes to determine quarterly seasonal indices based on last year's sales, which are shown in the table opposite.

Summer	*Autumn*	*Winter*	*Spring*
920	1085	1241	446

Solution

1 The seasons are quarters. Write the formula in terms of quarters.

$$\text{Seasonal index} = \frac{\text{value for season}}{\text{seasonal average}}$$

2 Find the quarterly average for the year.

$$\text{Quarterly average} = \frac{920 + 1085 + 1241 + 446}{4}$$
$$= 923$$

3 Work out the seasonal index (SI) for each time period.

$$SI_{Summer} = \frac{920}{923} = 0.997$$
$$SI_{Autumn} = \frac{1085}{923} = 1.176$$
$$SI_{Winter} = \frac{1241}{923} = 1.345$$
$$SI_{Spring} = \frac{446}{923} = 0.483$$

4 Check that the seasonal indices sum to 4 (the number of seasons). The slight difference is due to rounding error.

Check: $0.997 + 1.176 + 1.345 + 0.483 = 4.001$

5 Write out your answers as a table of the seasonal indices.

Seasonal indices

Summer	Autumn	Winter	Spring
0.997	1.176	1.345	0.483

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The next example illustrates how seasonal indices are calculated with 3 years' data. While the process looks more complicated, we just repeat what we did in Example 11 three times and average the results for each year at the end.

Example 11 Calculating seasonal indices (several years' data)

Suppose that Mikki has 3 years of data, as shown. Use the data to calculate seasonal indices, correct to two decimal places.

Year	*Summer*	*Autumn*	*Winter*	*Spring*
1	920	1085	1241	446
2	1035	1180	1356	541
3	1299	1324	1450	659

Solution

The strategy is as follows:

- Calculate the seasonal indices for years 1, 2 and 3 separately, as Example 10 (as we already have the seasonal indices for year 1 from Example 10, we will save ourselves some time by simply quoting the result).
- Average the three sets of seasonal indices to obtain a single set of seasonal indices.

1 Write down the result for year 1.

Year 1 seasonal indices:

Summer	Autumn	Winter	Spring
0.997	1.176	1.345	0.483

2 Now calculate the seasonal indices for year 2.

a The seasons are quarters. Write the formula in terms of quarters.

$$\text{Seasonal index} = \frac{\text{value for quarter}}{\text{quarterly average}}$$

b Find the quarterly average for the year.

$$\text{Quarterly average} = \frac{1035 + 1180 + 1356 + 541}{4} = 1028$$

c Work out the seasonal index (SI) for each time period.

$$SI_{\text{Summer}} = \frac{1035}{1028} = 1.007$$

$$SI_{\text{Autumn}} = \frac{1180}{1028} = 1.148$$

$$SI_{\text{Winter}} = \frac{1356}{1028} = 1.319$$

$$SI_{\text{Spring}} = \frac{541}{1028} = 0.526$$

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d Check that the seasonal indices sum to 4.

Check: $1.007 + 1.148 + 1.319 + 0.526 = 4.000$

e Write out your answers as a table of the seasonal indices.

Year 2 seasonal indices:

Summer	Autumn	Winter	Spring
1.007	1.148	1.319	0.526

3 Now calculate the seasonal indices for year 3.

a Find the quarterly average for the year.

$$\text{Quarterly average} = \frac{1299 + 1324 + 1450 + 659}{4} = 1183$$

b Work out the seasonal index (SI) for each time period.

$$SI_{Summer} = \frac{1299}{1183} = 1.098$$

$$SI_{Autumn} = \frac{1324}{1183} = 1.119$$

$$SI_{Winter} = \frac{1450}{1183} = 1.226$$

$$SI_{Spring} = \frac{659}{1183} = 0.557$$

c Check that the seasonal indices sum to 4.

Check: $1.098 + 1.119 + 1.226 + 0.557 = 4.000$

d Write out your answers as a table of the seasonal indices.

Year 3 seasonal indices:

Summer	Autumn	Winter	Spring
1.098	1.119	1.226	0.557

4 Find the 3-year averaged seasonal indices by averaging the seasonal indices for each season.

Final seasonal indices:

$$S_{Summer} = \frac{0.997 + 1.007 + 1.098}{3} = 1.03$$

$$S_{Autumn} = \frac{1.176 + 1.148 + 1.119}{3} = 1.15$$

$$S_{Winter} = \frac{1.345 + 1.319 + 1.226}{3} = 1.30$$

$$S_{Spring} = \frac{0.483 + 0.526 + 0.557}{3} = 0.52$$

5 Check that the seasonal indices sum to 4.

Check: $1.03 + 1.15 + 1.30 + 0.52 = 4.00$

6 Write out your answers as a table of the seasonal indices.

Summer	Autumn	Winter	Spring
1.03	1.15	1.30	0.52

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Interpreting the seasonal indices

Having calculated these seasonal indices, what do they tell us in the above situation?

The seasonal index of:

- 1.03 for summer tells us that summer sales are typically 3% above average
- 1.15 for autumn tells us that autumn sales are typically 15% above average
- 1.30 for winter tells us that winter sales are typically 30% above average
- 0.52 for spring tells us that spring sales are typically 48% below average.

Correcting for seasonality

Also, using the rule

$$\text{deseasonalised figure} = \frac{\text{actual figure}}{\text{seasonal index}}$$

we can work out how much we need to increase or decrease the actual sales figures to correct for seasonality.

For example, we see that for winter:

$$\begin{aligned}\text{deseasonalised figure} &= \frac{\text{actual figure}}{1.30}\\ &= 0.769\ldots \times \text{actual figure} \approx 77\% \text{ of the actual figures}\end{aligned}$$

Thus, to correct the seasonality in winter, we need to decrease the actual sales by about 23%.

Similarly we can show that, to correct for seasonality in spring ($SI_{\text{spring}} = 0.52$), we need to increase the actual spring sales figure by around 92% ($\frac{1}{0.52} \approx 1.92$).

Exercise 6D

Skillsheet

Basic skills and interpretation

Use the following information to answer Questions 1 to 8.

The table below shows the monthly sales figures (in $'000s) and seasonal indices (for January to November) for a product produced by the U-beaut company.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Seasonal index	1.2	1.3	1.1	1.0	1.0	0.9	0.8	0.7	0.9	1.0	1.1	
Sales ($'000s)	9.6	10.5	8.6		7.1	6.0	5.4		6.4	7.2	8.3	7.4

1 The seasonal index for December is:

A 0.8 **B** 0.9 **C** 1.0 **D** 1.1 **E** 1.2

2 The deseasonalised sales (in $'000s) for March is closest to:

A 7.7 **B** 7.8 **C** 8.6 **D** 9.5 **E** 10.3

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3 The deseasonalised sales (in $'000s) for June is closest to:

A 5.4 **B** 5.9 **C** 6.0 **D** 6.6 **E** 6.7

4 The deseasonalised sales (in $'000s) for August are 5.6. The actual sales are closest to:

A 2.7 **B** 3.9 **C** 5.6 **D** 5.9 **E** 7.3

5 The deseasonalised sales (in $'000s) for April are 6.9. The actual sales are closest to:

A 5.4 **B** 6.3 **C** 6.9 **D** 7.6 **E** 8.3

6 The seasonal index for February tells us that, over time, February sales tend to be greater than the average monthly sales by:

A 0% **B** 10% **C** 20% **D** 30% **E** 70%

7 The seasonal index for September tells us that, over time, September sales tend to be less than the average monthly sales by:

A 90% **B** 10% **C** 0% **D** 10% **E** 90%

8 The seasonal index for January is 1.2. To correct the actual monthly sales figure for seasonality we need to:

A decrease the actual sales figures by around 20%
B increase the actual sales figures by around 20%
C decrease the actual sales figures by around 17%
D increase the actual sales figures by around 17%
E increase the actual sales figures by around 80%

Use the following information to answer Questions 9 to 15.

The table below shows the quarterly newspaper sales (in $'000s) of a corner store. Also shown are the seasonal indices for newspaper sales for the first, second and third quarters.

	Quarter 1	*Quarter 2*	*Quarter 3*	*Quarter 4*
Sales		1060	1868	1642
Seasonal index	0.8	0.7	1.3	

9 The seasonal index for quarter 4 is:

A 0.8 **B** 0.7 **C** 1.0 **D** 1.2 **E** 1.3

10 The deseasonalised sales (in $'000s) for quarter 2 are closest to:

A 742 **B** 980 **C** 1060 **D** 1514 **E** 1694

11 The deseasonalised sales (in $'000s) for quarter 3 are closest to:

A 1437 **B** 1678 **C** 1868 **D** 2428 **E** 2567

12 The deseasonalised sales (in $'000s) for quarter 1 are 1256. The actual sales are closest to:

A 986 **B** 1005 **C** 1256 **D** 1570 **E** 1678

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Deseasonalising a time series

13 The following table shows the number of students enrolled in a 3-month computer systems training course along with some seasonal indices that have been calculated from the previous year's enrolment figures. Complete the table by calculating the seasonal index for spring and the deseasonalised student numbers for each course.

	Summer	*Autumn*	*Winter*	*Spring*
Number of students	56	125	126	96
Deseasonalised numbers				
Seasonal index	0.5	1.0	1.3	

14 The number of waiters employed by a restaurant chain in each quarter of 1 year, along with some seasonal indices that have been calculated from the previous year's data, are given in the following table.

	Quarter 1	*Quarter 2*	*Quarter 3*	*Quarter 4*
Number of waiters	198	145	86	168
Seasonal index	1.30		0.58	1.10

a What is the seasonal index for the second quarter?

b The seasonal index for quarter 1 is 1.30. Explain what this means in terms of the average quarterly number of waiters.

c Deseasonalise the data.

Calculating seasonal indices

15 The table below records quarterly sales (in \$'000s) for a shop.

Quarter 1	*Quarter 2*	*Quarter 3*	*Quarter 4*
60	56	75	78

Use the data to determine the seasonal indices for the four quarters. Give your results correct to two decimal places. Check that your seasonal indices add to 4.

16 The table below records the monthly sales (in \$'000s).

Jan	*Feb*	*Mar*	*Apr*	*May*	*Jun*	*Jul*	*Aug*	*Sep*	*Oct*	*Nov*	*Dec*
12	13	14	17	18	15	9	10	8	11	15	20

Use the data to determine the seasonal indices for the 12 months. Give your results correct to two decimal places. Check that your seasonal indices add to 12.

17 The table below records the monthly sales (in \$'000s).

Jan	*Feb*	*Mar*	*Apr*	*May*	*Jun*	*Jul*	*Aug*	*Sep*	*Oct*	*Nov*	*Dec*
22	19	25	23	20	18	20	15	14	11	23	30

Use the data to determine the seasonal indices for the 12 months. Give your results correct to two decimal places. Check that your seasonal indices add to 12.

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6E Fitting a trend line and forecasting

▶ Fitting a trend line

If there appears to be a linear trend, we can use the least squares method to fit a line to the data to model the trend.

Example 12 Fitting a trend line

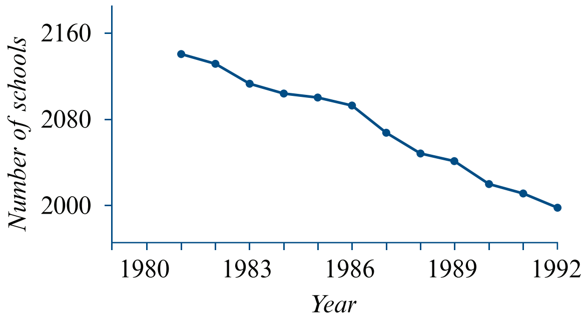
Fit a trend line to the data in the following table, which shows the number of government schools in Victoria over the period 1981–92, and interpret the slope.

Year	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992
Number	2149	2140	2124	2118	2118	2114	2091	2064	2059	2038	2029	2013

Solution

1 Construct a time series plot of the data to confirm that the trend is linear.

Note: For convenience we let 1981 = 1, 1982 = 2 and so on when entering the data into a calculator.



2 Fit a least squares line to the data with *year* as the EV. Write down its equation.

Number of schools = 2169 − 12.5 × year

3 Write down the slope and interpret.

Slope = −12.5

Over the period 1981–92 the number of schools in Victoria decreased at an average rate of 12.5 schools per year.

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Forecasting

Using a trend line fitted to a time series plot to make predictions about future values is known as **trend line forecasting**.

Example 13 Forecasting

How many government schools do we predict for Victoria in 2015 if the same decreasing trend continues? Give your answer correct to the nearest whole number.

Solution

Substitute the appropriate value for *year* in the equation determined using a least squares regression. Since 1981 was designated as year '1', then 2015 is year '35'.

$$\text{Number of schools} = 2169 - 12.5 \times \text{year}$$
$$= 2169 - 12.5 \times 35$$
$$\approx 1732$$

Note: As with any prediction involving extrapolation, the results obtained when predicting well beyond the range of the data should be treated with caution.

Forecasting taking seasonality into account

When time series data is seasonal, it is usual to deseasonalise the data before fitting the trend line.

Example 14 Fitting a trend line (seasonality)

The *deseasonalised* quarterly sales data from Mikki's shop are shown below.

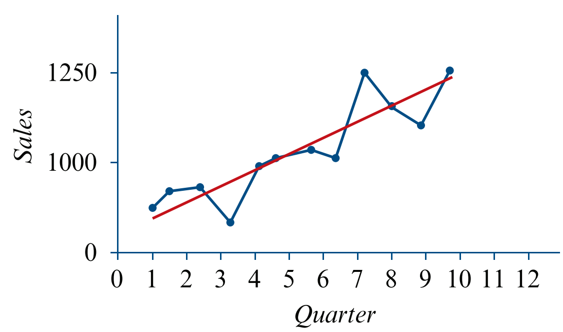
Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Sales	893	943	955	858	1005	1026	1043	1040	1261	1151	1115	1267

Fit a trend line and interpret the slope.

Solution

1 Plot the time series.

2 Using the calculator (with *Quarter* as the explanatory and *Sales* as the response variable), find the equation of the least squares regression line. Plot it on the time series.



3 Write down the equation of the least squares regression line.

$$\text{Sales} = 838.0 + 32.1 \times \text{quarter}$$

4 Interpret the slope in terms of the variables involved.

Over the 3-year period, sales at Mikki's shop increased at an average rate of 32 sales per quarter.

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▶ Making predictions with deseasonalised data

When using deseasonalised data to fit a trend line, you must remember that the result of any prediction is a deseasonalised value. To be meaningful, this result must then be reseasonalised by multiplying by the appropriate seasonal index.

Example 15 Forecasting (seasonality)

What sales do we predict for Mikki's shop in the winter of year 4? (Because many items have to be ordered well in advance, retailers often need to make such decisions.)

Solution

1 Substitute the appropriate value for the time period in the equation for the trend line. Since summer year 1 was designated as quarter '1', then winter year 4 is quarter '15'.

$$\text{Sales} = 838.0 + 32.1 \times \text{quarter}$$
$$= 838.0 + 32.1 \times 15$$
$$= 1319.5$$

Deseasonalised sales prediction for winter of year 4 = 1319.5

2 The value just calculated is the deseasonalised sales figure for the quarter in question.
To obtain the *actual* predicted sales figure we need to reseasonalise this predicted value. To do this, we multiply this value by the seasonal index for winter, which is 1.30.

Seasonalised sales prediction for winter of year 4 $= 1319.5 \times 1.30$
≈ 1715

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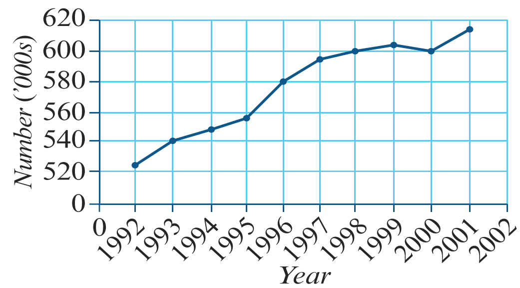
Exercise 6E

Fitting a least squares line to a time series plot (no seasonality)

1 The data show the number of students enrolled (in thousands) at university in Australia for the period 1992–2001.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
Number	525	539	545	556	581	596	600	603	600	614

The time series plot of the data is shown below.



a Comment on the plot.

b Fit a least squares regression trend line to the data, using 1992 as year 1, and interpret the slope.

c Use this equation to predict the number of students expected to enroll at university in Australia in 2020.

2 The table below shows the percentage of total retail sales that were made in department stores over an 11-year period:

Year	1	2	3	4	5	6	7	8	9	10	11
Sales (%)	12.3	12.0	11.7	11.5	11.0	10.5	10.6	10.7	10.4	10.0	9.4

a Construct a time series plot.

b Comment on the time series plot in terms of trend.

c Fit a trend line to the time series plot, find its equation and interpret the slope.

d Draw the trend line on your time series plot.

e Use the trend line to forecast the percentage of retails sales which will be made by department stores in year 15.

3 The average ages of mothers having their first child in Australia over the years 1989–2002 are shown below.

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Age	27.3	27.6	27.8	28.0	28.3	28.5	28.6	28.8	29.0	29.1	29.3	29.5	29.8	30.1

a Fit a least squares regression trend line to the data, using 1989 as year 1, and interpret the slope.

b Use this trend relationship to forecast the average ages of mothers having their first child in Australia in 2018 (year 30). Explain why this prediction is not likely to be reliable.

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Fitting a least-squares line to a time series with seasonality

4 **a** The table below shows the *deseasonalised* quarterly washing-machine sales of a company over 3 years. Use least squares regression to fit a trend line to the data.

	Year 1				*Year 2*				*Year 3*			
Quarter number	1	2	3	4	5	6	7	8	9	10	11	12
Deseasonalised	53	51	54	55	64	64	61	63	67	69	68	66

b Use this trend equation for washing-machine sales, with the seasonal indices below, to forecast the sales of washing machines in the fourth quarter of year 4.

Quarter	1	2	3	4
Seasonal index	0.90	0.81	1.11	1.18

5 The median duration of marriage to divorce (in years) for each year during the period 1992–2002 is given in the following table.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Duration	10.5	10.7	10.9	11.0	11.0	11.1	11.2	11.3	11.6	11.8	12.0

a Fit a least squares regression trend line to the data, using 1992 as year 1, and interpret the slope.

b Use this trend relationship to forecast the median duration of marriage to divorce in Australia in 2020. Explain why this prediction is not likely to be reliable.

6 The sale of boogie boards for a surf shop over a 2-year period is given in the table.

Year	*Quarter 1*	*Quarter 2*	*Quarter 3*	*Quarter 4*
1	138	60	73	230
2	283	115	163	417

The quarterly seasonal indices are given below.

Seasonal index	1.13	0.47	0.62	1.77

a Use the seasonal indices to calculate the deseasonalised sales figures for this period.

b Plot the actual sales figures and the deseasonalised sales figures for this period and comment on the plot.

c Fit a trend line to the deseasonalised sales data. Write the slope and intercept correct to three significant figures.

d Use the relationship calculated in **c**, together with the seasonal indices, to forecast the sales for the first quarter of year 4 (you will need to reseasonalise here).

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Key ideas and chapter summary

Time series data — **Time series data** are a collection of data values along with the times (in order) at which they were recorded.

Time series plot — A **time series plot** is a line graph where the values of the response variable are plotted in time order.

Features to look for in a time series plot
- Trend
- Seasonality
- Possible outliers
- Cycles
- Structural change
- Irregular (random) fluctuations

Trend — **Trend** is present when there is a long-term upward or downward movement in a time series.

Cycles — **Cycles** are present when there is a periodic movement in a time series. The period is the time it takes for one complete up and down movement in the time series plot. This term is generally reserved for periodic movements with a period greater than one year.

Seasonality — **Seasonality** is present when there is a periodic movement in a time series that has a calendar related period – for example, a year, a month, a week.

Structural change — **Structural change** is present when there is a sudden change in the established pattern of a time series plot.

Outliers — **Outliers** are present when there are individual values that stand out from the general body of data.

Irregular (random) fluctuations — **Irregular (random) fluctuations** are always present in any real-world time series plot. They include all of the variations in a time series that we cannot reasonably attribute to systematic changes like trend, cycles, seasonality, structural change or the presence of outliers.

Smoothing — **Smoothing** is a technique used to eliminate some of the irregular fluctuations in a time series plot so that features such as trend are more easily seen.

Moving mean smoothing — In **moving mean smoothing**, each original data value is replaced by the mean of itself and a number of data values on either side. When smoothing over an even number of data points, centring is required to ensure the smoothed mean is centred on the chosen point of time.

Moving median smoothing — **Moving median smoothing** is a graphical technique for smoothing a time series plot using moving medians rather than means.

Seasonal indices — **Seasonal indices** are used to quantify the seasonal variation in a time series.

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Deseasonalise	The process of accounting for the effects of seasonality in a time series is called **deseasonalisation**.
Reseasonalise	The process of a converting seasonal data back into its original form is called **reseasonalisation**.
Trend line forecasting	**Trend line forecasting** uses the equation of a trend line to make predictions about the future.

Skills check

Having completed this chapter you should be able to:

- recognise time series data
- construct a times series plot
- identify the presence of trend, cycles, seasonality, structural change and irregular (random) fluctuations in a time series plot
- smooth a time series to help identify any trend
- graphically smooth a time series plot using median smoothing
- calculate and interpret seasonal indices
- calculate and interpret a trend line for linear trends
- use a trend line to make forecasts.

Multiple-choice questions

1 The pattern in the time series in the graph shown is best described as:

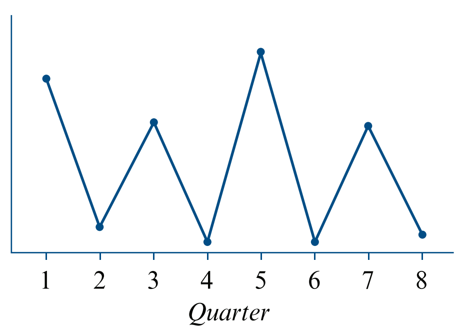
A trend

B cyclical but not seasonal

C seasonal

D irregular

E average



Use the following table to answer Questions 2 and 5.

Time period	1	2	3	4	5	6
Data value	2.3	3.4	4.4	2.7	5.1	3.7

2 The three-moving mean for time period 2 is closest to:

A 3.4 **B** 3.6 **C** 3.9 **D** 4.0 **E** 4.2

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3 The five-moving mean for time period 3 is closest to:

A 3.4 **B** 3.6 **C** 3.9 **D** 4.1 **E** 4.2

4 The two-moving mean for time period 5 with centring is closest to:

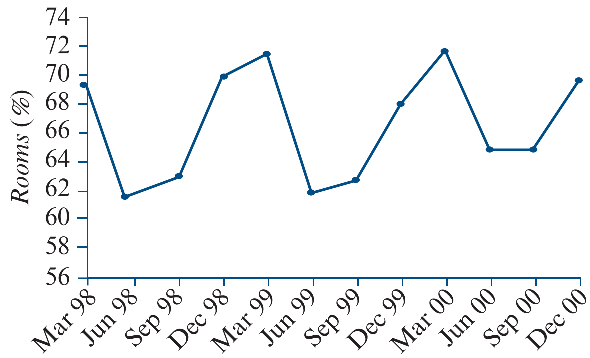
A 2.7 **B** 3.6 **C** 3.9 **D** 4.0 **E** 4.2

5 The four-moving mean for time period 4 with centring is closest to:

A 2.7 **B** 3.6 **C** 3.9 **D** 4.1 **E** 4.2

Use the following information to answer Questions 6 and 7.

The time series plot for hotel room occupancy rate (%) in Victoria over the period March 1998–December 2000 is shown below.



6 The five-median smoothed value for March 1999 was closest to:

A 62 **B** 63 **C** 66 **D** 68 **E** 69

7 The five-median smoothed value for September 1998 was closest to:

A 62 **B** 64 **C** 66 **D** 68 **E** 69

8 The seasonal indices for the number of customers at a restaurant are as follows.

Jan	*Feb*	*Mar*	*Apr*	*May*	*Jun*	*Jul*	*Aug*	*Sep*	*Oct*	*Nov*	*Dec*
1.0	p	1.1	0.9	1.0	1.0	1.2	1.1	1.1	1.1	1.0	0.7

The value of p is:

A 0.5 **B** 0.7 **C** 0.8 **D** 1.0 **E** 1.2

Use the following information to answer Questions 9 to 12.

The seasonal indices for the number of bathing suits sold at a surf shop are given in the table.

Quarter	Summer	Autumn	Winter	Spring
Seasonal index	1.8	0.4	0.3	1.5

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Review

9 The number of bathing suits sold one summer is 432. The deseasonalised number is closest to:

A 432 **B** 240 **C** 778 **D** 540 **E** 346

10 The *deseasonalised* number of bathing suits sold one winter was 380. The actual number was closest to:

A 114 **B** 133 **C** 152 **D** 380 **E** 1267

11 The seasonal index for spring tells us that, over time, the number of bathing suits sold in spring tends to be:

A 50% less than the seasonal average
B 15% less than the seasonal average
C the same as the seasonal average
D 15% more than the seasonal average
E 50% more than the seasonal average

12 To correct for seasonality, the actual number of bathing suits sold in Autumn should be:

A reduced by 60% **B** reduced by 40% **C** increased by 40%
D increased by 60% **E** increased by 250%

13 The number of visitors to an information centre each quarter was recorded for one year. The results are tabulated below.

Quarter	Summer	Autumn	Winter	Spring
Visitors	1048	677	593	998

Using this data, the seasonal index for autumn is estimated to be closest to:

A 0.25 **B** 1.0 **C** 1.23 **D** 0.82 **E** 0.21

Use the following information to answer Questions 14 and 15.

A trend line is fitted to a time series plot displaying the average age at marriage of males (in years) for the period 1995–2002.

The equation of this line is: $age = 27.1 + 0.236 \times year$

Here year 1 is 1995, year 2 is 1996, and so on.

14 Using this trend line, the average age of marriage of males in 2004 is forecasted to be:

A 29.0 **B** 29.2 **C** 29.5 **D** 29.7 **E** 500.0

15 From the slope of the trend line it can be said that:

A on average, the age of marriage for males is increasing by about 3 months per year
B on average, the age of marriage for males is decreasing by about 3 months per year

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Review

C older males are more likely to marry than younger males
D no males married at an age younger than 27 years
E on average, the age of marriage for males is increasing by 0.236 months per year

Questions 16 and 17 use the following information.

Suppose that the seasonal indices for the wholesale price of petrol are:

Day	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Index	1.2	1.0	0.9	0.8	0.7	1.2	1.2

The daily deseasonalised prices for a petrol outlet for a week (in cents/litre) are given in the following table.

Day	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Price	88.3	85.4	86.7	88.5	90.1	91.7	94.6

16 The equation of the least squares regression line that could enable us to predict the deseasonalised price is closest to:

A $price = 84.3 + 1.25 \times day$ **B** $price = -49.7 + 0.601 \times day$
C $price = 1.25 + 84.3 \times day$ **D** $price = 0.601 - 49.7 \times day$
E $price = 84.3 - 1.25 \times day$

17 The seven-mean deseasonalised smoothed price of petrol (in cents/litre) for this week was closest to:

A 87.4 **B** 88.3 **C** 88.5 **D** 89.3 **E** 90.0

18 The deseasonalised (in cents/litre) price on Thursday was 90.1. The actual price on Thursday was closest to:

A 63.1 **B** 75.6 **C** 110.8 **D** 128.7 **E** 135.4

Extended-response questions

1 Table 6.1 shows the Australian gross domestic product (GDP) per person, in dollars, at five yearly intervals for the period 1980 to 2005.

Table 6.1

Year	1980	1985	1990	1995	2000	2005
GDP	20 900	23 300	25 000	26 400	30 900	33 800

a Construct a times series plot.
b Briefly describe the general trend in the data.

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Table 6.2

Year	1980	1985	1990	1995	2000	2005
Time	0	5	10	15	20	25
GDP	20 900	23 300	25 000	26 400	30 900	33 800

Table 6.2, the variable *year* has been rescaled using 1980 = 0, 1985 = 5 and so on. The new variable is *time*.

c Use the variables *time* and *GDP* to write down the equation of the least squares regression line that can be used to predict *GDP* from *time*. Take *time* as the EV.

d In the year 2007, the GDP was $34 900. Find the error in the prediction if the least squares regression line calculated in **c** is used to predict GDP in 2007.

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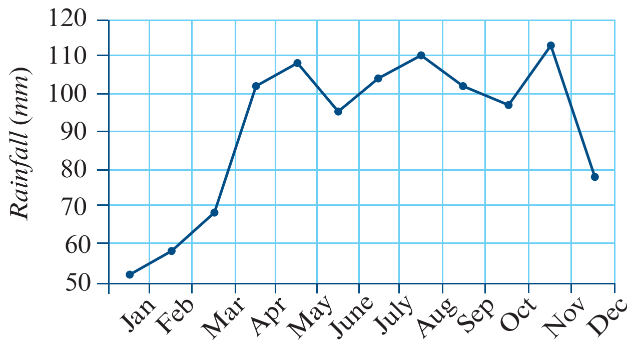
2 The time series plot below shows the rainfall (in mm) for each month during 2008.



a Which month had the highest rainfall?

b Use three-median smoothing to smooth the time series. Plot the smoothed time series on the plot above. Mark each smoothed data point with a cross (×).

c Describe the general pattern in rainfall that is revealed by the smoothed plot.

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3 The table below shows the average interest rate for the period 1987–97. Also shown are the three-mean smoothed interest rates but with one missing.

Year	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
Rate (%)	15.50	13.50	17.00	16.50	13.00	10.50	9.50	8.75	10.50	8.75	7.55
Three-mean (%)		15.33	15.67	15.50	13.33		9.58	9.58	9.33	9.93	

a Complete the table by showing that the three-mean smoothed interest rate for 1992 is 11.0%.

b Construct a time series plot for the average interest rate during the period 1987–97.

c Plot the smoothed interest rate data on the graph and comment on any trend.

ISBN 978-1-107-56757-3

7

Revision: Data analysis

ISBN 978-1-107-56757-3

Cambridge University Press

7A Multiple-choice questions: univariate data

Use the following information to answer Questions 1–3.

The following table shows the data collected from a sample of five senior students at a regional college. The variables in the table are:

- *sex* – the sex (F = female, M = male)
- *time* – the time in minutes each students took to get to school that day
- *transport* – mode of transport (1 = walked or rode a bike, 2 = car, 3 = public transport)
- *number* – number of siblings at the school
- *postcode* – postcode of place of residence.

Sex	*Time*	*Transport*	*Number*	*Postcode*
F	12	1	0	7267
M	35	3	2	7268
F	15	2	1	7268
F	43	3	0	7250
M	27	2	3	7267

1 The variables *sex* and *transport* are:

A both nominal variables
B a nominal and an ordinal variable respectively
C an ordinal and a nominal variable respectively
D a nominal and a discrete numerical variable respectively
E a nominal and a continuous numerical variable respectively

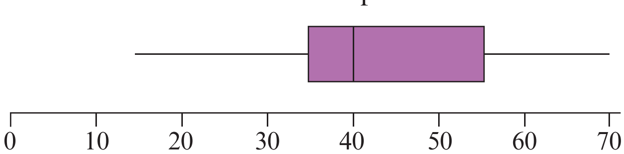
2 The number of numerical variables is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

3 The number of male students who used public transport to get to school is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

4 Consider this box-and-whisker plot. Which *one* of the following statements is true?



A The median is 45.
B Less than one-quarter of the observations are less than 30.
C Less than one-quarter of the observations are greater than 50.
D All of the observations are less than 60.
E More than half of the observations are less than 30.

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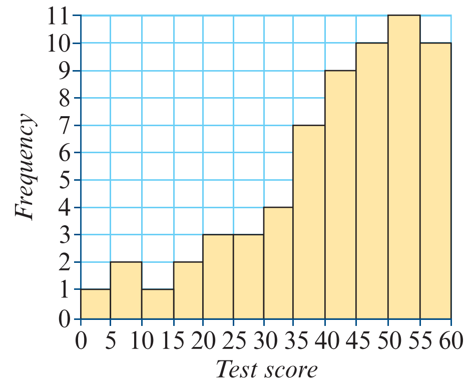
Use the following information to answer Questions 5 to 10.

A group of VCE mathematics students sat for a test. There were 63 students in the group. Their test scores are summarised opposite in the form of a histogram.



5 The distribution of test scores is:

A positively skewed

B negatively skewed

C symmetric

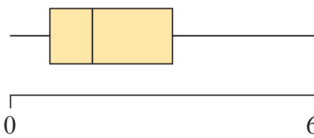
D symmetrically skewed

E symmetric with a clear outlier

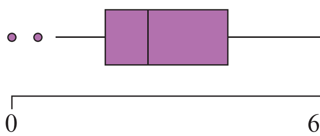
6 Displayed in the form of a box plot, the distribution of test scores would look like:

A

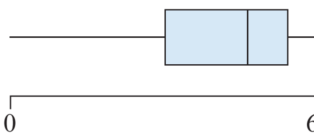
B



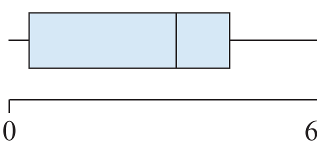
C



D



E



7 The pass mark on the test was 30. The percentage of students who failed the test is closest to:

A 12% **B** 19% **C** 30% **D** 50% **E** 63%

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8 The number of students who scored at least 30 but under 45 marks is:

A 4 **B** 7 **C** 19 **D** 11 **E** 20

9 The modal mark lies in the interval:

A 5–10 **B** 15–20 **C** 25–30 **D** 40–45 **E** 50–55

10 The median mark lies in the interval:

A 5–10 **B** 15–20 **C** 25–30 **D** 40–45 **E** 50–55

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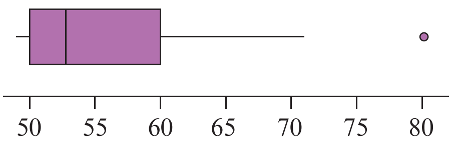
11 For the box plot opposite, outliers are defined as data values that are:

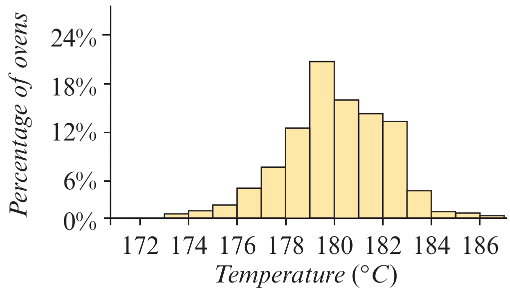


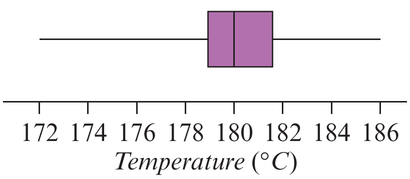
A less than 35 or greater than 75
B less than 48 or greater than 70
C less than 48 **D** greater than 70 **E** greater than 80

Use the following information to answer Questions 12 to 14.

To test the temperature control on an oven, the control is set to 180°C and the oven is heated for 15 minutes. The temperature of the oven is then measured.

The temperatures of 300 ovens tested in this way were recorded and are displayed opposite using both a histogram and a box plot.





12 A total of 300 ovens were tested and their temperatures were recorded.
The number of these temperatures that lie between 179°C and 181°C is closest to:

A 40 **B** 50 **C** 70 **D** 114 **E** 150

13 The interquartile range for temperature is closest to:

A 1.3°C **B** 1.5°C **C** 2.0°C **D** 2.7°C **E** 4.0°C

14 Using the 68–95–99.7% rule, the standard deviation for temperature is closest to:

A 1°C **B** 2°C **C** 3°C **D** 4°C **E** 6°C

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Use the following information to answer Questions 15–20.

The length of ants of a particular species is approximately normally distributed with a mean of 4.8 mm and a standard deviation of 1.2 mm.

15 From this information it can be concluded that around 95% of the lengths of the ants should lie between:

A 2.4 mm and 6.0 mm **B** 2.4 mm and 7.2 mm **C** 3.6 mm and 6.0 mm
D 3.6 mm and 7.2 mm **E** 4.8 mm and 7.2 mm

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16 The standardised ant length of $z = -0.5$ corresponds to an actual ant length of:

A 2.4 mm **B** 3.6 mm **C** 4.2 mm **D** 5.4 mm **E** 7.0 mm

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17 The percentage of ants with lengths less than 3.6 mm is closest to:

A 2.5% **B** 5% **C** 16% **D** 32% **E** 95%

18 The percentage of ants with lengths less than 6.0 mm is closest to:

A 5% **B** 16% **C** 32% **D** 68% **E** 84%

19 The percentage of ants with lengths greater than 3.6 mm and less than 7.2 mm is closest to:

A 2.5% **B** 18.5% **C** 68% **D** 81.5% **E** 97.5%

20 In a sample of 1000 ants, the number with a length between 2.4 mm and 4.8 mm is expected to be around:

A 3 **B** 50 **C** 475 **D** 975 **E** 997

Use the following information to answer Questions 21 and 22.

The mean mark for a class test is 65 with a standard deviation of 10.

21 A student's mark on the test is 45. Their standardised score is:

A −2.5 **B** −2.0 **C** 0 **D** 2 **E** 2.5

22 A student's standardised score on the test is 1.2 Their actual mark was:

A 52 **B** 53 **C** 65 **D** 77 **E** 78

23 A class of students sat for a biology test and a legal studies test. Each test had a possible maximum score of 100 marks. The table below shows the mean and standard deviation of the marks obtained in these tests.

	Subject	
	Biology	*Legal Studies*
Class mean	54	78
Class standard deviation	15	5

The class marks in each subject are approximately normally distributed.

Sashi obtained a mark of 81 in the biology test. The mark that Sashi would need to obtain on the legal studies test to achieve the same standard score for both legal studies and biology is:

A 81 **B** 82 **C** 83 **D** 87 **E** 95

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Revision

7B Multiple-choice questions: associations

1 The table below shows the percentage of students in two age groups (15–19 years and 20–24 years) who regularly use the internet at one or more of three locations:

- at home
- at an educational institution
- at work.

Location of internet use	*Age group* 15–19 years	*Age group* 20–24 years
At home	95%	95%
At an educational institution	85%	18%
At work	38%	74%

For the students surveyed, which of the following statements, by itself, supports the contention that the location of internet use is associated with the age group of the internet user?

A 85% of students aged 15–19 years used the internet at an educational institution.

B 95% of students aged 15–19 years used the internet at home, but only 38% of 15–19 year olds used it at work.

C 95% of students aged 15–19 years used the internet at home and 18% of 20–24 year olds used the internet at an educational institution.

D The percentage of students who used the internet at an educational institution decreased from 85% for those aged 15–19 years to 18% for those aged 20–24 years.

E The percentage of students who used the internet at home was 95% for those aged 15–19 years and 95% for those aged 20–24 years.

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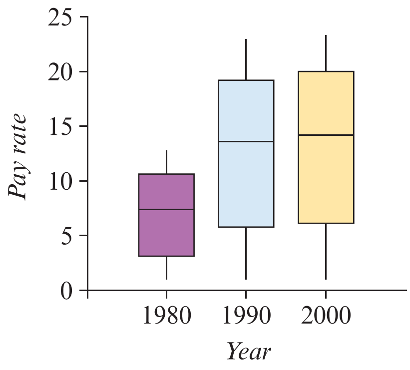
2 The box plots opposite display the distribution of the average pay rate, in dollars per hour, of workers in 35 countries for the years 1980, 1990 and 2000.

The aim is to investigate the association between *pay rate* and *year*.



Which of the following statements is *not* true?

A There is little change in the shape of the distribution of average pay rate with year.

B The median pay rate increased with year.

C The *IQR* increased with year

D *Pay rate* is the response variable.

E *Pay rate* is *not* associated with year.

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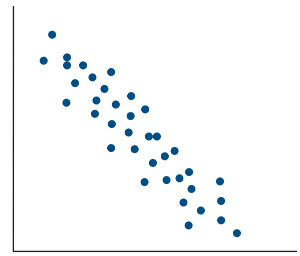
3 The value of r for the scatterplot is closest to:

A 0.8 **B** 0.5 **C** 0
D -0.5 **E** -0.9



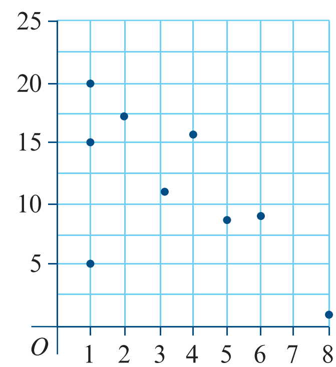
4 The association pictured in the scatterplot in the previous question is best described as:

A strong, positive, linear **B** strong, negative, linear **C** weak, negative, linear
D strong, negative, non-linear with an outlier **E** strong, negative, linear

5 Which of the following statistics can be either positive or negative?

A The standard deviation **B** The interquartile range
C The range **D** The correlation coefficient
E The coefficient of determination

6 When the correlation coefficient, r, was calculated for the data displayed in the scatterplot, it was found to be $r = -0.64$.
If the point (1, 5) was replaced with the point (6, 5) and the correlation coefficient, r, recalculated, then the value of r would be:



A unchanged **B** positive but closer to 1
C negative but closer to 0 **D** positive but closer to 0
E negative but closer to -1

7 The correlation between computer ownership (number/1000 people) and car ownership (number/1000 people) in six countries is $r = 0.92$, correct to two decimal places.
Based on this information, which of the following statements is *not* true?

A Computer ownership and car ownership are both numerical variables.
B Around 85% of the variation in computer ownership is explained by car ownership.
C Either *computer ownership* or *car ownership* could be the explanatory variable.
D For these countries, computer ownership increases as car ownership increases.
E For these countries, car ownership decreases as car ownership increases.

ISBN 978-1-107-56757-3

Cambridge University Press

Revision

8 The correlation between the score on a maths test and height for a group of primary school students is found to be 0.7. From this information, it is reasonable to conclude that:

A learning maths makes children grow taller
B there is no association between height and maths test scores
C a child's maths ability depends only on their height
D the children who obtained high maths test scores tended to be taller
E all tall children are better at maths than shorter children

9 A back-to-back stem plot is a useful tool for displaying the association between:

A weight (kg) and handspan (cm)
B height (cm) and age (years)
C handspan (cm) and eye colour (brown, blue, green)
D height in centimetres and sex (female, male)
E meat consumption (kg/person) population and country of residence

10 To explore the association between owning a mobile phone (yes or no) and sex (male or female), it would be best to use the data collected to construct:

A an appropriately percentaged table
B a back-to-back stem plot
C parallel box plots
D a scatterplot
E a histogram

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11 The association between the time taken to walk 5 km (in minutes) and fitness level (below average, average, above average) is best displayed using:

A a histogram
B a scatterplot
C a time series plot
D parallel box plots
E a back-to-back stem plot

7C Multiple-choice questions: regression and data transformation

1 Given that $r = 0.675$, $s_x = 2.567$ and $s_y = 4.983$ the slope of the least squares regression line $y = a + bx$, is closest to:

A 0.35 **B** 0.68 **C** 1.3 **D** 1.7 **E** 3.36

The following data relate to Questions 2 and 3.

Number of hot dogs sold	190	168	146	155	150	170	185
Temperature (°C)	10	15	20	15	17	12	10

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We wish to determine the equation of the least squares line for the data that will enable the number of hot dogs sold to be predicted from temperature.

2 The slope of the regression line will be closest to:

A –4.3 **B** –0.2 **C** 0.2 **D** 4.3 **E** 227

3 The coefficient of determination will be closest to:

A –0.94 **B** –0.89 **C** 0.21 **D** 0.89 **E** 0.94

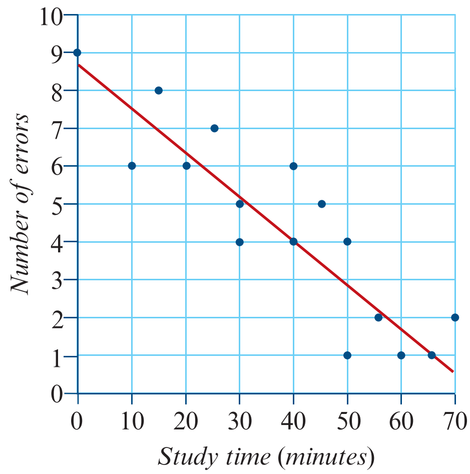
The following information relates to Questions 4 to 9.

Eighteen students sat for a 15-question multiple-choice test. In the scatterplot opposite, the number of errors made by each student on the test is plotted against the time they reported studying for the test.

A least squares regression line has been determined for the data and is also displayed on the scatterplot. The equation for the least squares regression line is:

$$number\ of\ errors = 8.8 - 0.12 \times study\ time$$

and the coefficient of determination is 0.8198.



4 The least squares regression line predicts that a student reporting a study time of 35 minutes would make:

A 4.3 errors **B** 4.6 errors **C** 4.8 errors **D** 5.0 errors **E** 13.0 errors

5 The value of Pearson's product moment correlation coefficient, r, is closest to:

A –0.91 **B** –0.82 **C** 0.67 **D** 0.82 **E** 0.91

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6 The student who reported a study time of 10 minutes made six errors. The predicted score for this student would have a residual of:

A –7.6 **B** –1.6 **C** 0 **D** 1.6 **E** 7.6

7 Which of the following statements that relate to the regression line are *not* true?

A The slope of the regression line is –0.12.

B The equation predicts that a student who spends 40 minutes studying will make around four errors.

C The least squares line does *not* pass through the origin.

D On average, a student who does not study for the test will make around 8.8 errors.

E The explanatory variable in the regression equation is *number of errors*.

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Revision

8 This regression line predicts that, on average, the number of errors made:

A decreases by 0.82 for each extra minute spent studying

B decreases by 0.12 for each extra minute spent studying

C increases by 0.12 for each extra minute spent studying

D increases by 8.8 for each extra minute spent studying

E decreases by 8.8 for each extra minute spent studying

9 Given that the coefficient of determination is 0.8198, we can say that close to:

A 18% of the variation in the number of errors made can be explained by the variation in the time spent studying

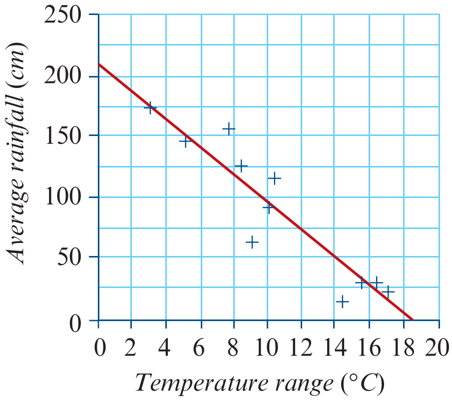
B 33% of the variation in the number of errors made can be explained by the variation in the time spent studying

C 67% of the variation in the number of errors made can be explained by the variation in the time spent studying

D 82% of the variation in the number of errors made can be explained by the variation in the time spent studying

E 95% of the variation in the number of errors made can be explained by the variation in the time spent studying

10 The average rainfall and temperature range at several locations in the South Pacific region are displayed in the scatterplot opposite.



A least squares regression line has been fitted to the data, as shown. The equation of this line is closest to:

A $average\ rainfall = 210 - 11 \times temperature\ range$

B $average\ rainfall = 210 + 11 \times temperature\ range$

C $average\ rainfall = 18 - 0.08 \times temperature\ range$

D $average\ rainfall = 18 + 0.08 \times temperature\ range$

E $average\ rainfall = 250 - 13 \times temperature\ range$

11 For a city, the correlation coefficient between:

- population density and distance from the city centre is $r = -0.563$
- house size and distance from the city centre is $r = 0.357$.

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Given this information, which *one* of the following statements is *true*?

A Around 31.7% of the variation observed in house size in the city is explained by the variation in distance from the city centre.

B Population density tends to increase as the distance from the city centre increases.

C House sizes tend to be larger as the distance from the centre of the city decreases.

D The slope of a least squares regression line relating population density to distance from the centre of the city is positive.

E Population density is more strongly associated with distance from the centre of the city than is house size.

based on VCAA (2013)

12 The following data were collected for two related variables x and y.

x	0.4	0.5	1.1	1.1	1.2	1.6	1.7	2.3	2.4	3.4	3.5	4.3	4.7	5.3
y	5.8	4.7	3.3	5.5	4.2	3.4	2.3	2.8	1.8	1.3	1.9	1.2	1.6	0.9

A scatterplot indicates a non-linear relationship. The data is linearised using a $1/y$ transformation. A least squares line is then fitted to the transformed data.

The equation of this line is closest to:

A $\frac{1}{y} = 0.08 + 0.16x$ **B** $\frac{1}{y} = 0.16 + 0.08x$ **C** $\frac{1}{y} = -0.08x + 5.23x$

D $\frac{1}{y} = 5.23 - 0.08x$ **E** $\frac{1}{y} = 1.44 + 1.96x$

13 The equation of a least squares line that has been fitted to transformed data is:

$$population = 58\,170 + 43.17 \times year^2$$

Using this equation, the predicted value of *population* when *year* = 10 is closest to:

A 9.2 **B** 9.9 **C** 10.6 **D** 62 417 **E** 62 487

14 The equation of a least squares line that has been fitted to transformed data is:

$$weight^2 = 52 + 0.78 \times area$$

Using this equation, the predicted value of *weight* when *area* = 8.8 is closest to:

A −7.7 **B** ±7.7 **C** 7.7 **D** ±58 **E** 58

15 The equation of a least squares regression line that has been fitted to transformed data is: $\log\ number = 1.31 + 0.083 \times month$

Using this equation, the predicted value of *number* when *month* = 6 is closest to:

A 1.8 **B** 6.0 **C** 18 **D** 64 **E** 650

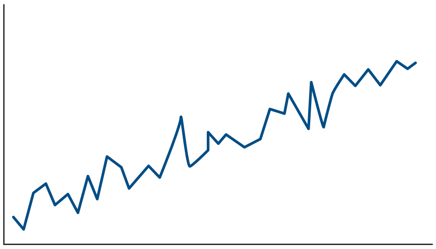
ISBN 978-1-107-56757-3

Revision

7D Multiple-choice questions: time series

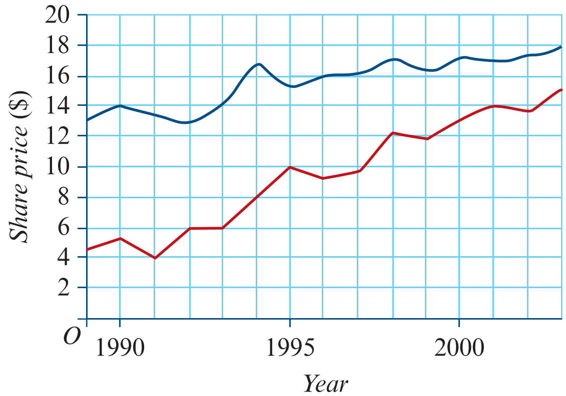
1 The pattern in the time series in the graph shown is best described as:

A trend **B** cyclical, but not seasonal
C seasonal **D** random **E** average



2 The time series plot shows the share price of two companies over a period of time.
From the plot, it can be concluded that over the interval 1990–2000, the *difference* in share price between the two companies has shown:

A a decreasing trend
B an increasing trend
C seasonal variation **D** a 5-year cycle **E** no trend



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Use the information in the table below to answer Questions 3 to 6.

t	1	2	3	4	5	6	7	8	9	10
y	4	5	4	4	8	6	9	10	9	12

3 The three-smoothed mean for $t = 2$ is closest to:

A 4.3 **B** 6.2 **C** 6.4 **D** 6.5 **E** 7.25

4 The five-smoothed mean for $t = 5$ is closest to:

A 4.3 **B** 6.2 **C** 6.4 **D** 6.5 **E** 7.25

5 The centred two-smoothed mean for $t = 6$ is closest to:

A 4.3 **B** 4.75 **C** 6.25 **D** 6.5 **E** 7.25

6 The centred four-smoothed mean for $t = 3$ is closest to:

A 4.3 **B** 4.75 **C** 6.25 **D** 7.25 **E** 9.75

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Use the following information to answer Questions 7 and 8.

The *long-term* quarterly sales figures of a car dealer are shown in the table. Also shown are the seasonal indices for the first and second quarters.

Quarter	1	2	3	4
Sales (number)	21	36	49	28
Seasonal index	0.6	1.0		

7 The car dealer sells 18 cars in the first quarter of this year. The deseasonalised number sold is:

A 11 **B** 13 **C** 18 **D** 20 **E** 30

8 The seasonal index for the fourth quarter is closest to:

A 0.6 **B** 0.8 **C** 1.1 **D** 1.5 **E** 4.0

Use the information below to answer Questions 9 to 11.

The quarterly sales figures for a soft drink company and the seasonal indices are as shown.

Quarter	1	2	3	4
Sales ($'000s)	1200	1000	800	1200
Seasonal index	1.1	0.90	0.8	

9 The deseasonalised figure (in $'000s) for quarter 3 is:

A 640 **B** 667 **C** 800 **D** 1000 **E** 1500

10 The seasonal index for quarter 4 is:

A 0.6 **B** 0.8 **C** 1.00 **D** 1.1 **E** 1.2

11 The deseasonalised sales (in dollars) for June were $91 564. The seasonal index for June is 1.45.

The *actual* sales for June were closest to:

A $41 204 **B** $61 043 **C** $63 148 **D** $91 564 **E** $132 768

12 Sales for a major department store are reported quarterly. The seasonal index for the third quarter is 0.85. This means that sales for the third quarter are typically:

A 85% below the quarterly average for the year

B 15% below the quarterly average for the year

C 15% above the quarterly average for the year

D 18% above the quarterly average for the year

E 18% below the quarterly average for the year

ISBN 978-1-107-56757-3

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13 The seasonal index for headache tablet sales in summer is 0.80.
To correct for seasonality, the headache sales figures for summer should be:

A reduced by 80% **B** reduced by 25% **C** reduced by 20%
D increased by 20% **E** increased by 25%

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14 The table below shows the number of broadband users in Australia for the years 2004 to 2008.

Year	2004	2005	2006	2007	2008
Number	1 012 000	2 016 000	3 900 000	4 830 000	5 140 000

A two-point moving mean with centring is used to smooth the time series.
The smoothed value for the number of broadband users in Australia in 2006 is:

A 2 958 000 **B** 3 379 600 **C** 3 455 500
D 3 661 500 **E** 3 900 000

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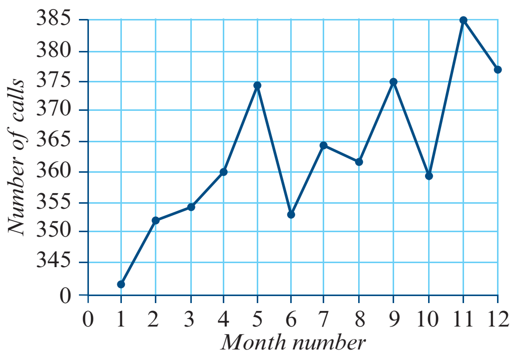
Use the information below to answer Questions 15 and 16.

The time series plot opposite shows the number of calls each month to a call-centre over a 12-month period.



15 The three-median smoothed number of calls for month 9 is closest to:

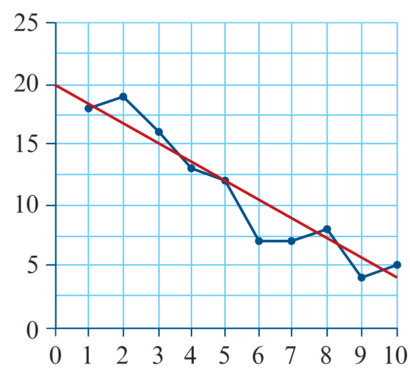
A 358 **B** 362 **C** 371 **D** 375 **E** 377

16 The five-median smoothed number of calls for month 10 is closest to:

A 358 **B** 362 **C** 371 **D** 375 **E** 377

17 A time series for y is shown in the graph, where t represents time. If a linear trend line is fitted to this data, as shown, then the equation of the line is closest to:

A $y = 20 - 1.6t$ **B** $y = -1.6t$
C $y = 20 + 1.6t$ **D** $y = 20 - 0.6t$
E $y = 20 + 0.6t$



ISBN 978-1-107-56757-3

7E Extended-response questions

AS

1 In a small survey, twenty-eight Year 8 girls were asked what they did (walked, sat, stood, ran) for most of the time during a typical school lunch time. Their responses are recorded below.

Sat	Stood	Sat	Ran	Sat	Ran	Walked
Walked	Walked	Sat	Walked	Ran	Sat	Ran
Sat	Walked	Walked	Walked	Ran	Ran	Sat
Walked	Ran	Walked	Ran	Walked	Ran	Sat

a Use the data to:

i complete the frequency table opposite.

Activity	*Frequency*
Walked	
Sat or stood	
Ran	
Total	28

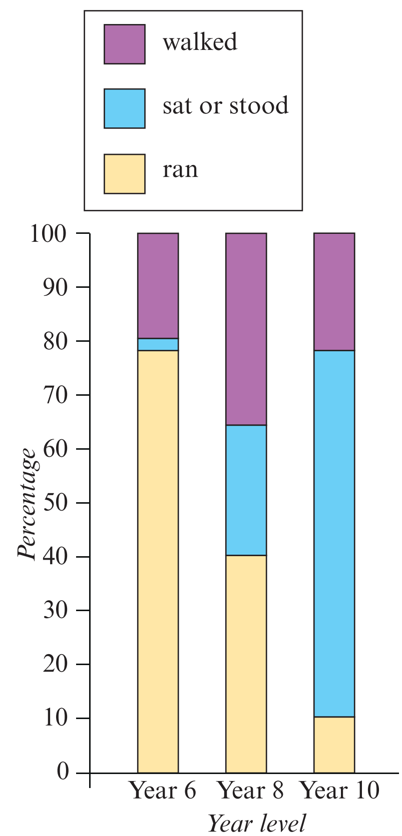
ii determine the percentage of Year 8 girls who ran for most of the time during a typical school lunch time.

b Is the variable *activity* (walked, sat or stood, ran) a nominal or ordinal variable?

2 In a larger survey, Years 6, 8 and 10 girls were asked what they did (walked, sat, stood, ran) for most of the time during a typical school lunch time. The results are displayed in the percentage segmented bar chart opposite.

a Does the percentage segmented bar chart support the opinion that, for these girls, the lunch time activity (walked, sat or stood, ran) undertaken is associated with year level? Justify your answer by quoting appropriate percentages.

b Is the variable *year level* (walked, sat or stood, ran) a nominal or ordinal variable?



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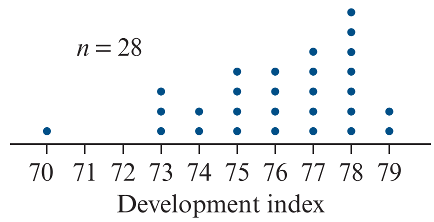
ISBN 978-1-107-56757-3

Revision

3 The development index for each country is a whole number between 0 and 100. The dot plot below displays the value of the development index for each of the 28 countries that have a high development index.



a Using the information in the dot plot, determine the mode and the range.

b Write down an appropriate calculation and use it to explain why the country with a development index of 70 for this group of countries is an outlier.

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4 The stem plot [below] shows the distribution of the average age, in years, that women first marry in 18 countries.

a For these countries, determine:

i the lowest average age of women at first marriage

ii the median average age of women at first marriage.

Key: 27|3 represents 27.3 years

24	
25	0
26	6
27	1 1 3 4 7
28	2 2 2 2 3 3 6
29	1 1
30	1 4

The stem plot [below] shows the distribution of the average age, in years, that men first marry in these 15 countries.

b For these countries, find the interquartile range (IQR) for the average age of men at marriage.

c If the data displayed [opposite] were used to construct a box plot showing outliers, then the country for which the average age of men at first marriage is 26.0 years would be shown as an outlier.

Explain why this is so. Show a calculation to support your explanation.

Key: 32|5 represents 32.5 years

25	
26	0
27	
28	9
29	0 9 9
30	0 0 3 5 6 7 9
31	0 0 2

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5 Table [7.1] shows information about a country town. It shows the percentage of women, by age at first marriage, for the years 1986, 1996 and 2006.

Table [7.1]

Age of women at first marriage	*Year of marriage*		
	1986	*1996*	*2006*
19 years and under	8.5%	3.7%	2.0%
20 to 24 years	42.1%	31.3%	21.5%
25 to 29 years	23.4%	31.7%	34.5%
30 years and over	26.0%	33.3%	42.0%

a Of the women who first married in 1986, what percentage were aged 20 to 29 years inclusive?

b Does the information in Table [7.1] support the opinion that, for the years 1986, 1996 and 2006, the age of women at first marriage was associated with year of marriage?

Justify your answer by quoting appropriate percentages. It is sufficient to consider one age group only when justifying your answer.

c If the aim of analysing this data is to predict the *age at first marriage* of women from their *year of marriage* which is the explanatory variable?

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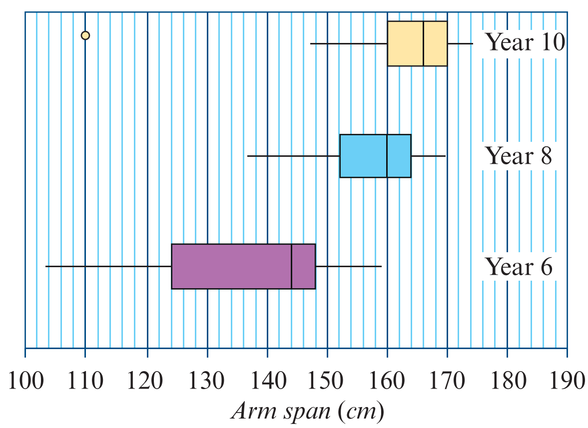
6 The arm spans (in cm) were recorded for each of the girls in Years 6, 8 and 10.

The results are summarised in the three parallel box plots displayed opposite.



a Complete the following sentence: 'The middle 50% of Year 6 students have an arm span between ______ and ______ cm.'

b The three parallel box plots suggest that arm span and year level are associated. Explain why.

c The 110 cm arm span of one of the Year 10 girls is shown as an outlier on the box plot. This value is an error. Her real arm span is 140 cm. If this error is corrected, would the girl's arm span still show as an outlier on the box plot? Give reasons for your answer, showing an appropriate calculation.

based on VCAA (2008)

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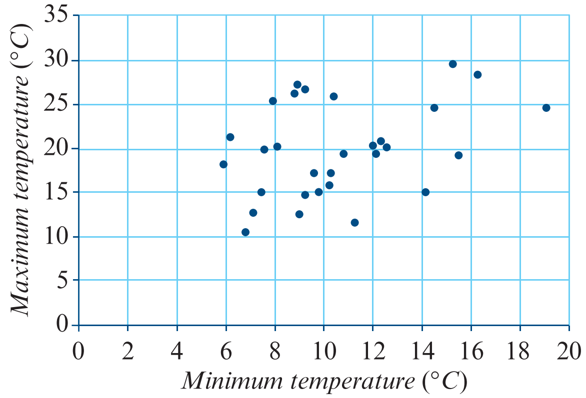
7 The maximum temperature and the minimum temperature at [a] weather station on each of the 30 days in November 2011 are displayed in the scatterplot below.



The correlation coefficient for this data set is $r = 0.630$.

The equation of the least squares regression line for this data set is:

$$\textit{maximum temperature} = 13 + 0.67 \times \textit{minimum temperature}$$

a Draw this least squares regression line on the scatterplot above.

b Interpret the vertical intercept of the least squares regression line in terms of maximum temperature and minimum temperature.

c Describe the association between the maximum temperature and the minimum temperature in terms of strength, direction and form.

d Interpret the slope of the least squares regression line in terms of maximum temperature and minimum temperature.

e Determine the percentage of variation in the maximum temperature that may be explained by the variation in the minimum temperature. Write your answer, correct to the nearest percentage.

f On the day that the minimum temperature was 11.1°C, the actual maximum temperature was 12.2°C.

Determine the residual value for this day if the least squares regression line is used to predict the maximum temperature. Write your answer, correct to the nearest degree.

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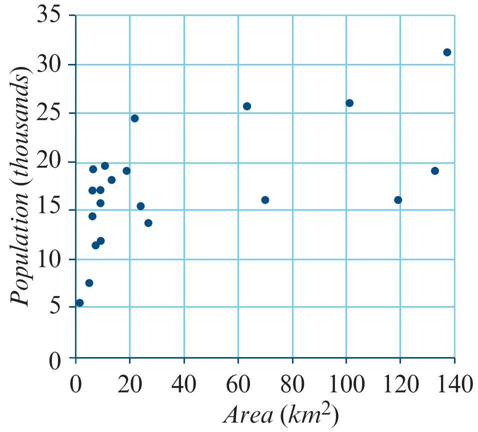
8 The scatterplot and table below show the population, in thousands, and the area, in square kilometres, for a sample of 21 outer suburbs of the same city.



Area (km^2)	*Population (thousands)*
1.6	5.2
4.4	14.3
4.6	7.5
5.6	11.0
6.3	17.1
7.0	19.4
7.3	15.5
8.0	11.3
8.8	17.1
11.1	19.7
13.0	17.9
18.5	18.7
21.3	24.6
24.2	15.2
27.0	13.6
62.1	26.1
66.5	16.4
101.4	26.2
119.2	16.5
130.7	18.9
135.4	31.3

In the outer suburbs, the relationship between population and area is non-linear.
A log transformation can be applied to the variable area to linearise the scatterplot.

a Apply the log transformation to the data and determine the equation of the least squares regression line that allows the population of an outer suburb to be predicted from the logarithm of its area.
Write the slope and intercept of this regression line in the boxes provided below.

$$population = \boxed{} + \boxed{} \times \log_{10}(area)$$

Write your answers, correct to one decimal place.

b Use this regression equation to predict the population of an outer suburb with an area of 90 km^2.
Write your answer, correct to the nearest one thousand people.

based on VCAA (2014)

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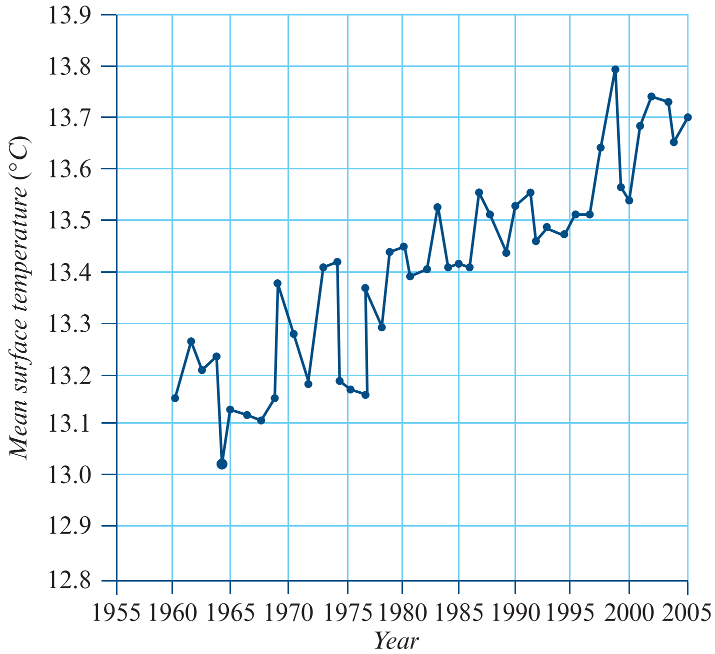
9 The mean surface temperature (in °C) of Australia for the period 1960 to 2005 is displayed in the time series plot below.



a In what year was the lowest mean surface temperature recorded?

The least squares method is used to fit a trend line to the time series plot.

b The equation of this trend line is found to be:

$$\textit{mean surface temperature} = -12.36 + 0.013 \times \textit{ year}$$

i Use the trend line to predict the mean surface temperature (in °C) for 2010. Write your answer correct to two decimal places.

The actual mean surface temperature in the year 2000 was 13.55°C.

ii Determine the residual value (in °C) when the trend line is used to predict the mean surface temperature for 2000. Write your answer correct to two decimal places.

iii By how many degrees does the trend line predict Australia's mean surface temperature will rise each year? Give your answer correct to three decimal places.

c Describe the pattern in the time series plot.

based on VCAA (2007)

ISBN 978-1-107-56757-3

Core: Recursion and financial modelling

8 Modelling growth and decay using recursion

ISBN 978-1-107-56757-3

Cambridge University Press

8A Sequences

A list of numbers, written down in succession, is called a **sequence**. Each of the numbers in a sequence is called a **term**. We write the terms of a sequence as a list, separated by commas. If a sequence continues indefinitely, or if there are too many terms in the sequence to write them all, we use an *ellipsis*, '. . .', at the end of a few terms of the sequence like this:

12, 22, 5, 6, 16, 43, . . .

The terms in this sequence of numbers could be the ages of the people boarding a plane.

The age of these people is random so this sequence of numbers is called a *random sequence*. There is no pattern or rule that allows the next number in the sequence to be predicted.

Some sequences of numbers do display a pattern. For example, this sequence

1, 3, 5, 7, 9, . . .

has a definite pattern and so this sequence is said to be *rule-based*.

The sequence of numbers has a starting value. We add 2 to this number to generate the term 3. Then, add 2 again to generate the term 5, and so on.

The rule is 'add 2 to each term'.

$$1 \xrightarrow{+2} 3 \xrightarrow{+2} 5 \xrightarrow{+2} 7 \xrightarrow{+2} 9 \quad \ldots$$

Example 1 Generating a sequence of terms recursively

Write down the first five terms of the sequence with a starting value of 6 and the rule 'add 4 to each term'.

Solution

1 Write down the starting value.	6
2 Apply the rule (add 4) to generate the next term.	$6 + 4 = 10$
3 Calculate three more terms.	$10 + 4 = 14$ $14 + 4 = 18$ $18 + 4 = 22$
4 Write your answer.	The sequence is 6, 10, 14, 18, 22, . . .

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Example 2 Generating a sequence of terms

Write down the first five terms of the sequence with a starting value of 10 and the rule 'double the number and then subtract 3'.

Solution

1. Write down the starting value. — 5
2. Apply the rule (double, then subtract 3) to generate the next term. — $2 \times 5 - 3 = 7$
3. Calculate three more terms. — $2 \times 7 - 3 = 11$; $2 \times 11 - 3 = 19$; $2 \times 19 - 3 = 35$
4. Write your answer. — The sequence is $5, 7, 11, 19, 35, \ldots$

▶ Using a calculator to generate a sequence of numbers from a rule

All of the calculations to generate sequences from a rule are repetitive. The same calculations are performed over and over again – this is called *recursion*. A calculator can perform recursive calculations very easily, because it automatically stores the answer to the last calculation it performed, as well as the method of calculation.

Example 3 Generating a sequence of numbers with a calculator

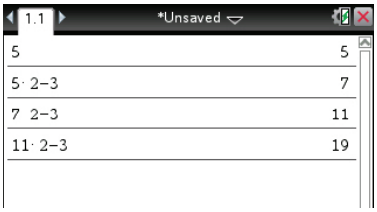
Use a calculator to generate the first five terms of the sequence with a starting value of 5 and the rule 'double and then subtract 3'.

Solution

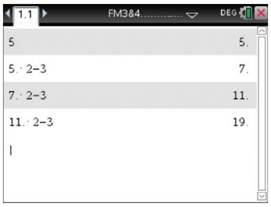
Steps

1. Start with a blank computation screen.
2. Type 5 and press enter or EXE.
3. Next type **×2 –3** and press enter or EXE to generate the next term in the sequence, '7' is generated and displayed on the right-hand side of the screen. Note that the computation generating this value is shown of the left side of the screen as '5.2–3' on the TI-Nspire and 'ans ×2 –3' on the ClassPad (here 'ans' represents the answer to the previous calculation).
4. Pressing enter or EXE repeatedly applies the rule '×2 –3' to the last calculated value, in the process generating successive terms of the sequence as shown.

TI-Nspire



ClassPad



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Exercise 8A

Generating a sequence recursively

1 Use the following starting values and rules to generate the first five terms of the following sequences recursively by hand.

a Starting value: 2
rule: add 6

b Starting value: 5
rule: subtract 3

c Starting value: 1
rule: multiply by 4

d Starting value: 10
rule: divide by 2

e Starting value: 6
rule: multiply by 2 add 2

f Starting value: 12
rule: multiply by 0.5 add 3

2 Use the following starting values and rules to generate the first five terms of the following sequences recursively using a calculator.

a Starting value: 4
rule: add 2

b Starting value: 24
rule: subtract 4

c Starting value: 2
rule: multiply by 3

d Starting value: 50
rule: divide by 5

e Starting value: 5
rule: multiply by 2 add 3

f Starting value: 18
rule: multiply by 0.8 add 2

8B Recurrence relations

A **recurrence relation** is a mathematical rule that we can use to generate a sequence. It has two parts:

1 a *starting point*: the value of one of the terms in the sequence

2 a *rule* that can be used to generate successive terms in the sequence.

For example, in words, a recursion rule that can be used to generate the sequence:

$$10, 15, 20, \ldots$$

can be written as follows:

1 Start with 10.

2 To obtain the next term, add 5 to the current term and repeat the process.

A more compact way of communicating this information is to translate this rule into symbolic form. We do this by defining a subscripted variable. Here we will use the variable V_n, but the V can be replaced by any letter of the alphabet.

Let V_n be the term in the sequence *after n* **iterations**.[1]

[1] Each time we apply the rule it is called an iteration.

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Using this definition, we now proceed to translate our rule written in words into a mathematical rule.

Starting value ($n = 0$)	*Rule for generating the next term*	*Recurrence relation* (*two parts: starting value plus a rule*)
$V_0 = 10$	$V_{n+1} = V_n + 5$ next term = current term + 5	$V_0 = 10,\quad V_{n+1} = V_n + 5$ starting value rule

Note: Because of the way we defined V_n, the starting value of n is 0. At the start there have been no applications of the rule. This is the most appropriate starting point for financial modelling.

The key step in using a recurrence relation to generate the terms of a sequence is to be able to translate the mathematical recursion rule into words.

Example 4 Generating a sequence from a recurrence relation

Write down the first five terms of the sequence defined by the recurrence relation

$$V_0 = 9, V_{n+1} = V_n - 4$$

showing the values of the first four iterations.

Solution

1 Write down the starting value.

$V_0 = 9$

2 Use the rule to find the next term, V_1.

$$\begin{aligned} V_1 &= V_0 - 4 \\ &= 9 - 4 \\ &= 5 \end{aligned}$$

3 Use the rule to determine three more terms.

$$\begin{aligned} V_2 &= V_1 - 4 \\ &= 5 - 4 \\ &= 1 \end{aligned} \qquad \begin{aligned} V_3 &= V_2 - 4 \\ &= 1 - 4 \\ &= -3 \end{aligned} \qquad \begin{aligned} V_4 &= V_3 - 4 \\ &= -3 - 4 \\ &= -7 \end{aligned}$$

4 Write your answer.

The sequence is 9, 5, 1, −3, −7, . . .

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Example 5 Using a calculator to generate sequences from recurrence relations

A sequence is generated by the recurrence relation $V_0 = 300$, $V_{n+1} = 0.5V_n - 9$.

Use your calculator to generate this sequence and determine how many terms of the sequence are positive.

Solution

1. Start with a blank computation screen.
2. Type **300** and press enter (or EXE).
3. Next type ×**0.5 –9** and press enter (or EXE) to generate the next term in the sequence, '141' is generated and displayed on the right-hand side of the screen.
4. Continue to press enter (or EXE) until the first negative terms appears.
5. Write your answer.

300	300.
$300 \cdot 0.5 - 9$	141.
$141 \cdot 0.5 - 9$	61.5
$61.5 \cdot 0.5 - 9$	21.75
$21.75 \cdot 0.5 - 9$	1.875
$1.875 \cdot 0.5 - 9$	–8.625

The first five terms of the sequence are positive.

Exercise 8B

Generating sequences using recurrence relations

1 Without using your calculator, write down the first five terms of the sequences generated by each of the recurrence relations below.

a $W_0 = 2$, $W_{n+1} = W_n + 3$
b $D_0 = 50$, $D_{n+1} = D_n - 5$
c $M_0 = 1$, $M_{n+1} = 3M_n$
d $L_0 = 3$, $L_{n+1} = -2L_n$
e $K_0 = 5$, $K_{n+1} = 2K_n - 1$
f $F_0 = 2$, $F_{n+1} = 2F_n + 3$
g $S_0 = -2$, $S_{n+1} = 3S_n + 5$
h $V_0 = -10$, $V_{n+1} = -3V_n + 5$

Generating sequences using recurrence relations and a calculator

2 Using your calculator, write down the first five terms of the sequence generated by each of the recurrence relations below.

a $A_0 = 12$, $A_{n+1} = 6A_n - 15$
b $Y_0 = 20$, $Y_{n+1} = 3Y_n + 25$
c $V_0 = 2$, $V_{n+1} = 4V_n + 3$
d $H_0 = 64$, $H_{n+1} = 0.25H_n - 1$
e $G_0 = 48\,000$, $G_{n+1} = G_n - 3000$
f $C_0 = 25\,000$, $C_{n+1} = 0.9C_n - 550$

Exploring sequences with a calculator

3 How many terms of the sequence formed from the recurrence relation below are positive?

$$F_0 = 150, \quad F_{n+1} = 0.6F_n - 5$$

4 How many terms of the sequence formed from the recurrence relation below are negative?

$$Y_0 = 30, \quad Y_{n+1} = 0.8Y_n + 2$$

ISBN 978-1-107-56757-3

8C Modelling linear growth and decay

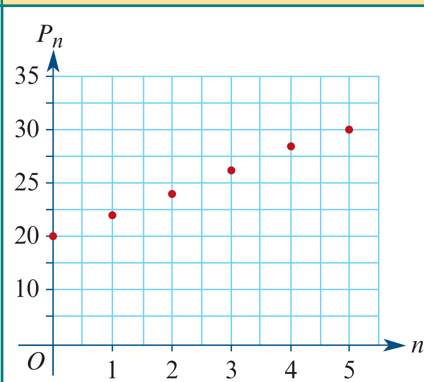
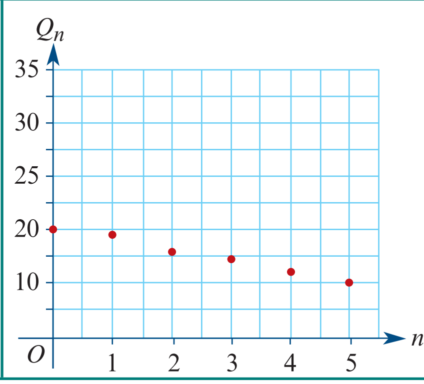
Linear growth and decay is commonly found around the world. They occur when a quantity increases or decreases by the same amount at regular intervals. Everyday examples include the paying of simple interest or the depreciation of the value of a new car by a constant amount each year.

▶ A recurrence model for linear growth and decay

The recurrence relations

$$P_0 = 20, P_{n+1} = P_n + 2 \qquad Q_0 = 20, Q_{n+1} = Q_n - 2$$

both have rules that generate sequences with linear patterns, as can be seen from the table below. The first generates a sequence whose successive terms have a linear pattern of growth, and the second a linear pattern of decay.

Recurrence relation	*Rule*	*Sequence*	*Graph*
$P_0 = 20, P_{n+1} = P_n + 2$	'add 2'	$20, 22, 24, \ldots$	
$Q_0 = 20, Q_{n+1} = Q_n - 2$	'subtract 2'	$20, 18, 16, \ldots$	

As a general rule, if D is a constant, a recurrence relation rule of the form:

- $V_{n+1} = V_n + D$ can be used to model **linear growth**.
- $V_{n-1} = V_n - D$ can be used to model **linear decay**.

We are now in a position to use this knowledge to **model** and investigate simple interest loans and investments, as well as flat rate depreciation and unit cost depreciation of assets, the topics of this section.

ISBN 978-1-107-56757-3

Simple interest loans and investments

If you deposit money into a bank account, the bank is effectively borrowing money from you. The bank will pay you a fee for using your money and this fee is called **interest**. If a fixed amount of interest is paid into the account at regular time periods, it is called a **simple interest** investment.

If you borrow money from the bank and are charged a fixed amount of interest after regular time periods, it is called a *simple interest loan*.

Simple interest is a special case of linear growth in which the starting value is the amount borrowed or invested. The amount borrowed or invested is called the **principal**. The amount added at each step is the interest and is usually a percentage of this principal.

Recurrence model for simple interest

Let V_n be the value of the loan or investment after n years.

Let r be the percentage **interest rate**.

The recurrence relation for the value of the loan or investment after n years is

$$V_0 = \text{principal},\ V_{n+1} = V_n + D$$

where $D = \dfrac{r}{100} \times V_0$.

Example 6 Modelling a simple interest with a recurrence relation

Cheryl invests \$5000 in an investment account that pays 4.8% per annum simple interest.

Model this simple investment using a recurrence relation of the form:

$$V_0 = \text{the principal}, \qquad V_{n+1} = V_n + D \qquad \text{where} \qquad D = \frac{r}{100}V_0.$$

Solution

1 Define the symbol V_n in this model.	V_n is the value of the investment after n years.
2 Write down the value of V_0, the principal of the investment.	$V_0 = 5000$
3 Write down the interest rate r and use it to determine the value of $D = \dfrac{r}{100}V_0$.	$r = 4.8\%$ $D = \dfrac{4.8}{100} \times 5000 = 240$
4 Use the values of V_0, and D to write down the recurrence relation.	$V_0 = 5000,\ \ V_{n+1} = V_n + 240.$

ISBN 978-1-107-56757-3

Once we have a recurrence relation, we can use it to determine things such as the value of the investment after a given number of years.

Example 7 Using a recurrence relation to analyse a simple interest investment

Cheryl's simple interest investment is modelled by

$$V_0 = 5000, \qquad V_{n+1} = V_n + 240$$

where V_n is the value of the investment after n years.

a Use the model to determine the value of Cheryl's investment after 3 years.

b When will Cheryl's investment first exceed \$6000, and what will its value be then?

Solution

a **i** Write down the recurrence relation.

$V_0 = 5000, V_{n+1} = V_n + 240$

ii On a blank calculation screen, type **5000** and press enter (Exec).

iii Type **+240** and press enter (Exec) three times to obtain the value of Cheryl's investment after 3 years. Write your answer.

b **i** Continue pressing enter (Exec) until the value of the investment first exceeds \$6000.

ii Write your answer.

5000	5000.
5000. + 240	5240.
5240. + 240	5480.
5480. + 240	5720.
5720. + 240	5960.
5960. + 240	6200.
|	

a \$5720

b After 5 years; \$6200.

▶ Depreciation

Over time, the value of large item gradually decreases. A car bought new this year will not be worth the same amount of money in a few years' time. A new television bought for \$2000 today is unlikely to be worth anywhere near this amount in 5 years.

ISBN 978-1-107-56757-3

Large equipment, machinery and other assets used in a business also lose value, or *depreciate*, over time. The depreciating value of equipment is often taken into account when calculating the actual costs of the business operations. It is important for businesses to be able to estimate the likely value of an asset after a certain amount of time. This is called the *future value* of the asset.

After a certain amount of time, or when the value of an item is depreciated to a certain amount, called its **scrap value**, the item will be sold or disposed of. At this point, the item has reached the end of its useful life and will be *written off*. This means the item is no longer an asset for the business.

Individuals often use the depreciation in value of equipment to calculate tax refunds. The taxation law allows individuals to buy equipment such as computers or other items necessary for their work and then to claim tax refunds based on the depreciating value of those assets.

There are a number of techniques for estimating the future value of an asset. Two of them, **flat-rate depreciation** and **unit-cost depreciation**, can be modelled using a linear decay recurrence relation.

Flat-rate depreciation

Flat-rate depreciation is very similar to simple interest, but instead of adding a constant amount of interest, a constant amount is *subtracted* to decay the value of the asset after every time period. This constant amount is called the **depreciation** amount and, like simple interest, it is often given as a percentage of the initial purchase price of the asset.

Recurrence model for flat-rate depreciation

Let V_n be the value of the asset after n years.

Let r be the percentage depreciation rate.

The recurrence relation for the value of the asset after n years is

$$V_0 = \text{initial value of the asset}, \; V_{n+1} = V_n - D$$

where $D = \dfrac{r}{100} \times V_0$.

Example 8 Modelling flat rate depreciation with a recurrence relation

A new car was purchased for $24 000 in 2014. The car depreciates by 20% of its purchase price each year. Model the depreciating value of this car using a recurrence relation of the form:

$$V_0 = \text{intial value}, \; V_{n+1} = V_n - D \text{ where } D = \frac{r}{100}V_0$$

Solution

1 Define the symbol V_n in this model.

V_n is the value of the car after n years depreciation.

ISBN 978-1-107-56757-3

2 Write down the value of V_0. Here, V_0 is the value of the car when new.

$V_0 = 24\,000$

3 Write down the annual rate of depreciation, r, and use it to determine the value of $D = \frac{r}{100}V_0$.

$r = 20\%$

$D = \frac{20}{100} \times 24\,000 = 4800$

4 Use the values of V_0 and D to write down the recurrence relation.

$V_0 = 24\,000, \quad V_{n+1} = V_n - 4800$

Once we have a recurrence relation, we can use it to determine things such as the value of an asset after a given number of years of flat rate depreciation.

Example 9 Using a recurrence relation to analyse flat rate depreciation

The flat rate depreciation of a car is modelled by

$$V_0 = 24\,000, \qquad V_{n+1} = V_n - 4800$$

where V_n is the value of the car after n years.

a Use the model to determine the value of the car after 2 years.

b In what year will the car's value depreciate to zero?

Solution

a i Write down the recurrence relation.

$V_0 = 24\,000, \; V_{n+1} = V_n - 4800$

ii On a blank calculation screen, type **24 000** and press enter (**Exec**).

iii Type **–4800** and press enter (**Exec**) twice to obtain the value of the car after 2 years' depreciation. Write your answer.

b i Continue pressing enter (**Exec**) until the car has no value.

ii Write your answer.

24000	24000.
24000. – 4800	19200.
19200. – 4800	14400.
14400. – 4800	9600.
9600. – 4800	4800.
4800. – 4800	0.

a \$14 400

b In 2019, after 5 years depreciation

Unit-cost depreciation

Some items lose value because of how often they are used, rather than because of their age. A photocopier that is 2 years old but has never been used could still be considered to be in 'brand new' condition and therefore worth the same, or close to, what it was 2 years ago. But if that photocopier was 2 years old and had printed many thousands of papers over those 2 years, it would be worth much less than its original value.

Cars can also depreciate according to their use rather than time. People often look at the number of kilometres a car has travelled before they consider buying it. An older car that

ISBN 978-1-107-56757-3

has travelled few kilometres overall could be considered a better buy than a new car that has travelled a large distance.

When the future value of an item is based upon use rather than age, we use a *unit-cost depreciation* method. Unit-cost depreciation can be modelled using a linear decay recurrence relation.

Recurrence model for unit-cost depreciation

Let V_n be the value of the asset after n units of use.

Let D be the cost per unit of use.

The recurrence relation for the value of the asset after n units of use is:

$$V_0 = \text{initial value of the asset}, V_{n+1} = V_n - D$$

Example 10 Modelling unit cost depreciation with a recurrence relation

A professional gardener purchased a lawn mower for \$270. The mower depreciates in value by \$3.50 each time it is used.

Model the depreciating value of this mower using a recurrence relation of the form:

$$V_0 = \text{intial value}, V_{n+1} = V_n - D \text{ where } D = \text{the depreciation in value per use}$$

Solution

1 Define the symbol V_n in this model.	V_n is the value of the mower after being used to mow n lawns.
2 Write down the value of V_0. Here, V_0 is the value of the mower when new.	$V_0 = 270$
3 Write down the unit cost rate of depreciation, D.	$D = 3.50$
4 Write your answer.	$V_0 = 270$ $V_{n+1} = V_n - 3.50$

Once we have a recurrence relation, we can use it to determine things such as the value of an asset after a given number of years flat rate depreciation.

Example 11 Using a recurrence relation to analyse unit cost depreciation

The depreciated value of the lawn mower is modelled by

$$V_0 = 270, V_{n+1} = V_n - 3.50$$

where V_n is the value of the mower after being used to mow n lawns.

a Use the model to determine the value of the mower after it has been used three times.

b How many times can the mower be used before its depreciated value is less than \$250?

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Solution

a **i** Write down the recurrence relation.

ii On a blank calculation screen, type **270** and press [enter] (Exec).

iii Type **–3.50** and press [enter] (Exec) three times to obtain the value of the mower after three mows. Write your answer.

b **i** Continue pressing [enter] (Exec) until the value of the lawn mower is first less than $250.

ii Write your answer.

$V_0 = 270, V_{n+1} = V_n - 3.50$

270	270.
270. – 3.5	266.5
266.5. – 3.5	263.
263. – 3.5	259.5
259.5 – 3.5	256.
256. – 3.5	252.5
252.5 – 3.5	249.

a $259.50

b After six mows

Exercise 8C

Modelling simple interest with recurrence relations

1 The following recurrence relation can be used to model a simple interest investment of $2000 paying interest at the rate of 3.8% per annum.

$$V_0 = 2000, V_{n+1} = V_n + 76$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b Use your calculator to determine how many weeks it takes for the value of the investment to be more than $3000.

c Write down a recurrence relation model if $1500 was invested at the rate of 6.0% per annum.

2 The following recurrence relation can be used to model a simple interest loan of $7000 charged interest at the rate of 7.4% per annum.

$$V_0 = 7000, V_{n+1} = V_n + 518$$

In the recurrence relation, V_n is the value of the loan after n years.

a Use the recurrence relation to find the value of the loan after 1, 2 and 3 years.

b Use your calculator to determine how many weeks it takes for the value of the loan to be more than $10 000.

c Write down a recurrence relation model if $12 000 was borrowed at the rate of 8.2% per annum.

ISBN 978-1-107-56757-3

3 The following recurrence relation can be used to model a simple interest investment. In the recurrence relation, V_n is the value of the loan after n years.

$$V_0 = 15\,000, V_{n+1} = V_n + 525$$

a **i** What is the principal of this investment?

ii How much interest is paid every year?

iii What is the annual interest rate of this investment?

b Use your calculator to determine how many weeks it takes for the value of the investment to be more than double the principal.

Modelling flat-rate depreciation with recurrence relations

4 The following recurrence relation can be used to model the depreciation of a computer with purchase price \$2500 and annual depreciation of \$400.

$$V_0 = 2500, V_{n+1} = V_n - 400$$

In the recurrence relation, V_n is the value of the computer after n years.

a Use the recurrence relation to find the value of the computer after 1, 2 and 3 years.

b Use your calculator to determine how many weeks it takes for the value of the computer to be less than \$1000.

c Write down a recurrence relation model if the computer was purchased for \$1800 and depreciated at \$350 per annum.

5 The following recurrence relation can be used to model the depreciation of a car purchased for \$23 000 and depreciated at 3.5% of its original value each year.

$$V_0 = 23\,000, V_{n+1} = V_n - 805$$

In the recurrence relation, V_n is the value of the car after n years.

a Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

b Use your calculator to determine how many weeks it takes for the value of the car to be less than \$10 000.

c Write down a recurrence relation model for a car purchased for \$37 000 and depreciated at \$700 per annum.

d Write down a recurrence relation model for a car purchased for \$12 000 and depreciated at 4.5% of its purchase price per annum.

6 The following recurrence relation can be used to model the depreciation of a television. In the recurrence relation, V_n is the value of the television after n years.

$$V_0 = 1500, V_{n+1} = V_n - 102$$

a **i** What is the purchase price of this television?

ii What is the depreciation of the television each year?

iii What is the annual percentage depreciation of the television?

b Use your calculator to determine a selling price if the television is sold after 8 years.

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Modelling unit-cost depreciation with recurrence relations

7 The following recurrence relation can be used to model the depreciation of a printer with purchase price \$450 and depreciated by 5 cents for every page printed.

$$V_0 = 450,\ V_{n+1} = V_n - 0.05$$

In the recurrence relation, V_n is the value of the printer after n pages printed.

a Use your calculator to find the value of the printer after 20 pages.

b Write down a recurrence relation model if the printer was purchased for \$300 and depreciated at 8 cents per page printed.

8 The following recurrence relation can be used to model the depreciation of a delivery van with purchase price \$48 000 and depreciated by \$200 for every 1000 kilometres travelled.

$$V_0 = 48\,000,\ V_{n+1} = V_n - 200$$

In the recurrence relation, V_n is the value of the delivery van after n lots of 1000 kilometres travelled.

a Use the recurrence relation to find the value of the van after 1000, 2000 and 3000 kilometres.

Skillsheet

b Use your calculator to determine the value of the van after 15 000 kilometres.

c Use your calculator to determine how many kilometres it takes for the value of the van to reach \$43 000.

8D Rules for the *n*th term in a sequence modelling linear growth or decay

While we can generate as many terms as we like in a sequence using a recurrence relation for linear growth and decay, it is possible to derive a rule for calculating any term in the sequence directly. This is most easily seen by working with a specific example.

For instance, if you invest \$2000 in a simple interest investment paying 5% interest per annum, your investment will increase by the same amount, \$100, each year.

If we let V_n be the value of the investment after n years, we can use the following recurrence relation to model this investment:

$$V_0 = 2000,\ V_{n+1} = V_n + 100$$

Using this recurrence relation we can write out the sequence of terms generated as follows:

$$\begin{array}{lll}
V_0 = 2000 & = V_0 + 0 \times 100 & \text{(no interest paid yet)} \\
V_1 = V_0 + 100 & = V_0 + 1 \times 100 & \text{(after 1 years' interest paid)} \\
V_2 = V_1 + 100 = (V_0 + 100) + 100 & = V_0 + 2 \times 100 & \text{(after 2 years' interest paid)} \\
V_3 = V_2 + 100 = (V_0 + 2 \times 100) + 100 & = V_0 + 3 \times 100 & \text{(after 3 years' interest paid)} \\
V_4 = V_3 + 100 = (V_0 + 3 \times 100) + 100 & = V_0 + 4 \times 100 & \text{(after 4 years' interest paid)}
\end{array}$$

and so on.

ISBN 978-1-107-56757-3

Following this pattern, after n years' interest has been added, we can write:

$$V_n = 2000 + n \times 100$$

With this rule, we can now predict the value of the nth term in the sequence without having to generate all of the other terms first.

For example, using this rule, the value of the investment after 20 years would be:

$$V_{20} = 2000 + 20 \times 100 = \$4000$$

This rule can be readily generalised to apply to any linear growth or decay situation.

For a recurrence rule of the form:

$$V_0 = \text{initial value}, V_{n+1} = V_n + D \qquad (D \text{ constant})$$

the value of the nth term generated by this recurrence relation is:

- $V_n = V_0 + nD$ for *linear growth*.
- $V_n = V_0 - nD$ for *linear decay*.

This general rule can then be applied to simple interest investments and loans, flat-rate depreciation and unit-cost depreciation.

Simple interest investments and loans

Let V_0 be the initial value of the simple interest investment or loan.

Let r be the annual interest rate.

The value of a simple interest investment or loan after n years is

$$V_n = V_0 + nD \quad \text{where} \quad D = \frac{r}{100}V_0 \qquad \text{or} \qquad V_n = V_0 + n \times \frac{r}{100} \times V_0$$

Flat rate of depreciation

Let V_0 be the initial value of the asset.

Let r be the flat rate of depreciation.

The value of the asset after n years is

$$V_n = V_0 - nD \quad \text{where} \quad D = \frac{r}{100}V_0 \qquad \text{or} \qquad V_n = V_0 - n \times \frac{r}{100} \times V_0$$

Unit-cost depreciation

Let V_0 be the initial value of the asset.

Let D be the cost per unit of use.

The value of the asset after n years is

$$V_n = V_0 - nD$$

ISBN 978-1-107-56757-3

Example 12 Using a rule to determine the value of a simple interest investment 1

The following recurrence relation can be used to model a simple interest investment:

$$V_0 = 3000, V_{n+1} = V_n + 260$$

where V_n is the value of the investment after n years.

a What is the principal of the investment? How much interest is added to the investment each year?

b Write down the rule for the value of the investment after n years.

c Use a rule to find the value of the investment after 15 years.

d Use a rule to find when the value of the investment first exceeds \$10 000.

Solution

a These values can be read directly from the recurrence relation.

a Principal: \$3000
Amount of interest = \$260

b Start with the general rule: $V_n = V_0 + nD$ and substitute $V_0 = 3000$ and $D = 260$.

b $V_n = 3000 + n \times 260$
$= 3000 + 260n$

c Substitute $n = 15$ into the rule to calculate V_{15}

c $V_{15} = 3000 + 260 \times 15$
$= \$4200$

d Substitute $V_n = 10\,000$ into the rule, and solve for n. Write your conclusion.

Note: Because the interest is only paid into the account after a whole number of years, any decimal answer will need to be *rounded up* to the next whole number.

d $10\,000 = 3000 + 260n$
so $7000 = 260n$
or $n = 7000/260$
$= 26.92\ldots$ years
The value of the investment will first exceed \$10 000 after 27 years.

Example 13 Using a rule to determine the value of a simple interest investment 2

Amie invests \$4000 in a simple interest investment of paying interest at the rate of 6.5% per year.

Use a rule to find the value of the investment after 10 years.

Solution

The amount invested and the annual interest rate are given so use the rule:

$$V_n = V_0 + n \times \frac{r}{100} \times V_0$$

Here $V_0 = 3000, n = 10$ and $r = 6.5\%$.
Calculate V_{10}.

$$V_{10} = 3000 + 10 \times \frac{6.5}{100} \times 3000$$
$$= \$4950$$

ISBN 978-1-107-56757-3

Example 14 Using a rule for the flat rate of deprecation of an asset

The following recurrence relation can be used to model the flat rate of deprecation of a set of office furniture:

$$V_0 = 12\,000,\ V_{n+1} = V_n + 1200$$

where V_n is the value of the furniture after n years.

a What is the initial value of the furniture? By how much does the furniture decrease in value each year?

b Write down the rule for the value of the investment after n years.

c Use a rule to find the value of the investment after 6 years.

d How long does it take for the furniture's value to decrease to zero?

e A photocopier in the office costs \$6000 when new. Its value depreciates at the flat rate of 17.5% of its new value each year. What is its value after 4 years?

Solution

a These values can be read directly from the recurrence relation.

a Initial value: \$12 000
Depreciation = \$1200

b Start with the general rule $V_n = V_0 - nD$ and substitute $V_0 = 12\,000$ and $D = 1200$.

b $V_n = 12\,000 - n \times 1200$
$= 12\,000 - 1200n$

c Use the rule to calculate V_6.

c $V_6 = 12\,000 - 1200 \times 6$
$= \$4800$

d Substitute $V_n = 0$, and solve for n. Write your conclusion.

d $0 = 12\,000 - n \times 1200$
so $n \times 1200 = 12\,000$
or $n = 10$ years
The value of the furniture will depreciate to zero after 10 years.

e In this exercise, the initial value of the asset and the depreciation interest rate are given so use the rule:
$$V_n = V_0 - n \times \frac{r}{100} \times V_0$$
Here $V_0 = 6000$, $n = 4$ and $r = 17.5\%$.
Calculate V_4.

e $V_4 = 6000 - 4 \times \frac{17.5}{100} \times 6000$
$= \$1800$

ISBN 978-1-107-56757-3

Example 15 Using a rule to determine the value of an asset with unit-cost depreciation

A hairdryer in a salon was purchased for \$850. The value of the hairdryer depreciates by 25 cents for every hour it is in use.

Let V_n be the value of the hairdryer after n hours of use.

a Write down a rule to find the value of the hairdryer after n hours of use.

b What is the value of the hairdryer after 50 hours of use?

c On average, the salon will use the hairdryer for 17 hours each week. How many weeks will it take the value of the hairdryer to halve?

Solution

a **1** Identify the values of V_0 and D. — $V_0 = 850$ and $D = 0.25$

2 Write down the rule for the value of the hairdryer after n hours of use. — $V_n = 850 - 0.25n$

b **1** Decide the value of n and substitute into the rule. — After 50 hours of use, $n = 50$.

$V_{50} = 850 - 0.25 \times 50$

$V_{50} = 837.50$

2 Write your answer. — After 50 hours of use, the hairdryer has a value of \$837.50.

c **1** Halving the value of the hairdryer means it will have a value of \$425. This is V_n. — $V_n = 425$

2 Write down the rule, with the value of the hairdryer, $V_n = 425$. — $425 = 850 - 0.25n$

3 Solve the equation for n.

$0.25n = 850 - 425$

$0.25n = 425$

$n = 1700$

4 Divide by 17 to get the number of weeks. — Number of weeks $= 100$

5 Write your answer. — After 100 weeks, the hairdryer is expected to halve in value.

ISBN 978-1-107-56757-3

Exercise 8D

Writing rules for the *n*th term of a sequence generated by a linear recurrence relation

1 Write down a rule for the value of the nth term of the sequence generated by each of the following recurrence relations. In each case calculate A_{20}.

a $A_0 = 4, A_{n+1} = A_n + 2$

b $A_0 = 10, A_{n+1} = A_n - 3$

c $A_0 = 5, A_{n+1} = A_n + 8$

d $A_0 = 300, A_{n+1} = A_n - 18$

Using a rule to find the value of a simple interest loan or investment after *n* years

2 The value of a simple interest loan after n years, V_n, can be calculated from the rule $V_n = 8000 + 512n$.

a What is the principal of this loan?

b How much interest is charged every year in dollars?

c Use the rule to find:

i the value of the loan after 12 years

ii how long it takes for the initial investment to double.

d The annual interest rate for this loan is 6.4%. Use this information to find the value of the loan after 15 years.

3 The value of a simple interest investment after n years, V_n, can be calculated from the rule $V_n = 2000 + 70n$.

a What is the principal of this investment?

b How much interest is charged every year in dollars?

c Use the rule to find:

i the value of the investment after 6 years

ii how long it takes for the initial investment to double.

d The annual interest rate for this loan is 3.5%. Use this information to find the value of the loan after 10 years.

4 Webster borrows \$5000 from a bank at an annual simple interest rate of 5.4%.

a Determine how much interest is charged each year in dollars.

b Write down:

i a recurrence relation to model the value of the loan, V_n, from year to year

ii a rule for the value of the loan, V_n, after n years.

c Use your rule in part **b ii** to find how much Webster will owe the bank after 9 years.

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5 Anthony borrows \$12 000 from a bank at an annual simple interest rate of 7.2%.

a Determine how much interest is charged each year in dollars.

b Write down:

i a recurrence relation to model the value of the loan, V_n, from year to year

ii a rule for the value of the loan, V_n, after n years.

c Use your rule in part **b ii** to find how much Anthony will owe the bank after 9 years.

Using a rule to find the value of an asset after flat-rate depreciation

6 The value of a sewing machine after n years, V_n, can be calculated from the rule $V_n = 1700 - 212.5n$.

a What is the purchase price of the sewing machine?

b By how much is the value of the sewing machine depreciated each year in dollars?

c Use the rule to find the value of the sewing machine after 4 years.

d The sewing machine depreciates by 12.5% of its purchase price each year. Use this information to determine its value after 8 years.

7 The value of a harvester after n years, V_n, can be calculated from the rule $V_n = 65000 - 3250n$.

a What is the purchase price of the harvester?

b By how much is the value of the harvester depreciated each year in dollars?

c What is the percentage depreciation for the harvester?

d Use the rule to find the value of the harvester after 7 years.

e How long does it take the harvester to reach a value of \$29 250?

8 A computer is purchased for \$5600 and is depreciated at a flat rate of 22.5% per year.

a Determine the annual depreciation in dollars.

b Write down:

i a recurrence relation to model the value of the computer, V_n, from year to year

ii a rule for the value of the computer, V_n, after n years.

c Use your rule in part **b ii** to find:

i the value of the computer after 3 years

ii the number of years it takes for the computer to be worth nothing.

9 A machine costs \$7000 new and depreciates at a flat rate of 17.5% per annum. The machine will be written off when its value is \$875.

a Determine the annual depreciation in dollars.

b Write down:

i a recurrence relation to model the value of the machine, V_n, from year to year

ii a rule for the value of the machine, V_n, after n years.

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c Use your rule in part **b ii** to find:

i the value of the machine after 2 years

ii the number of years the machine will be used.

Using a rule to find the value of an asset after *n* units of use

10 The value of a taxi after n kilometres, V_n, can be calculated from the rule $V_n = 29000 - 0.25n$.

a What is the purchase price of the taxi?

b By how much is the value of the taxi depreciated per kilometre of travel?

c What is the value of the taxi after 20 000 kilometres of travel?

d Find how many kilometres have been travelled if the taxi is valued at $5000.

11 A car is valued at $35 400 at the start of the year, and at $25 700 at the end of that year. During that year, the car travelled 25 000 kilometres.

a Find the total depreciation of the car in that year in dollars.

b Find the depreciation per kilometre for this car.

c Using $V_0 = 25\,700$, write down a rule for the value of the car, V_n, after n kilometres.

d How many kilometres are travelled when the value of the car is $6688?

e Including the first year of travel, how many kilometres in total is this car expected to drive before it has a value of zero?

12 A printing machine costing $110 000 has a scrap value of $2500 after it has printed 4 million pages.

a Find:

i the unit cost of using the machine

ii the value of the machine after printing 1.5 million pages

iii the annual depreciation of the machine if it prints 750 000 pages per year.

b Find the value of the machine after 5 years if it prints, on average, 750 000 pages per year.

c How many pages has the machine printed by the time the value of the machine is $70 000?

8E Modelling geometric growth and decay

Like linear growth and decay, **geometric growth** and **decay** are seen commonly around the world. Everyday examples include the payment of compound interest or the depreciation of the value a new car by a constant percentage each year. This sort of depreciation is commonly called reducing balance depreciation.

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▶ A recurrence model for geometric growth and decay

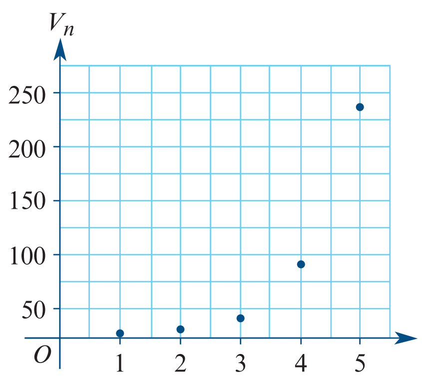
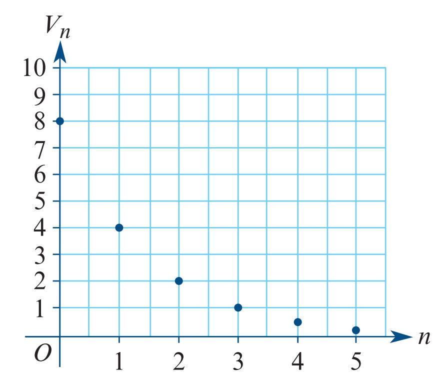
Geometric growth or decay in a sequence occurs when quantities increase or decrease by the same percentage at regular intervals.

The recurrence relations

$$V_0 = 1, V_{n+1} = 3V_n$$

$$V_0 = 8, V_{n+1} = 0.5V_n$$

both have rules that generate a geometric pattern, as can be seen from the table below. The first generates a sequence whose successive terms grow geometrically, the second decay geometrically.

Recurrence relation	*Rule*	*Sequence*	*Graph*
$V_0 = 1$, $V_{n+1} = 3V_n$	'multiply by 3'	$1, 3, 9, \ldots$	
$V_0 = 8$, $V_{n+1} = 0.5V_n$	'multiply by 0.5'	$8, 4, 2, \ldots$	

As a general rule, if R is a constant, a recurrence relation rule of the form:

- $V_{n+1} = RV_n$ for $R > 1$, can be used to model *geometric growth*.
- $V_{n+1} = RV_n$ for $R < 1$, can be used to model *geometric decay*.

We are now in a position to use this knowledge to model and investigate compound interest loans and investments, and reducing-balance depreciation, the topics of this section.

▶ Compound interest investments and loans

Most interest calculations are not as straightforward as simple interest. The more usual form of interest is **compound interest** where any interest that is earned after one time period is added to the principal and then contributes to the earning of interest in the next time period.

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This means that the value of the investment grows in every increasing amounts, or grows geometrically, instead of by the same amount as in simple interest.

Consider an investment of \$5000 that pays 8% interest per annum, compounding yearly. This means that the investment's value increases by 8% each year.

We can model the investment with a recurrence relation as follows:

Let V_n be the value of the investment after n years.

We can then write:

$$V_0 = \$2000$$

Each year, the investment will increase in value by 8%, so:

value next year = this year's value + interest earned

or symbolically:

$$V_{n+1} \quad = \quad V_n \quad + \quad 0.08 \times V_n$$

or more compactly:

$$V_{n+1} = 1.08V_n$$

We now have a recurrence rule that we can use to model and investigate the growth of investment over time. Unlike simple interest, compound interest loans or investments often accrue interest over periods less than a year and we need to take this into account when we formulate a general recurrence relation, as we have done below.

A recurrence model for compound interest investments and loans that compound yearly

Let V_n be the value of the investment after n years.

Let r be the percentage interest per compound period.

The recurrence model for the value of the investment after n compounding periods is:

$$V_0 = \text{principal}, V_{n+1} = RV_n \text{ where } R = 1 + \frac{r}{100}$$

Example 16 Modelling compound interest with a recurrence relation 1

The following recurrence relation can be used to model a compound interest investment of \$2000 paying interest at the rate of 7.5% per annum.

$$V_0 = 2000, V_{n+1} = 1.075 \times V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b Determine when the value of the investment will be first exceed \$2500.

c Write down the recurrence relation if \$1500 was invested at a compound interest rate of 6.0% per annum.

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Solution

a **1** Write down the principal of the investment, V_0.

$V_0 = 2000$

2 Use the recurrence relation to calculate V_1, V_2 and V_3. Use your calculator if you wish.

$V_1 = 1.075 \times 2000 = 2150$
$V_2 = 1.075 \times 2150 = 2311.25$
$V_3 = 1.075 \times 2311.25 = 2484.59$
(rounded to nearest cent)

b **Steps**

1 Type '2000' and press enter (or EXE).

2 Type **×1.075**.

3 Count how many times you press enter (or EXE) until the term value is greater than 2500.

2000	2000.
2000. · 1.075	2150.
2150. · 1.075	2311.25
2311.25 · 1.075	2484.59375
2484.59375 · 1.075	2670.93828125

4 Write your answer.

After 4 years, the investment will first exceed $2500.

c **1** Identify the value of V_0 and r.

$V_0 = 1500$ and $r = 6$

2 Calculate the value or R.

The value of the investment will grow over time, so $R = 1 + \frac{r}{100} = 1 + \frac{6}{100}$

$R = 1.06$

3 Write your answer.

$V_0 = 1500, \ V_{n+1} = 1.06 \times V_n$

Example 17 Compound interest with different compounding periods

Brian borrows $5000 from a bank. He will pay interest at the rate of 4.8% per annum.

Let V_n be the value of the loan after n compounding periods.

Write down a recurrence relation to model the value of Brian's loan if interest is compounded:

a yearly **b** quarterly **c** monthly.

Solution

a **1** Define the variable V_n. The compounding period is *yearly*.

Let V_n be the value of Brian's loan after n years.

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	2 Determine the value of R.	The interest rate is 4.5% per annum. $R = 1 + \frac{4.5}{100} = 1.045$
	3 Write the recurrence relation.	$V_0 = 5000,\ V_{n+1} = 1.045 \times V_n$
b	**1** Define the variable V_n. The compounding period is *quarterly*.	Let V_n be the value of Brian's loan after n quarters.
	2 Determine the value of R.	The interest rate is 4.5% per annum. The quarterly rate is $\frac{4.5}{4} = 1.125$. $R = 1 + \frac{1.125}{100} = 1.01125$
	3 Write the recurrence relation.	$V_0 = 5000,\ V_{n+1} = 1.01125 \times V_n$
c	**1** Define the variable V_n. The compounding period is *monthly*.	Let V_n be the value of Brian's loan after n months.
	2 Determine the value of R.	The interest rate is 4.5% per annum. The monthly rate is $\frac{4.5}{12} = 0.375$. $R = 1 + \frac{0.375}{100} = 1.00375$
	3 Write the recurrence relation.	$V_0 = 5000,\ V_{n+1} = 1.00375 \times V_n$

Reducing-balance depreciation

Earlier in the chapter, we studied two different methods for depreciating the value of an asset, both of which were examples of linear decay. **Reducing-balance depreciation** is another method of depreciation – one where the value of an asset decays geometrically. Each year, the value will be reduced by a percentage, $r\%$, of the previous year's value. The calculations are very similar to compounding interest, but with decay in value, rather than growth.

A recurrence model for reducing balance depreciation

Let V_n be the value of the asset after n years.

Let r be the annual percentage depreciation.

The recurrence model for the value of the investment after n years is:

$$V_0 = \text{initial value},\ V_{n+1} = RV_n \text{ where } R = 1 - \frac{r}{100}$$

ISBN 978-1-107-56757-3

Example 18 Modelling reducing-balance depreciation with recurrence relations

The following recurrence relation can be used to model the value of office furniture with a purchase price of \$6900, depreciating at a reducing-balance rate of 7% per annum.

$$V_0 = 6900, V_{n+1} = 0.93 \times V_n$$

In the recurrence relation, V_n is the value of the office furniture after n years.

a Use the recurrence relation to find the value of the office furniture, correct to the nearest cent, after 1, 2 and 3 years.

b Determine when the value of the investment will first be less than \$5000.

c Write down the recurrence relation if the furniture was initially valued at \$7500 and is depreciating at a reducing-balance rate of 8.4% per annum.

Solution

a **1** Write down the purchase price of the furniture, V_0.

$V_0 = 6900$

2 Use the recurrence relation to calculate V_1, V_2 and V_3. Use your calculator if you wish.

$V_1 = 0.93 \times 6900 = 6417$
$V_2 = 0.93 \times 6417 = 5967.81$
$V_3 = 0.93 \times 5967.81 = 5550.06$

b **Steps**

1 Type **6900** and press [enter] or [EXE].

2 Type **× 0.93**.

3 Count how many times you press [enter] until the term value is less than 5000.

6900	6900.
6900. · 0.93	6417.
6417. · 0.93	5967.81
5967.81 · 0.93	5550.0633
5550.0633 · 0.93	5161.558869
5161.558869 · 0.93	4800.24974817

4 Write your answer.

The value of the furniture drops below \$5000 after 5 years.

c **1** Identify the value of V_0.

$V_0 = 7500$

2 Calculate the value or R.

The depreciation rate is 8.4% per annum.

$R = 1 - \frac{8.4}{100}$ or $R = 0.916$

3 Write your answer.

$V_0 = 7500, \; V_{n+1} = 0.916 \times V_n$

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Exercise 8E

Modelling compound interest with recurrence relations

1 An investment of \$6000 earns compounding interest at the rate of 4.2% per annum. A recurrence relation that can be used to model the value of the investment after n years is shown below.

$$V_0 = 6000, V_{n+1} = 1.042V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b Determine how many years it takes for the value of the investment to first exceed \$8000.

c Write down a recurrence relation model for the value of an investment of \$5000 at a compounding interest rate of 6.8% per annum.

2 An loan of \$20 000 is charged compounding interest at the rate of 6.3% per annum. A recurrence relation that can be used to model the value of the loan after n years is shown below.

$$V_0 = 20\,000, V_{n+1} = 1.063V_n$$

In the recurrence relation, V_n is the value of the loan after n years.

a Use the recurrence relation to find the value of the loan after 1, 2 and 3 years.

b Determine how many years it takes for the value of the loan to first exceed \$30 000.

c Write down a recurrence relation model for the value of a loan of \$18 000 at a compounding interest rate of 9.4% per annum.

3 Wayne invests \$7600 with a bank. He will be paid interest at the rate of 6% per annum, compounding monthly. Let V_n be the value of the investment after n months.

a Write a recurrence relation to model Wayne's investment.

b How much is Wayne's investment worth after 5 months?

4 Jessica borrows \$3500 from a bank. She will be charged compound interest at the rate of 8% per annum, compounding quarterly. Let V_n be the value of the loan after n quarters.

a Write a recurrence relation to model the value of Jessica's loan from year to year.

b If Jessica pays back everything she owes to the bank after 1 year, how much money will she need?

Modelling reducing-balance depreciation with recurrence relations

5 A motorcycle, purchased new for \$9800 will be depreciated using a reducing-balance depreciation method with an annual depreciation rate of 3.5%. Let V_n be the value of the motorcycle after n years.

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a Write a recurrence relation to model the value of the motorcycle from year to year.

b Generate a sequence of numbers that represent the value of the motorcycle from year to year for 5 years in total. Write the values of the terms of the sequence correct to the nearest cent.

c What is the value of the motorcycle after 5 years?

d What is the depreciation of the motorcycle in the third year?

6 Office furniture was purchased new for \$18 000. It will be depreciated using a reducing-balance depreciation method with an annual depreciation rate of 4.5%. Let V_n be the value of the furniture after n years.

a Write a recurrence relation to model the value of the furniture from year to year.

b Generate a sequence of numbers that represent the value of the furniture from year to year for 5 years in total. Write the values of the terms of the sequence correct to the nearest cent.

c What is the value of the furniture after 3 years?

d What is the total depreciation of the furniture after 5 years?

8F Rules for the *n*th term in a sequence modelling geometric growth or decay

While we can generate as many terms as we like in a sequence using a recurrence relation for geometric growth and decay, it is possible to derive a rule for calculating any term in the sequence directly. This is most easily seen by working with a specific example.

We invest \$2000 in a compound interest investment paying 5% interest per annum, compounding yearly. If we let V_n be the value of the investment after n years, we can use the following recurrence relation to model this investment:

$$V_0 = 2000, V_{n+1} = 1.05V_n$$

Using this recurrence relation we can write out the sequence of terms generated as follows:

$$V_0 = 2000$$
$$V_1 = 1.05V_0$$
$$V_2 = 1.05V_1 = 1.05(1.05V_0) = 1.05^2V_0$$
$$V_3 = 1.05V_2 = 1.05(1.05^2V_0) = 1.05^3V_0$$
$$V_4 = 1.05V_3 = 1.05(1.05^3V_0) = 1.05^4V_0$$

and so on.

Following this pattern, after n year's interest has been added, we can write:

$$V_n = 1.05^nV_0$$

With this rule, we can now predict the value of the nth term in the sequence without having to generate all of the other terms first.

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For example, using this rule, the value of the investment after 20 years would be:

$$V_{20} = 1.05^{20} \times 2000 = \$5306.60 \text{ (to the nearest cent)}$$

This rule can be readily generalised to apply to any geometric growth or decay situation.

A rule for individual terms of a geometric growth and decay sequence

For a geometric growth or decay recurrence relation

$$V_0 = \text{starting value}, \; V_{n+1} = RV_n$$

the value of the nth term of the sequences is generated by the rule:

$$V_n = R^n \times V_0$$

Exactly the same rule will work for both growth and decay because growth or decay depends on the value of R, not the format of the calculation. This general rule can also be applied to compound interest loans and investment and reducing-balance depreciation.

Compound interest loans and investments

Let V_0 be the amount borrowed or invested (principal).
Let r be the annual interest rate of the loan or investment.
The value of a compound interest loan or investment after n compounding periods, V_n, is given by the rule

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

Reducing-balance depreciation

Let V_0 be the purchase price of the asset.
Let r be the annual percentage rate of depreciation.
The value of an asset after n years, V_n is given by the rule

$$V_n = \left(1 - \frac{r}{100}\right)^n \times V_0$$

Example 19 Using a rule to find the value of an investment

A principal value of \$10 000 is invested in an account earning compound interest at the rate of 9% per annum. The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.09^n \times 10\,000$$

a Find the value of the investment after 4 years, correct to the nearest cent.
b Find the amount of interest earned after 4 years, correct to the nearest cent.
c Find the amount of interest earned in the fourth year, correct to the nearest cent.
d If the interest compounds monthly instead of yearly, write down a rule for the value of the investment after *n months*.
e Use this rule to find the value of the investment after 4 years (48 months).

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Solution

a **1** Substitute $n = 4$ into the rule for the value of the investment.

$V_4 = 1.09^4 \times 10\,000$
$V_4 = 14\,115.816\ldots$

2 Write your answer, rounded to the nearest cent.

After 4 years, the value of the investment is \$14 115.82, correct to the nearest cent.

b To find the total interest earned in 4 years, subtract the principal from the value of the investment after 4 years.

Amount of interest
$= \$14\,115.82 - \$10\,000$
$= \$4115.82$
After 4 years, the amount of interest earned is \$4115.82.

c **1** The amount of interest earned in the fourth year is equal to 9% of the value of the loan after 3 years.

2 Calculate V_3 to the nearest cent.

$V_3 = 1.09^3 \times 10\,000$
$V_3 = 12\,950.29$ (nearest cent)

3 Calculate 9% of V_3.

9% of $V_3 = 0.09 \times 12\,950.29$
$= 1165.53$

4 Write your answer.

Interest of \$1165.53 was earned in the fourth year.

d **1** Define the symbol V_n.

Let V_n be the value of the investment after months.

2 The general rule for the nth term is $V_n = R^n V_0$ where $R = 1 + r/100$. In this investment, r is the *monthly* interest rate. Hence determine R.

$$r = \frac{9\%}{12} = 0.75\%$$

3 Determine the value of r. To convert to a monthly interest rate, divide the annual rate by 12.

$$R = 1 + \frac{0.75}{100}$$
$$= 1.0075$$

4 Substitute $R = 1.0075$ and $V_0 = 10\,000$ into the rule to find the rule for V_n.

$V_n = 1.0075^n \times 10\,000$

5 Substitute $n = 48$ (4 years = 48 months), $R = 1.0075$ and $V_0 = 10\,000$ into the rule.

$V_{48} = 1.0075^{48} \times 10\,000$
$= \$14214.05$

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Example 20 Calculating the value and total depreciation of an asset after reducing-balance depreciation

A computer system costs \$9500 to buy, and decreases in value with reducing-balance depreciation of 20% each year. A recurrence relation that can be used to model the value of the computer system after n years, V_n, is shown below.

$$V_0 = 9500, V_{n+1} = 0.8 \times V_n$$

a Write down the rule for the value of the computer system after n years.

b Use the rule to find the value of the computer system after 8 years. Write your answer, correct to the nearest cent.

c Calculate the total depreciation of the computer after 8 years.

Solution

a **1** Write down the values of V_0 and R.	$V_0 = 9500$ $R = 1 - \frac{20}{100} = 0.8$
2 Write down the rule.	$V_n = R^n \times V_0$ $V_n = 0.8^n \times 9500$
b **1** Substitute $n = 8$ into the rule.	$V_8 = 0.8^8 \times 9500$ $V_8 = 1593.835\ldots$
2 Write your answer, rounded as required.	After 8 years, the value of the computer system is \$1593.84, correct to the nearest cent.
c To find the total interest depreciation in 8 years, subtract the value of the computer system after 8 years from the original value of the computer system. Write your answer.	Depreciation = \$9500 − \$1593.84 = \$7906.16 After 8 years, the computer system has depreciated by \$7906.16.

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Example 21 Using a calculator to solve geometric growth and decay problems

How many years will it take an investment of \$2000, paying compound interest at 6% per annum, to grow above \$3000? Write your answer correct to the nearest year.

Solution

1 Write down the values of V_0, V_n, R.

$V_0 = 2000 \quad R = 1 + \frac{6}{100} = 1.06$

$V_n = 3000$

2 Substitute into the rule for the particular term of a sequence.

$V_n = R^n \times V_0$

$3000 = 1.06^n \times 2000$

3 Solve this equation for n using a CAS calculator.

solve $(3000 - (1.06)^n \cdot 2000, n)$

$n = 6.95851563317$

4 Write your answer.

After 6.95... years, the value of the investment is \$3000. It will grow above \$3000 after 7 years.

Example 22 Using a calculator to solve geometric growth and decay problems

An industrial weaving company purchased a new loom at a cost of \$56 000. It has an estimated value of \$15 000 after 10 years of operation. If the value of the loom is depreciated using a reducing-balance method, what is the annual rate of depreciation? Write your answer correct to one decimal place.

Solution

1 Write down the values of V_0, V_n, R and n.

$V_0 = 56\,000$

$V_n = 15\,000$

$R = 1 - \frac{r}{100}$

$n = 10$

2 Substitute into the rule for the nth term of a sequence.

$V_n = R^n \times V_0$

$15\,000 = \left(1 - \frac{r}{100}\right)^{10} \times 56\,000$

3 Solve this equation for r using a CAS calculator.

solve $\left(15000 - \left(1 - \frac{r}{100}\right)^{10} \cdot 56000,\ r\right)$

$r = 12.3422491484$ or

$r = 187.657750852$

Note: there are two answers. Choose the smaller of the two.

4 Write your answer.

The annual rate of depreciation is 12.3%, correct to one decimal place.

ISBN 978-1-107-56757-3

Exercise 8F

Writing rules for the *n*th term of a sequence generated from a geometric recurrence relation

1 Write down a rule for the value of the *n*th term of the sequences generated by the following recurrence relations. Use each rule to find the value of V_4.

a $V_0 = 6, V_{n+1} = 2V_n$

b $V_0 = 10, V_{n+1} = 3V_n$

c $V_0 = 1, V_{n+1} = 0.5V_n$

d $V_0 = 80, V_{n+1} = 0.25V_n$

Using a rule for compound interest

2 The value of an investment earning compound interest every year is modelled using the recurrence relation $V_0 = 3000, V_{n+1} = 1.1V_n$.

a **i** How much money was invested?

ii What is the annual interest rate for this investment?

b Write down a rule for the value of the investment after n years.

c Use the rule to find V_5.

d What does your answer in part **c** represent?

3 The value of a loan charged compound interest every year is modelled using the recurrence relation $V_0 = 2000, V_{n+1} = 1.06V_n$.

a **i** How much money was borrowed?

ii What is the annual interest rate for this investment?

b Write down a rule for the value of the loan after n years.

c Use the rule to find the value of the loan after 4 years.

d If the loan is fully repaid after 6 years, what is the total interest paid?

4 \$8000 is invested in an account earning 12.5% compound interest each year. Let V_n be the value of the investment after n years.

a Write down a rule for the value of the investment after n years.

b Use the rule to find the value of the investment after 3 years.

c How much interest has been earned over 3 years?

d How much interest was earned in the third year of the investment?

5 A loan of \$3300 is charged interest at the rate of 6% per annum, compounded monthly. Let V_n be the value of the loan after n months.

a Write down a rule for the value of the loan after n months.

b Use the rule to find the value of the loan after 10 months.

c How much interest has been charged over 10 months?

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Using a rule for reducing-balance depreciation

6 The value of a stereo system depreciating using reducing-balance depreciation is modelled using the recurrence relation $V_0 = 1200, V_{n+1} = 0.88V_n$.

a **i** What is the purchase price of the stereo system?

ii At what percentage rate is the stereo system being depreciated?

b Write down a rule for the value of the stereo system after n years.

c Use the rule to find the value of the stereo system after 7 years.

7 A car was purchased for \$38 500. It depreciates in value at a rate of 9.5% per year, using a reducing-balance depreciation method. Let V_n be the value of the car after n years.

a Write down a rule for the value of the car after n years.

b Use the rule to find the value of the car after 5 years.

c What is the total depreciation of the car over 5 years?

Using a CAS calculator to solve geometric growth and decay problems

8 Sarah invested \$3500 at 6.75% per annum. If the investment now amounts to \$5179.35, for how many years was it invested?

9 How long would it take for \$200 to exceed \$20 000 if it was invested at 4.75% per annum?

10 Suppose that an investment of \$1000 has grown to \$1601.03 after 12 years invested at r% per annum compound interest. Find the value of r.

11 What reducing-balance depreciation rate would cause the value of a car to drop from \$8000 to \$6645 in 3 years?

12 How much money must you deposit at a fixed rate of 6.8% per annum if you require \$12 000 in 4 years' time?

13 A machine has a book value after 10 years of \$13 770. If it has depreciated at a reducing-balance rate of 8.2% per annum, what was the initial cost of the machine?

8G Nominal and effective interest rates

Nominal interest rate

Compound interest rates are usually quoted as annual rates, or interest rate per annum. This rate is called the **nominal interest rate** for the investment or loan. Sometimes an annual rate might be quoted, but the interest can be calculated and paid according to a different time period, such as monthly. The time period after which compound interest is calculated and paid is called the **compounding period**.

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The terms of a compound interest loan or investment are usually quoted as a nominal interest rate followed by a compounding period. The interest rate for the compounding period is easily calculated using simple arithmetic.

It must be assumed that there are:

- 12 equal months in every year (even though some months have different numbers of days)
- 4 quarters in every year (a quarter is equal to 3 months)
- 26 fortnights in a year (even though there are slightly more than this)
- 52 weeks in a year (even though there are slightly more than this)
- 365 days in a year (ignore the existence of leap years).

A nominal interest rate is converted to a compounding interest rate by *dividing* by these numbers.

Example 23 Converting nominal interest rates to compounding interest rates

An investment account will pay interest at the rate of 3.6% per annum. Convert this interest rate to

a a monthly rate **b** a fortnightly rate **c** a quarterly rate.

Solution

Annual interest rates are converted to rates for other compounding periods by dividing by the relevant number.

a Divide by 12. Monthly interest rate $= \frac{3.6}{12} = 0.3\%$

b Divide by 26. Fortnightly interest rate $= \frac{3.6}{26} = 0.138\%$ to 3 d.p.

c Divide by 4. Quarterly interest rate $= \frac{3.6}{4} = 0.9\%$

Effective interest rates

As a general principle with compound interest, the more frequently interest is calculated and added to your investment or loan (the compounding period), the more rapidly the value of your investment or loan increases.

This is illustrated in the next table, which compares the value of a $5000 investment paying a nominal interest rate of 4.8% per annum with the value of the investment if the interest is calculated on a quarterly or monthly basis, rather than just yearly.

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Principal of investment: \$5000 Nominal annual interest rate: 4.8%			
	Value of investment for interest earned at the rate of:		
Month	*4.8% per annum*	*1.2% per quarter*	*0.4% per month*
0	5000.00	5000.00	5000.00
1			5020.00
2			5040.08
3		5060.00	5060.24
4			5080.48
5			5100.80
6		5120.72	5121.21
7			5141.69
8			5162.26
9		5182.17	5182.91
10			5203.64
11			5224.45
12	5240.00	5244.35	5245.35
Total interest earned*	240.00	244.35	245.35
Effective annual interest rate	4.80%	4.81%	4.91%

*note that the total interest earned is the value of the investment at the end of the year less the principal.

As you can see from the table, the more frequently interest is calculated and added, the greater the value of the investment at the end of the year. For this investment, the amounts ranged from \$240.00 when interest is calculated and added annually to \$245.35 when interest is calculated and added monthly.

What this means is that an investment where the interest is calculated and added monthly has a higher effective annual interest rate than an investment where interest is calculated and yearly. The effective interest rate is determined as follows.

The interest earned can be expressed as a percentage of the original investments. This value, called the **effective interest rate**, can be used to compare the investment performance.

$$\text{Compounding } \textit{yearly}\text{: effective rate} = \frac{240}{5000} \times 100\% = 4.80\% \text{ per annum}$$

$$\text{Compounding } \textit{quarterly}\text{: effective rate} = \frac{240.35}{5000} \times 100\% = 4.81\% \text{ per annum}$$

$$\text{Compounding } \textit{monthly}\text{: effective rate} = \frac{245.35}{5000} \times 100\% = 4.91\% \text{ per annum}$$

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The effective rate for an investment at 4.8% per annum, compounding quarterly, is 4.81%. This means that, after 1 year, the interest earned will be 4.81% of the amount invested. After 1 year, the interest earned by the monthly compounding investment will be 4.91% of the amount invested. The investment that compounds monthly earns more interest in a year.

Instead of writing a table for a whole year in order to calculate the effective interest rates for different loans or investments, we can use the following rule.

Effective interest rate

The effective interest rate of a loan or investment is the interest earned after one year expressed as a percentage of the amount borrowed or invested.

Let:

- r be the nominal interest rate per annum
- $r_{\text{effective}}$ be the effective annual interest rate
- n be the number of times the interest compounds each year.

The effective annual interest rate is given by: $r_{\text{effective}} = \left(\left(1 + \dfrac{r/n}{100}\right)^n - 1\right) \times 100\%$

Example 24 Comparing loans and investments with effective interest rates

Brooke would like to borrow \$20 000. She is deciding between two loan options:

- option A: 5.95% per annum compounding weekly
- option B: 6% per annum compounding quarterly.

a Calculate the effective interest rate for each investment.

b Which investment option is the best and why?

Solution

	Option A	Option B
a 1 Decide on the values of r and n for each option.	$r = 5.95$ There are 52 weeks in a year so $n = 52$.	$r = 6$ There are 4 quarters in a year so $n = 4$.
2 Apply the effective interest rate rule.	$r_{\text{effective}} = \left(\left(1 + \dfrac{5.95/52}{100}\right)^{52} - 1\right) \times 100\%$ $= 6.13\%$	$r_{\text{effective}} = \left(\left(1 + \dfrac{6/4}{100}\right)^{4} - 1\right) \times 100\%$ $= 6.14\%$

b Compare the effective interest rates.

Brooke is borrowing money, so the best option is the one with the lowest effective interest rate. She will pay less interest with option A.

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The amount of interest earned over a particular time period depends on the number of compounds within that time period. In a short time period, the number of compounds has very little effect on the total interest paid or earned; however, over long time periods the number of compounds can have a significant effect on the total interest earned or paid.

Remember

For any compound interest loan or investment, increasing the number of compounds per year will increase the total interest earned or paid.

Example 25 Calculating effective interest rates using a CAS calculator

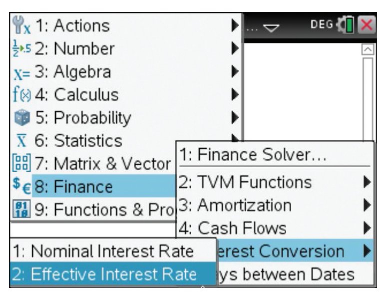
Marissa has \$10 000 to invest. She chooses an account that will earn compounding interest at the rate of 4.5% per annum, compounding monthly.

Use a CAS calculator to find the effective rate for this investment, correct to three decimal places.

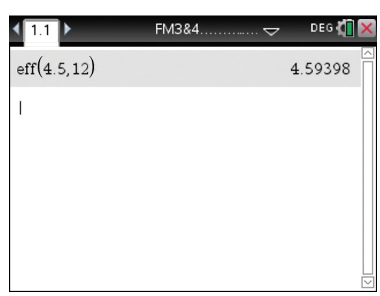
Solution

Steps

1 Press menu and then select
8: Finance ▶
5: Interest Conversion ▶
2: Effective interest rate
to paste in the **eff(...)** command.
The parameters of this function are
eff(nominal rate, number of compounds per year).



2 Enter the nominal rate (4.5) and compounds per year (12) into the function, separated by a comma. Press enter to get the effective interest rate.



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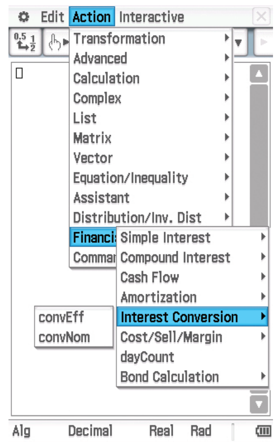
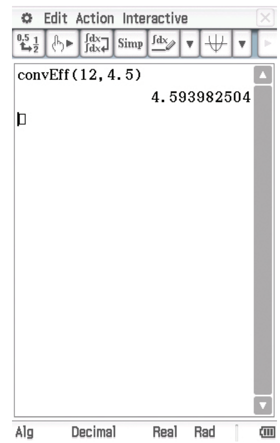
Solution for ClassPad

Steps

1 Select **Action**, **Financial**, **Interest Conversion**, **ConvEff** to paste in the **convEff(...)** command.
The parameters of this function are convEff(number of compounds per year, nominal rate).

2 Enter the nominal rate (4.5) and compounds per year (12) into the function, separated by a comma.
Press **EXE** to get the effective interest rate.





3 Write your answer.

The effective interest rate for this investment is 4.594%.

Exercise 8G

Interest rate conversions and effective interest rates

1 Convert each of the annual interest rates below to an interest rate for the given time period. Write your answers, correct to three significant figures.

a 4.95% per annum to monthly
b 8.3% per annum to quarterly
c 6.2% per annum to fortnightly
d 7.4% per annum to weekly
e 12.7% per annum to daily

2 Convert each of the interest rates below to an annual interest rate.

a 0.54% monthly
b 1.45% quarterly
c 0.57% fortnightly
d 0.19% weekly
e 0.022% daily

3 Use your calculator to determine the effective annual interest rate, correct to two decimal places, for the following nominal rates and compounding periods.

a 6.2% per annum compounding monthly
b 8.4% per annum compounding daily
c 4.8% per annum compounding weekly
d 12.5% per annum compounding quarterly
e 7.5% per annum compounding every 6 months

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Comparing loans and investments with effective interest rates

4 Brenda invests \$15 000 in an account earning nominal compound interest of 4.60% per annum, compounding quarterly.

a Explain why Brenda would be better off with *more frequent* compounds per year.

b Calculate the effective interest rate for the current loan with quarterly compounds, correct to two decimal places.

c Calculate the effective rate for this investment with monthly compounds, correct to two decimal places.

d Explain how these effective rates support your answer to part **a**.

5 Stella borrows \$25 000 from a bank and pays nominal compound interest of 7.94% per annum, compounding fortnightly.

a Explain why Stella would be better off with *less frequent* compounds per year.

b Calculate the effective rate for the current loan with fortnightly compounds, correct to two decimal places.

c Calculate the effective rate for this loan with monthly compounds, correct to two decimal places.

d Explain how these effective rates support your answer to part **a**.

6 Luke is considering a loan of \$35 000. His bank has two compound interest rate options:

A: 8.3% per annum, compounding monthly

B: 7.8% per annum, compounding weekly.

a Calculate the effective interest rate for each of the loan options.

b Calculate the amount of interest Luke would pay in the first year for each of the loan options.

c Which loan should Luke choose and why?

7 Sharon is considering investing \$140 000. Her bank has two compound interest investment options:

A: 5.3% per annum, compounding monthly

B: 5.5% per annum, compounding quarterly.

a Calculate the effective interest rate for each of the loan options.

b Calculate the amount of interest Sharon would earn in the first year for each of the loan options.

c Which investment option should Sharon choose and why?

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Review

Key ideas and chapter summary

Sequence A **sequence** is a list of numbers or symbols written in succession, for example: 5, 15, 25, . . .

Term Each number or symbol that makes up a sequence is called a **term**.

Recurrence relation A relation that enables the value of the next term in a sequence to be obtained by one or more current terms. Examples include ‘to find the next term, add two to the current term’ and ‘to find the next term, multiply the current term by three and subtract five’.

Modelling **Modelling** is the use of a mathematical rule or formula to represent or model real-life situations. Recurrence relations can be used to model situations involving the *growth* (increase) or *decay* (decrease) in values of a quantity.

Percentage growth and decay If a quantity grows by $r\%$ each year, then $R = 1 + \frac{r}{100}$.
If a quantity decays by $r\%$ each year, then $R = 1 - \frac{r}{100}$.

Principal The **principal** is the initial amount that is invested or borrowed.

Balance The **balance** is the value of a loan or investment at any time during the loan or investment period.

Interest The fee that is added to a loan or the payment for investing money is called the **interest**.

Simple interest **Simple interest** is a fixed amount of interest that is paid at regular time intervals. Simple interest is an example of linear growth.

Depreciation **Depreciation** is the amount by which the value of an item decreases after a period of time.

Scrap value **Scrap value** is the value of an item at which it is ‘written off’ or is considered no longer useful or usable.

Flat-rate depreciation **Flat-rate depreciation** is a constant amount that is subtracted from the value of an item at regular time intervals. It is an example of linear decay.

Unit-cost depreciation **Unit-cost depreciation** is depreciation that is calculated based on units of use rather than time. Unit-cost depreciation is an example of linear decay.

Compounding period Interest rates are usually quoted as annual rates (per annum). Interest is sometimes calculated more regularly than once a year, for example each quarter, month, fortnight, week or day. The time period for the calculation of interest is called the **compounding period**.

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Compound interest When interest is added to a loan or investment and then contributes to earning more interest, the interest is said to compound. **Compound interest** is an example of geometric growth.

Reducing-balance depreciation When the value of an item decreases as a percentage of its value after each time period, it is said to be depreciating using a reducing-balance method. **Reducing-balance depreciation** is an example of geometric decay.

Nominal interest rate A **nominal interest rate** is an annual interest rate for a loan or investment.

Effective interest rate The **effective interest rate** is the interest earned or charged by an investment or loan, written as a percentage of the original amount invested or borrowed. Effective interest rates allow loans or investments with different compounding periods to be compared. Effective interest rates can be calculated using the rule $r_{\text{effective}} = \left(\left(1 + \frac{r/n}{100}\right)^n - 1\right) \times 100\%$ where r is the nominal interest rate and n is the number of compounding periods in 1 year.

Skills check

Having completed this chapter, you should be able to:

- understand the concept of a first-order linear recurrence relation and its use in generating the terms in a sequence
- use a given first-order recurrence relation to generate the terms in a sequence
- understand the concepts of simple and compound interest
- use a recurrence relation to model compound interest investments and loans, and the flat-rate, unit-cost and reducing-balance methods for depreciating of assets
- understand the concepts nominal and effective interest rates as they apply to compound interest investments and loans
- use a rule for the future value of a compound interest investment or loan, or a depreciating asset, to solve practical problems.

Multiple-choice questions

1 The sequence generated by the recurrence relation $V_0 = 5,\ V_{n+1} = V_n - 3$ is:

A 5, 15, 45, 135, 405, . . .
B 5, 8, 11, 14, 17, . . .
C 5, 2, −1, −4, −7, . . .
D 5, 15, 45, 135, 405, . . .
E 5, −15, 45, −135, 405, . . .

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2 A sequence of numbers is generated by the recurrence relation $V_0 = 2,\ V_{n+1} = 2V_n + 8$. The value of the term V_4 in this sequence is:

A 12 **B** 18 **C** 32 **D** 72 **E** 152

3 A sequence of numbers is generated by the recurrence relation $V_0 = 5,\ V_{n+1} = 3$. The value of V_3 is:

A 5 **B** 9 **C** 21 **D** 57 **E** 165

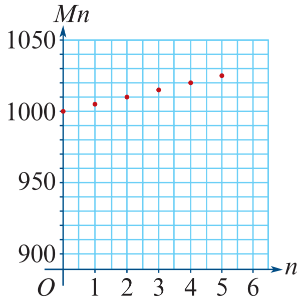
4 Brian has two trees in his backyard. Every month, he will plant three more trees. A recurrence model for the number of trees in Brian's backyard after n month is:

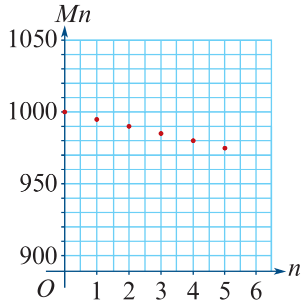
A $T_0 = 2,\ T_{n+1} = 3T_n$ **B** $T_0 = 2,\ T_{n+1} = 3T_n + 3$

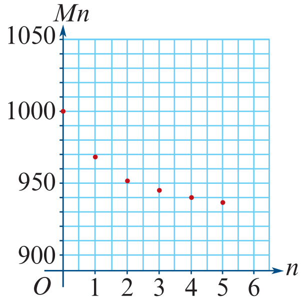
C $T_0 = 2,\ T_{n+1} = T_n + 3$ **D** $T_0 = 2,\ T_{n+1} = T_n - 3$

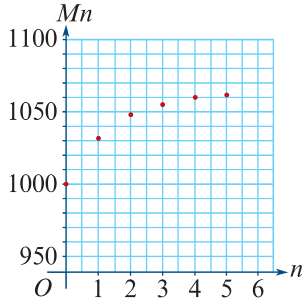
E $T_0 = 2,\ T_{n+1} = 3T_n - 3$

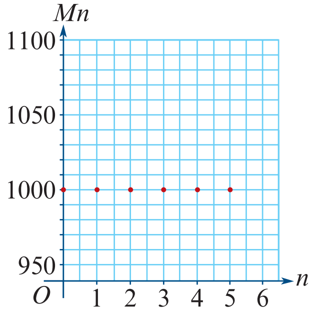
5 A graph that shows the value of a simple interest investment of $1000, earning interest of $5 per months is:

A


B


C


D


E


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Review

6 Arthur invests \$2000 with a bank. He will be paid simple interest at the rate of 5.1% per annum. If V_n is the value of Arthur's investment after n years, a recurrence model for Arthur's investment is:

A $V_0 = 2000,\ V_{n+1} = V_n + 5.1$
B $V_0 = 2000,\ V_{n+1} = 5.1V_n$
C $V_0 = 2000,\ V_{n+1} = 0.051V_n + 102$
D $V_0 = 2000,\ V_{n+1} = V_n + 102$
E $V_0 = 2000,\ V_{n+1} = 5.1V_n + 2000$

7 A nominal interest rate of 4.6% per annum is the same as a compounding interest rate of:

A 1.15% per quarter
B 0.35% per month
C 0.17% per week
D 0.18% per fortnight
E \$441.00 per month

8 The recurrence relation that generates a sequence of numbers representing the value of a car n years after it was purchased is $V_0 = 18\,000,\ V_{n+1} = V_n - 1098$. The car had a purchase price of \$18 000 and is being depreciated using:

A flat-rate depreciation at 6.1% of its value per annum
B flat-rate depreciation at \$6.10 per kilometre travelled
C flat-rate depreciation at \$1098 per kilometre travelled
D unit-cost depreciation at \$6.10 per kilometre travelled
E unit-cost depreciation at \$1098 per kilometre travelled

9 A sequence is generated from the recurrence relation $V_0 = 40,\ V_{n+1} = V_n - 16$. The rule for the value of the term V_n is:

A $V_n = 40n - 16$
B $V_n = 40 - 16n$
C $V_n = 40n$
D $V_n = 40 + 16n$
E $V_n = 40n - 16$

10 A computer is depreciated using a flat-rate depreciation method. It was purchased for \$2800 and depreciates at the rate of 8% per annum. The amount of depreciation after 4 years is:

A \$224 **B** \$448 **C** \$672 **D** \$896 **E** \$1904

11 A car is depreciated using a unit-cost depreciation method. It was purchased for \$18 990 and, after travelling a total of 20 000 kilometres, it has an estimated value of \$15 990. The depreciation amount, per kilometre, is:

A \$0.15 **B** \$0.80 **C** \$0.95 **D** \$6.67 **E** \$3000

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12 A population of penguins is decreasing by 8% every year. There are currently 2700 penguins in the population. A recurrence relation model for the number of penguins in the population after n years, P_n, is:

A $P_0 = 2700, V_{n+1} = 1.8 \times V_n$

B $V_0 = 2700, V_{n+1} = 1.08 \times V_n$

C $V_0 = 2700, V_{n+1} = 0.92 \times V_n$

D $V_0 = 2700, V_{n+1} = 1 + 8 \times V_n$

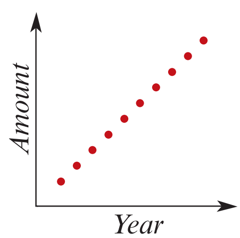
E $V_0 = 2700, V_{n+1} = 1.08 + V_n$

13 Sandra invests \$6000 in an account that pays compounding interest at the rate of 4.57% per annum. The number of years it takes the investment to exceed \$8000 is:

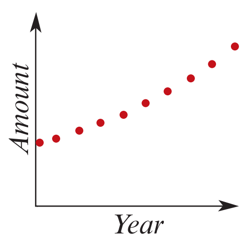
A 5 **B** 6 **C** 7 **D** 8 **E** 9

14 An investment of \$50 000 is made at a fixed rate of interest compounding annually over a number of years. Which graph best represents the value of the investment at the end of each year?

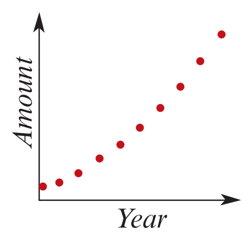
A



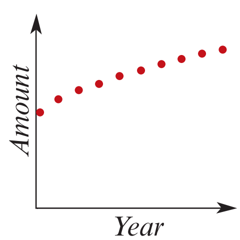
B



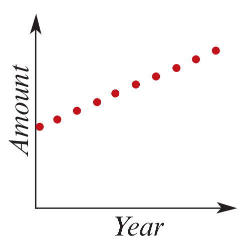
C



D



E



15 An item is depreciated using a reducing-balance depreciation method. The value of the item after n years, V_n, is modelled by the recurrence relation $V_0 = 4500, V_{n+1} = 0.86V_n$. The rule for the value of the item after n years is:

A $V_n = 0.86^n \times 4500$

B $V_n = 1.86^n \times 4500$

C $V_n = (1 + 0.86)^n \times 4500$

D $V_n = 0.86 \times n \times 4500$

E $V_n = (1 - 0.86)^n \times 4500$

Note: $V_0 = \$500, V_{n+1} = 0.86V_n$

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16 After 10 years, a compound interest investment of \$8000 earned a total of \$4000 in interest. The annual interest rate of this investment was closest to:

A 2.5% **B** 4.14% **C** 5.03% **D** 7.2% **E** 50%

17 The interest rate on a compound interest loan is 12.6% per annum, compounding monthly. The value of the loan after n years, V_n, is modelled by the recurrence relation $V_0 = 400$, $V_{n+1} = R \times V_n$.
The value of R in this model is:

A 0.874 **B** 1.00 **C** 1.0105 **D** 1.126 **E** 2.05

18 An amount of \$2000 is invested, earning compound interest at the rate of 5.4% per annum, compounding quarterly. The effective annual interest rate is closest to:

A 5.2% **B** 5.3% **C** 5.4% **D** 5.5% **E** 5.6%

Extended-response questions

1 A sequence is generated from the recurrence relation $H_0 = 800,\ H_{n+1} = 0.8H_n - 10$.

a Use your calculator to generate the first five terms of the sequence.

b How many iterations are required to generate the first negative term?

2 A sequence of numbers has starting value 30. Each term of the sequence is then generated using the rule 'subtract 3'.
Let A_n be the value of the nth term.

a Write a recurrence relation with $A_0 = 30$ that generates this sequence of numbers.

b Sketch a graph of the value of the terms of the sequence against the value of n, the number of steps of the recurrence relation.

c How many iterations will it take for the value of the number in the sequence to be zero?

3 Jack borrows \$20 000 from a bank and is charged simple interest at the rate of 9.4% per annum. Let V_n be the value of the loan after n years.

a Write down a recurrence relation model for the value of Jack's loan after n years.

b Use the model to find how much Jack will need to pay the bank after 5 years.

c Convert the annual interest rate to a weekly one, rounding your answer to two significant figures.

d How many weeks does it take the value of Jack's loan to reach \$20 612 ?

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Review

4 Kelly bought her current car 5 years ago for \$22 500.
Let V_n be the value of Kelly's car after n years.

a If Kelly assumes a flat-rate depreciation of 12% per annum:

i write down a recurrence model for the value of Kelly's car after n years

ii use the recurrence model to find the current value of Kelly's car.

b If Kelly assumes reducing value depreciation at 16% per annum:

i write down a recurrence model for the value of Kelly's car after n years

ii use the recurrence model to find the current value of Kelly's car using reducing-balance depreciation.

c On the same axes, sketch a graph of the value of Kelly's car against the number of years for both flat-rate and reducing-balance depreciation.

5 A commercial cleaner bought a new vacuum cleaner for \$650. The value of the vacuum cleaner decreases by \$10 for every 50 offices that it cleans.

a By how much does the cleaning of one office depreciate the value of the vacuum cleaner?

b Write down a recurrence model for the value of the vacuum cleaner after n offices have been cleaned.

c The cleaner has a contract to clean 10 offices, 5 nights a week for 40 weeks in a year. What is the value of the vacuum cleaner after 1 year?

6 Meghan has \$5000 to invest.

a Company A offers her an account paying 6.3% per annum simple interest. How much will she have in this account at the end of 5 years?

b Company B offers her an account paying 6.1% per annum compound interest. How much will she have in this account at the end of 5 years?

c Find, correct to one decimal place, the simple interest rate that company A should offer if the two investments are to have equal values after 5 years.

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7 A sum of \$30 000 is borrowed at an interest rate of 6.8% per annum, compounding monthly.

Let V_n be the value of the loan after n months.

a Write a recurrence model the value of this loan.

b Generate a sequence of numbers that represent the value of this loan after 5 months.

c What is the value of the loan after 1 year?

d If the loan is fully repaid after 18 months, how much money is needed?

8 Ilana uses her credit card to buy a dress costing \$300, knowing that she will not be able to pay it off for some time. If she is charged interest on the amount for 6 months, how much will the dress finally cost her? (Assume that interest is charged at 18% per annum, compounded monthly, and that no payments are made during the 6-month period.)

9 On the birth of his granddaughter, a man invests a sum of money at a fixed rate of 11.65% per annum compounded twice a year. On her 21st birthday he gives all the money in the account to his granddaughter. If she receives \$2529.14, how much did her grandfather invest?

10 Geoff invests \$18 000 in an investment account. After 2 years the investment account contains \$19 282.05. If the account pays r% interest per annum compounded quarterly, find the value of r, to one decimal place.

ISBN 978-1-107-56757-3

Chapter 9

9

Core: Recursion and financial modelling

Modelling and analysing reducing-balance loans and annuities

ISBN 978-1-107-56757-3

9A Combining geometric growth and decay

In the previous chapter, recurrence relations were used to model financial situations with linear and geometric growth and decay. These included simple and compound interest and the depreciation of assets. Recurrence relations can be used to model situations that involve elements of both linear and geometric growth.

The number of trout in a fish farm, for example, grows geometrically over time, but this growth is moderated by regular harvesting of mature trout that would be sold. Another example is a personal loan. The **principal** of the loan is the amount that is borrowed. The amount owed, or the **balance** of the loan, will increase over time because interest is charged and added to the loan. It is usual for regularly scheduled payments to be made on the loan so that the amount owed will gradually reduce to zero.

A recurrence relation of the form

$$V_0 = \text{starting value}, V_{n+1} = R \times V_n \pm D$$

can be used to model situations that involve both geometric and linear growth or decay.

The following worked examples will give you practice working with recurrence relations of this form before you learn to apply them to modelling financial situations with combined geometric and linear growth and decay.

Example 1 Generating a sequence from a recurrence relation of the form $V_{n+1} = R \times V_n \pm D$

Write down the sequence generated by the recurrence relation

$$V_0 = 3, V_{n+1} = 4V_n - 1$$

Solution

1 Write down the starting value.	3
2 Apply the rule (multiply 4, then subtract 1) to generate four more terms.	$4 \times 3 - 1 = 11$ $4 \times 11 - 1 = 43$ $4 \times 43 - 1 = 171$ $4 \times 171 - 1 = 683$
3 Write your answer.	The sequence is 3, 11, 43, 171, 683, . . .

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Example 2 **Modelling population change using a recurrence relation**

The number of trout in a fish farm pond after n months, T_n, can be modelled using the recurrence relation:

$$T_0 = 10\,000, T_{n+1} = 1.1 \times T_n - 3000$$

a Use the recurrence relation to determine the number of trout in the pond after 2 months.

b After how many months will there be no trout left in the pond?

Solution

a **1** Write down the starting value.

$T_0 = 10\,000$

2 Use the recurrence relation rule to calculate two more terms.

$T_1 = 1.1 \times 10000 - 3000 = 8000$

$T_2 = 1.1 \times 8000 - 3000 = 5800$

3 Write your answer.

After 2 months there will be 5800 trout in the pond.

b **1** Use your calculator to generate the terms of the sequence. Count the number times you press enter (TI-Nspire) or EXE (Casio) until zero or a negative number first appears.

10000	10000
10000 · 1.1–3000	8000
8000 · 1.1–3000	5800
5800 · 1.1–3000	3380
3380 · 1.1–3000	718
718 · 1.1–3000	–2210.2

2 Write your answer.

There will be no trout left in the pond after 5 months.

Exercise 9A

Generating a sequence using a recurrence relation

1 Write down the first five terms of the sequences generated by the following recurrence relations.

a $T_0 = 50, T_{n+1} = 2T_n - 10$

b $Z_0 = 128, Z_{n+1} = 0.5Z_n + 32$

c $P_0 = 1000, P_{n+1} = 1.02P_n - 20$

Modelling real-life situations using a recurrence relation rule of the form $V_{n+1} = R \times V_n \pm D$

2 Each day, 1.4% of the water in a swimming pool is lost due to evaporation. At the end of each day, 1000 litres of water are added to the pool. The following recurrence relation can be used to model the amount of water in the swimming pool after n days, W_n.

$$W_0 = 150\,000, W_{n+1} = 0.986W_n + 1000$$

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a How many litres of water, correct to the nearest litre, will there be in the swimming pool after 5 days?

b How many days will it take for the water in the swimming pool to be less than 130 000 litres?

3 The following recurrence relation can be used to model the number of kangaroos in a population after n years, K_n.

$$K_0 = 3000, K_{n+1} = 1.02K_n - 200$$

a How many kangaroos were initially in the population?

b Is the population of kangaroos growing or decaying?

c At what percentage rate is the kangaroo population growing or decaying?

Skillsheet

d How many kangaroos leave the population each year?

e Use the recurrence relation to find the nearest whole number of kangaroos expected to be in population after 5 years.

9B Analysing reducing-balance loans with recurrence relations

▶ Reducing-balance loans

When money is borrowed from a bank, it is very unusual for the borrower to wait until the term of the loan is complete before paying all of the money owed, including the interest, back to the bank. Generally loans are repaid by making regular payments to reduce the amount owed, the **balance** of the loan, to zero over time. The amount originally borrowed is called the **principal** of the loan.

This kind of loan is called a **reducing-balance loan**. A reducing-balance loan is effectively a compound interest loan, with regular payments. Personal loans and mortgages (home loans) are examples of reducing-balance loans.

Modelling reducing balance loans

Let V_n be the *balance* of the loan after n payments have been made.

Let r be the *interest rate* per compounding period.

Let D be the *payment* made.

A recurrence relation rule that can be used to model a reducing balance loan is

$$V_0 = \text{the principal}, \quad V_{n+1} = RV_n - D$$

where $R = 1 + \dfrac{r}{100}$

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Example 3 Modelling a reducing-balance loan with a recurrence relation

Alyssa borrows \$1000 at an interest rate of 15% per annum, compounding monthly.

She will repay the loan by making four monthly payments of \$257.85.

Construct a recurrence relation to model this loan, in the form

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

where V_n is the balance of the loan after n payments.

Solution

1 Define the symbol V_n.	Let V_n be the balance of the loan after n payments.
2 Write down the value of V_0. Here V_0 is the principal of the loan, the balance of the loan before any payments have been made.	$V_0 = 1000$
3 Because interest compounds monthly, calculate the monthly interest rate.	$r = \frac{15\%}{12} = 1.25\%$ or 0.0125
4 Determine the value of R. Either recognise that a 1.25% increase gives an R of 1.0125, or use the formula $R = 1 + r/100 = 1 + 1.25/100 = 1.0125$	$R = 1.0125$
5 Write down the value of D. In this context, D is the amount repaid each month.	$D = 257.85$
6 Use the values of V_0, R and D to write down the recurrence relation.	$V_0 = 1000$ $V_{n+1} = 1.0125V_n - 257.85$

Once we have a recurrence relation, we can use it to determine things such as the balance of a loan after a given number of payments.

Example 4 Using a recurrence relation to analyse a reducing-balance loan

Alyssa's loan can be modelled by the recurrence relation:

$$V_0 = 1000, \qquad V_{n+1} = 1.0125V_n - 257.85$$

a Use your calculator to determine recursively the balance of the loan after Alyssa has made each of the four payments.

b What is the balance of the loan (the amount she still owes) after she has made two payments? Give your answer to the nearest cent.

c Is the loan fully paid out after four payments have been made? If not, how much will the last payment have to be to ensure that the loan is fully repaid after four payments?

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Solution

a **i** Write down the recurrence relation.

$V_0 = 1000, V_{n+1} = 1.0125V_n - 257.85$

ii Type '1000' and press 'enter' or 'EXE'.

iii Type '× 1.0125-257.85' and press 'enter' (or EXE) 4 times to obtain the screen opposite.

1000	1000
1000 ·1.0125 – 257.85	754.65
754.65·1.0125 – 257.85	506.233
506.233125·1.0125 – 257.85	254.711
254.7110390625·1.0125 – 257.85	0.044927

b The balance of the loan after two payments corresponds to the third line on the screen. Round the answer to the nearest cent.

$506.23 (to the nearest cent)

c No. Alyssa still owes 4 cents. Add this amount to the regular monthly payment to find the value of the last payment that ensures that the loan is fully repaid after 4 payments.

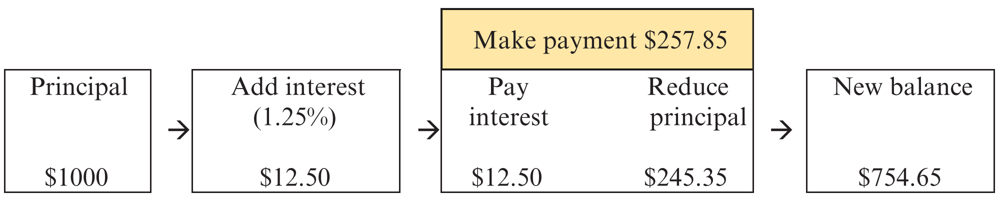
$$\text{Last payment} = 257.85 + 0.04$$
$$= \$257.89$$

Note: In Example 4, the balance of Alyssa's loan is reduced to around 4 cents. The loan is almost fully repaid after four payments, but not exactly. To repay the loan in exactly four payments, Alyssa would have to repay $258.861023 ... each month. In practice this amount must be rounded to the nearest cent, so it is not possible to repay the loan exactly by making four equal payments.

Amortisation tables

Loans that are repaid by making regular payments until the balance of the loan is zero are called **amortising** loans. In an amortising loan, part of each payment goes towards paying the interest owed on the unpaid balance of the loan with the remainder used to reduce the principal of the loan (the amount borrowed).

For example, consider Alyssa's loan from Example 4. Interest on the $1000 loan was charged at the rate of 1.25% per month and the loan was to be repaid with four monthly payments of $257.85. The first step in the amortisation process is shown in the diagram below.



The calculation of the values in this diagram are explained below.

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At the end of the first month, when the first payment is due:

- 1 month's interest is charged on the full \$1000; that is, 1.25% of \$1000 = \$12.50
- the interest owed is then deducted from the payment, leaving an amount of \$245.35 (\$257.85 – \$12.50)
- the \$257.85 is then used to reduce the balance of the loan to give a new balance of \$754.65 (\$1000 – \$257.85).

The process is then repeated when the next payment is made and continued until four payments have been made.

Balance	→	Add interest (1.25%)	→	Make payment \$257.85: Pay interest	Make payment \$257.85: Reduce principal	→	New balance
\$754.65		\$9.43		\$9.43	\$248.42		\$506.23

The results of this analysis can then be summarised in an **amortisation table**. The amortisation table shows the payment number, the payment made, the interest paid, the principal reduction and the balance of the loan after each payment has been made. The amortisation table for Alyssa's loan is shown below.

Payment number	*payment amount*	*Interest paid*	*Principal reduction*	*Balance of loan*
0	0	0	0	1000.00
1	257.85	12.50	245.35	754.65
2	257.85	9.43	248.42	506.23
3	257.85	6.33	251.52	254.71
4	257.85*	3.18	254.71	0.00
	Total	31.44	10 000.00	

Note: The last payment was increased by 4 cents so that the balance of the loan was reduced to exactly \$0.00 after 4 payments had been made. This was because the payment amount must be rounded to the nearest cent. Making such an adjustment is normal when amortising a loan.

The amortisation table above can be used to summarise the key properties of a reducing balance loan.

Properties of a reducing balance loan

At each step of the loan:

1 interest paid = interest rate per payment period × unpaid balance.

For example, when payment 1 is made, interest paid = 1.25% of 1000 = \$12.50

2 principal reduction = payment made – interest paid

For example, when payment 1 is made, principal reduction = 257.85 – 12.50 = \$245.35

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3 balance of loan = balance owing − reduction in balance

For example, when payment 1 is made, reduction in balance = 1000 − 245.35 = \$754.65.

4 cost of repaying the loan = the sum of the payments

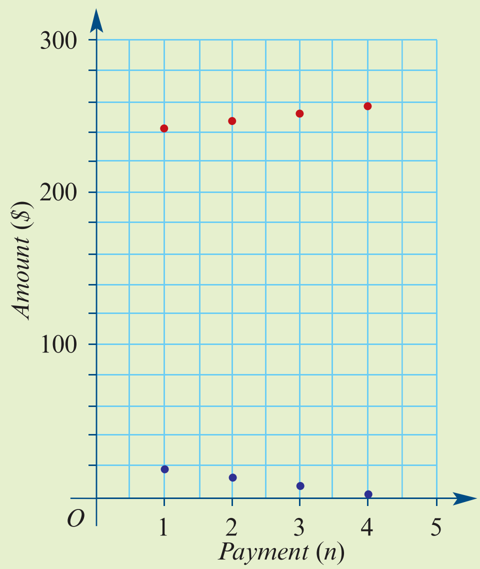
For this loan, the total cost of repaying the loan = 3 × 257.85 + 257.89 = \$1031.44.

5 total interest paid = total cost of repaying the loan − principal

For this loan, the total interest paid = 1031.44 − 1000 = \$31.44.

6 The amount of interest paid from each payment decreases with payment number while the amount of principal repaid increases.

For this loan, the graph on the right shows how the amount of interest paid each payment (blue dots) decreases with payment number, while the amount of principal paid increases (red dots).



Example 5 Reading and interpreting an amortisation table

A business borrows \$10 000 at a rate of 8% per annum. The loan is to be repaid by making four quarterly payments of \$2626.23. The amortisation table for this loan is shown below.

Payment number	*Payment*	*Interest*	*Principal reduction*	*Balance of loan*
0	0	0.00	0.00	10 000.00
1	2626.20		2426.20	7573.80
2	2626.20	151.48		5099.08
3	2626.20	101.98	2524.22	
4	2626.36*	51.50	2574.86	0
	Total		10 000.00	

*the final payment has been adjusted so that the final balance is zero.

a Determine, to the nearest cent:

i the quarterly interest rate

ii the interest paid when payment 1 is made using the quarterly interest rate

iii the principal repaid from payment 2

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iv the balance of the loan after payment 3 has been made

v the total cost of repaying the loan

b verify that the total amount of interest paid = total cost of the loan − principal

Solution

a **i** Quarterly rate = annual rate/4

Quarterly interest rate = 8/4
= 2% or 0.02

ii Interest paid = quarterly interest rate × unpaid balance

Interest paid = 2% of \$10 000 = \$200

iii Principal reduction = payment made − interest paid

Principal reduction = 2626.20 − 151.48
= \$2474.72

iv Balance of loan = balance owing − principal reduction

Balance of the loan after 3 payments
= 5099.08 − 2524.22 = \$2574.86

v The total cost of repaying the loan = sum of payments made

Total cost = 3 × 2626.20 + 2626.36
= \$10 504.96

b Use the rule: total interest paid = total cost of the loan − principal

Total interest = \$10 504.96 − 10 000
= \$504.96

Exercise 9B

Using a recurrence relation to analyse a reducing balance loan

1 A reducing-balance loan can be modelled by the recurrence relation

$$V_0 = 2500, \qquad V_{n+1} = 1.08V_n - 626.00$$

where V_n is the balance of the loan after n payments have been made.
The loan is to be fully repaid after five payments.

a Use your calculator to determine recursively the balance of the loan after three payments have been made. Give your answer to the nearest cent.

b Is the loan fully paid out after five payments of \$626 have been made? If not, how much will the last payment have to be to ensure that the loan is fully repaid after five payments?

2 A reducing-balance loan can be modelled by the recurrence relation:

$$V_0 = 5000, \qquad V_{n+1} = 1.01V_n - 862.70$$

The loan is to be fully repaid after six payments have been made.

a Use your calculator to determine recursively the balance of the loan after two payments have been made. Give your answer to the nearest cent.

b How much will the final payment have to be so that the loan is fully repaid after six payments?

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Modelling reducing-balance loans with recurrence relations

3 \$2000 is borrowed at an interest rate of 6% per annum compounding monthly. The loan will be repaid by making six monthly payments of \$339.
Let V_n be the balance of the loan after n payments have been made.

a Model this loan using a recurrence relation of the form:

$$V_0 = \text{the principal}, \qquad V_{n+1} = RV_n - D$$

b Use the recurrence relation to determine the balance of the loan after 4 months. Give your answer to the nearest cent.

c How much extra will the final payment need to be to ensure that the loan is fully repaid after 6 months?

4 \$10 000 is borrowed at an interest rate of 12% per annum compounding quarterly. The loan will be repaid the loan by making four quarterly payments of \$2690.27.
Let B_n be the balance of the loan after n payments have been made.

a Model this loan using a recurrence relation of the form:

$$B_0 = \text{the principal}, \qquad B_{n+1} = RB_n - D$$

b Use the recurrence relation to determine the balance of the loan after two payments have been made.

c To the nearest cent, does a quarterly payment of \$2690.27 ensure that the loan is fully repaid after four payments?

5 Let V_n be the balance of a loan after n payments have been made. Write down a recurrence relation model for the balance of a loan of:

a \$3500 borrowed at 4.8% per annum, compounding monthly, with payments of \$280 per month

b \$150 000 borrowed at 3.64% per annum, compounding fortnightly, with payments of \$650 per fortnight.

Reading and interpreting an amortisation table

6 A student borrows \$2000 at an interest rate of 12% per annum to pay off a debt. The loan is to be repaid by making six monthly payments of \$345.
The amortisation table for this loan is shown below.

Payment number	*Payment*	*Interest paid*	*Principal reduction*	*Balance of loan*
0	0	0.00	0.00	2000.00
1	345.00	20.00	325.00	1675.00
2	345.00		328.25	1346.75
3	345.00	13.47	331.53	1015.22
4	345.00	10.15		680.37
5	345.00	6.80	338.20	
6	345.60	3.42	342.18	0.00

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Determine, to the nearest cent:

a the monthly interest rate

b the interest paid when payment 2 is made using the monthly interest rate.

c the principal repaid from payment 4

d the balance of the loan after payment 5 has been made

e the total cost of repaying the loan

f determine the total amount of interest paid.

7 The amortisation table for a loan is shown below. The loan is to be repaid over 2 years by making eight quarterly payments.

Payment number	*Payment*	*Interest paid*	*Principal reduction*	*Balance of loan*
0	0	0.00	0.00	4000.00
1	557.85	100.00	457.85	3542.15
2	557.85	88.55	469.30	3072.85
3	557.85	76.82	481.03	2591.83
4	557.85	A	493.05	2098.77
5	557.85	52.47	B	1593.39
6	557.85	39.83	518.02	C
7	557.85	D	E	544.41
8	557.85	13.61	544.24	0.17

a Reading directly from the table:

i what is the principal of this loan?

ii what is the quarterly payment?

iii how much interest was paid from payment 1?

iv how much of the principal was repaid from payment 2?

v what is the balance of the loan after payment 3 is made?

vi what must the final payment for the loan to be fully repaid after 8 quarters?

b What is:

i the quarterly interest rate? **ii** the annual interest rate?

c Determine the values of A, B, C, D and E.

9C Using a finance solver to analyse reducing-balance loans

The recurrence relations that have been used to model reducing-balance loans are very convenient and easy to use, particularly if you need to analyse a reducing-balance loan that is repaid with only a small number of payments.

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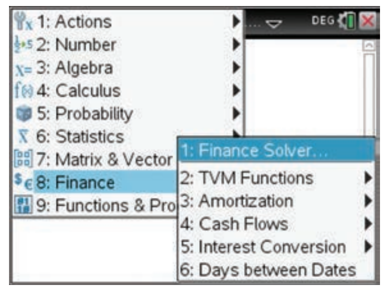
One of the most common types of reducing-balance loans are home loans. Large amounts of money are borrowed to buy property and then repaid with many payments over a number of years. A typical home loan would involve fortnightly or monthly payments for a period of 25 to 30 years. CAS calculators have a **Finance Solver** feature that makes analysing reducing-balance loans with a large number of payments much easier than by hand.

Using the Finance Solver on the TI-Nspire

Steps

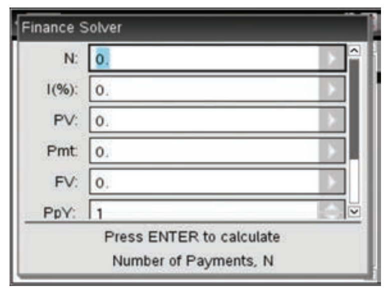
1 Start a new document by pressing ctrl + **N** (or on > **New Document**. You may be prompted to save your current document. (See Appendix A).

2 Select **Add Calculator**.
Press menu > **Finance** > **Finance Solver**.



3 To use Finance Solver you need to know the meaning of each of its symbols. These are as follows:

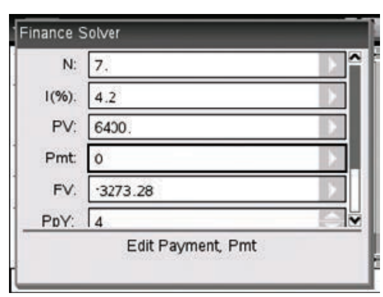
- **N** is the total number of payments.
- **I(%)** is the annual interest rate.
- **PV** is the present value of the loan.
- **Pmt** is the amount paid at each payment.
- **FV** is the future value of the loan.
- **PpY** is the number of payments per year.
- **CpY** is the number of times the interest is compounded per year. (It is almost always the same as **PpY**.)



Note: Use tab or to ▼ move down boxes. Press ▲ to move up. For **PpY** and **CpY** press tab to move down to the next entry box.

- **PmtAt** is used to indicate whether the interest is compounded at the end or at the beginning of the time period. Leave this set at **END**.

4 When using Finance Solver to solve loan and investment problems, there will be one unknown quantity. To find its value, move the cursor to its entry field and press enter to solve.
In the example shown pressing enter will solve for **Pmt**.



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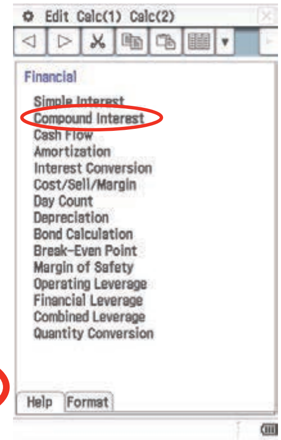
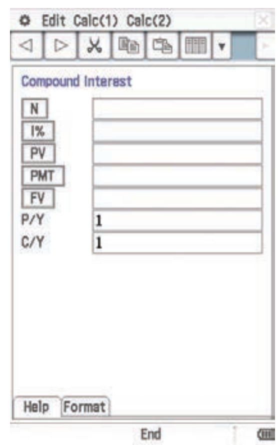
Using the Finance Solver on the Casio ClassPad

Steps

1. Tap **Financial** from the main menu screen.
2. Select the compound interest solver by tapping on **Compound Interest** from the solver screen.

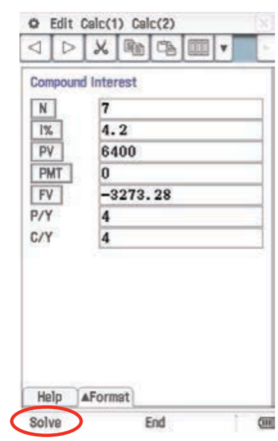


3. To use Finance Solver you need to know the meaning of each of its symbols. These are as follows:
   - **N** is the total number of payments.
   - **I%** is the annual interest rate.
   - **PV** is the present value of the loan or investment.
   - **PMT** is the amount paid at each payment.
   - **FV** is the future value of the loan or investment.
   - **ppY** is the number of payments per year.
   - **CpY** is the number of times interest is compounded per year. (It is almost always the same as **P/Y**.)



4. Tap **Format** and confirm that the setting for 'Odd Period' is set to 'off' and 'Payment Date' is set to 'End of period'.
5. When using Finance Solver to solve loan problems, there will be one unknown quantity. To find its value, tap its entry field and tap **Solve**.

   In the example shown tapping **Solve** will solve for **Pmt**.



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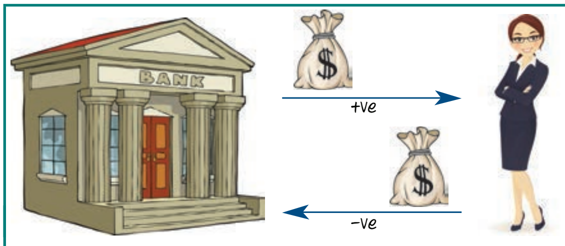
Using a finance solver to solve financial problems

A finance solver is a powerful is computation tool. However, you have to be very careful in the way you enter information because it needs to know which way the money is flowing. It does this by following a **sign convention**.

In general terms:

- if you receive money, or someone owes you money, we treat this as a positive (+ve)
- if you pay out money or you owe someone money, we treat this as a negative (−ve).

The sign convention for a reducing balance loan is summarised below. You will learn how to apply it in the following worked examples.

<table>
<tr><td colspan="2">
</td><td>Rules:
■ the bank gives you money: positive (+ve)
■ you give bank money: negative (−ve).</td></tr>
<tr><td>Reducing-balance loan</td><td>PV: positive
PMT: negative

FV: negative, zero or positive</td><td>Bank gives (lends) you money.
You repay the loan by making regular payments.
After the payment is made, you still owe the bank money (FV negative), the loan is fully paid out (FV is zero), or you have overpaid your loan and the bank needs to repay you some money (FV positive).</td></tr>
</table>

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Example 6 Determining the balance of a reducing-balance loan after a given number of payments

Andrew borrows $20 000 at an interest rate of 7.25% per annum, compounding monthly. This loan will be repaid over 4 years with payments of $481.25 each month.

a How much, correct to the nearest cent, does Andrew owe after 3 years?

b What is final payment amount Andrew must make to fully repay the loan after 4 years?

Solution

a 1 Open the Finance Solver on your calculator and enter the information below, as shown opposite.

- **N**: 36 (number of monthly payments in 3 years)
- **I%**: 7.25 (annual interest rate)
- **PV**: 20000 (positive to indicate that this is money received by Andrew from the lender)
- **Pmt** or **PMT**: −487.25 (negative to indicate that this is money that Andrew is giving back to the lender)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

N:	36
I%:	7.25
PV:	20000
Pmt or PMT:	-481.25
FV:	
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for the unknown future value (FV). On the:

- *TI-Nspire:* Move the cursor to the **FV** entry box and press to enter solve.
- *ClassPad:* Tap on the **FV** entry box and tap 'Solve'. The amount –5554.3626 . . . now appears in the **FV** entry box.

Note: A negative FV indicates that Andrew will still owe the lender money after the payment has been made.

N:	36
I%:	7.25
PV:	20000
Pmt or PMT:	-481.25
FV:	-5554.3626
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer, correct to the nearest cent

Andrew owes $5554.36.

b 1 Find the amount Andrew owes after 48 payments. Enter the information below, as shown opposite.

- **N**: 48 (number of monthly payments in 3 years)
- **I%**: 7.25 (annual interest rate)
- **PV**: 20000
- **Pmt** or **PMT** (the payment amount is negative): −481.25
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

N:	48
I%:	7.25
PV:	20000
Pmt or PMT:	-481.25
FV:	
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

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2 Solve for the unknown future value (FV). On the:

- *TI-Nspire:* Move the cursor to the **FV** entry box and press enter to solve.
- *ClassPad:* Tap on the **FV** entry box and tap **Solve**

The amount 0.1079... (11 cents) now appears in the **FV** entry box.

N:	48
I%:	7.25
PV:	20000
Pmt or PMT:	-481.25
FV:	0.107924
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Note: The FV is positive (+11 cents). This means that the bank owes Andrew 11 cents. Andrew has repaid the bank slightly more than required to repay the loan exactly. In this case the regular payment was rounded up to the nearest cent, meaning that each monthly payment was slightly more than required to repay the loan with exactly 48 payments. To compensate, Andrew's final payment will be reduced by 11 cents.

Final payment
$= \$481.25 - \$0.11$
$= \$481.14$

3 Write your answer.

Andrew's final payment will be \$481.14.

Example 7 Determining the payment amount, total cost and total amount of interest paid

Sipho borrows \$10 000 to be repaid in equal payments over a period of 5 years. Interest is charged at the rate of 8% per annum, compounding monthly.

Find:

a the monthly payment amount, correct to the nearest cent

b the total cost of repaying the loan to the nearest dollar

c the total amount of interest paid to the nearest dollar.

Solution

a **1** Open the Finance Solver on your calculator and enter the information below, as shown opposite.

- **N**: 60 (number of monthly payments in 5 years)
- **I%**: 8 (annual interest rate)
- **PV**: 10000
- **FV**: 0 (the balance will be zero when the loan is repaid)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

N:	60
I%:	8
PV:	10000
Pmt or PMT:	
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

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2 Solve for the unknown future value (Pmt or PMT).
On the:
- *TI-Nspire:* Move the cursor to the **Pmt** entry box and press enter to solve.
- *ClassPad:* Tap on the **PMT** entry box and tap **Solve**.

The amount –202.7639... now appears in the **Pmt** or **PMT** entry box.

Note: The sign of the payment is negative to indicate that this is money Sipho is giving back to lender.

N:	60
I%:	8
PV:	10000
Pmt or PMT:	-202.7639
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer.

Sipho repays \$202.76 every month.

b Total cost of repaying the loan = number of payments × payment amount

Note: Using the *unrounded* payment avoids the issue of having to separately calculate the last payment.

Total cost = number of payments × payment amount
= 60 × 202.7639 . . .
= 12 165.834
= \$12 166 (to the nearest dollar)

c Total interest paid = total cost of repaying the loan – the principal

Interest paid = total cost – principal
= 12 166 – 10 000
= \$2166 (to the nearest dollar)

Example 8 Reducing-balance loans with changing conditions

An amount of \$150 000 is borrowed for 25 years at an interest rate of 6.8% per annum, compounding monthly.

a What are the monthly payments for this loan?

b How much is still owing at the end of 3 years?

After 3 years, the interest rate rises to 7.2% per annum.

c What are the new monthly payments that will see the loan paid in 25 years?

d How much extra does it now cost (to the nearest 10 dollars) to repay the loan in total?

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Solution

a 1 Open the finance solver on your calculator and enter the information below, as shown opposite.

- **N**: 300 (number of monthly payments in 25 years)
- **I%**: 6.8 (annual interest rate)
- **PV**: 150000
- **FV**: 0 (the balance will be zero when the loan is repaid)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

Note: You can enter **N** as 25 × 12 (25 years of monthly payments). The finance solver will calculate this as 300 for you.

N:	300
I%:	6.8
PV:	150000
Pmt or PMT:	
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for **Pmt** or **PMT**.

N:	300
I%:	6.8
PV:	150000
Pmt or PMT:	-1041.1081
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer.

The payments on this loan are $1041.11 each month.

b 1 Change **N** to 3 × 12 or 36 (3 years of monthly payments).

N:	36
I%:	6.8
PV:	150000
Pmt or PMT:	-1041.1081
FV:	
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Clear **FV** and solve.

N:	36
I%:	6.8
PV:	150000
Pmt or PMT:	-1041.1081
FV:	-142391.91
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer.

After 3 years, there is $142 391.91 still owing on the loan.

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c If the loan is still to be repaid in 25 years, there are still 22 years left.

1 Change:

- **N** to 22 × 12 or 264 payments
- **I(%)** to 7.2 (the new interest rate)
- **PV** to 142391.91 (the balance after 3 years)
- **FV** to 0 (to pay out the loan).

N:	264
I%:	7.2
PV:	142391.91
Pmt or PMT:	
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Clear **Pmt** or **PMT** and solve.

N:	264
I%:	7.2
PV:	142391.91
Pmt or PMT:	-1076.1803
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer.

The new payments will be \$1076.18 each month.

d Work out the extra amount paid each month and multiply by 264, the number of months the higher interest rate applied. Round the answer to the nearest 10 dollars.

Extra paid each month:

$$\text{extra} = 1076.18 - 1041.11$$
$$= \$35.07$$

Total extra paid:

$$\text{total} = 264 \times 35.07$$
$$= \$9258.48$$
$$= \$9260$$

Exercise 9C

Determining the balance of a reducing-balance loan using a financial calculator

1 Use a financial calculator to find the balance, correct to the nearest cent, of each of the following reducing-balance loans after the given number of compounding periods.

	Principal	*Annual interest rate*	*Compounding*	*Payment per period*	*Balance after . . .*
a	\$8000	4.5%	Monthly	\$350	6 months
b	\$25 000	7.8%	Monthly	\$1200	1 year
c	\$240 000	8.3%	Quarterly	\$7900	5 years
d	\$75 000	6.9%	Quarterly	\$4800	2 years
e	\$50 000	4.6%	Weekly	\$350	1 year

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Determining the payment amount of a reducing-balance loan using a financial calculator

2 Use a financial calculator to find the payment amount, correct to the nearest cent, for each of the following reducing-balance loans.

	Principal	*Annual interest rate*	*Compounding*	*Time to fully repay the loan*
a	\$17 000	6.8%	Monthly	30 months
b	\$9500	4.2%	Monthly	2 years
c	\$2800	9.6%	Quarterly	6 quarters
d	\$140 000	8.6%	Quarterly	15 years
e	\$250 000	5.2%	Fortnightly	25 years

Determining the payment amount, total cost and total interest paid using a financial calculator

3 A loan of \$90 000 is to be repaid over a period of 30 years. Interest is charged at the rate of 11% per annum compounding monthly.
Find:

a the monthly payment correct to the nearest cent
b the total cost of paying off the loan to the nearest dollar
c the total amount of interest paid.

4 A building society offers \$240 000 home loans at an interest rate of 10.25% compounding monthly.

a If payments are \$2200 per month, calculate the amount still owing on the loan after 12 years. Write your answer, correct to the nearest cent.
b If the loan is to be fully repaid after 12 years, calculate:
  i the monthly payment, correct to the nearest cent
  ii the total amount repaid, correct to the nearest cent
  iii the total amount of interest paid, correct to the nearest cent.

5 Dan arranges to make payments of \$450 per month to repay a loan of \$20 000, with interest being charged at 9.5% per annum compounded monthly.
Find:

a the number of monthly payments required to pay out the loan (to the nearest month)
b the amount of interest charged.

6 Joan considers taking out a loan on the terms given in Question **5**. However, she decides that she can afford higher monthly payments of \$550.

a How long does it take her to pay off her loan (to the nearest month)?
b How much interest does Joan save by paying the higher monthly payment?

ISBN 978-1-107-56757-3

Reducing-balance loans with changing conditions

7 An amount of \$35 000 is borrowed for 20 years at 10.5% per annum compounded monthly. Write all answers in this question correct to the nearest cent.

a What are the payments for the loan?

b How much interest is paid on the loan over the 20-year period?

c How much is still owing at the end of 4 years?

After 4 years, the interest rate rises to 13.75% per annum.

d What are the new payments that will see the amount repaid in 20 years?

e How much extra must now be repaid on the loan over the term of 20 years?

8 A couple negotiates a 25-year mortgage of \$150 000 at a fixed rate of 7.5% per annum compounded monthly for the first 7 years, then at the market rate for the remainder of the loan. They agree to monthly payments of \$1100 for the first 7 years. Calculate:

a the amount still owing after the first 7 years

b the new monthly payments required to pay off the loan if after 7 years the market rate has risen to 8.5% per annum.

9 A couple puts a \$50 000 down-payment on a new home and arranges to pay off the rest in monthly payments of \$1384 for 30 years at a monthly compounded interest rate of 8.5% per annum. Write all answers to this question correct to the nearest cent.

a What was the selling price of the house?

b How much interest will they pay over the term of the loan?

c How much do they owe after 6 years?

After 6 years the interest rates increase by 0.9%. The couple must now extend the period of their loan in order to pay it back in full.

d How much will they still owe after the original 30-year period?

e Calculate the new monthly payment amount required if the couple still wishes to pay off the loan in 30 years.

ISBN 978-1-107-56757-3

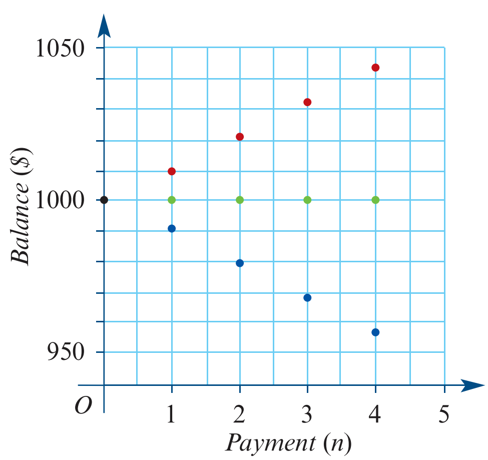
9D Interest-only loans

In an **interest-only loan**, the borrower repays only the interest that is charged. As a result, the value of the loan, that is the amount that must be paid back, remains the same for the duration of the loan. To understand how this happens, consider a loan of \$1000, charged 5% interest compounding yearly. The interest that is charged after 1 year will be 5% of \$1000, or \$50. If the borrower only pays the \$50 interest, the value of the loan will still be \$1000. The recurrence relation $V_0 = 1000,\ \ V_{n+1} = 1.05V_n - D$ can be used to model this loan.

The table below shows the balance of the loan over a 4-year period for three different payment amounts: $D = 40, D = 50$ and $D = 60$.

$D = 40$	$D = 50$	$D = 60$
$V_0 = 1000,$ $V_{n+1} = 1.05V_n - 40$	$V_0 = 1000,$ $V_{n+1} = 1.05V_n - 50$	$V_0 = 1000,$ $V_{n+1} = 1.05V_n - 60$
$V_0 = 1000$ $V_1 = 1010$ $V_2 = 1020.50$ $V_3 = 1031.525$ $V_4 = 1043.101\ldots$	$V_0 = 1000$ $V_1 = 1000$ $V_2 = 1000$ $V_3 = 1000$ $V_4 = 1000$	$V_0 = 1000$ $V_1 = 990$ $V_2 = 979.50$ $V_3 = 968.475$ $V_4 = 956.898\ldots$
The amount owed keeps growing.	The amount owed stays constant.	The amount owed keeps decaying.

We can plot these balances against the payment number.



If the periodic payments on this loan are smaller than \$50 (e.g. $D = 40$), the amount owed will increase over time. The balance of the loan is shown as red dots.

If the periodic payments on this loan are larger than \$50 (e.g. $D = 60$), the amount owed will decrease over time. The balance of the loan is shown as blue dots.

If the periodic payments on this loan are exactly \$50, then the amount owed on the loan will always be \$1000. The balance of the loan is shown as green dots.

This latter type of loan is called an interest-only loan. It can be useful in situations where the borrowed money is used to purchase an asset that is likely to increase in value, such as shares or property. Over time, the amount owed on the loan remains exactly the same as the borrowed amount, but if the asset bought with this money increases significantly in value, the whole amount can be repaid when it is sold – ideally with money left over!

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Interest-only loans

Let V_0 be the *principal* of the loan.

Let r be the interest *rate* per compounding period.

Let D be the regular payment per compounding period.

For interest-only loans:

payment = interest charged

so $$D = \frac{r}{100} \times V_0$$

Example 9 Repaying an interest-only loan

Jane borrows \$50 000 to buy some shares. Jane negotiates an interest-only loan for this amount, at an interest rate of 9% per annum, compounding monthly. What is the monthly amount Jane will be required to pay?

Solution

Use the rule $D = \frac{r}{100} \times V_0$.

Here

1 V_0 is the amount borrowed = \$50 000

2 r is the monthly interest rate $\left(\frac{9\%}{12} = 0.75\%\right)$

3 Evaluate the rule for these values and write your answer.

$V_0 = 50\,000$

$r = 0.75\%$

$D = \frac{0.75}{100} \times V_0$

$D = \frac{0.75}{100} \times 50\,000$

$D = 375$

Jane will need to repay \$375 every month on this interest-only loan.

Problems involving interest-only loans can be solved using a financial calculator.

Example 10 Repaying an interest-only loan using a financial calculator

Stuart borrows \$180 000 to buy a house. He negotiates an interest-only loan for this amount, at an interest rate of 7.6% per annum, compounding fortnightly. What is the fortnightly payment, correct to the nearest cent?

Solution

We will consider just one compounding period because all compounding periods will be identical.

1 Open the Finance Solver on your calculator and enter the information below, as shown opposite.

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- **N**: 1 (one compounding period)
- **I%**: 7.6 (annual interest rate)
- **PV**: 180000
- **FV**: −180000 (the amount owing will be the same after one payment)
- **Pp/Y**: 26 (fortnightly payments)
- **Cp/Y**: 26 (interest compounds fortnightly)

N:	1
I%:	7.6
PV:	180000
Pmt or PMT:	
FV:	-180000
Pp/y or P/Y:	26
Cp/Y or C/Y:	26

2 Solve for the unknown future value (Pmt or PMT). On the:

- *TI-Nspire:* Move the cursor to the **Pmt** entry box and press enter to solve.
- *ClassPad:* Tap on the **PMT** entry box and tap **Solve**.

The amount –526.1538… now appears in the **Pmt** or **PMT** entry box.

N:	1
I%:	7.6
PV:	180000
Pmt or PMT:	-526.15384
FV:	-180000
Pp/y or P/Y:	26
Cp/Y or C/Y:	26

3 Write your answer, rounding as required.

Stuart will need to repay $526.15 every fortnight.

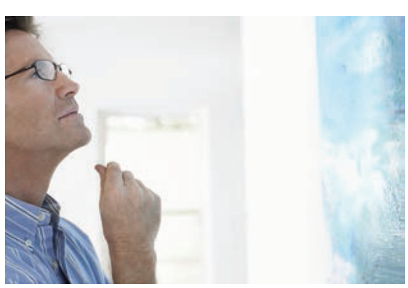
Exercise 9D

Interest-only loans

1 Georgia borrows $100 000 to buy an investment property. If the interest on the loan is 7.15% per annum, compounding monthly, what will be her monthly payment on an interest-only loan?

2 In order to invest in the stockmarket, Jamie takes out an interest-only loan of $50 000. If the interest on the loan is 8.15% per annum compounding monthly, what will be his monthly payments?

3 Jackson takes out an interest-only loan of $30 000 from the bank to buy a painting, which he hopes to resell at a profit in 12 months' time. The interest on the loan is 9.25% per annum, compounding monthly, and he makes monthly payments on the loan. How much will he need to sell the painting for in order not to lose money?



ISBN 978-1-107-56757-3

9E Annuities

The calculations used to model the values of reducing-balance loans and **annuities** are identical. The only difference is that the value of the loan represents how much is owed at any time, whereas the value of the annuity represents how much money that is left in the investment.

Modelling an annuity

Let r be the interest *rate* per compounding period.

Let D be the payment received.

A recurrence relation that can be used to model the value of an annuity after n payments, V_n, is

$$V_0 = \text{principal},\ V_{n+1} = RV_0 - D$$

where $R = 1 + \dfrac{r}{100}$

Example 11 Modelling an annuity with a recurrence relation

Reza plans to travel overseas for 6 months. He invests \$12 000 in annuity that earns interest at the rate of 6% per annum, providing him with a monthly income of \$2035 per month for 6 months.

Model this annuity using a recurrence relation of the form

$$V_0 = \text{the principal}, \quad V_{n+1} = RV_n - D$$

where V_n is the value of the annuity after n payments have been received.

Solution

1 Define the symbol V_n.

Let V_n be the value of the annuity after n payments have been received.

2 Write down the value of V_0. Here V_0 is the principal of the annuity, the original amount of money invested.

$V_0 = 12\,000$

3 Because interest compounds monthly, calculate the monthly interest rate.

$r = \dfrac{6\%}{12} = 0.5\%$ or 0.005

4 Determine the value of R. Either recognise that a 0.5% increase gives an R of 1.005 or use the formula $R = 1 + r/100 = R = 1 + 5/100 = 1.005$.

$R = 1.005$

5 Write down the value of D. In this context, D is the payment received each month.

$D = 2035$

6 Use the values of V_0, R and D to write down the recurrence relation.

$V_0 = 12\,000$

$V_{n+1} = 1.005V_n - 2035$

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Once we have a recurrence relation, we can use it to determine things such as the value of the annuity of a loan after making a given number of payments have been received.

Example 12 Using a recurrence relation to analyse an annuity

Reza's annuity can be modelled by the recurrence relation

$$V_0 = 12000, \qquad V_{n+1} = 1.005V_n - 2035$$

where V_n is the value of the annuity after n payments have been received.

a Use your calculator to determine recursively the value of the annuity after Reza has received three payments from the annuity.

b Is the annuity fully paid out after six monthly payments have been made? If not, how much will the last payment have to be to ensure that the annuity terminates after 6 months?

Solution

a **i** Write down the recurrence relation.

ii Type **12000** and press enter or EXE.

iii Type **× 1.005-2035** and press enter or EXE six times to obtain the screen opposite.

iv Read the value from the screen. Round the answer to the nearest cent.

$V_0 = 12000, \qquad V_{n+1} = 1.005V_n - 2035$

12000. · 1.005 – 2035	10025.
10025. · 1.005 – 2035	8040.13
8040.125 · 1.005 – 2035	6045.33
6045.325625 · 1.005 – 2035	4040.55
4040.552253125 · 1.005 – 2035	2025.76
2025.7550143906 · 1.005 – 2035	0.883789

\$6045.33 (to the nearest cent)

b No. There will still be 0.88 cents left in the annuity. To terminate the annuity, the 88 cents is added to Reza's final payment.

$$\text{Last payment} = 2035 + 0.88 = \$2035.88$$

ISBN 978-1-107-56757-3

Amortisation tables for annuities

An amortisation table can be used to summarise the key properties of an annuity.

An amortisation table for an annuity shows the payment number, the payment received, the interest earned, the principal reduction and the balance of the annuity after each payment has been received. The amortisation table for Reza's annuity is shown below.

Payment number	*Payment received*	*Interest earned*	*Principal reduction*	*Balance of annuity*
0	0	0.00	0.00	12000.00
1	2035.00	60.00	1975.00	10025.00
2	2035.00	50.13	1984.88	8040.13
3	2035.00	40.20	1994.80	6045.33
4	2035.00	30.23	2004.77	4040.55
5	2035.00	20.20	2014.80	2025.76
6	2035.88	10.13	2024.87	0.00

Note: The last payment was increased by 88 cents so that the balance of the annuity was reduced to exactly $0.00 after six payments had been made. This was because the payment amount must be rounded to the nearest cent. Making such an adjustment is normal when amortising an annuity.

The amortisation table above can be used to summarise the key properties of an annuity.

Properties of an annuity

At each step of the loan:

1 interest earned = interest rate per compounding period × balance.
For example, when payment 1 is received, interest paid = 0.5% of 12 000 = $60.

2 principal reduction = payment made − interest earned
For example, when payment 1 is made, principal reduction = 2035 − 60 = $1975.

3 balance of loan = previous balance − principal reduction
For example, when payment 1 is made, reduction in balance = 12000 − 1975
= $10 025

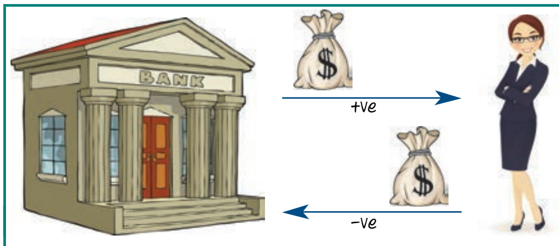
4 total return from the annuity = the sum of the payments made
For this annuity, the return from the investment = 5 × 2035 + 2035.88 = $12 210.90.
Note: This amount can also be obtained by summing the payment column.

5 total interest paid = total cost of repaying the loan − principal
For this annuity, the total interest paid = 12 210.90 − 12 000 = $210.90.
Note: This amount can also be obtained by summing the interest column.

ISBN 978-1-107-56757-3

Using a finance solver to analyse an annuity

The sign convention for an annuity is summarised below. You will learn how to apply the sign convention in the worked examples that follow.

<table>
<tr>
<td colspan="2">
</td>
<td>Rules:
■ the bank gives you money: positive (+ve)
■ you pay the bank money: negative (−ve).</td>
</tr>
<tr>
<td>Annuity</td>
<td>PV: negative

PMT: positive

FV: positive or zero</td>
<td>You buy an annuity. To do this you must give the bank some money.
You will receive regular payments from the bank.
After each payment is made, the bank will continue to make future payments to you (FV positive), or the annuity is fully paid out (FV is zero) and no further payments will be made.</td>
</tr>
</table>

Example 13 **Analysing an annuity using a finance solver**

Joe invests \$200 000 into an annuity, paying 5% compound interest per annum, compounding monthly.

1 If he wishes to be paid monthly payments for 10 years, how much will he receive each month?

2 If he receives a regular monthly payment of \$3000, long will the annuity last? Give your answer correct to the nearest month.

3 What interest rate, correct to one decimal place, would allow Joe to withdraw \$2500 each month for 10 years?

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Solution

a 1 Open the finance solver on your calculator and enter the information below, as shown.

- **N**: 120 (10 years)
- **I%**: 5 (annual interest rate)
- **PV**: −200000
- **FV**: 0 (the annuity will be exhausted after 10 years)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

N:	120
I%:	5
PV:	-200000
Pmt or PMT:	
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for **Pmt** or **PMT**.

Note: The sign of Pmt or PMT is positive, because it is money received.

N:	120
I%:	5
PV:	-200000
Pmt or PMT:	2121.3103
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer.

Joe will receive $2121.31 each month from the annuity.

b 1 Change the payment **Pmt** or **PMT** to 3000 and solve for **N**.

N:	78.2639745
I%:	5
PV:	-200000
Pmt or PMT:	3000
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Write your answer, round to the nearest month.

Joe's annuity will last 78 months.

c 1 Change the **Pmt** or **PMT** value to 2500 and the number of withdrawals, **N** to 120 (10 years). Solve for I%.

N:	120
I%:	8.68922416
PV:	-200000
Pmt or PMT:	2500
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Write your answer.

Joe would require an interest rate of 8.7% per annum to make monthly withdrawals of $2500 for 10 years.

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Exercise 9E

Using a recurrence relation to analyse an annuity

1 An annuity can be modelled by the recurrence relation

$$V_0 = 5000, \qquad V_{n+1} = 1.01V_n - 1030$$

where V_n is the balance of the annuity after n monthly payments have been made. The annuity is to be fully repaid after five payments.

a Use your calculator to determine recursively the balance of the annuity after three payments have been made. Give your answer to the nearest cent.

b Is the annuity fully paid out after five payments of \$1030 have been received? If not, how much will the last payment have to be to ensure that the balance of the annuity is zero after five payments?

2 An annuity can be modelled by the recurrence relation

$$V_0 = 6000, \qquad V_{n+1} = 1.005V_n - 1518$$

where V_n is the balance of the annuity after n payments have been made. The annuity is to be fully repaid after four payments.

a Use your calculator to determine recursively the balance of the annuity after the two payments have been made. Give your answer to the nearest cent.

b Is the annuity fully paid out after four payments of \$1518 have been received? If not, how much will the last payment have to be to ensure that the balance of the annuity is zero after five payments?

Using a recurrence relation to model an annuity

3 Helen invests \$40 000 in an annuity paying interest at the rate of 6% per annum, compounding quarterly. She receives a payment of \$10 380 each quarter for 1 year. Let V_n be the balance of the loan after n payments have been received.

a Model this loan using a recurrence relation of the form:

$$V_0 = \text{the principal}, \quad V_{n+1} = RV - D$$

b Use the recurrence relation to determine the balance of the annuity after 6 months. Give your answer to the nearest cent.

ISBN 978-1-107-56757-3

Reading and interpreting an amortisation table

4 A university student is given a living allowance of \$6000 for her first year of study. She invests the money in an annuity paying an interest rate of 3% per annum, compounding monthly. From this annuity, she receives a monthly payment of \$508. The amortisation table for this annuity is shown below.

Payment number	*Payment received*	*Interest earned*	*Principal reduction*	*Balance of annuity*
0	0	0.00	0.00	6000.00
1	508.00	15.00	493.00	5507.00
2	508.00	13.77	494.23	5012.77
3	508.00	12.53	495.47	4517.30
4	508.00	11.29	496.71	4020.59
5	508.00	10.05	497.95	3522.64
6	508.00	A	B	C
7	508.00	7.56	500.44	2523.01
8	508.00	6.31	501.69	2021.32
9	508.00	5.05	502.95	1518.37
10	508.00	3.80	504.20	1014.17
11	508.00	2.54	505.46	508.70
12	508.00	1.27	506.73	1.97

Determine, to the nearest cent:

a the monthly interest rate

b the interest earned when payment 1 is received

c the principal reduction when payment 3 is received

d the balance of the annuity after payment 5 has been received

e the values of A, B and C

f the value of the last payment if the balance of the annuity is to be zero after the 12th payment is received

g the total return from the annuity

h the total amount of interest earned.

Solving annuity problems with a financial solver

5 Leigh invests \$64 000 in an annuity, with interest of 6.25% per annum, compounding monthly. If he receives payments of \$1275 per month, how long will this annuity last? Write your answer, correct to the nearest month.

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6 Raj invests \$85 500 in an annuity, with interest of 7.25% per annum, compounding quarterly. If Raj receives a regular quarterly payment of \$5000, how long will the annuity last?
Write your answer, correct to the nearest quarter.

7 Stephanie invests \$40 000 in an annuity, with interest paid at 7.5% per annum compounded monthly. If she wishes to receive a monthly payment for 10 years, how much will she receive each month, correct to the nearest cent?

8 Helen has \$80 000 to invest. She chooses an annuity that pays interest at the rate of 6.4% per annum, compounding monthly. Helen expects her investment to be fully exhausted after 15 years.

- **a** Find the monthly withdrawal that Helen can make, correct to the nearest cent
- **b** Find the amount Helen has left in the investment after 2 years, correct to the nearest cent

After 2 years, the interest rate of Helen's investment was reduced to 6.2% per annum, compounding monthly.

- **c** If Helen continues to withdraw the same amount from the investment each month, how many more months, correct to the nearest month, will her investment last?
- **d** If Helen would like her investment to still be exhausted after 15 years in total, what is the new monthly withdrawal, correct to the nearest cent, that she can make?

9 Sameep has \$150 000 to invest. He deposits the money into a savings account earning 6.2% per annum, compounding monthly, for 10 years. He makes no withdrawals during that time.

- **a** How much money, correct to the nearest cent, is in this account after 10 years?
  Sameep invests the total amount in his savings account in an annuity investment. He will require this investment to provide monthly withdrawals of \$2800.
- **b** What annual interest rate, correct to two decimal places, is required if Sameep's investment is to be exhausted after 12 years?

ISBN 978-1-107-56757-3

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9F Perpetuities

An annuity will earn interest after every compounding period. If the payment received after each compounding period is smaller than this interest, the annuity will continue to grow. If the payment received after each compounding period is larger than this interest, the annuity will decay until nothing is left.

If the payment received is exactly the same as the interest earned in one compounding period, the annuity will maintain its value indefinitely. This type of annuity is called a **perpetuity**. Payments of the same amount as the interest earned can be made *in perpetuity*, or forever. Perpetuities have exactly the same relationship to annuities as interest-only loans have to reducing-balance loans.

Perpetuities

Let V_0 be the initial value of the annuity.

Let r be the interest *rate* per compounding period.

Let D be the regular payment received.

For perpetuities:

payment received = interest earned

so

$$D = \frac{r}{100} \times V_0$$

Example 14 Calculating the withdrawal from a perpetuity

Elizabeth invests her superannuation payout of $500 000 into a perpetuity that will provide a monthly income without using any of the initial investment.

If the interest rate for the perpetuity is 6% per annum, what monthly payment will Elizabeth receive?

Solution

1 Write down the values of V_0 and r.

Elizabeth invests $500 000 so V = 500 000.

The annual interest rate is 6% so $r = \frac{6}{12} = 0.5\%$.

2 Find the monthly interest earned.

$$D = \frac{r}{100} \times V_0$$

$$D = \frac{0.5}{100} \times 500000$$

$$D = 2500$$

3 Write your answer, rounding as required.

Elizabeth will receive $2500 every month from her investment.

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Example 15 Calculating the investment required to establish a perpetuity

How much money will need to be invested in a perpetuity account, earning interest of 4.2% per annum compounding monthly, if $200 will be withdrawn every month? Write your answer to the nearest dollar.

Solution

1 Write down the values of r and D.

The annual interest rate is 4.2%,

so $r = \frac{4.2}{12} = 0.35\%$.

The amount withdrawn each month is \$200 so $D = 200$.

2 Use the rule $D = \frac{r}{100} \times V_0$ to write down an expression that can be solved for V_0.

$$200 = 0.0035 \times V_0$$

$$V_0 = \frac{200}{0.0035}$$

$$= 57\,142.867$$

3 Write your answer.

\$57 143 will need to be invested to establish the perpetuity investment.

Problems involving perpetuities can be solved using a financial calculator.

Example 16 Calculating the interest rate of a perpetuity

A university mathematics faculty has $30 000 to invest. It intends to award an annual mathematics prize of $1500 with the interest earned from investing this money in a perpetuity.

What is the minimum interest rate that will allow this prize to be awarded indefinitely?

Solution

Using a financial solver

We will consider just one compounding period because all compounding periods will be identical.

1 Open the Finance Solver on your calculator and enter the information below.

- **N**: 1 (one payment)
- **PV**: –30 000
- **Pmt or PMT**: 1500 (prize is $1500 each year)
- **FV**: 30 000 (the balance will be the same after each payment)
- **Pp/Y**: 1 (yearly payment)
- **Cp/Y**: 1 (interest compounds yearly)

N:	1
I%:	
PV:	-30000
Pmt or PMT:	1500
FV:	30000
Pp/y or P/Y:	1
Cp/Y or C/Y:	1

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2 Solve for the unknown interest rate (**I%**). On the:

- *TI-Nspire:* Move the cursor to the **I%** entry box and press enter to solve.
- *ClassPad:* Tap on the **I%** entry box and tap **Solve** The amount **5** now appears in the **I%** entry box.

N:	1
I%:	5
PV:	-30000
Pmt or PMT:	1500
FV:	30000
Pp/y or P/Y:	1
Cp/Y or C/Y:	1

3 Write your answer, rounding as required.

The minimum annual interest rate to award this prize indefinitely is 5%.

Alternative method

The prize is equal to the $r\%$ of the amount invested.

1 Set up an equation with r.

Prize = $r\%$ of investment

$$1500 = r\% \times 30\,000$$

$$1500 = \frac{r}{100} \times 30\,000$$

2 Solve the equation for r.

$$1500 = r \times 300$$

$$r = \frac{1500}{300} = 5$$

3 Write your answer.

The minimum annual interest rate to award this prize indefinitely is 5%.

Exercise 9F

Calculating the principal and interest rate of a perpetuity

1 Geoff wishes to set up a fund so that every year \$2500 is donated to the RSPCA in his name.

a If the interest on his initial investment averages 2.5% per annum, compounding annually, how much should he invest?

b If Geoff has \$80 000 to invest, what is the minimum interest rate he requires to provide for the donation in perpetuity?

2 Barbara wishes to start a scholarship that will reward the top mathematics student each year with a \$500 prize.

a If the interest on the initial investment averages 2.7% per annum, compounding annually, how much should be invested? Write your answer, correct to the nearest dollar.

b Barbara has \$12 000 to invest in a perpetuity to provide this prize. What is the minimum interest rate, correct to one decimal place, that she requires in order to pay the prize in perpetuity?

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3 Cathy wishes to maintain an ongoing donation of \$5500 per year to the Collingwood Football Club. If the interest on the initial investment averages 2.75% per annum, compounding annually, how much should she invest?

Calculating the payment from a perpetuity

4 Craig wins \$1 000 000 in a lottery and decides to place it in a perpetuity that pays 5.75% per annum interest, compounding monthly. What monthly payment, correct to the nearest cent, does he receive?

5 Suzie invests her inheritance of \$642 000 in a perpetuity that pays 6.1% per annum compounding quarterly.

- **a** What quarterly payment does she receive?
- **b** After five quarterly payments, how much money remains invested in the perpetuity?
- **c** After 10 quarterly payments, how much money remains invested in the perpetuity?

9G Compound interest investments with regular additions to the principal (annuity investments)

Suppose you have some money invested in an account that pays **compound interest**. To increase the rate at which your investment grows, you decide to add to your investment by making additional payments on a regular basis.

Modelling compound interest investments with regular additions to the principal

Let r be the *interest rate* per compounding period.

Let D be the *payment* made.

A recurrence relation that can be used to model the value of a compound interest investment, V_n, after n additional payments have been made is

$$V_0 = \text{the principal}, \quad V_{n+1} = RV_n + D$$

where $R = 1 + \dfrac{r}{100}$.

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Example 17 Modelling compound interest investments with additions to the principal

Nor plans to travel overseas when she finishes her VCE. She has already saved $1200 and thinks that she can save an additional $50 each month that she plans to add to her savings account. The account pays interest at a rate of 3% per annum, compounding monthly.

Model this investment using a recurrence relation of the form

$$V_0 = \text{the principal}, \quad V_{n+1} = RV_n + D$$

where V_n is the value of the investment after n payments (additions to the principal) have been made.

Solution

1 Write down and define the variable V_n.	Let V_n be the value of the annuity after n payments.
2 Write down the value of V_0. Here, V_0 is the principal of the annuity, the original amount of money invested.	$V_0 = 1200$
3 Because interest compounds monthly, calculate the monthly interest rate.	$r = \frac{3\%}{12} = 0.25\%$ or 0.0025
4 Determine the value of R. Either recognise that a 0.25% increase gives an R of 1.0025, or use the formula $R = 1 + r/100$, so $R = 1 + 2.5/100 = 1.0025$.	$R = 1.0025$
5 Write down the value of D. In this context, D is the amount added each month.	$D = 50$
6 Use the values of V_0, R and D to write down the recurrence relation.	$V_0 = 50$ $V_{n+1} = 1.0025V_n + 50$

Once we have a recurrence relation, we can use it to determine things such as the value of the investment after a given number of additional payments have been made.

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Example 18 **Using a recurrence relation to analyse compound interest investments with additions to the principal**

Nor's investment can be modelled by the recurrence relation

$$V_0 = 1200, \qquad V_{n+1} = 1.0025V_n + 50$$

where V_n is the value of the investment after n payments have been received.

a Use your calculator to determine recursively the value of the investment after Nor has made five additional payments to her investment.

b What will be the value of her investment after 1 year?

Solution

a **i** Write down the recurrence relation.

$V_0 = 1200, \quad V_{n+1} = 1.0025V_n + 50$

ii Type 12000 and press enter (or EXE).

iii Type × 1.0025 + 50 and press enter (or EXE) three more times to obtain the screen opposite.

1200	1200
1200 · 1.0025 + 50	1253
1253 · 1.0025 + 50	1306.13
1306.1325 · 1.0025 + 50	1359.4

iv Write you answer.

After three additional payments, the value of Nor's investment is $1359.40.

b **i** Continue pressing enter or EXE nine more times to obtain the screen opposite. The value of the investment after 12 additional payments can now be read from the screen.

1466.3283166427 · 1.0025 + 50	1519.99
1519.9941374343 · 1.0025 + 50	1573.79
1573.7941227779 · 1.0025 + 50	1627.73
1627.7286080848 · 1.0025 + 50	1681.8
1681.797929605 · 1.0025 + 50	1736.
1736.002424429 · 1.0025 + 50	1790.34
1790.3424304901 · 1.0025 + 50	1844.82

ii Write your answer.

After 1 year, the value of Nor's investment is $1844.82.

Amortisation tables for compound interest investments with additions to the principal

An amortisation table for an annuity shows the payment number, the payment made, the interest earned, the principal increase and the balance of the investment after each payment has been received. The amortisation table for Nor's investment annuity follows.

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Payment number	*Payment made*	*Interest earned*	*Principal increase*	*Balance of investment*
0	0.00	0.00	0.00	1200.00
1	50.00	3.00	53.00	1253.00
2	50.00	3.13	53.13	1306.13
3	50.00	3.27	53.27	1359.40
4	50.00	3.40	53.40	1412.80
5	50.00	3.53	53.53	1466.33
6	50.00	3.67	53.67	1519.99
7	50.00	3.80	53.80	1573.79
8	50.00	3.93	53.93	1627.73
9	50.00	4.07	54.07	1681.80
10	50.00	4.20	54.20	1736.00
11	50.00	4.34	54.34	1790.34
12	50.00	4.48	54.48	1844.82

The amortisation table above can be used to summarise the key properties of a compound interest investment with addition payments.

Properties of an investment

At each step of the investment:

1 interest earned = interest rate per compounding period × previous balance

For example, when payment 1 is made, interest paid = 0.25% of 1200 = \$3.00.

2 principal increase = payment made + interest earned

For example, when payment 1 is made principal increase = 3.00 + 50.00 = \$53.00.

3 Balance of investment = previous balance + interest + principal increase

For example, when payment 1 is made, the new balance is 1200.00 + 3.00 + 50.00 = \$1253.00.

4 total interest earned = balance of loan − (principal + additional payments)

After 12 months, the total interest earned = 1844.82 − (1200 + 12 × 50) = \$44.82

Note: This amount can also be obtained by summing the interest column.

Using a financial solver to analyse a compound interest investment with regular additions to the principal

The sign convention for a compound interest loan with additions to the principal is summarised below. You will learn how to apply the sign convention in the worked example that follows.

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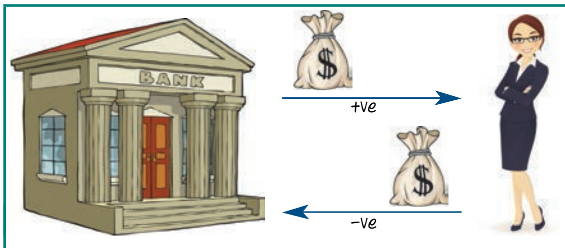
		Rules: ■ the bank gives you money: positive (+ve) ■ you pay the bank money: negative (−ve).
Adding to an investment	PV: negative	You make an investment. To do this you must give the bank some money (PV is negative).
	PMT: negative	You make regular payments to the bank to increase your investment. Again, you are giving money to the bank (PMT is negative).
	FV: positive	When your investment matures, the bank will return the money to you (FV positive).

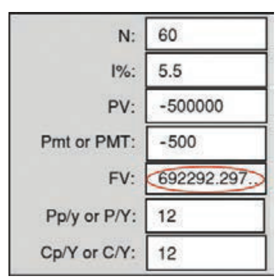
Example 19 Determining the value of an investment with regular additions made to the principal using a financial solver

Lars invests \$500 000 at 5.5% per annum, compounding monthly. He makes a regular deposit of \$500 per month into the account. What is the value of his investment after 5 years?

Solution

1 Open the Finance Solver on your calculator and enter the information below, as shown opposite.

- **N**: 60 (5 years)
- **I%**: 5.5
- **PV**: –500 000
- **PMT**: –500
- **FV**: to be determined
- **Pp/Y**: 12 payments per year
- **Cp/Y**: 12 compounding periods per year



2 Solve for FV and write your answer correct to the nearest cent.

After 5 years, Lar's investment will be worth \$692 292.30.

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Exercise 9G

Using a recurrence relation to model and analyse an investment with additions to the principal

1 A compound interest loan with regular additions to the principal can be modelled by the recurrence relation

$$V_0 = 2000, \qquad V_{n+1} = 1.08V_n + 1000$$

where V_n is the value of the investment after n yearly payments (additions to the principal) have been made.

- **a** Use your calculator to determine recursively the balance of the investment after 5 years. Give your answer to the nearest cent.
- **b** What is the principal of this investment?
- **c** How much is added to the principal each year?
- **d** What is the annual interest rate?

2 A compound interest loan with regular additions to the principal can be modelled by the recurrence relation

$$V_0 = 20\,000, \qquad V_{n+1} = 1.025V_n + 2000$$

where V_n is the value of the investment after n quarterly payments (additions to the principal) have been made.

- **a** Use your calculator to determine recursively the balance of the investment after three quarterly payments have been made. Give your answer to the nearest cent.
- **b** What is the principal of this investment?
- **c** How much is added to the principal each quarter?
- **d** What is the quarterly interest rate? What is the annual interest rate?

3 Sarah invests \$1500 at 9% per annum, compounding monthly. She plans to add an additional \$40 to her investment each month.
Let V_n be the value of the investment after Sarah has made n monthly payments to her investment.
Write down a recurrence relation to model Sarah's investment.

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Reading and interpreting an amortisation table

4 The amortisation table below charts the growth of a compound interest investment with regular additions made to the principal each month.

Payment number	*Payment received*	*Interest earned*	*Principal reduction*	*Balance of annuity*
0	0.00	0.00	0.00	5000.00
1	100.00	50.00	150.00	5150.00
2	100.00	51.50	151.50	5301.50
3	100.00	A	B	C
4	100.00	54.55	154.55	5609.06
5	100.00	56.09	156.09	5765.15
6	100.00	57.65	157.65	5922.80
7	100.00	59.23	159.23	6082.03
8	100.00	60.82	160.82	6242.85

a Write down:

- **i** the original amount invested
- **ii** the amount added to the principal each month
- **iii** the amount of interest earned in month 1
- **iv** the monthly interest rate
- **v** the annual interest rate
- **vi** the principal increase in month 2
- **vii** value of the investment after 8 months.

b Determine, to the nearest cent:

- **i** the values of A, B and C
- **ii** the total interest earned by the investment after 8 months.

Solving problems involving compound interest loans with additions to the principal using a financial solver

5 Lee invests \$12 000 at 5.7% per annum, compounding monthly. He makes a regular deposit of \$250 per month into the account. What is the value of his investment after 10 years?

6 A sporting club invests \$10 000 at 5.2% per annum, compounding monthly. The club plans to add \$200 to the loan each month. How long will it take for the investment to be worth \$12 000? Give your answer to the nearest month.

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7 Bree has \$25 000 in an account that pays interest at a rate of 6.15% per annum compounding monthly.

a If she makes monthly deposits of \$120 to the account, how much will she have in the account at the end of 5 years?

b If she makes monthly withdrawals of \$120 from the account, how much will she have in the account at the end of 5 years?

8 Jarrod saves \$500 per month in an account that pays interest at a rate of 6% per annum, compounding monthly.

a If he makes monthly deposits of \$500 to the account, how much will he have in the account at the end of 10 years?

b Suppose that, after 10 years of making deposits, Jarrod starts withdrawing \$500 each month from the account. How much will he have in the account at the end of another 10 years?

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Key ideas and chapter summary

Reducing-balance loan A **reducing-balance loan** is a loan that attracts compound interest but is reduced in value by making regular payments.
Each payment partly pays the interest that has been added and partly reduces the value of the loan.

Interest-only loan An **interest-only loan** is a loan where the regular payments made are equal in value to the interest charged. Interest-only loans have the same value after each payment is made.

Annuity An **annuity** is an investment that earns compound interest and from which regular payments are made.

Perpetuity A **perpetuity** is an annuity where the regular payments or withdrawals are the same as the interest earned. The value of a perpetuity remains constant.

Amortisation An amortising loan is one that is paid back with periodic payments. An amortising investment is one that is exhausted by regular withdrawals.
Amortisation of reducing-balance loans tracks the distribution of each periodic payment, in terms of the interest paid and the reduction in the value of the loan.
Amortisation of an annuity tracks the source of each withdrawal, in terms of the interest earned and the reduction in the value of the investment.

Amortisation table An **amortisation table** shows the amortisation (payment) of all or part of a reducing-balance loan or annuity. It has columns for the payment number, the payment amount, the interest paid or earned, the principal reduction and the balance after the payment has been made.

Finance Solver **Finance Solver** is a function on a CAS calculator that performs financial calculations. It can be used to determine any of the principal, interest rate, periodic payment, future value or number of payments given all of the other values.

Skills check

Having completed this chapter, you should be able to:

- use a given first-order linear recurrence relation to generate the terms in a sequence
- understand the concept of a first-order linear recurrence relation and its use in generating the terms in a sequence
- model problems of growth and decay in financial contexts using a first-order recurrence relation of the form $u_0 = a$, $u_{n+1} = bu_n + c$

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Review

- demonstrate the use of a recurrence relation to determine the depreciating value of an asset or the future value of an investment or a loan after n time periods
- use an amortisation table to investigate and analyse on a payment-by-payment basis the amortisation of a reducing-balance loan or an annuity
- with the aid of a financial solver, solve practical problems associated with reducing balance loans, annuities and perpetuities, and annuity investments.

Multiple-choice questions

1 The wombats in a national park are breeding at a rate that increases their population by 8% each year. On average, 60 wombats die each year. There are currently 490 wombats in the national park. A recurrence relation that can be used to estimate the number of wombats in the national park after n years is:

A $W_0 = 490,\ W_{n+1} = 1.08V_n - 4.8$
B $W_0 = 60,\ W_{n+1} = 1.08V_n - 39.2$
C $W_0 = 490,\ W_{n+1} = 1.08V_n - 60$
D $W_0 = 490,\ W_{n+1} = 1.12V_n - 60$
E $W_0 = 60,\ W_{n+1} = 1.12V_n - 490$

2 V_n is the value of an investment after n years. The value of this investment from *month to month* is modelled by the recurrence relation $V_0 = 25\,000,\ V_{n+1} = 1.007V_n - 400$. The annual interest rate for this investment is:

A 0.084% **B** 0.7% **C** 2.8% **D** 8.4% **E** 36.4%

3 An investment of \$18 000, earning compound interest at the rate of 6.8% per annum, compounding yearly, and with regular additions of \$2500 every year can be modelled with a recurrence relation. If V_n is the value of the investment after n years, the recurrence relation is:

A $V_0 = 18000, V_{n+1} = 1.006V_n - 2500$
B $V_0 = 2500, V_{n+1} = 1.068V_n - 18\,000$
C $V_0 = 18\,000, V_{n+1} = 1.068V_n - 2500$
D $V_0 = 18000, V_{n+1} = 1.068V_n - 2500$
E $V_0 = 2500, V_{n+1} = 1.006V_n - 18000$

Questions 4 and 5 relate to the following information.

A loan of \$28 000 is charged interest at the rate of 6.4% per annum, compounding monthly. It is repaid with regular monthly payments of \$1200.

4 Correct to the nearest cent, the value of the loan after 5 months is:

A \$21 611.35 **B** \$22 690.33 **C** \$23 763.59 **D** \$24 831.16 **E** \$31 363.91

5 The final payment on the loan, correct to the nearest cent, will be:

A \$1125.41 **B** \$1131.41 **C** \$1175.20 **D** \$1181.47 **E** \$1200

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6 The monthly payment on an interest-only loan of \$175 000, at an interest rate of 5.9% per annum, compounding monthly, is closest to:

A \$198 **B** \$397 **C** \$860 **D** \$1117 **E** \$2581

7 A scholarship will be set up to provide an annual prize of \$400 to the best mathematics student in a school. The scholarship is paid for by investing an amount of money into a perpetuity, paying interest of 3.4% per annum, compounding annually. The amount that needs to be invested to provide this scholarship is closest to:

A \$400 **B** \$800 **C** \$1176 **D** \$11 7645.00 **E** \$11 7647

8 A loan of \$6000, plus interest, is to be repaid in full in 12 quarterly payments. Interest at 10% per annum is calculated on the remaining balance each quarter. The amount of the payment required to pay out the loan is closest to:

A \$585 **B** \$550 **C** \$650 **D** \$527 **E** \$500

9 Monthly withdrawals of \$220 are made from an account that has an opening balance of \$35 300, invested at 7% per annum, compounding monthly. The balance of the account after 1 year is closest to:

A \$35 125 **B** \$40 578 **C** \$32 660 **D** \$33 500 **E** \$35 211

10 Paula borrows \$12 000 from a bank, to be repaid over 5 years. Interest of 12% per annum is charged monthly on the amount of money owed. If Paula makes monthly payments, then the amount she owes at the end of the second year is closest to:

A \$9120 **B** \$6410 **C** \$8040 **D** \$5590 **E** \$2880

Questions 11–14 refer to the following amortisation table for a reducing-balance loan.

Payment number	*Payment amount*	*Interest paid*	*Reduction in principal*	*Balance after payment*
0	0.00	0.00	0.00	40000.00
1	400.00	160.00	240.00	39760.00
2	400.00	159.04	240.96	39519.04
3	400.00	158.08	241.92	39277.12
4	400.00	157.11	242.89	39034.22
5	400.00	156.14	243.86	38790.36
6	400.00	155.16	244.84	38545.52

11 The principal of this loan is:

A \$20 000 **B** \$30 000 **C** \$40 000 **D** \$50 000 **E** \$60 000

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Review

12 The periodic payment amount on this loan is:

A \$80 **B** \$160 **C** \$240 **D** \$400 **E** \$560

13 The reduction in the value of the loan by the fifth payment is:

A \$156 **B** \$247.89 **C** \$243.80 **D** \$244.84 **E** \$400

14 As a percentage of the payment amount, the interest paid from the sixth payment is closest to:

A 1% **B** 39% **C** 63% **D** 65% **E** 96%

Extended-response questions

1 A population of fish in a pond is growing at the rate of 6% every month. On average, each month there are 80 fish caught by people fishing in the pond. Initially, there were 840 fish in the pond.

Let F_n be the number of fish in the pond after n months.

a Write down a recurrence model for the number of fish in the pond.

b Determine the number of fish in the pond after 5 months.

c After how many months will there be no fish left in the pond?

2 Barry is considering borrowing \$250 000 to buy a house. A home loan at his bank will charge interest at the rate of 4.9% per annum, compounding monthly. Barry can afford payments of \$1800 every month.

Let V_n be the value of Barry's loan after n months.

a Write down a recurrence model for the value of Barry's loan.

b After 12 months, how much would Barry owe on this loan?

c How many months would it take Barry to reduce the value of his loan below \$200 000?

d If Barry chose to take an interest-only loan for the first year:

i what would his monthly payments be?

ii how much interest in total would he pay in the first year?

iii how much would he owe after the 12th interest-only payment?

3 Samantha inherited \$150 000 from her aunt. She decides to invest this money into an account paying 6.25% per annum interest, compounding monthly.

a If Samantha deposited her money into a perpetuity, what monthly payment would she receive?

b If Samantha deposited her money into an annuity and withdrew \$1000 per month, how much would she have in the account after 1 year?

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c If Samantha deposited her money into an annuity and withdrew \$2000 per month, how long would it take for the value of her investment to drop below \$100 000?

d If Samantha deposited her money into an annuity and withdrew \$4000 per month:

i How long would her investment last?

ii Including interest, what would be the value of her last withdrawal?

4 A loan of \$10 000 is to be repaid over 5 years. Interest is charged at the rate of 11% per annum compounding quarterly.
Find:

a the quarterly payment, correct to the nearest cent

b the total cost of paying off the loan, to the nearest dollar

c the total amount of interest paid.

5 Interest on a reducing balance loan of \$65 000 is compounded quarterly at an interest rate of 12.75% per annum.
Calculate the quarterly payment if:

a the amount still owing after 10 years is \$25 000

b the amount still owing after 20 years is \$25 000

c the loan is fully repaid after 10 years

d the loan is fully repaid after 20 years.

6 The Andersons were offered a \$24 800 loan to pay the total cost of a new car. Their loan is to be repaid in equal monthly payments of \$750, except for the last month when less than this will be required to fully pay out the loan. They will pay 10.8% interest per annum, calculated monthly on the reducing balance.

a Calculate the least number of months needed to repay this loan plus interest.

b Calculate, to the nearest cent, the amount of the final payment.

c When the Andersons took out their loan they had the choice of making monthly payments of \$750 or quarterly payments of \$2250. They chose to make monthly payments of \$750. In either case they would have to pay 10.8% interest per annum calculated monthly on the reducing balance. In terms of the total amount of money they would have to pay to repay the loan, did they make the correct decision? Explain your answer without making any further calculations.

7 When a family bought their home they borrowed \$100 000 at 9.6% per annum compounded quarterly. The loan was to be repaid over 25 years in equal quarterly payments.

a How much of the first quarterly payment went towards paying off the principal?

b The family inherit some money and decide to terminate the loan after 10 years and pay what is owing in a lump sum. How much will this lump sum be?

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8 Helene has won \$750 000 in a lottery. She decides to place the money in an investment account that pays 4.5% per annum interest, compounding monthly.

a How much will Helene have in the investment account after 10 years?

b After the 10 years are up, Helene decides to use her money to invest in an annuity, which pays 3.5% per annum, compounding monthly. If Helen requires \$6000 per month for her living expenses, how long will the annuity last?

c Helene's accountant suggests that rather than purchase an annuity she places the money in a perpetuity so that she will be able to leave some money to her grandchildren. If the perpetuity pays 3.5% per annum compounding monthly, how much is the monthly payment that Helene will receive?

9 A home buyer takes out a reducing balance loan to purchase an apartment. The principal of the loan will be \$250 000 and the interest will be calculated and paid monthly at the rate of 6.25% per annum.

a The loan will be fully repaid in equal monthly payments over 20 years.

i Find the monthly payment amount, in dollars, correct to the nearest cent.

ii Calculate the total interest that will be paid over the 20-year term of the loan.

b After 60 monthly payments have been made, what will be the outstanding value of the loan? Write your answer, correct to the nearest dollar.

By making a lump sum payment after nine years, the home buyer is able to reduce the value of the loan to \$100 000. At this time, his monthly payment changes to \$1250. The interest rate remains at 6.25% per annum, compounding monthly.

c With these changes, how many months, in total, will it take the home buyer to fully repay the \$250 000 loan?

ISBN 978-1-107-56757-3

Chapter 10

10

Core: Recursion and financial modelling

Revision

Revision: Recursion and financial modelling

ISBN 978-1-107-56757-3

10A Multiple-choice questions

1 The sequence that follows the rule 'multiply by 3 and subtract 2' is:

A 5, 14, 41, 122, 365, . . . **B** 4, 10, 28, 82, 244, . . . **C** 10, 17, 31, 59, 115, . . .
D 2, 8, 26, 80, 242, . . . **E** 3, 11, 35, 107, 323, . . .

2 The recurrence relation with starting value 6 and rule 'multiply by 0.5 and add 4' is:

A $V_0 = 6,\ V_{n+1} = 4V_n + 0.5$ **B** $V_0 = 6,\ V_{n+1} = 0.5V_n - 4$
C $V_0 = 4,\ V_{n+1} = 0.5V_n + 6$ **D** $V_0 = 6,\ V_{n+1} = 0.5V_n + 4$
E $V_0 = 4,\ V_{n+1} = 0.5V_n - 6$

3 A sequence is generated by the recurrence relation $V_0 = 50,\ V_{n+1} = 0.84V_n + 15$.
After five steps, the value of the term, correct to two decimal places, is:

A 62.82 **B** 71.97 **C** 75.45 **D** 78.38 **E** 80.84

4 The terms of the sequence of numbers generated by the recurrence relation
$A_0 = 270,\ A_{n+1} = 0.92A_n - 8$ will always be larger than:

A −102 **B** −101 **C** −100 **D** −99 **E** −98

5 A recurrence relation that generates a sequence that grows only geometrically is:

A $A_0 = 4,\ A_{n+1} = 0.93\ A_n + 5$ **B** $A_0 = 5,\ A_{n+1} = 1.05\ A_n$
C $A_0 = 6,\ A_{n+1} = 0.94\ A_n$ **D** $A_0 = 7,\ A_{n+1} = A_n$
E $A_0 = 10, A_{n+1} = 1.04A_n - 2$

6 The recurrence relation $V_0 = 20\,000,\ V_{n+1} = 1.045V_n + 200$ could model a:

A compound interest loan with annual interest rate of 4.5% and annual payments of $200
B compound interest investment with annual interest rate of 4.5% and annual additions of $200
C reducing-balance depreciation at an annual rate of 4.5%
D compound interest investment with annual interest rate of 5.4% and monthly additions of $200
E compound interest loan with annual interest rate of 5.4% and monthly repayments of $200

7 The recurrence relation $V_0 = 150\,000,\ V_{n+1} = 1.0017V_n - 180$, is used to calculate the value of a reducing balance loan with fortnightly payments. The annual interest rate is closest to:

A 2.04% **B** 3.85% **C** 4.4% **D** 6.8% **E** 8.3%

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8 A car is depreciated using a reducing-balance depreciation method, at the rate of 4.5% of its value each year. It was purchased for \$24 990. A rule to calculate the value of the car after n years, V_n, is:

A $V_n = (1.045)^n \times 24990$ **B** $V_n = (1.00375)^n \times 24990$

C $V_n = (0.956)^n \times 24990$ **D** $V_n = (0.955)^n \times 24990$

E $V_n = (0.045)^n \times 24990$

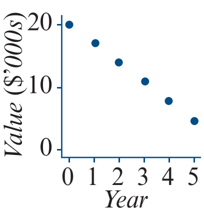
9 A company purchased a machine for \$5000 and depreciates its value at a rate of 20% of its purchase price per year. When the book value falls below \$1500 the company will write it off. The length of time that the machine will be in use is closest to:

A 2 years **B** $2\frac{1}{2}$ years **C** 3 years

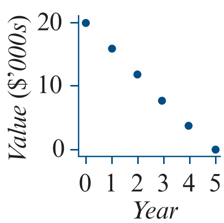
D $3\frac{1}{2}$ years **E** 4 years

10 A printing machine is bought for \$20 000. Using flat rate depreciation, its value after 5 years will be \$4000. Which of the following graphs best represents the value of the machine at the end of each year for this 5-year period?

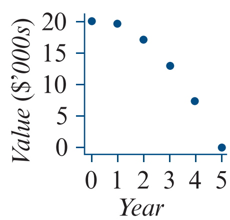
A



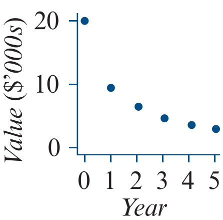
B



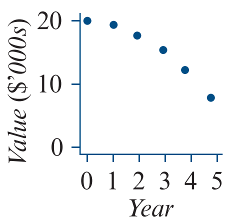
C



D



E



based on VCAA (2003)

11 Three thousand dollars is placed in an investment account compounding monthly at an interest rate of 6% per annum. The value of the investment in dollars, after n years, is given by:

A $1.005^n \times 3000$ **B** $1.005^{12n} \times 3000$ **C** $1.06^n \times 3000$

D $1.06^{12n} \times 3000$ **E** $0.06^n \times 3000$

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12 Kylie has \$5000 in an investment account paying interest at a rate of 6% per annum compounding monthly. If she makes monthly deposits of \$500 to the account, the value of the investment at the end of the second year is:

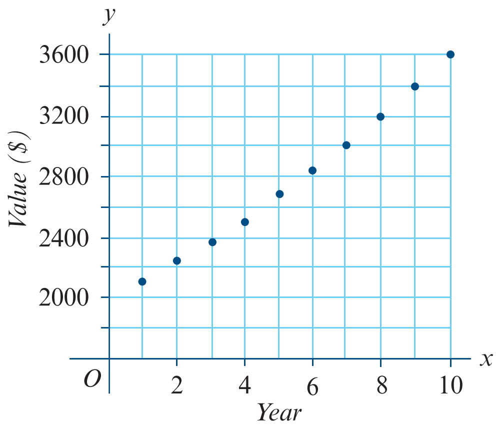
A \$17 000.00 **B** \$17 720.00 **C** \$18 351.78
D \$17 978.00 **E** \$19 655.34

13 Isabella invests \$6350 at 5.9% interest per annum compounding quarterly. The amount of interest she earns during the fifth year of the investment is:

A \$8510.50 **B** \$2160.50 **C** \$484.14
D \$8026.36 **E** \$1676.36

14 The graph represents the growth of an investment over several years. If V_n dollars is the value of the investment after n years, then a rule for describing the growth of this investment could be:

A $V_n = (1.06)^n \times 2000$
B $V_n = (0.06)^n \times 2000$
C $V_n = 1.06n \times 2000$
D $V_n = 0.06n \times 2000$
E $V_n = (1.06)^n + 2000$



15 Bennet invested \$15 000 for 3 years. He earned 6.5% per annum interest, compounding annually. The value of his investment at the end of 3 years, to the nearest dollar, is:

A \$15 004 **B** \$18 119 **C** \$18 220
D \$17 925 **E** \$35 850

16 A loan of \$40 000 is to be paid back over 15 years at an interest rate of 6.5% per annum on a reducing monthly balance. The monthly payment will be closest to:

A \$353.82 **B** \$348.44 **C** \$784.00
D \$1362.07 **E** \$6268.28

17 Zoltan is running a convenience store. He buys equipment for \$6500. It is anticipated that the equipment will last 5 years and have a depreciated value of \$2000. Assuming the flat rate method of depreciation, the equipment depreciates annually by:

A \$400 **B** \$900 **C** \$1027 **D** \$1300 **E** \$4500

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18 Peter borrows \$80 000 for 10 years at 5.6% per annum, compounding monthly, with monthly payments of \$555. Which one of the following statements is true?

A The loan will be fully paid out in 10 years.
B At the end of 5 years, the balance of the loan will be \$40 000.
C The amount of interest paid each month during the loan increases.
D Weekly payments of \$132 compounding weekly would reduce the period of the loan.
E If one extra payment of \$2000 is to be made, it would be better to make it at the end of year 8 than at the end of year 2.

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19 \$15 000 is invested for 12 months.
For the first 6 months the interest rate is 6.1% per annum compounding monthly.
After 6 months the interest rate increases to 6.25% per annum compounding monthly.
The total interest earned by this investment over 12 months is closest to:

A \$926 **B** \$935 **C** \$941 **D** \$953 **E** \$965

10B Extended-response questions

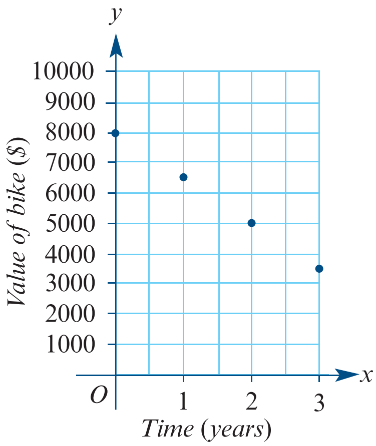
1 Hugo is a professional bike rider. The value of his bike will be depreciated over time using a flat-rate depreciation method. The graph below shows his bike's initial purchase price and its value after each year for a period of 3 years.

a What was the initial purchase price of the bike?

b What is the value of the annual depreciation?

c **i** Write down a recurrence relation model for the depreciation of the bike from year to year.

ii Use the recurrence relation to calculate the value of the bike after 5 years.



d The bike can also be depreciated using a unit-cost depreciation method. The value of the bike is reduced by \$0.25 for every kilometre it travels. After 2 years, the total depreciation will be the same as the flat-rate depreciation as described above. How many kilometres did the bike travel in the 2-year period?

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Hugo won \$5000 in a road race and invested this money in an account earning compound interest of 4.8% per annum, compounding monthly.

e If Hugo was to add \$200 to his investment every month:

i Write down a recurrence relation model for the value of his investment from month to month.

ii Use a calculator to determine how long it takes Hugo's investment to double.

iii How much interest will Hugo earn after 1 year?

based on VCAA (2012)

2 Daniel borrows \$50 000 at an interest rate of 6.4% per annum.

a What is the interest rate per quarter?

Daniel will pay back this loan with quarterly payments of \$4800.

b **i** Write down a recurrence relation model for the value of Daniel's loan from quarter to quarter.

ii Use the recurrence relation to determine how many years it will take Daniel to fully repay the loan.

3 Leanne currently owes \$138 500 on her home loan. She pays interest at the rate of 4.2% per annum, compounding monthly, and repays the loan with monthly payments of \$1200.

a Write down a recurrence model for the value of Leanne's loan.

b Use the recurrence relation model to write down a sequence of numbers representing the value of Leanne's loan for 6 iterations, correct to the nearest cent.

After 6 months, the interest rate of Leanne's loan increased to 4.35% per annum, compounding monthly.

c Write a new recurrence model for the value of Leanne's loan from month to month.

d Use the new model to determine the amount Leanne owes on her loan after a further 6 months, correct to the nearest cent.

Leanne decides to increase her payments to \$1500 per month.

e How much, correct to the nearest cent, will Leanne owe on her loan after a further 12 months?

4 Adele decides to spend her money as follows:

- \$40 000 on a new car
- \$40 000 on the latest computer equipment.

Adele knows that the car will depreciate by 25% per annum based on the reducing value of the car, whereas the computer equipment will depreciate at a flat rate of \$8000 per year.

a What is the value of the car after:

i 1 year? **ii** 3 years?

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b What is the value of the computer equipment after 2 years?

c After how many full years does the depreciated value of the car first exceed the depreciated value of the computer equipment?

d Determine the annual percentage flat rate depreciation applied to the computer equipment.

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5 Lucy wants to borrow $25 000. Interest is calculated quarterly on the reducing balance at an interest rate of 7.9%.

a If Lucy can afford to repay her loan at $1600 per quarter:

i How much of Lucy's first payment is interest?

ii Will payments of $1600 enable Lucy to repay the loan within 4 years? Explain.

b Suppose Lucy arranges to pay $1525 per quarter.

i How long will it take her to pay back the loan? Give your answer to the nearest quarter.

ii How much will the period of Lucy's loan be reduced if her payments are increased to $1745? Give your answer to the nearest quarter.

6 Roslyn earns an annual salary of $54 200, which is paid monthly. She did not join the superannuation fund until her 37th birthday and she now pays 7% of her gross salary to the superannuation fund. Her employer contributes a further 14%.

a What amount of money is placed each month into her superannuation fund?

b The superannuation fund pays 4.2% per annum compound interest, compounded monthly. Assuming that Roslyn's annual salary remains constant, what is the amount of superannuation she will have available at her 60th birthday?

c Suppose that when Roslyn retires at 60 years of age, she places her superannuation in a perpetuity that will provide a monthly income without using any of the principal. If the perpetuity pays 4.25% per annum compounding monthly, what monthly payment will Roslyn receive?

7 Shelly decides to sell her business and invest the proceeds in an investment account that pays 5.5% per annum interest, compounding monthly. She plans to continue to work for 5 more years and add another $1500 per month to the account, and then retire.

a If she makes a profit of $825 000 on her business, how much will Shelly have in the investment account when she retires?

b When she retires in 5 years, Shelly plans to use her money to buy an annuity, which pays 5.75% per annum compounding monthly. If she receives $8400 per month for her living expenses, how long will the annuity last?

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c Alternatively, Shelly could place the money in a perpetuity. If the perpetuity she selects pays 5.75% per annum compounding monthly, how much is the monthly payment that Shelly will receive?

8 Robyn invests \$100 000 to provide a scholarship valued at \$10 000 to the best mathematics student in the final year at her old school. She invests the money into an annuity at an interest rate of 8.25% per annum compound interest. She makes the payment to the winning student each year immediately after the interest is paid into the account.

a How much money is left in the account after the first two scholarships are awarded?

b Determine the amount that is left in Robyn's account after 10 years of awarding scholarships. Give your answer to the nearest cent.

c What would be the maximum value for each scholarship if they are to be awarded forever?

d How much would Robyn need to invest to be able to pay the \$10 000 scholarship in perpetuity? Give your answer to the nearest dollar.

ISBN 978-1-107-56757-3

11

Module 1: Matrices

Chapter 11

Matrices I

ISBN 978-1-107-56757-3

11A What is a matrix?

The table of data displays the heights, weights, ages and pulse rates of eight students.

Name	*Height*	*Weight*	*Age*	*Pulse rate*
Mahdi	173	57	18	86
Dave	179	58	19	82
Jodie	167	62	18	96
Simon	195	84	18	71
Kate	173	64	18	90
Pete	184	74	22	78
Mai	175	60	19	88
Tran	140	50	34	70

$$D = \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 167 & 62 & 18 & 96 \\ 195 & 84 & 18 & 71 \\ 173 & 64 & 18 & 90 \\ 184 & 74 & 22 & 78 \\ 175 & 60 & 19 & 88 \\ 140 & 50 & 34 & 70 \end{bmatrix}$$

If we extract the numbers from the table and enclose them in square brackets, we form a **matrix** (plural: matrices). We might call this matrix D (for data matrix). We use capital letters A, B, C, etc. to name matrices.

▶ Rows and columns

Rows and columns are the building blocks of matrices. We number rows from the top down: row 1, row 2, etc. Columns are numbered from the left across: column 1, column 2, etc.

$$\begin{array}{cc} & \begin{array}{cccc} & & \text{Col. 3} & \end{array} \\ \begin{array}{c} \\ \text{Row 2} \\ \\ \\ D = \\ \\ \\ \\ \end{array} & \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 167 & 62 & 18 & 96 \\ 195 & 84 & 18 & 71 \\ 173 & 64 & 18 & 90 \\ 184 & 74 & 22 & 78 \\ 175 & 60 & 19 & 88 \\ 140 & 50 & 34 & 70 \end{bmatrix} \end{array}$$

▶ Order of a matrix

In its simplest form, a matrix is just a rectangular array (rows and columns) of numbers.

The **order** (or size) of matrix D is said to be 8×4, read '8 by 4' because it has *eight rows* and *four columns*.

> **Order of a matrix**
>
> Order of a matrix = number of rows × number of columns

The numbers, or entries, in the matrix are called **elements**.

The number of elements in a matrix is determined by its order. For example, the number of elements in matrix D is 32 ($8 \times 4 = 32$).

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Row matrices

Matrices come in many shapes and sizes. For example, from this same set of data, we could have formed the matrix we might call K (for Kate's matrix):

$$K = \begin{bmatrix} 173 & 64 & 18 & 90 \end{bmatrix}$$

This matrix has been formed from just one row of the data: the data values for Kate. Because it only contains *one row* of numbers, it is called a **row matrix** (or **row vector**). It is a 1×4 matrix: one row by four columns. It contains $1 \times 4 = 4$ elements.

Column matrices

Equally, we could form a matrix called H (for height matrix). This matrix is formed from just one column of the data, the heights of the students.

Because it only contains *one column* of numbers, it is called a **column matrix** (or **column vector**). This is an 8×1 matrix: eight rows by one column. It contains $8 \times 1 = 8$ elements.

$$H = \begin{bmatrix} 173 \\ 179 \\ 167 \\ 195 \\ 173 \\ 184 \\ 175 \\ 140 \end{bmatrix}$$

Switching rows and columns: the transpose of a matrix

If you switch the rows and columns in a matrix you have what is called the **transpose** of the matrix.

For example, the *transpose* of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

The transpose of a row matrix is a column matrix and vice versa.

For example, the *transpose* of the matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

The symbol we use to indicate the transpose of a matrix is T.

Thus, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Note: The transpose of a 3×2 matrix is a 2×3 matrix because the rows and columns are switched.

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Example 1 The transpose of a matrix

a Write down the transpose of $\begin{bmatrix} 7 & 4 \\ 8 & 1 \end{bmatrix}$.

b Write down the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}^T$.

c If $A = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$, write down the matrix A^T.

Solution

a The transpose of the matrix is obtained by switching (interchanging) its rows and columns.

$$\begin{bmatrix} 7 & 8 \\ 4 & 1 \end{bmatrix}$$

b The symbol T is an instruction to transpose the matrix.

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}$$

c The symbol T is an instruction to transpose matrix A.

$$A^T = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

▶ Square matrices

As a final example, we could form a matrix we call M (for males). This matrix contains only the data for the males. As this matrix has four rows and four columns, it is a 4×4 matrix. It contains $4 \times 4 = 16$ elements.

$$M = \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 195 & 84 & 18 & 71 \\ 184 & 74 & 22 & 78 \end{bmatrix}$$

A matrix with an *equal* number of *rows* and *columns* is called a **square matrix**.

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Example 2 Matrix facts

For each of the matrices below, write down its type, order and the number of elements.

Solution

Matrix	*Type*	*Order*	*No. of elements*
$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 4 \\ 2 & -1 & 6 \end{bmatrix}$	Square matrix rows = columns	3×3 3 rows, 3 cols.	9 $3 \times 3 = 9$
$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	Column matrix single column	3×1 3 rows, 1 col.	3 $3 \times 1 = 3$
$C = \begin{bmatrix} 3 & 1 & 0 & 5 & -3 & 1 \end{bmatrix}$	Row matrix single row	1×6 1 row, 6 cols.	6 $1 \times 6 = 6$

Diagonal, symmetric and triangular matrices

Some square matrices occur so often in practice that they have their own names.

Diagonal matrices

A square matrix has two diagonals: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

In practice, the diagonal going downwards from left to right in the matrix (coloured red) turns out to be more important than the other diagonal (coloured blue), so we give it a special name: the *leading diagonal*.

A matrix whose elements are zero except in the leading diagonal is called a *diagonal matrix*.

The matrices opposite are all diagonal matrices: $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

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Identity matrices

Diagonal matrices in which all of the elements in the diagonal are of special importance. They are called identity or unit matrices and have their own name and symbol (I).

Every order of square matrix has its own **identity matrix**, three of which are shown opposite.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Symmetric matrices

A symmetric matrix is a square matrix that is unchanged by transposition (switching rows and columns). In a symmetric matrix, the elements above the leading diagonal are a mirror image of the elements below the diagonal. Three are shown.

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 & 6 \\ 2 & 1 & 5 & 7 \\ 4 & 5 & 3 & 8 \\ 6 & 7 & 8 & 5 \end{bmatrix}$$

Triangular matrices

Triangular matrices come in two types:

1 An upper triangular matrix is a square matrix in which all elements below the leading diagonal are zeros.

2 A lower triangular matrix is a square matrix in which all elements are above the leading diagonal are zeros.

Examples of triangular matrices are shown.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 6 & 5 & 4 & 0 \\ 0 & 9 & 8 & 7 \end{bmatrix}$$

upper triangular matrix   lower triangular matrix

Example 3 Types of matrices

Consider the following square matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 7 \\ 5 & 7 & 2 \end{bmatrix}$$

Write down:

a the upper triangular matrices

b the identity matrix

c the diagonal matrices

d the symmetric matrices.

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Solution

a All the zeros must be below the leading diagonal $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

b All elements in the diagonal matrix must be 1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c All elements other than those in the leading diagonal are zero. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

d The matrix must be its own transpose. $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 7 \\ 5 & 7 & 2 \end{bmatrix}$

Some notation

In some situations, we talk about a matrix and its elements without having specific numbers in mind. We can do this as follows.

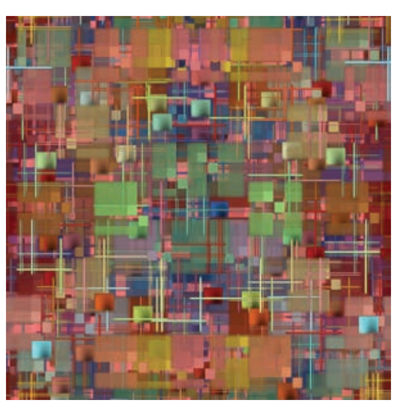
For the matrix A, which has n rows and m columns, we write:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix} \qquad a_{23}$$

row number column number

Thus:

- a_{21} represents the element in the second row and the first column
- a_{12} represents the element in the first row and the second column
- a_{22} represents the element in the second row and the second column
- a_{mn} represents the element in the mth row and the nth column.



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Example 4 Identifying the elements in a matrix

For the matrices A and B, opposite, write down the values of:

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -1 & 0 & 4 \\ 2 & -2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

a a_{12} **b** a_{21} **c** a_{33} **d** b_{31}.

Solution

a a_{12} is the element in the first row and the second column of A. $a_{12} = 5$

b a_{21} is the element in the second row and the first column of A. $a_{21} = -1$

c a_{33} is the element in the third row and the third column of A. $a_{33} = 6$

d b_{31} is the element in the third row and the first column of B. $b_{31} = 1$

In some instances, there is a rule connecting the value of each matrix with its row and column number. In such circumstances, it is possible to construct this matrix knowing this rule and the order of the matrix.

Example 5 Constructing a matrix given a rule for its *ij*th term

A is a 2×2 matrix. The element in row i and column j is given by $a_{ij} = i + j$. Construct the matrix.

Solution

1 The matrix is square and will have the form:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{22} \end{bmatrix}$$

Use the rule

$$a_{ij} = i + j$$

to generate the elements one by one.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where

$a_{11} = 1 + 1 = 2$ $a_{12} = 1 + 2 = 3$

$a_{21} = 2 + 1 = 3$ $a_{22} = 2 + 2 = 4$

2 Write down the matrix.

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

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▶ Entering a matrix into a graphics calculator

Later in this chapter, you will learn about matrix arithmetic: how to add, subtract and multiply matrices. While it is possible to carry out these tasks by hand, for all but the smallest matrices this is very tedious. Most matrix arithmetic is better done with the help of a graphics calculator. However, before you can perform matrix arithmetic, you need to know how to enter a matrix into your calculator.

How to enter a matrix on the TI-Nspire CAS

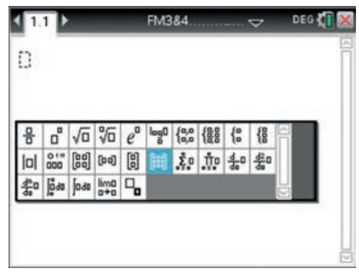
Enter the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and determine its transpose (A^T).

Steps

1 Press [⌂ on]>**New Document>Add Calculator**.

2 Press [templates] and use the cursor ▼ ▶ arrows to highlight the matrix template shown. Press [enter].

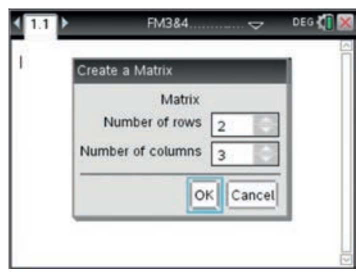
Note: **Math Templates** can also be accessed by pressing [ctrl]+[menu]>**Templates**.



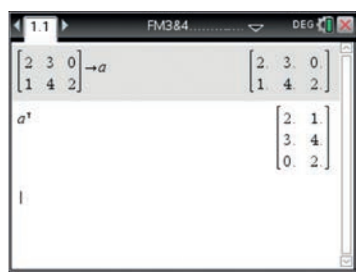
3 Use the ▼ arrow to select the **Number of rows** required (number of rows in this example is 2).

Press [tab] to move to the next entry and repeat for the **Number of columns** (the number of columns in this example is 3).

Press [tab] to highlight **OK**; press [enter].



4 Type the values into the matrix template. Use [tab] to move to the required position in the matrix to enter each value. When the matrix has been completed, press [tab] to move outside the matrix, press [ctrl] [var], followed by [A]. Press [enter]. This will store the matrix as the variable a.



5 When you type ***A*** (or ***a***) it will paste in the matrix $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$. Press [enter] to display.

6 To find a^T, type in **a** (for matrix A) and then [menu]>**Matrix & Vector>Transpose** as shown.

Note: Superscript T can also be accessed from the symbols palette ([ctrl] [symbols]).

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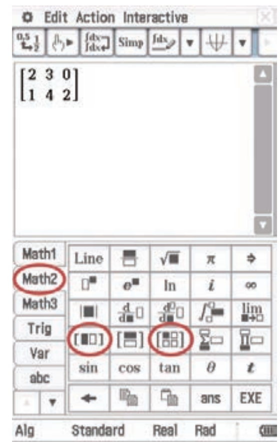
How to enter a matrix using the ClassPad

Enter the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and determine its transpose (A^T).

Steps

1 **a** Open the **Main** ($\sqrt{\alpha}$) application Press Keyboard to display the soft keyboard.

b Select the Math2 keyboard.



2 Tap the 2×2 matrix icon, followed by the 1×2 matrix icon. This will add a third column and create a 2×3 matrix.

3 Type the values into the matrix template.

Note: Tap at each new position to enter the new value or use the black cursor key on the hard keyboard to navigate to the new position.

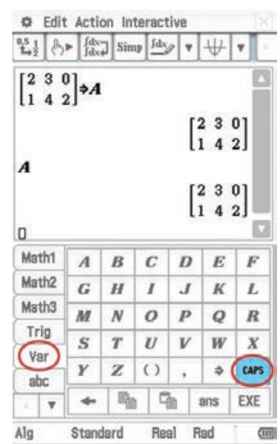
4 To assign the matrix the variable name $\boldsymbol{A}$:

a move the cursor to the very right-hand side of the matrix

b tap the variable assignment key ⇒ followed by var CAPS $\boldsymbol{A}$

c press EXE to confirm your choice.

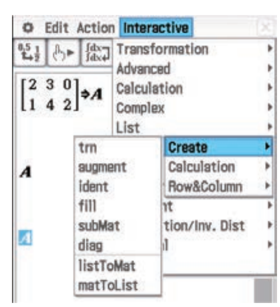
Note: Until it is reassigned, $\boldsymbol{A}$ will represent the matrix as defined above.

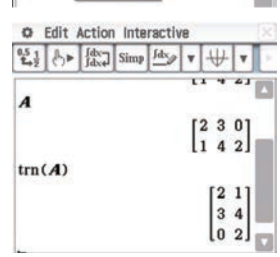


5 To calculate the transpose matrix $\boldsymbol{A}^T$:

a type and highlight $\boldsymbol{A}$ (by swiping with the stylus)

b select **Interactive** from the menu bar, tap **Matrix-Create** and then tap **trn**.





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Exercise 11A

Order of a matrix

1 How many elements are there in a 3×5 matrix?

2 A matrix has 12 elements. What are its possible orders? (There are six.)

The transpose of a matrix

3 **a** $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^T =$ **b** $\begin{bmatrix} 3 \\ 5 \end{bmatrix}^T =$ **c** $\begin{bmatrix} 9 & 1 & 0 & 7 \\ 8 & 9 & 1 & 5 \end{bmatrix}^T =$

Square matrices and their elements

4 Complete the sentences below that relate to the following matrices.

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 6 \\ -1 & 0 \\ 1 & 3 \\ 4 & -4 \end{bmatrix} \quad E = \begin{bmatrix} 4 & 3 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

a The square matrices are ☐ and ☐.

b Matrix B has rows ☐.

c The row matrix is ☐.

d The column matrix is ☐.

e Matrix D has ☐ rows and ☐ columns.

f The order of matrix E is ☐ $\times$ ☐.

g The order of matrix A is ☐ $\times$ ☐.

h The order of matrix B is ☐ $\times$ ☐.

i The order of matrix D is ☐ $\times$ ☐.

j There are ☐ elements in matrix E.

k There are ☐ elements in matrix A.

l $a_{14} =$ ☐ **m** $b_{31} =$ ☐ **n** $c_{11} =$ ☐ **o** $d_{41} =$ ☐

p $e_{22} =$ ☐ **q** $d_{32} =$ ☐ **r** $b_{11} =$ ☐ **s** $c_{12} =$ ☐

Types of matrices and their elements

5 Consider the following square matrices.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$$

Identify:

a the upper triangular matrices

b the identity matrix

c the diagonal matrices

d the symmetric matrices.

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Constructing a matrix given a rule for its *ij*th term

6 B is a 3×2 matrix. The element in row i and column j is given by $b_{ij} = i \times j$. Construct the matrix.

7 C is a 4×1 matrix. The element in row i and column j is given by $c_{ij} = i + 2j$. Construct the matrix.

Entering a matrix into a CAS calculator and determining the transpose

8 Enter the following matrices into your calculator and determine the transpose.

a $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ **b** $C = \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix}$ **c** $E = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ **d** $F = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

11B Using matrices to represent information

At the start of this chapter we used a matrix to store numerical information in a data table. Matrices can also be used to carry codes that encrypt credit-card numbers for internet transmission or to carry all the information needed to solve sets of simultaneous equations. A less obvious application is using matrices to represent network diagrams.

▶ Using a matrix to represent data tables

The numerical information in a data table is frequently presented in rows and columns. As such, it is relatively straight forward process to convert this information into matrix form.

Example 6 Representing information in a table by a matrix

The table opposite shows the three types of membership of a local gym and the number of males and females enrolled in each. Construct a matrix to display the numerical information in the table.

	Gym membership		
Gender	*Weights*	*Aerobics*	*Fitness*
Males	16	104	86
Females	75	34	94

Solution

1 Draw a blank (2×3) matrix.
Label the rows M for male and F for female.
Label the columns W for weights, A for aerobics and F for fitness.

$$\begin{array}{c c} & \begin{array}{ccc} W & A & F \end{array} \\ \begin{array}{c} M \\ F \end{array} & \begin{bmatrix} & & \\ & & \end{bmatrix} \end{array}$$

2 Fill in the elements of the matrix row by row, starting at the top left-hand corner of the table.

$$\begin{array}{c c} & \begin{array}{ccc} W & A & F \end{array} \\ \begin{array}{c} M \\ F \end{array} & \begin{bmatrix} 16 & 104 & 86 \\ 75 & 34 & 94 \end{bmatrix} \end{array}$$

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Example 7 Entering a credit card number into a matrix

Convert the 16-digit credit card number: 4454 8178 1029 3161 into a 2×8 matrix, listing the digits in pairs, one under the other. Ignore the spaces.

Solution

1 Write out the sequence of numbers.

Note: Writing the number down in groups of four (as on the credit card) helps you keep track of the figures.

4454 8178 1029 3161

2 Fill in the elements of the matrix row by row, starting at the top left-hand corner of the table.

$$\begin{bmatrix} 4 & 5 & 8 & 7 & 1 & 2 & 3 & 6 \\ 4 & 4 & 1 & 8 & 0 & 9 & 1 & 1 \end{bmatrix}$$

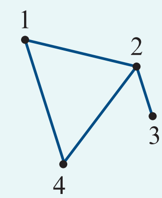
Using matrices to represent network diagrams

A less obvious use of matrices is to represent the information contained in network diagrams. Network diagrams consist of a series of numbered or labelled points joined in various ways. They are a powerful way of representing and studying things as different as friendship networks, airline routes, electrical circuits and road links between towns.

Example 8 Representing a network diagram by a matrix

Represent the network diagram shown opposite as a 4×4 matrix A, where the:

- matrix element = 1 if the two points are joined by a line
- matrix element = 0 if the two points are not connected.



Solution

1 Draw a blank 4×4 matrix, labelling the rows and columns 1, 2, 3, 4 to indicate the points.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

2 Fill in the elements of the matrix row by row, starting at the top left-hand corner:

- $a_{11} = 0$ (no line joining point 1 to itself)
- $a_{12} = 1$ (a line joining points 1 and 2)
- $a_{13} = 0$ (no line joining points 1 and 3)
- $a_{14} = 1$ (a line joining points 1 and 4)

and so on until the matrix is complete.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Note: If a network contains no 'loops' (lines joining points to themselves) the elements in the leading diagonal will always be zero. Knowing this can save a lot of work.

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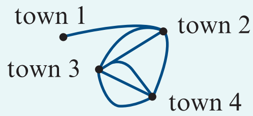
Example 9 Interpreting a matrix representing a network diagram

The diagram opposite shows the roads connecting four towns: town 1, town 2, town 3, and town 4. This diagram has been represented by a 4×4 matrix, A. The elements show the number of roads between each pair of towns.



a In the matrix A, $a_{24} = 1$. What does this tell us?

b In the matrix A, $a_{34} = 3$. What does this tell us?

c In the matrix A, $a_{41} = 0$. What does this tell us?

d What is the sum of the elements in row 3 of matrix A and what does this tell us?

e What is the sum of all the elements of matrix A and what does this tell us?

$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix} \end{array}$$

Solution

a There is one road between town 2 and town 4.

b There are three roads between town 4 and town 3.

c There is no road between town 4 and town 1.

d 5: the total number of roads between town 3 and the other towns in the network.

e 14: the total number of different ways you can travel between towns.

Note: For each road, there are two ways you can travel; for example, from town 1 to town 2 ($a_{12} = 1$) and from town 2 to town 1 ($a_{21} = 1$).

Exercise 11B

Representing a table of data in matrix form

1 The table opposite gives the number of residents, TVs and computers in three households.

Household	*Residents*	*TVs*	*Computers*
A	4	2	1
B	6	2	3
C	2	1	0

Use the table to:

a construct a matrix to display the numbers in the table. What is its order?

b construct a row matrix to display the numbers in the table relating to household B. What is its order?

c construct a column matrix to display the numbers in the table relating to computers. What is its order? What does the sum of its elements tell you?

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2 The table opposite gives the yearly car sales for two car dealers.

Car sales	*Small*	*Medium*	*Large*
Honest Joe's	24	32	11
Super Deals	32	34	9

Use the table to:

a construct a matrix to display the numbers in the table. What is its order?

b construct a row matrix to display the numbers in the table relating to Honest Joe's. What is its order?

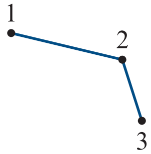
c construct a column matrix to display the numerical information in the table relating to small cars. What is its order? What does the sum of its elements tell you?

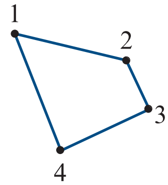
3 Convert the 16-digit credit card number 3452 8279 0020 3069 into a 2×8 matrix. List the digits in pairs, one under the other. Ignore any spaces.

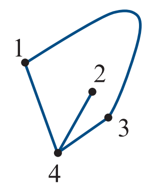
Representing the information in a network diagram in matrix form

4 Represent each of the following network diagrams by a matrix A using the rules:

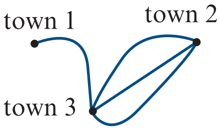
- matrix element = 1 if points are joined by a line
- matrix element = 0 if points are not joined by a line.

a 

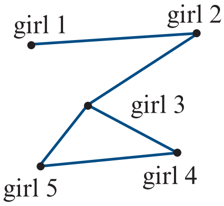
b 

c 

5 The diagram opposite shows the roads interconnecting three towns: town 1, town 2 and town 3. Represent this diagram with a 3×3 matrix where the elements represent the number of roads between each pair of towns.



6 The network diagram opposite shows a friendship network between five girls: girl 1 to girl 5.
This network has been represented by a 5×5 matrix, F, using the rule:



- element = 1 if the pair of girls are friends
- element = 0 if the pair of girls are not friends.

$$F = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a In the matrix F, $f_{34} = 1$. What does this tell us?

b In the matrix F, $f_{25} = 0$. What does this tell us?

c What is the sum of the elements in row 3 of the matrix and what does this tell us?

d Which girl has the least friends? The most friends?

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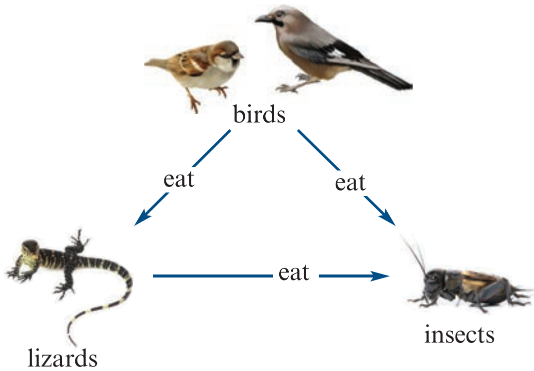
7 **a** The diagram below shows the feeding patterns for insects (I), birds (B) and lizards (L).

The matrix E below has been set up to represent the information in the diagram.

$$E = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc} & I & B & L \\ & & & \end{array}$$

$$E = \begin{array}{cc} \begin{array}{ccc} I & B & L \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} I \\ B \\ L \end{array} \end{array}$$



Referring to insects, birds or lizards:

i What does the '1' in column B, row L, of matrix E represent?

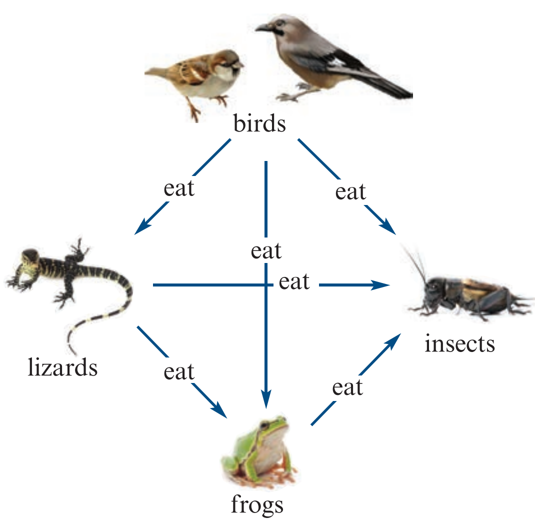
ii What does the row of zeroes in matrix E represent?

The diagram below shows the feeding patterns for insects (I), birds (B), lizards (L) and frogs (I).

The matrix Z below the diagram has been set up to represent the information in the diagram.

Matrix Z is incomplete.

$$Z = \begin{array}{cc} \begin{array}{cccc} I & B & L & F \end{array} & \\ \begin{bmatrix} 0 & 1 & \square & \square \\ 0 & 0 & \square & \square \\ 0 & 1 & \square & \square \\ \square & \square & \square & \square \end{bmatrix} & \begin{array}{c} I \\ B \\ L \\ F \end{array} \end{array}$$



b Complete matrix Z.

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11C Matrix arithmetic: addition, subtraction and scalar multiplication

Equality of two matrices

Skillsheet

Equal matrices have the same order and each corresponding element is identical in value. It is not sufficient for the two matrices to contain an identical set of numbers; they must also be in the same positions.

For example:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is equal to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ because the corresponding elements are equal.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is *not* equal to $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ because the numbers are in different positions.

Matrix addition and subtraction

Adding and subtracting matrices

If two matrices are of the same order (have the same number of rows and columns), they can be added (or subtracted) by adding (or subtracting) their corresponding elements.

Example 10 Adding two matrices

$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ $\quad$ Find $A + B$.

Solution

1 As the two matrices have the same order, 2×3, they can be added.

$$A + B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 3+2 & 0+3 \\ 1+2 & 4+(-2) & 2+1 \end{bmatrix}$$

2 Add corresponding elements.

$$= \begin{bmatrix} 3 & 5 & 3 \\ 3 & 2 & 3 \end{bmatrix}$$

Likewise, if we have two matrices of the same order (same number of rows and columns), we can subtract the two matrices by subtracting their corresponding elements.

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Example 11 Subtracting two matrices

$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ Find $A - B$.

Solution

1 As the two matrices have the same order, 2×3, they can be subtracted.

$$A - B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 3-2 & 0-3 \\ 1-2 & 4-(-2) & 2-1 \end{bmatrix}$$

2 Subtract corresponding elements.

$$= \begin{bmatrix} 1 & 1 & -3 \\ -1 & 6 & 1 \end{bmatrix}$$

▶ Multiplying matrices by a number (scalar multiplication)

Mutiplying a matrix by a number has the effect of multiplying each element in the matrix by that number.

Multiplying a matrix by a number is called **scalar multiplication**, because it has the effect of scaling the matrix by that number. For example, multiplying a matrix by 2 doubles each element in the matrix.

Example 12 Scalar multiplication

$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix}$ Find $3A$ and $0.5C$.

Solution

Multiplying a matrix by a number has the effect of multiplying each element by that number.

$$3A = 3 \times \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 3\times 2 & 3\times 3 & 3\times 0 \\ 3\times 1 & 3\times 4 & 3\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 & 0 \\ 3 & 12 & 6 \end{bmatrix}$$

$$0.5C = 0.5 \times \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 0.5\times 4 & 0.5\times(-4) \\ 0.5\times(-2) & 0.5\times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

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▶ The zero matrix

If $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $X - Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

A matrix of any order with *all zeros*, is known as a **zero matrix**. The symbol O is used to represent a zero matrix. The matrices below are all examples of zero matrices.

$$O = [0], \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 13 The zero matrix

$G = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}$ $\quad H = \begin{bmatrix} 9 & 0 \\ -6 & 3 \end{bmatrix}$ $\quad$ Show that $3G - 2H = O$.

Solution

$$3G - 2H = 3 \times \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} - 2 \times \begin{bmatrix} 9 & 0 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ -12 & 6 \end{bmatrix} - \begin{bmatrix} 18 & 0 \\ -12 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 18 - 18 & 0 - 0 \\ -12 - (-12) & 6 - 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore 3G - 2H = O$$

▶ Using a CAS calculator to perform matrix addition, subtraction and scalar multiplication

For small matrices, it is usually quicker to add, subtract or multiply a matrix by a number (scalar multiplication) by hand. However, if dealing with larger matrices, it is best to use a graphics calculator.

How to add, subtract and scalar multiply matrices using the TI-Nspire CAS

If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $3A - 2B$

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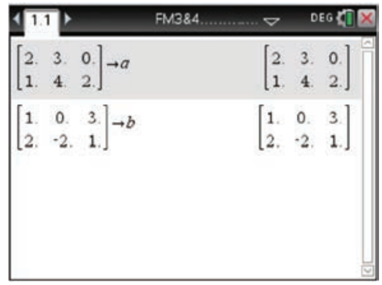
Steps

1 Press [⌂ on]>**New Document>Add Calculator.**

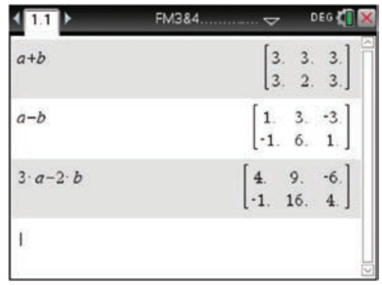
2 Enter the matrices A and B into your calculator.



a To determine $A + B$, type $\boldsymbol{a} + \boldsymbol{b}$.
Press [enter] to evaluate.

b To determine $A - B$, type $\boldsymbol{a} - \boldsymbol{b}$.
Press [enter] to evaluate.

c To determine $3A - 2B$, type $\mathbf{3}\boldsymbol{a} - \mathbf{2}\boldsymbol{b}$.
Press [enter] to evaluate.



How to add, subtract and scalar multiply matrices with the ClassPad

If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $3A - 2B$

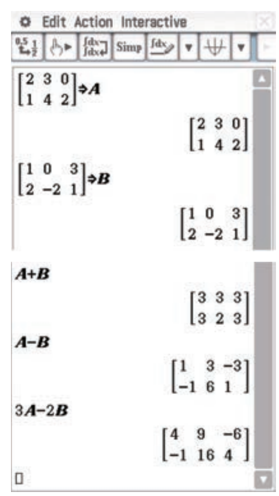
Steps

1 Enter the matrices A and B into your calculator using the [var] keyboard.

a To calculate $A + B$, type $\boldsymbol{A} + \boldsymbol{B}$ and then press [EXE] to evaluate.

b To calculate $A - B$, type $\boldsymbol{A} - \boldsymbol{B}$ and then press [EXE] to evaluate.

c To calculate $3A - 2B$, type $\mathbf{3}\boldsymbol{A} - \mathbf{2}\boldsymbol{B}$ and then press [EXE] to evaluate.



ISBN 978-1-107-56757-3

Cambridge University Press

Example 14 **Processing data using addition, subtraction and scalar multiplication**

The sales data for two used car dealers, Honest Joe's and Super Deals, are displayed below.

Car sales	*2014*			*2015*		
	Small	*Medium*	*Large*	*Small*	*Medium*	*Large*
Honest Joe's	24	32	11	26	38	16
Super Deals	32	34	9	35	41	12

Solution

a Construct two matrices, A and B, to represent the sales data for 2014 and 2015 separately.

$$A = \begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix}$$

b Construct a new matrix $C = A + B$. What does this matrix represent?

$$C = A + B$$
$$= \begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix} + \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix}$$
$$= \begin{bmatrix} 50 & 70 & 27 \\ 67 & 75 & 21 \end{bmatrix}$$

Matrix C represents the total sales for 2011 and 2012 for the two dealers.

c Construct a new matrix, $D = B - A$. What does this matrix represent?

$$D = B - A$$
$$= \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix} - \begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 6 & 5 \\ 3 & 7 & 3 \end{bmatrix}$$

Matrix D represents the increase in sales from 2011 and 2012 for the two dealers.

d Both dealers want to increase their 2015 sales by 50% by 2016. Construct a new matrix $E = 1.5B$. Explain why this matrix represents the planned sales figures for 2016.

$$E = 1.5B = 1.5 \times \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix}$$
$$= \begin{bmatrix} 1.5 \times 26 & 1.5 \times 38 & 1.5 \times 16 \\ 1.5 \times 35 & 1.5 \times 41 & 1.5 \times 12 \end{bmatrix}$$
$$= \begin{bmatrix} 39 & 57 & 24 \\ 52.5 & 61.5 & 18 \end{bmatrix}$$

Forming the scalar product $1.5B$ multiplies each element by 1.5. This has the effect of increasing each value by 50% per year.

ISBN 978-1-107-56757-3

Exercise 11C

Matrix addition, subtraction and scalar multiplication

1 The questions below relate to the following six matrices. Computations will be quicker if done by hand.

$$A = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

a Which matrices are equal?

b Which matrices have the same order?

c Which matrices can be added or subtracted?

d Compute each of the following, where possible.

i $A + B$ **ii** $D + E$ **iii** $C - F$ **iv** $A - B$ **v** $E - D$

vi $3B$ **vii** $4F$ **viii** $3C + F$ **ix** $4A - 2B$ **x** $E + F$

2 **a** $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$

b $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$

c $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + 2 \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$

d $\begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \end{bmatrix} =$

e $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$

f $3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} =$

g $\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 2 \end{bmatrix} =$

h $3\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} =$

i $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$

3 Use a calculator to evaluate the following.

a $2.2 \times \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 1.1 \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$

b $\begin{bmatrix} 1.2 & 0.2 \\ 4.5 & 3.3 \end{bmatrix} - 3.5 \times \begin{bmatrix} 0.4 & 4 \\ 1 & 2 \end{bmatrix} =$

c $5 \times \begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 2 \times \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ 0.5 & 0 & -2 \end{bmatrix} =$

d $0.8 \times \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 0 & -1 & 2 \end{bmatrix} + 0.2 \times \begin{bmatrix} -1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix} =$

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Applications of matrix addition, subtraction and scalar multiplication

4 The number of DVDs sold in a company's city, suburban and country stores for each 3-month period in a year is shown in the table.

	DVD sales (thousands)			
Store location	*Jan–March*	*April–June*	*July–Sept*	*Oct–Dec*
City	2.4	2.8	2.5	3.4
Suburban	3.5	3.4	2.6	4.1
Country	1.6	1.8	1.7	2.1

a Construct four 3×1 matrices A, B, C, and D that show the sales in each of the three-month periods during the year.

b Evaluate $A + B + C + D$. What does the sum $A + B + C + D$ represent?

5 The numbers of females and males enrolled in three different gym programs for 2014 and 2015, *Weights*, *Aerobics* and *Fitness*, are shown in the table.

	2014			*2015*		
Gym membership	*Weights*	*Aerobics*	*Fitness*	*Weights*	*Aerobics*	*Fitness*
Females	16	104	86	24	124	100
Males	75	34	94	70	41	96

a Construct two matrices, A and B, which represent the gym memberships for 2014 and 2015 separately.

b Construct a new matrix $C = A + B$. What does this matrix represent?

c Construct a new matrix $D = B - A$. What does this matrix represent? What does the negative element in this matrix represent?

d The manager of the gym wants to double her 2015 membership by 2018. Construct a new matrix E that would show the membership in 2018 if she succeeds with her plan. Evaluate.

ISBN 978-1-107-56757-3

11D Matrix arithmetic: the product of two matrices

The process of multiplying two matrices involves both multiplication and addition. The process can be illustrated using Australian Rules football scores.

An illustration of matrix multiplication

Two teams, the Ants and the Bulls, play each other. At the end of the game:

- the Ants had scored 11 goals 5 behinds
- the Bulls had scored 10 goals 9 behinds.

Now calculate each team's score in points:

- one goal = 6 points
- one behind = 1 point.

Thus we can write:

$11 \times 6 + 5 \times 1 = 71$ points

$10 \times 6 + 9 \times 1 = 69$ points

Matrix multiplication follows the same pattern.

$$\begin{array}{lccccccc} & \text{Goals} \quad \text{Behinds} & & \text{Point values} & & & & \text{Final points} \\ \text{Ants score:} & \multirow{2}{*}{$\begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$} & \times & \begin{bmatrix} 6 \\ 1 \end{bmatrix} & = & \begin{bmatrix} 11 \times 6 + 5 \times 1 \\ 10 \times 6 + 9 \times 1 \end{bmatrix} & = & \begin{bmatrix} 71 \\ 69 \end{bmatrix} \\ \text{Bulls score:} & & & & & & & \end{array}$$

▶ The order of matrices and matrix multiplication

Look at the **order** of each of the matrices involved in the **matrix multiplication** below.

$$\begin{array}{lccccccc} & \text{Goals} \quad \text{Behinds} & & \text{Point values} & & & & \text{Final points} \\ \begin{matrix}\text{Ants score:} \\ \text{Bulls score:}\end{matrix} & \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix} & \times & \begin{bmatrix} 6 \\ 1 \end{bmatrix} & = & \begin{bmatrix} 11 \times 6 + 5 \times 1 \\ 10 \times 6 + 9 \times 1 \end{bmatrix} & = & \begin{bmatrix} 71 \\ 69 \end{bmatrix} \\ \text{Order of matrices:} & 2 \times 2 & & 2 \times 1 & & & & 2 \times 1 \end{array}$$

Thus, multiplying a 2×2 matrix by a 2×1 matrix gives a 2×1 matrix.

Two observations can be made here:

1. To perform matrix multiplication, the *number of columns in the first matrix* (2) needs to be the same as the *number of rows in the second matrix* (2). For example, if there were three columns in the first matrix, there would not be enough elements in the second matrix to complete the multiplication. When this happens, we say that matrix multiplication is not defined.
2. The final result of multiplying the two matrices is a 2×1 matrix. For each row in the first matrix, there will be a row in the product matrix (there are two rows). For each column in the second matrix, there will be a column in the product matrix (there is one column).

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These observations can be generalised to give two important rules for matrix multiplication.

Rule 1: Condition for matrix multiplication to be defined

Matrix multiplication of two matrices requires the *number of columns*in the *first matrix* to *equal* the *number of rows* in the *second matrix*.

That is, if A is of order $m \times n$ and B is of order $r \times s$, then the product AB is only defined if $n = r$.

For example, if $A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, then:

- $AB = \underset{2 \times 3}{\begin{bmatrix} 3 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}} \times \underset{3 \times 1}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}$ is *defined*: columns in $A(3)$ = rows in $B(3)$
- $BC = \underset{3 \times 1}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}} \times \underset{2 \times 2}{\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}}$ is *not defined*: columns in $B(1) \neq$ rows in $C(2)$.

Example 15 **Is a matrix product defined?**

$A = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Which of the following matrix products are defined?

a AB **b** BC **c** AC

Solution

a Write down the matrix product. Under each matrix, write down its order (columns × rows)

$AB = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix}$; not defined

order: 2×2 1×2

b The matrix product is defined if the number of columns in matrix 1 = the number of rows in matrix 2.

$BC = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; defined

order: 1×2 2×1

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c Write down your conclusion.

$$AC = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{; defined}$$

order: $2 \times 2 \quad 2 \times 1$

Once we know that two matrices can be multiplied, we can use the order of the two matrices to determine the order of the resulting matrix.

Rule 2: Determining the order of the product matrix

If two matrices can be multiplied, then the *product matrix* will have the same *number of rows* as the *first matrix* and the same *number of columns* as the *second matrix*.

That is, if A is of order $m \times n$ and B is of order $n \times s$, then AB will be of order $m \times s$.

For example, if $A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, then:

- $AB = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ is *defined* and will be of order 2×1.

$$2 \times \underbrace{3 \qquad 3}_{\text{equal}} \times 1 \qquad\qquad 2 \times 1$$

- $AB = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ is *defined* and will be of order 2×3.

$$2 \times \underbrace{2 \qquad 2}_{\text{equal}} \times 3 \qquad\qquad 2 \times 3$$

Example 16 Determining the order of a matrix product

$$A = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The following matrix products are defined. What is their order?

a BA **b** BC **c** AC

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Solution

1. Write down the matrix product. Under each matrix, write down its order rows × columns.

a $BA = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}$; order of BA 1×2

order: $(1 \times 2)\ (2 \times 2)$

2. The order of the product matrix is given by rows in matrix 1 × columns in matrix 2.

b $BC = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; order of BC 1×1

order: $(1 \times 2)\ (2 \times 1)$

3. Write down the order.

c $AC = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; order of AC 2×1

order: $(2 \times 2)\ (2 \times 1)$

Order of multiplication is important when multiplying matrices

You might have noticed in Example 13 that while the matrix product BA was defined, the matrix product AB in Example 14 was not defined. Order is important in matrix multiplication. For example, if we have two matrices, M and N, and form the products MN and NM, frequently the products will be different. We will return to this point when in the next section.

Determining matrix products

For large matrices, the process of matrix multiplication is complex and can be error prone and tedious to do by hand. Fortunately, graphics calculators will do it for us, and that is perfectly acceptable.

However, before we show you how to use a graphics calculator to multiply matrices, we will illustrate the process by multiplying a row matrix by a column matrix and a rectangular matrix by a column matrix *by hand*.

In terms of understanding matrix multiplication, and using this knowledge to solve problems later in this module, these are the two most important worked examples in the chapter.

Example 17 **Multiplying a row matrix by a column matrix (by hand)**

Evaluate the matrix product AB, where $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

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Solution

1 Write down the matrix product and, above each matrix, write down its order. Use this information to determine whether the matrix product is defined and its order.

$$AB = \overset{1 \times 3}{\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}} \overset{3 \times 1}{\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}}$$

Defined: the number of columns in A equals the number of rows in B.
The order of AB is 1×1.

2 To determine the matrix product:

a multiply each element in the row matrix by the corresponding element in the column matrix

b add the results

c write down your answer.

$$\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = [1 \times 2 + 3 \times 4 + 2 \times 1]$$
$$= [16]$$
$$\therefore AB = [16]$$

Example 18 Multiplying a rectangular matrix by a column matrix (by hand)

Evaluate the matrix product AB, where $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Solution

1 Write down the matrix product and, above each matrix, write down its order. Use this information to determine whether the matrix product is defined and its order.

$$AB = \overset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}} \overset{2 \times 1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

Defined: the number of columns in A equals the number of rows in B.
The order of AB is 2×1.

2 To determine the matrix product:

a multiply each element in the row matrix by the corresponding element in the column matrix

b add the results

c write down your answer.

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 0 \times 3 \\ 2 \times 2 + 3 \times 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$
$$\therefore AB = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$

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▶ Using a CAS calculator to multiply two matrices

In principle, if you can multiply a row matrix by a column matrix, you can work out the product between any two matrices, provided it is defined. However, because you have to do it for every possible row/column combination, it soon gets beyond even the most patient and careful person. For that reason, in practice we use technology to do the calculation for us.

We will illustrate how to use a calculator to multiply matrices by evaluating the matrix product in the football score example given earlier.

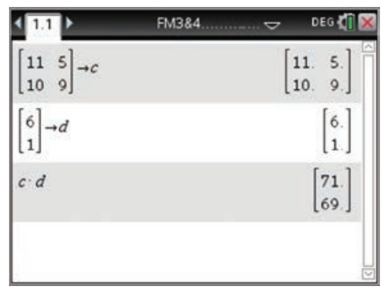
How to multiply two matrices using the TI-Nspire CAS

If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, find the matrix CD.

Steps

1. Press [⌂ on]>**New Document**>**Add Calculator**.
2. Enter the matrices C and D into your calculator.
3. To calculate matrix CD, type $\boldsymbol{c} \times \boldsymbol{d}$. Press [enter] to evaluate.

   Note: You must put a multiplication sign between the c and d.

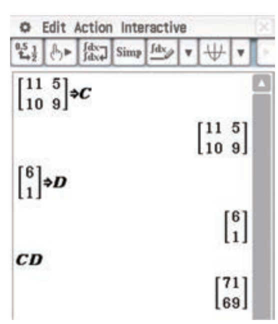


How to multiply two matrices using the ClassPad

If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ find the matrix product CD.

Steps

1. Enter the matrices C and D into your calculator.
2. To calculate $C \times D$, type ***CD*** and then press [EXE] to evaluate.



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Applications of the product of two matrices

Example 16 shows a practical application of matrix multiplication.

Example 19 A practical application of matrix multiplication

$$E = \begin{array}{c} \begin{bmatrix} 25 \\ 40 \end{bmatrix} \end{array} \begin{array}{l} Walk \\ Run \end{array}$$

Matrix E gives the energy in kilojoules consumed per minute when walking and running.

$$T = \begin{array}{cc} [20 & 40] \\ Walk & Run \end{array}$$

Matrix T gives the times (in minutes) a person spent walking and running in a training session.

Compute the matrix product TE and show that it gives the total energy consumed during the training session.

Solution

$$T \times E = \begin{bmatrix} 20 & 40 \end{bmatrix} \begin{bmatrix} 25 \\ 40 \end{bmatrix} = [20 \times 25 + 40 \times 40] = [2100]$$

The total energy consumed is:

20 minutes $\times$ 25 kJ/minute + 40 minutes $\times$ 40 kJ/minute = 2100 kJ

This is the value given by the matrix product TE.

You could work out the energy consumed on the training run for one person just as quickly without using matrices. However, the advantage of using a matrix formulation is that, with the aid of a calculator, you could almost as quickly have worked out the energy consumed by 10 or more runners, all with different times spent walking and running.

The summing matrix

A row or column matrix in which all the elements are 1 is called a *summing matrix*.

The matrices opposite are all examples of summing matrices.

The rules for using a summing matrix to sum the rows and columns of a matrix follow.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Using matrix multiplication to sum the rows and columns of a matrix

- To *sum* the *rows* of an $m \times n$ matrix, *post-multiply* the matrix by an $n \times 1$ summing matrix.
- To *sum the columns* of an $m \times n$ matrix, *pre-multiply* the matrix by a $1 \times m$ summing matrix.

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Example 20 Using matrix multiplication to sum the rows and columns of a matrix

Use matrix multiplication to generate a matrix that:

a displays row sums of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 7 \\ 3 & 0 & 1 \end{bmatrix}$

b displays the column sums of the matrix $\begin{bmatrix} 2 & 5 & -1 & -3 & 4 \\ 0 & 6 & 2 & -2 & 3 \end{bmatrix}$.

Solution

a To sum the rows of a 3×3 matrix, *post*-multiply a 3×1 summing matrix.

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 7 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \\ 4 \end{bmatrix}$$

b To sum the columns of a 2×5 matrix, *pre*-multiply a 1×2 summing matrix.

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & -1 & -3 & 4 \\ 0 & 6 & 2 & -2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 11 & 1 & -5 & 7 \end{bmatrix}$$

Displaying assets of simultaneous equations in matrix form

In the next chapter, you will learn to solve sets of simultaneous linear equations using matrices. The first step in this process is to write a set of simultaneous equations in matrix notation using matrix multiplication.

Example 21 Generating sets of simultaneous equations

Show that the matrix equation $\begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ generates the following pair of simultaneous linear equations:

$$4x + 2y = 5$$
$$3x + 2y = 2$$

Note: When we form the matrix product $A \times B$, we say we are *pre*-multiplying B by A. When we form the product $B \times A$, we say we are *post*-multiplying B by A.

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Solution

1 Write out the matrix equation and then multiply out the left-hand side.

$$\begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4 \times x + 2 \times y \\ 3 \times x + 2 \times y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\text{or} \begin{bmatrix} 4x + 2y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

2 For the matrices to be equal:
$4x + 2y$ must equal 5
and
$3x + 2y$ must equal 2.
This gives the required pair of simultaneous linear equations.

$$\therefore \begin{aligned} 4x + 2y &= 5 \\ 3x + 2y &= 2 \end{aligned}$$

Exercise 11D

Matrix multiplication

Skillsheet

1 The questions below relate to the following five matrices.

$$A = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

a Which of the following matrix products are defined?

i AB **ii** BA **iii** AC **iv** CE
v EC **vi** EA **vii** DB **viii** CD

b Compute the following products by hand.

i AB **ii** CE **iii** DB **iv** AD

c Enter the five matrices into your calculator and compute the following matrix expressions.

i AB **ii** EC **iii** $AB - 3CE$ **iv** $2AD + 3B$

2 Evaluate each of the following matrix products *by hand*.

a $\begin{bmatrix} 0 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \end{bmatrix} =$ **b** $\begin{bmatrix} 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$ **c** $\begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} =$

d $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$ **e** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$ **f** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} =$

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3 Evaluate each of the following matrix products using a CAS calculator.

a $\begin{bmatrix} 0.5 \\ -1.5 \\ 2.5 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} =$

b $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$

c $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}$

d $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 2 & 1 \\ -1 & 4 & 2 \\ -2 & 1 & 2 \end{bmatrix} =$

Summing matrices

4 Use matrix multiplication to construct a matrix that:

a displays row sums of the matrix $\begin{bmatrix} 2 & 4 & 1 & 7 & 8 \\ 1 & 9 & 0 & 0 & 2 \\ 3 & 4 & 3 & 3 & 5 \\ 2 & 1 & 1 & 1 & 7 \\ 5 & 3 & 6 & 7 & 9 \end{bmatrix}$

b displays the column sums of the matrix $\begin{bmatrix} 4 & 5 & 1 & 2 & 1 \\ 0 & 3 & 4 & 5 & 1 \\ 4 & 2 & 1 & 7 & 9 \end{bmatrix}$.

Practical applications of matrix multiplication

5 Six teams play an indoor soccer competition.

If a team:

- wins, it scores two points
- draws, it scores one point
- loses, it scores zero points.

This is summarised in the points matrix opposite.

$$P = \begin{array}{c} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} Win \\ Draw \\ Lose \end{array}$$

The results of the competition are summarised in the results matrix.

Work out the final points score for each team by forming the matrix product RP.

$$R = \begin{array}{c} \begin{array}{ccc} W & D & L \end{array} \\ \begin{bmatrix} 4 & 1 & 0 \\ 3 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} \end{array} \begin{array}{l} \\ Team\ 1 \\ Team\ 2 \\ Team\ 3 \\ Team\ 4 \\ Team\ 5 \\ Team\ 6 \end{array}$$

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6 Four people complete a training session in which they walked, jogged and ran at various times.
The energy consumed in kJ/minute when walking, jogging or running is listed in the energy matrix opposite.

$$E = \begin{array}{c} \begin{bmatrix} 25 \\ 40 \\ 65 \end{bmatrix} \end{array} \begin{array}{l} Walk \\ Jog \\ Run \end{array}$$

The time spent in each activity (in minutes) by four people is summarised in the time matrix opposite. Work out the total energy consumed by each person, by forming the matrix product TE.

$$T = \begin{array}{c} \begin{array}{ccc} W & J & R \end{array} \\ \begin{bmatrix} 10 & 20 & 30 \\ 15 & 20 & 25 \\ 20 & 20 & 20 \\ 30 & 20 & 10 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Person 1} \\ \text{Person 2} \\ \text{Person 3} \\ \text{Person 4} \end{array}$$

7 A manufacturer sells three products, A, B and C, through outlets at two shopping centres, Eastown (E) and Noxland (N).
The number of units of each product sold per month at each shopping centre is given by the matrix Q.

$$Q = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 2500 & 3400 & 1890 \\ 1765 & 4588 & 2456 \end{bmatrix} \end{array} \begin{array}{l} \\ E \\ N \end{array}$$

a Write down the order of matrix Q.

The matrix P, shown opposite, gives the selling price, in dollars, of products A, B and C.

$$P = \begin{bmatrix} 14.50 \\ 21.60 \\ 19.20 \end{bmatrix} \begin{array}{l} A \\ B \\ C \end{array}$$

b **i** Evaluate the matrix M, where $M = QP$.
ii What information do the elements of matrix M provide?

c Explain why the matrix PQ is not defined.

Using summing matrices to sum the rows and columns of matrices

8 For the matrix opposite, write down a matrix that can be used to: $\begin{bmatrix} 2 & 5 \\ -1 & 1 \\ 9 & 3 \end{bmatrix}$.

a sum its rows **b** sum its columns

9 Show how the matrix $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ can be used to sum the columns of $\begin{bmatrix} 9 & 0 & 2 \\ 1 & 7 & 3 \\ 8 & 3 & 4 \end{bmatrix}$.

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Matrix representation of a pair of simultaneous equations

10 Show that, by multiplying out the matrices, the matrix equation opposite generates the equations below.

$$\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 5 \end{bmatrix}$$

$$x + 3y = 16$$
$$2x - 4y = 5$$

11 Show that, by multiplying out the matrices, the matrix equation shown generates the set of equations below.

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

$$2x + y + 3z = 2$$
$$3x + 2y - z = 4$$
$$2x + 3z = 3$$

11E Matrix powers

Now that we can multiply matrices, we can also determine the **power of a matrix**. This is an important tool when we meet communication and dominance matrices in the next section and transition matrices in the next chapter.

▶ The power of a matrix

Just as we define

2^2 as 2×2,

2^3 as $2 \times 2 \times 2$,

2^4 as $2 \times 2 \times 2 \times 2$ and so on,

we define the various powers of matrices as

A^2 as $A \times A$,

A^3 as $A \times A \times A$,

A^4 as $A \times A \times A \times A$ and so on.

Only square matrices can be raised to a power.

Example 22 **Evaluating matrix expressions involving powers**

If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, determine:

a $2A + B^2 - 2C$ **b** $(2A - B)^2 - C^2$ **c** $AB^2 - 3C^2$

ISBN 978-1-107-56757-3

Cambridge University Press

Solution

1 Write down the matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix},\ B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

2 Enter the matrices A, B and C into your calculator.

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \rightarrow a \qquad \begin{bmatrix} 1 & 0 \\ 2. & -1. \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \rightarrow b \qquad \begin{bmatrix} -1. & 1. \\ 2. & 1. \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow c \qquad \begin{bmatrix} 0. & 1. \\ 1. & 1. \end{bmatrix}$$

$$2 \cdot a + b^2 - 2 \cdot c \qquad \begin{bmatrix} 5. & -2. \\ 2. & -1. \end{bmatrix}$$

$$(2 \cdot a - b)^2 - c^2 \qquad \begin{bmatrix} 6. & -1. \\ -1. & 5. \end{bmatrix}$$

$$a \cdot b^2 - 3 \cdot ^2 \qquad \begin{bmatrix} 0. & -3. \\ 3. & -9. \end{bmatrix}$$

3 Type in each of the expressions as written, and press to evaluate. Write down your answer.

a $2A + B^2 - 2C = \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix}$

b $(2A + B)^2 - C^2 = \begin{bmatrix} 6 & -1 \\ -1 & 5 \end{bmatrix}$

c $AB^2 - 3C^2 = \begin{bmatrix} 0 & -3 \\ 3 & -9 \end{bmatrix}$

Note: For CAS calculators you must use a multiplication sign between a and b^2 in the last example, otherwise it will be read as variable $(ab)^2$.

Exercise 11E

1 If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, determine A^2, A^3, A^4 and A^7.

2 If $A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$, determine A^4, A^5, A^6 and A^7.

3 If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$, evaluate:

a $A + 2B - C^2$ **b** $AB - 2C^2$ **c** $(A + B + 2C)^2$

d $4A + 3B^2 - C^3$ **e** $(A - B)^3 - C^3$

ISBN 978-1-107-56757-3

11F Binary, permutation, communication and dominance matrices

Binary matrices

A **binary matrix** is a special kind of matrix that has only 1s and zeros as its elements. You have met binary matrices earlier when representing information presented in diagrammatic form. In this section we will extend this work to investigate communication systems and the ranking of players in sporting competition.

The following matrices are examples of binary matrices.

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Binary matrices are at the heart of many practical matrix applications, including analysing communication systems and using the concept of dominance to rank players in sporting competitions.

Permutation matrices

A **permutation**[1] **matrix** is a square binary matrix in which there is only one '1' in each row and column.

The following matrices are examples of permutation matrices.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

An identity matrix is a special permutation matrix. A permutation matrix can be used to rearrange the elements in another matrix.

Example 23 Applying a permutation matrix

X is the column matrix $X = \begin{bmatrix} T \\ A \\ R \end{bmatrix}$. P is the permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

a Show that:

i pre-multiplying[2] X by P changes the matrix X to the matrix $Y = \begin{bmatrix} R \\ A \\ T \end{bmatrix}$

ii pre-multiplying X by P^2 leaves the matrix X unchanged.

b What can be deduced about P^2 from the result in **a ii**?

---

[1] The word 'permutation' means a rearrangement a group of objects, in this case the elements of a matrix, into a different order.

[2] When we form the matrix product $A \times B$, we say that we are pre-multiplying by A.

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Solution

a i Form the matrix product PX.

$$PX = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ A \\ R \end{bmatrix} = \begin{bmatrix} R \\ A \\ T \end{bmatrix}$$

ii Form the matrix product P^2X.

$$P^2X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2 \begin{bmatrix} T \\ A \\ R \end{bmatrix} = \begin{bmatrix} T \\ A \\ R \end{bmatrix}$$

b To leave the matrix X unchanged, P^2 must be an identity matrix.

P^2 is the identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

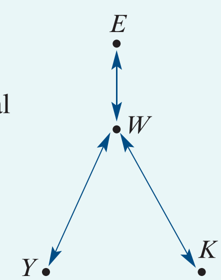
Communication matrices

A **communication matrix** is a square binary matrix in which the 1s represent the links in a communication system.

Example 24 Constructing a communication matrix

Eva, Wong, Yumi and Kim are students who are staying in a backpacker's hostel. Because they speak different languages they can have problems communicating. The situation they have to deal with is that:

- Eva speaks English only
- Yumi speaks Japanese only
- Kim speaks Korean only
- Wong speaks English, Japanese and Korean.



Conveniently, this information can be summarised in a network diagram, as shown above. In this diagram, the arrow linking Eva and Wong indicates that they can communicate directly because they both speak English.

The task is to construct a communication matrix.

Solution

1 There are four people so a 4×4 matrix is needed.
Label the columns and rows E, W, Y and K.

2 Label the rows 'Speaker' and the columns 'Receiver'.

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3 Designate each element as a '1' or '0' according to the following rules:

- the element = 1 if two people can communicate directly because they speak the same language
- the element = 0 if two people *cannot* communicate directly because they do not have a common language.

The completed matrix is shown opposite.

Receiver

$$\text{Speaker}\quad \begin{array}{c} \\ E \\ W \\ Y \\ K \end{array}\begin{array}{c} \begin{array}{cccc} E & W & Y & K \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

There is little point in having a matrix representation of a communication system if we already have a network diagram. However, several questions that are not so easily solved with a network diagram can be answered using a communication matrix.

For example, we can see from the network diagram that Eva, who speaks only English, cannot communicate directly with Yumi, who speaks only Japanese. We call this a *one-step communication link*.

However, Eva can communicate with Yumi by sending a message via Wong, who speaks both English and Japanese. In the language of communication systems we call this a *two-step communication link*.

The power of the matrix representation is that *squaring* the communication matrix generates a matrix that identifies all possible *two-step communication links* in a communication network. For example, if we call the communication matrix C, we have:

$$C^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

To understand what the numbers in the squared matrix are telling us, we need to look more closely at how they were generated.

For example, the 1 in row E and column Y is generated by multiplying row E by column Y:

$$\text{row } E\text{: } \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \times \overset{\text{column } Y}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} = 0 \times 0 + 1 \times 1 + 0 \times 0 + 0 \times 0 = 1$$

In row E, the 1 represents the *one-step communication link*: Eva → Wong.

In column Y, the 1 represents the *one-step communication link*: Wong → Yumi.

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Multiplying these two 1s together then represents the *two-step link*: Eva → Wong → Yumi.

That is, Eva can communicate with Yumi by using Wong as a go-between.

However, if we use the same process to help us interpret the ‘1’ in row E, column E, we will find that it represents the two-step link Eva → Wong → Eva. This is not a very useful thing to know. Two-step links that have the same sender and receiver are said to be **redundant communication links** because they do not contribute to the communication between different people.

Redundant communication links

A communication link is said to be redundant if the sender and the receiver are the same person.

All of the non-zero elements in the leading diagonal of a communication matrix, or its powers, represent redundant links in the matrix.

However, all of the remaining non-zero elements represent meaningful two-step communication links.

For example, the 1 in row Y, column K represents the two-step communication link that enables Yumi to send a message to Kim.

$$C^2 = \begin{array}{c c} \begin{array}{cccc} E & W & Y & K \end{array} & \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} & \begin{array}{c} E \\ W \\ Y \\ K \end{array} \end{array}$$

Finally, the total number of one and two-step links in a communication system, T, can be found by evaluating $T = C + C^2$.

$$T = C + C^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{array}{c c} \begin{array}{cccc} E & W & Y & K \end{array} & \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} E \\ W \\ Y \\ K \end{array} \end{array}$$

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Analysing communication matrices

- A communication matrix (C) is a square binary matrix in which the 1s are used to identify the direct (one-step) links in the communication system.
- The number of two-step links in a communication system can be identified by squaring its communication matrix.
- The total number of one and two-step links in a communication system can be found by evaluating the matrix sum $T = C + C^2$.

These statements can be readily generalised to include the determination of three (or more) step links by evaluating the matrices C^3, C^4, etc. However, unless the communication networks are extremely large, most of the multi-step links identified will be redundant.

Note: In all cases, the diagonal elements of a communication matrix (or its power) represent redundant communication links.

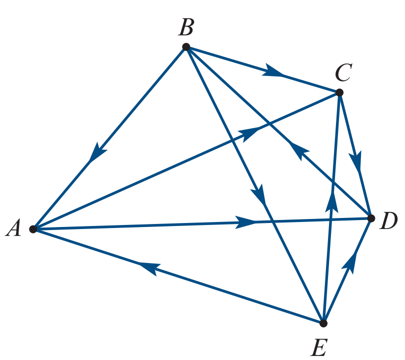
Dominance

In many group situations, certain individuals are said to be dominant. This is particularly true in sporting competitions. Problems of identifying dominant individuals in a group can be analysed using the same approach we used to analyse communication networks.

For example, five players – Anna, Birgit, Cas, Di and Emma – played in a **round-robin tournament**[3] of tennis to see who was the dominant (best) player.

The results were as follows:

- Anna defeated Cas and Di
- Birgit defeated Anna, Cas and Emma
- Cas defeated Di
- Di defeated Birgit
- Emma defeated Anna, Cas and Di.



We can use a network diagram to display the results graphically, as shown opposite. In this diagram, the arrow from B to A tells us that, when they played, Birgit defeated Anna.

Both Birgit and Emma had three wins each so there is a tie. How can we resolve this situation and see who is the best player? One way of doing this is to calculate a dominance score for each player. We do this by a constructing a series of dominance matrices.

[3] A round-robin tournament is one in which each of the participants play each other once.

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One-step dominances

The first **dominance matrix**, D, records the number of one-step dominances between the players.

For example, Anna has a one-step dominance over Cas because, when they played, Anna *beat* Cas.

$$D = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Dominance} \\ 2 \\ 3 \\ 1 \\ 1 \\ 3 \end{array}$$

This matrix can be used to calculate a one-step dominance score for each player, by summing each of the rows of the matrix. According to this analysis, B and E are equally dominant with a dominance score of 3.

Now let us take into account two-step dominances between players.

Two-step dominances

A two-step dominance occurs when a player beats another player who has beaten someone else. For example, Birgit has a two-step dominance over Di because Birgit defeated Cas who defeated Di.

Two-step dominances can be determined using the same technique used to obtain two-step links in a communication network. We simply square the one-step dominance matrix. The two-step dominances for these players are shown in matrix D^2.

$$D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Dominance} \\ 2 \\ 6 \\ 1 \\ 3 \\ 4 \end{array}$$

We can combine the information contained in both D and D^2 by calculating a new matrix $T = D + D^2$.

$$T = D + D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Total} \\ 4 \\ 9 \\ 2 \\ 4 \\ 7 \end{array}$$

Using these total dominance scores:

- Birgit is the top-ranked player with a total dominance score of 9
- Eva is second with a total score of 7
- Anne and Di are equal third with a total score of 4
- Cas is the bottom-ranked player with a total score of 2.

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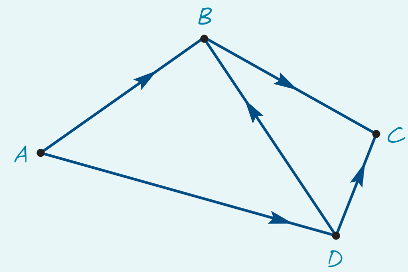
Example 25 Determining dominance

Four people, A, B, C and D, have been asked to form a committee to decide on the location of a new toxic waste dump.



From previous experience, it is known that:

- A influences the decisions of B and D
- B influences the decisions of C
- C influences the decisions of no one
- D influences the decisions of C and B.

a Use the graph to construct a dominance matrix that takes into account both one-step and two-step dominances.

b From this matrix, determine who is the most influential person on the committee.

Solution

a Construct the one-step dominance matrix D.

$$D = \begin{array}{c c c} & \begin{array}{cccc} A & B & C & D \end{array} & \text{One-step} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} 2 \\ 1 \\ 0 \\ 2 \end{array} \end{array}$$

Construct the two-step dominance matrix D^2.

$$D^2 = \begin{array}{c c c} & \begin{array}{cccc} A & B & C & D \end{array} & \text{Two-step} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} 3 \\ 0 \\ 0 \\ 1 \end{array} \end{array}$$

Form the sum $T = D + D^2$.

$$T = D + D^2 = \begin{array}{c c c} & \begin{array}{cccc} A & B & C & D \end{array} & \text{Total} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} & \begin{array}{c} 5 \\ 1 \\ 0 \\ 3 \end{array} \end{array}$$

b The person with the highest total dominance score is the most influential.

Person A is the most influential person with a total dominance score of 5.

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Cambridge University Press

Exercise 11F

Permutation matrices

1 Which of the following binary matrices are permutation matrices?

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2 X is the row matrix: $X = \begin{bmatrix} \mathrm{S} & \mathrm{H} & \mathrm{U} & \mathrm{T} \end{bmatrix}$

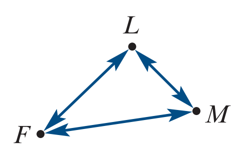
P is the permutation matrix: $P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

a What does matrix X change to if it is post-multiplied by P?

b For what value of n does XP^n first equal X?

Communication matrices

3 Freya (F), Lani (L) and Mei (M) are close friends who regularly send each other messages. The direct (one-step) communication links between the friends are shown in the diagram opposite.

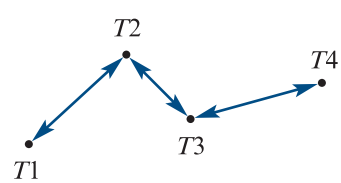


a Construct a communication matrix C from this diagram.

b Calculate C^2.

c How many different ways can Mei send a message to Freya?

4 Four fire towers $T1$, $T2$, $T3$ and $T4$, can communicate with one another as shown in the diagram opposite. In this diagram an arrow indicates that a direct channel of communication exists between a pair of fire towers.



For example, a person at tower 1 can directly communicate with a person in tower 2 and vice versa.

The communication matrix C can be also used to represent this information.

$$C = \begin{array}{c c} \begin{array}{cccc} T1 & T2 & T3 & T4 \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \square & 0 \\ 0 & 1 & 0 & 1 \\ 0 & \square & 1 & 0 \end{bmatrix} & \begin{array}{c} T1 \\ T2 \\ T3 \\ T4 \end{array} \end{array}$$

a Explain the meaning of a zero in the communication matrix.

b Which two towers can communicate directly with $T2$?

c Write down the values of the two missing elements in the matrix.

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The matrix C^2 is shown opposite.

$$C^2 = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} T1 \quad T2 \quad T3 \quad T4 \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ T1 \\ T2 \\ T3 \\ T4 \end{array}$$

d Explain the meaning of the 1 in row $T3$, column $T1$.

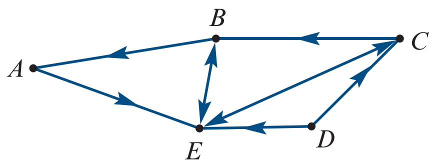
e How many of the two-step communication links shown in the matrix C^2 are redundant?

f Construct a matrix that shows all one and two-step communication links.

g Which of the four towers need a three-step link to communicate with each other?

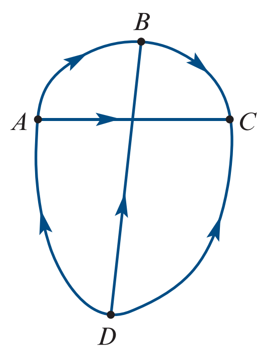
Representing a large network diagram by a matrix

5 Construct a 5×5 matrix to represent the communication network diagram opposite.



Dominance matrices

6 The results of a competition between teams A, B, C and D are displayed opposite. An arrow from D to C indicates that team D defeated team C.



a Construct a dominance matrix showing one-step dominance between the teams. Rank the teams according to one-step dominances.

b Construct a dominance matrix showing two-step dominances between the teams. Rank the teams, taking into account both one-step and two-step dominances.

7 Five students play each other at chess. The dominance matrix shows the winner of each game with a '1' and the loser or no match with a '0'. For example, row 2 indicates that B loses to A, D and E but beats C.

a Find the one-step dominance score for each student and use these to rank them.

b Calculate the two-step dominance matrix.

c Determine the matrix $T = D + D^2$ and use this matrix to rank the players.

$$\text{Winners} \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \text{Losers} \\ A \quad B \quad C \quad D \quad E \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

ISBN 978-1-107-56757-3

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Key ideas and chapter summary

Matrix A **matrix** is a rectangular array of numbers or symbols (elements) enclosed in brackets (plural: matrices).

Row matrix A **row matrix** contains a *single row* of elements.

Column matrix A **column matrix** contains a *single column* of elements.

Transpose The **transpose** of a matrix is obtained by interchanging its rows and columns.

Square matrix A **square matrix** has an **equal number of rows and columns**.

Zero matrix A **zero (null) matrix**, O, contains only zeros.

Identity matrix An **identity matrix**, I, is a square matrix in which the diagonal elements are 1s and the remaining elements are zeroes.

Binary matrix A **binary matrix** is a matrix whose elements are either zeros or ones.

Permutation matrix A **permutation matrix** is a square binary matrix in which there is only a single 1 in each row and column.

Order The **order** (or size) of a matrix is given by the number of rows and columns in that order.

Locating an element The location of each element in the matrix is specified by its row and column number in that order.

Equal matrices Matrices are *equal* when they have the *same order* and *corresponding elements* are *equal* in value.

Adding and subtracting matrices Two matrices of the same order can be added or subtracted, by adding or subtracting corresponding elements.

Scalar multiplication Multiplying a matrix by a number (**scalar multiplication**) multiplies every element in the matrix by that number.

Matrix multiplication **Matrix multiplication** is a process of multiplying rows by columns. To multiply a row matrix by a column matrix, each element in the row matrix is multiplied by each element in the column matrix and the results added. For example:

$$\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = [1 \times 4 + 0 \times 2 + 3 \times 5] = [19]$$

Power of a matrix The **power of a matrix** is defined in the same way as the powers of numbers: $A^2 = A \times A$, $A^3 = A \times A \times A$, and so on.
Only **square matrices** can be raised to a power.
A^0 is defined to be I, the **identity matrix**.

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Communication matrix A **communication matrix** is a square binary matrix in which the 1s represent direct (one-step) communication links.

Redundant communication link A communication link is said to be redundant if the sender and the receiver are the same people.

Round-robin tournament A **round-robin tournament** is one in which each of the participants plays every other competitar once.

Dominance matrix A **dominance matrix** is a square binary matrix in which the 1s represent one-step dominances between the members of a group.

Skills check

Having completed this chapter you should be able to:

- determine the order of a matrix
- recognise a row, column and square matrix
- recognise a zero, unit, binary and permutation matrix
- determine the transpose of a matrix
- when appropriate, add or subtract two matrices
- multiply a matrix by a number (scalar multiplication)
- when appropriate, multiply two matrices
- solve practical problems involving matrices, including the use of permutation, communication and dominance matrices.

Multiple-choice questions

The following matrices are needed for Questions 1 to 8.

$$U = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad V = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad W = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad X = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad Z = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

1 The row matrix is:

A U **B** V **C** W **D** X **E** Z

2 The square matrices are:

A U and V **B** X and Y **C** Y and W **D** U and Y **E** U, V and X

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Review

3 The order of matrix X is:

A 2×2 **B** 2×3 **C** 3×2 **D** 3×3 **E** 6

4 The following matrices can be added.

A U and V **B** V and W **C** X and Y **D** U and Y **E** none of the above

5 The following matrix product is *not* defined.

A WV **B** XZ **C** YV **D** XY **E** UY

6 $-2Y =$

A $\begin{bmatrix} 0 & -2 \\ 2 & -4 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -2 \\ -2 & 4 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} -2 & -1 \\ -4 & 2 \end{bmatrix}$

7 The order of matrix product XZ is:

A 1×3 **B** 2×1 **C** 3×1 **D** 3×2 **E** 3×3

8 $U^T =$

A $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

9 In the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 3 \\ -5 & -4 & 7 \end{bmatrix}$, the element $a_{23} =$

A -4 **B** -1 **C** 0 **D** 3 **E** 4

10 $2\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} =$

A $\begin{bmatrix} 5 & 0 \\ -4 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & 0 \\ 4 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 0 \\ -2 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 6 & 0 \\ 1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix}$

11 $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} =$

A $\begin{bmatrix} 10 \end{bmatrix}$ **B** $\begin{bmatrix} 12 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$ **E** not defined

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12 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$

A $\begin{bmatrix} 18 \end{bmatrix}$ **B** $\begin{bmatrix} 12 \end{bmatrix}$ **C** $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ **D** $\begin{bmatrix} 4 \\ 10 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}$

13 X is a 3×2 matrix. Y is a 2×3 matrix. Z is a 2×2 matrix. Which of the following matrix expressions is *not* defined?

A XY **B** YX **C** $XZ - 2X$ **D** $YX + 2Z$ **E** $XY - YX$

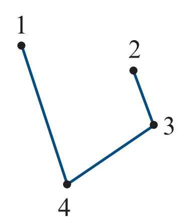
14 $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$.

The matrix expression that displays the mean of the numbers 3, 5, 2, 4 is:

A $\frac{1}{4}(A + B)$ **B** $\frac{1}{2}(A + B)$ **C** $\frac{1}{4}B$ **D** $\frac{1}{4}AB$ **E** $\frac{1}{4}BA$

15 The diagram opposite is to be represented by a matrix, A, where:

- element = 1 if the two points are joined by a line
- element = 0 if the two points are not connected.



The matrix A is:

A $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

16 The matrix equation $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ generates the following pair of simultaneous linear equations.

A $\begin{aligned} x + 4y &= 2 \\ 3x - 2y &= 2 \end{aligned}$ **B** $\begin{aligned} 3x - 2y &= 1 \\ x + 4y &= 2 \end{aligned}$ **C** $\begin{aligned} x + 2y &= 1 \\ 4x + y &= 2 \end{aligned}$

D $\begin{aligned} 3x + 2y &= 1 \\ x - 4y &= 2 \end{aligned}$ **E** $\begin{aligned} x + 2y &= 3 \\ 4x - y &= -2 \end{aligned}$

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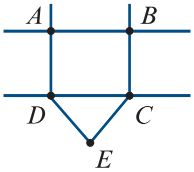
Review

17 A, B, C, D and E are five intersections joined by roads, as shown in the diagram opposite. Some of these roads are one-way only.



The matrix opposite indicates the direction that cars can travel along each of these roads.

In this matrix:

- the 1 in column A and row B indicates that cars can travel directly from A to B
- the 0 in column B and row A indicates that cars cannot travel directly from B to A (either it is a one-way road or no road exists).

From intersection

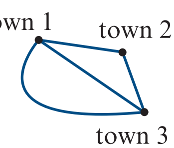
$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ & \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} A \\ B \\ C \ \textit{To intersection} \\ D \\ E \end{array} \end{array}$$

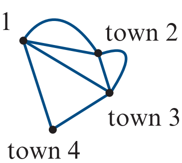
Cars can travel in both directions between intersections:

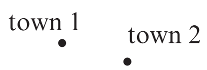
A A and D **B** B and C **C** C and D **D** D and E **E** C and E

Extended-response questions

1 The following diagrams represent the road network joining several towns. Represent each road by a matrix.

a 

b 

c 

2 Heights in feet and inches can be converted into centimetres using matrix multiplication. The matrix $C = \begin{bmatrix} 30.45 \\ 2.54 \end{bmatrix}$ can be used as a conversion matrix (1 foot = 30.45 cm and 1 inch equals 2.54 cm).

a What is the order of matrix C?

Jodie tells us that her height is 5 feet 4 inches. We can write her height as a matrix $J = \begin{bmatrix} 5 & 4 \end{bmatrix}$.

b What is the order of matrix J?

c Is the matrix product JC defined? Why?

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d Evaluate the matrix product JC. Explain why it gives Jodie's height in centimetres.

e Matrix $H = \begin{bmatrix} 5 & 8 \\ 6 & 1 \end{bmatrix}$ gives the heights in feet and inches of two other people.

Use the conversion matrix C and matrix multiplication to generate a matrix that displays the heights of these four people in centimetres.

3 Books can be classified as *fiction* or *non-fiction* and come in either *hardback* or *paperback* form. The table shows the number of book titles carried by two bookshops in each of the categories.

	Bookshop 1		*Bookshop 2*	
Number of titles	*Hardback*	*Paperback*	*Hardback*	*Paperback*
Fiction	334	876	354	987
Non-fiction	213	456	314	586

a How many non-fiction paperback titles does bookshop 1 carry?

b The matrix $A = \begin{bmatrix} 334 & 876 \\ 213 & 456 \end{bmatrix}$ displays the number of book titles available at bookshop 1 in all categories. What is the order of this matrix?

c Write down a matrix equivalent to matrix A that displays the number of book titles available at bookshop 2. Call this matrix B.

d Construct a new matrix, $C = A + B$. What does this matrix represent?

e The average cost of books is \$45 for a hardback and \$18.50 for a paperback. These values are summarised in the matrix $E = \begin{bmatrix} 45.00 \\ 18.50 \end{bmatrix}$.

i What is the order of matrix E?

ii Construct the matrix product AE and evaluate.

iii What does the product AE represent?

f Bookshop 1 plans to double the number of titles it carries in every category. Write down a matrix expression that represents this situation and evaluate.

4 Biology and Chemistry are offered in a first year university science course.

The matrix $N = \begin{array}{c} \begin{bmatrix} 460 \\ 360 \end{bmatrix} \end{array}\begin{array}{l} \textit{Biology} \\ \textit{Chemistry} \end{array}$ lists the number of students enrolled in each subject.

The matrix $P = \begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0.05 & 0.125 & 0.175 & 0.45 & 0.20 \end{bmatrix} \end{array}$ lists the proportion of these students expected to be awarded an A, B, C, D or E grade in each subject.

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Review

a Write down the order of matrix P.

b Let the matrix $R = NP$.

 i Evaluate the matrix R.

 ii Explain what the matrix element R_{24} represents.

c Students enrolled in Biology have to pay a laboratory fee of \$110, while students enrolled in Chemistry pay a laboratory fee of \$95.

 i Write down a clearly labelled row matrix, called F, that lists these fees.

 ii Show a matrix calculation that will give the total laboratory fees, L, paid in dollars by the students enrolled in Biology and Chemistry. Find this amount.

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5 In a game of basketball, a successful shot for goal scores one point, two points, or three points, depending on the position from which the shot is thrown. G is a column matrix that lists the number of points scored for
each type of successful shot.

$$G = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In one game, Oscar was successful with

- 4 one-point shots for goal
- 8 two-point shots for goal
- 2 three-point shots for goal.

a Write a row matrix, N, that shows the number of each type of successful shot for goal that Oscar had in that game.

b Matrix P is found by multiplying matrix N with matrix G so that $P = N \times G$. Evaluate matrix P.

c In this context, what does the information in matrix P provide?

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ISBN 978-1-107-56757-3

Chapter 12

12

Module 1: Matrices

Matrices II

ISBN 978-1-107-56757-3

Cambridge University Press

12A The inverse matrix

The inverse matrix A^{-1}

So far, you have been shown how to add, subtract and multiply matrices, but what about dividing them? As you might expect matrix division, like matrix multiplication, is a more complicated process than its equivalent process for dividing numbers.

The starting point for matrix division is the **inverse matrix**. You will see why as we proceed.

The inverse matrix A^{-1}

The inverse of a square matrix A is called A^{-1}.

The inverse matrix has the property $AA^{-1} = A^{-1}A = I$.

Having defined the inverse matrix, two questions immediately come to mind. Does the inverse of a matrix actually exist? If so, how can we calculate it?

First we will demonstrate that at least some matrices have inverses. We can do this by showing that two matrices, which we will call A and B, have the property $AB = I$ and $BA = I$, where I is the **identity matrix**. If this is the case, we can then say that $B = A^{-1}$.

Example 1 Demonstrating that two matrices are inverses

Show that the matrices $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ are inverses.

Solution

1 Write down A and B.

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

2 Form the product AB and evaluate. You can use your calculator to speed things up if you wish.

$$AB = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 5 + 3 \times (-3) & 2 \times (-3) + 3 \times 2 \\ 3 \times 5 + (-5) \times 3 & 3 \times (-3) + 5 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore AB = I$$

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3 Form the product BA and evaluate. You can use your calculator here to speed things up if you wish.

$$BA = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \times 2 + (-3) \times 3 & 5 \times 3 + (-3) \times 5 \\ (-3) \times 2 + 2 \times 3 & (-3) \times 3 + 2 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore BA = I$

4 Write down your conclusion.

Because $AB = I$ and $BA = I$, we conclude that A and B are inverses.

While Example 1 clearly demonstrates that the matrices $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ both have an inverse, many square matrices do *not* have inverses. To see why, we need to introduce another new matrix concept, the **determinant**, and see how it relates to finding the inverse of a square matrix. To keep things manageable, we will restrict ourselves initially to 2×2 matrices.

The determinant of a matrix

The determinant of a 2 × 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of matrix A is given by:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$$

Example 2 Finding the determinant of a 2 × 2 matrix

Find the determinant of the matrices:

a $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ **b** $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ **c** $C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$

Solution

1 Write down the matrix and use the rule $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$.

2 Evaluate.

a $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $\quad \therefore \det(A) = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 2 \times 5 - 3 \times 3 = 1$

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b $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ $\quad \therefore \ \det(B) = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 3 = 0$

c $C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ $\quad \therefore \ \det(C) = \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 4 = -2$

From Example 2, we can see that the determinant of a matrix is a number that can take on both positive and negative values as well as being zero. For a matrix to have an inverse, its determinant must be non-zero.

How to determine the inverse of a 2 × 2 matrix

Normally you will use a calculator to determine the inverse of a matrix, but we need to do the following example by hand to show you why some square matrices do not have an inverse. To do this we first need to consider the rule for finding the determinant of a 2×2 matrix.

The rule for finding the inverse of a 2 × 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its inverse, A^{-1}, is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $\dfrac{1}{ad - bc} \neq 0$; that is, provided $\det(A) \neq 0$.

The most important thing about this rule is that it shows immediately why you cannot calculate an inverse for some matrices. These are the matrices whose determinant is zero.

Example 3 Using the rule to find the inverse of a 2 × 2 matrix

Find the inverse of the following matrices.

a $A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

Solution

1 Write down the matrix and use the rule

$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$

to evaluate the determinant.

a $A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$

$\det(A) = \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = 2 \times 4 - 3 \times 2 = 2$

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Use the rule

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

to evaluate A^{-1}.

$$\therefore A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1.5 & 1 \end{bmatrix}$$

b

2 Write down the matrix and use the rule $\det(B) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$. to evaluate the determinant.

$$B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\therefore \det(B) = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 3 = 0$$

$\det(B) = 0$

$\therefore$ B does not have an inverse.

▶ Using a CAS calculator to determine the determinant and inverse of an $n \times n$ matrix

There are rules for finding the inverse of a square matrix of any size, but in practice,we tend to use a calculator. The same goes for calculating determinants, although the inverse and determinant of a 2×2 matrix is often computed more quickly by hand.

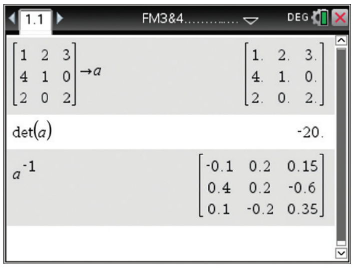
How to find the determinant and inverse of a matrix using the TI-Nspire CAS

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, find $\det(A)$ and A^{-1}.

Steps

1. Press [on] >**New Document>Add Calculator**.
2. Enter the matrix A into your calculator.
3. To calculate $\det(A)$, type **det(*a*)** and press [enter] to evaluate.

   Note: det() can also be accessed using [menu]>**Matrix & Vector>Determinant**.
4. To calculate the inverse matrix A^{-1} type **a ∧ – 1** and press [enter] to evaluate. If you want to see the answer in fractional form, enter as **exact (a∧ –1)** and press [enter] to evaluate.



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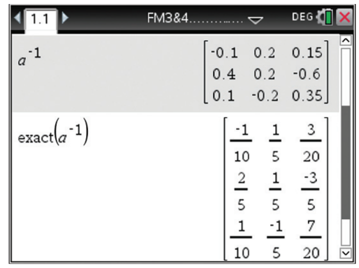
Note:

1 Long strings of decimals can be avoided by asking for an exact inverse. Type in **exact(a^{-1})**.

2 If the matrix has no inverse, the calculator will respond with the error message **Singular matrix**.



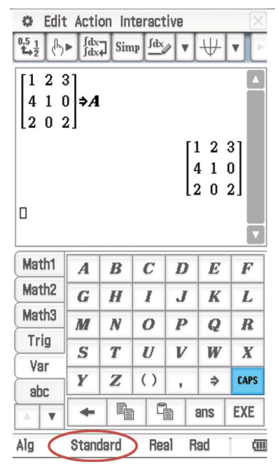
How to find the determinant and inverse of a matrix using the ClassPad

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, find det($A$) and A^{-1}.

Steps

1 Enter the matrix A into your calculator.

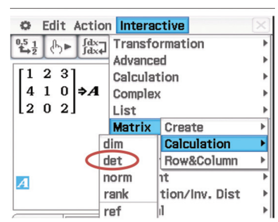
Note: Change the status of the calculator to **Standard** for fractions to be displayed. Tapping on **Decimal** will change the calculator to **Standard**.



2 To calculate det(A):

a type and highlight ***A*** (by swiping with the stylus)

b select **Interactive** from the menu bar, tap **Matrix-Calculation**, then tap **det**.

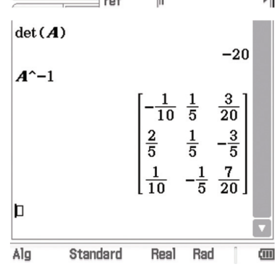


3 To calculate the inverse matrix A^{-1}:

a type ***A*^–1**

b press EXE to evaluate.

Note: If the matrix has no inverse, the calculator will respond with the message **Undefined**.



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Exercise 12A

Review of the properties of the identity matrix

1 **a** Write down the:

i 2×2 identity matrix **ii** 3×3 identity matrix **iii** 4×4 identity matrix.

b If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, show that $AI = IA = A$.

c If $C = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, show that $CI = IC = C$.

Demonstrating that one matrix is the inverse of the other

2 Show that each of the following pairs of matrices are inverses by multiplying one by the other. Use a calculator if you wish.

a $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -1.5 & 1 \end{bmatrix}$

c $\begin{bmatrix} 9 & 7 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} -3 & 7 \\ 4 & -9 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$

Calculating the determinant of a matrix

3 Determine (by hand) the value of the determinant for each of the following matrices.

a $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ **b** $B = \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}$ **c** $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ **d** $D = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$

Calculating the inverse of a matrix

4 Use a calculator to determine the inverse of each of the following matrices.

a $A = \begin{bmatrix} 1.1 & 2.2 \\ 0 & 3.0 \end{bmatrix}$ **b** $B = \begin{bmatrix} 0.2 & -0.1 \\ 10 & 4 \end{bmatrix}$ **c** $D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **d** $E = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

12B Applications of the inverse matrix: solving simultaneous linear equations

One application of the inverse matrix is to use it to help solve a system of simultaneous equations written in linear form.

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Solving a set of simultaneous equations written in matrix form

In the previous chapter, we saw that the pair of simultaneous linear equations

$$4x + 2y = 5$$
$$3x + 2y = 2$$

could be written as the matrix equation:

$$\begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

We will now show that inverse matrices can be used to solve this matrix equation.

Let $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $C = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

Then $AX = C$.

Assuming that A^{-1}, the inverse of A, is defined, we have:

$$A^{-1}AX = A^{-1}C$$

or $$IX = A^{-1}C$$

or $$X = A^{-1}C$$

Note: The order of multiplication is important here. It is $A^{-1}C$, *not* CA^{-1}.

Noting $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1.5 & 2 \end{bmatrix}$ and substituting $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $C = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1.5 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 5 + (-1) \times 2 \\ (-1.5) \times 5 + 2 \times 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.5 \end{bmatrix} \text{ (or use a calculator)}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3.5 \end{bmatrix} \text{ or } x = 3 \text{ and } y = -3.5$$

Thus, using the matrices, the process of solving a pair of simultaneous linear equations can be reduced to finding the inverse of a matrix and performing a matrix multiplication. While there is little to gain over algebraic methods for a pair of simultaneous equations, the same matrix method can be used to solve three, four or more simultaneous linear equations. Using algebraic methods is a very time-consuming task.

Before you go on to practise this method, you need to know when it does not work and why. It has to do with the fact that not all square matrices have inverses. First, we need to recognise that not all pairs of simultaneous linear equations have solutions. This can happen for two reasons:

1. the equations are **inconsistent**: their graphs are parallel and thus do not cross
2. the equations are **dependent**: the graphs coincide and thus do not cross at a single point.

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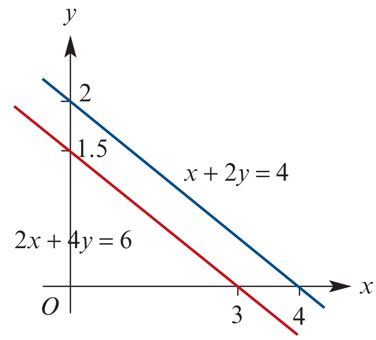
Inconsistent equations

The pair of simultaneous linear equations

$$x + 2y = 4$$
$$2x + 4y = 6$$

has *no solution*. We can see why, by plotting their graphs (opposite). When plotted, the two equations are *parallel lines* with no point of intersection. Sets of equations that have this property are said to be *inconsistent*.



If we try to solve these equations by solving the matrix equation:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

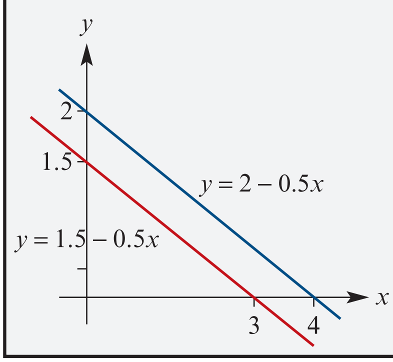
we find that it cannot be done. This is because the determinant of the coefficient matrix is:

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$
$$= 1 \times 4 - 2 \times 2$$
$$= 0$$

Calculator hint: To construct this plot on a graphics calculator, rewrite the equations in the form $y = a + bx$.

$x + 2y = 4$ *or* $y = 2 - 0.5x$

$2x + 4y = 6$ *or* $y = 1.5 - 0.5x$



Summary

As a general rule, if a pair of linear equations is *inconsistent*, $\det(A) = 0$. Inconsistent equations have *no* solution as can be seen from the graphs above.

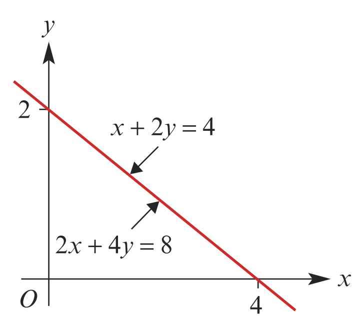
Dependent equations

The pair of simultaneous linear equations

$$x + 2y = 4$$
$$2x + 4y = 8$$

has *no solution*. We can see why, by plotting a graph (opposite).

When plotted, the graphs of the two equations *coincide*, with no single point of intersection. There is *no unique solution*.



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Rather, there are an infinite number of possible solutions; i.e. all the pairs of points that lie on the line. When this happens, we say that the equations are *dependent*.

If we try to solve these equations by solving the matrix equation

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

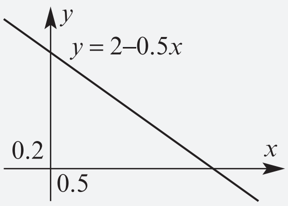
we find that it cannot be done. As before, the determinant of the coefficient matrix is:

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 2 = 0$$

Calculator hint: To construct this plot on a graphics calculator, rewrite the equations in the form $y = a + bx$.

$x + 2y = 4$ *or* $y = 2 - 0.5x$

$2x + 4y = 8$ *or* $y = 2 - 0.5x$



Summary

As a general rule, if a pair of linear equations is *dependent*, $\det(A) = 0$. Dependent equations have *no unique solution*.

Solving a set of linear equations

Provided that $\det(A) \neq 0$, the set of linear equations defined by the matrix equation $AX = C$ has the unique solution:

$$X = A^{-1}C$$

Note: The order of multiplication is important here: $X = A^{-1}C$, *not* CA^{-1}.

For the set of equations:

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } C = \begin{bmatrix} e \\ f \end{bmatrix}$$

For the set of equations:

$$\begin{aligned} ax + by + cz &= j \\ dx + ey + fz &= k \\ gx + hy + iz &= l \end{aligned} \qquad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } C = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

and so on.

Example 4 Solving a pair of simultaneous linear equations using the inverse matrix

Solve using matrix methods:

$$3x + 4y = 6$$
$$2x + 3y = 4$$

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Solution

1 Rewrite the equations in matrix form.

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

2 By identifying the matrices A, X and C, rewrite the matrix equation in the form $AX = C$.

Let $A \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $C = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

3 Provided that det $(A) \neq 0$, the solution in matrix form is $X = A^{-1}C$.

Then $AX = C$ or $X = A^{-1}C$ ($\det(A) \neq 0$).

4 Enter the matrices A and C into your calculator.

5 Solve the matrix equation by evaluating the matrix product $A^{-1}C$.

$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \to a$	$\begin{bmatrix} 3. & 4. \\ 2. & 3. \end{bmatrix}$
$\begin{bmatrix} 6 \\ 4 \end{bmatrix} \to c$	$\begin{bmatrix} 6. \\ 4. \end{bmatrix}$
$a^{-1}.c$	$\begin{bmatrix} 2. \\ 0. \end{bmatrix}$

6 Write down your answer.

Therefore, $X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ or $x = 2$ and $y = 0$.

However, not all pairs of simultaneous linear equations have unique solutions, as we will see in the next example.

Example 5 A set of two simultaneous linear equations with no unique solution

Solve using matrix methods:

$$3x + 4.5y = 9$$
$$2x + 3y = 4$$

Solution

1 Rewrite the equations in matrix form.

$$\begin{bmatrix} 3 & 4.5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

2 By identifying the matrices A, X and C, rewrite the matrix equation in the form $AX = C$.

Let $A = \begin{bmatrix} 3 & 4.5 \\ 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $C = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$.

3 Provided that $\det(A) \neq 0$, the solution in matrix form is $X = A^{-1}C$. In this case, $\det(A) = 0$ but let us see what happens.

Then $AX = C$ or $X = A^{-1}C$ ($\det(A) \neq 0$).

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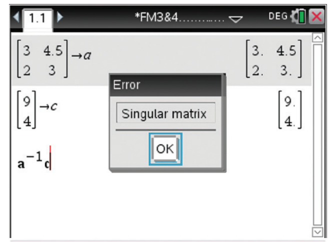
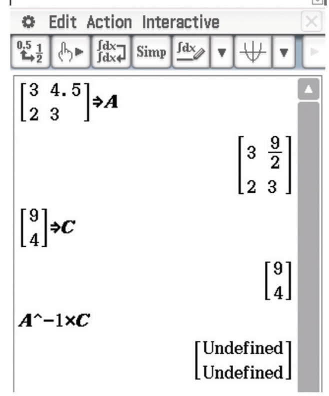
4 Enter the matrices A and C into your calculator.

5 Attempt to solve the matrix equation by evaluating the matrix product $A^{-1}C$.





6 The calculator gives an error message: **Singular matrix** (TI-Nspire) or **Undefined** (Classpad). This is because $\det(A) = 0$. The system of equations does not have a unique solution. Write down your conclusion.

There is no unique solution as $\det(A) = 0$.

The power of the matrix method for solving systems of linear equations becomes apparent when we solve a system of three or more equations.

Example 6 Solving a set of three simultaneous equations using the inverse matrix

Solve using matrix methods:

$$\begin{aligned} 3x + 4y - 2z &= -5 \\ 2x + 3y &= -1 \\ x + 2y + 3z &= 3 \end{aligned}$$

Solution

1 Rewrite the equations in matrix form.

$$\begin{bmatrix} 3 & 4 & -2 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$$

2 By identifying the matrices A, X and C, rewrite the matrix equation in the form $AX = C$.

Note: There is no z term in the second equation, so its coefficient is zero.

Let $A = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $C = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$

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3 Provided that $\det(A) \neq 0$, the solution in matrix form is $X = A^{-1}C$.

Then $AX = C$ or $X = A^{-1}C$ $(\det(A) \neq 0)$.

4 Enter the matrices A and C into your calculator.

5 Solve the matrix equation by evaluating the matrix product $A^{-1}C$.

$$\begin{bmatrix} 3 & 4 & 2 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix} \to a \qquad \begin{bmatrix} 3 & 4 & 2 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix} \to c \qquad \begin{bmatrix} -5 \\ -1 \\ 3. \end{bmatrix}$$

$$a^{-1} \cdot c \qquad \begin{bmatrix} -11 \\ -7 \\ 0 \end{bmatrix}$$

6 Write down your answer.

Therefore, $X = \begin{bmatrix} -11 \\ 7 \\ 0 \end{bmatrix}$ or $x = -11, y = 7$ and $z = 0$.

Example 7 A practical application

A manufacturer makes two sorts of orange-flavoured chocolates: House Brand and Orange Delights. The number of kilograms of House Brand (x) and the number of kilograms of Orange Delights (y) that can be made from 80 kg of chocolate and 120 kg of orange filling can be found by solving the following pair of equations:

$$0.3x + 0.5y = 80$$
$$0.7x + 0.5y = 120$$

Solve for x and y using matrix methods.

Solution

1 Rewrite equations in matrix form.

$$\begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 80 \\ 120 \end{bmatrix}$$

2 By identifying the matrices A, X and C, rewrite the matrix equation in the form $AX = C$.

Let $A = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $C = \begin{bmatrix} 80 \\ 120 \end{bmatrix}$.

3 Provided that $\det(A) \neq 0$, the solution in matrix form is $X = A^{-1}C$.

Then $AX = C$
or $X = A^{-1}C$ $(\det(A) \neq 0)$

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4 Enter the matrices A and C into your calculator and solve the matrix equation by evaluating the matrix product $A^{-1}C$.

$$\begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix} \rightarrow a \qquad \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 80 \\ 120 \end{bmatrix} \rightarrow c \qquad \begin{bmatrix} 80 \\ 120 \end{bmatrix}$$

$$a^{-1} \cdot c \qquad \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

5 Write down your answer.

Therefore, $X = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ or $X = 100$ and $Y = 100$.

Exercise 12B

Writing a set of simultaneous equations in matrix form

1 Write each of the following systems of linear equations in matrix form.

a $3x + 2y = 2$
$2x + 5y = 4$

b $3x + 5y = 6$
$2x + 4y = 3$

c $5x - 2z = 3$
$x - y + z = 2$
$x + y + z = 1$

Important concepts

2 Give two explanations of how a set of two linear equations can have no unique solution.

3 What is the condition for the matrix equation $AX = C$ *not* to have a unique solution?

Using the determinant test to identify simultaneous equations with no solution

4 By evaluating an appropriate determinant, determine which of the following matrix equations have *no* solution.

a $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **c** $\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Using the inverse matrix to solve matrix equations

5 Solve the following matrix equations. (Not all have solutions.)

a $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Solving a system of simultaneous equations using matrix methods

6 Solve the following systems of linear equations using matrix methods. (Not all have solutions.)

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a $3x - 5y = 9$
$2x - 3y = 12$

b $x + 3y = 4$
$x + 5y = 10$

c $3x - 2y = 4$
$9x - 6y = 19$

d $x + 4y - 2z = 8$
$2x + 3y + 2z = 4$
$x + 2y - z = 3$

Multiple-choice questions

7 How many of the following five sets of simultaneous linear equations have a unique solution?

$4x + 2y = 10$, $2x + y = 5$ $x = 0$, $x + y = 6$ $x + y = 3$, $x + y = 3$ $2x + y = 5$, $2x + y = 16$ $x = 8$, $y = 2$

A 1 **B** 2 **C** 3 **D** 4 **E** 5

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8 A system of three simultaneous linear equations is written in matrix form as follows.

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$$

One of the three linear equations is:

A $x - 2y + z = 4$ **B** $x + y + 3z = 11$ **C** $2x - y = -5$
D $x + 3z = 11$ **E** $3y - z = -5$

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9 The solution of the simultaneous equations opposite is given by:

$$x + z = 6$$
$$2y + z = 8$$
$$2x + y + 2z = 15$$

A $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 0 & 1 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

B $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

C $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1.5 & 0.5 & 0.5 \\ -1 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

D $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

E $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

Practical applications

10 An outdoor clothing manufacturer makes two sorts of jackets, Polarbear and Polarfox. The number of Polarbear jackets (x) and Polarfox jackets (y) that can be made from 150 metres of fabric and 150 hours of worker time can be found by solving the following system of equations:

$$2x + 2y = 150$$
$$2.4x + 1.8y = 150$$

Solve for x and y using matrix methods.

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11 An advertising company plans to run an advertising campaign in three regional cities. Advertisements will be run on TV, radio and in the local newspaper. The costs of running the same advertising program in the three cities are \$175 000, \$149 000 and \$183 500 respectively.
Find the number of advertisements that were run on TV (x), on radio (y) and in the local newspaper (z) by solving the following system of equations:

$$3000x + 1000y + 500z = 175\ 000$$

$$2500x + 900y + 600z = 149\ 000$$

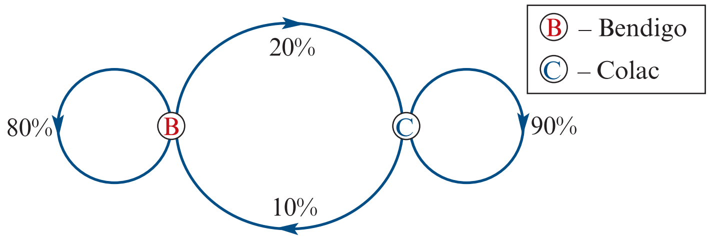
$$3100x + 1100y + 650z = 183\ 500$$

Solve for x, y and z using matrix methods.

12C Transition matrices and their applications

▶ Setting up a transition matrix

A car rental firm has two branches: one in Bendigo and one in Colac. Cars are usually rented and returned in the same town. However, a small percentage of cars rented in Bendigo each week are returned in Colac, and vice versa. The diagram below describes what happens on a weekly basis.



What does this diagram tell us?

From week to week:

- 0.8 (or 80%) of cars rented each week in Bendigo are returned to Bendigo
- 0.2 (or 20%) of cars rented each week in Bendigo are returned to Colac
- 0.1 (or 10%) of cars rented each week in Colac are returned to Bendigo
- 0.9 (or 90%) of cars rented each week in Colac are returned to Colac.

The percentages (written as proportions) are summarised in the form of the matrix below.

$$\begin{array}{ll} & \qquad\qquad \textit{Rented in} \\ & \quad \textit{Bendigo} \quad \textit{Colac} \\ \textit{Returned to} \begin{array}{r} \textit{Bendigo} \\ \textit{Colac} \end{array} & \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \end{array}$$

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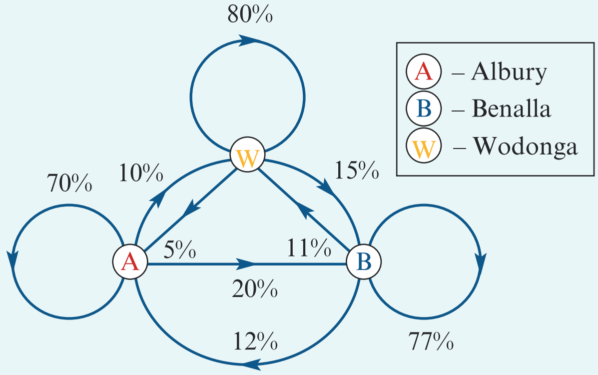
This matrix is an example of a **transition matrix (T)**. It describes the way in which transitions are made between two *states*:

- state 1: the rental car is based in Bendigo.
- state 2: the rental car is based in Colac.

Note: In this situation, where the total number of cars remains constant, the columns in a transitional matrix will always add to one (100%). For example, if 80% of cars are returned to Bendigo, then 20% must be returned to Colac.

Example 8 Setting up a transition matrix

The diagram gives the weekly return rates of rental cars at three locations: Albury, Wodonga and Benalla. Construct a transition matrix that describes the week-by-week return rates at each of the three locations. Convert the percentages to proportions.



Solution

1 There are three locations from which the cars can be rented and returned: Albury (A), Wodonga (W) and Benalla (B). To account for all the possibilities, a 3×3 matrix is needed. Construct a blank matrix labelling the rows and columns A, W and B, respectively. Column labels indicate where the car was rented. The row labels indicate where the cars were returned to.

$$\begin{array}{cc} & \text{Rented in} \\ & \begin{array}{ccc} A & W & B \end{array} \\ \text{Returned to} \begin{array}{c} A \\ W \\ B \end{array} & \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \end{array}$$

2 Complete the matrix by writing each of the percentages (converted to proportions) into the appropriate locations. Start with column A and write in values for each row: 0.7 (70%), 0.1 (10%) and 0.2 (20%).

$$\begin{array}{cc} & \begin{array}{ccc} A & W & B \end{array} \\ \begin{array}{c} A \\ W \\ B \end{array} & \begin{bmatrix} 0.7 & \quad & \quad \\ 0.1 & & \\ 0.2 & & \end{bmatrix} \end{array}$$

3 Mentally check your answer by summing columns; they should sum to 1.

$$\begin{array}{cc} & \begin{array}{ccc} A & W & B \end{array} \\ \begin{array}{c} A \\ W \\ B \end{array} & \begin{bmatrix} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{bmatrix} \end{array}$$

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Example 9 Setting up a transition matrix

A factory has a large number of machines. Machines can be in one of two states: operating or broken. Broken machines are repaired and come back into operation, and vice versa. On a given day:

- 85% of machines that are operational stay operating
- 15% of machines that are operating break down
- 5% of machines that are broken are repaired and start operating again
- 95% of machines that are broken stay broken.

Construct a transition matrix to describe this situation. Use the columns to define the situation at the 'Start' of the day and the rows to describe the situation at the 'End' of the day.

Solution

1 There are two machine states: operating (O) or broken (B). To account for all the possibilities, a 2×2 transition matrix is needed. Construct a blank matrix, labelling the rows and columns O and B, respectively.

$$\begin{array}{cc} & \text{Start} \\ & \begin{array}{cc} O \quad & \quad B \end{array} \\ \text{End} \begin{array}{c} O \\ B \end{array} & \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$

2 Complete the matrix by writing each of the percentages (converted to proportions) into the appropriate locations. Start with column O and write in the values for each row: 0.85 (85%) and 0.15 (15%).

$$\begin{array}{cc} & \begin{array}{cc} O \quad & \quad B \end{array} \\ \begin{array}{c} O \\ B \end{array} & \begin{bmatrix} 0.85 & \\ 0.15 & \end{bmatrix} \end{array}$$

3 Mentally check your answer by summing the columns; they should sum to 1.

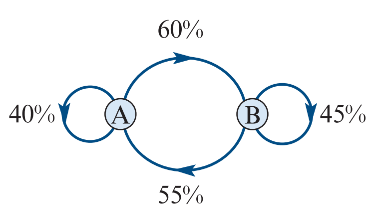
$$\begin{array}{cc} & \begin{array}{cc} O \quad & \quad B \end{array} \\ \begin{array}{c} O \\ B \end{array} & \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \end{array}$$

Exercise 12C-1

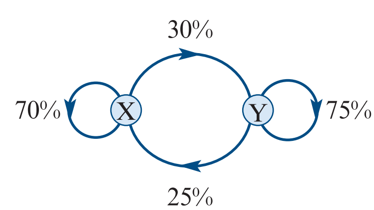
Setting up a transition matrix from a transition diagram

1 The diagrams below describe a series of transitions between the states indicated. Construct a transition matrix that can be used to represent each of these diagrams. Use columns to define the starting points. Convert the percentages to proportions.

a



b



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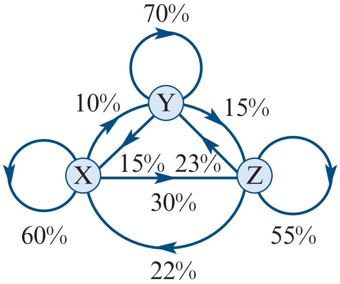
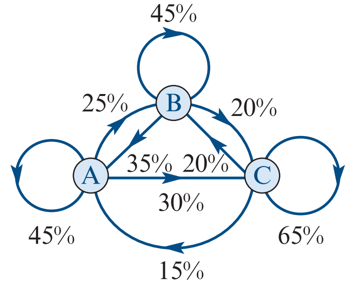
c



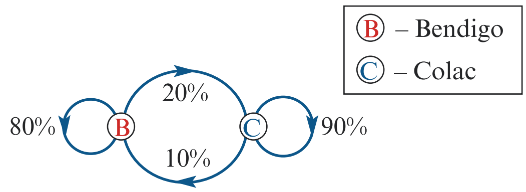
d



▶ Interpreting transition matrices

Let us return to the car rental problem at the start of this section. As we saw then, the following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns in Bendigo and Colac.

$$\begin{array}{c} \\ \\ T = \end{array} \begin{array}{c} \textit{Rented in} \\ \begin{array}{cc} B & C \end{array} \\ \begin{bmatrix} 0.80 & 0.10 \\ 0.20 & 0.90 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ B \\ C \end{array} \quad \textit{Returned to}$$



Using this information alone, a number of predictions can be made.

For example, if 50 cars are rented in Bendigo this week, the transition matrix predicts that:

- 80% or 40 of these cars will be returned to Bendigo next week ($0.80 \times 50 = 40$)
- 20% or 10 of these cars will be returned to Colac next week ($0.20 \times 50 = 10$).

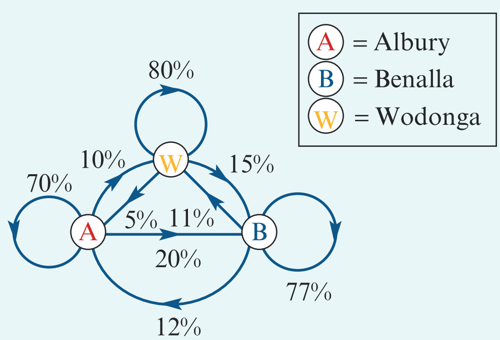
Further, if 40 cars are rented in Colac this week, the transition matrix predicts that:

- 10% or 4 of these cars will be returned to Bendigo next week ($0.10 \times 40 = 4$)
- 90% or 36 of these cars will be returned to Colac next week ($0.90 \times 40 = 36$).

Example 10 Interpreting a transition matrix

The following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns in three locations: Albury, Wodonga and Benalla.

$$T = \begin{array}{c} \begin{array}{ccc} A & W & B \end{array} \\ \begin{bmatrix} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{bmatrix} \end{array} \begin{array}{l} \\ A \\ W \\ B \end{array} \quad \textit{Returned to}$$



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Use the transition matrix T and its transition diagram to answer the following questions.

a What percentage of cars rented in Wodonga each week are predicted to be returned to:

i Albury? **ii** Benalla? **iii** Wodonga?

b Two hundred cars were rented in Albury this week. How many of these cars do we expect to be returned to:

i Albury? **ii** Benalla? **iii** Wodonga?

c What percentage of cars rented in Benalla each week are *not* expected to be returned to Benalla?

d One hundred and sixty cars were rented in Albury this week. How many of these cars are expected to be returned to either Benalla or Wodonga?

Solution

a **i** 0.5 or 5% **ii** 0.15 or 15% **iii** 0.80 or 80%

b **i** $0.70 \times 200 = 140$ cars **ii** $0.20 \times 200 = 40$ cars **iii** $0.10 \times 200 = 20$ cars

c $11 + 12 = 23\%$ or $100 - 77 = 23\%$

d 20% of $160 + 10\%$ of $160 = 48$ cars

Exercise 12C-2

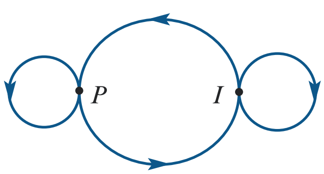
Interpreting transition matrices

1 When people go to the movies they buy either a bag of popcorn (P) or an ice cream (I). Experience has shown that:

- 85% of people who buy popcorn this time will buy popcorn next time
- 15% of people who buy popcorn this time will buy an ice cream next time
- 75% of people who buy an ice cream this time will buy an ice cream next time
- 25% of people who buy ice cream this time will buy popcorn next time.

a Construct a transition matrix and transition diagram that can be used to describe this situation. Use the models below.

$$\begin{array}{c} \\ \\ T = \end{array}\begin{array}{c} \textit{This time} \\ \begin{array}{cc} P & I \end{array} \\ \begin{bmatrix} & \\ & \end{bmatrix} \end{array}\begin{array}{l} \\ \\ \begin{array}{l} P \\ I \end{array} \textit{Next time} \end{array}$$



b Eighty people are seen buying popcorn at the movies. How many of these are expected to buy popcorn next time they go to the movies?

c Sixty people are seen buying an ice cream at the movies. How many of these are expected to buy popcorn next time they go to the movies?

d On another occasion, 120 people are seen buying popcorn and 40 are seen buying an ice cream. How many of these are expected to buy an ice cream next time they attend the movies?

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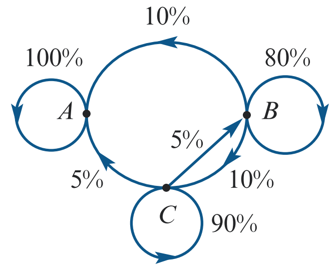
2 On Windy Island, sea birds are observed nesting at three sites: A, B and C. The following transition matrix and accompanying transition diagram can be used to predict the movement of these sea birds between these sites from year to year.

$$T = \begin{array}{c} \textit{This year} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.90 \\ 0 & 0.10 & 0.90 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \end{array} \quad \textit{Next year}$$



a What percentage of sea birds nesting at site B this year were expected to nest at:

i site A next year? **ii** site B next year? **iii** site C next year?

b This year, 850 sea birds were observed nesting at site B. How many of these are expected to:

i still nest at site B next year ? **ii** move to site A to nest next year?

c This year, 1150 sea birds were observed nesting at site A. How many of these birds are expected to nest at:

i site A next year? **ii** site B next year? **iii** site C next year?

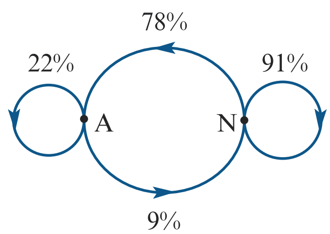
d What does the '1' in column A, row A of the transition matrix indicate?

3 A car insurance company finds that:

- 22% of car drivers involved in an accident this year (A) are also expected to be involved in an accident next year
- 9% of drivers who are *not* involved in an accident this year (N) are expected to be involved in an accident next year.

The transition diagram that can be used to describe this situation is shown below.

$$T = \begin{array}{c} \textit{This year} \\ \begin{array}{cc} A & N \end{array} \\ \begin{bmatrix} 0.22 & 0.09 \\ 0.78 & 0.91 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ N \end{array} \quad \textit{Next year}$$



a In 2015, 84 000 drivers insured with the company were *not* involved in an accident.

i How many of these drivers were *not* expected to be involved in an accident in 2016?

ii How many of these drivers were expected to be involved in an accident in 2016?

b In 2015, 25 000 drivers insured with the company were involved in an accident.

i How many of these drivers were expected to be involved in an accident in 2016?

ii How many of these drivers were expected to be involved in an accident in 2017?

iii How many of these drivers were expected to be involved in an accident in 2018?

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Multiple-choice questions

4 There are 30 children in a Year 6 class. Each week every child participates in one of three activities: cycling (C), orienteering (O) or swimming (S).

$$\begin{array}{c} \textit{This week} \\ \begin{array}{ccc} C & \quad O & \quad S \end{array} \end{array}$$
$$T = \begin{bmatrix} 0.5 & 0.3 & 0.3 \\ 0.1 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \begin{array}{c} C \\ O \\ S \end{array} \ \textit{Next week}$$

The activities that the children select each week change according to the transition matrix opposite.
From the transition matrix it can be concluded that:

A in the first week of the program, ten children do cycling, ten children do orienteering and ten children do swimming.

B at least 50% of the children do not change their activities from the first week to the second week.

C in the long term, all of the children will choose the same activity.

D orienteering is the most popular activity in the first week

E 50% of the children will do swimming each week.

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5 Wendy will have lunch with one of her friends each day of this week. Her friends are Angela (A), Betty (B), Craig (C), Daniel (D) and Edgar (E).
Based on the transition matrix, the order in which Wendy has lunch with her friends for the next 4 days is:

$$\begin{array}{c} \textit{Today} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \end{array}$$
$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \ \textit{Tomorrow}$$

A Angela, Betty, Craig, Daniel

B Daniel, Betty, Angela, Craig

C Daniel, Betty, Angela, Edgar

D Edgar, Angela, Daniel, Betty

E Edgar, Daniel, Betty, Angela

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▶ Using recursion to generate state matrices step-by-step

We return again to the car rental problem. The car rental firm now plans to buy 90 new cars. Fifty will be based in Bendigo and 40 in Colac.

Given this pattern of rental car returns, the first question the manager would like answered is:

> 'If we start with 50 cars in Bendigo, and 40 cars in Colac, how many cars will be available for rent at both towns after 1 week, 2 weeks, etc?'

You have met this type of problem earlier when doing financial modelling (Chapter 8). For example, if we invest \$1000 at an interest rate of 5% per annum, how much will we have after 1 year, 2 years, 3 years, etc?

We solved this type of problem by using a **recurrence relation** to model the growth in our investment year-by-year. We do the same with the car rental problem, the only difference being that we are now working with matrices.

Constructing a matrix recurrence relation

A recurrence relation must have a *starting point*.

In this case it is the **initial state matrix**: $S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$

Generating S_1

To find out the number of cars in Bendigo and Colac after 1 week, we use the transition matrix $T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ to generate the next **state matrix** in the sequence, S_1, as follows:

$$\begin{aligned} S_1 &= T\,S_0 \\ &= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 50 \\ 40 \end{bmatrix} = \begin{bmatrix} 0.8 \times 50 + 0.1 \times 40 \\ 0.2 \times 50 + 0.9 \times 40 \end{bmatrix} \\ \text{or } S_1 &= \begin{bmatrix} 44 \\ 46 \end{bmatrix} \end{aligned}$$

Thus, after 1 week we predict that there will be 44 cars in Bendigo and 46 in Colac.

Generating S_2

Following the same pattern, after 2 weeks;

$$S_2 = T\,S_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 44 \\ 46 \end{bmatrix} = \begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$$

Thus, after 2 weeks we predict that there will be 39.8 cars in Bendigo and 50.2 in Colac.

Generating S_3

After 3 weeks:

$$S_3 = T\,S_2 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix} = \begin{bmatrix} 36.9 \\ 53.1 \end{bmatrix}$$

Thus, after 3 weeks we predict that there will be 36.9 cars in Bendigo and 53.1 in Colac.

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A pattern is now emerging. So far we have seen that:

$$S_1 = TS_0$$
$$S_2 = TS_1$$
$$S_3 = TS_2$$

If we continue this pattern we have:

$$S_4 = TS_3$$
$$S_5 = TS_4$$

or, more generally, $S_{n+1} = TS_n$.

With this rule as a starting point, we now have a recurrence relation that will enable us to model and analyse the car rental problem on a step-by-step basis.

Recurrence relation

$$S_0 = \text{intial value}, \quad S_{n+1} = TS_n$$

Let us return to the factory problem in Example 9.

Example 11 Using a recursion relation to calculate state matrices step-by-step

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

At the start, 80 machines are operating and 20 are broken.

Use the recursion relation

$$S_0 = \text{intial value}, \; S_{n+1} = TS_n$$

where

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

to determine the number of operational and broken machines after 1 day and after 3 days.

Solution

1 Write down a column matrix with S_0 representing the initial operational state of the machines, and the transition matrix.

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \quad T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

2 Use the rule $S_{n+1} = TS_n$ to determine the operational state of the machines after one day by forming the product $S_1 = TS_0$ and evaluate.

$$S_1 = TS_0 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 69 \\ 31 \end{bmatrix}$$

After 1 day, 69 machines are operational and 31 are broken.

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3 To find the operational state of the machines after 3 days, we must first find the operating state of the machines after 2 days (S_2) and use this matrix to find S_3 using $S_3 = TS_2$.

$$S_2 = TS_1 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 69 \\ 31 \end{bmatrix} = \begin{bmatrix} 60.2 \\ 39.8 \end{bmatrix}$$

$$S_3 = TS_2 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 60.2 \\ 39.8 \end{bmatrix} = \begin{bmatrix} 53.16 \\ 46.84 \end{bmatrix}$$

After 3 days, 53 machines are operating and 47 are broken.

Calculator hint: In practice, generating matrices recursively is performed on your CAS calculator as shown opposite for the calculations performed in Example 11.

$\begin{bmatrix} 80 \\ 20 \end{bmatrix} \rightarrow s0$	$\begin{bmatrix} 80. \\ 20. \end{bmatrix}$
$\begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \rightarrow t$	$\begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$
t.s0	$\begin{bmatrix} 69. \\ 31. \end{bmatrix}$
$\begin{bmatrix} 69 \\ 31 \end{bmatrix}$	$\begin{bmatrix} 60 & 2 \\ 39 & 8 \end{bmatrix}$

and so on.

▶ A rule for determining the state matrix of a system after *n* steps

While we can use the recurrence relation:

$$S_0 = \text{intial value}, \; S_{n+1} = TS_n$$

to generate state matrices step-by-step, there is a more efficient method when need to determine the state matrix after a large number of steps.

If we follow through the process step-by-step we have:

$$S_1 = TS_0$$
$$S_2 = TS_1 = T(TS_0) = T^2S_0$$
$$S_3 = TS_2 = T(TS_1) = T^2S_1 = T^2(TS_0) = T^3S_0$$

Continuing the process

$$S_4 = T^4S_0$$
$$S_5 = T^5S_0$$

or more generally, $S_{\text{n}} = T^nS_0$.

We now have a simple rule for finding the value, S_n, of the state matrix after n steps.

A rule for finding the state matrix after *n* steps

If the recurrence rule for determining state matrices is

$$S_0 = \text{initial state matrix}, \; S_{n+1} = TS_n,$$

the state matrix after n steps (or transitions) is given by $S_n = T^nS_0$.

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Let us return to the factory problem we analysed in Example 9.

Example 12 Determining the nth state of a system using the rule $S_n = T^n S_0$

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

Initially, 80 machines are operating and 20 are broken, so:

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

Determine the number of operational and broken machines after 1 day and after 3 days.

Solution

1 Write down the transition matrix, T, and initial state matrix, S_0. Enter the matrices into your calculator. Use T and S.

$$T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \quad S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

2 To find out how many machines are in operation and how many are broken after 10 days, write down the rule $S_n = T^n S_0$ and substitute $n = 10$ to give $S_{10} = T^{10} S_0$.

$$S_n = T^n S_0$$
$$\therefore S_{10} = T^{10} S_0$$

3 Enter the expression $T^{10} S$ into your calculator and evaluate.

$$\begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \rightarrow t \qquad \begin{bmatrix} 0.85 & 0.05. \\ 0.15 & 0.95. \end{bmatrix}$$
$$\begin{bmatrix} 80 \\ 20 \end{bmatrix} \rightarrow s \qquad \begin{bmatrix} 80. \\ 20. \end{bmatrix}$$
$$t^{10} \cdot s \qquad \begin{bmatrix} 30.9056 \\ 69.0944 \end{bmatrix}$$

4 Write down your answer in matrix form and then in words.

$$S_{10} = \begin{bmatrix} 30.9 \\ 69.1 \end{bmatrix}$$

After 10 days, 31 machines will be operational and 69 broken.

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▶ The steady-state solution

A second question a manager might like answered about the car rental is as follows.

> 'Will the number of rental cars available from each location vary from week to week or will they settle down to some fixed value?'

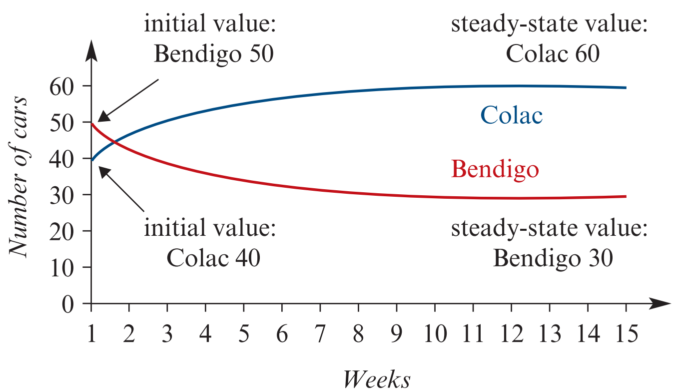
To investigate this question, we start by listing the state matrices from week 0 to week 15.

Week	0	1	2	3	4–11	12	13	14	15
State matrix	$\begin{bmatrix}50\\40\end{bmatrix}$	$\begin{bmatrix}44\\46\end{bmatrix}$	$\begin{bmatrix}39.8\\50.2\end{bmatrix}$	$\begin{bmatrix}36.9\\53.1\end{bmatrix}$	$\ldots$	$\begin{bmatrix}30.3\\59.7\end{bmatrix}$	$\begin{bmatrix}30.2\\59.8\end{bmatrix}$	$\begin{bmatrix}30.1\\59.9\end{bmatrix}$	$\begin{bmatrix}30.1\\59.9\end{bmatrix}$

What you should notice is that, as the weeks go by, the number of cars at each of the locations starts to settle down. We call this the *steady- or equilibrium- state solution*.

For the rental car problem, the *steady-state solution* is 30.1 (in practice, 30) cars at the Bendigo branch and 59.9 (in practice, 60) cars at the Colac branch, which means the numbers of cars at each location will *not* change from then on.

This can be seen more clearly in the graph below (the points have been joined to guide the eye).



In summary, even though the number of cars returned to each location varied from day to day, the numbers at each location eventually settled down to an equilibrium or steady-state solution. In the steady state, the number of cars at each location remained the same.

Important

1 In the *steady state*, cars are still moving between Bendigo and Colac, but the number of cars rented in Bendigo and returned to Colac is balanced by the number of cars rented in Colac and returned to Bendigo. Because of this balance, the steady state is also called the *equilibrium state*.

2 For a system to have a steady state, the transition matrix must be *regular* and the columns must add up to 1. A *regular matrix* is one whose powers never contain any zero elements. In practical terms, this means that every state represented in the transition matrix is accessible, either directly or indirectly from every other state.

ISBN 978-1-107-56757-3

A strategy for estimating the steady-state solution

In the car rental problem we found that, even though the number of cars returned to each location initially varied from day to day, it eventually settled down so the number of cars at each location remained the same.

Although we arrived at this conclusion by repeated calculations, we can arrive at the solution much faster by using the rule $S_n = T^n S_0$ to find the nth state.

Estimating the steady state solution

If S_0 is the initial state matrix, then the **steady-state matrix**, S, is given by

$$S = T^n S_0$$

as n tends to infinity (∞).

Note: While in practice we cannot evaluate T^n for $n = \infty$, we find that, depending on the circumstances, large values of n can often give a very close approximation to the steady-state solution.

Example 13 Estimating the steady-state solution

For the car rental problem:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Estimate the steady-state solution by calculating S_n for $n = 10, 15, 17$ and 18.

Solution

1 Write down the transition matrix T and initial state matrix S_0. Enter the matrices into your calculator. Use T and S.

$$T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

2 Use the rule $S_n = T^n S_0$ to write down the expression for the nth state for $n = 10$.

$$S_n = T^n S_0$$

$$\therefore S_{10} = T^{10} S_0 = \begin{bmatrix} 30.6 \\ 59.4 \end{bmatrix}$$

3 Enter the expression $T^{10}S$ into your calculator and evaluate.

4 Repeat the process for $n = 15, 17$ and 18.

$$t^{10} \cdot s \quad \begin{bmatrix} 30.565 \\ 59.435 \end{bmatrix}$$

$$t^{15} \cdot s \quad \begin{bmatrix} 30.095 \\ 59.905 \end{bmatrix}$$

$$t^{17} \cdot s \quad \begin{bmatrix} 30.047 \\ 59.953 \end{bmatrix}$$

$$t^{18} \cdot s \quad \begin{bmatrix} 30.033 \\ 59.967 \end{bmatrix}$$

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5 Write down your answer in matrix form and then in words. This result agrees with the graphical result arrived at earlier.

$S_{15} = \begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}$, $S_{17} = \begin{bmatrix} 30.0 \\ 60.0 \end{bmatrix}$, $S_{18} = \begin{bmatrix} 30.0 \\ 60.0 \end{bmatrix}$

The estimated steady-state solution is 30 cars based in Bendigo and 60 cars based in Colac.

Note: To establish a steady state to a given degree of accuracy, in this case one decimal place, at least two successive state matrices must agree to this degree of accuracy.

Exercise 12C-3

Calculating state matrices step-by-step and by rule

1 For the initial state matrix $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ and the transition matrix $T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$:

a use the recursion relation: S_0 = initial state matrix, $S_{n+1} = TS_n$, to determine:

i S_1 **ii** S_2 **iii** S_3

b determine the value of T^5

c use the rule $S_n = T^n S_0$ to determine:

i S_2 **ii** S_3 **iii** S_7

d by calculating $S_n = T^n S_0$ for $n = 10, 15, 21$ and 22, show that the steady-state matrix is close to $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

2 For the initial state matrix $S_0 = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$ and the transition matrix $T = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}$:

a use a recurrence relation to determine:

i S_1 **ii** S_2 **iii** S_3

b use the relationship $S_n = T^n S_0$ to determine:

i S_2 **ii** S_3 **iii** S_7

c by calculating $S_n = T^n S_0$ for $n = 10, 15, 17$ and 18, show that the steady-state matrix is close to $\begin{bmatrix} 247.1 \\ 129.4 \\ 223.5 \end{bmatrix}$.

Practical applications of transition matrices

3 Two fast-food outlets, Jill's and Pete's, are located in a small town.

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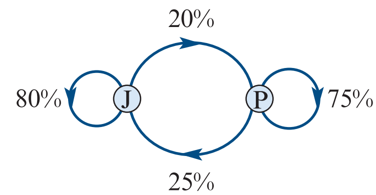
In a given week:



- 80% of people who go to Jill's return the next week
- 20% of people who go to Jill's go to Pete's the next week
- 25% of people who go to Pete's go to Jill's the next week
- 75% of people who go to Pete's return the next week.

a Construct a transition matrix to describe this situation. Call the matrix T.

b Initially, 400 people eat at Jill's and 400 eat at Pete's. Write down a column matrix S_0 that describes this situation.

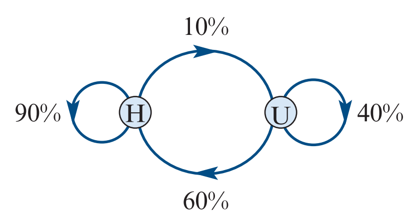
c How many of these people do we expect to go to Jill's the next week? How many to Pete's?

d How many do we expect to go to Jill's after 5 weeks? How many to Pete's?

e In the long term, how many do we expect to eat at Jill's each week? How many of these people do we expect to eat at Pete's?

4 Imagine that we live in a world in which people are either 'happy' or 'unhappy', but the way people feel can change from day to day.

In this world:



- 90% of people who are happy today will be happy tomorrow
- 10% of people who are happy today will be unhappy tomorrow
- 40% of people who are unhappy today will be happy tomorrow
- 60% of people who are unhappy today will be unhappy tomorrow.

a Construct a transition matrix to describe this situation. Call the matrix T.

b On a given day, out of 2000 people, 1500 are happy and 500 are unhappy. Write down a column matrix, S_0, that describes this situation.

c The next day, how many of these people do we expect to be 'happy' and how many 'unhappy'?

d After 4 days, how many of these people do we expect to be 'happy' and how many 'unhappy'?

e In the long term, how many people do we expect to be 'happy' and how many 'unhappy'?

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5 In another model of this world, people can be 'happy', 'neither happy nor sad', or 'sad', but the way people feel can change from day to day.

The transition matrix opposite shows how people's feelings may vary from day to day in this world, and the proportions of people involved.

$$T = \begin{array}{c} \\ H \\ N \\ S \end{array} \!\!\begin{array}{c} \begin{array}{ccc} H & N & S \end{array} \\ \begin{bmatrix} 0.80 & 0.40 & 0.35 \\ 0.15 & 0.30 & 0.40 \\ 0.05 & 0.30 & 0.25 \end{bmatrix} \end{array}$$

In the transition matrix, the columns define the situation today and the rows define the situation tomorrow.

a On a given day, out of 2000 people, 1200 are 'happy', 600 are 'neither happy nor sad' and 200 are 'sad'. Write down a matrix, S_0, that describes this situation.

b The next day, how many people do we expect to be happy?

c After 5 days, how many people do we expect to be happy?

d In the long term, how many of the 2000 people do we expect to be happy?

▶ Transition matrix modelling using the rule $S_{n+1} = TS_n + B$

To date, we have only considered matrix recurrence models of the form

$$S_0 = \text{ initial state matrix, } S_{n+1} = TS_n$$

This recurrence model can be used to model situations where the total number of objects in the system, like cars, machines, people or birds, remains unchanged. For example, in the car rental problem 90 cars are available for rental. But what happens if management wants to increase the total number of cars available for rent by adding, say, an extra car at each location each week?

To allow for this situation we need to use the matrix recurrence relation:

$$S_0 = \text{initial state matrix, } S_{n+1} = TS_n$$

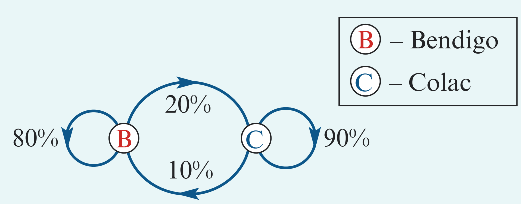
where B is a column matrix.

The next example applies this model to the rental car problem.

Example 14 Determining the nth state of a system using the rule $S_n = T^nS_0 + B$

A rental starts with 90 cars, 50 located at Bendigo and 40 located at Colac.

Cars are usually rented and returned in the same town. However, a small percentage of cars rented in Bendigo are returned in Colac and vice versa. The transition diagram opposite gives these percentages.



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To increase the number of cars, two extra cars are added to the rental fleet at each location each week. The recurrence relation that can be used to model this situation is:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}, \; S_{n+1} = TS_n + B \quad \text{where} \quad T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Determine the number of cars at Bendigo and Colac after:

a 1 week **b** 2 weeks.

Solution

a Use the rule $S_1 = TS_o + B$, to determine the state matrix after 1 week and write your conclusion.

$$S_1 = TS_0 + B$$
$$= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 50 \\ 40 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 44 \\ 46 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 46 \\ 48 \end{bmatrix}$$

Thus, we predict that there will be 46 cars in Bendigo and 48 cars in Colac.

b Use the rule $S_2 = TS_1 + B$, to determine the state matrix after 2 weeks and write your conclusion.

$$S_2 = TS_1 + B$$
$$= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 46 \\ 48 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 41.6 \\ 52.4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 43.6 \\ 54.4 \end{bmatrix}$$

Thus, we predict that there will be 43.6 cars in Bendigo and 54.4 cars in Colac.

Unfortunately, the recurrence rule $S_{n+1} = TS_n + B$ does not lead to a simple rule for the state matrix after n steps, so we need to work our way through this sort of problem step-by-step.

Exercise 12C-4

Using a recurrence rule to calculate state matrices

1 For the transition matrix $T = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$ and the state matrix $S_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$:

a use the recurrence rule $S_{n+1} = TS_n$ to determine:

i S_1 **ii** S_3

b use the recurrence rule $S_{n+1} = TS_n + R$, where $R = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, to determine:

i S_1 **ii** S_2

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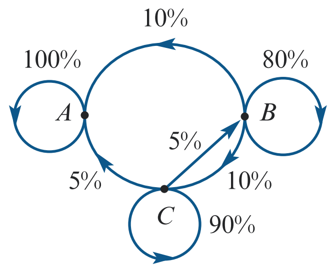
c use the recurrence rule $S_{n+1} = TS_n - B$, where $B = \begin{bmatrix} -20 \\ 20 \end{bmatrix}$, to determine:

i S_1 **ii** S_2

Practical application

2 On Windy Island, sea birds are observed nesting at three sites: A, B and C. The following transition matrix and accompanying transition diagram can be used to predict the movement of sea birds between these sites from year to year.

$$\begin{array}{cc} & \textit{This year} \\ & \begin{array}{ccc} A & B & C \end{array} \\ T = & \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix} \begin{array}{l} A \\ B \quad \textit{Next year} \\ C \end{array} \end{array}$$



Initially, 10 000 sea birds were observed nesting at each site, so $S_0 = \begin{bmatrix} 10\,000 \\ 10\,000 \\ 10\,000 \end{bmatrix}$.

a Use the recurrence rule $S_{n+1} = TS_n$ to:

i determine S_1, the state matrix after 1 year

ii predict the number of sea birds nesting at site B after 2 years.

b Without calculation, write down the number of sea birds predicted to nest at each of the three sites in the long term. Explain why this can be done without calculation.

To help solve the problem of having all the birds eventually nesting at site A, the ranger suggests that 2000 sea birds could be removed from site A each year and relocated in equal numbers to sites B and C.

The state matrix, S_2, is now given by

$$S_2 = TS_1 + N$$

where $S_1 = \begin{bmatrix} 10000 \\ 10000 \\ 10000 \end{bmatrix}$, $T = \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix}$ and $N = \begin{bmatrix} -2000 \\ 1000 \\ 1000 \end{bmatrix}$.

c Evaluate:

i S_2 **ii** S_3 (assuming that $S_3 = TS_2 + N$) **iii** S_4 (assuming that $S_4 = TS_3 + N$).

ISBN 978-1-107-56757-3

Review

Key ideas and chapter summary

Identity matrix An **identity matrix**, I, is a square matrix with 1s down the leading diagonal and zeros elsewhere.

Determinant The **determinant** of a *matrix*, A, is written as $\det(A)$.
Only **square matrices** have determinants.

If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ then $\det(A) = \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 3 \times 3 = -5$

For higher order matrices, a calculator is used to calculate the determinant.

Inverse The **inverse** of a matrix, A, is written as A^{-1} and has the property that $AA^{-1} = A^{-1}A = I$.
Only **square matrices** have inverses.
The *inverse* of a matrix is *not defined* if $\det(A) = 0$.
A calculator is used to determine the inverse of a matrix.

State matrix Transition matrixes A **state matrix** S_n is a column matrix whose elements represent the nth state of a dynamic system defined by a recurrence relation of the form: S_0 = initial state, $S_{n+1} = TS_n$. Here T is a square matrix called a **transition matrix**.

Steady-state matrix The **steady-state matrix**, S, represents the equilibrium state of a system. For regular matrices, this equilibrium state of a system can be estimated by calculating $T^n S_0$ for a large value of n.

Skills check

Having completed this chapter you should be able to:

- calculate the determinant of a matrix
- know the properties of an inverse matrix
- find the inverse of a square matrix using a calculator
- use determinants to test a system of linear equations for solutions
- use inverse matrices to solve systems of linear equations
- use of the matrix recurrence relation: S_0 = initial state matrix, $S_{n+1} = TS_n$, to generate a sequence of state matrices, including an informal identification of the equilibrium or steady-state matrix in the case of regular state matrices
- construct a transition matrix from a transition diagram and vice versa
- construct a transition matrix to model the transitions in a population
- use of the matrix recurrence relation S_0 = initial state matrix, $S_{n+1} = TS_n + B$ to model systems that include external additions or reductions at each step of the process.

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Multiple-choice questions

The following matrices are needed for Questions 1 to 7.

$$U = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad V = \begin{bmatrix} 43 \\ 45 \end{bmatrix} \quad W = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

1 The matrix that cannot be raised to a power is:

A U **B** V **C** W **D** X **E** Y

2 $\det(U) =$

A 1 **B** 0 **C** 1 **D** 2 **E** 4

3 $Y^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **D** $\frac{1}{8}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ **E** not defined

4 $U^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **D** $\frac{1}{8}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ **E** not defined

5 The matrix product that is defined is:

A UX **B** XY **C** VW **D** UW **E** WX

The following matrices are needed for Questions 6 and 7.

$$W = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

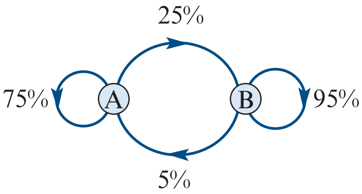
6 The lower triangular matrix is:

A W **B** X **C** Y **D** Z **E** none

7 The permutation matrix is:

A W **B** X **C** Y **D** Z **E** none

8 The transition matrix that can be used to represent the information in the diagram shown is:



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Review

A $\begin{array}{ccc} & & \textit{From} \\ & \text{A} & \text{B} \\ \textit{To}\ \text{A} & \left[\begin{matrix}0.75 \\ 0.05\end{matrix}\right. & \left.\begin{matrix}0.25 \\ 0.95\end{matrix}\right] \\ \text{B} & & \end{array}$

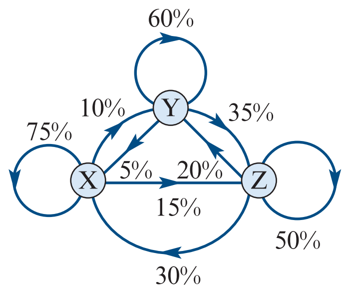
B $\begin{array}{ccc} & & \textit{From} \\ & \text{A} & \text{B} \\ \textit{To}\ \text{A} & \left[\begin{matrix}0.75 \\ 0.25\end{matrix}\right. & \left.\begin{matrix}0.05 \\ 0.95\end{matrix}\right] \\ \text{B} & & \end{array}$

C $\begin{array}{ccc} & & \textit{From} \\ & \text{A} & \text{B} \\ \textit{To}\ \text{A} & \left[\begin{matrix}0.75 \\ 0.05\end{matrix}\right. & \left.\begin{matrix}0.25 \\ 0.95\end{matrix}\right] \\ \text{B} & & \end{array}$

D $\begin{array}{ccc} & & \textit{From} \\ & \text{A} & \text{B} \\ \textit{To}\ \text{A} & \left[\begin{matrix}0.75 \\ 0.25\end{matrix}\right. & \left.\begin{matrix}0.95 \\ 0.05\end{matrix}\right] \\ \textit{B} & & \end{array}$

E $\begin{array}{ccc} & & \textit{From} \\ & \text{A} & \text{B} \\ \textit{To}\ \text{A} & \left[\begin{matrix}0.25 \\ 0.75\end{matrix}\right. & \left.\begin{matrix}0.05 \\ 0.95\end{matrix}\right] \\ \text{B} & & \end{array}$

9 The transition matrix that can be used to represent the information in the diagram shown is:



A $\begin{array}{cc} & \begin{matrix}\text{X} & \text{Y} & \text{Z}\end{matrix} \\ \textit{To}\ \begin{matrix}\text{X}\\ \text{Y}\\ \text{Z}\end{matrix} & \begin{bmatrix}0.75 & 0.05 & 0.30 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50\end{bmatrix}\end{array}$

B $\begin{array}{cc} & \begin{matrix}\text{X} & \text{Y} & \text{Z}\end{matrix} \\ \textit{To}\ \begin{matrix}\text{X}\\ \text{Y}\\ \text{Z}\end{matrix} & \begin{bmatrix}0.75 & 0.10 & 0.15 \\ 0.60 & 0.05 & 0.35 \\ 0.50 & 0.30 & 0.20\end{bmatrix}\end{array}$

C $\begin{array}{cc} & \begin{matrix}\text{X} & \text{Y} & \text{Z}\end{matrix} \\ \textit{To}\ \begin{matrix}\text{X}\\ \text{Y}\\ \text{Z}\end{matrix} & \begin{bmatrix}0.75 & 0.10 & 0.15 \\ 0.60 & 0.05 & 0.35 \\ 0.50 & 0.30 & 0.20\end{bmatrix}\end{array}$

D $\begin{array}{cc} & \begin{matrix}\text{X} & \text{Y} & \text{Z}\end{matrix} \\ \textit{To}\ \begin{matrix}\text{X}\\ \text{Y}\\ \text{Z}\end{matrix} & \begin{bmatrix}0.75 & 0.05 & 0.15 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50\end{bmatrix}\end{array}$

E $\begin{array}{cc} & \begin{matrix}\text{X} & \text{Y} & \text{Z}\end{matrix} \\ \textit{To}\ \begin{matrix}\text{X}\\ \text{Y}\\ \text{Z}\end{matrix} & \begin{bmatrix}0.75 & 0.05 & 0.15 \\ 0.15 & 0.35 & 0.50 \\ 0.10 & 0.60 & 0.20\end{bmatrix}\end{array}$

10 Which of the following systems of linear equations has a *unique* solution?

I $x - 3y = 6$
$2x + y = 3$

II $2x + 2y = 6$
$4x + 4y = 3$

III $4x - 3y = 6$
$8x - 12y = 3$

A I only **B** I and II only **C** II only **D** I and III only **E** all

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11 The linear equations $\begin{aligned} 2x - 3y &= 6 \\ 2x + y &= 3 \end{aligned}$ can be written in matrix form as:

A $\begin{bmatrix} 2 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

D $\begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

The following information is needed for Questions 12 to 17.

A system is defined by a transition matrix $T = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$ with $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$.

12 For this system, $S_1 =$

A $\begin{bmatrix} 60 \\ 200 \end{bmatrix}$ **B** $\begin{bmatrix} 140 \\ 160 \end{bmatrix}$ **C** $\begin{bmatrix} 160 \\ 140 \end{bmatrix}$ **D** $\begin{bmatrix} 166 \\ 144 \end{bmatrix}$ **E** $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$

13 For this system, T^2 is:

A $\begin{bmatrix} 0.36 & 0.25 \\ 0.16 & 0.25 \end{bmatrix}$ **B** $\begin{bmatrix} 0.56 & 0.55 \\ 0.44 & 0.45 \end{bmatrix}$ **C** $\begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$ **D** $\begin{bmatrix} 1.2 & 1.0 \\ 0.8 & 1.0 \end{bmatrix}$

E not defined

14 For this system, S_3 is closest to:

A $\begin{bmatrix} 160 \\ 140 \end{bmatrix}$ **B** $\begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$ **C** $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$ **D** $\begin{bmatrix} 640 \\ 560 \end{bmatrix}$ **E** $\begin{bmatrix} 400 \\ 800 \end{bmatrix}$

15 For this system, the steady-state matrix is closest to:

A $\begin{bmatrix} 166.5 \\ 133.5 \end{bmatrix}$ **B** $\begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$ **C** $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$ **D** $\begin{bmatrix} 166.8 \\ 133.2 \end{bmatrix}$ **E** $\begin{bmatrix} 166.9 \\ 133.1 \end{bmatrix}$

16 If $L_1 = TS_0 + B$, where $B = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then L_1 equals:

A $\begin{bmatrix} 70 \\ 220 \end{bmatrix}$ **B** $\begin{bmatrix} 150 \\ 180 \end{bmatrix}$ **C** $\begin{bmatrix} 170 \\ 160 \end{bmatrix}$ **D** $\begin{bmatrix} 176 \\ 164 \end{bmatrix}$ **E** $\begin{bmatrix} 210 \\ 120 \end{bmatrix}$

17 If $P_1 = TS_0 - 2B$, where $B = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then P_1 equals:

A $\begin{bmatrix} 140 \\ 100 \end{bmatrix}$ **B** $\begin{bmatrix} 170 \\ 100 \end{bmatrix}$ **C** $\begin{bmatrix} 170 \\ 100 \end{bmatrix}$ **D** $\begin{bmatrix} 170 \\ 160 \end{bmatrix}$ **E** $\begin{bmatrix} 180 \\ 180 \end{bmatrix}$

 ISBN 978-1-107-56757-3

The following information is needed for Questions 18 and 19.

A system of state matrices S_n is defined by the matrix equation $S_{n+1} = GS_n$ where

$G = \begin{bmatrix} 0 & -0.5 \\ 1.5 & 0.5 \end{bmatrix}$.

18 If $S_1 = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then S_2 equals:

A $\begin{bmatrix} -12.5 \\ -2.5 \end{bmatrix}$ **B** $\begin{bmatrix} -10 \\ 25 \end{bmatrix}$ **C** $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ **D** $\begin{bmatrix} 10 \\ 25 \end{bmatrix}$ **E** $\begin{bmatrix} 15 \\ 30 \end{bmatrix}$

19 $T = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}$ is a transition matrix. $S_5 = \begin{bmatrix} 22 \\ 18 \end{bmatrix}$ is a state matrix.

If $S_5 = TS_4$, then S_4 equals:

A $\begin{bmatrix} 18 \\ 22 \end{bmatrix}$ **B** $\begin{bmatrix} 20 \\ 20 \end{bmatrix}$ **C** $\begin{bmatrix} 21.8 \\ 18.2 \end{bmatrix}$ **D** $\begin{bmatrix} 22 \\ 18 \end{bmatrix}$ **E** $\begin{bmatrix} 18.2 \\ 21.2 \end{bmatrix}$

20 A large population of birds lives on a remote island. Every night each bird settles at either location A or location B.

On the first night the number of birds at each location was the same. On each subsequent night, a percentage of birds changed the location at which they settled.

The movement of birds between the two locations is described by the transition matrix T shown opposite. Assume this pattern of movement continues.

In the long term, the number of birds that settle at location A will:

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0 \\ 0.2 & 1 \end{bmatrix} \end{array}$$

A not change **B** gradually decrease to zero **C** gradually increase

D eventually settle at around 20% of the island's bird population

E eventually settle at around 80% of the island's bird population

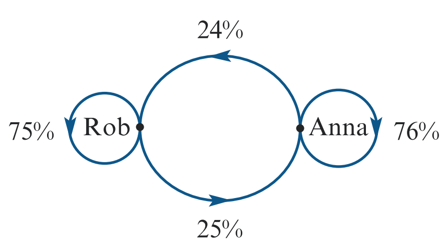
based on VCAA (2010)

Use the following information to answer Questions 21 and 22.

Two politicians, Rob and Anna, are the only candidates for a forthcoming election. At the beginning of the election campaign, people were asked for whom they planned to vote. The numbers were as per the table.

Candidate	*Number of people planing to vote for candidate*
Rob	5692
Anna	3450

During the election campaign, it is expected that people may change the candidate that they plan to vote for each week according to the transition diagram shown.



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Cambridge University Press

21 The total number of people who are expected to change the candidate that they plan to vote for 1 week after the election campaign begins is:

A 828 **B** 1423 **C** 2251 **D** 4269 **E** 6891

22 The election campaign will run for 10 weeks. If people continue to follow this pattern of changing the candidate they plan to vote for, the expected winner after 10 weeks will be:

A Rob by about 50 votes
B Rob by about 100 votes
C Rob by fewer than 10 votes
D Anna by about 100 votes
E Rob by about 200 votes

Extended-response question

1 The Dinosaurs (D) and the Scorpions (S) are two basketball teams that play in different leagues in the same city.

The matrix A_1 opposite is the attendance matrix for the first game. This matrix shows the number of people who attended the first Dinosaurs game and the number of people who attended the first Scorpions game.

$$A_1 = \begin{array}{c} \begin{bmatrix} 2000 \\ 1000 \end{bmatrix} \end{array} \begin{array}{c} D \\ S \end{array}$$

The number of people expected to attend the second game for each team can be determined using the matrix equation

$$A_2 = GA_1$$

$$\text{where } G \text{ is the matrix } G = \begin{array}{c} \textit{This game} \\ \begin{array}{cc} D & S \end{array} \\ \begin{bmatrix} 1.2 & -0.3 \\ 0.2 & 0.7 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ D \\ S \end{array} \ \textit{Next game}$$

a **i** Determine A_2, the attendance matrix for the second game.

ii Every person who attends either the second Dinosaurs game or the second Scorpions game will be given a free cap. How many caps are expected to be given away in total?

Assume that the attendance matrices for successive games can be determined as follows:

$$A_3 = GA_2, \quad A_4 = GA_3, \text{ and so on such that } A_{n+1} = GA_n$$

b Determine the attendance matrix (with the elements written correct to the nearest whole number) for game 10.

c Describe the way in which the number of people attending the Dinosaurs games is expected to change over the next 80 or so games.

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Review

The attendance at the first Dinosaurs game was 2000 people and the attendance at the first Scorpions game was 1000 people.

Suppose instead that 2000 people attend the first Dinosaurs game, and 1800 people attend the first Scorpions game.

d Describe the way in which the number of people attending the Dinosaurs games is expected to change over the next 80 or so games.

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13

Revision: Matrices

ISBN 978-1-107-56757-3

Revision

13A Multiple-choice questions

The following matrices are needed for Questions 1 to 9.

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \quad V = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 5 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

1 The column matrix is:

A U **B** V **C** W **D** X **E** Z

2 The square matrix is:

A U **B** V **C** X **D** Y **E** Z

3 The order of matrix Y is:

A 2×2 **B** 2×3 **C** 3×2 **D** 3×3 **E** 8

4 The following matrices can be added:

A U and V **B** V and W **C** W and Y **D** U and Y **E** none of these

5 The following matrix product is defined:

A UV **B** YU **C** VW **D** ZY **E** WZ

6 $-4V =$

A $[4 \quad 1 \quad 0]$ **B** $[-4 \quad -1 \quad 0]$ **C** $[-16 \quad -4 \quad 0]$

D $4\ [4 \quad 1 \quad 0]$ **E** $[0 \quad -3 \quad -4]$

7 The order of matrix product UY is:

A 1×3 **B** 2×1 **C** 3×1 **D** 3×2 **E** 3×3

8 The order of matrix Y^T is:

A 1×3 **B** 2×3 **C** 3×1 **D** 3×2 **E** 3×3

9 The binary matrix is:

A U **B** V **C** W **D** X **E** Z

10 In the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 3 \\ -5 & -4 & 7 \end{bmatrix}$ the element $a_{32} =$

A -4 **B** -1 **C** 0 **D** 3 **E** 4

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11 $\begin{bmatrix} \text{O} & \text{T} & \text{S} & \text{P} \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$

A $\begin{bmatrix} \text{S} & \text{T} & \text{O} & \text{P} \end{bmatrix}$ **B** $\begin{bmatrix} \text{P} & \text{O} & \text{S} & \text{T} \end{bmatrix}$ **C** $\begin{bmatrix} \text{T} & \text{O} & \text{P} & \text{S} \end{bmatrix}$
D $\begin{bmatrix} \text{O} & \text{P} & \text{S} & \text{T} \end{bmatrix}$ **E** $\begin{bmatrix} \text{T} & \text{O} & \text{S} & \text{P} \end{bmatrix}$

12 $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ equals

A $[0]$ **B** $[1]$ **C** $[2]$ **D** $[3]$ **E** $[5]$

13 R is a 3×3 matrix. The order of matrix R^2 is:

A 3 **B** 3×3 **C** 4×4 **D** 4 **E** 9×9

14 Matrix A has three rows and two columns.
Matrix B has four rows and three columns.
Matrix $C = B \times A$ has:

A two rows and three columns **B** three rows and two columns
C three rows and three columns **D** four rows and two columns
E four rows and three columns

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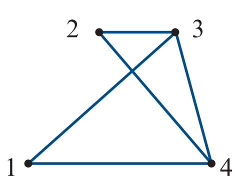
15 Matrix X is of order $p \times q$ and matrix Y is of order $q \times r$. The matrix products XY^{-1} and $X^{-1}Y$ are both defined:

A for no values of p, q or r **B** when $p = r$
C when $p = q = r$ only **D** when $p = q$ and $q = r$
E for all values of p, q and r

16 The diagram opposite is to be represented by a matrix A, where:

- element = 1 if the two points are joined by a line
- element = 0 if the two points are not connected.



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The matrix A is:

A $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

17 The matrix equation $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ generates the following pair of simultaneous linear equations:

A $x + y = 1$, $x + 3y = 4$
B $2x = 1$, $x + 3y = 4$
C $x + 4y = 0$, $x + 3y = 4$
D $x = 1$, $y = 4$
E $x + y = 1$, $3x + y = 4$

The following matrices are needed for Questions 18 to 22.

$$U = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix} \quad V = \begin{bmatrix} 0.75 & 0.35 \\ 0.15 & 0.45 \end{bmatrix} \quad W = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

18 The matrix that cannot be raised to a power is:

A U **B** V **C** W **D** X **E** Y

19 $\det(X) =$

A -1 **B** -0.25 **C** 0.25 **D** 1 **E** 0.75

20 $Y^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -1 \\ -0.5 & 0.5 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ **E** not defined

21 Which of the following matrix expressions is *not* defined?

A $VX - XV$ **B** $UW - W$ **C** VW **D** $Y^2 - VX^{-1}$ **E** $YW + WX$

22 $(U - W)^2 =$

A $\begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}$ **B** $\begin{bmatrix} 10 & 21 \\ 21 & 45 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 7 \\ 5 & 14 \end{bmatrix}$ **D** $\begin{bmatrix} 44 & 119 \\ 85 & 231 \end{bmatrix}$ **E** not defined

23 A and B are square matrices such that $AB = BA = I$, where I is an identity matrix. Which one of the following statements is **not** true?

A $ABA = A$ **B** $AB^2A = I$ **C** both A and B have inverses
D B must equal A **E** B is the inverse of A

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The following information is needed for Questions 24 to 26.

The recurrence relation $S_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, $S_{n+1} = TS_n$, where $T = \begin{bmatrix} 0.1 & 0.8 \\ 0.9 & 0.2 \end{bmatrix}$, can be used to generate a sequence of matrices. In this sequence:

24 $S_1 =$

A $\begin{bmatrix} 90 \\ 110 \end{bmatrix}$ **B** $\begin{bmatrix} 100 \\ 100 \end{bmatrix}$ **C** $\begin{bmatrix} 110 \\ 90 \end{bmatrix}$ **D** $\begin{bmatrix} 120 \\ 80 \end{bmatrix}$ **E** $\begin{bmatrix} 140 \\ 60 \end{bmatrix}$

25 S_5 is closest to:

A $\begin{bmatrix} 90 \\ 110 \end{bmatrix}$ **B** $\begin{bmatrix} 93.1 \\ 106.9 \end{bmatrix}$ **C** $\begin{bmatrix} 95.5 \\ 104.5 \end{bmatrix}$ **D** $\begin{bmatrix} 107.9 \\ 92.1 \end{bmatrix}$ **E** $\begin{bmatrix} 106.9 \\ 93.1 \end{bmatrix}$

26 The steady-state matrix is closest to:

A $\begin{bmatrix} 93.0 \\ 107.0 \end{bmatrix}$ **B** $\begin{bmatrix} 93.6 \\ 106.4 \end{bmatrix}$ **C** $\begin{bmatrix} 94.1 \\ 105.9 \end{bmatrix}$ **D** $\begin{bmatrix} 106.4 \\ 93.6 \end{bmatrix}$ **E** $\begin{bmatrix} 107 \\ 93 \end{bmatrix}$

27 Kerry sat for a multiple-choice test consisting of six questions.
Each question had four alternative answers, A, B, C or D.
He selected D for his answer to the first question.
He then determined the answers to the remaining questions using the transition matrix opposite.

$$\begin{array}{cc} & \textit{This question} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \textit{Next question} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The answers that he gave to the six test questions, starting with D, were:

A *DBCADB* **B** *DBCAAA* **C** *DBCACA* **D** *DACBDD* **E** *DCBABC*

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The following information relates to Questions 28 to 30.

A large population of mutton birds migrates each year to a remote island to nest and breed. There are four nesting sites on the island, A, B, C and D.

Researchers suggest that the following transition matrix can be used to predict the number of mutton birds nesting at each of the four sites in subsequent years. An equivalent transition diagram is also given.

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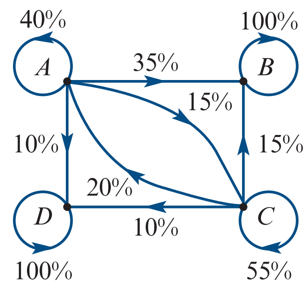
Revision

$$T = \begin{array}{c} \textit{This year} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.40 & 0 & 0.2 & 0 \\ 0.35 & 1 & 0.15 & 0 \\ 0.15 & 0 & 0.55 & 0 \\ 0.1 & 0 & 0.1 & 1 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array} \quad \textit{Next year}$$



28 Two thousand eight hundred mutton birds nest at site C in 2008.
Of these 2800 mutton birds, the number that nest at site A in 2009 is predicted to be:

A 560 **B** 980 **C** 1680 **D** 2800 **E** 3360

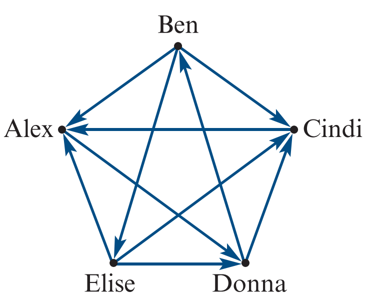
29 This transition matrix predicts that, in the long term, the mutton birds will:

A nest only at site A
B nest only at site B
C nest only at sites A and C
D nest only at sites B and D
E continue to nest at all four sites

30 Six thousand mutton birds nest at site B in 2008.
Assume that an equal number of mutton birds nested at each of the four sites in 2007.
The same transition matrix applies.
The total number of mutton birds that nested on the island in 2007 was:

A 6000 **B** 8000 **C** 12 000 **D** 16 000 **E** 24 000

31 The network diagram below shows the results of a chess competition between five players: Alex, Ben, Cindi, Donna and Elise. Each arrow indicates the winner of individual games. For example, the arrow from Alex to Donna indicates that Alex beat Donna in their game.



$$\textit{Winner} \begin{array}{c} \textit{Loser} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}$$

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In the matrix on the right 1s and 0s represent winners and losers respectively. When the sum of the one-step and two-step dominances is used to rank the players in this competition, the ranking is:

A Ben, Elise, Donna, Alex, Cindi
B Ben, Elise, Cindi, Donna, Alex
C Ben, Elise, Donna, Cindi, Alex
D Elise, Ben, Donna, Alex, Cindi
E Elise, Ben, Donna, Cindi, Alex

based on VCAA (2014)

32 A school has three computer classes, A, B and C. There are 15 students in each class. Each student is given a mark out of 100 based on their performance in a test. Matrix M below displays the marks obtained by these 45 students, listed by class.

$$M = \begin{bmatrix} 56 & 78 & 79 & 43 & 67 & 56 & 80 & 85 & 75 & 89 & 55 & 64 & 95 & 34 & 63 \\ 90 & 45 & 56 & 65 & 76 & 79 & 27 & 45 & 69 & 73 & 70 & 63 & 65 & 34 & 59 \\ 76 & 76 & 89 & 47 & 50 & 66 & 68 & 89 & 88 & 90 & 45 & 67 & 78 & 45 & 87 \end{bmatrix}$$

Two other matrices, S and R, are defined opposite.

Which one of the following matrix expressions can be used to generate a matrix that displays the mean mark obtained for each class?

A $\frac{1}{45}M$

B $\frac{1}{3}R \times M$

C $\frac{1}{3}R \times M \times S$

D $\frac{1}{15}M \times S$

E $\frac{1}{15}S \times R \times M$

$$S = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

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Revision

13B Extended-response questions

AS

1 At some stage, the matrix $C = \begin{bmatrix} 1.316 \\ 1.818 \\ 0.167 \end{bmatrix} \begin{array}{l} \textit{US dollar rate} \\ \textit{Euro rate} \\ \textit{HK dollar rate} \end{array}$

could be used to convert US dollars (US\$), European euro (€) and Hong Kong dollars (HK\$) into Australian dollars (A\$).

a What is the order of matrix C?

You return from an overseas trip with US\$102, €262 and HK\$516.

We can write this information as the matrix $H = \begin{bmatrix} 102 & 262 & 516 \end{bmatrix}$.

b What is the order of matrix H?

c Is the matrix product HC defined? Why?

d **i** Evaluate the matrix product HC.

ii What does the matrix product represent and why?

e Matrix $M = \begin{bmatrix} 125 & 216 & 54 \\ 0 & 34 & 453 \\ 0 & 356 & 0 \end{bmatrix}$ gives the amounts in US dollars, euros and HK dollars that three other people want to change into Australian dollars. The rows represent people. The columns indicate the amounts of each currency they have. Use the conversion matrix C and matrix multiplication to generate a matrix that displays the amounts of Australian currency that each person will receive.

2 We wish to solve the following system of linear equations

$x + 2y = -4$

$3x - 2y = 12$

using matrix methods.

a Write the equations in matrix form.

b The solution is given by $X = A^{-1}C$. Write down the matrices A, A^{-1}, X and C.

c Solve the equations.

d Use the determinant test to show that the following systems of linear equations do not have a unique solution. In each case, decide whether the equations are inconsistent or dependent.

i $\begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ **ii** $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$ **iii** $\begin{bmatrix} -2 & 10 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \end{bmatrix}$

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3 We wish to solve the system of linear equations below using matrix methods.

$$y + z = 2$$
$$3x + y - z = 2$$
$$x - y = 5$$

a Write the equations in matrix form.

b The solution is given by $X = A^{-1}C$. Write down the matrices A, A^{-1}, X and C.

c Solve the equations.

4 Lake Blue and Lake Green are two small lakes connected by a channel. This enables fish to move between the two lakes on a daily basis. Research has shown that each day:

- 67% of fish in Lake Blue stay in Lake Blue
- 33% of fish in Lake Blue move to Lake Green
- 72% of fish in Lake Green stay in Lake Green
- 28% of fish in Lake Green move to Lake Blue.

a Construct a transition matrix, T, of the form:

$$\begin{array}{ccc} & & \textit{From} \\ & & \begin{array}{cc} \textit{Blue} & \textit{Green} \end{array} \\ \textit{To} & \begin{array}{c} \textit{Blue} \\ \textit{Green} \end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$

to describe this situation.

b Today there are currently 4000 fish in Lake Blue and 6000 fish in Lake Green. Write down a column matrix, S_0, that describes this situation.

c How many fish do you expect to be in each lake tomorrow?

d How many fish do you expect to be in each lake in 3 days' time?

e In the long term, how many fish do you expect to be in each lake?

5 For the transition matrix $T = \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix}$ and an initial state matrix $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$:

a use the recurrence rule $S_{n+1} = T^n S_n$ to determine:

i S_1 **ii** S_2 **iii** S_3

b determine the value of T^6.

c use the recurrence rule $S_{n+1} = T^n S_n$ to determine:

i S_2 **ii** S_3 **iii** S_5

d calculate $S_n = T^n S_0$ for $n = 10, 15, 20$ and 21 to show that the steady-state solution is close to $\begin{bmatrix} 176.5 \\ 123.5 \end{bmatrix}$.

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6 The table below displays the energy content and amounts of fat, carbohydrate and protein contained in a serve of four foods: bread, margarine, peanut butter and honey.

Food	*Energy content (kilojoules/serve)*	*Fat (grams/serve)*	*Carbohydrate (grams/serve)*	*Protein (grams/serve)*
Bread	531	1.2	20.1	4.2
Margarine	41	6.7	0.4	0.6
Peanut butter	534	10.7	3.5	4.6
Honey	212	0	12.5	0.1

a Write down a 2×3 matrix that displays the fat, carbohydrate and protein content (in columns) of bread and margarine.

b A and B are two matrices defined as follows.

$$A = \begin{bmatrix} 2 & 2 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 531 \\ 41 \\ 534 \\ 212 \end{bmatrix}$$

i Evaluate the matrix product AB.

ii Determine the order of matrix product BA.

Matrix A displays the number of servings of the four foods (bread, margarine, peanut butter and honey) needed to make a peanut butter and honey sandwich.
Matrix B displays the energy content per serving of the four foods.

iii Explain the information that the matrix product AB provides.

c The number of serves of bread (b), margarine (m), peanut butter (p) and honey (h) that contain, in total, 53 grams of fat, 101.5 grams of carbohydrate, 28.5 grams of protein and 3568 kilojoules of energy can be determined by solving the matrix equation:

$$\begin{bmatrix} 1.2 & 6.7 & 10.7 & 0 \\ 20.1 & 0.4 & 3.5 & 12.5 \\ 4.2 & 0.6 & 4.6 & 0.1 \\ 531 & 41 & 534 & 212 \end{bmatrix} \begin{bmatrix} b \\ m \\ p \\ h \end{bmatrix} = \begin{bmatrix} 53 \\ 101.5 \\ 28.5 \\ 3568 \end{bmatrix}$$

Solve the matrix equation to find the values b, m, p and h.

7 To study the life-and-death cycle of an insect population, a number of insect eggs (E), juvenile insects (J) and adult insects (A) are placed in a closed environment.
The initial state of this population S_0 can be described by the column matrix [opposite].

$$S_0 = \begin{array}{c} \begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} E \\ J \\ A \\ D \end{array}$$

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Revision

A row has been included in the state matrix to allow for insects and eggs that die (D).

a What is the total number of insects in the population (including eggs) at the beginning of the study?

In this population:

- eggs may die, or they may live and grow into juveniles
- juveniles may die, or they may live and grow into adults
- adults will live a period of time but they will eventually die.

In this population, the adult insects have been sterilised so that no new eggs are produced.

In these circumstances, the life-and-death cycle of the insects can be modelled by the transition matrix [opposite].

$$\begin{array}{cc} & \textit{This week} \\ & \begin{array}{cccc} E & J & A & D \end{array} \\ T = & \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix} \begin{array}{c} E \\ J \\ A \\ D \end{array} \quad \textit{Next week} \end{array}$$

b What proportion of eggs turn into juveniles each week?

c **i** Evaluate the matrix product $S_1 = TS_0$.

ii Write down the number of live juveniles in the population after one week.

iii Determine the number of live juveniles in the population after four weeks. Write your answer correct to the nearest whole number.

iv After a number of weeks there will be no live eggs (less than one) left in the population. When does this first occur?

v Write down the exact long-term state matrix for this population.

d If the study is repeated with unsterilised adult insects, eggs will be laid and potentially grow into adults.

Assume that 30% of adults lay eggs each week.

The population matrix after 1 week, S_1, is now given by

$$S_1 = TS_0 + BS_0 \quad \text{where } B = \begin{bmatrix} 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} \begin{array}{c} E \\ J \\ A \\ D \end{array}$$

i Determine S_1.

This pattern continues. The population matrix after n weeks S_n, is given by:

$$S_{n+1} = TS_n + BS_n$$

ii Determine the number of live eggs in this insect population after two weeks.

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ISBN 978-1-107-56757-3

Cambridge University Press

Revision

8 The following transition matrix, T, is used to help predict class attendance of History students at the university on a lecture-by-lecture basis.

$$\begin{array}{cc} & \textit{This lecture} \\ & \begin{array}{cc} \textit{attend} & \textit{not attend} \end{array} \\ T = & \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \end{array} \begin{array}{l} \textit{attend} \\ \textit{not attend} \end{array} \quad \textit{Next lecture}$$

S_1 is the attendance matrix for the first History lecture.

$$S_1 = \begin{bmatrix} 540 \\ 36 \end{bmatrix} \begin{array}{l} \textit{attend} \\ \textit{not attend} \end{array}$$

S_1 indicates that 540 History students attended the first lecture and 36 History students did not attend the first lecture.

a Use T and S_1 to:

i to the nearest whole number, determine S_2, the attendance matrix for the second lecture

ii predict the number of History students attending the fifth lecture

b Write down a matrix equation for S_n in terms of T, n and S_1.

The History lecture can be transferred to a smaller lecture theatre when the number of students predicted to attend falls below 400.

c For which lecture can this first be done?

d In the long term, how many History students are predicted to attend lectures?

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9 The bookshop manager at a university has developed a matrix formula for determining the number of Mathematics and Physics textbooks he should order each year.
For 2009, the starting point for the formula is the column matrix S_{2008}. This matrix lists the number of Mathematics and Physics textbooks sold in 2008.

$$S_{2008} = \begin{array}{c} \begin{bmatrix} 456 \\ 350 \end{bmatrix} \end{array} \begin{array}{l} \textit{Mathematics} \\ \textit{Physics} \end{array}$$

O_{2009} is a column matrix listing the number of Mathematics and Physics textbooks to be ordered for 2009.
O_{2009} is given by the matrix formula:

$$O_{2009} = AS_{2008} + B \quad \text{where } A = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.68 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 18 \\ 12 \end{bmatrix}$$

a Determine O_{2009}.

The matrix formula above only allows the manager to predict the number of books he should order one year ahead. A new matrix formula enables him to determine the number of books to be ordered two or more years ahead.
The new matrix formula is:

$$O_{n+1} = CO_n - D$$

where O_n is a column matrix listing the number of Mathematics and Physics textbooks to be ordered for year n.

$$\text{Here} \quad C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \quad \text{and } D = \begin{bmatrix} 40 \\ 38 \end{bmatrix}$$

The number of books ordered in 2008 was given by:

$$O_{2008} = \begin{array}{c} \begin{bmatrix} 500 \\ 360 \end{bmatrix} \end{array} \begin{array}{l} \textit{Mathematics} \\ \textit{Physics} \end{array}$$

b Use the new matrix formula to determine the number of Mathematics textbooks the bookshop manager should order in 2010.

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Chapter 14

14

Module 2: Networks and decision mathematics

Graphs, networks and trees: travelling and connecting problems

ISBN 978-1-107-56757-3

14A Graphs and networks

Representing connections with graphs

There are many situations in everyday life that involve connections between people or objects. Towns are connected by roads, computers are connected to the internet and people connect to each other through being friends on social media. A diagram that shows these connections is called a **graph**.

Edges and vertices

Six people – Anna, Brett, Cora, Dario, Ethan and Frances – have connections on a social media website. The graph shows these connections.

Anna is a friend of Brett, Ethan and Frances.

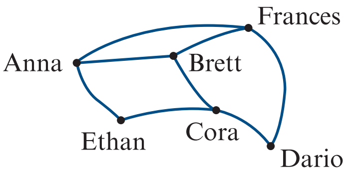
Brett is a friend of Anna, Cora and Frances.

Cora is a friend of Brett, Dario and Ethan.

Dario is a friend of Cora and Frances.

Ethan is a friend of Anna and Cora.

Frances is a friend of Anna, Brett and Dario.

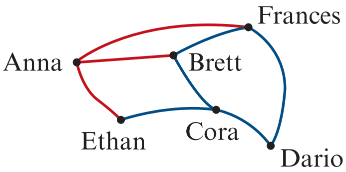


The graph shows each of the people as a dot called a **vertex**. The *vertices* (plural of vertex) are joined together by a line that indicates the social media friendship between the people. The lines that join the vertices in the graph are called **edges**.

Degree of a vertex

Anna has three friends. The vertex representing Anna has three edges attached to it. Each of these connects Anna to one of her friends.

The number of times edges attach to a vertex is called the **degree** of that vertex.



The degree of the vertex representing Anna is odd, because there is an odd number of attachments (three) by the edges. The degree of the vertex representing Dario is even because there is an even number of attachments (two) by edges to it.

In symbolic form, we can let the letter A represent the vertex for Anna. The degree of this vertex can be written as deg(A). In this graph, deg(A) = 3.

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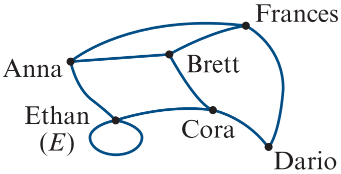
Loops

Imagine that Ethan is able to add himself as a friend on the social media website.

The edge representing this connection would connect the vertex representing Ethan, E, back to itself. This type of edge is called a **loop**.



A loop is attached twice to a vertex and so it will count as two degrees. So $\deg(E) = 4$.

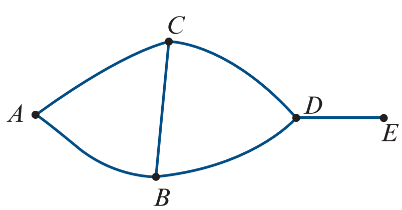
> **Edges, vertices and loops**
> - A graph consists of **vertices** joined by **edges**.
> - The number of edges attached to a vertex is called the degree of the vertex. The symbolic form for the degree of vertex A is $\deg(A)$.
> - A loop connects a vertex to itself. Loops count for two degrees for a vertex.

▶ Describing graphs

Graphs that represent connections between objects can take different forms and have different features. This means that there is a variety of ways to describe these graphs.

Simple graphs

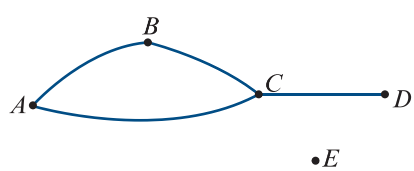
Simple graphs do not have any loops. There are no duplicate or **multiple edges** either.



Isolated vertex

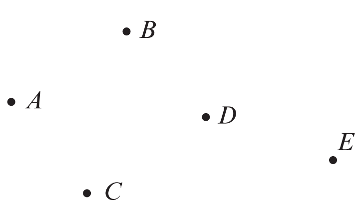
A graph has an **isolated vertex** if there is a vertex that is not connected to another vertex by an edge.

The isolated vertex in this graph is E, because it is not connected to any other vertex by an edge.



Degenerate graphs

Degenerate graphs have all vertices isolated. This means that there are no edges in the graph at all.



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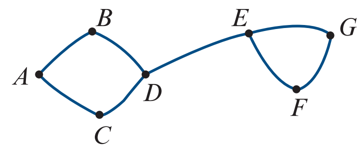
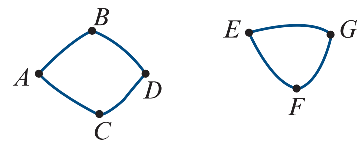
Connected graphs and bridges

A **connected graph** has every vertex connected to every other vertex, either directly or indirectly via other vertices. The graph on the right is connected. A **bridge** is an edge in a connected graph that, if removed, will cause the graph to be disconnected. The graph on the right has a bridge connecting vertex D to vertex E.

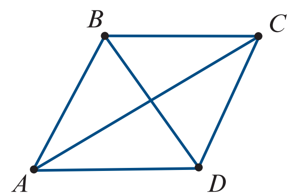


The graph on the right shows the bridge from vertex D to vertex E removed. There are now two separate sections of the graph that are not connected to each other.



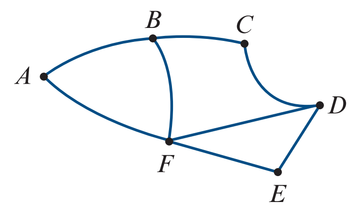
Complete graphs

If there is an edge between every pair of vertices, the graph is called a **complete graph**. Every vertex in the graph is connected directly by an edge to every other vertex in the graph.

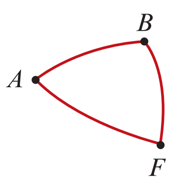


Subgraphs

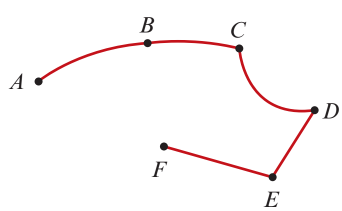
A **subgraph** is a part of a larger graph. All of the edges and vertices in the subgraph must exist in the original graph. If there are extra edges or vertices, the graph will not be a subgraph of the larger graph.



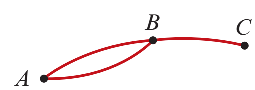
Graph 1



Graph 2



Graph 3



Graph 4

Graphs 2 and 3 above are subgraphs of graph 1. All of the vertices and edges in graphs 2 and 3 exist in graph 1.

Graph 4 above is not a subgraph of graph 1. There are two edges connecting vertex A to vertex B, but in graph 1 there is only one.

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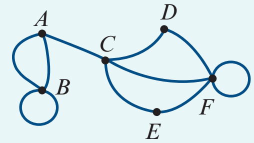
Example 1 Graphs

A connected graph is shown on the right.



a What is the degree of vertex C?

b Which vertices have a loop?

c What is the degree of vertex F?

d A bridge exists between two vertices. Which vertices are they?

e Draw a subgraph of this graph that involves only vertices A, B and C.

Solution

a Count the number of times an edge connects to vertex C. There are four connections.

The degree of vertex C is 4.
deg(C) = 4

b A vertex has a loop if an edge connects it to itself.

Vertex B and vertex F have loops.

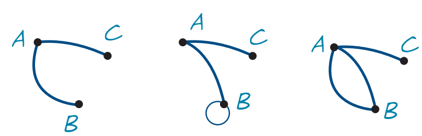
c Count the number of times an edge connects to vertex F. Remember that a loop counts as two degrees.

The degree of vertex F is 5.
deg(F) = 5

d Look for an edge that, if removed, would disconnect the graph.

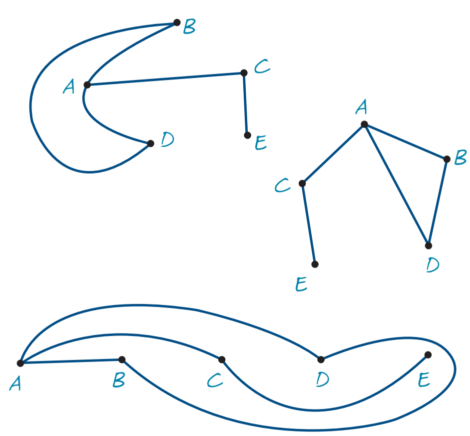
A bridge exists between vertex A and vertex C.

e There are a few possible answers for this question. Some are shown on the right.



Equivalent (Isomorphic) graphs

All of the graphs shown in the diagram below contain exactly the same information. For example, the edge between vertex E and C exists in all of them. The vertex A is connected to B, D and C as well.



The location of the vertices and edges in the diagram are unimportant. As long as the connections are all represented accurately, the graph can be drawn in any way that you prefer.

The first of the graphs has some curved edges and the second has all straight edges. The third has the vertices arranged in a straight line.

The style that they are drawn in and the position of the vertices relative to each other is unimportant. It *is* important that the information contained in the graph – the connections between the vertices – is correct.

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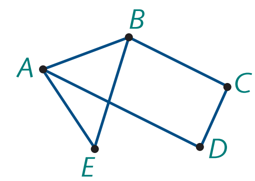
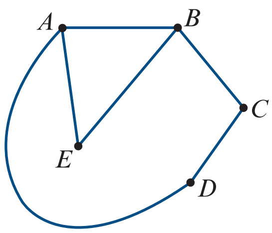
All of these graphs are considered to be *equivalent* to each other because they all contain identical information. Each has edges connecting the same vertices. Graphs that contain identical information like this are called **equivalent graphs** or **isomorphic graphs**.

Planar graphs



The graph opposite has two edges that overlap. It is important to note that there is *no vertex at the point of overlap of the edges.*

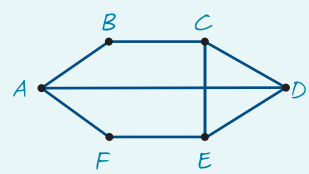
It can help to think of an edge as an insulated electrical wire. It is quite safe to cross two such electrical wires because the wires themselves never touch and never interfere with each other. The edges that cross over in this diagram are similar, in that they do not intersect and do not interfere with each other.



If a graph has edges that cross, it may be possible to redraw the graph so that the edges no longer cross. The edge between vertices A and D has been moved, but none of the information in the graph has changed. Graphs where this is possible are called **planar graphs**. If it is impossible to draw an equivalent graph without crossing edges, the graph is called a *non-planar graph*.

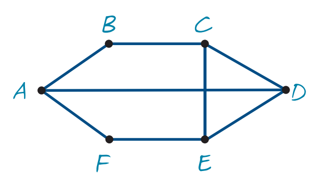
Example 2 Redrawing a graph in planar form

Show that this graph is planar by redrawing it so that no edges cross.

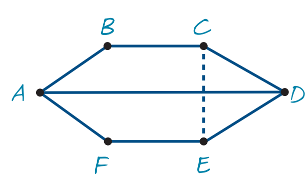


Solution

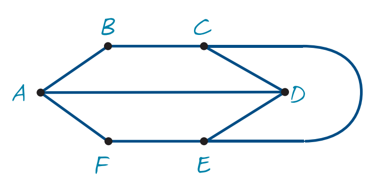
1 Choose one of the edges that crosses over another edge.



2 Remove it temporarily from the graph.



3 Redraw the edge between the same vertices but without crossing over another edge.



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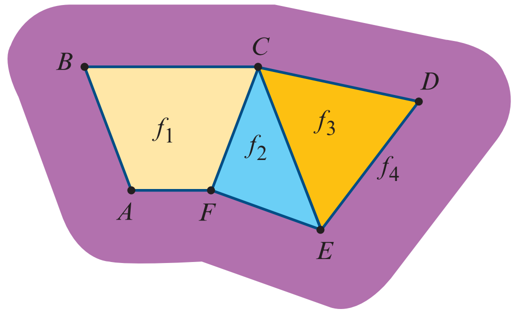
Euler's rule

Leonard Euler (pronounced 'oiler') was one of the most prolific mathematicians of all time. He contributed to many areas of mathematics and his proof of the rule named after him is considered to be the beginning of the branch of mathematics called topology.



Faces

A planar graph defines separate regions of the paper it is drawn on. These regions are enclosed spaces that you could colour in and these regions are called **faces**. An often-forgotten face of a graph is the space outside of the graph itself, covering the infinite space around it. This face is labelled f_4 in the graph above.

The number of faces for a graph can be counted. In the graph shown above, there are four faces, labelled f_1, f_2, f_3 and f_4.

Euler's rule

There is a relationship between the number of vertices, v, the number of edges, e, and the number of faces, f, in a connected planar graph.

In words: number of vertices − number of edges + number of faces = 2

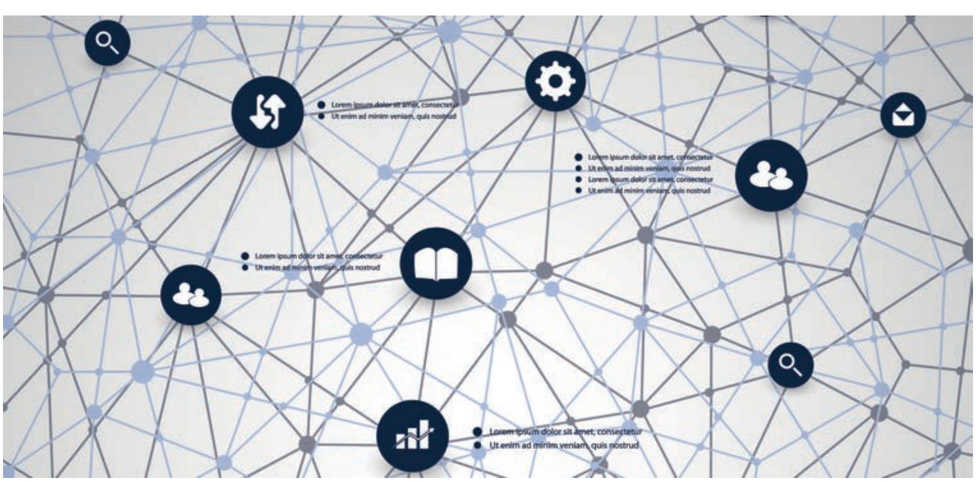
In symbols: $v - e + f = 2$

Euler's rule

For any planar graph:

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges and f is the number of faces in the graph.



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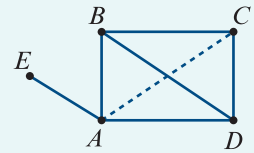
Example 3 Verifying Euler's rule

For the graph shown on the right:

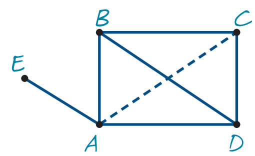
a redraw the graph into planar form

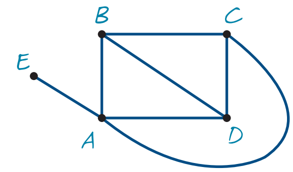
b verfiy Euler's rule for this graph.



Solution

a Temporarily remove an edge that crosses another edge and redraw it so that it does not cross another edge.





b Count the number of vertices, edges and faces.

In the planar graph there are five vertices, seven edges and four faces.

$v - e + f = 5 - 7 + 4 = 2$

Euler's rule is verified.

Example 4 Using Euler's rule

A connected planar graph has six vertices and nine edges. How many faces does the graph have? Draw a connected planar graph with six vertices and nine edges.

Solution

a Write down the known values.

$v = 6 \quad e = 9$

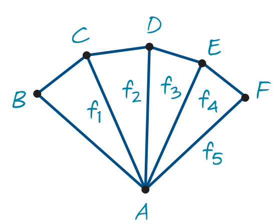
b Substitute into Euler's rule and solve for the unknown value.

$$\begin{aligned} v - e + f &= 2 \\ 6 - 9 + f &= 2 \\ f - 3 &= 2 \\ f &= 5 \end{aligned}$$

This graph has five faces, labelled f_1, f_2, f_3, f_4 and f_5.

c Sketch the graph.

Note: There are other possible graphs.



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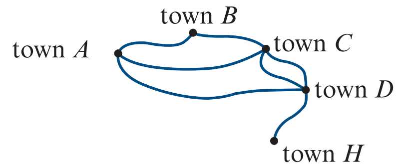
Exercise 14A

Drawing and describing graphs

1 This section of a road map can be considered as a graph, with towns as vertices and the roads connecting the towns as edges.



a Give the degree of:

i town A **ii** town B **iii** town H.

b What is the sum of the degrees of all the vertices of this graph?

c Draw a subgraph of this road map that contains only towns H, D and C.

2 Draw a graph that:

a has three vertices, two of which are odd

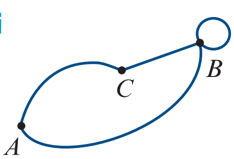
b has four vertices and five edges, one of which is a loop

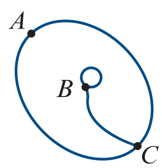
c has six vertices, eight edges and one bridge

d has six vertices, two of which are odd, and contains a subgraph that is a triangle.

Equivalent graphs

3 In each question below, three graphs are isomorphic and the fourth is not. Identify the graph which is not isomorphic to the others.

a **i**


ii


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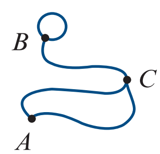

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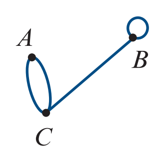
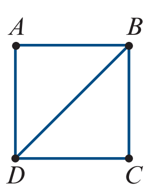
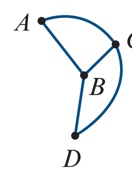


b **i**


ii


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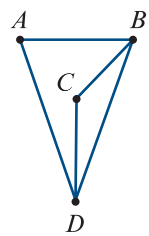
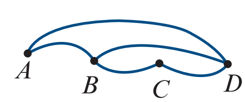
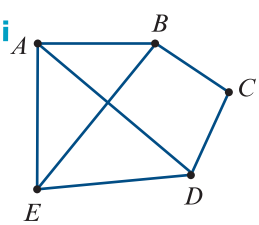
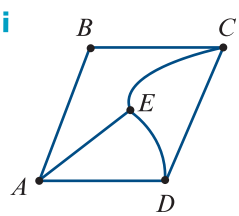


iv


c **i**


ii


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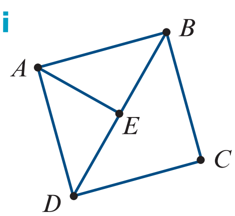

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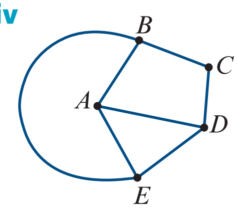

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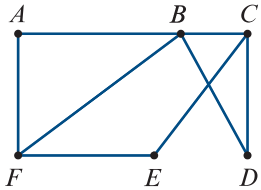
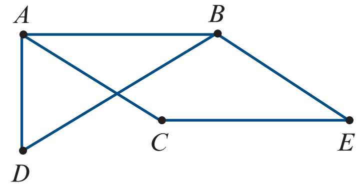
Drawing planar graphs

4 Where possible, show that the following graphs are planar by redrawing them in a suitable planar form.

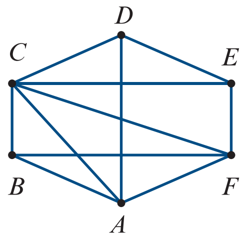
a



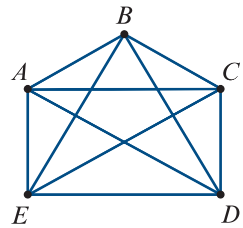
b



c



d

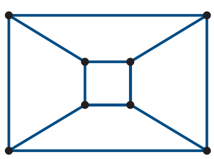


Euler's rule

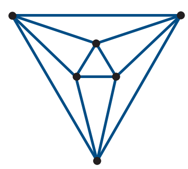
5 For each of the following graphs:

i state the values of v, e and f **ii** verify Euler's rule.

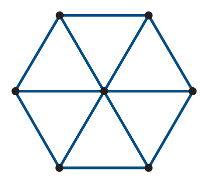
a



b



c



6 For a planar connected graph, find:

a f, if $v = 8$ and $e = 10$ **b** v, if $e = 14$ and $f = 4$ **c** f, if $v = 5$ and $e = 14$
d e, if $v = 10$ and $f = 11$.

14B Adjacency matrices

▶ Summarising the connections in a graph

A matrix can be used to summarise the information in a graph. A matrix that records the number of connections between vertices of a graph is called an **adjacency matrix**.

A graph and the adjacency matrix for that graph are shown overleaf.

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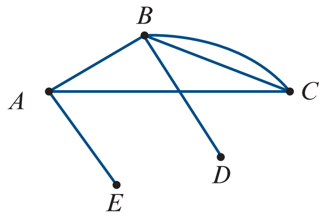


$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix has:

- five rows and five columns, one for each vertex in the graph
- row and column labels that match the vertices in the graph, A, B, C, D, E
- a '0' in the intersection of row A and column D because there is no edge connecting A to D
- a '0' in the intersection of row A and column A because there is no edge connecting A to itself; that is, there is no loop at vertex A
- a '1' in the intersection of row A column B because there is one edge connecting A to B
- a '2' in the intersection of row C and column B because there are two edges connecting C to B.

The number of edges between every other pair of vertices in the graph is recorded in the adjacency matrix in the same way.

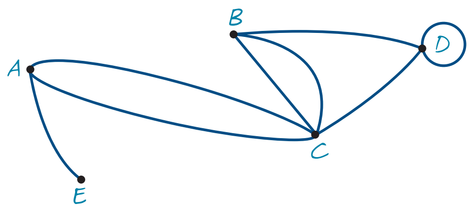
Example 5 Drawing a graph from an adjacency matrix

Draw the graph that has adjacency matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Solution

1. Draw a dot for each vertex and label A to E.
2. There is a '2' in the intersection of row A and column C. This means there are two edges connecting vertex A and vertex C. Add these to the graph.



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3 Note the '1' in the intersection of row D and column D. This shows that there is a loop at vertex D.

4 Look at every intersection of row and column and add edges to the graph, if they do not already exist.

Note: This graph is drawn as a planar graph, but this is not strictly necessary unless required by the question.

Adjacency matrices

The adjacency matrix A of a graph is an $n \times n$ matrix in which, for example, the entry in row C and column F is the number of edges joining vertices C and F.

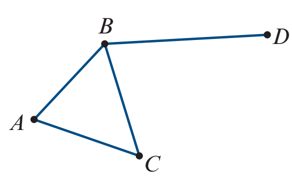
A loop is a single edge connecting a vertex to itself.

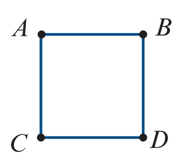
Loops are counted as one edge.

Exercise 14B

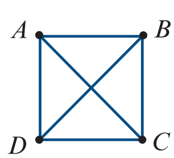
Writing adjacency matrices

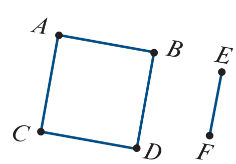
1 For each of the following graphs, write down the adjacency matrix.

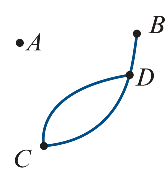
a


b


c


d


e


f


Drawing graphs from adjacency matrices

2 Draw a graph from each of the following adjacency matrices.

a $$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

b $$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

c $$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

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Miscellaneous

3 The adjacency matrix on the right has a row and column for vertex C that contains all zeros. What does this tell you about vertex C?

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

4 Every vertex in a graph has one loop. What feature of the adjacency matrix would tell you this information?

5 A graph has five vertices: A, B, C, D and E. It has no duplicate edges and no loops. If this graph is complete, write down the adjacency matrix for the graph.

14C Exploring and travelling

▶ Travelling

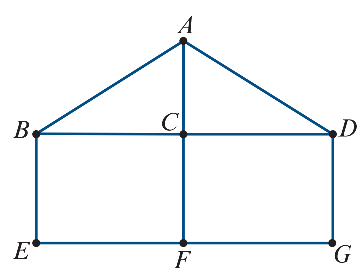
Graphs can be used to model and analyse problems involving exploring and **travelling**. These problems include minimising the distance travelled or time taken between different locations using different routes. For example, a courier driver would like to know the shortest route to use for deliveries, and a tour guide would like to know the quickest route that allows tourists to see a number of sights without retracing their steps.

To solve these types of problems, you will need to learn the language we use to describe the different ways of navigating through a graph, from one vertex to another.

▶ Walks, trails, paths, circuits and cycles

The different ways of navigating through graphs, from one vertex to another, are described as *walks, trails, paths, circuits* and *cycles*.

The graph opposite will be used to explain and define each of these terms.

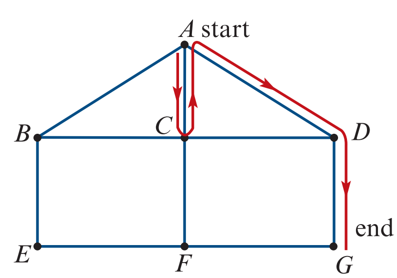


Walks

A **walk** is a sequence of edges, linking successive vertices in a graph.

A walk starts at one vertex and follows any route to finish at another vertex.

The red line in the graph opposite traces out a walk. This walk can be written down by listing the vertices in the order they are visited: $A–C–A–D–G$.



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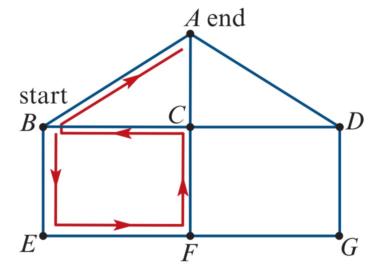
Trails

A **trail** is a walk with no repeated edges.

The red line in the graph opposite traces out a trail. This trail can be written down by listing the vertices in the order they are visited: $B–E–F–C–B–A$.

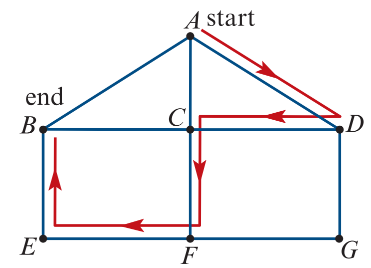


Note: There are *no repeated edges* in this trail, but one vertex (B) is repeated.

Paths

A **path** is a walk with no repeated edges and no repeated vertices.

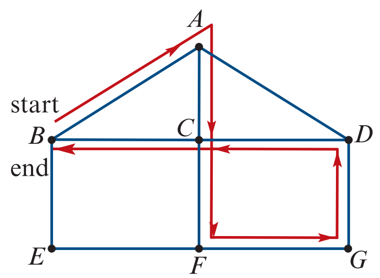
The red line in the graph opposite traces out a path. This path can be written down by listing the vertices in the order they are visited: $A–D–C–F–E–B$.



Circuits

A **circuit** is a trail (no repeated edges) that starts and ends at the same vertex. Circuits are also called *closed trails*.

The red line in the graph opposite traces out a circuit. This circuit can be written down by listing the vertices in the order they are visited: $A–C–F–G–D–C–B–A$.

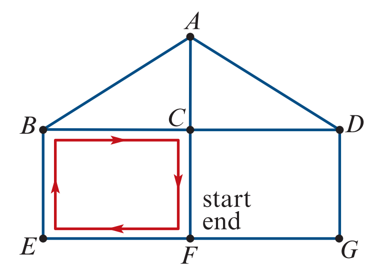


Note: There are *no repeated edges* in this circuit, but one vertex, C, is repeated. The start and end vertices are also repeated because of the definition of a circuit.

Cycles

A **cycle** is a path (no repeated edges, no repeated vertices) that starts and ends at the same vertex. The start and end vertex is an exception to repeated vertices. Cycles are also called *closed paths*.

The red line in the graph opposite traces out a cycle. This cycle can be written down by listing the vertices in the order they are visited: $F–E–B–C–F$.



Note: There are *no repeated edges* and *no repeated vertices* in this cycle, except for the start and end vertices.

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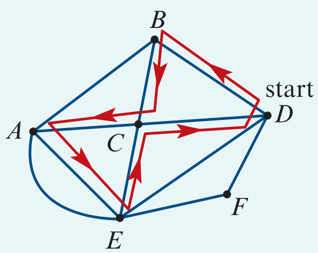
Walks, trails, paths, circuits and cycles

- A *walk* is a sequence of edges, linking successive vertices in a graph.
- A *trail* is a walk with no repeated edges.
- A *path* is a walk with no repeated edges (and no repeated vertices).
- A *circuit* is a trail (no repeated edges) that starts and ends at the same vertex.
- A *cycle* is a path (no repeated edges and no repeated vertices) that starts and ends at the same vertex.

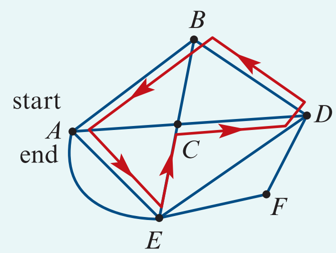
Example 6 Identifying types of walks

Identify the walk in each of graphs below as a trail, path, circuit, cycle or walk only.

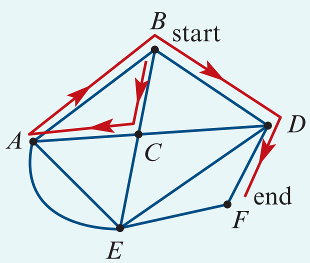
a



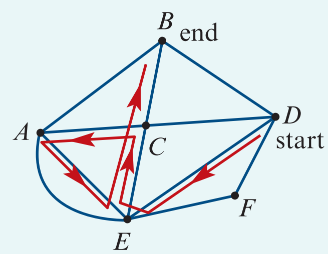
b



c



d



Solution

a This walk starts and ends at the same vertex so it is either a circuit or a cycle. The walk passes through vertex C twice without repeated edges, so it must be a circuit.

b This walk starts and ends at the same vertex so it is either a circuit or a cycle. The walk has no repeated vertex or edge so it is a cycle.

c This walk starts at one vertex and ends at a different vertex, so it is not a circuit or a cycle. There is one repeated vertex (B) and no repeated edge, so it must be a trail.

d This walk starts at one vertex and ends at a different vertex so it is not a circuit or a cycle. There are repeated vertices (C and E) and repeated edges (the edge between C and E), so it must be a walk only.

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Eulerian trails and circuits

Trails and circuits that follow every edge, without duplicating any edge, of a graph are called **eulerian trails** and **eulerian circuits**. Eulerian trails and circuits are important for some real-life applications. If, for example, a graph shows towns as vertices and roads as edges, then being able to identify a route through the graph that follows every road can be important for mail delivery, or for checking the condition of the roads.

Eulerian trails and circuits exist under easily identified conditions.

Eulerian trails and circuits

Eulerian trails

An eulerian trail follows every edge of a graph.

An eulerian trail will exist if the graph:

- is connected
- has exactly *two* vertices that have an *odd degree*.

Eulerian circuits

An eulerian circuit is an eulerian trail (follows every edge) that starts and ends at the same vertex.

An eulerian circuit will exist if the graph:

- is connected
- has vertices that *all* have an *even degree.*

An eulerian circuit can begin at *any* of the vertices.

Note: If a graph has more than two odd-degree vertices, neither an eulerian trail nor an eulerian circuit exists.

Hamiltonian paths and cycles

Paths and cycles that pass through every vertex of a graph only once are called **hamiltonian paths** and **hamiltonian cycles**, named after the mathematician William Rowan Hamilton.

Hamiltonian paths and cycles have real-life applications to situations where every vertex of a graph needs to be visited, but the route taken is not important. If, for example, the vertices of a graph represent people and the edges of the graph represent email connections between those people, a hamiltonian path would ensure that every person in the graph received a message intended for everyone.

Unlike eulerian trails and circuits, hamiltonian paths and cycles do not have a convenient rule or feature that identifies them. Inspection is the only way to identify them.

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Hamiltonian paths and cycles

Hamiltonian paths

A hamiltonian path visits every vertex of a graph.

Hamiltonian cycles

A hamiltonian cycle is a hamiltonian path (every vertex) that starts and ends at the same vertex.

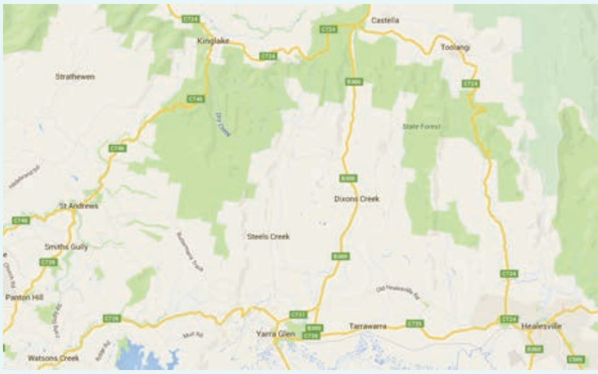
Note: Inspection is the only way to identify Hamilton paths and cycles.

Remember: Eulerian trails and circuits do not repeat edges. Hamiltonian paths and cycles do not repeat vertices.

Hint: To remember the difference between eulerian and hamiltonian travels, remember that eulerian refers to edges, and both start with 'e'.

Example 7 Eulerian and hamiltonian travel

A map showing the towns of St Andrews, Kinglake, Yarra Glen, Toolangi and Healesville is shown on the right.



a Draw a graph with a vertex representing each of these towns and edges representing the direct road connections between the towns.

b Explain why an eulerian trail, but not an eulerian circuit, is possible through this graph.

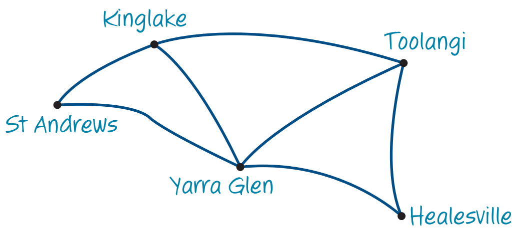
c Write down an eulerian trail that begins at Toolangi.

d Write down a hamiltonian cycle that begins at Healesville.

Solution

a A road connection exists between:

- St Andrews and Kinglake
- St Andrews and Yarra Glen
- Kinglake and Yarra Glen
- Kinglake and Toolangi
- Yarra Glen and Toolangi
- Yarra Glen and Healesville
- Healesville and Toolangi.



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b The graph has two odd-degree vertices (Toolangi and Kinglake).

There are exactly two odd-degree vertices in this graph. An eulerian trail will exist, but an eulerian circuit does not.

c There are a few different answers to this question. One of these is shown.

An eulerian trail, starting at Toolangi is: Toolangi–Healesville–Yarra Glen–Toolangi–Kinglake–Yarra Glen–St Andrews–Kinglake

d There are two different answers to this question. One of these is shown.

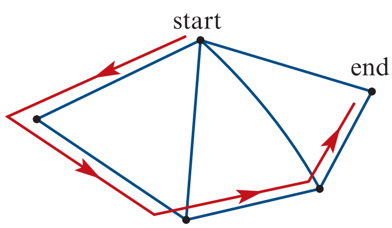
A hamiltonian cycle that begins at Healesville is: Healesville–Yarra Glen–St Andrews–Kinglake–Toolangi–Healesville

Exercise 14C

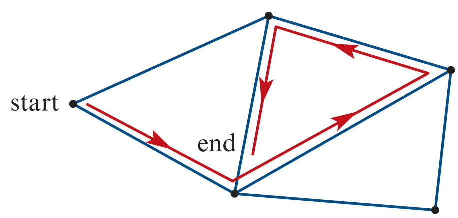
Describing travels

1 Identify the walk in each of the graphs below as a trail, path, circuit or walk only.

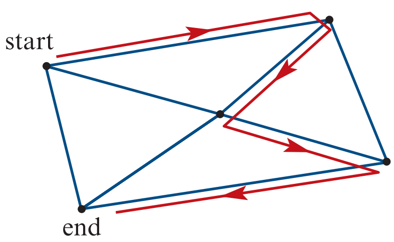
a



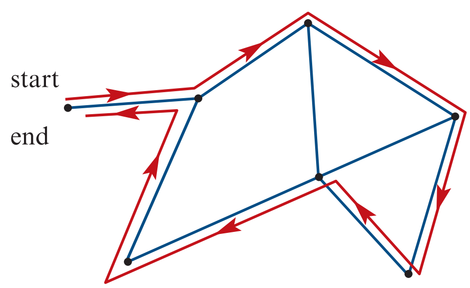
b



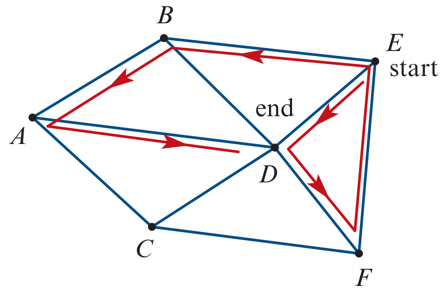
c



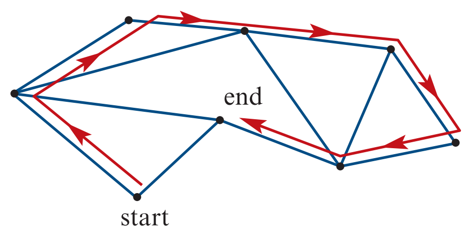
d



e



f



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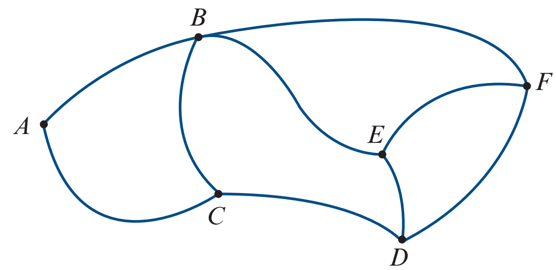
2 Using the graph opposite, identify the walks below as a trail, path, circuit, cycle or walk only.

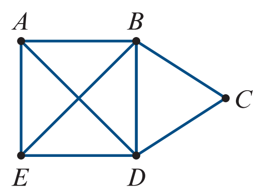


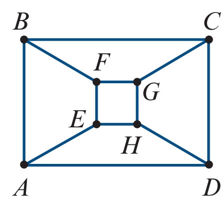
a $A–B–E–B–F$
b $B–C–D–E–B$
c $C–D–E–F–B–A$
d $A–B–E–F–B–E–D$
e $E–F–D–C–B$
f $C–B–E–F–D–E–B–C–A$

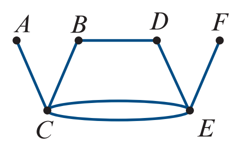
Eulerian trails and circuits

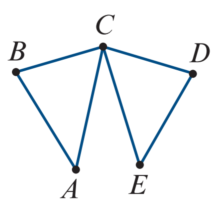
3 **i** Identify whether each graph below has an eulerian circuit, or an eulerian trail but not an eulerian circuit, or neither an eulerian circuit nor an eulerian trail.

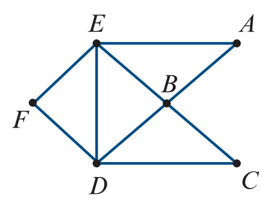
ii Name the eulerian circuits or trails found.

a


b


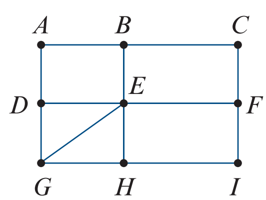
c


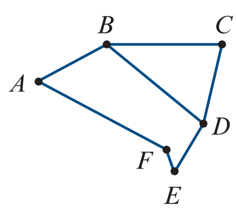
d


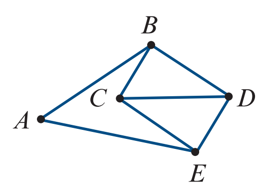
e


Hamiltonian paths and cycles

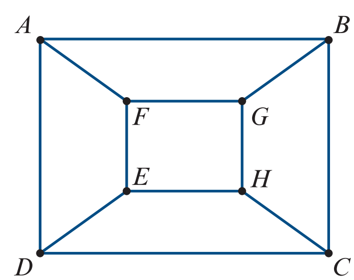
4 List a hamiltonian cycle for each of the following.

a


b


c


5 List a hamiltonian path for this graph, starting at F and finishing at G.



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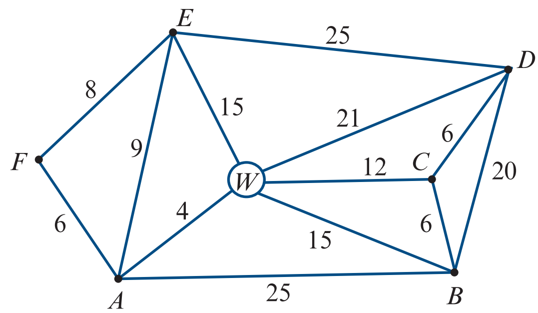
Applications

6 The network diagram below shows the location of a warehouse, W. This warehouse supplies equipment to six factories A, B, C, D, E and F. The numbers on the edges indicate the shortest distance (in kilometres) to drive along each of the connecting roads.



a What is the degree of vertex W?

b A salesman plans to leave factory E, first visit the warehouse, W, and then visit every other factory. He is to visit each location only once. He will not return to factory E.

i Write down the mathematical term used to describe the planned route.

ii Write down an order in which he can visit the factories.

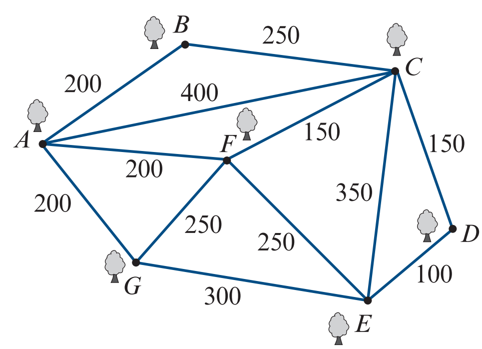
c The company plans to build an office along one of the roads in the graph. To inspect all possible locations, the manager wishes to drive along a route through the graph, which follows an eulerian circuit. She will start at the warehouse, W.

i Explain why the journey that the manager plans to take is **not** possible for this graph.

ii A journey that follows an eulerian trail, starting at the warehouse, W, is possible for this network. At which vertex will this eulerian trail end?

based on VCAA (2005)

7 A housing estate has large open parklands that contain seven large trees.
The trees are denoted as vertices A to G on the graph below.
Walking paths link the trees as shown.
The numbers on the edges represent the lengths of the paths in metres.



a Determine the sum of the degrees of the vertices of this graph.

b One day, Jamie decides to go for a walk that will take him along each of the paths between the trees. He would like to walk the minimum distance that is possible.

i Write down the vertex at which Jamie could begin his walk.

ii Determine the total distance, in metres, that Jamie will walk.

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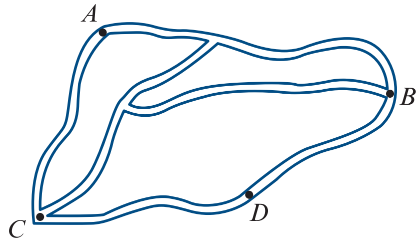
c Michelle is currently at F. She would like to follow a route that can be described as the shortest hamiltonian cycle.
Write down a route that Michelle can take.

based on VCAA (2007)

8 Four children each live in a different town. The diagram below is a map of the roads that link the four towns, A, B, C and D.



a How many different trails are there from town A to town D?

b How many different ways can a vehicle travel between town A and town B without visiting any other town?

c Draw this map as a graph by representing towns as vertices and each different route between two towns as an edge.

d Explain why a vehicle at A could not follow an eulerian circuit through this graph.

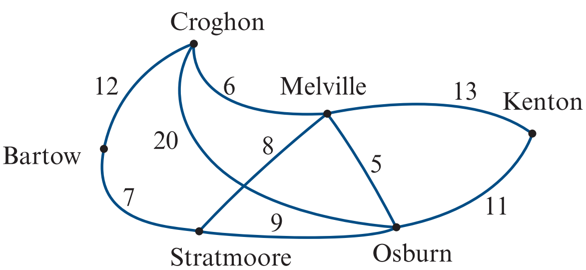
14D Weighted graphs and networks

▶ Weighted graphs

The edges of graphs represent connections between the vertices. Sometimes there is more information known about that connection. If the edge of a graph represents a road between two towns, we might also know the length of this road, or the time it takes to travel this road.

Extra numerical information about the edge that connects vertices can be added to a graph by writing the number next to the edge. Graphs that have a number associated with each edge are called **weighted graphs**.

The weighted graph in the diagram on the right shows towns, represented by vertices, and the roads between those towns, represented by edges. The numbers, or *weights*, on the edges are the distances along the roads.



Weighted graphs in which the weights are physical quantities, for example distance, time or cost, are called **networks**.

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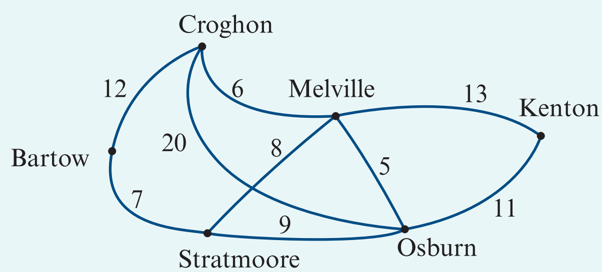
Shortest path problems

When we know numerical information about the connections, travelling through a graph will have extra considerations. If the weights of a network represent time, we can choose a route that will allow us to travel in the shortest time. If the weights represent distance, we can determine a route that will allow us to travel the shortest distance.

These types of problems involve finding the **shortest path** from one vertex to another. In networks that have only a few vertices, it is often easy to find the shortest path between two vertices by inspection. All of the possible route options should be listed, but it is sometimes obvious that certain routes are going to be much longer than others.

Example 8 Finding the shortest path from one vertex to another

Find the shortest path from Bartow to Kenton in the network shown on the right.



Solution

1 List options for travelling from Bartow to Kenton.

B–S–M–O–K
B–S–M–K
B–S–O–K

Note: Routes via Croghan are likely to be much longer than routes via Stratmoore because of the larger distance from Bartow to Croghan (12 km).

2 Add the weights for each route.

B–S–M–O–K $7 + 8 + 5 + 11 = 31$ km
B–S–M–K $7 + 8 + 13 = 28$ km
B–S–O–K $7 + 9 + 11 = 27$ km

3 Write your answer.

The shortest path from Bartow to Kenton is 27 km with route B–S–O–K.

Dijkstra's algorithm

Finding the shortest path from one vertex of a graph to another is easy to determine if the graph is small and does not have too many vertices and edges. When there are many vertices and many edges, a systematic method, called an **algorithm**, can be used to find the shortest path.

Dutch computer scientist, Edsger Wybe Dijkstra (pronounced 'Dyke-stra') developed an algorithm for determining the shortest path through a graph. This algorithm, and others like it, have important applications to computerised routing and scheduling programs, such as GPS navigation devices.

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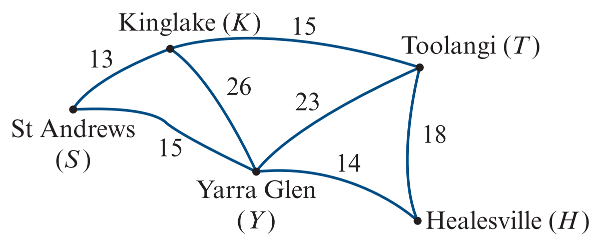
Using Dijkstra's algorithm to find the shortest path between two vertices

Dijkstra's algorithm can be used to find the shortest path through a network, from a starting vertex to a destination vertex.

The weights in the graph opposite show the distances, in kilometres, by road between the towns. The algorithm will be used to find the shortest path between St Andrews (S) and Toolangi (T).



Step 1: Create a table

1. Write the starting vertex as the first row vertex.
2. Write the other vertices in the network as column vertices (order is unimportant).

	K	Y	T	H
S				

Note: St Andrews (S) is the starting vertex and is the first row vertex. All the other towns are column vertices.

Step 2: Complete the first row

Look at the graph to find the distance from the starting vertex (the row vertex) to the vertices that are directly connected to it (a column vertex).

1. Write down the distance from the row vertex to a column vertex, directly underneath the column vertex as shown.
2. If a vertex is not directly connected to the starting vertex, mark a cross (×).

	K	Y	T	H
S	13	15	×	×

Notes:

1. Kinglake (K) is 13 km from St Andrews (S).
2. Yarra Glen (Y) is 15 km from St Andrews (S).
3. Toolangi (T) and Healsville (H) are not directly connected to St Andrews (S) and have a cross (×).

3. Look for the smallest number in the first row and draw a box around it. If there are two or more the same, any one can be chosen.
4. The column vertex for this boxed number becomes the next row vertex

	K	Y	T	H
S	$\boxed{13}$	15	×	×
K				

Notes:

1. Put a box around 13 because it is the smallest number in the row.
2. The column vertex for this number is K so this becomes the next row vertex.

Step 3: Complete further rows

1. Copy all boxed numbers into the next row.

	K	Y	T	H
S	$\boxed{13}$	15	×	×
K	$\boxed{13}$			

Note: 13 has a box so it is copied into the next row.

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2 For the remaining columns, add the boxed number for the row vertex to the distance from the row vertex to the column vertex.

a If the value is *greater than* the value above it in the column, ignore the new value and copy the existing one.

b If the value is *less than or equal to* the value above it in the column, write down the new value.

c If the row vertex is not directly connected to the column vertex, mark with a cross (×).

	K	Y	T	H
S	$\boxed{13}$	15	×	×
K	$\boxed{13}$	15	28	×

Notes:

1 The box number for column K is 13. This must be added to the distance from K to every other vertex.

2 For Y: $\boxed{13} + 26 = 39$.
This is larger than the existing 15, so 15 is copied into the new row.

3 For T: $\boxed{13} + 15 = 28$.
This is now a possible connection so copy 28 into the new row.

4 For H: There is no direct connection between K and H, so it stays as a cross (×).

3 Look for the smallest *unboxed* number in the row and draw a box around it.

4 The column vertex for this new boxed number becomes the next row vertex.

	K	Y	T	H
S	$\boxed{13}$	15	×	×
K	$\boxed{13}$	$\boxed{15}$	28	×
Y				

Notes:

1 Put a box around 15 because it is the smallest unboxed number in the row.

2 The 15 is in the column for Y, so Y is the next row vertex.

5 Repeat step 3 until the destination vertex value has a box around it.

	K	Y	T	H
S	$\boxed{13}$	15	×	×
K	$\boxed{13}$	$\boxed{15}$	28	×
Y	$\boxed{13}$	$\boxed{15}$	$\boxed{28}$	29

Notes:

1 The box number for column Y is 15. This must be added to the distance from Y to every other vertex.

2 For T: $\boxed{15} + 23 = 38$.
This is larger than the existing 28, so copy 28 into the new row.

3 For H: $\boxed{15} + 14 = 29$.
This is now a possible connection, so copy 29 into the new row.

4 The destination vertex (T) has a box around its value, so stop the algorithm.

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Step 4: Backtrack to identify the shortest path and its length

1 Start at the box value for the destination vertex. This is the length of the shortest path from the starting vertex to the destination vertex.

2 Draw a line up the column to the last number that is the same as the box number (it does not have to have a box around it).

3 Look at the row vertex for this number and draw a horizontal line to the column for this vertex.

4 Repeat until the starting vertex is reached.

5 The horizontal lines in the table indicate the shortest path.

	K	Y	T	H
S	$\boxed{13}$	15	×	×
K	$\boxed{13}$	$\boxed{15}$	28	×
Y	$\boxed{13}$	$\boxed{15}$	$\boxed{28}$	29

Notes:

1 The shortest path length from St Andrews to Toolangi is 28 kilometres.

2 Start at the value 28 under the column vertex for T.

3 Draw a line up to the last value of 28 in the column. This occurs on row K.

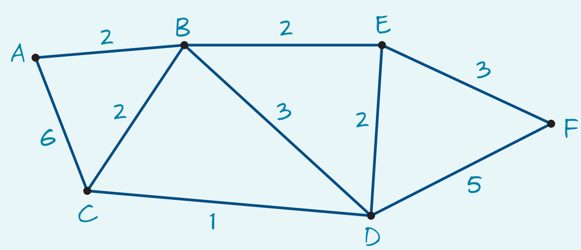
4 Draw a horizontal line to column K.

5 Draw a line up to the last value of 13 in the column. This occurs on row S.

6 There are horizontal lines between S and K and K and T so the shortest path is S–K–T.

Example 9 Finding the shortest path from one vertex to another using Dijkstra's algorithm

Find the shortest path from A to F in the weighted graph shown on the right.



Solution

Step 1

1 Create a table with the starting vertex (A) as the first row vertex.

2 Write the other vertices as column vertices

	B	C	D	E	F
A					

Step 2

1 Write down the distance from A to all the other vertices.

- A–B is distance 2.
- A–C is distance 6.

	B	C	D	E	F
A	$\boxed{2}$	6	×	×	×
B					

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2 Put a cross (×) if the vertices are not directly connected.

- A is not directly connected to D, E nor F so insert a cross (×).

3 Put a box around the smallest number in the row.

- Put a box around 2, the smallest value in the row.

4 Look at the column vertex for this box number. This becomes the next row vertex.

- Vertex B is the next row vertex.

Step 3

1 Copy the boxed numbers into the next row.

2 For the remaining columns, add the box number for the row vertex to the distance from the row vertex to the column vertex.

- For column C: $B–C$ is distance 2. New value is $\boxed{2} + 2 = 4$. This is less than 6, so write 4 in the row.
- For column D: $B–C$ is distance 3. New value is $\boxed{2} + 3 = 5$. Write 5 in the row.
- For column E: $B–E$ is distance 2. New value is $\boxed{2} + 2 = 4$. Write 4 in the row.
- For column F: $B–F$ has no direct connection. Write down a cross (×).

	B	C	D	E	F
A	$\boxed{2}$	6	×	×	×
		2+2= 4	2+3= 5	2+2= 4	
B	$\boxed{2}$	4	5	4	×

3 Put a box around the smallest unboxed number in the row.

4 Look at the column vertex for this box number. This becomes the next row vertex.

- Column C and column E both have the smallest number of 4. Either one can be boxed. Use E as the next row vertex.

	B	C	D	E	F
A	$\boxed{2}$	6	×	×	×
		$2+2=4$	$2+3=5$	$2+2=4$	
B	$\boxed{2}$	4	5	$\boxed{4}$	×
E					

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5 Repeat until the destination vertex (F) value has a box around it.

	B	C	D	E	F
A	$\boxed{2}$	6	×	×	×
B	$\boxed{2}$	$2+2=4$ 4	$2+3=5$ 5	$2+2=4$ $\boxed{4}$	×
E	$\boxed{2}$	$\boxed{4}$	$4+2=6$ 5	$\boxed{4}$	$4+3=7$ 7
C	$\boxed{2}$	$\boxed{4}$	$4+1=5$ $\boxed{5}$	$\boxed{4}$	7
D	$\boxed{2}$	$\boxed{4}$	$\boxed{5}$	$\boxed{4}$	$5+5=10$ $\boxed{7}$

Step 4

1 The shortest distance from A to F is the boxed number in column F; that is, 7. Start backtracking at this number.

2 Draw a vertical line up the column to the last number; that is also 7. This occurs on row E.

3 Draw a horizontal line across the table to column E. The value in this position is 4.

4 Draw a vertical line up the column to the last number that is also 4. This occurs on row B.

5 Draw a horizontal line across the table to column B. The value in this position is 2.

6 Draw a vertical line up the column to the last number that is also 2. This occurs on row A, the starting row.

	B	C	D	E	F
A	$\boxed{2}$	6	×	×	×
B	$\boxed{2}$	$2+2=4$ 4	$2+3=5$ 5	$2+2=4$ $\boxed{4}$	×
E	$\boxed{2}$	$\boxed{4}$	$4+2=6$ 5	$\boxed{4}$	$4+3=7$ 7
C	$\boxed{2}$	$\boxed{4}$	$4+1=5$ $\boxed{5}$	$\boxed{4}$	7
D	$\boxed{2}$	$\boxed{4}$	$\boxed{5}$	$\boxed{4}$	$5+5=10$ $\boxed{7}$

7 Write your answer.

The shortest path from A to F is A–B–E–F.

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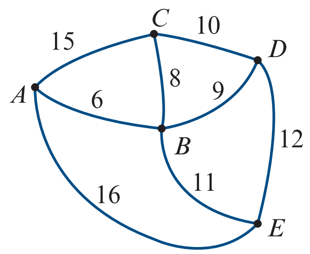
Exercise 14D

Weighted graphs and networks

1 The graph on the right shows towns A, B, C, D and E represented by vertices. The edges represent road connections between the towns. The weights on the edges are the average times, in minutes, it takes to travel along each road.



a Which two towns are 12 minutes apart by road?

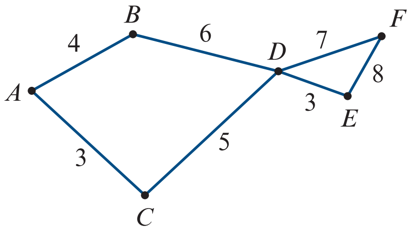
b How long will it take to drive from C to D via B?

c A motorist intends to drive from D to E via B. How much time will they save if they travel directly from D to E?

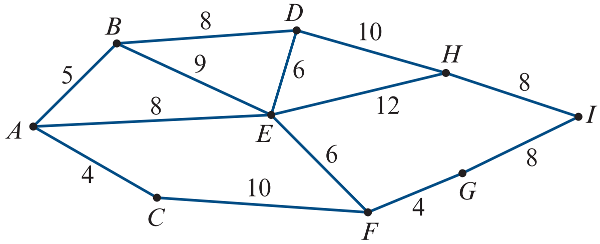
d Find the shortest time it would take to start at A, finish at E and visit every town exactly once.

Shortest path by inspection

2 By inspection, find the length of the shortest path from A to E.



3 The network on the right shows the distance, in kilometres, along walkways that connect landmarks A, B, C, D, E, F, G, H and I in a national park.



a What distance is travelled on the path A–B–E–H–I?

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b What distance is travelled on the circuit $F–E–D–H–E–A–C–F$?

c What is the distance travelled on the shortest cycle starting and finishing at E?

d Find the shortest path from A to I.

Calculations within Dijkstra's algorithm

4 The table contains the first line first of a Dijkstra's algorithm solution to a shortest path problem.

	B	C	D	E	F
A	5	2	×	×	6

a Which vertex is the starting vertex for the problem?

b Which two vertices are not directly connected to the starting vertex?

c Which vertex will be the next vertex in the algorithm?

d What is the distance between vertex A and F?

5 The table on the right contains the first two lines of a Dijkstra's algorithm solution to a shortest path problem. Vertex R is a distance of 5 from Q, a distance of 6 from S, a distance of 4 from T and a distance of 3 from U. Complete the row for vertex R.

	Q	R	S	T	U
P	3	$\boxed{1}$	×	4	×
R		$\boxed{1}$			

6 The table on the right contains the first two lines of a Dijkstra's algorithm solution to a shortest path problem.

	N	P	W	V	U
M	×	5	7	$\boxed{5}$	×
V	8	5	6	$\boxed{5}$	×

a Vertex V was chosen as the second vertex in the algorithm. Which other vertex could have been chosen instead?

b Which vertex is not directly connected to vertex V?

c What is the distance from vertex V to W?

7 A completed table of calculations for the shortest path through a network using Dijkstra's algorithm is shown on the right.

	B	C	D	E
A	$\boxed{8}$	14	×	×
B	$\boxed{8}$	$\boxed{10}$	11	×
C	$\boxed{8}$	$\boxed{10}$	$\boxed{11}$	16
D	$\boxed{8}$	$\boxed{10}$	$\boxed{11}$	$\boxed{16}$

a What is the length of the shortest path from A to C?

b What is the length of the shortest path from A to E?

c Write down the shortest path taken from A to E.

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8 A completed table of calculations for the shortest path through a network using Dijkstra's algorithm is shown on the right.

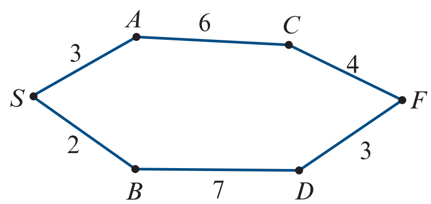
- **a** What is the length of the shortest path from A to G?
- **b** What is the shortest path from A to G?
- **c** What is the length of the shortest path from A to I?
- **d** What is the shortest path from A to I?

	B	C	D	E	F	G	H	I
A	2	$\boxed{1}$	×	5	×	×	×	×
C	$\boxed{2}$	$\boxed{1}$	×	5	4	×	×	×
B	$\boxed{2}$	$\boxed{1}$	6	5	$\boxed{4}$	×	×	×
F	$\boxed{2}$	$\boxed{1}$	6	$\boxed{5}$	$\boxed{4}$	×	8	×
E	$\boxed{2}$	$\boxed{1}$	$\boxed{6}$	$\boxed{5}$	$\boxed{4}$	7	7	×
D	$\boxed{2}$	$\boxed{1}$	$\boxed{6}$	$\boxed{5}$	$\boxed{4}$	$\boxed{7}$	7	×
G	$\boxed{2}$	$\boxed{1}$	$\boxed{6}$	$\boxed{5}$	$\boxed{4}$	$\boxed{7}$	$\boxed{7}$	8
H	$\boxed{2}$	$\boxed{1}$	$\boxed{6}$	$\boxed{5}$	$\boxed{4}$	$\boxed{7}$	$\boxed{7}$	$\boxed{8}$

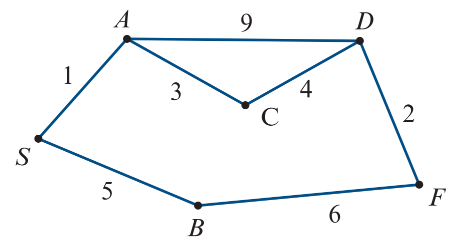
Dijkstra's algorithm

9 Use Dijkstra's algorithm to determine the *shortest path from S to F in the following networks.* Write down the length of the shortest path.

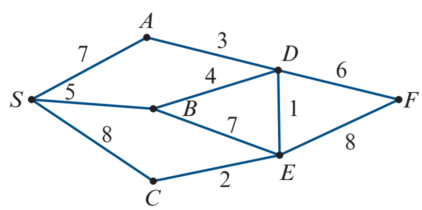
a



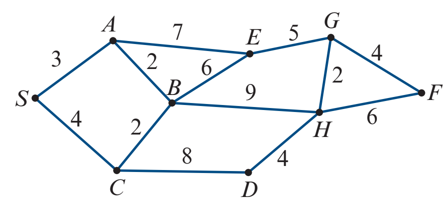
b



c

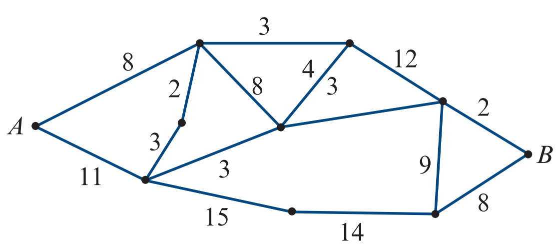


d



10 In the network below, the vertices represent small towns and the edges represent roads. The numbers on each edge indicate the distances (in kilometres) between towns.

Determine the length of the shortest path between the towns labelled A and B.



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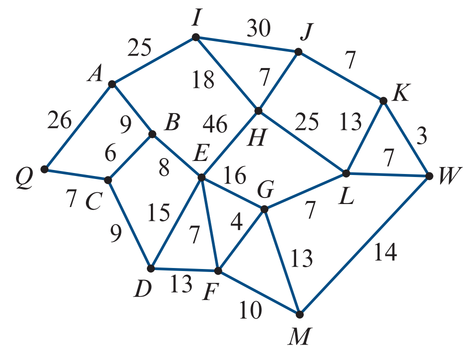
11 The network below shows the distances, in kilometres, along a series of roads that connect a quarry, Q, with worksites shown as vertices.

a Find the length of the shortest path from the quarry (Q) to worksite W.

b Write down the shortest path.



based on VCAA (2004)

14E Trees and minimum connector problems

In the previous applications of networks, the weights on the edges of the graph were used to determine a minimum pathway through the graph. In other applications, it is more important to minimise the number and weights of the edges in order to keep all vertices connected to the graph. For example, a number of towns might need to be connected to a water supply. The cost of connecting the towns can be minimised by connecting each town into a network or water pipes only once, rather than connecting each town to every other town.

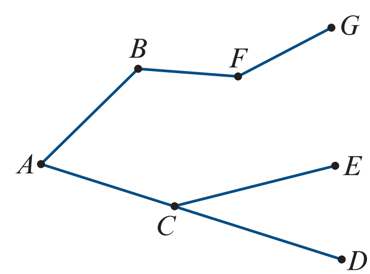
Problems of this type are called connector problems. In order to solve connector problems, you need to learn the language of networks that have as few edges as possible.

Trees

A **tree** is a connected graph that has no loops, multiple edges or cycles.

This tree has seven vertices and six edges.

The number of edges is always one less than the number of vertices.



Spanning trees

Every connected graph will have at least one subgraph that is a tree. A subgraph is a tree, and if that tree connects all of the vertices in the graph, then it is called a **spanning tree**.

> **Trees**
>
> A *tree* has no loops, multiple edges or cycles.
>
> If a tree has n vertices, it will have $n - 1$ edges.
>
> A *spanning tree* is a tree that connects all of the vertices of a graph.

There can be more than one spanning tree for any connected graph. The *total weight* of a spanning tree is the total of all the weights on the edges that make up the tree.

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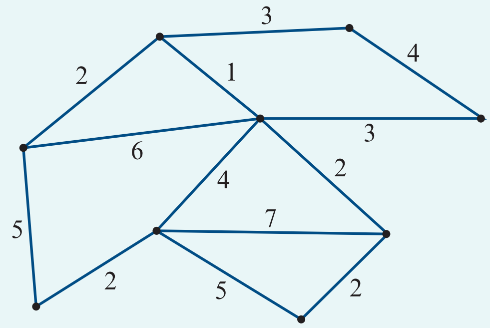
Example 10 Finding the weight of a spanning tree

a Draw one spanning tree for the graph shown.

b Calculate the weight of the spanning tree.



Solution

a **1** Count the number of vertices and edges in the graph.

There are 9 vertices and 13 edges.

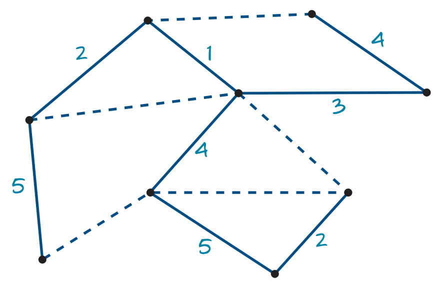
2 Calculate the number of edges in the spanning tree.

The spanning tree will have 8 edges.

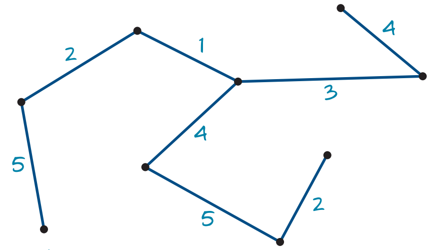
3 Calculate how many edges must be removed.

Remove $13 - 8 = 5$ edges.

4 Choose edges to remove.



b Add the weights of the remaining edges.



$$\text{Weight} = 5 + 2 + 1 + 4 + 5 + 2 + 3 + 4$$
$$= 26$$

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Minimum spanning trees

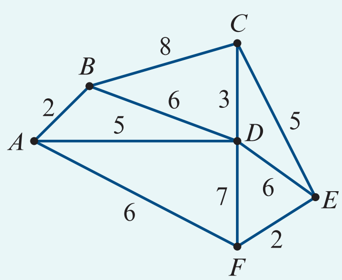
One of the spanning trees from a particular connected graph will have the *smallest* total weight. This tree is called the **minimum spanning tree**. Minimum spanning trees can be found using an algorithm called **Prim's algorithm**.

Prim's algorithm for finding a minimum spanning tree

- Choose a starting vertex (any will do).
- Inspect the edges starting from the starting vertex and choose the one with the lowest weight. (If there are two edges that have the same weight, it does not matter which one you choose). The starting vertex, the edge and the vertex it connects to form the beginning of the minimum spanning tree.
- Now inspect all of the edges starting from both of the vertices you have in the tree so far. Choose the edge with the lowest weight, ignoring edges that would connect the tree back to itself. The vertices and edges you already have, plus the extra edge and vertex it connects form the minimum spanning tree so far.
- Keep repeating this process until all of the vertices are connected.

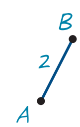
Example 11 Finding the minimum spanning tree

Apply Prim's algorithm to find the minimum spanning tree for the graph shown on the right. Write down the total weight of the minimum spanning tree.

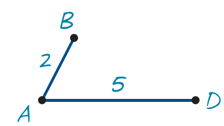


Solution

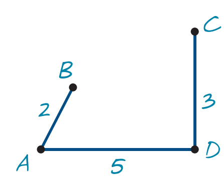
Start with vertex A.
The smallest weighted edge from vertex A is to B with weight 2.



Look at vertices A and B. The smallest weighted edge from either vertex A or vertex B is from A to D with weight 5.



Look at vertices A, B and D. The smallest weighted edge from vertix A, B or D is from D to C with weight 3.



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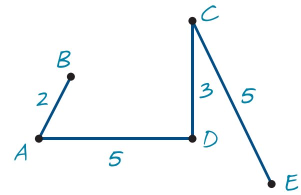
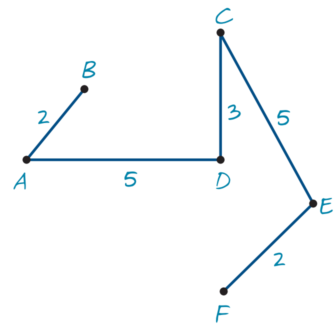
Look at vertices A, B, D and C. The smallest weighted edge from vertex A, B, D or C is from C to E with weight 5.



Look at vertices A, B, D, C and E. The smallest weighted edge from vertex A, B, D, C or E is from E to F with weight 2.
All vertices have been included in the graph. This is the minimum spanning tree.



Add the weights to find the total weight of the minimum spanning tree.

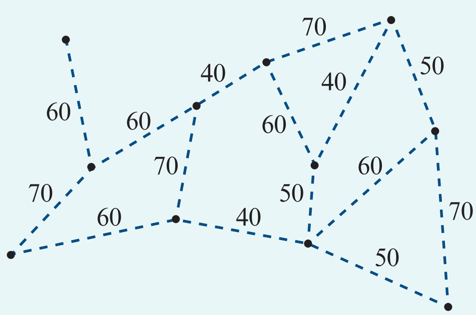
The total weight of the minimum spanning tree is $2 + 5 + 3 + 5 + 2 = 17$.

► Connector problems

Minimum spanning trees represent the least weight required to keep all of the vertices connected in the graph. If the edges of a graph represent the cost of connecting towns to a gas pipeline, then the total weight of the minimum spanning tree would represent the minimum cost of connecting the towns to the gas. This is an example of a *connector problem*, where the cost of keeping towns or other objects connected together is important to make as low as possible.

Example 12 Solving a connector problem

At a showgrounds, 11 locations require access to water. These locations are represented by vertices on the network diagram shown. The dashed lines on the network diagram represent possible water pipe connections between adjacent locations. The numbers on the dashed lines show the minimum length of pipe, in metres, required to connect these locations.



a On the diagram, show where these water pipes will be placed.

b Calculate the total length, in metres, of water pipe that is required.

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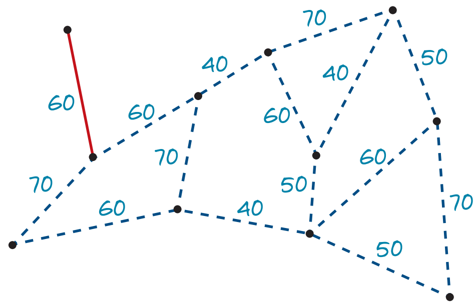
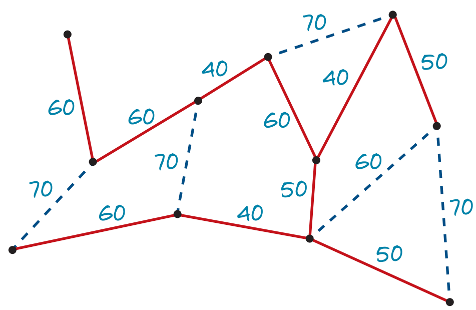
Solution

a **1** The water pipes will be a minimum length if they are placed on the edges of the minimum spanning tree for the network.

A good starting point for Prim's algorithm is the vertex that is connected by just one edge. This vertex must be connected to the minimum spanning tree by this edge.



2 Follow Prim's algorithm to find the minimum spanning tree.



b Add the weights of the minimum spanning tree. Write your answer

The length of water pipe required is $60 + 60 + 40 + 60 + 50 + 40 + 50 + 60 + 40 + 50 + 60 = 510$ metres

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Exercise 14E

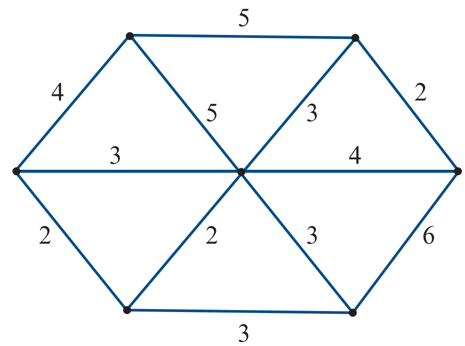
Spanning trees

1 A network is shown on the right.

a How many edges must be removed in order to leave a spanning tree?

b Remove some edges to form three different trees.

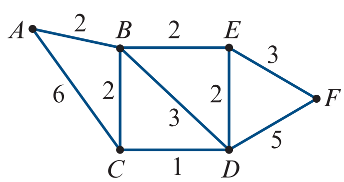
c For each tree in part **b**, find the total weight.



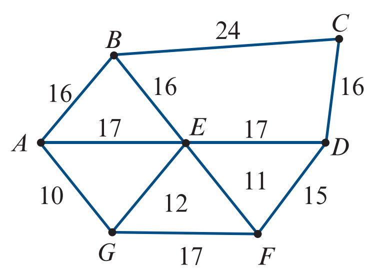
Minimum spanning trees and Prim's algorithm

2 Find a minimum spanning tree for each of the following graphs and give the total weight.

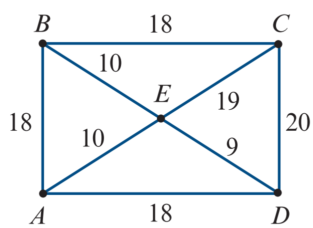
a



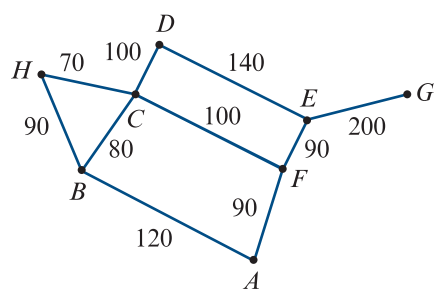
b



c

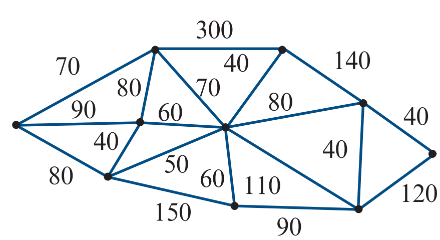


d



Connector problems

3 In the network opposite, the vertices represent water tanks on a large property and the edges represent pipes used to move water between these tanks. The numbers on each edge indicate the lengths of pipes (in m) connecting different tanks.



Determine the shortest length of pipe needed to connect all water storages.

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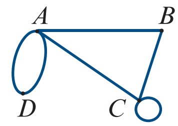
Review

Key ideas and chapter summary

Graph A **graph** is a diagram that consists of a set of points called **vertices** and a set of lines called **edges**. Each edge joins two vertices.



Edge In the graph above, the lines joining A, B, C and D are edges.

Vertex In the graph above, the points A, B, C, D are vertices.

Loop A **loop** is an edge that connects a vertex to itself. In the graph above there is a loop at vertex C.

Degree of a vertex The **degree of a vertex** is the number times edges attach to a vertex. The degree of vertex A is written as $\deg(A)$.
In the graph above, $\deg(A) = 4, \deg(B) = 2$ and $\deg(D) = 2$.
A loop has degree of 2. In the graph above, $\deg(C) = 4$.

Multiple edge Sometimes a graph has two or more identical edges. These are called **multiple edges**. In the graph above, there are multiple edges between vertex A and vertex D.

Simple graph **Simple graphs** are graphs that do not have loops and do not have multiple edges.

Isolated vertex An **isolated vertex** is one that is not connected to any other vertex. Isolated vertices have degree of zero.

Degenerate graph A **degenerate graph** has no edges. All of the vertices are isolated.

Connected graph A **connected graph** has no isolated vertex. There is a path between each pair of vertices.

Bridge A **bridge** is a single edge in a connected graph that, if it were removed, leaves the graph disconnected.

Complete graph A **complete graph** has every vertex connected to every other vertex by an edge.

Subgraph A **subgraph** is a graph that is part of a larger graph and has some of the same vertices and edges as that larger graph. A subgraph does not have any extra vertices or edges that do not appear in the larger graph.

Equivalent graph (isomorphic graph) Graphs that contain identical information (connections between vertices) to each other are **equivalent graphs** or **isomorphic graphs**.

Face An area in a graph or network that can only be reached by crossing an edge. One such area is always the area surrounding a graph.

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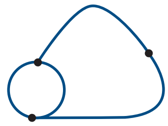
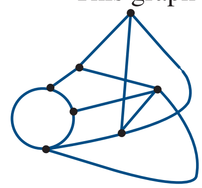
Planar graph

A **planar graph** can be drawn so that no two edges overlap or intersect, except at the vertices.

This graph is planar.



This graph is **not** planar.



Euler's rule

Euler's rule applies to planar graphs. It states:

'The number of vertices minus the number of edges plus the number of faces = 2.'

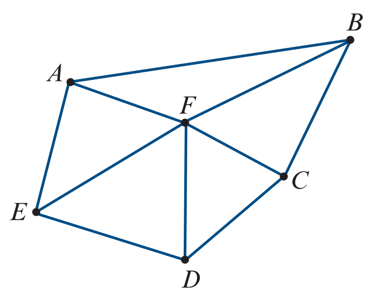
If v = the number of vertices, e = the number of edges and f = the number of faces, then $v - e + f = 2$

Adjacency matrix

An **adjacency matrix** is a square matrix that uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph.

Travelling

Movement through a graph from one vertex to another along the edges is called **travelling** through the graph.



Walk

A **walk** is a sequence of edges, linking successive vertices, in a graph.
In the graph above, $E–A–F–D–C–F–E–A$ is an example of a walk.

Trail

A **trail** is a walk with no repeated edges.
In the graph above, $A–F–D–E–F–C$ is an example of a trail.

Path

A **path** is a walk with no repeated vertices and no repeated edges.
In the graph above, $F–A–B–C–D$ is an example of a path.

Circuit

A **circuit** is a trail (no repeated edges) that starts and ends at the same vertex.
In the graph above, $A–F–D–E$ is an example of a circuit.

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Cycle A **cycle** is a path (no repeated edges nor vertices) that starts and ends at the same vertex. The start and end vertex is an exception to repeated vertices.
In the graph above, $B–F–D–C–B$ is an example of a cycle.

Eulerian trail An **eulerian trail** is a trail (no repeated edges) that includes all of the edges of a graph. Eulerian trails exist if the graph is connected and has exactly two vertices of odd degree. The remaining vertices have even degree. The eulerian trail will start at one of the odd vertices and end at the other.

Eulerian circuit An **eulerian circuit** is a trail (no repeated edges) that includes all of the edges of a graph and that starts and ends at the same vertex. Eulerian circuits exist if the graph is connected and has all of the vertices with an even degree.

Hamiltonian path A **hamiltonian path** is a path (no repeated edges or vertices) that includes all of the vertices of a graph.

Hamiltonian cycle A **hamiltonian cycle** is a path (no repeated edges or vertices) that starts and ends at the same vertex. The starting vertex is an exception to repeated vertices.

Weighted graph A **weighted graph** has numbers, called weights, associated with the edges of a graph. The weights often represent physical quantities as additional information to the edge, such as time, distance or cost.

Network A **network** is a weighted graph where the weights represent physical quantities such as time, distance or cost.

Shortest path The **shortest path** through a network is the path along edges so that the total of the weights of that path is the minimum for that network. Shortest path problems involve finding minimum distances, costs or times through a network. Shortest paths can be determined by inspection or by using Dijkstra's algorithm.

Dijkstra's algorithm Dijkstra's **algorithm** is an algorithm for determining the shortest path through a network from one vertex to another.

Tree A **tree** is a connected graph that contains no cycles, multiple edges or loops.
A tree with n vertices has $n - 1$ edges.

Spanning tree A **spanning tree** is a tree that connects every vertex of a graph.
A spanning tree is found by counting the number of vertices (n) and removing enough edges so that there are $n - 1$ edges left that connect all vertices.

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Minimum spanning tree A **minimum spanning tree** is a spanning tree for which the sum of the weights of the edges is as small as possible.

Prim's algorithm **Prim's algorithm** is an algorithm for determining the minimum spanning tree of a network.

Skills check

Having completed this chapter you should be able to:

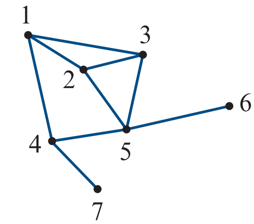
- identify edges, vertices and loops in a graph
- determine the degree of a vertex in a graph
- define and identify simple graphs, isolated vertices, degenerate graphs, connected graphs, bridges and subgraphs
- recognise isomorphic graphs
- define planar graphs
- redraw graphs in planar form
- use Euler's rule
- construct a graph from an adjacency matrix
- write an adjacency matrix from a graph
- define walks, trails, paths, circuits and cycles through a graph
- identify eulerian trails and circuits through graphs
- determine whether an eulerian trail or circuit exists in a graph
- identify hamiltonian paths and cycles through graphs
- define a weighted graph
- calculate the shortest path from one vertex to another by inspection
- calculate the shortest path from one vertex to another using Dijkstra's algorithm
- define tree, spanning tree, minimum spanning tree
- draw a minimum spanning tree using Prim's algorithm.

Multiple-choice questions

1 The minimum number of edges for a graph with seven vertices to be connected is:

A 4 **B** 5 **C** 6 **D** 7 **E** 21

2 Which of the following graphs is a spanning tree for the network shown?



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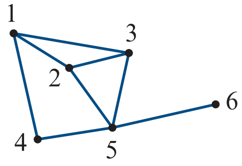
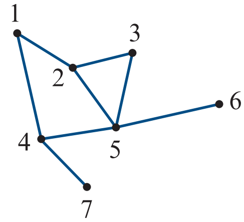
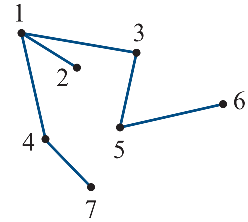
A



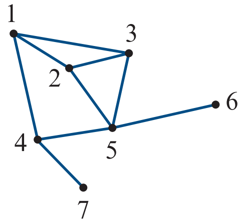
B



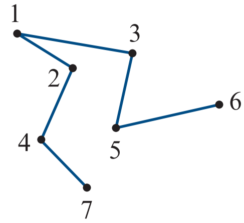
C



D

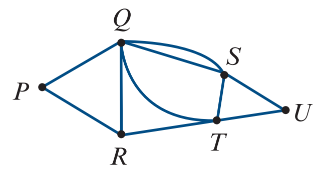


E



3 For the graph shown, which vertex has degree 5?

A Q **B** T **C** S **D** R **E** U

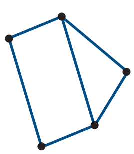


4 A connected graph with 15 vertices divides the plane into 12 regions. The number of edges connecting the vertices in this graph will be:

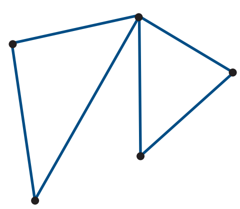
A 15 **B** 23 **C** 24 **D** 25 **E** 27

5 Which of the following graphs does *not* have an eulerian circuit?

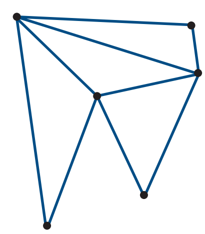
A



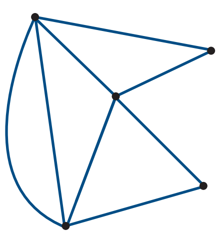
B



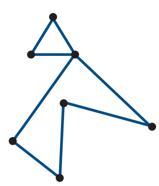
C



D



E



6 A connected planar graph divides the plane into a number of regions. If the graph has eight vertices and these are linked by 13 edges, then the number of regions is:

A 5 **B** 6 **C** 7 **D** 8 **E** 10

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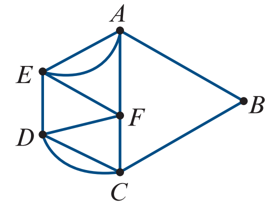
7 For the graph shown, which of the following paths is a hamiltonian cycle?



A $A–B–C–D–C–F–D–E–F–A–E–A$

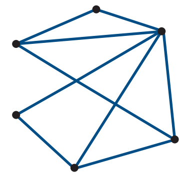
B $A–E–F–D–C–B–A$

C $A–F–C–D–E–A–B–A$

D $A–B–C–D–E–A$

E $A–E–D–C–B–A–F$

8 The graph opposite has:

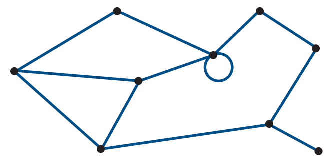


A four faces **B** five faces

C six faces **D** seven faces

E eight faces

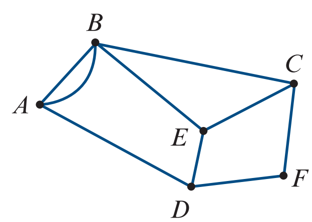
9 The sum of the degrees of the vertices on the graph shown here is:



A 20 **B** 21 **C** 22

D 23 **E** 24

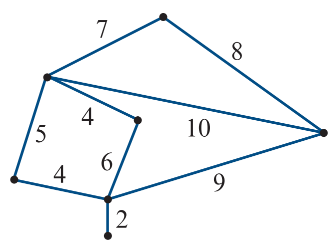
10 For the graph shown, which additional edge could be added to the network so that the graph formed would contain an eulerian trail?



A $A–F$ **B** $D–E$ **C** $A–B$

D $C–F$ **E** $B–F$

11 For the graph shown here, the minimum spanning tree has length:

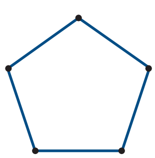


A 30 **B** 31 **C** 33

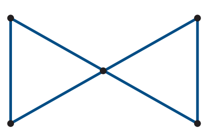
D 34 **E** 26

12 Of the following graphs, which one has both eulerian circuit and hamiltonian cycles?

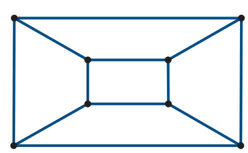
A



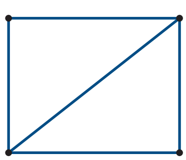
B



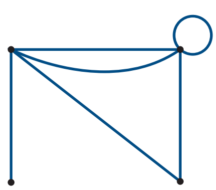
C



D



E



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Review

13 A *complete* graph with six vertices is drawn. This network would best represent:

A the journey of a paper boy who delivers to six homes covering the minimum distance

B the cables required to connect six houses to a pay television service that minimises the length of cables needed

C a six-team basketball competition where all teams play each other once

D a project where six tasks must be performed between the start and finish

E the allocation of different assignments to a group of six students

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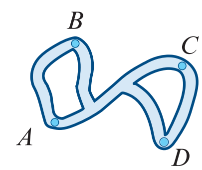
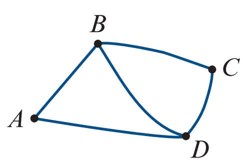
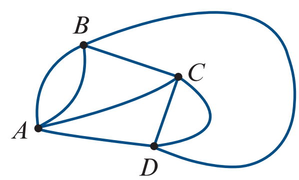
14 Four towns, A, B, C and D, are linked by roads as shown. Which of the following graphs could be used to represent the network of roads? Each edge represents a route between two towns.



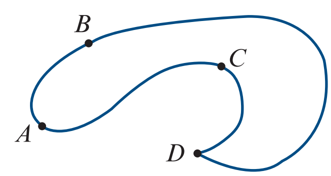
A



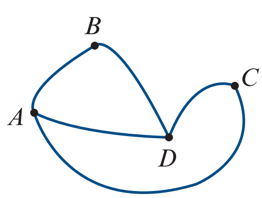
B



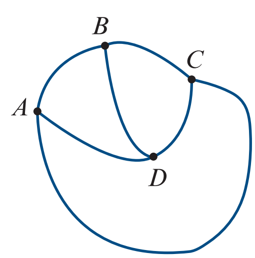
C



D

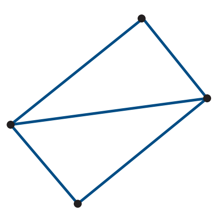


E

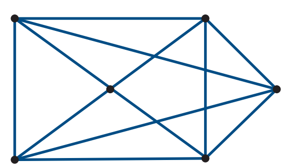


15 Which *one* of the following graphs has an eulerian circuit?

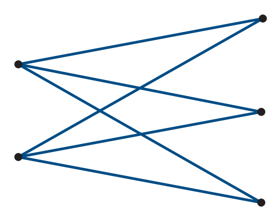
A



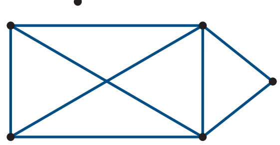
B



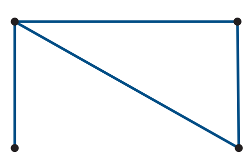
C



D



E



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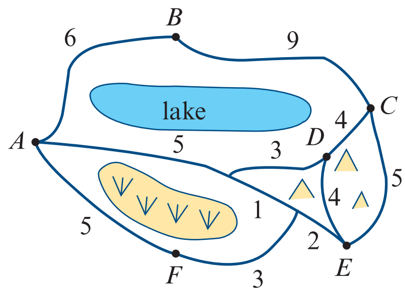
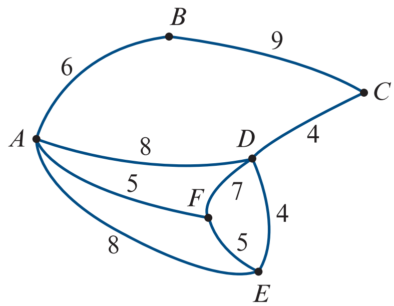
Extended-response questions

1 The map shows six campsites, A, B, C, D, E and F, which are joined by tracks. The numbers by the paths show lengths, in kilometres, of that section of track.



a **i** Complete the graph opposite, which shows the shortest direct distances between campsites. (The campsites are represented by vertices and tracks are represented by edges.)



ii A telephone cable is to be laid to enable each campsite to phone each other campsite. For environmental reasons, it is necessary to lay the cable along as few of the existing tracks as possible. What is the minimum length of cable necessary to complete this task?

iii Fill in the missing entries for the adjacency matrix shown for the completed graph formed above.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & - & - & - \\ 1 & 0 & - & - & - & - \\ 1 & 0 & - & - & - & - \end{bmatrix} \end{array}$$

b A walker follows the route $A–B–A–F–E–D–C–E–F–A$.

i How far does this person walk?

ii Why is the route *not* a hamiltonian cycle?

iii Write down a route that a walker could follow that is a hamiltonian cycle.

iv Find the distance walked in following this hamiltonian cycle.

c It is impossible to start at A and return to A by going along each track exactly once. An extra track joining two campsites can be constructed so that this is possible. Which two campsites need to be joined by a track to make this possible?

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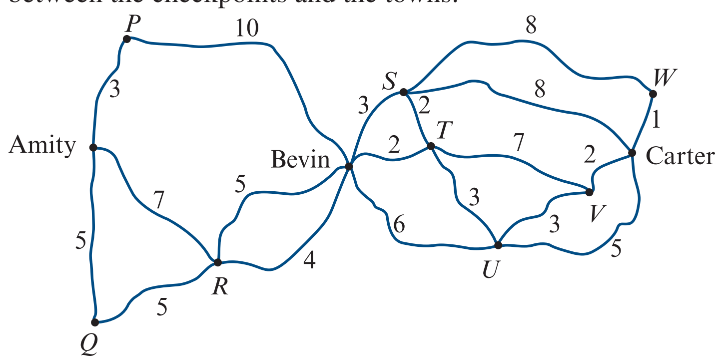
Review

2 The diagram below shows the roads that connect the towns of Amity, Bevin and Carter represented as edges of a network. The vertices of the network, labelled P, Q, R, S, T, U, V and W, are checkpoints for the Amity Cycling Club road race. The numbers on the edges of the network are the lengths, in kilometres, of the roads between the checkpoints and the towns.



a Find, by inspection, the length of the shortest path from Amity to Bevin.

The road race covers the full length of every road on the network in any order or direction chosen by the riders. A rider may pass through each checkpoint more than once, but must travel along each road exactly once.

b One competitor claims this cannot be done. Explain why it *is* possible to travel every road once only during this race.

c If the race begins at Amity, where must this race finish?

d One of the competitors is following this path: Amity–P–Bevin–T–S–Bevin. Which checkpoint should *not* be visited next by this competitor? Explain why.

A road race for junior riders begins at Amity and ends at Carter. Participants are allowed to take any route they prefer.

e Use Dijkstra's algorithm to find the shortest path from Bevin to Carter.

f Using your answers to parts **a** and **e**, what is the shortest distance from Amity to Carter?

The Water Authority wants to lay water mains along the roads in order to put a fire hydrant at the locations of the checkpoints in the diagram above. A minimal spanning tree will be used for these water mains.

g Draw the minimum spanning tree for the diagram above.

based on VCAA (2002)

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15

Flow, matching and scheduling problems

ISBN 978-1-107-56757-3

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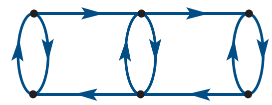
15A Flow problems

▶ Directed graphs

Skillsheet

In the previous chapter, graphs were used to represent connections between people, places or objects. The vertices of a graph represented objects, such as towns, and edges represented the conditions between them, such as roads. **Weighted graphs** included extra numerical information about the connections, such as distance, time or cost. When a graph has this numerical information we call it a **network**.

A **directed graph**, or **digraph**, records directional information on networks using arrows on the edges. The network on the right shows roads around a city. The vertices are the intersections of the roads and the edges are the possible road connections between the intersections. The arrows show that some of the roads only allow traffic in one direction, while others allow traffic in both directions.



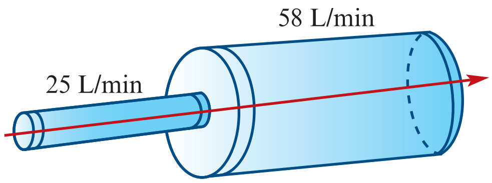
▶ Understanding minimum flow

One of the applications of directed graphs to real-life situations is flow problems. Flow problems involve the transfer or **flow** of material from one point, called the **source**, to another point called the **sink**. Examples of this include water flowing through pipes, or traffic flowing along roads.

source → flow through network → sink

Water flows through pipes in only one direction. In a digraph representing water flow, the vertices are the origin and destination of the water and the edges represent the pipes connecting them. The weights on the edges would be the amount of water that can flow through the pipe in a given time. The weights of flow problem directed graphs are called **capacities**.

The diagram on the right shows two pipes that are joined together, connecting the source of water to the sink. There is a small pipe with capacity 25 litres per minute joined to a large pipe with capacity 58 litres per minute.



Even though the large pipe has a capacity greater than 25 litres per minute, the small pipe will only allow 25 litres of water through each minute. The flow through the large pipe will never be more than 25 litres per minute. The large pipe will experience flow below its capacity.

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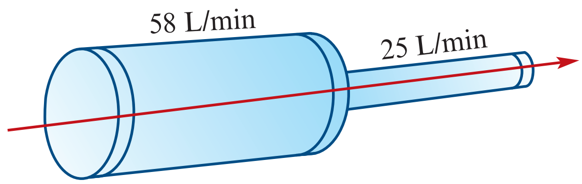
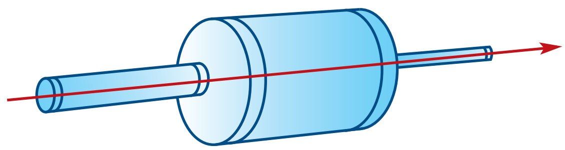
If we reverse the connection and direct water through the large capacity pipe into the smaller capacity pipe, there will be a 'bottleneck' of flow at the junction.



The large capacity pipe is delivering 58 litres of water every minute to the small pipe, but the small pipe will only allow 25 litres per minute to pass.

In both of these situations, the flow through the entire pipe system (both pipes from source to sink) is restricted to a maximum of 25 litres per minute. This is the capacity of the smallest pipe in the connection.

If we connect more pipes together, one after the other, we can calculate the overall capacity or **maximum flow** of the pipe system by looking for the *smallest capacity pipe* in that system.

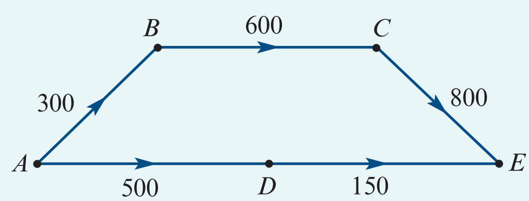


Maximum flow

If pipes of different capacities are connected one after the other, the *maximum flow* through the pipes is equal to the *minimum* capacity of the individual pipes.

Example 1 Calculating the maximum flow

In the digraph shown on the right, the vertices A, B, C, D and E represent towns. The edges of the graph represent roads and the weights of those edges are the maximum number of cars that can travel on the road each hour. The roads allow only one-way travel.



a Find the maximum traffic flow from A to E through town C.

b Find the maximum traffic flow from A to E overall.

c A new road is being built to allow traffic from town D to town C. This road can carry 500 cars per hour.

  i Add this road to the digraph.

  ii Find the maximum traffic flow from A to E overall after this road is built.

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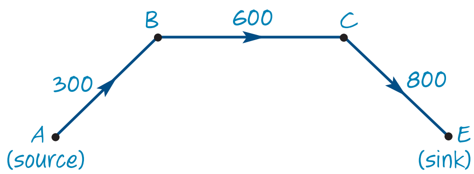
Solution

a Look at the subgraph that includes town C. The smallest capacity of the individual roads is 300 cars per hour. This will be the maximum flow through town C.



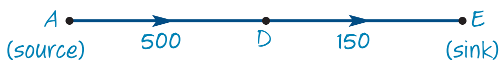
The maximum flow from A to E through town C is equal to the smallest capacity road along that route. The maximum flow is 300 cars per hour.

b Look at the two subgraphs from A to E.

The maximum flow through C will be 300 cars per hour.

The maximum flow through D will be 150 cars per hour (minimum capacity).

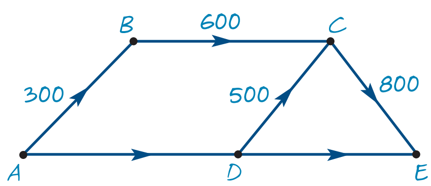
Add the maximum flow through C to the maximum flow through D.



The maximum flow from A to E overall is:

$300 + 150 = 450$ cars per hour

c **i** Add the edge to the diagram.



ii The maximum flow through A–B–C–E is 300. But C–E has capacity 800. If another 500 cars per hour come through D–C, they will be able to travel from C–D.

The new maximum flow is now 800 cars per hour.

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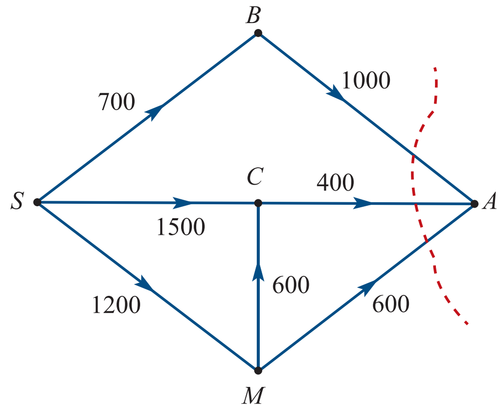
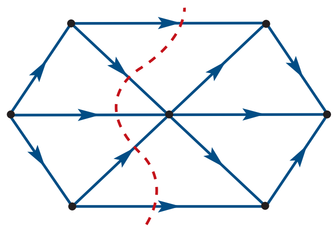
Cuts

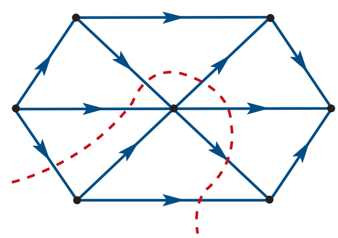
It is difficult to determine the maximum flow by inspection for directed networks that involve many vertices and edges. We can simplify the search for maximum flow by searching for **cuts** within the digraph.

A cut divides the network into two parts, completely separating the source from the sink. It is helpful to think of cuts as imaginary breaks within the network that completely block the flow through that network. For the network or water pipes shown in this diagram, the dotted line is a cut. This cut completely blocks the flow of water from the source (S) to the sink (A).





The dotted line on the graph above is a cut because it separates the source and the sink completely. No material can flow from the source to the sink.

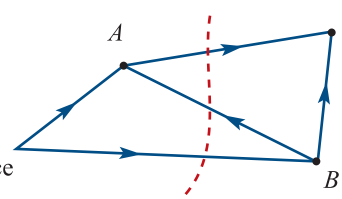


The dotted line on the graph above is *not* a cut because material can still flow from the source to the sink. Not all of the pathways from source to sink have been blocked by the cut.

Capacity of a cut

The **cut capacity** is the sum of all the capacities of the edges that the cut passes through, taking into account the direction of flow. The capacity of an edge is only counted if it flows from the source side to the sink side of the cut.

In the simple network shown, the cut passes through three edges. The edge B to A is not counted in the capacity of the cut because the flow for that edge is from the sink side to the source side of cut.



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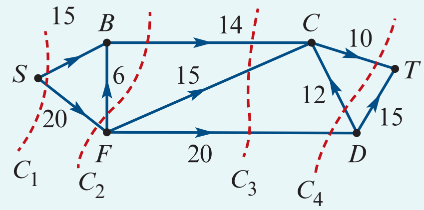
Example 2 Calculating cut capacity

Calculate the capacity of the four cuts shown in the network on the right. The source is vertex S and the sink is vertex T.



Solution

All edges in C_1 are counted.

The capacity of $C_1 = 15 + 20 = 35$

Note that the edge from F to B in not counted in C_2.

The capacity of $C_2 = 14 + 20 = 34$

All edges in C_3 are counted.

The capacity of $C_3 = 14 + 15 + 20 = 49$

Note that the edge from F to B in not counted in C_2.

The capacity of $C_4 = 20 + 10 = 30$

The capacity of a cut is important to help determine the maximum flow through any digraph. Look for the smallest, or minimum, cut capacity that exists in the graph. This will be the same as the maximum flow that is possible through that graph. This is known as the *maximum-flow minimum-cut theorem*.

Cut, cut capacity and minimum cut capacity

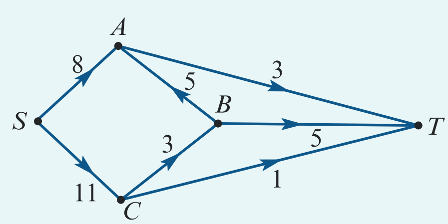
A *cut* is an imaginary line across a directed graph that completely separates the *source* (start of the flow) from the *sink* (destination of the flow).

The *cut capacity* is the sum of the capacities of the edges that are cut. Only edges that flow from the source side of the cut to the sink side of the cut are included in a cut capacity calculation.

The ***minimum cut*** *capacity* possible for a graph equals the *maximum flow* through the graph.

Example 3 Calculating maximum flow

Determine the maximum flow from S to T for the digraph shown on the right.



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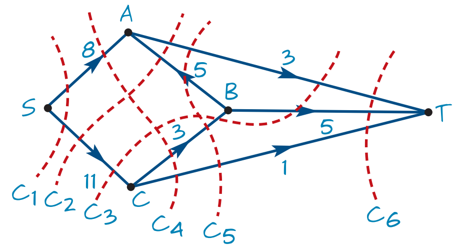
Solution

1 Mark in all possible cuts on the network.



2 Calculate the capacity of all the cuts.

The capacity of $C_1 = 8 + 11 = 19$
The capacity of $C_2 = 3 + 11 = 14$
The capacity of $C_3 = 3 + 5 + 11 = 19$
The capacity of $C_4 = 8 + 3 + 1 = 12$
The capacity of $C_5 = 3 + 3 + 1 = 7$
The capacity of $C_6 = 3 + 5 + 1 = 9$

3 Identify the minimum cut capacity and write your answer.

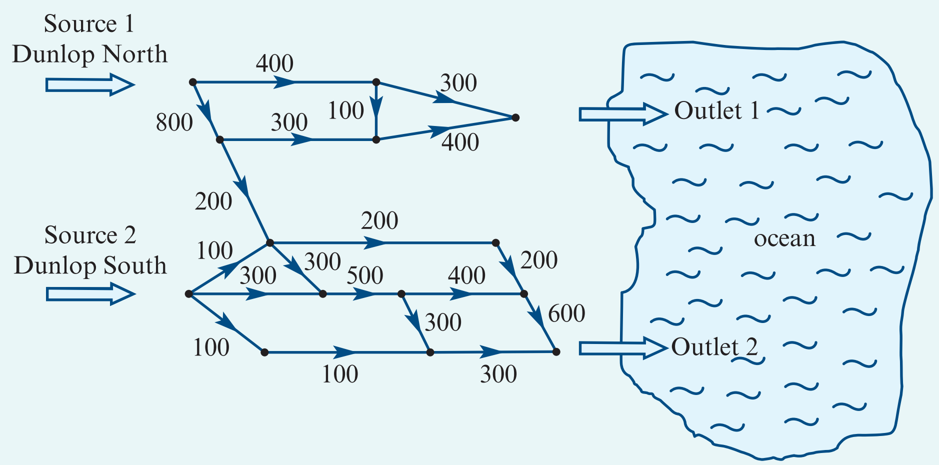
The minimum cut capacity is 7 so the maximum flow from S to T is 7.

Example 4 Calculating maximum flow

Storm water enters a network of pipes at either Dunlop North (Source 1) or Dunlop South (Source 2) and flows into the ocean at either Outlet 1 or Outlet 2.

On the network diagram below, the pipes are represented by straight lines with arrows that indicate the direction of the flow of water. Water cannot flow through a pipe in the opposite direction.

The numbers next to the arrows represent the maximum rate, in kilolitres per minute, at which storm water can flow through each pipe.



Determine the maximum rate, in kilolitres per minute, at which water can flow from these pipes into the ocean at Outlet 1 and Outlet 2.

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Solution

The outlets need to be considered separately.

Outlet 1

Look for the minimum cut that prevents water reaching Outlet 1.

Note: The pipe with capacity 200 leading towards Outlet 2 does not need to be considered in any cut because this pipe *always* prevents water from reaching Outlet 1.

The capacity of C_1 is: $400 + 800 = 1200$

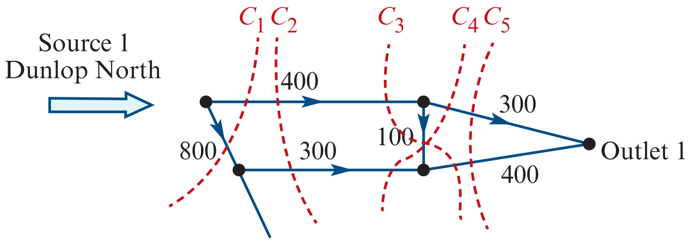
The capacity of C_2 is: $400 + 300 = 700$

The capacity of C_3 is: $400 + 400 = 800$

The capacity of C_4 is: $300 + 100 + 300 = 700$

The capacity of C_5 is: $300 + 400 = 700$

The minimum cut/maximum flow is 700 kilolitres per minute.



Outlet 2

Look for the minimum cut that prevents water reaching Outlet 2.

Note: The pipe with capacity 200 leading towards Outlet 2 will need to be considered in any cut because this pipe delivers water towards Outlet 2 and must be 'cut' like all the others. Other cuts are possible, but have not been included in the diagram.

The capacity of C_1 is: $200 + 100 + 300 + 100 = 700$

The capacity of C_2 is: $200 + 300 + 300 + 100 = 900$

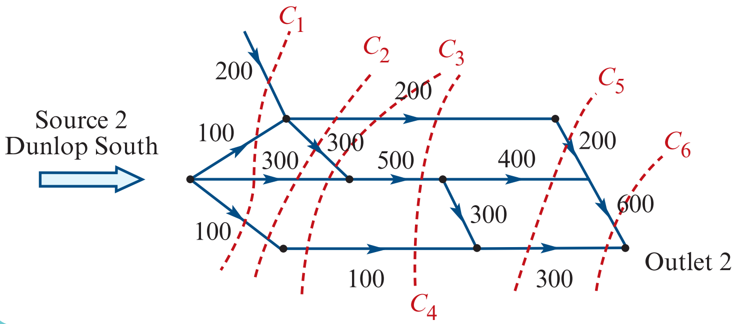
The capacity of C_3 is: $200 + 300 + 300 + 100 = 900$

The capacity of C_4 is: $200 + 500 + 100 = 800$

The capacity of C_5 is: $200 + 400 + 300 = 900$

The capacity of C_6 is: $600 + 300 = 900$

The minimum cut/maximum flow is 700 kilolitres per minute.



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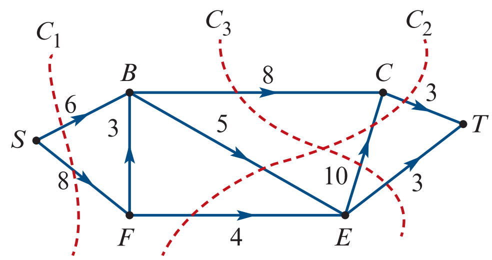
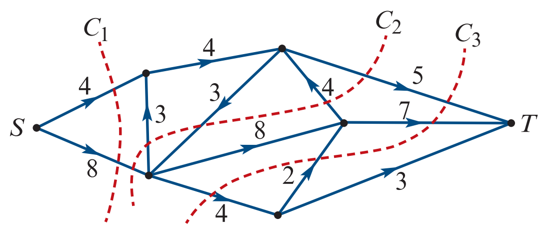
Exercise 15A

Cuts

1 Determine the capacity of each of the cuts in the digraph below. The source is vertex S and the sink is vertex T.



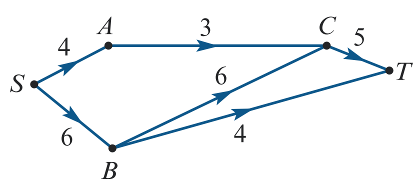
2 Determine the capacity of each of the cuts in the digraph below. The source is vertex S and the sink is vertex T.



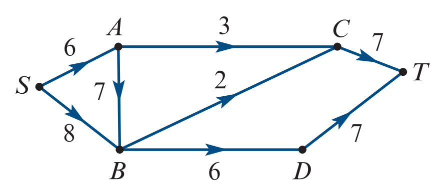
Minimum-cut maximum-flow

3 Find the maximum flow for each of the following graphs. The source is vertex S and the sink is vertex T.

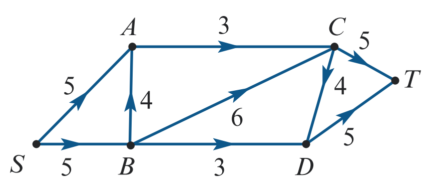
a



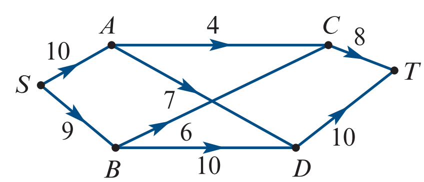
b



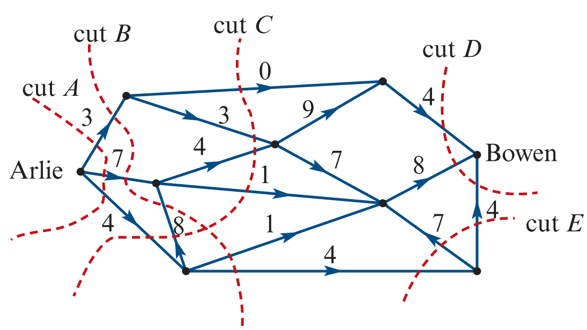
c



d



4 A train journey consists of a connected sequence of stages formed by edges on the following directed network from Arlie to Bowen. The number of available seats for each stage is indicated beside the corresponding edge, a shown on the diagram on the right.



The five cuts, A, B, C, D and E, shown on the network are attempts to find the maximum number of available seats that can be booked for a journey from Arlie to Bowen.

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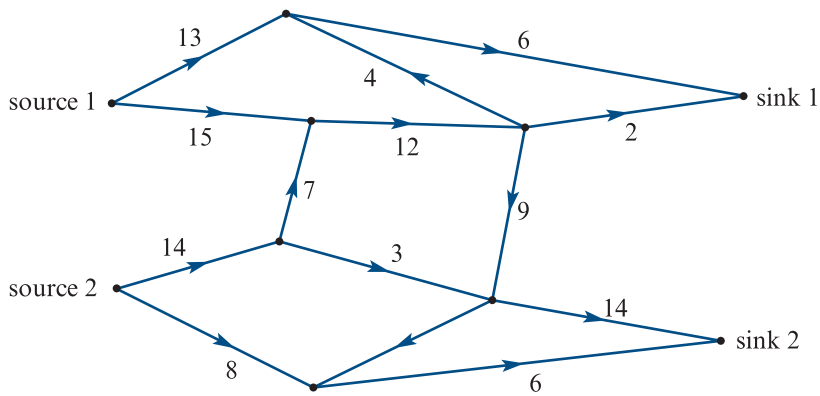
a Write down the capacity of cut A, cut B, cut C, cut D and cut E.

b Explain why cut E is not a valid cut when trying to find the minimum cut between Arlie and Bowen.

c Find the maximum number of available seats for a train journey from Arlie to Bowen.

5 Water pipes of different capacities are connected to two water sources and two sinks. Network of water pipes is shown in the diagram below. The numbers on the edges represent the capacities, in kilolitres per minute, of the pipes.

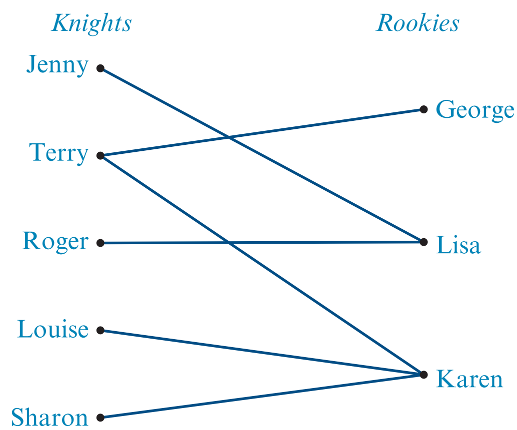


Find the maximum flow, in kilolitres per minute, to each of the sinks in this network .

15B Matching and allocation problems

▶ Bipartite graphs

In some situations, the vertices of a graph belong in two separate sets. Consider the members of two different chess teams. The Knights team has five players and the Rookies team has three players. The members of the teams are represented by a vertex, arranged vertically underneath the team name as shown in the diagram opposite. The edges of the diagram connect team members that have played a chess match.



This type of graph is called a **bipartite graph**. Each edge in a bipartite graph joins one vertex from one group to a vertex in the other group.

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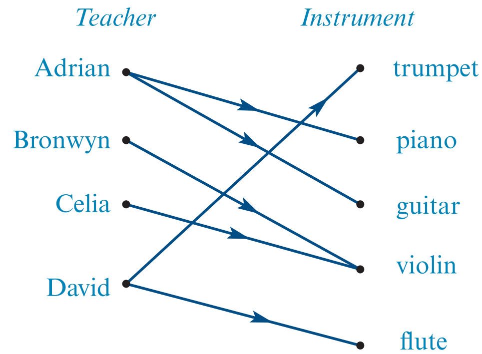
Directed bipartite graphs

Bipartite graphs have vertices that are divided into two separate groups. In matching problems, one vertex from one group is *matched* or *allocated* to one, or more, vertices of the second group and we use a directed bipartite graph to represent this matching.

Imagine a music school that has four teachers: Adrian, Bronwyn, Celia and David. These teachers, between them can teach five different instruments: trumpet, piano, guitar, violin and flute.



The school would need to match each teacher to one instrumental music class. The directed bipartite graph on the right graphically shows the instrument or instruments that each teacher can teach.

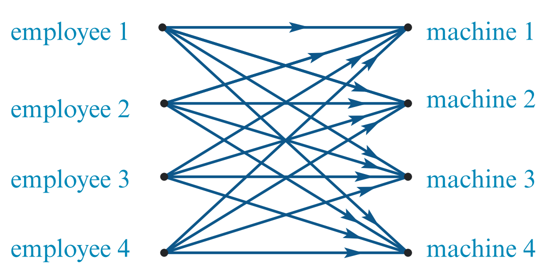
David is the only teacher who can teach trumpet, and he is also the only teacher who can teach flute. If each teacher has only one instrumental class, one of these instruments could not be taught. Bronwyn and Celia both teach violin and no other instrument. One of them will have no class to teach at all.

Under these conditions, the school cannot allocate each teacher to an instrumental class.

Complete weighted bipartite graphs

The **allocation** of teacher to class above would be greatly simplified if the school had five teachers to teach the five instruments. In addition, if every teacher could teach all of the instruments, then the graph used to represent the possible matching of teachers and classes would be a *complete bipartite graph* with an equal number of vertices in each group.

The graph on the right shows four employees in a factory. There are four different machines that are used in the production of an item. Every employee can use every machine and so the bipartite graph is complete. Employees and machines can be matched in many different ways.



Rather than just assign an employee to a machine randomly, the factory could use information about how well each employee uses each machine, perhaps in terms of how quickly each performs the task. The times taken would be the weights on the edges of the bipartite graph. Rather than writing all of the weights on a complete bipartite graph (which would be a very complicated diagram), we can summarise the time information in a table

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and then use an algorithm, called the **Hungarian algorithm**, to allocate employees to machines in order to minimise the time taken to finish the tasks.

The Hungarian algorithm

The table on the right shows the four employees: Wendy, Xenefon, Yolanda and Zelda. The machines in a factory are represented by the letters A, B, C and D.

The numbers in the table are the times, in minutes, it takes each employee to finish the task on each machine.

Employee	*A*	*B*	*C*	*D*
Wendy	30	40	50	60
Xenefon	70	30	40	70
Yolanda	60	50	60	30
Zelda	20	80	50	70

The table is called a **cost matrix**. Even though the numbers do not represent money value, this table contains information about the cost, in terms of time, of employees using each machine. The cost matrix can be used to determine the best way to allocate an employee to a machine so that the overall cost, in terms of the time taken to finish the work, is minimised. The Hungarian algorithm is used to do this.

Performing the Hungarian algorithm

Step 1: Subtract the lowest value in each row, from every value in that row.

- 30 has been subtracted from every value in the row for Wendy.
- 30 has been subtracted from every value in the row for Xenefon.
- 30 has been subtracted from every value in the row for Yolanda.
- 20 has been subtracted from every value in the row for Zelda.

Employee	*A*	*B*	*C*	*D*
Wendy	0	10	20	30
Xenefon	40	0	10	40
Yolanda	30	20	30	0
Zelda	0	60	30	50

Step 2: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 3.

- The zeros can be covered with three lines. This is less than the number of allocations to be made (4).
- Continue to step 3.

Employee	*A*	*B*	*C*	*D*
Wendy	0	10	20	30
Xenefon	40	0	10	40
Yolanda	30	20	30	0
Zelda	0	60	30	50

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Step 3: If a column does not contain a zero, subtract the lowest value in that column from every value in that column.

- Column C does not have a zero.
- 10 has been subtracted from every value in column C.

Employee	*A*	*B*	*C*	*D*
Wendy	0	10	10	30
Xenefon	40	0	0	40
Yolanda	30	20	20	0
Zelda	0	60	20	50

Step 4: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 5a.

- The zeros can be covered with three lines. This is less than the number of allocations to be made (4).
- Continue to step 5a.

Employee	*A*	*B*	*C*	*D*
Wendy	0	10	10	30
Xenefon	40	0	0	40
Yolanda	30	20	20	0
Zelda	0	60	20	50

Step 5a: Add the smallest uncovered value to any value that is covered by two lines. Subtract the smallest uncovered value from all the uncovered values.

- The smallest uncovered element is 10.
- 10 has been *added* to Xenefon–A and Xenefon–D because these values are covered by two lines.
- 10 has been *subtracted* from all the uncovered values.

Employee	*A*	*B*	*C*	*D*
Wendy	0	0	0	30
Xenefon	50	0	0	50
Yolanda	30	10	10	0
Zelda	0	50	10	50

Step 5b: Repeat from step 4.

- The zeros can be covered with a minimum of four lines. This is the same as the number or allocations to make.
- Continue to step 6.

Employee	*A*	*B*	*C*	*D*
Wendy	0	0	0	30
Xenefon	50	0	0	50
Yolanda	30	10	10	0
Zelda	0	50	10	50

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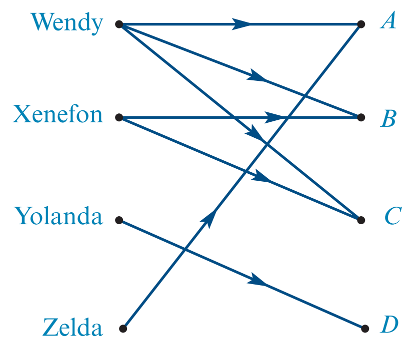
Step 6: Draw a directed bipartite graph with an edge for every zero value in the table.

In the bipartite graph:

- Wendy will be connected to *A, B* and *C*
- Xenefon will be connected to *B* and *C*
- Yolanda will be connected to *D*
- Zelda will be connected to *A*.



Step 7: Make the allocation and calculate minimum cost

- Zelda must operate machine *A* (20 minutes).
- Yolanda must operate machine *D* (30 minutes).
- Wendy can operate either machine *B* (40 minutes) or *C* (50 minutes).
- Xenefon can operate either machine *B* (30 minutes) or *C* (40 minutes).

Note: Because Wendy and Xenefon can operate either B or C, there are two possible allocations. Both allocations will have the same minimum cost.

The minimum time taken to finish the work = 20 + 30 + 50 + 30 = 130 minutes.

Exercise 15B

▶ Bipartite graphs

1 Gloria, Minh, Carlos and Trevor are buying ice-cream. They have a choice of five flavours: chocolate, vanilla, peppermint, butterscotch and strawberry. Gloria likes vanilla and butterscotch, but not the others. Minh only likes strawberry. Carlos likes chocolate, peppermint and butterscotch. Trevor likes all flavours.

a Explain why a bipartite graph can be used to display this information.

b Draw a bipartite graph with the people on the left and flavours on the light.

c What is the degree of the vertex representing Trevor?

▶ The Hungarian algorithm

2 **a** A cost matrix is shown. Find the allocation(s) by the Hungarian algorithm that will give the minimum cost.

	A	*B*	*C*	*D*
W	110	95	140	80
X	105	82	145	80
Y	125	78	140	75
Z	115	90	135	85

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b Find the minimum cost for the given cost matrix and give a possible allocation.

	A	*B*	*C*	*D*
W	2	4	3	5
X	3	5	3	4
Y	2	3	4	2
Z	2	4	2	3

3 A school is to enter four students in four track events: 100 m, 400 m, 800 m and 1500 m. The four students' times (in seconds) are given in the table. The rules permit each student to enter only one event. The aim is to obtain the minimum total time.

Student	*100 m*	*400 m*	*800 m*	*1500 m*
Dimitri	11	62	144	379
John	13	60	146	359
Carol	12	61	149	369
Elizabeth	13	63	142	349

Use the Hungarian algorithm to select the 'best' student for each event.

4 Three volunteer workers, Joe, Meg and Ali, are available to help with three jobs. The time (in minutes) in which each worker is able to complete each task is given in the table opposite. Which allocation of workers to jobs will enable the jobs to be completed in the minimum time?

	Job		
Student	*A*	*B*	*C*
Joe	20	20	36
Meg	16	20	44
Ali	26	26	44

5 A company has four machine operators and four different machines that they can operate. The table shows the hourly cost in dollars of running each machine for each operator. How should the machinists be allocated to the machines to maximise the hourly cost from each of the machines with the staff available?

	Machine			
Operator	*W*	*X*	*Y*	*Z*
A	38	35	26	54
B	32	29	32	26
C	44	26	23	35
D	20	26	32	29

6 A football association is scheduling football games to be played by three teams (the Champs, the Stars and the Wests) on a public holiday. On this day, one team must play at their Home ground, one will play Away and one will play at a Neutral ground.

The costs (in $'000s) for each team to play at each of the grounds are given in the table below.

Determine a schedule that will minimise the total cost of playing the three games and determine this cost.

Note: There are two different ways of scheduling the games to achieve the same minimum cost. Identify both of these.

Team	*Home*	*Away*	*Neutral*
Champs	10	9	8
Stars	7	4	5
Wests	8	7	6

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7 A roadside vehicle assistance organisation has four service vehicles located in four different places. The table below shows the distance (in kilometres) of each of these service vehicles from four motorists in need of roadside assistance.

	Motorist			
Service vehicle	*Jess*	*Mark*	*Raj*	*Karla*
A	18	15	15	16
B	7	17	11	13
C	25	19	18	21
D	9	22	19	23

Determine a service vehicle assignment that will ensure that the total distance travelled by the service vehicles is minimised. Determine this distance.

Note: There are two ways that the service vehicles can be assigned to minimise the total distance travelled. Identify both of these.

15C Precedence tables and activity networks

▶ Drawing activity networks from precedence tables

Building a house, manufacturing a product, organising a wedding and other similar projects all require many individual **activities** to be completed before the project is finished. The individual activities often rely upon each other and some can't be performed until other activities are complete.

In the organisation of a wedding, invitations would be sent out to guests, but a plan for seating people at the tables during the reception can't be completed until the invitations are accepted. When building a house, the plastering of the walls can't begin until the house is sealed from the weather.

For any project, if activity A must be completed before activity B can begin then activity A is said to be an **immediate predecessor** of activity B. The activities within a project can have multiple immediate predecessors and these are usually recorded in a table called a **precedence table**.

This precedence table shows some of the activities involved in a project and their immediate predecessors.

Activity	*Immediate predecessors*
A	–
B	–
C	*A*
D	*B*
E	*B*
F	*C*, *D*
G	*E*, *F*

The information in the precedence table can be used to draw a network diagram called an **activity network**.

Activity networks do not have labelled vertices, other than the *start* and *finish* of the project. The activities in the project are represented by the edges of the diagram and *so it is the edges that must be labelled*, not the vertices.

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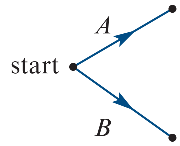
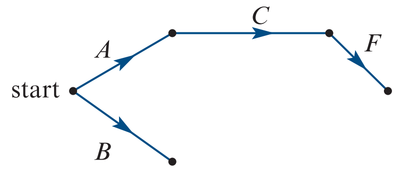
Activities A and B have no immediate predecessors.

These activities can start immediately and can be completed at the same time.



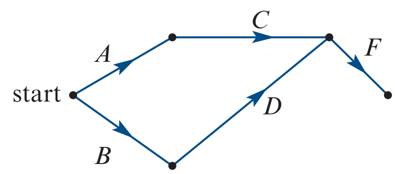
Activity A is an immediate predecessor of activity C, so activity C must be follow immediately after activity A.

Activity C is an immediate predecessor of activity F, so activity F must follow immediately after activity C.



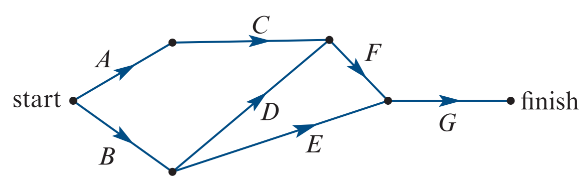
Activity D has immediate predecessor activity B so it follows immediately after activity B.

Activity D is also an immediate predecessor of activity F so activity F must follow immediately after activity D.



Activity E has immediate predecessor activity B so it will follows immediately after activity B.

Activity G has immediate predecessor activity F and activity E and so it must follow immediately after both of these activities.



Activity G is not an immediate predecessor for any activity and so the project is finished after this activity is complete.

Activity networks

When activity A must be completed before activity B can begin, activity A is called an immediate *predecessor* of activity B.

A table containing the activities of a project, and their immediate predecessors, is called a *precedence table*.

An *activity network* can be drawn from a precedence table. Activity networks have edges representing activities. The vertices are not labelled, other than the start and finish vertices.

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Example 5 Drawing an activity network from a precedence table.

Draw an activity network from the precedence table shown on the right.

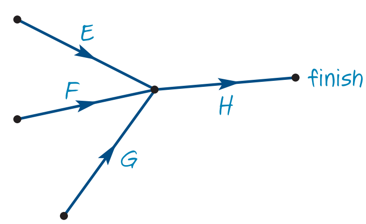
In this solution, the activity network will be drawn from the finish back to the start.

Activity	*Immediate predecessors*
A	–
B	A
C	A
D	A
E	B
F	C
G	D
H	E, F, G

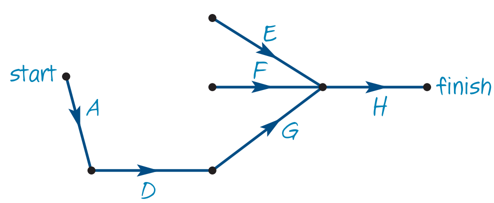
Solution

H is not an immediate predecessor for any other activity so it will lead to the finish of the project.

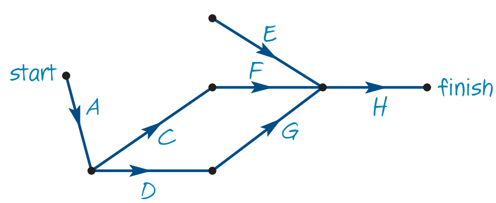
H has immediate predecessors E, F and G and so these three activities will lead into activity H.



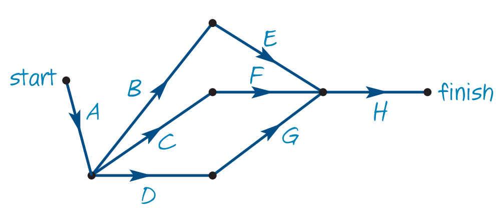
Activity D is an immediate predecessor of activity G *and* has immediate predecessor activity A. There will be a path through activity A, activity D and then activity G.



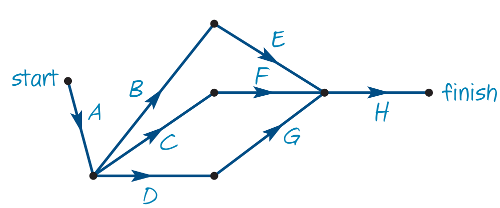
Activity C is an immediate predecessor of activity F *and* has immediate predecessor activity A. There will be a path through activity A, activity C and then activity F.



Activity B is an immediate predecessor of activity E *and* has immediate predecessor activity A. There will be a path through activity A, activity B and then activity E.



Activity A has no immediate predecessors, so it is the start of the project.



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Sketching activity networks

Activities that have no immediate predecessors follow from the start vertex.

Activities that are not immediate predecessors for other activities lead to the finish vertex.

For every other activity, look for:

- which activities for which it is an immediate predecessor
- which activities it has as immediate predecessors.

Construct the activity network from this information.

Dummy activities

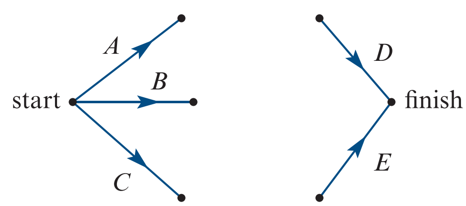
Sometimes two activities will have some of the same immediate predecessors, but not all of them. In this very simple precedence table, activity D and activity E share the immediate predecessor activity B, but they both have an immediate predecessor activity that the other does not.

Activity	*Immediate predecessors*
A	–
B	–
C	–
D	A, B
E	B, C

This overlap of predecessors presents some difficulty when constructing the activity network, but this difficulty is easily overcome.

Activity D and activity E are not immediate predecessors for any other activity, so they will lead directly to the finish vertex of the project.

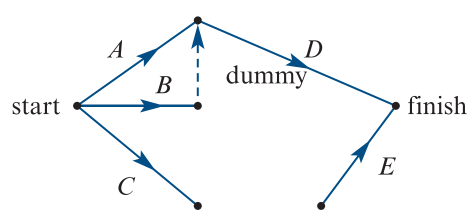
Activites A, B and C have no immediate predecessors, so they will follow directly from the start vertex of the project.



The start and finish of the activity network are shown in the diagram above. We need to use the precedence information for activity D and activity E to join these two parts together.

Activity D needs to follow directly from activity A and activity B, but we can only draw one edge for activity D. Activity E needs to follow directly from both activity B and activity C, but again we only have one edge for activity E, not two.

The solution is to draw the diagram with activity D starting after one of its immediate predecessors, and using a **dummy activity** for the other. The dummy activities are represented by dotted edges and are, in effect, imaginary. They are not real activities, but they allow all of the predecessors from the table to be correctly represented.



The dummy activity for D allows activity D to directly follow both activity A and B.

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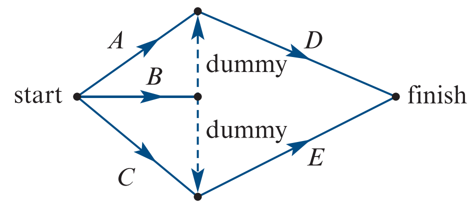
A dummy activity is also needed for activity E because it, too, has to start after two different activities, activity B and C.



Dummy activities

A dummy activity is required if two activities share some, but not all, of their immediate predecessors.

A dummy activity will be required *from the end* of each shared immediate predecessor *to the start* of the activity that has additional immediate predecessors.

Dummy activities are represented in the activity network using *dotted lines*.

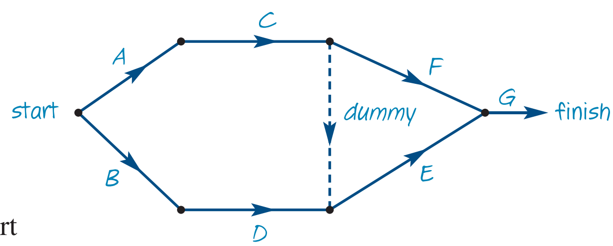
Example 6 Using a dummy activity in an activity network

Draw an activity network from the precedence table shown on the right.

Activity	*Immediate predecessors*
A	–
B	–
C	A
D	B
E	C, D
F	C
G	E, F

Solution

- A and B will lead from the start vertex.
- G will lead to the end vertex.
- A dummy will be required from the end of activity C (shared immediate predecessor) to the start of activity E (the activity with an additional immediate predecessor.



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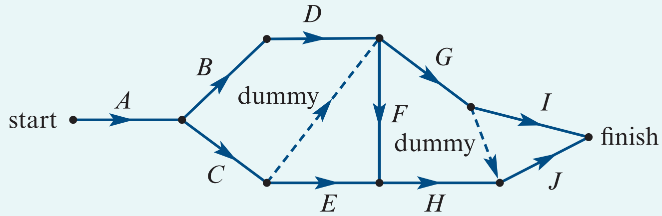
Example 7 Creating a precedence table from an activity network

Write down a precedence table for the activity network shown on the right.



Solution

1. Create a table with a row for each activity.
2. Look at the start of an activity. Write down all of the activities that lead directly to this activity in the immediate predecessor column.
3. The dummy activity makes activity C a predecessor of activities E, F and G.
4. The dummy activity makes activity G a predecessor of activities I and J.

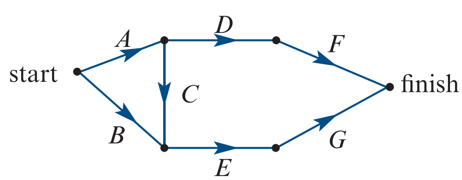
Activity	Immediate predecessors
A	–
B	A
C	A
D	B
E	C
F	D, C
G	D, C
H	E, F
I	G
J	G, H

Exercise 15C

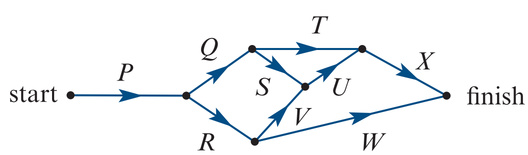
Constructing precedence tables from activity networks

1. Write down a precedence table for the activity networks shown below.

a



b



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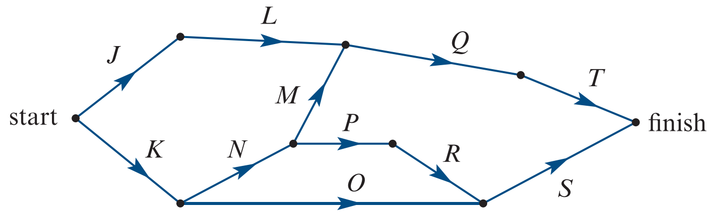
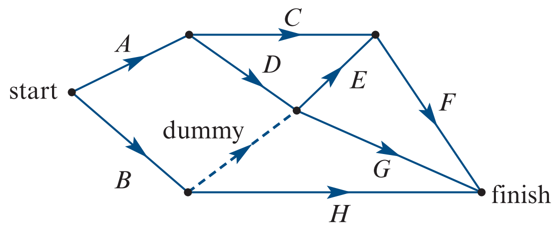
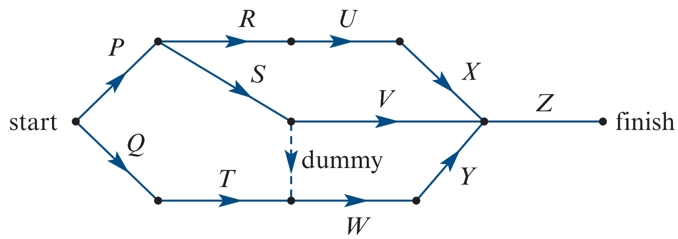
c



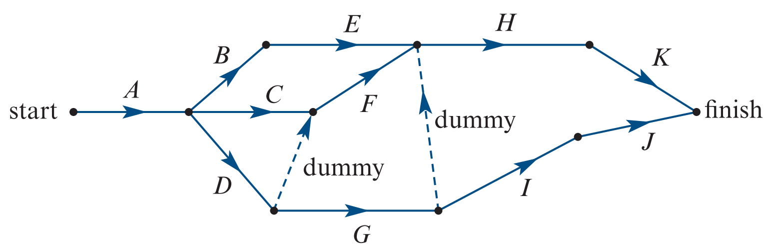
d



e



f



Constructing activity networks from precedence tables

2 Draw an activity network for each of the precedence tables below.

a

Activity	*Immediate predecessors*
A	–
B	A
C	A
D	B
E	C

b

Activity	*Immediate predecessors*
P	–
Q	–
R	P
S	Q
T	R, S

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c

Activity	Immediate predecessors
T	–
U	–
V	T
W	U
X	V, W
Y	X
Z	Y

d

Activity	Immediate predecessors
F	–
G	–
H	–
I	F
J	G, I
K	H, J
L	K

e

Activity	Immediate predecessors
K	–
L	–
M	K
N	M
O	N, L
P	O
Q	P
R	M
S	R, Q

f

Activity	Immediate predecessors
A	–
B	–
C	–
D	B
E	A, D
F	E, C
G	F
H	G
I	E, C
J	G
K	H, I

Constructing activity networks requiring dummy activities from precedence tables

3 Draw an activity network for the following precedence tables. Dummy activities will need to be used.

a

Activity	Immediate predecessors
F	–
G	–
H	F
I	H, G
J	G

b

Activity	Immediate predecessors
A	–
B	A
C	A
D	B
E	B, C

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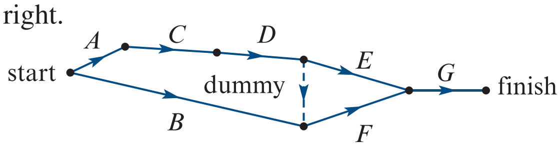
c

Activity	*Immediate predecessors*
P	–
Q	–
R	P
S	Q
T	Q
U	R, S
V	R, S, T

d

Activity	*Immediate predecessors*
A	–
B	A
C	A
D	B, C
E	C
F	E
G	D
H	F, G
I	H
J	I

4 The following activity network shows the activities in a project to repair a dent in a car panel. The activities are listed in the table on the right.



Activity	*Description*
A	Remove panel
B	Order component
C	Remove broken component
D	Pound out dent
E	Repaint
F	Install new component
G	Replace panel

a Which activity or activities are the immediate predecessors of the event 'remove broken component'?

b Which activities are the immediate predecessors of the activity 'install new component'?

15D Scheduling problems

▶ Scheduling

Projects that involve multiple activities are usually completed against a time schedule. Knowing how long individual activities within a project are likely to take allows managers of such projects to hire staff, book equipment and also to estimate overall costs of the project. Allocating time to the completion of activities in a project is called *scheduling*. Scheduling problems involve analysis to determine the minimum overall time it would take to complete a project.

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Weighted precedence tables

The estimated time to complete activities within a project can be recorded in a precedence table, alongside the immediate predecessor information.

A precedence table that contains the estimated duration, in days, of each activity is shown on the right.

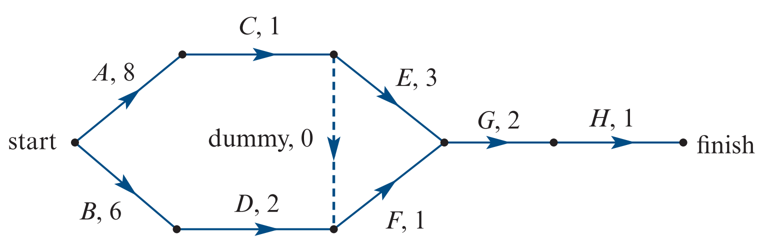
These estimated times are the weights for the edges of the activity network and need to be recorded alongside the name of the activity on the graph.

A dummy activity is required from the end of activity C (C is repeated in immediate predecessors) to the start of activity F (F is the activity that has an extra immediate predecessor).

Activity	*Estimated completion time (days)*	*Immediate predecessors*
A	8	–
B	6	–
C	1	A
D	2	B
E	3	C
F	1	C, D
G	2	E, F
H	1	G

The weight (duration) of dummy activities is always zero.

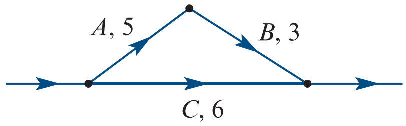
The activity network for this project is shown below.



Float times

The diagram below shows a small section of an activity network. There are three activities shown, with their individual durations, in hours.

Activity B and activity C are both immediate predecessors to the next activity, so the project cannot continue until both of these tasks are finished. Activity B cannot begin until activity A is finished.



Activity C can be completed at the same time as activity A and activity B.

Activity A and B will take a total of $5 + 3 = 8$ hours, while activity C only requires 6 hours. There is some flexibility around when activity C needs to start. There are $8 - 6 = 2$ hours spare for the completion of activity C. This value is called the **float time** for activity C.

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The flexibility around the starting time for activity C can be demonstrated with the following diagram.

	A	A	A	A	A	B	B	B
Start at same time	C	C	C	C	C	C	Slack	Slack
Delay C by 1 hour	Slack	C	C	C	C	C	C	Slack
Delay C by 2 hour	Slack	Slack	C	C	C	C	C	C

The five red squares represent the 5 hours it takes to complete activity A. The three green squares represent the 3 hours it takes to complete activity B.

The six yellow squares represent the 6 hours it takes to complete activity C. Activity C does not have to start at the same time as activity A because it has some slack time available (2 hours).

Activity C should not be delayed by more than 2 hours because this would cause delays to the project. The next activity requires B and C to be complete before it can begin.

Calculating and recording earliest starting times (EST)

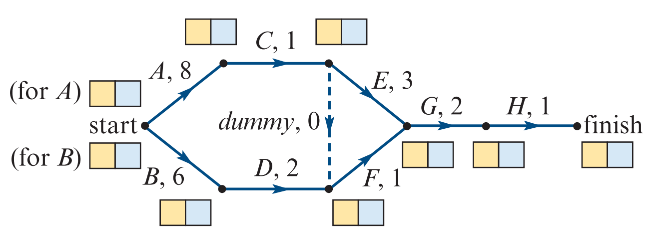
In order for a project to be completed in the shortest time possible, it is important that activities start at the earliest possible time. The **earliest starting time**, or **EST**, for each activity is the earliest time after the start of the entire project that the individual activity can start. An EST of 8 means an activity can start 8 hours (or whatever time period is given) after the start of the project.

The EST for each activity is found by a process called **forward scanning**.

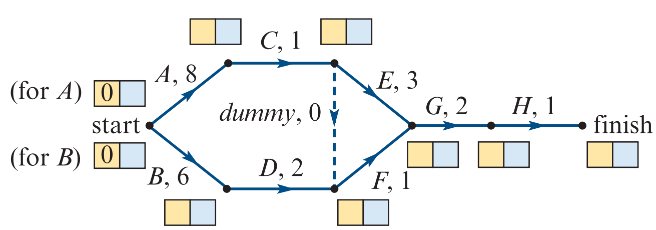
Forward scanning

Forward scanning will be demonstrated using the activity network below. The durations of each are in days.

1 Draw a box, split into two cells, next to each vertex of the activity network, as shown in the diagram opposite. If more than one activity begins at a vertex, draw a box for each of these activities.



2 Activities that begin at the start of the project have an EST of zero (0). Write this in the left box, shown shaded yellow in the diagram.



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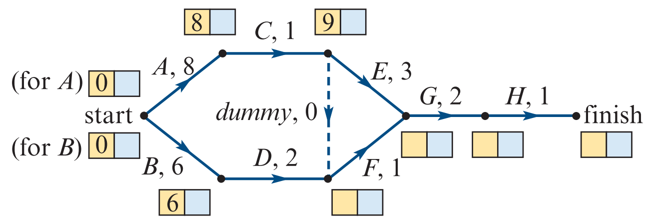
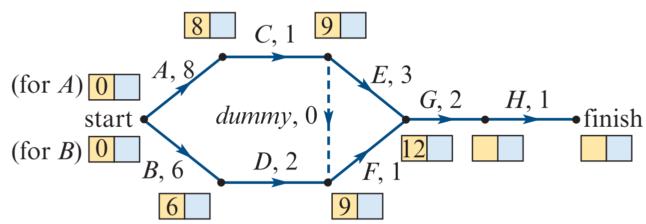
3 Calculate the EST of each activity of the project by adding the EST of the immediate predecessor to the duration of the immediate predecessor.



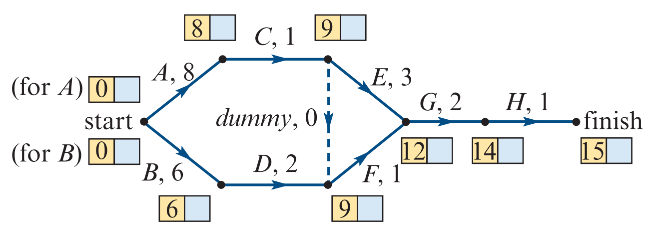
Notes:
EST of C = EST of A + duration of A (EST of $C = 0 + 8 = 8$)
EST of D = EST of B + duration of B (EST of $D = 0 + 6 = 6$)
EST of E = EST of C + duration of C (EST of $C = 8 + 1 = 9$)

4 If an activity has more than one predecessor, calculate the EST using each of the predecessors and choose the *largest* value.



Notes:
EST of $F = 6 + 2 = 8$ or EST of $F = 9 + 0 = 9$. Use 9.
EST of $G = 9 + 1 = 10$ or EST of $G = 9 + 3 = 12$. Use 12.

5 The EST value at the finish of the project is the minimum time it takes to complete the project.



Notes: The minimum time to complete this project is 15 days.

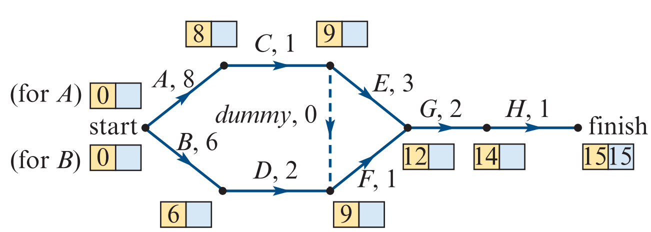
Calculating and recording latest starting times (LST)

Some activities, as we saw earlier, have some flexibility around the time that they can start. The **latest start time**, or LST, for each activity is the latest time after the start of the entire project that the individual activity can start. LSTs for each activity are calculated using the reverse of the process used to calculate the ESTs. This process is called **backward scanning**.

Backward scanning

Backward scanning will be demonstrated using the activity network with completed forward scanning from above.

1 Copy the minimum time to complete the project into the right cell shown shaded blue in the diagram.



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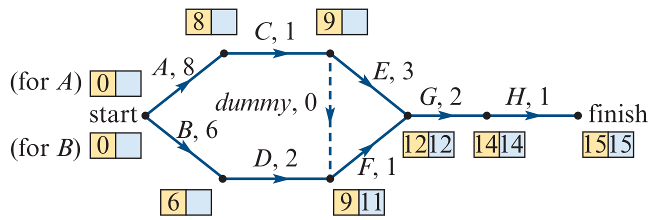
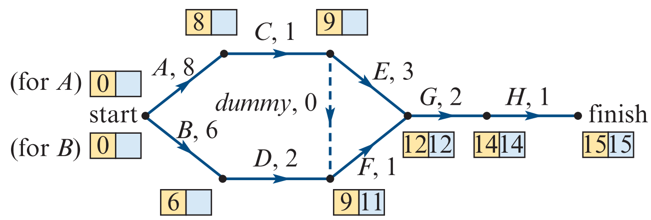
2 Calculate the LST for each activity by subtracting the duration of the activity from the LST of the following activity.



Notes:

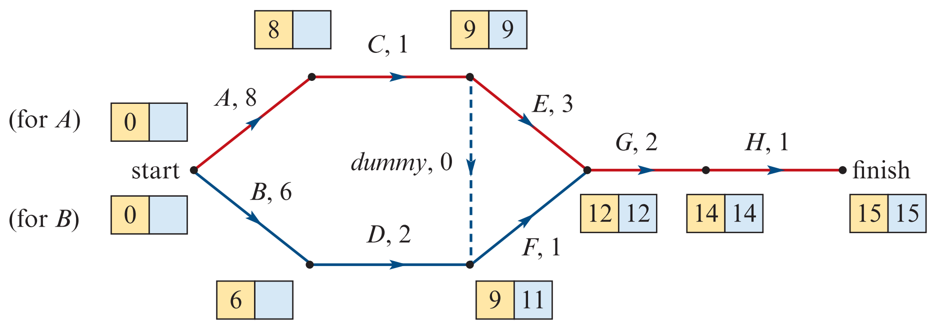
1 LST of H – LST of finish – duration of H (LST of H = 15 – 1 = 14).
2 LST of G – LST of H – duration of G (LST of G = 14 – 2 = 12).
3 LST of F – LST of G – duration of F (LST of F = 12 – 1 = 11).

3 If more than one activity have the same predecessor, calculate the LST using each of the activities that follow and choose the *smallest* value.



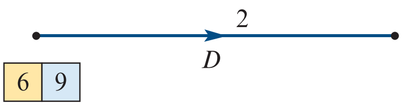
Note: LST of C = 12 – 3 = 9 (from E) or LST of C = 11 – 0 = 11 (from *dummy*. Use 9.

The completed activity network with all EST and LST is shown below.



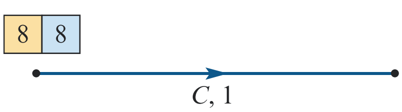
▶ Identifying float times and the critical path

The boxes at the vertices in the activity network above give the EST and LST for the activity that begins at that vertex.



The EST for activity D is 6 and the LST for activity D is 9. This means activity D has a float time of 9 – 6 = 3 hours.

Activity D can be delayed by 3 hours without delaying the rest of the project.



The EST for activity C is 8 and the LST for activity C is 8. This means activity C has a float time of 8 – 8 = 0 hours.

Activity C has no flexibility around its starting time at all. Any delay to the start of this activity will delay the whole project and extend the minimum time for completion.

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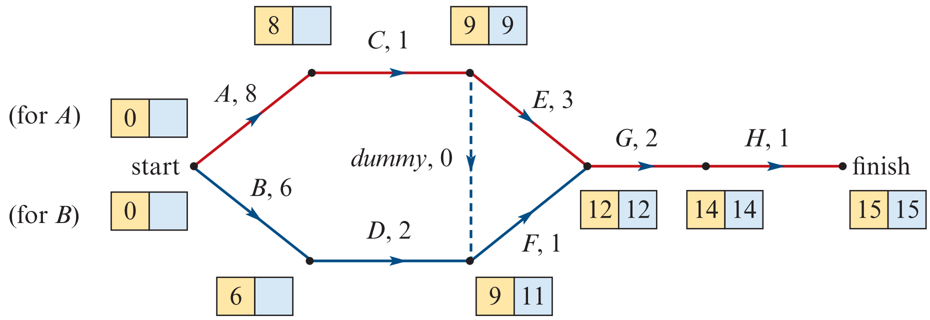
Activities that have no float time are critical ones for completion of the project. Tracking through the activity network along the edges of critical activities gives the **critical path** for the project. The critical path for this project is highlighted in red on the diagram below.



Critical path

The critical path is the sequence of activities that cannot be delayed without affecting the overall completion time of the project.

The process for determining the critical path is called **critical path analysis**.

Skillsheet

Critical path analysis

- Draw a box with two cells next to each vertex of the activity network.
- Calculate the EST for each activity by forward scanning:

  EST = EST of predecessor + duration of predecessor
- If an activity has more than one predecessor, the EST is the *largest* of the alternatives.
- The minimum overall completion time of the project is the EST value at the end vertex.
- Calculate the LST for each activity by backward scanning:

  LST = LST of following activity − duration of activity
- If an activity has more than one following activity, the LST is the *smallest* of the alternatives.
- Float = LST − EST
- If float time = 0, the activity is on the critical path.

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Example 8 Finding the critical path

A project has six activities as shown in the precedence table opposite.

a Draw an activity network for this project.

b Complete the critical path analysis to calculate the EST and LST for each activity.

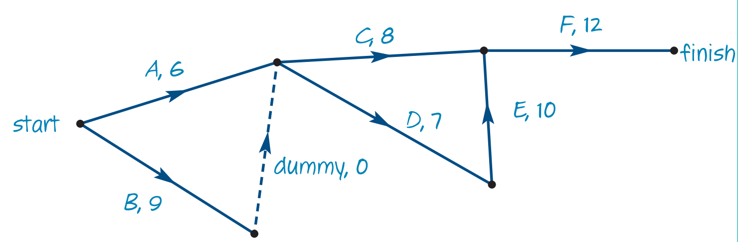
c Write down the critical path of this project.

d What is the minimum time required to complete the project?

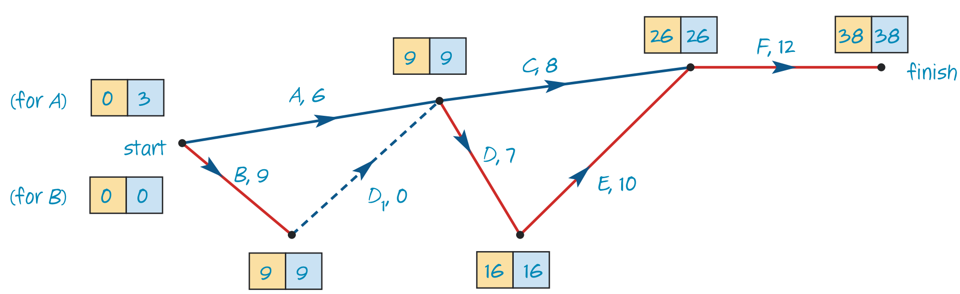
Activity	*Duration (days)*	*Immediate predecessors*
A	6	–
B	9	–
C	8	A, B
D	7	A, B
E	10	D
F	12	C, E

Solution

a A and B have no predecessors and so can begin at the same time. Since A and B are predecessors of C and D, a dummy activity will be required.



b



c The critical path is highlighted in red.
Note: The dummy is not included in the critical path.

The critical path of this project is $B \to D \to E \to F$.

d The minimum completion time is in EST of the end box.

The minimum completion time of this project is 38 days.

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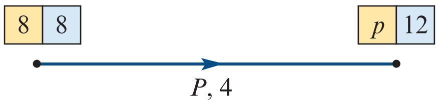
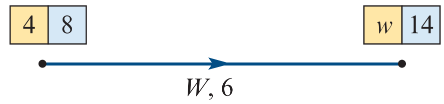
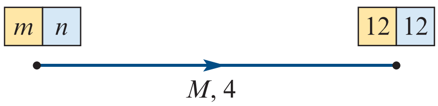
Exercise 15D

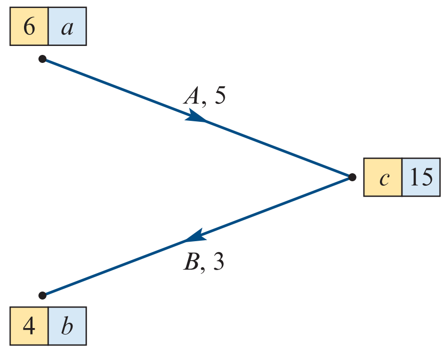
Calculations from elements of an activity network

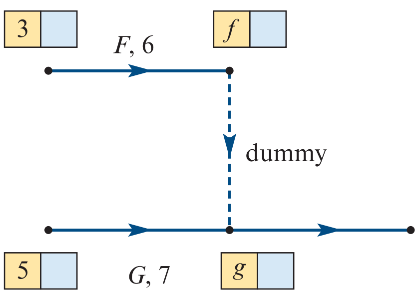
1 Write down the value of each pronumeral in the sections of activity networks below.

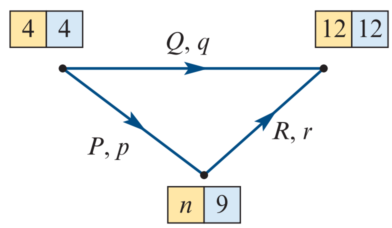
a 

b 

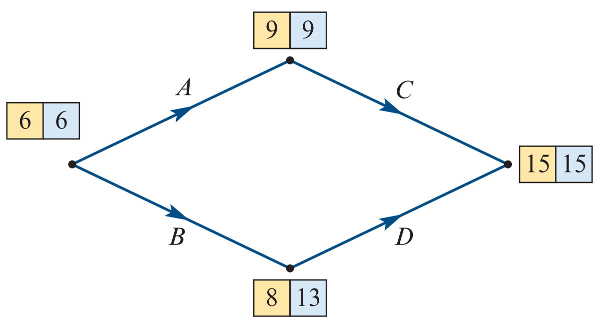
c 

d 

e 

f 

2 Consider the section of an activity network shown in the diagram below.



a What is the duration of activity A?

b What is the critical path through this section of the activity network?

c What is the float time of activity B?

d What is the latest time that activity D can start?

e What is the duration of activity D?

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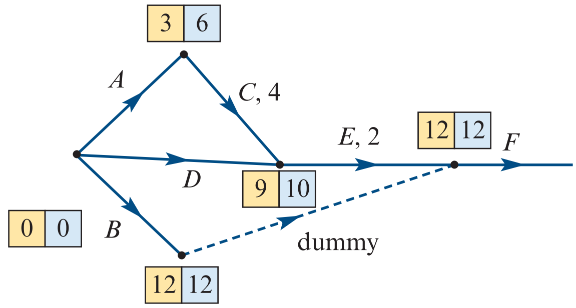
3 Consider the section of an activity network shown in the diagram below.



a What is the duration of activity B?

b What is the latest start time for activity E?

c What is the earliest time that activity E can start?

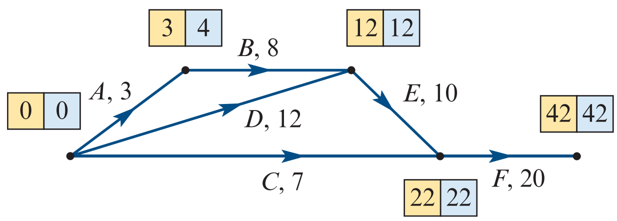
d What is the float time for activity E?

e What is the duration of activity A?

f What is the duration of activity D?

Completion of critical path analysis from a given activity network

4 An activity network is shown in the diagram below.



a Write down the critical path for this project.

b Calculate the float times for non-critical activities.

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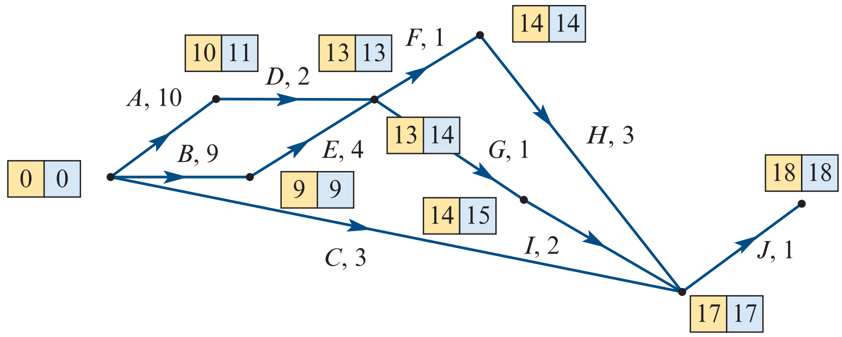
5 An activity network is shown in the diagram below.

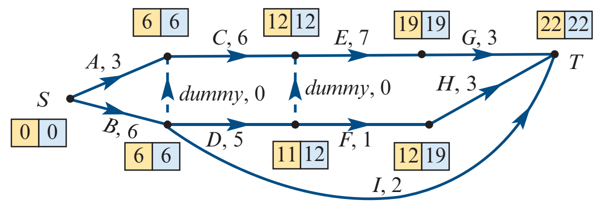


a Write down the critical path.

b Write down the float times for all non-critical activities.

6 A precedence table and activity network for a project are shown below. The precedence table is incomplete.

Activity	*Duration (weeks)*	*Immediate predecessors*
A	3	–
B	6	–
C	6	
D		*B*
E	7	
F	1	*D*
G		*E*
H	3	
I	2	*B*



a Complete the table above.

b Write down the critical path for this project.

Critical path analysis from precedence table only

7 The precedence table for a project is shown below.

a Draw an activity network for this project.

b Complete the critical path analysis to calculate the EST and LST for each activity.

c Write down the critical path of this project.

d What is the minimum time required to complete the project?

Activity	*Duration (weeks)*	*Immediate predecessors*
P	4	–
Q	5	–
R	12	–
S	3	*P*
T	6	*Q*
U	3	*S*
V	4	*R*
W	8	*R*, *T*, *U*
X	3	*V*
Y	6	*W*, *X*

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8 The precedence table for a project is shown below.

a Draw an activity network for this project.

b Complete the critical path analysis to calculate the EST and LST for each activity.

c Write down the critical path of this project

d What is the minimum time required to complete the project?

Activity	*Duration (weeks)*	*Immediate predecessors*
I	2	–
J	3	–
K	5	–
L	4	I
M	8	J, N
N	1	K
O	6	L, M
P	6	J, N
Q	7	J, N
R	5	K
S	1	O
T	9	Q, R

15E Crashing

Altering completion times

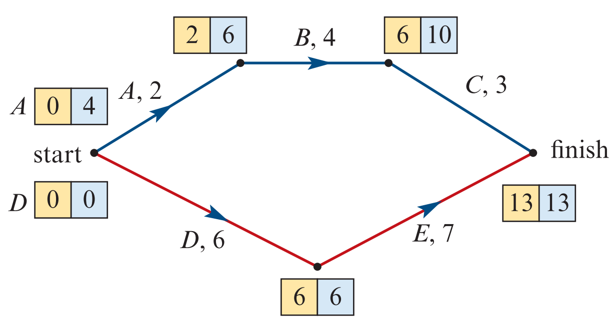
The minimum time it takes to complete a project depends upon the time it takes to complete the individual activities of the project, and upon the predecessors each of the activities have. Critical path analysis can be completed to find the overall minimum completion time.

Sometimes, the managers of a project might arrange for one or more activities within the project to be completed in a shorter time than originally planned. Changing the conditions of an activity within a project, and recalculating the minimum completion time for the project, is called **crashing**.

An individual activity could be crashed by employing more staff, sourcing alternate materials or simply because weather or other factors allow the activity to be completed in a shorter time than usual.

A simple crashing example

A simple activity network is shown in the diagram on the right. The forwards and backwards scanning processes have been completed and the critical path has been determined. The critical path is shown in red on the diagram.



The minimum time for completion is currently 13 hours. In order to reduce this overall time, the manager of the project should try to complete one, or more, of the activities in a shorter time than normal. Reducing the time taken to complete activity A, B or C would not

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achieve this goal however. These activities are not on the critical path and so they already have slack time. Reducing their completion time will not shorten the overall time taken to complete the project.

Activity D and E, on the other hand, lie on the critical path. Reducing the duration of these activities will reduce the overall time for the project. If activity D was reduced in time to 4 hours instead, the project will be completed in 11, not 13, hours.

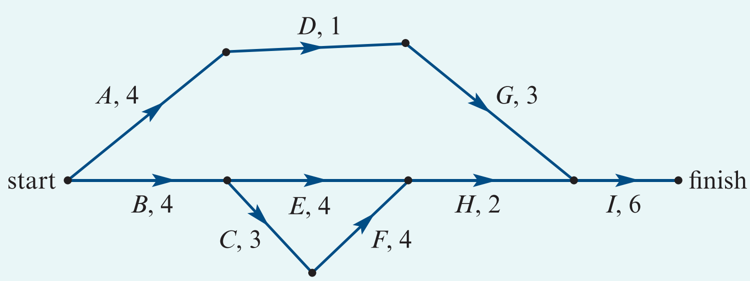
▶ Crashing with cost

Shortening the completion time for any individual activity could result in an extra cost for the project. In the simple example above, the cost of reducing the completion time of activity D by 1 hour is \$150, while the cost of reducing the completion time of activity E by 1 hour is \$18.

Clearly it is best to reduce the completion time, or crash, the activity that will cost the least.

Example 9 Crashing

A community centre is to be built on a new housing estate. Nine activities have been identified for this building project. The activity network shown on the right shows the activities and their completion times in weeks.



a Determine the minimum time, in weeks, to complete this project.

b Determine the slack time, in weeks, for activity D.

c Write down the critical path of this project.

The builders of the community centre are able to speed up the project.

Some of the activities can be reduced in time at an additional cost.

The activities that can be reduced in time are A, C, E, F and G.

d Which of these activities, if reduced in time individually, would not result in an earlier completion of the project?

The owner of the estate is prepared to pay the additional cost to achieve early completion.

The cost of reducing the time of each activity is \$5000 per week.

The maximum reduction in time for each one of the five activities, A, C, E, F and G is 2 weeks.

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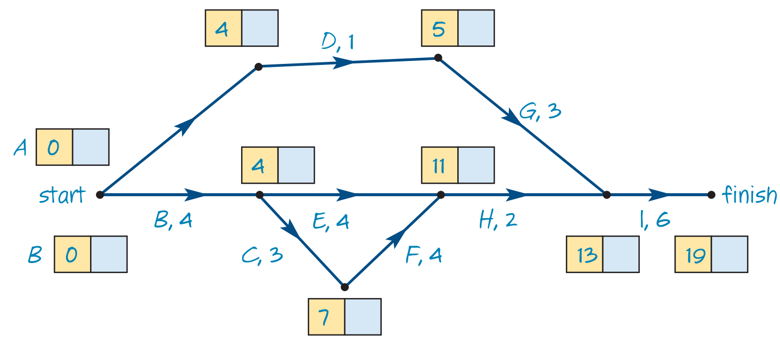
e Determine the minimum time, in weeks, for the project to be completed now that certain activities can be reduced in time.

f Determine the minimum additional cost of completing the project in this reduced time.

based on VCAA (2007)

Solution

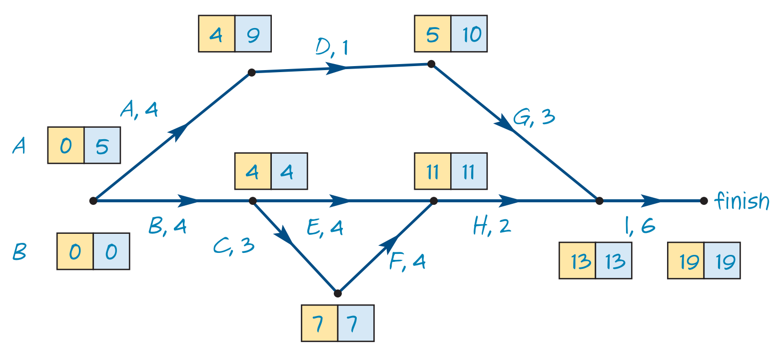
a **1** Perform forwards scanning to determine the EST for all activities and also the minimum time to complete the project.



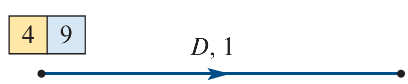
2 Write your answer.

The minimum completion time of this project is 19 weeks.

b **1** Perform backwards scanning to determine the LST for all activities.



2 Look at activity *D*. It has an EST of 4 and an LST of 9.



EST = 4 and LST = 9

Float = LST − EST

= 9 − 4

= 5 weeks

c Follow boxes that have EST and LST equal through the activity network.

The critical path of this project is $B \rightarrow C \rightarrow F \rightarrow H \rightarrow I$.

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d Look for an activity that is not on the critical path. Reducing this activity time will not affect the overall time for the project.

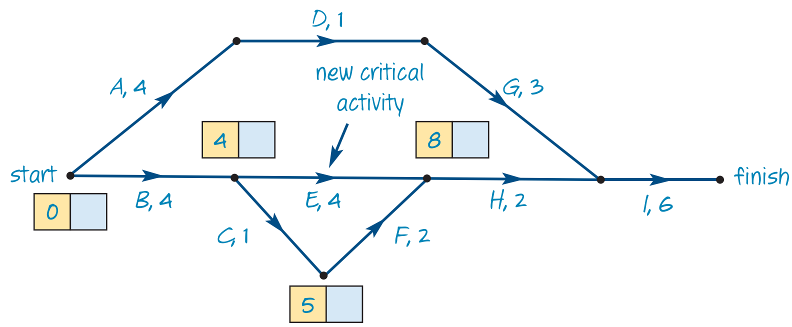
Activities A, E and G are not on the critical path. These activities would not result in an earlier completion of the project.

e 1 Maximum reduction is 2 weeks for any activity. Reduce the critical path activities by the maximum.

Activities C and F are on the critical path. Reduce these by 2 weeks each to reduce the path $B \to C \to F \to H \to I$ by 4 weeks, from 19 down to 15 weeks.

2 Check for change in the critical path.

This has resulted in a new critical path.



3 Adjust if possible to make lowest critical path length.

E can be reduced by 1 week to ensure overall minimum completion time of 15 weeks.

f Add the cost of reducing the activities.

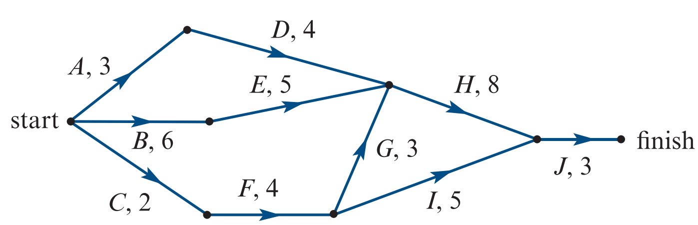
C and F have been reduced by 2 weeks. E has been reduced by 1 week. There is 5 weeks reduction at \$5000 per week.

Minimum cost is \$25 000.

Exercise 15E

Altering completion times

1 The activity network for project is shown in the diagram on the right. The duration for each activity is in hours.



a Identify the critical path for this project.

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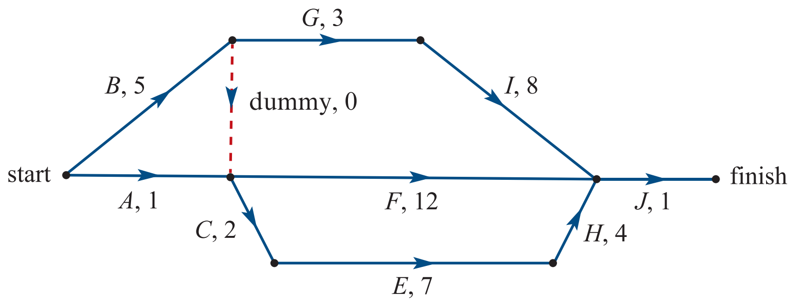
b What is the maximum number of hours that the completion time for activity E can be reduced without changing the critical path of the project?

c What is the maximum number of hours that the completion time for activity H can be reduced without affecting the critical path of the project?

d Every activity can be reduced in duration by a maximum of 2 hours.

If every activity was reduced by the maximum amount possible, what is the minimum completion time for the project?

2 The activity network for a project is shown in the diagram on the right. The duration for each activity is in hours.



a Identify the critical path for this project.

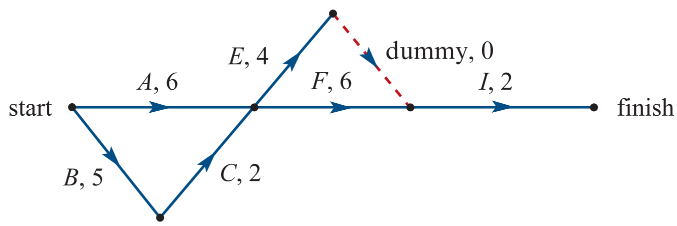
b How much time overall is saved if activity F is reduced in duration by 2 hours?

c If only one activity can have its duration reduced, what is the maximum possible reduction in the completion time of the project?

based on VCAA (pre-2002)

Crashing with cost

3 The activity network for a project is shown in the diagram on the right. The duration for each activity is in hours.



a Determine the shortest time in which this project can be completed.

b Write down the critical path for this project.

c The time it takes to complete activity A can be reduced by one hour at a cost of \$50. Explain why this will not affect the completion time of this project.

Activity B can be reduced in time at a cost of \$100 per hour. Activity F can be reduced in time at a cost of \$50 per hour.

d What is the cost of reducing the completion time of this project as much as possible?

based on VCAA (2004)

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Review

Key ideas and chapter summary

Weighted graph A **weighted graph** is a graph in which a number representing the size of same quantity is associated with each edge. These numbers are called weights.

Network A **network** is a set of points called vertices and connecting lines called edges, enclosing and surrounded by areas called faces.

Directed graph (digraph) A **directed graph** is a graph where direction is indicated for every edge. This is often abbreviated to **digraph**.

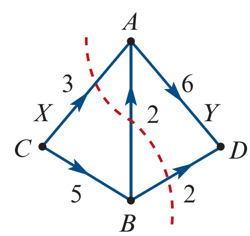
Flow The transfer of material through a directed network. **Flow** can refer to the movement of water or traffic.

Capacity The maximum flow of substance that an edge of a directed graph can allow during a particular time interval. The **capacity** of water pipes is the amount of water (usually in litres) that the pipe will allow through per time period (minutes, hours, etc.). Other examples of capacity are number of cars per minute or number of people per hour.

Source The **source** is the origin of the material flowing through a network.

Sink The **sink** is final destination of the material flowing through a network.

Cut A **cut** is a line dividing a directed graph into two parts (shown as a broken line dividing the graph below into two sections, labelled X and Y).



Cut capacity The sum of the capacities (weights) of the edges directed from X to Y that the cut passes through. For the weighted digraph shown, the capacity of the cut is 7.

Minimum cut The **minimum cut** is the cut with the minimum capacity. The cut must separate the source from the sink.

Maximum flow The **maximum flow** through a directed graph is equal to the capacity of the minimum cut.

Bipartite graph A **bipartite graph** has two distinct groups or categories for the vertices. Connections exist between a vertex or vertices from one group with a vertex or vertices from the other group. There are no connections between the vertices within a group.

Allocation An **allocation** is made when each of the vertices in one group from a bipartite graph are matched with one of the vertices in the other group from that graph. An allocation is possible when both groups have exactly the same number of vertices. The vertices in each group are matched to only one vertex from the other group.

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Cost matrix	A table that contains the costs of allocating objects from one group (such as people) to another (such as tasks). The 'cost' can be money, or other factors such as the time taken.
Hungarian algorithm	The **Hungarian algorithm** is an algorithm that is used to determine the best allocation to minimise the overall cost.
Activity network	An **activity network** is a directed graph that shows the required order of completing individual activities that make up a project.
Immediate predecessor	If activity A is an **immediate predecessor** to activity B, activity A must be completed before activity B can begin.
Precedence table	A **precedence table** is a table that records the activities of a project and their immediate predecessors. Precedence tables can also contain the duration of each activity.
Dummy activity	A **dummy activity** has zero cost. It is required if two activities share some, but not all, of the same immediate predecessors. It allows the network to show all precedence relationships in a project correctly.
Earliest starting time (EST)	**EST** is the earliest time an activity in a project can begin.
Latest starting time (LST)	**LST** is the latest time an activity in a project can begin, without affecting the overall completion time for the project.
Float (slack) time	**Float (slack) time** is the difference between the latest starting time and the earliest starting time. Float = LST − EST The float time is sometimes called the slack time. It is the largest amount of time that an activity can be delayed without affecting the overall completion time for the project.
Forward scanning	**Forward scanning** is a process of determining the EST for each activity in an activity network. The EST of an activity is added to the duration of that activity to determine the EST of the next activity. The EST of any activity is equal to the largest forward scanning value determined from all immediate predecessors.
Backward scanning	**Backward scanning** is a process of determining the LST for each activity in an activity network. The LST of an activity is equal to the LST of the activity that follows, minus the duration of the activity.
Critical path	The **critical path** is the series of activities that cannot be delayed without affecting the overall completion time of the project. Activities on the critical path have no slack time. Their EST and LST are equal.
Critical path analysis	**Critical path analysis** is a project planning method in which activity durations are known with certainty.

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Crashing — **Crashing** is the process of shortening the length of time in which a project can be completed by reducing the time required to complete individual activities. Reducing the individual activity completion times often costs money; this increases the overall cost of a project.

Skills check

Having completed this chapter you should be able to:

- define and describe a directed graph
- determine the shortest path through a directed graph using Dijkstra's algorithm
- describe the flow of material through a directed graph
- determine the maximum flow for any section of sequential edges of a directed graph
- determine cut capacities
- determine the maximum flow as equal to the minimum cut capacity
- draw directed and weighted bipartite graphs
- use the Hungarian algorithm to determine an optimum allocation in order to minimise cost
- create an activity network from a precedence table
- write down a precedence table from an activity network
- decide when to use dummy activities in an activity network
- use forward scanning to determine the earliest starting time of activities in an activity network
- use backward scanning to determine the latest starting time of activities in an activity network
- determine the float time for activities in an activity network
- determine the overall minimum completion time for a project using critical path analysis
- determine the critical path for an activity network
- use crashing to reduce the completion time of a project.

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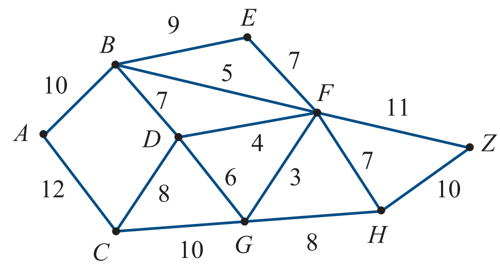
Multiple-choice questions

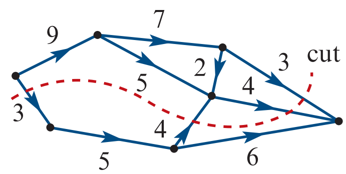
1 The shortest path from A to Z in the network on the right has length:

A 10 **B** 15
C 22 **D** 26
E 28



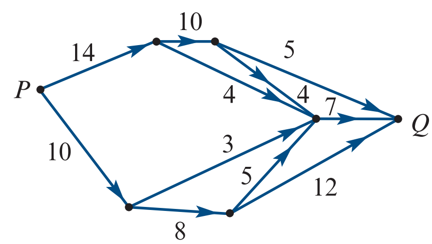
2 For the network shown on the right, the capacity of the cut is:

A 3 **B** 6 **C** 9
D 10 **E** 14



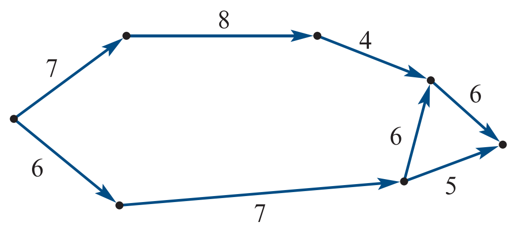
3 In the communications network shown, the numbers represent transmission capacities for information (data) in scaled units. What is the maximum flow of information from station P to station Q?

A 20 **B** 22
C 23 **D** 24
E 30



4 The maximum flow in the network opposite, from source to sink, is:

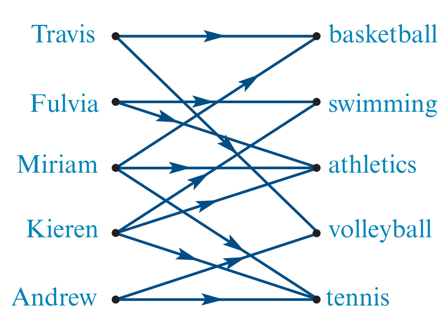
A 10 **B** 11
C 12 **D** 13
E 14



5 A group of five students represent their school in five different sports. The information is displayed in a bipartite graph. From this graph we can conclude that:

A Travis and Miriam played all the sports between them.

B In total, Miriam and Fulvia played fewer sports than Andrew and Travis.



C Kieren and Miriam each played the same number of sports.

D In total, Kieren and Travis played fewer different sports than Miriam and Fulvia.

E Andrew played fewer sports than any of the others.

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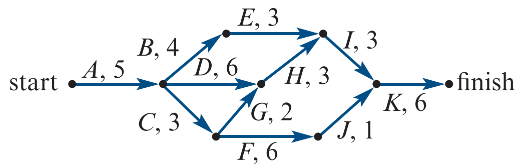
6 This activity network is for a project where the component times in days are shown. The critical path for the network of this project is given by:



A $A–B–E–I–K$

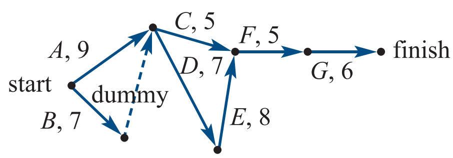
B $A–D–H–I–K$

C $A–C–G–H–I–K$

D $A–C–F–J–K$

E $A–D–G–F–J–K$

7 The activity network shown represents a project development with activities and their durations (in days) listed on the edges of the graph. Note that the dummy activity takes zero time.



The earliest time (in days) that activity F can begin is:

A 0 **B** 12 **C** 14 **D** 22 **E** 24

8 The table opposite lists the six activities in a project and the earliest start time, in hours, and the predecessor(s) of each task. The time taken for activity E is two hours. Without affecting the time taken for the entire project, the time taken for activity C could be increased by:

A 0 hours **B** 8 hours

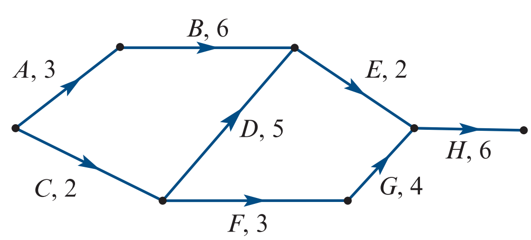
C 9 hours **D** 11 hours

E 27 hours

Task	*Immediate predecessor*	*EST*
A	–	0
B	–	0
C	*A*	8
D	*B*	15
E	*C*	22
F	*D*, *E*	35

based on VCAA (2004)

9 The edges in this activity network correspond to the tasks involved in the preparation of an examination. The numbers indicate the time, in weeks, needed for each task. The total number of weeks needed for the preparation of the examination is:



A 14 **B** 15 **C** 16 **D** 17 **E** 27

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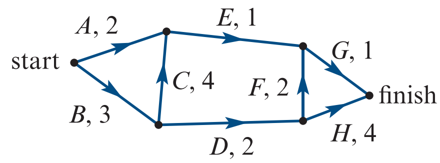
10 The activity network represents a manufacturing process with activities and their duration (in hours) listed on the edges of the graph. The earliest time (in hours) after the start that activity G can begin is:



A 3 **B** 5 **C** 6 **D** 7 **E** 8

Extended-response questions

1 Ann, Bianca, Con and David are four examination supervisors. There are four examinations venues: B, C, D and E. Each examination venue requires one examination supervisor.

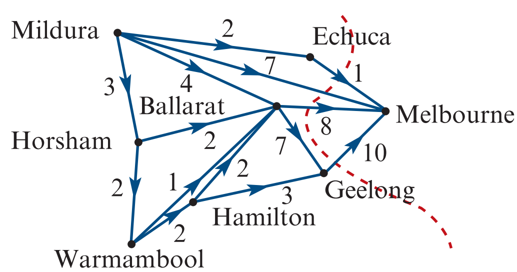
The table shows the times (in minutes) that the examination supervisors would take to travel from their home to each examination venue.

Determine an allocation of examination supervisor to examination venue in order to minimise the overall travel time for the supervisors.

Supervisor	*B*	*C*	*D*	*E*
Ann	25	30	15	35
Bianca	22	34	20	45
Con	32	20	33	35
David	40	30	28	26

based on VCAA (pre-2002)

2 WestAir Company flies routes in western Victoria. The network shows the layout of connecting flight paths for WestAir, which originate in Mildura and terminate in either Melbourne or on the way to Melbourne. On this network, the available spaces for passengers flying out of various locations on one morning are shown.



The network has one cut shown.

a What is the capacity of this cut?

b What is the maximum number of passengers who could travel from Mildura to Melbourne for the morning?

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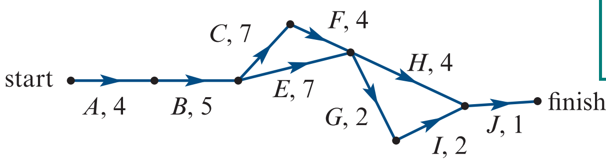
3 LiteAero Company designs and makes light aircraft for the civil aviation industry. They identify 10 activities required for production of their new model, the MarchFly. These, and the associated activity durations, are given in the table opposite.

Activity	*Duration (weeks)*	*Immediate predecessor(s)*
A	4	–
B	5	*A*
C	7	*B*
D	6	*B*
E	7	*B*
F	4	*C*
G	2	*E*, *F*
H	4	*F*
I	2	*D*, *G*
J	1	*H*, *j*

a An incomplete activity network for this project is shown below. Complete the network by drawing and labelling activity *D*.



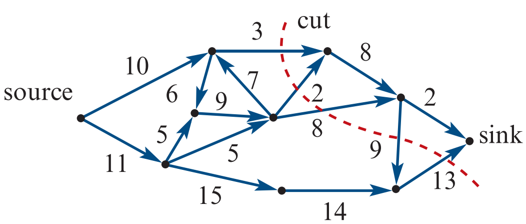
b What is the critical path for this network?

based on VCAA (pre-2002)

4 A school swimming team wants to select a 4 × 200 metre relay team. The fastest times of its four best swimmers in each of the strokes are shown in the table below. Which swimmer should swim which stroke to give the team the best chance of winning, and what would be their time to swim the relay?

Swimmer	*Backstroke*	*Breaststroke*	*Butterfly*	*Freestyle*
Rob	76	78	70	62
Joel	74	80	66	62
Henk	72	76	68	58
Sav	78	80	66	60

5 In the network opposite, the values on the edges give the maximum flow possible between each pair of vertices. The arrows show the direction of flow in the network. Also shown is a cut that separates the source from the sink.



a Determine the capacity of the cut shown.

b Determine the maximum flow through this network.

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16

Module 2: Networks and decision mathematics

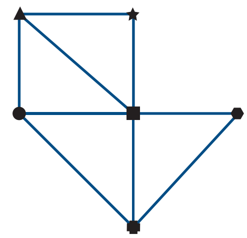
Revision: Networks and decision mathematics

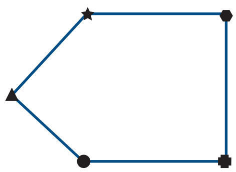
ISBN 978-1-107-56757-3

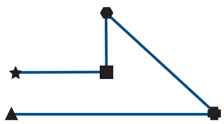
Cambridge University Press

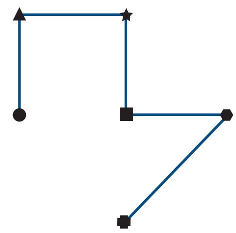
16A Multiple-choice questions

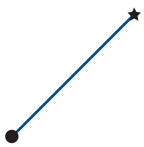
1 Consider this network graph.
A subgraph of this graph is:

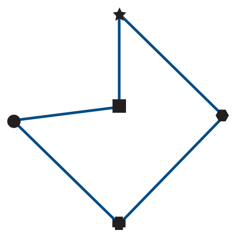


A


B


C


D


E


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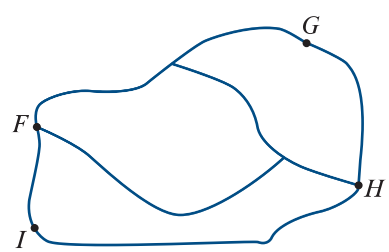
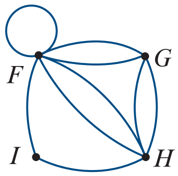
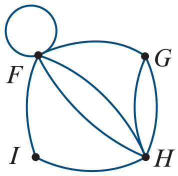
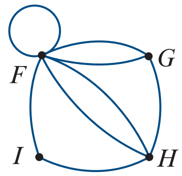
2 The diagram shows a map of the roads between four towns: F, G, H and I.

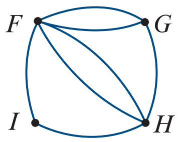


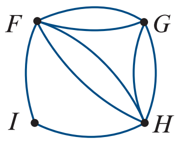
A network diagram that represents the connections between towns on the map is:

A


B


C


D


E


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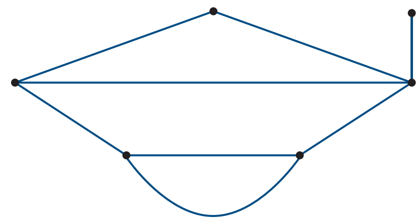
3 Five people are to be each allocated one of five tasks (A, B, C, D, E). The table shows the time, in hours, that each person takes to complete the tasks. The tasks must be completed in the least possible total amount of time. If no person can help another, Francis should be allocated task:

Name	*A*	*B*	*C*	*D*	*E*
Francis	12	15	99	10	14
David	10	9	10	7	12
Herman	99	10	11	6	12
Indira	8	8	12	9	99
Natalie	8	99	9	8	11

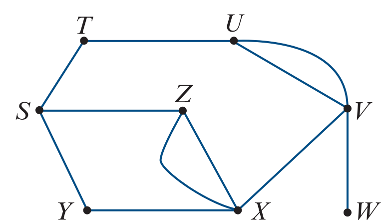
A A **B** B **C** C **D** D **E** E

4 The sum of the degrees of all the vertices in the network opposite is:



A 6 **B** 7 **C** 8
D 15 **E** 16

5 Adding which one of the following edges to the graph makes an eulerian trail possible?



A ST **B** SU **C** SX
D XW **E** ZY

6 A connected planar graph has an equal number of vertices and faces. If there are 20 edges in this graph, the number of vertices must be:

A 9 **B** 10 **C** 11 **D** 20 **E** 22

7 Underground water pipes are needed to water a new golf course. Water will be pumped from the dam in the back corner of the course. To find the smallest total length of water pipe needed, we must find:

A a critical path
B a minimal spanning tree
C the shortest eulerian circuit
D the shortest hamiltonian cycle
E the perimeter of the golf course

8 Which one of the following is a true statement about a critical path in a project?

A Knowledge of the critical path can be used to decide if any tasks in a project can be delayed without extending the length of time of the project.
B All tasks on the critical path must be completed before any other task in the same project can be started.
C Decreasing the times of tasks, not on the critical path, will decrease the length of time of the project.

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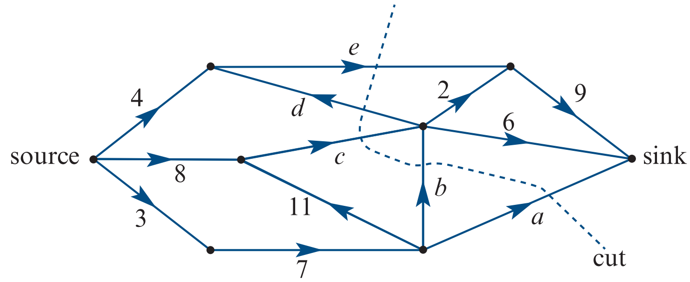
D The critical path must always include at least two tasks in a project.

E There is only one critical path in any project.

9 In the directed graph opposite, the weight of each edge is non-zero. The capacity of the cut shown is:



A $a + b + c + d + e$

B $a + c + d + e$

C $a + b + c + e$

D $a + b + c - d + e$

E $a - b + c - d + e$

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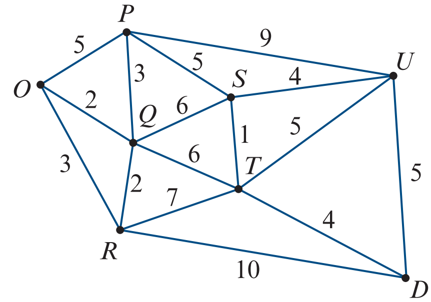
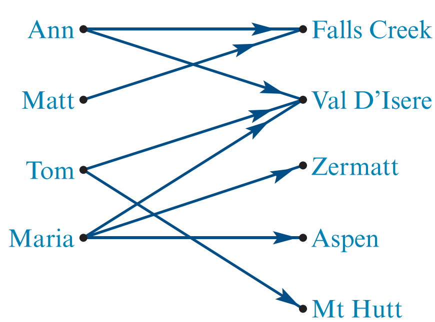
10 The length of the shortest path between the origin, O, and destination, D, in the network shown here is:



A 11 **B** 12 **C** 13

D 14 **E** 15

11 Four students, talking about five ski resorts they have visited, represented their information on the bipartite graph shown here. Which one of the following statements is implied by this bipartite graph?



A Ann and Maria between them have visited fewer ski resorts than Matt and Tom between them.

B Matt and Tom have been to four ski resorts between them.

C Maria has visited fewer ski resorts than any of the others.

D Ann and Maria between them have visited all five ski resorts discussed.

E Ann and Tom between them have visited fewer resorts than Matt and Maria between them.

12 A gas pipeline is to be constructed to link several towns in the country. Assuming the pipeline construction costs are the same everywhere in the region, the cheapest network formed by the pipelines and the towns as vertices would form:

A a hamiltonian cycle **B** an eulerian circuit **C** a minimum spanning tree

D a critical path **E** a complete graph

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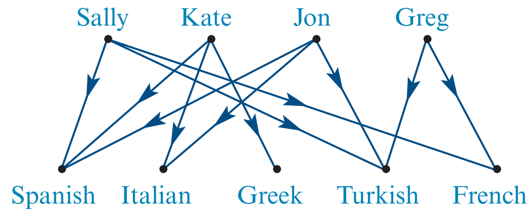
13 Which one of the following statements is not implied by this bipartite graph?



A There are more translators of French than Greek.

B Sally and Kate can translate five languages between them.

C Jon and Greg can translate four languages between them.

D Kate and Jon can translate more languages between them than can Sally and Greg.

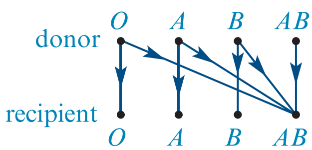
E Sally and Jon can translate more languages between them than can Kate and Greg.

14 There are four different human blood types: O, A, B and AB. The relationships between donor and recipients for these blood types are as follows:

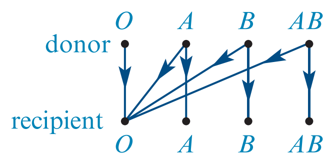
- Type O can donate blood to any type.
- Type AB can receive blood from any type.
- Each type can donate blood to its own type.
- Each type can receive blood from its own type.

Which one of the following donor–recipient bipartite graphs correctly represents this information?

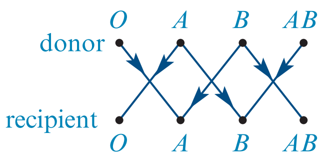
A



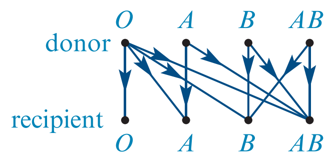
B



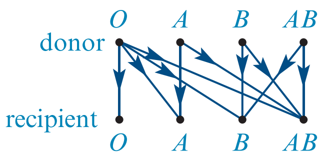
C



D



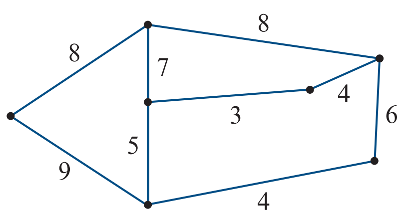
E



15 For the weighted graph shown, the length (total weight) of the minimum spanning tree is:

A 28 **B** 29 **C** 30

D 31 **E** 32



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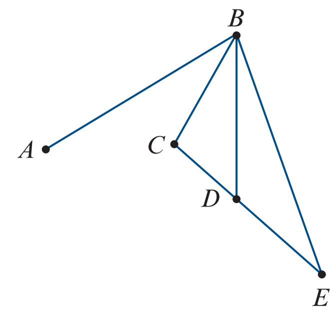
16 The number of edges for a complete graph with five vertices is:

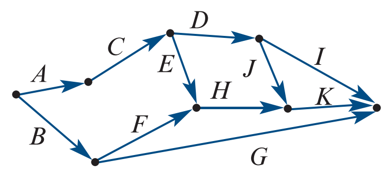
A 4 **B** 5 **C** 10 **D** 15 **E** 20

17 What additional edge could be added to the graph to ensure that the resulting graph would contain an Euler circuit?



A AB **B** AC **C** AD
D AE **E** BC

18 This network represents a project development with activities listed on the edges of the graph. Which of the following statements must be true?

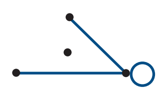


A A must be completed before B can start.
B A must be completed before F can start.
C E and F must start at the same time.
D E and F must finish at the same time.
E E cannot start until A is finished.

19 A connected graph with 12 edges divides a plane into four faces. The number of vertices in this graph will be:

A 6 **B** 10 **C** 12 **D** 13 **E** 14

20 An adjacency matrix for the graph opposite could be:



A $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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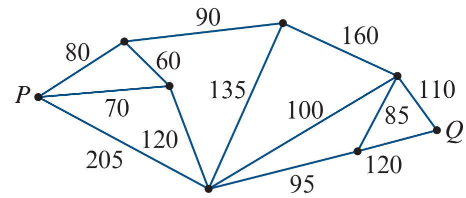
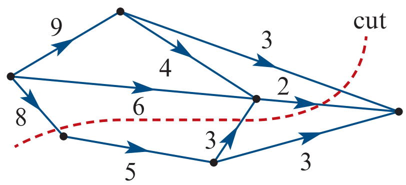
21 A vehicle is travelling from town P to town Q. The journey requires the vehicle to travel along a network linking suitable fuel stops. The cost, in dollars, of travel between these is shown on the network below, where the vertices represent fuel stops.



What is the minimum cost, in dollars, for the trip?

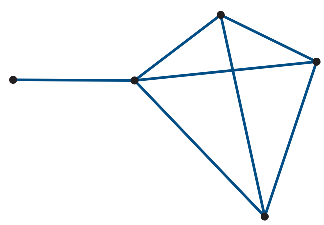
A 400 **B** 405 **C** 410 **D** 420 **E** 440

22 The capacity of the cut in the network flow diagram shown is:



A 0 **B** 2 **C** 10
D 13 **E** 16

23 The sum of the degrees of the vertices on the graph shown is:

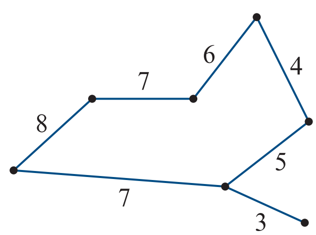


A 12 **B** 13 **C** 14
D 15 **E** 16

24 A connected planar graph divides the plane into a number of faces. If the graph has nine vertices and these are linked by 20 edges, then the number of faces is:

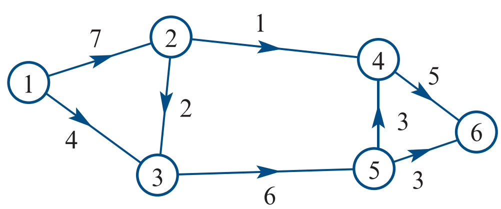
A 11 **B** 13 **C** 21 **D** 27 **E** 31

25 The sum of the weights of the minimum spanning tree of the weighted graph is:



A 2 **B** 30 **C** 32
D 33 **E** 35

The following graph relates to Questions 26 and 27.



26 The maximum flow in the network linking vertex 1 to vertex 6 is:

A 5 **B** 6 **C** 7 **D** 8 **E** 9

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27 The number of ways that vertex 6 can be reached from vertex 1 is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

28 An undirected connected graph has five vertices. Three of these vertices are of even degree and two of these vertices are of odd degree. One extra edge is added. It joins two of the existing vertices. In the resulting graph, it is *not* possible to have five vertices that are:

A all of even degree
B all of equal degree
C one of even degree and four of odd degree
D three of even degree and two of odd degree
E four of even degree and one of odd degree

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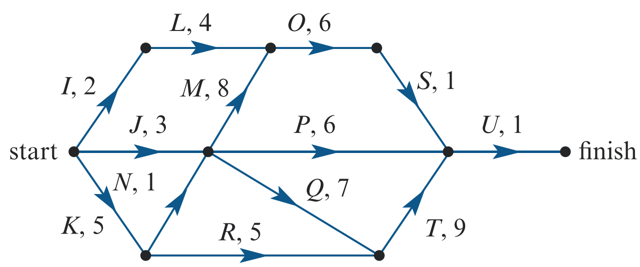
29 The following network gives the times in hours to complete the 12 tasks required to finish a project.



The critical path for this project is

A $J–P–U$ **B** $K–R–T–U$ **C** $J–M–O–S–U$
D $K–N–Q–T–U$ **E** $K–N–M–O–S–U$

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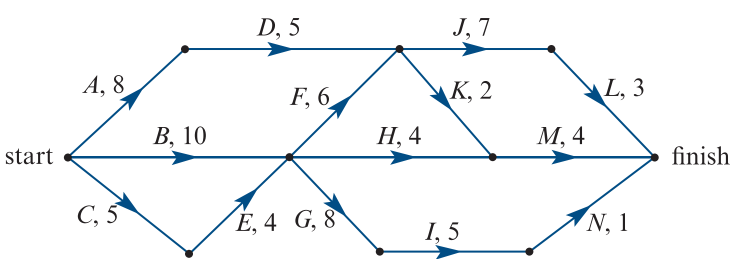
30 The activities and their completion times (days) needed to complete a project are shown in the digraph below.



For the network shown, the length of the critical path is

A 22 days. **B** 23 days. **C** 25 days. **D** 26 days. **E** 28 days.

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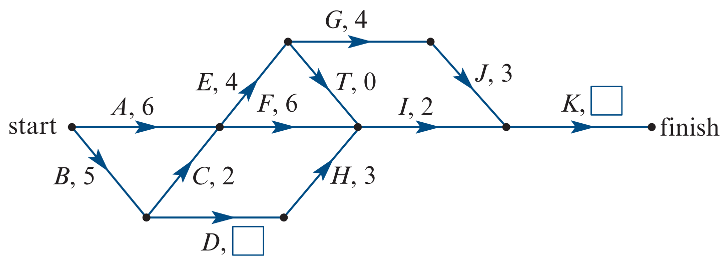
16B Extended-response questions

AS

1 All the activities and their durations (in hours) in a project at the quarry are shown in the network diagram below. The least time required for completing this entire project is 30 hours.



For each activity in this project, the table below shows the completion time, the earliest starting time and the latest starting time.

Activity	*Completion time (hours)*	*Earliest starting time (hours)*	*Latest starting time (hours)*
A	6	0	
B	5	0	0
C	2	5	5
D		5	9
E	4	7	7
F	6	7	
G	4	11	11
H	3	9	13
I	2	13	16
J	3	15	15
K		18	18

a Complete the missing times in the table.

b Write down the critical path for this project.

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2 A development project involves completing a number of activities as shown in the table. With each activity, there is the optimistic assessment of completion time (i.e. the shortest time likely to occur). Time is measured in days.

Activity	Optimistic time (days)	Predecessor(s)
A	4	–
B	2	–
C	1	A
D	6	B
E	5	B
F	7	B
G	5	C, D
H	1	E
I	2	F
J	10	G, H
K	6	I

The set of activities can be represented on a directed graph.

a Construct a graph for this project, labelling the activities on the edges with their associated shortest durations.

b Determine the earliest start time for each activity from your graph.

c How long is the estimated project time under this set of activity durations?

d Determine the latest start time for each activity from your graph.

e State the critical path.

f If the final activity, K, had to be delayed, how many days could this delay take before the project schedule was disrupted?

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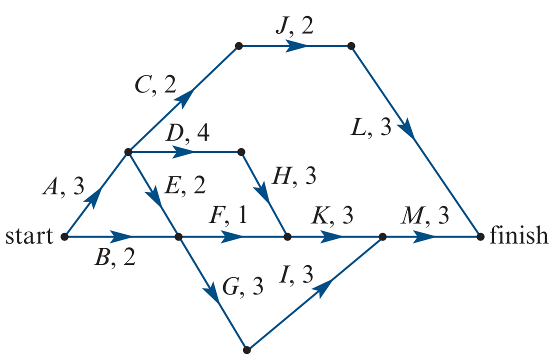
3 The Bowen Yard Buster team specialises in backyard improvement projects. The team has identified the activities required for a backyard improvement. The network diagram on the next page shows the activities identified and the actual times, in hours, needed to complete each activity, that is, the duration of each activity.



The table [overleaf] lists the activities, their immediate predecessor(s) and the earliest starting times (EST), in hours, of each of the activities. Activity X is not yet drawn on the network diagram.

a Use the information in the network diagram to complete the table.

b Draw and label activity X on the network diagram above, including its direction and duration.

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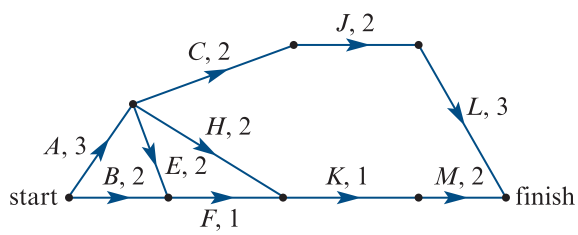
c The path $A–D–H–K–M$ is the only critical path in this project.

i Write down the duration of path $A–D–H–K–M$.

ii Explain the importance of the critical path in completing the project.

Activity	*Immediate predecessor(s)*	*EST*
A	–	0
B	–	0
C	A	3
D	A	3
E		3
F	B, E	5
G	B, E	5
H	D	7
I	G	
J	C, X	8
K	F, H	10
L	J	10
M	I, K	
X	D	7

d To save money, Bowen Yard Busters decide to revise the project and leave out activities D, G, I and X. This results in a reduction in the time needed to complete activities H, K and M as shown.



i For this revised project network, what is the earliest starting time for activity K?

ii Write down the critical path for this revised project network.

iii Without affecting the earliest completion time for this entire revised project, what is the latest starting time for activity M?

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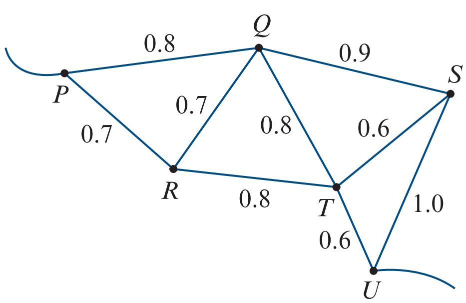
4 A rural town, built on hills, contains a set of roads represented by arcs in the network shown here. The numbers on the network refer to distances along the roads (in kilometres) and the letters refer to intersections of the roads. The arcs without endpoints refer to the two roads in and out of town.



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a **i** What is the length of the shortest route through the town from P *to* U?

ii A safety officer who enters the town at P needs to examine all intersections in the town before leaving from U to travel to the next town. To save time, she wants to pass through each intersection only once. State a path through the network of roads that would enable her to do this.

b A technician from the electricity company is checking the overhead cables along each street. The technician elects to follow an eulerian path through the network streets (ignoring the roads in and out of town) starting at R and finishing at S.

i Complete the following eulerian trail:

$R–Q–P–R–$ ☐ $–$ ☐ $–$ ☐ $–T–U–S$

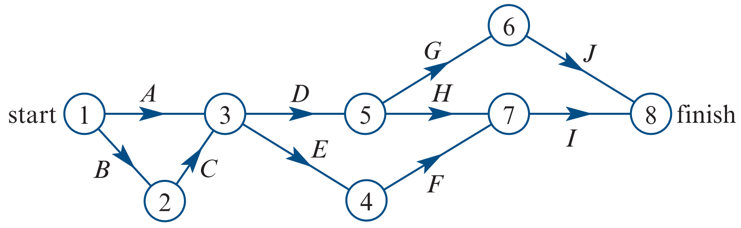
ii How would the technician benefit from choosing an eulerian path?

5 The local council plans to turn the main street of the town into a mall. The planning phase involves a number of activities whose normal completion times are supplied in Table 16.1. Also included in the table are the 'crash time' (possible time to which the activity time can be shortened) and the daily cost of this 'crashing'.

Table 16.1 Project completion times and costs

Activity	*Normal completion time (days)*	*Crash time (days)*	*Cost of crashing per day ($)*
A	10	8	400
B	5	5	–
C	3	2	400
D	5	4	600
E	4	4	–
F	6	5	500
G	6	4	200
H	7	5	300
I	5	5	–
J	4	3	400

The network for this project is shown below.



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Revision

a Using normal completion times as given in Table 16.1, determine the times missing from Table 16.2.

Table 16.2 Normal times for job starting

Activity	*Earliest start time (day)*	*Latest start time (day)*
A	0	0
B	0	2
C	5	7
D	10	10
E	10	12
F		16
G	15	
H	15	15
I	22	22
J	21	23

b **i** State the critical path in this network.

ii Determine the length of the critical path.

c **i** Complete Table 16.3, taking into account that some of the activities can be crashed, as shown in Table 1, to reduce the total completion time of the project.

ii Determine the shortest time in which the project can now be finished.

iii Apart from A, what three other activities must be shortened so the project is completed in minimum time?

iv What is the cost of achieving this time reduction for the whole project?

Table 16.3 Reduced times for job starting using crash data

Activity	*Earliest start time (days)*	*Latest start time (days)*
A	0	0
B	0	
C	5	6
D	8	8
E	8	8
F	12	12
G	12	15
H	12	12
I	17	17
J		

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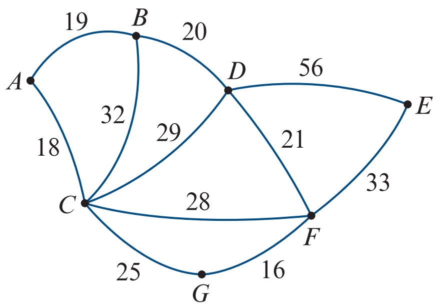
6 Seven towns on an island have been surveyed for transport and communications needs. The towns (labelled A, B, C, D, E, F, G) form the network shown here. The road distances between the towns are marked in kilometres. (All towns may be treated as points being of no size compared to the network lengths.)



a Explain what is meant by the description of the graph as 'planar'.

b The roads between the towns define boundaries used by the local authority to establish rural planning subregions. (That is, the section bounded by roads AB, AC and BC would be one subregion. These subregions are non-overlapping.) Treating the subregions as faces of the graph (with the exterior of the network as one subregion), the roads as edges and towns as vertices, show that Euler's rule linking the number of vertices, edges and faces in a planar graph, i.e. number of vertices + number of faces = number of edges + 2, is satisfied.

An inspector of roads is stationed at B. Starting from B, she must travel the complete network of roads to examine them.

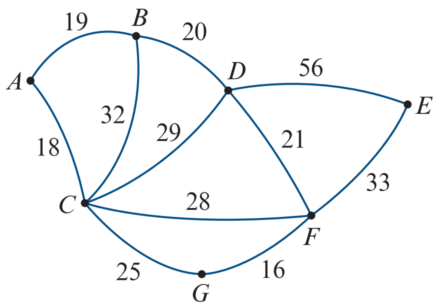
c If she wishes to travel the least distance where will she end up in the network?

d What will that distance be?

e Is the route unique? Briefly justify your answer.

f Determine the shortest distance that a fire truck stationed at E must travel to assist at an emergency at A.

g To establish a cable network for telecommunications on the island, it is proposed to put the cable underground beside the existing roads. What is the minimal length of cable required here if back-up links are not considered necessary; that is, there are no loops in the cable network?



The Island Bank has outlets in each of the towns. The regional assistant manager stationed at C must visit each outlet every second Friday and then return to the office at C.

h Treating the towns as vertices and roads as edges in a graph, what is the distance of a journey that forms a hamiltonian cycle in the graph?

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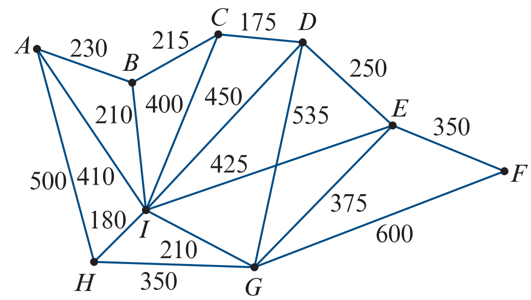
7 A company is building the new Bigtown University. The company has constructed nine new faculty buildings in a layout as shown. The minimum distances in metres between adjacent buildings in the university are also shown.



A computer network is to be built to serve the whole university.

a Draw a network that will ensure that all the buildings are connected to the network but that also minimises the amount of cable used. Label each node in the network.

b What is the minimum length of cable required?

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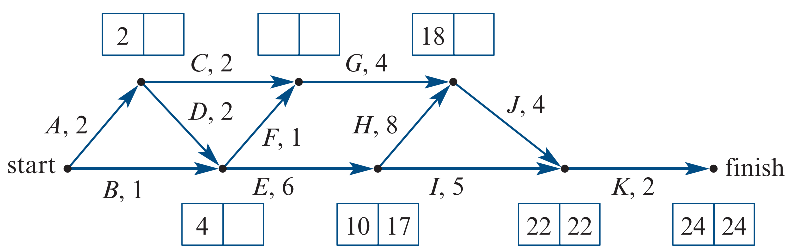
8 The assembly of machined parts in a manufacturing process can be represented by the following network. The activities are represented by the letters on the arcs and the numbers represent the time taken (in hours) for the activities scheduled.



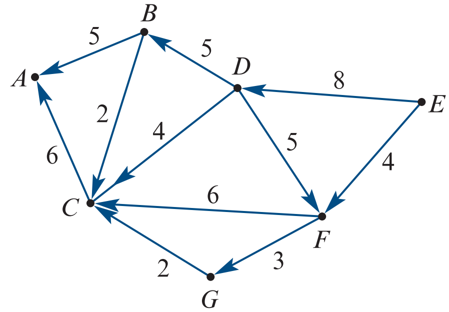
Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
EST	0	0	2	2	4	4		10	10	18	22

a The earliest start times (EST) for each activity except G are given in the table. Complete the table by finding the EST for G.

b What is the shortest time required to assemble the product?

c What is the float (slack time) for activity I?

9 A reservoir at E pumps water through pipes along the network routes shown. The capacities of the flow are given in the digraph shown here in megalitres per day. Occasionally, there are fire emergencies in the forest beside A and additional flow of water is used. What is the maximum flow that can reach A from E?



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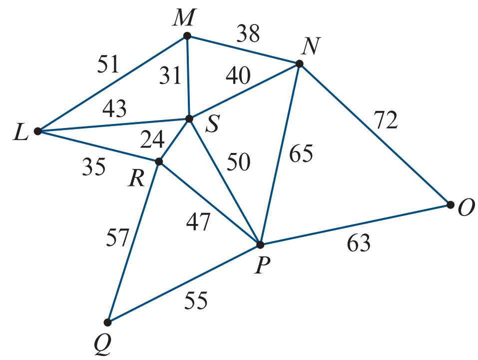
10 A number of towns need to be linked by pipelines to a natural gas supply. In the network shown, the existing road links between towns L, M, N, O, P, Q and R and to the supply point, S, are shown as edges. The towns and the gas supply are shown as vertices. The distances along roads are given in kilometres.



a What is the shortest distance along roads from the gas supply point S to the town O?

b The gas company decides to run the gas lines along the existing roads. To ensure that all nodes on the network are linked, the company does not need to place pipes along all the roads in the network.

i What is the usual name given to the network within a graph (here, the road system) which links all nodes (towns and supply) and which gives the shortest total length?

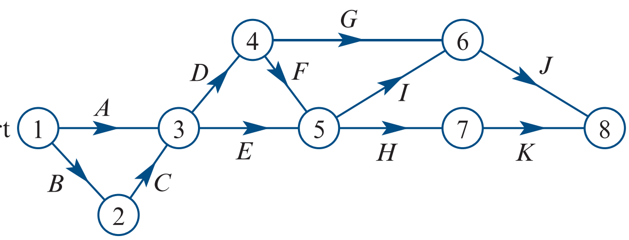
ii Sketch this network.

iii What is the minimum length of gas pipeline the company can use to supply all the towns by running the pipes along the existing roads?

c The gas company decides it wants to run the pipeline directly to any town which is linked by road to its supply at S. Towns not directly connected to S by road will be linked via other towns in the network.

What is the minimum length of pipeline that will enable all towns to be connected to the gas supply under these circumstances?

11 In laying a pipeline, the various jobs involved have been grouped into a set of specific tasks A–K, which are performed in the precedence described in the network below.



a List all the task(s) that must be completed before task E is started.

The durations of the tasks are given in Table 16.7.

b Use the information in Table 16.7 to complete Table 16.8.

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Table 16.7 Task durations

Task	*Normal completion time (months)*
A	10
B	6
C	3
D	4
E	7
F	4
G	5
H	4
I	5
J	4
K	3

Table 16.8 Starting times for tasks

Task	*EST*	*LST*
A	0	0
B	0	
C	6	7
D	10	10
E		11
F	14	14
G	14	18
H	18	20
I	18	
J	23	23
K	22	24

c For this project:

i write down the critical path

ii determine the length of the critical path (that is, the earliest time the project can be completed).

d If the project managers are prepared to pay more for additional labour and machinery, the time taken to complete task A can be reduced to 8 months, task E can be reduced to 5 months and task I can be reduced to 4 months.

Under these circumstances:

i what would be the critical path(s)?

ii how long would it take to complete the project?

12 The pipeline construction team needs tractors at four different worksites. Four tractors are available but these are in four different locations. The cost (in dollars) of providing a tractor at each of the sites from each of the locations is given in Table [16.9].

Table [16.9] Cost of providing tractors (in dollars)

	Tractor based at			
Assigned to	*Location 1*	*Location 2*	*Location 3*	*Location 4*
Site 1	1130	830	2010	1140
Site 2	1020	1100	690	850
Site 3	2010	1320	1150	1410
Site 4	960	1210	2100	1530

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a Use the Hungarian algorithm, or otherwise, to complete the following table. From each location show where the tractors should be sent to minimise the total cost of providing tractors to the pipeline construction team.

Tractor at	*Assign to*
Location 1	Site 4
Location 2	
Location 3	
Location 4	

b What is the minimum cost of providing tractors to the pipeline construction team?

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13 Camp sites A, B, C and D are to be supplied with food. Four local residents, W, X, Y and Z, offer to supply one campsite each. The cost in dollars of supplying one load of food from each resident to each campsite is tabulated.

Camp site	W	X	Y	Z
A	30	70	60	20
B	40	30	50	80
C	50	40	60	50
D	60	70	30	70

a Find the two possible matchings between campsites and residents so that the total cost is a minimum.

b State this minimum cost.

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Chapter 17

17

Module 3: Geometry and measurement

Geometry and measurement

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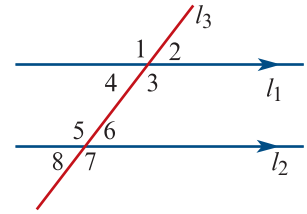
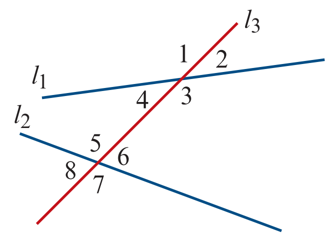
17A Properties of parallel lines: a review



- Angles 4 and 6 are called **alternate angles**.
- Angles 5 and 3 are called *alternate angles*.
- Angles 2 and 6 are called **corresponding angles**.
- Angles 1 and 5 are called *corresponding angles*.
- Angles 4 and 8 are called *corresponding angles*.
- Angles 3 and 7 are called *corresponding angles*.
- Angles 3 and 6 are called **co-interior angles**.
- Angles 4 and 5 are called *co-interior angles*.
- Angles 1 and 3 are called **vertically opposite angles** and are of equal magnitude.
- Other pairs of vertically opposite angles are 2 and 4, 5 and 7, and 6 and 8.
- Angles 1 and 2 are **supplementary**; that is, their magnitudes add to 180°.

Lines l_1 and l_2 are cut by a transversal l_3.



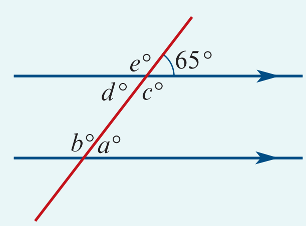
When lines l_1 and l_2 are parallel, *corresponding angles* are of equal magnitude, *alternate angles* are of equal magnitude and *co-interior angles* are *supplementary*.

Converse results also hold:

- If corresponding angles are equal, then l_1 is parallel to l_2.
- If alternate angles are equal, then l_1 is parallel to l_2.
- If co-interior angles are supplementary, then l_1 is parallel to l_2.

Example 1 **Parallel line properties**

Find the values of the pronumerals.



Solution

There are lots of ways of finding these values. One sequence of reasoning has been used here.

$a = 65$ (corresponding)
$d = 65$ (alternate with a)
$b = 115$ (co-interior with d)
$e = 115$ (corresponding with b)
$c = 115$ (vertically opposite e)

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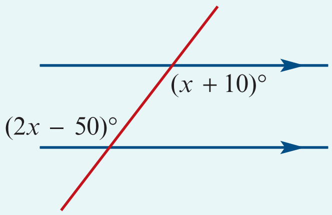
Example 2 Parallel line properties

Find the values of the pronumerals.

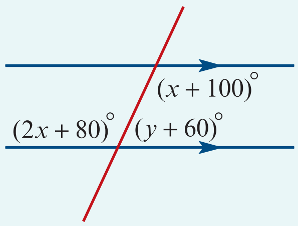


Solution

$$2x - 50 = x + 10 \quad \text{(alternate angles)}$$
$$\therefore 2x - x = 50 + 10$$
$$\therefore x = 60$$

Example 3 Parallel line properties

Find the values of the pronumerals.

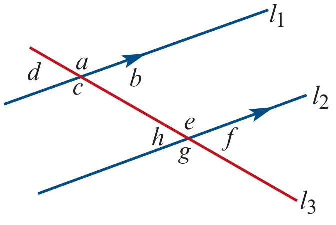


Solution

$$x + 100 = 2x + 80 \quad \text{(alternate angles)}$$
$$\therefore 100 - 80 = 2x - x$$
$$\therefore x = 20$$
Also $x + 100 + y + 60 = 180$ (co-interior)
and $x = 20$
$$\therefore y + 180 = 180 \text{ or } y = 0$$

Exercise 17A

Questions 1 to 5 apply to the following diagram.



1 Angles d and b are:

A alternate **B** co-interior **C** corresponding
D supplementary **E** vertically opposite

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2 Angles d and a are:

A alternate **B** co-interior **C** corresponding
D supplementary **E** vertically opposite

3 Angles c and h are:

A alternate **B** co-interior **C** corresponding
D supplementary **E** vertically opposite

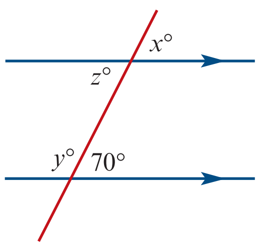
4 Angles b and f are:

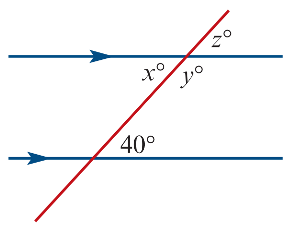
A alternate **B** co-interior **C** corresponding
D supplementary **E** vertically opposite

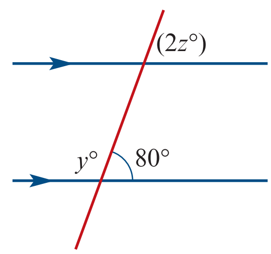
5 Angles c and e are:

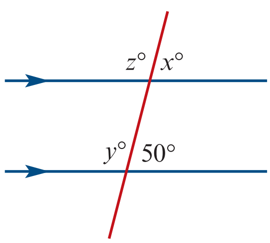
A alternate **B** co-interior **C** corresponding
D supplementary **E** vertically opposite

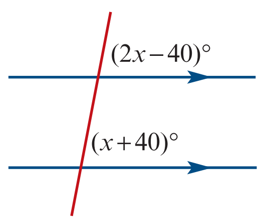
6 Find the values of the pronumerals in each of the following.

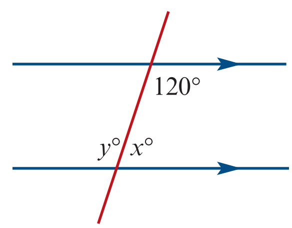
a


b


c


d


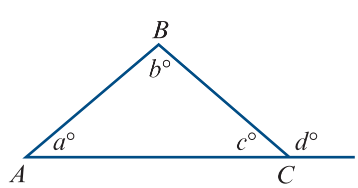
e


f


17B Properties of triangles: a review

$a°$, $b°$ and $c°$ are the magnitudes of the **interior angles** of the triangle ABC and $d°$ is the magnitude of an **exterior angle** at C.

- The *sum* of the magnitudes of the *interior angles* of a triangle is equal to *180°*: $a° + b° + c° = 180°$.
- $b° + a° = d°$. The magnitude of an *exterior angle* is equal to the *sum* of the magnitudes of the *two opposite interior angles*.



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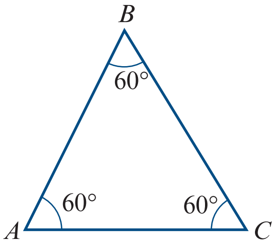
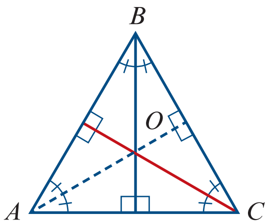
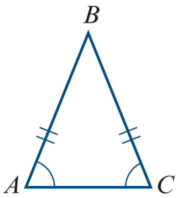
- A triangle is said to be *equilateral* if all its sides are of the same length: $AB = BC = CA$.
- The *angles* of an **equilateral triangle** are all of magnitude *60°*.



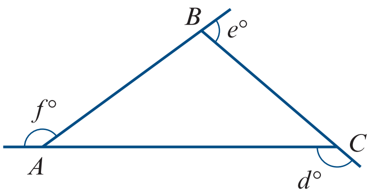
- The *bisector* of each of the angles of an *equilateral triangle* meets the opposite side at *right angles* and passes through the *midpoint* of that side.



- A triangle is said to be **isosceles** if it has two sides of equal length. If a triangle is isosceles, the angles opposite each of the equal sides are equal.



- The sum of the magnitudes of the exterior angles of a triangle is equal to 360°:
  $e° + d° + f° = 360°$
- A triangle is said to be a *right-angled triangle* if it has one angle of magnitude *90°*.



Example 4 Angle sum of a triangle

Find the values of the pronumerals.

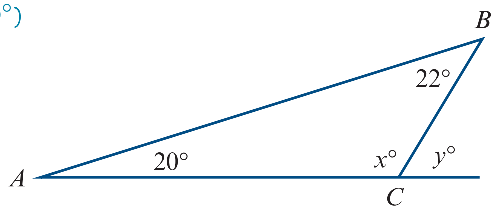
Solution

$20° + 22° + x° = 180°$ (sum angles $\Delta = 180°$)

$\therefore 42° + x° = 180°$ or $x = 138$

$138° + y° = 180°$ (sum angles = 180°)

$\therefore y = 42$



Or:

To find x: two of the angles sum to 42° and therefore the third angle is 138°.

To find y: the two angles sum to 180°. Therefore the second is 42°.

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Example 5 Angle sum of an isosceles triangle

Find the values of the pronumerals.

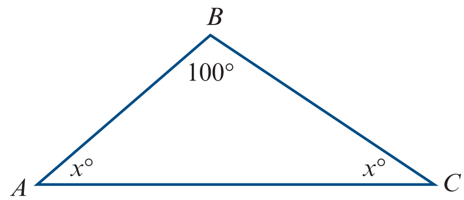
Solution

$100^\circ + 2x^\circ = 180^\circ$ (sum angles $\Delta = 180^\circ$)

$\therefore 2x^\circ = 80^\circ$ or $x = 40$

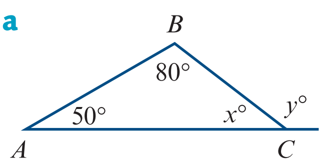
Or:

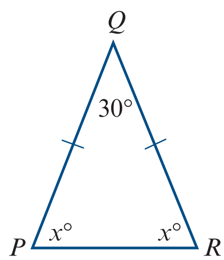
Observe the two unknown numbers are the same and must sum to 80°, so each of them has size 40°.

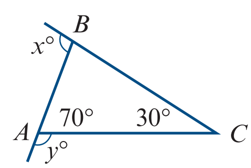


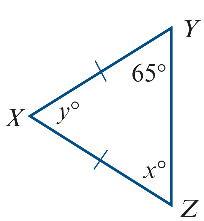
Exercise 17B

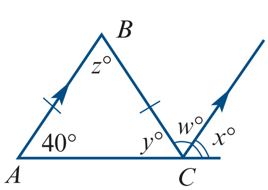
1 Find the values of the pronumerals in each of the following.

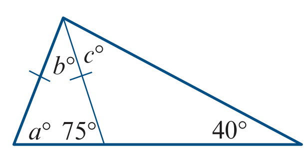
a


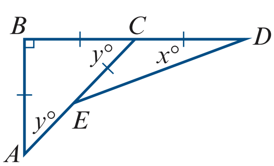
b


c


d


e


f


g


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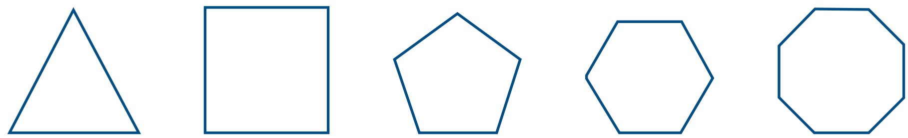
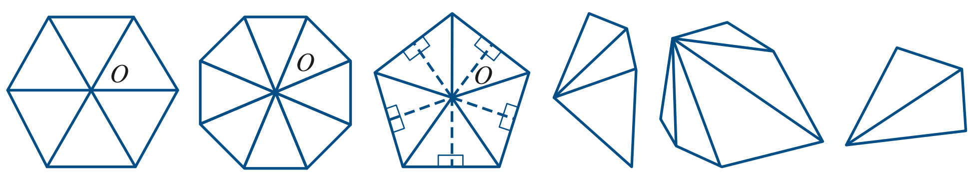
17C Properties of regular polygons: a review



- A **regular polygon** has *all sides of equal length* and *all angles of equal magnitude*.
- A **polygon** with *n sides* can be divided into *n triangles*. The first three polygons below are regular polygons.



- The angle **sum of the interior angles** of an n-sided **convex polygon** is given by the formula:

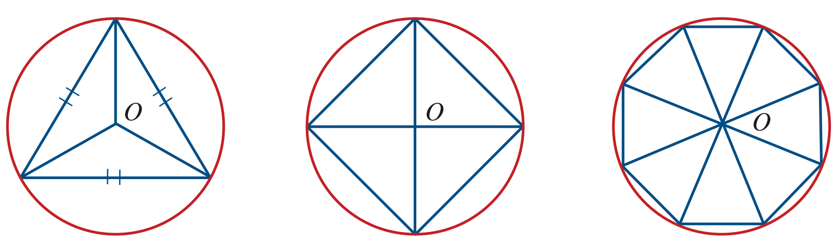
$$S = [180(n-2)]^\circ = (180n - 360)^\circ$$

The result holds for any convex polygon. *Convex* means that a line you draw from any vertex to another vertex lies inside the polygon.

- The magnitude of each of the *interior angles* of an n-sided polygon is given by:

$$x = \frac{(180n - 360)^\circ}{n}$$

- The *angle bisectors* of a *regular polygon* meet at a point O.
- For a regular polygon, a circle can be drawn with centre O on which all the vertices lie.



The *sum* of the *angles* at O of a *regular polygon* is *360*°.

- The *sum* of the *exterior angles* of a *regular polygon* is *360*°.

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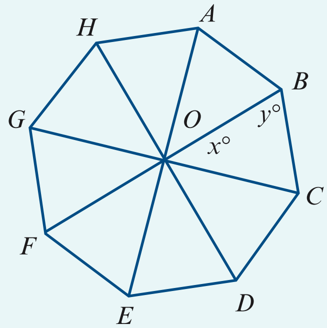
Example 6 Angle properties of an octagon

The diagram opposite shows a regular octagon.

a Show that $x = 45$.

b Find the size of angle y.



Solution

a $8x^\circ = 360^\circ$ (sum of angles at $O = 360^\circ$)

$\therefore x^\circ = \dfrac{360^\circ}{8} = 45^\circ$

$\therefore x = 45$

b $y^\circ + y^\circ + 45^\circ = 180^\circ$ (ΔOBC isosceles)

$\therefore 2y^\circ = 135$ or $y = 67.5$

Example 7 Angle sum of an octagon

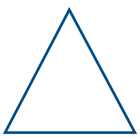
Find the sum of the interior angles of an eight-sided convex polygon (octagon).

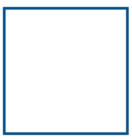
Solution

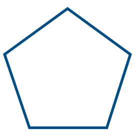
1 Use the rule for the sum of the interior angles of an n-sided polygon.	$S = (180n - 360)^\circ$
2 In this example, $n = 8$. Substitute and evaluate.	$n = 8$ $\therefore S = (180 \times 8 - 360)^\circ = 1080^\circ$
3 Write down your answer.	The sum of the interior angles is 1080°.

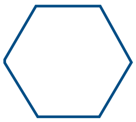
Exercise 17C

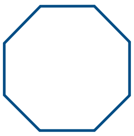
1 Name each of the following regular polygons.

a 

b 

c 

d 

e 

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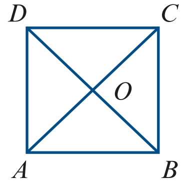
2 $ABCD$ is a square. BD and AC are diagonals that meet at O.

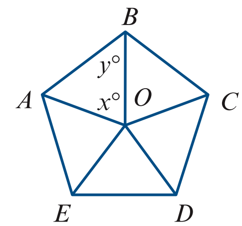


a Find the size of each of the angles at O.

b What type of triangle is:

i BAD? **ii** AOB?

3 $ABCDE$ is a regular pentagon.

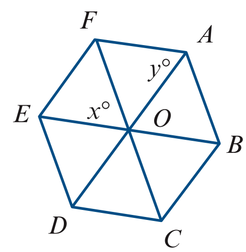


a Find the value of:

i x **ii** y

b Find the sum of the interior angles of the regular pentagon $ABCDE$.

4 $ABCDEF$ is a regular hexagon.
Find the value of:

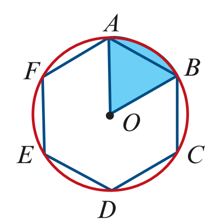


a x **b** y

5 State the sum of the interior angles of:
a a seven-sided regular polygon **b** a hexagon **c** an octagon.

6 The angle sum of a regular polygon is 1260°. How many sides does the polygon have?

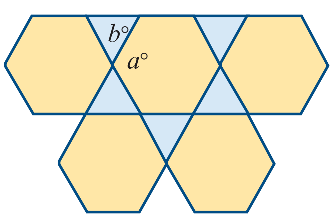
7 A circle is circumscribed about a hexagon $ABCDEF$.



a Find the area of the circle if $OA = 2$ cm.

b Find the area of the shaded region.

8 The diagram shows a tessellation of regular hexagons and equilateral triangles.
State the values of a and b and use these to explain the existence of the tessellation.



9 If the magnitude of each angle of a regular polygon is 135°, how many sides does the polygon have?

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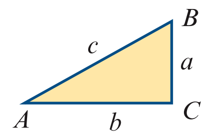
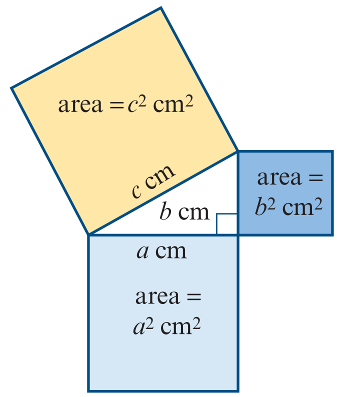
17D Pythagoras' theorem

Pythagoras' theorem

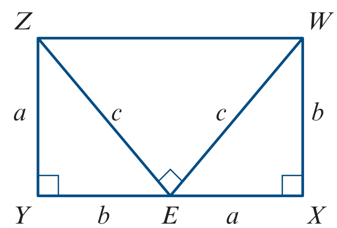
Pythagoras' theorem states that for a right-angled triangle ABC with side lengths a, b and c, as shown in the diagram, $a^2 + b^2 = c^2$.



Pythagoras' theorem can be illustrated by the diagram shown here. The sum of the areas of the two smaller squares is equal to the area of the square on the longest side (hypotenuse).



There are many different proofs of Pythagoras' theorem. A proof from the 20th President of the United States, James A. Garfield, is produced using the following diagram.



$$\text{Area of trapezium } WXYZ = \frac{1}{2}(a+b)(a+b)$$

$$\text{Area of } \Delta EYZ + \text{area of } \Delta EWX + \text{area of } \Delta EWZ$$

$$= \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab$$

$$= ab + \frac{1}{2}c^2$$

$$\therefore \frac{a^2}{2} + ab + \frac{b^2}{2} = ab + \frac{1}{2}c^2$$

$$\therefore a^2 + b^2 = c^2$$

Pythagorean triads

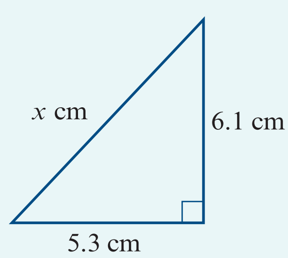
A triple of natural numbers (a, b, c) is called a Pythagorean triad if $c^2 = a^2 + b^2$.

The table presents the first six such 'primitive' triples. The adjective 'primitive' indicates that the highest common factor of the three numbers is 1.

a	3	5	7	8	9	11
b	4	12	24	15	40	60
c	5	13	25	17	41	61

Example 8 Pythagoras' theorem

Find the value, correct to two decimal places, of the unknown length for the triangle opposite.



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Solution

1 Using Pythagoras' theorem, write down an expression for x in terms of the two other sides of the right-angled triangle. Solve for x.

$x^2 = 5.3^2 + 6.1^2$ (Pythagoras)

$\therefore x = \sqrt{5.3^2 + 6.1^2} = 8.080\ldots$

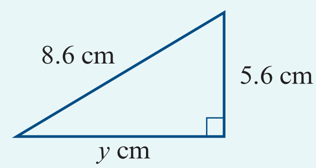
$\sqrt{(5.3)^2 + (6.1)^2}$ 8.08084

2 Write down your answer for the length, correct to two decimal places.

The length is 8.08 cm, correct to two decimal places.

Example 9 Pythagoras' theorem

Find the value, correct to two decimal places, of the unknown length for the triangle opposite.



Solution

1 Using Pythagoras' theorem, write down an expression for y in terms of the two other sides of the right-angled triangle. Solve for y.

$5.6^2 + y^2 = 8.6^2$ (Pythagoras)

$\therefore y^2 = 8.6^2 - 5.6^2$

$\therefore y = \sqrt{8.6^2 - 5.6^2} = 6.526\ldots$

2 Write down your answer for the length, correct to two decimal places.

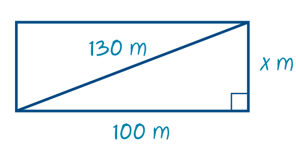
The length is 6.53 cm, correct to two decimal places.

Example 10 Pythagoras' theorem

The diagonal of a soccer ground is 130 m and the long side of the ground measures 100 m. Find the length of the short side, correct to the nearest cm.

Solution

1 Draw a diagram. Let x be the length of the short side.



2 Using Pythagoras' theorem, write down an expression for x in terms of the two other sides of the right-angled triangle. Solve for x.

$x^2 + 100^2 = 130^2$ (Pythagoras)

$\therefore x^2 = 130^2 - 100^2 = 6900$

$\therefore x = \sqrt{130^2 - 100^2} = 83.066\ldots$

3 Write down your answer, correct to the nearest cm.

Correct to the nearest centimetre, the length of the short side is 83.07 m.

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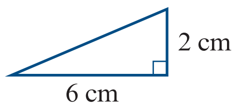
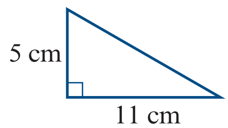
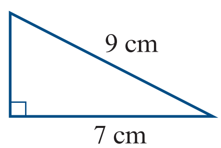
Exercise 17D

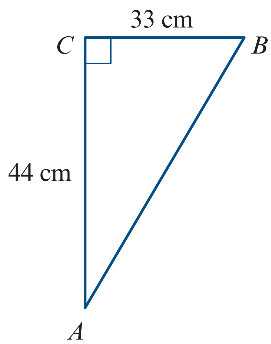
1 Find the length of the 'unknown' side for each of the following.

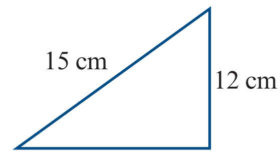
a 

b 

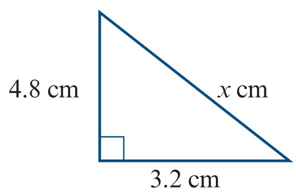
c 

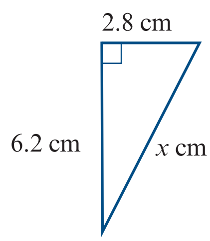
d 

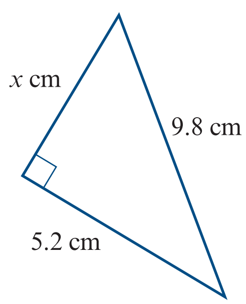
e 

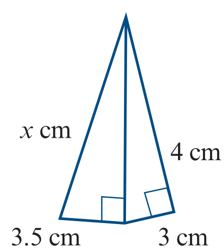
f 

2 In each of the following find the value of x, correct to two decimal places.

a 

b 

c 

d 

3 Find the value of x for each of the following ($x > 0$). Give your answers correct to two decimal places.

a $x^2 = 6^2 + 4^2$ **b** $52 + x^2 = 9^2$ **c** $4.62 + 6.1^2 = x^2$

4 In triangle VWX, there is a right angle at X. $VX = 2.4$ cm and $XW = 4.6$ cm. Find VW.

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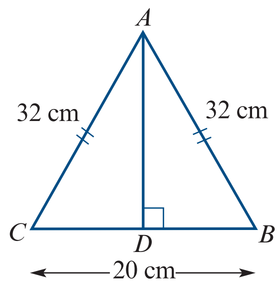
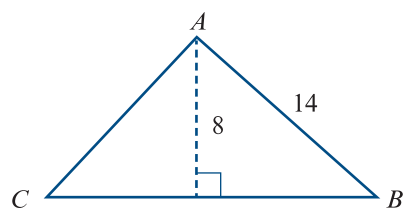
5 Find AD, the height of the triangle.



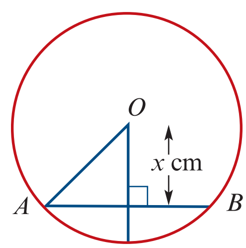
6 An 18 m ladder is 7 m away from the bottom of a vertical wall. How far up the wall does it reach?

7 Find the length of the diagonal of a rectangle with dimensions 40 m × 9 m.

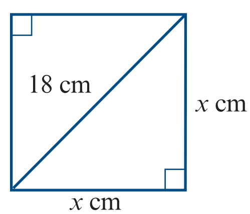
8 Triangle ABC is isosceles. Find the length of CB.



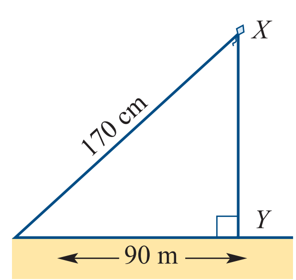
9 In a circle of centre O, a chord AB is of length 4 cm. The radius of the circle is 14 cm. Find the distance of the chord from O.



10 Find the value of x.



11 How high is the kite above the ground?



12 A square has an area of 169 cm^2. What is the length of the diagonal?

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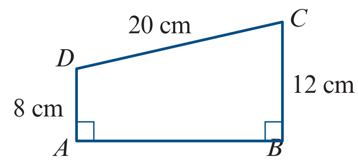
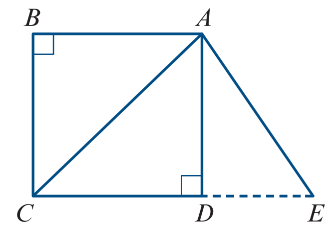
13 Find the area of a square with a diagonal of length:

a $8\sqrt{2}$ cm **b** 8 cm.

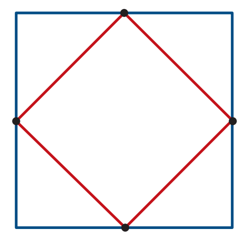
14 Find the length of AB.



15 $ABCD$ is a square of side length 2 cm. If $CA = CE$, find the length of DE.



16 The midpoints of a square of side length 2 cm are joined to form a new square. Find the area of the new square.



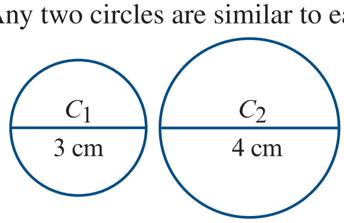
17E Similar figures

Similarity

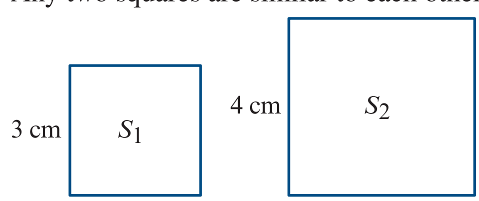
In this section we informally define two objects to be **similar figures** if they have the same shape but not the same size.

Examples

Any two circles are similar to each other.



Any two squares are similar to each other.



It is not true that any rectangle is similar to any other rectangle. For example, rectangle 1 (R_1) is not similar to rectangle 2 (R_2).

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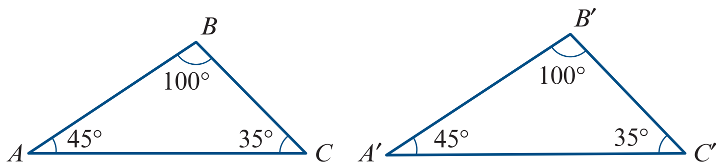
A rectangle similar to R_1 is:

So, for two rectangles to be similar, their corresponding sides must be in the same ratio $\left(\frac{8}{2} = \frac{4}{1}\right)$.

Similar triangles

Two triangles are similar if one of the following conditions holds:

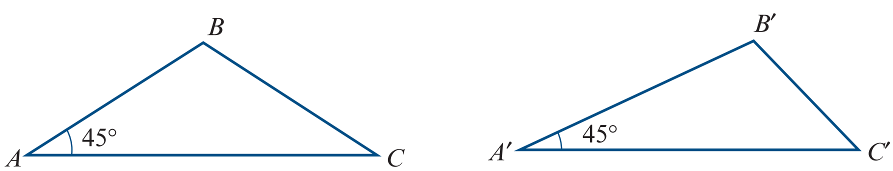
- corresponding angles in the triangles are equal



- corresponding sides are in the same ratio

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = k$$

  k is the scale factor.
- two pairs of corresponding sides have the same ratio and the included angles are equal.



$$\frac{AB}{A'B'} = \frac{AC}{A'C'}$$

If triangle ABC is similar to triangle XYZ this can be written symbolically as $\Delta ABC \approx \Delta XYZ$. The triangles are named so that angles of equal magnitude hold the same position; that is, A corresponds to X, B corresponds to Y, C corresponds to Z.

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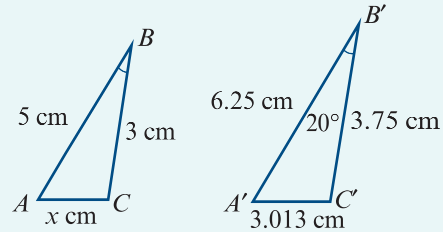
Example 11 Similar triangles

Find the value of length of side AC in ΔABC, correct to two decimal places.



Solution

1 Triangle ABC is similar to triangle $A'B'C'$: two pairs of corresponding sides have the same ratio $\left(\frac{5}{3} = \frac{6.25}{3.75}\right)$ and included angles (20°).

Triangles similar

2 For similar triangles, the ratios of corresponding sides are equal; for example, $\frac{AC}{A'C'} = \frac{AB}{A'B'}$. Use this fact to write down an expression involving x. Solve for x.

$$\therefore \frac{x}{3.013} = \frac{5}{6.25}$$
$$\therefore x = \frac{5}{6.25} \times 3.013$$
$$= 2.4104$$

3 Write down your answer, correct to two decimal places.

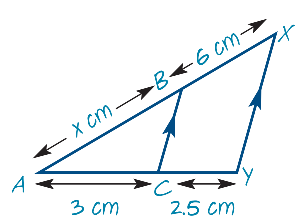
The length of side AC is 2.41 cm, correct to two decimal places.

Example 12 Similar triangles

Find the value of length of side AB in ΔABC.

Solution

1 Triangle ABC is similar to triangle AXY (corresponding angles are equal).



Triangles similar

2 For similar triangles, the ratios of corresponding sides are equal (for example, $\frac{AB}{AX} = \frac{AC}{AY}$). Use this fact to write down an expression involving x. Solve for x. Note that if $AB = x$, then $AX = x + 6$.

$$\therefore \frac{x}{x+6} = \frac{3}{5.5}$$
$$\therefore 5.5x = 3(x + 6)$$
$$= 3x + 18$$
$$\therefore 2.5x = 18 \text{ or } x = \frac{18}{2.5} = 7.2$$

3 Write down your answer.

The length of side AB is 7.2 cm.

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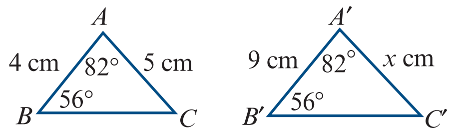
Exercise 17E

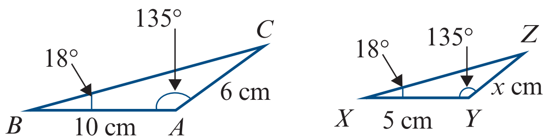
Skillsheet

1 Find the value of x for each of the following pairs of similar triangles.

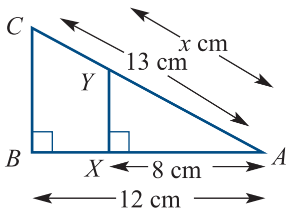
a



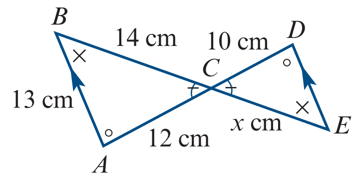
b



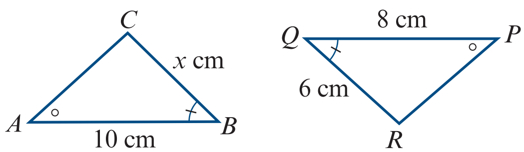
c



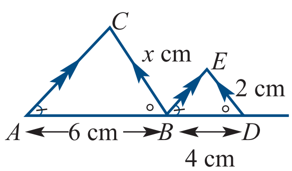
d



e



f



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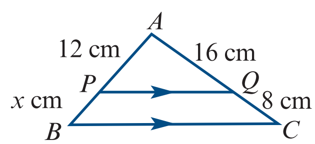

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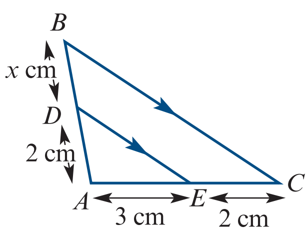
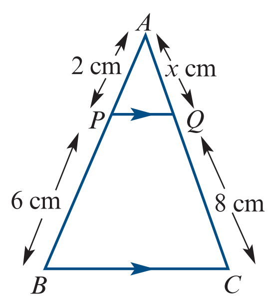
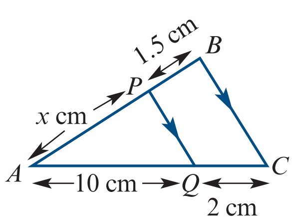


i



j



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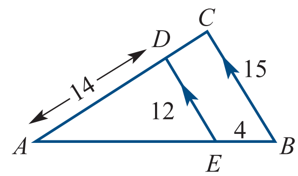
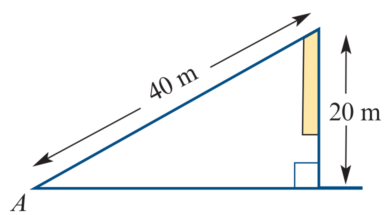
2 Given that $AD = 14$, $ED = 12$, $BC = 15$ and $EB = 4$, find AC, AE and AB.



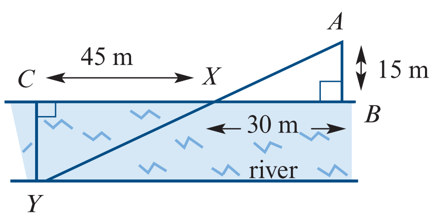
3 A tree casts a shadow of 33 m and at the same time a stick 30 cm long casts a shadow 24 cm long. How high is the tree?

4 A 20-metre high neon sign is supported by a 40 m steel cable as shown. An ant crawls along the cable starting at A. How high is the ant when it is 15 m from A?



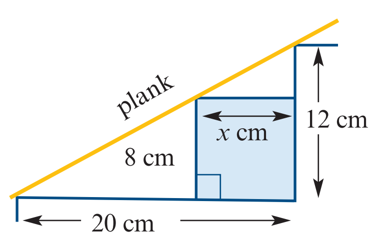
5 A hill has a gradient of 1 in 20; that is, for every 20 m horizontally there is a 1 m increase in height. If you go 300 m horizontally, how high up will you be?

6 A man stands at A and looks at point Y across the river. He gets a friend to place a stone at X so that A, X and Y are collinear. He then measures AB, BX and XC to be 15 m, 30 m and 45 m, respectively. Find CY, the distance across the river.



7 Find the height, h m, of a tree that casts a shadow 32 m long at the same time that a vertical straight stick 2 m long casts a shadow 6.2 m long.

8 A plank is placed straight up stairs that are 20 cm wide and 12 cm deep. Find x, where x cm is the width of the widest rectangular box of height 8 cm that can be placed on a stair under the plank.



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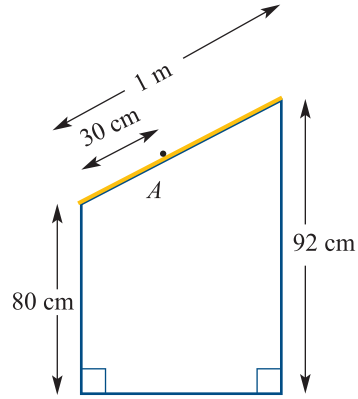
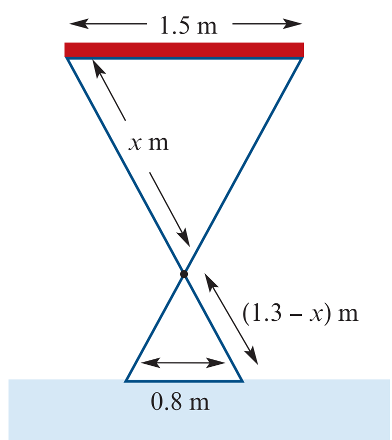
9 The sloping edge of a technical drawing table is 1 m from front to back. Calculate the height above the ground of a point A, which is 30 cm from the front edge.



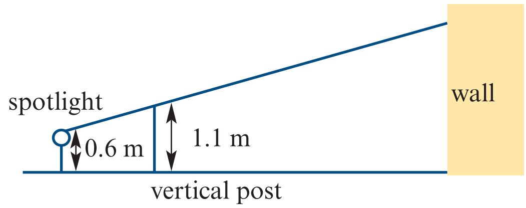
10 Two similar rods 1.3 m long have to be hinged together to support a table 1.5 m wide. The rods have been fixed to the floor 0.8 m apart. Find the position of the hinge by finding the value of x.



11 A man whose eyes are 1.7 m from the ground when standing 3.5 m in front of a wall 3 m high can just see the top of a tower that is 100 m away from the wall. Find the height of the tower.

12 A man is 8 m up a 10 m ladder, the top of which leans against a vertical wall and touches it at a height of 9 m above the ground. Find the height of the man above the ground.

13 A spotlight is at a height of 0.6 m above ground level. A vertical post 1.1 m high stands 3 m away, and 5 m further away there is a vertical wall. How high up the wall does the shadow reach?



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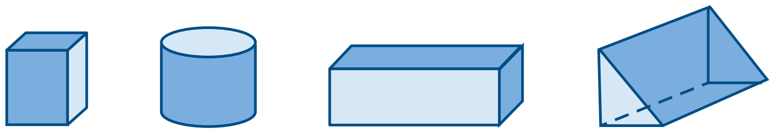
17F Volumes and surface areas

▶ Volume of a prism

A prism is a solid that has a constant cross-section. Examples are cubes, cylinders, rectangular prisms and triangular prisms.



The volume of a prism can be found by using its cross-sectional area.

$$\text{volume} = \text{area of cross} - \text{section} \times \text{height (or length)}$$
$$V = A \times h$$

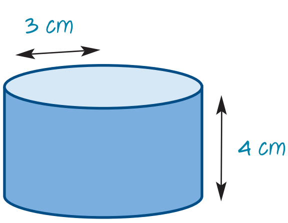
Answers will be in cubic units; that is, mm^3, cm^3, m^3, etc.

Example 13 Volume of a cylinder

Find the volume of this cylinder, which has radius 3 cm and height 4 cm, correct to two decimal places.

Solution

1 Find the cross-sectional area of the prism.



Area of cross-section $= \pi r^2 = \pi \times 3^2$
$= 28.27 \text{ cm}^2$

2 Multiply by the height.

3 Make sure that accuracy is given to the correct number of decimal places.

Volume $= 28.27 \times 4$
$= 113.10 \text{ cm}^3$ (correct to two decimal places)

The formulas for determining the volumes (V) of some 'standard' prisms are given overleaf.

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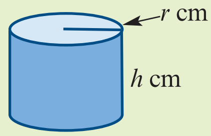
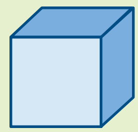
Volume of solids

Cylinder
(radius r cm, height h cm)



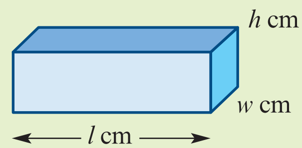
$V = \pi r^2 h$

Cube
(all edges x cm)



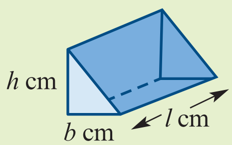
$V = x^3$

Rectangular prism
(length l cm, width w cm, height h cm)



$V = lwh$

Triangular prism



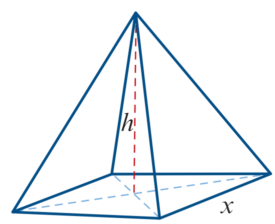
The triangular prism shown has a right-angled triangle base but the following formula holds for all triangular prisms.

$V = \frac{1}{2}bhl$

▶ Volume of a pyramid

The formula for finding the volume of a right pyramid can be stated as:

$$\text{volume of pyramid} = \frac{1}{3} \times \text{base area} \times \text{perpendicular height}$$



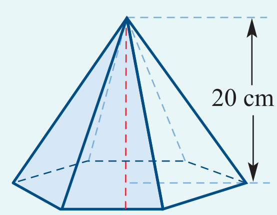
For the square pyramid shown:

$$V = \frac{1}{3}x^2 h$$

The term 'right' in this context means that the apex of the pyramid is directly over the centre of the base.

Example 14 Volume of a pyramid

Find the volume of this hexagonal pyramid with a base area of 40 cm^2 and a height of 20 cm. Give the answer correct to one decimal place.



Solution

$V = \frac{1}{3} \times A \times h$

$= \frac{1}{3} \times 40 \times 20$

$= 266.7$ cm^3 (correct to one decimal place)

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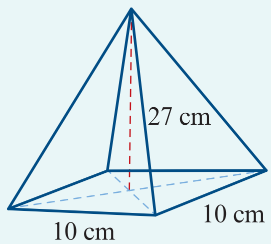
Example 15 **Volume of a pyramid**

Find the volume of this square pyramid with a square base with each edge 10 cm and a height of 27 cm.



Solution

$$V = \frac{1}{3}x^2h$$
$$= \frac{1}{3} \times 10 \times 10 \times 27$$
$$= 900 \text{ cm}^3$$

Volume of a cone

The formula for finding the volume of a cone can be stated as:

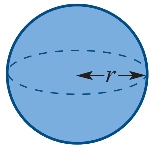
$$\text{volume of cone} = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$V = \frac{1}{3}\pi r^2 h$$

Volume and surface area of a sphere

The formulas for the volume and the **surface area** of a sphere are:

$$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$$

where r is the radius of the sphere.



Example 16 **Volume of a sphere and a cone**

Find the volume of a sphere with radius 4 cm and a cone with radius 4 cm and height 10 cm.

Solution

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times 4^3$$
$$= 268.08 \text{ cm}^3 \text{ (2 d.p.)}$$

$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 4^2 \times 10$$
$$= 167.55 \text{ cm}^3 \text{ (2 d.p.)}$$

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Example 17 **Surface area of a sphere**

Find the surface area of a sphere with radius 10 cm.

Solution

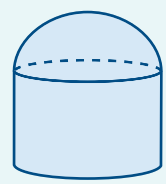
$$\begin{aligned}\text{Surface area of sphere} &= 4\pi r^2 \\ &= 4\pi \times 10^2 \\ &= 1256.64 \text{ cm}^2 \text{ (2 d.p.)}\end{aligned}$$

Composite shapes

Using the shapes above, composite shapes can be made. The volumes of these can be found by summing the volumes of the component shapes.

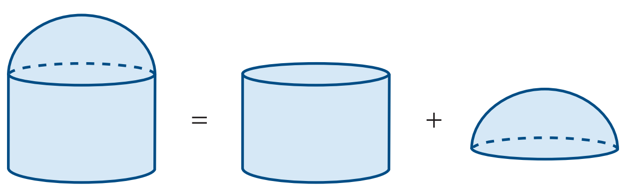
Example 18 **Volume of a composite shape**

A hemisphere is placed on top of a cylinder to form a capsule. The radius of both the hemisphere and the cylinder is 5 mm. The height of the cylinder is also 5 mm. What is the volume of the composite solid in cubic millimetres, correct to two decimal places?



Solution

1 The composite shape is made up from a cylindrical base plus a hemispherical top. The volume of the composite shape is the volume of the cylinder plus the volume of the hemisphere (half a sphere).



2 Use the formula $V = \pi r^2 h$ to find the volume of the cylinder.

$V_{cyl.} = \pi \times 5^2 \times 5 = 392.699\ldots \text{ mm}^3$

3 Find the volume of hemisphere noting that the volume of a hemisphere is $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$.

$V_{hem.} = \frac{1}{2} \times \frac{4}{3}\pi \times 5^3 = 261.799\ldots \text{ mm}^3$

4 Add the two together.

5 Write down your answer.

The volume of the composite $= 654.50 \text{ mm}^3$ (correct to two decimal places).

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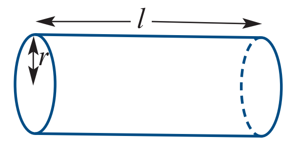
Surface area of three-dimensional shapes

The surface area of a solid can be found by calculating and totalling the area of each of its surfaces. The net of the cylinder in the diagram demonstrates how this can be done.

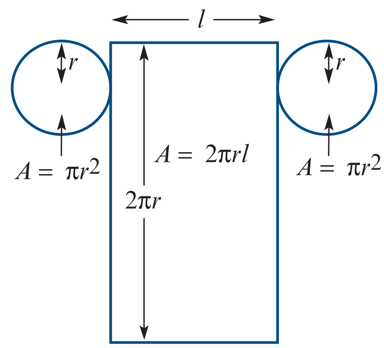


The surface area of the cylinder

= area of ends + area of curved surface

= area of two circles + area of rectangle

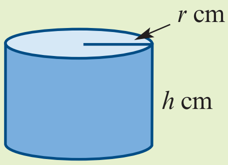
$= 2 \times \pi r^2 + 2\pi r \times l = 2\pi r^2 + 2\pi rl$



The formulas for the surface areas (S) of some common three-dimensional shapes follow.

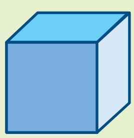
Surface area of solids

Cylinder (radius r cm, height h cm)



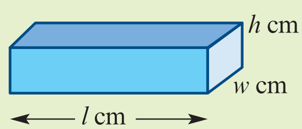
$S = 2\pi r^2 + 2\pi rh$

Cube (all edges x cm)



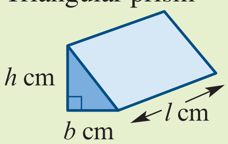
$S = 6x^2$

Rectangular prism (length l cm, width w cm, height h cm)



$S = 2(lw + lh + wh)$

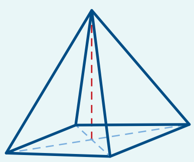
Triangular prism



$S = bh + bl + hl + l\sqrt{b^2 + h^2}$

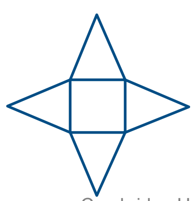
Example 19 Surface area of a right square pyramid

Find the surface of the right square pyramid shown if the square base has each edge 10 cm in length and the isosceles triangles each have height 15 cm.



Solution

1 Draw the net of the pyramid.



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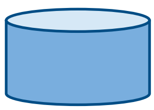
2 First determine the area of the square.	*Area of the square* $= 10^2 = 100\ cm^2$
3 Determine the area of one of the isosceles triangles.	*The area of one triangle is* $\frac{1}{2} \times 10 \times 15$ $= 75\ cm^2$
4 Find the sum of the areas of the four triangles and add to the area of the square.	*The surface area of the solid* $= 100 + 4 \times 75 = 100 + 300$ $= 400\ cm^2$

Exercise 17F

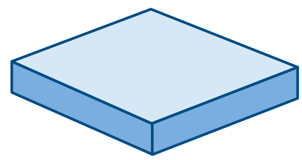
▶ Volumes

Skillsheet

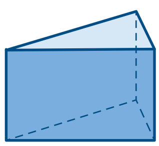
1 Find the volume in cm^3 of each of the following shapes, correct to two decimal places.

a 

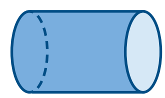
radius 6.3 cm, height 2.1 cm

b 

dimensions 2.1 cm, 8.3 cm and 12.2 cm

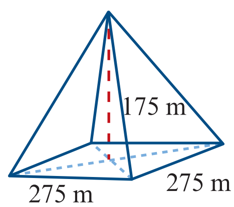
c 

area of cross-section = 2.8 cm^2,
height = 6.2 cm

d 

radius 2.3 cm, length 4.8 cm

2 Each side of the square base of one of the great Egyptian pyramids is 275 m long. It has a perpendicular height of 175 m. Calculate the volume of this pyramid, correct to the nearest cubic metre.



3 Find the volume, correct to one decimal place, of a:

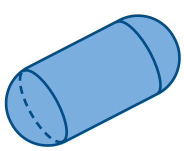
a sphere with radius 1.5 m

b cone with radius 6 cm and height 15 cm

c hemisphere of diameter 3.8 mm

d cone with diameter 15 mm and height 10 mm.

4 The diagram shows a capsule, which consists of two hemispheres, each of radius 2 cm, and a cylinder of length 5 cm and radius 2 cm. Find the volume of the capsule correct to the nearest cm^3.



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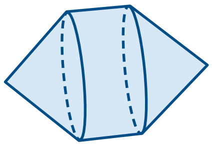
5 The diagram shows a composite shape made from a cylinder and two cones. Both the cylinder and the two cones have a radius of 12 cm. The length of the cylinder is 8 cm and height of the cones is 10 cm. Find the volume of the composite shape. Give your answer correct to the nearest cm^3.



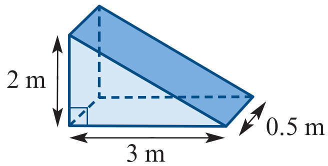
Surface areas

6 Find the total surface areas of shapes **a** and **b** of Question **1**. Give answers correct to the nearest cm^2.

7 For the triangular prism shown, find:

a the volume in m^3

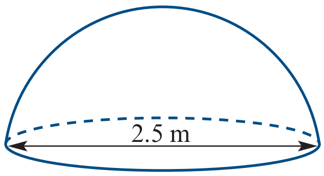
b the *total* surface area in m^2, correct to one decimal place.



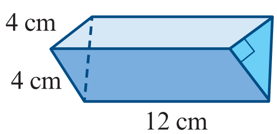
8 A hemispherical dome tent has a diameter of 2.5 m, as shown.

a Determine the volume enclosed by the tent, correct to the nearest m^3.

b Determine the total surface area of the tent (including its floor), correct to the nearest m^2.



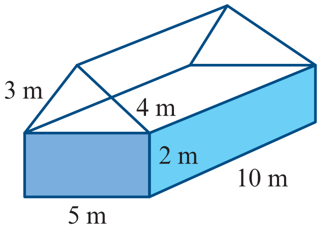
9 Find, correct to two decimal places, the surface area and volume of the solid shown given that the cross-section is a right-angled isosceles triangle.



10 Find:

a the surface area

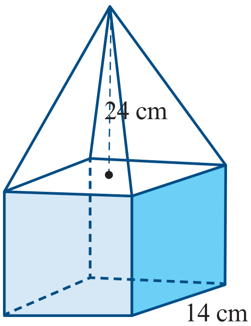
b the volume of the object shown.



11 The diagram opposite shows a right pyramid on a cube. Each edge of the cube is 14 cm. The height of the pyramid is 24 cm. Find:

a the volume of the solid

b the surface area of the solid.



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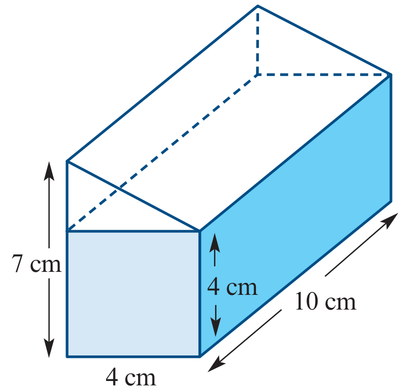
12 Find:

a the surface area

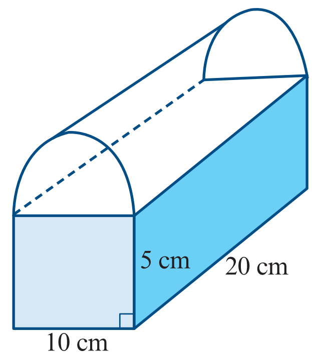
b the volume of the solid shown opposite.



13 The solid opposite consists of a half cylinder on a rectangular prism. Find correct to two decimal places:

a the surface area

b the volume.

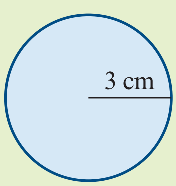


17G Areas, volumes and similarity

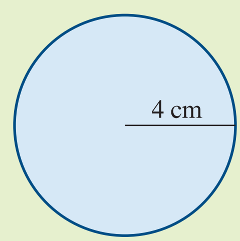
Areas

Some examples of similar shapes and the ratio of their areas are considered in the following.

Similar circles



$$\text{Scale factor} = k = \frac{\text{radius circle 2}}{\text{radius circle 1}} = \frac{4}{3}$$

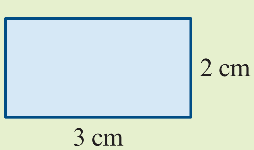


Area $= \pi \times 3^2$

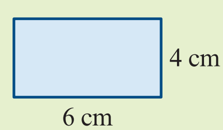
$$\text{Ratio of areas} = \frac{\pi \times 4^2}{\pi \times 3^2} = \frac{4^2}{3^2}\left(\frac{4}{3}\right)^2 = k^2$$

Area $= \pi \times 4^2$

Similar rectangles



$$\text{Scale factor} = k = \frac{\text{length rectangle 2}}{\text{length rectangle 1}} = \frac{6}{3} = 2$$



Area $= 3 \times 2$

$= 6 \text{ cm}^2$

$$\text{Ratio of areas} = \frac{24}{6} = 4 = (2)^2 = k^2$$

Area $= 6 \times 4$

$= 24 \text{ cm}^2$

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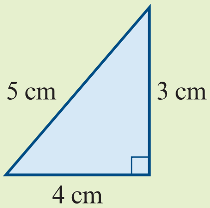
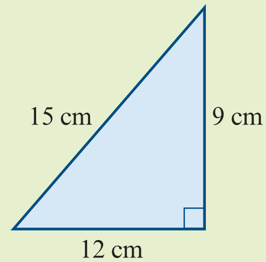
Similar triangles



$$\text{Scale factor} = k = \frac{\text{height triangle 2}}{\text{height triangle 1}} = \frac{9}{3} = 3$$



$$\text{Area} = \frac{1}{2} \times 4 \times 3 = 6 \text{ cm}^2$$

$$\text{Ratio of areas} = \frac{54}{6} = 9 = (3)^2 = k^2$$

$$\text{Area} = \frac{1}{2} \times 12 \times 9 = 54 \text{ cm}^2$$

A similar pattern emerges for other shapes. Scaling the linear dimension of a shape by a factor of k scales the area by a factor of k^2.

Scaling areas

If two *shapes are similar* and the *scale factor is* k, then the *area* of the similar shape = $k^2 \times$ area of the original shape.

Example 20 Using area scale factors with similarity

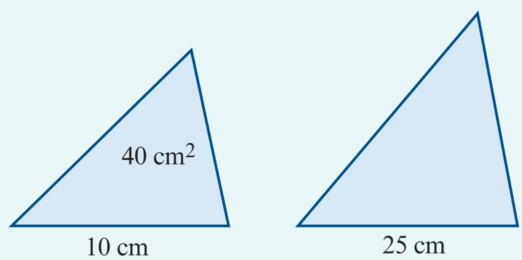
The two triangles shown are similar.

The base of the smaller triangle has a length of 10 cm.

Its area is 40 cm^2.

The base of the larger triangle has a length of 25 cm.

Determine its area.



Solution

1 Determine the scale factor k.

$$k = \frac{25}{10} = 2.5$$

2 Write down the area of the small triangle.

Area of small triangle = 40 cm^2

3 Area of larger triangle = k^2 × area of smaller triangle

Substitute the appropriate values and evaluate.

$$\therefore \text{area of larger triangle} = 2.5^2 \times 40 = 250$$

4 Write down your answer.

The area of the larger triangle is 250 cm^2.

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Example 21 Scale factors and area

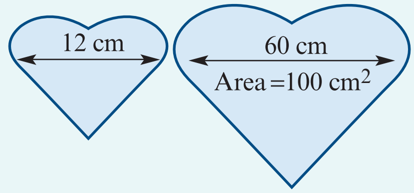
The two hearts shown are similar shapes.

The width of the larger heart is 60 cm.

Its area is 100 cm^2.

The width of the smaller heart is 12 cm.

Determine its area.



Solution

1 Determine the scale factor k. Note we are scaling down.

$k = \frac{12}{60} = 0.2$

2 Write down the area of the larger heart. Substitute the appropriate values and evaluate.

Area of larger heart = 100 cm^2

3 Area of smaller heart = k^2 × area of larger heart

$\therefore$ area of smaller heart $= 0.2^2 \times 100$
$= 4$

4 Write down your answer.

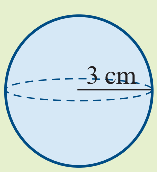
The area of the smaller heart is 4 cm^2.

Volumes

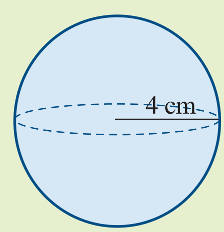
Two solids are considered to be similar if they have the same shape and the ratio of their corresponding linear dimensions is equal.

Some examples of similar volume and the ratio of their areas are considered in the following.

Similar spheres



$$\text{Scale factor} = k = \frac{\text{radius sphere 2}}{\text{radius sphere 1}} = \frac{4}{3}$$



$$\text{Volume} = \frac{4}{3}\pi \times 3^3$$
$$= 36 \text{ cm}^3$$

$$\text{Ratio of volumes} = \frac{\frac{256}{3}\pi}{36\pi} = \frac{256}{108_3}$$
$$= \frac{64}{27} = \left(\frac{4}{3}\right)^3 = k^3$$

$$\text{Volume} = \frac{4}{3}\pi \times 4^3$$
$$= \frac{256}{3}\pi \text{ cm}^3$$

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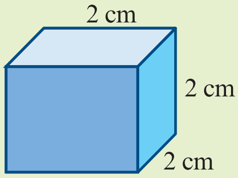
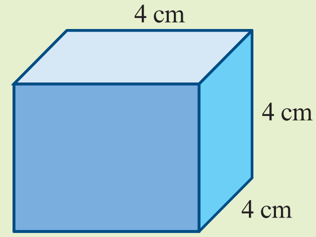
Similar cubes



$$\text{Scale factor} = k = \frac{\text{side length 2}}{\text{side length 1}}$$
$$= \frac{4}{3} = 2$$

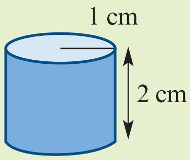


$$\text{Volume} = 2 \times 2 \times 2$$
$$= 8 \text{ cm}^3$$

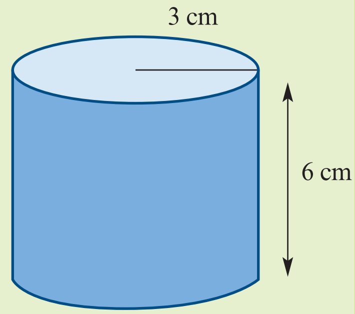
$$\text{Ratio of volumes} = \frac{64}{8} = 8$$
$$= (2)^3 = k^3$$

$$\text{Volume} = 4 \times 4 \times 4$$
$$= 64 \text{ cm}^3$$

Similar cylinders



$$\text{Scale factor} = k = \frac{\text{radius 2}}{\text{radius 1}}$$
$$= \frac{3}{1} = 3$$



$$\text{Volume} = \pi \times 1^2 \times 2$$
$$= 2\pi \text{ cm}^3$$

$$\text{Ratio of volumes} = \frac{54\pi}{2\pi}$$
$$= 27 = (3)^3 k^3$$

$$\text{Volume} = \pi \times 3^2 \times 6$$
$$= 54\pi \text{ cm}^3$$

A similar pattern emerges for other solids. Scaling the linear dimension of a solid by a factor of k scales the volume by a factor of k^3.

Scaling volumes

If two *solids are similar* and the *scale factor is* k, then the *volume* of the similar solid = k^3 × volume of the original solid.

Example 22 Similar solids

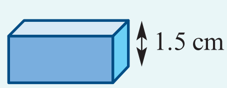
The two cuboids shown are similar solids.

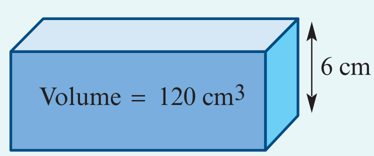
The height of the larger cuboid is 6 cm.

Its volume is 120 cm^3.

The height of the smaller cuboid is 1.5 cm.

Determine its volume.





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Solution

1 Determine the scale factor k. Note that we are scaling down.

$k = \frac{1.5}{6} = 0.25$

2 Write down the volume of the larger cuboid.

Volume larger cuboid = 120 cm^3

3 Volume smaller cuboid = $k^3 \times$ volume larger cuboid.

4 Substitute the appropriate values and evaluate.

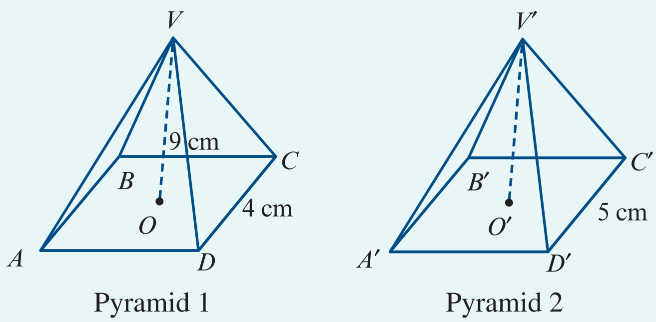
Volume smaller cuboid $= 0.25^3 \times 120$
$= 1.875$

5 Write down your answer.

The volume of the smaller cuboid is 1.875 cm^3.

Example 23 Similar solids

The two square pyramids shown are similar with a base dimensions 4 and 5 cm, respectively. The height of the first pyramid is 9 cm and its volume is 48 cm^3. Find the height and volume of the second pyramid.



Solution

1 Determine the scale factor, k. Use the base measurements.

$k = \frac{5}{4} = 1.25$

Height

2 Write down the height of pyramid 1.

Height 1 = 9 cm

3 Height pyramid 2 = $k \times$ height pyramid 1.

Substitute the appropriate values and evaluate.

$\therefore$ height 2 $= 1.25 \times 9 = 11.25$

4 Write down your answer.

The height of pyramid 2 is 11.25 cm.

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Volume

5 Volume pyramid 2 = $k^3 \times$ volume pyramid 1.

Substitute the appropriate values and evaluate.

Volume pyramid 1 = 48 cm³

Volume pyramid 2 = $1.25^3 \times 48 = 93.75$

6 Write down your answer.

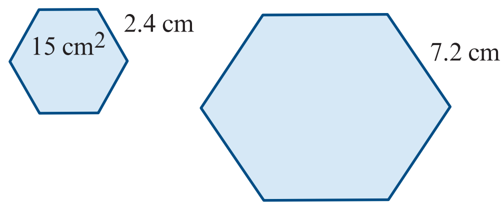
The volume of pyramid 2 is 93.75 cm³.

Exercise 17G

Areas

Skillsheet

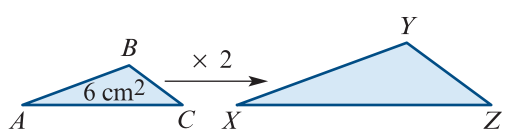
1 Two regular hexagons are shown. The side length of the smaller hexagon is 2.4 cm. The side length of the larger hexagon is 7.2 cm.



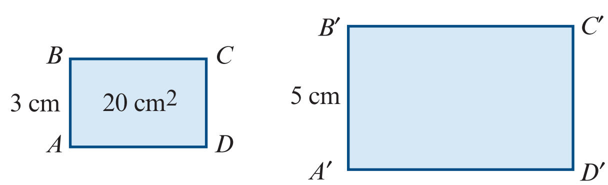
a Determine the length scale factor k for scaling up.

b The area of the smaller hexagon is 15 cm². Determine the area of the larger hexagon.

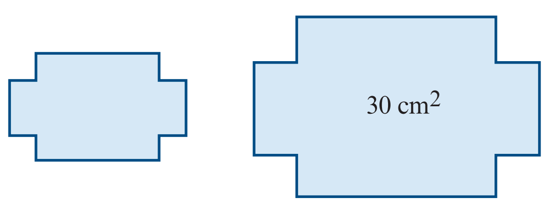
2 Triangle ABC is similar to triangle XYZ. The length scale factor $k = 2$. The area of triangle ABC is 6 cm². Find the area of triangle XYZ.



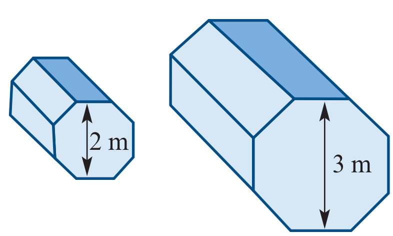
3 The two rectangles are similar. The area of rectangle $ABCD$ is 20 cm². Find the area of rectangle $A'B'C'D'$.



4 The two shapes shown are similar. The length scale factor for scaling down is $\frac{2}{3}$. The area of the shape to the right is 30 cm². What is the area of the shape on the left?



5 The octagonal prisms are similar. The height of the smaller prism is 2 m. The height of the larger prism is 3 m.
The surface area of the smaller prism is 18 m². Determine the surface area of the larger prism in m².



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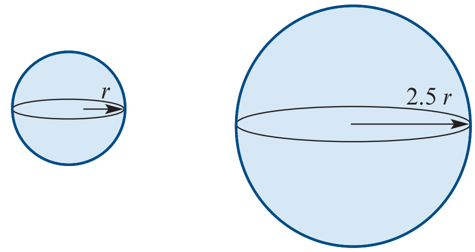
Volumes

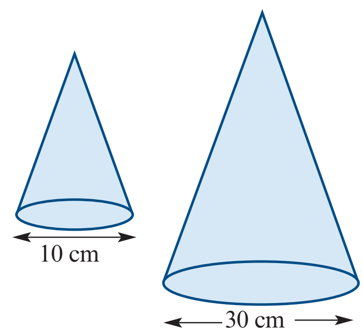
6 The radius of the larger sphere is 2.5 times the radius of the smaller sphere. The volume of the smaller sphere is 24 mm^3.



a Write down the length scale factor k for scaling up.

b Determine the volume of the larger sphere in mm^3.

7 The two cones shown are similar. The smaller cone has a diameter of 10 cm. The larger cone has a diameter of 30 cm.



a Determine the length scale factor k for scaling up.

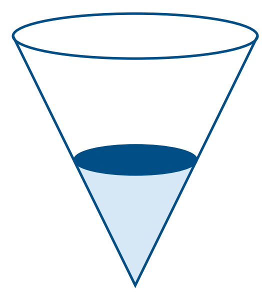
b What is the length scale factor k for scaling down?

c The height of the larger cone is 45 cm. Determine the height of the smaller cone.

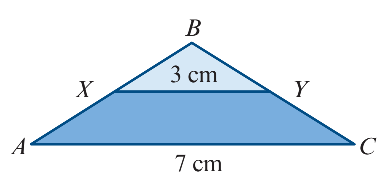
d The surface area of the smaller cone is 326.9 cm^2. Determine the surface area of the larger cone correct to the nearest cm^2.

e The volume of the smaller cone is 392.7 cm^3. What is the volume of the larger cone, correct to the nearest cm^3?

8 An inverted right circular cone of capacity 100 m^3 is filled with water to half its depth. Find the volume of water.



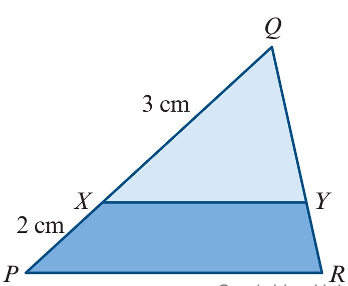
9 Triangles XBY and ABC are similar. The area of triangle XBY is 1.8 cm^2. Determine the area of triangle ABC.



10 Triangles XQY and PQR are similar. The area of triangle PQR is 7.5 cm^2.

a Determine k, the length scale factor for scaling down.

b Determine the area of triangle XQY.



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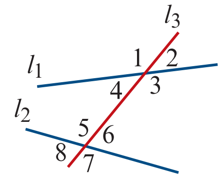
Key ideas and chapter summary

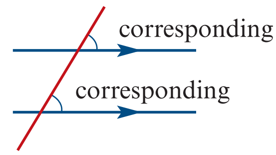
Alternate, corresponding, co-interior and vertically opposite angles

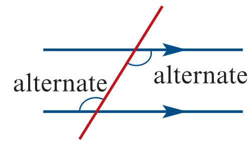
Angles 4 and 6 are examples of **alternate angles**.
Angles 2 and 6 are examples of **corresponding angles**.
Angles 3 and 6 are examples of **co-interior angles**.
Angles 1 and 3 are examples of **vertically opposite angles** and are of equal magnitude.



Angles associated with parallel inclines crossed by a transversal line

When lines l_1 and l_2 are parallel **corresponding angles** of equal magnitude, **alternate angles** are of equal magnitude and **co-interior angles** are supplementary.
Converse results also hold:





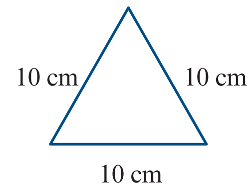
- If corresponding angles are equal, then l_1 is parallel to l_2.
- If alternate angles are equal, then l_1 is parallel to l_2.
- If co-interior angles are supplementary, then l_1 is parallel to l_2.

Angle sum of triangle

The sum of the magnitudes of the interior angles of a triangle is equal to 180°: $a° + b° + c° = 180°$.

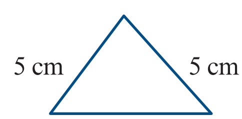
Equilateral triangle

A triangle is said to be *equilateral* if all of its sides are of the same length. The angles of an equilateral triangle are all of magnitude 60°.



Isosceles triangle

A triangle is said to be *isosceles* if it has two sides of equal length. If a triangle is isosceles, the angles opposite each of the equal sides are equal.



Interior angle

An **interior angle** is an angle associated with a geometric figure that lies within the figure.

Exterior angle

An **exterior angle** is an angle associated with a geometric figure that lies atside the figure.

Supplementary

Two angles are **supplementary** if the sum their sizes is 180°.

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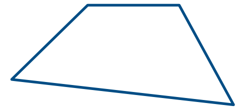
Review

Polygon

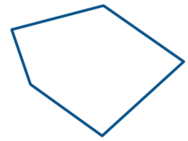
A **polygon** is a closed geometric shape with sides that are segments of straight lines. Examples are:

3 sides: triangle

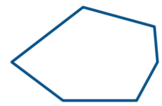
4 sides: quadrilateral



5 sides: pentagon



6 sides: hexagon



Convex polygon

A polygon is said to be **convex** if any diagonal lies inside the polygon.

Regular polygon

A **regular polygon** has all sides of equal length and all angles of equal magnitude.

Sum of the interior angles

The angle sum of the interior angles of an n-sided polygon is given by the formula: $S = (180n - 360)°$.

Pythagoras' theorem

Pythagoras' theorem states that for a right-angled triangle ABC with side lengths a, b and c, $a^2 + b^2 = c^2$, where c is the longest side.

Similar figures

We informally define two objects to be **similar** if they have the same shape but not the same size.

Conditions for similarity of triangles

- Corresponding angles in the triangles are equal.
- Corresponding sides are in the same ratio.

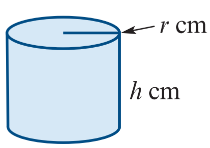
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = k$$

  where k is the scale factor.
- Two pairs of corresponding sides have the same ratio and the included angles are equal.

Volumes of solids

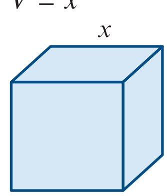
Cylinder:

$V = \pi r^2 h$



Cube:

$V = x^3$



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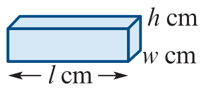
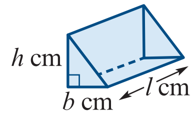
Rectangular prism:

$V = lwh$



Right-angled triangular prism

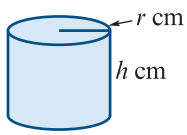
$V = \frac{1}{2}bhl$



Surface area of solids

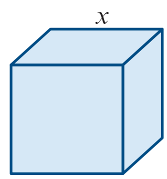
Cylinder:

$S = 2\pi r^2 + 2\pi rh$



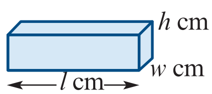
Cube:

$S = 6x^2$



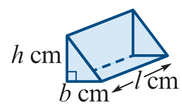
Rectangular prism:

$S = 2(lw + lh + wh)$



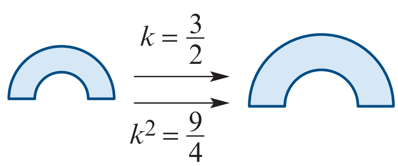
Right-angled triangular prism

$S = bh + bl + hl + l\sqrt{b^2 + h^2}$

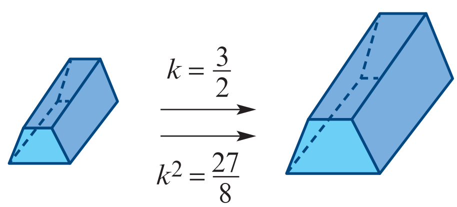


Scaling, areas and volumes

- If two shapes are similar and the scale factor is k, then the area of the similar shape $= k^2 \times$ area of the original shape.



- If two solids are similar and the scale factor is k, then the volume of the similar solid $= k^3 \times$ volume of the original solid.



 ISBN 978-1-107-56757-3

Review

Skills check

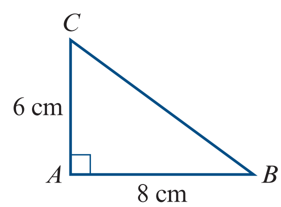
Having completed this chapter you should be able to:

- apply the properties of parallel lines and triangles and regular polygons to find the size of an angle, given suitable information
- find the size of each interior angle of a regular polygon with a given number of sides
- use the definition of objects such as triangles, quadrilaterals, squares, pentagons, hexagons, equilateral triangles, isosceles triangles to determine angles
- recognise when two objects are similar
- determine unknown lengths and angles through use of similar triangles
- find surface areas and volumes of solids
- use Pythagoras' theorem to find unknown lengths in right-angled triangles
- use similarity of two- and three-dimensional shapes to determine areas and volumes.

Multiple-choice questions

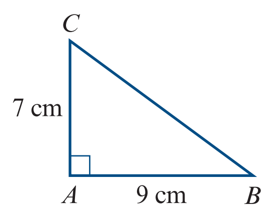
1 The triangle ABC has a right angle at A. The length of side BC, in cm, is:

A 10 **B** 14 **C** 9
D 9.8 **E** 11



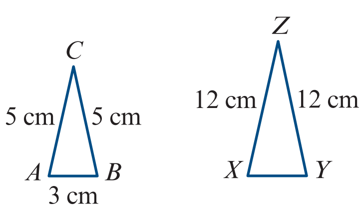
2 The triangle ABC has a right angle at A. The length of side BC, to the nearest cm, is:

A 10 **B** 14 **C** 9
D 12 **E** 11



3 Triangles ABC and XYZ are similar isosceles triangles. The length of XY is:

A 4 cm **B** 5 cm
C 4.2 cm **D** 8.5 cm
E 7.2 cm



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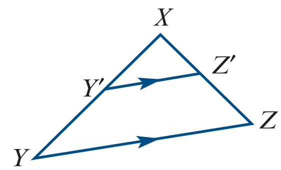
4 YZ is parallel to $Y'Z'$ and $Y'Y = \frac{1}{2}YX$.

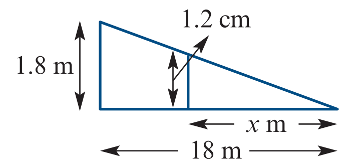
The area of triangle XYZ is 60 cm^2. The area of triangle $XY'Z'$ is:



A 20 cm^2 **B** 30 cm^2 **C** 15 cm^2

D $\frac{20}{3}$ cm^2 **E** $\frac{80}{3}$ cm^2

5 The value of x is:



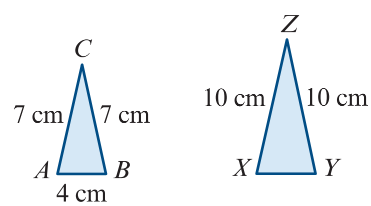
A 12 **B** 27 **C** 2.16

D 20.8 **E** 13.81

6 A regular convex polygon has 12 sides. The magnitude of each of its interior angles is:

A 30° **B** 45° **C** 60° **D** 150° **E** 120°

7 Triangles ABC and XYZ are similar isosceles triangles. The length of XY, correct to one decimal place, is:

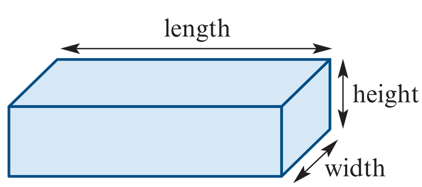


A 4.8 cm **B** 5.7 cm **C** 4.2 cm

D 8.5 cm **E** 8.2 cm

8 The rectangular prism shown has a volume of 12.8 cm^3. A second rectangular prism is made with half the length, four times the height and double the width.

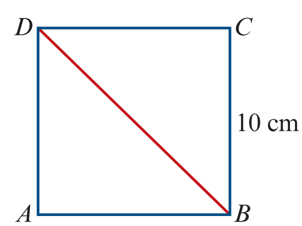
The volume of the second prism (in cm^3) is:



A 6.4 **B** 12.8 **C** 51.2

D 102.4 **E** 204.8

9 Each side length of a square is 10 cm. The length of the diagonal is:



A 10 **B** $5\sqrt{2}$ **C** $10\sqrt{2}$

D 8 **E** 1.4

10 To the nearest mm^2, the surface area of a sphere of radius of radius 8 mm is:

A 202 mm^2 **B** 268 mm^2 **C** 804 mm^2 **D** 808 mm^2 **E** 2145 mm^2

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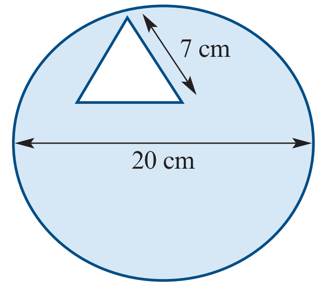
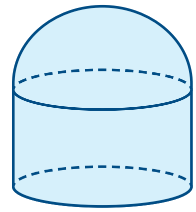
Review

11 An equilateral triangle of side length 7 cm is cut from a circular sheet of metal of diameter 20 cm. The area of the resulting shape (in cm^2) is closest to:



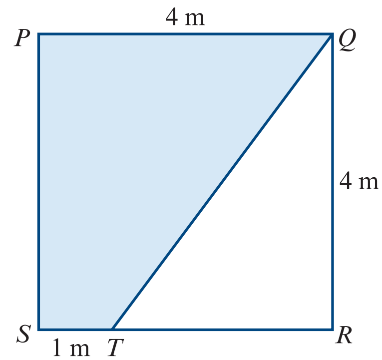
A 21 **B** 293 **C** 314
D 335 **E** 921

12 The diagram shows a composite shape that consists of a hemisphere of radius 6 cm placed on top of a cylinder of height 8 cm and radius 6 cm.
The total surface area of the composite shape (including the base) is closest to:



A 302 cm^2 **B** 452 cm^2 **C** 528 cm^2
D 641 cm^2 **E** 754 cm^2

13 *PQRS* is a square of side length 4 m as shown in the diagram below.
The distance *ST* is 1 m.
The shaded area *PQTS* shown in the diagram, in m^2, is closest to



A 6 **B** 8 **C** 9
D 10 **E** 12

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14 A cylinder of radius R and height H has volume V.
The volume of a cylinder with radius $3R$ and height $3H$ is:

A $3V$ **B** $6V$ **C** $9V$ **D** $27V$ **E** $81V$

ISBN 978-1-107-56757-3

Module 3: Geometry and measurement

Chapter 18

18

Applications of geometry and trigonometry

ISBN 978-1-107-56757-3

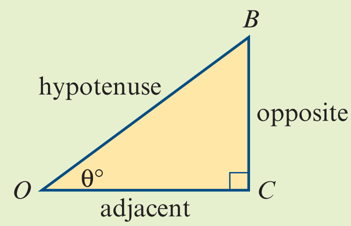
Cambridge University Press

18A Defining sine, cosine and tangent

$$\sin\theta^\circ = \frac{\text{opp}}{\text{hyp}} \qquad \left(\frac{\text{opposite}}{\text{hypotenuse}}\right)$$

$$\cos\theta^\circ = \frac{\text{adj}}{\text{hyp}} \qquad \left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)$$

$$\tan\theta^\circ = \frac{\sin\theta^\circ}{\cos\theta^\circ} = \frac{\text{opp}}{\text{adj}} \qquad \left(\frac{\text{opposite}}{\text{adjacent}}\right)$$



Also

$$\sin\theta^\circ = \sin(180 - \theta^\circ)$$

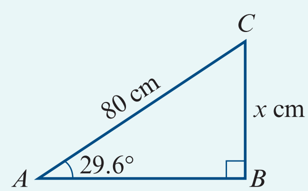
e.g. $\sin 45^\circ = \sin 135^\circ$ and

$$\cos\theta^\circ = -\cos(180 - \theta^\circ)$$

e.g. $\cos 45^\circ = -\cos 135^\circ$

Example 1 The use of sine

Find the value of x, correct to two decimal places.



Solution

1 The hypotenuse length and the angle A are given. The length of the side opposite the angle is to be found.

$$\frac{x}{80} = \sin 29.6^\circ$$

$$\therefore\ x = 80\sin 29.6^\circ$$

$$= 39.5153$$

2 Choose the appropriate **trigonometric ratio**. In this case sine, as $\sin = \dfrac{\text{opp}}{\text{hyp}}$.

3 Solve the equation using a calculator to evaluate $\sin 29.6^\circ$.

80·sin(29.6°)	39.5153

$\therefore\ x = 39.52$, correct to two decimal places.

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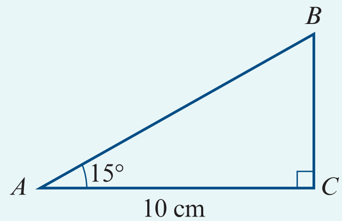
Example 2 The use of cosine

Find the length of the hypotenuse, correct to two decimal places.



Solution

1 The adjacent side length and the angle A are given. The length of the hypotenuse is to be found.

AB is the hypotenuse.

$$\frac{10}{AB} = \cos 15^\circ$$

2 Choose the appropriate trigonometric ratio. In this case cosine, as $\cos = \frac{\text{adj}}{\text{hyp}}$.

$$\therefore\ 10 = AB \cos 15^\circ$$

$$\therefore\ AB = \frac{10}{\cos 15^\circ}$$

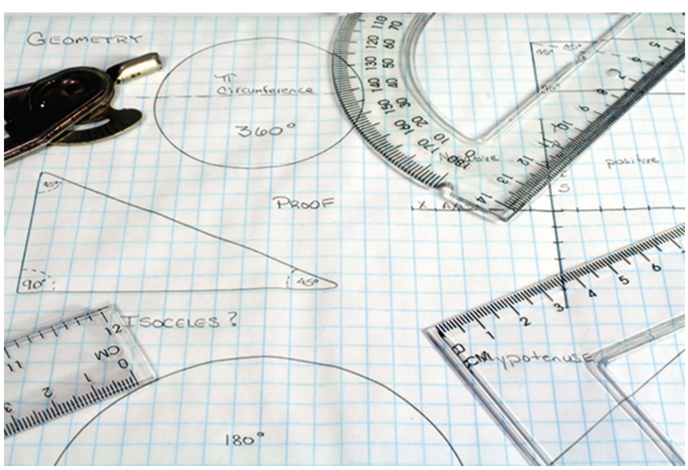
3 Solve the equation using a calculator to evaluate cos 15°.

$\frac{10}{\cos(15^\circ)}$	10.3528

4 Write down your answer.

The length of AB = 10.35 cm, correct to two decimal places.

Note: Make sure that your calculator is in degree and approximate or decimal mode. See the appropriate appendix for your type of calculator if help is needed.



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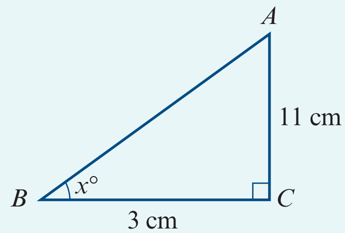
Example 3 The use of tan

Find the magnitude of $\angle ABC$.



Solution

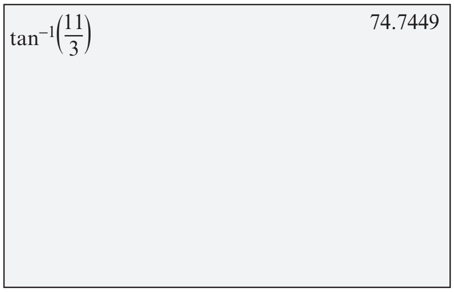
1 The adjacent side length and opposite side length are given. The size of the angle at B is to be found.

$\tan x° = \frac{11}{3}$

$\therefore x° = \tan^{-1}\left(\frac{11}{3}\right)$

2 Choose the appropriate trigonometric ratio. In this case tan, as $\tan(x) = \frac{\text{opp}}{\text{adj}}$.

3 Solve the equation using a calculator to evaluate $\tan^{-1}\left(\frac{11}{3}\right)$.

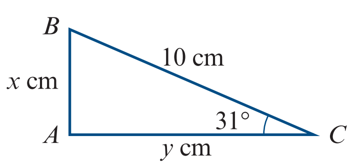


4 Write down your answer.

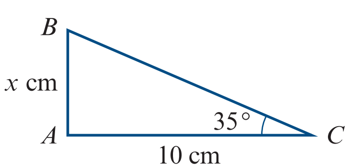
$\therefore x = 74.74$, correct to two decimal places

Exercise 18A

1 Use sine to find x and cosine to find y, correct to four decimal places.



2 Use tangent to find x, correct to four decimal places.



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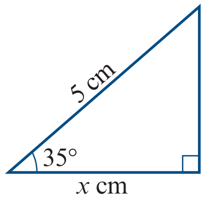
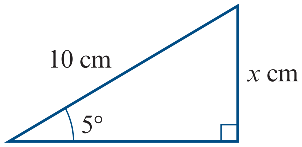
3 Find the value of x in each of the following.

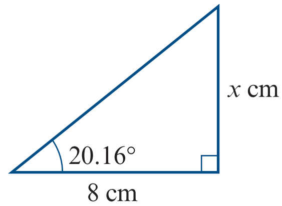
a



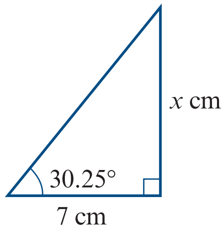
b



c



d



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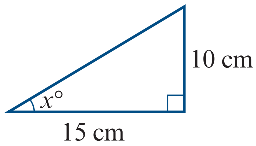

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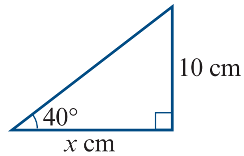
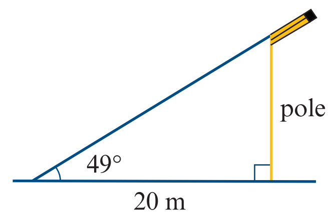


4 An equilateral triangle has altitudes of length 20 cm. Find the length of one side.

5 The base of an isosceles triangle is 12 cm long and the equal sides are 15 cm long. Find the magnitude of each of the three angles of the triangle.

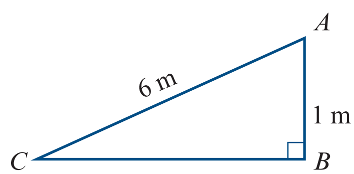
6 A pole casts a shadow 20 m long when the altitude of the Sun is 49°. Calculate the height of the pole.



7 This figure represents a ramp.

a Find the magnitude of angle ACB.

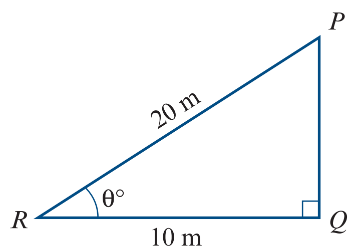
b Find the distance BC.



8 This figure shows a vertical mast PQ, which stands on horizontal ground. A straight wire 20 m long runs from P at the top of the mast to a point R on the ground which is 10 m from the foot of the mast.

a Calculate the angle of inclination, $\theta°$, of the wire to the ground.

b Calculate the height of the mast.



9 A ladder leaning against a vertical wall makes an angle of 26° with the wall. If the foot of the ladder is 3 m from the wall, calculate:

a the length of the ladder

b the height it reaches above the ground.

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10 An engineer is designing a straight concrete entry ramp, 60 m long, for a car park 13 m above street level. Calculate the angle of the ramp to the horizontal.

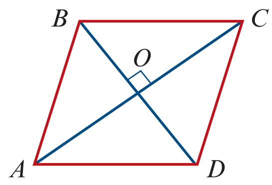
11 A vertical mast is secured from its top by straight cables 200 m long fixed at the ground. The cables make angles of 66° with the ground. What is the height of the mast?

12 A mountain railway rises for 400 m at a uniform slope of 16° with the horizontal. What is the distance travelled by a train for this rise?

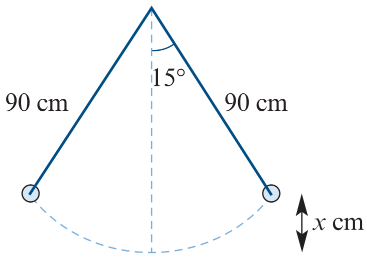
13 The diagonals of a rhombus bisect each other at right angles. If $BD = AC = 10$ cm, find:

a the length of the sides of the rhombus

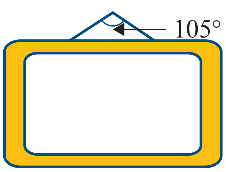
b the magnitude of angle ABC.



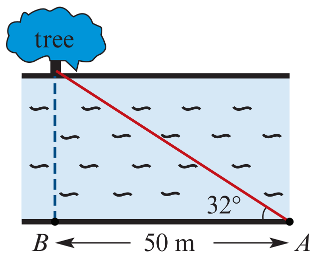
14 A pendulum swings from the vertical through an angle of 15° on each side of the vertical. If the pendulum is 90 cm long, what is the distance, x cm, between its highest and lowest points?



15 A picture is hung symmetrically by means of a string passing over a nail with its ends attached to two rings on the upper edge of the picture. The distance between the rings is 30 cm and the angle between the two portions is 105°. Find the length of the string.

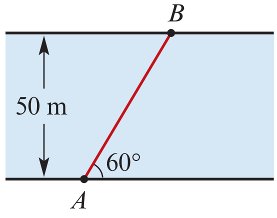


16 The distance $AB = 50$ m. If the line of sight of a person standing at A to the tree makes an angle of 32° with the bank, how wide is the river?



17 A ladder 4.7 m long is to be placed against a wall. The foot of the ladder must not be placed in a flower bed, which extends a distance of 1.7 m from the foot of the wall. How high up the wall can the ladder reach?

18 A river is known to be 50 m wide. A swimmer sets off at A to cross the river, and the path of the swimmer is shown. How far does the swimmer swim?



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18B The sine rule

In Section 18A, methods for finding unknown lengths and angles for right-angled triangles are discussed. In this section and the next, methods for finding unknown quantities in non-right-angled triangles are discussed.

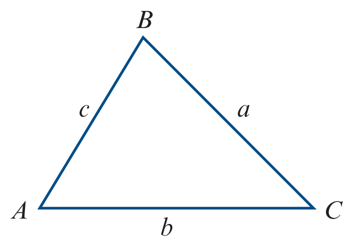
The **sine rule** is used to find unknown quantities in a triangle when one of the following situations arises:

- one side and two angles are given
- two sides and a non-included angle are given.

In the first of the two cases a unique triangle is defined, but for the second it is possible for two triangles to exist.

Labelling convention

The following convention is followed in the remainder of this module. Interior angles are denoted by upper-case letters and the length of the side opposite an angle is denoted by the corresponding lower-case letter.



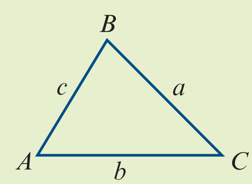
The magnitude of angle BAC is denoted by A.

The length of side BC is denoted by a.

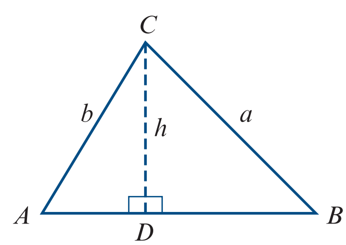
The sine rule

The sine rule states that for a triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



A proof will only be given for the acute-angled triangle case. The proof for obtuse-angled triangles is similar.



In triangle ACD, $\sin A = \dfrac{h}{b}$ $\qquad \therefore h = b \sin A$

In triangle BCD, $\sin B = \dfrac{h}{a}$ $\qquad \therefore h = a \sin B$

$$a \sin B = b \sin A$$

$$\text{i.e. } \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, starting with a perpendicular from A to BC would give:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

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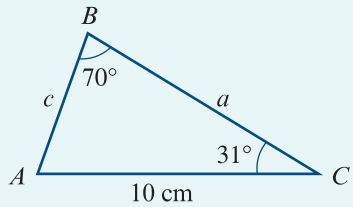
Example 4 Sine rule given two angles and a side

Find the length of AB.



Solution

1 Observe that the information given is two angles and a side. Therefore the sine rule is used. The known ratio is $\dfrac{b}{\sin B} = \dfrac{10}{\sin 70°}$.

2 The side to find is AB, which has length c, and the opposite angle is C, which has size 31°.

$$\frac{c}{\sin 31°} = \frac{10}{\sin 70°}$$

$$\therefore \quad c = \frac{10 \times \sin 31°}{\sin 70°}$$

$\dfrac{10 \cdot \sin(31°)}{\sin(70°)}$ 5.48092

The length of AB is 5.48 cm, correct to two decimal places.

Example 5 Sine rule given two sides and a non-included angle: ambiguous case

Find possible values for the magnitude of angle XZY in the triangle XYZ, given that $Y = 25°$, $y = 5$ cm and $z = 6$ cm.

Solution

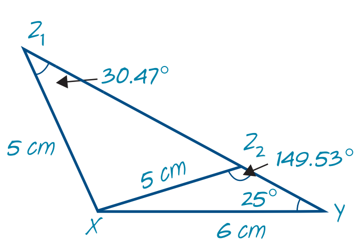
1 Observe that the information given is two sides and a non-included angle. Therefore the sine rule is used. The known ratio is $\dfrac{y}{\sin Y} = \dfrac{5}{\sin 25°}$.

2 The angle to find is XZY. The opposite side is XY has size 6 cm.

Remember: $\sin 180 - \theta° = \sin \theta°$

There are two solutions for the equation $\sin Z = 0.5071\ldots$

They are $Z° = 30.47°$ and $Z° = 149.53°$.



$$\frac{5}{\sin 25°} = \frac{6}{\sin z°}$$

$$\therefore \quad \frac{\sin z°}{6} = \frac{\sin 25°}{5}$$

$$\therefore \quad \sin z° = \frac{6 \times \sin 25°}{5}$$

$$= 0.5071\ldots$$

$$\therefore \quad z = \sin^{-1} 0.5071$$

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$\frac{6\cdot\sin(25°)}{5}$	0.507142
$\sin^{-1}(0.5071)$	30.4709
180°–30.4709°	149.529

$\therefore\ z° = 30.4736°\ldots$ or $180° - 30.4736°\ldots$

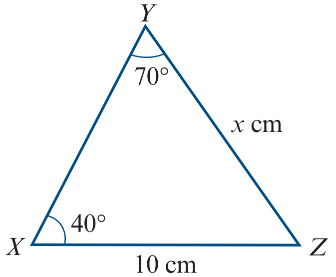
$\therefore\ z° = 30.47°$ or $z° = 149.53°$,
correct to two decimal places.

Exercise 18B

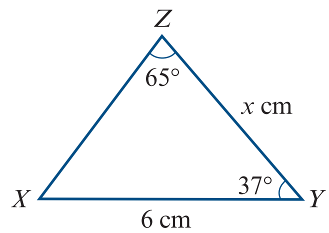
Skillsheet

1 Find the value of x in each of the following.

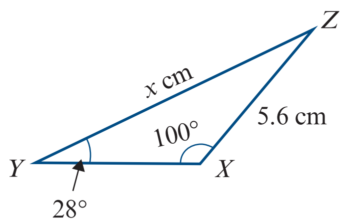
a



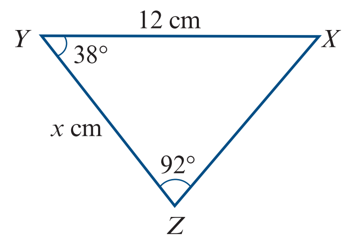
b



c

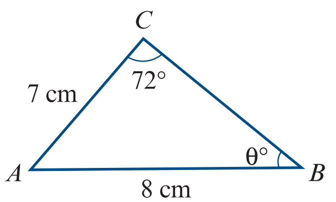


d

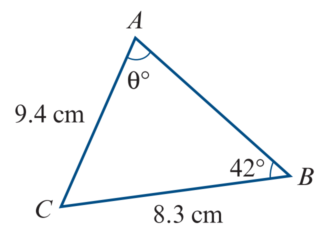


2 Find the value of θ for each of the following triangles.

a



b



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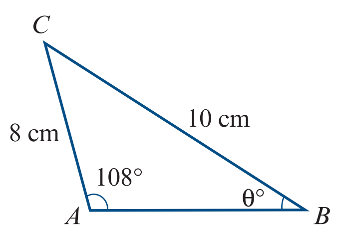
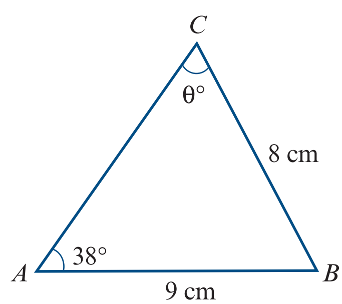


d



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3 Solve the following triangles (i.e. find all sides and angles).

a $a = 12, B = 59°, C = 73°$ **b** $A = 75.3°, b = 5.6, B = 48.25°$

c $A = 123.2°, a = 11.5, C = 37°$ **d** $A = 23°, a = 150, B = 40°$

e $B = 140°, b = 20, A = 10°$

4 Solve the following triangles (i.e. find all sides and angles).

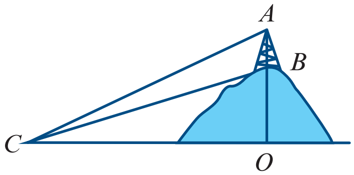
a $b = 17.6, C = 48.25°, c = 15.3$

b $B = 129°, b = 7.89, c = 4.56$

c $A = 28.25°, a = 8.5, b = 14.8$

5 A landmark A is observed from two points B and C, which are 400 m apart. The magnitude of angle ABC is found to be 68° and the magnitude of angle ACB is 72°. Find the distance of A from C.

6 AB is a tower 60 m high on top of a hill. The magnitude of ACO is 49° and the magnitude of BCO is 37°.

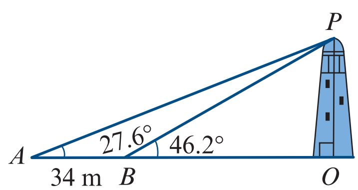


a Find the magnitude of angles ACB, CBO and CBA.

b Find the length of BC.

c Find the height of the hill, i.e. the length of OB.

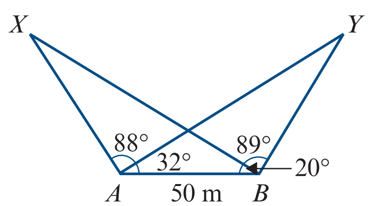
7 P is a point at the top of a lighthouse. Measurements of the length of AB and angles PBO and PAO are taken and are shown in the diagram. Find the height of the lighthouse.



8 A and B are two points on a coastline. They are 1070 m apart; C is a point at sea. The angles CAB and CBA have magnitudes of 74° and 69°, respectively. Find the distance of C from A.

9 Find:

a AX **b** AY.



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18C The cosine rule

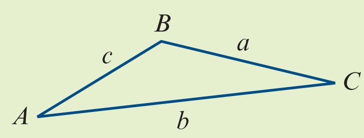
The **cosine rule** is used to find unknown quantities in a triangle when one of the following situations arises:

- two sides and an included angle are given
- three sides are given.

The cosine rule

The cosine rule states that for a triangle ABC
$a^2 = b^2 + c^2 - 2bc\cos A$ or, equivalently,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$



The symmetrical results also hold, i.e.:

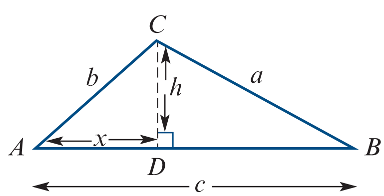
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$

The result will be proved for an acute-angled triangle. The proof for obtuse-angled triangles is similar. In triangle ACD:

$$b^2 = x^2 + h^2 \text{ (Pythagoras' theorem)}$$

$$\cos A = \frac{x}{b} \text{ and therefore } x = b\cos A$$

In triangle BCD, $a^2 = (c - x)^2 + h^2$ (Pythagoras' theorem). Expanding gives:



$$a^2 = c^2 - 2cx + x^2 + h^2$$

$$= c^2 - 2cx + b^2 \quad (\text{as } x^2 + h^2 = b^2)$$

$$a^2 = b^2 + c^2 - 2bc\cos A \quad (\text{as } x = b\cos A)$$

Rearranging gives:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

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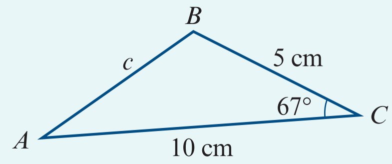
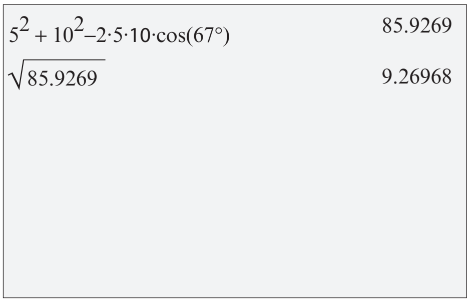
Example 6 Applying the cosine rule: two sides and included angle given

For triangle ABC, find the length of AB in centimetres, correct to two decimal places.



Solution

Strategy: Apply the cosine rule – two sides and an included angle are given.



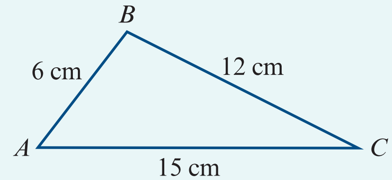
$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$
$$= 85.9268\ldots$$
$$\therefore\ c = 9.269\ldots$$

The length of AB is 9.27 cm, correct to two decimal places.

Example 7 Applying the cosine rule: three sides given

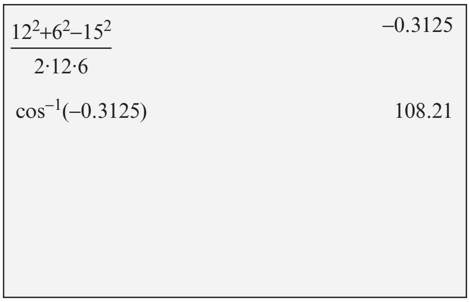
Find the magnitude of angle ABC for triangle ABC.

Give your answer correct to two decimal points.



Solution

Strategy: Apply the cosine rule – three sides are given.



$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$
$$= -0.3125$$
$$\therefore B = 108.2099\ldots$$

The magnitude of angle ABC is 108.21°, correct to two decimal places.

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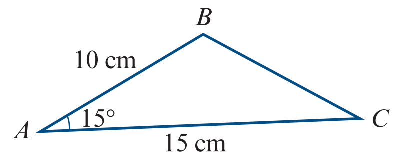
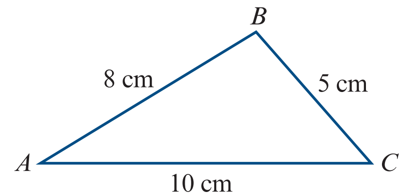
Exercise 18C

Skillsheet

1 Find the length of BC.



2 Find the magnitudes of angles ABC and ACB.



3 For triangle ABC with:

a $A = 60°$, $b = 16$, $c = 30$, find a.

b $a = 14$, $B = 53°$, $c = 12$, find b.

c $a = 27$, $b = 35$, $c = 46$, find the magnitude of angle ABC.

d $a = 17$, $B = 120°$, $c = 63$, find b.

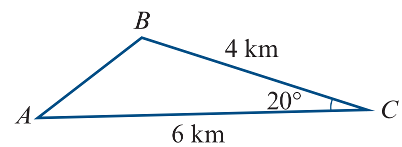
e $a = 31$, $b = 42$, $C = 140°$, find c.

f $a = 10$, $b = 12$, $c = 9$, find the magnitude of angle BCA.

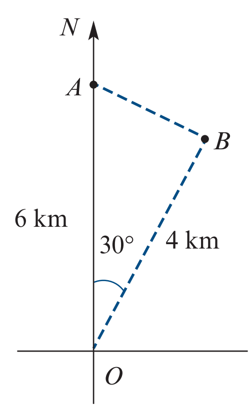
g $a = 11$, $b = 9$, $C = 43.2°$, find c.

h $a = 8$, $b = 10$, $c = 15$, find the magnitude of angle CBA.

4 A section of an orienteering course is shown. Find the length of leg AB.



5 Two ships sail from point O. At a particular time their positions, A and B, are shown. Find the distance between the ships at this time.



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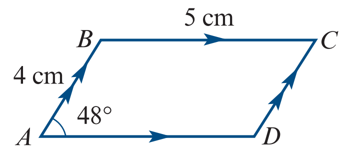
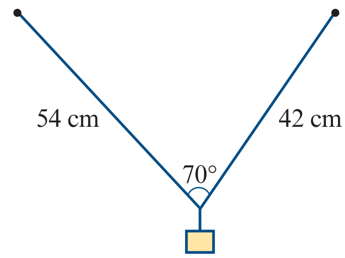
6 $ABCD$ is a parallelogram. Find the length of the diagonals:

a AC **b** BD.

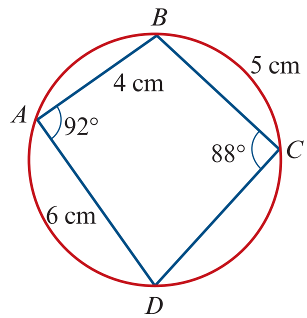


7 A weight is hung from two hooks in a ceiling by strings of length 54 cm and 42 cm, which are inclined at 70° to each other. Find the distance between the hooks.



8 **a** Find the length of diagonal BD.

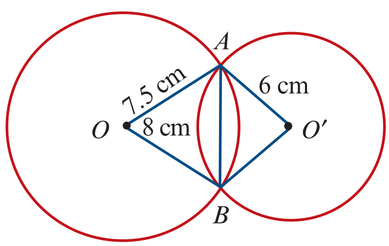
b Use the sine rule to find the length of CD.



9 Two circles of radius 7.5 cm and 6 cm have a common chord of length 8 cm.

a Find the magnitude of angle $AO'B$.

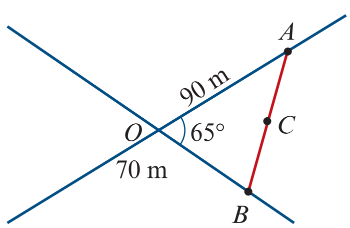
b Find the magnitude of angle AOB.



10 Two straight roads intersect at an angle of 65°. A point, A, on one road is 90 m from the intersection and a point, B, on the other road is 70 m from the intersection, as shown on the diagram.

a Find the distance of A from B.

b C is the midpoint of AB. Find the distance of C from the intersection.



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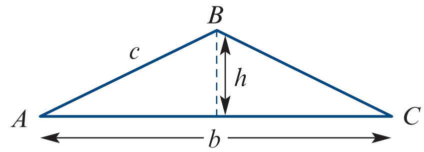
18D Area of a triangle

By observing that $h = c \sin A$ the following formula can be found.

It is known that the **area of a triangle** is given by the formula:

$$A = \frac{1}{2}bh$$



Area = $\frac{1}{2}$ × base length × height

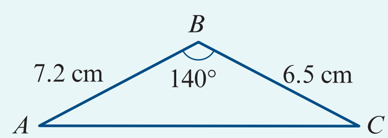
Area of a triangle

$$\text{Area} = \frac{1}{2}bc \sin A$$

The area is given by half the product of the length of two sides and the sine of the angle included between them.

Example 8 Determining the area of a triangle, $A = \frac{1}{2}ac \sin B$

Find the area of triangle ABC. Give your answer correct to two decimal places.



Solution

Strategy: Apply the formula for the area of a triangle, area or = $\frac{1}{2}ac \sin B$.

$\frac{1}{2} \cdot 7.2 \cdot 6.5 \cdot \sin(140°)$ 15.0412

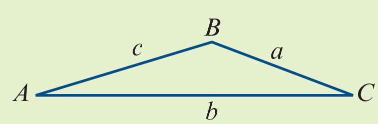
Area = $\frac{1}{2} \times 7.2 \times 6.5 \times \sin 140°$

= 15.04 cm^2

The area of triangle ABC is 15.04 cm^2, correct to two decimal places.

Heron's formula

Heron's formula provides a way of determining the area of a triangle given the lengths of three sides.



The area of a triangle with side lengths a, b and c is given by:

$A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter and $s = \dfrac{a+b+c}{2}$.

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Example 9 **Determining the area of a triangle using Heron's formula**

Find the area of the triangle with sides 6 cm, 4 cm and 4 cm. Give your answer correct to two decimal places.

Solution

1 Three sides given, so use Heron's formula.

$A = \sqrt{s(s-a)(s-b)(s-c)}$

2 Find the value of $s = \dfrac{a+b+c}{2}$. In this case, $a = 6$, $b = 4$ and $c = 4$.

$s = \dfrac{6+4+4}{2} = 7$

3 Substitute values into the formula.

$A = \sqrt{7 \times 1 \times 3 \times 3} = \sqrt{63} = 7.937\ldots$

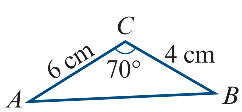
4 Write down your answer, correct to two decimal places.

The area of the triangle is 7.94 cm, correct to two decimal places.

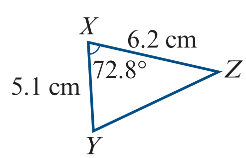
Exercise 18D

1 Find the area of each of the following triangles.

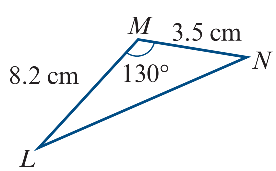
a



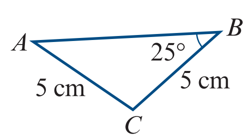
b



c



d



2 Find the area of an equilateral triangle with side length:

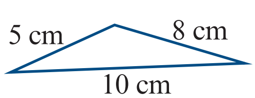
a 6.2 cm **b** 3.7 cm.

3 In triangle XYZ, $XY = 9$ cm and $YZ = 13$ cm. The area of the triangle is 13 cm^2. Find two possible values for the magnitude of $\angle XYZ$.

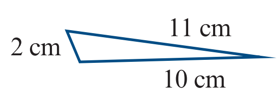
4 In triangle ABC, $A = 108.6°$, $b = 9$ cm and $c = 8$ cm. Find the area of the triangle.

5 Find the areas of each of the following triangles in cm^2, giving your answers correct to two decimal places.

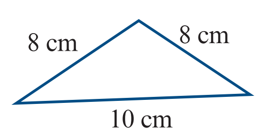
a



b



c



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6 The area of a triangle ABC is 6 cm^2; $AB = 3$ cm and $AC = 5$ cm.

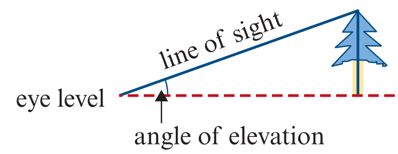
a Find two possible values for the magnitude of angle BAC.

b Find two possible lengths for BC.

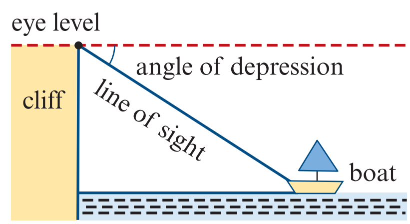
18E Angles of elevation and depression, bearings and triangulation

▶ Angles of elevation and depression

The **angle of elevation** is the angle between the horizontal and a direction above the horizontal.



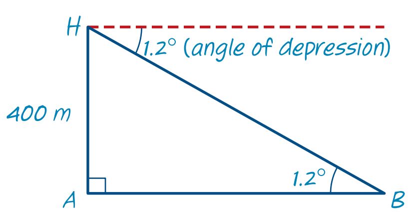
The **angle of depression** is the angle between the horizontal and a direction below the horizontal.



Example 10 Angle of depression

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2°. Draw a diagram and calculate the horizontal distance of the boat to the helicopter, correct to the nearest 10 metres.

Solution



$$\frac{AH}{AB} = \tan 1.2^\circ$$

$$\therefore \frac{400}{AB} = \tan 1.2^\circ$$

$$AB = \frac{400}{\tan 1.2^\circ}$$

$$AB = 19095.80056\ldots$$

The horizontal distance is 19100 m, to the nearest 10 metres.

Example 11 Angle of elevation

The light on a cliff-top lighthouse, known to be 75 m above sea level, is observed from a boat at an angle of elevation of 7.1°. Draw a diagram and calculate the distance of the boat from the lighthouse, to the nearest metre.

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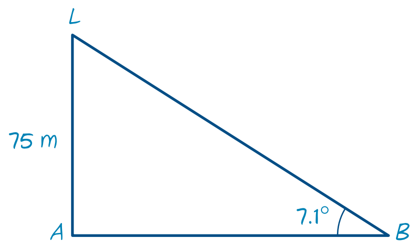
Solution



$$\frac{75}{AB} = \tan 7.1°$$

$$\therefore AB = \frac{75}{\tan 7.1°}$$

$$= 602.135$$

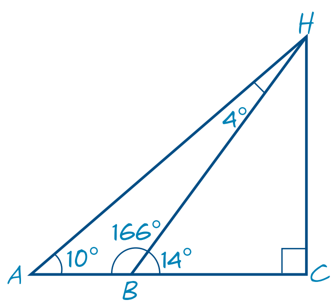
The distance of the boat from the lighthouse is 602 m, to the nearest metre.

Example 12 Applying geometry and trigonometry with angle of elevation

From a point A, a man observes that the angle of elevation of the summit of a hill is 10°. He then walks towards the hill for 500 m along flat ground. The summit of the hill is now at an angle of elevation of 14°. Draw a diagram and find the height of the hill above the level of A, to the nearest metre.

Solution

1 Draw a diagram.



2 Find all the unknown angles that will be required. This is done using properties of angles discussed in Chapter 17.

The magnitude of angle $HBA = (180 - 14)° = 166°$
The magnitude of angle $AHB = 180 - (166 + 10) = 4°$

3 You choose to work in particular triangles. In this case it is triangle ABH.

Using the sine rule in triangle ABH:

$$\frac{500}{\sin 4°} = \frac{HB}{\sin 10°}$$

$$\therefore HB = \frac{500 \times \sin 10°}{\sin 4°}$$

$$= 1244.67\ldots$$

4 The information found in triangle ABH is the length HB. This can now be used to find HC in triangle BCH.

In triangle BCH:

$$\frac{HC}{HB} = \sin 14°$$

$$\therefore HC = HB \sin 14°$$

$$= 301.11\ldots$$

5 Write down your answer.

The height of the hill is 301 m, to the nearest metre.

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Bearings

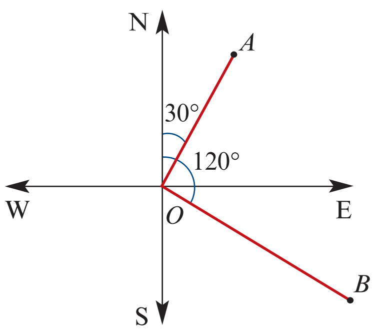
The **three-figure bearing** (or compass bearing) is the direction measured clockwise from north.

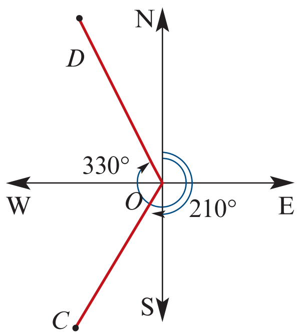
The bearing of A from O is 030°.

The bearing of B from O is 120°.

The bearing of C from O is 210°.

The bearing of D from O is 330°.

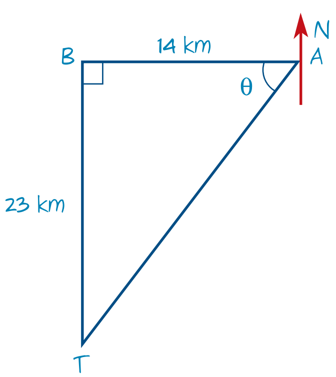




Example 13 Bearings and Pythagoras' theorem

The road from town A runs due west for 14 km to town B. A television mast is located due south of B at a distance of 23 km. Draw a diagram and calculate the distance of the mast from the centre of town A, to the nearest kilometre. Find the bearing of the mast from the centre of the town.

Solution



$$\tan\theta = \frac{23}{14}$$

$$\therefore\ \theta = 58.67° \text{(to two decimal places)}$$

$$\therefore \text{ bearing} = 180° + (90 - 58.67)°$$

$$= 211.33°$$

By Pythagoras' theorem:

$$AT^2 = AB^2 + BT^2$$

$$= 14^2 + 23^2$$

$$= 725$$

$$\therefore\ AT = 26.925\ldots$$

$\therefore$ the mast is 27 km from the centre of town A (to the nearest kilometre) and on a bearing of 211.33°.

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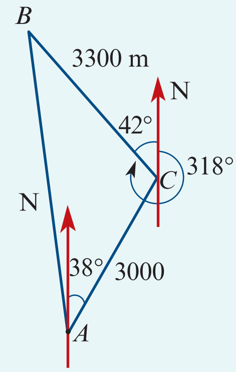
Example 14 Bearings and the cosine and sine rules

A yacht starts from a point A and sails on a bearing of 038° for 3000 m. It then alters its course to a bearing of 318°, and after sailing for 3300 m it reaches a point A.

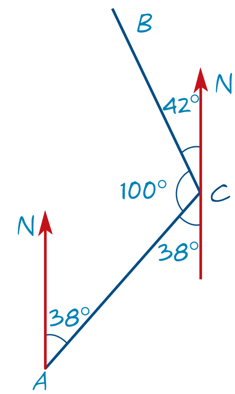
a Find the distance AB, correct to the nearest metre.

b Find the bearing of B from A, correct to the nearest degree.



Solution

a **1** To find the distance AB, the magnitude of angle ACB needs to be determined so that the cosine rule can be applied in triangle ABC. The magnitude of angle $ACB = [180 - (38 + 42)]° = 100°$



2 Apply the cosine rule in triangle ABC.

$AB^2 = 3000^2 + 3300^2 - 2 \times 3000 \times 3300 \times \cos 100° = 23\ 328\ 23.392\ldots$

$\therefore AB = 4829.93104\ldots$

3 Write down your answer.

The distance of B from A is 4830 m (to the nearest metre).

b To find the bearing of B from A, the magnitude of angle BAC must first be found. The sine rule can be used.

$$\frac{3300}{\sin A} = \frac{AB}{\sin 100°}$$

$$\therefore \sin A = \frac{3300 \times \sin 100°}{AB}$$

$\therefore \sin A = 0.6728\ldots$

$\therefore A = (42.288\ldots)°$

$\therefore$ the bearing of B from A

$= 306° - (42.29° - 38°)$

$= 355.71°$

The bearing of B from A is 356°, to the nearest degree.

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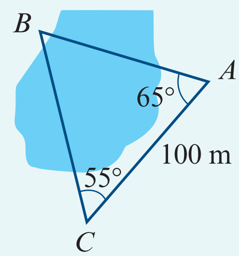
Triangulation

Surveyors sometimes need to measure distances to inaccessible points, or measure lengths that are impossible to measure directly. The method of *triangulation* involves using the theory of solving triangles developed in this module.

Example 15 Triangulation

Two points, A and B, are on opposite sides of a lake so that the distance between them cannot be measured directly. A third point, C, is chosen at a distance of 100 m from A and with angles BAC and BCA of 65° and 55°, respectively. Calculate the distance between A and B, correct to two decimal places.



Solution

The magnitude of angle ABC is 60°. Using the sine rule for triangle ABC:

$$\frac{100}{\sin 60^\circ} = \frac{AB}{\sin 55^\circ}$$

$$\therefore AB = \frac{100}{\sin 60^\circ} \times \sin 55^\circ$$

$$= 94.587\ldots$$

The length of AB is 94.59 m, correct to two decimal places.

Exercise 18E

Angles of depression and elevation

1. The angle of elevation of the top of an old chimney stack at a point 40 m from its base is 41°. Find the height of the chimney.

2. From the top of a vertical cliff 130 m high, the angle of depression of a buoy at sea is 18°. What is the distance of the buoy from the foot of the cliff?

3. A man standing on top of a mountain observes that the angle of depression to the foot of a building is 41°. If the height of the man above the foot of the building is 500 m, find the horizontal distance from the man to the building.

4. A man lying down on top of a cliff 40 m high observes the angle of depression to a buoy in the sea below to be 20°. If he is in line with the buoy, calculate the distance between the buoy and the foot of the cliff, which may be assumed to be vertical.

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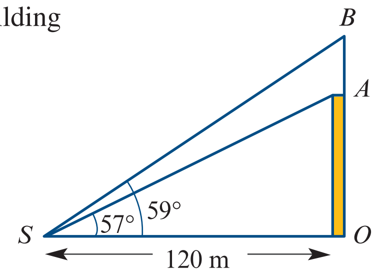
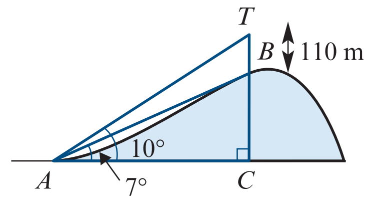
5 Point S is at a distance of 120 m from the base of a building On the building is an aerial, AB.
The angle of elevation from S to A is 57°.
The angle of elevation from S to B is 59°.



a Find the distance OA.
b Find the distance OB.
c Find the distance AB.

6 A tower 110 m high stands on the top of a hill. From a point A at the foot of the hill the angle of elevation of the bottom of the tower is 7°, and that of the top is 10°.



a Find the magnitude of angles TAB, ABT and ATB.
b Use the sine rule to find the length AB.
c Find CB, the height of the hill.

Bearings

7 The bearing of a point A from a point B is 207°. What is the bearing of B from A?

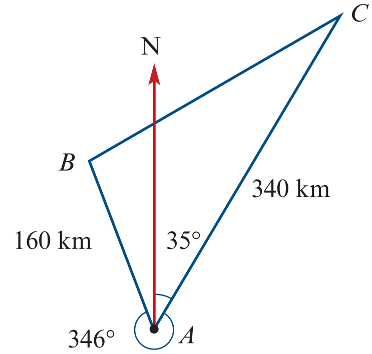
8 A ship sails 10 km north and then 15 km east. What is its bearing from the starting point?

9 A ship leaves port A and steams 15 km due east. It then turns and steams for 22 km due north.
a What is the bearing of the ship from A?
b What is the bearing of port A from the ship?

10 The bearing of a ship, S, from a lighthouse, A, is 055°. A second lighthouse, B, is due east of A. The bearing of S from B is 302°. Find the magnitude of angle ASB.

11 A yacht starts from L and sails 12 km due east to M. It then sails 9 km on a bearing of 142° to K. Find the magnitude of angle MLK.

12 The bearing of C from A is 035°.
The bearing of B from A is 346°.
The distance of C from A is 340 km.
The distance of B from A is 160 km.



a Find the magnitude of angle BAC.
b Use the cosine rule to find the distance of B to C.

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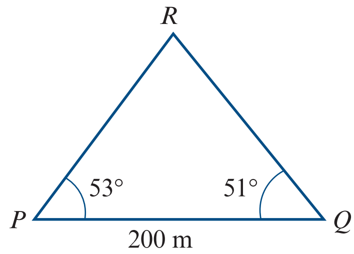
Triangulation

13 P and Q are points on the bank of a river. A tree is at a point, R, on the opposite bank such that $\angle QPR$ is 53° and $\angle RQP$ is 51°.

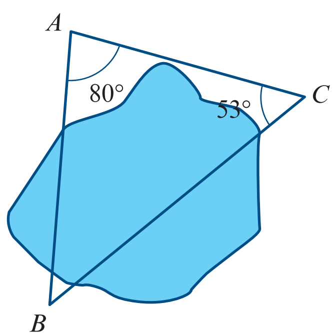
a Find:

i RP **ii** RQ.

b T is a point between P and Q such that $\angle PTR$ is a right angle. Find RT and hence the width of the river, correct to two decimal places.



14 Two points, A and B, are on opposite sides of a lake so that the distance between them cannot be measured directly. A third point, C, is chosen at a distance of 300 m from A and with angles BAC and BCA of 80° and 53°, respectively. Calculate the distance between A and B.



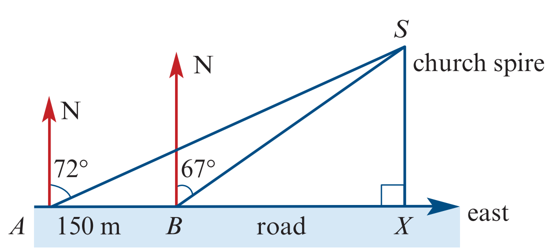
Mixed problems

15 A man walking due east along a level road observes a church spire from point A. The bearing of the spire from A is 072°. He then walks 150 m to point B where the bearing is 067°.

a Find the distance of the church spire from B (i.e. BS).

b Find the distance of the church spire from the road (i.e. SX).

16 From a ship, S, two other ships, P, and Q, are on bearings 320° and 075°, respectively. The distance PS = 7.5 km and the distance QS = 5 km. Find the distance PQ.



17 A yacht starts from point A and sails on a bearing of 035° for 2000 m. It then alters its course to one in a direction with a bearing of 320° and after sailing for 2500 m it reaches point B.

a Find the distance AB.

b Find the bearing of B from A.

18 A man standing on top of a cliff 50 m high is in line with two buoys whose angles of depression are 18° and 20°. Calculate the distance between the buoys.

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18F Problems in three dimensions

Problems in three dimensions are solved by picking out triangles from a main figure and finding lengths and angles through these triangles.

Example 16 Applications in three dimensions

ABCDEFGH is a cuboid. Find:

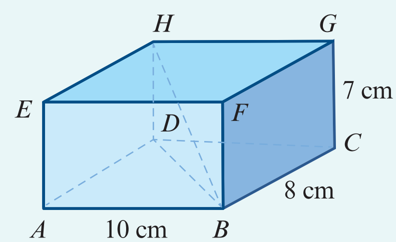
a distance *DB*

b distance *HB*

c the magnitude of angle *HBD*

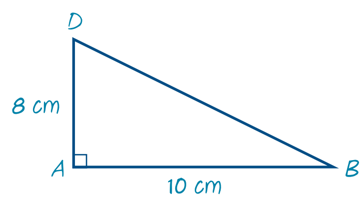
d distance *HA*

e the magnitude of angle *HBA*.



Solution

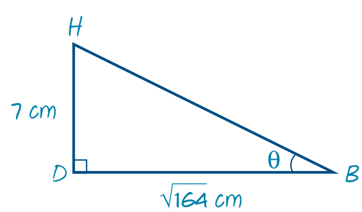
a *Strategy:* You always work in triangles to obtain values. Here we start with triangle *ABD*.



$$DB^2 = 8^2 + 10^2$$
$$= 164$$
$$\therefore\ DB = \sqrt{164}$$
$$\therefore\ DB = 12.806\ldots$$

The length of DB is 12.81 cm, correct to two decimal places.

b *Strategy:* Now use the information from part **a** in triangle *BDH*.



$$HB^2 = HD^2 + DB^2$$
$$= 7^2 + \left(\sqrt{164}\right)^2$$
$$= 213$$
$$\therefore\ HB = \sqrt{213}$$
$$= 14.59\ldots$$

The length of HB is 14.59 cm, correct to two decimal places.

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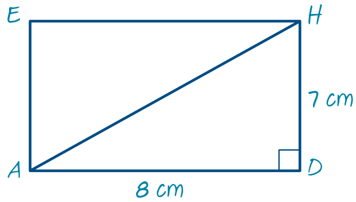
c *Strategy*: Triangle BDH is again used.

$$\tan\theta = \frac{HD}{BD} = \frac{7}{\sqrt{164}} = 0.5466\ldots$$

$\theta = 28.66°$, correct to two decimal places.

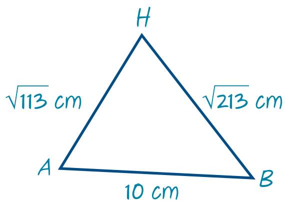
d *Strategy*: Triangle ABH is used to find HA.



$$\begin{aligned} HA^2 &= 8^2 + 7^2 \\ &= 64 + 49 \\ &= 113 \end{aligned}$$

$\therefore\ HA = \sqrt{113}$

e *Strategy*: Apply the cosine rule to triangle ABH to find the magnitude of angle HBA.



$$HA^2 = AB^2 + HB^2 - 2 \times AB \times HB \ \cos B$$

$$113 = 100 + 213 - 2 \times 10 \times \sqrt{213} \cos$$

$$\therefore \cos B = \frac{-200}{-20\sqrt{213}}$$

$$\therefore \cos B = 0.68518\ldots$$

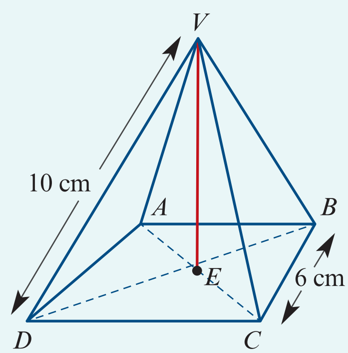
$B = 46.75°$ (correct to two decimal places)

Example 17 Using Pythagoras' theorem in three dimensions

The diagram shows a pyramid with a square base. The base has sides 6 cm long and the edges VA, VB, VC, VD are each 10 cm long.

a Find the length of DB.
b Find the length of BE.
c Find the length of VE.
d Find the magnitude of angle VBE.

Give all answers correct to two decimal places.



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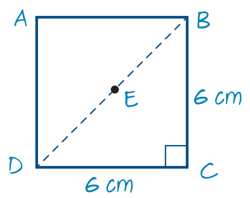
Solution

a



$DB^2 = 6^2 + 6^2$

$= 72$

$\therefore DB = 8.4852\ldots$

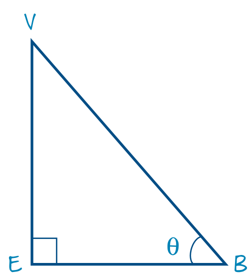
The length of DB is 8.49 cm, correct to two decimal places.

b $BE = \frac{1}{2}DB$

$\therefore BE = \frac{1}{2}\sqrt{72} = 4.2426\ldots$

The length of BE is 4.24 cm, correct to two decimal places.

c



$VE^2 = VB^2 = EB^2$

$= 100 - \frac{1}{4} \times 72$

$= 100 - 18 = 82$

$\therefore VE = \sqrt{82} = 9.0553\ldots$

The length of VE is 9.06 cm, correct to two decimal places.

d $\sin\theta = \frac{VE}{VB} = \frac{\sqrt{82}}{10}$

$= 0.9055\ldots$

$\therefore \theta = 64.90°$

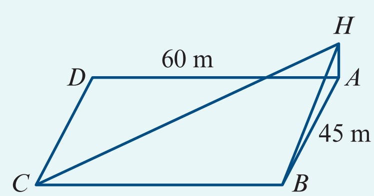
The magnitude of angle VBE is 64.90°, correct to two decimal places.

Example 18 Using Pythagoras' theorem and tan in three dimensions

A communications mast is erected at corner A of a rectangular courtyard $ABCD$ whose sides measure 60 m and 45 m. If the angle of elevation of the top of the mast from C is 12°, find:

a the height of the mast

b the angle of elevation of the top of the mast from B (where $AB = 45$ m).



Give answers correct to two decimal places.

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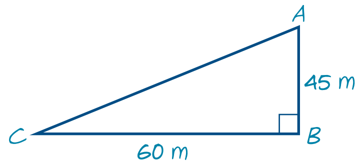
Solution

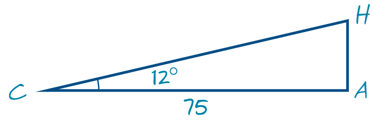
a



$AC^2 = AB^2 + CB^2$

$= 45^2 + 60^2 = 5625$

$\therefore\ AC = 75$



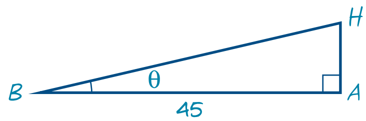
$\dfrac{HA}{75} = \tan 12°$

$\therefore\ HA = 75 \tan 12°$

$= 15.9417$

The height of the mast is 15.94 m, correct to two decimal places.

b



$\tan\theta = \dfrac{HA}{45}$

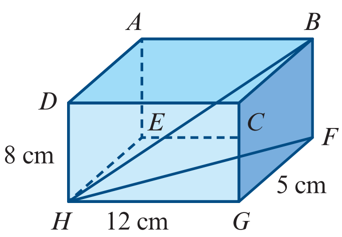
$= 0.3542\ldots$

$\therefore\ \theta \approx 19.51°$

The angle of elevation of the top of the mast, H, from B is 19.51°, correct to two decimal places.

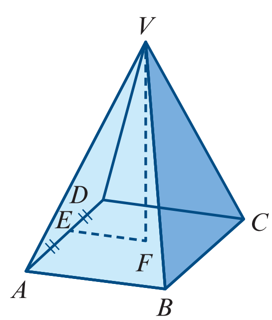
Exercise 18F

1 *ABCDEFGH* is a cuboid with dimensions as shown.



- **a** Find the length of *FH*.
- **b** Find the length of *BH*, correct to two decimal places.
- **c** Find the magnitude of angle *BHF*, correct to one decimal place.
- **d** Find the magnitude of angle *BHG*, correct to two decimal places.

2 *VABCD* is a right pyramid with a square base. The sides of the base are 8 cm in length. The height, *VF*, of the pyramid is 12 cm.



- **a** Find *EF*.
- **b** Find the magnitude of angle *VEF*.
- **c** Find the length of *VE*.
- **d** Find the length of a sloping edge.
- **e** Find the magnitude of angle *VAD*.
- **f** Find the surface area of the pyramid.

Where decimals are involved, give answers correct to two decimal places.

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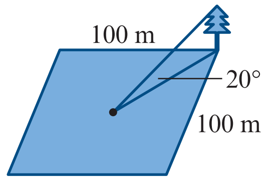
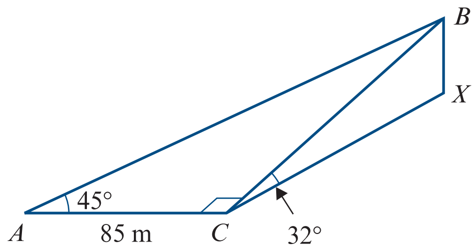
3 A tree stands at the corner of a square playing field. Each side of the square is 100 m long. At the centre of the field a tree subtends an angle of 20°. What angle does it subtend at each of the other three corners of the field, correct to the nearest degree?



4 Suppose that A, C and X are three points in a horizontal plane and B is a point vertically above X. If the length of $AC = 85$ m and the magnitudes of angles BAC, ACB and BCX are 45°, 90° and 32°, respectively, find:



a the distance CB, correct to the nearest metre.

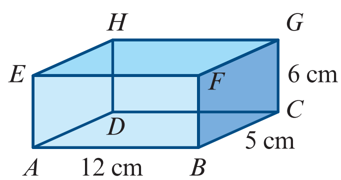
b the height XB, correct to the nearest metre.

5 Standing due south of a tower 50 m high, the angle of elevation of the top is 26°. What is the angle of elevation after walking a distance 120 m due east?

6 From the top of a cliff 160 m high, two buoys are observed. Their bearings are 337° and 308°. Their respective angles of depression are 3° and 5°. Calculate the distance between the buoys, correct to the nearest metre.

7 Find the magnitude of each of the following angles for the cuboid shown.

a ACE **b** HDF **c** ECH



8 From a point, A, due north of a tower, the angle of elevation to the top of the tower is 45°. From a point, B, 100 m on a bearing of 120° from A, the angle of elevation is 26°. Find the height of the tower.

9 A and B are two positions on level ground. From an advertising balloon at a vertical height of 750 m, A is observed in an easterly direction and B at a bearing 106°. The angles of depression of A and B, as viewed from the balloon, are 40° and 20°, respectively. Find the distance between A and B.

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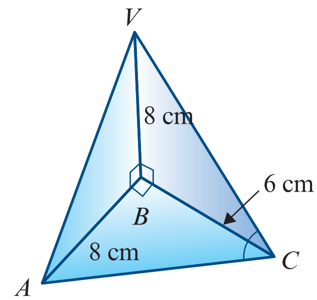
10 Angles VBC, VBA and ABC are right angles.

a Find the distance VA.

b Find the distance VC.

c Find the distance AC.

d Find the magnitude of angle VCA.

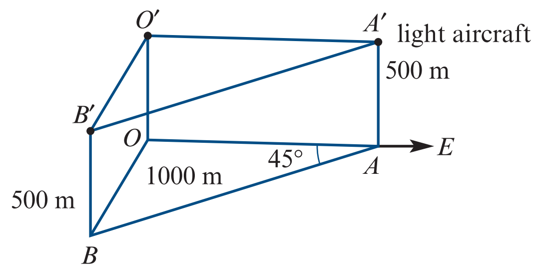


11 A right pyramid, height 6 cm, stands on a square base of side 5 cm. Find:

a the length of a sloping edge

b the area of a triangular face.

12 A light aircraft flying at a height of 500 m above the ground is sighted by an observer stationed at a point, O, on the ground, measured to be 1 km from the plane. The aircraft is flying south-west (along $A'B'$) at 300 km/h.



a How far will it travel in 1 minute?

b Find its bearing from OO' at this time.

c What will be its angle of elevation from O at this time?

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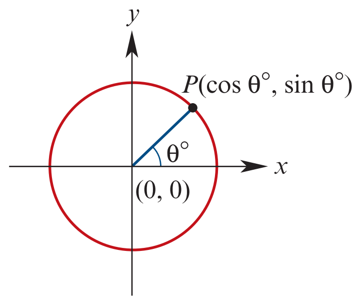
Key ideas and chapter summary

Sine and cosine

Sine and **cosine** may be defined for any angle through the unit circle. For an angle of $\theta°$, a point P on the unit circle is defined as illustrated. The angle is measured in an anticlockwise direction from the positive direction of the x-axis. $\cos\theta°$ is defined as the x-coordinate of the point P and $\sin\theta°$ is defined as the y-coordinate of P. Your calculator gives approximate values for these coordinates.



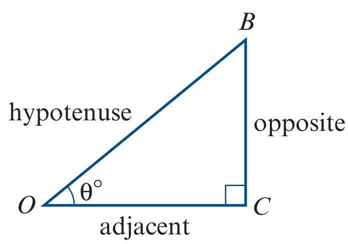
Trigonometric ratios

For a right-angled triangle the naming of sides with respect to an angle $\theta°$ is shown.

$$\sin\theta° = \frac{\text{opp}}{\text{hyp}}\left(\frac{\text{opposite}}{\text{hypotenuse}}\right)$$

$$\cos\theta° = \frac{\text{adj}}{\text{hyp}}\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)$$

$$\tan\theta° = \frac{\sin\theta°}{\cos\theta°} = \frac{\text{opp}}{\text{adj}}$$

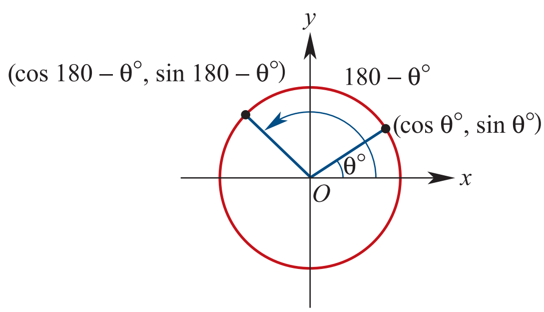


Obtuse angles with sine and cosine

From the unit circle, note that:

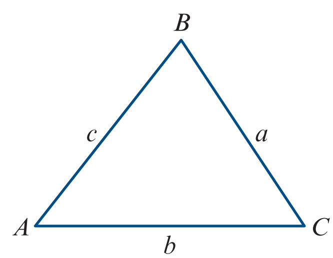
$\sin\theta° = \sin 180 - \theta°$ e.g. $\sin 45° = \sin 135°$

and $\cos\theta° = -\cos 180 - \theta°$ e.g. $\cos 45° = -\cos 135°$



Labelling convention

Interior angles are denoted by upper-case letters and the length of the side opposite an angle is denoted by the corresponding lower-case letter. The magnitude of angle BAC is denoted by A. The length of side BC is denoted by a.



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Review

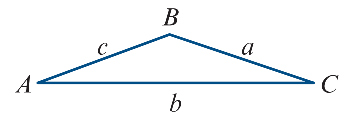
Sine rule, when to use
The **sine rule** is used to find unknown quantities in a triangle when one of the following situations arises:

- one side and two angles are given
- two sides and a non-included angle are given.

In the first of the two cases a unique triangle is defined but for the second it is possible for two triangles to exist.

Sine rule
The **sine rule** states that for triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



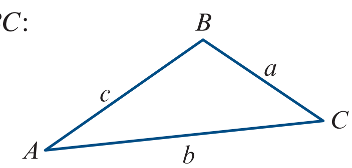
Cosine rule, when to use
The **cosine rule** is used to find unknown quantities in a triangle when one of the following situations arises:

- two sides and an included angle are given
- three sides are given.

Cosine rule
The **cosine rule** states that for triangle ABC:

$a^2 = b^2 + c^2 - 2bc\cos A$ or,

equivalently, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$



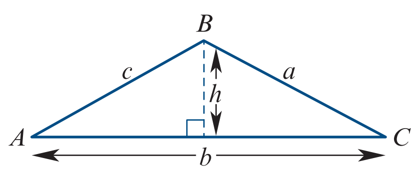
The symmetrical results also hold, i.e.:

- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$

Area of a triangle
The **area of a triangle** is given by the formula $A = \frac{1}{2}bh$.

By observing that $h = c\ \sin A$ the following formula can be found:

$A = \frac{1}{2}bc\sin A$



That is, the area is given by half the product of the length of two sides and the sine of the angle included between them.

Heron's formula
Heron's formula states that the area of a triangle with side lengths a, b and c is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter and $s = \dfrac{a+b+c}{2}$.

Angle of elevation
The **angle of elevation** is the angle between the horizontal and a direction above the horizontal.

Angle of depression
The **angle of depression** is the angle between the horizontal and a direction below the horizontal.

Three-figure bearing
The **three-figure bearing** is the direction measured from north clockwise (also called the compass bearing).

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Review

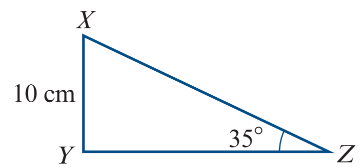
Skills check

Having completed this chapter you should be able to:

- apply the trigonometric ratios sine, cosine and tangent to right-angled triangles
- apply the sine rule to triangles to determine unknown lengths and angles
- apply the cosine rule to triangles to determine unknown lengths and angles
- determine the area of any triangle given
  - the base and height ($A = \frac{1}{2} \times \text{base} \times \text{height}$)
  - two sides and an included angle ($A = \frac{1}{2}ab\sin\theta$)
  - the lengths of three sides (Heron's formula)
- apply the idea of angle of depression
- apply the idea of angle of elevation
- apply the idea of a bearing.

Multiple-choice questions

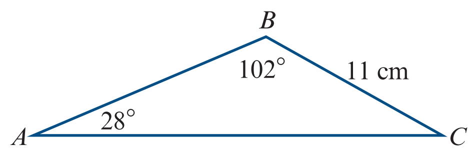
1 In the right-angled triangle XYZ, the right angle is at Y and $\angle YZX = 35°$. The length of $XY = 10$ cm. The length of XZ is given by:



A $10\cos 35°$ **B** $10\sin 35°$ **C** $\dfrac{10}{\sin 35°}$

D $\dfrac{10}{\cos 35°}$ **E** $10\tan 35°$

2 The length of AC, correct to one decimal place, is:



A 13.2 cm **B** 22.9 cm

C 23.4 cm **D** 40.4 cm

E 40.6 cm

3 In a triangle ABC, $a = 5.2$ cm, $b = 6.8$ cm and $c = 7.3$ cm. $\angle ACB$, correct to the nearest degree, is:

A 43 **B** 63° **C** 74° **D** 82° **E** 98°

4 In a triangle ABC, $a = 30$, $b = 21$ and $\cos C = \dfrac{51}{53}$. The value of c, to the nearest whole number, is:

A 9 **B** 10 **C** 11 **D** 81 **E** 129

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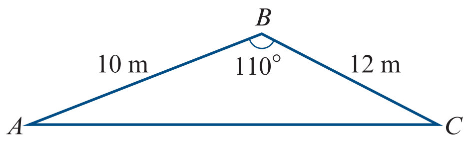
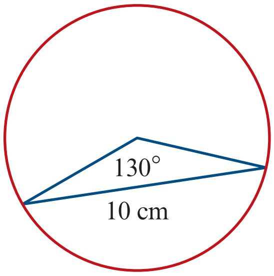
5 The length of AC, to the nearest metre, is:

A 16 **B** 18 **C** 22
D 36 **E** 326



6 The length of the radius of the circle shown, correct to two decimal places, is:

A 5.52 cm **B** 8.36 cm **C** 9.01 cm
D 12.18 cm **E** 18.13 cm



7 In a triangle XYZ, $x = 21$ cm, $y = 18$ cm and $\angle YXZ = 62°$. The size of $\angle XYZ$, correct to one decimal place, is:

A 0.4° **B** 0.8° **C** 1.0° **D** 49.2° **E** 53.1°

8 In a parallelogram $ABCD$, $AB = CD = 8$ cm and $BC = AD = 12$ cm. If $\angle BCD = 52°$, the length of the diagonal AC, to the nearest centimetre, is:

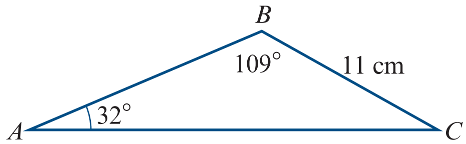
A 12 cm **B** 18 cm **C** 16 cm **D** 149 cm **E** 267 cm

9 The area of the triangle ABC, where $b = 5$ cm, $c = 3$ cm, $\angle A = 30°$ and $\angle B = 70°$, is:

A 2.75 cm^2 **B** 3.75 cm^2 **C** 6.50 cm^2
D 7.50 cm^2 **E** 8 cm^2

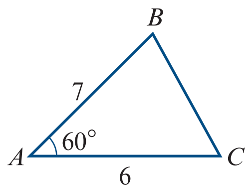
10 The length of AC, correct to one decimal place, is:

A 6.2 cm **B** 16.3 cm
C 19.6 cm **D** 40.4 cm
E 20.3 cm



11 The square of the length of side BC is:

A 36 **B** 85 **C** 49
D 42 **E** 43



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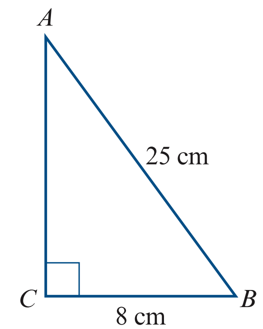
Review

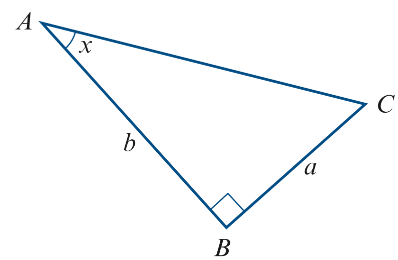
12 For the triangle shown, the value of the cosine of angle ABC is:



A $\frac{8}{25}$ **B** 74 **C** $\frac{5}{6}$

D $\frac{-5}{6}$ **E** 73

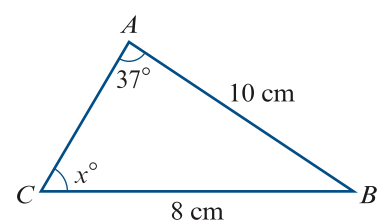
13 In the triangle ABC, $\cos x =$



A $\frac{a}{\sqrt{a^2+b^2}}$ **B** $\frac{b}{\sqrt{a^2+b^2}}$ **C** $\frac{a}{b}$

D $\frac{a}{b}$ **E** $\frac{\sqrt{a^2+b^2}}{a}$

14 In triangle ABC, $\sin x°$ is equal to:



A $1.25 \sin 37°$ **B** $\frac{1.25}{\sin 37°}$ **C** $\frac{\sin 37°}{1.25}$

D $0.8 \sin 37°$ **E** $\frac{0.8}{\sin 37°}$

15 A man walks 5 km due east followed by 7 km due south. The bearing he must take to return to the start is:

A 036° **B** 306° **C** 324° **D** 332° **E** 348°

16 A boat sails at a bearing of 215° from A to B. The bearing he would take from B to return to A is:

A 035° **B** 055° **C** 090° **D** 215° **E** 250°

17 From a point on a cliff 500 m above sea level, the angle of depression to a boat is 20°. The distance from the foot of the cliff to the boat, to the nearest metre, is:

A 182 m **B** 193 m **C** 210 m **D** 1374 m **E** 1834 m

18 A tower 80 m high is 1.3 km away from a point on the ground. The angle of elevation to the top of the tower from this point, correct to the nearest degree, is:

A 1° **B** 4° **C** 53° **D** 86° **E** 89°

19 A boat sails from a harbour on a bearing of 035° for 100 km. It then takes a bearing of 190° for 50 km. The distance from the harbour, correct to the nearest km, is:

A 51 km **B** 58 km **C** 59 km **D** 108 km **E** 3437 km

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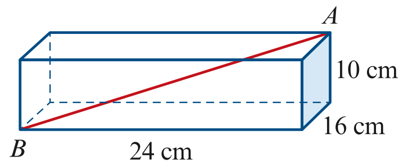
20 A hiker walks 5.2 km on a bearing of 160° and then takes a bearing of 035° and walks 7 km. The bearing he must take to return directly to the start is:

A 047° **B** 215° **C** 223° **D** 235° **E** 262°

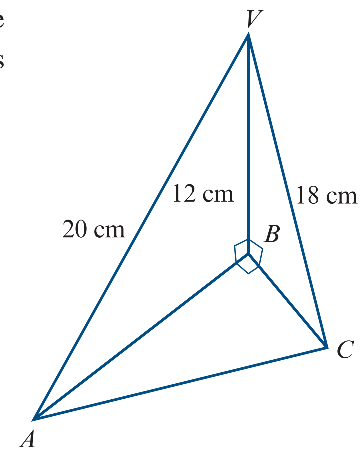
21 The diagram shows a rectangular prism (cuboid). The dimensions are as marked. The length of the diagonal is closest to:



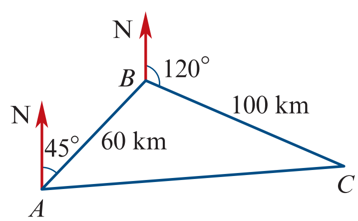
A 26 cm **B** 30 cm **C** 31 cm
D 34 cm **E** 39 cm

22 $VABC$ is a pyramid. V is vertically above B and triangle ABC has a right angle at B. The distance between points A and C is closest to:

A 12 cm **B** 21 cm **C** 26 cm
D 30 cm **E** 36 cm

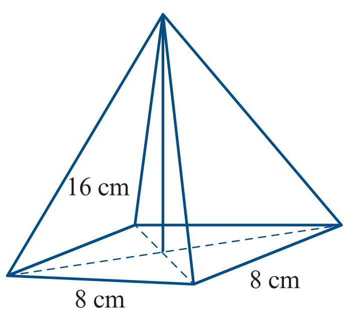


23 A ship travels from point A to point B on a bearing of 045° for a distance of 60 km. It then travels to point C on a bearing of 120° for a distance of 100 km. The distance of C to A is closest to:



A 123 km **B** 125 km **C** 127 km
D 129 km **E** 131 km

24 A right pyramid with a square base is shown in the diagram. Each edge of the square base has length 8 cm and the height of the pyramid is 16 cm. The length of a sloping edge of the pyramid, in centimetres, is:



A $\sqrt{288}$ **B** $\sqrt{155}$ **C** $\sqrt{125}$
D $\sqrt{324}$ **E** $\sqrt{425}$

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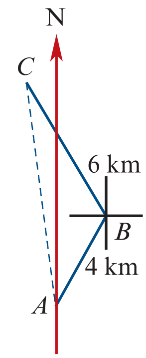
25 A hiker walks 4 km from point A on a bearing of 030° to point B, and then 6 km on a bearing of 330° to point C. The distance AC, in kilometres, is:

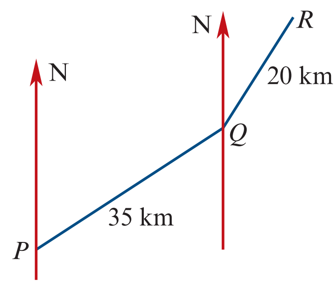


A $\dfrac{4}{\sin 30°}$ **B** $\sqrt{6^2 + 4^2 - 48\cos 120°}$

C $\sqrt{6^2 + 4^2 + 48\cos 120°}$ **D** $6\sin 60°$

E $\sqrt{52}$

26 A ship travels from P to Q on a course of 050° for 35 km and then from Q to R on a course of 020° for 20 km. How far east of P is R, in kilometres?



A $35\sin 50° + 20\sin 20°$

B $35\cos 50° + 20\cos 20°$

C $\dfrac{35}{\sin 40°} + \dfrac{50}{\sin 70°}$

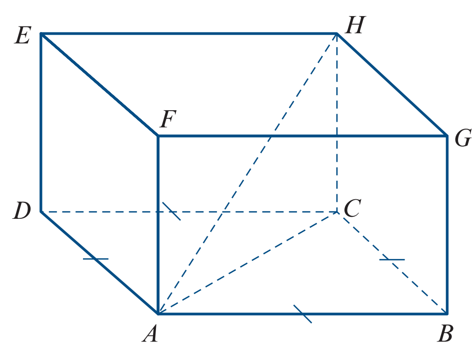
D $\dfrac{35}{\sin 40°} + \dfrac{20}{\cos 70°}$

E $55\cos 110°$

27 There are four telecommunications towers in a city. The towers are called Grey Tower, Black Tower, SilverTower and White Tower.
Grey Tower is 10 km due west of Black Tower.
Silver Tower is 10 km from Grey Tower on a bearing of 300°.
White Tower is 10 km due north of Silver Tower.
Correct to the nearest degree, the bearing of Black Tower from White Tower is:

A 051° **B** 129° **C** 141 **D** 309° **E** 321°

28 A rectangular prism with a square base, $ABCD$, is shown above.
The diagonal of the prism, AH, is 8 cm.
Height of the prism, HC, is 4 cm.
The volume of this rectangular prism is:



A 64 cm^3 **B** 96 cm^3

C 128 cm^3 **D** 192 cm^3

E 256 cm^3

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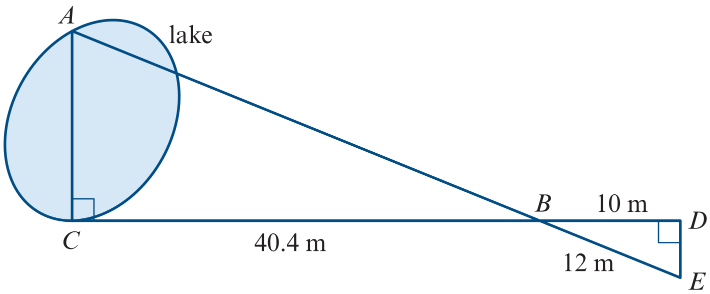
29 The distance, AC, across a small lake can be calculated using the measurements shown in the diagram below.



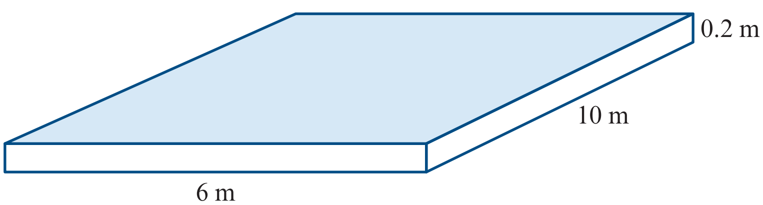
In this diagram, BCA and BDE are right-angled triangles, where $CB = 40.4$ m, $BD = 10$ m and $BE = 12$ m.

The distance between the points A and C, in metres, is closest to:

A 22.4 **B** 26 **C** 33.6 **D** 48.5 **E** 177.8

Extended-response questions

1 A shed is built on a concrete slab. The concrete slab is a rectangular prism 6 m wide, 10 m long and 0.2 m deep.



a Determine the volume of the concrete slab in m^3.

b On a plan of the concrete slab, a 3 cm line is used to represent a length of 6 m.

i What scale factor is used to draw this plan?

The top surface of the concrete slab shaded in the diagram above has an area of 60 m^2.

ii What is the area of this surface on the plan?

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Review

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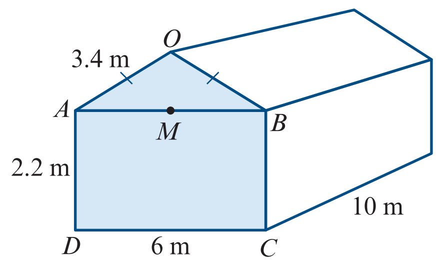
Review

2 The shed has the shape of a prism. Its front face, $AOBCD$, is shaded in the diagram below. $ABCD$ is a rectangle and M is the midpoint of AB.



a Show that the length of OM is 1.6 m.

b Show that the area of the front face of the shed, $AOBCD$, is 18 m^2.

c Find the volume of the shed in m^3.

d All inside surfaces of the shed, including the floor, will be painted.

i Find the total area that will be painted in m^2.

One litre of paint will cover an area of 16 m^2.

ii Determine the number of litres of paint that is required.

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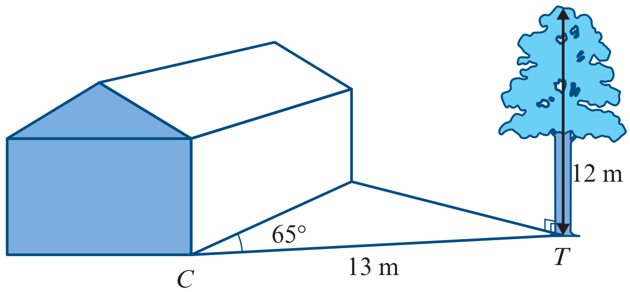
3 A tree, 12 m tall, is growing at point T near the shed. The distance, CT, from corner C of the shed to the centre base of the tree is 13 m.

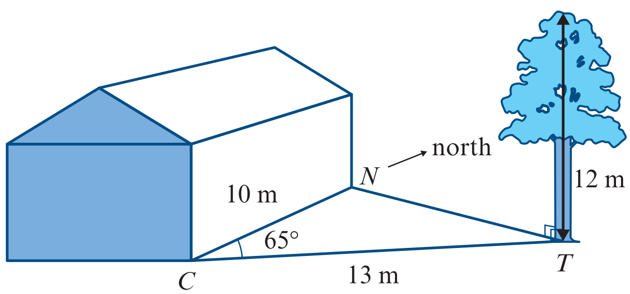


a Calculate the angle of elevation of the top of the tree from point C.

Write your answer, in degrees, correct to one decimal place.

N and C are two corners at the base of the shed. N is due north of C.

The angle, TCN, is 65°.



b Show that, correct to one decimal place, the distance, NT, is 12.6 m.

c Calculate the angle, CNT, correct to the nearest degree.

d Determine the bearing of T from N. Write your answer correct to the nearest degree.

e Is it possible for the tree to hit the shed if it falls? Explain your answer showing appropriate calculations.

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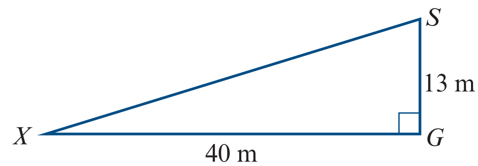
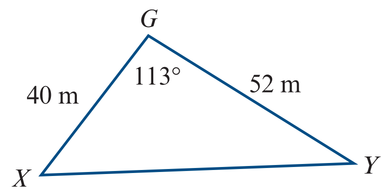
4 A spectator, S, in the grandstand of an athletics ground is 13 m vertically above point G.
Competitor X, on the athletics ground, is at a horizontal distance of 40 m from G.

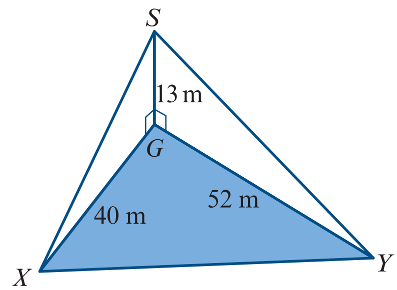


a Find the distance, SX, correct to the nearest metre.
Competitor X is 40 m from G and competitor Y is 52 m from G.
The angle XGY is $113°$.



b **i** Calculate the distance, XY, correct to the nearest metre.
ii Find the area of triangle XGY, correct to the nearest square metre.

c Determine the angle of elevation of spectator S from competitor Y, correct to the nearest degree. Note that X, G and Y are on the same horizontal level.



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Chapter 19

19

Module 3: Geometry and measurement

Spherical geometry

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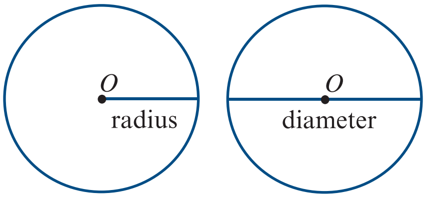
19A Arc length, area of a sector, area of a segment

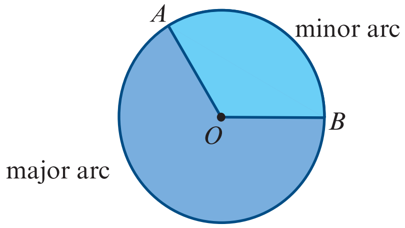
Arc length

Any line segment drawn from the centre of a given circle to any point on the circle is called a **radius** (plural: radii).

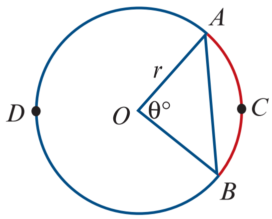
Any line segment joining two points on the circle and passing through the centre is called the *diameter* of the circle.



Any two points on a circle divide the circle into **arcs**. The shorter arc is called the *minor arc*, and the longer is the *major arc*.



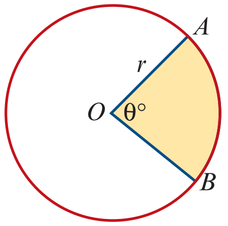
The arc ACB is said to subtend the angle $\angle AOB$ at the centre of the circle. If $\angle AOB = \theta°$ and radius length is r units, then s units, the length of arc ACB, will be a fraction of the circumference.



The fraction of the circumference will be $\frac{\theta}{360}$. Recall that the circumference C of a circle of radius r is given by $C = 2\pi r$.

Therefore the length s of an arc that subtends an angle of $\theta°$ at the centre is:

$$s = \frac{\theta}{360} \times 2\pi r$$



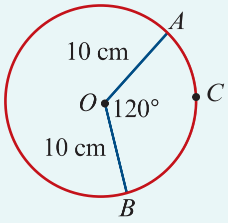
Length of an arc

The length s of an arc, of a circle of radius r, that subtends an angle of $\theta°$ at the centre is given by:

$$s = \frac{\pi r \theta}{180}$$

Example 1 Calculating the length of an arc

In this circle, centre O, radius length 10 cm, the angle subtended at O by arc ACB has magnitude. Find the length of the arc ACB, correct to one decimal place.



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Solution

1 Write down the formula.

$$s = \frac{\pi r \theta}{180}$$

2 Substitute $\theta = 120°$ and $r = 10$.

$$s = \frac{\pi \times 10 \times 120}{180}$$
$$= \frac{20\pi}{3}$$
$$\approx 20.9 \text{ cm (correct to one decimal place)}$$

Area of a sector

If $\angle AOB = \theta°$ the area of the sector is a fraction of the area of the circle.

The area A of a circle of radius r is given by $A = \pi r^2$.

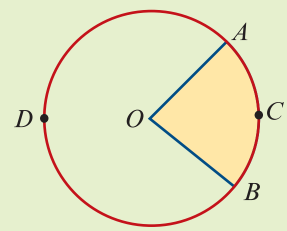
Minor and major sectors

In the diagram below with circle centre O, the yellow region is a *minor sector* and the unshaded region is a *major sector*.

Area of a sector

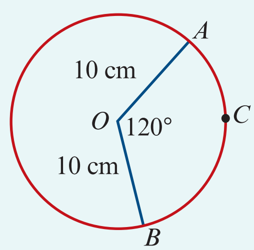
The area A of a sector, of a circle of radius r, where the arc of the sector subtends an angle of $\theta°$ at the centre is given by:

$$A = \frac{\pi r^2 \theta}{360}$$



Example 2 Calculating the area of a sector

In this circle, centre O, radius length 10 cm, the angle subtended at O by arc ACB has magnitude 120°.



Find:

a the area of the minor sector AOB

b the area of the major sector AOB.

Give your answer correct to two decimal places.

Solution

a 1 Write down the formula.

$$A = \frac{\pi r^2 \theta}{360}$$

2 Substitute $\theta = 120°$ and $r = 10$.

$$A = \frac{\pi \times 100 \times 120}{360}$$
$$= \frac{100\pi}{3}$$
$$\approx 104.72 \text{ cm}^2$$

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b **1** Write down the formula.

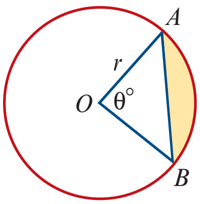
$$A = \frac{\pi r^2 \theta}{360}$$

2 Substitute $\theta = 240°$ and $r = 10$.

$$A = \frac{\pi \times 100 \times 240}{360}$$
$$= \frac{200\pi}{3}$$
$$\approx 209.44 \text{ cm}^2$$

▶ Area of a segment

Every chord divides the interior of a circle into two regions called segments. The smaller is called the *minor segment*, and the larger is the *major segment*. In the diagram shown the minor segment is shaded and the major segment is unshaded.



Area of segment shaded = area of minor sector OAB – area of ΔAOB

Area of a segment

The area A of a segment of a circle of radius r, where the arc of the sector subtends an angle of $\theta°$ at the centre is given by:

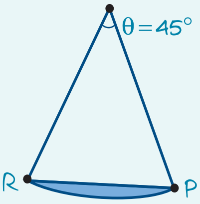
$$A = \text{area of minor sector } OAB - \text{area of } \Delta AOB$$
$$= \frac{\pi r^2 \theta}{360} - \frac{1}{2} r^2 \sin\theta$$
$$= r^2\left(\frac{\pi\theta}{360} - \frac{1}{2}\sin\theta\right)$$

Example 3 Calculating the area of a segment

Find the area A of the segment that is shaded blue.

The radius of the circle is 10 cm.

Give your answer, correct to two decimal places.



Solution

1 Write down the formula.

$$A = \frac{\pi r^2 \theta}{360} - \frac{1}{2} r^2 \sin\theta$$

2 Substitute $\theta = 45°$ and $r = 10$.

$$A = \frac{\pi \times 10^2 \times 45}{360} - \frac{1}{2} \times 10^2 \times \sin 45°$$
$$\approx 3.91 \text{ cm}^2$$

The area is 3.91 cm^2, correct to two decimal places.

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Exercise 19A

Skillsheet

Arc length

1 What is the circumference of each circle? Answer correct to two decimal places.

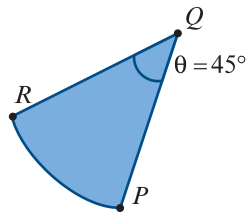
a Radius of 8 cm **b** Radius of 14 m **c** Radius of 45 mm
d Diameter of 12 mm **e** Diameter of 14 m

2 What fraction of a circle is each sector?

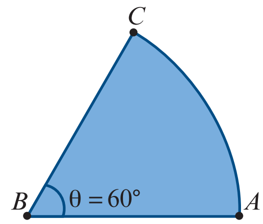
a Angle at the centre is 90°. **b** Angle at the centre is 270°.
c Angle at the centre is 30°. **d** Angle at the centre is 120°.
e Angle at the centre is 60°. **f** Angle at the centre is 150°.

3 Find the arc length of each sector. The radius is 10 cm. Answer correct to two decimal places.

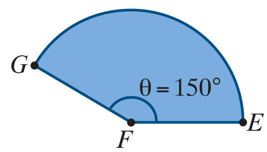
a



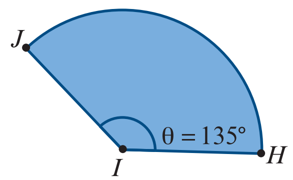
b



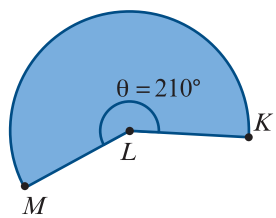
c



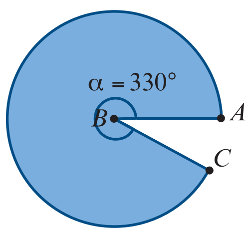
d



e



f



4 Find the arc length of the following arcs where the radius of the circle (given in cm) and the angle subtended at the centre are given. Give your answer correct to two decimal places.

a $r = 15,\ \theta = 50°$ **b** $r = 20,\ \theta = 15°$ **c** $r = 30,\ \theta = 150°$
d $r = 16,\ \theta = 135°$ **e** $r = 40,\ \theta = 175°$ **f** $r = 30,\ \theta = 210°$

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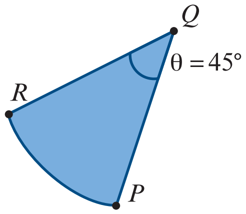
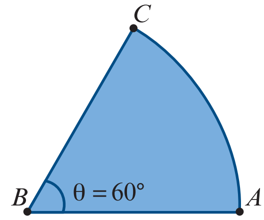
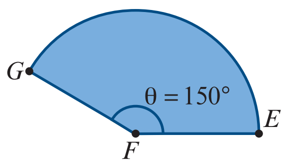
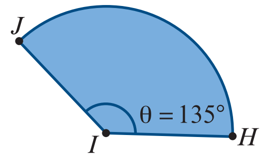
Area of a sector

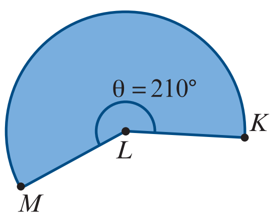
5 Find the area of each of the following sectors, correct to one decimal place. The radius of the circle is 10 cm.

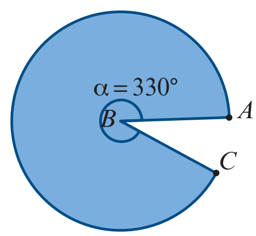
a


b


c


d


e


f


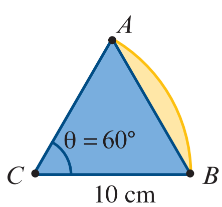
6 Find the area of each sector where the radius of the circle (given in cm) and the angle subtended at the centre are given. Give your answer correct to two decimal places.

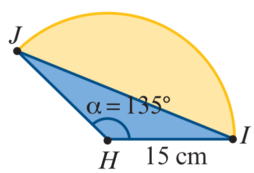
a $r = 10,\ \theta = 150°$ **b** $r = 40,\ \theta = 35°$ **c** $r = 45,\ \theta = 150°$

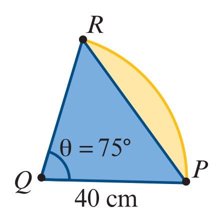
d $r = 16,\ \theta = 300°$ **e** $r = 50,\ \theta = 108°$ **f** $r = 30,\ \theta = 210°$

Area of a segment

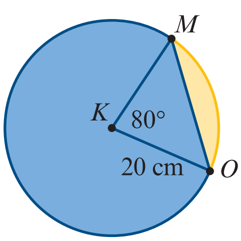
7 Find the area of each segment giving your answer correct to two decimal places.

a


b


c


8 Find the area of the major and minor segment. That is the yellow segment (the minor segment) and the blue segment, the major segment.



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9 Find the area of the major and minor segments where the radius of the circle (given in cm) and the angle subtended at the centre are given. Give your answer correct to two decimal places.

a $r = 10,\ \theta = 150°$ **b** $r = 40,\ \theta = 35°$ **c** $r = 45,\ \theta = 150°$

d $r = 16,\ \theta = 300°$ **e** $r = 50,\ \theta = 108°$ **f** $r = 30,\ \theta = 210°$

Mixed exercises

10 Find the arc length that subtends an angle of magnitude 105° at the centre of a circle of radius length 25 cm.

11 Find, correct to two decimal places, the size of the angle subtended at the centre of a circle of radius length 30 cm, by:

a an arc of length 50 cm **b** an arc of length 25 cm.

12 A chord of length 6 cm is drawn in a circle of radius 7 cm. Find:

a the length of the minor arc cut off by the chord

b the area of the smaller region inside the circle cut off by the chord.

13 Find the area of the region between an equilateral triangle of side length 10 cm and the circle that passes through the three vertices of the triangle).

Applications

14 A person stands on level ground 60 m from the nearest point of a cylindrical tank of radius length 20 m. Calculate:

a the circumference of the tank

b the percentage of the circumference that is visible to the person.

15 The minute hand of a large clock is 4 m long.

a How far does the tip of the minute hand move between 12:10 p.m. and 12:35 p.m?

b What is the area covered by the minute hand between 12:10 p.m. and 12:35 p.m?

16 Two circles of radii 3 cm and 4 cm have their centres 5 cm apart. Calculate the area of the region common to both circles.

17 A sector of a circle has perimeter of 32 cm and an area of 63 cm^2. Find the radius length and the magnitude of the angle subtended at the centre of the two possible sectors.

18 A frame in the shape of an equilateral triangle encloses three circular discs of radius length 5 cm so that the discs touch each other. Find:

a the perimeter of the smallest frame which can enclose the discs

b the area enclosed between the discs.

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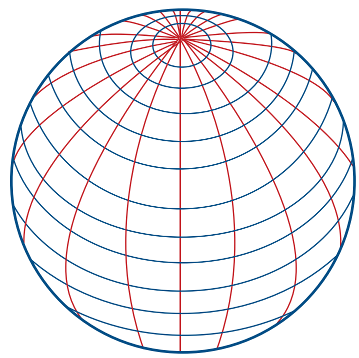
19B Latitude and longitude

The Earth can be modelled by a sphere of radius 6400 km. From your previous work in this module you could work out the surface area and volume of such a sphere.

In this section we are interested in describing the location of points on the surface of the Earth. We do this in a manner similar to how we described points in the plane with Cartesian coordinates.

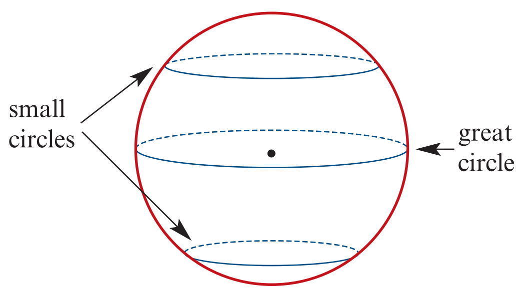
We do this by using a grid of lines as shown here. They are used to give *coordinates* called *latitude* and *longitude*. The red lines will be used for longitude and the blue lines for latitude.



▶ Great circles and small circles

A **great circle** is a section of a sphere that contains a diameter of the sphere. It contains the centre of the sphere. Sections of the sphere that do not contain a diameter are called **small circles**. A small circle does not contain the centre of the sphere.

The great circle shown is in the *plane of the equator*.



▶ Meridians and parallels

Meridians of longitude are semi-great circles which pass through the north and south poles. The red lines on the sphere above are meridians of longitude.

Parallels of latitude are small circles whose planes are parallel to that of the equator. The blue lines in the sphere above are parallels of latitude.

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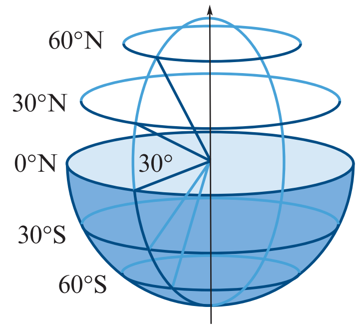
Latitude

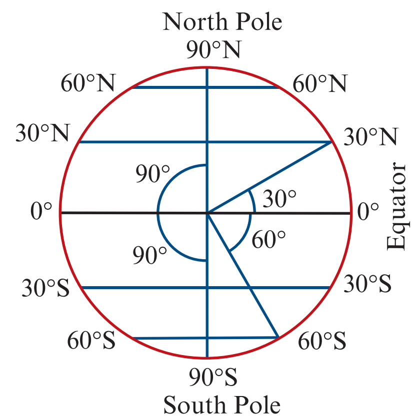
The blue lines in the diagram opposite are parallels of latitude. The latitude of a point on a sphere is the elevation of the point from the plane of the equator.



- The equator has latitude 0°N.
- The North Pole has latitude 90°N.
- The South Pole has latitude 90°S.

In the diagram the latitudes 60°N, 30°N, 60°S and 30° are shown.

In the diagram opposite the Earth has been sliced in half along a great circle. The vertical line through the poles is perpendicular, or at 90°, to the plane of the equator.

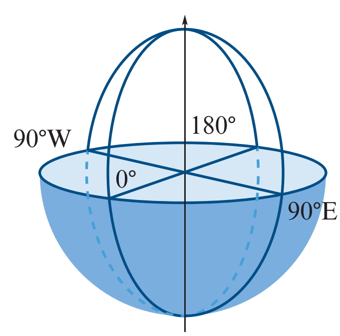


At the surface of the Earth, at a given latitude, draw a line from that location to the centre of the Earth. The angle between this line and the equator is the latitude measurement.

The diagram shows two examples: one for 30°N and one for 60°S.

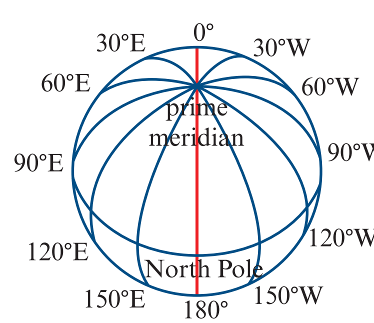
Longitude and the prime meridian

Lines of longitude are measured in degrees east or west of the *prime meridian* (0°). The lines of longitude shown in the diagram opposite are 0°, 90°E, 180° and 90°W. Note that you don't need to add E or W to the 0° or the 180°.



The prime meridian passes through Greenwich in England.

The diagram opposite is a diagram of the Earth looking down from the North Pole. The evident plane in the diagram above is the plane of the equator. The angle formed between the prime meridian and the line from the centre of the Earth to the point where the meridian of longititude meets that plane is the longitude.



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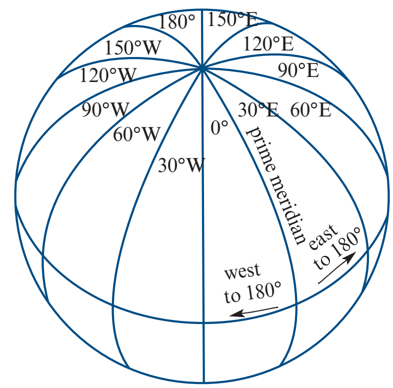
The meridian 120°W is on the same great circle as the meridian 60°E.

The meridian 30°W is on the same great circle as the meridian 150°E.

This is shown in a different way in the diagram opposite.



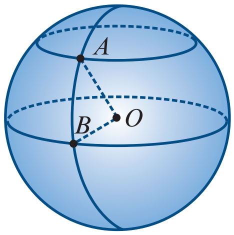
Latitude and longitude

Any point on the Earth's can be described by giving its latitude and longitude. For example, Melbourne has latitude 37.8°S and longitude 144.9°E. These are called the *coordinates of the location*.

Distance along a meridian

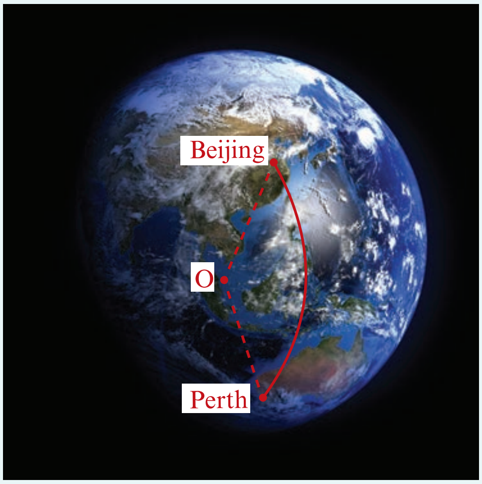
On a flat surface, the shortest distance between two points is a straight line. Since the Earth's surface is curved, the shortest distance between A and B is the arc length AB of the great circle that passes through A and B. This is called the *great circle distance*.

Each meridian is a great circle.



Example 4 **Finding a distance along a meridian**

Beijing, China and Perth, Australia have coordinates (40°N, 116°E) and (32°S, 116°E) respectively. Calculate the shortest distance between Beijing and Perth, to the nearest kilometre, given that the Earth's radius is 6400 km. Give your answer to the nearest kilometre.



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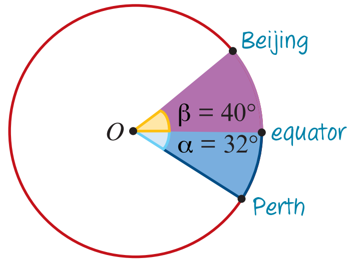
Solution

The two cities have the same longitude correct to the nearest degree. Therefore they are on the great circle which is the meridian of longitude 116°E. The angle subtended by the points representing the cities = $(40 + 32)^\circ = 72^\circ$.



The length of the arc

$$= \frac{\pi \times 6400 \times 72}{180}$$

$$\approx 8042 \text{ km}$$

Distance between two points on the equator

The equator is a great circle and therefore distance between points on the equator are possible to find using our knowledge of length of arcs.

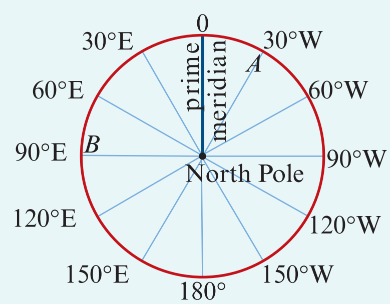
Example 5 Finding a distance between two points on the equator

Point A has longitude 30°W and latitude 0°.

Point B has longitude 90°E and latitude 0°.

Find the distance between the two points.

Give your answer to the nearest kilometre.



Solution

We use the formula for the length of an arc:

$$s = \frac{\pi r \theta}{180}$$

Here $r = 6400$ and $\theta = (30 + 90)^\circ = 120^\circ$.

Hence $s = \dfrac{\pi \times 6400 \times 120}{180}$

$\approx 13\,404$ km

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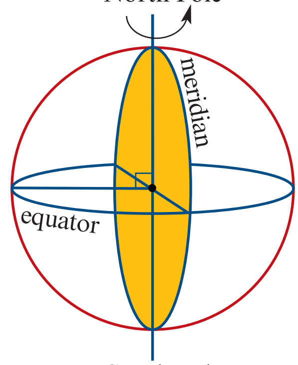
Distance from a pole or from the equator

Every meridian passes through the poles and a point on the equator. Therefore you can find the distance from any point on the surface of the Earth to a pole or the equator if we know its latitude.



Example 6 Distance to the equator or a pole

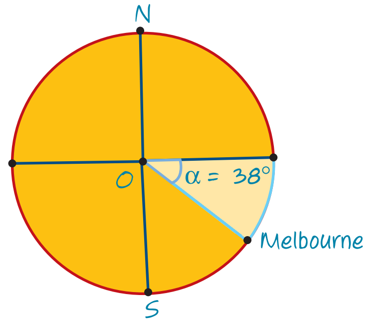
Melbourne has latitude 38°S and longitude 145°E. Find the distance of Melbourne to:

a the equator **b** the South Pole **c** the North Pole.

Give your answer to the nearest kilometre.

Solution

a The diagram shows the plane of the meridian 145°E.



We use the length of arc formula:

$$s = \frac{\pi r \theta}{180}$$

$$s = \frac{\pi \times 6400 \times 38}{180}$$

$$\approx 4245 \text{ km}$$

The distance from Melbourne to the equator along the great circle is 4245 km.

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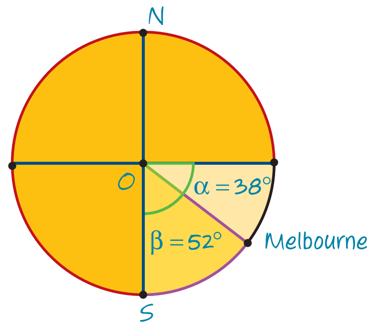
b The required angle $= (90 - 38)^\circ$

$= 52^\circ$



We use the length of arc formula:

$$s = \frac{\pi r \theta}{180}$$

$$s = \frac{\pi \times 6400 \times 52}{180}$$

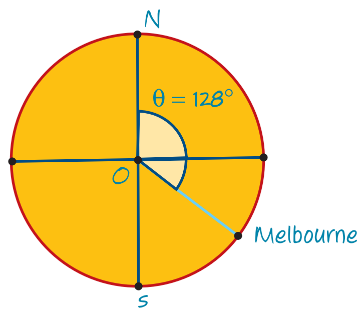
$$\approx 5808 \text{ km}$$

The distance from Melbourne to the South Pole along the great circle is 5808 km.

c The required angle $= (90 + 38)^\circ$

$= 128^\circ$

Note: The distance from the North Pole to the South Pole along a great circle $= \pi \times 6400 \approx 20\,106$ km.



$$s = \frac{\pi \times 6400 \times 128}{180}$$

$$\approx 14\,298 \text{ km}$$

The distance from Melbourne to the North Pole along the great circle is 14 298 km.

Distance along a parallel of latitude

In the following we use Pythagoras' theorem to find the radius of a small circle corresponding to a parallel of latitude. We will illustrate this by finding the distance between Kalgoorlie and Tamworth.

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Example 7 Distance along a parallel of latitude

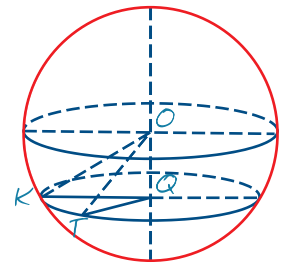
Kalgoorlie, WA (K) has latitude 31°S and longitude 122°E.

Tamworth, NSW (T) has latitude 31°S and longitude 151°E.

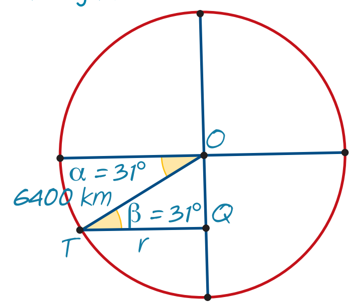
Find the distance along the parallel of latitude 31°S from Kalgoorlie to Tamworth. Give your answer to the nearest kilometre.

Solution

In the diagram opposite, the Earth is shown. K and T represent the locations of Kalgoorlie and Tamworth. The circle passing through K and T is the parallel of latitude 31°S.



First, find radius r km of small circle through K and T.



We need to find the radius of this circle and angle KQT. Then we can find the length of arc KT.

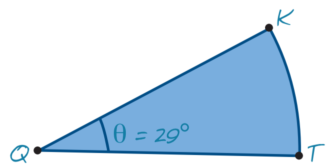
In right-angled triangle OTQ:

$$r = 6400 \cos 31^\circ \approx 5486 \text{ km}$$

In sector QKT we have

$QT = 5486$

and $\angle KQT = (151 - 122)^\circ = 29^\circ$



Length of arc $KT \approx 2777$ km

The distance around the parallel of latitude 31° S is 2777 km.

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Exercise 19B

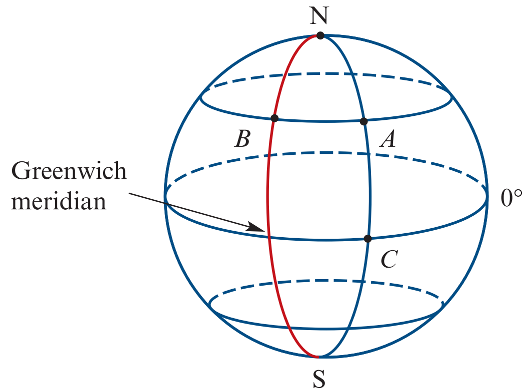
Skillsheet

Latitude and longitude

1 On the diagram opposite the latitude and longitude of point A are (65°N, 75°E).

a What are the coordinates of point B?

b What are the coordinates of point C?



2 Athens has the coordinates (38°N, 24°E) and Sofia has the coordinates (43°N, 23°E).

a What is the latitude and longitude of a point 20° due south of Athens?

b What is the latitude and longitude of a point 20° due east of Athens?

c What is the latitude and longitude of a point 60° due south of Sofia?

d What is the latitude and longitude of a point 60° due west of Sofia?

3 The following table shows the latitude and longitude of cities around the world.

City	*Latitude*	*Longitude*
Melbourne	38°S	145°E
Marseilles	43°N	5°E
Mexico city	19°N	99°W
Wellington	41°S	175°E
Zurich	47°N	8°E
London	52°N	0°
Lima	12°S	77°W
Plymouth	50°N	4°W
Rangoon	19°N	96°

Note: The centre of London is actually 0.1275° west of Greenwich.

Which city or cities:

a are closest to the following latitudes?

i 15°N **ii** 28°S

b are closest to the following longitudes?

i 122°E **ii** 20°W

iii Greenwich meridian

iv Longitude of Sofia

c are in the northern hemisphere?

d are in the western hemisphere?

e have the same latitude?

f have the same longitude?

g are closest to the North Pole?

h are closest to the South Pole?

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Distance between two points on the same meridian

4 Two places on the same meridian have latitudes 22°N and 35°S. Find their distance apart. Give your answer to the nearest kilometre.

5 Cairns 17°S and Melbourne 37.8°S are nearly on the same meridian. Pretend they are and find the distance between them. Give your answer to the nearest kilometre.

6 How far apart are Esperance, WA (34°S, 122°E) and Broome, WA (18°S, 122°E)?

7 Cairns, Queensland and Griffith, NSW are 1920 km apart and both are on the same meridian. If the latitude of Cairns is 17°S find the latitude of Griffith.

8 Calculate the shortest distance along a meridian in each of the following cases. Give your answer to the nearest kilometre.

- **a** Point X: lat. 10°N, long. 18°W, point Y: lat. 45°N, long. 18°W
- **b** Point X: lat. 14°N, long. 35°W, point Y: lat. 13°S, long. 35°W
- **c** Point X: lat. 23°S, long. 140°W, point Y: lat. 67°S, long. 140°W
- **d** Point X: lat. 15°N, long. 60°W, point Y: lat. 25°S, long. 60°W
- **e** Point X: lat. 15°N, long. 70°W point Y: lat. 15°S, long. 70°W

Distance between two points on the equator

9 The difference of longitude between two points on the equator is 32°. Find the distance between them in kilometres.

10 There are places in Ecuador, South America and Somalia, Africa which are on the equator. In Ecuador the longitude of a place X on the equator is 78°W and in Somalia the longitude of a place Y is 42°W. Find the distance between them in kilometres.

11 Calculate the shortest distance along the equator between places A and B having the following longitudes.

- **a** A 137°E B 87°E
- **b** A 57°E B 13°W
- **c** A 57°W B 27°W
- **d** A 140°E B 160°W
- **e** A 95°W B 113°E

12 The distance between two points on the equator is 600 km. What is the difference in their longitudes?

Distance between a point and a pole (north or south) and to the equator

13 New Orleans has latitude 30°N and longitude 90°W. Find the distance of New Orleans to:

- **a** the equator
- **b** the North Pole
- **c** the South Pole.

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14 Izmir has latitude 38°N and longitude 27°E. Find the distance of Izmir to

a the equator **b** the North Pole **c** the South Pole.

15 Ballarat has latitude 37.5500°S and longitude 143.8500°E. Find the distance of Ballarat to:

a the equator **b** the North Pole **c** the South Pole.

16 Find the distance from the given point to the equator and to each of the poles.

a Latitude 42°N, longitude 134°E **b** Latitude 55°N, longitude 45°W

c Latitude 15°S, longitude 35°E **d** Latitude 14°S, longitude 75°W

Distance around a parallel of latitude

17 **a** Find the radius of the small circle which is the parallel of latitude 15°S.

b Find the radius of the small circle which is the parallel of latitude 30°S.

c Find the radius of the small circle which is the parallel of latitude 45°S.

d Find the radius of the small circle which is the parallel of latitude 60°S.

18 For which parallels of latitude is the radius of the small circle half the radius of the equator?

19 Find the distance around the parallel of latitude for the following locations.

a X: latitude 22°N, longitude 134°E; Y: latitude 22°N, longitude 145°E

b X: latitude 32°S, longitude 50°E; Y: latitude 32°N, longitude 80°E

c X: latitude 12°S, longitude 30°E; Y: latitude 12°N, longitude 80°E

20 The position of Salzburg is 48°N, 13°E and the position of Seattle is 48°N, 122°W. Find the distance around the 48°N parallel of latitude between Seattle and Salzburg.

Mixed exercises

21 **a** An aircraft flies from Bairnsdale (38°S, 148°E) due north to Grenfell, NSW (34°S, 148°E). How far is this (great circle distance)?

b It then flies due west to Renmark SA (34°S, 141°E). How far is it from Grenfell to Renmark (the distance around the parallel of latitude)?

c What is the total distance flown?

22 **a** An aircraft flies from Warragul (38°S, 146°E) due north to Cairns, Qld (17°S, 146°E). How far is this (great circle distance)?

b It then flies due west to Derby WA (17°S, 124°E). How far is it from Cairns to Derby (distance around the parallel of latitude)?

c What is the total distance flown?

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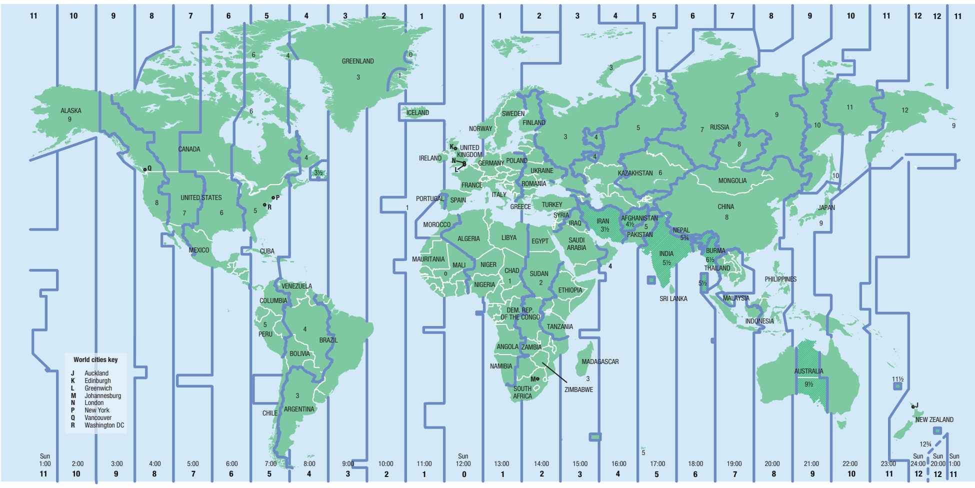
19C Time zones



The time zones are determined largely by the meridians of longitude. You can see from the map above that there are exceptions to this because of local requirements.

If it is 12:00 noon along a meridian, on the other side of the world along the meridian which makes up the other half of the great circle it is midnight. For example, when it is noon in Victoria on the 145°E meridian, it is midnight in the far east of Brazil on the 35°W meridian.

Since the Earth turns 360° in 24 hours, it turns 15° in 1 hour. For every 15° of longitude, the time difference is 1 hour and so for every 1° of longitude the time difference is 4 minutes.

15° longitude = 1 hour time difference; 1° longitude = 4 minutes time difference

Local times around the world are given relative to the time along the prime meridian. This is called Greenwich Mean Time (GMT). Places east of the prime meridian are ahead of GMT, while places west are behind GMT. In the following we do not take 'summer time' into account.

Example 8 Finding time differences

Singapore is located at 1°N 104°E and Sydney is located at 34°S 151°E. What is the time difference between Singapore and Sydney?

Solution

Calculate the difference in longitude using 1 hour for each 15° of longitude.	Difference in longitude $= 151^\circ - 104^\circ = 47^\circ$ $47 \div 15 = 3\frac{2}{15}$ Therefore the time difference is 3 hours. Sydney is 3 hours ahead.

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Exercise 19C

1 Give the time differences between the following places.

a X longitude 150°E, Y longitude 120°E

b X longitude 0°, Y longitude 75°W

c X longitude 0°, Y longitude 75°E

2 The following table shows the latitude and longitude of cities around the world.

City	*Latitude*	*Longitude*
Melbourne	38°S	145°E
Marseilles	43°N	5°E
Mexico city	19°N	99°W
Wellington	41°S	175°E
Zurich	47°N	8°E
London	52°N	0°
Lima	12°S	77°W
Plymouth	50°N	4°W
Yangon	19°N	96°
Washington	38°N	77°N

a Find the time difference between Melbourne and:

i Wellington **ii** Marseilles
iii Yangon **iv** London
v Mexico city **vi** Washington.

b If it is 6:00 a.m. in Melbourne what time is it in:

i Wellington? **ii** Marseilles?
iii Yangon? **iv** London?
v Mexico City? **vi** Washington?

3 The time in a certain Pacific island is 10 hours behind GMT.

a What is the longitude of the island?

b What is the time in London when it is 4 a.m. on the island?

c What is the time in on the island when it is 5:30 a.m. in London?

4 The longitude of Hanoi, Vietnam is 105°E while the longitude of Cape Howe NSW is 150°E. When the time in Hanoi is 2:30 p.m., what is the time at:

a Cape Howe? **b** London?

5 Kalgoorlie, WA has longitude 121°E while the Pacific island of Nauru has longitude 166°E.

a Calculate the difference in longitude between these two places.

b Calculate the time difference between the two places.

c What is the time in Nauru when it is 11:45 a.m. in Kalgoorlie?

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6 Dimitri is in Athens, which is 2 hours ahead of Greenwich Mean Time. Allan is in New York, which is 5 hours behind Greenwich Mean Time.
Dimitri is going to ring Allan at 10 a.m. on Wednesday, Athens time.

- **a** What day and time will it be in New York when he rings?
- **b** Allan is going to fly from New York to Athens. His flight will leave on Wednesday at 10 p.m., New York time, and will take 10 hours. What day and time will it be in Athens when he arrives?

7 Louise is in Dubai, which is 3 hours ahead of Greenwich Mean Time. David is in Sydney, which is 10 hours ahead of Greenwich Mean Time.
Louise is going to ring David at 10 a.m. on Wednesday, Sydney time.

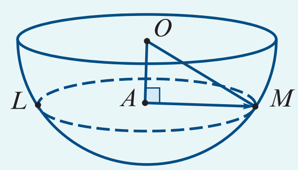
- **a** What day and time will it be in Dubai when he rings?
- **b** David is going to fly from Sydney to Dubai. His flight will leave on Wednesday at 10 p.m., Sydney time, and will take 14 hours. What day and time will it be in Dubai when he arrives?

19D Using Pythagoras' theorem in spheres

In this section we show how Pythagoras' theorem can be used to help solve problems involving spheres. Example 9 indicates the simple everyday day problems where such questions arise.

Example 9 Pond properties using Pythagoras

A pond in the shape of a hemisphere has radius 5 m contains water of depth 1 m.



Find:

- **a** the length of the water line
- **b** the area of the surface of the water
- **c** the shortest distance along the surface of the bowl between L and M where LM is a diameter of the water surface (use the great circle through L and M).

Solution

a Find OA.

Find the radius of the circle through M and L using Pythagoras' theorem.

$OA = 5 - 1 = 4$ m

$MA^2 = 25 - 16$

$= 9$

Therefore $MA = 3$.

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Find the length of the water line using:

$C = 2\pi r$

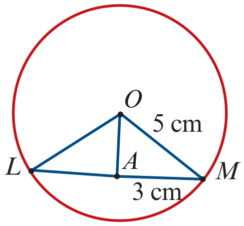
$C = 2\pi \times 3$
$= 6\pi$
≈ 18.85 m

b Find the area of the water surface using:

$A = \pi r^2$

$A = \pi \times 9 \approx 28.3\ \text{m}^2$

c This is along a great circle. The radius of this circle is 5 cm and the distance between L and M is 6 cm. The diagram here is a great circle through L and M.



$\sin(\angle AOM) = \frac{3}{5}$

Therefore $\angle AOM = 36.87°$.

Therefore $\angle LOM = 73.74°$.

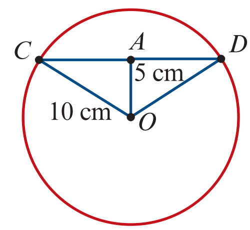
Therefore the length LM around the great circle $= \frac{5 \times 73.74\pi}{180}$

≈ 6.44 m

Note: We know that the distance around the water surface $= \frac{1}{2}$ of 6π
$= 3\pi$
≈ 9.42 m

Exercise 19D

1 A sphere has radius length 10 cm. Calculate the length of the diameter of the circle of a section of the sphere made by the plane at a distance 5 cm from the centre of the sphere. Give your answer correct to two decimal places.

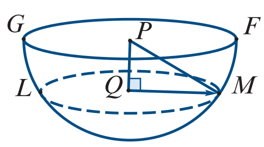


2 A hemispherical bowl of radius 26 cm contains water of depth 16 cm. Find:

a the length of the water line

b the area of the surface of the water

c the shortest distance along the surface of the bowl between L and M (use the great circle through L and M). LM is a diameter of the water surface.



3 A model globe of radius 50 cm rests in a circular hole of diameter 45 cm cut in a horizontal board. Two points, A and B, are marked at opposite ends of a diameter of this hole. Find the shortest distance from A to B along the surface of the sphere.

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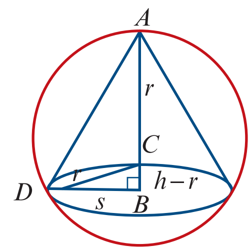
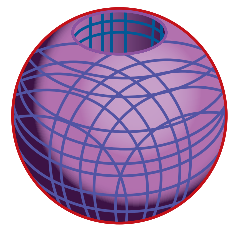
4 A cone of height h and radius s is inscribed in a sphere of radius r.
If $r = 10$ cm and $h = 15$ cm, find the radius s of the cone.



5 A cylinder is cut out of a sphere. The centre axis of the cylinder passes through the centre of the sphere. The radius of the sphere is 20 cm and the radius of the cylinder is 12 cm. Find the height of the cylinder and the height of the cup cut off.



6 A sphere has radius length 20 cm.

a Calculate the length of a diameter of the circle of a section of the sphere made by a plane at a distance of 7.5 cm from the centre of the sphere.

b Find the circumference and area or this circle.

7 Two locations A and B have the same latitude 20°S. The longitude of A is 25°E and the longitude of B is 45°E.

a Find the radius of the small circle of the 20°S latitude.

b Find the distance between A and B around the 20°S latitude.

c Find the distance around a great circle which passes through A and B.

8 Two locations X and Y have the same latitude 15°N. The longitude of X is 120°E and the longitude of Y is 150°E.

a Find the radius of the small circle of the 15°N latitude.

b Find the distance between X and Y around the 15°S latitude.

c Find the distance around a great circle which passes through X and Y.

9 A sphere of radius 5 cm and a cone of base radius 5 cm and height 10 cm rests on a horizontal table surface. A horizontal plane 3 cm above the surface cuts both solids.

a Find the radius of the circular cross-section of both the sphere and the cone.

b If points A and B are the end points of a diameter of a small circle, find the distance from A to B on a great circle which passes through A and B.

10 A hemispherical bowl of radius 12 cm sits on a horizontal table. It contains water with a depth of 7 cm. Find the surface area of the water.

11 A sphere of radius 8 cm is dropped into an inverted cone. The cone has an apex angle of 60° and the centre of the sphere sits below the top of the cone. Find the height of the top of the sphere above the apex of the cone.

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Review

Key ideas and chapter summary

Radius The **radius** is the distance from the centre to any point on the circle (sphere). It is half the diameter.

Arc An **arc** is the part of a circle between two given points on the circle.

Length of an arc The length s of an arc, of a circle of radius r, that subtends an angle of $\theta°$ at the centre is given by:

$$s = \frac{\pi r\theta}{180}$$

Area of a sector The area A of a sector, of a circle of radius r, where the arc of the sector subtends an angle of $\theta°$ at the centre is given by:

$$A = \frac{\pi r^2\theta}{360}$$

Area of a segment of a circle The area A of a segment of a circle of radius r, where the arc of the sector subtends an angle of $\theta°$ at the centre is given by:

$$\begin{aligned} A &= \text{area of minor sector } OAB - \text{area of } \Delta AOB \\ &= \frac{\pi r^2\theta}{360} - \frac{1}{2}r^2 \sin\theta = r^2\left(\frac{\pi\theta}{360} - \frac{1}{2}\sin\theta\right) \end{aligned}$$

Great circle A **great circle** is a section of a sphere that contains a diameter of the sphere. The section contains the centre of the sphere.

Small circles Sections of the sphere that do not contain a diameter are called **small circles**. A small circle does not contain the centre of the sphere.

Meridians of longitude **Meridians of longitude** are semi-great circles which pass through the north and south poles.

Parallels of latitude **Parallels of latitude** are small circles whose planes are parallel to that of the equator.

Skills check

Having completed this chapter you should be able to:

- find the length of an arc of a circle
- find the area of a sector of a circle
- find the area of a segment of a circle
- use the meridians of longitude and parallels of latitude to describe locations on Earth

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Review

- determine a distance around a parallel of latitude
- find the distance around a great circle of two points with the same longitude
- relate time zones to longitude.

Multiple-choice questions

1 An arc of length 9 cm subtends an angle of magnitude θ° at the centre of a circle of radius length 7 cm. The value of θ, to the nearest degree, is:

A 74 **B** $\frac{9}{7}$ **C** 1 **D** 45 **E** 63

2 If the area of the minor sector AOB in a circle of centre O and radius 35 m is 225 m^2, then the magnitude of $\angle AOB$, to the nearest degree, is:

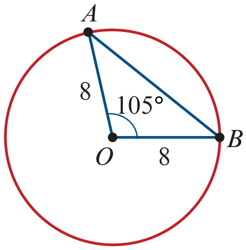
A 18 **B** 21 **C** 11 **D** 42 **E** 368

3 The arc length, in centimetres, which subtends an angle of magnitude 50° at the centre of a circle of radius length 15 cm is:

A 750 **B** $\frac{25\pi}{6}$ **C** $\frac{25}{6}$ **D** $\frac{25}{12}$ **E** $\frac{25\pi}{12}$

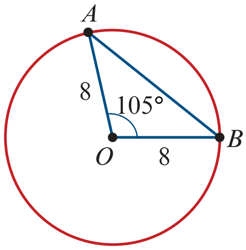
4 In the diagram opposite, the magnitude of the area of the minor sector AOB is:

A $32 \sin 105°$ **B** $\frac{136\pi}{3}$ **C** $\frac{14\pi}{3}$
D $\frac{34\pi}{3}$ **E** $\frac{56\pi}{3}$



5 In the diagram opposite, the magnitude of the area of the major sector AOB is:

A $32 \sin 105°$ **B** $\frac{14\pi}{3}$ **C** $\frac{34\pi}{3}$
D $\frac{136\pi}{3}$ **E** $\frac{56\pi}{3}$



6 A great circle on a newly found planet has a circumference of 11 000 km. The diameter of the planet is closest to:

A 320 km **B** 300 km **C** 3000 km **D** 250 km **E** 1100 km

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Review

7 The coordinates of two points M and N on the Earth's surface are (40°N; 40°E) and (25°S, 55°E). Which statement is most likely to be correct about the time difference?

A M is 5 hours behind N. **B** M is 1 hours behind N. **C** N is 5 hours behind M. **D** N is 1 hours behind M. **E** M is 15 hours ahead of N.

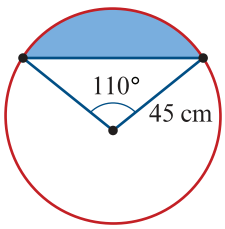
8 Two circles of radii 3 cm and 4 cm have their centres 6 cm apart. The area of the region common to both circles, to two decimal places, is:

A 5.71 cm^2 **B** 7.37 cm^2 **C** 4.30 cm^2 **D** 2.41 cm^2 **E** 6.37 cm^2

9 A child on a swing travels through an arc of length 3 m. If the ropes of the swing are 4 m in length, the angle which the arc makes at the top of the swing (where the swing is attached to the support) is best approximated by:

A 135° **B** 75° **C** 12° **D** 75° **E** 43°

10 The area of the shaded region in the diagram in cm^2 (to the nearest cm^2) is:



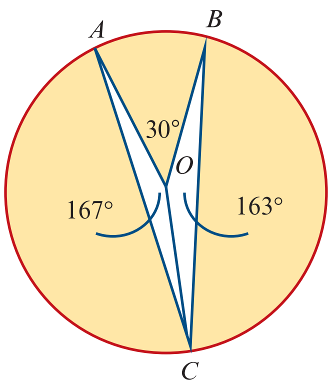
A 951 **B** 992 **C** 1944
D 2895 **E** 110424

11 Point X on the Earth's surface has coordinates (29°S, 32°E), while point Y is at (8°S, 32°E). The distance between X and Y is closest to:

A 2346 km **B** 750 km **C** 111 km **D** 1350 km **E** 2010 km

Extended-response questions

1 A logo for a Victorian team is shown here. O is the centre of the circle and A, B and C are points on the circle. $OC = OA = OB = 10$ cm.



a Find the length of the minor arc AB.

b The magnitude of $\angle AOC$ is 167° and the magnitude of $\angle BOC$ is 163°.
Find the length of chord BC, correct to two decimal places.

c Find correct to two decimal places:

- **i** the area of triangle BOC
- **ii** the area of triangle AOC
- **iii** the shaded area of the logo.

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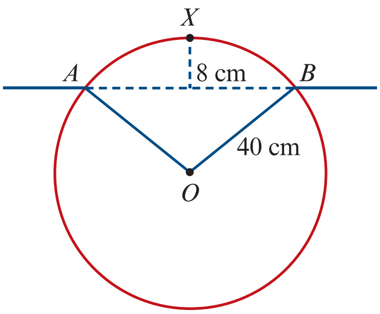
2 The figure shows the circular cross-section of a uniform log of radius 40 cm floating in water. The points A and B are on the surface of the water and the highest point X is 8 cm above the surface.



a Show that the magnitude of $\angle AOB$ is approximately 74°.

b Calculate:

i the length of arc AXB

ii the area of the cross-section below the surface

iii the percentage of the volume of the log below the surface.

3 The angle of a sector of a circle, centre O and radius length 12 cm, has magnitude 145°. The sector is folded so that OA and OA' are joined to form a cone. Calculate:

a the base radius length of the cone

b the curved surface area of the cone

c the shortest distance between two points diametrically opposed on the edge of the base.

4 Two locations X and Y have the same latitude 30°S. The longitude of X is 145°E and the longitude of Y is 130°E.

a Find the radius of the small circle of the 30°S parallel of latitude.

b Find the distance between X and Y around the parallel of latitude 30°S.

c Find the radius of the great circle through X and Y.

d Find the difference in distance between the distance around the great circle and the distance around the parallel of latitude.

5 The Tropic of Cancer is at latitude 23.5°N while the Tropic of Capricorn is at latitude 23.5°S.

a Calculate the distance between these two tropics along the same great circle in kilometres (correct to the nearest kilometre).

b Calculate the radius of the small circles of the Tropic of Capricorn and the Tropic of Cancer.

c Rockhampton in Queensland and Sao Paulo in Brazil are both on the tropic of Capricorn. Rockhampton has longitude 150.5°E and Sao Paolo has longitude 46.5°W. Find the distance around the tropic of Capricorn from Sao Paolo to Rockhampton.

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Module 3: Geometry and measurement

Revision: Geometry and measurement

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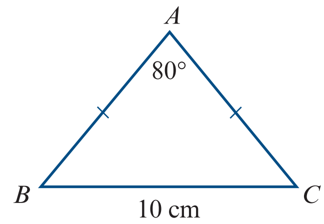
Revision

20A Multiple-choice questions

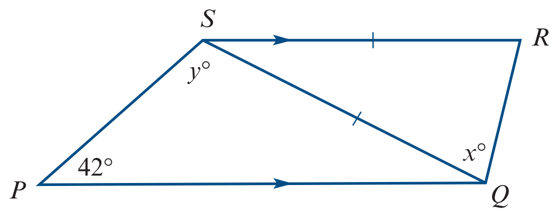
1 In triangle ABC, $AB = AC$. The length of AB and AC, in centimetres, is given by:



A $10 \sin 80°$ **B** $5 \cos 80°$ **C** $\dfrac{5}{\cos 50°}$

D $\dfrac{5}{\sin 50°}$ **E** $5 \sin 40°$

2 In this diagram, PQ and SR are parallel and $SR = SQ$. x and y satisfy the equation:



A $x = y$

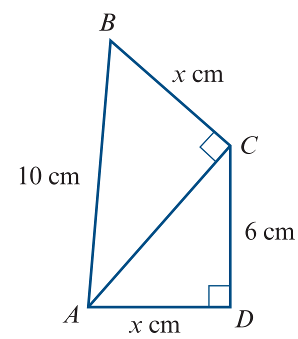
B $x + y = 138$

C $2x + y = 42$

D $x = y + 42$

E $2x - y = 42$

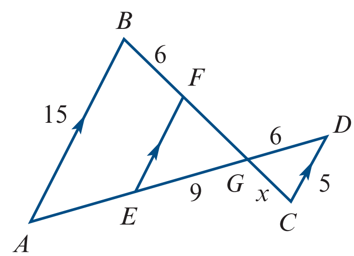
3 In this diagram, angles ACB and ADC are right angles. If BC and AD each have a length of x cm, then $x =$



A $2\sqrt{17}$ **B** 4 **C** 5

D $4\sqrt{2}$ **E** $5\sqrt{2}$

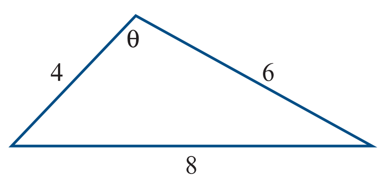
4 In this figure, $AB = 15$, $CD = 5$, $BF = 6$, $GD = 6$ and $EG = 9$. x is equal to:



A 3 **B** 4 **C** 4.5

D 4.75 **E** 5

5 For triangle ABC, $\cos \theta =$



A $-\dfrac{1}{4}$ **B** $-\dfrac{1}{2}$ **C** $\dfrac{1}{4}$

D $\dfrac{1}{2}$ **E** $\dfrac{3}{4}$

6 A pencil container is in the shape of a cylinder of length 30 cm and diameter 5 cm. The total surface area of the pencil container (including both ends) is closest to:

A 39 cm^2 **B** 157 cm^2 **C** 471 cm^2

D 491 cm^2 **E** 511 cm^2

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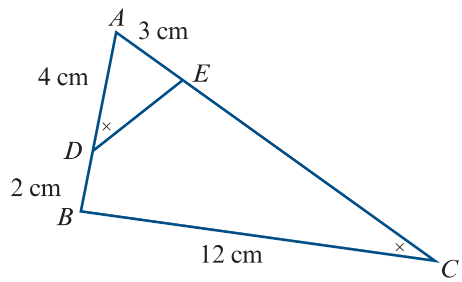
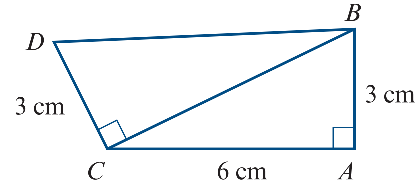
7 D and E are points on AB and AC, respectively. $AD = 4$ cm, $DB = 2$ cm, $AE = 3$ cm and $BC = 12$ cm. The magnitude of $\angle ADE$ = the magnitude of $\angle ACB$. The length DE, in centimetres, is:



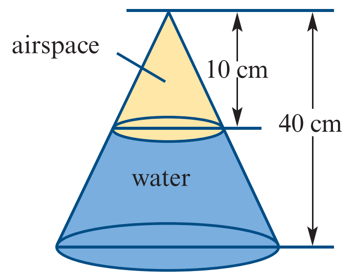
A 6 **B** $\frac{9}{2}$ **C** 9
D 10 **E** 11

8 In this figure the length of DB, in centimetres, is:



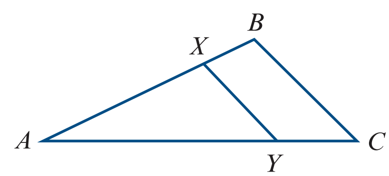
A 6 **B** 9 **C** $3\sqrt{5}$
D $3\sqrt{6}$ **E** $3\sqrt{7}$

9 A conical container is 40 cm tall and has a capacity of 6032 cm^3. Water is poured into the cone, leaving a conical airspace of height 10 cm, as shown in the diagram. The volume of the water, in cm^3, is closest to:



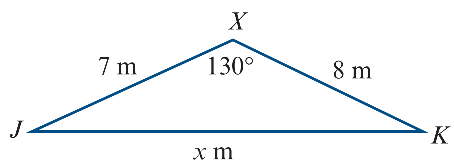
A 94 **B** 1508 **C** 4524
D 5655 **E** 5938

10 ΔABC is similar to ΔAXY and $AX = \frac{2}{3}AB$. The area of ΔABC is 108 cm^2. The area of ΔAXY is:



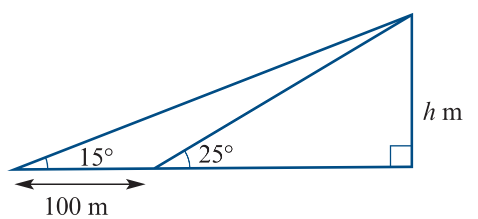
A 32 cm **B** 48 cm^2 **C** 54 cm^2
D 72 cm^2 **E** 81 cm^2

11 Which *one* of the following equations gives the correct value for x?



A $x^2 = 49 + 64 + 2(7)(8)\cos 50°$
B $x^2 = 49 + 64 + 2(7)(8)\cos 70°$
C $\dfrac{x}{\sin 130°} = \dfrac{8}{\sin 25°}$
D $\dfrac{x}{\sin 130°} = \dfrac{7}{\sin 25°}$
E $x^2 = 49 + 64 - 2(7)(8)\cos 50°$

12 The height, h m, of a television tower can be calculated by measuring the angles of elevation of the top of the tower from two points that are in line with the tower but that are 100 m apart. Which *one* of the following equations gives the correct value of h?



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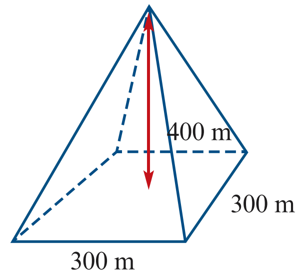
A $h = \dfrac{100 \sin 15° \tan 25°}{\sin 10°}$ **B** $h = \dfrac{100 \sin 15° \sin 25°}{\sin 15°}$

C $h = \dfrac{100 \sin 10° \sin 25°}{\sin 15°}$ **D** $h = \dfrac{100 \sin 10° \sin 25°}{\sin 15°}$

E $h = 100 \tan 15°$

13 The diagram shows a right square pyramid of height 400 m with its base a square of side 300 m. If θ is the angle between a sloping face and the base, which *one* of the following equations will give the correct value of θ?

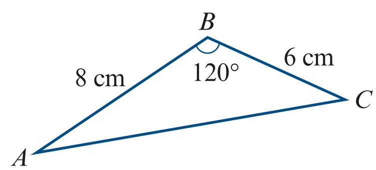


A $\tan\theta = \dfrac{50\sqrt{73}}{150}$ **B** $\tan\theta = \dfrac{400}{300\sqrt{2}}$

C $\tan\theta = \dfrac{400}{150\sqrt{2}}$ **D** $\tan\theta = \dfrac{400}{300}$

E $\tan\theta = \dfrac{400}{150}$

14 In ΔABC, the length of AC in centimetres is determined by evaluating:



A $\sqrt{100 + 96 \cos 120°}$

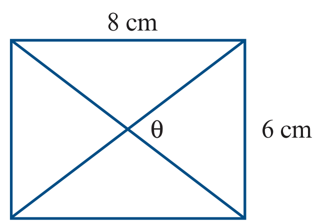
B $\sqrt{100 - 96 \cos 120°}$

C $\sqrt{100 - 96 \cos 60°}$

D $\sqrt{64 + 36 - 96}$

E $\sqrt{100\,(1 + 2 \cos 120°)}$

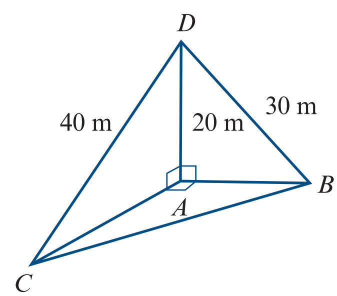
15 A rectangle is 8 cm long and 6 cm wide. The acute angle, θ, between its diagonals, correct to the nearest degree, is:



A 37° **B** 41° **C** 49°

D 74° **E** 83°

16 A vertical mast, AD, of height 20 m is supported by two cables attached to the ground at C and B, as shown. $\angle CAB$ is a right angle. Cable CD is of length 40 m and cable BD is of length 30 m. The distance CB in metres is:

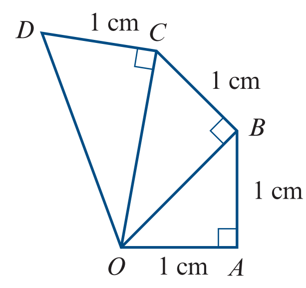


A $\sqrt{1700}$ **B** $\sqrt{2500}$

C $\sqrt{3300}$ **D** $\sqrt{2000} + \sqrt{1300}$

E $\sqrt{1200} + \sqrt{500}$

17 In the figure shown, the length of OD in centimetres is:



A $\sqrt{2}$ **B** $\dfrac{\sqrt{3}}{2}$ **C** $\sqrt{3}$

D 2 **E** 4

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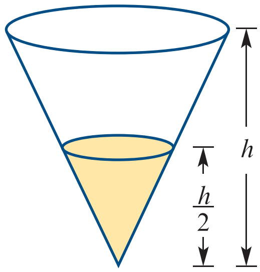
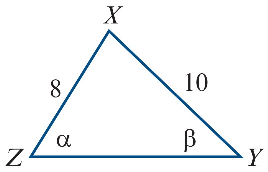
Revision

18 An inverted right circular cone of capacity 80 cm^3 is filled with water to half of its depth. The volume of water (in cm^3) is:



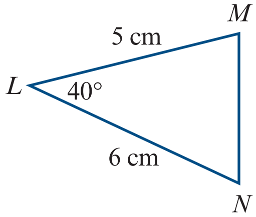
A $\sqrt[3]{80}$ **B** $\sqrt{80}$ **C** 10
D 20 **E** 40

19 Given that $XZ = 8$, $XY = 10$ and $\sin\alpha = \frac{2}{3}$, $\sin\beta$ equals:



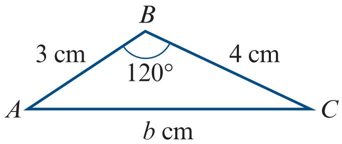
A $\frac{4}{15}$ **B** $\frac{8}{15}$ **C** $\frac{5}{15}$
D $\frac{4}{5}$ **E** $\frac{5}{6}$

20 The area of triangle LMN, in cm^2, is:



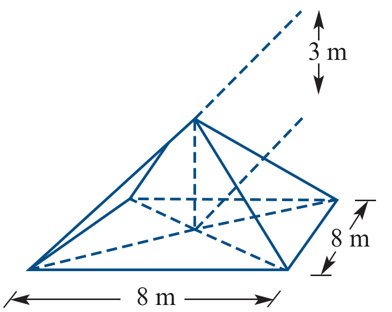
A $\frac{6}{5}\sin 40°$ **B** $\frac{6}{5\cos 40°}$ **C** $15\sin 40°$
D $30\cos 40°$ **E** $30\sin 40°$

21 In the diagram shown, the value of b is:



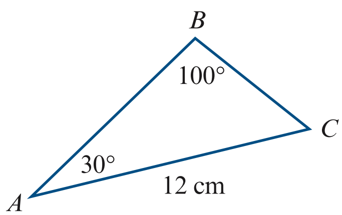
A $\sqrt{13}$ **B** 5 **C** $\sqrt{37}$
D 13 **E** 37

22 A right pyramid with a square base is shown. The height of the pyramid is 3 m and the square base has sides of length 8 m. The length of a sloping edge, in metres, is:



A $\sqrt{41}$ **B** $\sqrt{52}$ **C** $\sqrt{73}$
D $\sqrt{80}$ **E** $\sqrt{137}$

23 The length of AB, in centimetres, is equal to:



A $\frac{12}{\sin 100°} \times \sin 30°$
B $\sqrt{144 - 2 \times 12\cos 30°}$
C $\sqrt{30^2 + 100^2 - 2 \times 30 \times 100\cos 50^c}$
D $\frac{12}{\sin 100°} \times \sin 50°$
E $\frac{12}{\sin 50°} \times \sin 100°$

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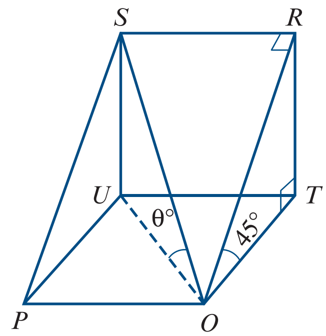
24 In this figure, $PQRS$ is a rectangle inclined at an angle of $45°$ to the horizontal plane $PQTU$. The magnitude of $\angle PQS = 60°$. Let $\theta°$ be the angle of inclination of QS to the horizontal plane. Then $\sin\theta =$

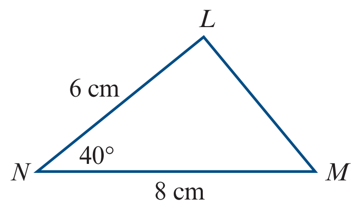


A $\dfrac{1}{4}$ **B** $\dfrac{\sqrt{2}}{4}$ **C** $\dfrac{\sqrt{2}}{2}$
D $\dfrac{\sqrt{3}}{2}$ **E** $\dfrac{\sqrt{6}}{4}$

25 The hexagonal prism shown has length 2 cm and volume 17.8 cm^3. Its cross-sectional area is closest to:

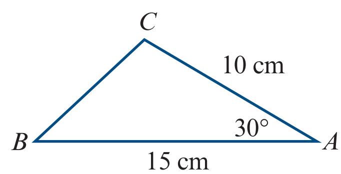
A 2.2 cm^2 **B** 4.5 cm^2 **C** 8.9 cm^2
D 35.6 cm^2 **E** 14.4 cm^2

26 The area of triangle LMN in square centimetres is:



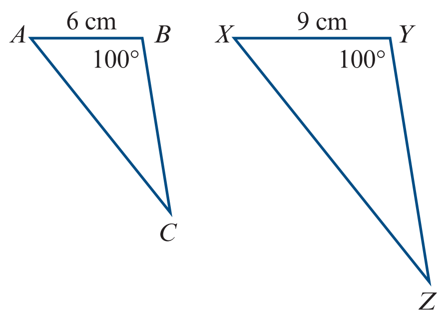
A $\dfrac{6}{5}\sin 40°$ **B** $\dfrac{6}{5\cos 40°}$ **C** $15\sin 40°$
D $24\cos 40°$ **E** $24\sin 40°$

27 The area of triangle ABC, in square centimetres, is:



A 15 **B** 37.5 **C** 75
D 90 **E** 150

28 The area of triangle ABC is 20 cm^2. Triangle XYZ is similar to triangle ABC. The area of triangle XYZ, in square centimetres, is:



A 30 **B** 35 **C** 40
D 45 **E** 50

29 A vegetable garden has an area of 324 m^2. When shown on a plan, the vegetable garden has an area of 36 cm^2. On this plan, 1 cm represents an actual distance of:

A 1 m **B** 2 m **C** 3 m
D 9 m **E** 18 m

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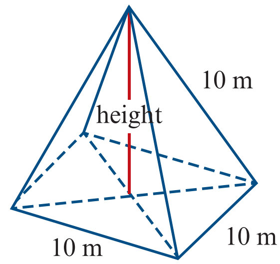
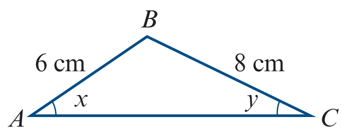
Revision

30 A right pyramid with a square base is shown. The square base has sides of length 10 m. The length of each sloping edge is also 10 m. The height of the pyramid, in metres, is:



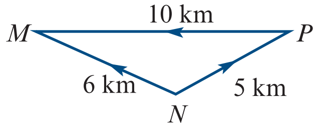
A $\sqrt{40}$ **B** $\sqrt{50}$ **C** $\sqrt{60}$
D $\sqrt{200}$ **E** $\sqrt{1000}$

31 In triangle ABC as shown, $\sin x = \frac{3}{7}$. The value of $\sin y$ is:



A $\frac{1}{7}$ **B** $\frac{9}{28}$ **C** $\frac{1}{2}$
D $\frac{4}{7}$ **E** $\frac{3}{4}$

32 A yacht follows a triangular course, MNP, as shown. The largest angle between any two legs of the course is closest to:

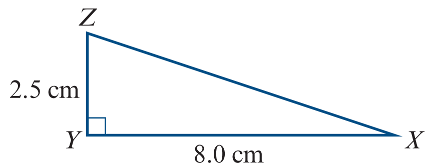


A 50° **B** 70° **C** 120°
D 130° **E** 140°

33 A hiker travels a distance of 5 km from point P to point Q on a bearing of 030°. She then travels from point Q to point R on a bearing of 330° for 10 km. How far *west* of P is R in kilometres?

A 2.5 **B** 5 **C** 7.5 **D** 10 **E** 15

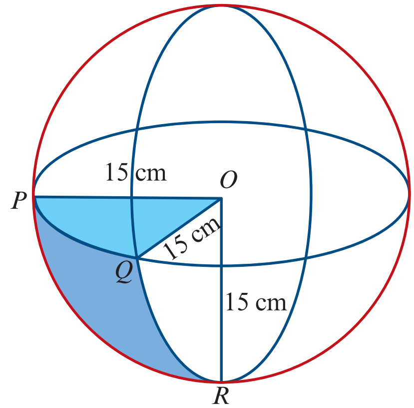
34 In right-angled triangle XYZ, $XY = 8.0$ cm and $YZ = 2.5$ cm as shown. The length of ZX, in centimetres, correct to one decimal place, is:



A 5.5 **B** 7.6 **C** 8.2
D 8.4 **E** 10.5

35 The solid $OPQR$, as shown opposite, is one-eighth of a sphere of radius 15 cm. The point O is the centre of the sphere and the points P, Q and R are on the surface of the sphere.

$$\angle POQ = \angle QOR = \angle ROP = 90°$$



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The total surface area of the solid $OPQR$, in cm^2, is closest to:

A 619 **B** 648 **C** 706
D 884 **E** 1767

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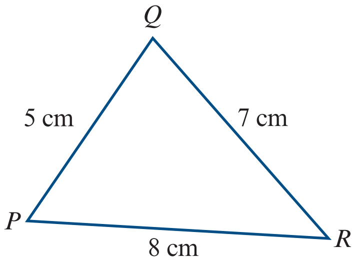
36 In the triangle shown, angle PQR, correct to the nearest degree, equals:

A 38° **B** 60° **C** 73°
D 82° **E** 98°

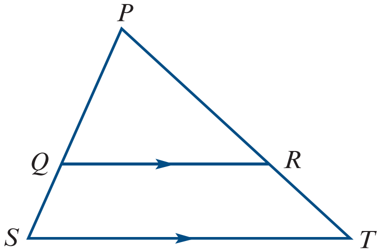


37 The diameter of a large sphere is four times the diameter of a smaller sphere. It follows that the ratio of the volume of the large sphere to the volume of the smaller sphere is:

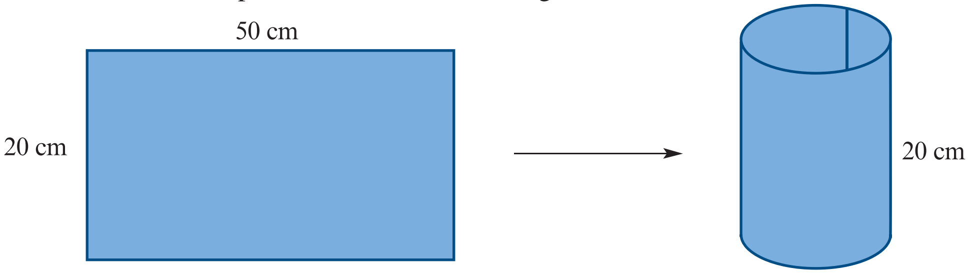
A 4 : 1 **B** 8 : 1 **C** 16 : 1 **D** 32 : 1 **E** 64 : 1

38 QR is parallel to ST and $PQ = 2\,QS$. Given that the area of triangle PST is 18 square centimetres, the area of triangle PQR in square centimetres is:

A 2 **B** 6 **C** 8
D 9 **E** 12



39 A rectangular sheet of cardboard has length 50 cm and width 20 cm.
This sheet of cardboard is made into an open-ended cylinder by joining the two shorter sides, with no overlap. This is shown in the diagram below.



The radius of this cylinder, in cm, is closest to:

A 6.4 **B** 8.0 **C** 15.6 **D** 15.9 **E** 17.8

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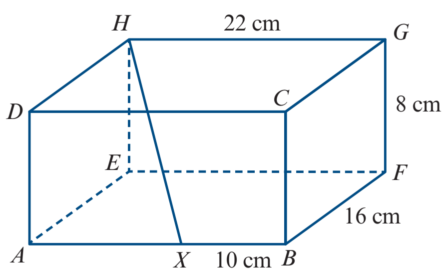
40 A rectangular box, $ABCDEFGH$, is 22 cm long, 16 cm wide and 8 cm high, as shown.



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A thin rod is resting in the box. One end of the rod sits at X and the other end of the rod sits at H.
The point X lies on the line AB at a distance of 10 cm from B.
The length of the rod, in centimetres, is closest to:

A 17.89 **B** 18.87 **C** 20.00 **D** 21.54 **E** 26.83

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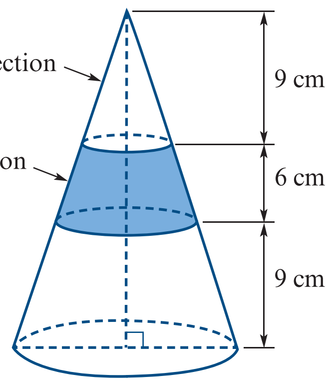
41 The middle section of a cone is shaded, as shown in the diagram opposite.



The surface area of the unshaded top section of the cone is 180 cm^2.
The surface area of the middle section of the cone, in square centimetres, is:

A 80 **B** 120 **C** 300 **D** 320 **E** 500

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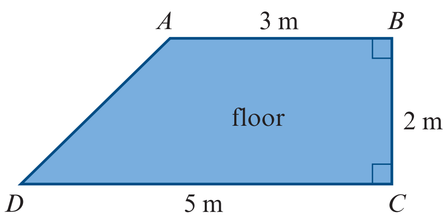
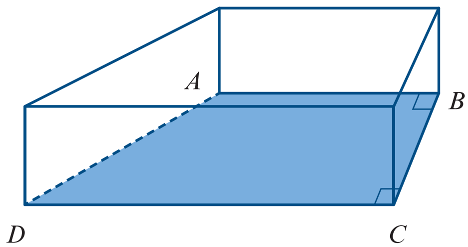
42 Two places with the same longitude have latitude 36° N and 24°S. The distance between them in kilometres is closest to:

A 6702 **B** 1340 **C** 4180 **D** 6402 **E** 5370

20B Extended-response questions

1 The floor of a chicken coop is in the shape of a trapezium. The floor, $ABCD$, and the chicken coop are shown below.





$AB = 3$ m, $BC = 2$ m and $CD = 5$ m

a What is the area of the floor of the chicken coop? Write your answer in square metres.

b What is the perimeter of the floor of the chicken coop? Write your answer in metres, correct to one decimal place.

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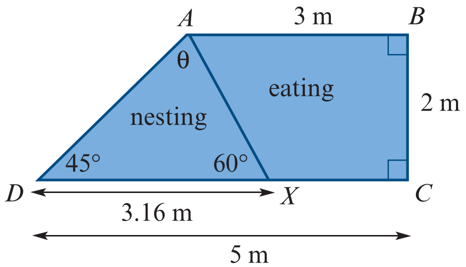
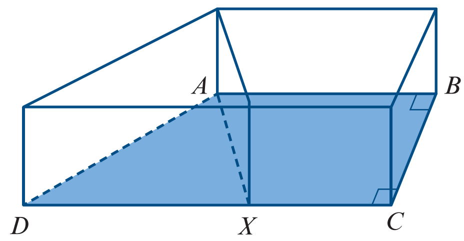
2 The chicken coop has two spaces, one for nesting and one for eating.
The nesting and eating spaces are separated by a wall along the line AX, as shown in the diagrams below.





$DX = 3.16$ m, $\angle ADX = 45°$ and $\angle AXD = 60°$.

a Write down a calculation to show that the value of θ is 75°.

b Use the sine rule to find the length of AX. Write your answer in metres, correct to two decimal places.

c Calculate the area of the floor of the nesting space, ADX. Write your answer in square metres, correct to one decimal place.
The height of the chicken coop is 1.8 m. Wire mesh will cover the roof of the eating space.
The area of the walls along the lines AB, BC and CX will also be covered with wire mesh.

d What total area, in square metres, will be covered by wire mesh? Write your answer, correct to the nearest square metre.

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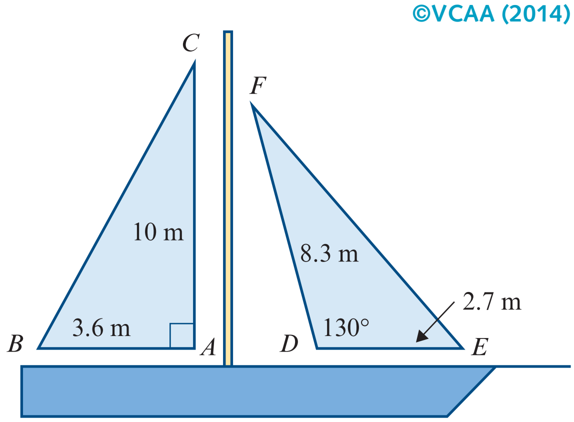
3 A yacht has two flat triangular sails, as shown in the diagram. The sail ABC is in the shape of a right-angled triangle. The height AC is 10 metres and the length AB is 3.6 metres.



a Calculate angle ABC. Write your answer correct to the nearest degree.

b Calculate the length BC. Write your answer in metres, correct to one decimal place.

The sail DEF has side lengths $DE = 2.7$ metres and $DF = 8.3$ metres. The angle EDF is 130°.

c Calculate the length EF. Write your answer in metres, correct to one decimal place.

d Calculate the area of the sail DEF. Write your answer in square metres, correct to one decimal place.

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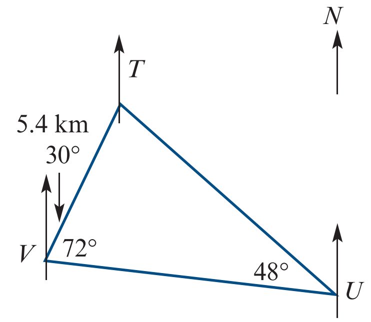
Revision

4 A course for a yacht race is triangular in shape and is marked by three buoys, T, U and V. Starting from buoy V, the yachts sail 5.4 km on a bearing of 030° to buoy T. They then sail to buoy U and back to buoy V. The angle TVU is 72° and the angle TUV is 48°.

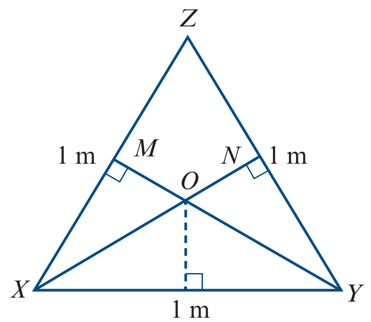


a Determine the bearing of V from U.

b Determine the distance TU. Write your answer in kilometres, correct to one decimal place.

c Determine the shortest distance to complete the race. Write your answer in kilometres, correct to one decimal place.

5 A navigational marker, XYZ, is in the shape of an equilateral triangle with side length of 1 metre. It is located in the vicinity of a yacht race.

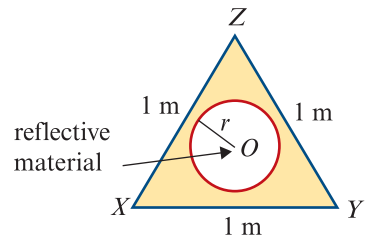


a Write down the size of angle XYZ.

Point O is the centroid (centre) of the triangle. Points M and N are the midpoints of sides XZ and YZ, respectively.

b Calculate the shortest distance from point O to side XY. Write your answer in metres, correct to three decimal places.

A piece of reflective material in the shape of a circle is attached to the centre of the navigational marker at the centroid O. The ratio of the area of the shaded region of the navigational marker XYZ to the area of the reflective material is 2:1.



c Determine the radius, r, of the circle. Write your answer in metres, correct to three decimal places.

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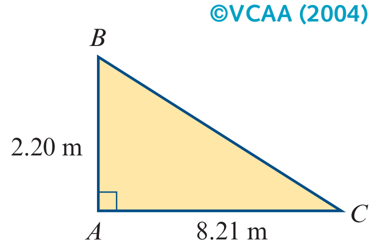
6 Jane is landscaping her garden. A piece of shade cloth, ABC, has the dimensions as shown.



a Determine the length BC in metres. Write your answer correct to two decimal places.

b Determine the angle ACB. Write your answer correct to the nearest degree.

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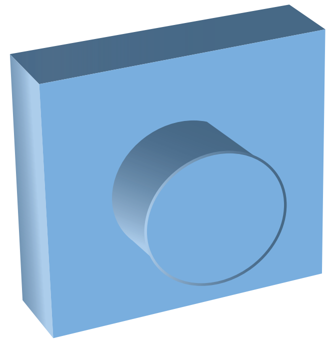
Revision

7 A wooden cylinder of radius 6 cm and height 5 cm is attached to a square flat wooden board of side length 12 cm and thickness 5 cm.

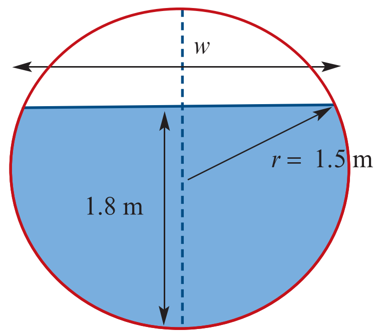


a Determine the volume of the composite shape correct to the nearest cm^3.

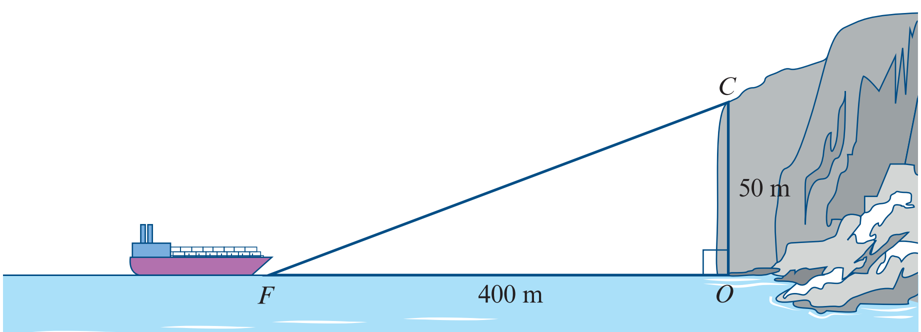
b Determine the total surface area of the composite shape correct to the nearest cm^2.

8 Two gliders travel in different directions from the same control tower.
Glider A travels 80 km on a bearing of 145°.
Glider B travels 50 km on a bearing of 055°.
Determine the bearing of glider A from glider B, correct to the nearest degree.

9 The cross-section of a waste water pipe is circular with a radius of 1.5 m, as shown. The water in the pipe is 1.8 m deep. Determine the width of the horizontal surface of the water (w). Give the answer in metres, correct to one decimal place.



10 A ferry, F, is 400 metres from point O at the base of a 50 metre high cliff, OC.



a Show that the gradient of the line FC in the diagram is 0.125.

b Calculate the angle of elevation of point C from F. Write your answer in degrees, correct to one decimal place.

c Calculate the distance FC, in metres, correct to one decimal place.

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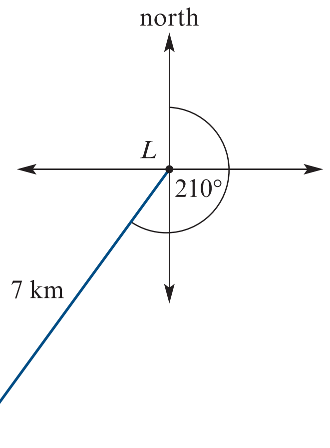
Revision

11 A yacht, Y, is 7 km from a lighthouse, L, on a bearing of 210° as shown in the diagram opposite.



a A ferry can also be seen from the lighthouse. The ferry is 3 km from L on a bearing of 135°. On the diagram opposite, label the position of the ferry, F, and show an angle to indicate its bearing.

b Determine the angle between LY and LF.

c Calculate the distance, in km, between the ferry and the yacht correct to two decimal places.

d Determine the bearing of the lighthouse from the ferry.

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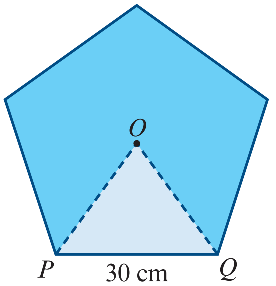
12 The ferry has a logo painted on its side. The logo is a regular pentagon with centre O and side length 30 cm. It is shown in the diagram opposite.



a Show that angle POQ is equal to 72°.

b Show that, correct to two decimal places, the length OP is 25.52 cm.

c Find the area of the pentagon. Write your answer correct to the nearest cm^2.

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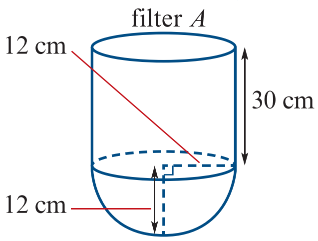
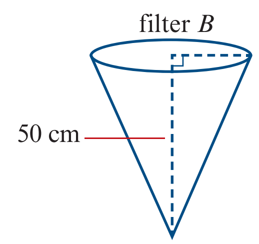
13 The ferry has two fuel filters, A and B. Filter A has a hemispherical base with radius 12 cm. A cylinder of height 30 cm sits on top of this base.



a Calculate the volume of filter A. Write your answer correct to the nearest cm^3.

Filter B is a right cone with height 50 cm.



b Originally filter B was full of oil, but some was removed. If the height of the oil in the cone is now 20 cm, what percentage of the original volume of oil was removed?

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Module 4: Graphs and relations

Chapter 21

21 Constructing graphs

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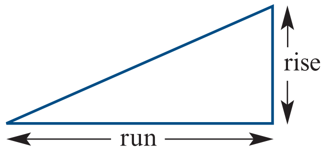
21A The gradient of a straight line

Through any two points it is possible to draw only a single straight line. Therefore a straight line is defined by any two points on the line.

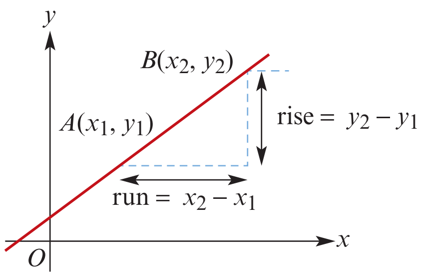
From previous work, you should be familiar with the concept of the **gradient** or slope of a line. The symbol used for gradient is m. This may be defined as:



$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$

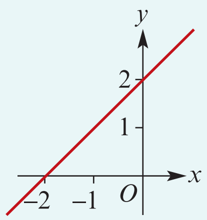
Hence, given any two points on the line, $A(x_1, y_1)$ and $B(x_2, y_2)$, the gradient of the line can be found.

$$\text{gradient, } m = \frac{y_2 - y_1}{x_2 - x_1}$$



Example 1 Finding the gradient of a line given two points

Find the gradient of this line.



Solution

1 From the graph, identify two points where the coordinates can be easily determined.

In this case the axes intercepts are chosen: $(-2, 0)$ and $(0, 2)$.

2 Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Let $(x_2, y_2) = (0, 2)$ and let $(x_1, y_1) = (-2, 0)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2 - 0}{0 - (-2)}$$

$$\text{Gradient} = \frac{2}{2} = 1$$

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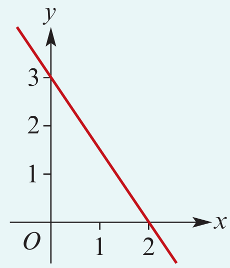
Example 2 Finding the gradient of a line given two points

Find the gradient of this line.



Solution

1 From the graph, identify two points where the coordinates can be easily determined.

In this case the axes intercepts are chosen (2, 0) and (0, 3).

2 Use $m = \dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{3}{2}$.

Let $(x_1, y_1) = (0, 3)$ and $(x_2, y_2) = (2, 0)$.

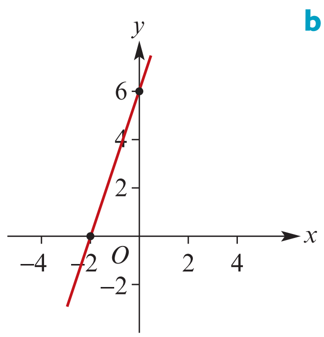
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 3}{2 - 0} = -\frac{3}{2} = -1.5$$

- The gradient of a line that slopes *upwards* from left to right is *positive*, and the gradient of a line that slopes *downwards* from left to right is *negative*.
- The gradient of a *horizontal* line is zero, since $y_2 - y_1 = 0$.
- The gradient of a *vertical* line is undefined, since $x_2 - x_1 = 0$.

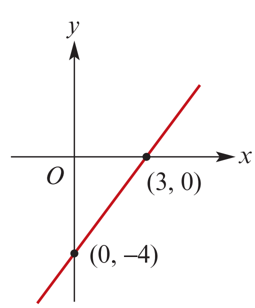
Exercise 21A

1 Calculate the gradient of each of the following.

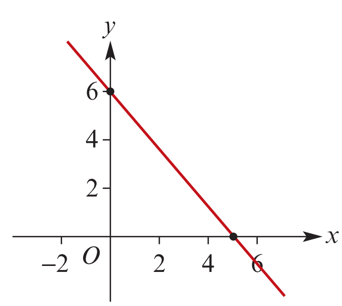
a



b



c



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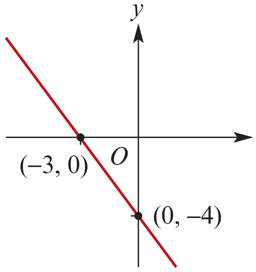
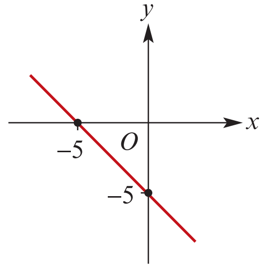
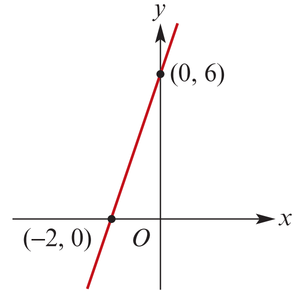
d



e



f



2 Sketch a graph of a line with gradient 1.

3 Sketch a graph of a line with gradient 0.

4 Find the gradients of the lines that pass through each of the following pairs of points.

a (12, 6), (4, 8)   **b** (−6, 8), (2, −12)   **c** (3, 3.5), (5.5, −1.5)

d (10, 16), (12, 0)   **e** (3, 0), (−3, 0)   **f** (0, −3), (−3, 0)

g (3, 9), (4, 16)   **h** (−5, 25), (−8, 64)

21B The general equation of a straight line

The general equation of a straight line is $y = mx + c$, where m is the gradient of the line. This form, expressing the relation in terms of y, is called the *gradient form*.

If we let $x = 0$, we get:

$$y = m(0) + c$$
$$y = c$$

That is, the *y-axis intercept is equal to c*.

Example 3 Interpreting the equation of a line given the form $y = mx + c$

Find the gradient and y-axis intercept of the graph of $y = 3x - 4$.

Solution

The equation of the line is in the form $y = mx + c$ where m is the gradient and c the y-axis intercept.

The value of m is 3 and the value of c is -4.

Therefore the gradient of this line is 3 and the y-axis intercept is -4.

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If we are given the rule of a straight line, we can sketch the graph using the gradient and the y-axis intercept.

Example 4 **Sketching straight-line graphs**

Sketch the graph of $y = 3x - 1$.

Solution

1 The equation is written in the form $y = mx + c$. Hence the gradient and y-axis intercept can be read directly.

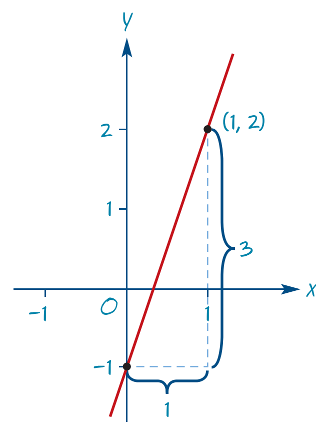
Gradient $= 3$; i.e. $\frac{\text{rise}}{\text{run}} = \frac{3}{1}$

y-axis intercept $= -1$

2 Plot the point $(0, -1)$, the y-axis intercept.

3 A gradient of 3 means that for every unit across in the positive x-axis direction you go up 3 in the positive y-axis direction.
From the y-axis intercept, move *across* 1 (run) and *up* 3 (rise) to plot the point $(1, 2)$.

4 Draw a line through the points $(0, -1)$ and $(1, 2)$ to sketch the graph.
If the equation for the straight line is not written in *gradient form*, to use the above method for sketching a graph the equation must first be transposed into *gradient form*.



Example 5 **Sketching a straight-line graph**

Sketch the graph of $3y + 6x = 9$.

Solution

1 First rearrange the equation into gradient form: $y = mx + c$

$3y + 6x = 9$

$\therefore 3y = 9 - 6x$ (subtracting $6x$ from both sides)

$y = \frac{9 - 6x}{3}$ (dividing both sides by 3)

$y = 3 - x$

$\therefore y = -2x + 3$

2 The equation is now written in the form $y = mx + c$. Therefore $m = -2$ and $c = 3$.

3 A gradient of -2 means that for every unit across in the positive x-axis direction, you go down 2 in the negative y-axis direction. From the y-intercept $(0, 3)$, move across 1 (run) and down 2 (rise) to plot the point $(1, 1)$.

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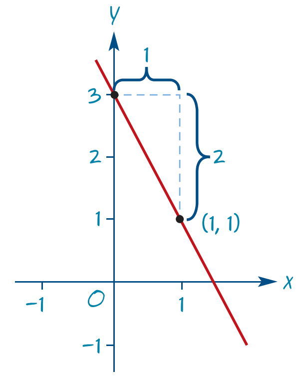
4 Draw a line through the points (0, 3) and (1, 1) to sketch the graph.



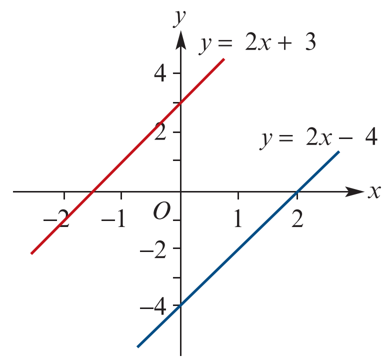
Parallel lines

If the value of m is the same for two rules, then the lines are parallel.

For example, consider the lines with rules:

$$y = 2x + 3$$
$$y = 2x - 4$$

They are parallel because they both have the same gradient, $m = 2$.



Exercise 21B

1 Sketch the graph of each of the following.

a $y = x + 1$ **b** $y = 2x + 1$ **c** $y = 3x - 6$
d $y = 4 - x$ **e** $y = 4 - 2x$ **f** $y = -x + 6$

2 For which of the following equations do the lines pass through the origin?

a $y + x = 1$ **b** $y + x = 0$ **c** $2y = 3x$
d $y + 3x = 3(x + 1)$ **e** $x - y = 1$

3 Give the gradient of each of the lines in Question **2**.

4 Sketch the graph of each of the following by first expressing it in the form $y = mx + c$.

a $4x + 2y = 12$ **b** $6x + 3y = 12$ **c** $6y - 4x = 24$
d $x + y = 3$ **e** $y - 2x = 6$ **f** $2y + 3x = 6$
g $x + 2y = 1$ **h** $y - 2x = 1$ **i** $2x - 3y = 6$

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21C Finding the equation of a straight line

The equation of a straight line can be found if the gradient and y-axis intercept are known.

Example 6 Finding the equation of a straight line

Write down the equation of the straight line if $m = -3$ and $c = 10$.

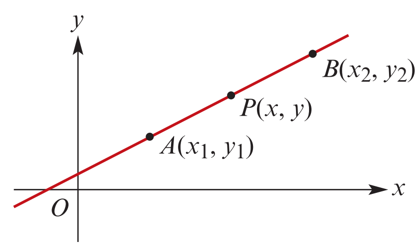
Solution

The equation is $y = -3x + 10$.

If, however, we are not told the gradient and/or the y-axis intercept, they can be determined from other information.

Case (i): We are given any two points, $A(x_1, y_1)$ and $B(x_2, y_2)$. Using these two points, we can first calculate the gradient of the line AB:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



Using the general point $P(x, y)$, also on the line, we have:

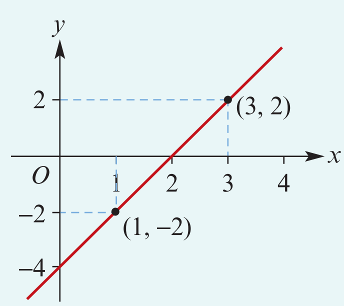
$$m = \frac{y - y_1}{x - x_1}$$

Therefore the equation of the line is:

$$y - y_1 = m(x - x_1) \text{ where } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 7 Finding the equation of a straight line

Find the equation of the straight line passing through the points $(1, -2)$ and $(3, 2)$.



Solution

1 First find the gradient using:

$$m = \frac{y_2 - y_1}{y_2 - x_1}$$

$x_1 = 1$, $y_1 = -2$, $x_2 =$ and $y_2 = 2$

$$\frac{2 - (-2)}{3 - 1} = \frac{4}{2} = 2$$

$$y - (-2) = 2(x - 1)$$

$$\therefore y = 2x - 4$$

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2 Now use $y - y_1 = m(x - x_1)$ to determine the equation.

3 Write down your answer. The equation of the straight line is $y = 2x - 4$.

Case (ii): We are given the gradient m and one other point, $A(x_1, y_1)$.
Since we already know the gradient m, we can find the rule using:

$$y - y_1 = m(x - x_1)$$

Example 8 Finding the equation of a straight line

Find the equation of the line that passes through the point (3, 2) and has a gradient of −2.

Solution

1 Since we already know the gradient m and a point on the line, we can find the rule, using $y - y_1 = m(x - x_1)$. Write down the values of m, x_1 and y_1. Substitute and reorganise the equation to make y the subject.

$m = -2, x_1 = 3$ and $y_1 = 2$

$$y - 2 = -2(x - 3)$$
$$y - 2 = -2x + 6$$
$$\therefore y = -2x + 8$$

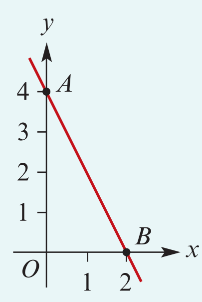
2 Write down your answer. The equation of the line is $y = -2x + 8$.

Note: The equation could also be expressed as $y + 2x - 8 = 0$.

The equation of a straight line can also be found from the graph by reading off two points and using them to find the equation, as outlined above.

Example 9 Finding the equation of a line given its graph

Find the equation of the line shown in the graph.



Solution

1 The equation of a straight line is $y = mx + c$.

2 c is the y-axis intercept. Read from the graph. The y-axis intercept is (0, 4), so $c = 4$.

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3 m is the gradient. Calculate using the two points A and B.

The coordinates of A and B are (0, 4) and (2, 0).

$$\text{gradient, } m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{0-4}{2-0} = -\frac{4}{2} = -2$$

4 Write down your answer.

The equation of the line is $y = -2x + 4$.

Vertical and horizontal lines

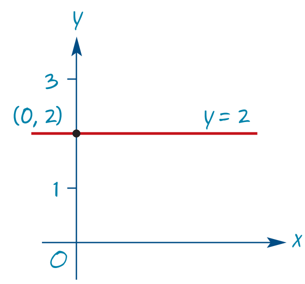
If $m = 0$, then the line is *horizontal* and the equation is simply $y = c$, where c is the y-axis intercept.

Example 10 Horizontal lines

Sketch the graph of $y = 2$.

Solution

1. The equation of a straight line is $y = mx + c$.
2. For this equation the y-axis intercept is 2 and the gradient is 0. The line is parallel to the x-axis.
3. Draw a horizontal line passing through (0, 2).
   If the line is *vertical*, the gradient is undefined and its rule is given as $x = a$, where a is the x-axis intercept.

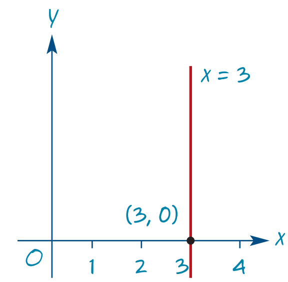


Example 11 Vertical lines

Sketch the graph of $x = 3$.

Solution

1. Note that the equation of a vertical line is not in the form $y = mx + c$.
2. For this equation, the graph is the line that consists of all points with coordinates $(3, y)$ and y varies.
3. Draw a vertical line passing through (3, 0).



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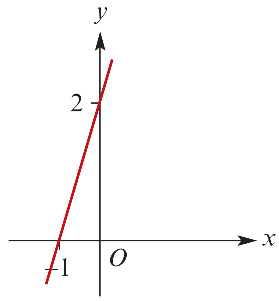
Exercise 21C

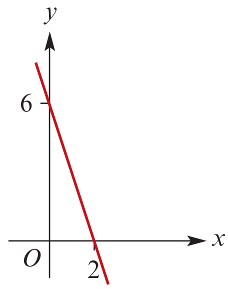
Skillsheet

1 Find the equation of each of the following straight lines.

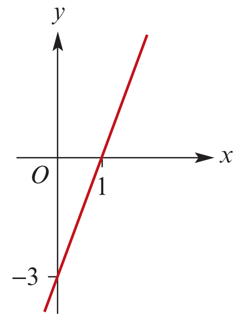
a



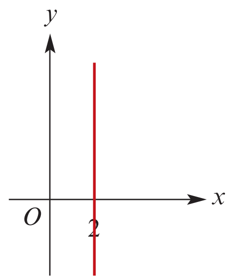
b



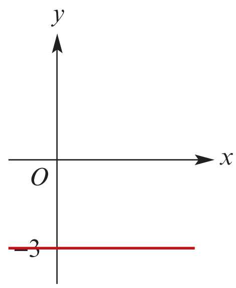
c



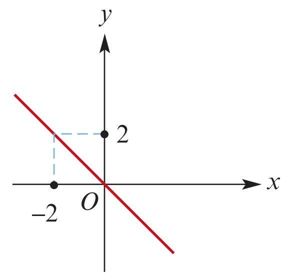
d



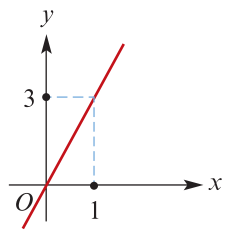
e



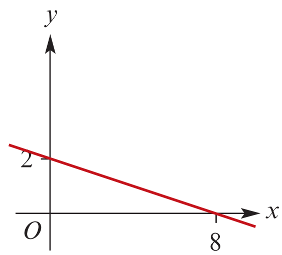
f



g



h



2 Find the equation of each of the following straight lines defined by:

a gradient $\frac{1}{2}$, passing through (0, 1) **b** gradient 2, passing through (1, 4)

c gradient 0, passing through (0, 6) **d** gradient 1, passing through (2, −4)

e gradient −2, passing through (1, 3)

3 Write in the form $y = mx + c$ the equation of the line that has the given gradient and passes through the given point.

a $\frac{3}{2}$, $(0, 1)$ **b** $\frac{1}{2}$, $(0, 6)$ **c** -3, $(1, 4)$

d -1, $(0, 2)$ **e** -2, $(-1, 4)$ **f** -1, $(3, 7)$

4 Find the equation of the line that passes through each of the following pairs of points.

a $(0, 3)$, $(3, 0)$ **b** $(-3, 0)$, $(0, -6)$ **c** $(0, 4)$, $(4, 2)$

d $(0, 1)$, $(-1, 0)$ **e** $(1, 4)$, $(5, 6)$

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21D Equation of a straight line in intercept form

Often we encounter a linear relation that is not expressed in the form $y = mx + c$. An alternative standard notation is:

$$ax + by = c$$

This is sometimes referred to as **intercept form**.

While it is necessary to transpose into gradient form if you wish to find the gradient, it is often convenient to work with linear relations in the intercept form.

Sketching graphs in intercept form

A convenient way to sketch graphs of straight lines is to plot the two axes intercepts.

Example 12 Sketching the graph of a line given in intercept form

Sketch the graph of $2x + 4y = 10$.

Solution

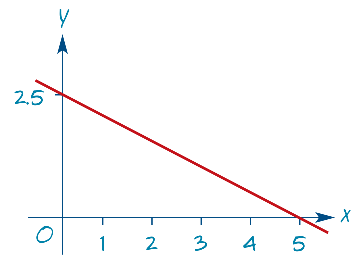
1 Find the x-axis intercept.

x-axis intercept ($y = 0$):
$\therefore 2x + 4(0) = 10$ or $x = 5$

2 Find the y-axis intercept.

y-axis intercept ($x = 0$):
$\therefore 2(0) + 4y = 10$ or $y = 2.5$

3 Use the two points to sketch the line.
When finding the equation of a straight line, it is also sometimes more convenient to express it in intercept form.



Example 13 Finding the equation of a line given two points

Find the equation of the line passing through the points $A(2, 5)$ and $B(6, 8)$.

Solution

1 Find the gradient of the line.

$$m = \frac{8-5}{6-2} = \frac{3}{4}$$

2 Use $y - y_1 = m(x - x_1)$ to find the equation.
Here $y_1 = 5$ and $x_1 = 2$. Substitute for m, x_1 and y_1, and rearrange into the intercept form.

$$y - 5 = \frac{3}{4}(x - 2)$$
$$\therefore 4(y - 5) = 3(x - 2)$$
$$\therefore 4y - 20 = 3x - 6$$
$$4y - 3x = 14$$

3 Write down your answer.

The equation of the line is:
$4y - 3x = 14$ or $-3x + 4y = 14$

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Exercise 21D

1 Sketch the graph of each of the following linear relations.

a $2x - 3y = 12$ **b** $4x - y = 8$ **c** $3x - 4y = 24$

d $2x - 5y = 20$ **e** $-3x + 4y = 15$ **f** $7x - 2y = 14$

2 Transpose each of the following linear relations from intercept form to gradient form and hence state the gradient.

a $2x - y = 6$ **b** $x + 4y = 12$ **c** $-x - 2y = 6$

d $5x - 2y = 10$ **e** $x - 5y = 10$ **f** $-x + 2y = 8$

21E Linear models

In many practical situations a linear rule can be used.

Example 14 Application of linear rules and graphs

Austcom's rates for local calls from private telephones consist of a quarterly rental fee of \$40 plus 25c for every call. Construct a linear rule that describes the quarterly telephone bill and sketch the graph.

Solution

Strategy: Construct a linear rule of the form $C = mn + c$ by finding the gradient m and the C-axis intercept.

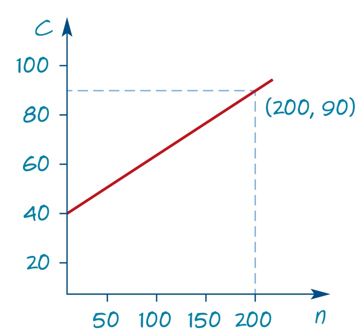
Let C = cost (\$) of quarterly telephone bill

n = number of calls

C-axis intercept = 40

Gradient, $m = 0.25$

$C = 0.25n + 40$



Note: The graph should be a series of discrete points rather than a continuous line. With the scale used it is not practical to show it correctly.

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Example 15 Application of linear rules and graphs

The tyres on a racing car had lost 3 mm of tread after completing 250 km of a race and 4 mm of tread after completing 1000 km. Assuming that the loss of tread was proportional to the distance covered, find the total loss of tread, d mm, after s km from the start of the race. What would be the tread loss by the end of a 2000 km race? Give your answer correct to one decimal place.

Solution

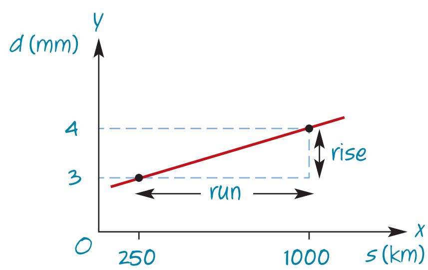
Strategy: Construct a rule of the form $d = ms + c$ by finding the gradient, m, and the d-axis intercept.

Rule: $d = ms + c$

Gradient, $m = \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{4 - 3}{1000 - 250} = \dfrac{1}{750}$

$\therefore d = \dfrac{1}{750}s + c$



When $s = 250, d = 3$

$\therefore$ $3 = \dfrac{1}{3} + c$ or $c = 2.67$ (2 d.p.)

$\therefore$ total loss of tread after s km, $d \approx \dfrac{1}{750}s + 2.67$

When $s = 2000$

$d \approx \dfrac{1}{750} \times 2000 + 2.67$

$= 5.3$ (1 d.p.)

The loss of tread at the end of a 2000 km race is 5.3 mm, correct to one decimal place.

Exercise 21E

1 The weekly wage, $\$w$, of a vacuum cleaner salesperson consists of a fixed sum of \$350 plus \$20 for each cleaner sold. If n cleaners are sold per week construct a rule that describes the weekly wage of the salesperson.

2 The reservoir feeding an intravenous drip contains 500 mL of a saline solution. If the drip releases the solution into a patient at the rate of 2.5 mL/min:

a construct a rule that relates v, the amount of solution left in the reservoir after time, t minutes

b state the possible values of t and v

c sketch the graph of the relation.

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3 The cost ($\$C$) of hiring a taxi consists of two elements: a fixed flagfall and a figure that varies with the number (n) of kilometres travelled. If the flagfall is \$2.60 and the cost per kilometre is \$1.50, determine a rule that gives C in terms of n.

4 A car rental company charges \$85, plus an additional 24 cents per kilometre.

a Write a rule to determine the total charge, $\$C$, for hiring a car and travelling x kilometres.

b What would be the cost to travel 250 kilometres?

5

Extension (x) cm	0	1	2	3	4	5	6
Weight (w) g	50	50.2	50.4	50.6	50.8	51.0	51.2

The table shows the extension of a spring when weights are attached to it.

a Sketch a graph to show the relationship between x and w.

b Write a rule that describes the graph.

c What will the extension be if $w = 52.5$ g?

6 A printing firm charges \$35 for printing 600 sheets of headed notepaper and \$47 for printing 800 sheets.

a Find a formula, assuming the relationship is linear, for the charge, $\$C$, in terms of number of sheets printed, n.

b How much would they charge for printing 1000 sheets?

7 An electronic bank teller registered \$775 after it had counted 120 notes and \$975 after it had counted 160 notes.

a Find a formula for the sum registered ($\$C$) in terms of the number of notes (n) counted. (Assume the formula connecting C and n is of the form $C = an + b$.)

b Was there a sum already on the register when counting began? If so, how much?

8 An electrician charges \$50 to call and \$80 per hour. Assuming that the relationship between charge and time is linear:

a find a formula for $\$C$, the charge in terms of the time spent (t hours)

b draw a graph of C against t

c calculate the cost of a job that takes 2 hours 45 minutes to complete.

9 The table shows the conversion from British pounds to Australian dollars for various amounts of money.

Australian dollars (\$)	46	92	184
British pounds (£)	20	40	80

a Plot these points and draw a straight line through them.

b What does the gradient of the straight line represent?

c State the gradient.

d Find a formula for $\$A$ in terms of $£P$, British pounds.

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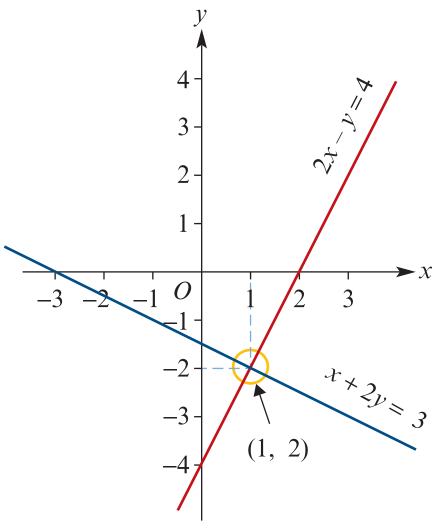
21F Simultaneous equations

A linear equation that contains two unknowns (e.g. $2y + 3x = 10$) does not have a single solution. Such an equation actually expresses a relationship between pairs of numbers, x and y, that satisfy the equation. If we represent graphically all the possible pairs of numbers (x, y) that will satisfy the equation, the result is a straight line – hence the name *linear relation*.

If the graphs of two such equations are drawn on the same set of axes, and they are non-parallel, the lines will intersect at one point only. Hence there is one pair of numbers that will satisfy both equations simultaneously.



Finding the intersection of two straight lines can be done graphically; however, the accuracy of the solution will depend on the accuracy of the graphs. Alternatively, this point of intersection can be found algebraically by solving the pair of **simultaneous equations**.

We will look first at two techniques for solving simultaneous equations.

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Example 16 Solving a pair of simultaneous equations

Solve the equations $2x - y = 4$ and $x + 2y = -3$.

Solution

By substitution

1 Write down the equations and label them 1 and 2.

$2x - y = 4 \quad (1)$

$x + 2y = -3 \quad (2)$

2 Express one unknown from either equation in terms of the other unknown.

From equation (2) we get:

$x = -3 - 2y$

3 Substitute this expression into the other equation, reducing it to one equation in one unknown.

Equation (1) then becomes:

$$2(-3 - 2y) - y = 4$$

$$\therefore \quad -6 - 4y - y = 4$$

$$\therefore \quad -5y = 10$$

4 Solve this equation.

$$\therefore \quad y = -2$$

5 Substitute the value of y into (2).

$$x + 2(-2) = -3$$

$$\therefore \quad x = 1$$

6 Check in equation 1.

Check in (1) LHS $= 2(1) - (-2) = 4$

RHS $= 4$

Note: This means that the point $(1, -2)$ is the point of intersection of the graphs of the two linear relations.

By elimination

Strategy: If the coefficient of one of the unknowns is the same in both equations, we can eliminate that unknown by subtracting one equation from the other. It may be necessary to multiply one of the equations by a constant to make the coefficients of x or y the same for the two equations.

1 Write down the equations and label them 1 and 2.

$2x - y = 4 \quad (1)$

$x + 2y = -3 \quad (2)$

$2x + 4y = -6 \quad (2')$

2 To eliminate x, multiply equation (2) by 2 and subtract the result from equation (1).

3 Subtract(1) – (2′).

$\therefore \quad -5y = 10 \quad (1) - (2')$

or $y = -2$

4 Now substitute for y in (1) to find x.

$\therefore \quad 2x - (-2) = 4$

5 Check as in the substitution method.

or $x = 1$

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How to solve simultaneous equations using the TI-Nspire CAS

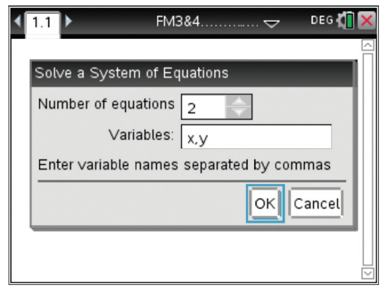
Solve the following pair of simultaneous equations.

$$24x + 12y = 36$$

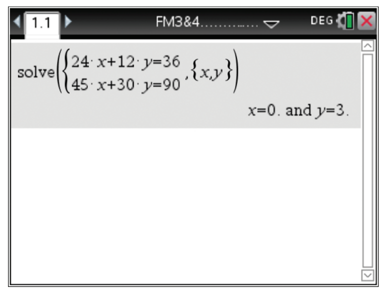
$$45x + 30y = 90$$

Steps

1. Start a new document and select **Add Calculator.**
2. Press menu **>Algebra>Solve System of Equations>Solve System of Equations** and complete the pop-up screen as shown (the default settings are for two equations with variables x and y).
   A **simultaneous equation template** will be pasted to the screen.



3. Enter the equations into the template, as shown.
4. Press enter to display the solution, $x = 0$ and $y = 3$.



5. The solution $x = 0$ and $y = 3$ can be checked by substitution.

$$24 \times 0 + 12 \times 3 = 36 \checkmark$$

$$45 \times 0 + 30 \times 3 = 90 \checkmark$$

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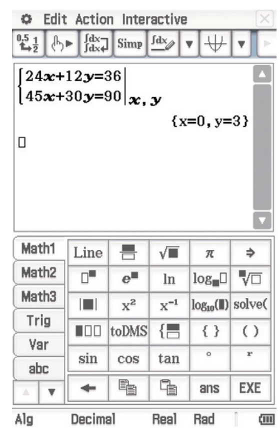
How to solve simultaneous linear equations algebraically using the ClassPad

Solve the following pair of simultaneous equations.

$$24x + 12y = 36$$
$$45x + 30y = 90$$

Steps



1. Open the **Main** application, Main $\sqrt{\alpha}$.
2. Press Keyboard button to display the soft keyboard.
3. In Math1 tap the simultaneous equation template.
4. Enter the information:

$$\begin{cases} 24x + 12y = 36 \\ 45x + 30y = 90 \end{cases}\Bigg|_{x,y}$$

Press EXE to display the solution $x = 0$ and $y = 3$.

5. The solution $x = 0$ and $y = 3$ can be checked by substitution.

$$24 \times 0 + 12 \times 3 = 36$$
$$45 \times 0 + 30 \times 3 = 90$$

Exercise 21F

1 Solve each of the following pairs of simultaneous linear equations using either the substitution or elimination method. Check your answers with your calculator.

a $x + 2y = 6$, $x + 3y = 4$ **b** $7x + 6y = 0$, $5x - 6y = 48$ **c** $3x + 2y = 12$, $x + 2y = 8$

d $x - 2y = 6$, $4x + 2y = 14$ **e** $x + 3y = 12$, $x + y = 8$ **f** $9x + 2y = 48$, $x - 2y = 2$

g $2x + 3y = 13$, $2x + 5y = 21$ **h** $3x - y = 10$, $x + y = -2$ **i** $3p + 5q = 17$, $4p + 5q = 16$

j $2x + 3y = 12$, $5x + 4y = 23$ **k** $5x + 4y = 21$, $3x + 6y = 27$ **l** $9x + 8y = 17$, $2x - 6y = -4$

m $y = 6 - x$, $2x + y = 8$ **n** $9 + x = y$, $x + 2y = 12$ **o** $2y = 4 + x$, $y = x + 8$

p $x + 4 = y$, $y = 10 - 2x$ **q** $y = 4 - x$, $y = x + 6$ **r** $y = 4 + 2x$, $y - 2x = 6$

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21G Problems involving simultaneous linear equations

In this section we see how to solve worded problems by forming simultaneous equations.

Example 17 Application of simultaneous linear equations

There are two possible methods for paying gas bills:

- method A: a fixed charge of \$25 per quarter + 50c per unit unit of gas used
- method B: a fixed charge of \$50 per quarter + 25c per unit unit of gas used.

Determine the number of units that must be used before method B becomes cheaper than method A.

Solution

1 Set up the equations.

Let C_1 = charge in \$ using method A

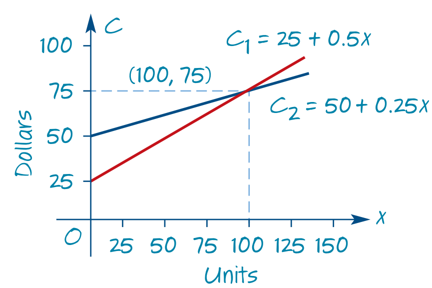
C_2 = charge in \$ using method B

x = number of units of gas used

Now $C_1 = 25 + 0.5x$

$C_2 = 50 + 0.25x$

2 Graph the two straight lines and find the point of intersection.



3 Write down your answer.

Note: The solution could also be obtained by solving simultaneous linear equations:

$$C_1 = C_2$$
$$25 + 0.5x = 50 + 0.25x$$
$$0.25x = 25$$
$$\therefore x = 100$$

It can be seen from the graph that if the number of units exceeds 100, then method B is cheaper.

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Example 18 Application of simultaneous linear equations

If 3 kg of jam and 2 kg of butter cost \$29, and 6 kg of jam and 3 kg of butter cost \$54, find the cost of 1 kg of jam and 1 kg of butter.

Solution

1 Set up the equations.

Let the cost of 1 kg of jam = x dollars and the cost of 1 kg of butter = y dollars.

Then $3x + 2y = 29$ (1)

and $6x + 3y = 54$ (2)

Multiply (1) by 2: $6x + 4y = 58$ (1′)

Subtract (1′) from (2): $-y = -4$

$y = 4$

Sustituting in (2) gives: $6x + 3(4) = 54$

$6x = 42$

$\therefore x = 7$

2 Solve the equations, or use your calculator.

3 Write down your answer.

∴ the jam costs \$7 per kilogram and the butter \$4 per kilogram.

Exercise 21G

Skillsheet

1 Find two numbers whose difference is 9 and whose sum is 11.

2 The sum of two numbers is 73 and their difference is 37. Find the numbers.

3 The sum of two numbers is 20 and their difference is 4. Find the numbers.

4 Find two numbers where twice the first added to the second is 26 and the first added to three times the second is 28.

5 A cup and a saucer cost \$5.25 together. A cup and two saucers cost \$7.50. Find the cost of a cup.

6 A shop sells bread rolls. If five brown rolls and six white rolls cost \$2.94 and three brown rolls and four white rolls cost \$1.86, find the cost of each type of roll.

7 In a test, the sum of Anne's mark and David's mark is 42. Sheila has twice as many marks as David and the sum of Anne's and Sheila's marks is 52. What are the marks of the three students?

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8 A herbalist wishes to prepare a mixture from two herbs, A and B. The mixture costs \$6 per kilogram. If herb A costs \$5 per kilogram and herb B costs \$8 per kilogram and 60 kilograms of the mixture is to be prepared, how many kilograms of each herb should be used?

9 The cost (\$$C$) of printing a pamphlet is given by the expression $C = a + bn$, where n is the number of pamphlets, \$$a$ is the fixed cost of printing and \$$b$ is the cost per pamphlet.

a If it costs \$460 to print 80 pamphlets and \$1060 to print 200 pamphlets, find:

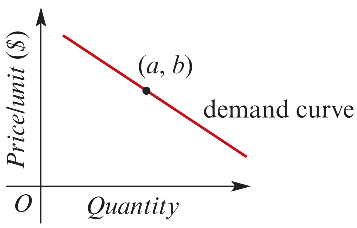
i a **ii** b.

b Find the cost of printing 500 pamphlets.

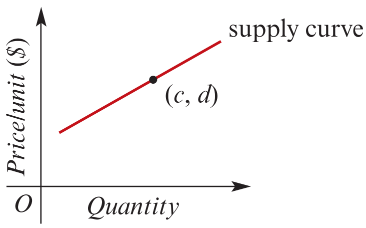
c Sketch the graph of C against n.

21H Break-even analysis

For each price of a product, a corresponding quantity of the product will be demanded (purchased) during some time interval. The quantities of a product that can be demanded or supplied can be represented graphically.



The price of \$$b$ for a units.



If c units are produced, the price is \$$d$.

Example 19 Break-even analysis

A firm sells its product at \$20 per unit. The cost of production (\$$C$) is given by the rule $C = 4x + 48$, where x is the number of units produced. Find the value of x for which the cost of the production of x units is equal to the revenue received by the firm for selling x units.

Solution

1 Set up the equations.

Let the revenue for producing x units be \$$R$.

$R = 20x$

$C = 4x + 48$

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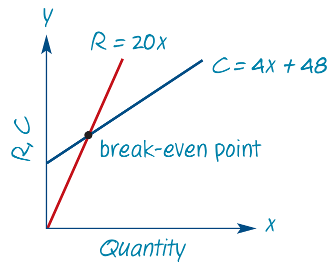
2 Draw the graphs.



3 Find the point of intersection.

When $C = R$

$20x = 4x + 48$

$\therefore \quad x = 3$

From the graph:

- if less than three item are produced, $C > R$
- if more than three item are produced $C < R$
- the point where $R = C$ is called the break-even point.

Exercise 21H

1 Find the break-even point if sales revenue (\$$S$) is given by the rule $S = 0.75x$, where x is the number of items sold and the cost (\$$C$) is given by the rule $C = 0.25x + 100$.

2 Find the break-even point for each of the following, where \$$C$ is the cost of producing x units, and the sales price per unit is \$$n$.

a $C = 2x + 16$, $n = 10$
b $C = 12x + 84$, $n = 40$
c $C = 90x + 60$, $n = 100$
d $C = 3x + 39$, $n = 16$
e $C = 30x + 350$, $n = 100$
f $C = 400x + 800$, $n = 500$

3 A manufacturer sells his product at \$72 per unit, selling all that he produces. His fixed cost is \$45 000 and the cost per unit is \$22. Find:

a the break-even point
b the profit when 1800 units are produced
c the loss when 450 units are produced
d the sales volume required in order to obtain a profit of \$90 000.

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4 A manufacturer of a product sells all that she produces. If her total revenue is given by $R = 7x$ (dollars) and her total cost is given by $C = 6x + 800$ (dollars), where x represents the number of units produced and sold then:

a determine the level of production at the break-even point

b sketch the graph of C and R against x

c determine the level of production at the break-even point if the total cost increases by 5%.

5 For each of the following, $\$R$ is the total revenue and $\$C$ is the total cost. If x represents both the number of units produced and the number of units sold, determine the break-even quantity and sketch a graph of R and C against x for each.

a $R = 3x$
$C = 2x + 4500$

b $R = 14x$
$C = \frac{22}{3}x + 1200$

c $R = 1.05x$
$C = 0.85x + 600$

d $R = 0.25x$
$C = 0.16x + 360$

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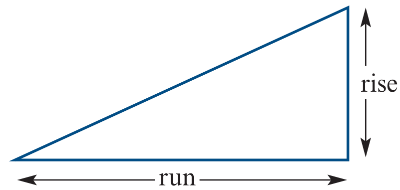
Key ideas and chapter summary

Gradient

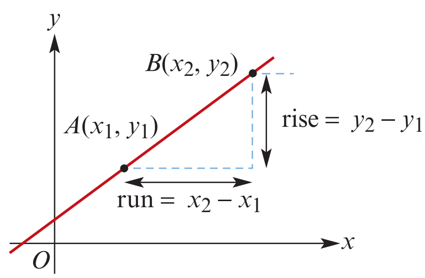
Gradient, $m = \dfrac{\text{rise}}{\text{run}}$



Gradient, coordinate definition

Given any two points on the line, $A(x_1, y_1)$ and $B(x_2, y_2)$, the gradient of the line can be found.

$$\text{gradient, } m = \frac{y_2 - y_1}{x_2 - x_1}$$



Gradient, sign of

The gradient of a line that slopes upwards from left to right is positive, and the gradient of a line that slopes downwards from left to right is negative.

Gradient–axis intercept form

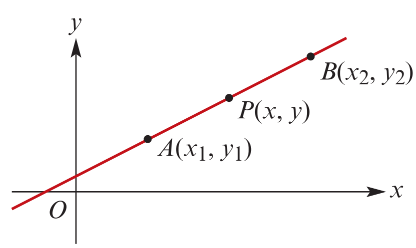
The general equation of a straight line is $y = mx + c$, where m is the gradient of the line. This form, expressing the relation in terms of y, is called the gradient–axis intercept form. The y-axis intercept is c.

Finding equation of straight line given two points

Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the equation of the line is:

$$y - y_1 = m(x - x_1)$$

$$\text{where } m = \frac{y_2 - y_1}{x_2 - x_1}$$



Finding equation of straight line given gradient and a point

If we are given the gradient, m, and one other point, $A(x_1, y_1)$, we can find the rule using:

$$y - y_1 = m(x - x_1)$$

Horizontal lines

If $m = 0$, then the line is horizontal and the equation is simply $y = c$, where c is the y-axis intercept.

Vertical lines

If the line is vertical, the gradient is undefined and its rule is given as $x = a$, where a is the x-axis intercept.

Intercept form of a straight line

$ax + by = c$

This is sometimes referred to as intercept form.

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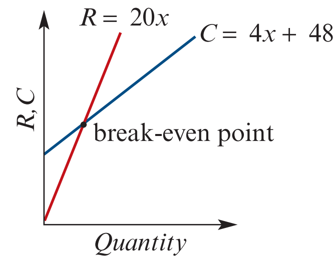
Review

Simultaneous equations — **Simultaneous equations** are equations of two or more lines or curves in a Cartesian plane, the solution of which is the point of intersection of the pairs of lines or curves.

Break-even point — The **break-even point** of a production process is where the cost of production is equal to the income received.



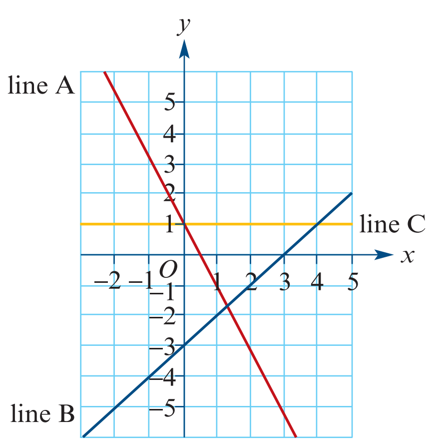
Skills check

Having completed this chapter you should be able to:

- sketch a straight line given the coordinates of two points
- sketch a straight line given gradient and a point
- interpret linear graphs that model real-life situations
- solve linear simultaneous equations by both algebraic and graphical techniques
- use simultaneous equations to solve problems
- use break-even analysis to solve problems.

Multiple-choice questions

Questions 1 to 4 refer to the graph below.



1 The slope of line A is:

A $-\frac{1}{2}$ **B** -2 **C** 2 **D** $-\frac{3}{2}$ **E** -5

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Review

2 The equation of line B is:

A $y - 2x = 3$ **B** $x - y = 3$ **C** $y - x = 3$
D $-2y - x = 6$ **E** $y + x = 3$

3 The equation of line C is:

A $y = -x$ **B** $y = -1$ **C** $x = 1$ **D** $x = -y$ **E** $y = 1$

4 Lines B and C intersect at the point:

A $(1, 4)$ **B** $(4, 1)$ **C** $(-4, -1)$ **D** $(-1, -4)$ **E** $(0, 1)$

5 The solution of the simultaneous equations $2x - y = 10$ and $x + 2y = 0$ is:

A $x = -2$ and $y = 3$ **B** $x = 2$ and $y = -3$ **C** $x = 4$ and $y = -2$
D $x = 6$ and $y = 2$ **E** $x = 1$ and $y = -8$

6 A straight line has gradient 3 and passes through the point with coordinate (1, 9). The equation of the line is:

A $y = x + 9$ **B** $y = 3x + 9$ **C** $y = 3x + 6$
D $y = -\frac{1}{3}x + 1$ **E** $y = -\frac{1}{3}x + 6$

7 If two lines, $5x - y + 7 = 0$ and $ax + 2y - 11 = 0$, are parallel, then a equals:

A -5 **B** 5 **C** -10 **D** 10 **E** $-\frac{1}{2}$

8 The cost ($\$C$) of hiring a car is given by the formula $C = 2.5x + 65$, where x is the number of kilometres travelled. A person is charged \$750 for the hire of the car. The number of kilometres travelled was:

A 65 **B** 145 **C** 160 **D** 200 **E** 274

9 If Sue buys three packets of chips and two bottles of cola, it will cost her \$13.20. However, if she buys two packets of chips and three bottles of cola it will cost her only \$11.80. The price of each item is:

A cola = \$3.20, chips = \$1.80 **B** cola = \$2.10, chips = \$3
C cola = \$1.80, chips = \$3.20 **D** cola = \$2, chips = \$2.90
E cola = \$1.20, chips = \$2

10 The equation of the line passing through (5, 9) and parallel to the line $y = 3x + 7$ is:

A $y = -\frac{1}{3}x + 7\frac{1}{3}$ **B** $y = -\frac{1}{3}x + 7$ **C** $y = 3x - 6$
D $y = 3x + 7$ **E** $y = 3x + 9$

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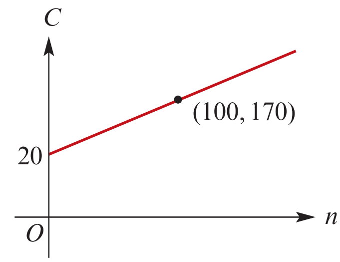
11 The graph opposite shows the cost, $\$C$, of making n apple pies.
The profit from the sale of 80 apple pies is \$100.
The selling price of one apple pie is:

A \$1.50 **B** \$1.75 **C** \$2.50
D \$2.75 **E** \$3.75



12 At a convenience store, one doughnut costs \$2.40 and one drink costs \$3.00.
A customer purchased five doughnuts and a number of drinks at a total cost of \$24.00.
The number of drinks purchased was:

A 4 **B** 5 **C** 6 **D** 9 **E** 10

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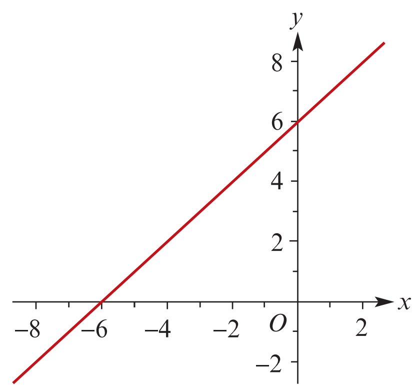
13 The equation of the line shown on the graph is:

A $y = x - 6$ **B** $y = x + 6$
C $y = 6 - x$ **D** $y = -6$
E $y = 6$



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14 A point that lies on the graph of $3x - 2y = -5$ is:

A $(3, -2)$ **B** $(1, 1)$ **C** $(1, -1)$ **D** $(2, -3)$ **E** $(-1, 1)$

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15 The Blue Caps cricket club has different prices for its junior and senior subscriptions.
The total cost for two junior subscriptions and one senior subscription is \$225.
The cost of a senior subscription is three times the cost of a junior subscription.
The cost of a senior subscription is:

A \$45 **B** \$75 **C** \$90 **D** \$135 **E** \$180

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16 The vertical line that passes through the point (3, 2) has the equation:

A $x + y = 5$ **B** $xy = 6$ **C** $3y = 2x$ **D** $y = 2$ **E** $x = 3$

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17 A line passes through the points (−1, 1) and (3, 5). Another point that lies on this line is:

A (0, 1) **B** (1, 3) **C** (2, 6) **D** (3, 4) **E** (4, 7)

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Extended-response question

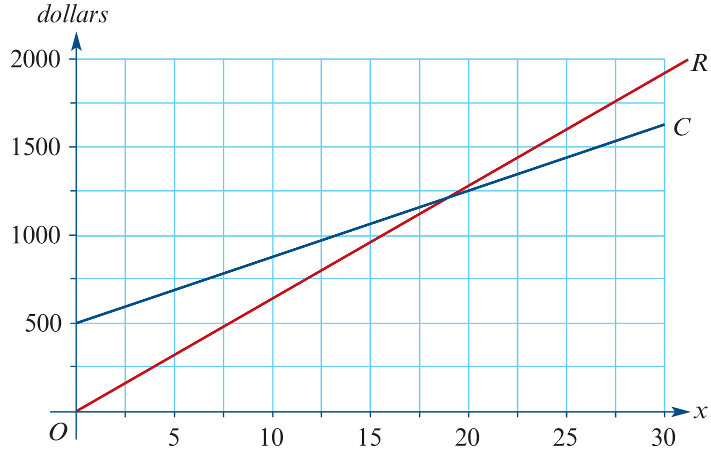
1 Anne sells Softsleep pillows for \$65 each.

a Write an equation for the revenue, R dollars, that Anne receives from the sale of x Softsleep pillows.

b The cost, C dollars, of making Softsleep pillows is given by $C = 500 + 40x$. Find the cost of making 30 Softsleep pillows.

The revenue, R, from the sale of x Softsleep pillows is graphed below. Also shown is the graph of $C = 500 + 40x$.

c How many Softsleep pillows will Anne need to sell to break even?



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22

Module 4: Graphs and relations

Chapter 22

Analysing and interpreting graphs

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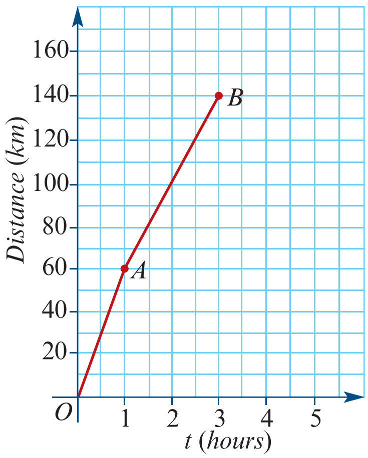
22A Line segment graphs

A car travels 60 km in 1 hour and then 80 km in the following 2 hours. We can represent this situation graphically, as shown opposite.



The line segment OA represents the journey for the first hour and the line segment AB represents the journey for the second and third hours.

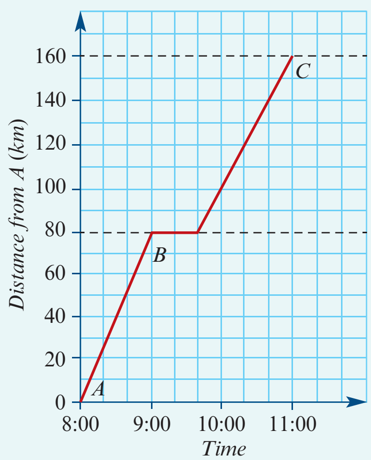
The gradient of line segment OA is 60. This gradient indicates that the speed of the car (assumed constant) for the first hour is 60 km/h.

The gradient of line segment AB is 40.

For the second section of the journey the constant speed is 40 km/h.

Example 1 Interpreting line segment graphs

The graph shows the journey of a car through three towns, A, B and C, on a highway. B is 80 km from A.



a Where was the car at:

  i 9:00? **ii** 9:30?

b What was the average speed of the car between:

  i A and B? **ii** B and C?

c For how long does the car stop at B?

d How long did the journey take?

e What was the average speed of the car for the whole journey? Give your answer correct to the nearest whole number.

Solution

a **i** At 9:00 the car was 80 km from A.

ii At 9:30 the car was 80 km from A.

b **i** The average speed between A and B

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{80 - 0}{1} = 80 \text{ km/h}$$

ii The average speed between B and C

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{160 - 80}{\frac{4}{3}} = 60 \text{ km/h (to 1 d.p.)}$$

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c The car stopped for 40 minutes = $\frac{2}{3}$ hour at B.

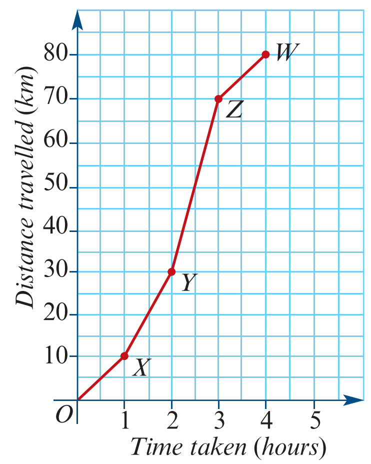
d The journey took 3 hours.

e The average speed $\frac{160}{3}$ = 53 km/h, correct to the nearest km/h.

The graph in Example 1 is called a **line segment graph** because it is made up of different straight line segments. The graphs are also called *piecewise linear* graphs.

Exercise 22A

1 The graph shows a train journey in four stages.



a How far has the train travelled after:

i 2 hours? **ii** $1\frac{1}{2}$ hours?

iii 3 hours?

b How long does the train take to travel:

i 5 km? **ii** 60 km?

iii 20 km?

c What is the speed of the train:

i from O to X?

ii from X to Y?

iii from Y to Z?

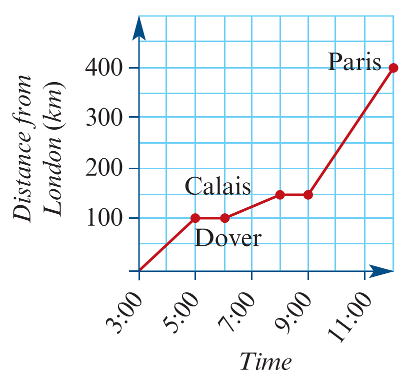
iv from Z to W?

2 A cyclist leaves home and rides at 15 km/h for 3 hours. He rests for 1 hour and then continues riding away from home at 12 km/h for 2 hours. He has another rest of half an hour before returning home without stopping at 20 km/h.

a Draw a travel graph to show the journey (use 1 cm for 1 hour and 1 cm for 10 km).

b How long has the cyclist been away from home?

3 The graph shows a man's journey by train, boat and train from London to Paris.



a At what time does he:

i arrive at Dover?

ii arrive in Paris?

iii leave Calais?

iv leave Dover?

b For how long does he stop at Dover?

c At what time is he exactly halfway between Calais and Paris?

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4 Water is poured into a tank at a rate of 10 L/min for 5 minutes and then the rate is increased to 15 L/min for another 10 minutes.

a Draw a line segment graph representing the volume of water in the tank at time t.

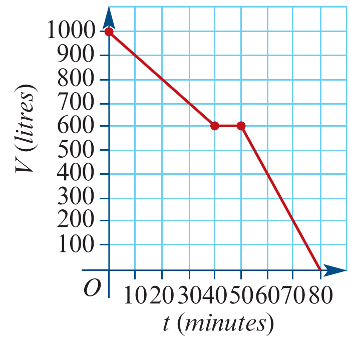
b Find the equation for the line segment representing:

i the amount of water in the tank for the first 5 minutes

ii the amount of water in the tank for the 10-minute period following the first 5 minutes.

c How much water has been poured into the tank?

5 The graph represents the volume of water in litres in a tank at time t minutes.



a Give the rate in litres/minute at which water is flowing out of the tank for the first 40 minutes.

b Give the equation for the volume, V, in terms of t for the first 40 minutes.

c Give the rate in litres/minute at which water is flowing out of the tank for $50 \le t \le 80$.

6 Two towns, A and B, are 150 km apart. David leaves town A and travels towards town B. Sue leaves town B at the same time and travels towards town A. The graph represents both of their journeys.

a At what speed does David travel?

b How long does Sue take for the journey?

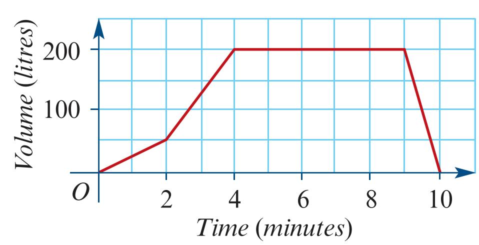
c At what time do their paths cross?

d Where do they pass each other?

e For how long does Sue stop on the way?

f At what speed does Sue travel for the last hour of her journey?

7 This graph shows the volume of water, in litres, in a bath.



a How long did it take the bath to fill to 200 litres?

b At what rate was the bath being filled for the first 2 minutes?

c At what rate was the bath emptied?

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22B Step graphs

A **step graph** is a line segment graph where each segment is horizontal.

Example 2 Constructing a step graph

This table shows the airmail rates for articles in a particular country.

a Write a cost function, i.e. C ($) in terms of $M(g)$.

b Sketch a graph of the cost function.

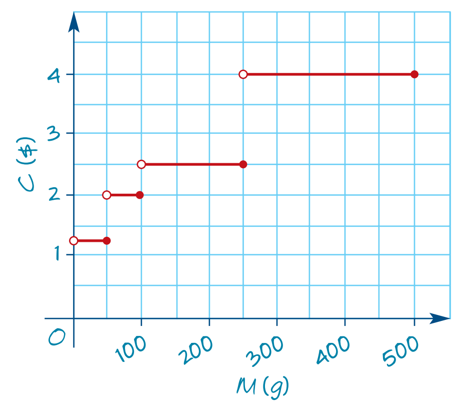
Mass M(g)	*Cost C ($)*
Up to 50 g	$1.20
Over 50 g up to 100 g	$2.00
Over 100 g up to 250 g	$2.50
Over 250 g up to 500 g	$4.00

Solution

a

$$C = \begin{cases} 1.20 & \text{for } M \le 50 \\ 2.00 & \text{for } 50 < M \le 100 \\ 2.50 & \text{for } 50 < M \le 250 \\ 4.00 & \text{for } 250 < M \le 500 \end{cases}$$

b



Note:

- ○ means the value is not included in the interval.
- ● means the value is included in the interval.

Example 3 Constructing a step graph

Sophie invests $10 000, for which interest will be 12% per annum. The interest is calculated at the end of each year and added to the amount invested; i.e. $10 000 is invested at 12% per annum compound interest.

The amount of money she has in the account for the first 5 years is shown in the table. Sketch the graph of the amount in the account against the year.

Year (interest paid at the end of each year)	*Amount ($) (amount given to the nearest dollar)*
0–1	10 000
1–2	11 200
2–3	12 544
3–4	14 049
4–5	15 735

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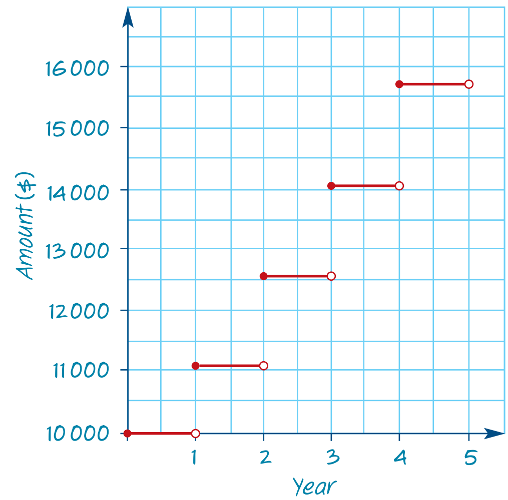
Solution

For each time interval, draw the appropriate line segment. In this case, indicates that change takes place at the end of the year.



The graphs in Examples 2 and 3 are called step graphs.

Exercise 22B

1 The postal rates for letters for a particular country are shown in this table.

a Sketch a step graph representing this information.

b Find the cost of sending a letter weighing:

i 370 g **ii** 201 g **iii** 200 g.

Weight not over	*Rate*	*Weight not over*	*Rate*
60 g	34c	350 g	$1.22
100 g	48c	450 g	$1.38
150 g	62c	500 g	$1.56
200 g	76c	750 g	$2.56
250 g	90c	1000 g	$3.40
300 g	$1.06		

2 A multistorey car park has tariffs as shown. Sketch a step graph showing this information.

First 2 hours	$5.00
2–3 hours	$7.50 (more than 2, less than or equal to 3)
3–4 hours	$11.00 (more than 3, less than or equal to 4)
4–8 hours	$22.00 (more than 4, less than or equal to 8)

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3 Suppose that Australia Post charged the following rates for airmail letters to Africa: \$1.20 up to 20 g; \$2.00 over 20 g and up to 50 g; \$3.00 over 50 g and up to 150 g.

a Write a cost function, C (\$), in terms of the mass, M (g), for letters up to 150 g.

b Sketch the graph of the function.

4 Telenet listed the following scale of charges for a 3-minute STD call between the hours of 6 p.m. and 10 p.m., Monday to Friday.

Distance, d (kilometres)	Up to 25 (including 25)	25–50 (not including 25)	50–85 (not including 50)	85–165 (not including 85)	165–745 (not including 165)	Over 745 (not including 745)
Cost, C (dollars)	0.30	0.40	0.70	1.05	1.22	1.77

a Write a cost function, C(\$), in terms of distance, d (km).

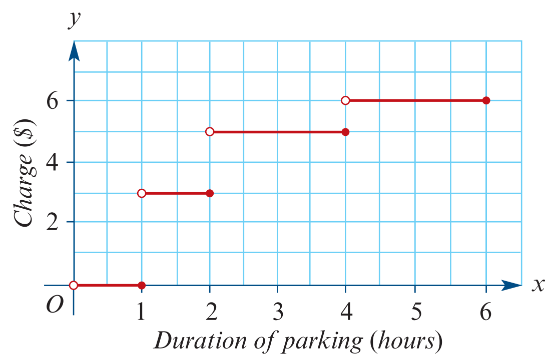
b Sketch the graph of the function.

5 This step graph shows the charges for a market car park.

a How much does it cost to park for 40 minutes?

b How much does it cost to park for 2 hours?

c How much does it cost to park for 3 hours?



22C Non-linear graphs

In the previous work in this chapter, all the graphs were linear or consisted of linear sections. This is not the case for many situations.

Example 4 Interpreting non-linear graphs

The graph represents the speed against time of a particular make of car.

a What was the speed of the car after 15 seconds?

b After how many seconds did the car reach the following speeds?

i 30 km/h **ii** 105 km/h **iii** 120 km/h

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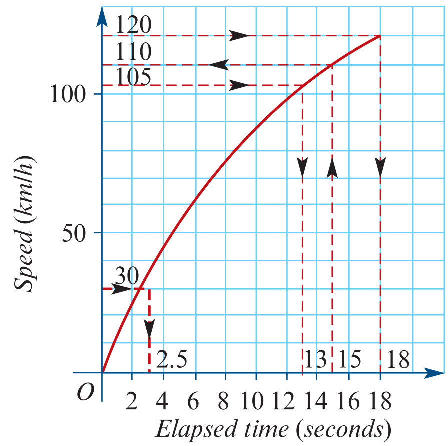
Solution

a From the graph: 110 km/h

b **i** 2.5 seconds **ii** 13 seconds
iii 18 seconds



Example 5 Interpreting real-world graphs

The population (in millions) of an island country for the years 1811 to 1951 is given in this table. A census is taken every 10 years (but not during war years).

Year	1811	1821	1831	1841	1851	1861	1871	1881	1891	1901	1911	1921	1931	1951	1961
Population	10.2	12.0	13.9	15.9	17.9	20.1	22.7	26.0	29.0	32.5	36.1	37.9	40.0	43.8	46.1

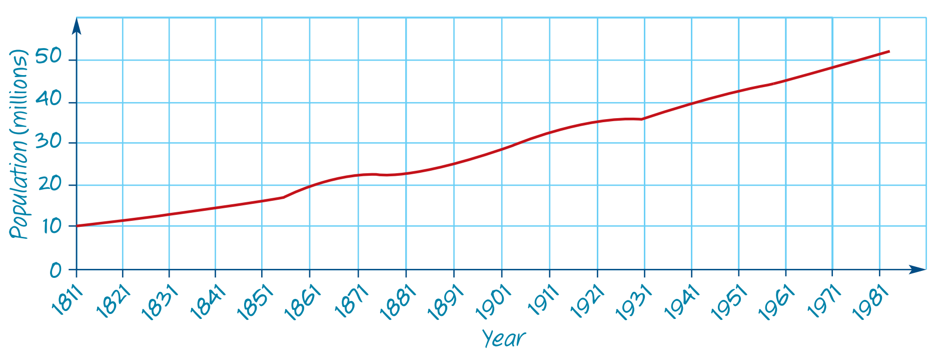
a Plot the graph.

b Estimate the population in:

i 1905 **ii** 1971.

Solution

a



b **i** From the graph, the population in 1905 is estimated to be approximately 34 million.

ii From the graph, the population in 1971 is estimated to be 48 million.

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Interpolation and extrapolation

The method of estimation of the population in 1905 in Example 5 is called **interpolation**. The year 1905 lies in the interval 1811 to 1961.

Finding a value from a graph that lies within the range of values used to construct the graph is called *interpolation*.

The method of estimation of the population in 1971 in Example 5 is called **extrapolation**. The year 1971 does not lie in the interval 1811 to 1961.

Finding a value from a graph that lies outside the range of values used to construct the graph is called *extrapolation*.

In general, extrapolation is a less reliable predictor than interpolation.

Exercise 22C

1 The forestry commission has collected data for a particular type of tree. The dataset gives the girth of a tree in centimetres at a particular age.

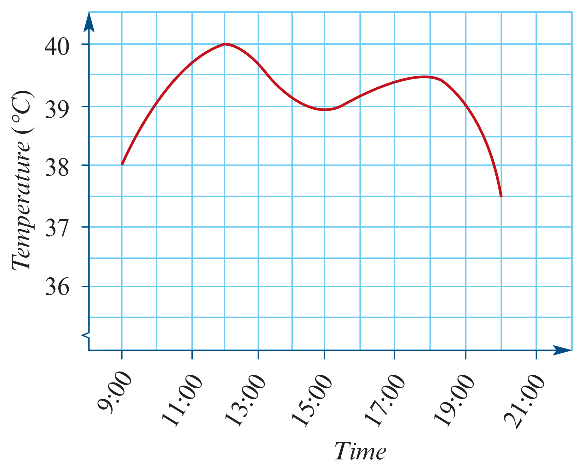
Age of tree (years)	1	1.6	2.4	3.7	5.6	6.9	8
Girth of tree (centimetres)	31.6	41.8	53	66.4	82.4	91.8	99

a Draw the graph of girth versus age.

b From the graph, find the girth of a tree in centimetres when it is 4 years old.

c At what age will the girth of the tree be 80 cm?

2 The graph shows the temperature of a person admitted to hospital at 9:00 and the temperature being recorded between 9:00 and 20:00.



a For how long did the person's temperature rise before reaching a maximum?

b What was the maximum temperature recorded?

c At what time(s) was the temperature 39°C?

d What was the temperature at 18:00?

e Assuming that the temperature continues to fall after 20:00, at what time would you expect the temperature to reach 37°C?

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ISBN 978-1-107-56757-3

Cambridge University Press

3 Plot these points on graph paper. Sketch in the curve of best fit. (Extend your curve to $x = 70$.)

x	0	10	20	30	40	50
y	5550	4280	3300	2550	1960	1510

a Find by interpolation a y-value corresponding to $x = 25$.

b Find by extrapolation a y-value corresponding to $x = 60$.

4 Plot these points on graph paper. Sketch in the curve of best fit. (Extend your curve to $x = 100$.)

x	0	20	40	60	80
y	200	190	161	117	62

a Find by interpolation a y-value corresponding to $x = 30$.

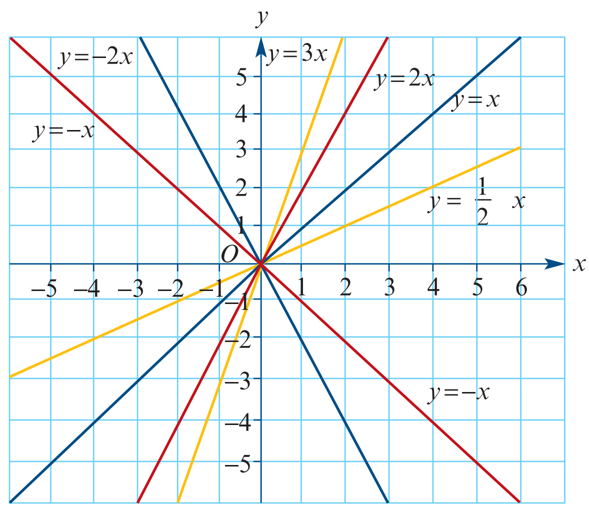
b Find by extrapolation a y-value corresponding to $x = 90$.

22D Relations of the form $y = kx^n$ for $n = 1, 2, 3, -1, -2$

▶ Graphs of relations of the form $y = kx^n$

Straight-line graphs

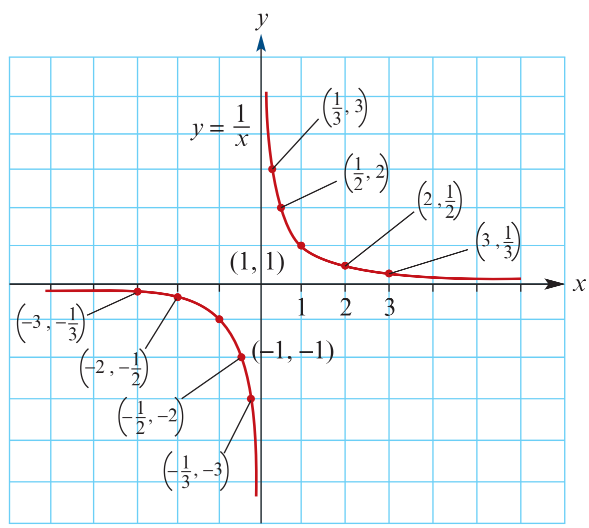
***n* = 1**; $y = kx$. These are **straight lines through the origin**.



Hyperbolas

***n* = −1**; $y = \frac{k}{x}$. These graphs are **hyperbolas**. For example, for $k = 1$, $y = \frac{1}{x}$.

The graph approaches both the x-axis (the line $y = 0$) and the y-axis (the line $x = 0$) but does not cross either of these lines. We refer to these lines as *asymptotes*.



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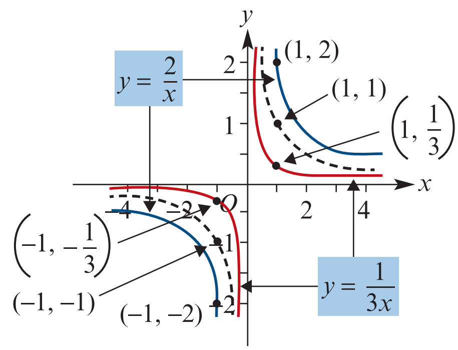
In the diagram on the right, the graphs of $y = \frac{1}{x}$, $y = \frac{2}{x}$ and $y = \frac{1}{3x}$ are shown.

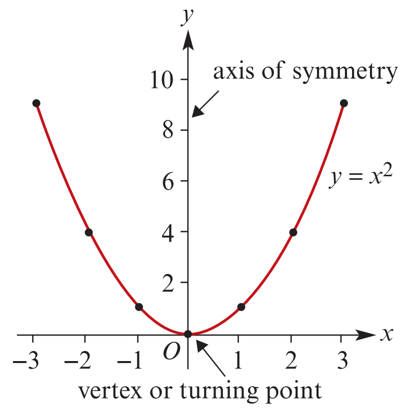
As can be seen from the diagram, the graphs of $y = \frac{2}{x}$ and $y = \frac{1}{3x}$ have the same 'shape' and asymptotes as $y = \frac{1}{x}$, but have been 'stretched'.



Parabolas

$\boldsymbol{n = 2}$; $y = kx^2$. These graphs are **parabolas** with the vertex as the origin. When $k = 1$, $y = x^2$.

- The graph is called a *parabola*.
- The possible y-values are all positive real numbers and 0.
- It is symmetrical about the y-axis. The line about which the graph is symmetrical is called the *axis of symmetry*.
- The graph has a *vertex*, or *turning point*, at the origin $(0, 0)$.



- The minimum value of y is 0 and it occurs at the turning point.

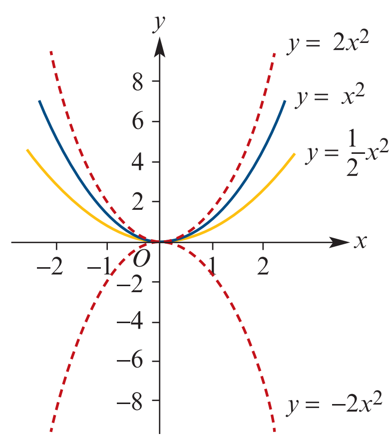
Consider graphs of the form $y = kx^2$. In the basic graph of $y = kx^2$, k is equal to 1. On the same set of axes we can sketch the graphs of:

$y = x^2$

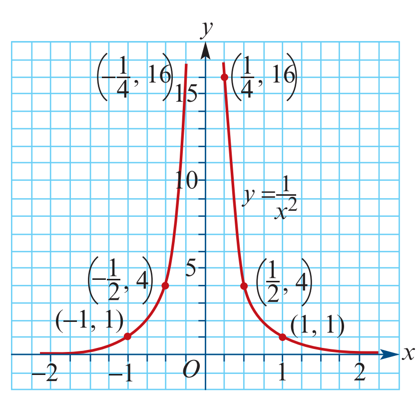
$y = 2x^2 \quad (k = 2)$

$y = \frac{1}{2}x^2 \quad (k = \frac{1}{2})$

$y = -2x^2 \quad (k = -2)$



$\boldsymbol{n = -2}$; $y = kx^{-2} = \frac{k}{x^2}$



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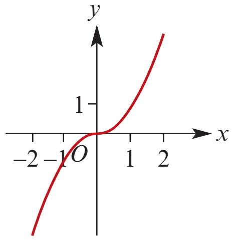
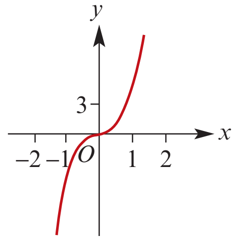
$\boldsymbol{n = 3}$; $y = kx^3$

A family of basic cubics is formed; for example:

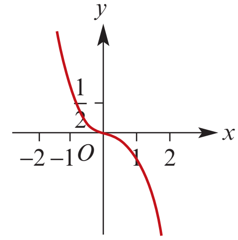
$y = x^3$



$y = 3x^3$



$y = -\frac{1}{2}x^3$



Example 6 Plotting a graph of $y = kx^n$

Plot the graph of $y = \frac{1}{3}x^3$ for $-3 \le x \le 3$.

Solution

Strategy: Form a table of values and plot, or use a graphics calculator.

When $x = -3, y = \frac{1}{3} \times (-3)^3 = \frac{1}{3} \times -27 = -9$

When $x = -2, y = \frac{1}{3} \times (-2)^3 = \frac{1}{3} \times -8 = \frac{-8}{3}$

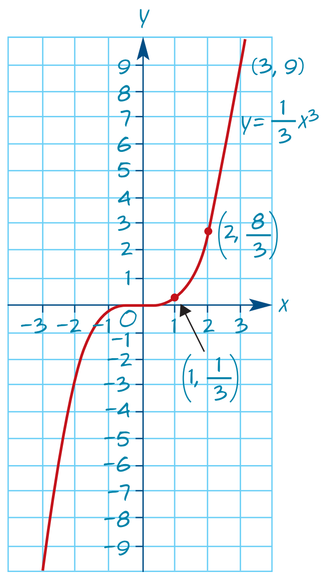
When $x = -1, y = \frac{1}{3} \times (-1)^3 = \frac{1}{3} \times -1 = \frac{-1}{3}$

When $x = 3, y = 9$

When $x = 0, y = \frac{1}{3} \times 0^3 = 0$

When $x = 1, y = \frac{1}{3}$

When $x = 2, y = \frac{8}{3}$



x	−3	−2	−1	0	1	2	3
y	−9	$-\frac{8}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{8}{3}$	9

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Example 7 Finding the value of k from the graph of $y = kx^n$

The point $(2, 50)$ lies on the graph of $y = kx^2$. Find the value of k.

Solution

1 Substitute $x = 2$ and $y = 50$ into the equation.

$50 = k \times 2^2$

2 Solve the equation for k.

$\therefore k = \dfrac{50}{4}$

$= 12.5$

Exercise 22D

Skillsheet

1 Prepare a table of values and plot the graph of each of the following for $-3 \le x \le 3$. Use a graphics calculator to help you.

a $y = \dfrac{1}{2}x^2$ b $y = -\dfrac{1}{2}x^2$ c $y = \dfrac{4}{x^2}$

d $y = -\dfrac{4}{x^2}$ e $y = -\dfrac{2}{x}$ f $y = x^2$

g $y = -\dfrac{1}{2}x^3$ h $y = -3x$ i $y = -\dfrac{2}{3}x$

2 The point $(1, 5)$ lies on the graph of $y = kx^3$. Find the value of k.

3 The point $(2, 30)$ lies on the graph of $y = kx^2$. Find the value of k.

4 The point $(16, 4)$ lies on the graph of $y = \dfrac{k}{x}$. Find the value of k.

5 The point $(2, 1)$ lies on the graph of $y = \dfrac{k}{x^2}$. Find the value of k.

6 The point $(3, -10)$ lies on the graph of $y = kx$. Find the value of k.

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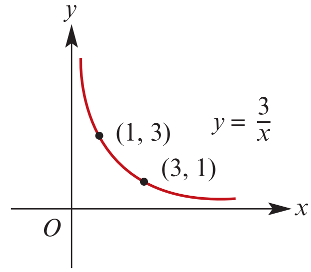
22E Linear representation of non-linear relations

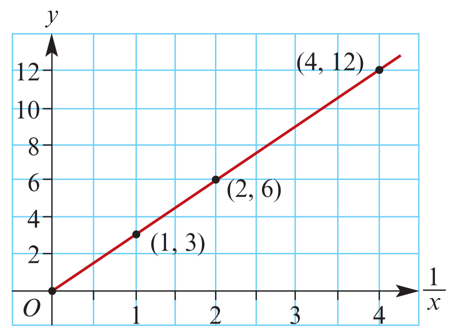
Consider the relation with rule $y = \frac{3}{x}$ for $x > 0$.

A linear graph can be obtained by graphing y against $\frac{1}{x}$.



x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4
$\frac{1}{x}$	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
y	12	6	3	$\frac{3}{2}$	1	$\frac{3}{4}$

The gradient of the resulting straight line is 3.



Example 8 Linearising graphs

A table of values is shown.

x	1	2	3	4
y	1.5	6	13.5	24

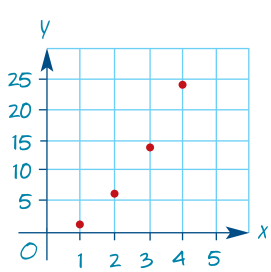
a Plot the points determined by the table.

b Plot graphs of y against x^2 and y against x^3.

c Determine the rule connecting x and y given that the rule is of the form $y = kx^2$ or $y = kx^3$.

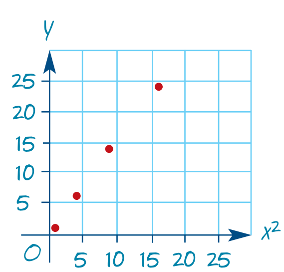
Solution

a



b

x	1	2	3	4
x^2	1	4	9	16
x^3	1	8	27	64
y	1.5	6	13.5	24



c For y plotted against x^2 the points lie in a straight line, which passes through the origin. Therefore the rule is of the form $y = kx^2$.

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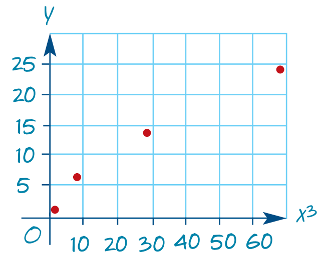
Since $\dfrac{y}{x^2} = k$

$\dfrac{1.5}{1} = k = 1.5$

The rule is $y = 1.5x^2$.

Note that the gradient of the line for the graph of y against x^2 is 1.5. From the graph of y against x^3 it is clear that the rule is not of the form $y = kx^3$.



Example 9 Determining a rule from a table of values

The table of values is determined by a rule of the form $y = kx^2 + c$.

a Plot the graph of y against x^2.

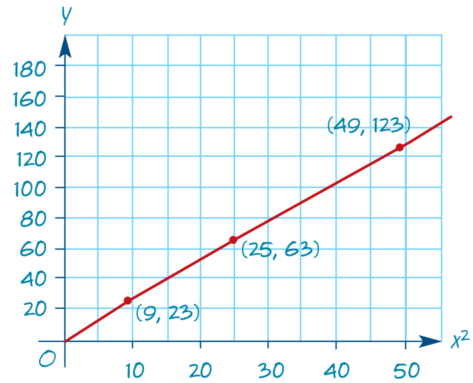
b Determine the values of k and c.

x	0	1	3	5	7
y	0.5	3	23	63	123

Solution

a

x	0	1	3	5	7
x^2	0	1	9	25	49
y	0.5	3	23	63	123



b The y-axis intercept is 0.5.

The gradient of the straight line is 2.5.

The equation for the data is

$y = 2.5x^2 + 0.5$.

Thus, the value of k is 2.5 and the value of c is 0.5.

How to find a rule connecting *x* and *y* using the TI-Nspire CAS

Find a rule connecting x and y.

x	0	1.2	1.3	1.5	1.8	2	2.2
y	0	6.0480	7.6895	11.8130	20.4120	28.0000	37.268

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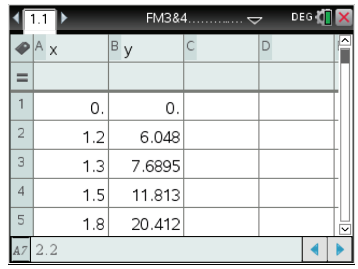
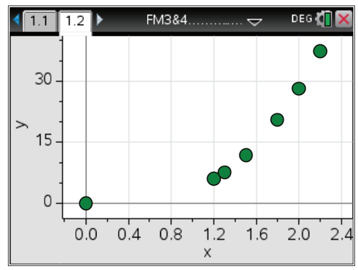
Steps

1 Start a new document by pressing ctrl + N, and select **Add Lists & Spreadsheet.**

Enter the data into lists named x and y.



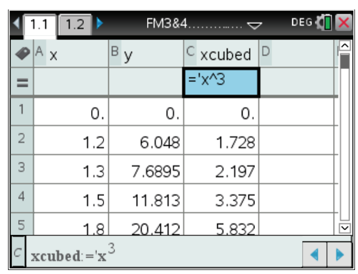
2 Press ctrl + I and select **Add Data & Statistics**. Construct a scatterplot of y against x. Let x be the explanatory (independent) variable and y the response (dependent) variable. The graph suggests that the rule is of the form $y = kx^n$.



To find the value of n we plot y against x^n. We are looking for a value of n that linearises the graph of y against x^n. We will guess that the rule is $y = kx^3$. To test, plot y against x^3.

3 Return to **Lists & Spreadsheet** by pressing ctrl + tab.

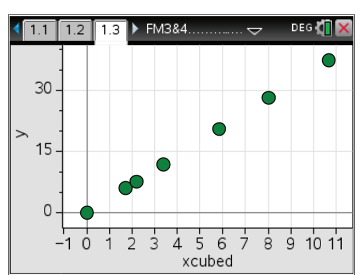
a Move the cursor to the top of column C and type ***xcubed*** (this list name will represent x-cubed). Press enter.



b Move the cursor to the grey cell immediately below the *xcubed* heading and type =. Then press var and highlight the variable x and press enter to paste into the formula line. Press ˆ3, then press enter to calculate and display the cubed values.

Note: You can also type in the variable x and then select **Variable Reference** when prompted. This prompt occurs because ***x*** can also be a column name.

4 Construct a scatterplot of y against x^3. Use ctrl + ▶ to return to the scatterplot created earlier and change the explanatory variable to *xcubed*. The plot of y against x^3 is linear, suggesting that our guess that the rule is of the form $y = kx^3$ is correct.



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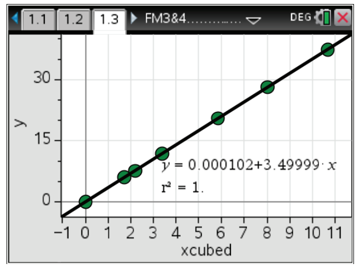
5 To find the value of n we need to fit a line to our plot and find its slope, y against x^3. We do this by fitting a least squares (regression) line to the data. Press menu>**Analyze>Regression>Show Linear (a + bx)**.

The equation is $y = 3.5x^3$, correct to one decimal place.



Note: To show the r^2 value when doing a linear regression press menu>**Settings** and check the **Diagnostic** field to activate this functionality.

How to find a rule connecting *x* and *y* using the ClassPad

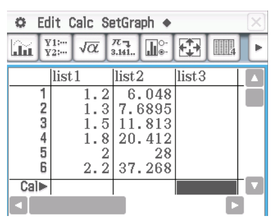
Find a rule connecting x and y.

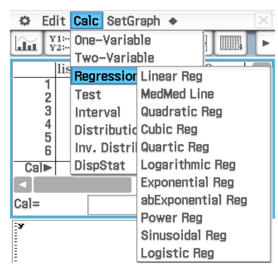
x	0	1.2	1.3	1.5	1.8	2	2.2
y	0	6.0480	7.6895	11.8130	20.4120	28.0000	37.268

Steps

1 Open the STATISTICS application

2 Enter the data, except (0, 0), into your calculator using the **Statistics** application.

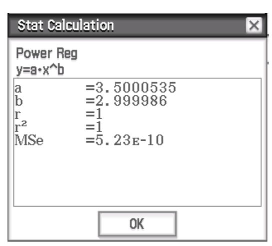


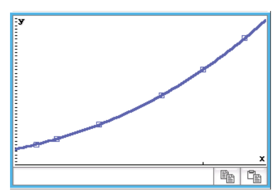


3 In the menu bar tap **Calc**, **Regression**, **Power Reg** and tap **OK**.

4 The calculator finds the equation to be in the form $y = ax^b$ where $a = 3.5$ and $b = 3$.

5 Tap OK to view the graph of $y = 3.5x^3$ passing through each point.





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Exercise 22E

Skillsheet

1 This set of data is known to satisfy a relationship of the form $y = kx^3$.

a Find the value of k.

b Complete the table.

x	0	2	3	6	7
y	0	20	67.5		

2 This set of data is known to satisfy a relationship of the form $y = \dfrac{k}{x^2}$.

a Find the value of k.

b Complete the table.

x	1	2	4	8	12	
y		6	1.5			

3 This set of data is known to satisfy a relationship of the form $y = kx^3$.

a Find the value of k.

b Complete the table.

c Sketch the graph of y against x for $0 \leq x \leq 12$.

x	2	6	4	3		
y	4	108	32		256	

4 This table of values is determined by a rule of the form $y = kx^2$.

a Complete the table of values and find k.

b Plot the graph of y against x^2.

x	0		0.5	1	2	
y		2	4			200

5 The price per 10 metres of rope depends on the diameter of the rope, as shown.

a Plot the graph of P against d^2.

b Find the rule for P in terms of d.

Diameter (d)	5	10	15	20	25
Price (P)	75	300	675	1200	1875

6 The pressure and volume of a fixed mass of gas when the temperature is constant is recorded in the table.

Volume (V)	1	5	10	25	50	100
Pressure (P)	100	20	10	4	2	1

a Sketch the graph of P against $\dfrac{1}{V}$.

b Find a rule for P in terms of V.

7 A stone falls from rest down a mine shaft. The distance it falls, d metres, at time t seconds, is given by a rule of the form $d = kt^2$.

a Given that the stone falls 39.2 m in the first 2 seconds, find the value of k.

b Given that it takes 5 seconds to reach the bottom of the shaft, find the depth of the shaft.

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8 Given that $y = kx^n$, find the result of doubling x if n is equal to:

a -2 **b** -1 **c** 1 **d** 2 **e** 3

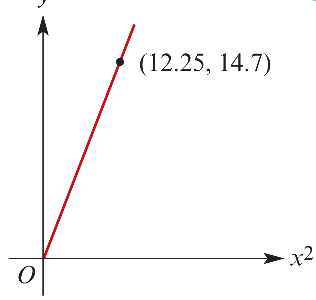
9 Mathematically similar jugs have volumes V cm^3 given by $V = kh^3$, where h is the height of the jug in centimetres. If a jug 10 cm high holds 0.125 litres, find:

a the volume of a jug that is 12 cm high

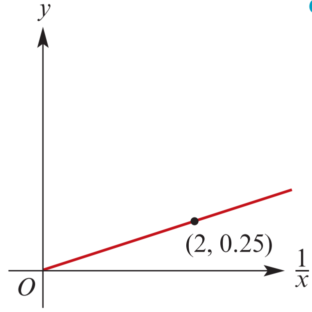
b the height of a jug that will hold 1 litre.

10 State the rule connecting y and x for each of the following graphs.

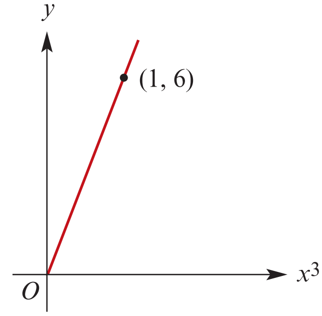
a



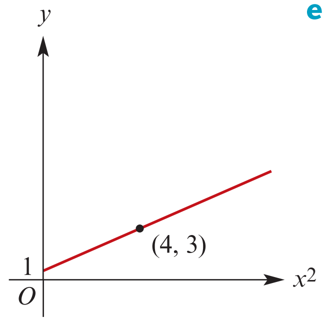
b



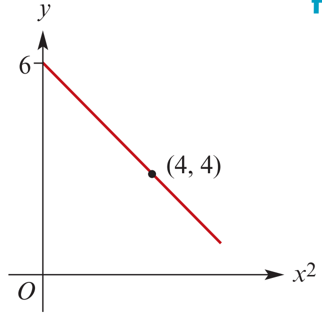
c



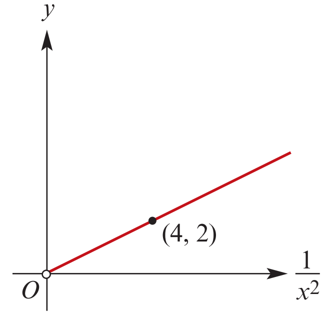
d



e



f



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Review

Key ideas and chapter summary

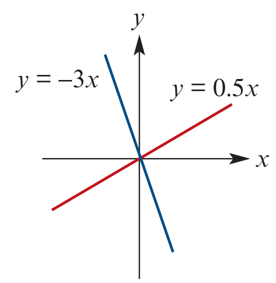
Line segment graph A **line segment graph** is a graph that consists of line segments (usually continuous).

Step graph A **step graph** can consist of straight line segments parallel to the horizontal axis.

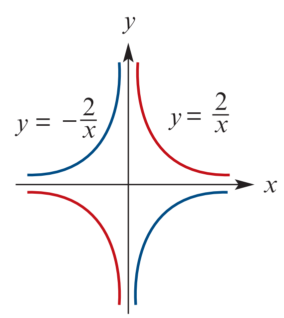
Interpolation In general, finding a value from a graph that lies within the range of values used to construct the graph is called **interpolation**.

Extrapolation In general, finding a value from a graph that lies outside the range of values used to construct the graph is called **extrapolation**.

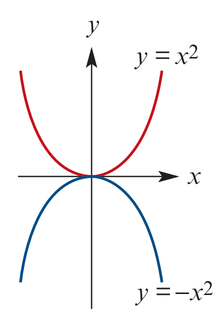
Straight lines through the origin Graphs of relations of the form $y = kx$, $k \neq 0$ are **straight lines through the origin**.



Hyperbolas Graphs of relations of the form $y = \frac{k}{x}$, $k \neq 0$ are **hyperbolas**.



Parabolas Graphs of the form $y = kx^2$, $k \neq 0$ are **parabolas**. For the basic graph of $y = x^2$, k is equal to 1. When k is negative, the graph is reflected in the x-axis.



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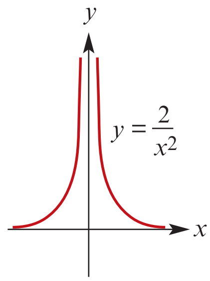
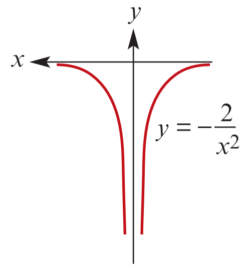
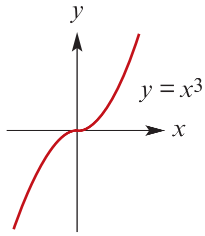
Graphs of the form $y = \frac{k}{x^2}$, $x \neq 0$





Graphs of the form $y = kx^3$, $k \neq 0$:



Determining rule A rule for y in terms of x^n can be found by plotting the graph of y against x^n and finding the gradient and y-axis intercept.

Skills check

Having completed this chapter you should be able to:

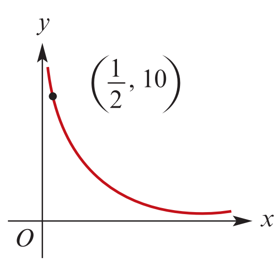
- construct and interpret line segment graphs
- construct and interpret step graphs
- interpret non-linear graphs and construct simple non-linear graphs
- by a suitable choice of a new independent variable, represent a non-linear relation as a straight line.

Multiple-choice questions

1 A possible equation for the graph shown ($x > 0$) is:

A $y = \frac{1}{5x}$ **B** $y = \frac{3}{5x}$ **C** $y = \frac{5}{x}$

D $y = \frac{5}{x^2}$ **E** $y = \frac{2}{5x^2}$



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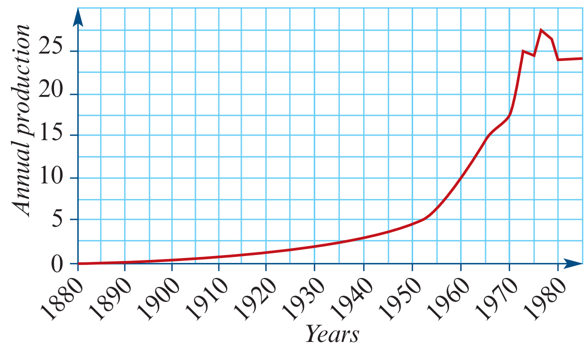
Review

2 This graph shows the growth of world oil production (measured in billions of barrels per year) over the period 1880–1984. From the graph, it can be seen that world oil production increased most rapidly during the period:



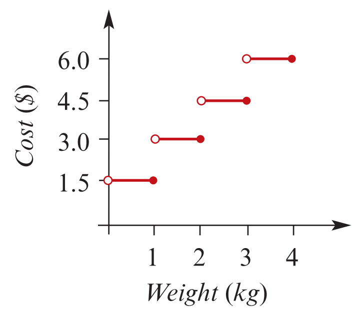
A 1920–1929
B 1930–1939
C 1940–1949
D 1950–1959
E 1960–1969

3 A table of values for y against x is shown. Given that $y = kx^n$, the values of k and n are nearest to:

x	1	2	3	4
y	4.2	33.6	113.4	268.8

A $n = 1,\ k = 4.2$ **B** $n = 2,\ k = 4.2$ **C** $n = 3,\ k = 4$
D $n = 2,\ k = 8.4$ **E** $n = 3,\ k = 4.2$

4 The graph shows the cost of posting parcels of various weights. A parcel weighing 3 kilograms will cost:

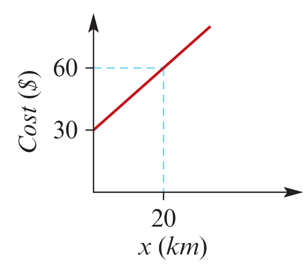


A \$4.50 **B** \$6.00 **C** \$5.00
D \$4.00 **E** \$5.25

5 The graph shown illustrates the cost of renting a car.

There is an initial charge and then a cost per kilometre.

If C is the cost in dollars and x is the distance traveled the formula connecting C and x is:



A $C = 1.3x + 30$ **B** $C = 4x + 30$ **C** $C = 1.5x + 30$
D $C = 20x + 30$ **E** $C = 2x + 30$

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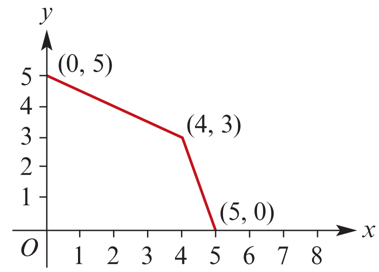
6 The line segment graph shown represents the function with the rule:



A $y = \begin{cases} \frac{1}{2}x + 5 & 0 \le x \le 4 \\ x - 5 & 4 < x \le 5 \end{cases}$

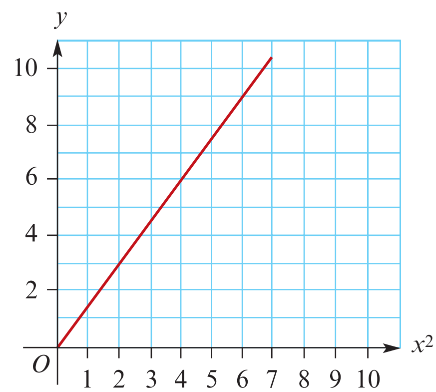
B $y = \begin{cases} \frac{1}{2}x + 5 & 0 \le x \le 4 \\ -2x + 11 & 4 < x \le 5 \end{cases}$

C $y = \begin{cases} -\frac{1}{2}x + 5 & 0 \le x \le 4 \\ -2x - 11 & 4 < x \le 5 \end{cases}$

D $y = \begin{cases} -\frac{1}{2}x + 5 & 0 \le x \le 4 \\ -3x + 15 & 4 < x \le 5 \end{cases}$

E $y = \begin{cases} x + 5 & 0 \le x \le 4 \\ -3x + 15 & 4 < x \le 5 \end{cases}$

7 The graph of y against x^2 is shown. The rule connecting x and y is:



A $y = 3x$ **B** $y = 3x^2$ **C** $y = \frac{2}{2}x^2$

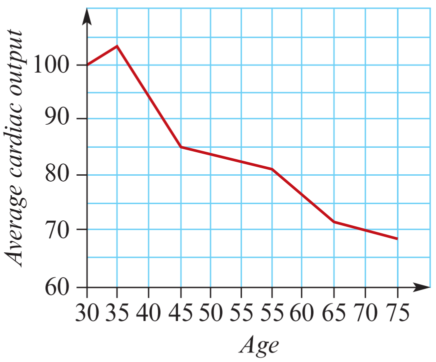
D $y = \frac{3}{2}x$ **E** $y = \frac{9}{4}x^2$

8 To rent a power tool from ACE Hire, there is an initial charge of \$15 and then an additional charge of \$2.50 per hour. If C is the cost in dollars and t is the time in hours, then the formula connecting C and t is:

A $C = 15 + t$ **B** $C = 2.5t$ **C** $C = 17.5t$

D $C = 15 + 2.5t$ **E** $C = 2.5(15 + t)$

9 The graph shows the average cardiac output for different ages. Average cardiac output is scaled to be 100 units at age 30. From the graph it can be seen that average cardiac output is at a maximum at the age of:



A 30 **B** 35 **C** 40

D 55 **E** 75

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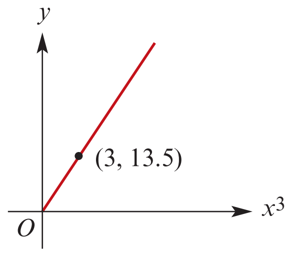
Review

10 The straight line graph shown is obtained by plotting y against x^3. It passes through the point with coordinates (3, 13.5). The rule giving y in terms of x is:



A $y = \frac{9x^3}{2}$ **B** $y = \frac{9x}{2}$ **C** $y = \frac{x^3}{3}$

D $y = \frac{x^3}{2}$ **E** $y = \frac{x^3}{8}$

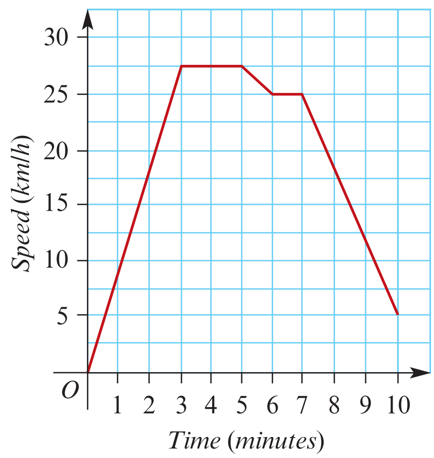
11 At a plant farm, petunias cost \$3.00 a punnet and marigolds cost \$2.50 a punnet. The total number of punnets sold was 150 and the total price paid was \$410. If x is the number of punnets of petunias sold and y is the number of punnets of marigolds sold, which one of the following pairs of equations is satisfied by x and y?

A $\begin{aligned} 2x + 2y &= 300 \\ 5x + 6y &= 410 \end{aligned}$ **B** $\begin{aligned} x + y &= 150 \\ x + y &= 410 \end{aligned}$ **C** $\begin{aligned} 6x + 5y &= 820 \\ x + y &= 150 \end{aligned}$

D $\begin{aligned} 5x + 6y &= 820 \\ x + y &= 150 \end{aligned}$ **E** $\begin{aligned} 6x + 5y &= 820 \\ 2x + 2y &= 150 \end{aligned}$

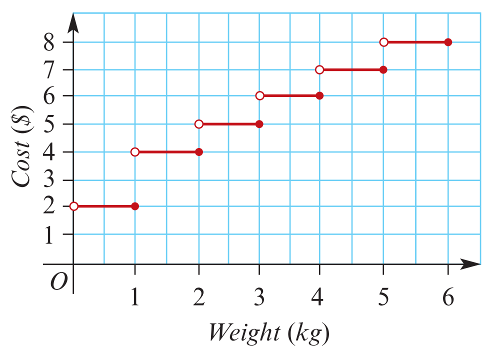
12 The graph shows the speed of a cyclist over 10 minutes of a journey. The cyclist's speed decreased most rapidly in the period between:

A 0 and 3 minutes
B 3 and 5 minutes
C 5 and 6 minutes
D 6 and 7 minutes
E 7 and 10 minutes



13 The graph shows the cost of posting parcels of various weights. A person posts two parcels, one weighing 3 kg and the other 1.5 kg. If each parcel is charged for separately, the cost of sending the two parcels is:

A \$4.50 **B** \$5.00 **C** \$7.00
D \$9.00 **E** \$10.00



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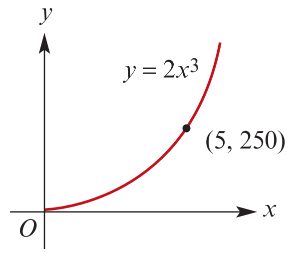
14 The cost ($\$C$) of renting a car is given by $C = an + b$ where:

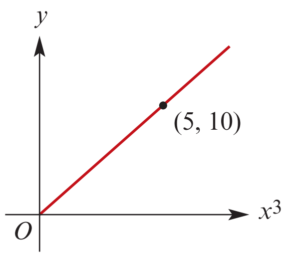
- n is the number of kilometers travelled
- a is the cost per kilometres travelled, and
- b is a fixed cost.

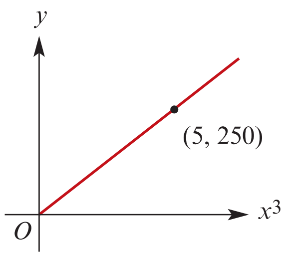
For a person travelling 200 km, the cost of car rental is \$430. For a person travelling 315 km, the cost of car rental is \$660. The values of a and b are:

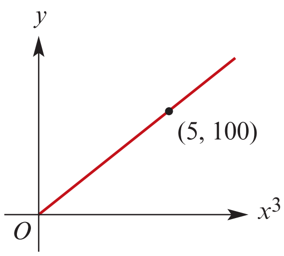
A $a = 1,\ b = 230$ **B** $a = 2,\ b = 30$ **C** $a = 3,\ b = 10$
D $a = 200,\ b = 430$ **E** $a = 315,\ b = 660$

15 The graph of $y = 2x^3$ is shown opposite. Which of the graphs A to C can also be used to represent this relationship between x and y?



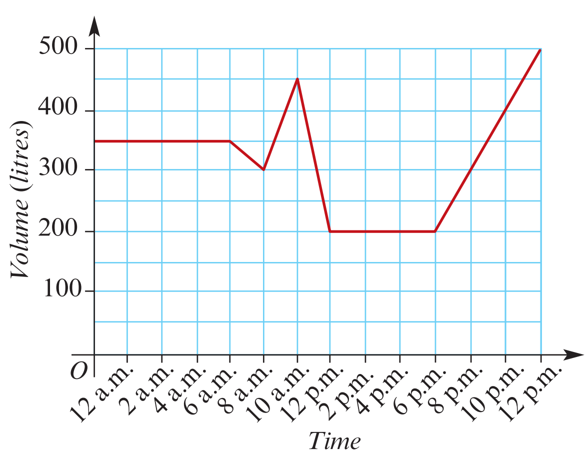
A


B


C


The following information is needed for Questions 16 and 17.

The volume of water that is stored in a tank over a 24-hour period is shown in the graph.



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Review

16 What is the difference in the volume of water (in litres) in the tank between 8 a.m. and 6 p.m.?

A 50 **B** 100 **C** 120 **D** 200 **E** 400

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17 The rate of increase in the volume of water in the tank (in litres/hour) between 8 a.m. and 10 a.m. is:

A 37.5 **B** 50 **C** 75 **D** 125 **E** 150

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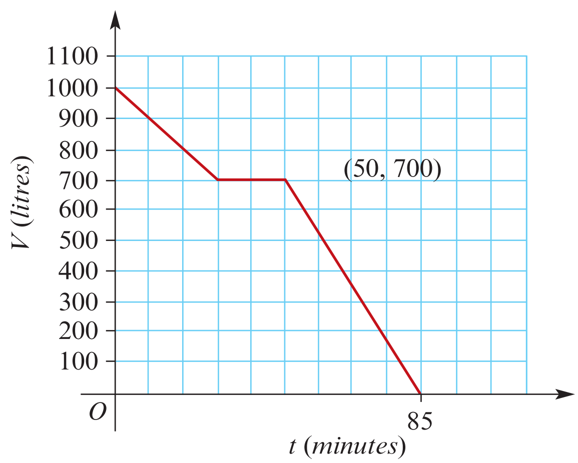
The following graph is needed for Questions 18 and 19.



18 The graph above shows the volume of water, V litres, in a tank at time t minutes.

The equation of this line between $t = 50$ and $t = 85$ minutes is:

A $V = 1700 - 20t$ **B** $V = 700 - 20t$ **C** $V = 20t + 1700$

D $V = 20t + 700$ **E** $V = 35t - 700$

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19 During the 85 minutes that it took to empty the tank, the volume of water in the tank first decreased at the rate of 15 litres per minute and then did not change for a period of time.

The period of time, in minutes, for which the volume of water in the tank did not change is:

A 15 **B** 20 **C** 30 **D** 50 **E** 85

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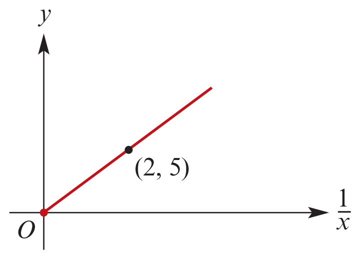
Review

20 The rule connecting x and y is:



A $y = \dfrac{5}{2x}$ **B** $y = \dfrac{5x}{2}$ **C** $y = \dfrac{2}{5x}$

D $y = \dfrac{2}{5}x$ **E** $y = \dfrac{10}{x}$

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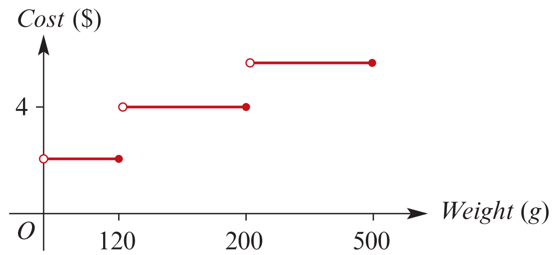
21 The cost of posting a parcel that weighs 500 grams or less depends on its weight, as shown on the graph.



The total cost of posting three separate parcels weighing respectively 100 grams, 150 grams and 210 grams is $11.

The total cost of posting three separate parcels weighing respectively 60 grams, 110 grams and 350 grams is $8.

The total cost of posting two separate parcels weighing respectively 170 grams and 500 grams is:

A $5 **B** $7 **C** $8 **D** $10 **E** $12

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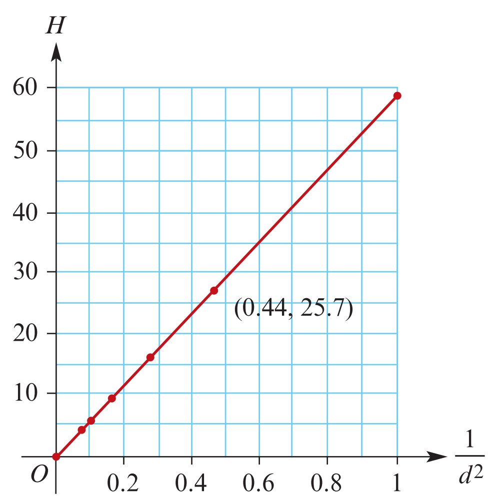
22 The heat intensity of a fire, H, is recorded at different distances, $\dfrac{1}{d^2}$, from the fire.

When H is plotted against $\left(\dfrac{1}{d^2}\right)$, the data points lie on a straight line, as shown.

The point (0.44, 25.7) lies on the line.

Given this information, the rule that relates the intensity of the fire, H, to the distance, d, from the fire is closest to:



A $H = \dfrac{58.4}{d^2}$ **B** $H = \dfrac{38.7}{d^2}$ **C** $H = \dfrac{4.98}{d^2}$

D $H = 38.7d$ **E** $H = 58.4$

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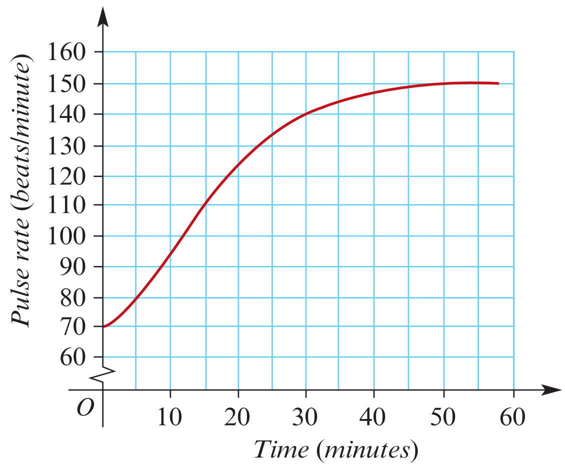
Review

Extended-response questions

1 Tiffany's pulse rate (in beats/minute) during the first 60 minutes of a long-distance run is shown in the graph.



a What was Tiffany's pulse rate (in beats/minute) 15 minutes after she started her run?

b By how much did Tiffany's pulse rate increase over the first 60 minutes of her run? Write your answer in beats/minute.

c The recommended maximum pulse rate for adults during exercise is determined by subtracting the person's age in years from 220.

i Write an equation in terms of the variables maximum pulse rate and age that can be used to determine a person's recommended maximum pulse rate from his or her age.

The target zone for aerobic exercise is between 60% and 75% of a person's maximum pulse rate.

Tiffany is 20 years of age.

ii Determine the values between which Tiffany's pulse rate should remain so that she exercises within her target zone. Write your answers correct to the nearest whole number.

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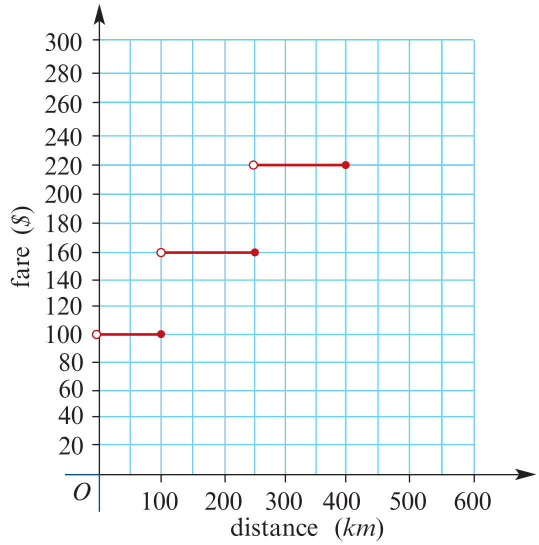
2 Fair Go Airlines offers air travel between destinations in regional Victoria. The table shows the fares for some distances travelled.

Distance (*km*)	*Fare*
$0 < distance \le 100$	\$100
$100 < distance \le 250$	\$160
$250 < distance \le 400$	\$220



a What is the maximum distance a passenger could travel for \$160? The fares for the distances travelled in [the table] are graphed [opposite].

b The fare for a distance longer than 400 km, but not longer than 550 km, is \$280. Draw this information on a copy of the graph above.

Fair Go Airlines is planning to change its fares.
A new fare will include a service fee of \$40, plus 50 cents per kilometre travelled.
An equation used to determine this new fare is given by $fare = 40 + 0.5 \times distance$.

c A passenger travels 300 km. How much will this passenger save on the fare calculated using the equation above compared to the fare shown in the table?

d At a certain distance between 250 km and 400 km, the fare, when calculated using either the new equation or [the table], is the same. What is this distance?

e An equation connecting the maximum distance that may be travelled for each fare in the table can be written as

$$fare = a + b \times maximum\ distance$$

Determine a and b.

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3 The Winkle Winery makes white wine that is sold at the cellar door for \$15 per bottle. The winery has fixed costs that total \$3200 per week. In addition, it costs \$7 to produce each bottle of wine.

a Find the total cost (including the fixed costs) of producing 200 bottles of white wine in a week.

b Find the revenue obtained by the winery for the sale of 200 bottles of white wine.

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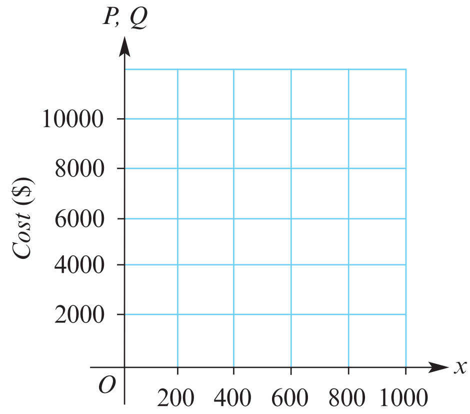
Review

c Draw up a grid like the one shown, and draw the graphs of the rules for:

i $\$P$, the total cost of producing x bottles of white wine in a week

ii $\$Q$, the total amount of money received by selling x bottles of white wine in a week

d Using the graph, or otherwise, find the smallest number of bottles of wine that the Winkle Winery needs to sell in a week in order to break even (to cover all costs).



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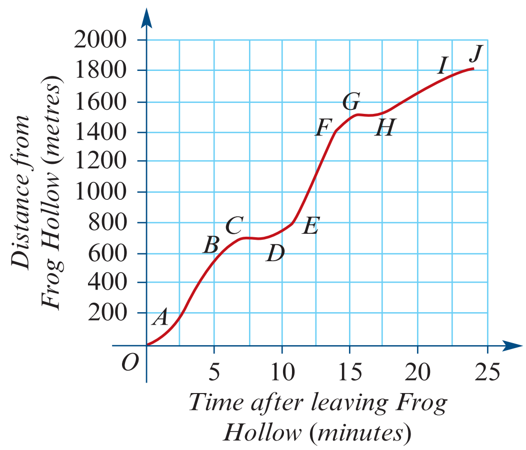
4 At the Gum Flat Fun Park there are many attractions. One that appeals especially to the younger visitors is the train Puffing Polly. The distance–time graph represents a train trip for Puffing Polly from Frog Hollow to Eagle Hill, stopping at two stations on the way.



a What is the total time for which Puffing Polly is stopped at the two stations?

b **i** Which line segment of the graph represents the section of the trip when Puffing Polly is travelling fastest?

ii Find Puffing Polly's speed for this section of the trip, clearly stating the units used in your answer.

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Chapter 23

23

Module 4: Graphs and relations

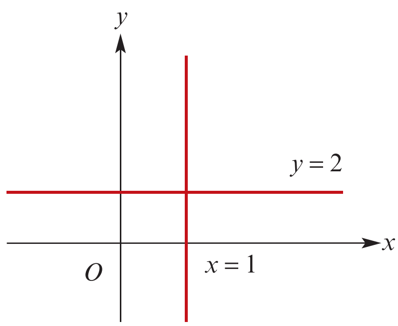
Linear programming

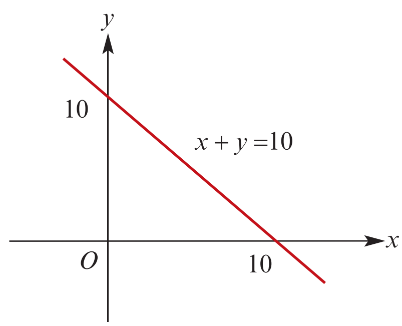
ISBN 978-1-107-56757-3

23A Regions defined by an inequality

Linear programming is used to solve problems in a wide variety of areas, such as planning, design and finance. There are now many computer programs available for solving linear programming problems, and these are used extensively in industry. Linear programming is often used to determine a solution to a problem to maximise profit or minimise cost. Such problems are called *optimisation* problems.

In Chapter 21, lines in the Cartesian plane were discussed. The lines are described by equations; for example:

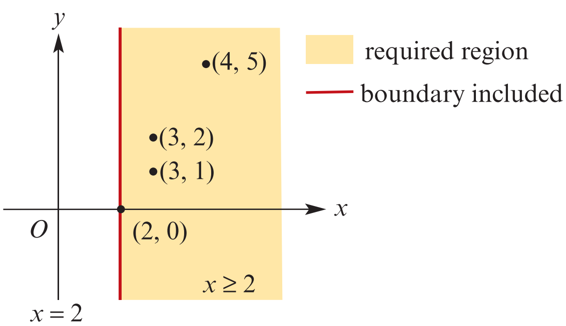




Each line is an infinite set of points. The position of each point can be described by coordinates. For example, points with the coordinates $(0, 2)$, $(1, 2)$, $(1.5, 2)$ and $(\sqrt{2}, 2)$ all lie on the line with the equation $y = 2$. Points with the coordinates $(2, 8)$, $(3, 7)$, $(0, 10)$ and $(3.6, 6.4)$ all lie on the line with the equation $x + y = 10$.

▶ Inequalities

Each point in the shaded region has an x-coordinate greater than 2. The shaded region is described as the region for which $x \geq 2$ and formally $\{(x, y): x \geq 2\}$. The boundary (the line with the equation $x = 2$) is included. The convention is that, if the boundary is included, it is represented with a continuous line and if it is not included it is represented with a dotted line.



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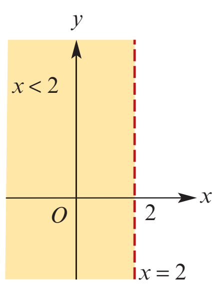
Example 1 **Sketching regions with horizontal or vertical lines as boundaries**

Represent each of the following regions with a suitable graph.

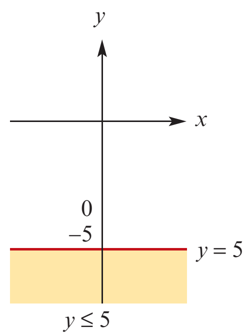
a $x < 2$ **b** $y \le -5$ **c** $y > 4$

Solution

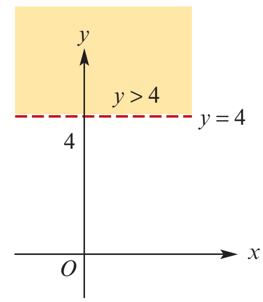
a



b



c

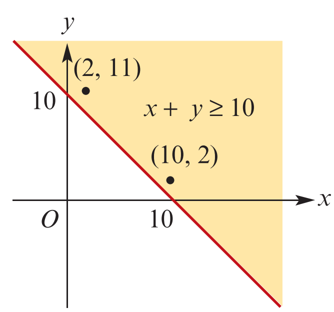


The line $x + y = 10$ separates the plane into two regions. Each point in the shaded region has coordinates whose sum is greater than or equal to 10; for example:

$(2, 11)\ 2 + 11 = 13 \ge 10$

$(10, 2)\ 10 + 2 = 12 \ge 10$

$(10, 0)\ 10 + 0 = 10 \ge 10$



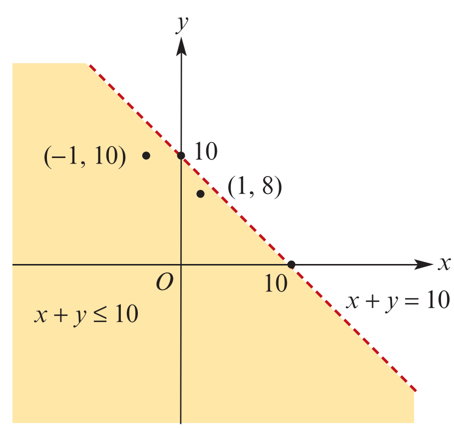
In the diagram opposite, each point in the shaded region has coordinates whose sum is less than 10; for example:

$-1 + 10 < 10$

$1 + 8 < 10$

$0 + 0 < 10$

The region to be shaded can always be established by testing one point. The origin is often the easiest point to check.



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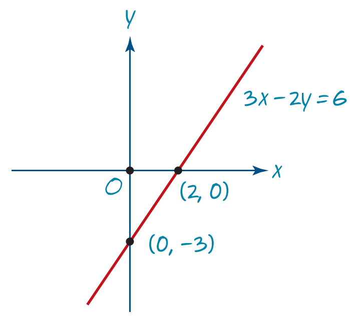
Example 2 Sketching regions defined by an inequality

Sketch the graph of the region $3x - 2y \leq 6$.

Solution

1 Sketch the graph of $3x - 2y = 6$. The straight line is drawn as a continuous line as the boundary is included.

when $x = 0$, $-2y = 6$
$\therefore y = -3$
when $y = 0$, $3x = 6$
$x = 2$

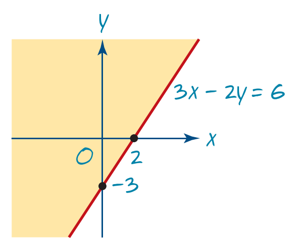


2 Check the origin (0, 0).

$$3 \times 0 - 2 \times 0 = 0 \leq 6$$

(0, 0) is in the required region.
$\therefore$ the region above the line is required.

3 Shade the required region.



Exercise 23A

1 Calculate the value of $5x + 6y$ corresponding to each of the given points, and state whether they are:

A in the region $5x + 6y > 22$ **B** in the region $5x + 6y < 22$
C on the line with equation $5x + 6y = 22$

a (2, 2) **b** (3, 1) **c** (4, 2) **d** (0, 0)

2 Sketch the graph of each of the following regions.

a $x \geq 5$ **b** $x < 4$ **c** $x \leq -2$
d $y > 4$ **e** $x > -5$ **f** $y \leq 6$

3 Sketch the graph of each of the following regions.

a $y \geq 2x + 5$ **b** $3x - 2y < 6$ **c** $x + y \leq 10$ **d** $x - y \geq 10$
e $2x - 6y \leq 3$ **f** $y \leq 2x$ **g** $x - y > -3$ **h** $x - 2y > -3$
i $2x + y \geq 12$ **j** $3x - 4y < 12$

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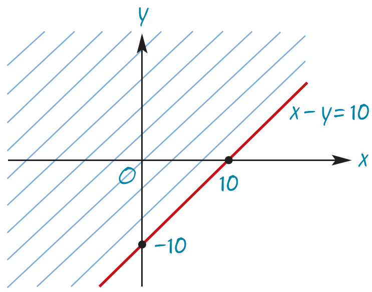
23B Regions defined by two inequalities

Example 3 Sketching a region defined by two inequalities

Sketch the graph of the region that is defined by the inequalities $x - y \leq 10$ and $y \leq x$.

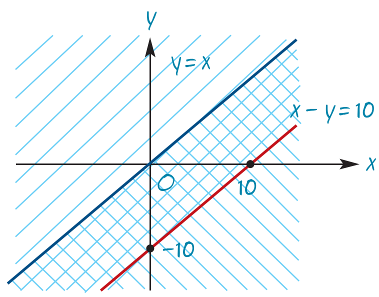
Solution

1 Sketch the graph of the region $x - y \leq 10$.



2 On the same set of axes, sketch the graph of $y \leq x$.

The region required is the region with overlap of the two shadings.



Example 4 Sketching the graph of a region

Sketch the graph of the region that is defined by the inequalities $3x + 2y \leq 6$ and $y - x \leq 1$.

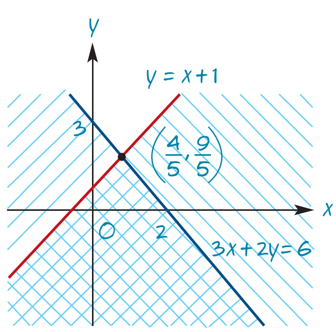
Solution

1 Draw the two straight lines.

2 Find the point of intersection of the two lines by solving the simultaneous equations or using a graphics calculator.

3 Shade each of the required regions.

4 The defined region is shown by the overlap.



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Exercise 23B

Skillsheet

1 Sketch the region defined by each of the following sets of inequalities and find the coordinates of the point of intersection.

a $y \geq -2x + 10$, $y \geq x - 2$

b $3x - 2y < 6$, $x - 3y > 9$

c $2x + 3y > -6$, $3x - y < 6$

d $2x + 3y \leq 6$, $x \geq 0$

e $2y - 3x < 6$, $x < 0$

f $y - 3x < 6$, $x - y > -3$

g $x - y < 1$, $y - x < 1$

h $2x - 2 \geq y$, $2x \leq 3 - 2y$

i $2y < 4x + 2$, $y < 2x + 1$

23C Feasible regions

Example 5 Shading a feasible region

A hardware shop has room for no more than 100 cans of paint of brand X and brand Y. It is known that at least 30 cans of brand X and no more than 50 cans of brand Y will be required.

By shading a suitable region of the Cartesian plane, indicate the possible number of brand X and brand Y that can be stored.

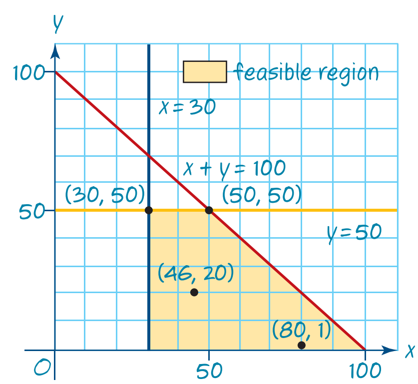
Solution

Let x be the number of cans of brand X.
Let y be the number of cans of brand Y.
The constraints can be expressed by the following inequalities.

$x \geq 30$

$0 \leq y \leq 50$

$x + y \leq 100$



The shaded region in Example 5 is called the **feasible region**. It is the graph of all possible pairs (x, y) that satisfy the inequalities.

Points with coordinates such as (30, 50), (46, 20), (50, 50), and (80, 1) all lie within the feasible region. Points on the boundary are included in the feasible region.

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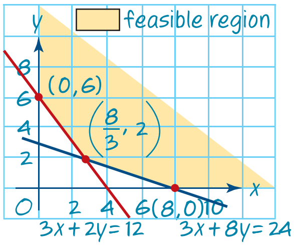
Example 6 Shading a feasible region

Indicate, by shading, the region that satisfies all the following constraints. Mark in the coordinates of the 'corner' points of the feasible region:

$$3x + 8y \geq 24 \qquad 3x + 2y \geq 12 \qquad x \geq 0 \qquad y \geq 0$$

Solution



Exercise 23C

1 For each of the following, indicate, by shading, the region that satisfies all of the constraints. Mark in the coordinates of the 'corner' points of the feasible region.

a $x + y \leq 7$
$5x - 3y \leq 15$
$x \geq 0$
$y \geq 0$

b $-x + 2y \leq 12$
$y \geq 2x - 9$
$x \geq 2$
$y \geq 0$

c $3x + 4y \leq 60$
$y \geq 15 - 3x$
$x \geq 0$
$y \geq 3$

d $x + 2y \geq 10$
$y \leq 2x$
$0 \leq x \leq 6$
$0 \leq y \leq 14$

e $-4x + 5y \leq 20$
$2x + 3y \geq 6$
$x \geq 0$
$y \geq 0$

f $x \leq 7$
$y \geq 4$
$x + y \leq 9$
$24x + 30y \geq 360$

g $x + y \geq 6$
$2x + 5y > 20$
$x \leq 4$
$y < 5$

h $x \geq 0$
$y \geq 0$
$3x + y \geq 15$
$2x + 3y \geq 36$

i $x \geq 0$
$y \geq 0$
$x + y \leq 4$
$x \leq y$
$y \leq 3$

23D Objective functions

An **objective function** is used to describe a quantity that you are trying to make as large as possible (for example, profits) or as small as possible (for example, the amount of paint needed to paint a house) subject to some constraints.

ISBN 978-1-107-56757-3

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Consider the following situation. A hardware shop has room for no more than 100 cans of paint of brand X and brand Y. At least 30 cans of brand X and no more than 50 cans of brand Y will be required. The profit on each can of brand X is \$2.00 and \$2.50 for each can of brand Y. If the entire stock can be sold, how many of each brand should there be to yield maximum profit?

Let x be the number of cans of brand X and let y be the number of cans of brand Y.

The constraints for this situation can be expressed by the following inequalities:

$x \geq 30$ (number of cans of brand X required)

$0 \leq y \leq 50$ (number of cans of brand Y required)

$x + y \leq 100$ (total number of cans of paint required)

Profit function

The profit function with the rule $P = 2x + 2.5y$ is an example of an **objective function**. In linear programming problems, the aim is to find the maximum or minimum value of an objective function for a given feasible region.

Solution method 1: sliding-line

To be a solution, (x, y) must be in the feasible region. Consider the point (60, 10).

For $x = 60$, $y = 10$, $P = 2 \times 60 + 2.5 \times 10 = 145$

However, there are other values of x and y that yield a profit of \$145,

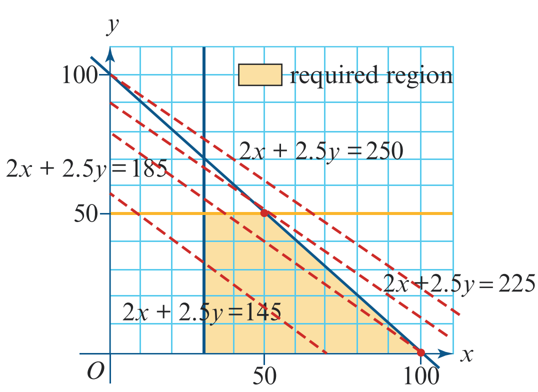
e.g. $x = 55$, $y = 14$; $x = 50$, $y = 18$; $x = 45$, $y = 22$; $x = 30$, $y = 34$.

All these points lie on the straight line $2x + 2.5y = 145$ (a profit of \$145).

For a profit of \$185, the straight line is $2x + 2.5y = 185$.

For a profit of \$225, the straight line is $2x + 2.5y = 225$.

All these lines are parallel. They have been added to the graph of the feasible region.



- It can be seen that the maximum profit possible is \$225 and this is achievable at only one point in the feasible region, i.e. (50, 50). See the red dot.
- It can be seen that a larger value of P (e.g. $P = 250$) will not yield points in the feasible region.

Solution method 2: corner point principle

A move commonly used method for finding the maximum or minimum value of an objective function uses the **corner point principle**.

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The corner point principle

In linear programming problems, the maximum or minimum value of a linear objective function will occur at one of the corners of the feasible region.

Note: If two corners give the same maximum or minimum value, then all points along a line joining the two points will also have the same maximum or minimum values. This occurs when the family of lines produced by the objective function is parallel to one of the boundaries of the feasible region.

This means that we need to evaluate the objective function only at each of the corner points to find the **maximum or minimum value** of an objective function.

Example 7 Finding the minimum value of an objective function

Find the minimum value of the objective function Z = $3x + 2y$ subject to the constraints:

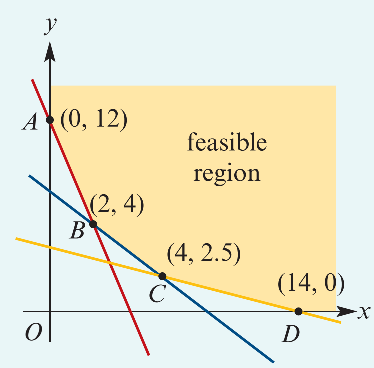
$x \geq 0, y \geq 0$

$4x + y \geq 12$

$x + y = 14$

$3x + 4y \geq 22$

as displayed in the feasible region opposite.



Solution

1 Write down the objective function.

$Z = 3x + 2y$

2 Evaluate the objective function at each of the corners.

$A(0, 12): Z = 3 \times 0 + 2 \times 12 = 24$

$B(2, 4): Z = 3 \times 2 + 2 \times 4 = 14$

$C(4, 2.5): Z = 3 \times 4 \times 2 \times 2.5 = 17$

$D(14, 0): Z = 3 \times 14 + 2 \times 0 = 42$

3 Identify the corner point that gives the minimum value and write down your answer.

The minimum value is $Z = 14$ when $x = 2$ and $y = 4$.

Integer solutions

Sometimes we are only interested in obtaining *integer* solutions for a problem. In the diagram that follows, the feasible region is defined by:

$x \leq 4$

$y \leq 7$

$2x + y \geq 9$

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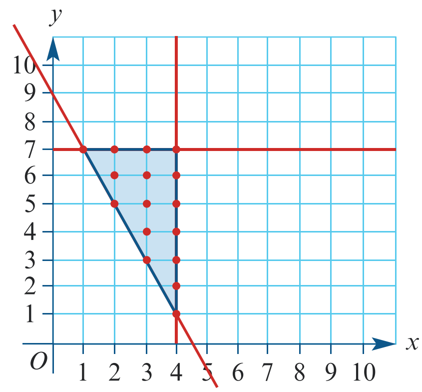
The 'integer valued points' in the feasible region are indicated by red dots on the graph opposite.

There are 16 dots, so there are 16 possible integer solutions.

Example 8 shows a practical problem that has only integer solutions.



Example 8 Finding integer solutions

A courier has to transport 900 parcels using large vans, which can take 150 parcels, and smaller vans, which can take 60 parcels. The costs of each journey are \$5.00 by large van and \$4.00 by small van. The total cost must be less than \$44.

a Represent the feasible region for these constraints on a graph.

b Indicate the possible values of x and y on the diagram by using a cross (×).

c What is:

i the largest number of vehicles that could be used?

ii the smallest number of vehicles that could be used?

Solution

a Let x be the number of large vans and y the number of small vans.

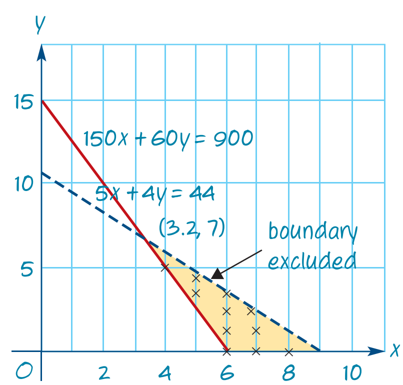
Constraints:

$x \geq 0$

$y \geq 0$

$150x + 60y \geq 900 \quad (5x + 2y \geq 30)$

$5x + 4y < 44$



b The crosses in the feasible region indicate the possible numbers of small and large vans.

c **i** The largest number of vehicles that could be used in total is 9. This could be 5 large and 4 small, or 4 large and 5 small, or 6 large and 3 small, or 7 large and 2 small.

ii The smallest number of vehicles that could be used is six large vans.

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Example 9 Setting up and solving a linear programming problem

A business produces imitation antique vases in two styles: Ming Dynasty vases and Geometric Period Greek vases. Each vase requires:

- potters to make the vase
- artists to decorate the vase.

During 1 week the business employs 10 potters and four artists. Each employee works for a total of 40 hours from Monday to Friday. The time spent making each vase is shown in the table.

Employee	*Ming*	*Geometric*
Potter	8 hours	4 hours
Artist	2 hours	2 hours

Let x represent the number of Ming Dynasty vases made in a week and let y represent the number of Geometric Period vases made in a week.

To meet regular orders, the business must make at least 10 Ming vases and 20 Geometric vases each week. These constraints can be written as:

$x \geq 10$ and $y \geq 20$

The total time available to the *potters* to make vases is 400 hours. This constraint can be written as:

$8x + 4y \leq 400$

a The total time available to the *artists* is 160 hours. Write an equality to represent this constraint.

b Draw the graphs of the four inequalities in part **a**.

c Shade in the feasible region on the graph.

The profit is \$50 on each Ming vase and \$30 on each Geometric vase.

d All vases produced in a week are sold. Write down an expression in terms of x and y for the total profit, \$$P$, that the business will receive.

e Determine the number of each type of vase that should be produced in a week to result in the maximum profit.

Due to increased costs, the profit made on each Geometric vase is decreased from \$30 to \$25. As a result, the profit from making x Ming vases and y Geometric vases is now given by $P = 50x + 25y$.

f Determine the number of each type of vase that should now be produced to result in the maximum profit.

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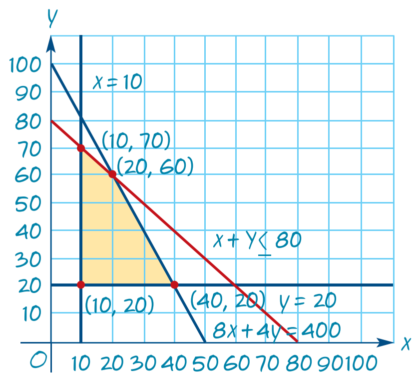
Solution

a $2x + 2y \le 160$ or, equivalently, $x + y \le 80$

b, c



d $P = 50x + 30y$

e Evaluate at each of the vertices of the feasible region:

$(10, 20)$ $P = 50 \times 10 + 30 \times 20 = 1100$

$(10, 70)$ $P = 50 \times 10 + 30 \times 70 = 2600$

$(20, 60)$ $P = 50 \times 20 + 30 \times 60 = 2800$

$(40, 20)$ $P = 50 \times 40 + 30 \times 20 = 2600$

To maximise the profit, 20 Ming vases and 60 Geometric vases should be produced.

f $P = 50x + 25y$

Evaluate P at each of the vertices of the feasible region:

$(10, 20)$ $P = 50 \times 10 + 25 \times 20 = 1000$

$(10, 70)$ $P = 50 \times 10 + 25 \times 70 = 2250$

$(20, 60)$ $P = 50 \times 20 + 25 \times 60 = 2500$

$(40, 20)$ $P = 50 \times 40 + 25 \times 20 = 2500$

The maximum value of $P = 2500$ occurs at two vertices: $(20, 60)$ and $(40, 20)$.

However, the corner point principle tells us that any point on the line joining these two vertices is also a solution. But, because we can only accept integer solutions, we can see from the graph that there is only one other integer solution and that occurs at the point $(30, 40)$. Evaluating P at this point confirms that the value of P is also maximised at this point:

$$P = 50 \times 30 + 25 \times 40 = 2500$$

Thus the profit can be maximised in three ways: by producing 20 Ming vases and 60 Geometric vases, as before; by producing 40 Ming vases and 20 Geometric vases; or by producing 30 Ming vases and 20 Geometric vases.

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Exercise 23D

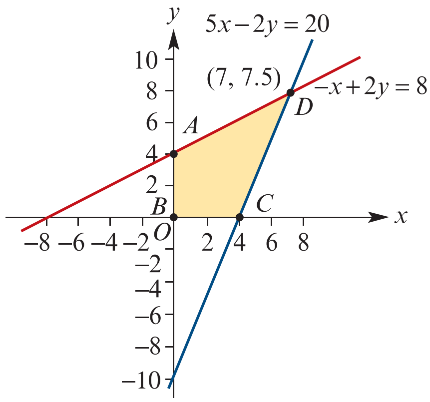
Constructing feasible regions and optimising objective functions

Skillsheet

1 The region that satisfies all of the following constraints

$$5x - 2y \le 20$$
$$-x + 2y \le 8$$
$$x \ge 0$$
$$y \ge 0$$

is shown.

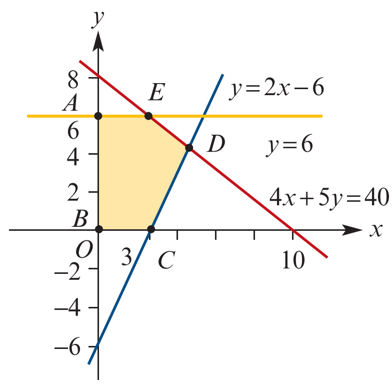


a Write down the values of the coordinates of A, B, C and D.

b Find the maximum value of $z = x + 2y$ subject to the set of constraints above.

2 The region that satisfies all the following constraints is shown.

$$4x + 5y \le 40$$
$$y \ge 2x - 6$$
$$x \ge 0$$
$$0 \le y \le 6$$



a Find the coordinates of A, B, C, D and E.

b Find the maximum value of $z = 2x + y$ subject to the set of constraints above.

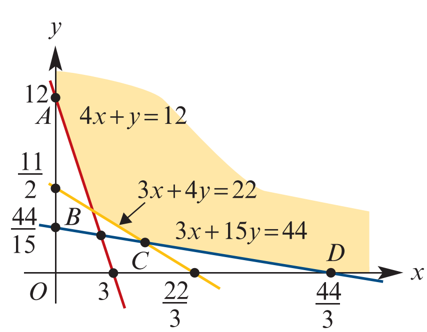
3 **a** Illustrate the region that satisfies all the following constraints.

$$x + 3y \le 17 \qquad 5x + 3y \ge 25 \qquad 0 \le x \le 8 \qquad 0 \le y \le 6$$

b Find the maximum value of $z = x + 3y$ subject to the set of constraints in part **a**.

4 The region that satisfies the following constraints is shown.

$$4x + y \ge 12$$
$$3x + 4y \ge 22$$
$$3x + 15y \ge 44$$
$$x \ge 0$$
$$y \ge 0$$



a Find the coordinates of points A, B, C and D.

b Find the minimum value of $z = 3x + 2y$ subject to the set of constraints above.

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5 **a** Illustrate the region that satisfies all the following constraints.

$$4x + 5y \geq 52 \qquad y \geq 0.5x \qquad y \leq 1.8x \qquad x \geq 4 \qquad y \geq 0$$

b Find the minimum value of $z = 4x + 10y$ subject to the set of constraints in **a**.

Integer valued solutions

6 **a** Write down the ten pairs of integers that satisfy:

$$x + 3y \leq 6$$

$$3x + 4y \leq 12$$

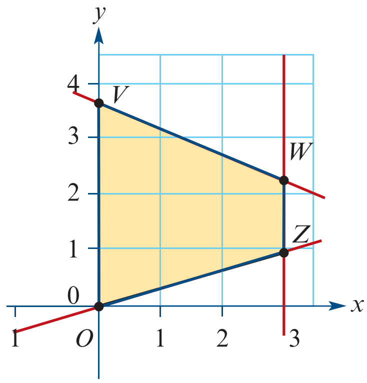
$$x \geq 0 \text{ and } y \geq 0$$

b Find the maximum value of $P = x + 2y$ and state the values of x and y that give this maximum value.

7 The feasible region shown opposite is defined by the constraints:

$x \geq 0$, $x \leq 3$, $y \leq 0.3x$ and $3x + 6y \leq 22$

x and y are integers



a Find the maximum value of the objective function $P = 2x + 3y$.

b Find the maximum value of the objective function $P = 5x + 3y$.

Setting up and solving linear programming problems (constraints given)

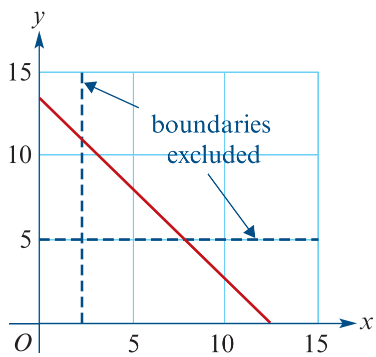
8 Samantha wants to buy some CDs that cost \$13 each and some books that cost \$12 each. She wants to buy more than two CDs and more than five books, but can spend no more than \$156.

Let x be the number of CDs and y be the number of books that Samantha buys. The three constraints than x and y must satisfy are:

constraint 1: $13x + 12y \leq 156$

constraint 2: $x > 2$

constraint 3: $y > 5$



a Explain the meaning of constraint 1 in the context of this problem. The lines defining these constraints have been plotted on the graph shown.

b Identify the feasible region by shading.

c Write down the coordinates of the six points in the feasible region that satisfy these constraints.

d Why is buying two CDs and eight books not a possible solution?

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e Samantha wants to maximise T, the total number of CDs and books she buys. Write down an expression for T in terms of x and y.

f Noting that x and y can take only integer values, determine the maximum total number of CDs and books Samantha can buy. List all possible combinations of CDs and books she can buy to give this maximum.

9 An ice-cream manufacturer makes just two flavours: chocolate superb and vanilla royal. Past experience shows that she needs to make at least 1200 litres of chocolate ice-cream and at least 600 litres of vanilla ice-cream each day. The maximum amount of ice-cream she can make each day (chocolate + vanilla) is 2000 litres.
Let x be the amount (in litres) of chocolate ice-cream she makes each day and y be the amount (in litres) of vanilla ice-cream she makes each day.
The three constraints that x and y must satisfy are:

constraint 1: $x \geq 1200$

constraint 2: $y \geq 600$

constraint 3: $x + y \leq 2000$

a Explain the meaning of constraint 3 in the context of this problem.

b Graph these three constraints and identify the feasible region. Determine the coordinates of each of the corner points of the feasible region.

The profit from the sale of one litre of chocolate ice-cream is \$1.10 and the profit from the sale of one litre of vanilla ice-cream is \$0.95.

c Write down an expression for the profit, \$$P$, the ice-cream manufacturer will make by selling x litres of chocolate ice-cream and y litres of vanilla ice-cream.

d Determine the amounts of chocolate and vanilla ice-cream she should make each day to maximise her profit. Determine this profit.

The ice-cream manufacturer changes her prices so that the profit she makes on both types of ice-cream is \$1.00.

e Write down an expression for the profit, \$$P$, the ice-cream manufacturer will now make by selling x litres of chocolate ice-cream and y litres of vanilla ice-cream.

f Determine the amounts of chocolate and vanilla ice-cream she should now make each day to maximise her profit and determine this profit.

10 A mining company is required to move 200 workers and 36 tonnes of equipment by air. It is able to charter two aircraft: a Hawk, which can accommodate 20 workers and 6 tonnes of equipment; and an Eagle, which can accommodate 40 workers and 4 tonnes of equipment. Let x denote the number of trips made by the Hawk aircraft and let y denote the number of trips made by the Eagle aircraft. The four constraints on the values x and y can take are:

constraint 1: $x \geq 0$ and $y \geq 0$

constraint 2: $20x + 40y \geq 200$

constraint 3: $6x + 4y \geq 36$

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a Graph these constraints and identify the feasible region.

Hawk aircraft cost \$3000 per trip while Eagle aircraft cost \$4000 per trip.

b Write down an expression for the cost, $\$C$, of making x trips with a Hawk aircraft and y trips with an Eagle aircraft.

c Determine the number of trips that should be made by each of the aircraft to minimise the total cost. Determine this cost.

11 For a journey across Ellesmere Island in the Arctic Circle, an explorer wishes to travel as lightly as possible. He can obtain supplies of two types of lightweight food, type X and type Y, with energy, protein and carbohydrate contents shown in the table.

Food	*Energy per serve (units)*	*Protein per serve (units)*	*Carbohydrate per serve (units)*
Type X	600	3.0	20
Type Y	400	3.5	35

The explorer estimates that his minimum daily requirements of energy, protein and carbohydrate will be 2600 units, 19 units and 150 units, respectively. Each serve of type X food weighs 36 g and each serve of type Y food weighs 56 g.
If x and y are the number of serves per day of type X and type Y foods, respectively, that the explorer takes, four of the constraints on x and y are:

$$x \geq 0 \quad y \geq 0$$

$$600x + 400y \geq 2600$$

$$3x + 3.5y \geq 19$$

a Give the constraint determined by the amount of carbohydrate required.

b If W grams is the weight per day of food that the explorer takes, write down the rule for W in terms of x and y.

c Determine the number of serves per day of each food type the explorer should take to minimise weight while still satisfying his daily dietary requirements.

Setting up constraints and objective functions

12 A market gardener decides to buy fertiliser to spread on her garden beds. Two types of fertiliser are available: Fast Grow and Easy Grow. Each fertiliser contains three different types of nutrients (A, B, and C) that promote plant growth but in different amounts.

Nutrient	*Fast Grow*	*Easy Grow*	*Minimum nutrient requirement*
A	3 units/bag	2 units/bag	160 units
B	5 units/bag	2 units/bag	200 units
C	1 unit/bag	2 units/bag	80 units

Let x be the number of bags of Fast Grow and y be the number of bags of Easy Grow.

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a There must be at least 160 units of nutrient A. Write down an inequality in terms of x and y that can be used to represent this constraint.

b There must be at least 200 units of nutrient B. Write down an inequality in terms of x and y that can be used to represent this constraint.

c There must be at least 80 units of nutrient C. Write down an inequality in terms of x and y that can be used to represent this constraint.

d The cost of a bag of Fast Grow is \$4.00 while the cost of a bag of Easy Grow is \$3.00. Write down an expression for the cost, \$$C$, of buying x bags of Fast Grow and y bags of Easy Grow.

13 A small service station sells petrol and diesel. The owner needs to refill his tanks. There are a number of constraints that determine how much petrol and diesel he orders. Let x be the volume (in litres) of diesel and y be the volume (in litres) of petrol he orders.

Constraints 1 and 2: The maximum amount of diesel he can store in his tanks is 15 000 litres. The maximum amount of petrol he can store in his tanks is 20 000 litres.

a Write down two inequalities in terms of x and y that can be used to represent these constraints.

Constraint 3: Experience shows that the demand for petrol is at least double the demand for diesel. That is, for every litre of diesel he has in his tanks, he needs to have at least 2 litres of petrol.

b Write down an inequality in terms of x and y that represents this constraint.

Constraint 4: One litre of diesel costs the service station \$1.20 while one litre of petrol costs the service station \$1.12. The service station can afford a maximum of \$25 000 in total to pay for the fuel.

c Write down an inequality in terms of x and y that represents this constraint.

Objective function: The profit made from the sale of one litre of diesel is 5.5 cents while the profit made from the sale of one litre of petrol is 4.6 cents.

d Write down an expression for the profit, \$$P$, the service station will make by selling x litres of diesel and y litres of petrol.

14 A factory makes two products, Widgets and Gidgets, each of which is constructed using different numbers of three different component parts, A, B and C.

Component part	*Widgets*	*Gidgets*	*Number in stock*
A	12	9	280
B	8	11	260
C	10	13	320

The number of Widgets and Gidgets that can be made is constrained by the number of As, Bs and Cs currently in stock. These numbers are also shown in the table.

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Let x be the number of Widgets made each month.

Let y be the number of Gidgets made each month.

One of the three constraints that x and y must satisfy is:

constraint 1: $12x + 9y \leq 280$

a Explain the meaning of constraint 1 in the context of this problem.

Constraint 2 arises because the total number of Bs used cannot exceed 260.

Constraint 3 arises because the total number of Cs used cannot exceed 320.

b Write down two inequalities in terms of x and y that can be used to represent constraints 2 and 3.

The profit from the sale of one Widget is \$84 while the profit from the sale of one Gidget is \$72.

c Write down an expression for the profit, \$$P$, the factory will make from producing x Widgets and y Gidgets.

By using new technology, the factory is able to increase the profit it makes from the sale of one Widget to \$120 and the profit it makes from the sale of one Gidget to \$90.

d Write down an expression for the profit, \$$P$, the factory will make from producing x Widgets and y Gidgets under these new circumstances.

Setting up and solving linear programming problems

15 An outdoor clothing manufacturer has 520 metres of polarfleece material. The manufacturer will use it to make jackets of two types, Polarbear and Polarfox, to sell to retailers. For each jacket of either type, 2.0 metres of material is required. However, the Polarbear is simpler in design, requiring 2.4 hours each in the production process, while each Polarfox requires 3.2 hours. There are 672 hours available.

From past experience of demand, the manufacturer has decided to make no more than half as many Polarfox jackets as Polarbear jackets. If the profit on each Polarbear jacket is \$36 and the profit on each Polarfox jacket is \$42, use a graphical method to find how many of each type should be made in order to maximise profit. What is this maximum profit?

16 The army is required to airlift 450 people and 36 tonnes of baggage by helicopter. There are 9 Redhawk helicopters and 6 Blackjet helicopters available. Each Redhawk can carry 45 passengers and 3 tonnes of baggage, while each Blackjet can carry 30 passengers and 4 tonnes of baggage. Running costs per hour are \$1800 for each Redhawk and \$1600 for each Blackjet.

If the army wishes to minimise the cost of the airlift per hour, use a graphical method to find how many of each helicopter should be used.

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Review

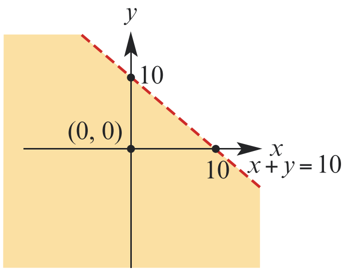
Key ideas and chapter summary

Procedure for sketching an inequality

The procedure for representing an inequality in a plane is as follows.

- Sketch the boundary line. The line is dotted if the boundary line is not included and continuous if the boundary line is included.
- Test with a point to determine which side of the line to shade.

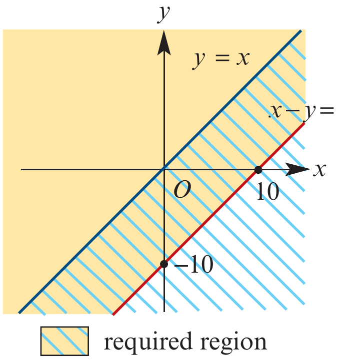
Example: $x + y < 10$
For the point $(0,\ 0)$ $0 + 0 < 0$
$\therefore$ shade underneath the boundary line.



Procedure for sketching the region defined by a system of inequalities

The procedure for representing a system of inequalities for the plane is as follows.

- Graph each of the inequalities using a different shading for each one.
  e.g. and
- The overlap determines the required region. *Example:* The region of points that satisfy $y \le x$ and $x - y \le 10$.

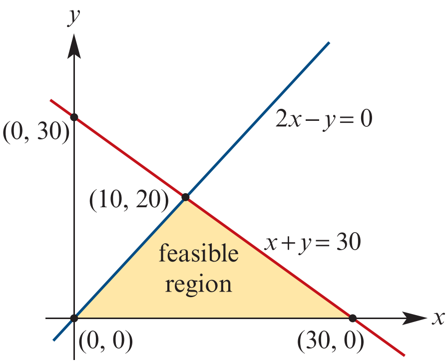


Feasible region

The **feasible region** is the set of points that can be considered for the given inequalities. For example, for the set of inequalities

$$y \ge 0$$
$$2x - y \le 0$$
$$x + y \le 30$$

the feasible region is shown.



Linear programming

Linear programming involves maximising or minimising a linear quantity subject to the constraints represented by a set of linear inequalities. The constraints define the feasible region in which the quantity is to be maximised or minimised.

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Objective function The **objective function** is a linear expression representing the quantity to be maximised or minimised in a linear programming problem. For example, $P = 3x + 5y$ could be an objective function representing profit.

Corner point principle The **corner point principle** states that, in linear programming problems, the maximum or minimum value of a linear objective function will occur at one of the corners of the feasible region, or on the line on the boundary of the feasible region joining two of the corners.

Maximum or minimum value The **maximum (or minimum) value** of the objective function is found by evaluating its values at the vertices (or along boundaries) of the feasible region. (See the corner point principle.)

Skills check

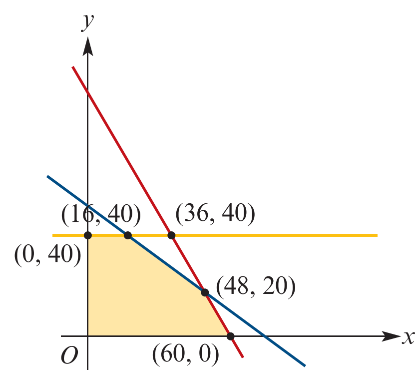
Having completed this chapter you should be able to:

- sketch the region defined by inequalities
- determine the feasible region defined by several inequalities
- find the maximum or minimum value of an objective function.

Multiple-choice questions

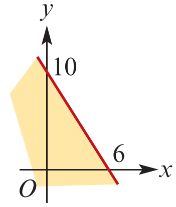
1 In the graph shown, the shaded region with boundary included is the feasible region for a linear programming problem that involves an objective function $P = 2x + 3y$. The maximum value of P for this feasible region occurs at a point with coordinates:

A (0, 40) **B** (16, 40)
C (36, 40) **D** (48, 20)
E (60, 0)

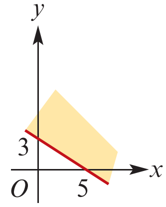


2 In which of the following graphs does the shaded region represent the points (x, y) such that $5x + 3y \le 30$? (In each case the boundary is included.)

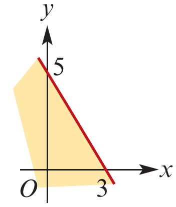
A



B



C



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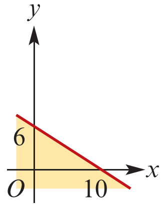
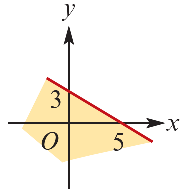
D

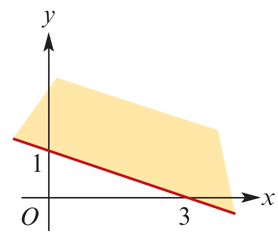


E



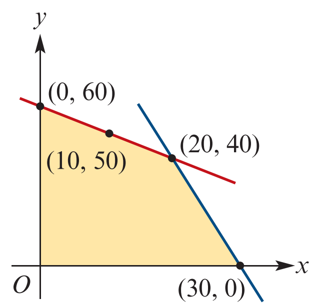
3 The shaded region of the graph is described by:

A $y + 3x \geq 0$ **B** $y \geq x + 1$ **C** $3y - x \geq 0$
D $3y + x \geq 3$ **E** $3x - y \geq 3$



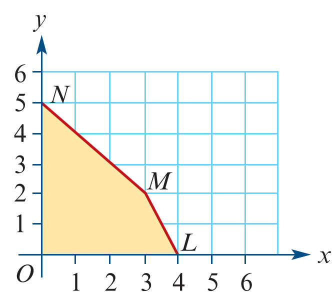
4 For the graph shown, the shaded region (boundaries included) is the feasible region for a linear programming problem that involves an objective function $T = 2x + 3y$. The maximum value of T for this region occurs at the point with coordinates:

A (0, 60) **B** (0, 50) **C** (20, 40)
D (30, 0) **E** (10, 60)



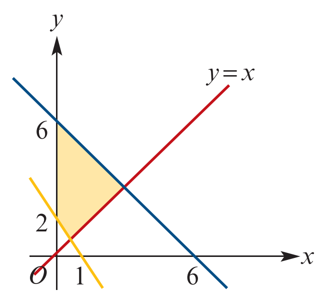
5 The region $LMNO$ is shown in the diagram. If (x, y) is a point in the region $LMNO$ (boundaries included), then the greatest value of $P = 2x + 3y$ is:

A 8 **B** 12 **C** 13
D 15 **E** 16



6 Which of the following sets of inequalities describes the shaded region in the diagram?

A $x, y \geq 0$ $2x + y \geq 2$ $y \leq x$ $x + y \leq 6$
B $x, y \geq 0$ $2x + y \leq 2$ $y \geq x$ $x + y \leq 6$
C $x, y \geq 0$ $2x + y \geq 2$ $y \geq x$ $x + y \leq 6$
D $x, y \geq 0$ $2x + y \leq 2$ $y \leq x$ $x + y \leq 6$
E $x, y \geq 0$ $2x + y \geq 2$ $y \geq x$ $x + y \geq 6$



Review

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Review

7 The shaded region is described by:

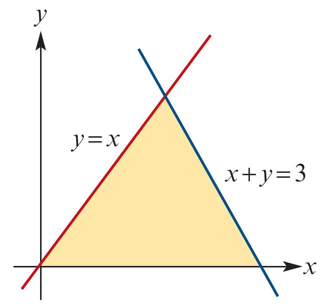
A $x \le 3,\ y \le 3,\ x \ge 0$

B $x + y \ge 3,\ x \ge y,\ x \ge 0$

C $x + y \ge 3,\ x \ge y,\ x \ge 0$

D $x + y \le 3,\ x \ge y,\ x \ge 0$

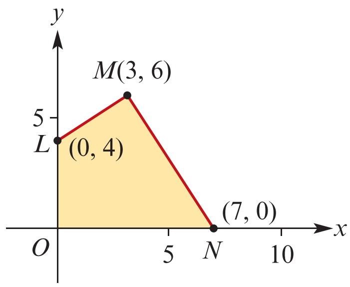
E $x + y \le 3,\ x \le y,\ x \ge 0$



8 For point (x, y) in the region $LMNO$ (boundaries included), as shown, the smallest value of $3x - 4y + 25$ is:

A 4 **B** 9 **C** 10

D 25 **E** 46



9 Which one of the following sets of inequations describes the shaded region (boundaries included) on the graph shown?

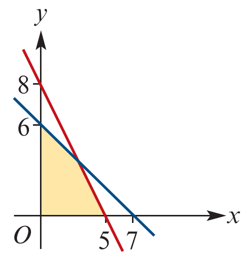
A $x \ge 0, y \ge 0$
$5x + 8y = 40$
$7x + 6y = 42$

B $x \ge 0, y \ge 0$
$8x + 5y = 40$
$6x + 7y = 42$

C $x \ge 0, y \ge 0$
$5x + 8y \le 40$
$7x + 6y \le 42$

D $x \ge 0, y \ge 0$
$8x + 5y \le 40$
$6x + 7y \le 42$

E $x \ge 0, y \ge 0$
$8x + 5y \ge 40$
$6x + 7y \ge 42$

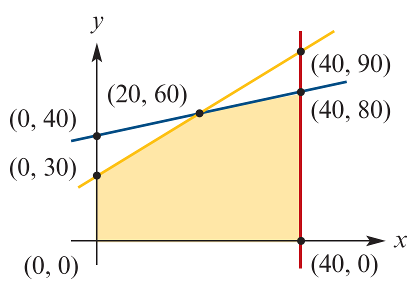


10 In the graph shown, the shaded region, with boundary included, is the feasible region for a linear programming problem. The equation of the objective function is $P = 2x + y$. The maximum value of P for this feasible region occurs at the point with coordinates:

A (0, 30) **B** (40, 80)

C (40, 90) **D** (20, 60)

E (40, 0)



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11 The constraints of a linear programming problem are given by the following set of inequalities.

$$x + y \leq 8$$
$$3x + 5y \leq 30$$
$$x \geq 0, y \geq 0$$

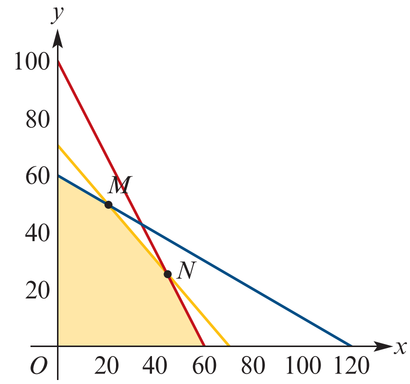
The coordinates of the points that define the boundaries of the feasible region for this linear programming problem are:

A (0, 0), (0, 6), (3, 5), (8, 0)
B (0, 0), (0, 6), (5, 3), (8, 0)
C (0, 0), (0, 6), (5, 3), (10, 0)
D (0, 0), (0, 8), (5, 3), (8, 0)
E (0, 0), (0, 8), (5, 3), (10, 0)

12 The shaded region in the graph represents the feasible region for a linear programming problem.

An objective function $Z = ax + by$ has its value maximised at both vertex M and vertex N.

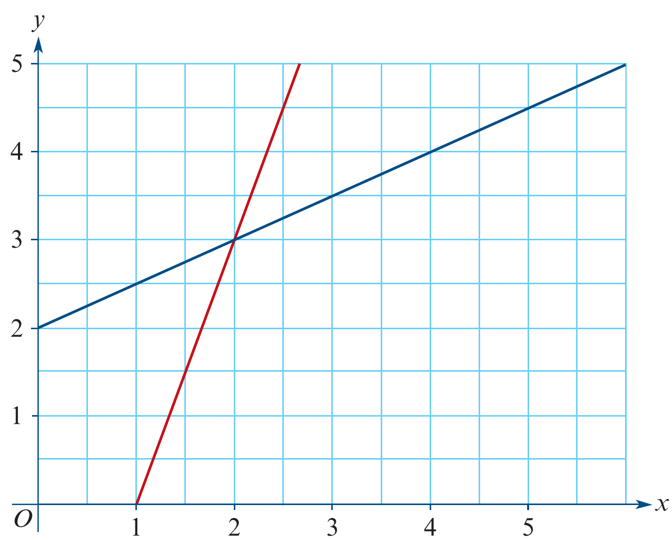
The values of a and b could be:



A $a = 15$ and $b = -15$
B $a = 15$ and $b = 15$
C $a = 15$ and $b = 25$
D $a = 25$ and $b = 50$
E $a = 50$ and $b = -25$

13 The graphs of the linear relations $x - 2y = -4$ and $3x - y = 3$ are shown.

A point that satisfies both the inequalities
$x - 2y \geq -4$ and
$3x - y \geq 3$ is:



A (1, 2)
B (1, 2.5)
C (2, 4)
D (3, 2)
E (3, 4)

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Review

Extended-response questions

1 Harry offers dog washing and dog clipping services.
Let x be the number of dogs washed and y be the number of dogs clipped in one day.
It takes 20 minutes to wash a dog and 25 minutes to clip a dog. There are 200 minutes available each day to wash and clip dogs. This information is represented by Inequalities 1–3.
Inequality 1 : $x \geq 0$ Inequality 2: $y \geq 0$ Inequality 3: $20x + 25y \leq 200$
In any one day the number of dogs clipped is *at least* twice the number of dogs washed.

a Write an inequality (see the skillsheet) to describe this information in terms of x and y.

b **i** Using graph paper, draw in the boundaries of the feasible region defined by Inequalities 1–4.

ii On a day when exactly five dogs are clipped, what is the maximum number of dogs that could be washed?

The *profit* from washing one dog is \$40 and the profit from clipping one dog is \$30.
Let P be the total profit obtained in one day from washing and clipping dogs.

c Write an equation for the total profit, P, in terms of x and y.

d **i** Determine the number of dogs that should be washed and the number of dogs that should be clipped in one day in order to maximise the total profit.

ii What is the maximum total profit generated by washing and clipping dogs in one day?

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2 A company repairs phones and laptops.
Let x be the number of phones repaired each day and y be the number of laptops repaired each day.
It takes 35 minutes to repair a phone and 50 minutes to repair a laptop.
The constraints on the company are as follows:

constraint 1: $x \geq 0$
constraint 2: $y \geq 0$
constraint 3: $35x + 50y \leq 1750$
constraint 4: $y \leq \frac{4}{5}x$

a Explain the meaning of constraint 3 in terms of the time available to repair phones and laptops.

b Constraint 4 describes the maximum number of phones that may be repaired relative to the number of laptops repaired.
Use this constraint to complete the following sentence:
For every 10 phones repaired, at most ☐ laptops may be repaired.

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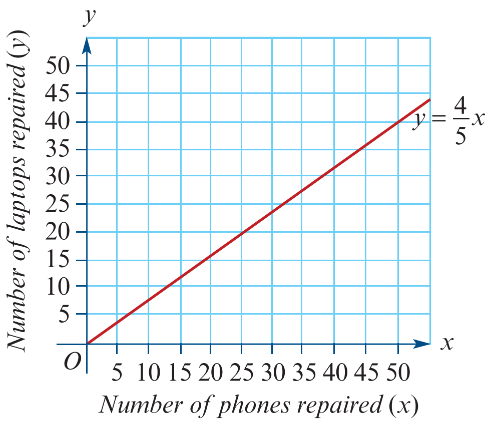
c The line $y = \frac{4}{5}x$ is drawn on the graph.
Redraw the graph and also draw the line:

$$35x + 50y = 1750$$



d Within constraints 1 to 4, what is the maximum number of laptops that can be repaired each day?

e On a day in which exactly nine laptops are repaired, what is the maximum number of phones that can be repaired?

The profit from repairing one phone is \$60 and the profit from repairing one laptop is \$100.

f **i** Determine the number of phones and the number of laptops that should be repaired each day in order to maximise the total profit.

ii What is the maximum total profit per day that the company can obtain from repairing phones and laptops?

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Module 4: Graphs and relations

Revision: Graphs and relations

ISBN 978-1-107-56757-3

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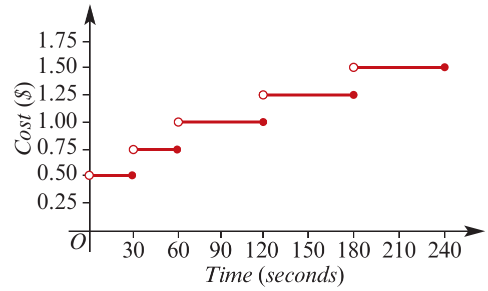
Revision

24A Multiple-choice questions

1 The graph shows the cost (dollars) of mobile telephone calls up to 240 seconds long. The cost of making a 90-second call followed by a 30-second call is:



A \$1.00
B \$1.20
C \$1.25
D \$1.50
E \$1.75

2 The point (2, 1) lies on the line $y = 3x + c$. The value of c is:

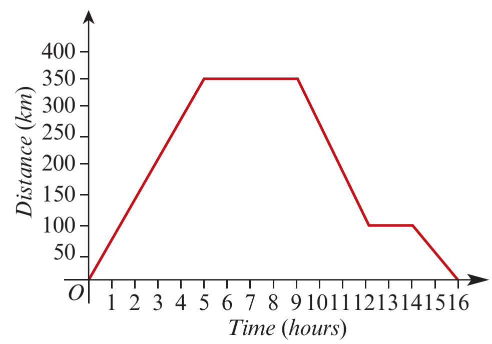
A −7 **B** −5 **C** −1 **D** 5 **E** 7

3 The lines $y + 8 = 0$ and $x - 12 = 0$ intersect at the point:

A $(-12, 8)$ **B** $(-8, 12)$ **C** $(0, 0)$ **D** $(8, -12)$ **E** $(12, -8)$

Questions 4 and 5 refer to the following graph.

The graph shows a distance–time graph for a car travelling from home along a long straight road over a 16-hour period.



4 In which one of the time intervals is the speed of the car greatest?

A 0 to 5 hours **B** 5 to 9 hours
C 9 to 12 hours **D** 12 to 14 hours
E 14 to 16 hours

5 After 12 hours the car has travelled a total distance of:

A 100 km **B** 350 km **C** 450 km **D** 600 km **E** 700 km

6 The cost, \$$C$, of hiring a boat for x hours is given by the equation $C = ax + b$, where a is the hourly rate and b is a fixed booking fee.
When the boat is hired for 4 hours the cost is \$320.
When the boat is hired for 6 hours the cost is \$450.
When the boat is hired for 1 hour the cost is:

A \$65 **B** \$75 **C** \$77 **D** \$80 **E** \$125

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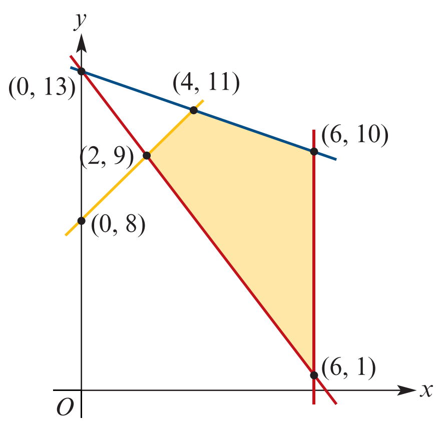
Revision

7 The shaded region shown in the graph (with boundaries included) represents the feasible region for a linear programming problem. The maximum value of the objective function $z = 2x + y$, for this feasible region, is:



A 18 **B** 22 **C** 25
D 27 **E** 33

8 If the two equations $2x - y = 2$ and $6x - 2y = 5$ are simultaneously true, the ratio $\frac{y}{x}$ equals:

A -2 **B** -1 **C** $-\frac{1}{2}$ **D** $\frac{1}{2}$ **E** 1

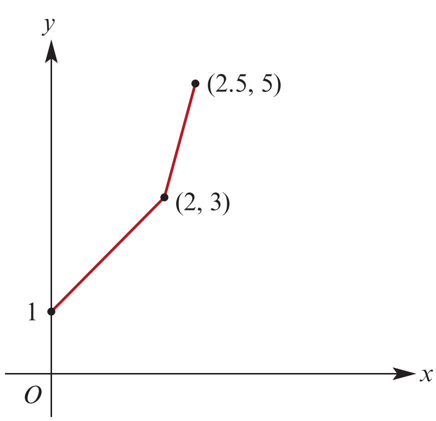
9 If the two lines $2x - y + 3 = 0$ and $ax + 3y - 1 = 0$ are parallel, then a is equal to:

A -6 **B** -2 **C** 2 **D** 3 **E** 6

10 The straight lines with equations $x + 2y = -2$ and $x - 6y = 18$ intersect at point P. The x coordinate of point P is:

A 3 **B** 4 **C** 6 **D** -6 **E** -2.5

11 The line segment graph shown represents the hybrid function with rule:



A $y = \begin{cases} x + 1 & 0 \le x \le 2 \\ 4x - 5 & 2 < x \le 2.5 \end{cases}$

B $y = \begin{cases} 3x + 1 & 0 \le x \le 2 \\ 2x - 1 & 2 < x \le 2.5 \end{cases}$

C $y = \begin{cases} 2x - 1 & 0 \le x \le 2 \\ 5x - 7 & 2 < x \le 2.5 \end{cases}$

D $y = \begin{cases} 2x - 1 & 0 \le x \le 2 \\ 5x - 7 & 2 < x \le 2.5 \end{cases}$

E $y = \begin{cases} x + 1 & 0 \le x \le 2 \\ 4x + 1 & 2 < x \le 2.5 \end{cases}$

12 If the line $2x + 3y + c = 0$ passes through the point $(-1, -1)$, then c is equal to:

A -5 **B** -1 **C** 0 **D** 1 **E** 5

ISBN 978-1-107-56757-3

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13 The straight lines with equations $x + 3y = -4$ and $x + y = 2$ intersect at the point P. The x coordinate of P is:

A -3 **B** 1 **C** 3 **D** 4 **E** 5

14 The equation of the straight line that passes through the points $(6, 10)$ and $(-4, 8)$ is:

A $5y - x = -40$ **B** $5y - x = 40$ **C** $5y + x = 44$
D $5y - x = 44$ **E** $5y - x = -44$

15 The line $3y = 3x - 4$ has x-axis intercept:

A $\frac{4}{3}$ **B** $\frac{3}{4}$ **C** -4 **D** 3 **E** 4

16 The gradient of the line $3y = 3x - 4$ is:

A -4 **B** $-\frac{4}{3}$ **C** 1 **D** 3 **E** 4

17 Which of the given lines is parallel to the line $2y = 3x - 4$?

A $y = 3x + 4$ **B** $y = 3x - 4$ **C** $2y = 2x - 1$
D $2y = 3x + 1$ **E** $3y = 3x + 4$

18 The gradient of the line joining the points $(1, 2)$ and $(3, -4)$ is:

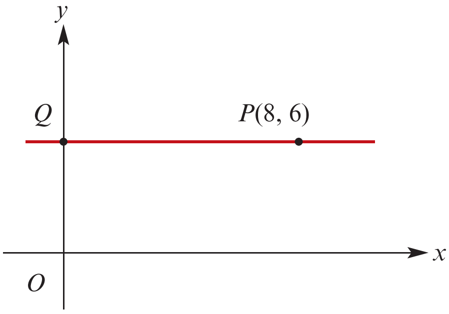
A -3 **B** -1 **C** $-\frac{1}{2}$ **D** $\frac{1}{3}$ **E** $\frac{2}{3}$

19 Which of the following lines is parallel to the y-axis?

A $y = x$ **B** $y = -x$ **C** $y = 2$ **D** $x = -3$ **E** $x = y + 1$

20 The diagram shows a line parallel to the x-axis and the point $P(8, 6)$ is on this line. The coordinates of the point Q, where the line meets the y-axis, are:

A $(8, 0)$ **B** $(0, 8)$ **C** $(6, 0)$
D $(0, 6)$ **E** $(6, 8)$



21 Given the linear relation $2x - 5y = 7$, which one of the following statements is true?

A As x increases y decreases.

B The relation has a straight line graph of slope $\frac{2}{5}$.

C The relation has a straight line graph of negative slope.

D The graph of the relation intersects the x-axis at $x = 2$ and the y-axis at $y = -5$.

E The graph of the relation is parallel to the line with equation $2x + 5y = 7$.

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22 Using the table of data, the rule giving y in terms of x is closest to:

x	1.0	1.2	1.4	1.6
y	4.50	7.78	12.35	18.43

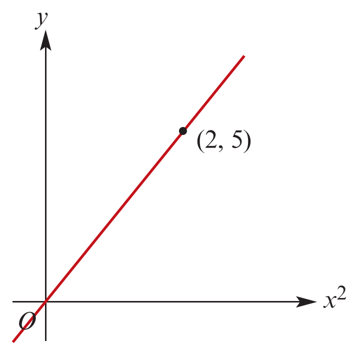
A $y = 4.5x$ **B** $y = 4.5x^2$
C $y = 5.4x^2$ **D** $y = 4.5x^3$
E $y = 6.5x^3$

23 The solution to the set of linear equations $20x - 14y = 63$ and $5x + 6y = 68$ is:

A $x = 5\frac{3}{5}, y = 3\frac{1}{2}$ **B** $x = 4, y = 8$ **C** $x = 8, y = 4$
D $x = 7, y = 5\frac{1}{2}$ **E** $x = 5, y = 7\frac{1}{2}$

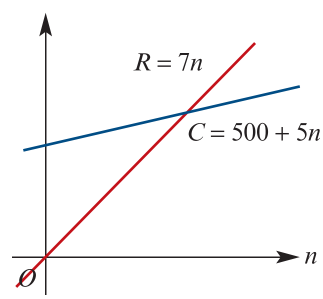
24 A graph of y against x^2 is shown. The rule connecting x and y is:

A $y = \frac{5x}{2}$ **B** $y = \frac{25x^2}{4}$ **C** $y = 5x^2$
D $y = 5x$ **E** $y = \frac{5x^2}{2}$



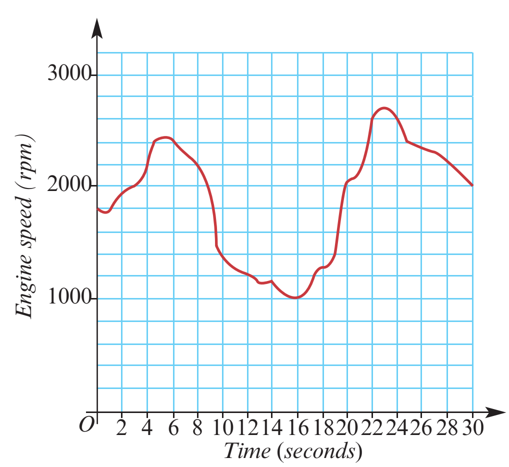
25 For the graph opposite, R is the revenue, in dollars, earned by a mushroom farmer selling n kilograms of mushrooms, and C is the cost, in dollars, of producing n kilograms of mushrooms. To make a profit, the number of kilograms of mushrooms the farmer should produce must exceed:

A 25 **B** 50 **C** 100
D 250 **E** 500



26 The graph shows the engine speed of a car measured in revolutions per minute (rpm) over a period of 30 seconds. The total time that the engine speed was above 2200 rpm is:

A 4 seconds
B 6 seconds
C 12 seconds
D 20 seconds
E 24 seconds



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27 If the point (3, −2) lies on the curve with equation $y = \frac{k}{x^2}$, then the value of k is:

A −18 **B** −12 **C** −6 **D** 12 **E** 18

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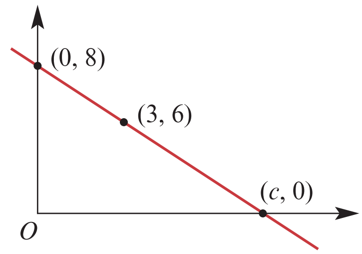
28 For this straight line graph, the value of c is:

A 8 **B** 11 **C** 12
D 14 **E** 16



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The following information refers to Questions 29 and 30.

A publisher produces a restaurant guide each year.

To produce x copies, the cost is C dollars, where $C = 15\ 000 + 15x$.

If all x copies produced are sold, then the revenue gained is R dollars, where $R = 25x$.

29 Which one of the following statements is not true?

A The cost and revenue equations are linear.
B The selling price for each copy of the guide is \$25.
C It will cost \$30 000 to produce 1000 copies of the guide.
D The revenue from selling 1000 copies of the guide is \$15 000.
E The revenue is more than the cost if 1600 copies of the guide are sold.

30 If x copies of the guide are produced and sold, then the profit made is P dollars, where P is given by:

A $P = 15000 - 10x$ **B** $P = 10x - 15000$ **C** $P = 15x - 15000$
D $P = 40x - 15000$ **E** $P = 15000 - 40x$

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31 For the pair of simultaneous equations $2x - 3y = 7$ and $3x = 5 - y$ the solution is:

A $x = -2, y = -1$ **B** $x = -1, y = -3$ **C** $x = -1, y = 2$
D $x = 2, y = -3$ **E** $x = 2, y = -1$

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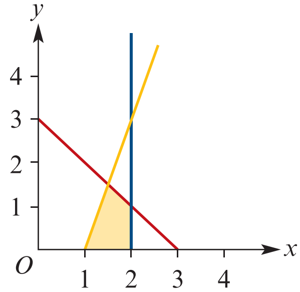
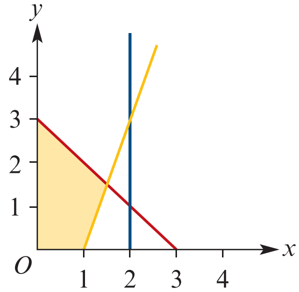
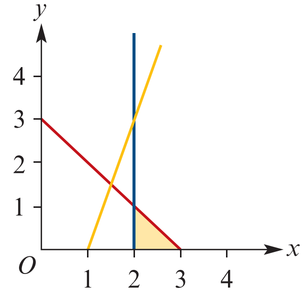
32 The following inequalities define a region in the x–y plane.

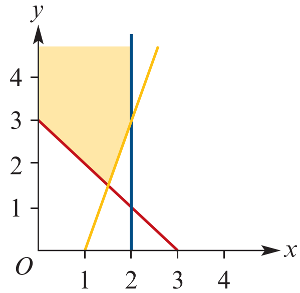
$$x \geq 0$$
$$x \leq 2$$
$$y \geq 0$$
$$3x - y \geq 3$$
$$x + y \leq 3$$

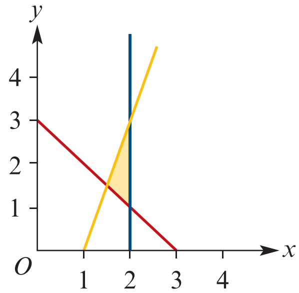
Which one of the following diagrams represents this region?

A


B


C


D


E


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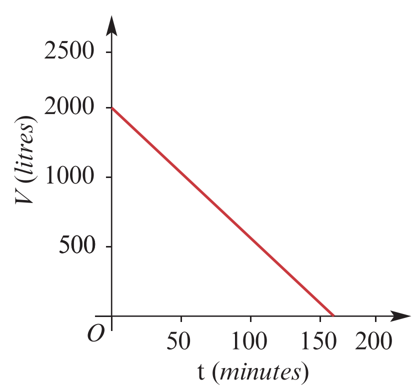
33 A full tank holds 2000 litres of water. Water is pumped out of the tank at a constant rate.
The graph opposite shows how the volume of water in the tank, V, changes with time, t.
The constant rate, in litres per minute, at which the water is being pumped out of the tank is:

A 0.8 **B** 2.0 **C** 12.5
D 80.0 **E** 160.0



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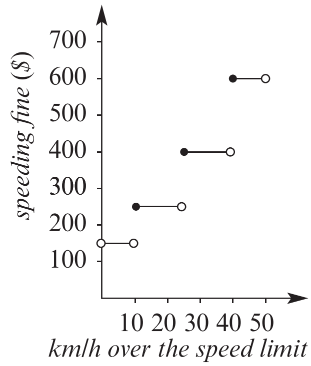
34 The step graph shows the speeding fines that are given for exceeding the speed limit by different amounts.
A driver was fined for driving at a speed of 65 km/h in a zone with a speed limit of 40 km/h.
The fine given was:

A $65
B $150
C $250
D $400
E $600



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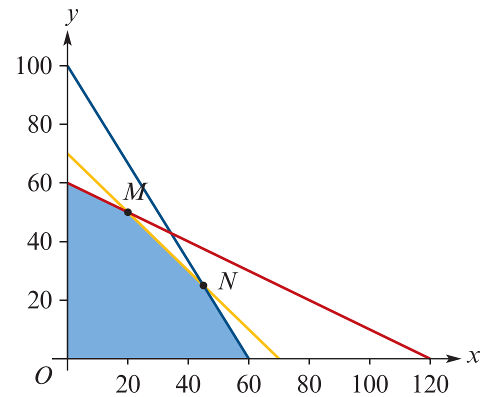
35 The shaded region in the graph represents the feasible region for a linear programming problem.
An objective function $Z = ax + by$ has its value maximised at both vertex M and vertex N.
The values of a and b could be:

A $a = 15$ and $b = -15$
B $a = 15$ and $b = 15$
C $a = 15$ and $b = 25$
D $a = 25$ and $b = 50$
E $a = 50$ and $b = -25$



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36 A cafe sells the first 200 cups of hot chocolate each day at a special price of $3.00 per cup. After that, a cup of hot chocolate will be sold for $4.50.
The revenue, R, in dollars, made from selling n cups of hot chocolate each day is given by the rule:

$$R = \begin{cases} 3n & 0 \le n \le 200 \\ 4.50n - 300 & n \ge 200 \end{cases}$$

The cost, C, in dollars, of making n cups of hot chocolate each day is:

$$C = 500 + 1.30n$$

To break even, the number of cups of hot chocolate that must be sold each day is:

A 63 **B** 200 **C** 250 **D** 295 **E** 300

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37 The manager of an office is ordering finger food for an office party.
Hot item cost \$2.15 each and cold item cost \$1.50 each.
Let x be the number of hot item ordered.
Let y be the number of cold item ordered.
The manager can spend no more than \$5 for each of the 200 employees.
An inequality that can be used to represent this constraint is:

A $1.5x + 2.15y \leq 5$
B $1.5x + 2.15y \leq 200$
C $1.5x + 2.15y \leq 1000$
D $2.15x + 1.5y \leq 200$
E $2.15x + 1.5y \leq 1000$

38 The cost in dollars, C, of making n pottery mugs is given by the equation:

$$C = 150 + 6n$$

A loss will result from selling:

A 60 mugs at \$9.00 each
B 70 mugs at \$8.50 each
C 80 mugs at \$7.50 each
D 90 mugs at \$8.00 each
E 100 mugs at \$9.50 each

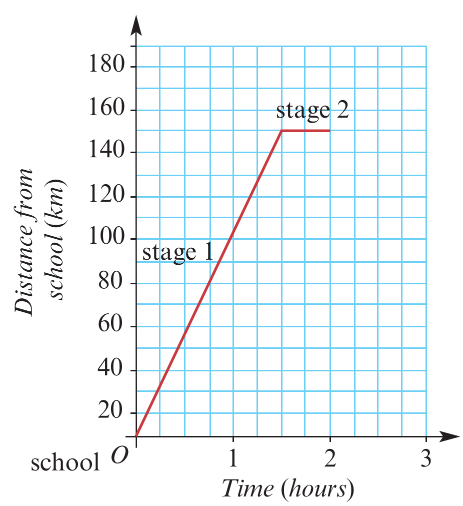
24B Extended-response questions

AS

1 The distance–time graph opposite shows the first two stages of a bus journey from a school to a camp.

a At what constant speed, in kilometres per hour, did the bus travel during stage 1 of the journey?

b For how many minutes did the bus stop during stage 2 of the journey?



The third stage of the journey is missing from the graph.
During stage 3, the bus continued its journey to the camp and travelled at a constant speed of 60 km/h for one hour.

c Copy the graph above on a grid and draw a line segment on the graph to represent stage 3 of the journey.

d Find the average speed of the bus over the three hours. Write your answer in kilometres per hour.

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The distance, D km, of the bus from the school t hours after departure is given by:

$$D = \begin{cases} 100t & 0 \le t \le 1.5 \\ 150 & 1.5 \le t \le 2 \\ 60t + k & 2 \le t \le 3 \end{cases}$$

e Determine the value of k.

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2 Students at the camp can participate in two different watersport activities: canoeing and surfing. The cost of canoeing is \$30 per hour and the cost of surfing is \$20 per hour.

The budget allows each student to spend up to \$200, in total, on watersport activities. The way in which a student decides to spend the \$200 is described by the following inequality.

$$30 \times hours\ canoeing + 20 \times hours\ surfing \le 200$$

a Hillary wants to spend exactly two hours canoeing during the camp. Calculate the maximum number of hours she could spend surfing.

b Dennis would like to spend an equal amount of time canoeing and surfing.

If he spent a total of \$200 on these activities, determine the maximum number of hours he could spend on each activity.

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3 A rock-climbing activity will be offered to students at the camp on one afternoon. Each student who participates will pay \$24.

The organisers have to pay the rock-climbing instructor \$260 for the afternoon. They also have to pay an insurance cost of \$6 per student.

Let n be the total number of students who participate in rock climbing.

a Write an expression for the profit that the organisers will make in terms of n.

b The organisers want to make a profit of at least \$500.

Determine the minimum number of students who will need to participate in rock climbing.

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4 The school group may hire two types of camp sites: powered sites and unpowered sites.

Let x be the number of powered camp sites hired

y be the number of unpowered camp sites hired.

Inequality 1 and inequality 2 give some restrictions on x and y.

Inequality 1 $\quad x \le 5$

Inequality 2 $\quad y \le 10$

There are 48 students to accommodate in total.

A powered camp site can accommodate up to six students and an unpowered camp site can accommodate up to four students.

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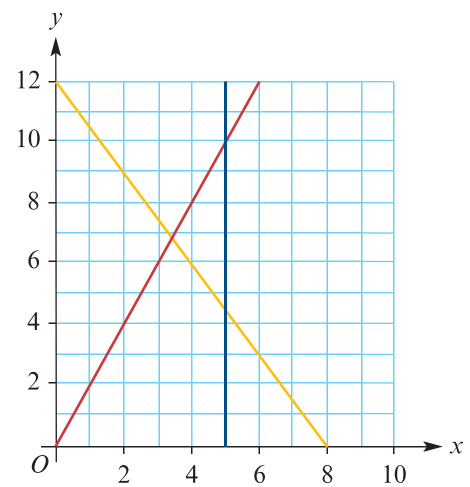
Inequality 3 gives the restrictions on x and y based on the maximum number of students who can be accommodated at each type of camp site.

Inequality 3 $\qquad ax + by \geq 48$

a Write down the values of a and b in inequality 3.

The graph opposite shows the three lines that represent the boundaries of inequalities 1, 3 and 4.



b From the graph list the coordinates of the points that satisfy inequalities 1, 2, 3 and 4.

c Determine the minimum number of camp sites that the school would need to hire.

d The cost of each powered camp site is \$60 per day and the cost of each unpowered camp site is \$30 per day.

i Find the minimum cost per day, in total, of accommodating 48 students.

School regulations require boys and girls to be accommodated separately.
The girls must all use one type of camp site and the boys must all use the other type of camp site.

ii Determine the minimum cost per day, in total, of accommodating the 48 students if there is an equal number of boys and girls.

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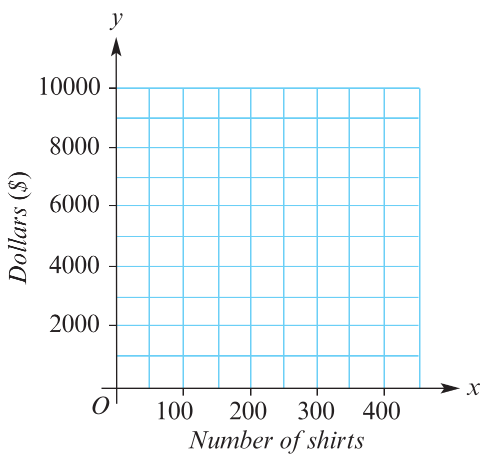
5 A clothing manufacturer finds that the cost, C dollars, of producing x shirts is given by the equation $C = 8x + 2400$.

a Determine the cost of producing 400 shirts.

b Determine the maximum number of shirts that can be produced for \$3000.

c Assuming that all the shirts are sold, the revenue, R dollars, from the sale of x shirts produced is given by the equation $R = 23x$.



On a set of axes similar to the one shown, draw a graph of the revenue equation $R = 23x$ for $0 \leq x \leq 400$.

On these same axes, draw a graph of the cost equation $C = 8x + 2400$ for $0 \leq x \leq 400$.

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d Determine the number of shirts that need to be produced and sold for the manufacturer to break even.

e Given that the cost equation is $C = 8x + 2400$ and the revenue equation is $R = 23x$, write an equation for the profit, P dollars, from the production and sale of x shirts.

f Calculate the profit from the production and sale of 345 shirts.

g The manufacturer also produces jackets. An order is received for 250 jackets. The cost of producing the 250 jackets is \$4800. Determine the selling price per jacket to achieve an overall profit of \$3000.

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6 Singlets are produced and sold in larger quantities. The revenue, R_S dollars, generated from the sale of x singlets is given by the equation

$$R_S = \begin{cases} 10x & x \le 500 \\ 6x + 2000 & x > 500 \end{cases}$$

a Calculate the revenue, R_S, generated by the sale of 620 singlets.

b Sketch a graph of the revenue, R_S, for $0 \le x \le 1000$ on a set of axes similar to that used in Question [10] (with x from 0 to 1000).

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7 Two bushwalkers, Malinda and Christos, set out to walk from Fishbone Creek to Snake Gully, a distance of 20 km. They start at the same time and follow the same route.

a Malinda walks at a constant speed of 4 km/h for the entire journey and takes no rest periods. How far does she travel in 1.5 hours?

b The distance walked by Malinda from Fishbone Creek, in kilometres, is given by the equation $D_m = 4t$ for $0 \le t \le 5$, where t is the time in hours since she began walking.

Draw and label the graph of D_m against t.

Christos started walking at the same time as Malinda and followed the same route. At the start he walked at a constant speed of 6 km/h. However, after walking at this speed for two hours he developed sore feet. Rather than stopping, he slowed down to a constant speed of 2 km/h for the remainder of the trip.

c Let D_C represent the distance walked by Christos. Draw and label the graph of D_C against t on the set of axes used in part **b**.

d Malinda eventually catches up to Christos. How many hours after they start walking does this happen?

e The equations below give the distance, D_C, in kilometres, walked by Christos at any time t hours.

$$D_c = \begin{cases} at & 0 \le t \le 2 \text{ hours} \\ bt + h & 2 < t \le d \text{ hours} \end{cases}$$

Determine the values for a, b, h and d.

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8 Malinda began the walk with 2000 millilitres of water in her bottle. It was a hot day and she sipped small amounts of water from her bottle frequently to ensure that she would not dehydrate.

t (*hours*)	0	2	4	5
t^2 (*hours*^{2})	0			
V (*millilitres*)	0	300	1200	1875

The table shows the total volume, V, of water, in millilitres, that Malinda had drunk after t hours.

a **i** Complete the table.

ii On a set of axes, plot the four points (t^2, V) from your completed table (with V from 0 to 2000 and t^2 from 0 to 25).

b Assume that this data set is modelled by the relation $V = kt^2$. Use the graph or the table to determine the value of k.

c Use your answer from part **b** to determine the volume of water Malinda drank during the first three hours. Write your answer correct to the nearest millilitre.

d At some point during the walk, Malinda had drunk half the water in her bottle.

i How long after the start of the walk did this happen? Write your answer in hours, correct to two decimal places.

ii Determine how far apart Malinda and Christos were at this time. Write your answer in kilometres, correct to one decimal place.

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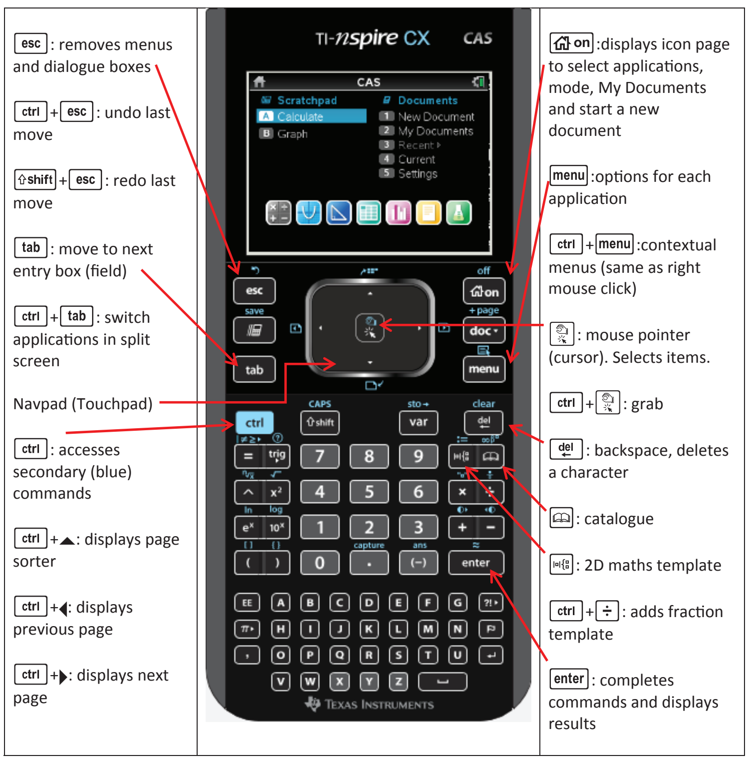
Appendix A: TI-Nspire CAS CX with OS4.0

Keystroke actions and short cuts for the TI-Nspire CAS CX



Mode Settings

How to set in Degree mode

For Further Mathematics it is necessary to set the calculator to **Degree** mode right from the start. This is very important for the Trigonometry topic. The calculator will remain in this mode unless you change the setting again or update the operating system.

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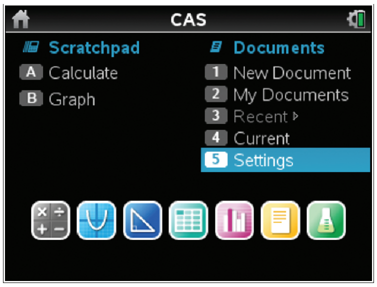
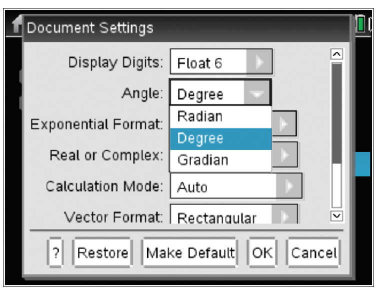
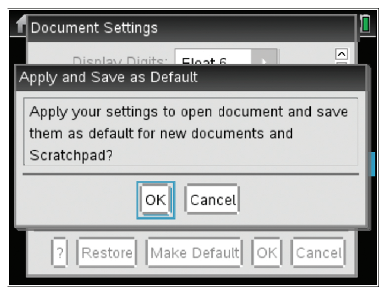
Press ⌂on and move to **Settings>Document Settings**, arrow down to the down to the **Angle** field, press ▶ and select **Degree** from the list, and then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note that there is a separate settings menu for the **Graphs Geometry** pages. These settings are accessed from the relevant pages. For Mathematics it is not necessary for you to change these settings.

Note: When you start your new document you will see **DEG** in the top status line.





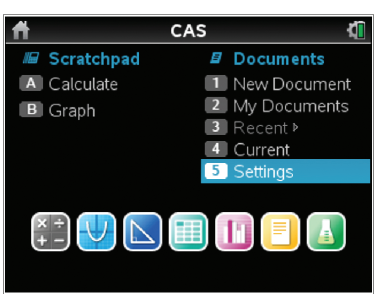


How to set in Approximate (Decimal) mode

For Further Mathematics it is useful to set the calculator to **Approximate (Decimal)** mode right from the start. The calculator will remain in this mode unless you change the setting again or update the operating system.

Press ⌂on and move to **Settings>Document Settings**, arrow down to the **Calculation Mode** field, press ▶ and select **Approximate** from the list, and then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note that you can make both the **Degree** and **Approximate Mode** selections at the same time if desired.



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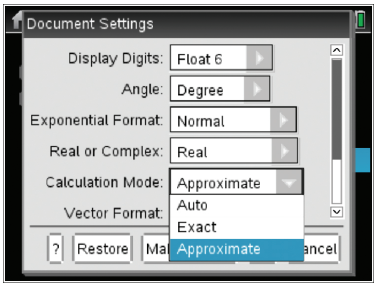
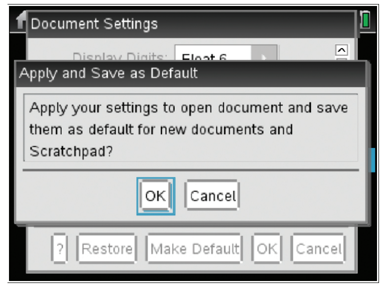




The home screen is divided into two main areas: **Scratchpad** and **Documents**.

All instructions given in the text, and in Appendix A, are based on the **Documents** platform.

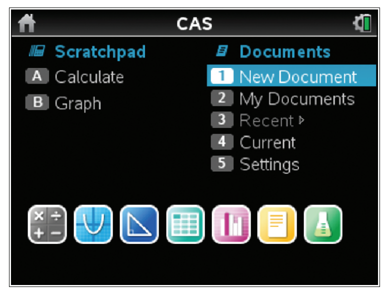
Documents

Documents can be used to access all the functionality required for Further Mathematics including all calculations, graphing, statistics and geometry.

Starting a new document

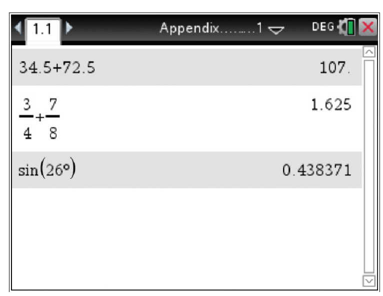
1. To start a new document, press ⌂on and select **New Document**.
2. If prompted to save an existing document move the cursor to **No** and press enter.

   Note: Pressing ctrl + N will also start a new document.



A: Calculator page – this is a fully functional CAS calculation platform that can be used for calculations such as arithmetic, algebra, finance, trigonometry and matrices. When you open a new document select **Add Calculator** from the list.

1. You can enter fractions using the fraction template if you prefer. Press ctrl ÷ to paste the fraction template and enter the values. Use the tab key or arrows to move between boxes. Press enter to display the answer. Note that all answers will be either whole numbers or decimals because the mode was set to approximate (decimal).



2. For problems that involve angles (e.g. evaluate sin(26°)) it is good practice to include the degree symbol even if the mode is set to degree (DEG) as recommended.

   Note: If the calculator is accidentally left in radian (RAD) mode the degree symbol will override this and compute using degree values.

   The degree symbol can be accessed using ?!▸. Alternatively select from the **Symbols** palette ctrl .To enter trigonometry functions such as *sin*, *cos*, etc., press the trig key or just type them in with an opening parenthesis.

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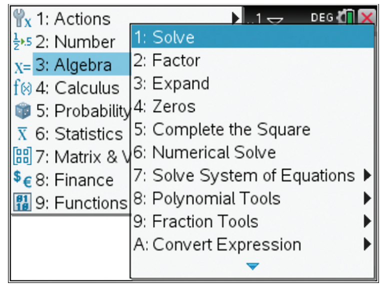
Solving equations

Using the **Solve command**, solve $2y + 3 = 7$ for y.

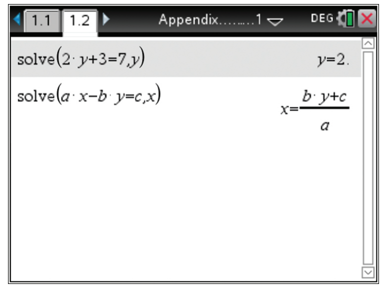
In a **Calculator** page press menu**>Algebra>Solve** and complete the **Solve** command as shown opposite. You must include the variable you are making the subject at the end of the command line.



Hint: You can also type in **solve(** directly from the keypad but make sure you include the opening bracket.

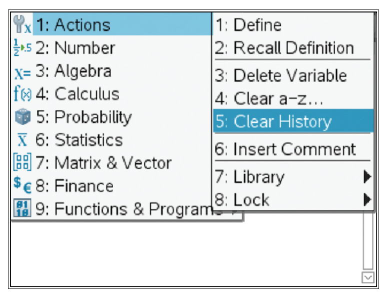
Literal equations such as $ax - by = c$ can be solved in a similar way.

Note that you must use a multiplication sign between two letters.



Clearing the history area

Once you have pressed enter the computation becomes part of the **History** area. To clear a line from the history area, press ▲ repeatedly until the expression is highlighted and press enter. To completely clear the History Area, press menu**>Actions>Clear History** and press enter again.



Alternatively press ctrl+menu to access the contextual menu.

It is also useful occasionally to clear any previously stored values. Clearing **History** does not clear stored variables.

Pressing menu**>Actions>Clear a-z...** will clear any stored values for single letter variables that have been used.

Use menu**>Actions>Delete Variable** if the variable name is more than one letter. For example, to delete the variable perimeter, then use **DelVar *perimeter***.

Note: When you start a new document any previously stored variables are deleted.

ISBN 978-1-107-56757-3

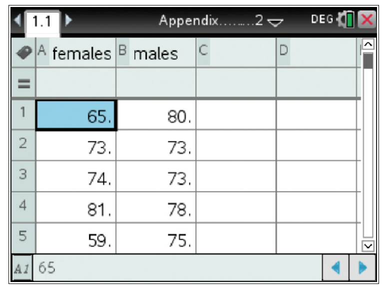
How to construct parallel box plots from two data lists

Construct parallel box plots to display the pulse rates of 23 adult females and 23 adult males.

Pulse rate (beats per minute)	
Females	*Males*
65 73 74 81 59 64 76 83 95 70 73 79 64 77 80 82 77 87 66 89 68 78 74	80 73 73 78 75 65 69 70 70 78 58 77 64 76 67 69 72 71 68 72 67 77 73

Steps

1 Start a new document: ctrl + N

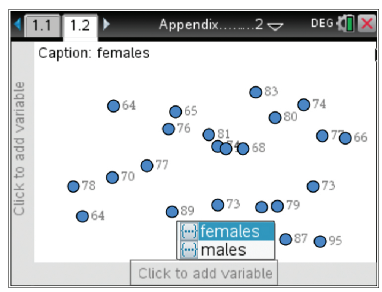


2 Select **Add Lists & Spreadsheet**. Enter the data into lists called ***females*** and ***males*** as shown.

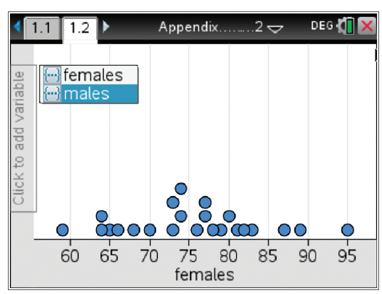
3 Statistical graphing is done through the **Data & Statistics** application.
Press ctrl + I and select **Add Data & Statistics**.

(or press on and arrow to and press enter).

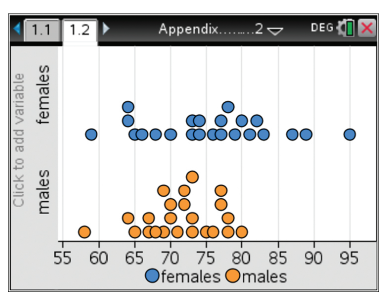
Note: A random display of dots will appear – this indicates list data are available for plotting. It is not a statistical plot.



a Press tab, or navigate and click on the 'Click to add variable' box to show the list of variables. Select the variable, ***females***. Press enter or to paste the variable to the x-axis. A dot plot is displayed by default as shown.



b To add another variable to the x-axis press menu**>Plot Properties>Add X Variable**, then enter. Select the variable ***males***. Parallel dot plots are displayed by default.



c To change the plots to box plots press menu**>Plot Type>Box Plot**, and then press enter. Your screen should now look like that shown opposite.

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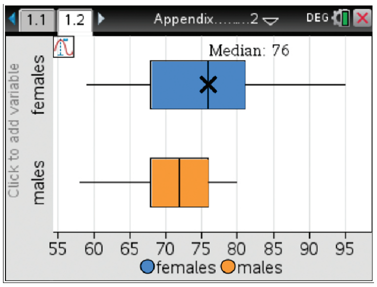
4 Data analysis

Use menu **>Analyze>Graph Trace** and use the cursor arrows to navigate through the key points. Alternatively just move the cursor over the key points. Starting at the far left of the plots, we see that, for females, the:

- minimum value is 59: **MinX** = **59**
- first quartile is 68: **Q1**= **68**
- median is 76: **Median** = **76**
- third quartile is 81: **Q3** = **81**
- maximum value is 95: **MaxX** = **95**

and for males, the:

- minimum value is 58: **MinX** = **58**
- first quartile is 68: **Q1** = **68**
- median is 72: **Median** = **72**
- third quartile is 76: **Q3** = **76**
- maximum value is 80: **MaxX** = **80**



Use ▼ to trace the other plot.

Press esc to exit the **Graph Trace** tool.

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Appendix B: Casio ClassPad II

Operating system

Written for operating system 2.0 or above.

Terminology

Some of the common terms used with the ClassPad are:

The menu bar

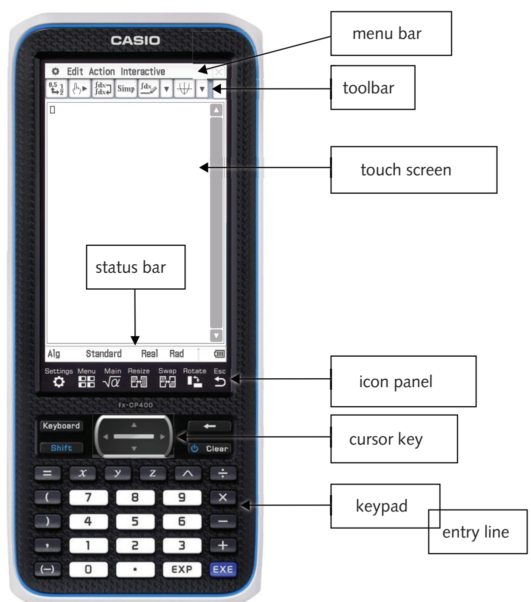
The toolbar

The touch screen contains the work area where the input is displayed on the left and the output is displayed on the right. Use your finger or stylus to tap and perform calculations.

The icon panel contains seven permanent icons that access settings, applications and different view settings. Press **escape** to cancel a calculation that causes the calculator to freeze.

The cursor key works in a similar way to a computer cursor keys

The keypad refers to the hard keyboard



Calculating

Tap Menu from the **icon panel** to display the application menu if it is not already visible.

Tap Main to open the **Main** application.

Note: There are two application menus. Alternate between the two by tapping on the screen selector at the bottom of the screen.

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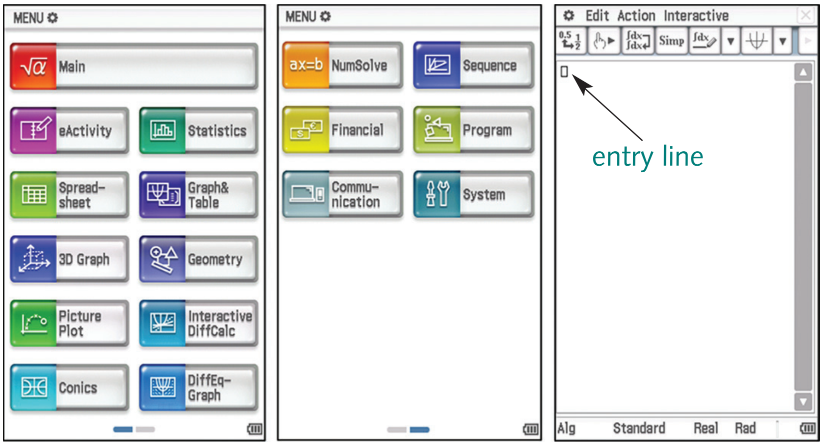
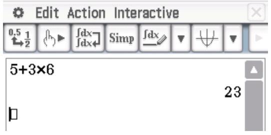
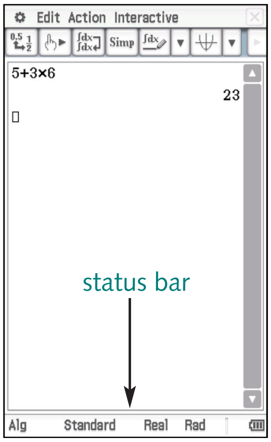
1 The main screen consists of an entry line (which is recognised by a flashing vertical line (cursor) inside a small square). The history area, showing previous calculations, is above the entry line.



2 To calculate, enter the required expression in the entry line and press EXE. For example, if we wish to evaluate $5 + 3 \times 6$, type the expression in the entry line and press EXE.
You can move between the entry line and the history area by tapping or using the cursor keys (i.e. ▼, ◀, ▲, ▶).



3 The ClassPad gives answers in either exact form or as a decimal approximation. Tapping settings in the **status bar** will toggle between the available options.



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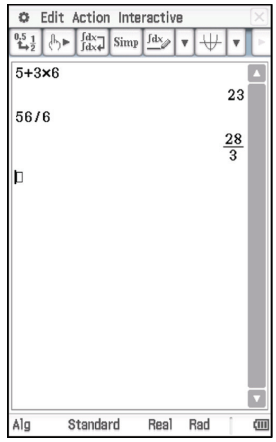
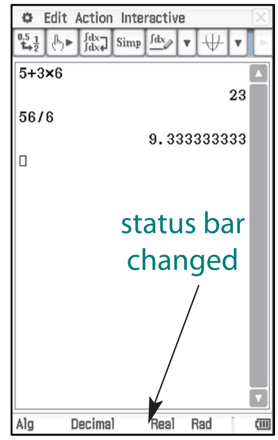
4 For example, if an exact answer is required for the calculation 56 ÷ 6, the **Standard** setting must be selected.

5 If a decimal approximation is required, change the **Standard** setting to **Decimal** by tapping it and press EXE.

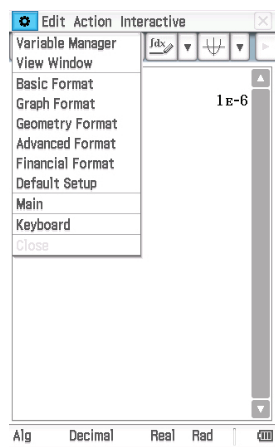




Extremely large and extremely small numbers

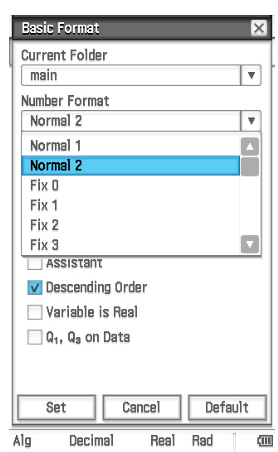
When solving problems that involve large or small numbers the calculator's default setting will give answers in scientific form.

For example, one millionth, or $\frac{1}{1000000}$, in scientific form is written as 1×10^{-6} and the calculator will present this as 1E-6.



To change this setting, tap on the settings icon and select **Basic Format.**

Under the Number Format select **Normal 2** and tap SET.



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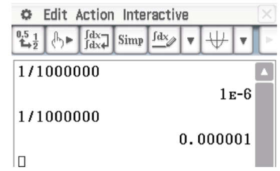
In the Main screen type $\frac{1}{1000000}$ and press EXE.

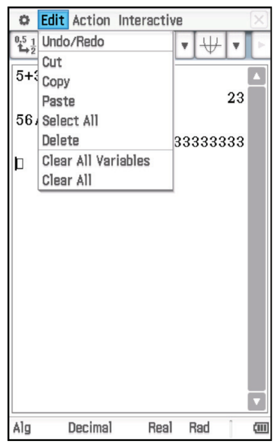
The answer will now be presented in decimal form 0.000001.

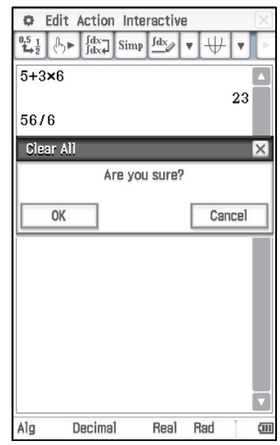
This setting will remain until the calculator is reset.



Clearing the history screen

To clear the **Main** application screen, select **Edit** from the menu bar and then tap **Clear All**. Confirm your selection by tapping **OK**. The entire screen is now cleared. To clear the entry line only, press Clear on the front calculator.

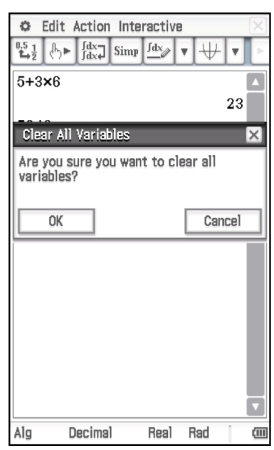


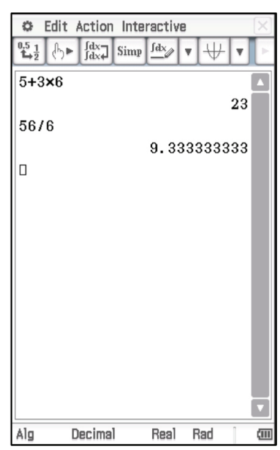


Clearing variables

To clear stored variable values, select **Edit** from the menu bar and then tap **Clear All Variables**. Confirm your selection by tapping OK.

The variables are cleared but the history created on the main screen is kept.





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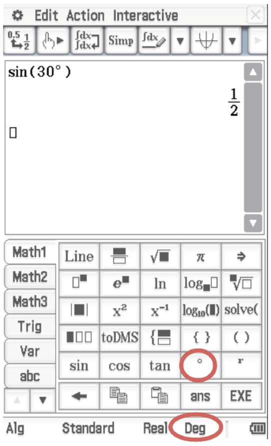
Degree mode

When solving problems in trigonometry, your calculator should be kept in **Degree** mode.

In the main screen, the status bar displays the angle mode.

To change the angle mode, tap on the angle unit in the status bar until **Deg** is displayed.



In addition, it is recommended that you always insert the degree symbol after any angle. This overrides any mode changes and reminds you that you should be entering an angle, not a length.

The degree symbol is found in the Math1 keyboard.

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Glossary

68–95–99.7% rule: [p. 75] A rule for determining the percentage of values that lie within one, two and three deviations of the mean in a normal distribution.

A

Activity (CPA): [p. 528] A task to be completed as part of a project. Activities are represented by the edges in the project diagram.

Activity network: [p. 528] An **activity network** is a weighted directed graph that shows the required order of completion of the activities that make up a project. The weights indicate the durations of the activities they represent.

Acute angle: An angle less than 90°.

Adding to the principal: see **annuity investment**.

Adjacency matrix: [p. 477] A square matrix showing the number of edges joining each pair of vertices in a graph.

Algorithm: [p. 489] A step-by-step procedure for solving a particular problem that involves applying the same process repeatedly. Examples include Prim's algorithm and the Hungarian algorithm.

Allocation: [p. 523] **Allocation** is the process of assigning a series of tasks to a different members of a group in a way that enables the tasks to be completed for the minimum time or cost.

Alternate angles: [p. 577] Two angles that lie between two lines on either side of a transversal that cuts the lines.

Amortisation: [p. 311] **Amortisation** is the repayment of a loan or an investment with regular payments made over a period of time.

Amortisation table: [p. 312] An **amortisation table** charts the amortisation (repayment) of a reducing balance loan or annuity on a step-by-step (payment-by payment) basis.

Angle of depression: [p. 631] The angle between the horizontal and a direction below the horizontal.

Angle of elevation: [p. 631] The angle between the horizontal and a direction above the horizontal.

Angle sum of a triangle: [p. 580] In triangle ABC, $\angle A + \angle B + \angle C = 180°$.

Annuity: [p. 330] An **annuity** is a compound interest investment from which regular payments are made.

Arc: [p. 655] The part of a circle between two given points on the circle. The length of the arc of a circle is given by $s = r(\frac{\theta}{180})\pi$, where r is the radius of the circle and θ is the angle in degrees subtended by the arc at the centre of the circle.

Area of a triangle: [p. 629] *See also* **Heron's formula**.

B

Backward scanning: [p. 539] **Backward scanning** is the process of determining the LST for each activity in a project activity network.

Balance: [p. 307] The **balance** of a loan or investment is the amount owed or accrued after a period of time.

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Bar chart: [p. 7] A statistical graph used to display the frequency distribution of categorical data.

Binary matrix: [p. 399] A matrix whose elements are either zero or one.

Bipartite graph (bigraph): [p. 522] A graph whose set of vertices can be split into two subsets, X and Y, in such a way that each edge of the graph joins a vertex in X and a vertex in Y.

Bivariate data: [p. 92] Data in which each observation involves recording information about two variables for the same person or thing. An example would be data recording the height and weight of the children in a preschool.

Box plot: [p. 53] A graphical display of the five-number summary of a data set showing outliers if present. *See* **outliers**.

Break-even analysis: [p. 713] Finding the point where the revenue of a business first equals the costs of running the business. Past this point, the business is running at a profit: profit = revenue – costs.

Bridge: [p. 471] A single edge in a connected graph that, if removed, leaves the graph disconnected.

C

Capacities (flow network) [p. 514] The weights of the directed edges in a flow network are called **capacities**. They give the maximum amount that can move between the two points in the flow network represented by these vertices in a particular time interval. This could be, for example, the maximum amount of water in litres per minute or the maximum number of cars per hour.

Categorical variable: [p. 2] Categorical variables are used to represent characteristics of individuals, for example place of birth, house number. Categorical variables come in types, nominal and ordinal.

Centre of a distribution: [pp. 21, 46] A measure of location of a distribution. Measures of centre include the median and the mean.

Centring: [p. 207] If smoothing takes place over an even number of data values, the smoothed values do not align with an original data value. A second stage of smoothing is carried out to centre the smoothed values at an original data value.

Circle of transformations: [p. 166] Provides guidance in choosing the transformations that can be used to linearise various forms of scatterplots.

Circuit: [p. 481] A **walk** with no repeated edges that starts and ends at the same vertex. *See also* **cycle (graphs)**.

Coefficient of determination (r^2): [p. 117] A coefficient which gives a measure of the predictive power of a regression line. It gives the percentage of variation in the RV that can be explained by the variation in the EV.

Co-interior angles: [p. 577] A pair of angles lying between two lines on the same side of a transversal (a line that cuts two lines).

Column matrix: [p. 365] A matrix with only one column.

Column vector: [p. 365] *see* **column matrix**.

Communication matrix [p. 400] A square binary matrix in which the 1s represent direct (one-step) communication links.

Complete graph: [p. 471] A graph with edges connecting all pairs of vertices.

Compound interest: [p. 279] Where the interest paid on a loan or investment is added to the principal and subsequent interest is calculated on the total.

Compound interest investments with periodic additions: [p. 341] A **compound interest investments with periodic payments** is an investment to which additions are made to the principal on a regular basis. Also known as 'adding to the principal'.

Compounding period: [p. 291] The **compounding period** is the time period for the calculation of interest for an investment or loan. Typical compounding periods are yearly, quarterly, monthly or daily.

Connected graph: [p. 471] A graph in which there is a path between every pair of vertices.

Continuous variable: [p. 4] A numerical variable that represents a quantity that is measured rather than counted, for example the weights of people in kilograms.

Convex polygon: [p. 582] A polygon in which any diagonal lies inside the polygon.

Corner point principle: [p. 758] A principle used to help locate the maximum or minimum value of a linear objective function in linear programming.

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Correlation coefficient *r*: [p. 110] A statistical measure of the strength of the linear association between two numerical variables.

Corresponding angles: [p. 577] When two lines are both cut by a third line, called a transversal, it forms two intersections, and the pairs of angles in the same relative positions on the intersections (the matching angles) are called corresponding angles. If the two lines are parallel then the corresponding angles are equal.

Cosine rule: [p. 625] In non-right-angled triangles, the cosine rule is used to find:

- the third side, given two sides and an included angle
- an angle, given three sides.

For triangle *ABC* and side *a*, the rule is: $a^2 = b^2 + c^2 - 2ab\cos(A)$. Similar rules exist for sides *b*and *c*.

Cost matrix: [p. 524] A **cost matrix** is a table that contains the cost of allocating objects from one group, such as people, to objects from another group, such as tasks. The cost can be money, or other factors such as the time taken to complete the project.

Crashing: [p. 546] **Crashing** is the process of shortening the length of time taken to complete a project by reducing the time required to complete individual activities.

Critical path: [p. 541] The project path that has the longest completion time.

Critical path analysis: [p. 541] A project planning method in which activity durations are known with certainty.

Cut: [p. 517] A line dividing a directed (flow) graph into two parts in a way that separates all 'sinks' from their 'sources'.

Cut capacity: [p. 517] The **capacity of a cut** is the sum of the capacities of the cuts passing through the cut that represents flow from the source to the sink. Edges that represent flow from the sink to the source do not contribute to the capacity of the cut.

Cycle (graphs): [p. 481] A **walk** with no repeated vertices that starts and ends at the same vertex. *See also* **circuit**.

Cycle (time series): [p. 196] Periodic movement in a time series but over a period greater than a year.

D

Data transformation: [p. 166] Using a mathematical rule to change the scale on either the *x*- or *y*-axis in order to linearise a non-linear scatterplot.

Degenerate graph: [p. 470] A graph in which no vertex is connected to any other vertex. All the vertices are isolated.

Degree of a vertex (deg(*A*)): [p. 469] The number of edges attached to the vertex. The degree of vertex *A* is written as deg(*A*).

Dependent linear equations: [p. 422] A set of linear equations with an infinite number of solutions.

Depreciation: [p. 266] The reduction in value of an item over time.

Deseasonalise: [p. 217] The process of removing seasonality in time series data.

Determinant: [p. 417] A number associated with square matrices. The determinant of a matrix A, written det(*A*), is used to decide if the matrix has an inverse. If $\det(A) = 0$, the matrix has no inverse; it is singular.

Dijkstra's algorithm: [p. 490] An algorithm for finding the shortest path between two vertices in a weighted graph. Pronounced 'Di-stra': 'Di' as in 'die' and 'stra' as in 'car'.

Directed graph (digraph): [p. 514] A graph or network in which directions are associated with each of the edges.

Discrete variable: [p. 3] A **numerical variable** that represents a quantity that is determined by counting; for example, the number of people waiting in a queue is a discrete variable.

Dominance matrix: [p. 404] A square binary matrix in which the 1s represent one-step dominances between the members of a graph.

Dominant category: [p. 10] The modal category in a categorical dataset. There may be more than one.

Dot plot: [p. 42] A statistical graph that uses dots to display individual data values on a number line; suitable for small sets of data only.

Dummy activity: [p. 531] An artificial activity of zero time duration added to a project diagram to ensure that all predecessor activities are properly accounted for.

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E

Earliest starting time (EST): [p. 533] The earliest time an activity in a project can be started.

Edge: [p. 469] A line joining one vertex in a graph or network to another vertex or itself (a loop).

Effective interest rate: [p. 293] Used to compare the interest paid on loans (or investments) with the same annual nominal interest rate r but with different compounding periods (daily, monthly, quarterly, annually, other).

Elements: [p. 364] The numbers or symbols displayed in a matrix.

Equal matrices: [p. 379] Matrices that have the same order and identical elements in identical positions.

Equilateral triangle: [p. 580] A triangle in which all sides are of the same length. The angles of an equilateral triangle are all of magnitude 60°.

Equivalent graph: [p. 473] *see* **isomorphic graphs**.

Eulerian circuit: [p. 483] An eulerian walk that starts and finishes at the same vertex. To have an eulerian circuit, a network must be connected and all vertices must be of even degree.

Eulerian trail: [p. 483] A walk in a graph or network that includes every edge just once (but does not start and finish at the same vertex).

To have an eulerian walk (but not an eulerian circuit), a network must be connected and have exactly two vertices of odd degree, with the remaining vertices having even degree.

Euler's rule: [p. 474] The formula $v - e + f = 2$, which relates the number of vertices, edges and faces in a connected graph.

Explanatory variable: [p. 92] When investigating associations in **bivariate data**, the explanatory variable (EV) is the variable used to explain or predict the value of the **response variable** (RV).

Exterior angle: [p. 579] An angle associated with a geometric figure that lies outside the figure.

Extrapolation: [pp. 144, 729] Using a mathematical model to make a prediction *outside* the range of data used to construct model.

F

Face: [p. 474] An area in a graph or network that can only be reached by crossing an edge. One such area is always the area surrounding a graph.

Feasible region: [p. 756] The region in the coordinate plane that contains all possible solutions to a set of inequalities. Also known as the solution region.

Finance Solver: [p. 317] A **finance solver** is a computer/calculator application that automates the computations associated with analysing a reducing balance loan, an annuity or an annuity investment.

Five-number summary [p. 53] A list of the five key points in a data distribution: the minimum value (min), the first quartile (Q_1), the median (M), the third quartile (Q_3) and the maximum value (max).

Flat-rate depreciation: [p. 266] Depreciation where the value of an item is reduced by the same amount each year. Flat-rate depreciation is equivalent, but opposite, to simple interest.

Float (slack) time: [p. 537] The amount of time available to complete a particular activity that does not increase the total time taken to complete the project.

Flow: [p. 514] **Flow** is the movement of something from a source to a sink.

Forward scanning: [p. 538] **Forward scanning** is the process of determining the EST for each activity in a project activity network.

Frequency table: [p. 6] A listing of the values a variable takes in a data set along with how often (frequently) each value occurs. Frequency can be recorded as a count or as a percentage.

G

Geometric decay [p. 278] When a recurrence rule involves multiplying by a factor less than one, the terms in the resulting sequence are said to decay geometrically.

Geometric growth [p. 278] When a recurrence rule involves multiplying by a factor greater than one, the terms in the resulting sequence are said to grow geometrically.

Gradient of a straight line: [p. 694] *See* **slope** of a straight line.

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Graph or network: [pp. 469, 488] A collection of points called vertices and a set of connecting lines called edges.

Great circle [p. 661] A circle on a sphere whose plane passes through the centre of the sphere. The shortest distance between two points on a sphere is along an arc of the great circle passing through the two points. *See also* **small circle**.

H

Hamiltonian cycle: [p. 483] A hamiltonian path that starts and finishes at the same vertex.

Hamiltonian path: [p. 483] A path through a graph or network that passes through each vertex exactly once. It may or may not start and finish at the same vertex.

Heron's formula: [p. 629] A rule for calculating the area of a triangle from its three sides. Using Heron's formula, the area, A, of the triangle ABC is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

Histogram: [p. 14] A statistical graph used to display the frequency distribution of a numerical variable; most suitable for medium to large sized data sets.

Hungarian algorithm: [p. 524] An algorithm for solving allocation (assignment) problems.

Hyperbolas: [p. 730] Graphs of relations of the form $y = \frac{k}{x}$, $k \neq 0$ are hyperbolas.

I

Identity matrix (*I*): [pp. 368, 416] A matrix that behaves like the number one in arithmetic. Any matrix multiplied by an identity matrix remains unchanged. An identity matrix is represented by the symbol I.

Immediate predecessor: [p.528] An activity that must be completed immediately before another one can start.

Inconsistent linear equations: [p. 422] A set of linear equations that have no solution.

Initial state matrix: [p. 437] A column matrix used to represent the starting state of a dynamic system.

Intercept: [p. 134] *see* **x-intercept**, **y-intercept**

Intercept form: [p. 703] A linear equation written in the form, $y = a + bx$, where a and b are constants. In this equation, a represents the y-intercept and b represents the slope. For example, $y = 5 - 2x$ is the equation of a straight line with y intercept of 5 and the slope of -2.

Interest: [p. 264] The amount of money paid (earned) for borrowing (lending) money over a period of time.

Interest-only loans: [p. 328] A loan on which only the interest is paid. At the end of the loan, the principal must be repaid in full.

Interest rate: [pp. 197, 264] The rate at which interest is charged or paid. Usually expressed as a percentage of the money owed or lent.

Interior angle: [p. 579] An angle associated with a geometric figure that lies within the figure.

Interpolation: [pp. 144, 729] Using a regression line to make a prediction *within* the range of values of the explanatory variable.

Interquartile range (*IQR*): [pp. 46, 71] The interquartile range is defined as $IQR = Q_3 - Q_1$. IQR gives the spread of the middle 50% of data values.

Inverse matrix: [p. 416] A matrix which, when multiplied by the original matrix, gives the identity matrix (I). For a matrix A, the inverse is written as A^{-1} and has the property that $A^{-1}A = AA^{-1} = I$.

Irregular (random) fluctuations [p. 200] Unpredictable fluctuations in a time series. Always present in any real world time series plot.

Isolated vertex: [p. 470] A vertex that is not connected to any other vertex. Its degree is zero.

Isomorphic graphs: [p. 473] Equivalent graphs. Graphs that have the same number of edges and vertices that are identically connected.

Isosceles triangle: [p. 580] A triangle is said to be isosceles if it has two sides of equal length. If a triangle is isosceles, then the angles opposite the equal sides are equal.

Iteration [p. 260] Each application of a recurrence rule to calculate a new term in a sequence is called an iteration.

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L

Latest start time (LST): [p. 539] The latest time an activity in a project can begin, without affecting the overall completion time for the project.

Least squares method: [p. 133] One way of finding the equation of a regression line. It minimises the sum of the squares of the residuals. It works best when there are no outliers.

Linear decay [p. 263] When a recurrence rule involves subtracting a fixed amount, the terms in the resulting sequence are said to decay linearly.

Linear growth [p. 263] When a recurrence rule involves adding a fixed amount, the terms in the resulting sequence are said to grow linearly.

Linear programming: [p. 752] A technique for maximising or minimising the value of a linear quantity subject to a set of constraints.

Linear regression: [p. 133] The process of fitting a straight line to bivariate data.

Line segment graph: [p. 723] A graph made up of two or more parts of different straight-line graphs, each representing different intervals of the x-axis. Sometimes called a **piecewise linear graph**.

Log scale [p. 27] A scale used to transform a strongly skewed histogram to symmetry or linearise a scatterplot.

Logarithmic transformations (log x or log y): [p. 174] Transformations that linearise a scatterplot by compressing the upper end of the scale on an axis.

Loop: [p. 470] An edge in a graph or network that joins a vertex to itself.

Lower fence: [p. 55] *See* **outliers**.

M

Matrix: [p. 364] A rectangular array of numbers or symbols set out in rows and columns within square brackets (plural: matrices).

Matrix multiplication: [p. 386] The process of multiplying a matrix by a matrix.

Maximum flow (graph): [p. 515] The capacity of the 'minimum' cut.

Maximum or minimum value of the objective function: [p. 759] The value found by evaluating the objective function's value at the vertices or a long boundaries of the **feasible region**

Mean ($\bar{x}$): [p. 67] The balance point of a data distribution. The mean is given by $\bar{x} = \frac{\Sigma x}{n}$, where Σx is the sum of the data values and n is the number of data values. The mean is given by $\bar{x} = \frac{\sum x}{n}$, where $\sum x$ is the sum of the data values and n is the number of data values. Best used for symmetric distributions.

Median: [p. 21, 46] The median (M) is the middle value in a data distribution. It is the midpoint of a distribution dividing an ordered data set into two equal parts. Can be used for skewed or symmetric distributions.

Meridians of longitude: [p. 661] Semi-great circles which pass through the north and south poles.

Minimum cut (graph): [p. 518] The cut through a graph or network with the minimum capacity.

Minimum spanning tree: [p. 500] The spanning tree of minimum length. For a given connected graph, there may be more than one minimum spanning tree.

Modal category or modal interval: [p. 9] The category or data interval that occurs most frequently in a data set. *See* **dominant category**.

Mode: [p. 9] The most frequently occurring value in a data set. There may be more than one.

Modelling: [p. 263] Mathematical **modelling** is the use of a mathematical rule or formula to represent real-life situations.

Moving mean smoothing: [p. 205] In three-moving mean smoothing, each original data value is replaced by the mean of itself and the value on either side. In five-moving mean smoothing, each original data value is replaced by the mean of itself and the two values on either side.

Moving median smoothing: [p. 212] **Moving median smoothing** is a graphical technique for smoothing a time series plot using moving medians rather than moving means.

Multiple edge: [p. 470] Where more than one edge connects the same two vertices in a graph.

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N

Negatively skewed distribution: [p. 62] A data distribution with a long tail to the left.

Network: [pp. 488, 514] A set of points called vertices and connecting lines called edges, enclosing and surrounded by areas called faces.

Nominal interest rate: [p. 291] The annual interest rate for a loan of investment that assumes the compounding period is 1 year. If the compounding period is less than a year, for example monthly, the actual or **effective interest rate** will be greater than r.

Nominal variable: [p. 3] A categorical variable that generates data values that can be used be name only – for example, eye colour: blue, green, brown.

Normal distribution: [p. 74] A data distribution that has a bell shape. For normal distributions, the 68–95–99.7% rule can be used to relate the mean and standard deviation to percentages in the distribution.

Numerical variable: [p. 3] A variable used to represent quantities that are counted or measured. For example, the number of people in a queue, the heights of these people cm. Numerical variables come in types: **discrete** and **continuous**.

O

Objective function: [p. 757] A linear expression representing the quantity to be maximised or minimised in a linear programming problem. For example, $P = 3x + 5y$ could be an objective function representing profit.

Order: [p. 364] Used to indicate the size and shape of a matrix. For a matrix with n rows and m columns, the order of a matrix is written as $(n \times m)$.

Ordinal variable: [p. 3] **A categorical variable** that generates data values that can be used to both name and order, for example house number.

Outliers: [pp. 19, 200] Data values that appear to stand out from the main body of a data set. In a box plot, possible outliers are defined as data values greater than the upper fence $(Q_3 + 1.5 \times IQR)$ or less than the lower fence $(Q_1 - 1.5 \times IQR)$.

P

Parabolas: [p. 731] Graphs of the form $y = kx^2$, $k \neq 0$.

Parallel box plots: [p. 99] A statistical graph in which two or more box plots are drawn side-by-side so that the distributions can be compared.

Parallels of latitude: [p. 661] Small circles whose planes are parallel to that of the equator.

Path [p. 481] A **walk** with no repeated vertices. *See also* **trail**.

Percentage frequency: [p. 6] Frequency expressed as a percentage.

Permutation matrix: [p. 399] A square binary matrix in which there is only a single one in each row and column.

Perpetuity: [p. 338] An investment where an equal amount is paid out on a regular basis forever.

Planar graph: [p. 473] A graph that can be drawn in such a way that no two edges intersect, except at the vertices.

Polygon: [p. 582] A closed geometric shape constructed from straight line segments. In **regular polygons**, the sides are of equal lengths.

Positively skewed distribution: [p. 62] A data distribution that has a long tail to the right.

Power of a matrix: [p. 397] Defined in the same way as the powers of numbers: $A^2 = A \times AA^3 = A \times A \times A$, etc. Only square matrices can be raised to a power. A^0 is defined to be I, the identity matrix.

Precedence table: [p. 528] A table that records the activities of a project, their immediate predecessors and often the duration of each activity.

Prim's algorithm: [p. 500] An algorithm for determining a minimum spanning tree in a connected graph.

Principal (P): [pp. 264, 309] The initial amount borrowed, lent or invested.

Pythagoras' theorem: [p. 585] A rule for calculating the third side of a right-angled triangle given the length of the other two sides. In triangle ABC, the rule is $c^2 = a^2 + b^2$, where c is the length of the hypotenuse.

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Q

Quartiles (Q_1, Q_2, Q_3): [p. 49] Summary statistics that divide an ordered data set into four equal sized groups.

R

Radius: [p. 655] The distance from the centre to any point on the circle (sphere). Half the diameter.

Range (*R*): [pp. 22, 46] The difference between the smallest and the largest observations in a data set; a measure of spread.

Reciprocal transformations (1/x or 1/y): [p. 181] Transformations that linearise a scatterplot by compressing the upper end of the scale on an axis to a greater extent than the log transformation.

Recurrence relation: [pp. 260, 437] A relation that enables the value of the next term in a sequence to be obtained by one or more current terms. Examples include 'to find the next term, add two to the current term' and 'to find the next term, multiply the current term by three and subtract five'.

Reducing-balance depreciation: [p. 282] When the value of an item is reduced by the same percentage each year. Reducing-balance depreciation is equivalent to, but opposite to, compound interest.

Reducing-balance loan: [p. 309] A loan that attracts compound interest, but where regular repayments are also made. In most instances the repayments are calculated so that the amount of the loan and the interest are eventually repaid in full.

Redundant communication link: [p. 402] A communication link is said to **redundant** if the sender and the receiver are the same people.

Regular polygon: [p. 582] *See* **polygon**.

Reseasonalise [p. 217] The process of a converting seasonal data back into its original form.

Residual: [p. 133] The vertical distance from a data point to a straight line fitted to a scatterplot is called a residual:

$$\text{residual} = \text{actual value} - \text{predicted value}$$

Residuals are sometimes called *errors of prediction*.

Residual plot: [p. 147] A plot of the residuals against the explanatory variable. Residual plots can be used to investigate the linearity assumption.

Response variable [p. 92] The variable of primary interest in a statistical investigation.

Round-robin tournament: [p. 403] A tournament in which each participant plays once.

Row matrix: [p. 365] A matrix with only one row. A row matrix is also called a row vector.

Row vector: [p. 365] *See* **row matrix**.

S

Scalar multiplication: [p. 380] The multiplication of a matrix by a number.

Scatterplot: [p. 103] A statistical graph used for displaying bivariate data. Data pairs are represented by points on a coordinate plane, the EV is plotted on the horizontal axis and the RV is plotted on the vertical axis.

Scrap value: [p. 266] The value at which an item is no longer of use to a business.

Seasonal indices: [p. 217] Indices calculated when the data shows seasonal variation. Seasonal indices quantify seasonal variation. A seasonal index is defined by the formula:

$$\text{seasonal index} = \frac{\text{value for season}}{\text{seasonal average}}$$

For seasonal indices, the average is 1 (or 100%).

Seasonality: [p. 199] The tendency for values in the time series to follow a seasonal pattern, increasing or decreasing predictably according to time periods such as time of day, day of the week, month, or quarter.

Segmented bar chart: [pp. 8, 125] A statistical graph used to display the information contained in a two-way frequency table. It is a useful tool for identifying associations between two categorical variables.

Sequence: [p. 258] A list of numbers or symbols written down in succession, for example 5, 15, 25, . . .

Shape of a distribution: [p. 19] The general form of a data distribution described as symmetric, positively skewed or negatively skewed.

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Cambridge University Press

Shortest path: [p. 489] The path through a graph or network with minimum length.

Similar figures: [p. 589] Figures that have exactly the same shape but that differ in size.

Simple graph: [p. 470] A graph with no loops or multiple edges.

Simple interest: [p. 264] Interest that is calculated for an agreed period and paid only on the original amount invested or borrowed.

Simultaneous equations: [p. 707] Two or more linear equations in two or more variables, for values that are common solutions to all equations. For example, $3x - y = 7$ and $x + y = 5$ are a pair of simultaneous linear equations in x and y with a common solution, $x = 3$ and $y = 2$.

Sine rule: [p. 621] In non-right-angled triangles, the sine rule is used to find:

- an unknown side given the angle opposite and another side and its opposite angle
- an unknown angle given the side opposite and another side and its opposite angle.

For triangle ABC the rule is:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Singular matrix: [p. 420] A matrix that does not have an inverse; its determinant is zero.

Sink: [p. 514] *See* **sink and source**.

Sink and source: [p. 514] In a flow network, a **source** generates flow while a **sink** absorbs the flow.

Slope (of a straight line): [p. 134] The slope of a straight line is defined to be: slope $= \dfrac{\text{rise}}{\text{run}}$. The slope is also known as the **gradient**.

Small circle [p. 661] Any circle on as sphere whose plane does not pass through the centre of the sphere. *See also* **great circle**.

Smoothing: [p. 205] A technique used to eliminate some of the variation in a time series plot so that features such as seasonality or trend are more easily identified.

Source [p. 514] *See* **sink and source**.

Spanning tree: [p. 498] A subgraph of a connected graph that contains all the vertices of the original graph, but without any multiple edges, circuits or loops.

Spread of a distribution: [p. 19] A measure of the degree to which data values are clustered around some central point in the distribution. Measures of spread include the standard deviation (s), the interquartile range (IQR) and the range (R).

Square matrix: [p. 366] A matrix with the same number of rows as columns.

Squared transformations (x^2 or y^2): [p. 167] Transformations that linearise a scatterplot by stretching out the upper end of the scale on an axis.

Standard deviation (s): [p. 71] A summary statistic that measures the spread of the data values around the mean. The standard deviation is given by $s = \sqrt{\dfrac{\sum(x - \bar{x})^2}{n - 1}}$.

Standardised (z) scores: [p. 79] The value of the standard score gives the distance and direction of a data value from the mean in terms standard deviations.
The rule for calculating a standardised scores is:

$$\text{standardised score} = \frac{\text{actual score} - \text{mean}}{\text{standard deviation}}$$

State matrix: [p. 437] A column matrix that represents the starting state of a dynamic system.

Statistical question: [p. 10] A question that depends on data for its answer.

Steady-state matrix: [p. 442] A column matrix that represents the final state of a dynamic system. Also called the equilibrium state.

Stem plot (stem-and-leaf plot): [p. 43] A method for displaying data in which each observation is split into two parts, a 'stem' and a 'leaf'. A stem plot is an alternative display to a histogram; suitable for small to medium sized data sets. When data values are tightly clustered, stems can be split to give finer detail.

Step graph: [p. 725] A graph consisting of straight-line segments parallel to the horizontal axis; looks like a set of steps.

Straight lines through the origin: [p. 730] Graphs of relations of the form $y = kx$, $k \neq 0$ are straight lines through the origin.

Strength of a linear relationship: [p. 110] Classified as weak, moderate or strong. Determined by observing the degree of scatter in a scatterplot or calculating a correlation coefficient.

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Structural change (time series) [p. 200] A sudden change in the established pattern of a time series plot.

Subgraph: [p. 471] Part of a graph that is also a graph in its own right.

Sum of the interior angles: [p. 582] The angle sum of the interior angles of an n-sided polygon is given by the formula: $S = (180n - 360)°$

Summary statistics [p. 42] **Statistics** that give numerical values to special features of a data distribution, such as centre and spread. Summary statistics include the mean, median, range, standard deviation and IQR.

Supplementary angles: [p. 577] Two angles are supplementary if the sum of their sizes is 180°.

Surface area: [p. 597] The total of the areas of each of the surfaces of a solid.

Symmetric distribution: [p. 20] A data distribution in which the data values are evenly spread out around the mean. In a symmetric distribution, the mean and the median are the same.

T

Three-figure bearing: [p. 633] An angular direction, measured clockwise from north and written with three digits, for example, 060° and 324°.

Time series data: [p. 194] A collection of data values along with the times (in order) at which they were recorded.

Time series plot: [p. 194] A line graph where the values of the response variable are plotted in time order.

Trail [p. 481] A walk with no repeated edges. *See also* **path**.

Transition matrix (T): [p. 431] A square matrix that describes the transitions made between the states of a system.

Transpose: [p. 365] The transpose of a matrix is obtained by interchanging its rows and columns.

Tree: [p. 498] A connected graph with no circuits, multiple edges or loops.

Trend: [p. 199] The tendency for values in the time series to generally increase or decrease over a significant period of time.

Trend line forecasting: [p. 227] Using a line fitted to an increasing or decreasing time series to predict future values.

Triangular matrix: [p. 368] An **upper triangular matrix** is a square matrix in which all elements below the leading diagonal are zeros. A **lower triangular matrix** is a square matrix in which all elements are above the leading diagonal are zeros.

Trigonometric ratios: [p. 616] In right-angled triangles, the ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Two-way frequency table: [p. 94] A frequency table in which subjects are classified according to two categorical variables. Two-way frequency tables are commonly used to investigate the associations between two categorical variables.

U

Unit-cost depreciation: [p. 266] Depreciation based on how many units have been produced or consumed by the object being depreciated. For example, a machine filling bottles of drink may be depreciated by 0.001 cents per bottle it fills.

Univariate data: [p. 35] Data associated with a single variable.

Upper fence: [p. 55] *See* **outliers**.

V

Vertex (graph): [p. 469] The points in a graph or network (*pl* vertices).

Vertically opposite angles: [p. 577]

Volume formulas: [pp. 595–597] Formulas used to calculate the volumes of regular solids including cubes, cuboids, triangular prisms, cones, pyramids and spheres.

W

Walk [p. 480] Any continuous sequence of edges, linking successive vertices, that connects two different vertices in a graph. *See also* **trail** and **path**.

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Weighted graph: [p. 488] A graph in which a number representing the size of some quantity is associated with each edge. These numbers are called weights.

X

***x*-intercept:** The point at which a graph cuts the x-axis.

Y

***y*-intercept:** The point at which a graph cuts the y-axis.

Z

Zero matrix (O): [p. 381] A matrix that behaves like zero in arithmetic. Represented by the symbol O. Any matrix with zeros in every position is a zero matrix.

ISBN 978-1-107-56757-3

Answers

Chapter 1

Exercise 1A

1 a Categorical variables are used to represent qualities, e.g. *sex, hair colour, house numbers,* etc.

b Numerical variables are used to represent quantities, e.g. *height* (in cm), *number of pages* in a book, etc.

2 nominal (e.g. *hair colour*) and ordinal (e.g. *anxiety level*)

3 continuous (e.g. *height* in cm) and discrete (e.g. *number of pages* in a book)

4 a numerical **b** numerical **c** categorical
d categorical **e** numerical **f** numerical
g categorical **h** categorical

5 a nominal **b** nominal **c** ordinal
d ordinal **e** ordinal **f** nominal

6 a discrete **b** discrete **c** continuous
d continuous **e** discrete **f** continuous

Exercise 1B

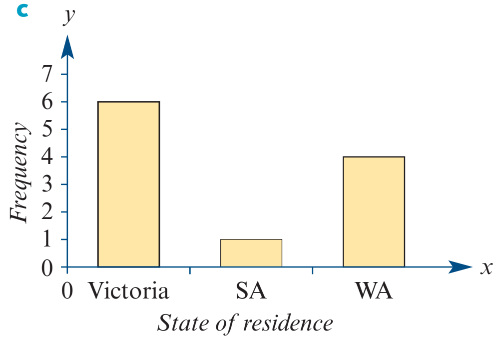
1 a most frequently occurring value
b i B **ii** 8

2 a categorical

b

State of residence	*Frequency* Count	*Frequency* Per cent
Victoria	6	54.5
SA	1	9.1
WA	4	36.4
Total	11	100.0

c

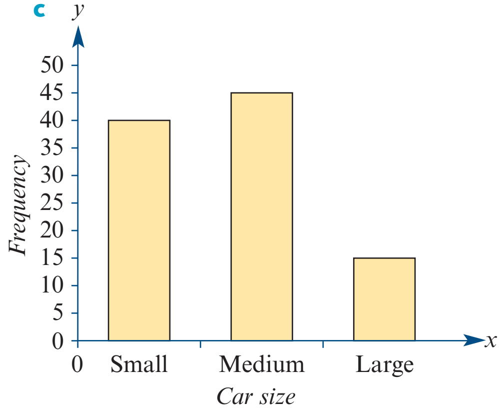


3 a categorical

b

Car size	*Frequency* Count	*Frequency* Per cent
Small	8	40
Medium	9	45
Large	3	15
Total	20	100

c



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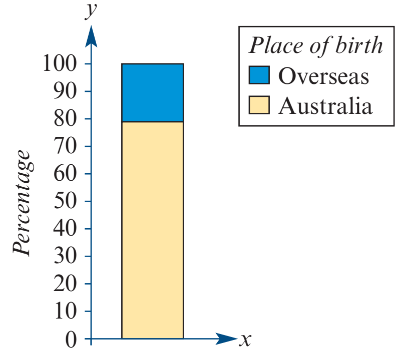
4 a nominal

b

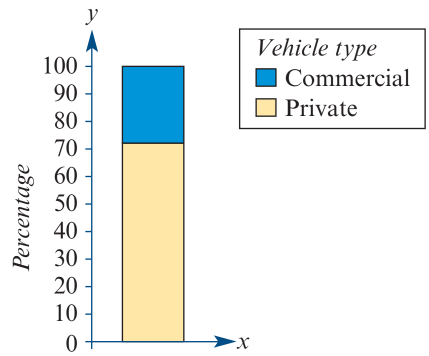


5 a nominal

b

Type of vehicle	*Frequency*	
	Count	*Per cent*
Private	132 736	73
Commercial	49 109	27
Total	181 845	100

c



6 a 20, 55 **b** 5 **c** 20 **d** 55%

e Report: 20 schools were classified according to school type. The majority of these schools, 55%, were found to be government schools. Of the remaining schools, 25% were independent while 20% were Catholic schools.

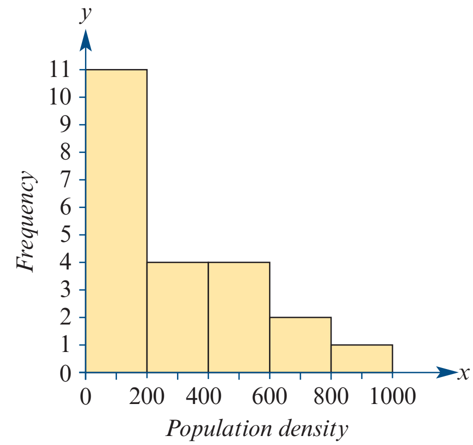
7 a 7, 45.5, 100.0

b Report: When 22 students were asked the question, 'How often do you play sport', the dominant response was 'sometimes', given by 45.5% of the students. Of the remaining students, 31.8% of the students responded that they played sport 'rarely' while 22.7% said that they played sport 'regularly'.

8 Report: The eye colours of 11 children were recorded. The majority, 54.5%, had brown eyes. Of the remaining children, 27.3% had blue eyes and 18.2% had hazel eyes.

Exercise 1C

1



2 a i 17% **ii** 13% **iii** 46% **iv** 33%

b i 6 **ii** 4

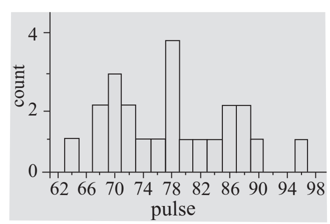
c 15–19 words/sentence

3 a 21

b i 13 **ii** 8 **iii** 5 **iv** 0

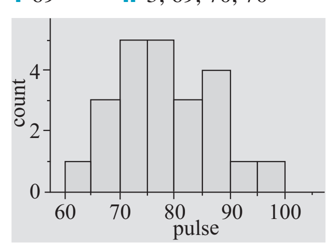
c i 4.8% **ii** 57.1%

4 a



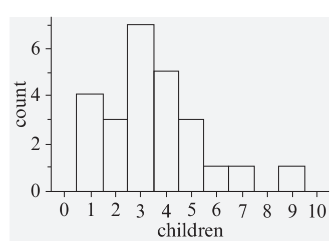
b i 69 **ii** 3; 69, 70, 70

c



d 3

5 a



b 3.5, 5

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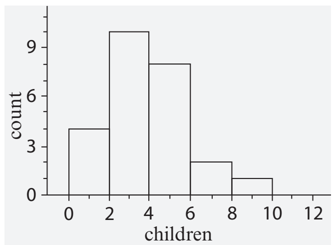
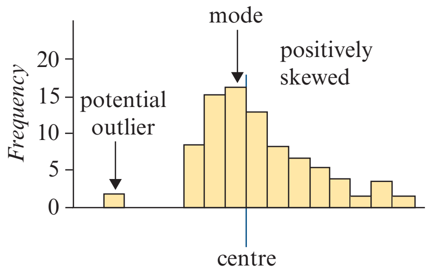
c



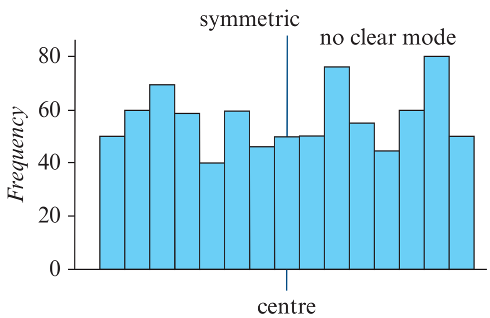
d **i** 2 **ii** 6 and 7

6 **a**



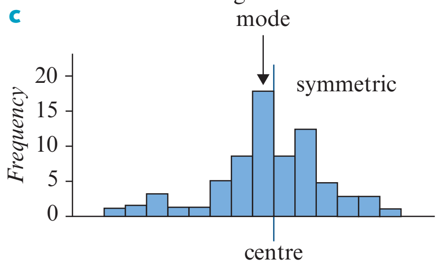
Histogram A

b



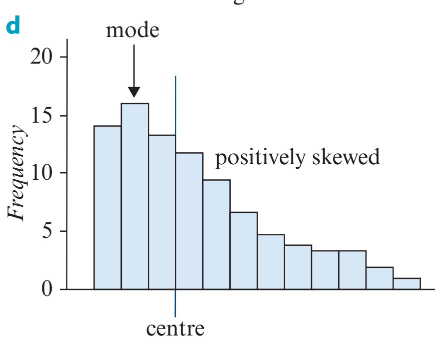
Histogram B

c



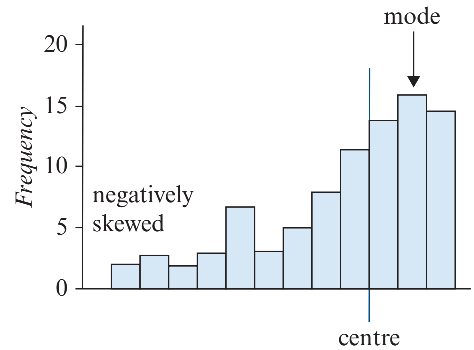
Histogram C

d



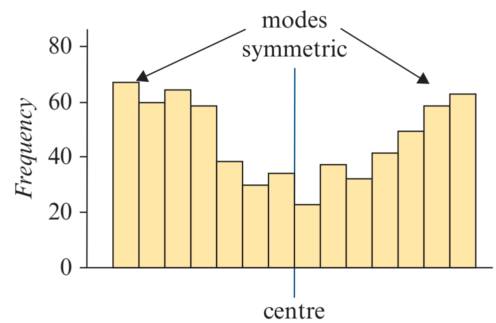
Histogram D

e



Histogram E

f



Histogram F

7 **a** approximately symmetric **b** no outliers
c A: 8–10; B: 24–26; C: 40–42 **d** B: 8
e C: 18

8 Report: For the 28 students, the distribution of pulse rates is approximately symmetric with an outlier. The centre of the distribution lies between 75–80 beats per minute and the spread of the distribution is 55 beats per minute. The outlier has a value somewhere between 110–115 beats per minute.

9 Report: For the 42 journeys the distribution of travel times is positively skewed with no outliers. The centre of the distribution lies is between 65 and 70 minutes and the spread of the distribution is 40 minutes

Exercise 1D

1 **a** 0.4 **b** 1.4 **c** 2.4 **d** 3.4
e −0.3 **f** −1.3 **g** −2.3 **h** −3.3

2 **a** 0.0032 **b** 0.032 **c** 0.32 **d** 1

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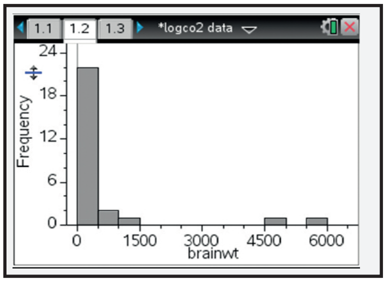
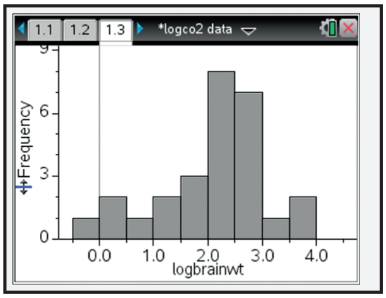
3 **a**



positively skewed with outliers

b



approximately symmetric

4 **a** -0.4 **b** 3.8 **c** 100 g **d** 0.1 g
e **i** 5 **ii** 12 **iii** 24

Chapter 1 review

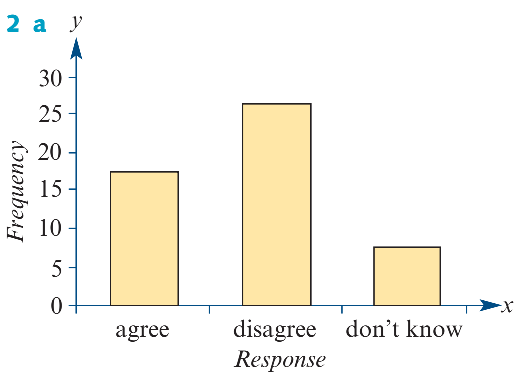
Multiple-choice questions

1 D	**2** E	**3** B	**4** C
5 C	**6** A	**7** C	**8** B
9 C	**10** D	**11** D	**12** B
13 E	**14** C	**15** C	**16** D
17 E			

Extended-response questions

1 **a** 19% **b** 23% **c** Sport, 35

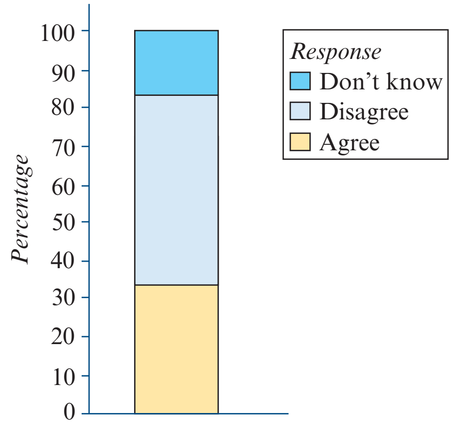
2 **a**



b

Should be legalised	Frequency	
	Count	Percentage
Agree	18	34.6
Disagree	26	50.0
Don't know	8	15.4
Total	52	100.0

c



d Report: In response to the question, 'Do you agree that the use of marijuana should be legalised?', 50% of the 52 students disagreed. Of the remaining students, 34.6% agreed, while 15.4% said that they didn't know.

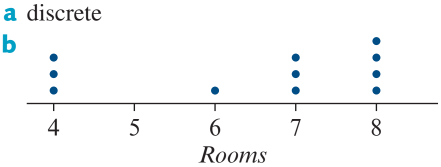
3 **a** **i** 50 **ii** 5
b \$105 – \$110 **c** 28 **d** 16%
e **i** approximately symmetric
ii \$110 – 115 **iii** \$50

4 Report: For the 34 cars, the distribution of waiting times is negatively skewed with an outlier. The distribution is centred between 25 and 30 seconds and has a spread of 55 seconds. There is an outlier that lies somewhere between 50 and 55 seconds.

Chapter 2

Exercise 2A

1 **a** discrete
b



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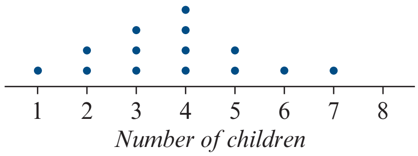
2 a discrete

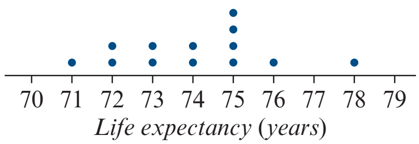
b



c 4; mode is the most frequently occurring number of children for these families

3 a continuous

b



c 75; mode is the most common life expectancy for these countries

4 a continuous

b Key: 1|6 = 16

0	3 3 6 9 9
1	2 2 6 7
2	0 2 2 5 7 8 9
3	1 5
4	
5	4 6
6	
7	
8	
9	9 9
10	0

c Modes: 3, 9,12, 22, 99. The mode represents the most common rate of urbanisation across the 23 countries.

5 a continuous

b i Key:16|5 = 16.5

16	579
17	0 1 2 3 6 6 7
18	2 4 5
19	3 9

ii Key:16|5 = 16.5

16	5 7 9
17	0 1 2 3
17	6 6 7
18	2 4
18	5
19	3
19	9

Exercise 2B

1 a The range is the difference between the largest and smallest value in a dataset.

b The median is the middle value; it divides the distribution into two equal parts.

c Quartiles divide a distribution into four equal parts.

d The IQR is the spread of the middle 50% of values.

2 a 5 **b** 12

3 \$850

4 a $M = 2$ **b** $Q_1 = 1,\ Q_3 = 3$
c $IQR = 2,\ R = 7$

5 a positively skewed with an outlier, the game in which 6 goals were scored
b $M = 0$
c $IQR = 1$ **d** $R = 6$

6 a $M = 11$ **b** $Q_1 = 10,\ Q_3 = 15$
c $IQR = 5,\ R = 18$

7 a approximately symmetric with no outliers.
b $M = 26$ **c** $Q_1 = 17.5, Q_3 = 30.5$
d $IQR = 13,\ R = 29$

8 a $M = 20$ **b** $Q_1 = 8,\ Q_3 = 26$
c $IQR = 18,\ R = 54$

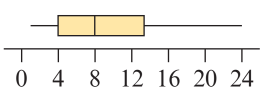
Exercise 2C

1

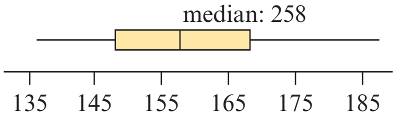
Minimum	4
First quartile (Q_1)	5
Median (M)	6
Third quartile (Q_3)	7
Maximum	9

2 136, 148, 158, 169, 189

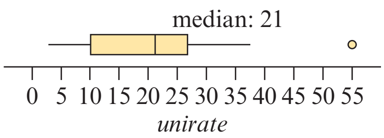
3 a



b

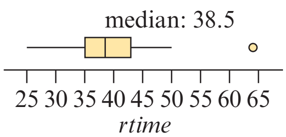


4 a



b 3, 10.5, 21, 26.5, 55; 55 is a possible outlier

5 a



b 25, 35, 38.5, 43, 64; 64 is a possible outlier.

6 a i 10 **ii** 5, 21 **iii** 16 **iv** 0, 45
v none

b i 27 **ii** 12, 42 **iii** 30 **iv** 5, 50
v none

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c **i** 38 **ii** 32, 42 **iii** 10 **iv** 5, 50
v 5

d **i** 16 **ii** 14, 21 **iii** 7
iv 1.5, 50 **v** 1.5, 3, 36, 40, 50

7 **a** **i** 55 **ii** 15
b **i** 100 **ii** 20

8 **a** 30;
b no, 31 > 30, so inside the lower fence

9 **a** 25% **b** 75% **c** 25% **d** 50%
e 75%

10 **a** 25% **b** 25% **c** 50% **d** 25%
e 75% **f** 50%

Exercise 2D

Box plot 1 matches histogram B.
Box plot 2 matches histogram D.
Box plot 3 matches histogram C.
Box plot 4 matches histogram A.

Exercise 2E

1 **a** The distribution is negatively skewed with an outlier. The distribution is centred at 39, the median value. The spread of the distribution, as measured by the IQR, is 10 and, as measured by the range, 45. There is an outlier at 5.

b The distribution is positively skewed with outliers. The distribution is centred at 16, the median value. The spread of the distribution, as measured by the IQR, is 6 and, as measured by the range, 35. The outliers are at 5, 8, 36 and 40.

c The distribution is negatively skewed with no outliers. The distribution is centred at 41.5, the median value. The spread of the distribution, as measured by the IQR, is 15 and, as measured by the range, 47.

d The distribution is approximately symmetric but with outliers. The distribution is centred at 41, the median value. The spread of the distribution, as measured by the IQR, is 7 and, as measured by the range, 36. The outliers are at 10, 15, 20 and 25.

2 **a** The distributions of pulse rates are approximately symmetric for both men and women. There are no outliers. The median pulse rate for females (M = 76 beats/minute) is greater than for males (M = 73 beats/minute). The IQR is also greater for females (IQR = 14 beats/minute) than males (IQR = 8 beats/minute). The range of pulse rates is also greater for females (R = 30 beats/minute) than males (R = 19 beats/minute).

b For this group of males and females, the females generally had higher and more variable pulse rates.

3 **a** The distribution for brand A is approximately symmetric with outliers. The distribution for brand B is positively skewed. The median battery lifetime for brand A (M = 34 hours) is greater than for brand B (M = 28 hours). The IQR for brand A (IQR = 14 hours) is less than for brand B (IQR = 14 hours). The range of lifetimes for brand A (R = 26 hours) is also less than for brand B (R = 39 hours). There are no outliers.

b On average, brand A batteries have longer and less variable lifetimes.

Exercise 2F-1

1 **a** median **b** symmetric **c** mean
d median, less affected by extremely high salaries

2 **a** $n = 4,\ \Sigma x = 12,\ \bar{x} = 3$
b $n = 5,\ \Sigma x = 104,\ \bar{x} = 20.8$
c $n = 7,\ \Sigma x = 21,\ \bar{x} = 3$

3 **a** $\bar{x} = 3,\ M = 3$, Mode = 2
b $\bar{x} = 5,\ M = 5$, Mode = 5

4 You will know when you have a correct answer.

5 **a** **i** mean = 36.1 **ii** median = 36.0
b The mean and median almost coincide because the distribution is approximately symmetric.

6 **a** **i** mean = \$3.65 **ii** median = \$1.70
b The median. The mean is inflated because of the one large sale and not representative of the sales in general.

7 **a** strongly negatively skewed distribution
c outliers
d strongly positively skewed distribution
e positively skewed distribution with outliers

8 **a** symmetric; either
b mean = 82.8 median= 83

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Exercise 2F-2

1 **a** IQR **b** range **c** standard deviation
2 7.1, 0 **3** b, d, f
4 **a** 20.1, 1.8 **b** symmetric
5

	TVs	*Cars*	*Alcohol*
Calculated mean	450	376	14.9
Calculated SD	100	107	5.1

6 females: $\bar{x} = 75.4$, $s = 8.8$ to 1 d.p.
males: $\bar{x} = 71.4$, $s = 5.3$ to 1 d.p.

Exercise 2G

1 **a** 114 and 154 **b** 94 and 174 **c** 74 and 194
d 154 **e** 94 **f** 74 **g** 134
2 **a** 68% **b** 99.7% **c** 16% **d** 2.5%
e 0.15% **f** 50%
3 **a** **i** 84% **ii** 50% **iii** 47.5% **b** 25
4 **a** **i** 99.7% **ii** 2.5% **iii** 81.5% **b** 800
5 **a** **i** 50% **ii** 34% **iii** 81.5% **b** 1994

Exercise 2H

1 **a** $z = 1$ **b** $z = 2$ **c** $z = -1$ **d** $z = 0$
e $z = -3$ **f** $z = 0.5$
2 **a** 120 **b** 116 **c** 142 **d** 100
e 72 **f** 50
3

Subject	*z-score*	*Rating*
English	2.25	Top 2.5%
Biology	3	Top 0.15%
Chemistry	0	Exactly average
Further Maths	1.1	Top 16%
Psychology	−2.25	Bottom 2.5%

4 **a** 0.2 **b** 46.5 **c** 2.5% **d** 34%
e 16% **f** 97.5%

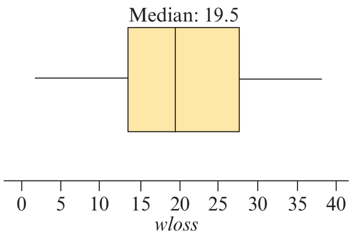
Chapter 2 review

Multiple-choice questions

1 E **2** B **3** A **4** C
5 E **6** B **7** A **8** C
9 B **10** B **11** A **12** D
13 D **14** A **15** B **16** B
17 A **18** D **19** B **20** D
21 B **22** C **23** D **24** E
25 E **26** A **27** A **28** E
29 C

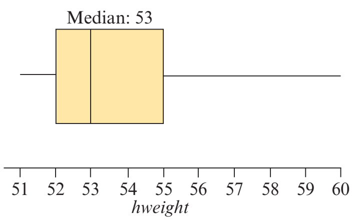
Extended-response questions

1 **a**



b $M = 19.5$, $Q_1 = 13.5$, $Q_3 = 27.5$
c 13.5, 27.5, 13.5
d The distribution is approximately symmetric with no outliers. The distribution is centred at 19.5 kg, the median value. The spread of the distribution, as measured by the IQR, is 14 kg and, as measured by the range, 36 kg.

2 **a**



b 51, 52, 53, 56, 60 **c** 4
d The distribution is positively skewed with no outliers. The distribution is centred at 53 kg, the median value. The spread of the distribution, as measured by the IQR, is 4 kg and, as measured by the range, 9 kg.

3 **a** overestimate
b less variable
c The distribution of the predicted marks is negatively skewed with no outliers. The distribution of actual marks is approximately symmetric with no outliers. The median predicted mark ($M = 38$) is greater than the median actual mark ($M = 35$). The IQR for the predicted marks ($IQR = 12$) is less than the IQR for the actual marks ($IQR = 13$). The range for the predicted marks ($R = 27$) is also less than the range for the actual marks ($R = 39$).
d The teacher's predicted marks are, on average, higher than the students' actual marks and less variable. This suggests that the teacher overestimated her students' performances.

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4 **a** suburb A **b** suburb C

c High family incomes that do not fall into the normal pattern for that suburb.

d **i** true **ii** true **iii** false **iv** true

5 **a** true **b** true **c** false **d** false

e false

Chapter 3

Exercise 3A

1 **a** EV: *colour*; RV: *toxicity*

b EV: *diet*; RV: *weight loss*

c EV: *age*; RV: *price*

d EV: *type of fuel*; RV: *cost of heating*

e EV: *post code*; RV: *house price*

2 **a** *age*

b *years of education*

c *temperature*

d *time of year*

e *age group*

f *state of residence*

Exercise 3B

1 **a** enrolment status

b No. The percentage of full-time and part-time students who drank alcohol is similar: 80.5% to 81.8%. This indicates that drinking behaviour is not related to enrolment status.

2 **a** *handedness*

b

	Sex (%)	
Handedness	*Male*	*Female*
Left	9.0	9.8
Right	91.0	90.2
Total	100.0	100.0

c No, there is little difference in the percentage of males and females who are left handed, 9.0% compared to 9.8%.

3 **a** *sex* **b** ordinal **c** 54.9%

d There are several ways you can answer the question.

Focusing on the category ‘rarely’.
Yes; the percentage of males who rarely exercised (28.8%) was significantly less that the percentage of females who rarely exercised (39.2%).
or
Yes; the percentage of males who exercised regularly (18.6%) was significantly higher than the percentage of females who exercised regularly (5.9%).
Note: For the category ‘sometimes’, there is no association between level of exercise and sex.

4 **a** 11.9% **b** 52.3% **c** marital status

d ordinal

e Yes. There are several ways that this can be seen. For example, by comparing the married and widowed groups, we can see that a smaller percentage of those widowed found life exciting (33.8%) compared to those who were married (47.6%). Or: a bigger percentage of widowed people found life pretty routine (54.3% to 48.7%) and dull (11.9% to 3.7%) compared to those who were married.

Exercise 3C

1 **a** *lifetime*: numerical; *price*: categorical

b Yes, in two ways: the median lifetime increases with price while the variability in lifetime decreases with price.

2 Yes. From the parallel dot plots we can see that median number of days spent away from home for Japanese tourists (M = 16 days) is considerably higher than for Australian tourists (M = 7 days).
From this information it can be concluded that the number of days these tourists spend away from home was associated with their country of origin.

3 Yes. From the parallel back-to-back stem plot we can see that the median age of the females (34 years) admitted to the hospital was considerably higher than the median age of males (25.5 years).
From this information it can be concluded that the median age of the people admitted to the hospital during this week was associated with their age.

Exercise 3D

1 **a** number of seats

b numerical

c 8 aircraft

d around 800 km/h

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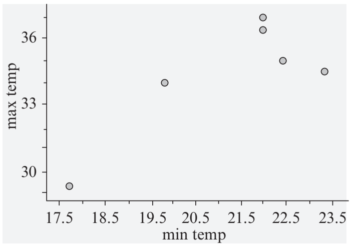
2



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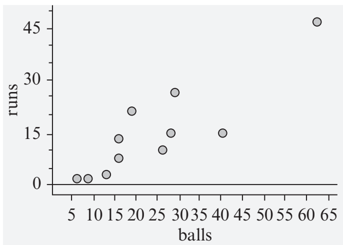

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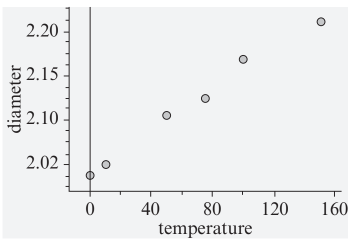
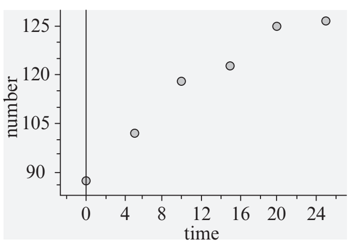


5



Exercise 3E

1 1: variables numerical; 2: association linear; 3: no clear outliers

2 Note: There are no absolute right or wrong answers to these questions as answering them requires a degree of personal judgment.

a no association **b** yes, positive
c yes, positive **d** yes, positive
e yes, negative **f** yes, negative

3 a i positive association
ii negative association
iii positive association
iv no association

b i $r \approx +0.7$; moderate positive
ii $r \approx -0.4$; weak negative
iii $r \approx +0.9$; strong positive
iv $r \approx 0$; no association

Exercise 3F

1 a A: strong, positive, non-linear relationship with no outliers
B: strong, negative, linear relationship with an outlier
C: weak, negative, linear relationship with no outliers

b A: non-linear
B: outlier

2 $r = 0.73$

3 a–d Answers given in question.

Exercise 3G

1 a 45.6% **b** 11.9% **c** 32.1% **d** 45.3%
e 1.5%

2 a $r = 0.906$ **b** $r = -0.353$

3 a The coefficient of determination is $r^2 = (-0.611)^2 = 0.373$ or 37.3%; that is, 37.3% of the variation observed in hearing test scores can be explained by variation in age.

b The coefficient of determination is $r^2 = (0.716)^2 = 0.513$ or 51.3%; that is, 51.3% of the variation observed in mortality rates can be explained by variation in smoking rates.

c The coefficient of determination is $r^2 = (-0.807)^2 = 0.651$ or 65.1%; that is, 65.1% of the variation observed in life expectancies can be explained by variation in birth rates.

d The coefficient of determination is $r^2 = (0.818)^2 = 0.669$ or 66.9%; that is, 66.9% of the variation observed in daily maximum temperature is explained by the variability in daily minimum temperatures.

e The coefficient of determination is $r^2 = (0.8782)^2 = 0.771$ or 77.1%; that is, 77.1% of the variation in the runs scored by a batsman is explained by the variability in the number of balls they face.

Exercise 3H

Note: These answers are for guidance only. Alternative explanations for the source of an association may be equally acceptable as the variables suggested.

1 Not necessarily. In general, older children are taller and have been learning mathematics longer. Therefore they tend to do better on mathematics tests. Age is the probable common cause for this association.

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2 Not necessarily. While one possible explanation is that religion is encouraging people to drink, a better explanation might be that towns with large numbers of churches also have large populations, thus explaining the larger amount of alcohol consumed. Town size is the probable common cause for this association.

3 Probably not. The amount of ice-cream consumed and the number of drownings would both be affected by weather conditions. Weather conditions are the probable common cause.

4 Maybe but not necessarily. Bigger hospitals tend to treat more people with serious illnesses and these require longer hospital stays. A common cause could be the type of patients treated at the hospital.

5 Not necessarily. Possible confounding variables include age and diet.

6 There is no logical link between eating cheese and becoming tangled in bed sheets and dying. The correlation is probably spurious and the result of coincidence.

7 Not necessarily. For example, the more serious the fire, the more fire trucks in attendance and the greater the fire damage. A possible common cause is the severity of the fire.

Exercise 3I

1 a segmented bar chart **b** scatterplot
c parallel box plots **d** scatterplot
e scatterplot **f** segmented bar chart
g segmented bar chart **h** parallel box plots

2 E

Chapter 3 review

Multiple-choice questions

1 A	**2** D	**3** B	**4** D
5 C	**6** E	**7** D	**8** E
9 C	**10** C	**11** D	**12** A
13 E	**14** B	**15** C	**16** E
17 C			

Extended-response questions

1 a *Number of accidents* and *age*; both categorical variables

b RV: *Number of accidents*; EV: *age*

c 470

d

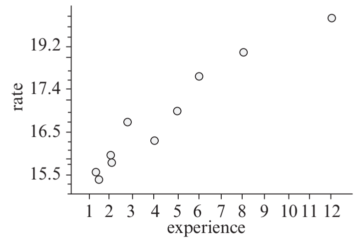
Number of accidents	*Age* < 30	*Age* ≥ 30
At most one accident	21.7%	42.5%
More than one accident	78.3%	57.5%

e The statement is correct. Of drivers aged less than 30, 78.3% had more than one accident compared to only 57.5% of drivers in the older category.

2 a Numerical: *age at marriage*. Categorical: *sex*

b Yes, the median age at marriage of men and women differs. This indicates that age at marriage is related to sex.

3 a



'Rate' is the dependent variable.

b There is a strong positive linear relationship; that is, people with more experience are generally being paid a higher starting pay rate. There are no apparent outliers. $r \approx +0.9$.

c 0.967

d Coefficient of determination = 0.934; that is, 93.4% of variation in pay rate is explained by the variation in experience.

Chapter 4

Exercise 4A

1 A residual is the difference between a data value and its value predicted by a regression line.

2 C

3 The data is numerical; the association is linear; there are no clear outliers.

Exercise 4B

1 a RV: pollution level; EV: traffic volume

b pollution level = −330 + 49 × traffic volume

2 a RV: life expectancy; EV: birth rate

b life expectancy = 110 − 1.5 × birthrate

3 a RV: distance travelled; EV: age of car

b distance travelled = 16 + 11 × age

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4 a r is also negative.
b Slope is zero: regression line is horizontal.
c intercept = $\bar{y}$ (mean of RV)

5–6 Answers given in question.

7 a Answers given in question.
b $runs = -2.6 + 0.73 \times balls$

8 a RV: number of TVs
b Answer given in question.
c $number\ of\ TVs =$ $61.2 + 0.930 \times number\ of\ cars$

Exercise 4C

1 $mark = 80 - 4.3 \times days\ absent$

2 a height **b** 0.33, 2.9 **c** 55.7
d 2.8

3 a fuel consumption **b** 0.01, −0.1
c 9.7 **d** −0.8

4 a 14.7,27.8 **b** 14.7 **c** 0.87
d 75.7, fat content
e i 145.4 **ii** −13.4

5 a −0.278: the slope predicts that success rate decreases by 27.8% for each additional metre the golfer is from the hole.
b 73.5 **c** 3.54 m **d** −0.705
e 49.7%: 49.7% of the variation in success rate in putting is explained by the variation in the distance the golfer is from the hole.

6 A: clear curved pattern in the residuals (not random)
C: clear curved pattern in the residuals (not random)

7 a yes, linear relationship
b 0.9351 or 93.5%
c 93.5%, pay rate, experience
d $pay\ rate = 8.56 + 0.289 \times experience$
e the pay rate for a worker with no experience
f On average, the pay rate increases by 29 cents per hour for each additional year of experience.
g i \$10.87 **ii** \$0.33
h yes; no clear pattern in the residual plot

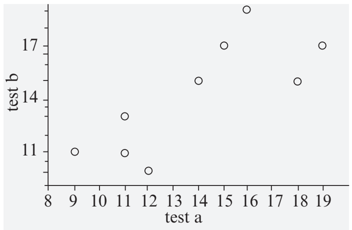
8 a $r = -0.608$
b 37% of the variation in the hearing test score is explained by the variation in age.
c $score = 4.9 - 0.043 \times age$
d decrease, 0.043
e i 4.04 **ii** −2.04
f i 0.3 **ii** −0.4
g yes; no clear pattern in the residual plot

9 negative, drug dose, −0.9492; 55.9; −9.3; decreases, 9.3; 55.9; 90.1, response time, drug dose; clear pattern

10 The scatterplot shows that there is a strong positive linear relationship between radial length and femur length: $r = 0.9879$. There are no outliers.
The equation of the least squares regression line is:
radial length = −7.2 + 0.74 × femur length
The slope of the regression line predicts that, on average, radial length increased by 0.74 cm for each centimetre increase in femur length.
The coefficient of determination indicates that 97.6% of the variation in radial lengths can be explained by the variation in femur lengths.
The residual plot shows no clear pattern, supporting the assumption that the relationship between radial and femur length is linear.

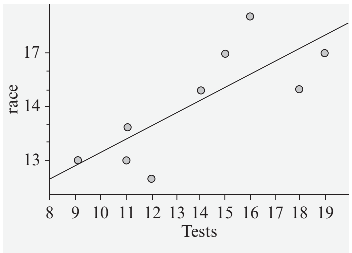
Exercise 4D

1 a

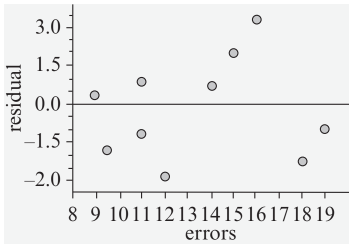


b $test\ B\ score = 4.2 + 0.72 \times$ $test\ A\ score,\ r = 0.78,\ r^2 = 0.61$

c



d



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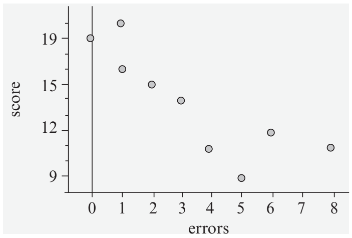
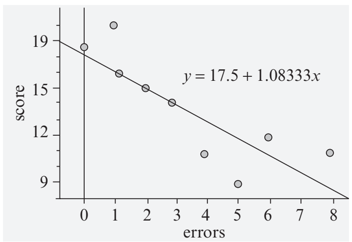
2 a

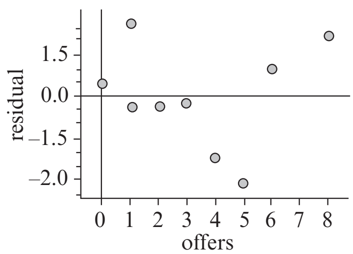


b *score* = 17.5 − 1.08 × error,
$r = -0.841$, $r^2 = 0.707$

c

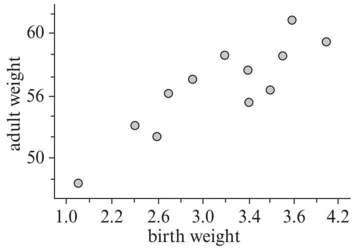


d



3 a RV: adult weight; EV: birth weight

b



c **i** strong positive linear association with no outliers

ii your judgment

d *adult weight* = 38.4 + 5.86 × *birth weight*,
$r^2 = 0.765$, $r = 0.875$

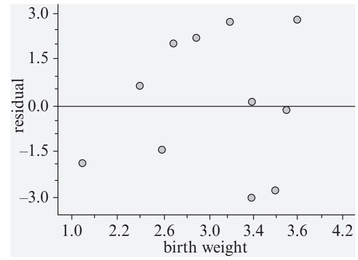
e 76.5% of the variation in the adult weight is explained by the variation in birth weight.

f On average, adult weight increases by 5.9 kg for each additional kilogram of birth weight.

g **i** 56.0 **ii** 53.1 **iii** 61.3

h Yes. 76.5% of the variation in the adult weight is explained by the variation in birth weight.

i



The lack of a clear pattern in the residual plot supports the assumption that the relationship between adult weight and birth weight is linear.

Chapter 4 review

Multiple-choice questions

1 C	**2** D	**3** A	**4** B
5 B	**6** D	**7** C	**8** A
9 A	**10** A	**11** D	**12** E
13 A	**14** C	**15** A	

Extended-response questions

1 a days of rain **b** −6.88, 2850 **c** 2024

d decrease, 6.88 **e** −0.696

f 48.4, days of rain **g** **i** 1873 **ii** −483

h interpolation

2 a *cost*

b Answer given in question.

c **i** \$182.30, interpolating

ii \$125.60, extrapolating

d **i** 81.5: the fixed costs of preparing meals is \$81.50.

ii \$2.10: The slope of the regression line predicts that, on average, meal preparation costs increase by \$2.10 for each additional meal produced.

e 0.956; 95.6% of the variation in the cost of preparing meals is explained by the variation in the number of meals produced.

3 a *male income*

b \$350

c **i** \$18 250

ii Making the prediction involves going well beyond the data used to determine the regression equation (extrapolation). We have no way of knowing whether the same relationship between male and female incomes still applies outside of this data.

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4 **a** RV: *height*; EV: *femur length*
b $height = 36.3 + 5.35 \times femur\ length$
c On average, height increases by 5.35 cm for each cm increase in femur length.
d $r^2 = 0.988$; that is, 98.8% of the variation in height is explained by the variation in femur length.

5 **a** RV: *height*; EV: *age*
b Answer given in question.
c 83 cm, extrapolation
d On average, height increases by 6.4 cm for each extra year.
e $r^2 = 0.995$; that is, 99.5% of the variation in height is explained by the variation in age.
f **i** 140.3 cm **ii** −0.7 cm
g **i** Answer given in question.
ii clear curved pattern

Chapter 5

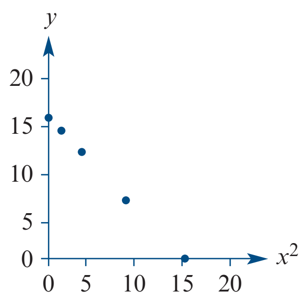
Exercise 5A

1 **a** $\log y,\ \frac{1}{y},\ \log x,\ \frac{1}{x}$
b None; trend needs to be consistently increasing or decreasing.
c $\log y,\ \frac{1}{y},\ x^2$
d x^2, y^2

Exercise 5B

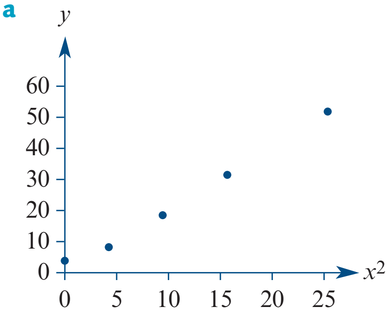
1 **a** 19.5 **b** 11.7 **c** 23.8 **d** 126.7

2 **a**



b $y = 16 - x^2$
c when $x = -2, y = 12$

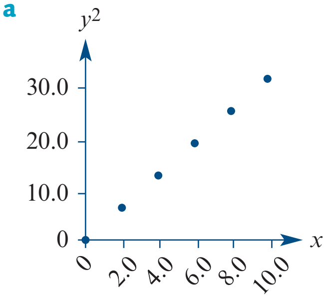
3 **a**



b $y = 1 + 2x^2$ **c** when $x = 6,\ y = 73$

4 **a** ±4.7 **b** ±1.3 **c** 6 **d** −8

5 **a**



b $y^2 = 1.5 + 3.1x$
c $y = \pm 5.4$, but only the positive solution applies here because the model is only defined for $y > 0$.

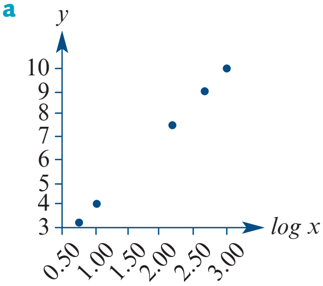
6 **a** $number\ of\ people = 4.1 \times diameter^2$
b 7

7 **a** $time^2 = 18 - 9.3 \times dose$ **b** 3.8 min

Exercise 5C

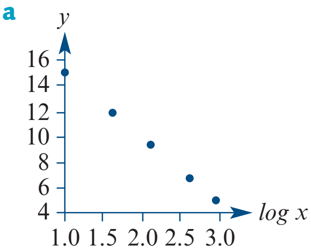
1 **a** 6.6 **b** 1.1 **c** −3.1 **d** 138.5

2 **a**



b $y = 1 + 3 \log x$ **c** 7

3 **a**



b $y = 20 - 5 \log x$ **c** 5

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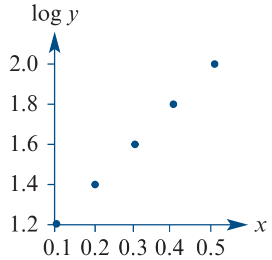
4 **a** 2 **b** 218.8 **c** 1 000 000
d 0.82

5 **a**



b $\log y = 1 + 2x$ **c** 158.5

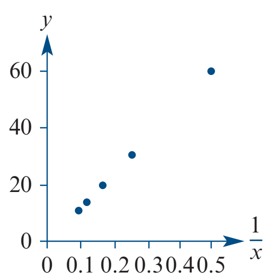
6 **a** *level* = 1.8 + 2.6 log (*time*) to 2 sig. figs
b 2.8 to 1 d.p.

7 **a** log(*number*) = 1.314 + 0.08301 × *month* to 4 sig. figs
b 135 to nearest whole number

Exercise 5D

1 **a** 13.3 **b** 2.8 **c** 4.9 **d** 309.5

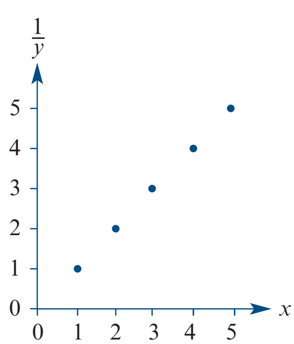
2 **a**



b $y = \dfrac{120}{x}$ **c** 24

3 **a** 0.17 **b** 0.07 **c** 0.16 **d** 0.06

4 **a**



linear

b $1/y = x$
c 4

5 **a** $horsepower = 22.1 + \dfrac{690}{consumption}$
b 99 to nearest whole number

6 **a** $\dfrac{1}{errors} = 0.050 \times times$ to 2 sig. figs
b 3 to nearest whole number

Chapter 5 review

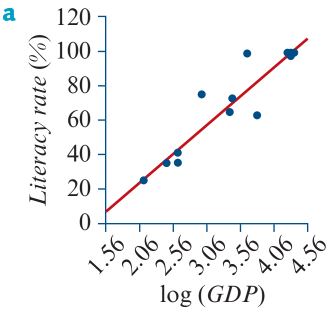
Multiple-choice questions

1 A **2** D **3** B **4** A
5 B **6** E **7** B **8** C
9 D **10** D

Extended-response questions

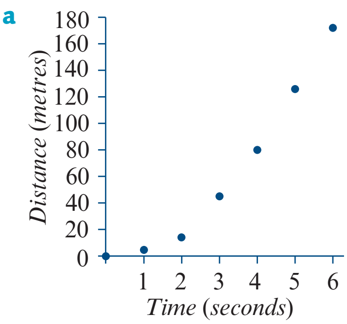
1 **a** *average age* = 2.39 + 5.89 log (*income*)
b 27.7 years

2 **a**



b *literacy rate* = −44.2 + 33.3 log (*GDP*)
c *slope* = 33.3; *intercept* = 7.75 **d** 89%

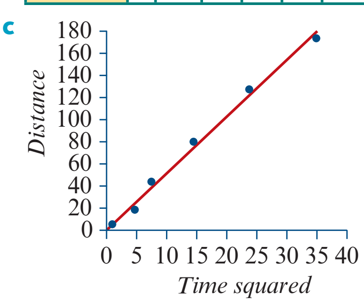
3 **a**



b

Time	0	1	2	3	4	5	6
Distance	0	5.2	18	42	79	128	168
Time^2	0	1	4	9	16	25	36

c



d *distance* = 0.45 + 4.8 × *time*^{2}
e 236 metres

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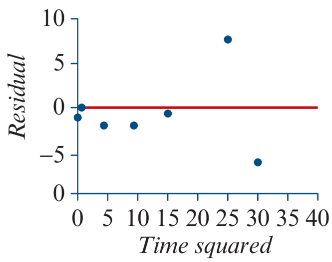
f



The residual plot is now essentially a random array of points indicating that the association between *distance* and *time*^{2} is justified.

Chapter 6

Exercise 6A

1

Feature	*Plot A*	*Plot B*	*Plot C*
Irregular fluctuations	✓	✓	✓
Increasing trend			✓
Decreasing trend	✓		
Outlier			✓

2

Feature	*Plot A*	*Plot B*	*Plot C*
Irregular fluctuations	✓	✓	✓
Increasing trend			✓
Decreasing trend	✓		
Cycles		✓	
Seasonality	✓		✓

3

Characteristic	*Plot A*	*Plot B*	*Plot C*
Irregular fluctuations	✓		✓
Structural change	✓		
Increasing trend	✓		
Decreasing trend	✓		
Seasonality			✓

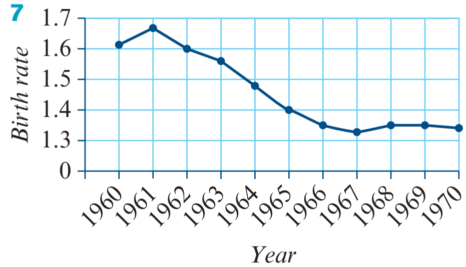
4 The demand for structural change appears to be seasonal, at its lowest in the June quarter and peaking in the March quarter each year. The graph does not show any clear trend.

5 **a** The percentage of males who smoke has consistently decreased since 1945, while the percentage of females who smoke increased from 1945 to 1975 but then decreased at a similar rate to males over the period 1975–1992.

b decrease

6 The number of whales caught increased rapidly between 1920 and 1930 but levelled off during the 1930s. In the period 1940–1945 there was a rapid decrease in the number of whales caught and numbers fell to below the 1920 catch. In the period 1945–1965 the numbers increased again but then fell again until 1985 when numbers were back to around the 1920 level.

7



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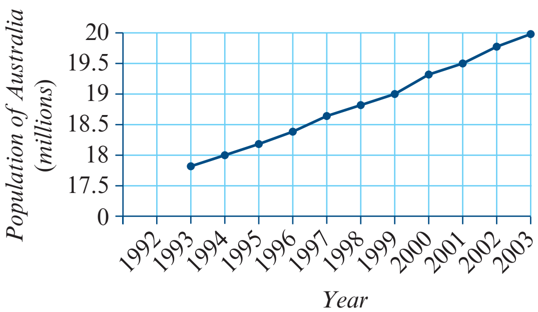

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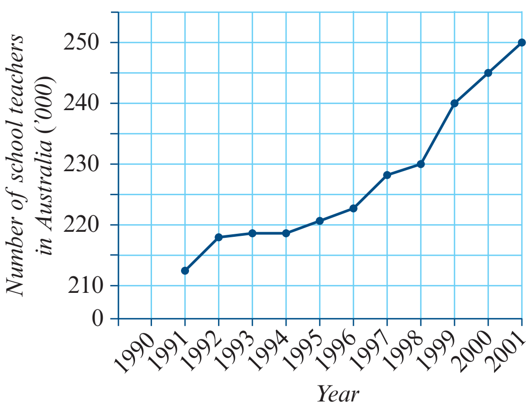

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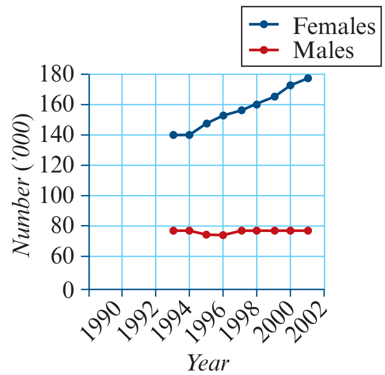
10 a



b The number of male school teachers has remained relatively constant over the years 1993–2001, whereas the number of female school teachers has increased over this time.

Exercise 6B

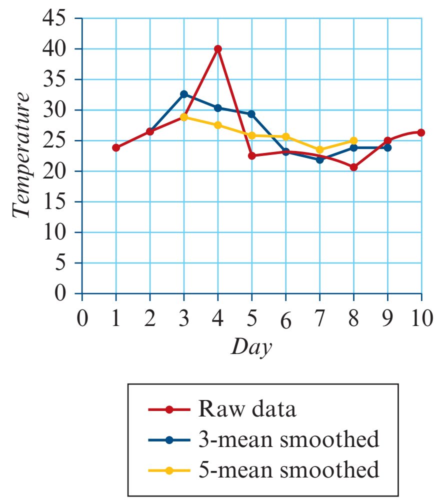
1 C **2** A **3** B **4** D **5** A

6 a 3 **b** 1 **c** 4 **d** 3.2
e 1.2 **f** 2.2 **g** 3.75 **h** 2.0
i 3.25 **j** 1.5

7

t	1	2	3	4	5	6	7	8	9
y	10	12	8	4	12	8	10	18	2
3-mean	–	10	8	8	8	10	12	10	–
5-mean	–	–	9.2	8.8	8.4	10.4	10	–	–

8 a, c

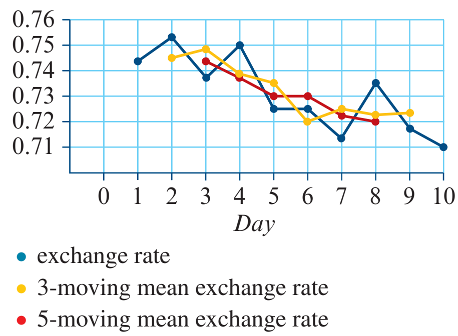


The smoothed plots show that the 'average' maximum temperature changes relatively slowly over the 10-day period (the 5-day average varies by only 5°) when compared to the daily maximum, which can vary quite widely (for example, nearly 20° between the fourth and fifth day) over the same period of time.

b

Day	*Temperature (C°)*	*3-moving mean*	*5-moving mean*
1	24	–	–
2	27	26.3	–
3	28	31.7	28.2
4	40	30.0	28.0
5	22	28.3	27.0
6	23	22.3	25.6
7	22	22.0	22.6
8	21	22.7	23.4
9	25	24.0	–
10	26	–	–

9 a, c



The exchange rate has a downward trend over the 10-day period. This is most obvious from the smoothed plots, particularly the 5-moving mean plot.

b

Day	*Exchange rate*	*3-moving mean*	*5-moving mean*
1	0.743	–	–
2	0.754	0.745	–
3	0.737	0.747	0.742
4	0.751	0.737	0.738
5	0.724	0.733	0.730
6	0.724	0.720	0.729
7	0.712	0.724	0.722
8	0.735	0.721	0.720
9	0.716	0.721	–
10	0.711	–	–

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10

Month	*Number of births*	*2-moving mean*	*2-moving mean with centring*
January	10	–	–
February	12	11	10
March	6	9	7.25
April	5	5.5	9.5
May	22	13.5	16.75
June	18	20	17.75
July	13	15.5	12.75
August	7	10	9
September	9	8	8.75
October	10	9.5	9.25
November	8	9	10.25
December	15	11.5	

11

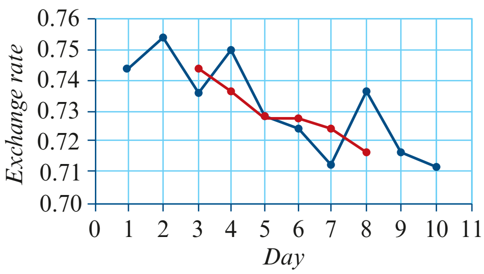
Month	*Internet usage*	*4-moving mean*	*4-moving mean with centring*
April	21	–	–
May	40	–	–
June	52	38.75	43.375
July	42	48	52.875
August	58	57.5	61.375
September	79	65	66.5
October	81	68	67
November	54	66	–
December	50		

Exercise 6C

1 **a** (3, 3) **c** (2.5, 2.5) **c** (3, 2) **d** (3, 3)

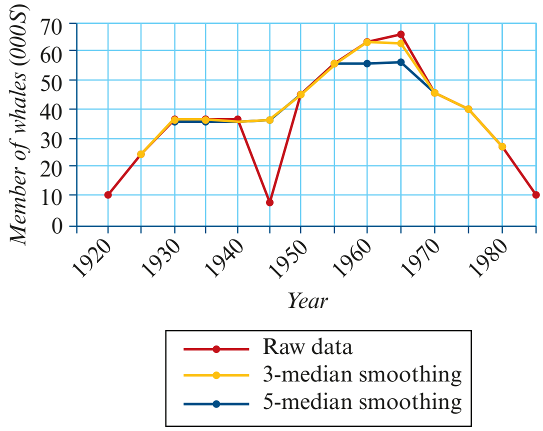
2 **a** **i** 30°C **ii** 25°C
b **i** 25°C **ii** 25°C

3

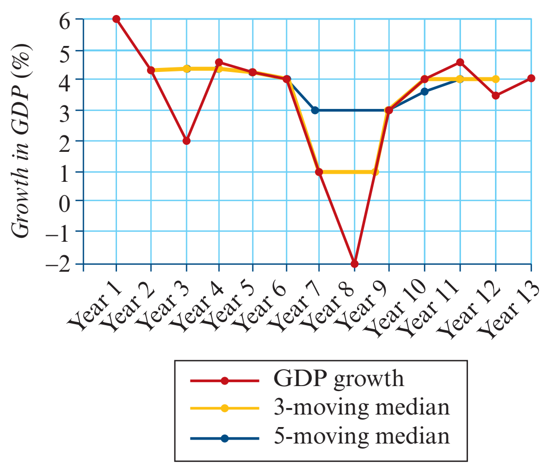


The smoothed plot shows that there was a general decreasing trend in the exchange rate over this period.

4



5 **a, b**



c The plot of the raw data, which is essentially a plot of GDP growth smoothed over 1 year, shows a great deal of variability over the whole time period. No clear trend is apparent. When smoothed over a 3-year period, GDP growth is still variable but to a lesser extent. No clear trend is apparent, but GDP appears to be going through a period of below average growth during the time period Year 7 to Year 9.
When smoothed over a 5-year period, GDP growth is much less variable but clearly shows the period of below average growth during the period Year 7 to Year 9.

Exercise 6D

1 C	**2** B	**3** E	**4** B
5 C	**6** D	**7** E	**8** C
9 D	**10** D	**11** A	**12** B

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13

Number of students:	56	125	126	96
Deseasonalised numbers:	112	125	97	80
Seasonal index:	0.5	1.0	1.3	1.2

14 a, c

Deseasonalised:	152	142	148	153
Seasonal index:	1.30	1.02	0.58	1.1

b In quarter 1 the restaurant chain employs 30% more waiters than the number employed in an average quarter.

15

Q1	*Q2*	*Q3*	*Q4*
0.89	0.83	1.12	1.16

16

Jan	*Feb*	*Mar*	*April*	*May*	*June*
0.89	0.96	1.04	1.26	1.33	1.11

July	*Aug*	*Sept*	*Oct*	*Nov*	*Dec*
0.67	0.74	0.59	0.81	1.11	1.48

17

Jan	*Feb*	*Mar*	*April*	*May*	*June*
1.1	0.95	1.25	1.15	1	0.9

July	*Aug*	*Sept*	*Oct*	*Nov*	*Dec*
1	0.75	0.7	0.55	1.15	1.5

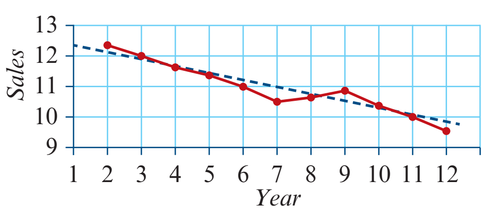
Exercise 6E

1 a There was a general increasing trend in the number of university students in Australia during the period 1992–2002.

b *students* ('000s) = 520 + 10.1 × *year*, On average, the number of university students in Australia has increased by 10 100 per year.

c 2020 is Year 29. Predicted number in 2020 = 813 000 (to nearest thousand)

2 a, d



b general decreasing trend in the percentage of retail sales made in department stores

c *sales* = 12.5 – 0.258 × *year* (to 3 sig. figs) The percentage of total retail sales that are made in department stores is decreasing by approximately 0.3% per year.

e 8.6%

3 a *age* = 27.2 + 0.199 × *year*; the average age of mothers having their first child increased by around 0.2 years each year between 1989 and 2002.

b 33.2 years; We are extrapolating 16 years beyond the period in which the data were collected.

4 a deseasonalised number = 50.9 + 1.59 × quarter number

b deseasonalised number = 76.34 reseasonalised (actual) number = 90 (to the nearest whole number)

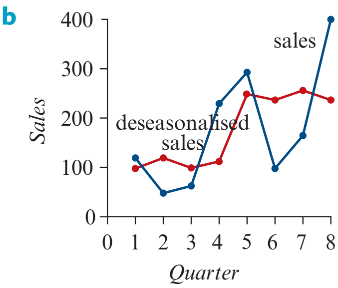
5 a *duration* = 10.4 + 0.135 × *year*, The median time to divorce increased by around 0.135 each year between 1989 and 2002.

b 14.3 years; we are extrapolating 18 years beyond the period in which data were collected.

6 a

Year	*Quarter 1*	*Quarter 2*	*Quarter 3*	*Quarter 4*
1	122	128	118	130
2	250	245	263	236

b



The deseasonalised sales appear to show an increasing trend over time.

c deseasonalised sales = 81.1 + 23.3 × *quarter*

d forecasted actual sales = 384 × 1.13 = 434

Chapter 6 review

Multiple-choice questions

1 C	**2** A	**3** B	**4** E
5 C	**6** B	**7** E	**8** C
9 B	**10** A	**11** E	**12** E
13 D	**14** C	**15** A	**16** A
17 D	**18** A		

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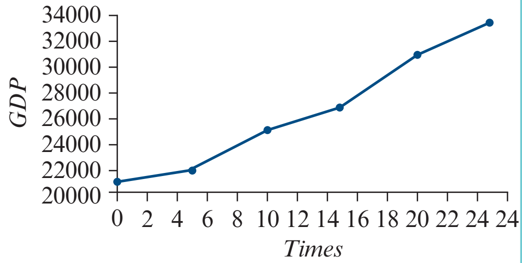
Extended-response questions

1 a



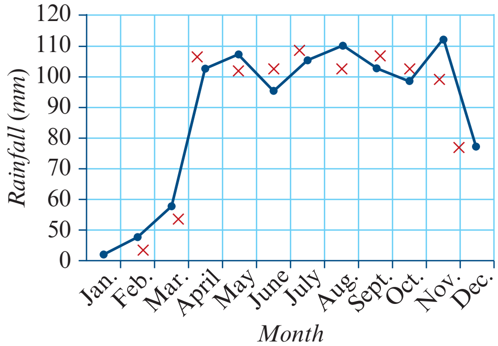
b an increasing trend

c $GDP = 20\,400 + 507 \times time$

d \$811

2 a November

b

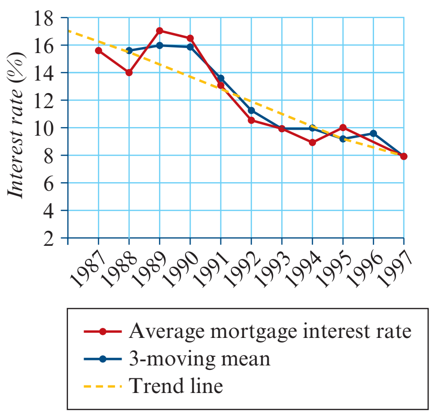


c In the smoothed time series, there are two key trends. Until April, there is an increase in monthly rainfall. It then remains relatively constant for the remainder of the year.

3 a 3-mean smoothed value for 1992

$$= \frac{13.0 + 10.5 + 9.5}{3} = \frac{33}{3} = 11\%$$

b, c



c Both the raw data and the smoothed data reveal a steadily decreasing trend.

Chapter 7

Exercise 7A

Univariate data

1 A	**2** C	**3** B	**4** B
5 B	**6** D	**7** B	**8** E
9 E	**10** D	**11** A	**12** D
13 D	**14** B	**15** B	**16** C
17 C	**18** E	**19** D	**20** C
21 B	**22** D	**23** D	

Exercise 7B

Associations

1 D	**2** E	**3** E	**4** B
5 D	**6** E	**7** E	**8** D
9 D	**10** A	**11** D	

Exercise 7C

Regression and data transformation

1 C	**2** A	**3** D	**4** B
5 A	**6** B	**7** E	**8** B
9 D	**10** A	**11** E	**12** A
13 E	**14** C	**15** D	

Exercise 7D

Time series

1 A	**2** A	**3** A	**4** B
5 E	**6** B	**7** E	**8** B
9 D	**10** E	**11** E	**12** B
13 E	**14** D	**15** B	**16** D
17 A			

Exercise 7E

Extended-response questions

1 a i 10, 9, 9 **ii** 32.1%

b nominal

Note: If the object was to measure the level of exercise, it could be classified as an ordinal variable.

2 Note: This question can be answered by focusing on any *one* category that shows a change across the year levels. We will focus on the category 'ran'.

a Yes; the percentage of students who ran decreased from 78% in Year 6, to 40% in Year 8, to 8% in Year 10.

b ordinal

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3 **a** 78, 9 **b** in a box plot
Lower fence = 75 − 1.5 × 3 = 70.5
Because 70 < 70.5, a development index of 70 is an outlier.

4 **a** **i** 25.0 years **ii** M = 28.2 years
b $IQR = Q_3 - Q_1 = 30.9 - 29.9 = 1.0$
c In a box plot, data values below the lower fence $Q_1 - 1.5 \times IQR$ are classified as outliers.
Here, the lower fence = $Q_1 - 1.5 \times IQR$ = 29.9 − 1.5 × 1.0 = 28.4.
As 26.0 < 28.4, 26 is an outlier.

5 **a** 65.5%
b Yes; for example, the percentage of women who were first married when 30 years and over increased from 26.0% in 1986 to 33.3% in 1996 to 42.0% in 2006.
c year of marriage

6 **a** 124, 148
b E.g. arm span and year level are associated because median arm span increases with year level.
c In a box plot, data values below the lower fence are classified as outliers.
Here, the lower fence = $Q_1 - 1.5 \times IQR$ = 160 − 1.5 × 10 = 145.
As 140 < 145, 140 is an outlier.

7 **a** A correctly drawn graph will pass through the points (0, 13) and (20, 26.4).
b The intercept predicts that, when the minimum is 0°C, the maximum temperature will be 13°C (or equivalent).
c There is moderate, increasing linear association between maximum and minimum temperatures.
d The slope predicts that, on average, the maximum temperature increases by 0.67°C for each 1°C increase in minimum temperature.
e 40%
f residual = −8°C

8 **a** $population = 7.7 + 7.7 \times \log_{10}\ (area)$
b 23 or 23000

9 **a** 1964
b **i** 13.77°C **ii** −0.09°C
iii 0.013°C/year
c increasing trend with a possible outlier, the year 1998

Chapter 8

Exercise 8A

1 **a** 2, 8, 14, 20, 26, ...
b 5, 2, −1, −4, −7, ...
c 1, 4, 16, 64, 256, ...
d 10, 5, 2.5, 1.25, 0.625, ...
e 6, 14, 30, 62, 126, ...
f 12, 9, 7.5, 6.75, 6.375, ...

2 **a** 4, 6, 8, 10, 12, ...
b 24, 20, 16, 12, 8, ...
c 2, 6, 18, 54, 162, ...
d 50, 10, 2, 0.4, 0.08, ...
e 5, 13, 29, 61, 125, ...
f 18, 16.4, 15.12, 14.096, 13.2768, ...

Exercise 8B

1 **a** 2, 5, 8, 11, 14, ...
b 50, 45, 40, 35, 30, ...
c 1, 3, 9, 27, 81, ...
d 3, -6, 12, -24, 48, ...
e 5, 9, 17, 33, 65, ... **f** 2, 7, 17, 37, 77, ...
g −2, −1, 2, 11, 38, ...
h −10, 35, −100, 305, −910, ...

2 **a** 12, 57, 327, 1947, 11667, ...
b 20, 85, 280, 865, 2620, ...
c 2, 11, 47, 191, 767, ...
d 64, 15, 2.75, −0.3125, −1.078125, ...
e 48 000, 45 000, 42 000, 39 000, 36 000, ...
f 25 000, 21 950, 19 205, 16 734.5, 14 511.05, ...

3 6

4 7

Exercise 8C

1 **a** \$2076, \$2152, \$2228 **b** 14
c $V_0 = 1500,\ V_{n+1} = V_n + 90$

2 **a** \$7518, \$8036, \$8554 **b** 6
c $V_0 = 12000,\ V_{n+1} = V_n + 984$

3 **a** **i** \$15 000 **ii** \$525 **iii** 3.5%
b 29

4 **a** \$2100, \$1700, \$1300 **b** 4
c $V_0 = 1800,\ V_{n+1} = V_n - 350$

5 **a** \$22 195, \$21 390, \$20 585 **b** 17
c $V_0 = 37\,000,\ V_{n+1} = V_n - 700$
d $V_0 = 1200,\ V_{n+1} = V_n - 540$

6 **a** **i** \$1500 **ii** \$102 **iii** 6.8%
b \$684

7 **a** \$449 **b** $V_0 = 300,\ V_{n+1} = V_n - 0.08$

8 **a** \$47 800, \$47 600, \$47 400 **b** \$45 000
c 25 000

ISBN 978-1-107-56757-3

Exercise 8D

1 a $A_n = 4 + 2n$, 44 b $A_n = 10 + 3n$, −50
c $A_n = 5 + 8n$, 165
d $A_n = 300 - 18n$, −60

2 a $8000 b $512
c i $14 144 ii 16 years d $15 680

3 a $2000 b $70
c i $2420 ii 29 years d $2700

4 a $270
b i $V_0 = 5000$, $V_{n+1} = V_n + 270$
ii $V_n = 5000 + 270n$
c $7430

5 a $864
b i $V_0 = 12\,000$, $V_{n+1} = V_n + 864$
ii $V_n = 12\,000 + 864n$
c $19 776

6 a $1700 b $212.50 c $850 d $0

7 a $65 000 b $3250 c 5% d $42 250
e 11 years

8 a $1260
b i $V_0 = 5600$, $V_{n+1} = V_n - 1260$
ii $V_n = 5600 - 1260n$
c i $1820 ii 5 years

9 a $1225
b i $V_0 = 7000$, $V_{n+1} = V_n - 1225$
ii $V_n = 7000 - 1225n$
c i $4550 ii 5 years

10 a $29 000 b $0.25 (25 cents) c $24 000
d 96 000 km

11 a $9700 b $0.388c
c $V_n = 25\,700 - 0.388n$
d 74 000 km e 91 237 km

12 a i $0.026875 ii $69 687.50
iii $20 156.25
b $9218.75
c 1 488 373

Exercise 8E

1 a $6252, $6514.58, $6788.20 b 7 years
c $V_0 = 5000$, $V_{n+1} = 1.068V_n$

2 a $21 260, $22 599.38, $24 023.14
b 7 years c $V_0 = 18\,000$, $V_{n+1} = 1.094V_n$

3 a $V_0 = 7600$, $V_{n+1} = 1.005V_n$
b $7791.91

4 a $V_0 = 3500$, $V_{n+1} = 1.02V_n$
b $3788.51

5 a $V_0 = 9800$, $V_{n+1} = 0.965V_n$
b $9800, $9457, $9126, $8806.59, $8498.36
c $8200.92 d $319.41

6 a $V_0 = 18\,000$, $V_{n+1} = 0.955V_n$
b $18 000, $17 190, $164 16.45, $156 77.71, $14 972.21
c $15 677.71 d $3701.54

Exercise 8F

1 a $V_n = 2^n \times 6$ b $V_n = 3^n \times 10$
c $V_n = 0.5^n \times 1$ d $V_n = 0.25^n \times 80$

2 a i 3000 ii 10%
b $V_n = 1.1^n \times 3000$ c $4831.53
d the value of the investment after 5 years

3 a i $2000 ii 6%
b $V_n = 1.06^n \times 2000$ c $2524.95
d $837.04

4 a $V_n = 1.125^n \times 8000$ b $11 390.63
c $3390.63 d $1265.63

5 a $V_n = 1.005^n \times 3300$ b $3468.76
c $168.76

6 a i $1200 ii 12%
b $V_n = 0.88^n \times 1200$ c $490.41

7 a $V_n = 0.905^n \times 38\,500$
b $23 372.42 c $15 127.58

8 6 years 9 100 years 10 4
11 6% 12 $9223.51 13 $32 397.17

Exercise 8G

1 a 0.413% b 2.08% c 0.238% d 0.142%
e 0.0357%

2 a 6.48% b 5.8% c 14.82% d 9.88%
e 8.03%

3 a 6.38% b 8.76% c 4.91% d 13.10%
e 7.64%

4 a More frequent compounds earn more interest.
b 4.68% c 4.70%
d The higher effective interest rate occurs with more frequent compounds.

5 a Less frequent compounds mean less interest is paid.
b 8.25% c 8.24%
d The lowest effective interest rate occurs with less frequent compounds.

6 a A – 8.62%, B – 8.11%
b A – $3017, B – $2838.50
c B – this loan will be charged less interest

7 a A – 5.43%, B – 5.61%
b A – $7602, B – $7854
c B – this investment will earn more interest

ISBN 978-1-107-56757-3

Chapter 8 review

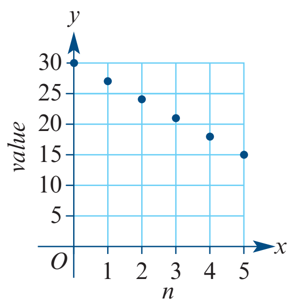
Multiple-choice questions

1 C	**2** E	**3** D	**4** C
5 A	**6** D	**7** A	**8** A
9 B	**10** D	**11** A	**12** C
13 C	**14** B	**15** A	**16** B
17 D	**18** D		

Extended-response questions

1 a 800, 630, 494, 385.2, 298.16, 228.528
b 13

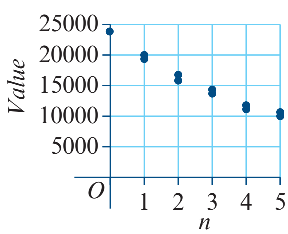
2 a $A_0 = 30,\ A_{n+1} = A_n - 3$
b



c 10

3 a $V_0 = 20\,000,\ V_{n+1} = V_n + 1880$
b \$29 400 **c** 0.0018 **d** 17

4 a i $V_0 = 22\,500,\ V_{n+1} = V_n - 2700$
ii \$9000
b i $V_0 = 22\,500,\ V_{n+1} = 0.84V_n$
ii \$9409.77
c



5 a \$0.20
b Let V_n be the value of the vacuum cleaner after cleaning n offices.
$V_0 = 650,\ V_{n+1} = V_n - 0.20$
c \$250

6 a \$6575 **b** \$6722.75 **c** 6.9%

7 a $V_0 = 30\,000,\ V_{n+1} = (1 + 6.8/12/100) \times V_n$
b 30 000, 30 170, 30 340.96, 30 512.90, 30 685.80, . . .
c \$32 117.57 **d** \$33 231.76

8 \$328.03 **9** \$250 **10** 3.5

Chapter 9

Exercise 9A

1 a 50, 90, 170, 330, 650
b 120, 92, 78, 71,67.5
c 1000, 1000, 1000, 1000, 1000

2 a 144652 L **b** 21 days

3 a 3000 **b** decaying **c** 2% growth
d 200 **e** 2271

Exercise 9B

1 a \$1117.03 **b** no – \$626.83 cents

2 a \$14 187.30 **b** \$862.96

3 a $V_0 = 2000,\ V_{n+1} = 1.005V_n - 339$
b \$674.10 **c** \$1.16

4 a $B_0 = 10\,000,\ B_{n+1} = 1.03V_n - 2690.27$
b \$5147.75
c Yes – balance rounded to the nearest cent is zero.

5 a $V_0 = 3500,\ V_{n+1} = 1.004V_n - 280$
b $V_0 = 150\,000,\ V_{n+1} = 1.0014V_n - 650$

6 a 1% **b** \$16.75 **c** \$334.85 **d** 342.17
e \$2070.60
f \$70.60 or \$70.59 depending on method used. Difference due to accumulation of rounding errors.

7 a i \$4000 **ii** \$557.85 **iii** \$100
iv \$469.30 **v** \$2591.83
vi \$558.02
b i 2.5% **ii** 10%
c A = \$64.80, B = \$505.38, C = \$1075.37, D = \$26.89, E = \$530.96

Exercise 9C

1 a \$6061.91 **b** \$12 095.13
c \$16 8519.40 **d** \$45 196.78
e \$33 735.99

2 a \$617.80 **b** \$413.38 **c** \$506.64
d \$4175.11 **e** \$687.65

3 a \$857.09 **b** \$308 552
c \$218 552.40

4 a \$1 977 933.85
b i \$2902.96 **ii** \$418 026.24
iii \$178 026.24

5 a 55 **b** \$4750

6 a 43 **b** \$1100

7 a \$349.43 **b** \$48 863.20
c \$32 437.73 **d** \$418.65
e \$13 290.24

ISBN 978-1-107-56757-3

8 **a** \$132 119.82 **b** \$1196.29

9 **a** \$229 994.24 **b** \$268 245.76
c 169 798.80 **d** \$111 567.70
e \$1487.29

Exercise 9D

1 \$595.83 **2** \$339.58 **3** \$32 775

Exercise 9E

1 **a** \$2030.50 **b** no; \$1.02

2 **a** \$3016.56 **b** no; \$3.21

3 **a** $V_0 = 40\,000$, $V_{n+1} = 1.015V_n - 10380$
b \$20 293.30

4 **a** 0.25% **b** \$15.00 **c** \$495.473
d \$3522.64
e A = 8.81, B = 499.19, B = 3023.01
f \$509.97 **g** \$6097.98 **h** \$97.98

5 58 months

6 21 quarters

7 \$474.81

8 **a** \$692.50 **b** \$73 213.07
c 153 months **d** \$684.73

9 **a** \$278 394.49 **b** 6.57%

Exercise 9F

1 **a** \$100 000 **b** 3.125%

2 **a** \$18 519 **b** 4.2%

3 \$200 000 **4** \$4791.68

5 **a** \$9790.50 **b** \$642 000
c \$642 000

Exercise 9G

1 **a** \$8805.26 **b** \$2000 **c** \$1000
d 8%

2 **a** \$27 689.06 **b** \$20 000 **c** \$2000
d 2.5% quarterly, 10% annually

3 $V_0 = 1500$, $V_{n+1} = 1.0075V_n + 40$

4 **a** **i** \$5000 **ii** \$100 **iii** \$50
iv 1% **v** 12%
vi \$151.50 **vii** \$6242.86
b **i** $A = 53.02$, $B = 153.02$, $C = 5454.52$
ii \$442.86

5 \$15 136.46

6 \$9 months

7 **a** \$26 253.41 **b** \$25 041.04

8 **a** \$5114.01 **b** \$261.53

Chapter 9 review

Multiple-choice questions

1 C	**2** D	**3** C	**4** B
5 D	**6** A	**7** D	**8** D
9 A	**10** C	**11** C	**12** E
13 C	**14** B		

Extended-response questions

1 **a** $F_0 = 840$, $F_{n+1} = 1.06\,F_n - 80$, $n \geq 0$
b 840, 810.4, 779.024, 745.77, 710.51
c 18

2 **a** $V_0 = 250\,000$
$V_{n+1} = (1 + 4.9/12/100)\,V_n - 1800$
b \$240 437.13 **c** 58
d **i** \$1020.83 **ii** \$12 250
iii \$250 000

3 **a** \$781.25 **b** \$147 298.48
c 38 months
d **i** 43 months **ii** \$3323.07

4 **a** \$656.72 **b** \$13 134.35
c \$3134.35

5 **a** \$2580.21 **b** \$2184.63
c \$2897.92 **d** \$2255.11

6 **a** 40 **b** \$317.92
c Yes, paying monthly will reduce the balance more often so less interest is paid.

7 **a** \$247 **b** \$83 713

8 **a** \$1 326 442.66 **b** 355 months
c \$3868.79

9 **a** **i** \$1827.32 **ii** \$188.557
b \$213 118 **c** 212

Chapter 10

Exercise 10A

1 B	**2** D	**3** C	**4** C
5 B	**6** B	**7** C	**8** D
9 E	**10** A	**11** A	**12** C
13 C	**14** A	**15** B	**16** B
17 B	**18** D	**19** D	

Exercise 10B

1 **a** \$8000 **b** \$1500
c **i** Let V_n be the value of the bike after n years.
$V_0 = 8000$, $V_{n+1} = V_n - 1500$, $n \geq 0$
ii \$500
d 12 000

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e **i** $V_0 = 5000, V_{n+1} = 1.004\ V_n + 200$
ii 22 months **iii** \$298.86

2 **a** 1.6%
b **i** Let V_n be the value of the loan after n quarters. $V_0 = 50\,000$
$V_{n+1} = 1.016\ V_n - 4800$
ii 3 years

3 **a** Let V_n be the value of Leanne's loan after n months.
$V_0 = 138\ 500$
$V_{n+1} = 1.0035V_n - 1200$
b 138 500, 137 784.75, 137 067, 136 364.73, 135 623.94, 134 898.63, 134 170.77
c $V_0 = 134\ 170.77$
$V_{n+1} = 1.003625\ V_n - 1200$
d \$129 849.99 **e** \$117 249.21

4 **a** **i** \$30 000 **ii** \$16 875
b \$24 000 **c** 3 years **d** 20%

5 **a** **i** \$493.75
ii No. After 4 years (16 quarters) Lucy still owes \$4420.27.
b **i** 21 quarters **ii** 3 quarters

6 **a** \$948.50 **b** \$439 829.26 **c** \$1557.73

7 **a** \$1 188 776.85 **b** 238 months
c \$5696.22

8 **a** \$96 355.63 **b** \$74 345.55 **c** \$8250
d \$121 212

Chapter 11

Exercise 11A

1 15

2 $1 \times 12,\ 12 \times 1,\ 6 \times 2,\ 2 \times 6,\ 4 \times 3,\ 3 \times 4$

3 **a** $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 5 \end{bmatrix}$ **c** $\begin{bmatrix} 9 & 8 \\ 1 & 9 \\ 0 & 1 \\ 7 & 5 \end{bmatrix}$

4 **a** C, E **b** 3 **c** A **d** B
e 4, 2 **f** 3, 3 **g** 1, 5 **h** 3, 1
i 4, 2 **j** 9 **k** 5 **l** 0
m 1 **n** 0 **o** 4 **p** −1
q 3 **r** 3 **s** 1

5 **a** $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ **b** none
c $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ **d** all but $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

6 $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$ **7** $\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$

8 **a** $\begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & -2 \\ -4 & 6 \end{bmatrix}$
c $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 3 \end{bmatrix}$

Exercise 11B

1 **a** $\begin{bmatrix} 4 & 2 & 1 \\ 6 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$, 3×3
b $\begin{bmatrix} 6 & 2 & 3 \end{bmatrix}$, 1×3
c $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, 3×1; the total number of computers owned by the three households

2 **a** $\begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix}$, 2×3;
b $\begin{bmatrix} 24 & 32 & 11 \end{bmatrix}$, 1×3
c $\begin{bmatrix} 24 \\ 32 \end{bmatrix}$, 2×1; the total number of small cars sold by the car dealers

3 $\begin{bmatrix} 3 & 5 & 8 & 7 & 0 & 2 & 3 & 6 \\ 4 & 2 & 2 & 9 & 0 & 0 & 0 & 9 \end{bmatrix}$

4 **a** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$
c $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

5 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

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6 a Girls 3 and 4 are friends.

b Girls 2 and 5 are not friends.

c 3: girl 3 has three friends.

d girl 1, girl 3

7 a i Birds eat lizards.

ii Nothing eats birds.

b
$$\begin{array}{c} \\ z = \end{array} \begin{array}{c} \begin{array}{cccc} I & B & L & F \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ I \\ B \\ L \\ F \end{array}$$

Exercise 11C

1 a C, F

b A and B, C and F, D and E

c A and B, C and F, D and E

d i $\begin{bmatrix} 4 & 4 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

iii $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **iv** $\begin{bmatrix} -2 & 2 \end{bmatrix}$

v $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$ **vi** $\begin{bmatrix} 9 & 3 \end{bmatrix}$

vii $\begin{bmatrix} 0 & 4 & 16 \\ 12 & 8 & 4 \end{bmatrix}$ **viii** $\begin{bmatrix} 0 & 4 & 16 \\ 12 & 8 & 4 \end{bmatrix}$

ix $\begin{bmatrix} -2 & 10 \end{bmatrix}$ **x** not defined

2 a $\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 9 & 8 \\ 6 & 7 \end{bmatrix}$

d $\begin{bmatrix} 0 & 0 \end{bmatrix}$ **e** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **f** $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

g not defined **h** $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

i not defined

3 a $\begin{bmatrix} -2.2 & 1.1 \\ 7.7 & 4.4 \end{bmatrix}$ **b** $\begin{bmatrix} -0.2 & -13.8 \\ 1 & -3.7 \end{bmatrix}$

c $\begin{bmatrix} 5 & 8 & 7 \\ 16 & 0 & 3 \\ -1 & 5 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} 0.6 & 2 & 1 & 3.2 \\ 1 & 0 & -0.6 & 2 \end{bmatrix}$

4 a $A = \begin{bmatrix} 2.4 \\ 3.5 \\ 1.6 \end{bmatrix}$ $B = \begin{bmatrix} 2.8 \\ 3.4 \\ 1.8 \end{bmatrix}$ $C = \begin{bmatrix} 2.5 \\ 2.6 \\ 1.7 \end{bmatrix}$ $D = \begin{bmatrix} 3.4 \\ 4.1 \\ 2.1 \end{bmatrix}$

b $\begin{bmatrix} 11.1 \\ 13.6 \\ 7.2 \end{bmatrix}$; the total (yearly) DVD sales for each store

5 a $A = \begin{bmatrix} 16 & 104 & 86 \\ 75 & 34 & 94 \end{bmatrix}$ $B = \begin{bmatrix} 24 & 124 & 100 \\ 70 & 41 & 96 \end{bmatrix}$

b $C = \begin{bmatrix} 40 & 228 & 186 \\ 145 & 75 & 190 \end{bmatrix}$; the total number of females and males enrolled in each of the three programs for the two years

c $D = \begin{bmatrix} 8 & 20 & 14 \\ -5 & 7 & 2 \end{bmatrix}$; the increase in the number of females and males enrolled in each of the three programs for the two years; a decrease in the number of men enrolled in weights classes

d $E = \begin{bmatrix} 48 & 248 & 200 \\ 140 & 82 & 192 \end{bmatrix}$

Exercise 11D

1 a i, ii, iv, v, vi, vii

b i [6] **ii** [2] **iii** $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **iv** $\begin{bmatrix} -3 & 7 \end{bmatrix}$

c i [6] **ii** $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ **iii** [0]

iv undefined

2 a [0] **b** [1] **c** [3] **d** $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

e $\begin{bmatrix} 5 & 5 \\ 1 & 2 \end{bmatrix}$ **f** $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$

3 a $\begin{bmatrix} 1 & 2 & 3 \\ -3 & -6 & -9 \\ 5 & 10 & 15 \end{bmatrix}$ **b** $\begin{bmatrix} 6 & 7 \\ 19 & 18 \end{bmatrix}$

c $\begin{bmatrix} 4 & 0 \\ 2 & -2 \\ 1 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} -5 & 15 & 9 \\ -2 & 8 & 4 \\ -5 & 8 & 7 \end{bmatrix}$

4 a $\begin{bmatrix} 22 \\ 12 \\ 18 \\ 12 \\ 30 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 10 & 6 & 14 & 11 \end{bmatrix}$

5 $\begin{bmatrix} 9 \\ 7 \\ 6 \\ 4 \\ 3 \\ 1 \end{bmatrix}$ **6** $\begin{bmatrix} 3000 \\ 2800 \\ 2600 \\ 2200 \end{bmatrix}$

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7 a 2×3

b i $\begin{bmatrix} 145978.00 \\ 171848.5 \end{bmatrix}$

ii the total revenue from selling products A, B and C at Eastown and Noxland respectively

c number of columns in $P \neq$ number of rows in Q

8 a $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

9

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 2 \\ 1 & 7 & 3 \\ 8 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 10 & 9 \end{bmatrix}$$

10, 11 Teacher to check.

Exercise 11E

1 $\begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}, \begin{bmatrix} 15 & 20 \\ 20 & 35 \end{bmatrix}, \begin{bmatrix} 50 & 75 \\ 75 & 125 \end{bmatrix}, \begin{bmatrix} 2250 & 3625 \\ 3625 & 5875 \end{bmatrix}$

2 $\begin{bmatrix} 5 & 7 \\ 7 & 26 \end{bmatrix}, \begin{bmatrix} 2 & 19 \\ 19 & 59 \end{bmatrix}, \begin{bmatrix} 17 & 40 \\ 40 & 137 \end{bmatrix}, \begin{bmatrix} 23 & 97 \\ 97 & 314 \end{bmatrix}$

3 a $\begin{bmatrix} -1 & 5 \\ 5 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} -3 & 8 \\ 6 & -3 \end{bmatrix}$

c $\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ **d** $\begin{bmatrix} 16 & 2 \\ 2 & 39 \end{bmatrix}$

e $\begin{bmatrix} 29 & -5 \\ -5 & 13 \end{bmatrix}$

Exercise 11F

1 B only

2 a $X = \begin{bmatrix} \text{H} & \text{U} & \text{T} & \text{S} \end{bmatrix}$ **b** $n = 4$

3 a $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **b** $C^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

c 2

4 a There is no direct communication link between the towers.

b $T1$ and $T3$

c 1, 0

d There is a 2-step communication link between $T3$ and $T1$.

e 6

f

$$T = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc} T1 & T2 & T3 & T4 \end{array}$$

$$T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{array}{c} T1 \\ T2 \\ T3 \\ T4 \end{array}$$

g $T1$ and $T4$

5

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

6 a

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

D, A, B, C

b

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

D, A, B, C

7 a

$$\begin{array}{ccc} & & \text{losers} \\ & & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \text{winners} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$A : 4,\ B : 1,\ C : 0,\ D : 2,\ E : 2$

Rank A, D and E equal; B, C tie for second place.

b

$$D^2 = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

c

$$T = \begin{bmatrix} 0 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{bmatrix} \begin{array}{c} 9 \\ 1 \\ 0 \\ 3 \\ 5 \end{array}$$

The tie can be broken using two-step dominances to give the ranking $A,\ E,\ D,\ B,\ C$.

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Chapter 11 review

Multiple-choice questions

1 C	**2** D	**3** B	**4** D
5 D	**6** A	**7** B	**8** C
9 D	**10** E	**11** A	**12** D
13 E	**14** D	**15** C	**16** B
17 E			

Extended-response questions

1 a $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2 a 2×1 **b** 1×2

c Yes; number of columns in C equals the number of rows in J.

d $[162.41]$; $5 \times 30.45 + 4 \times 2.54 = 162.41$

e $\begin{bmatrix} 172.57 \\ 185.24 \end{bmatrix}$

3 a 456 **b** 2×2

c $B = \begin{bmatrix} 354 & 987 \\ 314 & 586 \end{bmatrix}$

d $C = \begin{bmatrix} 688 & 1863 \\ 527 & 1042 \end{bmatrix}$; the total number of books of each type in the two stores

e i 2×1

ii $\begin{bmatrix} 31\,236 \\ 18\,021 \end{bmatrix}$

iii total value of fiction and non-fiction books at bookshop 1

f $2A = \begin{bmatrix} 668 & 1752 \\ 426 & 912 \end{bmatrix}$

4 a 1×5

b i $M = \begin{bmatrix} 23 & 57.5 & 80.5 & 207 & 92 \\ 18 & 45 & 63 & 162 & 72 \end{bmatrix}$

ii the number of students expected to get a D in Chemistry

c i $F = \begin{array}{c} \begin{array}{cc} B & C \end{array} \\ \begin{bmatrix} 110 & 95 \end{bmatrix} \end{array} \text{ s}$

ii $FN = \begin{bmatrix} 110 & 95 \end{bmatrix} \begin{bmatrix} 460 \\ 360 \end{bmatrix} = [84800]$

5 The total fees paid are \$84 800.

a $N = \begin{bmatrix} 4 & 8 & 2 \end{bmatrix}$ **b** $NG = [26]$

c total number of points scored by Oscar

Chapter 12

Exercise 12A

1 a i $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b $AI = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$;

$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$\therefore AI = IA = A$

c $CI = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$;

$IC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$CI = IC = C$

2 Teacher to check.

3 a 3 **b** -3 **c** 0 **d** -8

4 a $\begin{bmatrix} \frac{10}{11} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

b $\begin{bmatrix} \frac{20}{9} & \frac{1}{18} \\ \frac{-50}{9} & \frac{1}{9} \end{bmatrix}$

c Matrix has no inverse, det $(D) = 0$; calculator indicates this with the message: ERR: SINGULAR MATRIX (TI) or Undefined (Casio)

d $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise 12B

1 a $\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

c $\begin{bmatrix} 5 & 0 & -2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

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Cambridge University Press

2 Equations are either inconsistent or dependent.

3 $\det(A) = 0$ **4** b

5 **a** $x = 5,\ y = -3$ **b** no solution
c $x = -3,\ y = -5$

6 **a** $x = 33,\ y = 18$ **b** $x = -5, y = 3$
c no solution
d $x = -2, y = 2.6,\ z = 0.14$ (to 2 sig. figs)

7 B **8** D **9** A

10 $x = 25, y = 50$

11 $x = 50, y = 20, z = 10$

Exercise 12C–1

1 **a**
$$\begin{array}{cc} & \begin{array}{cc} A & \quad B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.40 & 0.55 \\ 0.60 & 0.45 \end{bmatrix} \end{array}$$

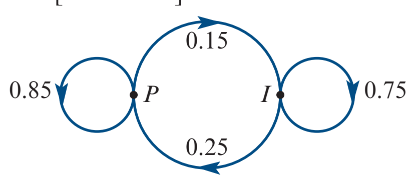
b
$$\begin{array}{cc} & \begin{array}{cc} X & \quad Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{bmatrix} 0.70 & 0.25 \\ 0.30 & 0.75 \end{bmatrix} \end{array}$$

c
$$\begin{array}{cc} & \begin{array}{ccc} X & \quad Y & \quad Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.6 & 0.15 & 0.22 \\ 0.1 & 0.7 & 0.23 \\ 0.3 & 0.15 & 0.55 \end{bmatrix} \end{array}$$

d
$$\begin{array}{cc} & \begin{array}{ccc} A & \quad B & \quad C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.45 & 0.35 & 0.15 \\ 0.25 & 0.45 & 0.20 \\ 0.30 & 0.20 & 0.65 \end{bmatrix} \end{array}$$

Exercise 12C–2

1 **a**
$$T = \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix}$$



b $0.85 \times 80 = 68$ **c** $0.25 \times 60 = 15$
d $0.15 \times 120 + 0.75 \times 40 = 48$

2 **a** **i** 10% **ii** 80% **iii** 10%
b **i** 680 **ii** 85
c **i** 1150 **ii** 0 **iii** 0
d All (100%) of the sea birds who nest at site A this year will nest at site A next year.

3 **a** **i** 76440 **ii** 7560
b **i** 5500 **ii** 1210 **iii** 266

4 B

5 E

Exercise 12C–3

1 **a** **i** $S_1 = \begin{bmatrix} 130 \\ 170 \end{bmatrix}$ **ii** $S_2 = \begin{bmatrix} 151 \\ 149 \end{bmatrix}$
iii $S_3 = \begin{bmatrix} 165.7 \\ 134.3 \end{bmatrix}$

b $T^5 = \begin{bmatrix} 0.72269 & 0.55462 \\ 0.27731 & 0.44538 \end{bmatrix}$

c **i** $S_2 = \begin{bmatrix} 151 \\ 149 \end{bmatrix}$ **ii** $S_3 = \begin{bmatrix} 165.7 \\ 134.3 \end{bmatrix}$
iii $S_7 = \begin{bmatrix} 191.8 \\ 108.2 \end{bmatrix}$

d Teacher to check.

2 **a** **i** $S_1 = \begin{bmatrix} 180 \\ 130 \\ 290 \end{bmatrix}$ **ii** $S_2 = \begin{bmatrix} 207 \\ 136 \\ 257 \end{bmatrix}$
iii $S_3 \begin{bmatrix} 225 \\ 132.1 \\ 242.9 \end{bmatrix}$

b **i** $S_2 \begin{bmatrix} 207 \\ 136 \\ 257 \end{bmatrix}$ **ii** $S_3 = \begin{bmatrix} 225 \\ 132.1 \\ 242.9 \end{bmatrix}$
iii $S_7 = \begin{bmatrix} 224.9 \\ 129.7 \\ 225.4 \end{bmatrix}$

c Teacher to check.

3 **a** $T = \begin{bmatrix} 0.80 & 0.25 \\ 0.20 & 0.75 \end{bmatrix}$

b $S_0 = \begin{bmatrix} 400 \\ 400 \end{bmatrix}$

c $S_1 = \begin{bmatrix} 420 \\ 380 \end{bmatrix}$, 420 to Jill's and 380 to Pete's

d $S_5 = \begin{bmatrix} 442.2 \\ 357.8 \end{bmatrix}$, 442 to Jill's and 358 to Pete's

e steady state solution: $S_s = \begin{bmatrix} 444.4 \\ 355.6 \end{bmatrix}$, 444 to Jill's and 356 to Pete's

4 **a** $T = \begin{bmatrix} 0.90 & 0.60 \\ 0.10 & 0.40 \end{bmatrix}$

b $S_0 = \begin{bmatrix} 1500 \\ 500 \end{bmatrix}$

c $S_1 = \begin{bmatrix} 1650 \\ 350 \end{bmatrix}$, 1630 are happy and 350 are unhappy

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d $S_4 = \begin{bmatrix} 1712.55 \\ 287.45 \end{bmatrix}$, 1713 are happy and 287 are unhappy

e steady state solution: $S = \begin{bmatrix} 1714.3 \\ 285.7 \end{bmatrix}$, 1714 are happy and 286 are unhappy

5 a $S_0 = \begin{bmatrix} 1200 \\ 600 \\ 200 \end{bmatrix}$

b $S_1 = \begin{bmatrix} 1270 \\ 440 \\ 290 \end{bmatrix}$, 1270 are happy

c $S_5 = \begin{bmatrix} 1310.33 \\ 429.82 \\ 259.85 \end{bmatrix}$, 1310 are happy

d steady state solution: $\begin{bmatrix} 1311.7 \\ 429.1 \\ 259.1 \end{bmatrix}$, 1312 are happy

Exercise 12C–4

1 a i $\begin{bmatrix} 80 \\ 120 \end{bmatrix}$ **ii** $\begin{bmatrix} 68.8 \\ 131.2 \end{bmatrix}$

b i
$$S_2 = TS_1 + R = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 90 \\ 125 \end{bmatrix}$$

ii
$$S_2 = TS_1 + R = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 90 \\ 125 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} 79 \\ 136 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 89 \\ 141 \end{bmatrix}$$

c i
$$S_2 = TS_1 - B = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix}$$
$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

ii
$$S_2 = TS_1 - B = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix}$$
$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

2 a i $S_1 = \begin{bmatrix} 11500 \\ 8500 \\ 10000 \end{bmatrix}$ **ii** 7300

b *A: 30 000, B: 0, C: 0*; While the sea birds move between nesting sites each year, the '1' in the transition matrix indicates that, once a sea bird nests at site *A*, it continues to nest at this site. Meanwhile, some of the birds who nest at sites *B* and *C* each year will move to site *A* until, in the long term, all birds are nesting at site *A*.

c i $\begin{bmatrix} 9500 \\ 9500 \\ 11000 \end{bmatrix}$ **ii** $\begin{bmatrix} 9000 \\ 9150 \\ 11850 \end{bmatrix}$ **iii** $\begin{bmatrix} 8507.5 \\ 8912.5 \\ 12\,580 \end{bmatrix}$

Chapter 12 review

Multiple-choice questions

1 B	**2** D	**3** E	**4** A
5 D	**6** A	**7** D	**8** B
9 A	**10** D	**11** D	**12** C
13 B	**14** B	**15** C	**16** C
17 A	**18** B	**19** B	**20** B
21 C	**22** E		

Extended-response question

1 a i $\begin{bmatrix} 2100 \\ 1100 \end{bmatrix}$ **ii** 3200

b $\begin{bmatrix} 2613 \\ 1613 \end{bmatrix}$

c Dinosaur attendance is expected to increase to around 3000 and stay at that level.

d Dinosaur attendance is expected to decrease to around 600 and stay at that level.

Chapter 13

Exercise 13A

1 E	**2** A	**3** C	**4** E
5 E	**6** C	**7** D	**8** B
9 C	**10** A	**11** A	**12** B
13 B	**14** D	**15** C	**16** C
17 B	**18** C	**19** C	**20** E
21 E	**22** E	**23** D	**24** A
25 B	**26** C	**27** B	**28** A
29 D	**30** C	**31** B	**32** A

Exercise 13B

1 a 3×1 **b** 1×3

c HC; because the number of columns in H equals the number of rows in CC

d i [696.72]

ii the number of Australian dollars ($696.72) that you would receive by converting your foreign currency into Australian dollars; $HC =$ $102 \times 1.316 + 262 \times 1.818 + 516 \times 0.167$

ISBN 978-1-107-56757-3

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e $\begin{bmatrix} 566.21 \\ 137.46 \\ 647.21 \end{bmatrix}$

2 a $\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$

b $A = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 0.25 & 0.25 \\ 0.375 & -0.125 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \end{bmatrix}$, $C = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$

c $X = A^{-}C = \begin{bmatrix} 0.25 & 0.25 \\ 0.375 & -0.125 \end{bmatrix}\begin{bmatrix} -4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$;

$x = 2,\ y = -3$

d i det(A) = 0; inconsistent, parallel lines

ii det(A) = 0; dependent, same equations

iii det(A) = 0; inconsistent, parallel lines

3 a $\begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$

b $A = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$,

$A^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0.4 \\ 0.2 & 0.2 & -0.6 \\ 0.8 & -0.2 & 0.6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$

c $X = A^{-1}C = \begin{bmatrix} 0.2 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0.6 \\ 0.8 & -0.2 & -0.6 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$

$$= \begin{bmatrix} \frac{14}{5} \\ \frac{-11}{5} \\ \frac{21}{5} \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.2 \\ 4.2 \end{bmatrix}$$

$x = 1,\ y = -4,\ z = -3$

4 a $\begin{bmatrix} 0.67 & 0.28 \\ 0.33 & 0.72 \end{bmatrix}$

b $S_0 = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$

c $S_1 = \begin{bmatrix} 4360 \\ 5640 \end{bmatrix}$; 4360 fish in Lake Blue and 5640 fish in Lake Green

d $S_3 = \begin{bmatrix} 4555.156 \\ 5444.844 \end{bmatrix}$; 4555 fish in Lake Blue and 5445 fish in Lake Green

e $S_s = \begin{bmatrix} 4590.2 \\ 5409.8 \end{bmatrix}$; 4590 fish in Lake Blue and 5410 fish in Lake Green

5 a i $S_1 = \begin{bmatrix} 126 \\ 174 \end{bmatrix}$ **ii** $S_2 = \begin{bmatrix} 143.16 \\ 156.84 \end{bmatrix}$

iii $S_3 = \begin{bmatrix} 154.4856 \\ 145.5144 \end{bmatrix}$

b $T_6 = \begin{bmatrix} 0.62 & 0.54 \\ 0.38 & 0.46 \end{bmatrix}$ (to 2 d.p.)

c i $S_2 = \begin{bmatrix} 143.16 \\ 156.84 \end{bmatrix}$ **ii** $S_3 = \begin{bmatrix} 154.4856 \\ 145.5144 \end{bmatrix}$

iii $S_5 = \begin{bmatrix} 166.9 \\ 133.1 \end{bmatrix}$ (to 1 d.p.)

d Answer given in question.

6 a $\begin{bmatrix} 1.2 & 20.1 & 4.2 \\ 6.7 & 0.4 & 0.6 \end{bmatrix}$

b i [1890]

ii 4×4

iii The total energy content of all these four foods in the same sandwich.

c $b = 4$, $m = 4$, $p = 2$, $h = 1$

7 a 700 **b** 0.5

c i $\begin{bmatrix} 160 \\ 280 \\ 180 \\ 80 \end{bmatrix}$ **ii** 280 **iii** 56

iv 7 weeks

v $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 700 \end{bmatrix}$ (they are all dead!)

d i $\begin{bmatrix} 190 \\ 280 \\ 180 \\ 80 \end{bmatrix}$ **ii** 130

8 a i $\begin{bmatrix} 493 \\ 83 \end{bmatrix}$ **ii** 421

b $S_n = T^{n-1} S_1$ **c** lecture 8 **d** 384

9 a $O_{2009} = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.68 \end{bmatrix}\begin{bmatrix} 456 \\ 350 \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \end{bmatrix} = \begin{bmatrix} 360 \\ 250 \end{bmatrix}$

b 248

$O_{2009} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 500 \\ 360 \end{bmatrix} - \begin{bmatrix} 40 \\ 38 \end{bmatrix} = \begin{bmatrix} 360 \\ 250 \end{bmatrix}$

$O_{2010} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 360 \\ 250 \end{bmatrix} - \begin{bmatrix} 40 \\ 38 \end{bmatrix} = \begin{bmatrix} 248 \\ 162 \end{bmatrix}$

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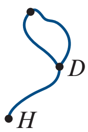
Chapter 14

Exercise 14A

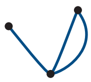
1 a i 3 **ii** 2 **iii** 1

b 14

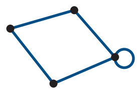
c



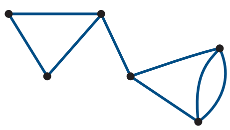
2 a



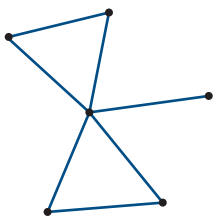
b



c

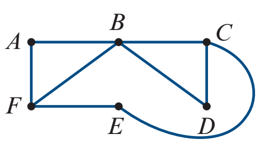


d

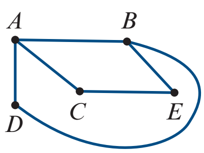


3 a i **b** ii **c** ii

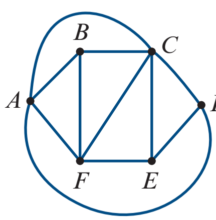
4 a



b



c



d not possible

5 a i $v = 8,\ f = 6,\ e = 12$

b i $v = 6,\ f = 8,\ e = 12$

c i $v = 7,\ f = 7,\ e = 12$

6 a 4 **b** 12 **c** 11 **d** 19

Exercise 14B

1 a

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A\\ B\\ C\\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A\\ B\\ C\\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

c

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A\\ B\\ C\\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

d

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A\\ B\\ C\\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

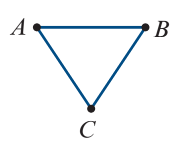
e

$$\begin{array}{cc} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A\\ B\\ C\\ D\\ E\\ F \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

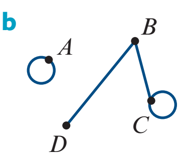
f

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A\\ B\\ C\\ D \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

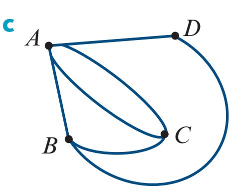
2 a



b



c



3 C is an isolated vertex.

4 Leading diagonals will all be ‘1’.

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5
$$\begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Exercise 14C

1 a path **b** trail **c** path **d** walk
e trail **f** path

2 a walk **b** cycle **c** path **d** walk
e path **f** walk

3 a i Euler trail
ii $A–B–E–D–B–C–B–A–E–A$
b neither
c i Euler trail
ii $A–C–E–C–B–D–E–F$
d i Euler circuit
ii $A–B–C–E–D–C–A$
e i Euler circuit
ii $E–F–D–E–A–B–D–C–B–E$

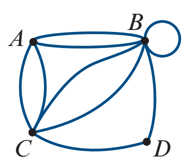
4 a $A–B–C–F–I–H–E–G–D$
b $A–B–C–D–E–F$ **c** $A–B–D–C–E–A$

5 $F–A–B–C–D–E–H–G$

6 a 5
b i Hamilton path
ii $E–W–D–C–B–A–F$
c i There are two odd vertices so trail is possible but circuit is not.
ii C

7 a 24 **b i** C or G **ii** 2800
c $F–G–A–B–C–D–E–F$

8 a 7 **b** 2 **c**



d Vertices are not all even.

Exercise 14D

1 a $D–E$ **b** 17 minutes
c 8 minutes **d** 36 minutes

2 11

3 a 34 **b** 56 **c** $E–B–A$, 22
d $A–E–F–G–I$ or $A–C–F–G–I$

4 a A **b** D, E **c** C **d** 6

5 R|3 $\boxed{1}$ 7 4 4

6 a P **b** U **c** 1

7 a 10 **b** 16 **c** $A–B–C–E$

8 a 7 **b** $A–E–G$ **c** 8
d $A–E–G–I$

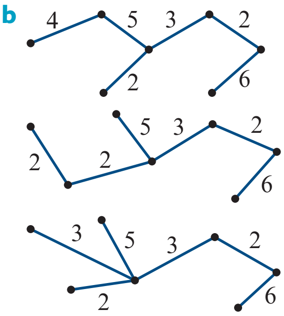
9 a $S–B–D–F$, 12 **b** $S–A–C–D–F$, 10
c $S–B–D–F$, 15 **d** $S–A–E–G–F$, 19

10 19 km

11 a 46 **b** $Q–C–B–E–G–L–W$

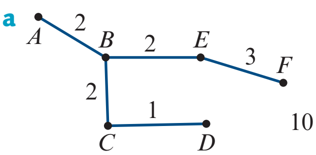
Exercise 14E

1 a 6
b

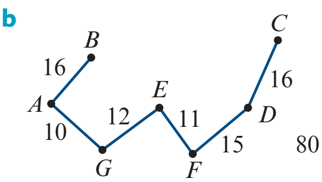


c 22, 20, 21

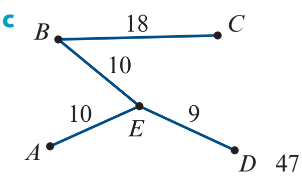
2 a



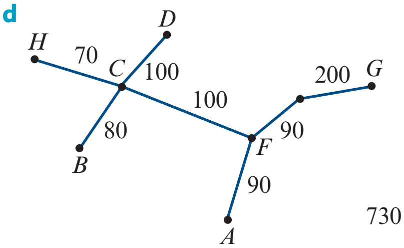
b



c



d



3 490 m

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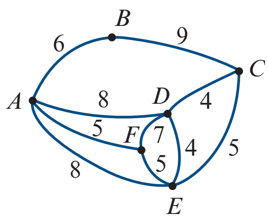
Chapter 14 review

Multiple-choice questions

1 C	**2** C	**3** A	**4** D
5 A	**6** C	**7** B	**8** B
9 E	**10** B	**11** A	**12** A
13 C	**14** B	**15** B	

Extended-response questions

1 a i



ii 24

iii

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b i 45 km (at minimum)

ii Some vertices are visited more than once.

iii F–E–D–C–B–A–F

iv 33 km (for route above; other answers possible)

c F and C

2 a 11 km

b There are exactly two odd-degree vertices in the network.

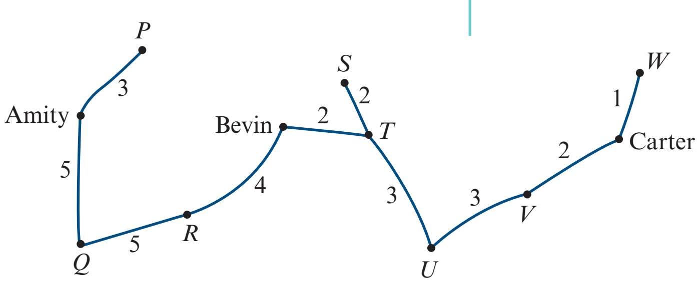
c Checkpoint V.

d Checkpoint U. If they do, they will have to travel one of the roads to Bevin a second time.

e Bevin–T–U–V–Carter or Bevin–T–U–Carter

f 21 km

g



Chapter 15

Exercise 15A

1 $C_1 = 14,\ C_2 = 12,\ C_3 = 21$

2 $C_1 = 12,\ C_2 = 16,\ C_3 = 16$

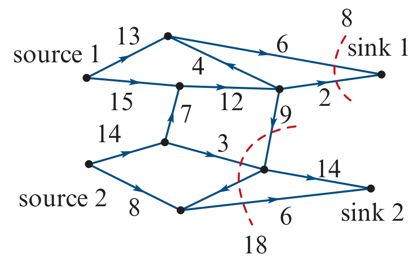
3 a 9 **b** 11 **c** 8 **d** 18

4 a A, 14; B, 23; C, 12; D, 16; E, 11

b It does not prevent flow from source (Arlie) to sink (Bowen).

c 12

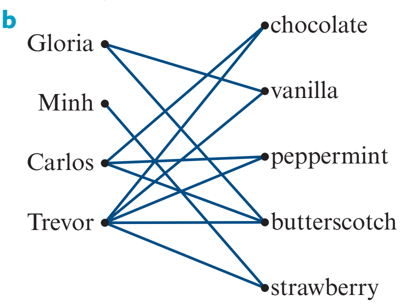
5 sink 1 = 8, sink 2 = 18



Exercise 15B

1 a two distinct groups of vertices (people and flavours)

b



c 5

2 a W–D, X–A, Y–B, Z–C

b E.g., minimum cost is 11; W–A, X–B, Y–D, Z–C

3 Dimitri 800 m, John 400 m, Carol 100 m, Elizabeth 1500 m

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4 Joe C, Meg A, Ali B

5 A–Y, B–Z, C–X, D–W

6 Champs Home, Starts Away, Wests Neutral; or Champs Neutral, Starts Away, Wests Home. Cost = $20 000

7 A Mark, B Karla, C Raj, D Jess; or A Karla, B Raj, C Mark, D Jess

Exercise 15C

1 a

Activity	Immediate predecessors
A	–
B	–
C	A
D	A
E	B, C
F	D
G	E

b

Activity	Immediate predecessors
P	–
Q	P
R	P
S	Q
T	Q
U	S, V
V	R
W	R
X	T, U

c

Activity	Immediate predecessors
J	–
K	–
L	J
M	N
N	K
O	K
P	N
Q	L, M
R	P
S	O, R
T	Q

d

Activity	Immediate predecessors
A	–
B	–
C	A
D	A
E	D, B
F	C, E
G	D, B
H	B

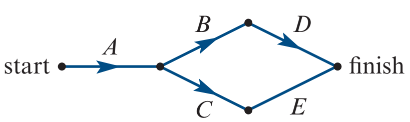
e

Activity	Immediate predecessors
P	–
Q	–
R	P
S	P
T	Q
U	R
V	S
W	S, T
X	U
Y	W
Z	V, X, Y

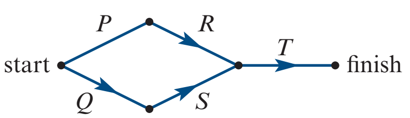
f

Activity	Immediate predecessors
A	–
B	A
C	A
D	A
E	B
F	C, D
G	D
H	E, F, G
I	G
J	I
K	H

2 a



b



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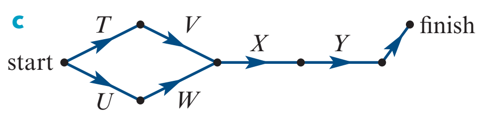
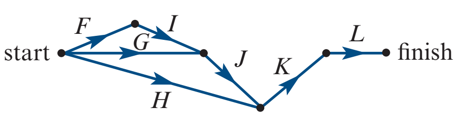
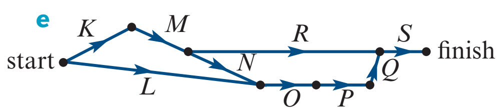


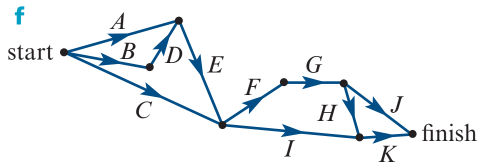
d



e



f



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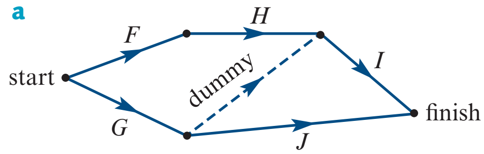
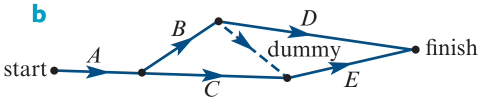
3

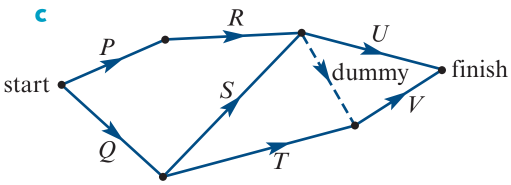
a



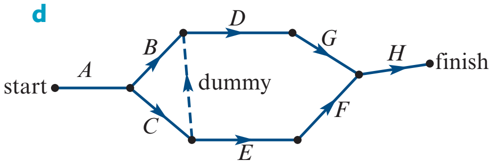
b



c



d



4 **a** Remove panel.
b 'Order component' and 'Pound out dent'

Exercise 15D

1 **a** $p = 12$ **b** $w = 10$ **c** $m = 8,\ n = 8$
d $a = 10,\ b = 12,\ c = 11$
e $f = 9,\ g = 12$
f $q = 8,\ r = 3,\ p = 2,\ n = 6$

2 **a** 3 **b** A–C **c** 5 **d** 13
e 2

3 **a** 12 **b** 10 **c** 9 **d** 1
e 3 **f** 9

4 **a** D–E–F **b** A: 1, B: 1, C: 15

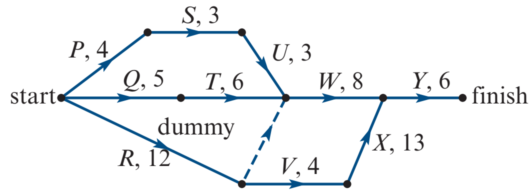
5 **a** B–E–F–H–J
b A: 1, C: 14, D: 1, G: 1, I: 1

6 **a**

Activity	*Duration (weeks)*	*Immediate predecessors*
A	3	–
B	6	–
C	6	A, B
D	5	B
E	7	C, D
F	1	D
G	3	E
H	3	F
I	2	B

b A–C–E–G

7 **a**

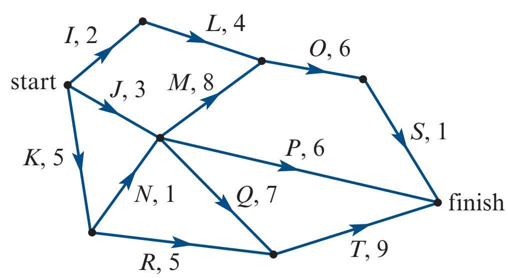


b

Activity	*EST*	*LST*
P	0	2
Q	0	1
R	0	0
S	4	6
T	5	6
U	7	9
V	12	13
W	12	12
X	16	17
Y	20	20

c R–W–Y **d** 26 hours

8 **a**



b

Activity	*EST*	*LST*
I	0	9
J	0	4
K	0	0
L	2	11
M	6	7
N	5	5
O	14	15
P	6	16
Q	6	6
R	5	8
S	20	21
T	13	13

c K–N–Q–T **d** 22 hours

Exercise 15E

1 **a** B–E–H–J **b** 2 hours
c 6 hours **d** 14 hours

2 **a** B–C–E–H–J
b none, F is not on the critical path.
c 4 hours

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3 a 15 hours

b $B–C–F–I$

c A is not on the critical path. It already has slack time and reducing it further has no effect.

d \$200

Chapter 15 review

Multiple-choice questions

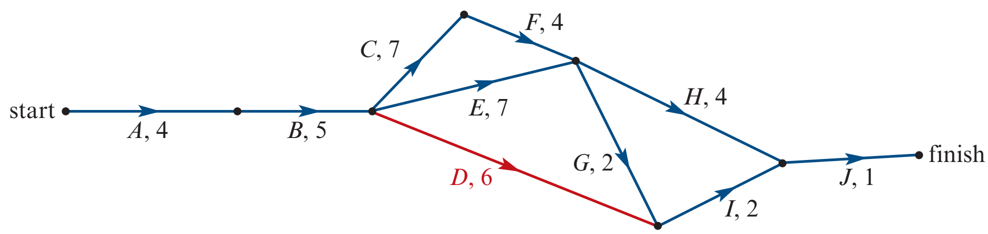
1 D	**2** D	**3** A	**4** B
5 C	**6** B	**7** E	**8** D
9 D	**10** E		

Extended-response questions

1 Ann–D, Bianca–B, Con–C, David–E

2 a 26 **b** 15

3 a



b $A–B–C–F–H–J$

4 Rob – breaststroke, Joel – backstroke, Henk – freestyle, Sav – butterfly

5 a 31 **b** 15

Chapter 16

Exercise 16A

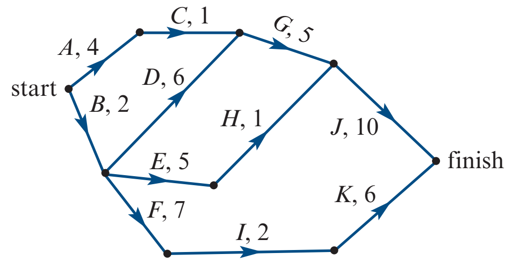
1 C	**2** A	**3** E	**4** E
5 B	**6** C	**7** B	**8** A
9 C	**10** B	**11** E	**12** C
13 D	**14** D	**15** D	**16** C
17 C	**18** E	**19** B	**20** B
21 A	**22** D	**23** C	**24** B
25 C	**26** C	**27** E	**28** E
29 D	**30** D		

Exercise 16B

1 a $A \to 1, D \to 4, F \to 10, K \to 12,$

b $B–C–E–G–J–K$

2 a



b A 0, B 0, C 4, D 2, E 2, F 2, G 8, H 7, I 9, J 13, K 11

c 23 days

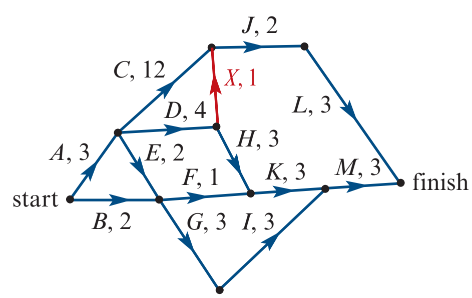
d A 3, B 0, C 7, D 2, E 7, F 8, G 8, H 12, I 15, J 13, K 17

e $B–D–G–J$

f 6 days

3 a Immediate predecessor of E is A. EST for I is 8. EST for M is 13.

b



c **i** 16 hours

ii Critical path is the sequence of activities that cannot be delayed without delaying the entire project.

d **i** 6 hours **ii** $A–C–J–L$ **iii** 8 hours

4 a **i** 2.1 km

ii $PQRTSU$ or $PRQSTU$ or $PRQTSU$ or $PRTQSU$

b **i** $R–Q–P–R–T–Q–S–T–U–S$ or $R–Q–P–R–T–S–Q–T–U–S$

ii travel each road only once

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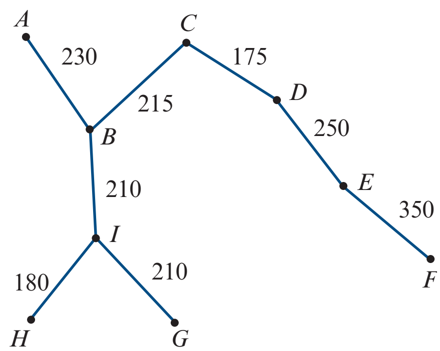
5 a EST for F is 14, LST for G is 17
b i A–D–H–I ii 27 days
c i EST for J is 16, LST for J is 19, LST for B is 1 ii 22 days
iii D, H, F iv \$2500

6 a None of the edges overlap.
b $7 + 6 - 11 = 2$ c C d 297 km
e no f 79 km g 127 km h 187 km

7 a

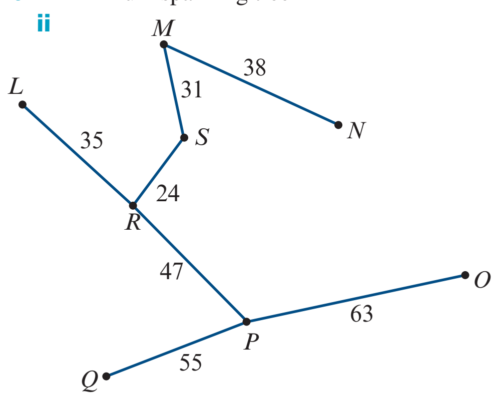


b 1820 m

8 a 5 b 24 hours c 7 hours

9 11 megalitres/day

10 a 112 km
b i minimum spanning tree
ii



iii 293 km
c 306 km

11 a A, B, C
b LST for B is 1, EST for E is 10, LST for I is 18
c i $A–D–F–I–J$ ii 27 months
d i $B–C–D–F–I–J$ ii 25 months

12 a Location 2 – Site 1, Location 3 – Site 3, Location 4 – Site 2
b \$3790

13 a $A–Z$, $B–W$, $C–X$, $D–Y$, or $A–Z$, $B–X$, $C–W$, $D–Y$
b \$130

Chapter 17

Exercise 17A

1 E 2 D 3 B 4 C 5 A

6 a $x = 70$ (corresponding)
$y = 110$ (straight line)
$z = 70$ (co-interior with y)
b $x = 40$ (alternate)
$y = 140$ (co-interior)
$z = 40$ (corresponding)
c $y = 100$ (straight line)
$z = 40$ (corresponding)
d $x = 50$ (corresponding)
$y = 130$ (straight line)
$z = 130$ (corresponding)
e $x = 80$ (corresponding)
f $x = 60$ (co-interior)
$y = 120$ (alternate)

Exercise 17B

1 a $x = 50, y = 130$ b $x = 75$
c $x = 100, y = 110$
d $x = 65°, y = 130°$
e $w = 100, x = 40, y = 40, z = 100$
f $a = 75, b = 30, c = 35$ g $x = 22.5, y = 45$

Exercise 17C

1 a triangle b square c pentagon
d hexagon e octagon

2 a 90°
b i right-angled isosceles
ii right-angled isosceles

3 a i $x = 72$ ii $y = 54$ b 540°

4 a 60° b 60°

5 a 900° b 720° c 1080°

6 9 sides 7 a 12.6 cm^2 b 2.1 cm^2

8 $a = 120, b = 60$. The interior angles add up to 360° around a point so they tessellate.

9 8 sides

Exercise 17D

1 a 6.32 cm b 12.1 cm c 10.4 cm d 5.7 cm
e 55 cm f 9 cm

2 a 5.77 cm b 6.80 cm c 8.31 cm d 6.10 cm

3 a $x = 7.21$ b $x = 5.39$ c $x = 6.47$

4 $VW = 5.2$ cm 5 $AD = 30.4$ cm

6 16.6 m 7 41 m 8 $CB = 23.0$

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9 13.9 cm **10** $x = 12.7$ cm
11 144.2 m **12** 18.4 cm
13 a 64 cm^2 **b** 32 cm^2
14 $AB = 19.6$ cm **15** $DE = 0.8$ cm
16 2 cm^2

Exercise 17E

1 a $x = 11.25$ **b** 3 **c** $\frac{26}{3}$
d $x = 11.7$ **e** $x = 7.5$ **f** $x = 3$
g $x = 6$ **h** $x = \frac{4}{3}$ **i** $x = \frac{8}{3}$ **j** $x = 7.5$
2 $AC = 17.5, AE = 16, AB = 20$
3 41.25 m **4** 7.5m **5** 15 m **6** 22.5 m
7 10.3 m **8** $x = 6.7$ **9** 83.6 cm
10 $x = 0.85$ **11** 40.14 m
12 7.2 m **13** 1.9 m

Exercise 17F

1 a 261.85 cm^3 **b** 212.65 cm^3
c 17.36 cm^3 **d** 79.77 cm^3
2 4 411 458 m^3
3 a 14.1 m^3 **b** 565.5cm^3
c 14.4 mm^3 **d** 589.0 mm^3
4 96 cm^3
5 6635 cm^3
6 a $2 \times \pi \times 6.3^2 + 2 \times \pi \times 6.3 \times 2.1 \approx 333$ cm^2
b $2 \times 8.3 \times 12.2 + 2 \times 2.1 \times 8.3 + 2 \times 2.1 \times 12.2 \approx 289$ cm^2
7 a 1.5 m^3
b $2 \times 1/2 \times 3 \times 2 + 3 \times 0.5 + 2 \times 0.5 + \sqrt{(2^2 + 3^2)} \times 0.5 \approx 10.3$ m^2
8 a 4 m^3
b $1/2 \times 4\pi \times 1.25^2 + \pi \times 1.25^2 \approx 15$ m^2
9 179.88 cm^2, 96 cm^3
10 a surface area = 192 m^2
b volume = 160 m^3
11 b volume = 4312 cm^3
a surface area = 1680 cm^2
12 a surface area = 244 cm^2
b volume = 220 cm^3
13 a surface area = 892.70 cm^2
b volume = 1785.40 cm^3

Exercise 17G

1 a 3 **b** $3^2 \times 15 = 135$ cm^2
2 24 cm^2 **3** 55.6 cm^2 (to 1 d.p.)
4 13.3 cm^2 (to 1 d.p.) **5** 40.5 m^2
6 a 2.5 **b** 375 cm^3
7 a 3 **b** $\frac{1}{3}$ **c** 15 cm
d 2942 cm^2 **e** 10 603 cm^3
8 12.5 m^3 **9** 9.8 cm^2
10 a $\frac{3}{5} = 0.6$ **b** 2.7 cm^2

Chapter 17 review

Multiple-choice questions

1 A **2** E **3** E **4** C
5 A **6** D **7** B **8** C
9 C **10** C **11** B **12** D
13 D **14** D

Chapter 18

Exercise 18A

1 $x = 5.1504, y = 8.5717$ **2** 7.0021
3 a 4.10 **b** 0.87 **c** 2.94 **d** 4.08
e 33.69° **f** 11.92
4 23.09 cm **5** 66.42°, 66.42°, 47.16° **6** 23 m
7 a 9.6° **b** 5.9 m
8 a 60° **b** 17.3 m
9 a 6.8 m **b** 6.2 m
10 12.51° **11** 183 m **12** 1451 m
13 a 7.07 cm **b** 90° ($ABCD$ is a square)
14 3.1 cm **15** 37.8 cm **16** 31 m
17 4.4 m **18** 58 m

Exercise 18B

1 a 6.84 **b** 6.48 **c** 11.75 **d** 9.20
2 a 56.32° **b** 36.22° **c** 49.54°
d 43.84° or 136.16°
3 a $A = 48°,\ b = 13.84,\ c = 15.44$
b $C = 56.45°,\ a = 7.26,\ c = 6.26$
c $B = 19.8°,\ b = 4.66,\ c = 8.27$
d $C = 117°,\ b = 246.76,\ c = 342.05$
e $C = 30°,\ a = 5.40,\ c = 15.56$
4 a $A = 72.63°, B = 59.12°, a = 19.57$ or $A = 10.87, B = 120.88°, a = 3.57$
b $A = 24.31°, C = 26.69°, a = 4.18$
c $C = 96.25°, B = 55.50°, c = 17.85$ or $C = 27.25°, B = 124.50, c = 8.22$
5 577 m
6 a $\angle ACB = 12°, \angle CBO = 53°, \angle CBA = 127°$
b 189.3 m **c** 113.9 m
7 35.6 m **8** 1660 m
9 a 26.60 m **b** 75.12 m

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Exercise 18C

1 5.93 cm

2 $\angle ABC = 97.90°$, $\angle ACB = 52.41°$

3 **a** 26 **b** 11.74 **c** 49.29° **d** 73 **e** 68.70 **f** 47.22° **g** 7.59 **h** 38.05°

4 2.6 km **5** 3.2 km

6 **a** 8.23 cm **b** 3.77 cm **7** 56 cm

8 **a** 7.33 cm **b** 5.53 cm

9 **a** 83.62° **b** 64.46°

10 **a** 87.6 m **b** 67.7 m

Exercise 18D

1 **a** 11.28 cm^2 **b** 15.10 cm^2 **c** 10.99 cm^2 **d** 9.58 cm^2

2 **a** 16.65 cm^2 **b** 5.93 cm^2

3 32° or 148° **4** 34.12 cm^2

5 **a** 19.81 cm^2 **b** 9.05 cm^2 **c** 31.22 cm^2

6 **a** 53.13° or 126.87° **b** 4.00 cm or 7.21 cm

Exercise 18E

1 34.8 m **2** 400 m **3** 575 m **4** 110 m

5 **a** 185 m **b** 200 m **c** 15 m

6 **a** $\angle TAB = 3°$, $\angle ATB = 80°$, $\angle ABT = 97°$ **b** 2070 m **c** 252 m

7 027° **8** 056°

9 **a** 034° **b** 214°

10 113° **11** 22°

12 **a** 49° **b** 264 km

13 **a** **i** 160.19 m **ii** 164.62 m **b** 127.93 m

14 327.6 m **15** **a** 531.8 m **b** 208 m

16 10.6 km **17** **a** 3583 m **b** 353°

18 16.5 m

Exercise 18F

1 **a** 13 cm **b** 15.26 cm **c** 31.6° **d** 38.17°

2 **a** 4 cm **b** 71.57° **c** 12.65 cm **d** 13.27 cm **e** 72.45° **f** 266.4 cm^2

3 14°, 14°, 10° **4** **a** 85 m **b** 45 m

5 17.58° **6** 1703 m

7 **a** 24.78° **b** 65.22° **c** 20.44°

8 42.4 m **9** 1946 m

10 **a** 11.31 cm **b** 10 cm **c** 10 cm **d** 68.90°

11 **a** 6.96 cm **b** 16.25 cm^2

12 **a** 5 km **b** 215.65° **c** 6.56°

Chapter 18 review

Multiple-choice questions

1 C	**2** B	**3** C	**4** C
5 B	**6** A	**7** D	**8** B
9 B	**10** C	**11** E	**12** A
13 B	**14** A	**15** C	**16** A
17 D	**18** B	**19** C	**20** E
21 C	**22** B	**23** D	**24** A
25 B	**26** A	**27** B	**28** B
29 B			

Extended-response questions

1 **a** 12 m^3
b **i** 1 : 200 **ii** 15 cm^2

2 **a** $AM = 3$

$\angle OMA = 90°$

$OM = 1.6$ (pythagoras)

b $(2.2 \times 6) + (3 \times 1.6) = 18\ m^2$ **c** 180 m^3

d **i** 208 m^2 **ii** 13 L

3 **a** 42.7° **b** Teacher to check **c** 69°
d 111°
e No. Shortest distance from the tree to the shed is at the midpoint of CN (12 m using Pythagoras).

4 **a** 42 m **b** **i** 77 m **ii** 957.33 m^2
c 14°

Chapter 19

Exercise 19A

1 **a** 50.27 cm **b** 87.96 m **c** 282.74 mm **d** 37.7 mm **e** 43.98 m

2 **a** $\frac{1}{4}$ **b** $\frac{3}{4}$ **c** $\frac{1}{12}$ **d** $\frac{1}{3}$ **e** $\frac{1}{6}$ **f** $\frac{5}{12}$

3 **a** 7.85 cm **b** 10.47 cm **c** 26.18 cm **d** 23.56 cm **e** 36.65 cm **f** 57.60 cm

4 **a** 13.09 cm **b** 5.24 cm **c** 78.54 cm **d** 37.70 cm **e** 122.17 cm **f** 109.96 cm

5 **a** 39.27 cm^2 **b** 52.36 cm^2 **c** 130.9 cm^2 **d** 117.81 cm^2 **e** 183.26 cm^2 **f** 287.98 cm^2

6 **a** 130.9 cm^2 **b** 488.69 cm^2 **c** 2650.72 cm^2 **d** 670.21 cm^2 **e** 2356.19 cm^2 **f** 1649.34 cm^2

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7 **a** 9.06 cm^2 **b** 185.52 cm^2
c 274.46 cm^2

8 82.29 cm^2; 1174.35 cm^2

9 **a** 208.26 cm^2; 105.90 cm^2
b 29.83 cm^2; 4996.72 cm^2
c 2144.47 cm^2; 4217.26 cm^2
d 23.19 cm^2; 781.06 cm^2
e 1167.37 cm^2; 6686.61 cm^2
f 953.10 cm^2; 1874.34 cm^2

10 45.81 cm

11 **a** 95.5° **b** 4775°

12 **a** 6.20 cm **b** 2.73 cm^2

13 61.42 cm^2 **14** **a** 125.66 m **b** 31%

15 **a** 10.47 m **b** 20.94 m^2

16 6.64 cm^2

17 $r = 7, 147.33°$; $r = 9$, 89.13°

18 **a** 81.96 cm **b** 4.03 cm^2

Exercise 19B

1 **a** (65°N, 0°) **b** (0°, 75°E)

2 **a** (18°N, 24°E) **b** (38°N, 44°E)
c (47°S, 23°E) **d** (43°N, 37°W)

3 **a** **i** Mexico city, Rangoon
ii Melbourne
b **i** Melbourne **ii** Plymouth
iii Plymouth **iv** Zurich
c Marseilles, Mexico city, Zurich, London, Plymouth, Rangoon
d Mexico city, Lima, Plymouth, London (just)
e Mexico city and Rangoon
f None of these **g** London **h** Wellington

4 6367 km **5** 2234 km

6 1787 km **7** 34°S

8 **a** 3910 km **b** 3016 km **c** 4915 km
d 4468 km **e** 3351 km

9 3574 km **10** 4021 km

11 **a** 5585 km **b** 7819 km **c** 3351 km
d 6702 km **e** 16 979 km

12 5.4°

13 **a** 3351 km **b** 6702 km **c** 13 404 km

14 **a** 4244 km **b** 5808 km **c** 14 298 km

15 **a** 4194 km **b** 14 247 km **c** 5859 km

16 **a** equator: 4691 km; North Pole: 5361 km; South Pole: 14 745 km
b equator: 6143 km; North Pole: 3910 km; South Pole: 16 197 km
c equator: 1676 km; North Pole: 11 729 km; South Pole: 8378 km
d equator: 1564 km; North Pole: 11 617 km; South Pole: 8489 km

17 **a** 6182 km **b** 5543 km
c 4525 km **d** 3200 km

18 60°N and 60°S

19 **a** 1139 km **b** 2841 km **c** 5463 km

20 10 090 km

21 **a** 447 km **b** 648.23 km
c 1095 km

22 **a** 2346 km **b** 2350 km
c 4696 km

Exercise 19C

1 **a** X 2 hours ahead of Y
b X 5 hours ahead of Y
c X 5 hours behind Y

2 **a** **i** Melbourne is 2 hours behind.
ii Melbourne is 9 hours ahead.
iii Actually Melbourne is 3 hours 30 minutes ahead.
iv Melbourne is 10 hours ahead.
v Melbourne is 16 (17) hours ahead.
vi Melbourne is 15 hours ahead.
b **i** 8:00 a.m.
ii 9:00 pm. the day before
iii 2:30 am
iv 8:00 pm
v 2:00 pm the day before
vi 3:00 pm the day before

3 **a** 150°W **b** 2:00 pm
c 7:30 pm the day before

4 **a** 5:30 pm **b** 7:30 am

5 **a** 45° **b** 3 hours **c** 2:45 pm

6 **a** 3:00 am **b** 1:00 am the next day

7 **a** 3:00 am **b** 5:00 am

Exercise 19D

1 17.32 cm

2 **a** 48π cm $\approx$ 150.8 cm
b 576π cm^2 $\approx$ 1809.56 cm^2
c 61.15 cm

3 28.27 cm

4 $5\sqrt{3}$ cm $\approx$ 8.67 cm

5 height = 32 cm, height of cap = 4 cm

6 **a** 37.08 cm **b** 116.5 cm

7 **a** 6014 km **b** 2099 km
c 2098 km

8 **a** 6182 km **b** 3237 km
c 3234 km

ISBN 978-1-107-56757-3

9 a sphere: $\sqrt{21} \approx 4.58$ cm; cone: 3.5 cm
b 9.273 cm

10 373.85 cm^2 **11** 24 cm

Chapter 19 review

Multiple-choice questions

1 A	**2** B	**3** B	**4** E
5 D	**6** C	**7** B	**8** D
9 E	**10** B	**11** A	

Extended-response questions

1 a $5\pi/3 \approx 5.24$ cm
b 19.78 cm
c i 14.62 cm^2 **ii** 11.25 cm^2
iii 288.29 cm^2

2 b i 51.48 cm **ii** 4764.95 cm^2
iii 94.8%

3 a 4.83 cm **b** 182.21cm^2 **c** 9.67 cm

4 a 5542.56 km **b** 1451 km
c 6400 km **d** 1.04 km

5 a 5250 km **b** 5869 km **c** 14 270 km

Chapter 20

Exercise 20A

1 C	**2** E	**3** D	**4** B
5 A	**6** E	**7** A	**8** D
9 E	**10** B	**11** A	**12** B
13 E	**14** B	**15** D	**16** A
17 D	**18** C	**19** B	**20** C
21 C	**22** A	**23** D	**24** E
25 C	**26** E	**27** B	**28** D
29 C	**30** B	**31** B	**32** D
33 A	**34** D	**35** D	**36** D
37 E	**38** C	**39** B	**40** D
41 D	**42** A		

Exercise 20B

1 a 8 m^2 **b** 12.8 m

2 a $\theta = 180 - (45 + 60) = 75°$ **b** 2.31 m
c 3.16 m^2 **d** 17 m^2

3 a 70° **b** 10.6 m **c** 10.2 m **d** 8.6 m^2

4 a 282° **b** 6.9 km **c** 18.6 km

5 a 60 **b** 0.289 m **c** 0.214 m

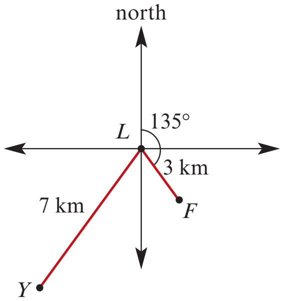
6 a 8.50 m **b** 15°

7 a 1285 cm^3 **b** 716 cm^2

8 177° **9** 2.9 m

10 b 7.12° **c** 403 m

11 a



b 75° **c** 6.87 km **d** 235.2°

12 c 1548 cm^2

13 a 17 191 cm^3 **b** 93.6%

Chapter 21

Exercise 21A

1 a 3 **b** $1\frac{1}{3}$ **c** $-\frac{6}{5}$ **d** $-1\frac{1}{3}$
e -1 **f** 3

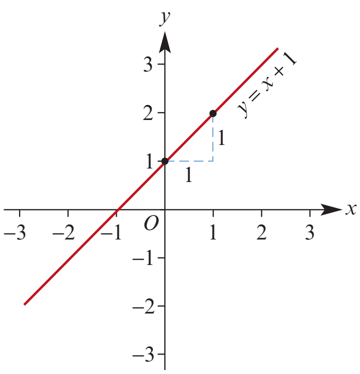
2 Any line at 45° to positive direction of x-axis will have gradient = 1.

3 Any horizontal line will have gradient = 0.

4 a $-\frac{1}{4}$ **b** $-2\frac{1}{2}$ **c** -2 **d** -8
e 0 **f** -1 **g** 7 **h** -13

Exercise 21B

1 a



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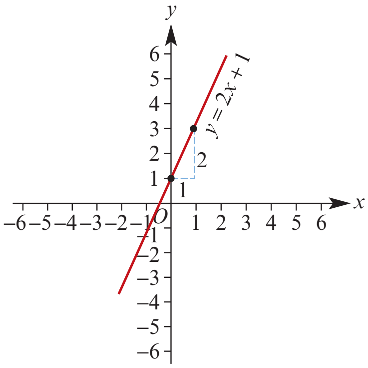

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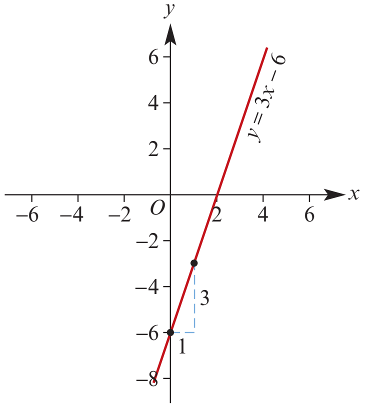
c



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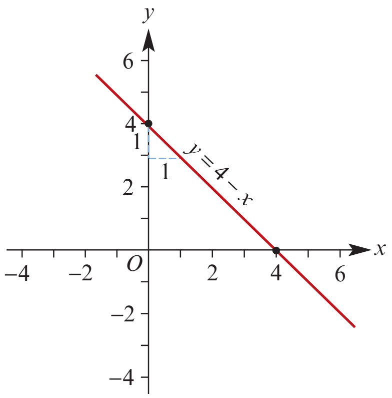

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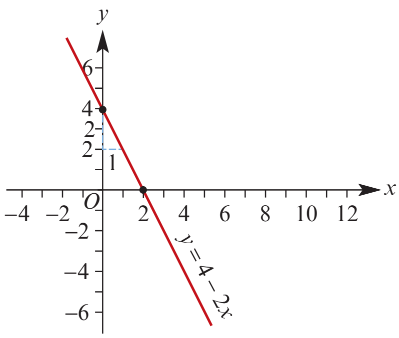

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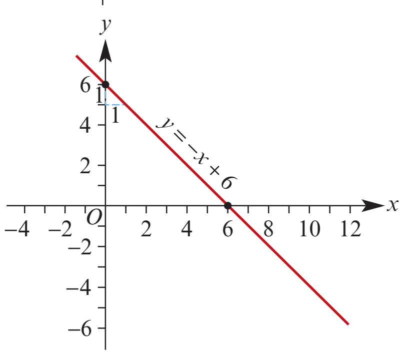
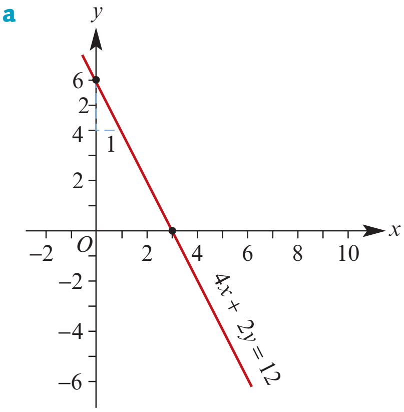


2 b $y + x = 0$ and **c** $2y = 3x$

3 a -1 **b** -1 **c** $\frac{3}{2}$ **d** 0 **e** 1

4 a



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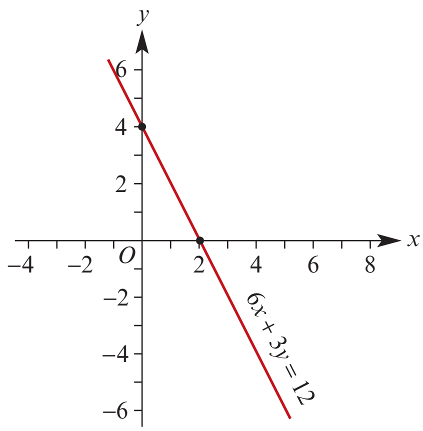

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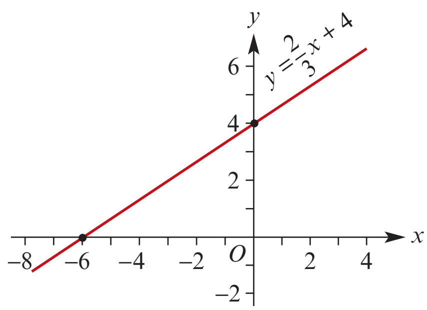

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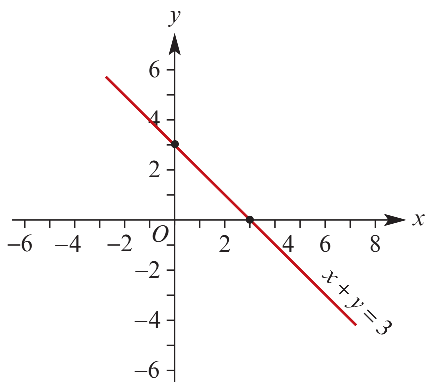

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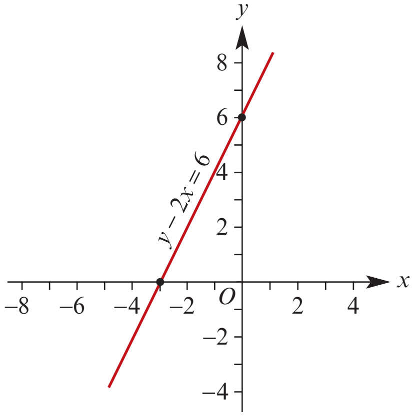

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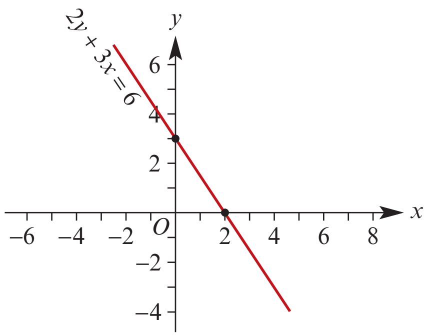

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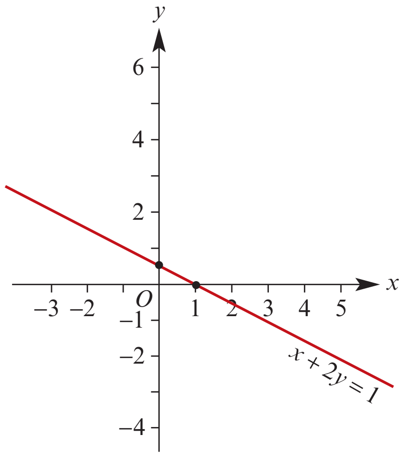

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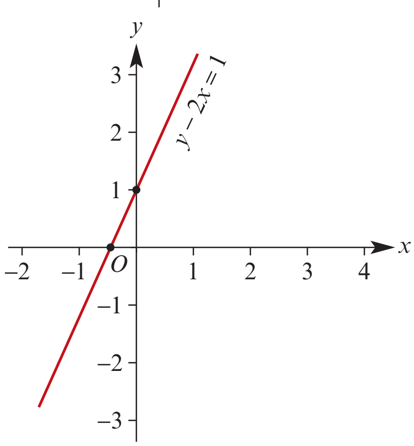

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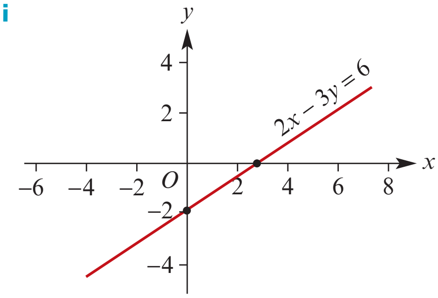


Exercise 21C

1 **a** $y = 2x + 2$ **b** $y = -3x + 6$
c $y = 3x - 3$ **d** $x = 2$
e $y = -3$ **f** $y = -x$
g $y = 3x$ **h** $y = -\frac{1}{4}x + 2$

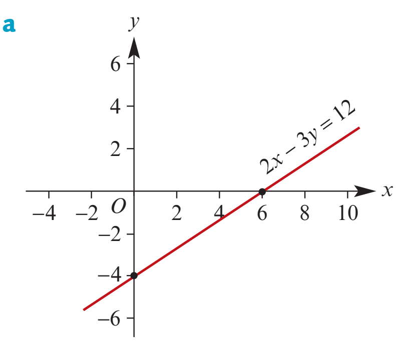
2 **a** $y = \frac{1}{2}x + 1$ **b** $y = 2x + 2$ **c** $y = 6$
d $y = x - 6$ **e** $y = -2x + 5$

3 **a** $y = \frac{3}{2}x + 1$ **b** $y = \frac{1}{2}x + 6$
c $y = -3x + 7$ **d** $y = -x + 2$
e $y = -2x + 2$ **f** $y = -x + 10$

4 **a** $y = -x + 3$ **b** $y = -2x - 6$
c $y = -\frac{1}{2}x + 4$ **d** $y = x + 1$
e $y = \frac{1}{2}x + 3\frac{1}{2}$

Exercise 21D

1 **a**



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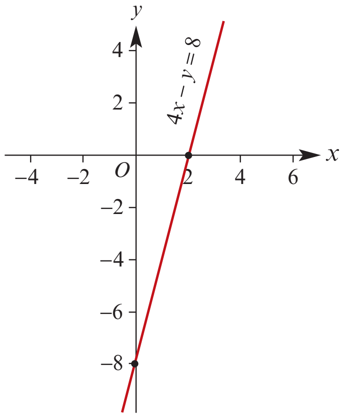
b



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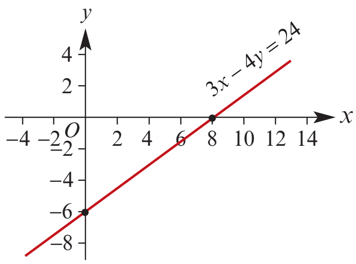
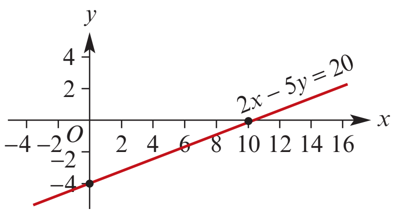


d



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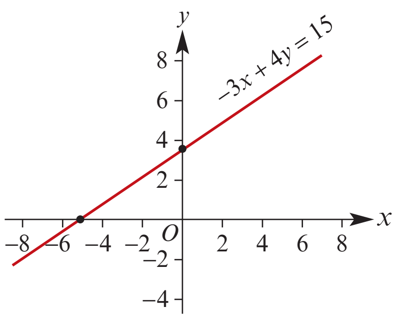

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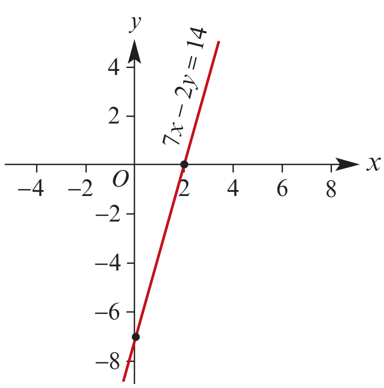


2 a $y = 2x - 6; m = 2$

b $y = -\frac{1}{4}x + 3; m = -\frac{1}{4}$

c $y = -\frac{1}{2}x - 3; m = -\frac{1}{2}$

d $y = \frac{5}{2}x - 5; m = \frac{5}{2}$

e $y = \frac{1}{5}x - 2; m = \frac{1}{5}$

f $y = \frac{1}{2}x + 4; m = \frac{1}{2}$

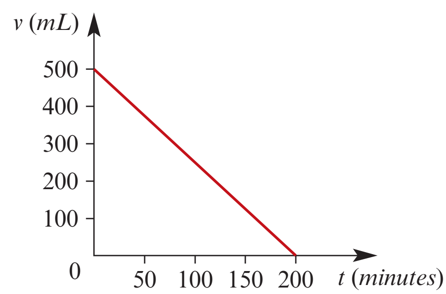
Exercise 21E

1 $w = 350 + 20n$, where w = weekly wage ($) and n = number of cleaners sold per week

2 a $v = 500 - 2.5t$, where v = amount of solution left in reservoir (mL) and t = time in minutes

b $0 \leq v \leq 500$, $0 \leq t \leq 200$

c

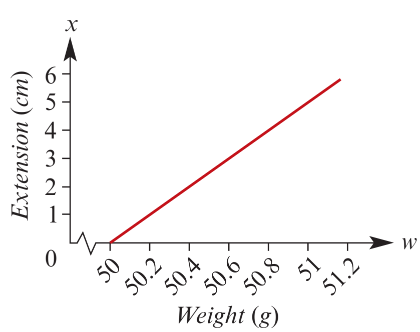


3 $C = 2.6 + 1.5n$, where C = cost of hiring taxi ($) and n = number of km travelled

4 a $C = 85 + 0.24x$, where
C = cost of hiring car ($) and
x = number of km travelled

b $145

5 a



b $x = 5w - 250$ **c** 12.5 cm

6 a $C = 0.06n - 1$, where C = charge ($) and n = number of sheets printed

b $59

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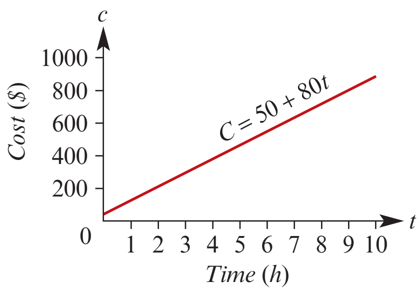
7 a $C = 5n + 175$, where C = sum on register ($) and n = number of notes counted

b yes, $175

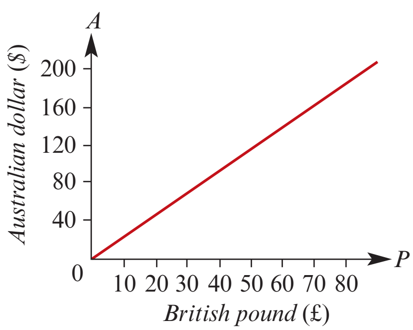
8 a $C = 50 + 80t$, where C = charge ($) and t = time (h)

b



c $270

9 a



b exchange rate between Australian dollar and British pound, i.e. number of Australian dollars equal to 1 British pound

c 2.3

d $A = 2.3P$, where A = Australian ($) and P = British pound (£)

Exercise 21F

1 a $x = 10, y = -2$ **b** $x = 4, y = \frac{-14}{3}$

c $x = 2, y = 3$ **d** $x = 4, y = -1$

e $y = 2, x = 6$ **f** $x = 5, y = \frac{3}{2}$

g $y = 4, x = \frac{1}{2}$ **h** $x = 2, y = -4$

i $p = -1, q = 4$ **j** $x = 3, y = 2$

k $x = 1, y = 4$ **l** $x = 1, y = 1$

m $x = 2, y = 4$ **n** $x = -2, y = 7$

o $x = -12, y = -4$ **p** $x = 2, y = 6$

q $x = -1, y = 5$

r infinitely many solutions

Exercise 21G

1 10 and 1 **2** 55 and 18 **3** 12 and 8

4 10 and 6 **5** $3.00

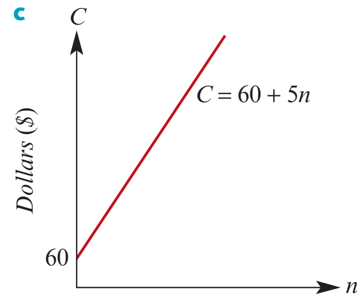
6 brown 30c, white 24c

7 David 10, Sheila 20, Anne 32

8 20 kg of B and 40 kg of A

9 a i $a = 60$ **ii** $b = 5$ **b** $2560

c



Exercise 21H

1 $x = 200$

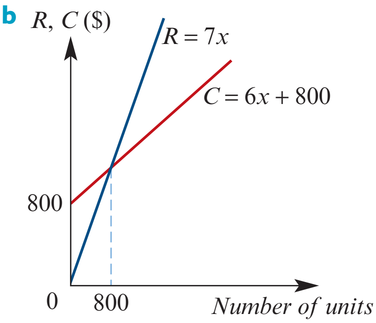
2 a $x = 2$ **b** $x = 3$ **c** $x = 6$ **d** $x = 3$

e $x = 5$ **f** $x = 8$

3 a 900 **b** $45 000 **c** $22 500 **d** 2700

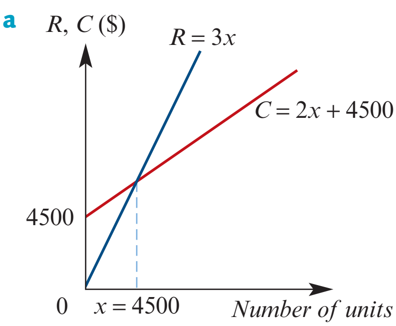
4 a $x = 800$

b



c 1200

5 a



$x = 4500$

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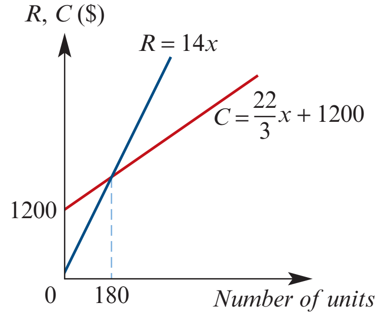
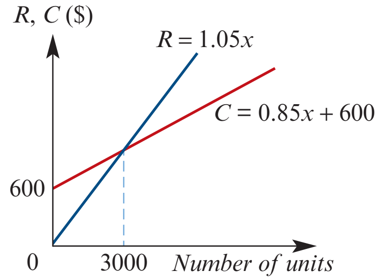
b



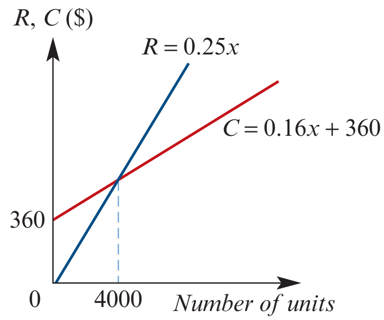
$x = 180$

c



$x = 3000$

d



$x = 4000$

Chapter 21 review

Multiple-choice questions

1 B	**2** B	**3** E	**4** B
5 C	**6** C	**7** C	**8** E
9 C	**10** C	**11** D	**12** A
13 B	**14** E	**15** D	**16** E
17 B			

Extended-response question

1 a $R = 65x$ **b** 1700 **c** 20

Chapter 22

Exercise 22A

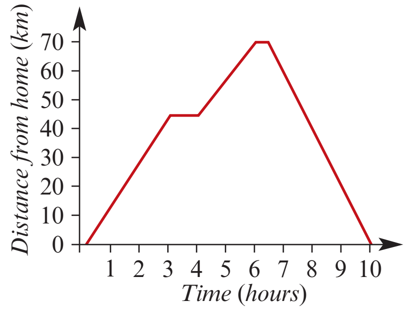
1 a i 30 km **ii** 20 km **iii** 70 km

b i 30 min **ii** $2\frac{3}{4}$ hours **iii** $1\frac{1}{2}$ hours

c i 10 km/h **ii** 20 km/h

iii 40 km/h **iv** 10 km/h

2 a

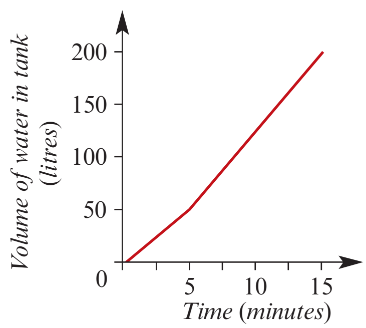


b 10 hours

3 a i 5:00 **ii** 12:00 **iii** 9:00 **iv** 6:00

b 1 hour **c** 10:30

4 a



b i $V = 10t$, V = amount of water in tank, t = time in minutes

ii $V = 15t - 25$

c 200 litres

5 a 10 L/min **b** $V = 1000 - 10t$ **c** 20 L/min

6 a 50 km/hour **b** 5 hours **c** 2 hours

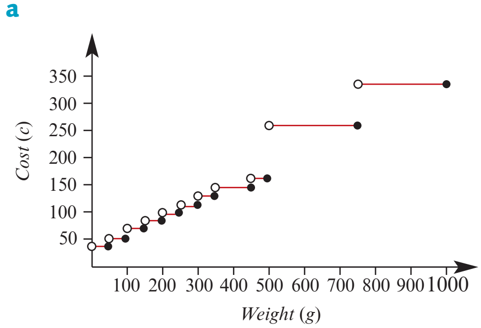
d 100 km from town A or 50 km from town B

e 30 min **f** 60 km/hour

7 a 4 min **b** 25 L/min **c** 200 L/min

Exercise 22B

1 a



b i \$1.38 **ii** 90c **iii** 76c

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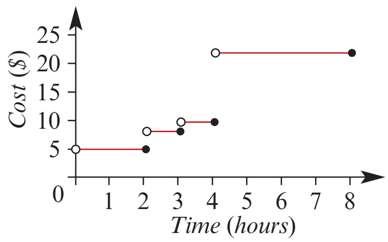
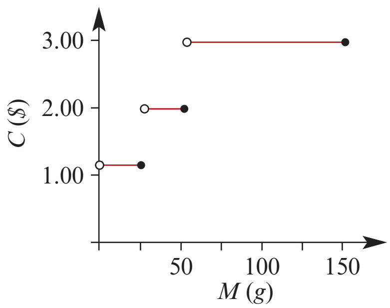
2



3 a

$$C = \begin{cases} \$1.20 \text{ for } 0 \leq M \leq 20 \\ \$2.00 \text{ for } 20 < M \leq 50 \\ \$3.00 \text{ for } 50 < M \leq 150 \\ \text{where } C \text{ is in dollars, } m \text{ is in grams} \end{cases}$$

b

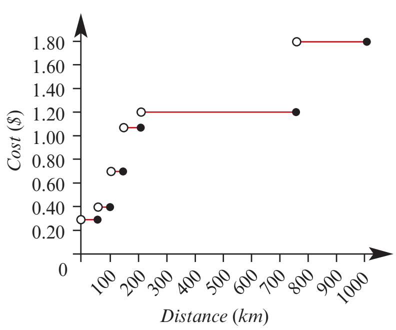


4 a

$$C = \begin{cases} \$0.30 \text{ for } d \leq 25 \\ \$0.40 \text{ for } 25 < d \leq 50 \\ \$0.70 \text{ for } 50 < d \leq 85 \\ \$1.05 \text{ for } 85 < d \leq 165 \\ \$1.22 \text{ for } 165 < d \leq 745 \\ \$1.77 \text{ for } d > 745 \end{cases}$$

where C is in dollars, d is in km

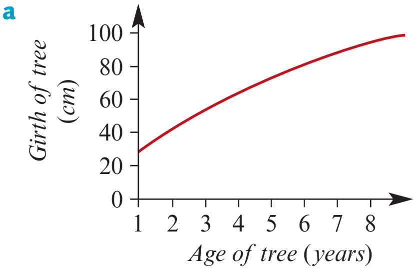
b



5 a free **b** \$3.00 **c** \$5.00

Exercise 22C

1 a



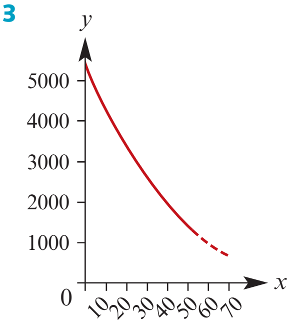
b 70 cm **c** $5\frac{1}{4}$ years

2 a 3 hours **b** 40°C

c 10:00, 14:00, 16:00, 18:50

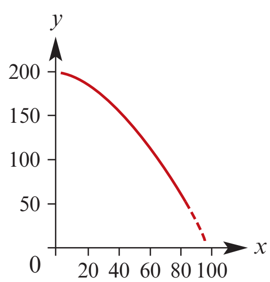
d 39.5°C **e** 19:45

3



a $y = 2800$ **b** $y = 1250$

4

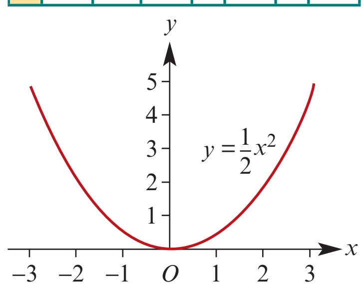


a $y = 176$ **b** $y = 30$

Exercise 22D

1 a

x	−3	−2	−1	0	1	2	3
y	4.5	2	0.5	0	0.5	2	4.5



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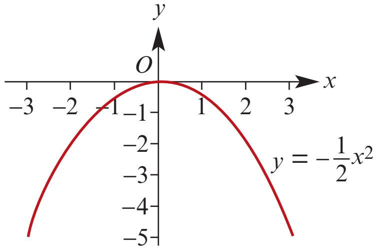
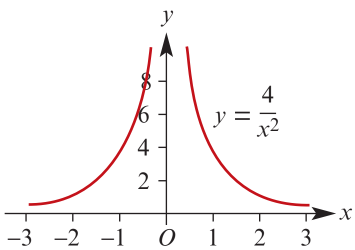
b

x	−3	−2	−1	0	1	2	3
y	4.5	−2	−0.5	0	−0.5	−2	4.5



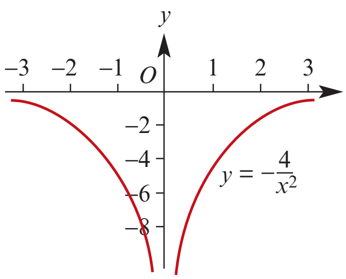
c

x	−3	−2	−1	0	1	2	3
y	$\frac{4}{9}$	1	4	–	4	1	$\frac{4}{9}$



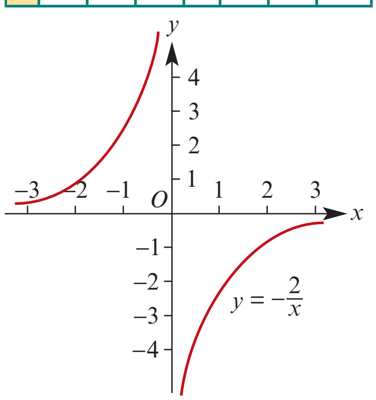
d

x	−3	−2	−1	0	1	2	3
y	$-\frac{4}{9}$	−1	−4	–	−4	−1	$-\frac{4}{9}$



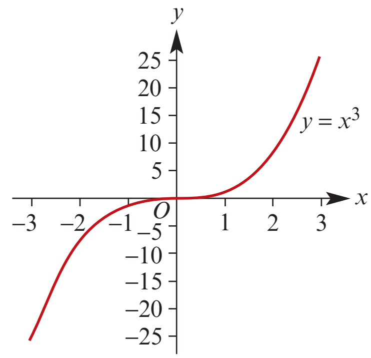
e

x	−3	−2	−1	0	1	2	3
y	$\frac{2}{3}$	1	2	–	−2	−1	$-\frac{2}{3}$



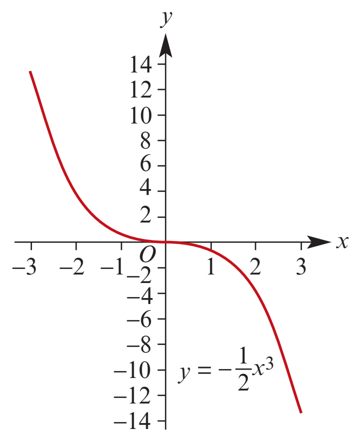
f

x	−3	−2	−1	0	1	2	3
y	−27	−8	−1	0	1	8	27



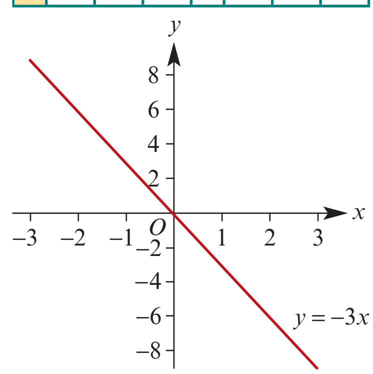
g

x	−3	−2	−1	0	1	2	3
y	13.5	4	0.5	0	−0.5	−4	−13.5



h

x	−3	−2	−1	0	1	2	3
y	9	6	3	0	−3	−6	−9



i

x	−3	−2	−1	0	1	2	3
y	2	$\frac{4}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	$-\frac{4}{3}$	−2

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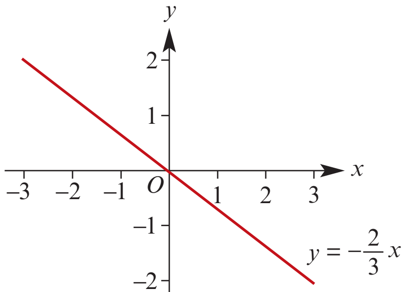


2 $k = 5$ **3** $k = 7.5$ **4** $k = 64$

5 $k = 4$ **6** $k = -3\frac{1}{3}$

Exercise 22E

1 **a** $k = 2.5$

b

x	0	2	3	6	7
y	0	20	67.5	540	857.5

2 **a** $k = 24$

b

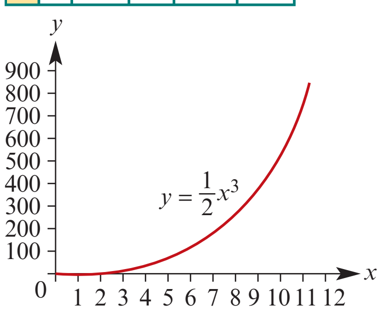
x	1	2	4	8	12
y	24	6	1.5	0.375	0.167

3 **a** $k = \frac{1}{2}$

b

x	2	6	4	3	8
y	4	108	32	13.5	256

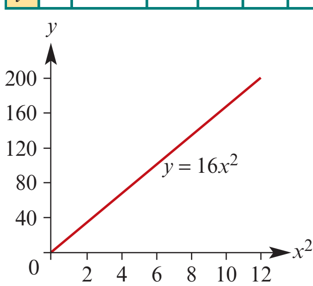
c



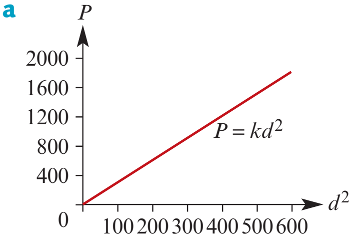
4 **a** $k = 16$

x	0	0.354	0.5	1	2	3.536
y	0	2	4	16	64	200

b

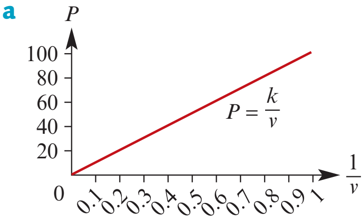


5 **a**



b $P = 3d^2$

6 **a**



b $P = \frac{100}{v}$

7 **a** $k = 9.8$ **b** 245 m

8 **a** y is multiplied by $\frac{1}{4}$.

b y is multiplied by $\frac{1}{2}$.

c y is multiplied by 2.

d y is multiplied by 4.

e y is multiplied by 8.

9 **a** 0.216 L **b** 20 cm

10 **a** $y = 1.2x^2$ **b** $y = \frac{1}{8x}$

c $y = 6x^3$ **d** $y = \frac{1}{2}x^2 + 1$

e $y = -\frac{1}{2}x^2 + 6$ **f** $y = \frac{1}{2x^2}$

Chapter 22 review

Multiple-choice questions

1 C **2** D **3** E **4** A **5** C **6** E
7 C **8** D **9** B **10** A **11** C **12** E
13 D **14** B **15** A **16** B **17** C **18** A
19 B **20** A **21** D **22** A

Extended-response questions

1 **a** 110 beats/min **b** 80 beats/min

c **i** max. pulse rate = 220 – age

ii 120–150 beats/min

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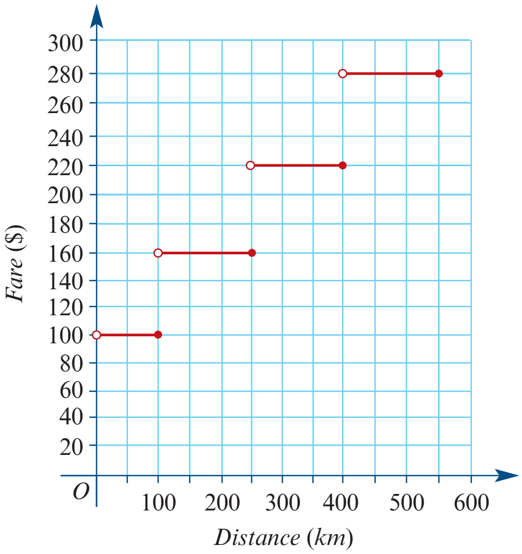
2 a 250 km

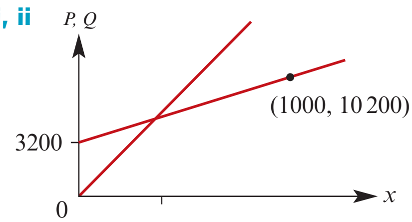
b



c \$30 **d** 360 km **e** $a = 60,\ b = \frac{2}{5}$

3 a \$4600 **b** \$3000

c i, ii



$P = 3200 + 7x \quad Q = 15x$

d 400

4 a 3.5 minutes

b i EF **ii** 200 metres per minute

Chapter 23

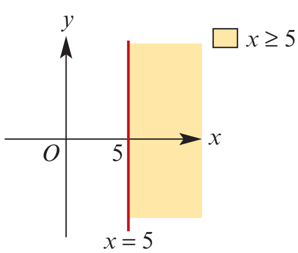
Exercise 23A

1 a C: on the line with equation $5x + 6y = 22$

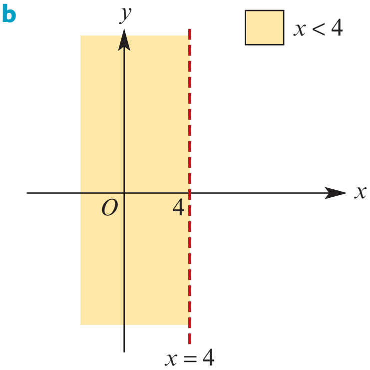
b B: $5x + 6y < 22$ **c** A: $5x + 6y > 22$

d B: $5x + 6y < 22$

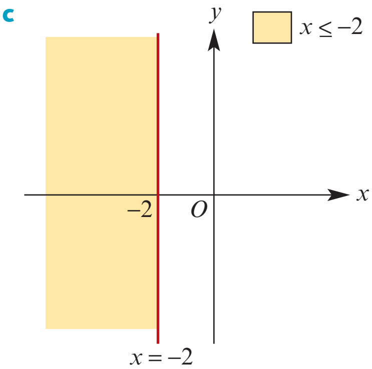
2 a



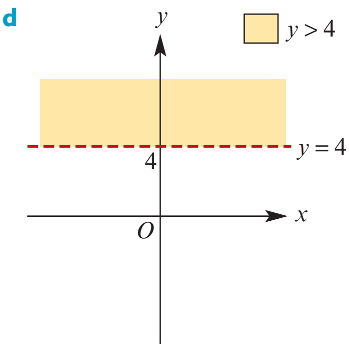
b



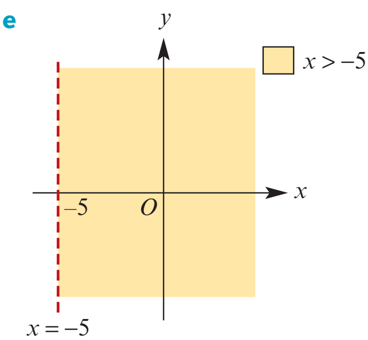
c



d



e



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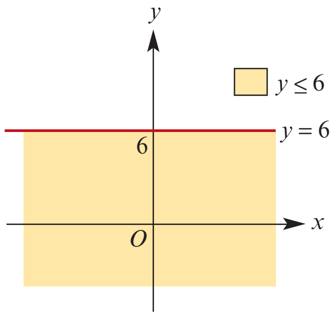
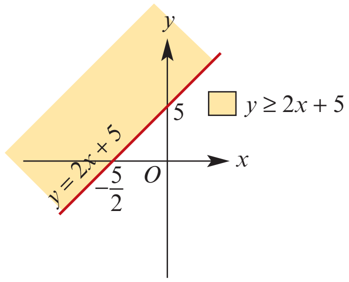
f



3 a



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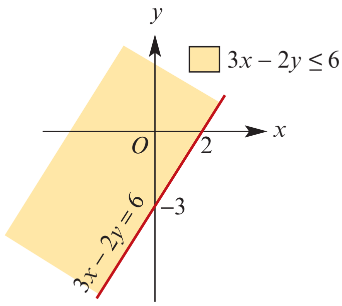

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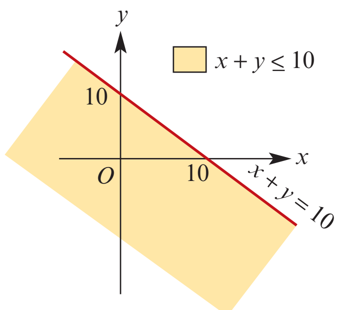
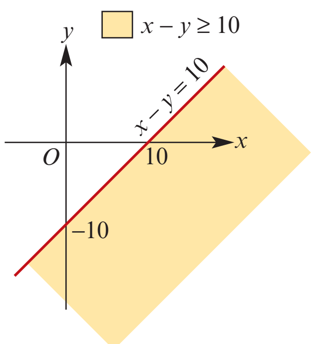


d



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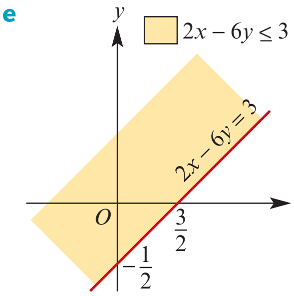

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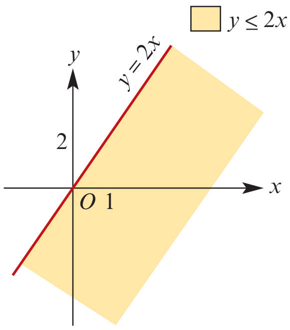
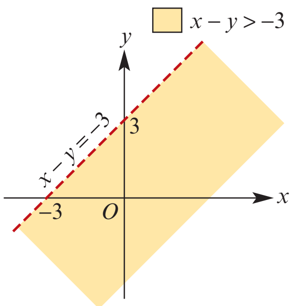
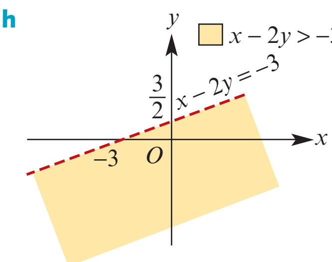


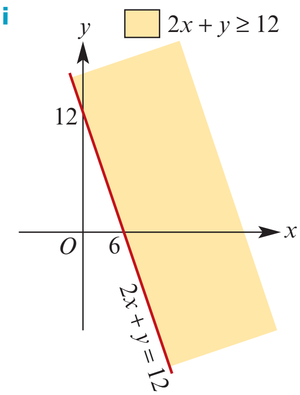
g



h



i



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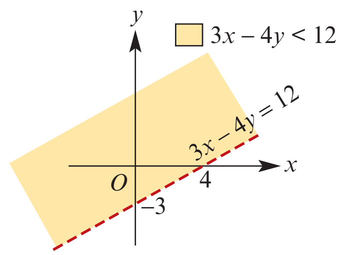
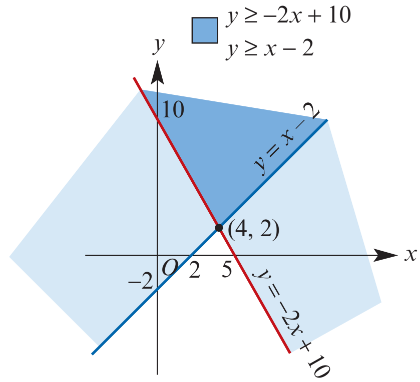
j



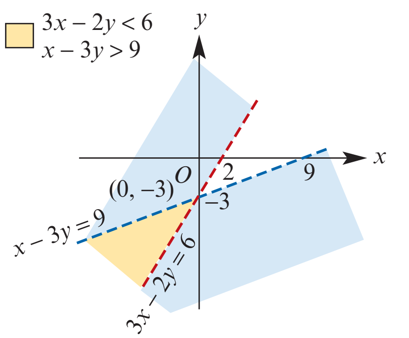
Exercise 23B

1 a



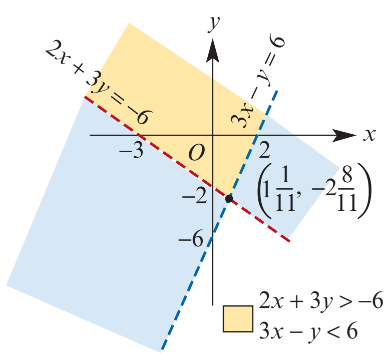
Point of intersection: (4, 2)

b



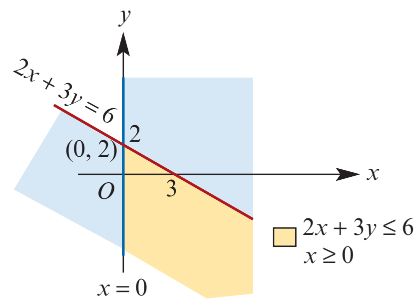
Point of intersection: (0, −3)

c



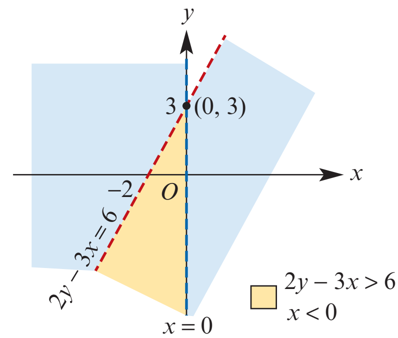
Point of intersection: $\left(1\frac{1}{11}, -2\frac{8}{11}\right)$

d



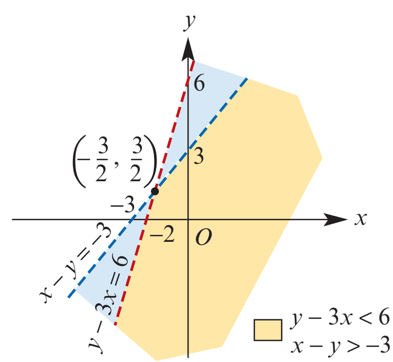
Point of intersection: (0, 2)

e



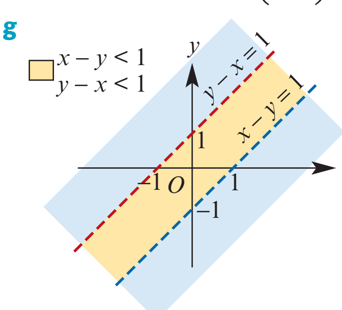
Point of intersection: (0, 3)

f



Point of intersection: $\left(-\frac{3}{2}, \frac{3}{2}\right)$

g



No point of intersection
(lines are parallel)

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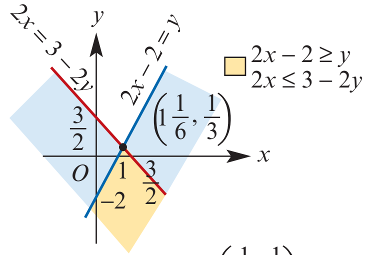
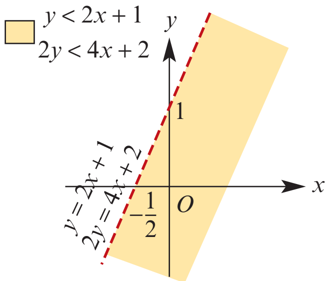
h



Point of intersection: $\left(1\frac{1}{6}, \frac{1}{3}\right)$

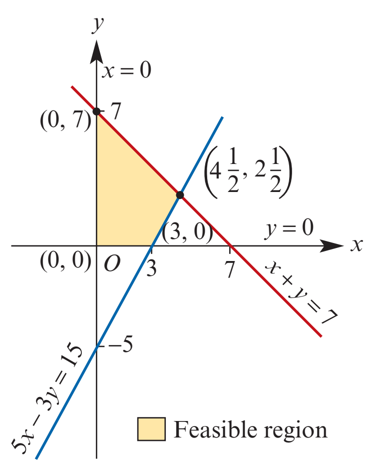
i



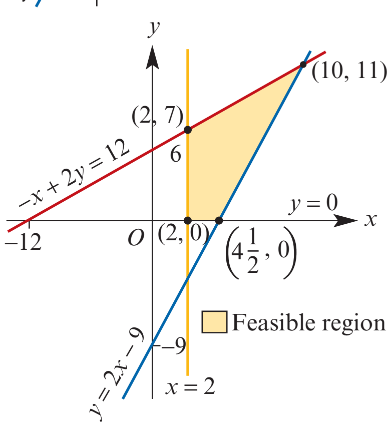
Both linear equations describe the same line

Exercise 23C

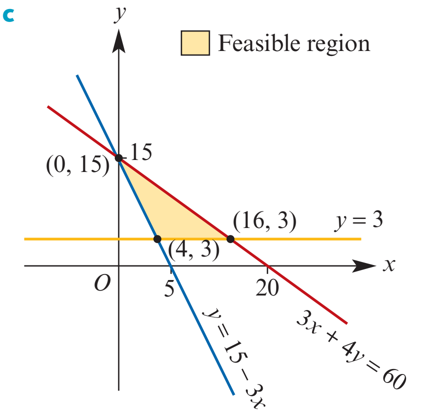
1 a



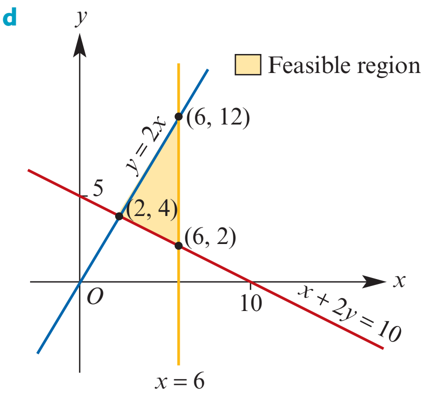
b



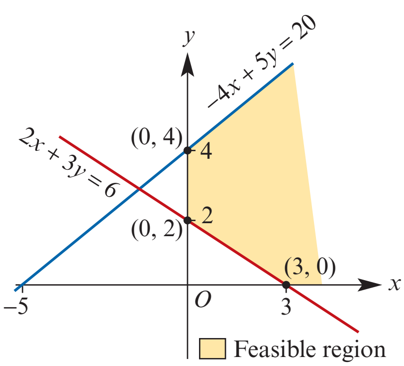
c



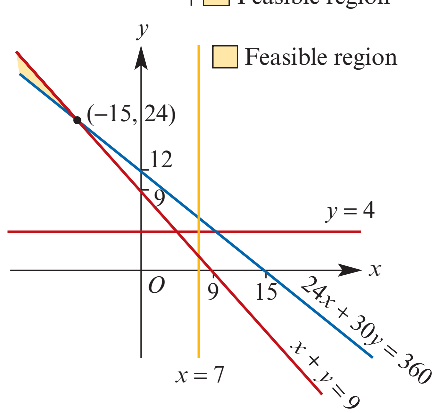
d



e



f



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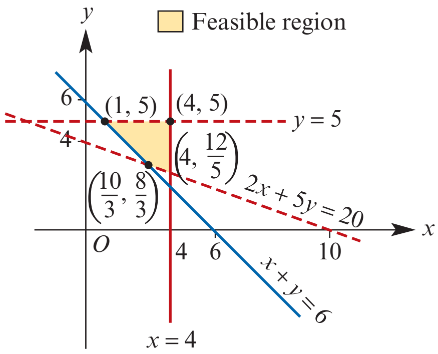
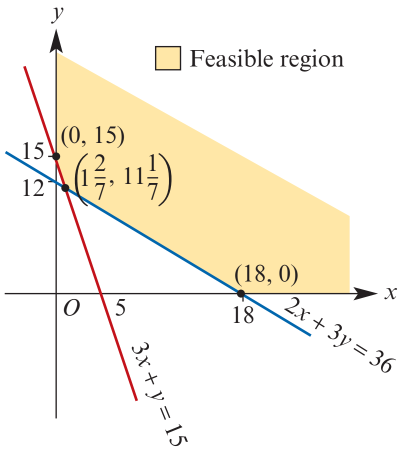
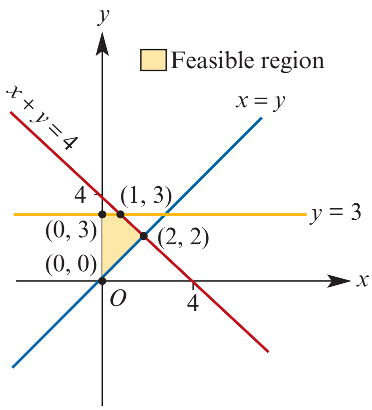
g



h



i



Exercise 23D

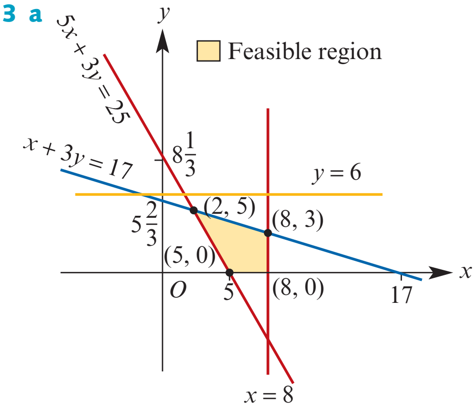
1 a $A(0,\ 4)$, $B(0,\ 0)$, $C(4,\ 0)$, $D(7,\ 7.5)$

b $z = 22$

2 a $A(0,\ 6)$, $B(0,\ 0)$, $C(3,\ 0)$, $D(5,\ 4)$, $E(2.5,\ 6)$

b $z = 14$

3 a

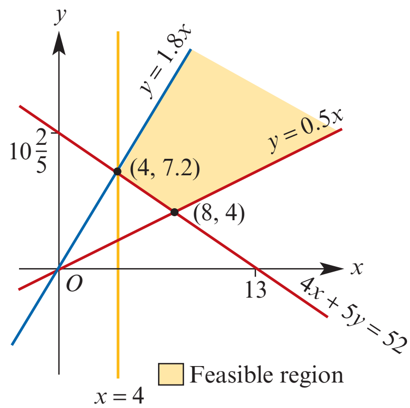


b $z = 17$

4 a $A(0,\ 12), B(2,\ 4),\ C\left(\frac{14}{3},\ 2\right),\ D\left(\frac{44}{3},\ 0\right)$

b $z = 14$

5 a



b $z = 72$

6 a (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1), (4, 0)

b $p = 7, (x, y) = (3, 1)$

7 a $P = 11$ **b** $P = 16$

8 a The total cost of buying x CDs and y books must not exceed $156.

b The enclosed triangular region excluding the vertical and horizontal boundaries should be shaded.

c (3, 7), (3, 8), (3, 9), (4, 7), (4, 8), (5, 7)

d The point (2, 8) lies on a boundary that is excluded.

e $T = x + y$

f 11 items in total: could be 3 CDs and 8 books, 4 CDs and 7 books or 5 CDs and 6 books

9 a The total amount of ice-cream made must be 2000 litres or less.

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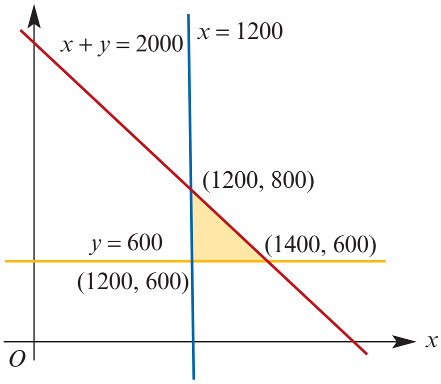
b

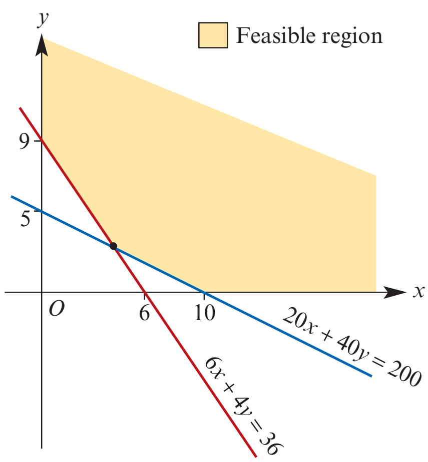


c $P = 1.10x + 0.95y$ **d** \$2110

e $P = 1.00x + 1.00y$

f \$2000; any point on the line $x + y = 2000$ for which $1200 \le x \le 1400$

10 a



b $C = 3000x + 4000y$

c minimum cost = \$24000; 4 Hawk aircraft; 3 Eagle aircraft

11 a $20x + 35y \ge 150$ **b** $W = 36x + 56y$

c 4 serves of X; 2 serves of Y

12 a $3x + 2y \ge 160$ **b** $5x + 2y \ge 200$

c $x + 2y \ge 80$ **d** $C = 4x + 3y$

13 a $x \le 15\,000$, $y \le 20\,000$ **b** $y \ge 2x$

c $1.2x + 1.12y \le 25\,000$

d $P = 0.055x + 0.046y$

14 a The total number of *A*s used cannot exceed 280.

b C2 : $8x + 11y \le 260$, C3 : $10x + 13y \le 320$

c $P = 84x + 72y$

d $P = 120x + 90y$

15 200 polar bears, 60 Polar foxes; max. profit = \$9720

16 8 Redhawk, 3 Blackjet

Chapter 23 review

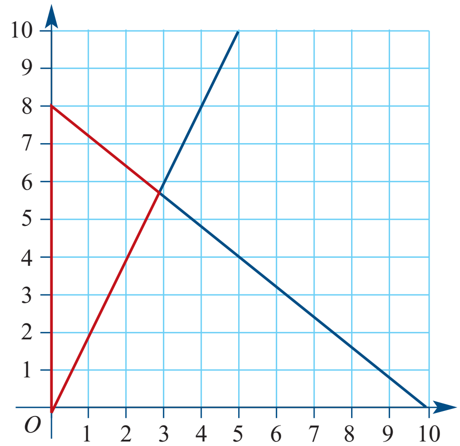
Multiple-choice questions

1 D **2** A **3** D **4** A
5 D **6** C **7** D **8** B
9 D **10** B **11** B **12** B
13 D

Extended-response questions

1 a $y \ge 2x$

b i



ii 2 dogs

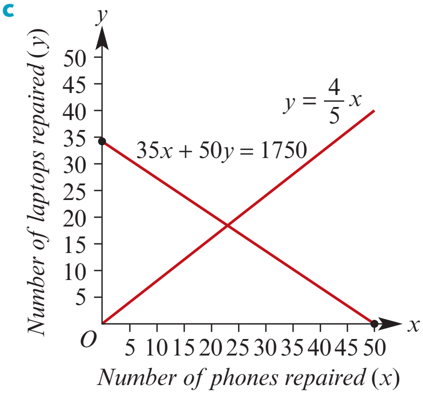
c $P = 40x + 30y$

d i 2 washes and 6 clips **ii** \$260

2 a less than 29.1 hours available for repairs

b 8

c



d 18 **e** 11

f i 23 phones, 18 laptops **ii** \$3180

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Chapter 24

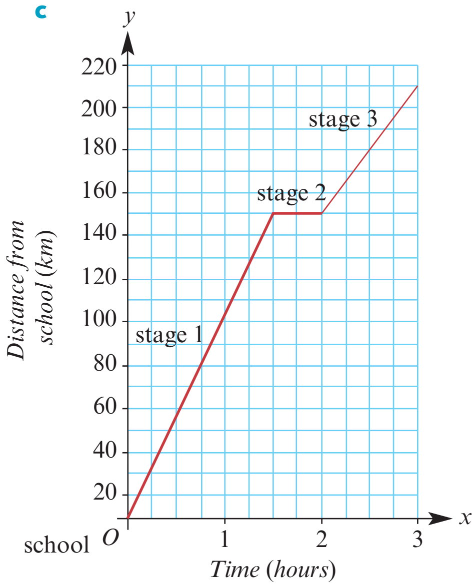
Exercise 24A

1 D	**2** B	**3** E	**4** C
5 D	**6** E	**7** B	**8** A
9 A	**10** A	**11** A	**12** E
13 E	**14** D	**15** A	**16** C
17 D	**18** A	**19** D	**20** D
21 B	**22** D	**23** D	**24** E
25 D	**26** C	**27** A	**28** D
29 D	**30** B	**31** E	**32** A
33 C	**34** D	**35** B	**36** C
37 E	**38** C		

Exercise 24B

1 a 100 km/h **b** 30 minutes

c



d 70 km/h **e** 30

2 a 7 hours **b** 4 hours

3 a $P = 18n - 260$ **b** 43 students

4 a $a = 6,\ b = 4$

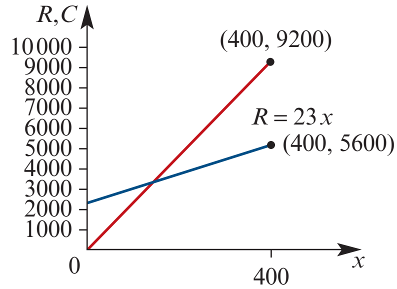
b (5, 10), (5, 9), (5, 8), (5, 7), (5, 6), (5, 5), (4, 8), (4, 7), (4, 6)

c 10

d i \$420 **ii** \$420

5 a \$5600 **b** 75 shirts

c

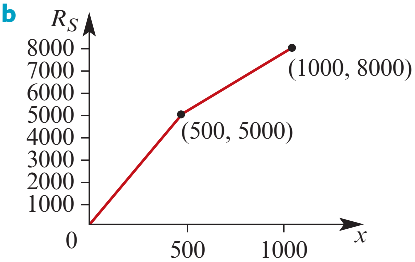


d 160 **e** $P = 15x - 2400$ **f** \$2775

g \$31.20

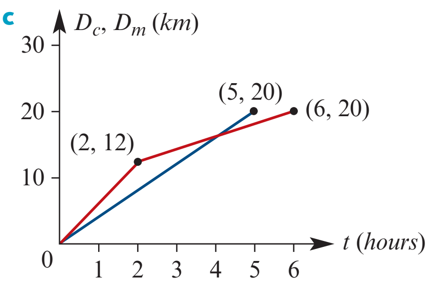
6 a \$5720

b



7 a 6 km

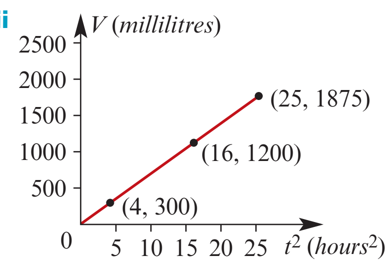
b, c



d 4 hours **e** $a = 6,\ b = 2,\ h = 8,\ d = 6$

8 a i 4, 16, 25

ii



b $k = 75$ **c** 675 millilitres

d i 3.65 hours **ii** 0.7 km

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