CAMBRIDGE

GENERAL MATHEMATICS

VCE UNITS 1 & 2

CAMBRIDGE SENIOR MATHEMATICS **VCE**

SECOND EDITION

PETER JONES | KAY LIPSON | DAVID MAIN | BARBARA TULLOCH
ROSE HUMBERSTONE | PETER KARAKOUSSIS | KYLE STAGGARD

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Introduction and overview

Cambridge General Mathematics VCE Units 1&2 Second Edition provides a complete teaching and learning resource for the VCE Study Design **to be first implemented in 2023**. It has been written with understanding as its chief aim, and with ample practice offered through the worked examples and exercises. The work has been trialled in the classroom, and the approaches offered are based on classroom experience and the responses of teachers to earlier editions of this book and the requirements of the new Study Design.

The course is designed as preparation for General Mathematics Units 3 and 4. The textbook and its resources provide a comprehensive coverage of the assumed knowledge and skills required.

General Mathematics Units 1 and 2 provide an introductory study of topics in the investigation and comparison of data distributions, arithmetic and geometric sequences, first-order linear recurrence relations and financial mathematics, linear functions, matrices, relationships between two numerical variables, graphs and networks, variation, measurement and trigonometry.

In keeping with the requirements of the new course, a rich and varied range of **Investigations** is provided in the Unit 1 and Unit 2 Revision chapters. Multiple-choice and extended-response questions, covering the year's course, are also included in the **Revision chapters**.

Key features of *Cambridge General Mathematics VCE Units 1&2 Second Edition*

For each topic within a chapter:

- The topic starts with a clear outline of its **Learning intentions**
- Every example is followed by an associated **Now try this** task, with appropriate hints so that students and their teacher can be confident the skill has been mastered
- The **Section summary** provides a clear and concise outline of the topic
- The questions in each **Exercise** are presented in three groups:
  Building understanding: warm up questions covering basic skills and building confidence
  Developing understanding: providing valuable practice applying the broad range of skills encountered in the topic
  Testing understanding: challenging questions designed to extend students' interest in the topic.

The **Review** section at the end of each chapter:

- Presents a **Key ideas and chapter summary**
- A **Skills checklist** provides a question to check mastery of each skill listed in the **Learning intentions**
- **Multiple-choice, short-answer** and **written-response** questions review the skills and concepts of the chapter.

The textbook is supported by an extensive range of online teacher and student resources.

The TI-Nspire calculator examples and instructions have been completed by Peter Flynn, and those for the Casio ClassPad by Mark Jelinek, and we thank them for their helpful contributions.

Overview of the print book

1 Learning Intentions added at the beginning of each section to clearly outline the goals of the lesson.

2 Graded step-by-step worked examples with precise explanations (and video versions) encourage independent learning, and are linked to exercise questions.

3 ‘Now Try This’ questions follow worked examples to give students immediate practice.

4 Section summaries provide important concepts in boxes for easy reference.

5 Additional linked resources in the Interactive Textbook are indicated by icons, such as skillsheets and video versions of examples.

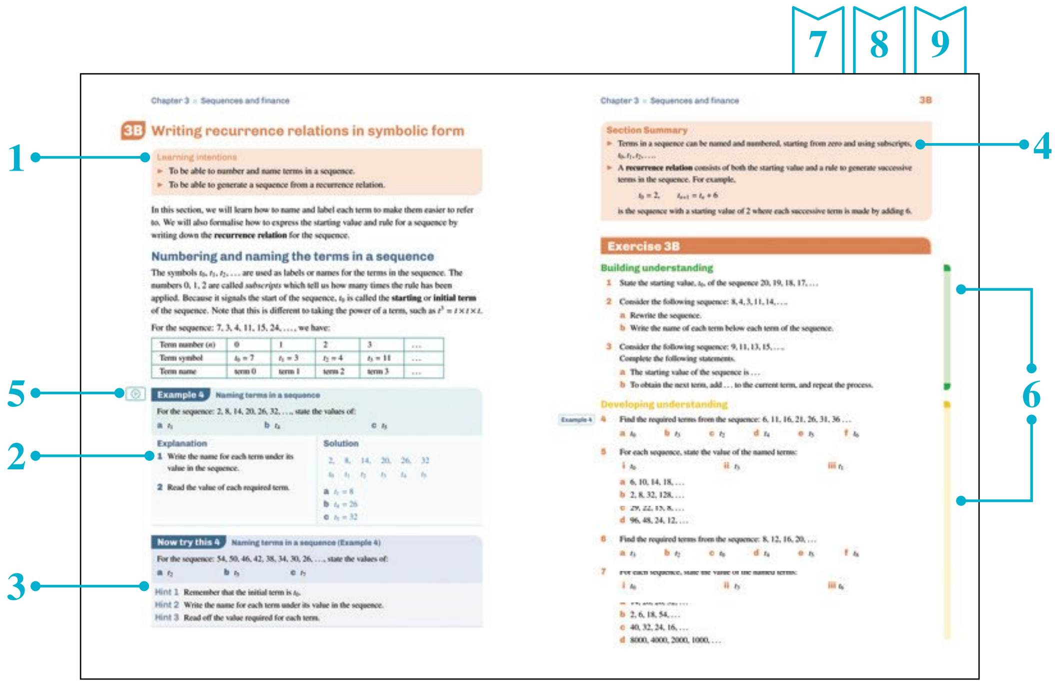
6 Exercises are divided up into three levels of questions – Building Understanding, Developing Understanding and Testing Understanding – to support differentiation.

7 Chapter reviews contain a chapter summary, a Skills Checklist with example questions, and multiple-choice and short-answer review questions.

8 Revision chapters provide comprehensive revision and preparation for assessment, including practice Investigations.

9 The glossary includes page numbers of the main explanation of each term.

Numbers refer to descriptions above.



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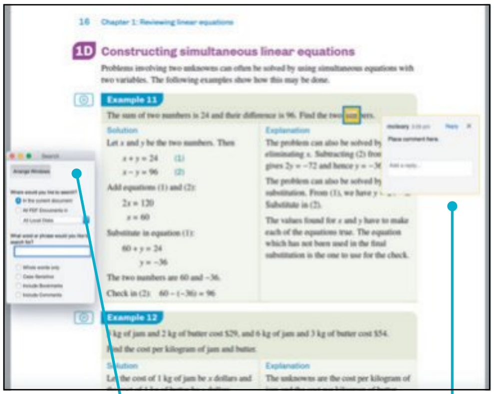
Overview of the downloadable PDF textbook

10 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.

11 PDF annotation and search features are enabled.



11 11

Overview of the Interactive Textbook

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

12 The material is formatted for on screen use with a convenient and easy-to-use navigation system and links to all resources.

13 **Workspaces** for all questions, which can be enabled or disabled by the teacher, allow students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploading images of writing or drawing done on paper.

14 **Self-assessment tools** enable students to check answers, mark their own work, and rate their confidence level in their work. This helps develop responsibility for learning and communicates progress and performance to the teacher. Student accounts can be linked to the learning management system used by the teacher in the Online Teaching Suite, so that teachers can review student self-assessment and provide feedback or adjust marks.

15 All worked examples have **video versions** to encourage independent learning.

16 **Worked solutions** are included and can be enabled or disabled in the student ITB accounts by the teacher.

17 An expanded and revised set of **Desmos interactives** and activities based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics.

18 The **Desmos graphics calculator, scientific calculator**, and **geometry tool** are also embedded for students to use for their own calculations and exploration.

19 **Revision of prior knowledge** is provided with links to diagnostic tests and Year 10 HOTmaths lessons.

20 **Quick quizzes** containing automarked multiple-choice questions have been thoroughly expanded and revised, enabling students to check their understanding.

21 **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

22 Messages from the teacher assign tasks and tests.

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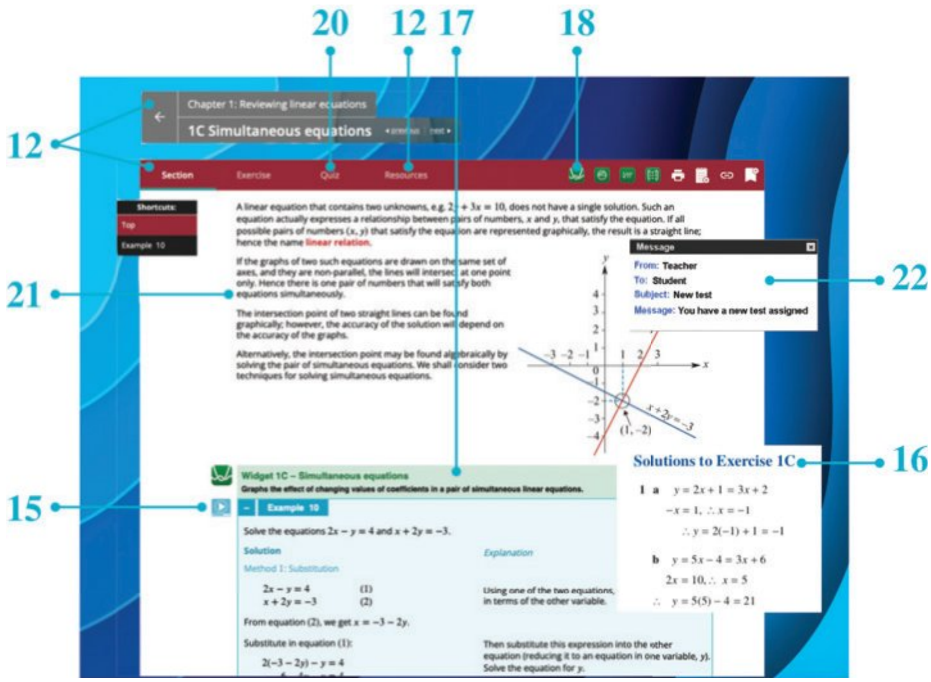
INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

A selection of features is shown. Numbers refer to the descriptions on pages xi–xii. HOTmaths platform features are updated regularly



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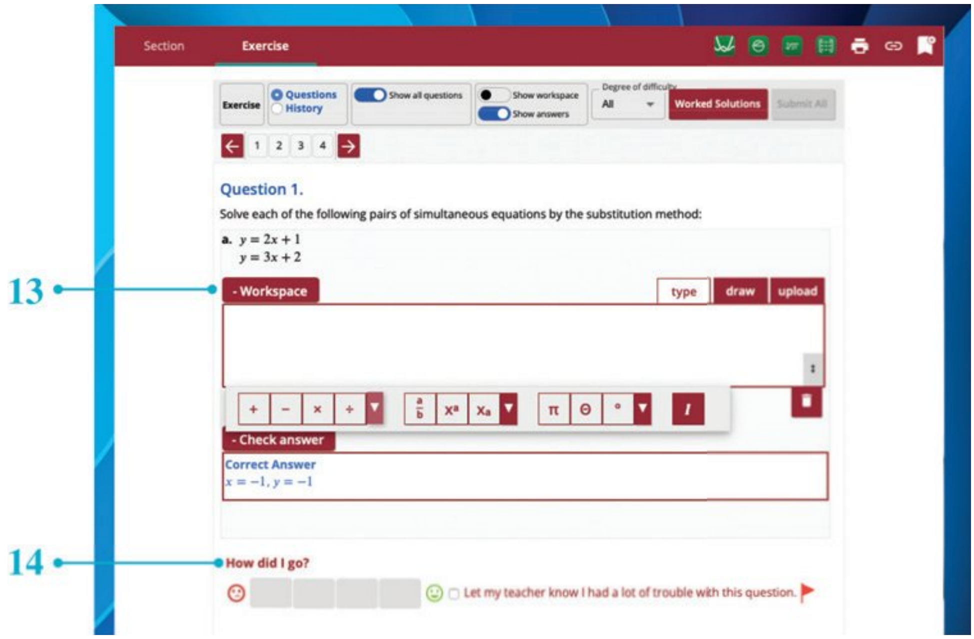

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Overview of the Online Teaching Suite powered by the HOTmaths platform

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

23 The HOTmaths learning management system with class and student analytics and reports, and communication tools.
24 Teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had diffculty.
25 A HOTmaths-style test generator.
26 An expanded and revised suite of chapter tests, assignments and sample investigations.
27 Editable curriculum grids and teaching programs.
28 A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

More about the Exam Generator

The Online Teaching Suite includes a comprehensive bank of VCAA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- VCAA marking scheme
- Multiple-choice exams can be auto-marked if completed online, with filterable reports
- All custom exams can be printed and completed under exam-like conditions or used as revision.

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Chapter 1

Review of percentages and ratios

Chapter questions

- What is a percentage?
- How do we convert percentages to fractions or decimals?
- How do we convert fractions or decimals to percentages?
- How do we calculate percentage increase, percentage decrease and percentage change?
- What is GST and how do we calculate it?
- How do we express ratios in their simplest form?
- How do we divide quantities in given ratios?
- What is the unitary method?
- How do we solve practical problems involving ratios, percentages and the unitary method?

This chapter revises percentages and ratios. An understanding of these topics will be useful for our study of data (Chapter 2) and finance (Chapter 3).

1A Percentages

Learning intentions

- To be able to express fractions and decimals as percentages.
- To be able to convert percentages to fractions or decimals.
- To be able to find a percentage of a quantity.
- To be able to compare two quantities.

Per cent is an abbreviation of the Latin words *per centum*, meaning 'by the hundred'.
A **percentage** is a rate or a proportion expressed as a part of one hundred. The symbol used to indicate percentage is %. Percentages can be expressed as common fractions or as decimals.

For example: 17% (17 per cent) means 17 parts out of every 100.

$$17\% = \frac{17}{100} = 0.17$$

Conversions

1. To convert a fraction or a decimal to a percentage, multiply by 100.
2. To convert a percentage to a decimal or a fraction, divide by 100.

Example 1 Converting a fraction to a percentage

Express $\frac{36}{90}$ as a percentage.

Explanation	Solution
Method 1 (by hand)	
1 Multiply the fraction $\frac{36}{90}$ by 100.	$\frac{36}{90} \times 100$
2 Evaluate and write your answer.	$= 40\%$
Method 2 (using CAS)	
1 Enter 36 ÷ 90 on the calculator.	36/90% 40
2 Press % sign and EXE (Casio) or ENTER (Ti-Nspire).	
3 Write your answer.	Expressed as a percentage, $\frac{36}{90}$ is 40%.

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Now try this 1 Converting a fraction to a percentage (Example 1)

Express $\frac{14}{70}$ as a percentage.

Hint 1 Multiply the fraction by 100.

Example 2 Converting a decimal to a percentage

Express 0.75 as a percentage.

Explanation	Solution
1 Multiply 0.75 by 100.	0.75×100
2 Evaluate and write your answer.	$= 75\%$

Now try this 2 Converting a decimal to a percentage (Example 2)

Express 0.25 as a percentage.

Hint 1 Multiply the decimal by 100.

Example 3 Converting a percentage to a fraction

Express 62% as a common fraction.

Explanation	Solution
1 As 62% means 62 out of 100, this can be written as a fraction $\frac{62}{100}$.	$62\% = \frac{62}{100}$
2 Simplify the fraction by dividing both the numerator and the denominator by 2.	$= \frac{62 \div 2}{100 \div 2}$ $= \frac{31}{50}$

Now try this 3 Converting a percentage to a fraction (Example 3)

Express 78% as a fraction.

Hint 1 Remember that % means "out of 100".

Hint 2 Simplify the fraction.

Example 4 Converting a percentage to a decimal

Express 72% as a decimal.

Explanation	Solution
1 Write 72% as a fraction out of 100 and express this as a decimal.	$\frac{72}{100} = 0.72$

Now try this 4 Converting a percentage to a decimal (Example 4)

Express 45% as a decimal.

Hint 1 Start by writing it as a fraction out of 100.

Finding a percentage of a quantity

To find a percentage *of* a number or a quantity, remember that, in mathematics, *of* means *multiply*.

Example 5 Finding a percentage of a quantity

Find 15% of \$140.

Explanation	Solution
Method 1	
1 Write out the problem and rewrite 15% as a fraction out of 100.	15% of 140 $= \frac{15}{100}$ of 140
2 Change *of* to *multiply*.	$= \frac{15}{100} \times 140$
3 Perform the calculation and write your answer. Note: The above calculation can be performed on the ClassPad calculator.	$= 21$ 15% of \$140 is \$21
Method 2 (using CAS)	
1 Enter 15%140 on the calculator.	15%140 21
2 Press EXE (Casio) or ENTER (Ti-Nspire).	
3 Write your answer and include the \$ sign.	\$21

Now try this 5 Finding a percentage of a quantity (Example 5)

Find 30% of \$90.

Hint 1 Write 30% as a fraction out of 100.

Hint 2 Remember *of* means *multiply*.

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Comparing two quantities

One quantity or number may be expressed as a percentage of another quantity or number (both quantities must always be in the same units). Divide the quantity by what you are comparing it with and then multiply by 100 to convert it to a percentage.

Example 6 Expressing a quantity as a percentage of another quantity

There are 18 girls in a class of 25 students. What percentage of the class are girls?

Explanation	Solution
1 Work out the fraction of girls in the class.	Girls $= \frac{18}{25}$
2 Convert the fraction to a percentage by multiplying by 100.	$\frac{18}{25} \times 100$
3 Evaluate and write your answer.	$= 72$ 72% of the class are girls.

Now try this 6 Expressing a quantity as a percentage of another quantity (Example 6)

There are 10 faulty batteries in a box of 25 batteries. What percentage of batteries are faulty?

Hint 1 Work out the fraction of faulty batteries in the box.

Hint 2 Convert to a percentage by multiplying by 100.

Example 7 Expressing a quantity as a percentage of another quantity with different units

Express 76 mm as a percentage of 40 cm.

Explanation	Solution
1 First, convert 40 centimetres to millimetres by multiplying by 10, as there are 10 millimetres in 1 centimetre.	$40\text{ cm} = 40 \times 10$ $= 400\text{ mm}$
2 Write 76 millimetres as a fraction of 400 millimetres.	$\frac{76}{400}$
3 Multiply by 100 to convert to a percentage.	$\frac{76}{400} \times 100$
4 Evaluate and write your answer.	$= 19\%$

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Now try this 7 Expressing a quantity as a percentage of another quantity with different units (Example 7)

Express 18 mm as a percentage of 60 cm.

Hint 1 Convert the 60 cm to mm by multiplying by 10.

Hint 2 Write 18 mm as a fraction of 600 mm and multiply by 100.

Section Summary

- To convert a fraction or a decimal to a percentage, multiply by 100.
- To convert a percentage to a decimal or a fraction, divide by 100.
- To find a percentage of a quantity, write the percentage as a fraction out of 100 and multiply by the quantity.
- To compare two quantities, make sure that they have the same units.

Skill-sheet

Exercise 1A

Building understanding

1 Express the following percentages as fractions out of 100.

a 17% **b** 94% **c** 71%

2 Write the following percentages as fractions, simplifying where possible.

a 7% **b** 13% **c** 50% **d** 10% **e** 20%

3 Rewrite the following fractions as percentages.

a $\frac{11}{100}$ **b** $\frac{23}{100}$ **c** $\frac{79}{100}$

4 Convert the following fractions to percentages by multiplying by 100.

a $\frac{1}{2}$ **b** $\frac{2}{5}$ **c** $\frac{1}{4}$

5 Convert the following decimals to percentages by multiplying by 100.

a 0.78 **b** 0.37 **c** 0.561

Developing understanding

Example 1
Example 2

6 Express the following as percentages.

a $\frac{1}{4}$ **b** $\frac{4}{5}$ **c** $\frac{3}{20}$ **d** $\frac{7}{10}$

e 0.19 **f** 0.79 **g** 2.15 **h** 39.57

i 0.073 **j** 1

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Example 3
Example 4

7 Express the following as:

i common fractions, in their lowest terms

ii decimals.

a 25% **b** 50% **c** 75% **d** 68%

e 5.75% **f** 27.2% **g** 0.45% **h** 0.03%

i 0.0065% **j** 100%

Example 5

8 Find the following correct to three significant figures.

a 15% of \$760 **b** 22% of \$500 **c** 17% of 150 m

d $13\frac{1}{2}$% of \$10 000 **e** 2% of 79.34 cm **f** 19.6% of 13.46

g 0.46% of 35€ **h** 15.9% of \$28 740 **i** 22.4% of \$346 900

j 1.98% of \$1 000 000

Example 6

9 From a class, 28 out of 35 students wanted to take part in a project. What percentage of the class wanted to take part?

10 A farmer lost 450 sheep out of a flock of 1200 during a drought. What percentage of the flock were lost?

11 In a laboratory test on 360 light globes, 16 globes were found to be defective. What percentage were satisfactory to one decimal place?

12 After three rounds of a competition, a basketball team had scored 300 points, and 360 points had been scored against them. Express the points scored by the team as a percentage of the points scored against them. Give the answer to two decimal places.

13 In a school of 624 students, 125 are in Year 10. What percentage of the students are in Year 10? Give your answer to the nearest whole number.

Example 7

14 Express 75 cm as a percentage of 2 m.

15 In a population of $3\frac{1}{4}$million people, 2 115 000 are under the age of 16. Calculate the percentage, to two decimal places, of the population who are under the age of 16.

16 The cost of producing a chocolate bar that sells for \$1.50 is 60c. Calculate the profit made on a bar of chocolate as a percentage of the production cost of a bar of chocolate.

Testing understanding

17 Kate and Tim were distributing letters. Tim had 5% more letters than Kate. Kate had 160 letters. How many did Tim have?

18 Martin and Simon were comparing their marble collections. Martin had 20% less marbles than Simon. Simon had 70 marbles. How many marbles did Martin have?

19 Ruby wanted to decorate the local hall for her party. She had 15 balloons. This was only 30% of the balloons that she required. How many balloons did she need in total?

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1B Percentage increase and decrease

When increasing or decreasing a quantity by a given percentage, the percentage increase or decrease is always calculated as a percentage of the *original* quantity.

Example 8 **Calculating the new price following a percentage increase**

Sally's daily wage of \$175 is increased by 15%. Calculate her new daily wage.

Explanation	Solution
Method 1	
1 First find 15% of \$175 by rewriting 15% as a fraction out of 100 and changing *of* to *multiply* (or use a calculator).	15% of 175 $= \frac{15}{100} \times 175$
2 Perform the calculation and write your answer.	$= 26.25$
3 As \$175 is to be increased by 15%, add \$26.25 to the original amount of \$175.	$175 + 26.25$ $= 201.25$
4 Write your answer in a sentence.	Sally's new daily wage is \$201.25.
Method 2	
1 An increase of 15% means that the new amount will be the original amount (in other words, 100%) plus an extra 15%. Find 115% of 175.	115% of 175 $= \frac{115}{100} \times 175$ $= 201.25$
2 Perform the calculation.	
3 Write your answer in a sentence.	Sally's new daily wage is \$201.25.

Now try this 8 **Calculating the new price following a percentage increase (Example 8)**

Tom's weekly wage of \$950 has increased by 10%. Calculate his new weekly wage.

Hint 1 Find 10% of \$950.

Hint 2 Add this amount to \$950.

Hint 3 **Alternatively**, an increase of 10% means the new weekly wage is (100+10)%=110% of the original weekly wage.

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Example 9 Calculating the new amount following a percentage decrease

A primary school's fun run distance of 2.75 km is decreased by 20% for students in Years 2 to 4. Find the new distance.

Explanation	Solution
Method 1	
1 First find 20% of 2.75 by writing 20% as a fraction out of 100 and changing *of* to *multiply* (or use a calculator).	20% of 2.75 $= \frac{20}{100} \times 2.75$
2 Evaluate and write your answer.	$= 0.55$
3 As 2.75 km is to be decreased by 20%, subtract 0.55 km from the original 2.75 km.	$2.75 - 0.55$ $= 2.2$
4 Write your answer in a sentence.	The new distance is 2.2 km.
Method 2	
1 A decrease of 20% means that the new amount will be the original amount (100%) minus 20%. Find 80% of 2.75.	80% of 2.75 $= \frac{80}{100} \times 2.75$ $= 2.2$
2 Perform the calculation.	
3 Write your answer in a sentence.	The new distance is 2.2 km.

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Now try this 9 Calculating the new amount following a percentage decrease (Example 9)

The local council's fun run of 12 km is decreased by 8%. Find the new distance.

Hint 1 Find 8% of 12 km.

Hint 2 Subtract this amount from 12 km.

Hint 3 **Alternatively**, a decrease of 8% means the new distance is (100 – 8)% = 92% of the original distance.

Example 10 Calculating a new price with a percentage discount

If a shop offers a discount of 15% on items in a sale, what would be the sale price of a pair of jeans originally priced at $95?

Explanation	Solution
Method 1	
1 Find 15% of 95.	$15\% \text{ of } 95 = \frac{15}{100} \times 95$ $= 14.25$
2 As jeans are discounted by 15%, this is a decrease, so we need to subtract the discounted price of $14.25 from the original price of $95.	$95 - 14.25$ $= 80.75$
3 Write your answer in a sentence.	The sale price would be $80.75.
Method 2	
1 A discount of 15% means that the new amount is 85% of 95.	$85\% \text{ of } 95$ $= \frac{85}{100} \times 95$
2 Perform the calculation.	$= 80.75$
3 Write your answer in a sentence.	The sale price would be $80.75.

Now try this 10 Calculating a new price with a percentage discount (Example 10)

A shoe shop offers a discount of 20%. What would be the sale price of a pair of shoes originally priced at $150?

Hint 1 Find 20% of $150.

Hint 2 Subtract this amount from $150 to obtain the new price.

Hint 3 **Alternatively**, 20% discount means the new price of the shoes is (100 – 20)% = 80% of the original price.

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Finding a percentage change

If we are given the original price and the new price of an item, we can find the percentage change. To find the percentage change, we compare the change (increase or decrease) with the original number.

Percentage change

$$\text{Percentage change} = \frac{\text{change}}{\text{original}} \times 100$$

Thus:

$$\text{Percentage discount} = \frac{\text{discount}}{\text{original}} \times 100$$

$$\text{Percentage increase} = \frac{\text{increase}}{\text{original}} \times 100$$

Example 11 Calculating a percentage increase

A university increased its total size at the beginning of an academic year by 3000 students. If the previous number of students was 35 000, by what percentage, to two decimal places, did the student population increase?

Explanation	Solution
1 To find the percentage increase, use the formula: $\text{Percentage increase} = \frac{\text{increase}}{\text{original}} \times 100$	$\text{Percentage increase} = \frac{\text{increase}}{\text{original}} \times 100$
Substitute increase as 3000 and original as 35 000.	$= \frac{3000}{35\,000} \times 100$
2 Evaluate.	$= 8.5714\ldots$
3 Write your answer to two decimal places.	Student population increased by 8.57%.

Now try this 11 Calculating a percentage increase (Example 11)

The number of students studying VCE at Hambrook High School increased by 70 students in one year. If the previous number of students studying VCE was 365, by what percentage, to two decimal places, did the VCE student population increase?

Hint 1 Use the formula for percentage increase and substitute in relevant values.

Hint 2 $\text{Percentage increase} = \frac{\text{increase}}{\text{original}} \times 100$

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Example 12 Calculating the percentage discount

Calculate the percentage discount obtained when a calculator with a normal price of $38 is sold for $32, to the nearest whole per cent.

Explanation

1 Find the amount of discount given by subtracting the new price, $32, from the original price, $38.

2 To find the percentage discount, use the formula:

$$\text{Percentage discount} = \frac{\text{discount}}{\text{original}} \times 100$$

Substitute discount as 6 and original as 38, and evaluate.

3 Write your answer to the nearest whole per cent.

Solution

$$\text{Discount} = \$38 - \$32$$
$$= \$6$$

$$\text{Percentage discount} = \frac{\text{discount}}{\text{original}} \times 100$$
$$= \frac{6}{38} \times 100$$
$$= 15.7895\ldots$$

The percentage discount is 16%.

Now try this 12 Calculating the percentage discount (Example 12)

Calculate the percentage discount obtained when an iPad with a normal price of $599 is sold for $529. Give your answer to the nearest whole per cent.

Hint 1 Find the amount of discount in $.

Hint 2 Use the formula for % discount. Percentage discount $= \frac{\text{discount}}{\text{original}} \times 100$

Section Summary

- When increasing or decreasing a quantity by a certain percentage, calculate the percentage of the original quantity.
- The formula for % change is: Percentage change $= \frac{\text{change}}{\text{original}} \times 100$.

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Exercise 1B

Building understanding

1 A $190 watch is for sale with a 20% discount.

a By finding 20% of $190, work out how much the discount is.

b What will be the new sale price of the watch?

Example 8

2 Sally's hourly work rate of $30 is increased by 10%.

a By how much will her hourly rate be increased?

b What is her new hourly rate?

Example 9

3 The price of a $25 book is decreased by 5%.

a By how much is it decreased?

b What is the new price of the book?

Developing understanding

4 A jewellery store has a promotion of 20% discount on all watches.

a How much discount will you get on a watch marked $185?

b What is the sale price of the watch?

Example 10

5 A store gave different savings discounts on a range of items in a sale. Copy and complete the following table. (Round to the nearest cent.)

	Normal price	% Discount	Saving	Sale price
a	$89.99	5		
b	$189.00	10		
c	$499.00	15		
d	$249.00	20		
e	$79.95	22.5		
f	$22.95	25		
g	$600.00	27.5		
h	$63.50	30		
i	$1000.00	33		

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6 In a particular shop the employees are given a $12\frac{1}{2}\%$ discount on any items they purchase. Calculate the actual price an employee would pay for each of the following:

a $1200 laptop **b** $1400 smart television

c $246 mini drone **d** $750 digital camera

7 A clothing store offers 6% discount for cash sales. A customer who paid cash purchased the following items:

- A pair of jeans at $95.95
- A leather belt at $29.95
- Two jumpers at $45 each.

Calculate, to the nearest cent:

a the total savings **b** the actual amount paid for the goods.

8 The number of cars of a particular model produced by a company is increased from 14 000 by 6% over a 12-month period. In 12 months' time, what is the new production figure?

9 If a new car is sold for $23 960, and three years later it is valued at $18 700, calculate the percentage decrease (called depreciation) to two decimal places.

10 A leading tyre manufacturer claims that a new tyre will average 12% more life than a previous tyre. The owner of a taxi fleet finds that the previous tyre averaged 24 000 km before replacement. How many kilometres should the new tyre average?

Example 12

11 Calculate the percentage discount for each of the following, to the nearest whole number.

	Normal price	Selling price	% Discount
a	$60.00	$52.00	
b	$250.00	$185.00	
c	$5000.00	$4700.00	
d	$3.80	$2.90	
e	$29.75	$24.50	
f	$12.95	$10.00	

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12 A second-hand car advertised for sale at $13 990 was sold for $13 000. Calculate, to two decimal places, the percentage discount obtained by the purchaser.

13 A sport shop advertised the following items in their end-of-year sale. Calculate the percentage discount for each of the items, to the nearest whole number.

		Normal price	Selling price	% Discount
a	Shoes	$79.99	$65.00	
b	12-pack of golf balls	$29.99	$19.99	
c	Exercise bike	$1099.00	$599.00	
d	Basketball	$49.99	$39.99	
e	Sports socks	$14.95	$10.00	
f	Hockey stick	$299.00	$250.00	

Example 11

14 Find the percentage increase that has been applied in each of the following:

a a book that is increased from $20 to $25

b an airfare that is increased from $300 to $420

c accommodation costs that are increased from $540 to $580.50.

Testing understanding

15 Which results in the larger sum of money, increasing $50 by 10% or decreasing $60 by 8%?

16 William is 1.86 m tall and Jonathon is 1.92 m tall. What percentage taller is Jonathon than William? Give your answer to two decimal places.

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1C Goods and Services Tax (GST)

Learning intentions

- To be able to calculate GST.
- To be able to find the cost of items that include GST.
- To be able to find the cost of items prior to GST being added.

The goods and services tax (GST) is a tax of 10% that is added to the price of most goods and services in Australia. GST is an example of percentage increase.

To find the GST payable, you find 10% of the original amount.

$$10\% = \frac{10}{100} = \frac{1}{10}$$

so finding 10% of the original amount is the same as dividing the original amount by 10.

Thus, Amount of GST $= \dfrac{\text{cost without GST}}{10}$

Example 13 Calculating the amount of GST when the cost without GST is known

If the cost of electricity in one quarter is $288.50, how much GST will be added?

Explanation	Solution
1 Use the rule for finding the amount of GST when cost without GST is known.	Amount of GST $= \dfrac{\text{cost without GST}}{10}$
2 Substitute in $288.50 for cost without GST. Or, use a calculator to find 10% of $288.50.	Amount of GST $= \dfrac{288.50}{10} = 28.85.$
3 Write your answer.	GST is $28.85.

Now try this 13 Calculating the amount of GST when the cost without GST is known (Example 13)

The cost of installing a sliding door is $850. How much GST will be added to this cost?

Hint 1 Use the rule, amount of GST $= \dfrac{\text{cost without GST}}{10}$ and substitute the known value.

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Example 14 Calculating the new price with GST

The bill for the services that a plumber provides is $650. GST needs to be added. What is the amount of the final bill with GST included?

Explanation	Solution
Method 1	
1 GST is 10%. Find 10% of $650.	10% of 650 $= \frac{10}{100} \times 650 = 65$
2 As $650 is to be increased by 10%, add $65 to the original amount of $650.	$650 + 65 = 715$
3 Write your answer.	Amount of final bill with GST is $715.
Method 2	
1 An increase of 10% means that the new amount will be the original amount (in other words, 100%) plus an extra 10%. Find 110% of 650.	110% of 650 $= \frac{110}{100} \times 650 = 715$
2 Write your answer.	Amount of final bill with GST is $715.

Now try this 14 Calculating the new price with GST (Example 14)

The price of a washing machine is $1400. GST needs to be added. What is the price of the washing machine with GST added?

Hint 1 Find 10% of $1400.

Hint 2 Add this amount to $1400.

Hint 3 **Alternatively**, an increase of 10% means that the new price is (100+10)% = 110% of the initial price.

Another way to evaluate cost prices with GST is to multiply the original amount by 1.1 as this is the same as multiplying by $\frac{110}{100}$ (finding 110%). Thus:

cost with GST = cost **without** GST × 1.1

We can then rearrange this equation to find the cost without GST:

$$\textbf{cost without GST} = \frac{\text{cost } \textbf{with} \text{ GST}}{1.1}$$

This can be summarised as follows:
To find the cost with GST, multiply the original cost (cost without GST) by 1.1.
To find the original cost (cost without GST), divide the cost with GST by 1.1.

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Example 15 Calculating the original cost (cost without GST)

An electric guitar costs \$2299 including GST. What is the price without GST?

Explanation	Solution
1 To find the original cost, divide the cost with GST by 1.1.	$2299 \div 1.1 = 2090$
2 Write your answer.	The price without GST is \$2090.

Now try this 15 Calculating the original cost (cost without GST) (Example 15)

A drum kit sells for \$737 including GST. What is the price without GST?

Hint 1 Divide the original price by 1.1.

We have seen that we can calculate the amount of GST from the original cost (cost without GST).

$$\textbf{Amount of GST} = 10\% \text{ of the original cost (cost without GST)}$$
$$= \frac{\text{cost without GST}}{10}$$

We can also find the amount of GST when we know the final cost (cost with GST).

Remember that: Original cost + 10% of the original cost = cost with GST

so, 110% of the original cost (cost without GST) = cost with GST.

This can be written as: $\frac{110}{100} \times \text{cost without GST} = \text{cost with GST}$

or, $\frac{11}{10} \times \text{cost without GST} = \text{cost with GST}$.

Dividing both sides by 11 gives:

$$\frac{\text{cost without GST}}{10} = \frac{\text{cost with GST}}{11}$$

But remember: $\frac{\text{cost } \textbf{without} \text{ GST}}{10} = \text{Amount of GST}$

So, $\textbf{Amount of GST} = \frac{\text{cost } \textbf{with} \text{ GST}}{11}$

We can now find the amount of GST added when we know either the original amount (cost without GST) or when we know the cost with GST.

$$\text{Amount of GST} = \frac{\text{cost } \textbf{without} \text{ GST}}{10} \quad \text{and} \quad \text{Amount of GST} = \frac{\text{cost } \textbf{with} \text{ GST}}{11}$$

Example 16 Calculating the amount of GST when the cost with GST is known

A desktop computer sells for \$935, including GST. What is the amount of GST that has been added?

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Explanation	Solution
1 Use the rule for finding the amount of GST when cost with GST is known.	$\text{Amount of GST} = \dfrac{\text{cost with GST}}{11}$
2 Substitute in \$935 for cost with GST.	$\text{Amount of GST} = \dfrac{935}{11} = 85$
3 Write your answer.	GST is \$85

Now try this 16 Calculating the amount of GST when the cost with GST is known (Example 16)

An iPad sells for \$1155, including GST. What is the amount of GST that has been added?

Hint 1 Use the rule: $\text{Amount of GST} = \dfrac{\text{cost with GST}}{11}$ and substitute the known value.

Section Summary

- GST stands for goods and services tax and is a tax of 10%.
- To find the amount of GST when the cost without GST is known, divide by 10.
- To find the amount of GST when the cost with GST is known, divide by 11.
- To calculate the cost with GST, multiply the original cost by 1.1.
- To calculate the original cost, divide the cost with GST by 1.1.

Skill-sheet

Exercise 1C

Building understanding

1 Find 10% of the following:

a \$900 **b** \$760 **c** \$599 **d** \$65 **e** \$2572 **f** \$48 755

Example 13

2 Find the GST payable on each of the following (give your answer to the nearest cent).

a a gas bill of \$121.30 **b** an electrician's bill of \$367.50

c a TV costing \$1085.50 **d** gardening services of \$395

Developing understanding

Example 14

3 The following prices are without GST. Find the price after GST has been added.

a a dress worth \$139 **b** a bedroom suite worth \$2678

c a home movie theatre worth \$9850 **d** painting services of \$1395

Example 15

4 If a computer is advertised for \$2399 including GST, how much would the computer have cost without GST to the nearest cent?

Example 16

5 What is the amount of the GST that has been added if the price of a car is advertised as \$39 990 including GST to the nearest cent?

6 A gas bill is \$109.78 after GST is added.

a What was the price before GST was added?

b How much GST must be paid?

Testing understanding

7 **a** How much was the GST on a car that sold for \$57 300 (to the nearest cent)?

b What was the pre-GST price of the car (to the nearest cent)?

8 Chris buys a camera for \$1599, including GST. How much of this price is GST (to the nearest cent)?

9 John sees a lawn mower advertised for \$765, including GST. He then sees another lawn mower for \$695 which has not as yet had GST added. Which lawn mower should he buy and how much will he save?

1D Ratio and proportion

Ratios are used to numerically compare the values of two or more quantities.
A **ratio** can be written as $a : b$ (read as 'a to b'). It can also be written as a fraction $\frac{a}{b}$.
The order of the numbers in a ratio is important. $a : b$ is *not* the same as $b : a$.

Example 17 Expressing quantities as a ratio

In a Year 10 class of 26 students, there are 14 girls and 12 boys. Express the number of girls to boys as a ratio.

Solution

As there are 14 girls and 12 boys, the ratio of girls to boys is 14 : 12.

Note: This could also be written as a fraction $\frac{14}{12}$.

Example 18 Expressing more than two quantities as a ratio

A survey of the same group of 26 students showed that 10 students walked to school, 11 came by public transport and 5 were driven by their parents. Express as a ratio the number of students who walked to school, to the number of students who came by public transport, to the number of students who were driven to school.

Solution

The order of the numbers in a ratio is important.

10 students walked, 11 used public transport and 5 were driven, so the ratio is 10 : 11 : 5.

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Now try this 18 **Expressing more than two quantities as a ratio (Example 18)**

It was found that, out of a group of 20 students, English was the favourite subject of 8 students, Mathematics was the favourite subject of 11 students and Computer Science was the favourite subject of 1 student. Express as a ratio the number of students whose favourite subject was English, to the number of students whose favourite subject was Mathematics, to the number of students whose favourite subject was Computer Science.

Hint 1 Express the numbers as a ratio, ensuring that the order is correct.

Expressing ratios in their simplest form

Ratios can be simplified by dividing through by a common factor or by multiplying each term as required.

Example 19 **Simplifying ratios**

Simplify the following ratios.

a $15:20$ **b** $0.4:1.7$ **c** $\frac{3}{4}:\frac{5}{3}$

Explanation	Solution
a 1 Divide both 15 and 20 by 5.	$15 : 20$ $= \frac{15}{5} : \frac{20}{5}$
2 Evaluate and write your answer.	$= 3 : 4$
b 1 Multiply both 0.4 and 1.7 by 10 to give whole numbers.	$0.4 : 1.7$ $= 0.4 \times 10 : 1.7 \times 10$
2 Evaluate and write your answer.	$= 4 : 17$
c Method 1	
1 Multiply both fractions by 4.	$\frac{3}{4} \times 4 : \frac{5}{3} \times 4$ $= 3 : \frac{20}{3}$
2 Multiply both sides of the ratio by 3.	$= 3 \times 3 : \frac{20}{3} \times 3$
3 Write your answer.	$= 9 : 20$
c Method 2	
1 Multiply both $\frac{3}{4}$ and $\frac{5}{3}$ by the lowest common multiple (LCM) of 3 and 4, which is 12, to eliminate fractions.	$\frac{3}{4} : \frac{5}{3}$ $= \frac{3}{4} \times 12 : \frac{5}{3} \times 12$
2 Evaluate and write your answer.	$= 9 : 20$

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In each of the above examples, the ratios are equivalent and the information is unchanged. For example, the ratio:

12 : 8 is equivalent to the ratio 24 : 16 (multiply both 12 and 8 by 2)

and

12 : 8 is also equivalent to the ratio 3 : 2 (divide both 12 and 8 by 4).

Now try this 19 **Simplifying ratios (Example 19)**

Express 14 : 49 as a ratio in simplest form.

Hint 1 Find the largest common factor of 14 and 49, and divide both 14 and 49 by this factor.

Example 20 **Simplifying ratios with different units**

Express 15 cm to 3 m as a ratio in its simplest form.

Explanation	Solution
1 Write down the ratio.	15 cm : 3 m
2 Convert 3 m to cm by multiplying 3 m by 100, so that both parts of the ratio will be in the same units.	15 cm : 3 × 100 cm = 15 cm : 300 cm
3 Simplify the ratio by dividing both 15 and 300 by 15.	= 15 : 300 $= \frac{15}{15} : \frac{300}{15}$
4 Write your answer.	= 1 : 20

Now try this 20 **Simplifying ratios with different units (Example 20)**

Express 52 mm to 8 cm as a ratio in simplest form.

Hint 1 Convert 8 cm to mm so that both parts of the ratio are in the same units.

Hint 2 Simplify the ratio.

Example 21 **Finding missing values in a ratio**

Find the missing value for the equivalent ratios 3 : 7 = ☐ : 28.

Explanation	Solution
1 Let the unknown value be x and write the ratios as fractions.	$3 : 7 = x : 28$ $\frac{3}{7} = \frac{x}{28}$
2 Solve for x.	

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Method 1 (by hand)

1 Multiply both sides of the equation by 28.

$$\frac{3}{7} \times 28 = \frac{x}{28} \times 28$$

$$x = 12$$

2 Evaluate and write your answer.

$$3 : 7 = 12 : 28$$

Method 2 (using CAS)

Use the solve function.

solve $\left(\frac{3}{7} = \frac{x}{28}, x\right)$

$x = 12$

Now try this 21 Finding missing values in a ratio (Example 21)

Find the missing value for the equivalent ratios 4 : 11 = ☐ : 88.

Hint 1 Let the unknown value be x and write the ratios as fractions.

Hint 2 Solve for x.

Section Summary

- When ratios are written in terms of the smallest possible whole numbers, they are expressed in their simplest form. (The highest common factor of the two numbers is 1.)
- The order of the numbers in a ratio is important. 3 : 5 is not the same as 5 : 3.
- Both parts of a ratio must be expressed in the same unit of measurement.

Skill-sheet

Exercise 1D

Building understanding

1 A shopper buys 17 wholemeal rolls and 9 sourdough rolls. Find:

a the ratio of the number of wholemeal rolls to the number of sourdough rolls

b the ratio of the number of sourdough rolls to the number of wholemeal rolls

c the ratio of the number of wholemeal rolls to the total number of rolls.

2 Simplify the following ratios.

a 25 : 75 b 0.2 : 0.5 c 32 mm : 5 cm

Developing understanding

Example 17

3 A survey of a group of 50 Year 11 students in a school showed that 35 of them have a part-time job and 15 do not. Express the number of students having a part-time job to those who do not as a ratio.

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Example 18

4 The table opposite shows the average life expectancy of some animals.

Animal	Life expectancy
Chimpanzee	40 years
Elephant	70 years
Horse	40 years
Kangaroo	9 years
Tortoise	120 years
Mouse	4 years
Whale	80 years

Find the ratios between the life expectancies of the following animals.

a Whale to horse
b Elephant to kangaroo
c Whale to tortoise
d Chimpanzee to mouse
e Horse to mouse to whale

Example 19

5 Express the following ratios in their simplest forms.

a 12 : 15 **b** 10 : 45 **c** 22 : 55 : 33 **d** 1.3 : 3.9
e 2.7 : 0.9 **f** $\frac{5}{3} : \frac{1}{4}$ **g** 18 : 8

Example 20

6 Express the following ratios in their simplest form after making sure that each quantity is expressed in the same units.

a 60 L to 25 L **b** \$2.50 to \$50
c 75 cm to 2 m **d** 5 kg to 600 g
e 15 mm to 50 cm to 3 m **f** 1 km to 1 m to 1 cm
g 5.6 g to 91 g **h** \$30 to \$6 to \$1.20 to 60c

Example 21

7 For each of the following equivalent ratios, find the missing value.

a 1 : 4 = ☐ : 20
b 15 : 8 = 135 : ☐
c 600 : 5 = ☐ : 1
d 2 : 5 = 2000 : ☐
e 3 : 7 = ☐ : 56

8 Which of the following statements are true and which are false? For those that are false, suggest a correct replacement statement, if possible.

a The ratio 4 : 3 is the same as 3 : 4.
b The ratio 3 : 4 is equivalent to 20 : 15.
c 9 : 45 is equivalent to 1 : 5.
d The ratio 60 to 12 is equivalent to 15 to 3, which is the same as 4 to 1.
e If the ratio of a father's age to his daughter's age is 7 : 1, then the girl is 7 years old when her father is 56.
f If my weekly allowance is $\frac{5}{8}$ of that of my friend, then the ratio of my monthly allowance to the allowance of my friend is 20 : 32.

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Testing understanding

9 The following recipe is for Anzac biscuits.

Anzac biscuits (makes 25)

100 grams rolled oats	60 grams desiccated coconut
175 grams plain flour	125 grams soft brown sugar
125 grams butter	3 tablespoons boiling water
2 tablespoons golden syrup	1 teaspoon bicarb soda

a What is the unsimplified ratio of rolled oats : coconut : flour : brown sugar : butter?

b Simplify the ratio from part **a**.

c You want to adapt the recipe to make 75 biscuits. What quantity of each ingredient do you need?

1E Dividing quantities in given ratios

If a quantity is divided in the ratio 5 : 3, there are 5 + 3 = 8 parts in total. After division, one allocation has 5 parts and the other 3. That is, if you divide $8 between two people in the ratio 5 : 3, one person gets $5 and the other $3.

Example 22 Dividing quantities in given ratios

Calculate the number of students in each class if 60 students are divided into classes in the following ratios.

a 1 : 3 **b** 1 : 5 **c** 1 : 2 : 7

Explanation	Solution
a **1** Add up the total number of parts. (Remember that a 1 : 3 ratio means that there is 1 part for every 3 parts).	The total number of parts is 1 + 3 = 4.
2 Divide the number of students (60) by the number of parts (4) to give the number of students in one group.	60 ÷ 4 = 15 One group of students will have 1 × 15 = 15 students.
3 Work out how many students are in the other group by multiplying the number of parts (3) by the number of students in one group (15).	The other group will have 3 × 15 = 45 students.
4 Check this gives a total of 60 students and write your answer.	15 + 45 = 60 The two groups will have 15 and 45 students.

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b 1 Add up the total number of parts. (Remember that a 1 : 5 ratio means that there is 1 part for every 5 parts.)	The total number of parts is $1 + 5 = 6$.
2 Divide the number of students (60) by the number of parts (6) to give the number of students in one group.	$60 \div 6 = 10$ One group of students will have $1 \times 10 = 10$ students.
3 Work out how many students in the other group by multiplying the number of parts (5) by the number of students in one group (10).	The other group will have $5 \times 10 = 50$ students.
4 Check this gives a total of 60 students and write your answer.	$10 + 50 = 60$. The two groups will have 10 and 50 students.
c 1 To divide 60 students into classes in the ratio 1 : 2 : 7, first add up the total number of parts.	The total number of parts is $1 + 2 + 7 = 10$.
2 Divide the number of students (60) by the number of parts (10) to give the number of students in one group.	$60 \div 10 = 6$ One group of students will have $1 \times 6 = 6$ students.
3 Work out how many students in the other two groups by multiplying the number of parts (2) and (7) by the number of students in one group (6).	The other groups will have $2 \times 6 = 12$ students and $7 \times 6 = 42$ students.
4 Check that this gives 60 students and write your answer.	$6 + 12 + 42 = 60$ The three groups will have 6, 12 and 42 students.

Now try this 22 **Dividing quantities in given ratios (Example 22)**

Divide \$36 in the ratio 1 : 2 : 3.

Hint 1 Add up the total number of parts.
Hint 2 Divide \$36 by this total to give the amount of one part.
Hint 3 Multiply this amount by each quantity in the ratio.

Section Summary

- To divide a quantity in a given ratio **a** : **b**, divide the quantity by the sum of **a** and **b**. This gives the allocation of one part. Multiply this one part value by **a** and then by **b** to give the required allocations.

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Exercise 1E

Building understanding

1 Divide 96 in each of the following ratios.

a 1:1 **b** 2:1 **c** 5:3 **d** 19:5

Developing understanding

Example 22a, b

2 If a 40 m length of rope is cut in the following ratios, what will be the lengths of the individual pieces of rope?

a 4 : 1 **b** 1 : 7 **c** 6 : 2 **d** 4 : 4

Example 22c

3 If a sum of $500 were shared among a group of people in the following ratios, how much would each person receive?

a 6 : 4 **b** 1 : 4 : 5 **c** 1 : 8 : 1 **d** 8 : 9 : 8 **e** 10 : 5 : 4 : 1

4 A basket contains bananas, mangos and pineapples in the ratio 10 : 1 : 4. If there are 20 pineapples in the basket, calculate:

a the number of bananas
b the number of mangos
c the total number of pieces of fruit in the basket.

5 7.5 litres of a mixed cordial drink is required for a children's party. If the ratio of cordial to water is 1 : 4,

a how many litres of cordial are required?
b how many litres of water are required?

6 The scale on a map is 1 : 20 000 (in cm). If the measured distance on the map between two historical markers is 15 centimetres, what is the actual distance between the two markers in kilometres?

Testing understanding

7 Parents divide their lottery winnings of $3 600 000 in the ratio 4 : 3 : 3 amongst their daughter and two sons. How much does each receive?

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1F Unitary method

Ratios can be used to calculate unit prices, i.e. the price of one item. This method is known as the *unitary method* and can be used to solve a range of ratio problems.

Example 23 Using the unitary method

If 24 golf balls cost $86.40, how much do 7 golf balls cost?

Explanation	Solution
1 Find the cost of one golf ball by dividing $86.40 (the total cost) by 24 (the number of golf balls).	$\$86.40 \div 24 = \$3.60$
2 Multiply the cost of one golf ball ($3.60) by 7.	$\$3.60 \times 7 = \$25.20$
3 Write your answer.	7 golf balls cost $25.20.

Now try this 23 Using the unitary method (Example 23)

If 8 pens cost $32, how much do 15 pens cost?

Hint 1 Divide $32 by 8 to find the cost of one pen.
Hint 2 Multiply this cost by 15.

Section Summary

The unitary method first identifies the total number of parts involved, then determines the allocation to one part by division and then finds the allocation to a greater number of parts by multiplication.

For example: If 15 items cost $45, then one item costs $\$45 \div 15 = \$3$, and 3 items cost $9.

Exercise 1F

Building understanding

1 **a** Twelve mangos cost $36.

- **i** How much does one mango cost?
- **ii** How much do 11 mangos cost?
- **iii** How many mangos could I buy for $18?

b If 14 oranges cost $12, how many oranges can I buy for:

i $6 **ii** $36 **iii** $48

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Developing understanding

Example 23

2 Use the unitary method to answer the following questions.

a If 12 cakes cost \$14.40, how much do 13 cakes cost?

b If a clock gains 20 seconds in 5 days, how much does the clock gain in three weeks?

c If 17 textbooks cost \$501.50, how much would 30 textbooks cost?

d If an athlete can run 4.5 kilometres in 18 minutes, how far could she run in 40 minutes at the same pace?

3 If one tin of red paint is mixed with four tins of yellow paint, it produces five tins of orange paint. How many tins of the red and yellow paint would be needed to make 35 tins of the same shade of orange paint?

4 If a train travels 165 kilometres in 1 hour 50 minutes at a constant speed, calculate how far it could travel in:

a 3 hours

b $2\frac{1}{2}$ hours

c 20 minutes

d 70 minutes

e 3 hours and 40 minutes

f $\frac{3}{4}$ hour

5 Ice creams are sold in two different sizes. A 35 g cone costs \$1.25 and a 73 g cone costs \$2.00. Which is the better buy?

6 A shop sells 2 L containers of Brand A milk for \$2.99, 1 L of Brand B milk for \$1.95 and 600 mL of Brand C milk for \$1.42. Calculate the best buy.

7 You need 6 eggs to bake 2 chocolate cakes. How many eggs will you need to bake 17 chocolate cakes?

8 A car uses 45 litres of petrol to travel 495 kilometres. Under the same driving conditions, calculate:

a how far the car could travel on 50 litres of petrol

b how much petrol the car would use to travel 187 kilometres.

Testing understanding

9 If 110% of an amount is \$330, how much is 100%?

10 At a certain time, one Australian dollar bought US\$0.72. Determine how many Australian dollars you would get for US\$180.

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Key ideas and chapter summary

Assign-ment

Percentages To convert a fraction or a decimal to a percentage, multiply by 100.
To convert a percentage to a decimal or a fraction, divide by 100.

$$\text{Percentage change} = \frac{\text{change}}{\text{original quantity or price}} \times 100$$

GST GST stands for goods and services tax and is a tax of 10%.
To find the amount of GST when the cost without GST is known, divide by 10.
To find the amount of GST when the cost with GST is known, divide by 11.
To calculate the cost with GST, multiply the original cost by 1.1.
To calculate the original cost, divide the cost with GST by 1.1.

Ratios Order of figures in a ratio is important. 4 : 3 is not the same as 3 : 4.
Ratios can be simplified, e.g. 6 : 2 = 3 : 1.

Unitary method The unitary method first identifies the total number of parts involved, then determines the allocation to one part by division and then finds the allocation to a greater number of parts by multiplication.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

1A **1** I can convert fractions to percentages. ☐

e.g. What is $\frac{3}{4}$ as a percentage?

1A **2** I can convert decimals to percentages. ☐

e.g. What is 0.67 as a percentage?

1A **3** I can convert percentages to fractions. ☐

e.g. What is 40% as a fraction?

1A **4** I can convert percentages to decimals. ☐

e.g. What is 85% as a decimal?

1A **5** I can find a percentage of a quantity. ☐

e.g. What is 30% of $50?

1B **6** I can increase or decrease a value by a percentage. ☐

e.g. Decrease $150 by 20%.

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1B	7	I can find percentage increase or decrease. e.g. A kg of bananas rose in value from \$4.50 to \$7.00. What was the % increase?	☐
1C	8	I can calculate GST. e.g. Find the amount of GST that will be added to the price of a new car costing \$60 000.	☐
1C	9	I can find the original cost when I know the final cost with GST added. e.g. The price of a dishwasher with GST added is \$1098. Find the original price.	☐
1C	10	I can find the GST added to an item when the cost with GST is known. e.g. A laptop costs \$1595 including GST. How much is the GST?	☐
1D	11	I can express quantities as a ratio. e.g. In a class there are ten students who are 16 years old and twenty students who are 17 years old. Express the ratio of the number of 16-year-olds to 17-year-olds.	☐
1D	12	I can express a ratio in simplest form. e.g. Simplify 64 : 96.	☐
1E	13	I can divide a quantity in a given ratio. e.g. Divide 1024 in the ratio 1 : 3 : 12.	☐
1F	14	I can use the unitary method to solve problems. e.g. If 50 items cost \$2500, how much do 10 items cost?	☐

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Multiple-choice questions

1 56% as a fraction in its simplest form is:

A 0.56 **B** $\frac{56}{100}$ **C** $\frac{0.56}{100}$ **D** $\frac{14}{25}$ **E** $\frac{28}{50}$

2 15% of $1600 is equal to:

A $24 **B** $150 **C** $240 **D** $1840 **E** $24 000

3 An item with a cost price of $450 is marked up by 30%. Its selling price is:

A $585 **B** $135 **C** $480 **D** $1350 **E** $463.50

4 A motorbike's original price is $19 990. The cost with GST added is:

A $1817.27 **B** $1999 **C** $18 172.72 **D** $20 000 **E** $21 989

5 A box contains 5 green marbles, 7 blue marbles and 3 yellow marbles. The ratio of blue marbles to total marbles is:

A 7 : 5 : 3 **B** 7 : 8 **C** 7 : 15 **D** 5 : 7 : 3 **E** 5 : 7 : 3 : 15

6 $750 is divided in the ratio 1 : 3 : 2. The smallest share is:

A $250 **B** $125 **C** $375 **D** $750 **E** $150

7 In simplest ratio form, the ratio of 450 grams to 3 kilograms is:

A 3 : 20 **B** 450 : 3 **C** 9 : 60 **D** 150 : 1 **E** 15 : 100

Short-answer questions

1 Express the following percentages as decimals.

a 75% **b** 40% **c** 27.5%

2 Express the following percentages as fractions in their lowest terms.

a 10% **b** 20% **c** 22%

3 Evaluate the following.

a 30% of 80 **b** 15% of $70 **c** $12\frac{1}{2}$% of $106

4 A new smart TV was valued at $1038. During a sale it was discounted by 5%.

a What was the amount of discount?

b What was the sale price?

5 Tom's weekly wage of $750 is increased by 15%. What is his new weekly wage?

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6 A 15-year-old girl working at a local bakery is paid \$12.50 per hour. Her pay will increase to \$15 per hour when she turns 16. What will be the percentage increase to her pay (to the nearest whole number)?

7 A leather jacket is reduced from \$516 to \$278. Calculate the percentage discount (to the nearest whole number).

8 Find the GST that needs to be added to the following items.

a a bookcase costing \$279

b a lawn mower costing \$445

9 The item costs below include GST. How much is the GST for each item?

a a hedge trimmer for \$198

b a sewing machine for \$2090

10 True or false?

a The ratio 3 : 2 is the same as 2 : 3.

b 1 : 5 = 3 : 12

11 If a sum of \$800 were to be shared among a group of people in the following ratios, how much would each person receive?

a 4 : 6 b 1 : 4 c 2 : 3 : 5 d 2 : 2 : 4

12 A recipe for pizza dough requires 3 parts wholemeal flour for each 4 parts of plain flour. How many cups of wholemeal flour are needed if 24 cups of plain flour are used?

13 The scale on a map is 1 : 1000 (in cm). Find the actual distance (in metres) between two markers if the distance between the two markers on a map is:

a 2.7 cm b 140 mm

14 If 5 kilograms of mincemeat costs \$50, how much does 2 kilograms cost?

15 A truck uses 12 litres of petrol to travel 86 km. How far will it travel on 42 litres?

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Chapter 2

Investigating and comparing data distributions

Chapter questions

- What are categorical and numerical data?
- What is a bar chart and when is it used?
- What is a histogram and when is it used?
- What are dot and stem plots and when are they used?
- What are the mean, median, range, interquartile range and standard deviation?
- What is the five-number summary?
- What is an outlier?
- How do we construct and interpret boxplots?
- How do we use stem plots and boxplots to compare two or more groups?

In this information age, we increasingly have to interpret data. This data may be presented in charts, diagrams or graphs, or it may simply be lists of words or numbers. There may be a lot of relevant information embodied in the data, but the story it has to tell will not always be immediately obvious. Various statistical procedures are available which will help us extract the relevant information from data sets. In this chapter, we will look at some of the techniques that help us to answer real-world questions when the data are collected from a **single variable**.

2A Classifying and displaying categorical data

Learning intentions

- To be able to classify data as categorical or numerical.
- To be able to further classify categorical data as nominal or ordinal.
- To be able to further classify numerical data as discrete or continuous.
- To be able to construct frequency and percentage frequency tables for categorical data.
- To be able to construct bar charts and percentage bar charts from frequency tables.
- To be able to interpret and describe frequency tables and bar charts.

Consider the following situation. In completing a survey, students are asked to:

- indicate their gender by circling an 'F' for female or an 'M' for male on the form
- indicate their preferred coffee cup size when buying takeaway coffee as 'small', 'medium' or 'large'
- write down the number of brothers they have, and
- measure and write their hand span in centimetres.

The information collected from four students is displayed in the table below.

Since the answers to each of the questions in the survey will vary from student to student, each question defines a different **variable** namely: *gender*, *coffee size*, *number of brothers* and *hand span*.

Gender	*Coffee size*	*Number of brothers*	*Hand span (in cm)*
M	Large	0	23.6
F	Small	2	19.6
F	Small	1	20.2
M	Large	1	24.0

The values we collect for each of these variables are called **data**.
The variables in the table, and hence the data collected from that variable, fall into two broad types: **categorical** or **numerical**.

Categorical data

The type of data arising from the students' responses to the first and second questions in the survey are called **categorical data** because the data values can be used to place the person into one of several groups or categories. However, the properties of the data generated by these two questions differ slightly.

- The question asking students to use an 'M' or 'F' to indicate their gender will prompt a response of either M or F. This identifies the respondent as either male or female but tells us no more. We call this **nominal data** because the values of the variable are simply names.

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- However, the question with responses ‘small’, ‘medium’ and ‘large’ that indicates the students’ preferred coffee size tells us two things. Firstly, it names the coffee size, but secondly, it enables us to order the students according to their preferred coffee sizes. We call this **ordinal data** because it allows us to both name and order their responses.

Numerical data

The type of data arising from the responses to the third and fourth questions in the survey are called **numerical data** because they have values for which arithmetic operations, such as adding and averaging, make sense. However, the properties of the data generated by these questions differs slightly.

- The question asking students to write down the number of brothers they have will prompt whole number responses like 0, 1, 2, ... Because the data can only take particular numerical values, it is called **discrete data**. Discrete data arises in situations where counting is involved.
- In response to the hand span question, students who wrote 24.1 cm could have an actual hand span of anywhere between 23.05 and 24.04 cm, depending on the accuracy of the measurement and how the student rounded their answer. This is called **continuous data** because the variable we are measuring, in this case *hand span*, can take any numerical value within a specified range. Continuous data are often generated when measurement is involved.

A quite different distinction is sometimes made between numerical data which is **interval** and numerical data which is **ratio**.

- An **interval** scale requires only that the differences between successive steps on the scale are equal. The temperature scale is an example of an interval scale; the amount of heat needed to raise the temperature from 13°C to 14°C is the same as that needed to raise the temperature from 14°C to 15°C and so on. There are, however, two limitations to an interval scale. The first is that zero on the scale does not mean absence of the quantity being measured: 0°C does not mean complete absence of heat. Secondly, we cannot make simple ratio statements such as a day of 40°C is twice as hot as a day of 20°C.
- A **ratio** scale has all the properties of an interval scale but has the additional properties that zero means complete absence of what is being measured, and ratio statements, such as ‘half as many’ and ‘fifty times as much,’ can be made. Ratio scales are those with which you are probably most familiar. When the number of brothers are counted or a person’s hand span is measured in cm, you are using a ratio scale. On a ratio scale, zero means complete absence of the quantity, for example, zero brothers. With a true zero point, you can make true ratio statements; a person with 4 brothers has twice as many brothers as someone who has 2 brothers.

For statistical purposes it is generally not necessary to distinguish between interval and ratio scales as both give numbers on which we can perform the same statistical analyses.

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Types of variables

Categorical variables

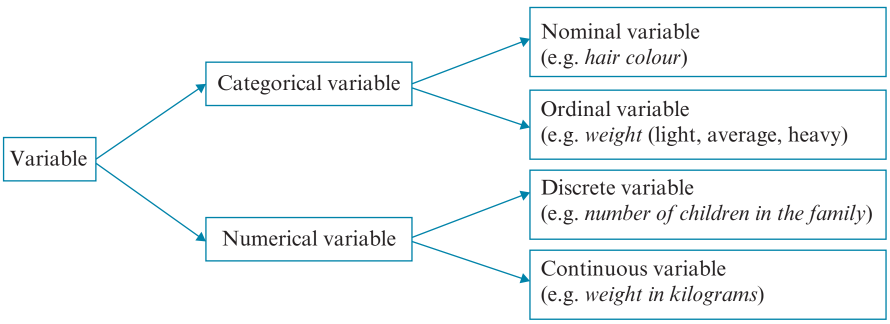
Variables that generate categorical data are called **categorical variables**. We can further separate categorical variables into **nominal** or **ordinal** variables. For example, *gender* is a nominal variable, while *coffee size* is an ordinal variable.

Numerical variables

Variables that generate numerical data are called **numerical variables**. We can further separate numerical variables into discrete or continuous variables. For example, *number of brothers* is a **discrete variable**, while *hand span* is a **continuous variable**.

Numerical or categorical?

The relationship between categorical variables (nominal or ordinal) and numerical variables (discrete or continuous) is displayed in the following diagram:



Example 1 **Categorical and numerical variables**

Classify the following variables as categorical or numerical.

a Students choose their favourite pet from ‘dog’, ‘cat’, ‘bird’ or ‘other’.

b The time, in seconds, taken to solve a puzzle is recorded.

Solution

a **Categorical**, as the values of the variable are categories of pet.

b **Numerical**, as the data takes values which represent the amount of time taken.

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Example 2 Nominal and ordinal variables

Classify the following variables as nominal or ordinal.

a A group of people record their level of happiness as 'very happy', 'happy', 'not too happy' or 'very unhappy'.

b Students select their favourite country to visit.

Solution

a **Ordinal**, as the data takes values which represent the person's level of happiness, and there is an order to the categories.

b **Nominal**, as the data takes values which are names of countries.

Example 3 Discrete and continuous variables

Classify the following numerical variables as discrete or continuous.

a The number of children in the family is recorded for all the students in a school.

b The birth weight of babies, measured in grams, is recorded at a hospital.

Solution

a **Discrete**, as the number of children will only take whole number values.

b **Continuous**, as the data can take any value, limited only by the accuracy to which the weight can be measured.

Example 4 Classifying variables

Classify the following numerical variables as nominal, ordinal, discrete or continuous.

a The number of students in each of 10 classes is counted.

b The time taken for 20 mice to each complete a maze is recorded in seconds.

c Diners at a restaurant were asked to rate how they felt about their meal: 1= Very satisfied, 2 = Satisfied, 3 = Indifferent, 4 = Dissatisfied, 5 = Very dissatisfied.

d Students choose a colour from a list: 1= Blue, 2 = Green, 3 = Red, 4 = Yellow.

e Students' heights were classified as 'less than 160 cm', '160 cm - 180 cm' or 'more than 180 cm'.

Solution

a **Discrete**, as the number of students will only take whole number values.

b **Continuous**, as the data can take any value, limited only by the accuracy to which the time can be measured.

c **Ordinal**, as the numbers in this data do not represent quantities, they represent each diner's level of approval of the meal.

d **Nominal**, as the numbers do not represent quantities, they represent colours.

e **Ordinal**, as each student's height is recorded into 3 categories which can be ordered.

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To make sense of data, we first need to organise it into a more manageable form. For categorical data, frequency tables and bar charts are used for this purpose.

The frequency table

Frequency

A **frequency table** is a listing of the values a variable takes in a data set, along with how often (frequently) each value occurs.

Frequency can be recorded as a:

- **frequency**: the number of times a value occurs
- **percentage frequency**: the percentage of times a value occurs, where:

$$\text{percentage frequency} = \frac{\text{count}}{\text{total}} \times 100$$

- **frequency distribution**: a listing of the values a variable takes, along with how frequently each of these values occurs.

Note that it is quite common for percentages to add to 99.9% or 100.1%. This is due to the rounding of each of the individual percentages in the table and is not a concern.

Example 5 Constructing a frequency table for categorical data

Thirty children chose a sandwich, a salad or a pie for lunch, as follows:

sandwich, salad, salad, pie, sandwich, sandwich, salad, salad, pie, pie, pie, salad, pie, sandwich, salad, pie, salad, pie, sandwich, sandwich, pie, salad, salad, pie, pie, pie, salad, pie, sandwich, pie

Construct a table for the data showing both frequency and percentage frequency.

Explanation

1 Set up a table as shown. The variable *Lunch choice* has three categories: 'sandwich', 'salad' and 'pie'.

2 Count the number of children choosing a sandwich, a salad or a pie. Record in the 'Number' column.

3 Add the frequencies to find the total number.

Solution

	Frequency	
Lunch choice	Number	%
Sandwich	7	23.3
Salad	10	33.3
Pie	13	43.3
Total	30	99.9

4 Convert the frequencies into percentages and record in the '%' column. For example, percentage frequency for pies equals $\frac{13}{30} \times 100 = 43.3\%$.

5 Total the percentages and record. Note that the percentages add up to 99.9%, not 100%, because of rounding.

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Now try this 5 **Constructing frequency table for categorical data (Example 5)**

Twenty-five students were asked how they usually travelled to school. They answered as follows:

walk, car, car, other, car, car, walk, car, bus, bus, car, bus, walk, walk, bus, bus, bus, car, car, car, car, car, car, bus, bus

Construct a table for the data, showing both frequency and percentage frequency.

Hint 1 Determine the possible values of the variable *Travel mode* - there are four.

Hint 2 Check that the frequencies add to 25.

Hint 3 Check that the percentage frequencies add to 100%.

Bar charts

When there are a lot of data, a frequency table can be used to summarise the information, but we generally find that a graphical display is also useful. When the data are categorical, the appropriate display is a **bar chart**.

Bar charts

In a bar chart:

- frequency or percentage frequency is shown on the vertical axis
- the variable being displayed is plotted on the horizontal axis
- the height of the bar (column) gives the frequency (or percentage)
- the bars are drawn with gaps to indicate that each value is a separate category
- there is one bar for each category.

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Example 6 Constructing bar and percentage bar charts from a frequency table

Use the frequency table for *Lunch choice* from Example 5 to construct:

a a bar chart **b** a percentage bar chart.

Explanation

a **1** Label the horizontal axis with the variable name, *Lunch choice*. Mark the scale off into three equal intervals and label them 'Pie', 'Salad' and 'Sandwich'.

2 Label the vertical axis 'Frequency'. Insert a scale allowing for the maximum frequency of 13. Up to 15 would be appropriate. Mark the scale in intervals of 5.

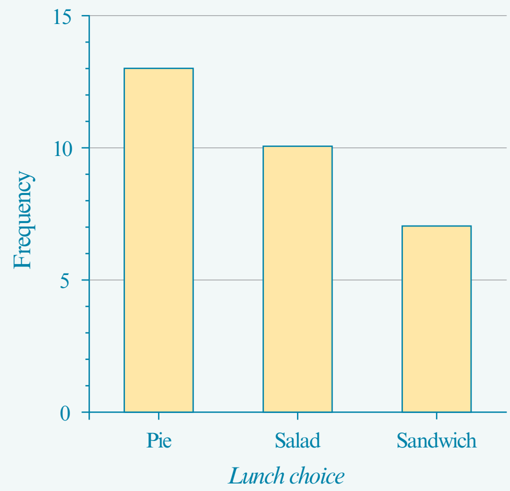
3 For each interval, draw in a bar as shown. Make the width of each bar less than the width of the category intervals, to show that the categories are quite separate. The height of each bar is equal to the frequency.

b **1** To construct a percentage bar chart of the *Lunch choice* data, follow the same procedure as above but label the vertical axis 'Percentage'. Insert a scale allowing for a maximum percentage frequency up to 45%.

2 Mark the vertical scale in intervals of 5%.

The height of each bar is equal to the percentage frequency.

Solution



Note: For nominal variables it is common but not necessary to list categories in decreasing order by frequency. This makes later interpretation easier.

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Now try this 6 **Constructing bar and percentage bar charts from a frequency table (Example 6)**

Use the frequency table for *Travel mode* from Now Try This (Example 5) to construct:

a a bar chart

b a percentage bar chart.

Hint 1 The horizontal axis for both charts is labelled with the values of the variable *Travel mode*.

Hint 2 The vertical axis of the bar chart is labelled Frequency. The scale should start at 0 and extend slightly more than the maximum frequency.

Hint 3 The vertical axis of the percentage bar chart is labelled Percentage. The scale should start at 0 and extend slightly more than the maximum percentage frequency.

The mode or modal category

One of the features of a data set that is quickly revealed with a bar chart is the **mode** or **modal category**. This is the most frequently occurring category. In a bar chart, this is given by the category with the tallest bar. For the bar chart in Example 5, the modal category is clearly ‘pie’. That is, the most frequent or popular lunch choice was a pie.

When is the mode useful?

The mode is most useful when a single value or category in the frequency table occurs more often (frequently) than the others. Modes are of particular importance in popularity polls, answering questions like ‘Which is the most frequently watched TV station between the hours of 6 p.m. and 8 p.m.?’ or ‘When is a supermarket in peak demand?’

Section Summary

- Data can be classified as **categorical** (nominal or ordinal), or **numerical** (discrete or continuous).
  - **Nominal data** takes values which are simply the **names** of categories.
  - **Ordinal data** takes values which both **name** and **order** categories.
  - **Discrete** data can only take particular numerical values, often whole numbers.
  - **Continuous** data can take any numerical value within a specified range.
- **Categorical** data can be summarised in a **frequency table** or a **percentage frequency table**.
- A frequency table can be displayed in a **bar chart**.
- A percentage frequency table can be displayed in a **percentage bar chart**.
- The value of a categorical variable with the highest frequency is called the **mode**.

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Skill-sheet

Exercise 2A

Building understanding

Example 1

1 Classify the data generated in each of the following as categorical or numerical.

a Kindergarten pupils bring along their favourite toys, and they are grouped together under the headings 'dolls', 'soft toys', 'games', 'cars' and 'other'.

b The number of students on each of 20 school buses are counted.

c A group of people each write down their favourite colour.

d Each student in a class is weighed in kilograms.

e People rate their enthusiasm for a certain rock group as 'low', 'medium' or 'high'.

Example 2

2 Classify the categorical data arising from people answering the following questions as either nominal or ordinal.

a What is your favourite football team?

b How often do you exercise? Choose one of 'never', 'once a month', 'once a week', 'every day'.

c Indicate how strongly you agree with 'alcohol is the major cause of accidents' by selecting one of 'strongly agree', 'agree', 'disagree', 'strongly disagree'.

d What language will you study next year, 'French', 'Chinese', 'Spanish' or 'none'?

Example 3

3 Classify the numerical variables identified below (in italics) as discrete or continuous.

a The *number of pages* in a book.

b The *price* paid to fill the tank of a car with petrol.

c The *volume* of petrol (in litres) used to fill the tank of a car.

d The *time* between the arrival of successive customers at an ATM.

e The *number of people* at a football match.

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Example 4

4 Classify the data generated in each of the following situations as nominal, ordinal or numerical (discrete or continuous).

a The different brand names of instant soup sold by a supermarket are recorded.
b A group of people are asked to indicate their attitude to capital punishment by selecting a number from 1 to 5, where 1 = strongly disagree, 2 = disagree, 3 = undecided, 4 = agree and 5 = strongly agree.
c The number of computers per household was recorded during a census.
d The annual salaries of the employees of a certain company were recorded as 'less than $60 000', '$60 000 - $90 000', 'more than $90 000'.

Developing understanding

Example 5

5 A group of people were asked to identify their gender as F = female or M = Male. The following data was collected:

F M M M F M F F M M M F M M M

a Is the variable *gender* nominal or ordinal?
b Construct a frequency table for the data including frequency and percentage frequency.

6 A group of 18-year-old males were asked to give their shoe size. The following data was collected:

8	9	9	10	8	8	7	9	8	9
10	12	8	10	7	8	8	7	11	11

a Is the variable *shoe size* nominal or ordinal?
b Construct a frequency table for the data including frequencies and percentages.

Example 6

7 The table shows the frequency distribution of the favourite type of fast food (*Food type*) of a group of students.

a Complete the table.
b Is the variable *Food type* nominal or ordinal?
c How many students preferred Chinese food?
d What percentage of students chose chicken as their favourite fast food?
e What was the favourite type of fast food for this group of students?
f Construct a frequency bar chart.

	Frequency	
Food type	Number	%
Hamburgers	23	33.3
Chicken	7	10.1
Fish and chips	6	
Chinese	7	10.1
Pizza	18	
Other	8	11.6
Total		99.9

8 The following responses were received to a question regarding the return of capital punishment.

a Complete the table.

b Is the data used to generate this table nominal or ordinal?

c How many people said 'Strongly agree'?

d What percentage of people said 'Strongly disagree'?

e What was the most frequent response?

f Construct a frequency bar chart.

Attitude to capital punishment	Frequency	
	Number	%
Strongly agree	21	8.2
Agree	11	4.3
Don't know	42	
Disagree		
Strongly disagree	129	50.4
Total	256	100.0

9 A bookseller noted the types of books purchased during a particular day, with the following results.

a Complete the table.

b Is the variable *Type of book* nominal or ordinal?

c How many books purchased were classified as 'Fiction'?

d What percentage of books were classified as 'Children'?

e How many books were purchased in total?

f Construct a bar chart of the percentage frequencies (%).

	Frequency	
Type of book	Number	%
Children	53	22.8
Fiction	89	
Cooking	42	18.1
Travel	15	
Other	33	14.2
Total	232	

10 The members of a club were classified according to the following age groups.

a How many people are in the club?

b Is the variable *Age group* nominal, ordinal or numerical?

c What percentage of people in the club were aged 35-44 years?

d What is the modal age category?

e Construct a bar chart of the percentage frequencies (%).

	Frequency	
Age group	Number	%
Under 18	84	42
18 - 24	26	13
25 - 34	46	23
35 - 44	24	12
45 -54	8	4
55 or over	12	6
Total	200	100

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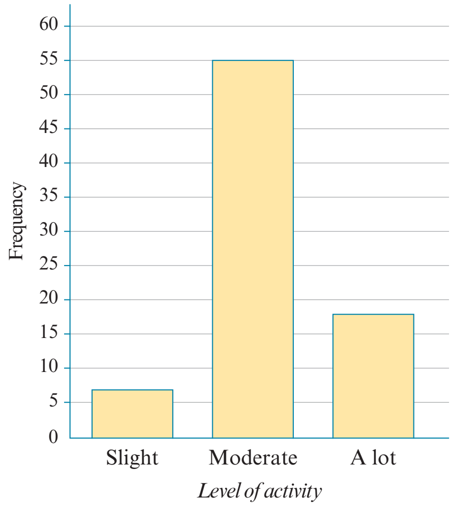
Testing understanding

11 In a survey, people were asked to select their level of activity as ‘slight’, ‘moderate’ or ‘a lot’. Their responses are summarised in the following bar chart.



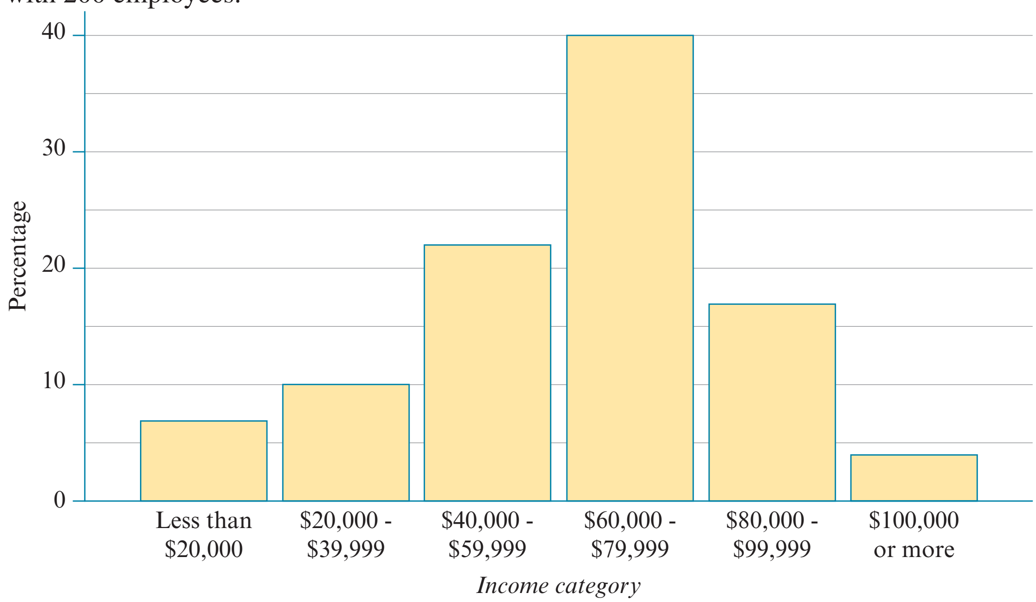
a Give the name of the variable, and classify it as nominal, ordinal or numerical.

b How many people responded to this question in the survey?

c How many people responded that they exercise ‘a lot’?

d What is the modal category for the variable, and what percentage of people chose that response?

12 The following percentage bar chart shows the incomes of people in a company with 200 employees.



a Is the variable *Income category* nominal, ordinal or numerical?

b How many employees earned salaries in the modal income category?

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2B Interpreting and describing frequency tables and bar charts

Learning intentions

- To be able to explain the features of a categorical variable by interpreting frequency tables and bar charts.
- To be able to write a report which communicates your findings.

As part of this topic, you will be expected to complete a statistical investigation. Under these circumstances, constructing a frequency table or a bar chart is not an end in itself. It is merely a means to an end. The end is being able to understand something about the variables you are investigating that you didn't know before.

To complete the investigation, you will need to communicate this finding to others. To do this, you will need to know how to describe and interpret any patterns you observe in the context of your data investigation in a written report that is both systematic and concise. The purpose of this section is to help you develop such skills.

Some guidelines for describing the distribution of a categorical variable and communicating your findings

- Briefly summarise the context in which the data were collected including the number of people (or things) involved in the study.
- If there is a clear modal category, make sure that it is mentioned.
- Include relevant counts or percentages in the report.
- If there are a lot of categories (more than 3), it is not necessary to mention every category.
- Either counts or percentages can be used to describe the distribution.

These guidelines are illustrated in the following examples.

Example 7 **Describing the distribution of a categorical variable from a frequency table**

A group of 30 children were offered a choice of a sandwich, a salad or a pie for lunch, and their responses were collected and summarised in the frequency table opposite.

Lunch choice	Frequency
Sandwich	7
Salad	10
Pie	13
Total	30

Use the frequency table to report on the relative popularity of the three lunch choices, quoting appropriate frequencies to support your conclusions.

Continued

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Solution

Report

A group of 30 children were offered a choice of a sandwich, a salad or a pie for lunch. The most popular lunch choice was a pie, chosen by 13 of the children. Ten children chose a salad. The least popular option was a sandwich, chosen by only 7 of the children.

Now try this 7 Describing the distribution of a categorical variable from a frequency table (Example 7)

A group of 25 kindergarten children were asked to choose an activity from painting, story time and playdough. Their choices are summarised in the frequency table opposite.

Activity choice	Frequency
Painting	8
Story time	3
Playdough	14
Total	25

Use the frequency table to report on the relative popularity of the three activities, quoting appropriate frequencies to support your conclusions.

Hint 1 Make sure you mention the modal value as ‘the most popular choice’ and the lowest frequency as ‘the least popular choice’.

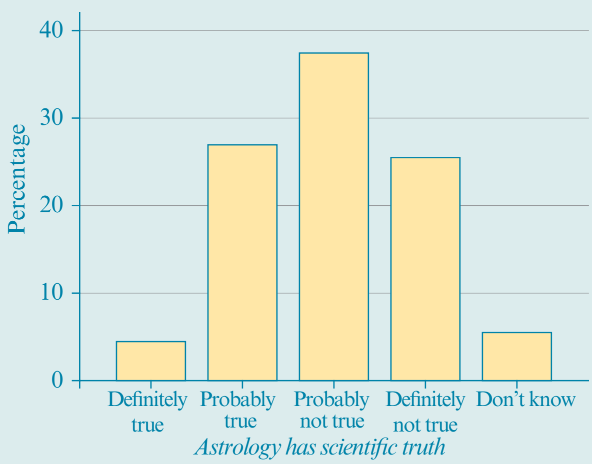
Hint 2 Write as if you are explaining the children’s choices to a friend.

Example 8 Describing the distribution of a categorical variable from a frequency table and bar chart

A sample of 200 people were asked to comment on the statement ‘Astrology has scientific truth’ by selecting one of the options ‘definitely true’, ‘probably true’, ‘probably not true’, definitely not true’ or ‘don’t know’.

The data are summarised in the following frequency table and bar chart (in a definite order because the data are ordinal).

Astrology has scientific truth	Frequency	
	Number	%
Definitely true	9	4.5
Probably true	54	27.0
Probably not true	75	37.5
Definitely not true	51	25.5
Don’t know	11	5.5
Total	200	100.0



Write a report using the frequency table and bar chart.

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Solution

Report

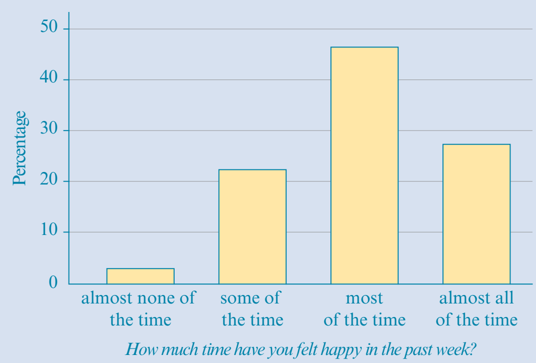
A sample of two hundred people were asked to respond to the statement 'Astrology has scientific truth'.

The majority of respondents did not agree, with 37.5% responding that they believed that this statement was probably not true, and another 25.5% declaring that the statement was definitely not true. Over one quarter (27%) of the respondents thought that the statement was probably true, while only 4.5% thought that the statement was definitely true.

Now try this 8 Describing the distribution of a categorical variable from a frequency table and bar chart (Example 8)

A sample of 66 people were asked to respond to the question 'How much time have you felt happy in the past week' by selecting one of the options 'almost none of the time', 'some of the time', 'most of the time' or 'almost all of the time'. The data are summarised in the frequency table and bar on the following page. Write a report summarising the findings of this investigation, quoting appropriate percentages to support your conclusions.

How much happy time?	Frequency	
	Number	%
almost none	2	3.0
some	15	22.7
most	31	47.0
almost all	18	27.3
Total	66	100.0



Hint 1 Compare the percentage frequencies associated with each value of the variable from highest to lowest.

Hint 2 Make sure you use comparative terms such as 'more' and 'less' - don't just state the percentages.

Hint 3 Write as if you are explaining the responses to this question to another person.

Section Summary

- Examination of frequency tables and bar charts can help us to understand the distribution of a categorical variable.
- Important features such as the values of the variable with the highest and lowest frequencies should be included in a written report.
- Be sure to always include the total number of data values.

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Exercise 2B

Building understanding

Example 7

1 A group of 69 students were asked to nominate their preferred type of fast food. The results are summarised in the percentage frequency table opposite. Use the information in the table to complete the report below by filling in the blanks.

Fast food type	*%*
Hamburgers	33.3
Chicken	10.1
Fish and chips	8.7
Chinese	10.1
Pizza	26.1
Other	11.6
Total	99.9

Report

A group of ____ students were asked their favourite type of fast food. The most popular response was ____ (33.3%), followed by pizza (____). The rest of the group were almost evenly split between chicken, fish and chips, Chinese and other, all around 10%.

2 Two hundred and fifty-six people were asked whether they agreed that there should be a return to capital punishment in their state. Their responses are summarised in the table opposite. Use the information in the table to complete the report below.

Capital punishment	*%*
Strongly agree	8.2
Agree	4.3
Don't know	16.4
Disagree	20.7
Strongly disagree	50.4
Total	100.0

Report

A group of 256 people were asked whether they agreed that there should be a return to capital punishment in their state. The majority of these people ____ (50.4%), followed by ____ who disagreed. Levels of support for return to capital punishment were quite low, with only 4.3% agreeing and 8.2% strongly agreeing. The remaining ____ said that they didn't know.

Developing understanding

3 A group of 200 students were asked how they prefer to spend their leisure time. The results are summarised in the frequency table opposite.

Use the information in the table to write a brief report on the results of this investigation.

Leisure activity	*%*
Internet and digital games	42
Read	13
Listen to music	23
Watch TV or go to movies	12
Phone friends	4
Other	6
Total	100

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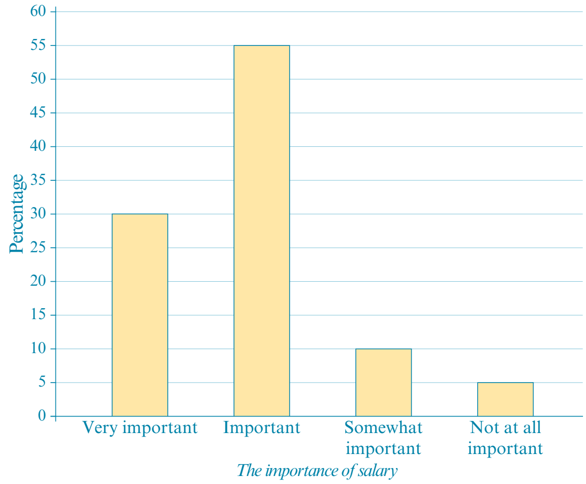
Example 8

4 A group of 600 employees from a large company were asked to rate the importance of salary in determining how they felt about their job. Their responses are shown in the following bar chart.



Write a report describing how these employees rated the importance of salary in determining how they felt about their job.

Testing understanding

5 Ask your class (or a convenient group of people) to respond to the question 'How concerned are you about climate change?' by selecting one of a range of responses (ensure that you have about four or five options).

a Summarise the responses in a frequency table with percentages and with a bar chart.

b Write a report based on your findings.

2C Displaying and describing numerical data

Learning intentions

- To be able to construct frequency tables for discrete numerical data.
- To be able to construct frequency tables for grouped numerical data (discrete and continuous).
- To be able to construct a histogram from frequency tables for numerical data.
- To be able to construct a histogram from numerical data using a CAS calculator.

Frequency tables can also be used to organise numerical data. For a discrete variable which only takes a small number of values, the process is the same as that for categorical data, as shown in the following example.

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Discrete data

Example 9 **Constructing a frequency table for discrete numerical data, taking a small number of values**

The number of brothers and sisters (siblings) reported by each of the 30 students in Year 11 are as follows:

2 3 4 0 3 2 3 0 4 1 0 0 1 2 3
0 2 1 1 4 5 3 2 5 6 1 1 1 0 2

Construct a table for these data showing both frequency and percentage frequency.

Explanation

1 Find the maximum and the minimum values in the data set. Here the minimum is 0 and the maximum is 6.

2 Construct a table as shown, including all the values between the minimum and the maximum.

3 Count the number of 0s, 1s, 2s, etc. in the data set. For example, there are seven 1s. Record these values in the number column.

4 Add the frequencies to find the total.

5 Convert the frequencies to percentages, and record in the per cent (%) column.

6 Total the percentages and record.

Solution

Number of siblings	Frequency	
	Number	%
0	6	20.0
1	7	23.3
2	6	20.0
3	5	16.7
4	3	10.0
5	2	6.7
6	1	3.3
Total	30	100.0

For example, percentage of 1s equals $\frac{7}{30} \times 100 = 23.3\%$.

Now try this 9 **Constructing a frequency table for discrete numerical data, taking a small number of values (Example 9)**

The number of faulty widgets produced each hour over a 24-hour period by a certain machine are as follows:

0 0 1 0 3 2 0 0 1 1 5 0 0 1 1 1 2 2 3 2 4 0 1 1

Hint 1 Determine the minimum and maximum values which the variable *faulty widgets* can take from the data set.

Hint 2 When you have counted the frequencies for each value, check that they add to 24.

Hint 3 Check that the percentage frequencies add to 100%.

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Grouping data

Some discrete variables can only take on a limited range of values, for example, the variable *number of children in a family*. For these variables, it makes sense to list each of these values individually when forming a frequency distribution.

In other cases, when the variable can take on a large range of values (e.g. age from 0 to 100 years) or when the variable is continuous (e.g. response times measured in seconds to two decimal places), we **group the data** into a small number of convenient intervals.

These grouping intervals should be chosen according to the following principles:

- Every data value should be in an interval.
- The intervals should not overlap.
- There should be no gaps between the intervals.

The choice of intervals can vary but there are some guidelines.

- A division which results in about 5 to 15 groups is preferred.
- Choose an interval width that is easy for the reader to interpret, such as 10 units, 100 units or 1000 units (depending on the data).
- By convention, the beginning of the interval is given the appropriate exact value, rather than the end. As a result, intervals of 0–49, 50–99, 100–149 would be preferred over the intervals 1–50, 51–100, 101–150 etc.

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Grouped discrete data

Example 10 Constructing a grouped frequency table for a discrete numerical variable

A group of 20 people were asked to record how many cups of coffee they drank in a particular week, with the following results:

2	0	9	10	23	25	0	0	34	32
5	0	17	14	3	6	0	33	23	0

Construct a grouped frequency table of these data showing both frequency (count) and percentage frequency.

Explanation

1 The minimum number of cups of coffee drunk is 0 and the maximum is 34. Intervals beginning at 0 and ending at 34 would ensure that all the data are included. Interval width of 5 will mean that there are 7 intervals. Note that the endpoints are within the interval, so that the interval 0 – 4 includes 5 values: 0, 1, 2, 3 and 4.

2 Set up the table as shown.

3 Count the data values in each interval to complete the number column.

4 Convert the frequencies into percentages and record in the per cent (%) column. For example, for the interval 5–9: % frequency $= \frac{3}{20} \times 100 = 15\%$.

5 Total the percentages and record.

Solution

Cups of coffee	Frequency	
	Number	%
0–4	8	40
5–9	3	15
10–14	2	10
15–19	1	5
20–24	2	10
25–29	1	5
30–34	3	15
Total	20	100

Now try this 10 Constructing a grouped frequency table for a discrete numerical variable (Example 10)

A group of thirty people were each asked the number of times they dined in a restaurant in the last 3 months, with the following results:

1	6	9	12	13	29	0	5	44	52
7	0	10	17	3	6	0	53	23	9
5	2	18	21	6	9	2	63	58	0

Construct a grouped frequency table of these data, showing both frequency (count) and percentage frequency.

Hint 1 Group the data in intervals of width 10.
Hint 2 When you have counted the frequencies for each group, check that they add to 30.
Hint 3 Check that the percentage frequencies add to 100%.

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Grouped continuous data

Example 11 Constructing a frequency table for a continuous numerical variable

The following are the heights of the 41 players in a basketball club, in centimetres.

178.1	185.6	173.3	193.4	183.1	184.6	202.4	170.9	183.3	180.3
185.8	189.1	178.6	194.7	185.3	191.1	189.7	191.1	180.4	180.0
193.8	196.3	189.6	183.9	177.7	178.9	193.0	188.3	189.5	182.0
183.6	184.5	188.7	192.4	203.7	180.1	170.5	179.3	184.1	183.8
174.7									

Construct a frequency table and a percentage frequency table for these data.

Explanation

1 Find the minimum and maximum heights, which are 170.5 cm and 203.7 cm. A minimum value of 170 and a maximum of 204.9 will ensure that all the data are included.

2 Interval width of 5 cm will mean that there are 7 intervals from 170 to 204.9, which is within the guidelines of 5–15 intervals.

3 Set up the table as shown. All values of the variable that are from 170 to 174.9 have been included in the first interval. The second interval includes values from 175 to 179.9, and so on for the rest of the table.

4 The number of data values in each interval is then counted to complete the number column of the table.

5 Convert the frequencies into percentages and record in the per cent (%) column.

6 Total the percentages and record.

Solution

	Frequency	
Height	Number	%
170–174.9	4	9.8
175–179.9	5	12.2
180–184.9	13	31.7
185–189.9	9	22.0
190–194.9	7	17.1
195–199.9	1	2.4
200–204.9	2	4.9
Total	41	100.1

For example, for the interval 175.0–179.9:

$\% \text{ frequency} = \frac{5}{41} \times 100 = 12.2\%.$

The interval that has the highest frequency is called the **modal interval**. In the example above, the modal interval is 180.0–184.9, as 13 players (31.7%) have heights that fall into this interval.

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Histograms

As with categorical data, we would like to construct a visual display of a frequency table for numerical data. The graphical display of a frequency table for a numerical variable is called a **histogram**. A histogram looks similar to a bar chart but because the data is numerical there is a natural order to the plot and the bar widths depend on the data values.

Histograms

In a histogram:

- frequency (number or percentage) is shown on the vertical axis
- the values of the variable being displayed are plotted on the horizontal axis
- each column corresponds to a data value, or a data interval if the data is grouped; alternatively, for ungrouped discrete data, the actual data value is located at the middle of the column
- the height of the column gives the frequency (number or percentage).

Example 12 Constructing a histogram for ungrouped discrete data

Construct a histogram for the data in the frequency table.

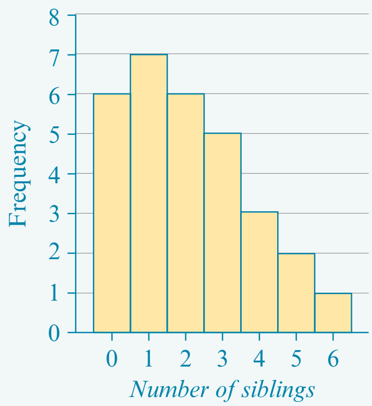
Number of Siblings	Frequency
0	6
1	7
2	6
3	5
4	3
5	2
6	1
Total	30

Explanation

1 Label the horizontal axis with the variable name *Number of siblings*. Mark in the scale in units that include all possible values.

2 Label the vertical axis 'Frequency'. Insert a scale allowing for the maximum frequency of 7. Up to 8 would be appropriate. Mark the scale in units.

Solution



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3 For each value of the variable, draw in a column. The data is discrete, so make the width of each column 1, starting and ending halfway between data values. For example, the column representing 2 siblings starts at 1.5 and ends at 2.5. The height of each column is equal to the frequency.

Now try this 12 Constructing a histogram for ungrouped discrete data (Example 12)

Use the frequency table for *Faulty widgets* from Now Try This (Example 9) to construct a histogram for the data.

Hint 1 The horizontal axis is labelled *Number of faulty widgets*.

Hint 2 The vertical axis of the histogram is labelled Frequency. The scale should start at 0 and extend slightly more than the maximum frequency.

Example 13 Constructing a histogram for continuous data

Construct a histogram for the data in the frequency table.

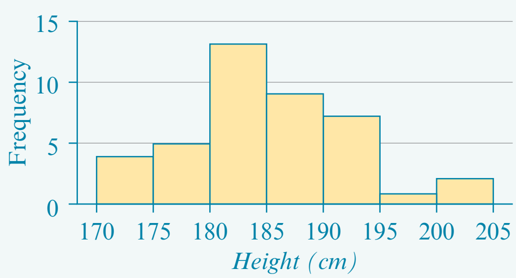
Height (cm)	Frequency
170.0–174.9	4
175.0–179.9	5
180.0–184.9	13
185.0–189.9	9
190.0–194.9	7
195.0–199.9	1
200.0–204.9	2
Total	41

Explanation

1 Label the horizontal axis with the variable name *Height (cm)*. Mark in the scale using the beginning of each interval as the scale points; that is, 170, 175, . . .

2 Label the vertical axis ‘Frequency’. Insert a scale allowing for the maximum frequency of 13. Up to 15 would be appropriate. Mark the scale in units.

Solution



3 For each interval, draw in a column. Each column starts at the beginning of the interval and finishes at the beginning of the next interval. Make the height of each column equal to the frequency.

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Now try this 13 **Constructing a histogram for continuous data (Example 13)**

The number of hours per week spent on email by a group of 150 people are summarised in this frequency table. Use it to construct a histogram for the data.

Hours on email	Frequency
0.0–4.9	47
5.0–9.9	52
10.0–14.9	21
15.0–19.9	9
20.0–24.9	7
25.0–29.9	5
30.0–34.9	5
35.0–39.9	2
40.0–44.9	0
45.0–49.9	2
Total	150

Hint 1 Label the horizontal axis with the variable name *Hours on email*. Mark in the scale using the beginning of each interval as the scale points; that is, 0, 5, 10, ...

Hint 2 Label the vertical axis ‘Frequency’. The scale should start at 0 and extend slightly more than the maximum frequency.

Hint 3 There should be no gaps between the columns in the histogram.

Constructing a histogram using a CAS calculator

It is relatively quick to construct a histogram from a frequency table. However, if you only have the data (as you mostly do), it is a very slow process because you have to construct the frequency table first. Fortunately, a CAS calculator will do this for us.

How to construct a histogram using the TI-Nspire CAS

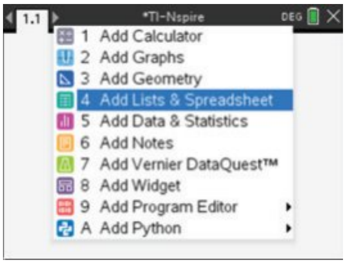
Display the following set of 27 marks in the form of a histogram.

16 11 4 25 15 7 14 13 14 12 15 13 16 14
15 12 18 22 17 18 23 15 13 17 18 22 23

Steps

1 Start a new document: Press [on] and select **New** (or use [ctrl] + [N]). If prompted to save an existing document, move the cursor to **No** and press [enter].

2 Select **Add Lists & Spreadsheet**.
Enter the data into a list named *marks*.



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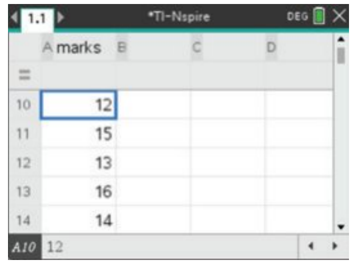
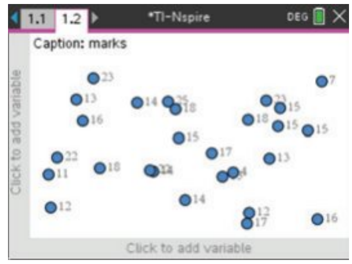
a Move the cursor to the name cell of column A (or any other column) and type in *marks* as the list variable. Press enter.

b Move the cursor down to row 1, type in the first data value and press enter. Continue until all the data has been entered. Press enter after each entry.



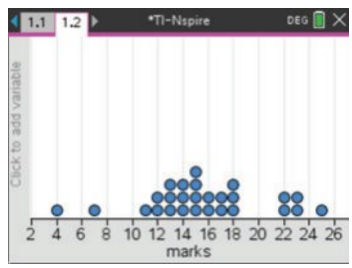
3 Statistical graphing is done through the **Data & Statistics** application.
Press ctrl + doc▼ (or alternatively press ctrl + I) and select **Add Data & Statistics** (or press ⌂ on, arrow to , and press enter).

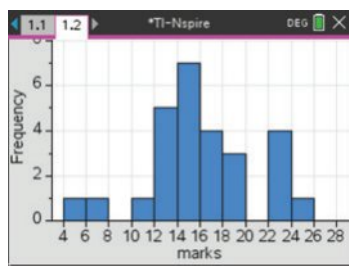
Note: A random display of dots will appear – this is to indicate that data are available for plotting. It is not a statistical plot.



a Press tab to show the list of variables that are available. Select the variable *marks*. Press enter to paste the variable *marks* to that axis.

b A dot plot is displayed as the default plot. To change the plot to a histogram, press menu **>Plot Type>Histogram** and then press enter or 'click' (press).
Your screen should now look like that shown opposite. This histogram has a column (or bin) width of 2 and a starting point of 4.

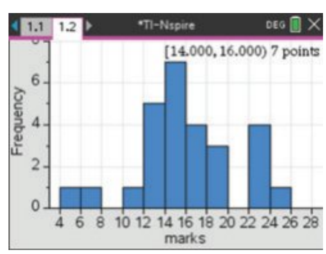




4 Data analysis

a Move the cursor onto any column. A will appear and the column data will be displayed, as shown opposite.

b To view other column data values, move the cursor to another column.



Note: If you click on a column it will be selected. To deselect any previously selected columns, move the cursor to the open area and press .

Hint: If you accidentally move a column or data point, press ctrl + esc to undo the move.

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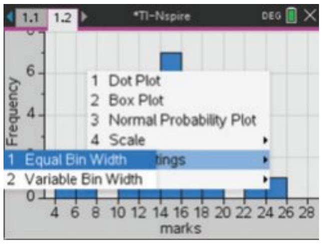
5 Change the histogram column (bin) width to 4 and the starting point to 2.

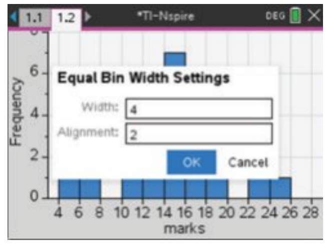
a Press ctrl + menu to access the context menu as shown (below left).

Hint: Pressing ctrl + menu with the cursor on the histogram gives you access to a context menu that enables you to do things that relate only to histograms.

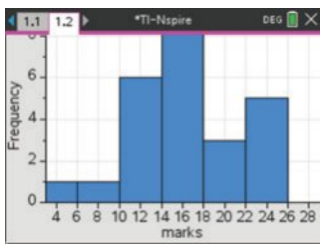
b Select **Bin Settings>Equal Bin Width**.

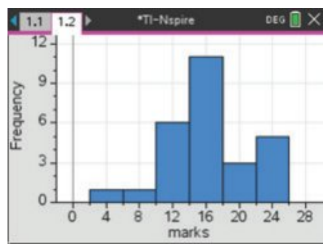
c In the settings menu (below right) change the **Width** to **4** and the **Starting Point (Alignment)** to **2** as shown. Press enter.



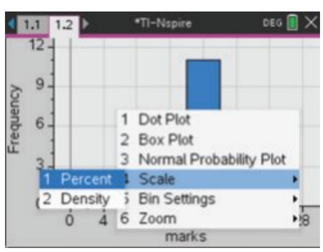


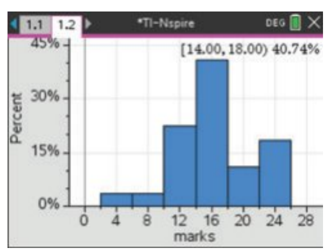
d A new histogram is displayed with a column width of 4 and a starting point of 2 but it no longer fits the viewing window (below left). To solve this problem, press ctrl + menu **>Zoom>Zoom-Data** and enter to obtain the histogram, as shown below right.





6 To change the frequency axis to a percentage axis, press ctrl + menu**>Scale>Percent** and then press enter.





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How to construct a histogram using the ClassPad

Display the following set of 27 marks in the form of a histogram.

16 11 4 25 15 7 14 13 14 12 15 13 16 14
15 12 18 22 17 18 23 15 13 17 18 22 23

Steps

1 From the application menu screen, locate the **Statistics** application.

Tap [Statistics] to open.

Note: Tapping [Menu] from the icon panel (just below the touch screen) will display the application menu if it is not already visible.

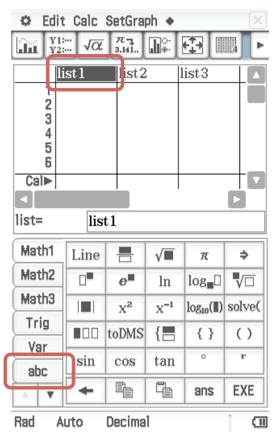
2 Enter the data into a list named *marks*.

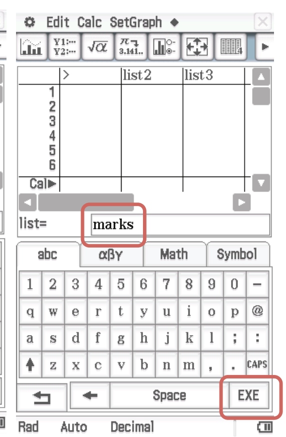
a Tap on the column heading list 1.

b Press [Keyboard] and tap [abc].

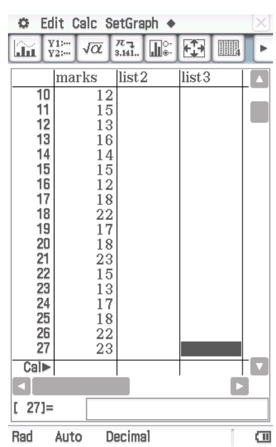
c Type *marks* and press [EXE].

d Starting in row 1, type in each data value. Press [EXE] or [▼] to move down the list.





Your screen should be like the one shown at right.



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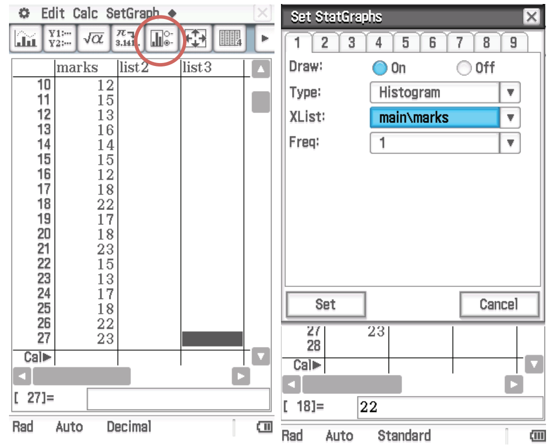
3 To plot a statistical graph:

a Tap at the top of the screen. This opens the **Set StatGraphs** dialog box.

b Complete the dialog box. For:

- **Draw:** select **On**
- **Type:** select **Histogram** (▼)
- **XList:** select **main\marks** (▼)
- **Freq:** leave as **1**.

c Tap Set to confirm your selections.



Note: To make sure only this graph is drawn, select **SetGraph** from the menu bar at the top and confirm there is a tick only beside **StatGraph1** and no other box.

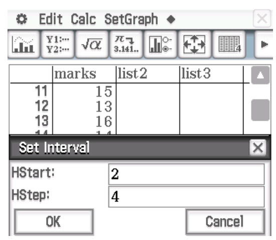
4 To plot the graph:

a Tap in the toolbar.

b Complete the **Set Interval** dialog box as given below. For:

- **HStart:** type in **2**
- **HStep:** type in **4**.

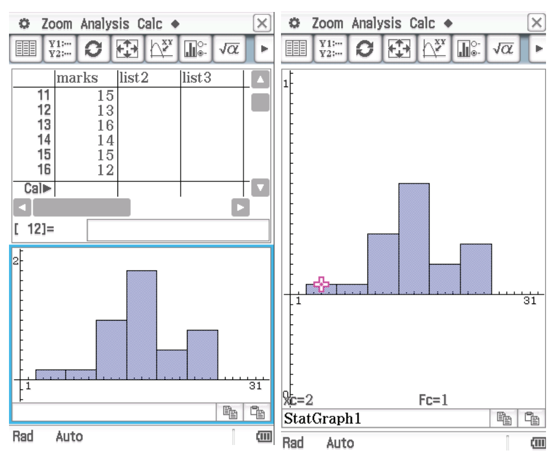
c Tap **OK**.



5 The screen is split in two. Tapping from the icon panel will allow the graph to fill the entire screen. Tap to return to half-screen size.

6 Tapping places a marker on the first column of the histogram and tells us that:

- the first interval begins at 2 ($\boldsymbol{x_c = 2}$)
- for this interval, the frequency is 1 ($\mathbf{F_c = 1}$).



To find the frequencies and starting points of the other intervals, use the arrow (▶) to move from interval to interval.

Section Summary

- Numerical data (both discrete and continuous) can be summarised in frequency tables (counts or percentages).
- When the data is **discrete, but can take many values**, the data should be **grouped** to form the frequency table.
- When the data is **continuous**, the data should be **grouped** to form the frequency table.
- A **histogram** is a display of a frequency table of numerical data.
- A CAS calculator can be used to construct a histogram.

Exercise 2C

Building understanding

Example 9

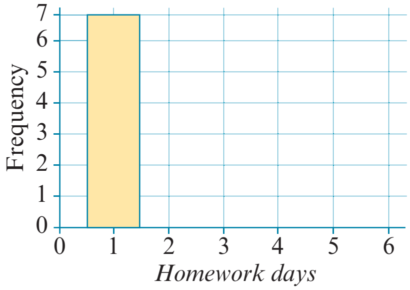
1 A group of 20 students were surveyed about the number of days they completed homework over a one-week period. The data is as shown.

1 5 3 2 1 1 2 4 3 1 4 2 4 5 3 2 2 1 1 1

a Use the data to complete the frequency table.

Homework days	Frequency
1	7
2	
3	
4	
5	
Total	20

b Use the information in the frequency table to complete the histogram shown.



Example 10
Example 12

2 The following are the heights, in centimetres, of 25 players in a women's football team.

a Use the data to complete the grouped frequency table.

188	175	176	161	183
169	171	176	165	166
162	170	174	168	178
169	180	173	163	179
163	170	163	175	177

Height (cm)	Frequency
160–164	5
165–169	
170–174	
175–179	
180–184	
185–190	
Total	25

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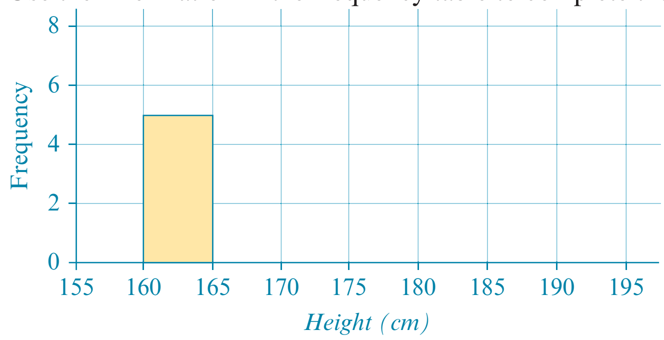
b Use the information in the frequency table to complete the histogram shown.



Developing understanding

3 The number of magazines purchased in a month by 15 different people was as follows:

0 5 3 0 1 0 2 4 3 1 0 2 1 2 1

Construct a frequency table for the data, including both the frequency and percentage frequency.

Example 11

4 The amount of money carried by 20 students is as follows:

$4.55 $1.45 $16.70 $0.60 $5.00 $12.30 $3.45 $23.60 $6.90 $4.35
$0.35 $2.90 $1.70 $3.50 $8.30 $3.50 $2.20 $4.30 $0.00 $11.50

Construct a frequency table for the data, including both the number and percentage in each category. Use intervals of $5, starting at $0.

Example 13

5 A group of 28 students were asked to draw a line that they estimated to be the same length as a 30 cm ruler. The results are shown in the frequency table, below.

a How many students drew a line with a length:

i from 29.0 to 29.9 cm?

ii of less than 30 cm?

iii of 32 cm or more?

b What percentage of students drew a line with a length:

i from 31.0 to 31.9 cm?

ii of less than 31 cm?

iii of 30 cm or more?

	Frequency	
Length of line (*cm*)	Number	%
28.0–28.9	1	3.6
29.0–29.9	2	7.1
30.0–30.9	8	28.6
31.0–31.9	9	32.1
32.0–32.9	7	25.0
33.0–33.9	1	3.6
Total	28	100.0

c Use the table to construct a histogram, using the counts.

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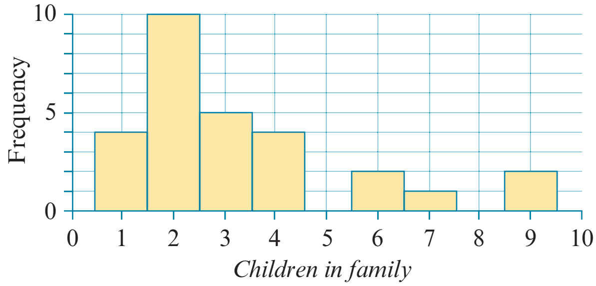
6 The number of children in the family for each student in a class is shown in the histogram.



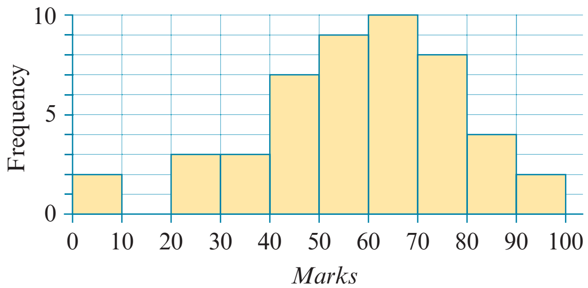
a How many students are the only child in a family?

b What is the most common number of children in a family?

c How many students come from families with 6 or more children?

d How many students are there in the class?

7 The following histogram gives the scores on a general knowledge quiz for a class of Year 11 students.



a How many students scored from 10 to 19 marks?

b How many students attempted the quiz?

c What is the modal interval?

d If a mark of 50 or more is designated as a pass, how many students passed the quiz?

e Of this group of students, what percentage did not pass the quiz? Round your answer to the nearest whole number.

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8 A student purchased 21 new textbooks from a schoolbook supplier with the following prices (in dollars):

41.65 34.95 32.80 27.95 32.50 53.99 63.99 17.80 13.50 18.99 42.98
38.50 59.95 13.20 18.90 57.15 24.55 21.95 77.60 65.99 14.50

a Use a CAS calculator to construct a histogram with a column width of 10 and a starting point of 10. Name the variable *Price*.

b For this histogram:

i what is the range of the third interval?

ii what is the 'frequency' for the third interval?

iii what is the modal interval?

9 The maximum temperatures for several capital cities around the world on a particular day, in degrees Celsius, were:

17 26 36 32 17 12 32 2 16 15 18 25
30 23 33 33 17 23 28 36 45 17 19 37

a Use a CAS calculator to construct a histogram with a column width of 2 and a starting point of 0. Name the variable *Max temp*.

b For this histogram:

i what is the starting point of the second column?

ii what is the 'frequency' for this interval?

c Use the window menu to redraw the histogram with a column width of 5 and a starting point of 0.

d For this histogram:

i how many cities had maximum temperatures from 20°C to 25°C?

ii what is the modal interval?

Testing understanding

10 The number of mistakes made on a test by each of 30 students is as follows.

1 2 3 5 1 9 3 2 2 3
3 2 6 6 8 5 9 9 9 8
3 5 6 1 1 4 2 8 9 4

a Organise the data into a frequency table.

b Construct a histogram of the data from the frequency table.

c What is the mode?

d A student who makes five or more errors is required to re-sit the test. What percentage of students would be required to re-sit?

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11 The following data gives the waiting time, in minutes, for a group of 30 people who attended an emergency department of a hospital.

36	76	14	26	11	90	32	2	16	125
12	45	59	75	17	5	18	22	45	28
15	23	9	53	17	13	31	78	12	51

a Use the data to construct a frequency table.

b Construct a histogram of the data from the frequency table.

c What is the modal value for the variable, and what percentage of people chose that response?

2D Characteristics of distributions, dot plots and stem plots

Learning intentions

- ► To be able to identify the key characteristics of a data distribution.
- ► To be able to construct a dot plot and a stem plot for numerical data.

Characteristics of a distribution

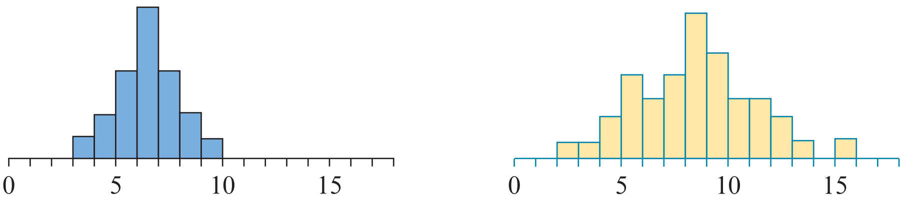
Distributions of numerical data are characterised by their shape and special features such as location (also referred to as the 'centre') and spread.

Shape of a distribution

Symmetry

A distribution is said to be **symmetric** if it forms a mirror image of itself when folded in the 'middle' along a vertical axis.

Histogram A below is exactly symmetric, while Histogram B shows a distribution that is approximately symmetric. In practice it is rare to find a histogram which is exactly symmetric, and approximate symmetry is enough for us to classify a histogram as symmetric.



Histogram A   Histogram B

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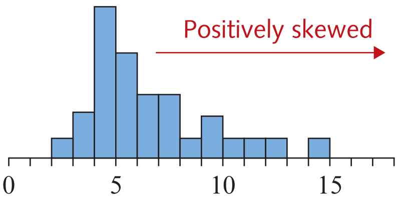
Positive and negative skew

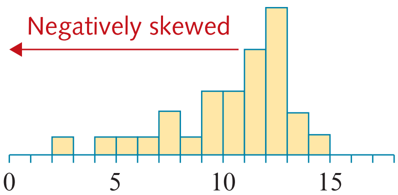
A histogram may be positively or negatively skewed.

- It is **positively skewed** if it has a short tail to the left and a long tail pointing to the right (because of the many values towards the positive end of the distribution).
- It is **negatively skewed** if it has a short tail to the right and a long tail pointing to the left (because of the many values towards the negative end of the distribution).

Histogram C is an example of a positively skewed distribution, and Histogram D is an example of a negatively skewed distribution.



Histogram C



Histogram D

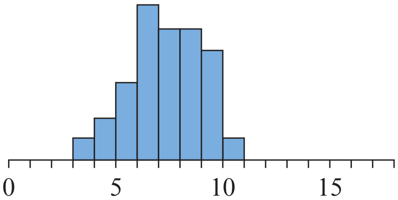
Knowing whether a distribution is skewed or symmetric is important, as this gives considerable information concerning the choice of appropriate summary statistics, as will be seen in the next section.

Centre

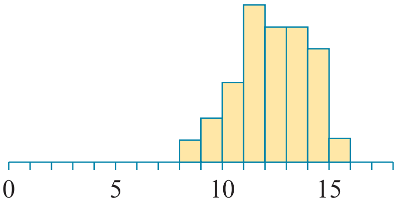
Comparing centre

Two distributions are said to differ in **centre** if the values of the data in one distribution are generally larger than the values of the data in the other distribution.

Consider, for example, the following histograms, shown on the same scale. Histogram F is identical in shape and width to Histogram E but is moved horizontally several units to the right, indicating that these distributions differ in location.



Histogram E



Histogram F

Spread

Comparing spread

Two distributions are said to differ in **spread** if the values of the data in one distribution tend to be more variable (spread out) than the values of the data in the other distribution.

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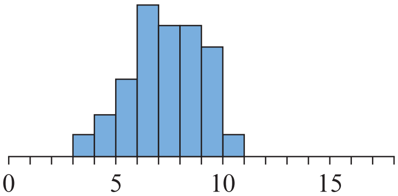
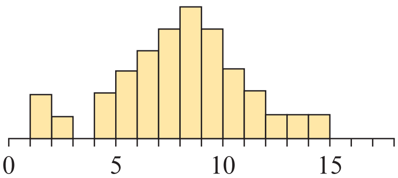
Histograms G and H illustrate the difference in spread. While both are centred at about the same place, Histogram H is more spread out.



Histogram G



Histogram H

As we have seen here, a histogram enables us to identify and compare the characteristics of a data distribution (shape, centre and spread). These key features can also be seen in two other plots which you are already familiar with, the dot plot and the stem and leaf plot.

Dot plots

The simplest display of numerical data, and an alternative to a frequency histogram, is the **dot plot**.

Dot plot

A dot plot consists of a number line with each data point marked by a dot. When several data points have the same value, the points are stacked on top of each other.

Dot plots display fairly small data sets where the data takes a limited number of values.

Example 14 Constructing a dot plot

The number of hours worked by each of 10 students in their part-time jobs is as follows:

6 9 5 8 6 4 6 7 6 5

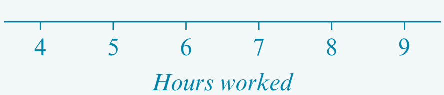
Construct a dot plot of these data.

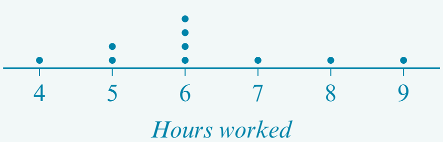
Explanation

1 Draw in a number line, scaled to include all data values. Label the line with the variable being displayed.

2 Plot each data value by marking in a dot above the corresponding value on the number, as shown.

Solution





In the same way that the shape of a distribution can be identified from a histogram, it can also be identified from the dot plot (if there are enough data values). When there is a longer tail on the dot plot in the positive direction, then the distribution would be positively skewed, while if the tail on the dot plot is in the negative direction, then the distribution would be negatively skewed. The dot plot in Example 14 would be considered to be approximately symmetric, although when there are so few data values it is difficult to comment on shape with any certainty.

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Now try this 14 Constructing a dot plot (Example 14)

The number of pages written for an assignment by a group of 12 students is as follows:

2 3 2 5 5 6 7 3 4 4 5 9

Construct a dot plot of these data.

Hint 1 Construct a number line spanning the minimum and maximum data values.

Hint 2 Ensure the vertical distance between dots is consistent.

Stem plots

The **stem plot** or **stem and leaf plot** is another very useful plot for displaying small numerical data sets.

Stem plot

A stem plot is a display where each data value is split into a **stem** (usually the leading digit or digits) and a **leaf** (usually the last digit). For example, 45 is split into 4 (stem) and 5 (leaf).

The stem values are listed vertically, and the leaf values are listed horizontally next to their stem. The stem is usually separated from the leaves by a vertical line.

Example 15 Constructing a stem plot

The following is a set of marks obtained by a group of students on a test:

15	2	24	30	25	19	24	33	18	60	42	37	28
28	17	19	52	55	27	5	7	19	45	19	25	

Display the data in the form of an ordered stem plot, and comment on the shape of the distribution.

Explanation

1 The data set has values in the units, tens, twenties, thirties, forties, fifties and sixties. Thus, appropriate stems are 0, 1, 2, 3, 4, 5 and 6. Write these down in ascending (smallest to largest) order, followed by a vertical line.

Solution

0	
1	
2	
3	
4	
5	
6	

2 Now attach the leaves. The first data value is 15. The stem is 1 and the leaf is 5. Opposite the 1 in the stem, write the number 5, as shown.

0	
1	5
2	
3	
4	
5	
6	

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The second data value is 2. The stem is 0 and the leaf is 2. Opposite the 0 in the stem, write the number 2, as shown.

0	2
1	5
2	
3	
4	
5	
6	

Continue systematically working through the data, following the same procedure, until all points have been plotted. You will then have the *unordered* stem plot, as shown.

0	2 5 7
1	5 9 8 7 9 9 9
2	4 5 4 8 8 7 5
3	0 3 7
4	2 5
5	2 5
6	0

3 Ordering the leaves in increasing value as they move away from the stem gives the *ordered* stem plot, as shown. Write the name of the variable being displayed (*Marks*) at the top of the plot, and add a key (1|5 means 15 marks).

Marks key: 1 | 5 = 15 marks

0	2 5 7
1	5 7 8 9 9 9 9
2	4 4 5 5 7 8 8
3	0 3 7
4	2 5
5	2 5
6	0

4 Looking at the stem plot, we can see that the tail of the distribution is longer in the direction of the increasing test scores (if this isn't clear then turn the stem plot on its side), so we can say this distribution is positively skewed.

Note that the stem is defined as the leading digit or digits and the leaf as the final digit. That is why it is always necessary to include a key, so that we know how to interpret the stem and the leaves.

Now try this 15 **Constructing a stem plot (Example 15)**

The weights of each of 22 pumpkins (in kg) grown in Essie's garden are as follows:

1.9	4.2	2.4	3.0	2.9	3.4	4.4	3.3	4.1	6.0	5.5
2.5	2.8	3.8	3.7	2.8	4.5	2.0	6.8	7.0	4.3	4.5

Display the data in the form of an ordered stem plot.

Hint 1 Choose values for the stem which span the minimum and maximum data values.

Hint 2 Make sure you include a key.

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Choosing between plots

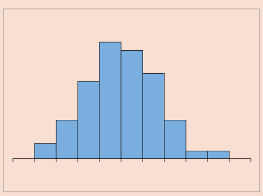
We now have three different plots that can be used to display numerical data: the histogram, the dot plot and the stem and leaf plot. They all allow us to make judgements concerning the important features of the distribution of the data, so how would we decide which one to use?

While there are no hard and fast rules, the following guidelines are often used.

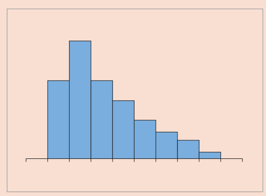
Plot	Used best when	How usually constructed
Dot plot	small data sets (say $n < 30$) discrete data	by hand or with technology
Stem plot	small data sets (say $n < 50$)	by hand
Histogram	large data sets (say $n > 30$)	with technology

Section Summary

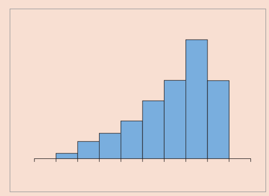
- **Numerical data distributions** are characterised by shape (symmetric or skewed), location and spread.
- The following shapes are commonly seen in data distributions.



symmetric



positively skewed



negatively skewed

- A **dot plot** is an appropriate display for a data distribution when there are a small number of data values.
- A **stem plot** is an appropriate display for a data distribution when there are a small to medium number of data values.
- A **histogram** is an appropriate display for a data distribution when there are a medium to large number of data values, and technology is used.
- The characteristics of a data distribution (shape, centre and spread) can be visually identified from all of these plots.

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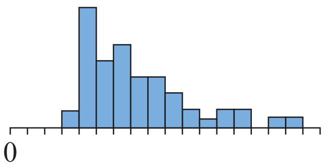
Exercise 2D

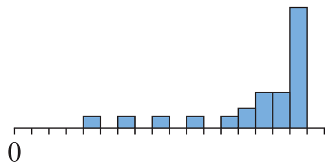
Building understanding

1 Describe the shape of each of the following distributions (negatively skewed, positively skewed or approximately symmetric).

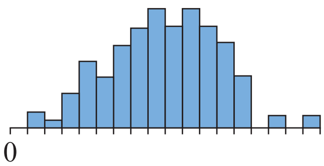
a



b

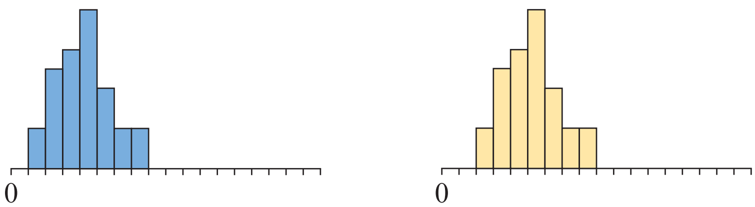


c

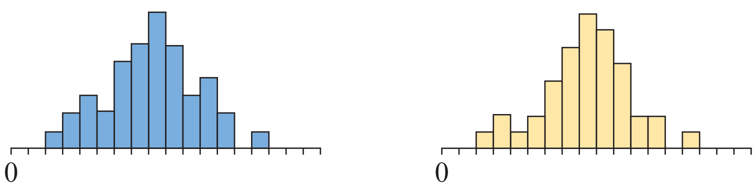


2 Do the following pairs of distributions differ in spread, centre, both or neither? Assume that each pair of histograms is drawn on the same scale.

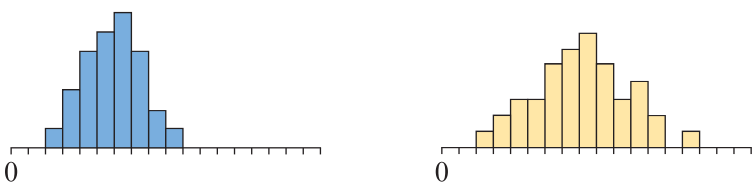
a



b



c



Developing understanding

Example 14

3 The number of children in each of 15 families is as follows:

0 7 2 2 2 4 1 3 3 2 2 2 0 0 1

a Construct a dot plot of the number of children.

b What is the mode of this distribution?

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4 A group of 20 people were asked how many times in the last week they had shopped at a particular supermarket. Their responses were as follows:

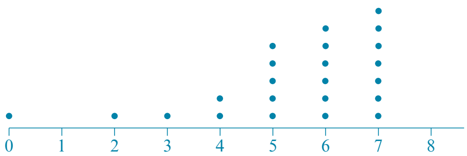
0	1	1	0	0	6	0	1	2	2
3	4	0	0	1	1	2	3	2	0

a Construct a dot plot of this data.

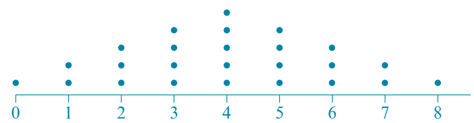
b How many people did not shop at the supermarket in the last week?

5 Describe the shape of each of the following distributions (negatively skewed, positively skewed or approximately symmetric).

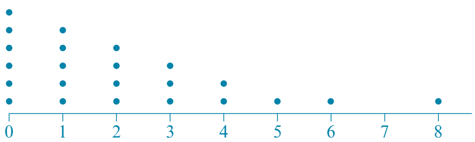
a



b



c



6 The ages of each member of a football team are as follows:

22	20	20	21	21	22	24	25	25	24	26
22	19	28	24	25	30	21	27	24	25	26

a Construct a dot plot of the ages of the players.

b What is the mode of this distribution?

c What is the shape of the distribution of player ages?

d What percentage of players are younger than 25?

7 In a study of the service offered at her cafe, Amanda counted the number of people waiting in the queue every 5 minutes from 12 noon until 1 p.m.

Time	12:00	12:05	12:10	12:15	12:20	12:25	12:30	12:35	12:40	12:45	12:50	12:55	1:00
Number	0	2	4	4	7	8	6	5	0	1	2	1	1

a Construct a dot plot of the number of people waiting in the queue.

b When does the peak demand at the cafe seem to be?

Example 15

8 The marks obtained by a group of students on an English examination are as follows:

92	65	35	89	79	32	38	46	26	43	83	79
50	28	84	97	69	39	93	75	58	49	44	59
78	64	23	17	35	94	83	23	66	46	61	52

a Construct a stem plot of the marks.

b What percentage of students obtained a mark of 50 or more?

c What was the lowest mark?

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9 Describe the shape of each of the following distributions (negatively skewed, positively skewed or approximately symmetric).

a key: 3|4 represents 34

2	2
3	4 5
4	4 5 6 7
5	0 5 5 6 7 8 8 9
6	0 0 2 3 4 5 7 9
7	0 1 4 5 6
8	0 3 5 7
9	5 9

b key: 3|1 represents 31

3	1
4	6
5	5 6
6	2 3 7 7 8
7	1 1 2 3 4 5 6 7
8	2 3 3 5 6 6 7 7 8 9
9	0 1

c key: 1|0 represents 10

0	0 1 1 2 3 4 5 6 6 7 7
1	0 1 2 3 4 5 5 9
2	1 2 7
3	0 1
4	0

10 The stem plot on the right shows the ages, in years, of all the people attending a meeting.

Age (years) key: 1 | 2 means 12 years

0	2 7
2	1 4 5 5 7 8 9
3	0 3 4 4 5 7 8 9
4	0 1 2 2 3 3 4 5 7 8 8 8
5	2 4 5 6 7 9
6	3 3 3 8
7	0

a How many people attended the meeting?

b What is the shape of the distribution of ages?

c How many of these people were less than 43 years old?

11 An investigator recorded the amount of time for which 24 similar batteries lasted in a toy. Her results (in hours) were:

26	40	30	24	27	31	21	27	20	30	33	22
4	26	17	19	46	34	37	28	25	31	41	33

a Make a stem plot of these times.

b How many of the batteries lasted for more than 30 hours?

12 The amount of time (in minutes) that a class of students spent on homework on one particular night was:

10	27	46	63	20	33	15	21	16	14	15
39	70	19	37	56	20	28	23	0	29	10

a Make a stem plot of these times.

b How many students spent more than 60 minutes on homework?

c What is the shape of the distribution?

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13 The prices of a selection of shoes at a discount outlet are as follows:

$49 $75 $68 $79 $75 $39 $35 $52 $149 $84
$36 $95 $28 $25 $78 $45 $46 $76 $82

a Construct a stem plot of this data.

b What is the shape of the distribution?

Testing understanding

14 **a** The minimum daily temperature over a two-week period in a certain town was recorded as follows:

1 2 5 3 2 3 1 1 2 6 7 4 4 5

Construct a dot plot of the data.

b The maximum daily temperature over the same two-week period in that town was also recorded as follows:

12 14 13 13 13 15 15 17 16 13 13 12 13 13

Construct a dot plot of the data, using the same scale for the axes as in part **a**.

c How do the two distributions compare in terms of centre and spread?

15 A researcher determined the percentage body fat for 30 adult males, as follows:

5 12 9 17 19 5 7 21 30 24 9 21 10 20 17
18 28 11 14 17 13 4 20 17 25 29 17 26 10 13

a Construct a stem plot of the data.

b After the researcher had finished collecting the data, she noted that there had been an error in the data collection which could be fixed by subtracting 2 from each data value. How do you think the distribution of the corrected data would compare to that shown in part **a** in terms of centre and spread?

16 State whether you would predict the data distribution of the following variables to be symmetric, positively skewed or negatively skewed. Give a reason for your choice.

a The height in cm of 16-year-old boys.

b The purchase price, in dollars, of a house in Melbourne.

c The gestation period for a human baby in weeks.

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2E Measures of centre

Learning intentions

- ► To be able to understand the mean and the median as measures of centre.
- ► To be able to know whether to use the median or mean as a measure of centre for a particular distribution.

A statistic is any number computed from data. Certain special statistics are called **summary statistics** because they numerically summarise important features of the data set. Of course, whenever any set of data is summarised into just one or two numbers, much information is lost. However, if a summary statistic is well chosen, it may reveal important information hidden in the data set.

In this section we will consider some summary statistics which are measures of centre.

The mean

The most commonly used measure of the centre of a distribution of a numerical variable is the **mean**. The mean is calculated by summing the data values and then dividing by their number. The mean of a set of data is commonly referred to as the 'average'.

The mean

$$\text{mean} = \frac{\text{sum of data values}}{\text{total number of data values}}$$

For example, consider the set of data: 1, 5, 2, 4

$$\text{Mean} = \frac{1+5+2+4}{4} = \frac{12}{4} = 3$$

Some notation

Because the rule for the mean is relatively simple, it is easy to write in words. However, later you will meet other rules for calculating statistical quantities that are extremely complicated and hard to write out in words. To overcome this problem, we use a shorthand notation that enables complex statistical formulas to be written out in a compact form.

In this notation we use:

- the Greek capital letter sigma, $\sum$, as a shorthand way of writing 'sum of'
- a lower case x, to represent a data value
- a lower case x with a bar, (pronounced 'x bar'), to represent the mean of the data values
- n to represent the total number of data values.

The rule for calculating the mean then becomes: $\bar{x} = \dfrac{\sum x}{n}$

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Example 16 Calculating the mean

The following data set shows the number of premierships won by each of the current AFL teams until the end of 2021. Find the mean of the number of premierships won. Round your answer to one decimal place.

Team	Premierships
Carlton	16
Essendon	16
Collingwood	15
Melbourne	13
Hawthorn	13
Richmond	13
Brisbane Lions	11
Geelong	9
Sydney	5

Team	Premierships
Kangaroos	4
West Coast	4
Adelaide	2
Western Bulldogs	2
Port Adelaide	1
St Kilda	1
Fremantle	0
Gold Coast	0
GWS	0

Explanation

1 Write down the formula and the value of n.

2 Substitute into the formula and evaluate.

3 We do not expect the mean to be a whole number, so give your answer to one decimal place.

Solution

$\bar{x} = \dfrac{\sum x}{n} \qquad n = 18$

$$\bar{x} = \frac{16 + 16 + 15 + \ldots + 1 + 1 + 0 + 0 + 0}{18}$$

$$= \frac{125}{18}$$

$$= 6.9$$

The median

Another useful measure of the centre of a distribution of a numerical variable is the middle value, or **median**. To find the value of the median, all the observations are listed in order, and the middle one is the median.

For example, the median of the following data set is 6, as there are five observations on either side of this value when the data are listed in order.

median = 6
↓
2 3 4 5 5 6 7 7 8 8 11

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When there is an even number of data values, the median is defined as the midpoint of the two middle values. For example, the median of the following data set is 6.5, as there are six observations on either side of the median value when the data are listed in order.

median = 6.5
↓

2	3	4	5	5	6	7	7	8	8	11	11

Returning to the premiership data; since the data are already given in order, it only remains to determine the middle observation.

Since there are 18 entries in the table there is no actual middle observation, so the median is chosen as the value halfway between the two middle observations, in this case the ninth and tenth values (5 and 4).

$$\text{median} = \frac{1}{2}(5 + 4) = 4.5$$

The interpretation here is that, of the teams in the AFL, half (or 50%) have won the premiership 5 or more times and half (or 50%) have won the premiership 4 or fewer times.

The following rule is useful for locating the median in a larger data set.

Determining the median

To compute the median of a distribution:

- arrange all the observations in ascending order according to size
- if n, the number of observations, is odd, then the median is the $\left(\frac{n+1}{2}\right)$th observation from the end of the list
- if n, the number of observations, is even, then the median is found by averaging the two middle observations in the list. That is, to find the median, the $\frac{n}{2}$th and the $\left(\frac{n}{2}+1\right)$th observations are added together and divided by 2.

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Example 17 Determining the median when n is odd

Find the median age for the 23 people whose ages are displayed in the ordered stem plot.

Age (years) key: 1 | 2 means 12 years

0	2 5
2	1 4 5 8
3	0 3 4 6
4	0 1 2 5 7
5	2 4 5 8
6	3 5 9 9

Explanation

As the data are already given in order, it only remains to determine the middle observation.

1. Determine the number of observations.
2. Since there are an odd number of values, the median is located at the $\frac{n+1}{2}$th position.

Solution

$n = 23$

Median is at the $\frac{23+1}{2} = 12$th position.

Thus the median age is 41 years.

Note: We can check to see whether we are correct by counting the number of data values either side of the median. They should be equal.

Example 18 Determining the median when n is even

Find the median age for a group of people whose ages are displayed in this ordered stem plot.

Age (years) 1 | 2 represents 12 years

0	5 9
2	1 3 5 8
3	0 0 4 9 9
4	0 4 5 8
5	3 7
6	3

Explanation

Again, the data are already given in order, so it only remains to determine the middle observation.

1. Determine the number of observations.
2. Since there are an even number of observations, to find the median the $\frac{n}{2}$th and the $\left(\frac{n}{2}+1\right)$th observations are added together and divided by 2.

Solution

$n = 18$

Median is the average of the values in the $\frac{18}{2} = 9$th and $\frac{18}{2} + 1 = 10$th positions.

Thus the median age is $\frac{34 + 39}{2} = 36.5$ years.

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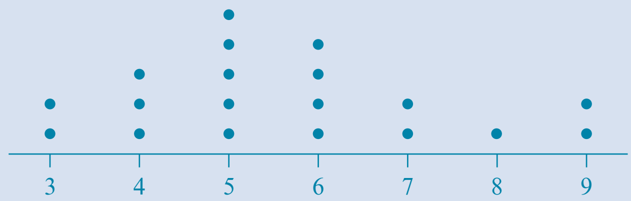
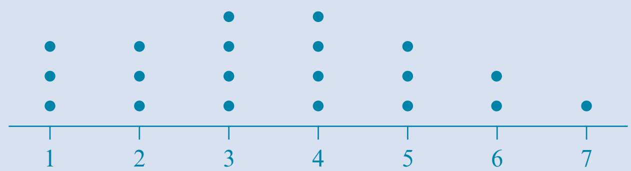
Now try this 18 **Determining the median from a dot plot (Examples 17 and 18)**

1 Find the median of the data displayed in the dotplot shown.



2 Find the median of the data displayed in the dotplot shown.



Hint 1 Determine the number of data values in the dot plot.

Hint 2 Locate the median using the appropriate formula for an odd or even number of data values.

Comparing the mean and median

In Example 16 we found that the mean number of premierships won by the 18 AFL clubs was $\bar{x} = 6.9$. We also found that the median number of premierships won was 4.5.

These two values are quite different, and the interesting question is: Why are they different, and which is the better measure of centre in this situation?

To help us answer this question, consider a stem plot of these data values.

Premierships won

0	0 0 0 1 1 2 2 4 4
0	5 9
1	1 3 3 3
1	5 6 6

From the stem and leaf plot it can be seen that the distribution of premierships won is positively skewed. This example illustrates a property of the mean. When the distribution is skewed or if there are one or two very extreme values, then the mean is pulled towards the tail of the distribution and the extreme values, giving us a value for the mean which may be far from the centre. Since the median is not so affected by unusual observations and always gives the middle value, then the median is the preferred measure of centre for a skewed distribution or one with outliers. (Outliers are really large or really small numbers compared to the rest of the data.) When the distribution is symmetric, and there are no outliers, then either measure is appropriate, although we tend to prefer the mean as it is easier to calculate and more familiar to most people.

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Section Summary

- **Summary statistics** are numbers calculated from a data set which represent important features of that data set.
- Two very useful summary statistics which give numerical values for the centre of a distribution are the **mean** and the **median**.
- The **mean** is denoted $\bar{x}$ and the mean $= \dfrac{\text{sum of data values}}{\text{total number of data values}}$
  $= \dfrac{\sum x}{n}$
- The **median** is the middle value in the ordered data, which is the $\left(\dfrac{n+1}{2}\right)$th observation from the end of the list when n is odd, or the average of the $\dfrac{n}{2}$th and the $\left(\dfrac{n}{2}+1\right)$th observations when n is even.
- When the distribution is skewed, the **median** is the preferred measure of centre.

Exercise 2E

Building understanding

Example 16

1 For the following data set:

1 2 1 0 2 3 1 1 2 6 7

a Find the sum of the data values.

b Hence, find the mean of the data set.

Example 17

2 For the following data set:

21 12 35 53 32 63 11 19 62 17 24 34 95

a Order the data from smallest to largest value.

b Hence, find the median of the data set.

Developing understanding

3 Find, without using a calculator, the mean for each of these data sets.

a	2	5	7	2	9	
b	4	11	3	5	6	1
c	15	25	10	20	5	
d	101	105	98	96	97	109
e	1.2	1.9	2.3	3.4	7.8	0.2

Example 18

4 Find, without using a calculator, the median for each of these ordered data sets.

a	2	2	5	7	9	11	12	16	23		
b	1	3	3	5	6	7	9	11	12	12	
c	21	23	24	25	27	27	29	31	32	33	
d	101	101	105	106	107	107	108	109			
e	0.2	0.9	1.0	1.1	1.2	1.2	1.3	1.9	2.1	2.2	2.9

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5 Without a calculator, find the median of the data displayed in the following stem plots.

a *Monthly rainfall (mm)*

4|8 represents 48 mm

Stem	Leaf
4	8 9 9
5	0 2 7 7 8 9 9
6	0 7

b *Battery time (hours)*

1|7 represents 17 hours

Stem	Leaf
0	4
1	7 9
2	0 1 2 4 5 6 6 7 7 8
3	0 0 1 1 3 3 4 7
4	0 1 6

6 The following data gives the area, in hectares, of each of the suburbs of a city:

3.6 2.1 4.2 2.3 3.4 40.3 11.3 19.4 28.4 27.6 7.4 3.2 9.0

a Find the mean and the median areas.

b Which is a better measure of centre for this data set? Explain your answer.

7 Find the mean of each of the data sets displayed in the stem plots in Question 5.

8 The prices, in dollars, of apartments sold in a particular suburb during one month were:

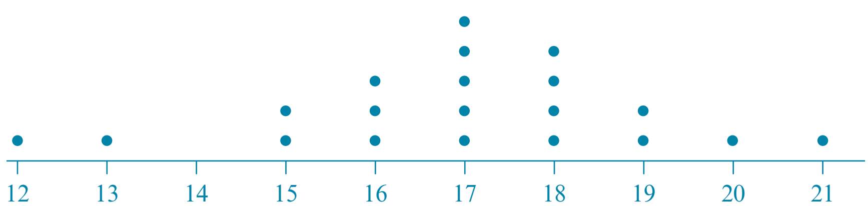
\$387 500 \$329 500 \$293 400 \$600 000 \$318 000 \$368 000 \$750 000

\$333 500 \$335 500 \$340 000 \$386 000 \$340 000 \$404 000 \$322 000

a Find the mean and the median of the prices.

b Which is a better measure of centre for this data set? Explain your answer.

9 Find the mean and median of the data set displayed in the following dot plot.



10 Suppose that the mean of a data set with 8 values is 23.6, and that when another data value is added, the mean becomes 24.9. What is the data value that has been added to the data set?

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Testing understanding

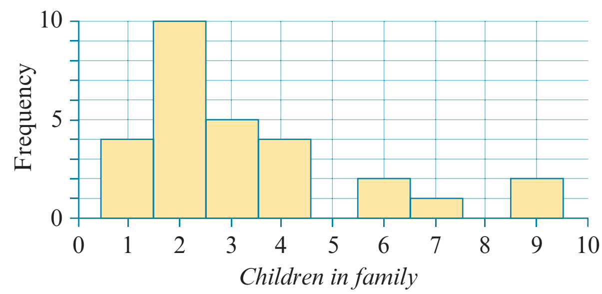
11 The mean score for Mr Miller's class in the Mathematics test was 67. The mean score for Mrs Lacey's class was 72. If 23 students took the test in Mr Miller's class, and 20 took the test in Mrs Lacey's class, what was the mean score across both classes?

12 Jessie is completing a science experiment. She has 20 data values and has calculated the mean of the data set to be 15.6 and the median to be 15.0 When she is typing up her results, she mistypes the smallest data value, typing 1.6 instead of 10.6.

a What would the new mean of the data set be?

b What would the new median of the data set be?

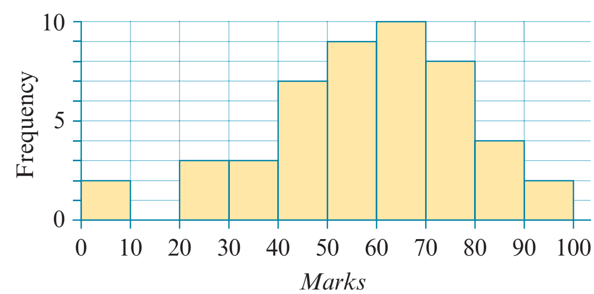
13 The number of children in the family for each student in a class is shown in the histogram.



a What is the median number of children in the family for this class?

b What is the mean number of children in the family for this class?

14 The following histogram gives the scores on a general knowledge quiz for a class of Year 11 students.



a What can you say about the median mark on the quiz?

b Can you identify the mean mark from this histogram? Justify your answer.

15 The following percentage frequency table gives the heights of members of a sporting team:

a What can you say about the median height of the team members?

b Can you identify the mean height from this percentage frequency table? Justify your answer.

Height (cm)	Percentage Frequency
170.0–174.9	5
175.0–179.9	22
180.0–184.9	35
185.0–189.9	26
190.0–194.9	10
195.0–199.9	2
Total	100

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2F Measures of spread

Learning intentions

- To be able to understand the range, interquartile range and standard deviation as measures of spread.
- To be able to know when to use each as a measure of spread for a particular distribution.
- To be able to learn to use a CAS calculator to calculate summary statistics.

A measure of spread is calculated in order to judge the **variability** of a data set. That is, are most of the values clustered together, or are they rather spread out?

The range

The simplest measure of spread can be determined by considering the difference between the smallest and the largest observations. This is called the **range**.

The range

The range (R) is the simplest measure of spread of a distribution.

The range is the difference between the largest and smallest values in the data set.

R = largest data value − smallest data value

Example 19 Finding the range

Consider the marks, for two different tasks, awarded to a group of students:

Task A

2	6	9	10	11	12	13	22	23	24	26	26	27	33	34
35	38	38	39	42	46	47	47	52	52	56	56	59	91	94

Task B

11	16	19	21	23	28	31	31	33	38	41	49	52	53	54
56	59	63	65	68	71	72	73	75	78	78	78	86	88	91

Find the range of each of these distributions.

Explanation	Solution
For Task A, the minimum mark is 2 and the maximum mark is 94.	Range for Task A = 94 − 2 = 92
For Task B, the minimum mark is 11 and the maximum mark is 91.	Range for Task B = 91 − 11 = 80

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Now try this 19 **Finding the range (Example 19)**

The weights of 19 cats are displayed in this ordered stem plot. Find the range.

Weight (kg) key: 1 | 2 represents 1.2 kg

0	5 9
2	1 3 5 8
3	0 0 4 9 9
4	0 4 5 8
5	3 7
6	3 4

Hint 1 Make sure that you carefully look at the key to the data values.

In Example 19, the range for Task A is greater than the range for Task B. Is the range a useful summary statistic for comparing the spread of the two distributions? To help make this decision, consider the stem plots of the data sets:

Task A key: 1 | 2 represents 12 marks

0	2 6 9
1	0 1 2 3
2	2 3 4 6 6 7
3	3 4 5 8 8 9
4	2 6 7 7
5	2 2 6 6 9
6	
7	
8	
9	1 4

Task B key: 1 | 2 represents 12 marks

0	
1	1 6 9
2	1 3 8
3	1 1 3 8
4	1 9
5	2 3 4 6 9
6	3 5 8
7	1 2 3 5 8 8 8
8	6 8
9	1

From the stem and leaf plots of the data it appears that the spread of marks for the two tasks is not really described by the range. It is clear that the marks for Task A are more concentrated than the marks for Task B, except for the two unusual values for Task A.

Another measure of spread is needed, one which is not so influenced by these extreme values. The statistic we use for this task is the **interquartile range**.

The interquartile range

Determining the interquartile range

To find the interquartile range of a distribution:

- arrange all observations in order according to size
- divide the observations into two equal-sized groups, and if n is odd, omit the median from both groups
- locate Q_1, the **first quartile**, which is the median of the lower half of the observations, and Q_3, the **third quartile**, which is the median of the upper half of the observations.

The interquartile range (IQR) is then: $IQR = Q_3 - Q_1$.

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We can interpret the interquartile range as follows:

- Since Q_1, the first quartile, is the median of the lower half of the observations, then it follows that 25% of the data values are less than Q_1, and 75% are greater than Q_1.
- Since Q_3, the third quartile, is the median of the upper half of the observations, then it follows that 75% of the data values are less than Q_3, and 25% are greater than Q_3.
- Thus, the interquartile range (IQR) gives the spread of the middle 50% of data values.

Definitions of the quartiles of a distribution sometimes differ slightly from the one given here. Using different definitions may result in slight differences in the values obtained, but these will be minimal and should not be considered a difficulty.
Note that the *median* which has 50% of the data below and 50% of the data values above, is denoted as Q_2, but we don't commonly use this notation.

Example 20 Finding the interquartile range (IQR)

Find the interquartile range for Task A and Task B in Example 19 and compare.

Explanation	Solution
1 There are 30 values in total. This means that there are fifteen values in the lower 'half', and fifteen in the upper 'half'. The median of the lower half (Q_1) is the 8th value.	Task A Lower half: 2 6 9 10 11 12 13 22 23 24 26 26 27 33 34 $Q_1 = 22$
2 The median of the upper half (Q_3) is the 8th value.	Upper half: 35 38 38 39 42 46 47 47 52 52 56 56 59 91 94 $Q_3 = 47$
3 Determine the IQR.	$IQR = Q_3 - Q_1 = 47 - 22 = 25$
4 Repeat the process for Task B.	Task B $Q_1 = 31$ $Q_3 = 73$ $IQR = Q_3 - Q_1 = 73 - 31 = 42$
5 Compare the IQR for Task A to the IQR for Task B.	The IQR shows that the variability of Task A marks is smaller than the variability of Task B marks.

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Now try this 20 **Finding the interquartile range (IQR) (Example 20)**

Find the interquartile range of the weights of the 19 cats whose weights are displayed in this ordered stem plot.

Weight (kg) 1 | 2 represents 1.2 kg

0	5 9
2	1 3 5 8
3	0 0 4 9 9
4	0 4 5 8
5	3 7
6	3 4

Hint 1 Locate the median.

Hint 2 Since there are an uneven number of values, omit the median. The upper and lower halves of the data will have 9 data values in each.

The interquartile range describes the range of the middle 50% of the observations. It measures the spread of the data distribution around the median (M). Since the upper 25% and the lower 25% of the observations are discarded, the interquartile range is generally not affected by outliers in the data set, which makes it a reliable measure of spread for any distribution, whether skewed or symmetric.

The standard deviation

The **standard deviation (*s*)** measures the spread of a data distribution about the mean ($\bar{x}$).

The standard deviation

The standard deviation is defined to be:

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$$

where n is the number of data values (sample size) and $\bar{x}$ is the mean.

Although it is not easy to see from the formula, the standard deviation is an average of the squared deviations of each data value from the mean. We work with the *squared* deviations because the sum of the deviations around the mean will always be zero. For theoretical reasons (not important here) we average by dividing by $n - 1$, not n.

Normally, you will use your calculator to determine the value of a standard deviation. However, to understand what is involved when your calculator is doing the calculation, you should know how to calculate the standard deviation from the formula.

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Example 21 **Calculating the standard deviation**

Use the formula:

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$$

to calculate the standard deviation of the data set: 2, 3, 4.

Explanation

1 To calculate s, it is convenient to set up a table with columns for:
x, the data values
$(x - \bar{x})$ the deviations from the mean
$(x - \bar{x})^2$ the squared deviations.

2 First find the mean ($\bar{x}$) and then complete the table as shown.

3 Substitute the required values into the formula and evaluate.

Solution

	x	$(x - \bar{x})$	$(x - \bar{x})^2$
	2	−1	1
	3	0	0
	4	1	1
Sum	9	0	2

$$\bar{x} = \frac{\sum x}{n} = \frac{2 + 3 + 4}{3} = \frac{9}{3} = 3$$

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}} = \sqrt{\frac{2}{3 - 1}} = 1$$

So, the standard deviation is 1.

Now try this 21 **Calculating the standard deviation (Example 21)**

Use the formula above to calculate the standard deviation for this data set:

0 1 3 5.

Hint 1 Set up a table as in Example 21, with 4 rows for the data values.

Hint 2 Remember when any negative number is squared, the result is positive.

Using a CAS calculator to calculate summary statistics

As you can see, calculating the various summary statistics you have encountered in this section is sometimes rather complicated and generally time consuming. Fortunately, it is no longer necessary to carry out these computations by hand, except in the simplest cases.

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How to find measures of centre and spread using the TI-Nspire CAS

The table shows the monthly rainfall figures for a year in Melbourne.

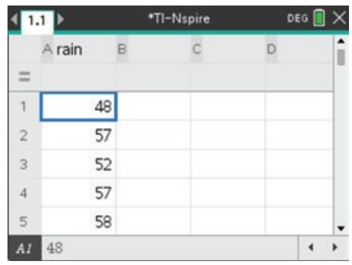
Month	J	F	M	A	M	J	J	A	S	O	N	D
Rainfall (mm)	48	57	52	57	58	49	49	50	59	67	60	59

Determine the mean and standard deviation, median and interquartile range, and range.

Steps

1 Start a new document: Press [on] and select **New** (or press [ctrl] + [N]).

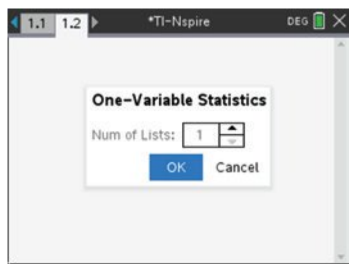
2 Select **Add Lists & Spreadsheet**.
Enter the data into a list named **rain** as shown.
Statistical calculations can be done in the **Lists & Spreadsheet** application or in the **Calculator** application.



3 Press [ctrl] + [doc▾] and select **Add Calculator** (or press [on] and arrow to the calculator icon and press [enter]).

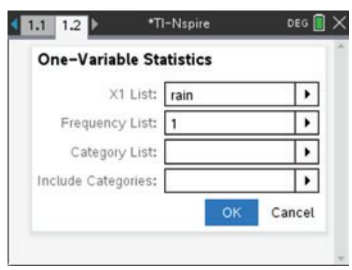
a Press [menu] **>Statistics>Stat Calculations> One-Variable Statistics**

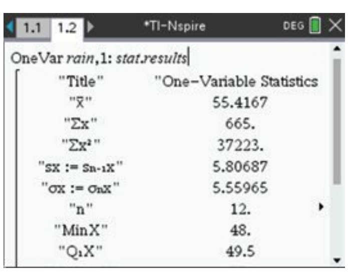
b Press [enter]

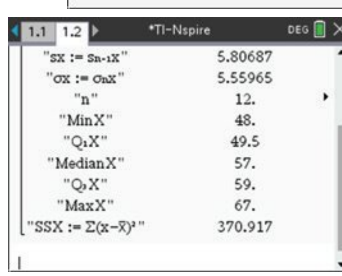


c Use the [tab] key to highlight **OK** and press [enter] to generate the statistical results screen below.

Notes: 1 The sample standard deviation is **sx**.
2 Use the ▲ arrows to scroll through the results screen to see the full range of statistics calculated.







4 Write the answers rounded to one decimal place.

$\bar{x} = 55.4, S = 5.8 \quad M = 57$

$IQR = Q_3 - Q_1 = 59 - 49.5 = 9.5$

$R = \max - \min = 67 - 48 = 19$

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How to calculate measures of centre and spread using the ClassPad

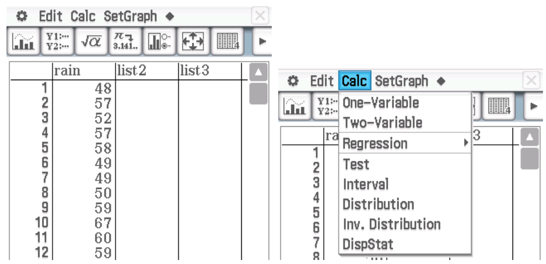
The table shows the monthly rainfall figures for a year in Melbourne.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Rainfall (mm)	48	57	52	57	58	49	49	50	59	67	60	59

Determine the mean and standard deviation, median and interquartile range, and range.

Steps

1 Open the **Statistics** application and enter the data into the column labelled *rain*.

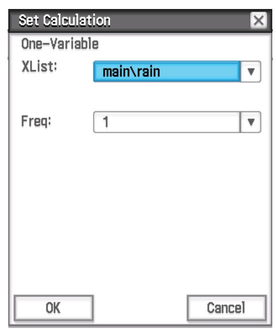


2 To calculate the mean, median, standard deviation and quartiles:

- Select **Calc** from menu bar.
- Tap **One-Variable**.

3 Complete the Set Calculation dialog box. For:

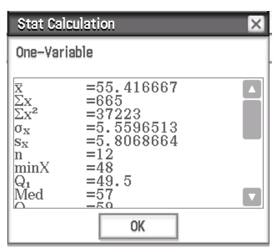
- **XList:** select **main \ rain (▼)**
- **Freq:** leave as **1**.

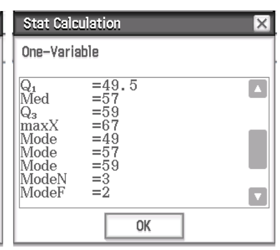


4 Tap **OK** to confirm your selections.

Notes: 1 The sample standard deviation is given by s_x.

2 Use the ▲▼ side-bar arrows to scroll through the results screen for additional statistics if required.





5 Write the answers rounded to one decimal place.

$\bar{x} = 55.4$, S = 5.8, M = 57

IQR = $Q_3 - Q_1 = 59 - 49.5 = 9.5$

R = max – min = 67 – 48 = 19

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Section Summary

- The **range** of a data set can be used to indicate the spread of the data set. The range is the difference between the largest and smallest values in the data set.

  range = largest data value − smallest data value

- The **quartiles** of a data set are values which divide the data set into quarters. The **first quartile** is the median of the lower half of the observations, and the **third quartile** is the median of the upper half of the observations.
- The **interquartile range or IQR** is a measure of spread of a data set

  $\text{IQR} = Q_3 - Q_1$

- The **standard deviation** is another measure of spread of a data set. It measures the variation of the data values around the mean.
- When the distribution is skewed or has outliers, the **interquartile range** is the preferred measure of spread.
- After entering data into a CAS spreadsheet, the mean, standard deviation, minimum, maximum, Q_1 and Q_3 can easily be determined.

Exercise 2F

Building understanding

Example 19
Example 20

1 For the following data set:

1 2 1 0 2 3 1 2 6 7

a Order the data from smallest to largest value.

b Find the minimum and maximum values, and hence find the value of the range.

c Find the value of Q_1, the median of the lower half of the data values.

d Find the value of Q_3, the median of the upper half of the data values.

e Hence, find the value of the interquartile range (IQR).

Example 21

2 For the following data set:

1 2 3 6

a Find the value of the mean.

b Use a table to calculate the value of the standard deviation, using the formula:

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$$

Developing understanding

3 Find, without using a calculator, the IQR and range of each of these ordered data sets.

a 2 2 5 7 9 11 12 16 23

b 1 3 3 5 6 7 9 11 12 12

c 21 23 24 25 27 27 29 31 32 33

d 101 101 105 106 107 107 108 109

e 0.2 0.9 1.0 1.1 1.2 1.2 1.3 1.9 2.1 2.2 2.9

4 Without a calculator, determine the IQR for the data displayed in the following stem plots.

a *Monthly rainfall (mm)*

key: 4|8 represents 48 mm

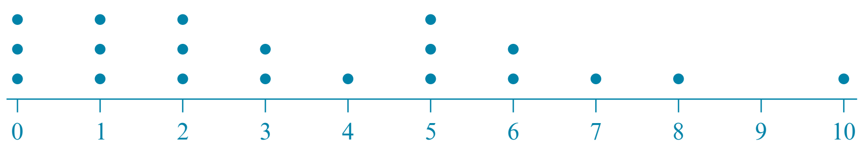
4	8 9 9
5	0 2 7 7 8 9 9
6	0 7

b *Battery time (hours)*

key: 1|7 represents 17 hours

0	4
1	7 9
2	0 1 2 4 5 6 6 7 7 8
3	0 0 1 1 3 3 4
4	0 1 6

5 Without using a calculator, find the median and quartiles for the data displayed in the following dot plot:



6 A manufacturer advertised that a can of soft drink contains 375 mL of liquid. A sample of 16 cans yielded the following contents (in mL):

357	375	366	360	371	363	351	369
358	382	367	372	360	375	356	371

Find the mean and standard deviation, median and IQR, and range for the volume of drink in the cans. Give answers rounded to one decimal place.

7 The serum cholesterol levels for a sample of 18 people are:

231	159	203	304	248	238	209	193	225
190	192	209	161	206	224	276	196	189

Find the mean and standard deviation, median and IQR, and range of the serum cholesterol levels. Give answers rounded to one decimal place.

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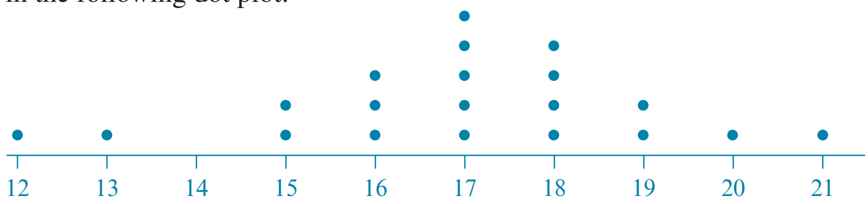
8 Find the range, interquartile range and standard deviation of the data set displayed in the following dot plot:



9 Twenty babies were born at a local hospital on one weekend. Their birth weights are given in the stem plot.

Birth weight (kg) 3 | 6 represents 3.6 kg

2	1 5 7 9 9
3	1 3 3 4 4 5 6 7 7 9
4	1 2 2 3 5

Find the mean and standard deviation, median and IQR, and range of the birth weights.

Testing understanding

10 The results of a student's chemistry experiment were as follows:

7.3 8.3 5.9 7.4 6.2 7.4 5.8 6.1 6.0

a **i** Find the mean and the median of the results.

ii Find the IQR and the standard deviation of the results.

b Unfortunately, when the student was transcribing his results into his chemistry book, he made a small error and wrote:

7.3 8.3 5.9 7.4 6.2 7.4 5.8 6.1 60

i Find the mean and the median of these results.

ii Find the interquartile range and the standard deviation of these results.

c Describe the effect the error had on the summary statistics in parts **a** and **b**.

2G Percentages of data lying within multiple standard deviations of the mean

Learning intentions

- ► To be able to consider the percentage of data lying within several standard deviations of the mean for a range of distributions.
- ► To be able to apply our knowledge of the mean and standard deviation to understand a data distribution.
- ► To be able to introduce the 68 - 95 - 99.7% rule for symmetric and bell shaped data distributions.

In the previous section we defined the interquartile range, IQR, and could readily interpret this statistic as the spread of the middle 50% of the data values in a distribution.

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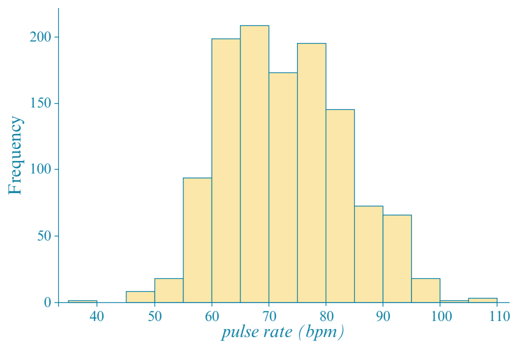
We also defined the values of the mean and the standard deviation for a distribution. Can we use these two statistics in combination to tell us a bit more about the distribution of the random variables we are exploring?

It turns out that when the data distribution is **symmetric** and approximately **bell shaped**, we can estimate the percentage of the data lying within several standard deviations of the mean.

We will explore this idea using data collected from 1000 people for the variable *pulse rate*, measured in beats per minute, displayed in the following histogram.



From the histogram we can see that this distribution is approximately symmetric and bell shaped.

From the data, the mean *pulse rate* is $\bar{x} = 72.31$ and the standard deviation is $s = 10.29$. We can use this information to construct intervals which are one, two and three standard deviations either side from the mean, as follows:

One SD: $(\bar{x} - s, \bar{x} + s) = (72.31 - 10.29, 72.31 + 10.29) = (62.02, 82.60)$
Two SD: $(\bar{x} - 2s, \bar{x} + 2s) = (72.31 - 2 \times 10.29, 72.31 + 2 \times 10.29) = (51.73, 92.89)$
Three SD: $(\bar{x} - 3s, \bar{x} + 3s) = (72.31 - 3 \times 10.29, 72.31 + 3 \times 10.29) = (41.11, 103.18)$

We can now go back to the original 1000 data values and determine the percentage of the data which lies within each of the three intervals. When we do this we find that for this particular set of 1000 values of *pulse rate*:

- 68% of the data values lie within 1 standard deviation of the mean
- 95% of the data values lie within 2 standard deviations of the mean
- 99.7% of the data values lie within 3 standard deviations of the mean.

If we repeat this analysis for another set of values for *pulse rate*, we find that we get very similar results. In fact, we are able to show theoretically that there is a general rule which can be applied here.

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The 68 - 95 - 99.7% rule

For data distribution which is approximately symmetric and bell shaped, approximately:

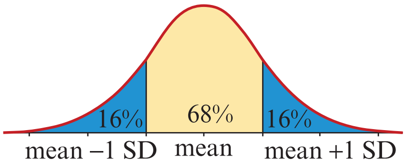
- 68% of the data will lie within one standard deviation of the mean.
- 95% of the data will lie within two standard deviations of the mean.
- 99.7% of the data will lie within three standard deviations of the mean.

Of course, to be able to show that these rules apply empirically we would require a large data set (preferably at least 200 values).

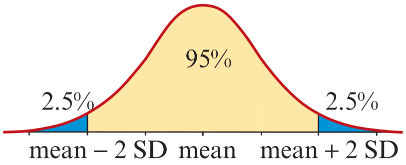
The 68–95–99.7% rule in graphical form

It is helpful to also illustrate this property of the standard deviation graphically. In the following diagrams a smooth curve has been used to imply that the underlying data distribution is generally symmetric and bell shaped.

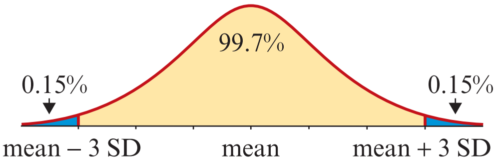
- Around *68%* of the data values will lie within one standard deviation (SD) of the mean.



- Around *95%* of the data values will lie within two standard deviations of the mean.



- Around *99.7%* of the data values will lie within three standard deviations of the mean.



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Example 22 Percentages of data lying within 1, 2 or 3 standard deviations of the mean

The distribution of the examination scores for a very large statewide examination is approximately symmetric and bell shaped, with a mean of 65 and with a standard deviation of 10.

a Approximately what percentage of students scored between 55 and 75?

b Approximately what percentage of students scored between 45 and 85?

c Approximately what percentage of students scored between 35 and 95?

Explanation	Solution
a A score of 55 is 1SD below the mean of 65 and a score of 75 is 1SD above the mean.	Approximately 68% of the scores are between 55 and 75.
b A score of 45 is 2SD below the mean of 65 and a score of 85 is 2SD above the mean.	Approximately 95% of the scores are between 45 and 85.
c A score of 35 is 3SD below the mean of 65 and a score of 95 is 3SD above the mean.	Approximately 99.7% of the scores are between 35 and 95.

Example 23 Finding the interval for a given percentage

The distribution of the diameter of bolts produced in a factory is approximately symmetric and bell shaped, with a mean of 5 mm and with a standard deviation of 0.01mm.

a If approximately 68% of the bolts measure between a and b, what are possible values for a and b?

b If approximately 95% of the bolts measure between c and d, what are possible values for c and d?

c If approximately 99.7% of the bolts measure between e and f, what are possible values for e and f?

Explanation	Solution
a The interval which contains 68% of the bolts is 1SD either side of the mean.	$a = 5 - 0.01 = 4.99$ mm $b = 5 + 0.01 = 5.01$ mm
b The interval which contains 95% of the bolts is 2SD either side of the mean.	$c = 5 - 2 \times 0.01 = 4.98$ mm $d = 5 + 2 \times 0.01 = 5.02$ mm
c The interval which contains 99.7% of the bolts is 3SD either side of the mean.	$e = 5 - 3 \times 0.01 = 4.97$ mm $f = 5 + 3 \times 0.01 = 5.03$ mm

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Section Summary

- Knowing the mean and standard deviation of a distribution allows us to make predictions about the percentage of the data that lies within specific intervals.
- If the data distribution is symmetric and bell shaped then approximately:
  - 68% of the data will lie within one standard deviation of the mean.
  - 95% of the data will lie within two standard deviations of the mean.
  - 99.7% of the data will lie within three standard deviations of the mean.

Skill-sheet

Exercise 2G

Building understanding

1 Suppose that the mean of a large data set is 15.8, and the standard deviation is 2.3. Find the interval which is:

a One standard deviation either side of the mean.

b Two standard deviations either side of the mean.

c Three standard deviations either side of the mean.

2 Suppose that the mean of a large data set is 435.6, and the standard deviation is 53.3. Find the interval which is:

a One standard deviation either side of the mean.

b Two standard deviations either side of the mean.

c Three standard deviations either side of the mean.

Developing understanding

Example 22

3 Suppose the distribution of height for females in a certain country is approximately symmetric and bell shaped, with a mean of 163 cm and a standard deviation of 8 cm.

a Approximately what percentage of females are between 147 cm and 179 cm tall?

b Approximately what percentage of females are between 139 cm and 187 cm tall?

4 The distribution of IQ scores in a certain country is approximately symmetric and bell shaped, with a mean of 100 and a standard deviation of 15.

a Approximately what percentage of people have an IQ score between 55 and 145?

b Approximately what percentage of people have an IQ score between 70 and 130?

5 The distribution of weights of eggs from a farm is approximately symmetric and bell shaped, with a mean of 60 gm and a standard deviation of 3 gm.

a Approximately what percentage of eggs have a weight between 57 gm and 63 gm?

b Approximately what percentage of eggs have a weight between 51 gm and 69 gm?

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6 The distribution of times taken for people to solve a puzzle is approximately symmetric and bell shaped, with a mean of 24 seconds and a standard deviation of 4 seconds.

a Approximately what percentage of people take between 20 seconds and 28 seconds to solve the puzzle?

b Approximately what percentage of people take between 16 seconds and 32 seconds to solve the puzzle?

Example 23

7 The distribution of the volume of soft drink in a 1 litre bottle is approximately symmetric and bell shaped, with a mean of 1.0 litre and a standard deviation of 2 mL. If approximately 95% of the bottle contains between a litres and b litres, what are possible values for a and b?

8 The distribution of salaries in a certain country is approximately symmetric and bell shaped, with a mean of \$91 000 per year and a standard deviation of \$26 500.

a If approximately 95% of people have salaries between a and b, what are possible values for a and b?

b If approximately 99.7% of people have salaries between c and d, what are possible values for c and d?

Testing understanding

9 Suppose the number of hours of exercise per week undertaken by people in a certain country is approximately symmetric and bell shaped, with a mean of 6.2 hours and a standard deviation of 1.6 hours.

a If approximately 95% of people exercise between a and b hours per week, what are possible values for a and b?

b If 99.7% of people exercise between c and d hours per week, what are possible values for c and d?

c If approximately 50% of people exercise for more than e hours per week, what is the value of e?

d Approximately what percentage of people exercise for between 7.8 and 9.4 hours per week?

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2H Boxplots

Learning intentions

- To be able to introduce the boxplot as a plot for displaying the distribution of a numerical data.
- To be able to introduce an exact definition of an **outlier**.
- To be able to define a **simple boxplot** and a **boxplot with outliers**.
- To be able to determine the characteristics of centre and spread from a boxplot.

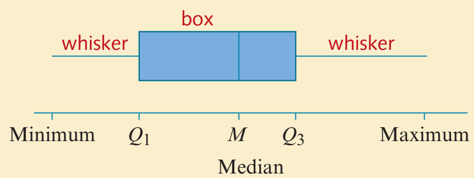
We already have three plots which we can use to display the distribution of a numerical variable, namely the histogram, the dot plot, and the stem and leaf plot. In this section we introduce another plot for displaying numerical data, the **boxplot**. The boxplot is extremely useful as it allows us to summarise quite large data sets into concise plots.

The simple boxplot

Knowing the median and quartiles of a distribution means that quite a lot is known about the central region of the data set. If something is known about the tails of the distribution as well, then a good picture of the whole data set can be obtained. This can be achieved by knowing the **maximum** and **minimum** values of the data.

When we list the median, the quartiles and the maximum and minimum values of a data set, we have what is known as a **five-number summary**. Its pictorial (graphical) representation is called a **boxplot** or a box-and-whisker plot.

Boxplots



- A boxplot is a graphical representation of a five-number summary.
- A box is used to represent the middle 50% of scores.
- The median is shown by a vertical line drawn within the box.
- Lines (whiskers) extend out from the lower and upper ends of the box to the smallest and largest data values of the data set, respectively.

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Example 24 Constructing a boxplot from a five-number summary

The following are the monthly rainfall figures for a year in Melbourne.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Rainfall (mm)	48	57	52	57	58	49	49	50	59	67	60	59

Construct a boxplot to display this data, given the five-number summary:

Min = 48 $Q_1 = 49.5$ $M = 57$ $Q_3 = 59$ Max = 67

Explanation

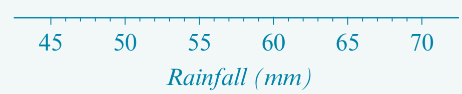
1 Draw in a labelled and scaled number line that covers the full range of values.

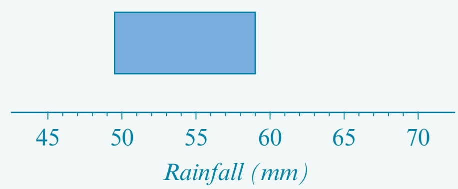
2 Draw in a box starting at $Q_1 = 49.5$ and ending at $Q_3 = 59$.

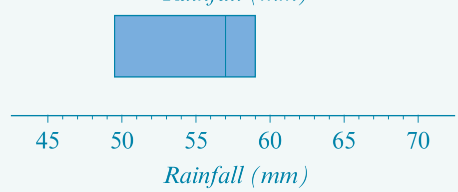
3 Mark in the median value with a vertical line segment at $M = 57$.

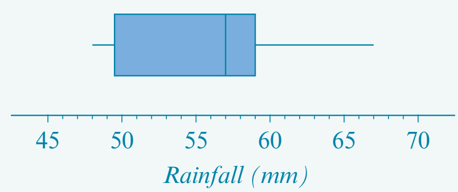
4 Draw in the whiskers, which are lines joining the midpoint of the ends of the box to the minimum and maximum values: 48 and 67, respectively.

Solution









Now try this 24 Constructing a boxplot from a five-number summary (Example24)

Construct a boxplot to display the following five-number summary:

Min = 7 $Q_1 = 14$ $M = 19$ $Q_3 = 24$ Max = 34

Hint 1 Make sure the scale on the number line spans the maximum and minimum values.

Hint 2 Locate each value of the five-number summary with a dot before you construct the boxplot.

Boxplots with outliers

An extension of the boxplot can also be used to identify possible outliers in a data set.

Sometimes it is difficult to decide whether or not an observation is an outlier. For example, a boxplot might have one extremely long whisker. How might we explain this?

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- One explanation is that the data distribution is extremely skewed, with lots of data values in its tail.
- Another explanation is that the long whisker hides one or more outliers.

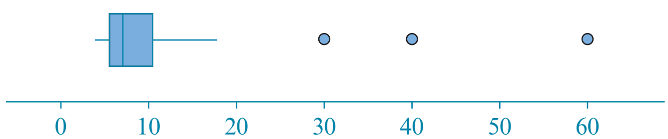
By modifying the boxplots, we can decide which explanation is most likely, but first we need a more exact definition of an outlier.

Defining outliers

Outlier

An outlier in a distribution is any data point that lies more than 1.5 interquartile ranges below the first quartile or more than 1.5 interquartile ranges above the third quartile.

To be more informative, the boxplot can be modified so that the outliers are plotted individually in the boxplot with a dot or cross, and the whisker now ends only at the largest or smallest data value that is not outside these limits. An example of a boxplot displaying outliers is shown below.



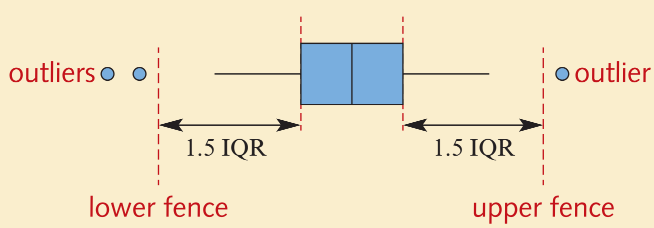
Upper and lower fences

When constructing a boxplot to display outliers, we must first determine the location of what we call the *upper and lower fences*. These are imaginary lines drawn one and a half times the interquartile range (or box widths) above and below the ends of the box. Data values outside these fences are classified as possible outliers and plotted separately. Note that if a data point lies exactly on an upper or lower fence, then it is not considered an outlier.

Using a boxplot to display possible outliers

In a boxplot, possible outliers are defined as those values that are:

- greater than $Q_3 + 1.5 \times \text{IQR}$ (upper fence)
- less than $Q_1 - 1.5 \times \text{IQR}$ (lower fence).



When drawing a boxplot, any observation identified as an outlier is indicated by a dot. The whiskers then end at the smallest and largest values that are not classified as outliers.

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Example 25 Constructing a boxplot showing outliers

The number of hours that each of 33 students spent on a school project is shown below.

2	3	4	9	9	13	19	24	27	35	36
37	40	48	56	59	71	76	86	90	92	97
102	102	108	111	146	147	147	166	181	226	264

Construct a boxplot for this data set that can be used to identify possible outliers.

Explanation

1 From the ordered list, state the minimum and maximum values. Find the median, the $\frac{1}{2}(33 + 1)$th = 17th value.

2 Determine Q_1 and Q_3. There are 33 values, so Q_1 is halfway between the 8th and 9th values, and Q_3 is halfway between the 25th and the 26th values.

3 Determine the IQR.

4 Determine the upper and lower fences.

5 Locate any values outside the fences, and the values that lie just inside the limits (the whiskers will extend to these values).

Solution

Minimum = 2 hours
Maximum = 264 hours
Median = 71 hours

First quartile, $Q_1 = \frac{24 + 27}{2} = 25.5$

Third quartile, $Q_3 = \frac{108 + 111}{2} = 109.5$

$IQR = Q_3 - Q_1 = 109.5 - 25.5 = 84$

$$\text{Lower fence} = Q_1 - 1.5 \times IQR = 25.5 - 1.5 \times 84 = -100.5$$

$$\text{Upper fence} = Q_3 + 1.5 \times IQR = 109.5 + 1.5 \times 84 = 235.5$$

There is one outlier: 264 hours.
The largest value that is not an outlier is 226 hours.

6 The boxplot can now be constructed as shown below. The upper whisker extends to the second highest value, and the circle denotes the outlier. The fences are not shown on the boxplot.

There is one possible outlier, the student who spent 264 hours on the project.

Now try this 25 **Constructing a boxplot showing outliers (Example 25)**

The number of hours that each of a sample of 22 people spent in paid employment last week is shown here.

32	16	40	20	35	40	40	43	40	40	35
45	40	72	75	30	60	60	40	55	48	40

Construct a boxplot with outliers for this data set.

Hint 1 Start by putting the data in order, and then determining the five-number summary.

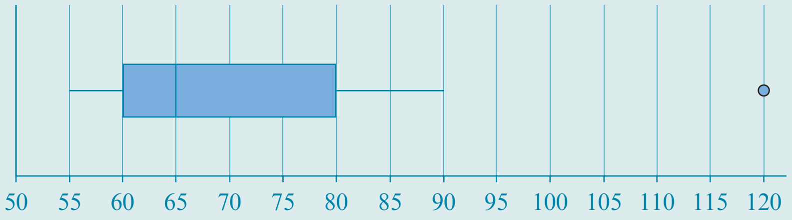
Hint 2 Determine the IQR, and hence the values for the lower and upper fences.

Hint 3 Remember the whiskers join the ends of the box to the smallest and largest values which are not outliers. They do not extend to the upper and lower fences.

We can use a boxplot to tell us a lot about the distribution of a data set, as is shown in the following example.

Example 26 **Estimating percentages from a boxplot**

For the boxplot shown, estimate the percentage of values which are:



a less than 60 **b** less than 65 **c** more than 80
d between 60 and 80 **e** between 60 and 120

Explanation	**Solution**
a 60 is the first quartile (Q_1).	25% of the data values are less than 60.
b 65 is the median (M, or sometimes Q_2).	50% of the data values are less than 65.
c 80 is the third quartile (Q_3).	75% of the data values are less than 80, so 25% of the data values are greater than 80.
d 75% of the data values are less than 80, and 25% of the data values are less than 60.	Thus 50% of the values are between 60 and 80.
e 100% of the data values are less than 120, and 25% of the data values are less than 60.	Thus 75% of the values are between 60 and 120.

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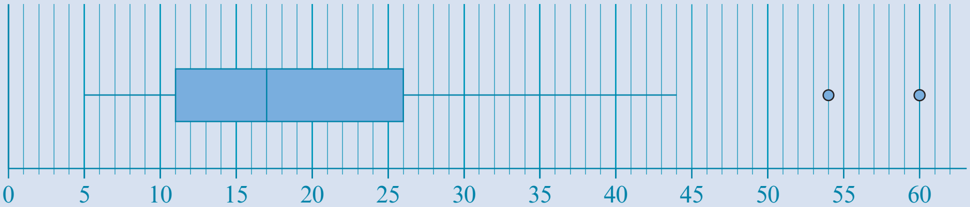
Now try this 26 **Estimating percentages from a boxplot (Example 26)**

For the boxplot shown, estimate the percentage of values which are:



a less than 26 **b** less than 5 **c** less than 11
d between 11 and 26 **e** between 5 and 26

Hint 1 Start by identifying the values of the five-number summary from the boxplot.

It is clearly very time consuming to construct boxplots displaying outliers by hand. Fortunately, your CAS calculator will do it for you automatically as we will see.

How to construct a boxplot using the TI-Nspire CAS

The number of hours that each of 33 students spent on a school project is shown below.

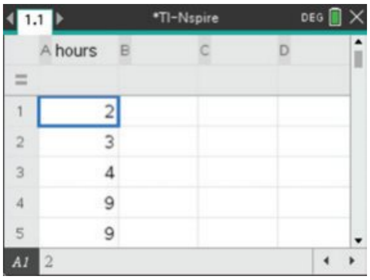
2	3	4	9	9	13	19	24	27	35	36
37	40	48	56	59	71	76	86	90	92	97
102	102	108	111	146	147	147	166	181	226	264

Construct a boxplot for this data set that can be used to identify possible outliers.

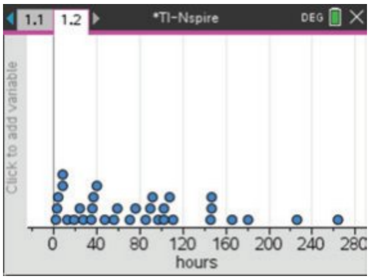
Steps

1 Press [on] and select **New** (or use [ctrl] + [N]).

2 Select **Add Lists & Spreadsheet**.
Enter the data into a list called *hours* as shown.



3 Statistical graphing is done through the **Data & Statistics** application. Press [ctrl] + [doc▾] and select **Add Data & Statistics** (or press [on], arrow to , and press [enter]).
Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.



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a Press tab to show the list of variables. Select the variable *hours*. Press enter to paste the variable *hours* to that axis. A dot plot is displayed as the default plot.

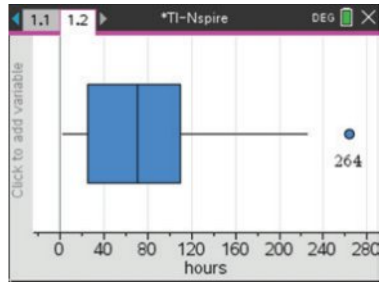
b To change the plot to a boxplot press menu**>Plot Type>Box Plot**, then enter or 'click' (press). Outliers are indicated by a dot(s).

4 Data Analysis

Move the cursor over the plot to display the key values (or use menu**>Analyze>Graph Trace**).

Starting at the far left of the plot, we see that the:

- minimum value is 2: **minX** = **2**
- first quartile is 25.5: $\mathbf{Q_1}$ = **25.5**
- median is 71: **Median** = **71**
- third quartile is 109.5: $\mathbf{Q_3}$ = **109.5**
- maximum value is 264: **maxX** = **264**. It is also an outlier.



How to construct a boxplot using the ClassPad

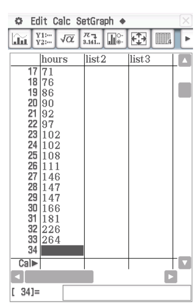
The number of hours that each of 33 students spent on a school project is shown below.

2	3	4	9	9	13	19	24	27	35	36
37	40	48	56	59	71	76	86	90	92	97
102	102	108	111	146	147	147	166	181	226	264

Construct a boxplot for this data set that can be used to identify possible outliers.

Steps

1 Open the **Statistics** application and enter the data into a column labelled *hours*.



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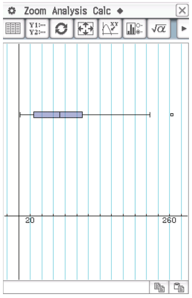
2 Open the **Set StatGraphs** dialog box by tapping in the toolbar. Complete the dialog box as shown, right. For:

- **Draw:** select **On**
- **Type:** select **MedBox** (▼)
- **XList:** select **main\hours** (▼)
- **Freq:** leave as **1**.

Tap the **Show Outliers** box.
Tap Set to exit.

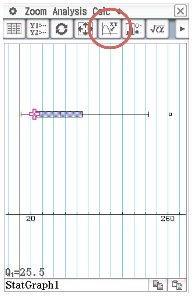
3 Tap to plot the boxplot.

4 Tap Resize to obtain a full-screen display.



Note: In the screen shot shown, the window parameters were adjusted to display vertical grid lines, no scale along the *y*-axis and less space below the *x*-axis. This was achieved by tapping on and selecting 20 for the *x* scale, 0 for the *y* scale and reducing the *y*min value.

5 Key values can be read from the boxplot by tapping .
Use the arrows (◀ and ▶) to move from point to point on the boxplot.
Starting at the far left of the plot, we see that the:



- minimum value is 2 (**minX** = **2**)
- first quartile is 25.5 (**Q**$_1$ = **25.5**)
- median is 71 (**Median** = **71**)
- third quartile is 109.5 (**Q**$_3$ = **109.5**)
- maximum value is 264 (**maxX** = **264**). It is also an outlier.

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Section Summary

- A boxplot is a useful plot for displaying the distribution of a numerical data set.
- A **simple boxplot** is a pictorial representation of the five-number summary.
- The five-number summary consists of the minimum, the first quartile Q_1, the median, the third quartile Q_3 and the maximum.
- From the boxplot we can see:
  - 25% of the data is between the minimum and Q_1.
  - 25% of the data is between Q_1 and the median.
  - 25% of the data is between the median and Q_3.
  - 25% of the data is between Q_3 and the maximum.
- A **boxplot with outliers** shows the five-number summary as well as any **outliers**.
- The **lower fence** is $Q_1 - 1.5 \times \text{IQR}$.
- The **upper fence** is $Q_3 + 1.5 \times \text{IQR}$.
- Any data value less than the lower fence, or greater than the upper fence, is shown on the boxplot as an outlier.

Skill-sheet

Exercise 2H

Building understanding

Example 24

1 The five-number summary for a data set is:

Min = 5 $Q_1 = 10$ $M = 20$ $Q_3 = 25$ Max = 45

a Construct a number line starting at 0, ending at 50, and marked off in units of 10.

b Mark in each of the values of the five-number summary on the number line.

c Hence construct a simple boxplot.

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Developing understanding

2 The data shows how many hours each of a group of forty-one players spent at training in a particular week.

24	11	5	7	4	15	13	4	12	14	3	12	4	4
3	10	17	8	6	2	18	15	5	6	9	14	4	5
14	12	16	11	6	7	12	4	16	2	8	10	1	

a Find the five-number summary for this data.

b Use this five-number summary to construct a simple boxplot by hand.

3 The five-number summary for a data set is:

Min = 0 $Q_1 = 13$ $M = 16$ $Q_3 = 19$ Max = 35

Determine the values of the lower and upper fences.

Example 25

4 The five-number summary for a data set is:

Min = 14 $Q_1 = 45$ $M = 55$ $Q_3 = 65$ Max = 99

a Determine the values of the upper and lower fences.

b The smallest three values in the data set are 14, 18, 34 and the largest are 90, 94, 99. Which of these are outliers?

5 The amount of pocket money paid per week to a sample of Year 8 students is:

\$5.00	\$10.00	\$12.00	\$8.00	\$7.50	\$12.00	\$15.00
\$10.00	\$10.00	\$0.00	\$5.00	\$10.00	\$20.00	\$15.00
\$26.00	\$13.50	\$15.00	\$5.00	\$15.00	\$25.00	\$16.00

The five-number summary is:

Min = 0 $Q_1 = 7.75$ $M = 12,$ $Q_3 = 15$ Max = 26

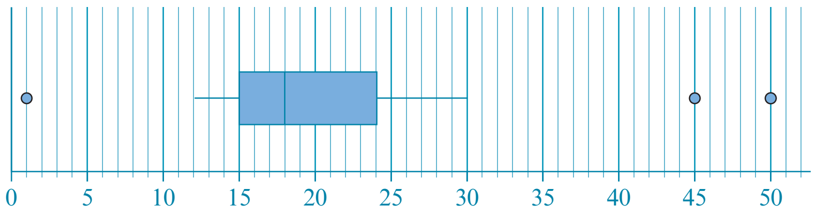
a Use the information from the five-number summary to determine the values of the lower and upper fences.

b Without using a calculator, determine the value(s) of any outliers.

c Without using a calculator, construct a boxplot showing any outliers.

Example 26

6 Use the boxplot below to estimate the percentage of values that are:



a less than 18 **b** less than 12 **c** between 12 and 24 **d** more than 24

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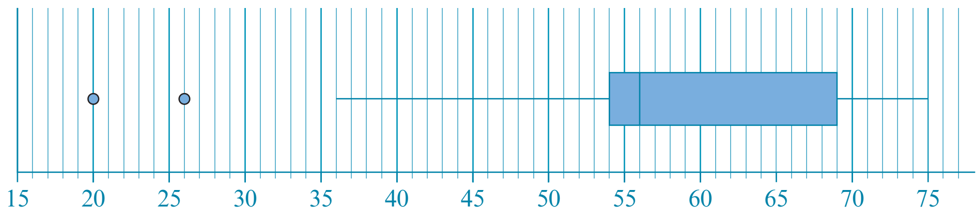
7 The boxplot below displays the scores on an exam for a group of students. Use it to estimate the percentage of students who had scores that are:



a less than 69 **b** less than 54 **c** between 54 and 69
d more than 75 **e** between 20 and 69 **f** between 56 and 69

8 The length of time, in years, that employees have been employed by a company is:

5	1	20	8	6	9	13	15	4	2
15	14	13	4	16	18	26	6	8	2
6	7	20	2	1	1	5	8		

Use a CAS calculator to construct a boxplot.

9 The times, in seconds, that 35 children took to tie a shoelace are:

8	6	18	39	7	10	5	8	6	14	11	10
8	35	6	6	14	15	6	7	6	5	8	11
8	15	8	8	7	8	8	6	29	5	7	

a Use a CAS calculator to construct a boxplot.
b If 25% of the children took less than k seconds to tie their shoelaces, what is the value of k?

10 A researcher is interested in the number of books people borrow from a library. She selected a sample of 38 people and recorded the number of books each person had borrowed in the previous year. Here are her results:

7	28	0	2	38	18	0	0	4	0	0	5	13
2	13	1	1	14	1	8	27	0	52	4	11	0
0	12	28	15	10	1	0	2	0	1	11	0	

a Use a CAS calculator to construct a boxplot of the data.
b Use the boxplot to identify any possible outliers, and write down their values.
c How many books were borrowed by the top 25% of library users?

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11 The following table gives the prices for houses sold in a particular suburb in one month (in thousands of dollars):

356	366	375	389	432
445	450	450	495	510
549	552	579	585	590
595	625	725	760	880
940	950	1017	1180	1625

a Use a CAS calculator to construct a boxplot of the data.
b Use the boxplot to identify any possible outliers, and write down their values.
c What is the price range of the top 50% most expensive houses?

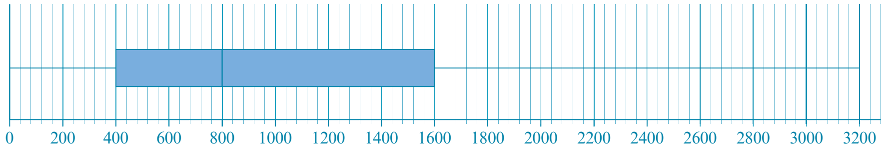
12 The time taken, in seconds, for a group of children to complete a puzzle is:

8	6	18	39	7	10	5	8	6	14	11	5
10	8	60	6	6	14	15	6	7	6	5	7
8	11	8	15	8	8	7	8	8	6	29	

a Use a CAS calculator to construct a boxplot of the data.
b Use the boxplot to identify any possible outliers, and write down their values.
c What is the slowest time for the fastest 25% of children?

Testing understanding

13 The following boxplot summarises the weekly income (in dollars) for a large sample of people. Use the boxplot to answer the following questions.



a What is the approximate value of the median income?
b What is the approximate value of the interquartile range of incomes?
c What is the minimum salary for a person who is in the top 25% of earners?
d The maximum weekly income for this sample of people is $3200. Confirm that it is not an outlier.

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14 The table shows the percentage of people using the internet in 23 countries in 2020.

Country	Internet users (%)	Country	Internet users (%)
Afghanistan	18.8	Malaysia	67.5
Argentina	78.6	Morocco	61.6
Australia	85.1	New Zealand	86.6
Brazil	70.2	Saudi Arabia	88.6
Bulgaria	53.1	Singapore	82.0
China	59.3	Slovenia	72.7
Colombia	63.2	South Africa	53.1
Greece	59.9	United Kingdom	94.7
Hong Kong (China)	80.5	United States	88.5
Iceland	96.5	Venezuela	61.5
India	40.6	Vietnam	66.3
Italy	92.9		

a Use a CAS calculator to construct a boxplot of the data.

b Use the boxplot to identify any possible outliers, and write down their values.

c What is the minimum internet usage for the top 75% of countries?

2I Comparing the distribution of a numerical variable across groups

Learning intentions

- To be able to use back-to-back stem plots to compare the distributions of two numerical variables.
- To be able to use parallel boxplots to compare the distributions of two or more numerical variables.
- To be able to write a report which communicates these comparisons.

It makes sense to compare the distributions of data sets when they are concerned with the same numerical variable, say *height*, measured for different groups of people, for example, a basketball team and a gymnastics team.

For example, it would be useful to compare the distributions for each of the following:

- the maximum daily temperatures in Melbourne in March and the maximum daily temperatures in Sydney in March
- the test scores for a group of students who had not had a revision class and the test scores for a group of students who had a revision class.

In each of these examples, we can actually identify two variables. One is a numerical variable and the other is a categorical variable.

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For example:

- The variable *maximum daily temperature* is numerical while the variable *city*, which takes the values 'Melbourne' or 'Sydney,' is categorical.
- The variable *test score* is numerical while the variable *attended a revision class*, which takes the values 'yes' or 'no,' is categorical.

Thus, when we compare two data sets in this section, we will be actually investigating the relationship between two variables: a numerical variable and a categorical variable.

The outcome of these investigations will be a brief written report that compares the distribution of the numerical variable across two or more groups, defined as categorical variables. The starting point for these investigations will be, as always, a graphical display of the data. To this end you will meet and learn to interpret two new graphical displays: the **back-to-back stem plot** and **parallel boxplots**.

Comparing distributions using back-to-back stem plots

A back-to-back stem plot differs from the stem plots you have met in the past in that it has a single stem with two sets of leaves, one for each of the two groups being compared.

Example 27 Comparing distributions using back-to-back stem plots

The following back-to-back stem plot displays the distributions of life expectancies for males (in years) in several countries in the years 1970 and 2010.

In this situation, *Male life expectancy* is the numerical variable. *Year*, which takes the values 1970 and 2010, is the categorical variable.

Male life expectancy (in years)

1970 8 | 3 = 38 years		2010 5 | 8 = 58 years
8	3	
	4	
9 4 2	5	8
9 9 9 8 7 7 1 0	6	9
4	7	1 2 4 4 6 8 9 9
	8	0 0 0
M = 67 years IQR = 12.5 years		M = 76 years IQR = 8 years

Use the back-to-back stem plot and the summary statistics provided to compare these distributions in terms of centre and spread, and draw an appropriate conclusion.

Explanation	Solution
1 Centre: Write a sentence using the medians to compare centres.	The median life expectancy of males in 2010 (M = 76 years) was higher than in 1970 (M = 67 years).

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2 Spread: Write a sentence using the IQRs to compare spreads.	The spread of life expectancies of males in 2010 (IQR = 8 years) was lower than the spread in 1970 (IQR = 12.5 years).
3 Conclusion: Use the above observations to add a general conclusion.	In conclusion, the median life expectancy for men in these countries has increased over the last 40 years, and the variability in male life expectancy has decreased over this time interval.

Now try this 27 **Comparing distributions using back-to-back stem plots (Example 27)**

The following back-to-back stem plot displays the distributions of the number of hours per week spent relaxing by a group of 17 males and 16 females.

Number of hours per week spent relaxing

1|2 = 21 hours; 1|6 = 16 hours

Males		Females
9 7	0	1 8 9
8 1	1	4 7
5 3 1 1	2	3 4 4 5
9 5 3 1 1 0 0	3	0 1 2 7 8 9
5 4	4	7

Males	Females
M = 30 hours	M = 24.5 hours
IQR = 14.5 hours	IQR = 19 hours

Use the back-to-back stem plot and the summary statistics provided to compare these distributions in terms of centre and spread, and draw an appropriate conclusion.

Hint 1 Make sure that you **compare** the summary statistics using terms such as 'more than' or 'less than', don't just state their values.

Hint 2 Ensure that your final comparison statement is clear and informative.

Comparing distributions using parallel boxplots

Back-to-back stem plots can be used to compare the distribution of a numerical variable across two groups when the data sets are small. Parallel boxplots can also be used to compare distributions. Unlike back-to-back stem plots, parallel boxplots can also be used when there are more than two groups.

By drawing parallel boxplots on the same axis, both the centre and spread for the distributions are readily identified and can be compared visually.

When comparing distributions of a numerical variable across two or more groups using parallel boxplots, the report should address the key features of:

- centre (the median)
- spread (the IQR)
- possible outliers

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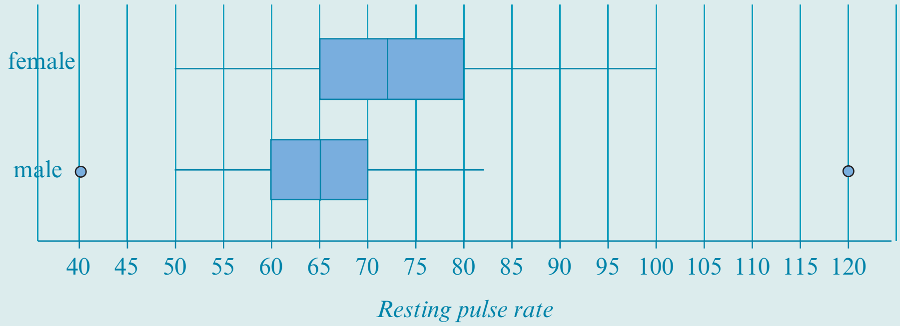
Example 28 Comparing distributions across two groups using parallel boxplots

The following parallel boxplots display the distribution of pulse rates (in beats/minute) for a group of female students and a group of male students.

Use the information in the boxplots to write a report comparing these distributions in terms of centre, spread and outliers in the context of the data.



Explanation	Solution
1 Centre: Determine values of the medians from the plot (the vertical lines in the boxes), and write a sentence comparing these values.	The median pulse rate for females (M = 72 beats/minute) is higher than that for males (M = 65 beats/minute).
2 Spread: Determine the spread of the two distributions using IQRs (the widths of the boxes), and write a sentence comparing these values.	The spread of pulse rates for females (IQR = 15) is higher than for males (IQR = 10).
3 Outliers: Locate any outliers and write a sentence describing these.	There are no female pulse rate outliers. The males with pulse rates of 40 and 120 were outliers.
4 Conclusion: Add a general conclusion based on these comparisons.	In conclusion, the median pulse rate for females was higher than for males, and female pulse rates were generally more variable than male pulse rates.

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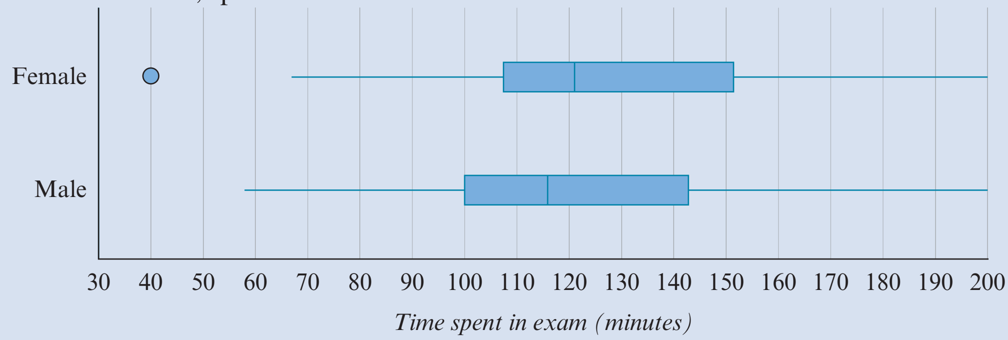
Now try this 28 **Comparing distributions across two groups using parallel boxplots (Example 28)**

The following parallel boxplots display the distribution of time (in minutes) spent in an exam for a group of female students and a group of male students.

Use the information in the boxplots to write a report comparing these distributions in terms of centre, spread and outliers in the context of the data.



Hint 1 Make sure that you **compare** the summary statistics using terms such as ‘more than’ or ‘less than,’ don’t just state their values.

Hint 2 Ensure that your final comparison statement is clear and informative.

Section Summary

- Comparing numerical distributions across groups can be considered as investigating the association between a **numerical** variable and a **categorical** variable.
- Where there are only two groups, and the data sets are small, then **back-to-back stem plots** can be used to display the data.
- Where there are more than two groups, or the data sets are larger, then **parallel boxplots** can be used to display the data.
- The data distributions should be compared in terms of both **centre** and **spread**, quoting the median and IQR for each group.
- The values for any **outliers** should be mentioned.

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Exercise 2I

Building understanding

Example 27

1 The following back-to-back stem plot displays the distribution of class scores for each of two tests (A and B).

Scores

Test A (1|4 = 41 marks)		Test B (5|8 = 58 marks)
6 3	3	
8 5 3 1	4	
9 8 7 7 4 2	5	8
9 7 1 0	6	3 4 5 7 9 9
9 4	7	1 2 4 4 6 8 9 9
	8	0 0 0

a Find the median score for each test.

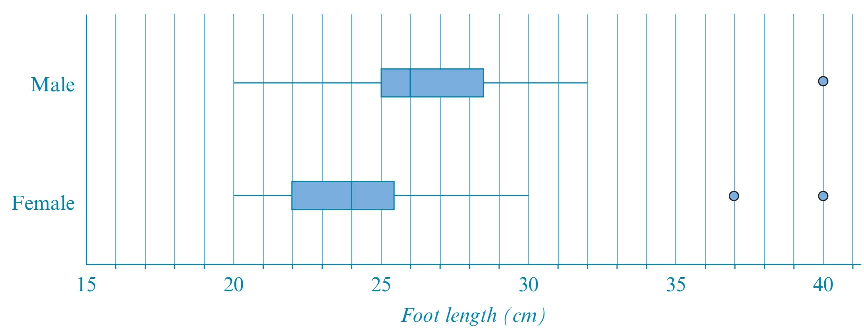
b Complete the sentence by choosing the correct alternative: 'The median score on Test A was (higher/lower) than the median score on Test B'.

c Find the IQR for each test.

d Complete the sentence by choosing the correct alternative: 'The scores on Test A were (more/less) variable than the scores on Test B'.

Example 28

2 The length (in cm) of the right foot for a group of 20 male and 20 female students in Year 9 are summarised in the following parallel boxplots.



a Complete the sentence by entering the values of the medians and choosing the correct alternative: 'The foot length for males (median = ☐ cm) is (longer/shorter) than the foot length for females (median = ☐ cm)'.

b Complete the sentence by entering the values of the IQR and choosing the correct alternative: 'The foot length for males (IQR = ☐ cm) is (more variable/less variable/similar in variability) to the foot length for females (IQR = ☐ cm)'.

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Developing understanding

Comparing groups using back-to-back stem plots

3 The stem plot displays the age distribution of ten females and ten males admitted to a regional hospital on the same day.

Age females		Age males
7 | 2 = 27 years 9	0	4 | 0 = 40 years
5 0	1	3 6
7	2	1 4 5 6 7
7 1	3	4
3 0	4	0 7
0	5	
	6	
9	7	

a Calculate the median and the IQR for the ages of the females and males in this sample.

b Write a report comparing these distributions in terms of centre and spread.

4 The stem plot opposite displays the mark distribution of students from two different mathematics classes (Class A and Class B) who sat the test. The test was marked out of 100.

Marks

Class B (9 | 6 = 69 marks)		Class A (7 | 1 = 71 marks)
3 2	1	9
	2	2
	3	9
	4	5 7 8
	5	5 8
9	6	5 8
6 4 3 3 2 2 1 0 0	7	1 6 7 9 9
8 8 4 4 3 2 1 1 0 0	8	0 1 2 2 5 5 9
8 1	9	1 9

a How many students in each class scored less than 50?

b Determine the median and the IQR for the marks obtained by the students in each class.

c Write a report comparing these distributions in terms of centre and spread in the context of the data.

5 The following table shows the number of nights spent away from home in the past year by a group of 20 Australian tourists and by a group of 20 Japanese tourists:

Australian

3	14	15	3	6	17	2	7	4	8
23	5	7	21	9	11	11	33	4	5

Japanese

14	3	14	7	22	5	15	26	28	12
22	29	23	17	32	5	9	23	6	44

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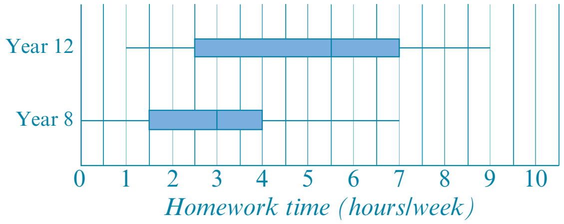
a Construct a back-to-back stem plot of these data sets.

b Determine the median and IQR for the two distributions.

c Write a report comparing the distributions of the number of nights spent away by Australian and Japanese tourists in terms of centre and spread.

6 The boxplots below display the distributions of homework time (in hours per week) of a sample of Year 8 and a sample of Year 12 students.



a Estimate the median and IQRs from the boxplots.

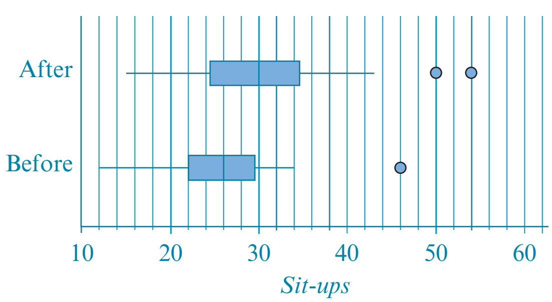
b Use the medians and IQRs to write a report comparing these distributions in terms of centre and spread in the context of the data.

7 The boxplots below display the distribution of smoking rates (in percentages) of males and females from several countries.

a Estimate the median and IQRs from the boxplots.

b Use the information in the boxplots to write a report comparing these distributions in terms of centre and spread in the context of the data.

8 The boxplots below display the distributions of the number of sit-ups a person can do in one minute, both before and after a fitness course.



a Estimate the median, IQRs and the values of any outliers from the boxplots.

b Use these medians and IQRs to write a report comparing these distributions in terms of centre and spread in the context of the data.

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9 To test the effect of alcohol on coordination, twenty randomly selected participants were timed to complete a task with both 0% blood alcohol and 0.05% blood alcohol. The times taken (in seconds) are shown in the accompanying table.

0% blood alcohol									
38	36	35	35	43	46	42	47	40	48
35	34	40	44	30	25	39	31	29	44

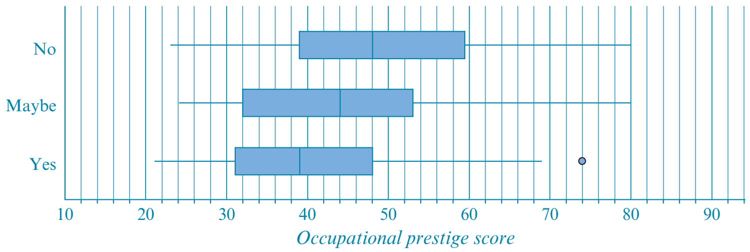
0.05% blood alcohol									
39	32	35	39	36	34	41	64	44	38
43	42	46	46	50	32	32	41	40	50

a Draw boxplots for each of the sets of scores on the same scale.

b Use the information in the boxplots to write a report comparing the distributions of the times taken to complete a task with 0% blood alcohol and 0.05% blood alcohol in terms of centre (medians), spread (IQRs) and outliers.

Testing understanding

10 Occupational prestige is a numerical measure used to describe the respect that a particular occupation holds in a society. It is measured on a scale from 0 - 100. Researchers asked a sample of 332 people if they were thinking about changing to a different kind of work, to which they could respond yes, maybe or no. They also collected the occupational prestige scores for their current occupations. The data they collected is summarised in the following parallel boxplots:



a Use the information in the boxplots to write a report comparing the occupational prestige scores for each group with whether someone is thinking of changing jobs (yes, maybe or no) in terms of centre (medians), spread (IQRs) and outliers.

b Does there seem to be an association between occupational prestige score and whether someone is thinking of changing to a different kind of work?

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Review

Key ideas and chapter summary

Assignment

Types of data	Data can be classified as **categorical** or **numerical**.
Categorical data	**Categorical data** arises when classifying or naming some quality or attribute. Categorical data can be **nominal** and **ordinal**.
Nominal data	**Nominal** data is a type of categorical data where the values of the variable are the names of groups.
Ordinal data	**Ordinal data** is a type of categorical data where there is an inherent order in the categories.
Numerical data	**Numerical data** arises from measuring or counting some quantity. Numerical data can be **discrete** or **continuous**.
Discrete data	**Discrete data** can only take particular numerical values, usually whole numbers, and often arises from counting.
Continuous data	**Continuous data** describes numerical data that can take any value, sometimes in an interval, and often arises from measuring.
Frequency table	A **frequency table** is a listing of the values that a variable takes in a data set, along with how often (frequently) each value occurs. **Frequency** can be recorded as the number of times a value occurs or as a **percentage**; the percentage of times a value occurs.
Bar chart	A **bar chart** uses bars to display the frequency distribution of a categorical variable.
Mode, modal category/modal interval	The **mode** (or modal category) is the value of a variable (or the category) that occurs most frequently. The **modal interval**, for grouped data, is the interval that occurs most frequently.
Histogram	A **histogram** uses columns to display the frequency distribution of a numerical variable: suitable for medium to large-sized data sets.
Stem plot	A **stem plot** is a visual display of a numerical data set, formed from the actual data values: suitable for small to medium-sized data sets.
Dot plot	A **dot plot** consists of a number line with each data point marked by a dot. Suitable for small to medium-sized data sets.
Describing the distribution of a numerical variable	The **distribution of a numerical variable** can be described in terms of **shape** (**symmetric** or **skewed**: positive or negative), **centre** (the middle of the distribution) and **spread**.

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Summary statistics — **Summary statistics** are numerical values for special features of a data distribution such as centre and spread.

Mean — The **mean** ($\bar{x}$) is a summary statistic that can be used to locate the centre of a symmetric distribution. The value of the mean is determined from the formula: $\bar{x} = \frac{\sum x}{n}$

Range — The **range** (***R***) is the difference between the smallest and the largest data values. It is the simplest measure of spread.

Standard deviation — The **standard deviation** (***s***) is a summary statistic that measures the spread of the data values around the mean. The value of the standard deviation is determined from the formula:

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$$

The 68-95-99.7% rule — For a data distribution which is approximately symmetric and bell shaped, approximately:

- 68% of the data will lie within one standard deviation of the mean.
- 95% of the data will lie within two standard deviations of the mean.
- 99.7% of the data will lie within three standard deviations of the mean.

Median — The **median** (***M***) is a summary statistic that can be used to locate the centre of a distribution. It is the midpoint of a distribution, so that 50% of the data values are less than this value and 50% are more. It is sometimes denoted as Q_2.

Quartiles — **Quartiles** are summary statistics that divide an ordered data set into four equal groups.

Interquartile range — The **interquartile range (IQR)** gives the spread of the middle 50% of data values in an ordered data set. It is found by evaluating $IQR = Q_3 - Q_1$.

Five-number summary — The median, the first quartile and the third quartile, along with the minimum and the maximum values in a data set, are known as a **five-number summary**.

Outliers — **Outliers** are data values that appear to stand out from the rest of the data set. They are values that are less than the **lower fence** or more than the **upper fence**.

Lower and upper fences — The **lower fence** is equal to $Q_1 - 1.5 \times IQR$.
The **upper fence** is equal to $Q_3 + 1.5 \times IQR$.

Boxplot — A **boxplot** is a visual display of a five-number summary with adjustments made to display outliers separately when they are present.

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Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

2A **1 I can differentiate between nominal, ordinal, discrete and continuous data.** ☐

e.g. Classify the following data as nominal, ordinal, discrete or continuous:

- **a** The time between people arriving at the coffee shop.
- **b** The number of people in the queue at the coffee shop.
- **c** Customer's coffee preference (latte, cappuccino, black).
- **d** Customer's rating of the coffee (excellent, quite good, not that good).

2A **2 I can construct a frequency table and a percentage frequency table.** ☐

e.g. Twenty students rated their school canteen as bad, ok or good, giving the following data: bad, bad, ok, ok, good, bad, good, ok, ok, ok, bad, ok, bad, good, good, good, good, bad, ok, ok.
Construct a percentage frequency table.

2A **3 I can identify the mode from a frequency table and interpret it.** ☐

e.g. From the frequency table for the canteen rating (above), identify the mode.

2A **4 I can construct a bar chart from a frequency table.** ☐

e.g. Construct a bar chart from the frequency table for the canteen rating (above).

2B **5 I can interpret and describe a frequency table and bar chart.** ☐

e.g. Use the information in the table above to report on the students' ratings of their canteen.

2C **6 I can construct a histogram from raw data using a CAS calculator.** ☐

e.g. The following data gives the number of games played in total by each member of an AFL club:

266	259	238	227	210	160	160	159	155	145	133
99	91	80	80	75	73	58	42	36	32	25
32	25	25	21	17	13	10	9	9	4	2

Use a CAS calculator to construct a histogram starting at 0 with column width 20.

2D **7 I can recognise symmetric, positively skewed and negatively skewed distributions.** ☐

e.g. Describe the shape of the histogram of the number of games played (above).

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2D **8** I can construct a dot plot from raw data. ☐

e.g. Construct a dot plot of the following data:

1 2 3 5 5 7 10 0 6 6 8 1 0 2 1 0 3 2 4 5

2D **9** I can construct a stem plot from raw data. ☐

e.g. Construct a stem plot of the ages of a group of people at a party:

19 21 22 56 34 19 27 28 34 17
66 37 20 20 21 45 26 19 23 29

2E **10** I can determine the median and mean as measures of centre for a data set, and identify when it is more appropriate to use the median. ☐

e.g. Determine the mean and median of the ages of the people at the party (from above). Which is the preferred measure of centre?

2F **11** I can determine the quartiles of a data set and hence calculate the IQR. ☐

e.g. Determine the quartiles and hence the IQR of the ages of the people at the party (from above).

2F **12** I can find the standard deviation of a data set. ☐

e.g. Find the standard deviation of the ages of the people at the party (from above).

2G **13** I can use the 68-95-99.7% rule to estimate the percentages of a data distribution that are within one, two or three standard deviations of the mean. ☐

e.g. Suppose the distribution of weights for a type of cat is approximately symmetric and bell shaped with a mean of 3.5 kg and a standard deviation of 0.5 kg. Approximately what percentage of cats weigh between 2 kg and 5 kg?

2H **14** I can produce a five-number summary from a set of data and use it to construct a boxplot. ☐

e.g. Complete the five-number summary for the ages of the people who attended the party (from above) and use it to construct a boxplot (without outliers).

2H **15** I can determine the values of any outliers in a data set. ☐

e.g. Calculate the values of the upper and lower fences, and use these to find any outliers in the ages of the people who attended the party (from above).

2I **16** I can use a back-to-back stem plot or a boxplot to compare distributions in terms of centre, spread and outliers, and communicate this information in a written report. ☐

e.g. See Example 27 and Example 28.

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Multiple-choice questions

1 In a survey, a number of people were asked to indicate how much they exercised by selecting one of the options: 'never', 'seldom', 'sometimes' or 'regularly'. The type of data generated is:

A variable
B numerical
C nominal
D ordinal
E discrete

2 For which of the following variables is a bar chart an appropriate display?

A Weight (kg)
B Age (years)
C Distance between towns (km)
D Hair colour
E Reaction time

3 For which of the following variables is a histogram an appropriate display?

A Hair colour
B Sex (male, female)
C Distance between towns (km)
D Postcode
E Weight (low, average, high)

4 In an experiment, researchers were interested in the effect of sunlight on the growth of plants. They exposed groups of plants to three levels of sunlight each day (3 - 5 hours, 6 - 8 hours, 9 - 11 hours) and then measured their growth in centimetres after three months. The variables *levels of sunlight* and *growth* are:

A both ordinal variables
B a numerical variable and an ordinal variable respectively
C an ordinal variable and a numerical variable respectively
D an ordinal variable and a nominal variable respectively
E both numerical

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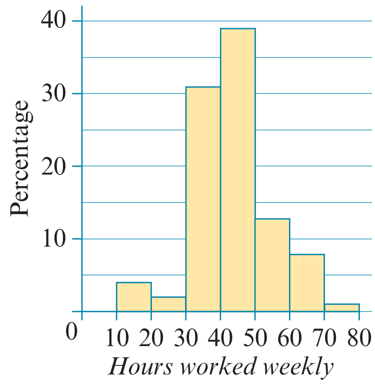
The following information relates to Questions 5 to 8.

The number of hours worked per week by employees in a large company is shown in the following percentage frequency histogram.



5 The percentage of employees who work from 20 to less than 30 hours per week is closest to:

A 1% **B** 2% **C** 6% **D** 10% **E** 33%

6 The percentage of employees who worked *less* than 30 hours per week is closest to:

A 2% **B** 3% **C** 4% **D** 6% **E** 30%

7 The modal interval for hours worked is:

A 10 to less than 20 **B** 20 to less than 30 **C** 30 to less than 40
D 40 to less than 50 **E** 50 to less than 60

8 The median number of hours worked is in the interval:

A 10 to less than 20 **B** 20 to less than 30 **C** 30 to less than 40
D 40 to less than 50 **E** 50 to less than 60

The following information relates to Questions 9 to 11.

A group of 18 employees of a company were asked to record the number of meetings they had attended in the last month.

1 1 2 3 4 5 5 6 7 9 10 12 14 14 16 22 23 44

9 The range of meetings is:

A 22 **B** 23 **C** 24 **D** 43 **E** 44

10 The median number of meetings is:

A 6 **B** 7 **C** 7.5 **D** 8 **E** 9

11 The interquartile range (IQR) of the number of meetings is:

A 0 **B** 4 **C** 9.5 **D** 10 **E** 14

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12 The heights of six basketball players (in cm) are:

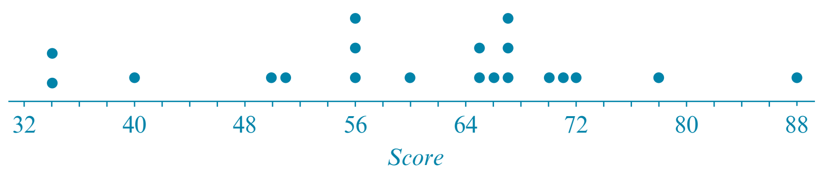
178.1 185.6 173.3 193.4 183.1 193.0

The mean and standard deviation are closest to:

A $\bar{x} = 184.4$; $s = 8.0$ **B** $\bar{x} = 184.4$; $s = 7.3$ **C** $\bar{x} = 182.5$; $s = 7.3$
D $\bar{x} = 182.5$; $s = 8.0$ **E** $\bar{x} = 183.1$; $s = 7.3$

The following information relates to Questions 13 and 14.

The dot plot below gives the scores in a mathematics test for a group of 20 students.



13 The number of students who scored 56 on the examination is:

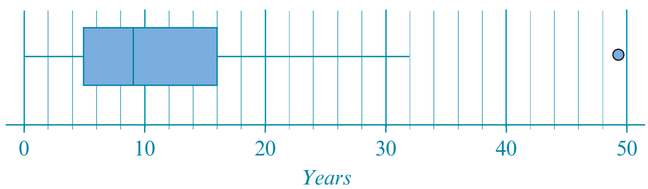
A 1 **B** 2 **C** 3 **D** 4 **E** 5

14 The percentage of students who scored between 40 and 80 on the exam is closest to:

A 50% **B** 70% **C** 80% **D** 90% **E** 100%

The following information relates to Questions 15 to 18.

The number of years for which a sample of people have lived at their current address is summarised in the boxplot.



15 The range is closest to:

A 15 **B** 20 **C** 25 **D** 30 **E** 50

16 The median number of *years lived at this address* is closest to:

A 5 **B** 9 **C** 12 **D** 15 **E** 47

17 The interquartile range of the number of *years lived at this address* is closest to:

A 5 **B** 10 **C** 15 **D** 20 **E** 45

18 The percentage who have lived at this address for more than 16 years is closest to:

A 10% **B** 25% **C** 50% **D** 60% **E** 75%

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Review

The following information relates to Questions 19 and 20.

The distribution of marks on a certain examination is approximately symmetric and bell shaped, with a mean of 65 and a standard deviation of 8.

19 Approximately what percentage of students score between 49 and 81?

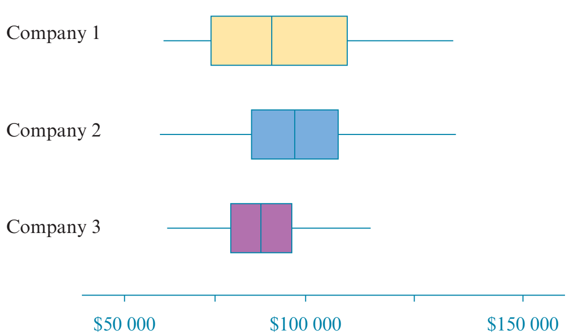
A 50% **B** 68% **C** 95% **D** 99.7% **E** 100%

20 If 99.7% of students score between a and b marks, then possible values of a and b are:

A $a = 53, b = 77$ **B** $a = 57, b = 73$ **C** $a = 49, b = 81$
D $a = 41, b = 89$ **E** $a = 0, b = 100$

The following information relates to Questions 21 to 23.

The amount paid per annum to the employees of each of three large companies is shown in the boxplots.



21 The company with the lowest median wage is:

A company 1 **B** company 2 **C** company 3
D company 1 and company 2 **E** company 2 and company 3

22 The company with the largest general spread (IQR) in wages is:

A company 1 **B** company 2 **C** company 3
D company 1 and company 2 **E** company 2 and company 3

23 Which of the following statements is *not* true?

A All workers in company 3 earned less than $125 000 per year.
B More than half of the workers in company 2 earned less than $100 000 per year.
C 75% of workers in company 2 earned less than the median wage in company 3.
D More than half of the workers in company 1 earned more than the median wage in company 3.
E More than 25% of the workers in company 1 earned more than the median wage in company 2.

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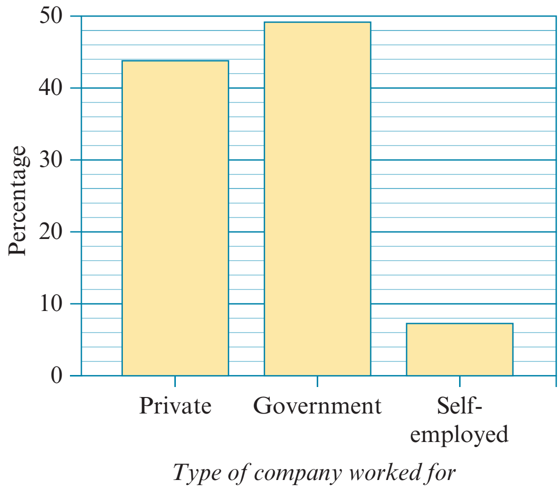
Short-answer questions

1 Classify the data that arises from the following situations as nominal, ordinal, discrete or continuous.

a The number of phone calls a hotel receptionist receives each day.

b Interest in politics on a scale from 1 to 5, where 1 = very interested, 2 = quite interested, 3 = somewhat interested, 4 = not very interested and 5 = uninterested.

2 The bar chart, shown opposite, shows the percentage of working people in a certain town who are: employed in private companies, work for the government or are self-employed.



a Is the data categorical or numerical?

b Approximately what percentage of the people are self-employed?

c Given that 80 people were included in this study, how many identified as being employed in private companies?

3 A researcher asked a group of 30 people to record how many cigarettes they had smoked on a particular day. Here are her results:

0	0	9	10	23	25	0	0	34	32	0	0	30	0	4
5	0	17	14	3	6	0	33	23	0	32	13	21	22	6

a Using class intervals of width 5, construct a histogram of this data.

b Describe the shape of the histogram.

c Within which interval is the median number of cigarettes smoked?

4 A teacher recorded the time taken (in minutes) by each of a class of students to complete a test:

56	57	47	68	52	51	43	22	59	51	39
54	52	69	72	65	45	44	55	56	49	50

a Make a dot plot of the times taken.

b Make a stem plot of the times taken.

c Use this stem plot to find the median and quartiles for the time taken.

d Are there any outliers in this data? Justify your reasoning.

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5 The monthly phone bills, in dollars, for a group of people are given below:

285 185 210 215 320 680 280 265 300 210 270 190 245 315

Find the mean and standard deviation, the median and the IQR, and the range of the monthly phone bills.

6 A group of students was asked to record the number of SMS messages that they sent in one 24-hour period. Use the following five-number summary to construct a boxplot.

Min = 0, $Q_1 = 3$, $M = 5$, $Q_3 = 12$, Max = 24

7 The following data gives the number of students absent from a large secondary college on each of 36 randomly chosen school days:

7	22	12	15	21	16	23	23	17	23	8	16	7	3	21	30	13	2
7	12	18	14	14	0	15	16	13	21	10	16	11	4	3	0	31	44

a Construct a boxplot of this data.

b What was the median number of students absent each day during this period?

c On what percentage of days were more than 20 students absent?

Written-response questions

1 A group of 500 people was asked to describe their general health by choosing one of the following responses: very healthy, pretty healthy, a little healthy, very unhealthy. The data is summarised in the following frequency table:

General health	Frequency	
	Number	%
very healthy	255	51.0
pretty healthy		35.6
a little unhealthy		
very unhealthy	29	
Total	500	100.0

a What type of data has been collected: nominal, ordinal, discrete or continuous?

b Complete the frequency table and the percentage frequency table.

c Construct a percentage bar chart from the table.

d Write a report summarising the responses, quoting appropriate percentages to support your conclusion.

2 The divorce rates (in percentages) of 19 countries are:

27	18	14	25	28	6	32	44	53	0
26	8	14	5	15	32	6	19	9	

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a Is the data categorical or numerical?

b Construct a dot plot of divorce rates.

c What shape is the distribution of divorce rates?

d What percentage of the 19 countries have divorce rates greater than 30%?

e Calculate the mean and median of the distribution of divorce rates.

f Use your calculator to construct a histogram of the data with class intervals of width 10.

  i What is the shape of the histogram?

  ii How many of the 19 countries have divorce rates from 10% to less than 20%?

3 Metro has decided to improve its service on the Lilydale line. Trains were timed on the run from Lilydale to Flinders Street and their times recorded over a period of six weeks at the same time each day. The journey times are shown below (in minutes):

60	61	70	72	68	80	76	65	69	79	82	90	59	86
70	77	64	57	65	60	68	60	63	67	74	78	65	68
82	89	75	62	64	58	64	69	59	62	63	89	74	60

a Use your CAS calculator to construct a histogram of the times taken for the journey from Lilydale to Flinders Street.

  i On how many days did the trip take 65–69 minutes?

  ii What shape is the histogram?

b Use your calculator to determine the following summary statistics for the *time* taken (rounded to two decimal places): $\bar{x}$, s, Min, Q_1, M, Q_3, Max

c Use the summary statistics to complete the following report.

  i The mean time taken from Lilydale to Flinders Street was ☐ minutes.

  ii 50% of the trains took more than ☐ minutes to travel from Lilydale to Flinders Street.

  iii The range of travelling times was ☐ minutes, while the interquartile range was ☐ minutes.

  iv 25% of trains took more than ☐ minutes to travel to Flinders Street.

  v The standard deviation of travelling times was ☐ minutes.

d Summary statistics for the year before Metro took over the Lilydale line from Connex are: Min = 55, Q_1 = 65, M = 70, Q_3 = 89, Max = 99
Construct boxplots for the last year Connex ran the line and for the data from Metro on the same plot.

e Use the information from the boxplots to write a report comparing the distribution of travelling times for the two transport corporations in terms of centre (medians) and spread (IQRs).

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Chapter 3

Sequences and finance

Chapter questions

- What is a sequence?
- How do we generate a sequence of numbers from a starting value and a rule?
- How do we identify particular terms in a sequence?
- What is recursion?
- What is an arithmetic sequence?
- What is a geometric sequence?
- How can I tabulate and graph an arithmetic or geometric sequence?
- How can I find a particular term of an arithmetic or geometric sequence using recursion?
- How can I generate an arithmetic or geometric sequence using recursion?
- How can recurrence relations be used to model simple interest, flat rate depreciation and unit-cost depreciation?
- How can recurrence relations be used to model compound interest and reducing-balance depreciation?
- How can a rule be used to find particular terms for linear growth and decay models?

In this chapter, we will be investigating sequences that can be generated by a rule. Some sequences make each new term by adding a constant amount. Others multiply each term by a fixed number to make the next term. These ideas have many applications to financial mathematics. In particular, the ideas of sequences are applied to calculating interest and loan repayments and the depreciation of assets.

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3A Number patterns

Learning intentions

- To be able to determine a simple rule for a sequence of numbers.
- To be able to generate a sequence from a starting number and a simple rule.

Sequences

A **sequence** is a list of numbers or symbols in a particular order. The numbers or items in a sequence are called the **terms** of the sequence. Each term is separated by a comma. If the sequence continues indefinitely, or if there are too many terms in the sequence to write them all, we use an *ellipsis* '. . .' at the end of a few terms of the sequence, like this:

$$7, 3, 4, 11, 15, 24, \ldots$$

Sequences may be either generated randomly or by **recursion**, using a rule. For example, recording the numbers obtained while tossing a die would give a randomly generated sequence, such as:

$$3, 1, 2, 2, 6, 4, 3, \ldots$$

Because there is no pattern in the sequence, there is no way of predicting the next term in the sequence. Consequently, random sequences are of no relevance to this chapter.

When there is a pattern to the sequence, sequences can exhibit different behaviours such as terms increasing, decreasing or being constant (e.g. $3, 3, 3, 3, \ldots$). Sequences can also oscillate (meaning change or alternate) between two or more values (e.g. $1, -1, 1, -1, \ldots$) or have a limiting value where the sequence approaches a particular value. For example,

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

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Example 1 **Identifying behaviour of sequences**

Consider the following sequences and identify their behaviour as increasing, decreasing, constant or oscillating. Also state whether the sequence has a limiting value.

a $-3, -3, -3, -3, \ldots$ **b** $100, 90, 80, 70, \ldots$ **c** $1000, 100, 10, 1, 0.1, \ldots$

d $1, 4, 9, 25, 36, \ldots$ **e** $10, -8, 10, -8, \ldots$

Explanation	Solution
a Each term has the same value.	The sequence exhibits constant behaviour.
b The value of each term is decreasing by 10.	The sequence exhibits decreasing behaviour.
c The value of each term is divided by 10 and so the sequence is getting closer to zero.	The sequence exhibits decreasing behaviour and has a limiting value of zero.
d The value of each term is increasing.	The sequence exhibits increasing behaviour.
e The value of each term is alternating between 10 and −8.	The sequence exhibits oscillating behaviour.

Now try this 1 **Identifying behaviour of sequences (Example 1)**

Consider the following sequences and identify their behaviour as increasing, decreasing, constant or oscillating. Also state whether the sequence has a limiting value.

a $8, 4, 8, 4, 8, \ldots$ **b** $65, 70, 75, 80, \ldots$ **c** $1.1, 1.01, 1.001, 1.0001, \ldots$

Hint 1 Determine what the relationship is between each term.

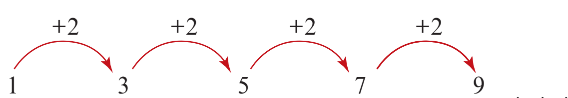
Hint 2 Identify the general pattern of the sequence.

An introduction to recursion

Some sequences of numbers do display a pattern. For example, if our sequence starts with the number one, and we have the rule:

‘add 2 to the current number,’

then we get the sequence:



For example, to find the term after 9, just add 2 to 9, to get $9 + 2 = 11$.

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Recursion is a process of generating a sequence of terms from a given starting point and a rule. Different rules can be used, such as adding or subtracting a term, multiplying or dividing by a particular number, squaring numbers or even combining previous terms to find new terms. Knowing the starting term and the rule means that the next term can be found easily.

In this chapter we will look at sequences that can be generated by a rule.

Example 2 Looking for a recursive rule for a sequence of numbers

Look for a pattern or rule in each sequence and find the next number.

a 2, 8, 14, 20, . . .

b 5, 15, 45, 135, . . .

c 7, 4, 1, −2, . . .

Explanation

a 2, 8, 14, 20, . . .

1 Add 6 to make each new term.

2 Add 6 to 20 to make the next term, 26.

b 5, 15, 45, 135, . . .

1 Multiply by 3 to make each new term.

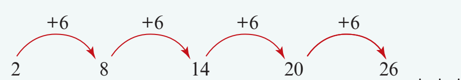
2 Multiply 135 by 3 to make the next term, 405.

c 7, 4, 1, −2, . . .

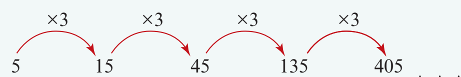
1 Subtract 3 each time to make the next term.

2 Subtract 3 from −2 to get the next term, −5.

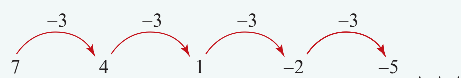
Solution



The next number is 26.



The next number is 405.



The next number is −5.

Now try this 2 Looking for a recursive rule for a sequence of numbers (Example 2)

Look for a pattern or rule in each sequence and find the next number.

a 25, 22, 18, 15, . . .

b 1000, 200, 40, . . .

c 3, 6, 9, 12, 15, . . .

Hint 1 Calculate the difference between each consecutive term.

Hint 2 Determine the relationship between two consecutive terms.

Hint 3 Apply the rule that you find, so that you can determine the next term.

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Generating sequences from a starting value and a rule

Sequences can be generated from a starting value and a rule that tells us how to find the next value in the sequence.

Example 3 Finding a sequence from a starting value and rule

Write down the first five terms of the sequence with a starting value of 5 and the rule 'add 3 to each term'.

Explanation	Solution
1 Write down the starting value.	5
2 Apply the rule (add 3) to generate the next term.	$5+3=8$
3 Calculate three more terms.	$8+3 = 11$ $11+3 = 14$ $14+3 = 17$
4 Write your answer.	The sequence is 5, 8, 11, 14, 17.

Now try this 3 Finding a sequence from a starting value and rule (Example 3)

Write down the first five terms of the sequence with a starting value of 7 and the rule 'add 5 to each term'.

Hint 1 Write down the starting value.

Hint 2 Apply the rule to generate the next term.

Using repeated addition on a CAS calculator to generate a sequence

As we have seen, a recursive rule based on repeated addition, such as 'to find the next term, add 6', is a quick and easy way of generating the next few terms of a sequence. However, it becomes tedious to do by hand if we want to find, say, the next 20 terms.

Fortunately, your CAS calculator can semi-automate the process.

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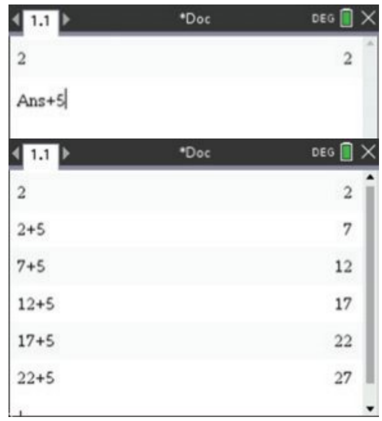
Using TI-Nspire CAS to generate the terms of an arithmetic sequence recursively

Generate the first six terms of the arithmetic sequence: 2, 7, 12, . . .

Steps

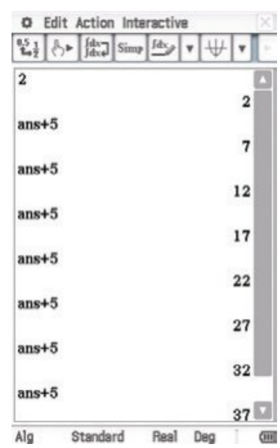


1. Press [on] >**New > Add Calculator**.
2. Enter the value of the first term, 2. Press [enter]. The calculator stores the value, 2, as Answer.
3. The common difference for the sequence is 5. So, type in +**5**.
4. Press [enter]. The second term in the sequence, 7, is generated.
5. Pressing [enter] again generates the next term, 12. Keep pressing [enter] until the desired number of terms is generated.
6. Write down the terms. The first 6 terms of the sequence are: 2, 7, 12, 17, 22, 27.

Using Class Pad to generate the terms of an arithmetic sequence recursively

Generate the first six terms of the arithmetic sequence: 2, 7, 12, . . .

Steps



1. Tap Main $\sqrt{\alpha}$ to open the Main application.
2. Starting with a clean screen, enter the value of the first term, **2**. Press [EXE]. The calculator stores the value, 2, as **ans**.
3. The common difference for this sequence is 5. So, type **+ 5**. Then press [EXE]. The second term in the sequence, **7**, is displayed.
4. Pressing [EXE] again generates the next term, **12**. Keep pressing [EXE] until the required number of terms is generated.
5. Write down the terms. The first 6 terms of the sequence are: 2, 7, 12, 17, 22, 27.

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Section Summary

- A **sequence** is a list of numbers or symbols in a particular order.
- Each number or symbol that makes up a sequence is called a **term**.
- **Recursion** involves repeating the same calculation over and over, using the previous result to calculate the next result.
- Sequences can exhibit different behaviours such as increasing, decreasing, constant, oscillating or limiting values.

Exercise 3A

Building understanding

1 Fill in the boxes for the following sequences.

a $2, 5, \square, 11, \square, \ldots$

b $1, 3, 9, \square, 81, \ldots$

2 State the starting value, and determine the rule in the following sequences.

a $10, 8, 6, 4, 2, 0, -2, \ldots$

b $a, d, g, j, m, \ldots$

3 Consider the following sequence:

$$5, 11, 17, 23, 29, \ldots$$

a State the starting value of the sequence.

b Determine the rule for the sequence.

c If the sequence were to continue according to the rule, determine the value of the next three terms.

Developing understanding

Example 1 **4** Consider the following sequences, and identify their behaviour as increasing, decreasing, constant or oscillating. Also state whether the sequence has a limiting value.

a $4, 4, 4, 4, 4, \ldots$

b $1, 1, 2, 3, 5, \ldots$

c $100, 10, 100, 10, 100, \ldots$

d $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

e $-4, -6, -8, -10, -12, \ldots$

f $-20, -16, -12, -8, -4, \ldots$

Example 2 **5** Find the next term in each sequence.

a 3, 7, 11, 15, ...

b 10, 9, 8, 7, ...

c 1, 2, 1, 2, ...

d 31, 24, 17, 10, ...

e 16, 24, 32, 40, ...

f -2, 0, 2, -2, 0, ...

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6 Find the next term in each sequence.

a 4, 8, 16, 32, . . . **b** 48, 24, 12, 6, . . . **c** 1, 3, 9, 27, . . .

d 243, 81, 27, 9, . . . **e** 1000, 500, 250, 125, . . . **f** 2, 6, 18, 54, . . .

7 Find the next term in each sequence.

a January, February, . . . **b** a, c, e, g, . . . **c** ♣, ♦, ♥, ♠, ♣, . . .

d ⇒, ⇐, ⇒, ⇐, . . . **e** Monday, Tuesday, . . . **f** ↑, →, ↓, ←, . . .

8 Describe how terms are generated in each number sequence, and give the next two terms.

a 5, 8, 11, 14, . . . **b** 19, 28, 37, 46, . . . **c** 38, 34, 30, 26, . . .

d 66, 58, 50, 42, . . . **e** 3, 6, 12, 24, . . . **f** 4, 12, 36, 108, . . .

g 128, 64, 32, 16, . . . **h** 3, –6, 12, –24, . . . **i** 1, 2, 3, 5, . . .

Example 3 **9** Write down the first five terms of the sequence with a starting value of 3 and the rule ‘add 2 to each term’.

10 Write down the first five terms of the sequence with a starting value of 90 and the rule ‘subtract 6 from each term’.

11 Write down the first five terms of the sequence with a starting value of 5 and the rule ‘multiply each term by 2, and then add 1’.

Testing understanding

12 Find the next term in each sequence.

a 1, 8, 27, 64, . . .

b 1, 2, 6, 24, . . .

c $-1, -2, -6, -24, -120, \ldots$

13 Describe how the terms are generated in each number sequence, and give the next two terms.

a 2, 4, 12, 48, . . .

b 1, 4, 9, 16, 25, . . .

c 1, 9, 25, 49, . . .

14 Consider the following sequence, and determine the next five terms.

$$1, -2, 4, -8, 16, \ldots$$

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3B Writing recurrence relations in symbolic form

Learning intentions

- To be able to number and name terms in a sequence.
- To be able to generate a sequence from a recurrence relation.

In this section, we will learn how to name and label each term to make them easier to refer to. We will also formalise how to express the starting value and rule for a sequence by writing down the **recurrence relation** for the sequence.

Numbering and naming the terms in a sequence

The symbols $t_0, t_1, t_2, \ldots$ are used as labels or names for the terms in the sequence. The numbers 0, 1, 2 are called *subscripts* which tell us how many times the rule has been applied. Because it signals the start of the sequence, t_0 is called the **starting** or **initial term** of the sequence. Note that this is different to taking the power of a term, such as $t^3 = t \times t \times t$.

For the sequence: 7, 3, 4, 11, 15, 24, . . . , we have:

Term number (n)	0	1	2	3	. . .
Term symbol	$t_0 = 7$	$t_1 = 3$	$t_2 = 4$	$t_3 = 11$	. . .
Term name	term 0	term 1	term 2	term 3	. . .

Example 4 Naming terms in a sequence

For the sequence: 2, 8, 14, 20, 26, 32, . . ., state the values of:

a t_1 **b** t_4 **c** t_5

Explanation

1 Write the name for each term under its value in the sequence.

2 Read the value of each required term.

Solution

2,	8,	14,	20,	26,	32
t_0	t_1	t_2	t_3	t_4	t_5

a $t_1 = 8$

b $t_4 = 26$

c $t_5 = 32$

Now try this 4 Naming terms in a sequence (Example 4)

For the sequence: 54, 50, 46, 42, 38, 34, 30, 26, . . ., state the values of:

a t_2 **b** t_5 **c** t_7

Hint 1 Remember that the initial term is t_0.

Hint 2 Write the name for each term under its value in the sequence.

Hint 3 Read off the value required for each term.

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Recurrence relations

A **recurrence relation** is a mathematical rule that is used to generate a sequence. It has two parts:

- a *starting point:* the value of the term at the start of the sequence
- a *rule*, that can be used to generate successive terms in the sequence.

For example, a recursion rule for the sequence: 2, 8, 14, 20, . . . , can be written as follows:

- Start with 2.
- To obtain the next term, add 6 to the current term, and repeat the process.

The recursion rule can be written in a more compact way using variables with subscripts.

Let t_n be the term in the sequence after n applications of the rule. Using this definition, the recurrence relation can be written as:

Starting value ($n = 0$)	Rule for generating the next term	Recurrence relation (Starting value and rule)
$t_0 = 2$	$t_{n+1} = t_n + 6$	$t_0 = 2, \quad t_{n+1} = t_n + 6$

Note that t can be replaced by any letter of the alphabet, and that each application of the rule is called an **iteration**.

Example 5 Generating a sequence from a recurrence relation

Write down the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 30, \quad t_{n+1} = t_n - 5$$

Explanation	Solution
1 Write down the starting value.	$t_0 = 30$
2 Use the rule to find the next term, t_1.	$t_1 = 30 - 5 = 25$
3 Use the rule to determine three more terms.	$t_2 = t_1 - 5 = 25 - 5 = 20$ $t_3 = t_2 - 5 = 20 - 5 = 15$ $t_4 = t_3 - 5 = 15 - 5 = 10$
4 Write your answer.	The first five terms are: 30, 25, 20, 15, 10

Now try this 5 Generating a sequence from a recurrence relation (Example 5)

Write down the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 12, \quad t_{n+1} = t_n + 5$$

Hint 1 Write down the starting value.

Hint 2 Use the rule to find the next term, t_1.

Hint 3 Use the rule to determine three more terms.

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Section Summary

- Terms in a sequence can be named and numbered, starting from zero and using subscripts, $t_0, t_1, t_2, \ldots$.
- A **recurrence relation** consists of both the starting value and a rule to generate successive terms in the sequence. For example,

$$t_0 = 2, \qquad t_{n+1} = t_n + 6$$

is the sequence with a starting value of 2 where each successive term is made by adding 6.

Exercise 3B

Building understanding

1 State the starting value, t_0, of the sequence 20, 19, 18, 17, . . .

2 Consider the following sequence: 8, 4, 3, 11, 14, ...

a Rewrite the sequence.

b Write the name of each term below each term of the sequence.

3 Consider the following sequence: 9, 11, 13, 15, ...
Complete the following statements.

a The starting value of the sequence is . . .

b To obtain the next term, add . . . to the current term, and repeat the process.

Developing understanding

Example 4

4 Find the required terms from the sequence: 6, 11, 16, 21, 26, 31, 36 . . .

a t_0 b t_3 c t_2 d t_4 e t_5 f t_6

5 For each sequence, state the value of the named terms:

i t_0 ii t_3 iii t_1

a 6, 10, 14, 18, . . .

b 2, 8, 32, 128, . . .

c 29, 22, 15, 8, . . .

d 96, 48, 24, 12, . . .

6 Find the required terms from the sequence: 8, 12, 16, 20, . . .

a t_3 b t_2 c t_0 d t_4 e t_5 f t_8

7 For each sequence, state the value of the named terms:

i t_0 ii t_3 iii t_6

a 14, 20, 26, 32, . . .

b 2, 6, 18, 54, . . .

c 40, 32, 24, 16, . . .

d 8000, 4000, 2000, 1000, . . .

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Example 5

8 Write down the first five terms of the sequence defined by the following recurrence relations, showing the values of the first four iterations.

a $t_0 = 1, \quad t_{n+1} = t_n + 2$

b $V_0 = 100, \quad V_{n+1} = V_n - 10$

c $P_0 = 52, \quad P_{n+1} = P_n + 12$

9 Rewrite the following recursion relations in symbolic form, where V_n represents the value after n applications of the rule.

a The starting value is 3, and the rule is 'add 7 to the current term and repeat the process'.

b The starting value is 9, and the rule is 'add 4 to the current term and repeat the process'.

c The starting value is 16, and the rule is 'subtract 3 from the current term and repeat the process'.

10 Generate the first 5 terms for each of the sequences listed in the question above.

11 State the recurrence relation in symbolic form for the following sequences.

a $11, 15, 19, 23, \ldots$

b $43, 39, 35, 31, \ldots$

c $3, -1, -5, -9, \ldots$

Testing understanding

12 The following recurrence relation can generate a sequence of numbers.

$$T_0 = 20, \quad T_{n+1} = T_n + 3$$

a State the term name for the term that has a value of 23.

b State the term name for the term that has a value of 32.

c State the term name for the term that has a value of 44.

13 The first five terms of a sequence are: 4, 9, 19, 34, 54, ...

a State the starting value of the sequence.

b Explain why the next term cannot be generated by adding or subtracting a value to or from the current term.

c Explain why the next term cannot be generated by multiplying or dividing the current term by a value.

d Determine the rule, in words, to generate the sequence.

e State the recurrence relation in symbolic form to generate the sequence, where V_n represents the value after n iterations.

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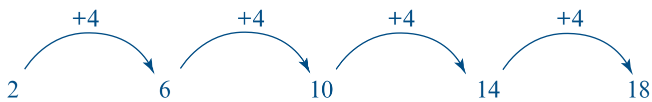
3C An introduction to arithmetic sequences

Learning intentions

- To be able to find the common difference in an arithmetic sequence.
- To be able to identify an arithmetic sequence.
- To be able to tabulate and graph an arithmetic sequence.

A sequence that can be generated by adding or subtracting a fixed amount to or from the previous term is called an **arithmetic sequence**. For example, the sequence 2, 6, 10, 14, 18, . . . is arithmetic because each successive term can be found by adding 4.



Other examples include:

Sequence	Rule
2, 7, 12, . . .	'to find the next term in the sequence, add 5 to the current term'
200, 300, 400, . . .	'to find the next term in the sequence, add 100 to the current term'
50, 45, 40, . . .	'to find the next term in the sequence, subtract 5 from the current term'
100, 99, 98, . . .	'to find the next term in the sequence, subtract 1 from the current term'

The common difference

The fixed amount we add or subtract to form an arithmetic sequence recursively is called the **common difference**. The symbol D is often used to represent the common difference.

If the sequence is *known* to be arithmetic, the common difference can be calculated by simply computing the difference between any pair of successive terms.

Common difference, *D*

In an **arithmetic sequence**, the fixed number added to (or subtracted from) each term to make the next term is called the **common difference**, D, where:

$$D = \text{any term} - \text{previous term}$$

For example, the common difference for the arithmetic sequence: 30, 25, 20, . . . is:

$$D = t_1 - t_0 = 25 - 30 = -5.$$

Often it is not necessary to calculate the common difference in this formal way. It may be easy to see what amount has been repeatedly added (or subtracted) to make each new term.

Example 6 Finding the common difference in an arithmetic sequence

Find the common difference in the following arithmetic sequences and use it to find the next term in each of the sequences below:

a 2, 5, 8, . . .

b 25, 23, 21, . . .

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Explanation

1 Because we know the sequence is arithmetic, all we need to do is find the difference in value between term t_0 and t_1.

2 To find t_3, add the common difference to t_2.

Solution

a $D = t_1 - t_0 = 5 - 2 = 3$

$t_3 = t_2 + D = 8 + 3 = 11$

b $D = t_1 - t_0 = 23 - 25 = -2$

$t_3 = t_2 + D = 21 + (-2) = 19$

Now try this 6 **Finding the common difference in an arithmetic sequence (Example 6)**

Find the common difference in the following arithmetic sequence and use it to find the next term in the sequence.

31, 27, 23, 19, 15, 11, . . .

Hint 1 Find the common difference by finding the difference between two consecutive terms.
Hint 2 Apply the common difference to the last term to find the next term.

Identifying arithmetic sequences

If a sequence is arithmetic, the difference between successive terms will be constant. We can use this idea to see whether or not a sequence is arithmetic.

Example 7 **Identifying an arithmetic sequence**

Which of the following sequences is arithmetic?

a 21, 28, 35, 42, . . .

b 2, 6, 18, 54, . . .

Explanation

a **1** Determine whether the difference between successive terms is constant.

2 Write your conclusion.

b 2, 6, 18, 54, . . .

1 Determine whether the difference between successive terms is constant.

2 Write your conclusion.

Solution

a 21, 28, 35, 42, . . .

Differences:

$28 - 21 = 7$

$35 - 28 = 7$

$42 - 35 = 7$

As the differences between successive terms are constant, the sequence is arithmetic.

b Differences:

$6 - 2 = 4$

$18 - 6 = 12$

$54 - 18 = 36$

As the differences between successive terms are not constant, the sequence is not arithmetic.

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Now try this 7 **Identifying an arithmetic sequence (Example 7)**

Which of the following sequences is arithmetic?

a 2, 4, 8, 16, . . .

b 58, 53, 48, 43, . . .

Hint 1 Determine the difference between successive terms.

Hint 2 If the difference is constant (the same), then the sequence is arithmetic.

Tables and graphs of arithmetic sequences

The terms of a sequence can be tabulated, highlighting that each input (n value) has a corresponding value (t_n). This tells us that a sequence can be thought of as a **function**. For example, the sequence 3, 6, 9, 12, 15, . . . can be tabulated as shown below:

Term number, n	0	1	2	3	4	5	6	7	8	9
Term value, t_n	3	6	9	12	15	18	21	24	27	30

If we plot the values of the terms of an arithmetic sequence (t_n) against their number (n) or the number of applications of the rule, we will find that the points lie on a straight line. We could anticipate this because, as we progress through the sequence, the value of successive terms increases by the same amount (the common difference, D).

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The advantage of graphing a sequence is that the straight line required for an arithmetic sequence is immediately obvious, and any exceptions would stand out very clearly. An upward slope indicates **linear growth** and a downward slope reveals **linear decay**.

Example 8 Graphing the terms of an increasing arithmetic sequence ($D > 0$)

The sequence 4, 7, 10, ... is arithmetic with common difference $D = 3$.

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Explanation

a Show the term numbers and values of the first four terms in a table.

1 Write the term numbers in the top row of the table.

2 Write the values of the terms in the bottom row.

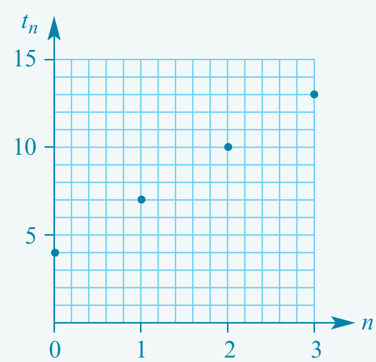
Solution

Term number, n	0	1	2	3
Term value, t_n	4	7	10	13

b Use the table to plot the graph.

1 Use the horizontal axis, n, for the term numbers.
Use the vertical axis for the value of each term, t_n.

2 Plot each point from the table.



c Describe the graph.

1 Are the points along a straight line or a curve?

2 Is the line of the points rising (positive slope) or falling (negative slope)?

The points of an arithmetic sequence with $D = 3$ lie along a rising straight line.
The line has a positive slope.

Now try this 8 Graphing the terms of an increasing arithmetic sequence ($D > 0$) (Example 8)

The sequence 2, 8, 14, ... is arithmetic with common difference $D = 6$.

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Hint 1 Plot each of the points on the graph carefully.

Hint 2 If a point does not lie on the straight line, check your work carefully.

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Example 9 Graphing the terms of a decreasing arithmetic sequence ($D < 0$)

The sequence 9, 7, 5, . . . is arithmetic with common difference $D = -2$.

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Explanation

a Show the term numbers and values of the first four terms in a table.

1 Write the term numbers in the top row of the table.

2 Write the values of the terms in the bottom row.

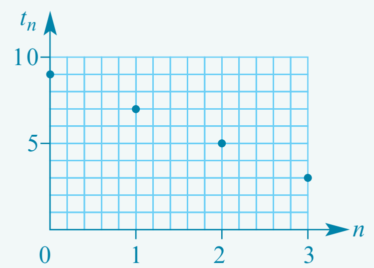
Solution

Term number, n	0	1	2	3
Term value, t_n	9	7	5	3

b Use the table to plot the graph.

1 Use the horizontal axis, n, for the term numbers. Use the vertical axis for the value of each term, t_n.

2 Plot each point from the table.



c Describe the graph.

1 Are the points along a straight line or a curve?

2 Is the line of the points rising (positive slope) or falling (negative slope)?

The points of an arithmetic sequence with $D = -2$ lie along a falling straight line. The line has a negative slope.

Now try this 9 Graphing the terms of a decreasing arithmetic sequence ($D < 0$) (Example 9)

The sequence 10, 7, 4, . . . is arithmetic with common difference $D = -3$.

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Hint 1 Plot each of the points on the graph carefully.

Hint 2 If a point does not lie on the straight line, check your work carefully.

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Graphs of arithmetic sequences are:

- points along a line with positive slope, when a constant amount is added ($D > 0$)
- points along a line with negative slope, when a constant amount is subtracted ($D < 0$).

A line with **positive** slope rises from left to right. A **negative** slope falls from left to right.

Section Summary

- A sequence is **arithmetic** if it is generated by adding or subtracting a fixed amount to or from the current term.
- In an arithmetic sequence, the **common difference**, D, is the difference between any two successive terms.
- Graphs of arithmetic sequences are:
  - points along a line with positive slope (rising from left to right), when a constant amount is added ($D > 0$),
  - points along a line with negative slope (falling from left to right), when a constant amount is subtracted ($D < 0$).

Skill-sheet

Exercise 3C

Building understanding

1 What is the value of term t_3 in each of the following sequences:

a $2, 4, 6, 8, \ldots$

b $1, 4, 9, 16, \ldots$

c $51, 48, 45, 42, \ldots$

2 For each of the following sequences, find the difference between t_0 and t_1 and the difference between t_1 and t_2.

a $11, 15, 19, 23, 27, 31, \ldots$

b $3, 1, -1, -3, -5, \ldots$

3 Consider the following sequence:

$$4, 11, 18, 25, 32, \ldots$$

a Copy down the sequence and label each term.

b State the value of t_3.

c Determine the difference between each pair of consecutive terms, and decide if the sequence is arithmetic.

d If the sequence were to continue, determine the value of t_5 and t_6.

Developing understanding

Example 6

4 For each of these arithmetic sequences, find the common difference and the value of t_3.

a $5, 11, 17, 23, \ldots$ **b** $17, 13, 9, 5, \ldots$ **c** $11, 15, 19, 23, \ldots$

d $8, 4, 0, -4, \ldots$ **e** $35, 30, 25, 20, \ldots$ **f** $1.5, 2, 2.5, 3, \ldots$

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5 Give the next two terms in each of these arithmetic sequences.

a 17, 23, 29, 35, . . . **b** 14, 11, 8, 5, . . . **c** 2, 1.5, 1.0, 0.5, . . .
d 27, 35, 43, 51, . . . **e** 33, 21, 9, −3, . . . **f** 0.8, 1.1, 1.4, 1.7, . . .

Example 7

6 Find out which of the sequences below is arithmetic. Give the common difference for each sequence that is arithmetic.

a 8, 11, 14, 17, . . . **b** 7, 15, 22, 30, . . . **c** 11, 7, 3, −1, . . .
d 12, 9, 6, 3, . . . **e** 16, 8, 4, 2, . . . **f** 1, 1, 1, 1, . . .

7 Use your CAS calculator to generate the first five terms of the following arithmetic sequences.

a 3, 8, . . . **b** 16, 9, . . . **c** 1.6, 3.9, . . .
d 8.7, 5.6, . . . **e** 293, 226, . . .

8 Using your CAS calculator:

a generate the first 7 terms of the arithmetic sequence 1, 6, . . . and write down t_6.
b generate the first 13 terms of the arithmetic sequence 45, 43, . . . and write down t_{12}.
c generate the first 11 terms of the arithmetic sequence 15, 14, . . . and write down t_{10}.
d generate the first 16 terms of the arithmetic sequence 0, 3, . . . and write down t_{15}.

9 Initially, Fumbles Restaurant had 320 wine glasses. After one week, they only had 305 wine glasses. On average, 15 glasses are broken each week.
Using your CAS calculator, determine how many weeks it takes at that breakage rate for there to be only 200 glasses left?

10 Elizabeth intially had 100 songs on her Favourites playlist. Each month she added seven more songs.
Using your CAS calculator:

a determine the number of songs she had in her Favourites playlist after each of the first 4 months.
b determine the number of songs she had in her Favourites playlist by the end of the year.

Example 8
Example 9

11 For the following arithmetic sequences:

a 3, 5, 7, . . .
b 11, 8, 5, . . .

i write down the next term.
ii write down the first four terms in a table, indicating the value n associated with each term.
iii use the table to plot a graph.
iv describe the graph.

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Testing understanding

12 Find the missing terms in the following arithmetic sequences.

a 8, 13, 18, 23, □, □, . . .
b 14, 8, 2, −4, □, □, . . .
c 6, 15, □, □, 42, . . .
d 23, 18, □, □, 3, −2, . . .
e 3, □, □, 27, 35, 43, . . .
f □, □, 29, 37, 45, 53, . . .
g □, □, 7, −4,−15, −26, . . .
h 36, □, 22, □, 8, 1, . . .
i 15, □, 31, □, 47, □, . . .

13 Consider the following sequence:

$$-2, -5, -8, -11, \ldots$$

a Calculate the common difference, D, for the arithmetic sequence.
b If you were to calculate t_{100}, how much would you need to add or subtract from the starting term?
c Determine t_{100}.
d Determine t_{200}.

3D Arithmetic sequences using recursion

Learning intentions

- To be able to generate an arithmetic sequence using a recurrence relation.
- To be able to use the rule for the nth term to solve problems involving arithmetic sequences.

Using a recurrence relation to generate and analyse an arithmetic sequence

Consider the arithmetic sequence below:

$$10, 15, 20, 25, 30, \ldots$$

In words, the recursion relation that can be used to generate this sequence is:

'start the sequence with 10'

'to find the next term, add 5 to the current term, and keep repeating the process'.

Labelling the terms $t_0, t_1, t_2, \ldots$ and following this process we have:

$t_0 = 10$	initial or starting value
$t_1 = 10 + 5 = 15$	after 1 application of the rule
$t_2 = 15 + 5 = 20$	after 2 applications of the rule
$t_3 = 20 + 5 = 25$	after 3 applications of the rule
$t_4 = 25 + 5 = 30$	after 4 applications of the rule

and so on until we have the rule $t_{n+1} = t_n + 5$ after n applications of the rule.

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The **recurrence relation** is a precise and compact way of expressing the starting value and rule that generates this sequence. For this example,

$$t_0 = 10, \qquad t_{n+1} = t_n + 5 \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

The recurrence relation for generating an arithmetic sequence

The recurrence relation:

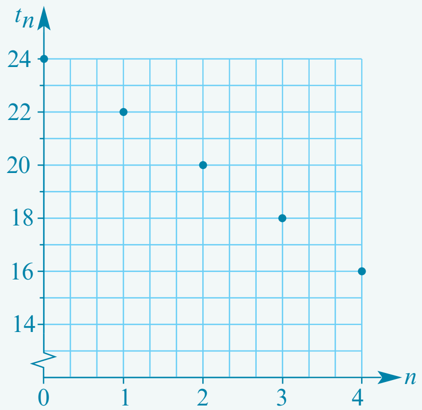
$$t_0 = a, \qquad t_{n+1} = t_n + D$$

can be used to generate an arithmetic sequence with first term $t_0 = a$ and common difference, D.

Example 10 Using a recurrence relation to generate an arithmetic sequence

Generate and graph the first five terms of the arithmetic sequence defined by the recurrence relation:

$$t_0 = 24, \qquad t_{n+1} = t_n - 2$$

Explanation	Solution
1 Write down the recurrence relation.	$t_0 = 24, \quad t_{n+1} = t_n - 2$
2 Write down the starting term.	$t_0 = 24$
3 Use the rule, which translates into 'to find the next term, subtract two from the previous term', to generate the first five terms in the sequence.	$t_1 = t_0 - 2 = 24 - 2 = 22$ $t_2 = t_1 - 2 = 22 - 2 = 20$ $t_3 = t_2 - 2 = 20 - 2 = 18$ $t_4 = t_3 - 2 = 18 - 2 = 16$
4 To graph the terms, plot t_n against n.	

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Now try this 10 **Using a recurrence relation to generate an arithmetic sequence (Example 10)**

Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 7, \qquad t_{n+1} = t_n + 3$$

Hint 1 Write down the starting term and then use the rule to find the next four terms.

Hint 2 To graph the terms, plot t_n against n.

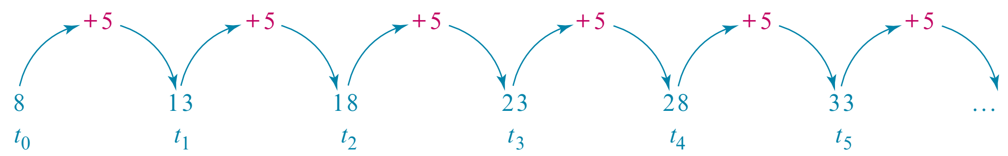
Finding the *n*th term in an arithmetic sequence

Repeated addition can be used to find each new term in an arithmetic sequence, but this process is very tedious for finding a term such as t_{50}. Instead, a general rule can be found to calculate any term, t_n, using n, the number of times the recursion rule is applied, the value of the first term, a, and the common difference, D.

Consider the arithmetic sequence: 8, 13, 18, 23, 28, 33, . . . which is defined by:

$t_0 = 8, \qquad t_{n+1} = t_n + 5 \qquad$ for $n = 0, 1, 2, 3, \ldots.$

This is illustrated pictorially in the diagram below.



Using the information from the diagram, we can write recursively:

$t_0 = 8 + 0 \times 5 = 8$ after 0 applications of the rule

$t_1 = 8 + 1 \times 5 = 13$ after 1 application of the rule

$t_2 = 8 + 2 \times 5 = 18$ after 2 applications of the rule

$t_3 = 8 + 3 \times 5 = 23$ after 3 applications of the rule.

A pattern emerges which suggests that after n applications of the recursion rule,

$t_n = 8 + n \times 5$ after n applications of the rule.

Using this rule, t_n can be found without having to find all previous values in the sequence.

Thus, using the rule:

$$t_{50} = 8 + 50 \times 5 = 258$$

This rule can be generalised to apply to any situation involving the recursive generation of an arithmetic sequence.

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Rule for finding the *n*th term of an arithmetic sequence

The recurrence relation:

$$t_0 = a, \qquad t_{n+1} = t_n + D$$

can be used to generate an arithmetic sequence with a starting value, $t_0 = a$, and a common difference, D.

The rule for directly calculating the term t_n in this sequence is generated from the recurrence relation:

$$t_n = a + nD$$

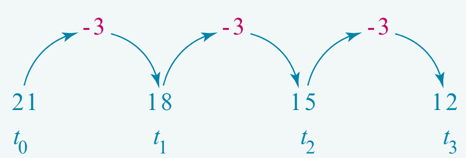
where n is the term number, $n = 0, 1, 2, 3, \ldots$.

Example 11 Finding term *n* of an arithmetic sequence

Consider the recurrence relation:

$$t_0 = 21, \qquad t_{n+1} = t_n - 3$$

Find t_{20}.

Explanation	Solution
1 State the initial value, a, and the common difference, D.	$a = 21, D = -3$
2 The term t_{20} requires 20 applications of the rule. 	$n = 20$
3 Substitute the values of a, D and n into $t_n = a + nD$.	$t_n = a + nD$ $t_{20} = 21 + 20(-3)$
4 Evaluate.	$= -39$
5 Write your answer.	$t_{20} = -39$

Now try this 11 Finding term *n* of an arithmetic sequence (Example 11)

Consider the recurrence relation:

$t_0 = 54, \qquad t_{n+1} = t_n + 7.$

Find t_{50}.

Hint 1 State the initial term of the sequence, a.
Hint 2 State the common difference, D.
Hint 3 Use the rule to find t_{50} ($n = 50$).

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Section Summary

- The recurrence relation:

$$t_0 = a, \qquad t_{n+1} = t_n + D$$

can be used to generate an arithmetic sequence with starting value $t_0 = a$ and common difference, D.

- The rule for directly calculating term t_n in this sequence is:

$$t_n = a + n \times D$$

where n is the term number 0, 1, 2, 3, ...

Skill-sheet

Exercise 3D

Building understanding

1 Give the value of a and D in each of the following arithmetic sequences.

a 7, 11, 15, 19, . . . **b** 8, 5, 2, −1, . . . **c** 14, 23, 32, 41, . . .
d 62, 35, 8, −19, . . . **e** −9, −4, 1, 6, . . . **f** −13, −19, −25, . . .

2 For each of the arithmetic sequences in Question 1, use the value of a and D that you found and the rule $t_n = a + n \times D$ to find the value of t_5.

3 For the following sequence, state the value of a, D and n required to find t_{20}.

12, 22, 32, 42,. . .

4 For the recurrence relation: $t_0 = 7, t_{n+1} = t_n + 3$, complete the blanks using the rule: $t_n = t_0 + n \times D$ to find the value of t_2, t_3, t_4 and t_{20}.

$$t_1 = t_0 + 1 \times 3 = 7 + 1 \times 3 = 10$$
$$t_2 = t_0 + \ldots \times 3 = \ldots + \ldots \times 3 = \ldots$$
$$t_3 = t_0 + \ldots \times 3 = \ldots + \ldots \times 3 = \ldots$$
$$t_{20} = t_0 + \ldots \times 3 = \ldots + \ldots \times 3 = \ldots$$

Developing understanding

Example 10

5 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:
$t_0 = 15, \qquad t_{n+1} = t_n + 5 \qquad$ where $n \geq 0$.
b Calculate the value of t_{44} in the sequence.

6 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:
$t_0 = 60, \qquad t_{n+1} = t_n - 5 \qquad$ where $n \geq 0$.
b Calculate the value of t_{10} in the sequence.

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7 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:
$t_0 = 15, \quad t_{n+1} = t_n + 35 \quad$ where $n \geq 0$.
b Calculate the value of t_{15} in the sequence.

Example 11

8 Find the required term in each of these arithmetic sequences.

a 18, 21, 24, 27, . . . Find t_{35}.
b −14, −6, 2, 10, . . . Find t_{41}.
c 27, 14, 1, −12, . . . Find t_{37}.
d 16, 31, 46, 61, . . . Find t_{29}.
e −19, −23, −27, −31, . . . Find t_{26}.
f 0.8, 1.5, 2.2, 2.9, . . . Find t_{36}.
g 82, 68, 54, 40, . . . Find t_{21}.
h 9.4, 8.8, 8.2, 7.6, . . . Find t_{29}.

9 Find t_{40} in an arithmetic sequence that starts at 11 and has a common difference of 8.

10 The first term in an arithmetic sequence is 27 and the common difference is 19. Find t_{100}.

11 A sequence started at 100 and had 7 subtracted each time to make new terms. Find t_{20}.

Testing understanding

12 When planted, a tree was initially 1.50 m high. It grew 0.75 m in each subsequent year. How high was the tree 18 years after it was planted?

13 In an arithmetic sequence, $t_4 = 10$ and $t_8 = 18$.
a Using the equation $t_n = a + nD$, substitute in t_4 and $n = 4$ to form an equation in terms of a and D.
b Using the equation $t_n = a + nD$, substitute in t_8 and $n = 8$ to form an equation in terms of a and D.
c Use simultaneous equations to solve the two equations you found in part **a** and **b** to find a and D.
d Check your answer by finding t_5 and t_9 using the rule and the values of a and D.
e Hence, write down the first three terms of the arithmetic sequence.

14 Consider the following arithmetic sequence:

$$5, 7, 9, 11, \ldots$$

How many terms in this sequence are less than 25?

3E Finance applications using arithmetic sequences and recurrence relations

Learning intentions

- To be able to use a recurrence relation to model simple interest.
- To be able to use a recurrence relation to model flat rate depreciation.
- To be able to use a recurrence relation to model unit cost depreciation.
- To be able to use a rule to determine the nth term for linear growth or decay.

This section is concerned with the use of arithmetic sequences and recurrence relations to model simple interest, flat rates and unit cost depreciating value of assets. The skill sheet available for this section through the Interactive Textbook also contains non-financial applications.

Using recurrence relations to model simple interest

Linear growth in a sequence occurs when a quantity increases by the same amount at regular intervals, for example, the payment of simple interest on an investment or the amount that is owed on a simple-interest loan.

Simple interest

Simple interest is usually determined by multiplying the annual **interest rate**, $r\%$, by the original amount borrowed (invested) called the **principal**, P, for each year of the loan.

The value after $n + 1$ years is the value after n years plus the interest in the previous year. This gives the recurrence relation:

$$V_0 = P, \qquad V_{n+1} = V_n + D$$

where V_n is the value after n years and $D = \frac{r}{100} \times V_0$.

Note: Remember that the interest rate is a percentage, so we write this as $r\% = \frac{r}{100}$.

Simple interest is usually applied to smaller loans, such as car loans, or investments that are short term. Compound interest, discussed later in this chapter, is used for larger loans over a long period, such as for mortgages.

Example 12 Finding the amount of simple interest each year

An investment of \$3500 pays interest at the rate of 4.2% per annum in the form of simple interest. Find the amount of interest paid each year.

Explanation	Solution
1 Define the symbol V_n in this model.	V_n is the value of the investment after n years.
2 Write down the value of V_0, the principal of the investment.	$V_0 = 3500$
3 Write down the interest rate and use it to calculate the value of $D = \frac{r}{100} \times V_0$.	$r = 4.2\%$ $D = \frac{4.2}{100} \times 3500 = \$147$
4 Determine the interest paid each year.	Thus, each year, the investment earns \$147 in simple interest.

Now try this 12 Finding the amount of simple interest each year (Example 12)

An investment of \$4600 pays interest at the rate of 5.1% per annum in the form of simple interest. Find the amount of interest paid each year.

Hint 1 State the value of the annual interest rate, r, and the principal, P.
Hint 2 Calculate $D = \frac{r}{100} \times V_0$.

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If the investment was for part of the year, for example, 6 months, then we would multiply $\frac{r}{100} \times Vo$ by the fraction of the year required, for example, $\frac{1}{2}$. In Example 12, interest over 6 months would then be \$73.50.

Example 13 Using a recurrence relation to model linear growth: simple interest

The following recurrence relation can be used to model a simple interest investment of \$2000, paying interest at the rate of 7.5% per annum:

$$V_0 = 2000, \qquad V_{n+1} = V_n + 150$$

where V_n is the value of the investment after n years.

Note: The amount of interest that is paid each year is found by multiplying the annual interest rate by the principal:

$$\frac{r}{100} \times V_0 = \frac{7.5}{100} \times 2000 = 150$$

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach \$2750 in value?

Explanation

a 1 Write down the recurrence relation. The recurrence relation tells you that: 'to find the next value, add 150 to the current value'.

2 With $V_0 = 2000$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

This can also be done on your CAS calculator as shown opposite.

b This question is best answered using your CAS calculator to generate the values of successive terms and counting the number of steps (years) until the value of the investment is \$2750.

Write your conclusion.

Note: Count the number of times 150 has been added, *not* the number of terms on the calculator screen.

Solution

a 1 $V_0 = 2000, \qquad V_{n+1} = V_n + 150$

2
$V_0 = 2000$
$V_1 = V_0 + 150 = 2000 + 150 = \$2150$
$V_2 = V_1 + 150 = 2150 + 150 = \$2300$
$V_3 = V_2 + 150 = 2300 + 150 = \$2450$

2000	2000
2000 + 150	2150
2150 + 150	2300
2300 + 150	2450

b

2000	2000
2000 + 150	2150
2150 + 150	2300
2300 + 150	2450
2450 + 150	2600
2600 + 150	2750

The investment will have a value of \$2750 after 5 years.

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Now try this 13 **Using a recurrence relation to model linear growth: simple interest (Example 13)**

The following recurrence relation can be used to model a simple interest investment of \$3000, paying interest at the rate of 5.2% per annum.

$$V_0 = 3000, \qquad V_{n+1} = V_n + 156$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach \$4092 in value?

Hint 1 Make sure you understand the recurrence relation and how 156 was calculated.

Hint 2 Use the starting value and the recurrence relation to generate the terms V_1, V_2 and V_3.

Hint 3 To find when the investment will reach the required value, continue to add \$156, and then count the number of times that it was added to the initial starting value.

Using recurrence relations to model flat rate depreciation

Like linear growth, linear decay can be represented as a recurrence relation. For example, a car that is purchased for \$80 000 will be worth far less in a few years' time. One way that depreciation is calculated is based on the age of the asset. This is called **flat rate** or **fixed rate** depreciation. The value of an asset, V_n, in period n, reduces in value by a fixed amount, D, each period. The fixed amount may be given or it may be calculated based on a percentage rate, r, of the original value, V_0. Thus, $D = \frac{r}{100} \times V_0$.

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Example 14 Using a recurrence relation to model linear decay: flat rate depreciation

The following recurrence relation can be used to model the flat rate depreciation of a car purchased for \$18 500, depreciating at a flat rate of 10% per year.

$$V_0 = 18\,500, \qquad V_{n+1} = V_n - 1850$$

In the recurrence relation, V_n is the value of the car after n years.

Note: The car depreciates \$1850 per year since the annual interest rate (r = 10%) multiplied by the principal (P = \$18 500) is $\frac{10}{100} \times \$18\,500 = \$1850$.

a Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

b How long will it take for the car's value to depreciate to zero?

Explanation

a 1 Write down the recurrence relation. The recurrence relation tells you that: 'to find the next value, subtract 1850 from the current value'.

2 With $V_0 = 18\,500$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

This can also be done on your CAS calculator.

Solution

$V_0 = 18\,500, \quad V_{n+1} = V_n - 1850$

$V_0 = 18\,500$

$V_1 = V_0 - 1850 = 18\,500 - 1850 = \$16\,650$

$V_2 = V_1 - 1850 = 16\,650 - 1850 = \$14\,800$

$V_3 = V_2 - 1850 = 14\,800 - 1850 = \$12\,950$

18500	18500
18500 – 1850	16650
16650 – 1850	14800
14800 – 1850	12950

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b This question is best answered using your calculator and counting the number of steps (years) until the depreciated value of the car is \$0.

18500	18500
18500 – 1850	16650
16650 – 1850	14800
14800 – 1850	12950
12950 – 1850	11100
11100 – 1850	9250
9250 – 1850	7400
7400 – 1850	5550
5550 – 1850	3700
3700 – 1850	1850
1850 – 1850	0

Write your conclusion.

Note: Count the number of times 1850 has been subtracted, *not* the number of terms on the calculator screen.

The car will have a value of zero after 10 years.

Now try this 14 Using a recurrence relation to model linear decay: flat rate depreciation (Example 14)

The following recurrence relation can be used to model the flat rate depreciation of office furniture, purchased for \$3700, depreciating at a flat rate of 7% per year.

$$V_0 = 3700, \quad V_{n+1} = V_n - 259$$

In the recurrence relation, V_n is the value of the office furniture after n years.

a Use the recurrence relation to find the value of the office furniture after 1, 2 and 3 years.

b How long will it take for the office furniture's value to depreciate to less than \$1000?

Hint 1 Use the starting value and the recurrence relation to generate the terms V_1, V_2 and V_3.

Hint 2 To find when the investment will reach the required value, continue to subtract \$259, and then count the number of times that it was subtracted from the initial starting value.

Using recurrence relations to model unit cost depreciation

A second way that depreciation can be calculated is based on how often the asset is used, rather than based on its age. In the instance of a car, the value of the car may depreciate based on the number of kilometres that the car has travelled. This is called **unit cost depreciation**. In this method, V_n represents the value of the asset after n uses, and D is the cost per unit of each use.

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Example 15 Using a recurrence relation to model linear decay: unit cost depreciation

A car with a purchase price of $32 000 depreciates at a unit cost of $150 per 1000 kilometres.

a State the recurrence relation for unit cost depreciation of the car, where V_n represents the value of the car after n thousand kilometres.

b Use the recurrence relation to find the value of the car after 1000, 2000 and 3000 kilometres.

c How many kilometres is the car expected to travel before its value depreciates by at least $1000?

Explanation	Solution
a 1 State the starting value and the common difference for each unit of 1000 kilometres.	$V_0 = 32\,000, \quad D = 150$
2 Use the common difference, $D = 150$, and the starting value, $V_0 = \$32\,000$, to write down the recurrence relation.	$V_0 = 32\,000, \quad V_{n+1} = V_n - 150$
b With $V_0 = 32\,000$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.	$V_0 = \$32\,000$ $V_1 = V_0 - 150 = 32\,000 - 150 = \$31\,850$ $V_2 = V_1 - 150 = 31\,850 - 150 = \$31\,700$ $V_3 = V_2 - 150 = 31\,700 - 150 = \$31\,550$
This can also be done on your CAS calculator.	32000 → 32000 32000 – 150 → 31850 31850 – 150 → 31700 31700 – 150 → 31550
c This question is best answered using your calculator and counting the number of steps until the depreciated value of the car is less than $31 000.	32000 → 32000 32000 – 150 → 31850 31850 – 150 → 31700 31700 – 150 → 31550 31550 – 150 → 31400 31400 – 150 → 31250 31250 – 150 → 31100 31100 – 150 → 30950
Write your conclusion. Note: Count the number of times 150 has been subtracted, *not* the number of terms on the calculator screen.	The car will have depreciated by at least $1000 after 7000 kilometres.

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Now try this 15 **Using a recurrence relation to model linear decay: unit cost depreciation (Example 15)**

An office printer purchased for $9200 depreciates at a unit cost of $15 per 100 000 sheets printed.

a State the recurrence relation for unit cost depreciation for the office printer, where V_n represents the value of the office printer after n hundred thousand sheets have been printed.

b Use the recurrence relation to find the value of the office printer after 100 000, 200 000 and 300 000 sheets.

c How many sheets can be printed before the office printer's value depreciates to less than $9000?

Hint 1 Find the starting value and common difference for the office printer to state the recurrence relation.

Hint 2 Use the starting value and the recurrence relation to generate the terms V_1, V_2 and V_3.

Hint 3 To find when the printer will reach the required value, continue to subtract the common difference and then count the number of times that it was subtracted from the initial starting value.

A rule for term *n* in a sequence modelling linear growth or decay recursively

While we can generate as many terms as we like in a sequence using a recurrence relation for linear growth and decay, it is possible to derive a rule for calculating any term in the sequence directly, using the rule established earlier.

For example, if $1000 is invested in a simple-interest investment paying 5% interest per annum, the value of the investment increases by $50 per year. This means that the value of the investment, V_n, after n years is $V_n = 1000 + n \times 50$.

This rule can be readily generalised to apply to any situation involving linear growth or decay, as follows:

Rule for term *n* in a sequence used to model linear growth or decay

Let V_n be the value of the nth term of the sequence used to model linear growth or decay.

The value of the nth term in this sequence generated by the recurrence relation:

$$V_0 = \text{starting or initial value}, \quad V_{n+1} = V_n + D$$

is given by:

$$V_n = V_0 + n \times D$$

For linear growth, $D > 0$.

For linear decay, $D < 0$.

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Example 16 Using a rule for determining the *n*th term for linear growth or decay

The following recurrence relation can be used to model a simple-interest investment of $4000, paying interest at the rate of 6.5% per year.

$$V_0 = 4000, \qquad V_{n+1} = V_n + 260$$

a How much interest is added to the investment each year?

b Use a rule to find the value of the investment after 15 years.

c Use a rule to find when the value of the investment first exceeds $10 000.

Explanation

a This value can be read directly from the recurrence relation or calculated by finding 6.5% of $4000 = 0.065 × 4000 = $260.

b Because it is linear growth, use the rule:
$V_n = V_0 + n \times D$
Here $V_0 = 4000$, $n = 15$ and $D = 260$.

c Substitute $V_n = 10\,000$, $V_0 = 4000$ and $D = 260$ into the rule: $V_n = V_0 + n \times D$, and solve for n.

Write your conclusion.

Note: Because the interest is only paid into the account after a whole number of years, any decimal answer will need to be *rounded up* to the next whole number.

Solution

a $260

b
$$V_n = V_0 + n \times D$$
$$V_{15} = 4000 + 15 \times 260 = \$7900$$

c
$$10\,000 = 4000 + n \times 260$$
$$6000 = n \times 260$$
$$n = \frac{6000}{260} = 23.07\ldots \text{ years}$$

The value of the investment will first exceed $10 000 after 24 years.

Now try this 16 Using a rule for determining the *n*th term for linear growth or decay (Example 16)

The following recurrence relation can be used to model a simple-interest investment of $12 000, paying interest at the rate of 4.8% per year.

$$V_0 = 12\,000, \qquad V_{n+1} = V_n + 576$$

a How much interest is added to the investment each year?

b Use a rule to find the value of the investment after 10 years.

c Use a rule to find when the value of the investment first exceeds $20 000.

Hint 1 To find the amount of interest each year, you should find 4.8% of the initial investment, $12 000.

Hint 2 Construct the rule using the initial value, the common difference and the value of $n = 10$, to find the value of the investment after 10 years.

Hint 3 To find the value of n, substitute in the value $V_n = 20\,000$.

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Section Summary

- **Linear growth and decay** can be modelled by $V_0 =$ initial or starting value, $V_{n+1} = V_n + D$, where V_n is the value after n years and D is the amount that is added each time period. Thus, $V_n = V_0 + n \times D$. For linear growth, $D > 0$. For linear decay, $D < 0$.
- **Simple interest** is an example of linear growth, where a fixed amount of interest is earned each period and found by multiplying the interest rate, r, by the initial amount, V_0. That is, $D = \frac{r}{100} \times V_0$.
- **Linear depreciation** is an example of linear decay, where the value of the asset declines by a fixed amount. It can be calculated based on the age of the asset (**flat rate** or **fixed rate**) or the use of the asset (**unit cost**).

Skill-sheet

Exercise 3E

Building understanding

1 An investment of $5000 is made that pays interest of 4% per annum. How much interest does the investment pay each year?

2 A computer initially cost $3000 but depreciates at a flat rate of 12% per year. How much does the computer depreciate by each year?

3 Let V_n be the value of term n in the sequence used to model linear growth or decay, and D be the common difference.
A recurrence relation for simple interest is given by $V_0 =$ starting value and $V_{n+1} = V_n + D$.
Using this recurrence relation, term n in the sequence can be given by:

$$V_n = \ldots + \ldots \times D$$

Developing understanding

Example 12 **4** Find the amount of simple interest that is paid each year for the following investments.

a $10 000 investment at 7.3% per annum.

b $16 500 investment at 3.8% per annum.

c $214 600 investment at 5.4% per annum.

Example 13 **5** The following recurrence relation can be used to model a simple-interest investment of $10 000, paying interest at the rate of 4.5% per year.

$$V_0 = 10\,000, \qquad V_{n+1} = V_n + 450$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach $14 500 in value?

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6 The following recurrence relation can be used to model a simple-interest investment.

$$V_0 = 8000, \qquad V_{n+1} = V_n + 400$$

In the recurrence relation, V_n is the value in dollars of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach \$12 000 in value?

c How much was invested at the start?

d What was the interest rate?

Example 14

7 A tractor costs \$90 000 when new. Its value depreciates at a flat rate of 10% or \$9000 per year.
Let V_n be the value (in dollars) of the tractor after n years.
A recurrence relation that models the depreciating value of this tractor over time is:

$$V_0 = 90\,000, \qquad V_{n+1} = V_n - 9000$$

a Use the recurrence relation to find the value of the tractor after three years.

b The tractor will be sold after 5 years. How much will it be worth then?

c If the tractor continues to depreciate at the same rate for the rest of its life, how many years will it take to have zero value?

d A different model of tractor costs \$95 000 and depreciates at a flat rate of 12% of its original value per year. Write down a recurrence relation to model this situation.

8 Let V_n be the value (in dollars) of a computer after n years.
A recurrence relation that models the depreciating value of this computer over time is:

$$V_0 = 2400, \qquad V_{n+1} = V_n - 300$$

a What was the value of the computer when it was new?

b By how much (in dollars) did the computer depreciate each year?

c What was the percentage flat rate of depreciation?

d After how many years will the value of the computer be \$600?

e When will the computer devalue to half of its original price?

Example 15 **9** A commercial sewing machine costs \$18 000 when new. Its value depreciates by \$200 for every 1000 pairs of jeans that are sewn using it.

Let V_n be the value (in dollars) of the sewing machine after n thousand pairs of jeans have been sewn. A recurrence relation that models the depreciating value of the sewing machine over time is:

$$V_0 = 18\,000, \qquad V_{n+1} = V_n - 200$$

a Use the recurrence relation to find the value of the sewing machine after it has sown 5000 pairs of jeans.

b The sewing machine will be sold after it has produced 20 000 pairs of jeans. How much will it be worth then?

c If the sewing machine continues to depreciate at the same rate for the rest of its life, how many pairs of jeans will it take to have zero value?

d A different model of sewing machine costs \$20 000 and depreciates by \$250 for every 1000 pairs of jeans that are sewn using it. Write down a recurrence relation to model this situation.

10 Let V_n be the value (in dollars) of a scissor-lift after n thousand uses.

A recurrence relation that models the depreciating value of the scissor-lift over time is:

$$V_0 = 26\,500, \qquad V_{n+1} = V_n - 70$$

a What was the value of the scissor-lift when it was new?

b By how much (in dollars) did the scissor-lift depreciate after every 1000 uses?

c After how many uses will the scissor-lift devalue to less than half of its original price?

Example 16 **11** The following recurrence relation can be used to model a simple-interest investment of \$32 000, paying interest at the rate of 2.5% per year. Let V_n be the value of the investment after n years.

$$V_0 = 32\,000, \qquad V_{n+1} = V_n + 800$$

a How much interest is added to the investment each year?

b Use a rule to find the value of the investment after 15 years.

c Use a rule to find when the value of the investment first reaches \$40 000.

12 The following recurrence relation can be used to model the flat rate depreciation of a motorbike purchased for \$4000, depreciating at a flat rate of 12.5% per year. Let V_n be the value of the motorbike after n years.

$$V_0 = 4000, \qquad V_{n+1} = V_n - 500$$

a How much does the motorbike depreciate by each year?

b Use a rule to find the value of the motorbike after 4 years.

c Use a rule to find when the depreciated value of the bike is zero.

Testing understanding

13 Bruce invests a certain amount of money and receives \$459 each year in simple interest. After 5 years, Bruce has \$10 795 including the initial amount and 5 years of interest. Determine how much Bruce initially invested.

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3F An introduction to geometric sequences

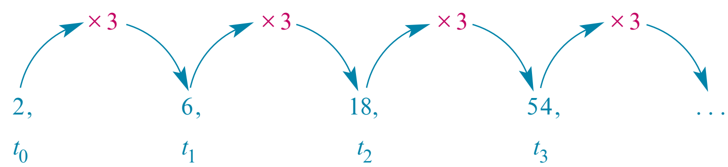
Learning intentions

- To be able to find the common ratio in a geometric sequence.
- To be able to identify a geometric sequence.
- To be able to use a CAS calculator to generate a geometric sequence.
- To be able to graph an increasing or decreasing geometric sequence.

The common ratio, *R*

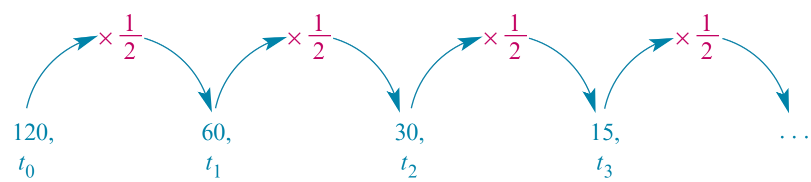
In a **geometric sequence**, each new term is made by multiplying the previous term by a fixed number, called the **common ratio**, ***R***. This repeating or recurring process is another example of a sequence generated by recursion.

In the sequence:



each new term is made by multiplying the previous term by 3. The common ratio is 3.

In the sequence:



each new term is made by halving the previous term. In this sequence we are multiplying each term by $\frac{1}{2}$, which is equivalent to dividing by 2. The common ratio is $\frac{1}{2}$.

New terms in a geometric sequence $t_0, t_1, t_2, t_3, \ldots$ are made by multiplying the previous term by the common ratio, R.

Common ratio, *R*

In a **geometric sequence**, the **common ratio**, R, is found by dividing the next term by the current term.

$$\text{Common ratio } R = \frac{\text{any term}}{\text{the previous term}} = \frac{t_1}{t_0} = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \ldots$$

As preparation for our study of growth and decay, we will be using common ratios which are greater than zero: $R > 0$.

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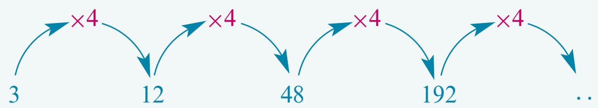
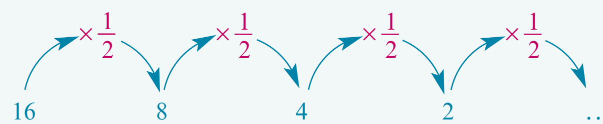
Example 17 Finding the common ratio, *R*

Find the common ratio in each of the following geometric sequences.

a 3, 12, 48, 192, . . . **b** 16, 8, 4, 2, . . .

Explanation	Solution
a 3, 12, 48, 192, . . .	
1 The common ratio is equal to any term divided by the previous term.	Common ratio, $R = \frac{t_1}{t_0} = \frac{12}{3} = 4$
2 Check that multiplying by 4 makes each new term.	
3 Write your answer.	The common ratio is 4.
b 16, 8, 4, 2, . . .	
1 Find the common ratio, *R*.	Common ratio, $R = \frac{t_1}{t_0} = \frac{8}{16} = \frac{1}{2}$
2 Check that multiplying by $\frac{1}{2}$ makes each new term.	
3 Write your answer.	The common ratio is $\frac{1}{2}$.

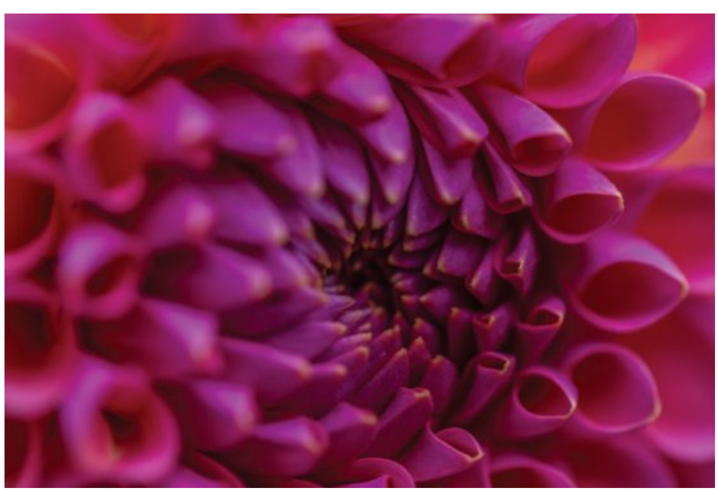
Now try this 17 Finding the common ratio, *R* (Example 17)

Find the common ratio in each of the following geometric sequences.

a 4, 20, 100, 500, . . . **b** 243, 81, 27, 9, . . .

Hint 1 To find the common ratio, calculate any term divided by the previous term.

Hint 2 Check your answer by multiplying each term by the common ratio.



Identifying geometric sequences

To identify a sequence as geometric, it is necessary to find a common ratio between successive terms.

Example 18 Identifying a geometric sequence

Which of the following is a geometric sequence?

a 2, 10, 50, 250, . . .   **b** 3, 6, 18, 36, . . .

Explanation	Solution
a 2, 10, 50, 250, . . .	
1 Find the ratio (multiplier) between successive terms.	$R = \frac{t_1}{t_0}$ $R = \frac{t_2}{t_1}$ $R = \frac{t_3}{t_2}$ $R = \frac{10}{2}$ $R = \frac{50}{10}$ $R = \frac{250}{50}$ $R = 5$ $R = 5$ $R = 5$
2 Check that the ratios are the same.	The common ratio is 5.
3 Write your conclusion.	The sequence is geometric.
b 3, 6, 18, 36, . . .	
1 Find the ratio between successive terms.	$R = \frac{t_1}{t_0}$ $R = \frac{t_2}{t_1}$ $R = \frac{t_3}{t_2}$ $R = \frac{6}{3}$ $R = \frac{18}{6}$ $R = \frac{36}{18}$ $R = 2$ $R = 3$ $R = 2$
2 Are the ratios the same?	The ratios are not the same.
3 Write your conclusion.	The sequence is not geometric.

Now try this 18 Identifying a geometric sequence (Example 18)

Which of the following is a geometric sequence?

a 3, 9, 12, 15, . . .   **b** 2, 4, 8, 16, . . .

Hint 1 Find the ratio between successive terms.
Hint 2 Check if the ratios are the same.
Hint 3 If the ratios are the same then the sequence is a geometric sequence.

The common ratio between two terms can also be based on a percentage. That is, terms in a sequence can increase or decrease by a percentage.

For example, successive terms can increase by 20%, meaning that to find the next term, we multiply by $1 + \frac{20}{100} = 1.2$, so the common ratio is 1.2.

Thus, the following sequence has an increase by 20%: 10, 12, 14.4, 17.28, 20.736,. . .

Using repeated multiplication on a CAS calculator to generate a geometric sequence

Using a recursive rule based on repeated multiplication, such as 'to find the next term, multiply by 2', is a quick and easy way of generating the next few terms of a geometric sequence. It would be tedious to find the next 50 terms.

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Fortunately, your CAS calculator can semi-automate the process of performing multiple repeated multiplications and do this very quickly.

How to use recursion to generate the terms of a geometric sequence with the TI-Nspire CAS

Generate the first six terms of the geometric sequence: 1, 3, 9, 27, ...

Steps

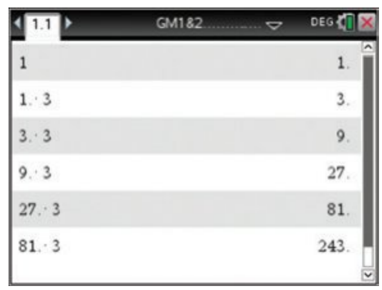
1 Press [home on] >**New Document> Add Calculator**.

2 Enter the value of the first term **1**. Press [enter].
The calculator stores the value, 1, as Answer (you cannot see this yet).

3 The common ratio for the sequence is 3. So, type in **×3**.

4 Press [enter]. The second term in the sequence, 3, is generated.

5 Pressing [enter] again generates the next term, 9. Keep pressing [enter] until the desired number of terms is generated.



6 Write down the first six terms of the sequence.

The first six terms of the sequence are: 1, 3, 9, 27, 81, 243.

How to use recursion to generate the terms of a geometric sequence with the ClassPad

Generate the first six terms of the geometric sequence: 1, 3, 9, 27, ...

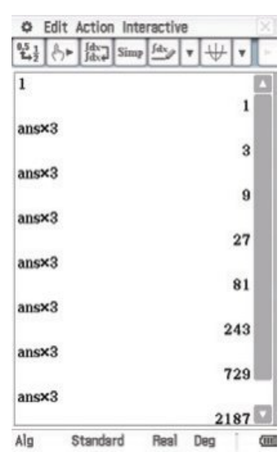
Steps

1 Tap [Main √α] to open the **Main** application.

2 Starting with a clean screen, enter the value of the first term, **1**. Press [EXE].
The calculator stores the value, 1, as **answer**. (You can't see this yet.)

3 The common ratio for this sequence is 3. So, type **× 3**. Then press [EXE]. The second term in the sequence (i.e. **3**) is displayed.

4 Pressing [EXE] again generates the next term, **9**. Keep pressing [EXE] until the required number of terms is generated.



5 Write down the first six terms of the sequence.
The first six terms of the sequence are: 1, 3, 9, 27, 81, 243.

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Graphs of geometric sequences

In contrast with the straight-line graph of an arithmetic sequence, the values of a geometric sequence lie along a curve. Graphing the values of a sequence is a valuable tool for understanding applications involving growth and decay.

The graph of a geometric sequence clearly displays a curve of increasing values associated with growth or decreasing values indicating decay. As we will see, this depends on the value of the common ratio, R.

Example 19 Graphing an increasing geometric sequence ($R > 1$)

Consider the geometric sequence: 2, 6, 18, . . .

a Find the next term.

b Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

c Use the table to plot the graph.

d Describe the graph.

Explanation

a **1** Find the common ratio using $R = \frac{t_1}{t_0}$.

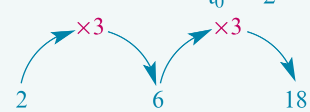
2 Check that this ratio makes the given terms.

3 Multiply 18 by 3 to make the next term, 54.

4 Write your answer.

Solution

Common ratio, $R = \frac{t_1}{t_0} = \frac{6}{2} = 3$



The next term is 54.

Explanation

b **1** Number the positions along the top row of the table.

2 Write the terms in the bottom row.

Solution

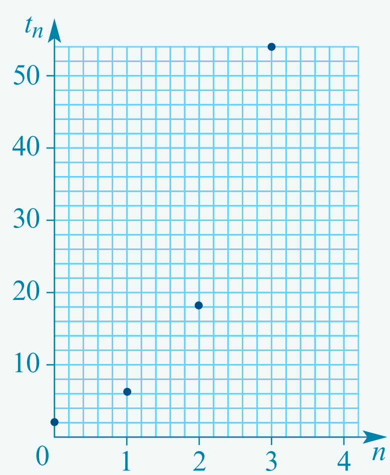
Position, n	0	1	2	3
Term, t_n	2	6	18	54

Explanation

c **1** Use the horizontal axis, n, for the position of each term.
Use the vertical axis, t_n, for the value of each term.

2 Plot each point from the table.

Solution



Explanation

d Describe the pattern revealed by the graph.

Solution

The values lie along a curve, and they are increasing.

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Now try this 19 Graphing an increasing geometric sequence ($R > 1$) (Example 19)

Consider the geometric sequence: 1, 3, 9, . . .

a Find the next term.

b Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

c Use the table to plot the graph.

d Describe the graph.

Hint 1 Find the common ratio so you can find the next term.

Hint 2 When graphing the points, remember to use the horizontal axis for the position of each term, n, and the vertical axis for t_n.

Example 20 Graphing a decreasing geometric sequence ($0 < R < 1$)

Consider the geometric sequence: 32, 16, 8, . . .

a Find the next term.

b Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

c Use the table to plot the graph.

d Describe the graph.

Explanation

a **1** Find the common ratio, using $R = \frac{t_1}{t_0}$.

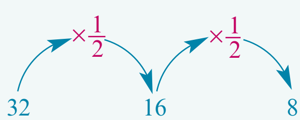
2 Check that this ratio makes the given terms.

3 Multiply 8 by $\frac{1}{2}$ to make the next term, 4.

4 Write your answer.

Solution

Common ratio, $R = \frac{t_1}{t_0} = \frac{16}{32} = \frac{1}{2}$



The next term is 4.

Explanation

b **1** Number the positions along the top row of the table.

2 Write the terms in the bottom row.

Solution

Position, n	0	1	2	3
Term, t_n	32	16	8	4

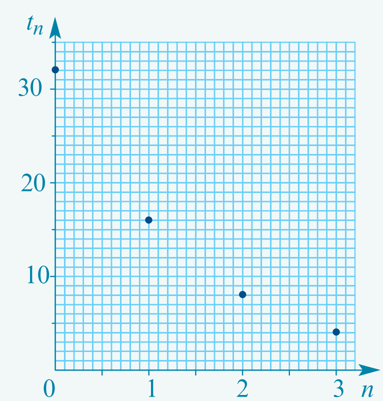
Explanation

c **1** Use the horizontal axis, n, for the position of each term.

Use the vertical axis, t_n, for the value of each term.

2 Plot each point from the table.

Solution



Explanation

d Describe the pattern revealed by the graph.

Solution

The graph is a curve with values decreasing and approaching zero.

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Now try this 20 **Graphing a decreasing geometric sequence ($0 < R < 1$) (Example 20)**

Consider the geometric sequence: 64, 16, 4, ...

a Find the next term.

b Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

c Use the table to plot the graph.

d Describe the graph.

Hint 1 Find the common ratio so you can find the next term.

Hint 2 When graphing the points, remember to use the horizontal axis for the position of each term, n, and the vertical axis for t_n.

Graphs of geometric sequences (for *R* positive)

Graphs of geometric sequences for $R > 0$ are:

- *increasing* when R is greater than 1 ($R > 1$),
- *decreasing* towards zero when R is less than 1 and greater than zero ($0 < R < 1$).

Section Summary

- In a geometric sequence, the common ratio is found by dividing the next term by the current term.

$$\text{Common ratio, } R = \frac{\text{any term}}{\text{the previous term}} = \frac{t_1}{t_0} = \frac{t_2}{t_1} = \ldots$$

- A geometric sequence is increasing if $R > 1$ or is decreasing towards zero if $0 < R < 1$.

Exercise 3F

Building understanding

1 Simplify the following fractions.

a $\frac{3}{6}$ **b** $\frac{81}{3}$ **c** $\frac{27}{9}$ **d** $\frac{4}{64}$

2 Calculate the ratio between the first two numbers in the following geometric sequences.

a 4, 8, 16, 32, ...

b 1, 3, 9, 27, ...

c 16, 8, 4, 2, ...

d 48, 24, 12, 6, ...

3 Decide if the following sequences are arithmetic or geometric.

a 2, 4, 6, 8, ...

b 1, 4, 16, 64, ...

c 729, 243, 81, 27, ...

d 100, 95, 90, 85, ...

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Developing understanding

Example 17

4 Find the common ratio for each of the following geometric sequences.

a 3, 6, 12, 24, ...
b 64, 16, 4, 1, ...
c 6, 30, 150, 750, ...
d 2, 8, 32, 128, ...
e 32, 16, 8, 4, ...
f 2, 12, 72, 432, ...
g 10, 100, 1000, 10 000, ...
h 3, 21, 147, 1029, ...

Example 18

5 Identify which of the following sequences are geometric. Give the common ratio for each sequence that is geometric.

a 4, 8, 16, 32, ...
b 1, 3, 9, 27, ...
c 5, 10, 15, 20, ...
d 5, 15, 45, 135, ...
e 24, 12, 6, 3, ...
f 3, 6, 12, 18, ...
g 4, 8, 12, 16, ...
h 27, 9, 3, 1, ...
i 2, 4, 8, 16, ...

6 Find the missing terms in each of these geometric sequences.

a 7, 14, 28, □, □, ...
b 3, 15, 75, □, □, ...
c 4, 12, □, □, 324, ...
d □, □, 20, 40, 80, ...
e 2, □, 32, 128, □, ...
f 3, □, 27, □, 243, 729, ...

7 Use your CAS calculator to generate each sequence, and find t_6, the sixth term.

a 7, 35, 175, ...
b 3, 18, 108, ...
c 96, 48, 24, ...
d 4, 28, 196, ...
e 160, 80, 40, ...
f 11, 99, 891, ...

Example 19

8 Consider each of the geometric sequences below. For each one:

i Find the next term.
ii Show the first four terms in a table.
iii Use the table to plot a graph.
iv Describe the graph.

a 3, 6, 12, ...
b 2, 10, 50, ...

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Example 20

9 Consider each of the geometric sequences below. For each one:

i Find the next term.
ii Show the first four terms in a table.
iii Use the table to plot a graph.
iv Describe the graph.

a 8, 4, 2, . . .
b 81, 27, 9, . . .

10 Consider a geometric sequence that starts with the value of 20 where each subsequent term increases by 10%.

a State what each term must be multiplied by to find the next term.
b Calculate the next three terms.

11 Consider a geometric sequence that starts with the value of 100 where each subsequent term decreases by 10%.

a State what each term must be multiplied by to find the next term.
b Calculate the next three terms.

12 Consider the geometric sequence 10 000, 12 000, 14 400, 17 280, ...

a Calculate the ratio between the first two numbers of the sequence.
b Calculate the ratio between t_2 and t_1, and between t_3 and t_2.
c State the percentage increase between t_0 and t_1.
d Use the percentage increase (found in **c**) or the common ratio (found in part **a** or **b**) to calculate the next term.

Testing understanding

13 A sequence is generated from the recurrence relation $V_0 = 500, V_{n+1} = 0.4V_n - 5$.

a Use your CAS calculator to generate the first five terms of the sequence.
b Identify whether the sequence is arithmetic or geometric or neither.
c Explain why a term greater than 800 will never occur in the sequence.
d How many iterations are required to generate the first negative term?

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3G Recursion with geometric sequences

Learning intentions

- To be able to generate a geometric sequence using a recurrence relation.
- To be able to find the *n*th term in a geometric sequence using a recurrence relation.

Using a recurrence relation to generate and analyse a geometric sequence

Consider the geometric sequence below:

$2, 6, 18, \ldots$

We can continue to generate the terms of this sequence by recognising that it uses the rule:

'start the sequence with 2'
'to find the next term, multiply the current term by 3, and keep repeating the process.'

Label the terms $t_0, t_1, t_2, \ldots$ and following this process we have:

$t_0 = 2$	initial or starting value
$t_1 = 3 \times t_0 = 6$	after 1 application of the rule
$t_2 = 3 \times t_1 = 18$	after 2 applications of the rule
$t_3 = 3 \times t_2 = 54$	after 3 applications of the rule
$t_4 = 3 \times t_3 = 162$	after 4 applications of the rule

and so on, until we have the rule $t_{n+1} = 3 \times t_n$ after n applications of the rule.

A **recurrence relation** is a way of expressing the starting value and the rule that generates this sequence in precise mathematical language.

The recurrence relation that generates the sequence 2, 6, 18, ... is:

$$t_0 = 2, \qquad t_{n+1} = 3t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots.$$

The rule tells us that:
'the first term is 2, and each subsequent term is equal to the current term multiplied by 3.'

The recurrence relation for generating a geometric sequence

The recurrence relation:

$$t_0 = a, \qquad t_{n+1} = Rt_n$$

can be used to generate a geometric sequence with the first term, $t_0 = a$, and the common ratio, R.

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Example 21 **Using a recurrence relation to generate a geometric sequence**

Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 5, \qquad t_{n+1} = 2t_n$$

Explanation

1 Write down the recurrence relation.

2 Write down the first term.

3 Use the rule, which translates into: 'to find the next term, multiply the previous term by 2' to generate the first five terms in the sequence.

4 To graph the terms, plot t_n against n for $0 \le n \le 4$.

Solution

$t_0 = 5, \qquad t_{n+1} = 2t_n$

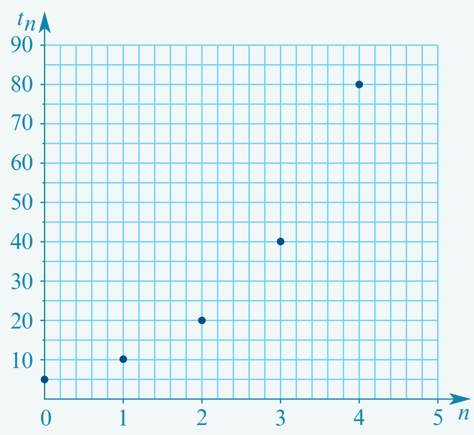
$t_0 = 5$

$t_1 = 2t_0 = 2 \times 5 = 10$

$t_2 = 2t_1 = 2 \times 10 = 20$

$t_3 = 2t_2 = 2 \times 20 = 40$

$t_4 = 2t_3 = 2 \times 40 = 80$



Now try this 21 **Using a recurrence relation to generate a geometric sequence (Example 21)**

Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 1000, \qquad t_{n+1} = 0.1t_n$$

Hint 1 Write down the starting term and then use the rule to find the next four terms.

Hint 2 To graph the terms, plot t_n against n.

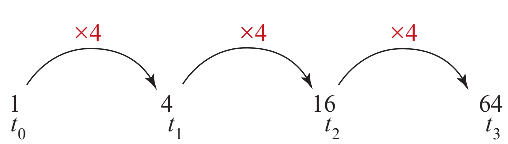
Finding the *n*th term in a geometric sequence

Repeated multiplication can be used to find each new term in a geometric sequence, but this process is very tedious for finding a term such as t_{50}. Instead, a general rule can be found to calculate any term, t_n, using: n, the number of times the recursion rule is applied, the value of the first term, a, and the common ratio, R.

Consider the geometric sequence: $1, 4, 16, 64, \ldots$ which is defined by

$$t_0 = 1, \qquad t_{n+1} = 4t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots.$$

This is illustrated pictorially in the diagram below:



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Using the information from the diagram, we can write recursively:

$t_0 = 1$	after 0 applications of the rule
$t_1 = 4 \times t_0 = 4^1 \times t_0 = 4$	after 1 application of the rule
$t_2 = 4 \times 4 \times t_0 = 4^2 \times t_0 = 16$	after 2 applications of the rule
$t_3 = 4 \times 4 \times 4 \times t_0 = 4^3 \times t_0 = 64$	after 3 applications of the rule

A pattern emerges which suggests that, after n applications of the recursion rule:

$t_n = 4^n \times t_0$ after n applications of the rule.

Using this rule, t_n can be found without having to find all previous values in the sequence. Thus, using the rule:

$$t_{10} = 4^{10} \times 1 = 1\ 048\ 576$$

This rule can be generalised to apply to any situation involving the recursive generation of a geometric sequence.

Rule for finding the *n*th term of a geometric sequence

The recurrence relation:

$$t_0 = a, \qquad t_{n+1} = R \times t_n$$

can be used to generate a geometric sequence with a starting value, $t_0 = a$, and a common ratio, R.

The rule for directly calculating term t_n in this sequence is generated by a recurrence relation:

$$t_n = R^n \times a$$

where n is the term number, $n = 0, 1, 2, 3, \ldots$.

Example 22 Finding the *n*th term of a geometric sequence

Consider the following recurrence relation:

$$t_0 = 2, \qquad t_{n+1} = 3t_n$$

Find t_{12}.

Explanation	Solution
1 State the initial value, a, and the common ratio, R.	$a = 2 \qquad R = 3$
2 The term t_{12} requires 12 applications of the rule.	$n = 12$
3 Substitute the values of a, R and n into $t_n = R^n \times a$.	$t_n = R^n \times a$ $t_{12} = 3^{12} \times 2$
4 Evaluate.	$= 1\ 062\ 882.$
5 Write your answer.	$t_{12} = 1\ 062\ 882.$

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Now try this 22 **Finding the *n*th term of a geometric sequence (Example 22)**

Consider the following recurrence relation:

$$t_0 = 5, \qquad t_{n+1} = 2t_n$$

Find t_{20}.

Hint 1 Write down the starting term and the common ratio.

Hint 2 Substitute the values of a, R and n into $t_n = R^n \times a$.

Section Summary

- The recurrence relation:

  $$t_0 = a, \qquad t_{n+1} = R \times t_n$$

  can be used to generate a geometric sequence with a starting value, $t_0 = a$, and a common ratio of R.
- The rule for directly calculating term t_n in this sequence is:

  $$t_n = R^n \times t_0$$

  where n is the term number, $n = 0, 1, 2, 3, \ldots$

Skill-sheet

Exercise 3G

Building understanding

1 Give the value of a and R in each of the following geometric sequences.

a $2, 6, 18, 54, \ldots$

b $5, 20, 80, 320, \ldots$

c $5, 10, 20, 40, \ldots$

d $3, 12, 48, 192, \ldots$

2 For each of the geometric sequences in Question 1, use the value of a and R that you found and the rule: $t_n = R^n \times t_0$, to find the value of t_5.

3 For the recurrence relation: $t_0 = 6, t_{n+1} = 2 \times t_n$, complete the blanks using the rule: $t_n = R^n \times t_0$, to find the value of t_1, t_2, t_3 and t_{20}.

$$t_1 = R^1 \times t_0 = 2^1 \times 6 = 2 \times 6 = 12$$
$$t_2 = R^2 \times t_0 = 2^{\cdots} \times 6 = \ldots \times \ldots = \ldots$$
$$t_3 = R^3 \times t_0 = 2^{\cdots} \times 6 = \ldots \times \ldots = \ldots$$
$$t_{20} = R^{\cdots} \times t_0 = \ldots^{\cdots} \times 6 = \ldots \times \ldots = \ldots$$

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Developing understanding

Example 21 **4** **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$t_0 = 2 \qquad t_{n+1} = 2t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

b Calculate the value of t_{10}.

5 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$t_0 = 3 \qquad t_{n+1} = 2t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

b Calculate the value of t_{12}.

6 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$t_0 = 4 \qquad t_{n+1} = 3t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

b Calculate the value of t_{10}.

7 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$t_0 = 100 \qquad t_{n+1} = \frac{1}{10}t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

b Calculate the value of t_{10}.

8 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$t_0 = 100 \qquad t_{n+1} = \frac{1}{2}t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

b Calculate the value of t_{15}.

Example 22 **9** Consider the following recurrence relation:

$t_0 = 10, \qquad t_{n+1} = 2t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

Use your CAS calculator to find the value of the following terms:

a t_3 **b** t_5 **c** t_{12} **d** t_{15} **e** t_{20} **f** t_{25}

10 Use your CAS calculator to find t_{30} for each of the following recurrence relations:

a $t_0 = 6, \qquad t_{n+1} = 2t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

b $t_0 = 4, \qquad t_{n+1} = 3t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

c $t_0 = 2000, \qquad t_{n+1} = 0.5t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

d $t_0 = 10\,000, \qquad t_{n+1} = 0.5t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

11 Find t_{20} in a geometric sequence that starts at 4 and has a common ratio of 2.

12 The first term in a geometric sequence is 5 and has a common ratio of 2. Find t_{10}.

13 A sequence starts at 5000 and is divided by 2 each time to make a new term. Find t_{20}.

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Testing understanding

14 A piece of paper had an area of 1m^2. It was cut in half each day with one half thrown away and the other half retained for the next day. What was the area of the paper that was retained at the end of the 7th day, in m^2?

15 In a geometric sequence, $t_2 = 12$ and $t_5 = 96$.

a Using the equation $t_n = R^n \times t_0$, substitute $t_2 = 12$ and $n = 2$ to form an equation in terms of R and t_0.

b Using the equation $t_n = R^n \times t_0$, substitute $t_5 = 96$ and $n = 5$ to form an equation in terms of R and t_0.

c Use simultaneous equations to solve the two equations you found in part **a** and **b** to find R and t_0.

d Hence, write down the recurrence relation.

e Check your answer by finding t_3 and t_4 using the rule and the values of R and t_0.

f Hence, write down the first six terms (from t_0) of the sequence.

3H Finance applications using geometric sequences and recurrence relations

Learning intentions

- To be able to use a recurrence relation to model compound interest for investments or loans.
- To be able to write a recurrence relation to model a loan that compounds with a different compounding period.
- To be able to use a recurrence relation to model reducing-balance depreciation.

The skill sheet available for this section through the Interactive Textbook also contains non-financial applications.

Geometric growth and decay

Geometric growth and **decay** are also commonly found in the world around us. Geometric growth or decay in a sequence occurs when the quantity being modelled increases or decreases by the same percentage at regular intervals. Everyday examples include the payment of compound interest or the depreciation of the value of a new car by a constant percentage of its depreciated value each year. This method of depreciation is commonly called **reducing-balance depreciation**.

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A recurrence model for geometric growth and decay

Geometric growth or decay in a sequence occurs when quantities increase or decrease by the same percentage at regular intervals.

Recall that the recurrence relation for a geometric sequence consists of the starting value and a rule to generate the next term. Consider the following two recurrence relations:

$$V_0 = 10, \qquad V_{n+1} = 5V_n$$
$$V_0 = 10, \qquad V_{n+1} = 0.5V_n$$

Both of these rules generate a geometric sequence, but the first relation generates a sequence that *grows* geometrically while the second relation generates a sequence that *decays* geometrically.

As a general rule, if R is a constant, the recurrence relation rule:

- $V_{n+1} = RV_n$ for $R > 1$, can be used to generate *geometric growth*.
- $V_{n+1} = RV_n$ for $0 < R < 1$, can be used to generate *geometric decay*.

Compound interest investments and loans

While interest is sometimes calculated using simple interest models, it is more commonly calculated using **compound interest**, where any interest earned after one period is added to the principal and then contributes to the earnings of interest in the next time period.

This means that the amount of interest earned in each period increases over time. The value of the investment grows geometrically.

For example, the investment of \$8000 that pays 5% interest per annum, compounding yearly, increases in value by 5% each year. This can be modelled using a recurrence relation as follows:

$$V_0 = 8000, \qquad V_{n+1} = V_n + 0.05V_n$$

or more compactly,

$$V_0 = 8000, \qquad V_{n+1} = 1.05V_n$$

where V_n is the value of the investment after n years. Note that in this example, $R = 1.05 > 1$, telling us that we have a model of geometric growth.

The recurrence relation for compounding interest investments and loans that compound yearly

Let V_n be the value of the investment after n years.

Let r be the percentage interest per compound period.

The recurrence model for the value of the investment after n compounding periods is:

$$V_0 = \text{ principal}, \qquad V_{n+1} = R \times V_n$$

where:

$$R = 1 + \frac{r}{100}$$

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Example 23 Using a recurrence relation to model geometric growth: a compound interest investment (1)

The following recurrence relation can be used to model a compound interest investment of \$1000, paying interest at the rate of 8% per annum.

$$V_0 = 1000, \qquad V_{n+1} = 1.08V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years, to the nearest cent.

b Determine when the value of the investment will first exceed \$1500.

Explanation

a 1 Write down the recurrence relation.

2 With $V_0 = 1000$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

This can also be done with your CAS calculator.

b This is best done using your CAS calculator and counting the number of steps (years) until the value of the investment first exceeds \$1500.

Write your conclusion.

Note: Count the number of times the value of the investment is increased by 8%.

Solution

a $V_0 = 1000 \quad V_{n+1} = 1.08V_n$

$V_0 = 1000$
$V_1 = 1.08V_0 = 1.08 \times 1000 = \$1080$
$V_2 = 1.08V_1 = 1.08 \times 1080 = \$1166.40$
$V_3 = 1.08V_2 = 1.08 \times 1166.40 = \$1259.71$

1000	1000
1000 · 1.08	1080
1080 · 1.08	1166.4
1166.4 · 1.08	1259.71

b

1000	1000
1000 · 1.08	1080
1080 · 1.08	1166.4
1166.4 · 1.08	1259.71
1259.712 · 1.08	1360.49
1360.48896 · 1.08	1469.33
1469.3280768 · 1.08	1586.87

The investment will first exceed \$1500 after 6 years.

Now try this 23 Using a recurrence relation to model geometric growth: a compound interest investment (Example 23)

The following recurrence relation can be used to model a compound interest investment of \$3000, paying interest at the rate of 4% per annum.

$$V_0 = 3000, \qquad V_{n+1} = 1.04V_n$$

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In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years, to the nearest dollar.

b Determine when the value of the investment will first exceed \$3600.

Hint 1 Write down the recurrence relation and use this to calculate V_1, V_2 and V_3.

Hint 2 Use your CAS calculator to find how many times the rule needs to be applied to get to at least \$3600.

Often, compound interest investments and loans accrue over periods of less than a year, and so this must be taken into account when we formulate a general recurrence relation.

Example 24 Using a recurrence relation to model geometric growth: a compound interest investment (2)

Sally borrows \$6000 from a bank. Interest will accrue at the rate of 4.2% per annum.

Let V_n be the value of the loan after n compounding periods.

Write down a recurrence relation to model the value of Sally's loan if interest is compounded:

a yearly **b** quarterly **c** monthly

Explanation	Solution
a **1** Define the variable V_n. The compounding period is **yearly**.	Let V_n be the value of Sally's loan after n years.
2 Determine the value of R.	The interest rate is 4.2% per annum. $R = 1 + \frac{4.2}{100} = 1.042$
3 Write the recurrence relation.	$V_0 = 6000, \quad V_{n+1} = 1.042 \times V_n$
b **1** Define the variable V_n. The compounding period is **quarterly**.	Let V_n be the value of Sally's loan after n quarters.
2 Determine the value of R.	The interest rate is 4.2% per annum. The quarterly interest rate is $\frac{4.2}{4} = 1.05$ $R = 1 + \frac{1.05}{100} = 1.0105$
3 Write the recurrence relation.	$V_0 = 6000, \quad V_{n+1} = 1.0105 \times V_n$
c **1** Define the variable V_n. The compounding period is **monthly**.	Let V_n be the value of Sally's loan after n months.
2 Determine the value of R.	The interest rate is 4.2% per annum. The monthly interest rate is $\frac{4.2}{12} = 0.35$ $R = 1 + \frac{0.35}{100} = 1.0035$
3 Write the recurrence relation.	$V_0 = 6000, \quad V_{n+1} = 1.0035 \times V_n$

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Now try this 24 **Using a recurrence relation to model geometric growth: a compound interest investment (2) (Example 24)**

Valentina borrows \$25 000 from a bank. Interest accrues at the rate of 4.8% per annum.

Let V_n be the value of the loan after n compounding periods.

Write down a recurrence relation to model the value of Valentina's loan if interest is compounded:

a yearly **b** quarterly **c** monthly

Hint 1 Define the variable V_n given the compounding period.

Hint 2 Determine the value of R.

Hint 3 Write the recurrence relation.

Reducing-balance depreciation

We have already considered two different methods of depreciation - flat rate depreciation and unit cost depreciation - both examples of linear decay. **Reducing-balance depreciation** is another method of depreciation where the value of an asset decays geometrically. Each year, the value will be reduced by a percentage, $r\%$, of the previous year's value. The calculations are very similar to compounding interest, but with decay in value rather than growth.

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The recurrence relation for reducing-balance depreciation

Let V_n be the value of the asset after n years.

Let r be the annual percentage depreciation.

The recurrence model for the value of the asset after n years is:

$$V_0 = \text{ initial value}, \qquad V_{n+1} = R \times V_n$$

where

$$R = 1 - \frac{r}{100}$$

Example 25 Using a recurrence relation to model geometric decay: reducing-balance depreciation

A car is purchased for \$18 500. The following recurrence relation can be used to model the car's value as it depreciates by 10% of its value each year.

$$V_0 = 18\,500, \qquad V_{n+1} = 0.9 \times V_n$$

In the recurrence relation, V_n is the value of the car after n years.

a Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

b When will the value of the car first be worth less than \$10 000?

Explanation	Solution
a 1 Write down the recurrence relation.	$V_0 = 18\,500, \quad V_{n+1} = 0.90V_n$
2 The recurrence relation tells you that: 'to find the next value, multiply the current value by 0.90'. With $V_0 = 18\,500$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.	$V_0 = 18\,500$ $V_1 = 0.9V_0 = 0.9 \times 18\,500 = \$16\,650$ $V_2 = 0.9V_1 = 0.9 \times 16\,650 = \$14\,985$ $V_3 = 0.9V_2 = 0.9 \times 14\,985$ $= \$13\,486.50$
This can also be done with your calculator.	18500 18500 18500 · 0.9 16650 16650 · 0.9 14985 14985 · 0.9 13486.5
b This is best done using your calculator and counting the number of times the car's value has been decreased by 10% until the value of the car is first less than \$10 000.	18500 18500 18500 · 0.9 16650 16650 · 0.9 14985 14985 · 0.9 13486.5 13486.5 · 0.9 12137.85 12137.85 · 0.9 10924.07 10924.065 · 0.9 9831.66
Write your conclusion.	The value of the car is first less than \$10 000 after 6 years.

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Now try this 25 **Using a recurrence relation to model geometric decay: reducing-balance depreciation (Example 25)**

A ute is purchased for \$53 800. The following recurrence relation can be used to model the ute's value as it depreciates by 8% of its value each year.

$$V_0 = 53\,800, \qquad V_{n+1} = 0.92 \times V_n$$

In the recurrence relation, V_n is the value of the ute after n years.

a Use the recurrence relation to find the value of the ute after 1, 2 and 3 years to the nearest cent.

b When will the value of the ute first be worth less than \$30 000?

Hint 1 Write down the recurrence relation.

Hint 2 Note that depreciation of 8% leaves you with 92% of the value.

Hint 3 Apply the rule to find the value of V_1, V_2 and V_3.

Hint 4 Use your CAS calculator to apply the rule to find the first time that the value of the asset falls below \$30 000.

Section Summary

- If R is a constant, the recurrence relation rule:
  - $V_{n+1} = RV_n$ for $R > 1$, can be used to generate *geometric growth*.
  - $V_{n+1} = RV_n$ for $0 < R < 1$, can be used to generate *geometric decay*.
- Let V_n be the value of the investment after n years and r be the percentage interest per compound period. The recurrence model for the value of the investment after n compounding periods is:

$$V_0 = \text{principal} \qquad V_{n+1} = R \times V_n, \text{ where } R = 1 + \frac{r}{100}$$

- Let V_n be the value of the asset after n years and r be the annual percentage depreciation. The recurrence model for the value of the asset after n periods is:

$$V_0 = \text{initial value} \qquad V_{n+1} = R \times V_n, \text{ where } R = 1 - \frac{r}{100}$$

Skill-sheet

Exercise 3H

Building understanding

1 Determine which of the following recurrence relations model geometric growth, geometric decay or neither.

a $V_0 = 5, \quad V_{n+1} = 4V_n \quad$ for $n = 0, 1, 2, 3, \ldots$

b $V_0 = 30, \quad V_{n+1} = \frac{1}{5}V_n \quad$ for $n = 0, 1, 2, 3, \ldots$

c $V_0 = 2, \quad V_{n+1} = V_n - 2 \quad$ for $n = 0, 1, 2, 3, \ldots$

d $V_0 = 100, \quad V_{n+1} = V_n + 10 \quad$ for $n = 0, 1, 2, 3, \ldots$

e $V_0 = 10, \quad V_{n+1} = 0.2V_n \quad$ for $n = 0, 1, 2, 3, \ldots$

f $V_0 = 3, \quad V_{n+1} = 1.1V_n \quad$ for $n = 0, 1, 2, 3, \ldots$

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2 The following recurrence relation is used to model a compound interest investment.

$$V_0 = 5000, \qquad V_{n+1} = 1.05V_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

where V_n is the value of the investment after n years.

a State the amount of money that is initially invested.

b State the annual interest rate that is applied to the investment.

c State the value of the investment after 1 year.

3 The following recurrence relation is used to model balance-reducing depreciation on an asset.

$$V_0 = 40\,000, \qquad V_{n+1} = 0.9V_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

where V_n is the value of the asset after n years.

a State the initial value of the asset.

b State the annual percentage depreciation of the asset.

c State the value of the asset after 1 year.

4 Consider an investment with an interest rate of 4.5% per annum. Calculate the interest rate for the compounding period if interest is compounded:

a quarterly

b monthly

Developing understanding

Example 23

5 The following recurrence relation can be used to model a compound interest investment of \$10 000, paying interest at the rate of 4.5% per year.

$$V_0 = 10\,000, \qquad V_{n+1} = 1.045V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years. Round your answers to the nearest cent.

b When will the value of the investment first exceed \$12 000 in value?

6 The following recurrence relation can be used to model a compound interest investment of \$200 000, paying interest at the rate of 5.2% per year.

$$V_0 = 200\,000, \qquad V_{n+1} = 1.052V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years. Round your answers to the nearest cent.

b When will the value of the investment first exceed \$265 000 in value?

7 The following recurrence relation can be used to model a compound interest investment.

$$V_0 = 8000, \qquad V_{n+1} = 1.075V_n$$

In the recurrence relation, V_n is the value, in dollars, of the investment after n years.

a How much was invested at the start?

b Use the recurrence relation to determine when the value of the investment first exceeds \$10 000.

c What was the annual interest rate?

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8 The following recurrence relation can be used to model a compound interest investment of \$100 000, paying interest at the rate of 6.3% per year.

$$V_0 = 100\ 000, \quad V_{n+1} = 1.063V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years. Round your answer to the nearest cent.

b When will the value of the investment be more than double the initial investment?

Example 24

9 Simon invests \$80 000 with a bank. He will be paid interest at the rate of 6% per year, compounding monthly.
In the recurrence relation, V_n is the value of the investment after n months.

a Determine the monthly interest rate.

b Write a recurrence relation to model Simon's investment.

c Use the recurrence relation to find the value of the investment after 1, 2 and 3 months.

Example 25

10 A tractor costs \$90 000 when new. Its value depreciates at a reducing-balance rate of 15% per year.
Let V_n be the value (in dollars) of the tractor after n years.
A recurrence relation that models the depreciating value of this tractor over time is:

$$V_0 = 90\ 000, \quad V_{n+1} = 0.85V_n$$

a Use the recurrence relation to find the value of the tractor at the end of each year for the first five years. Round your answers to the nearest cent.

b The tractor will be sold after 8 years. How much will it be worth then? Round your answer to the nearest cent.

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11 Let V_n be the value (in dollars) of a refrigerator after n years.
A recurrence relation that models the depreciating value of this refrigerator over time is:

$$V_0 = 1200, \quad V_{n+1} = 0.56V_n$$

- **a** What was the value of the refrigerator when it was new?
- **b** After how many years will the value of the refrigerator first be less than $200?
- **c** When will the refrigerator devalue to less than half of its new price?
- **d** What was the percentage rate of depreciation?

12 A car, purchased new for $84 000, will be depreciated using a reducing-balance depreciation method with an annual depreciation rate of 3.5%. Let V_n be the value (in dollars) of the car after n years.

- **a** What was the value of the car when it was new?
- **b** How much is the car worth after one year?
- **c** Write a recurrence relation to model the value of the car from year to year.
- **d** Confirm that your recurrence relation is correct by calculating the value of the car after one year, using the recurrence relation.
- **e** Generate a sequence of numbers that represents the value of the car from year to year for 5 years in total, starting with the initial value. Write the values of the terms of the sequence to the nearest cent.
- **f** How much has the value of the car declined by after five years?

Testing understanding

13 Jackson has $20 000 to invest at the bank for 5 years. The bank offers a number of investment options.

- **A** Jackson can invest his money and receive 7% simple interest per annum.
- **B** Jackson can invest his money and receive 6.5% interest per annum, compounded each year.
- **C** Jackson can invest his money and receive 6% interest per annum, compounded each month.

Determine which of these options gives Jackson the largest value at the end of five years, and how much interest he will earn from this option.

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3I Finding term *n* in a sequence modelling geometric growth and decay

Learning intentions

- To be able to find the value of an investment with compounding interest or an asset with reducing-balance depreciation after a certain length of time.
- To be able to find the amount of interest earned in a time period on an investment with compounding interest.

Earlier in this chapter, we found that the *n*th term of a geometric sequence is generated by the rule:

$$V_n = R^n \times V_0$$

This rule works for both geometric growth or decay because growth or decay depends on the value of R rather than the specification of the rule. This general rule can be applied to compound interest loans and investment as well as reducing-balance depreciation.

Compound interest loans and investment

Let V_0 be the amount borrowed or invested (principal).

Let r be the interest rate per compounding period.

The value of a compound-interest loan or investment after n compounding periods, V_n, is given by the rule:

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

Reducing-balance depreciation

Let V_0 be the purchase price of the asset.

Let r be the annual percentage rate of depreciation.

The value of an asset after n years, V_n, is given by the rule:

$$V_n = \left(1 - \frac{r}{100}\right)^n \times V_0$$

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Example 26 Using a rule for determining the value of an investment with reducing-balance depreciation over time

a The following recurrence relation can be used to model a compound interest investment of \$1000, paying interest at the rate of 10% per year.

$$V_0 = 1000, \qquad V_{n+1} = 1.1V_n$$

Use a rule to find the value of the investment after 15 years, to the nearest dollar.

b The following recurrence relation can be used to model the reducing-balance depreciation of a car purchased for \$18 500, where the value of the car depreciates at the rate of 10% per year.

$$V_0 = 18\ 500, \qquad V_{n+1} = 0.9V_n$$

Use a rule to find the value of the car after 12 years, to the nearest dollar.

Explanation	Solution
a Use the rule: $V_n = R^n V_0$, where $V_0 = 1000$, $n = 15$ and $R = 1.1$	$V_n = R^n \times V_0$ $= 1.1^{15} \times 1000$ $= \$4177.24\ldots$ $= \$4177$ (nearest dollar)
b Use the rule: $V_n = R^n V_0$, where $V_0 = 18\ 500$, $n = 12$ and $R = 0.90$	$V_n = R^n \times V_0$ $= 0.9^{12} \times 18\,500$ $= \$5224.94\ldots$ $= \$5225$ (nearest dollar)

Now try this 26 Using a rule for determining the value of an investment with reducing-balance depreciation over time (Example 26)

a The following recurrence relation can be used to model a compound interest investment of \$5000, paying interest at the rate of 6% per year.

$$V_0 = 5000, \qquad V_{n+1} = 1.06V_n$$

Use a rule to find the value of the investment after 20 years, to the nearest dollar.

b The following recurrence relation can be used to model the reducing-balance depreciation of a car purchased for \$37 500, where the value of the car depreciates at the rate of 15% per year.

$$V_0 = 37\ 500, \qquad V_{n+1} = 0.85V_n$$

Use a rule to find the value of the investment after 10 years, to the nearest dollar.

Hint 1 Write down the rule $V_n = R^n V_0$.
Hint 2 Substitute in the value for n, R and V_0.
Hint 3 Remember to round your answer to the nearest dollar.

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Example 27 Using a rule to analyse an investment

A principal value of $20 000 is invested in an account, earning compound interest at the rate of 6% per annum. The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.06^n \times 20\,000$$

a Find the value of the investment after 5 years, to the nearest cent.
b Find the amount of interest earned after 5 years, to the nearest cent.
c Find the amount of interest earned in the fifth year, to the nearest cent.
d If the interest compounds monthly instead of yearly, write down a rule for the value of the investment after n months.
e Use this rule to find the value of the investment after 5 years (60 months).

Explanation	Solution
a **1** Substitute $n = 5$ into the rule for the value of the investment.	$V_5 = 1.06^5 \times 20\,000$ $= 26\,764.511552$
2 Write your answer, rounded to the nearest cent.	After 5 years, the value of the investment is $26 764.51, to the nearest cent.
b To find the total interest earned in 5 years, subtract the principal from the value of the investment after 5 years.	Amount of interest $= 26\,764.51 - 20\,000$ $= 6764.51$ After 5 years, the amount of interest earned is $6764.51.
c **1** The amount of interest earned in the fifth year is equal to the difference between the value of the investment at the end of the fourth and fifth year.	
2 Calculate V_4 to the nearest cent.	$V_4 = 1.06^4 \times 20\,000$ $V_4 = 25\,249.54$ to the nearest cent.
3 Calculate the difference between V_4 and V_5.	$V_5 - V_4 = 26\,764.51 - 25\,249.54$ $= 1514.97$
4 Write your answer.	Interest earned in the fifth year was $1514.97.

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d **1** Define the symbol V_n.	Let V_n be the value of the investment after n months.
2 Determine the value of r. To convert to a monthly interest rate, divide the annual rate by 12.	$r = \frac{6}{12} = 0.5\%$
3 The general rule for the nth term is: $V_n = R^n V_0$, where $R = 1 + r/100$. In this investment, r is the monthly interest rate. Hence, determine R.	$R = 1 + \frac{0.5}{100}$ $= 1.005$
4 Substitute $R = 1.005$ and $V_0 = 20\,000$ into the rule to find the value for V_n.	$V_n = 1.005^n \times 20\,000$
e Substitute $n = 60$ (5 years = 60 months), $R = 1.005$ and $V_0 = 20\,000$ into the rule to find the value for V_{60}.	$V_{60} = 1.005^{60} \times 20\,000 = \$26\,977.00$

Now try this 27 Using a rule to analyse an investment (Example 27)

A principal value of \$40 000 is invested in an account earning compound interest at the rate of 3% per annum. The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.03^n \times 40\,000$$

a Find the value of the investment after 10 years, to the nearest cent.

b Find the amount of interest earned after 10 years, to the nearest cent.

c Find the amount of interest earned in the tenth year, to the nearest cent.

d If the interest compounds monthly instead of yearly, write down a rule for the value of the investment after n months.

e Use this rule to find the value of the investment after 10 years (120 months).

Hint 1 Substitute $n = 10$ into the rule: $V_n = R^n \times V_0$.

Hint 2 Remember that the interest earned is the change in the value of the investment.

Hint 3 When interest is earned at different time periods, the effective interest rate, r, must be calculated.

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Credit cards

Calculating credit card debt is an application of **compound interest**, where interest is calculated **daily**.

Calculating credit card debt

If a credit card debt of $\$P$ accumulates at the rate of $r\%$ per annum, compounding daily, then the amount of debt accumulated, A, after n days is given by:

$$A = P\left(1 + \frac{r/365}{100}\right)^n = P\left(1 + \frac{r}{36\,500}\right)^n$$

and the amount of interest payable after n days is given by:

$$I = A - P$$

Note: To determine the daily interest rate, the annual interest rate, r, is divided by 365. (1 year = 365 days)

Example 28 Calculating credit card interest

Determine how much interest is payable on a credit card debt of \$5630 at an interest rate of 17.8% per annum, compounding daily, for 27 days. Give your answer to the nearest cent.

Explanation	Solution
1 Calculate the value of debt after 27 days using the rule: $A = P\left(1 + \frac{r}{36\,500}\right)^n$	$P = \$5630$, $r = 17.8\%$ and $n = 27$ $A = P\left(1 + \frac{r}{36\,500}\right)^n$ $= 5630\left(1 + \frac{17.8}{36\,500}\right)^{27}$ $= \$5704.60$ to the nearest cent
2 The amount of interest payable is obtained by subtracting the original debt, P, from the value of the debt after 27 days.	$I = A - P$ $= \$5704.60 - \$5630.00$ $= \$74.60$

Now try this 28 Calculating credit card interest (Example 28)

Determine how much interest is payable on a credit card debt of \$3120 at an interest rate of 15.2% per annum, compounding daily, for 35 days.

Hint 1 Calculate the value of the debt for 35 days using the rule $A = P\left(1 + \frac{r}{36\,500}\right)^n$.

Hint 2 Calculate the amount of interest by finding the difference between A and P.

Most credit cards offer an interest-free period, which means that if you pay for your purchase within that time you won't pay any interest. This includes the statement period and some additional days to pay the full balance before an interest rate applies. The actual number of interest-free days varies, depending on when you make your purchase and the number of days remaining in your statement period.

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Example 29 Calculating credit card interest with an interest-free period

Janelle pays for her holiday, costing \$1500, to Bali using her credit card. Her bank offers a 30-day statement period plus a further 25 days interest free. After that time, the bank charges interest at a rate of 20% per annum, compounding daily.

Janelle makes the purchase on 17 August, which is day 10 of her statement period. She intends to pay off the credit card on 1 November. At this date, how much will she need to pay back? (Assume no interest is payable on the last day).

Explanation	Solution
1 Determine the number of interest-free days.	Janelle has 20 + 25 days = 45 interest-free days.
2 Determine the number of days for which interest is payable. Since the purchase was made on 17 August, start counting from 18 August.	August: 18th–31st = 14 days September: 1st–30th = 30 days October: 1st–31st = 31 days Total days = 14 + 30 + 31 = 75 Interest payable days = 75 – 45 = 30
3 Calculate the amount payable using the rule: $A = P\left(1 + \frac{r}{36\,500}\right)^n$.	$P = \$1500, r = 20\%, n = 30$ $A = 1500\left(1 + \frac{20}{36\,500}\right)^{30}$ $= \$1524.85$ to the nearest cent Janelle pays back \$1524.85.

Now try this 29 Calculating credit card interest with an interest-free period (Example 29)

Lars buys a new camera that costs \$5300 using his credit card. His bank offers a 30-day statement period plus a further 15 days interest free. After that time, the bank charges interest at a rate of 17% per annum, compounding daily.

Lars purchases the camera on 18 July, which is day 20 of his statement period. He intends to pay off the credit card on 14 September. At this date, how much will he need to pay back?

Hint 1 Determine the number of interest-free and interest-payable days.

Hint 2 Calculate the total amount payable using the rule: $A = P\left(1 + \frac{r}{36\,500}\right)^n$.

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Purchase and investment options

There are often several options available to consumers to pay for a product. This includes paying cash, taking out a loan, using a credit card or using a 'buy-now, pay-later' option.

To determine the best option, the overall cost of each should be calculated. For example, paying cash means not bearing any interest rates or fees but requires the individual to have the cash available. Taking out a loan will require the individual to pay interest, and a 'buy-now, pay-later' option requires regular payments with fees.

Example 30 Purchase options

A purchase of $2000 is made. Calculate the cost of the purchase under each of the following options.

a Cash: Pay the amount of the purchase in cash at the time of the purchase.

b Simple-interest loan: A simple-interest loan with annual interest rate of 8%, with the interest and principal paid back after two years.

c Compound-interest loan: A compound-interest loan with annual interest rate of 6%, with interest accruing annually and interest and principal payable after two years.

d Credit card: Use a credit card with a statement length of 30 days plus an additional 15 days, after which an interest rate of 20% per annum, compounding daily, is applied. The purchase is made on day 15 of the statement, and the full amount plus interest is repaid after two years.

e Buy-now, pay-later: An initial fee of $30 and a monthly fee of $10, paid each month for two years when the principal will be paid back.

f Establish which option is cheapest when cash is not available.

Explanation	Solution
a Only the purchase price is required.	Pay $2000 at the time of purchase.
b 1 Calculate the amount of interest paid annually, using $\frac{r}{100} \times V_0$.	$D = \frac{8}{100} \times \$2000 = \$160$
2 Calculate the total amount payable at the end of two years ($n = 2$), using the rule $V_n = 2000 + 160n$.	$V_2 = 2000 + 160 \times 2$ $= \$2320$ The total cost of a simple-interest loan is $2320.
c 1 Calculate the common ratio using $1 + \frac{r}{100}$.	With an interest rate of 6%, the common ratio is $R = 1.06$.
2 Calculate the total amount payable at the end of two years, using the rule $V_n = 1.06^n \times 2000$.	$V_2 = 1.06^2 \times 2000$ $= \$2247.20$ The total cost of a compound-interest loan is $2247.20.

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d **1** Calculate the number of days that interest is payable (2 years = 730 days).	730 days - (15 days left on statement + 15 days of interest free) = 700 days
2 Calculate the amount payable using the rule: $A = P\left(1 + \frac{r}{36\,500}\right)^n$.	$A = 2000\left(1 + \frac{20}{36\,500}\right)^{700}$ $= \$2934.70$ The total cost of using the credit card is \$2934.70.
e **1** Calculate the total fees payable.	$30 + 24 \times 10 = \$270$
2 Calculate the total amount payable at the end of two years by adding the total fees and the purchase price.	$\$270 + \$2000 = \$2270$ The total cost of buy-now, pay-later is \$2270.
f Compare the cost of each option.	The best option is a compound-interest loan, costing \$2247.20.

Now try this 30 Purchase options (Example 30)

A purchase of \$3000 is made. Calculate the cost of the purchase under each of the following options.

a Cash: Pay the amount of the purchase in cash at the time of the purchase.

b Simple-interest loan: A simple-interest loan with annual interest rate of 7%, with the interest and principal paid back after two years.

c Compound-interest loan: A compound-interest loan with annual interest rate of 6%, with interest accruing annually and the interest and principal payable after two years.

d Credit card: Use a credit card with a statement length of 30 days plus an additional 12 days, after which an interest rate of 19% per annum, compounding daily, is applied. The purchase is made on day 20 of the statement, and the full amount plus interest is repaid 100 days later.

e Buy-now, pay-later: An initial fee of \$20 and a monthly fee of \$12, paid each month for two years when the principal will be paid back.

f Establish which option is cheapest when cash is not available.

Hint 1 Calculate the total cost of each of the five options.

Hint 2 Determine which is the cheapest, other than cash.

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Inflation: Effect on prices and purchasing power

Inflation is a term that describes the continuous upward movement in the general level of prices. This has the effect of steadily *reducing* the **purchasing power** of your money; that is, what you can actually buy with your money.

In the early 1970s, inflation rates were very high, up to around 16% and 17%. Between 2009 and 2020, inflation in Australia has been low but continued to fluctuate, ranging from 0.9% and 3.3%.

Example 31 Determining the effect of inflation on prices over a short period of time

Suppose that inflation is recorded as 2.7% in 2022 and 3.5% in 2023, and that a loaf of bread costs $2.20 at the end of 2021. If the price of bread increases with inflation, what will be the price of the loaf at the end of 2023?

Explanation	Solution
1 Determine the increase in the price of the loaf of bread at the end of 2022 after a 2.7% increase.	Increase in price (2022): $= 2.20 \times \frac{2.7}{100} = 0.06$
2 Calculate the price at the end of 2022.	$\text{Price}_{2022} = 2.20 + 0.06 = \$2.26$
3 Determine the increase in the price of the loaf of bread at the end of 2023 after a further 3.5% increase.	Increase in price (2023) $= 2.26 \times \frac{3.5}{100} = 0.08$
4 Calculate the price at the end of 2023.	$\text{Price}_{2023} = 2.26 + 0.08 = \$2.34$

Now try this 31 Determining the effect of inflation on prices over a short period of time (Example 31)

Suppose that inflation is recorded as 1.3% in 2021 and 2.0% in 2022, and that a meat pie costs $4.10 at the end of 2020. If the price of a meat pie increases with inflation, what will be the price of the pie at the end of 2022?

Hint 1 Determine the price of the pie at the end of 2021 after the increase of 1.3%.

Hint 2 Determine the price of the pie at the end of 2022 after a further increase of 2.0%.

While the difference in price in one or two years does not seem like a lot, over the long term, the impact of inflation on prices can be significant.

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Example 32 Determining the effect of inflation on prices over a long period

Suppose that a one-litre carton of milk costs \$1.70 today.

a What will be the price of the one-litre carton of milk in 20 years' time if the average annual inflation rate is 2.1%?

b What will be the price of the one-litre carton of milk in 20 years' time if the average annual inflation rate is 6.8%?

Explanation	Solution
a 1 This is the equivalent of investing \$1.70 at 2.1% interest, compounding annually, so we can use the compound interest formula.	$A = P \times \left(1 + \frac{r}{100}\right)^n$
2 Substitute $P = 1.70, n = 20$ and $r = 2.1$ in the formula to find the price in 20 years.	$\text{Price} = 1.70 \times \left(1 + \frac{2.1}{100}\right)^{20}$ $= \$2.58$ to the nearest cent
b Substitute $P = 1.70, n = 20$ and $r = 6.8$ in the formula and evaluate.	$\text{Price} = 1.70 \times \left(1 + \frac{6.8}{100}\right)^{20}$ $= \$6.34$ to the nearest cent

Now try this 32 Determining the effect of inflation on prices over a long period (Example 32)

Suppose that a movie ticket costs \$12 today.

a What will be the price of the movie ticket in 20 years' time if the average annual inflation rate is 1.8%?

b What will be the price of the movie ticket in 20 years' time if the average annual inflation rate is 6.3%?

Hint 1 Use the rule: $A = P \times \left(1 + \frac{r}{100}\right)^n$

Another way of looking at the effect of inflation on our money is to consider what a sum of money today would buy in the future.

If you put \$100 in a box under the bed and leave it there for 10 years, what could you buy with the \$100 in 10 years' time? To find out, we need to 'deflate' this amount back to current-day purchasing power dollars, which can be done using the compound-interest formula.

Suppose there has been an average inflation rate of 4% over the 10-year period. Substituting $A = 100, r = 4$ and $n = 10$ gives:

$$100 = P \times \left(1 + \frac{4}{100}\right)^{10} = P \times (1 + 0.04)^{10}$$

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Rearranging this equation or using your CAS calculator to solve it, gives:

$$P = \frac{100}{(1 + 0.04)^{10}} = \$67.56 \text{ to the nearest cent}$$

That is, the money that was worth \$100 when it was put away has a purchasing power of only \$67.58 after 10 years if the inflation rate has averaged 4% per annum.

Example 33 Investigating purchasing power

If savings of \$100 000 are hidden in a mattress in 2021, what is the purchasing power of this amount in 8 years' time if the average inflation rate over this period is 3.7%? Give your answer to the nearest dollar.

Explanation

1 Write the compound-interest formula with P (the purchasing power, which is unknown), $A = 100\,000$ (current value), $r = 3.7$ and $n = 8$.

2 Use your CAS calculator to solve this equation for P, and write your answer.

Solution

$$A = P \times \left(1 + \frac{r}{100}\right)^n$$

$$100\,000 = P \times \left(1 + \frac{3.7}{100}\right)^8$$

The purchasing power of \$100 000 in 8 years is \$74 777, to the nearest dollar.

Now try this 33 Investigating purchasing power (Example 33)

If savings of \$60 000 are hidden in a mattress in 2022, what is the purchasing power of this amount in 10 years' time if the average inflation rate over this period is 2.2%? Give your answer to the nearest dollar.

Hint 1 Use the rule: $P = \dfrac{A}{\left(1 + \frac{r}{100}\right)^n}$

Section Summary

- Let V_0 be the amount borrowed or invested (principal) and r be the interest rate per compounding period. The value of a compound-interest loan or investment after n compounding periods, V_n, is given by:

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

- Let V_0 be the purchase price of the asset and r be the annual percentage rate of depreciation. The value of an asset after n years, V_n, is given by the rule:

$$V_n = \left(1 - \frac{r}{100}\right)^n \times V_0$$

- If a credit card debt of \$$P$ accumulates at the rate of r% per annum, compounding daily, then the amount of debt accumulated after n days is given by:

$$A = P\left(1 + \frac{r/365}{100}\right)^n = P\left(1 + \frac{r}{36\,500}\right)^n$$

  and the amount of interest payable after n days is given by: $I = A - P$.
- Inflation is a term that describes the continuous upward movement in the general level of prices.

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Exercise 3I

Building understanding

1 Consider the following recurrence relation.

$$t_0 = 5, \qquad t_{n+1} = 3t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

Find the value of:

a t_1 **b** t_3 **c** t_5 **d** t_7

2 Use the rule to find the value of $V_4 = R^4 V_0$.

a $V_0 = 5, V_{n+1} = 3V_n$ **b** $V_0 = 10, V_{n+1} = 2V_n$
c $V_0 = 1, V_{n+1} = 0.5V_n$ **d** $V_0 = 200, V_{n+1} = 0.25V_n$

3 For the following recurrence relation:

$$V_0 = 5000, \qquad V_{n+1} = 1.05V_n \qquad \text{for } n = 0, 1, 2, 3, \ldots,$$

use the rule: $V_n = R^n \times V_0$, to find the following to two decimal places.

a V_6 **b** V_{10} **c** V_{100}

Developing understanding

Example 26

4 The following recurrence relation can be used to model a compound-interest investment: $V_0 = 10\,000, V_{n+1} = 1.1V_n$, where V_n is the value of the investment after n years.

a How much money was initially invested?
b What was the annual interest rate for this investment?
c Write down a rule for the value of the investment after n years.
d Use the rule to find V_5, giving your answer to the nearest dollar.
e What does your answer to part **d** tell you?

5 The following recurrence relation can be used to model a compound-interest investment: $V_0 = 12\,000, V_{n+1} = 1.08V_n$, where V_n is the value of the investment after n years.

a How much money was initially invested?
b What was the annual interest rate for this investment?
c Write down a rule for the value of the investment after n years.
d Use the rule to find the value of the investment after 4 years, giving your answer to the nearest dollar.

6 The following recurrence relation can be used to model reducing-balance depreciation of a car: $V_0 = 18\,500, V_{n+1} = 0.9V_n$, where V_n is the value of the car after n years.

a How much was the car initially worth?
b What was the annual interest percentage depreciation?
c Write down a rule for the value of the car after n years.
d Use the rule to find V_5, giving your answer to the nearest dollar.
e What does your answer to part **d** tell you?

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7 The following recurrence relation can be used to model reducing-balance depreciation of a boat: $V_0 = 9500, V_{n+1} = 0.95V_n$, where V_n is the value of the boat after n years.

a How much was the boat initially worth?

b What was the annual interest percentage depreciation?

c Write down a rule for the value of the boat after n years.

d Use the rule to find V_{10}, giving your answer to the nearest dollar.

e What does your answer to part **d** tell you?

8 The following recurrence relation can be used to model a loan with compound interest: $V_0 = 520\,000, V_{n+1} = 0.9965V_n$, where V_n is the value of the investment after n months.

a How much money was initially borrowed?

b What was the monthly interest rate for this loan?

c Write down a rule for the value of the loan after n months.

d Use the rule to find V_6, giving your answer to the nearest dollar.

e What does your answer to part **d** tell you?

Example 27

9 A principal value of \$10 000 is invested in an account earning compound interest at the rate of 4.5% per annum. The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.045^n \times 10\,000$$

a Find the value of the investment after 5 years to the nearest cent.

b Find the amount of interest earned after 5 years to the nearest cent.

c Find the amount of interest earned in the fifth year to the nearest cent.

d Let I_n be the value of the investment after n months, when the investment compounds monthly instead of yearly. Write down a rule for the value of the investment after n months.

e Use this rule to find the value of the investment after 5 years (60 months) to the nearest cent.

10 A principal value of \$300 000 is invested in an account earning compound interest at the rate of 9% per annum. The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.09^n \times 300\,000$$

a Find the value of the investment after 10 years to the nearest cent.

b Find the amount of interest earned after 10 years to the nearest cent.

c Find the amount of interest earned in the tenth year to the nearest cent.

d Let I_n be the value of the investment after n months, when the investment compounds monthly instead of yearly. Write down a rule for the value of the investment after n months.

e Use this rule to find the value of the investment after 10 years (120 months) to the nearest cent.

11 A commercial printing machine was purchased (new) for \$24 000. It depreciates in value at a rate of 9.5% per year, using a reducing-balance depreciation method. Let V_n be the value of the printer after n years.

a Write down a rule for the value of the printer after n years.

b Use the rule to find the value of the printer after 5 years to the nearest cent.

c What is the total depreciation of the printer over 5 years?

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Example 30 **12** An item that costs \$5000 is to be purchased, and the customer can choose between four different payment options including a simple-interest loan, a compound-interest loan, a credit card or a buy-now, pay-later scheme as listed below.

a Calculate the total cost of a simple-interest loan with interest of 6% per annum, paid at the end of three years. Give your answer to two decimal places.

b Calculate the total cost of a compound-interest loan with interest of 5.5% per annum, paid at the end of three years. Give your answer to two decimal places.

c Credit card: Use a credit card with a statement length of 30 days plus an additional 12 days, after which an interest rate of 20% per annum, compounding daily, is applied. The purchase is made on day 18 of the statement, and the full amount plus interest is repaid 800 days later.

d Calculate the total cost of a buy-now, pay-later option with an initial fee of \$50 and a monthly fee of \$8 over three years. Give your answer to two decimal places.

e Determine which option is cheapest for the customer.

Example 28 **13** Determine the amount of interest payable on the following credit card debts.

a \$2000 at an interest rate of 18.9% per annum for 52 days

b \$785 at an interest rate of 24% per annum for 200 days

c \$12 000 at an interest rate of 22.5% per annum for 60 days

d \$837 at an interest rate of 21.7% per annum for 90 days

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Example 29 **14** Matt has two credit cards, each with different borrowing terms.

- Credit card A charges 22% p.a. interest and offers up to 60 days interest free.
- Credit card B charges 19% p.a. but only offers 40 days interest free.

He wishes to buy an item costing \$2000 on his credit card which he will purchase at the beginning of the statement period, whichever card he uses, so as to have the maximum interest-free days. Which credit card should he use:

a if he is going to pay off the card 30 days after purchase?
b if he is going to pay off the card 60 days after purchase?
c if he is going to pay off the card 90 days after purchase?
d if he is going to pay off the card 240 days after purchase?

Example 31 **15** Suppose that inflation is recorded as 2.7% in 2017 and 3.5% in 2018, and that a magazine costs \$3.50 at the end of 2016. Assume that the price increases with inflation.

a What will be the price of the magazine at the end of 2017?
b What will be the price of the magazine at the end of 2018?

Example 32 **16** Suppose that the cost of petrol per litre is \$1.80 today.

a What will be the price of petrol per litre in 20 years' time if the average annual inflation rate is 1.9%?
b What will be the price of petrol per litre in 20 years' time if the average annual inflation rate is 7.1%?

Example 33 **17** If savings of \$200 000 are hidden in a mattress today, what is the purchasing power of that money in 10 years' time:

a if the average inflation rate over the 10 year period is 3%?
b if the average inflation rate over the 10 year period is 13%?

Testing understanding

18 Carina invested \$7500 at 6.25% per annum, compounding each year. If the investment now amounts to \$10 155.61, to the nearest cent, for how many years was it invested?

19 Tyson's grandparents invested \$5000 on his behalf when he was born. If it was invested at 4.75% per annum, compounding yearly, how old would Tyson be when the investment first exceeds \$20 000?

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Key ideas and chapter summary

Assign-ment

Sequence A **sequence** is a list of numbers or symbols in a particular order.

Term Each number or symbol that makes up a sequence is called a **term**.

Arithmetic sequence In an **arithmetic sequence**, each new term is made by adding or subtracting a fixed number, called the **common difference**, D, to or from the previous term.

Recurrence relation for an arithmetic sequence A **recurrence relation for an arithmetic sequence** has the form

$$t_0 = a, \quad t_{n+1} = t_n + D$$

where D = common difference, and a = first term.
The rule for finding t_n, the nth term in an arithmetic sequence, is:

$$t_n = a + nD$$

Graph of arithmetic sequence For an arithmetic sequence graph:

- the values lie along a straight line
- there are increasing values when $D > 0$ (positive slope)
- there are decreasing values when $D < 0$ (negative slope).

Linear growth An arithmetic sequence can be used to model **linear growth** where the value is increasing. That is, $V_0 =$ starting value, $V_{n+1} = V_n + D$, where V_n is the value after n years, and D is the amount that is added each time period. Thus, $V_n = V_0 + n \times D$.

Linear decay An arithmetic sequence can be used to model **linear decay** where the value is decreasing. That is, $V_0 =$ starting value, $V_{n+1} = V_n - D$, where V_n is the value after n years, and D is the amount that is subtracted each time period. Thus, $V_n = V_0 - n \times D$.

Principal The **principal** is the initial amount that is invested or borrowed.

Balance The **balance** is the value of a loan or investment at any time during the loan or investment period.

Interest The fee that is added to a loan or investment for borrowing or investing is called the **interest**.

Simple interest **Simple interest** is an example of linear growth where a fixed amount of interest is earned each period and found by calculating $\frac{r}{100} \times V_0$.

Depreciation **Depreciation** occurs when the value of an asset decreases over a period of time.

Flat rate depreciation **Flat rate depreciation** is an example of linear decay where a constant amount is subtracted from the value of an item at regular intervals of time.

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Unit-cost depreciation

Unit-cost depreciation is an example of linear decay that is calculated based on units of use rather than by time. The value of the asset declines by a constant amount for each unit of use (e.g. per 100 kilometres).

Geometric sequence

In a **geometric sequence**, each term is made by multiplying the previous term by a fixed number, called the common ratio, R.
For example: 5, 20, 80, 320, . . . is made by multiplying each term by 4.
The common ratio, R, is found by dividing any term by its previous term, e.g. $\frac{t_2}{t_1}$.
In our example: $R = \frac{20}{5} = 4$

Recurrence relation for a geometric sequence

A **recurrence relation for a geometric sequence** has the form

$$t_0 = a, \qquad t_{n+1} = R \times t_n$$

where R = common ratio, and a = first term.
The rule for finding t_n, the nth term, in a geometric sequence is:

$$t_n = R^n \times t_0$$

Graph of geometric sequence

For a graph of a geometric sequence:

- the values are increasing values when $R > 1$
- the values decrease towards zero when $0 < R < 1$.

Geometric growth and decay

A geometric sequence can be used to model **geometric growth** (when $R > 1$) or **geometric decay** (when $0 < R < 1$).

Reducing-balance depreciation

When the value of an item decreases as a percentage of its value after each time period, it is said to be depreciating using a reducing-balance method. **Reducing-balance depreciation** is an example of geometric decay.

Compounding period

Interest rates are usually quoted as annual rates (per annum). Interest is sometimes calculated more regularly than once a year, for example, each quarter, month, fortnight, week or day. The time period for the calculation is called the **compounding period**.

Credit

Credit is an advance of money from a financial institution, such as a bank, that does not have to be paid back immediately but which attracts interest after an interest-free period.

Inflation

Inflation is the continuous upward movement of the economy that increases prices over time or, conversely, decreases the spending power of money over time.

Purchasing power

Purchasing power describes what you can actually buy with your money.

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Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

3A **1 I can identify the behaviour of a sequence.** ☐

e.g. Consider the following sequence and identify its behaviour as increasing, decreasing, constant, oscillating or having a limiting value: 100, 90, 80, 70, ...

3A **2 I can determine a simple rule for a sequence of numbers.** ☐

e.g. Look for a pattern or rule in the following sequence, and hence find the next number: 3, 7, 11, 15, 19, ...

3A **3 I can generate a sequence from a starting number and a simple rule.** ☐

e.g. Write down the first five terms of the sequence with a starting value of 3 and the rule 'add 2 to each term'.

3B **4 I can number and name terms in a sequence.** ☐

e.g. For the sequence: 4, 8, 12, 16, . . ., state the values of t_0 and t_3.

3B **5 I can generate a sequence from a recurrence relation.** ☐

e.g. Write down the first five terms of the sequence defined by the recurrence relation: $t_0 = 12$ and $t_{n+1} = t_n + 7$, showing the values of the first four iterations.

3C **6 I can find the common difference in an arithmetic sequence.** ☐

e.g. Find the common difference for the arithmetic sequence: 22, 25, 28, ..., and use it to find the next term in the sequence.

3C **7 I can identify an arithmetic sequence.** ☐

e.g. Determine whether the following sequence is an arithmetic sequence: 1, 3, 9, 27, ...

3C **8 I can tabulate and graph an arithmetic sequence.** ☐

e.g. The sequence, 5, 8, 11, ... is an arithmetic sequence with a common difference of 3. Construct a table showing the term number, n, and its value, t_n, for the first four terms in the sequence. Use the table to plot the graph.

3D **9 I can generate an arithmetic sequence using a recurrence relation.** ☐

e.g. Generate and graph the first four terms of the sequence defined by the recurrence relation: $t_0 = 7, t_{n+1} = t_n + 3$.

3D **10 I can find the *n*th term to solve problems involving arithmetic sequences.** ☐

e.g. Consider the recurrence relation: $t_0 = 40$, $t_{n+1} = t_n - 3$. Find t_{20}.

3E **11 I can find the amount of simple interest each year.** ☐

e.g. An investment of $2200 pays interest at the rate of 5% per annum in the form of simple interest. Find the amount of interest paid each year.

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Review

3E **12** I can use a recurrence relation to model simple interest. ☐

e.g. The following recurrence relation can be used to model a simple-interest investment of $1600, paying interest at a rate of 5% per annum:

$V_0 = 1600, \qquad V_{n+1} = V_n + 80.$

Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

3E **13** I can use a recurrence relation to model flat rate depreciation. ☐

e.g. The following recurrence relation can be used to model a simple flat rate depreciation of a jet ski, purchased for $18 000, depreciating at a flat rate of 15% per year.

$V_0 = 18\,000, \qquad V_{n+1} = V_n - 2700.$

Use the recurrence relation to find the value of the jet ski after 1, 2 and 3 years.

3E **14** I can use a recurrence relation to model unit cost depreciation. ☐

e.g. The following recurrence relation can be used to model the unit cost depreciation of a ute, purchased for $22 000, depreciating at a unit cost of $200 per 10 000 km:

$V_0 = 22\,000, \qquad V_{n+1} = V_n - 200.$

Use the recurrence relation to find the value of the ute after 10 000, 20 000 and 30 000 kilometres.

3E **15** I can use a rule to determine the *n*th term for linear growth or decay. ☐

e.g. The following recurrence relation can be used to model a simple-interest investment of $300 000, paying interest at the rate of 3.5% per year:

$V_0 = 300\,000, \qquad V_{n+1} = V_n + 10\,500.$

Determine how much interest is added to the investment each year, and then use the rule to find the value of the investment after 12 years.

3F **16** I can find the common ratio in a geometric sequence. ☐

e.g. For the sequence: 4, 8, 16, 32, . . ., state the common ratio.

3F **17** I can identify whether a sequence is a geometric sequence. ☐

e.g. Determine whether the following sequence is a geometric sequence: 1, 3, 9, 27, ...

3F **18** I can use a CAS calculator to generate a geometric sequence. ☐

e.g. Use your CAS calculator to generate the first 5 terms of a sequence that starts with 4 and has a common ratio of 2.

3F **19** I can graph an increasing or decreasing geometric sequence. ☐

e.g. Consider the geometric sequence: 2, 10, 50, 250, ... Find the next term and construct a table for the first 5 terms. Use the table to plot the graph of the sequence.

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3G **20** I can generate a geometric sequence using a recurrence relation. ☐

e.g. Write down the first five terms in the geometric sequence generated by the following recurrence relation: $t_0 = 8, \quad t_{n+1} = 1.5 \times t_n$.

3G **21** I can find the nth term in a geometric sequence using a recurrence relation. ☐

e.g. Find the fifth term in the geometric sequence generated by the following recurrence relation: $t_0 = 10, \quad t_{n+1} = 2t_n$.

3H **22** I can use a recurrence relation to model compound interest for investments or loans. ☐

e.g. Sam invests \$5000, paying interest at a rate of 6% per annum, compounding annually. Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

3H **23** I can write a recurrence relation to model a loan that compounds with a different compounding period. ☐

e.g. Jack borrows \$2000. He will pay interest at the rate of 4.2% per annum. Write down a recurrence relation to model the value of Jack's loan if interest is compounded quarterly.

3H **24** I can use a recurrence relation to model reducing-balance depreciation. ☐

e.g. Tyson purchases a car for \$27 500. The following recurrence relation can be used to model the car's value as it depreciates by 10% of its value each year:

$$V_0 = 27\,500, \quad V_{n+1} = 0.9V_n.$$

Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

3I **25** I can use the rule for determining the value of an investment with compounding interest or an asset with reducing-balance depreciation after a certain length of time. ☐

e.g. Adrian invests \$250 000 in an account earning compound interest at the rate of 6% per annum. Use a rule to find the value of the investment after 15 years.

3I **26** I can use a rule to analyse an investment. ☐

e.g. Rohan invests \$100 000 in an account earning compound interest at the rate of 3.5% per annum. Find the amount of interest earned in the fifth year of the investment.

3I **27** I can find the interest paid on a credit card. ☐

e.g. Determine how much interest is payable on a credit card debt of \$4780 at an interest rate of 15.3% per annum for 24 days. Give your answer to the nearest cent.

3I **28** I can calculate credit card interest with an interest-free period. ☐

e.g. Jane pays \$2700 for a holiday using her credit card. Her bank offers a 30-day statement period plus a further 15 days interest-free period. After that, the bank charges 18% per annum, compounding daily. If she pays for the holiday on day 12 of her statement period, how much does she pay back if she pays 84 days later?

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Review

3I **29** I can determine the best purchase option. ☐

e.g. A purchase of $2000 is made. Calculate the cost of a simple-interest loan with an annual interest rate of 8%, paid back after 2 years, and the cost of a compound-interest loan with annual interest rate of 6.5% per annum, compounding annually, and repaid after 2 years. Which is cheapest?

3I **30** I can determine the effect of inflation on prices over a short period of time. ☐

e.g. Suppose inflation is recorded as 2.2% in 2020 and 3.1% in 2021, and that a certain chocolate bar cost $2.40 at the start of 2020. If the price of the chocolate bar increases with inflation, what will be the price at the end of 2021?

3I **31** I can determine the effects of inflation on prices over a long period of time. ☐

e.g. Suppose that one litre of milk costs $1.90 today. What will be the price in 20 years' time if the average annual inflation is 3.8%.

3I **32** I can find the purchasing power of money. ☐

e.g. What is the purchasing power of $80 000 in 10 years' time if the average inflation rate over this period is 3.2%?

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Multiple-choice questions

1 In the sequence: 1, 4, 7, 10, 13, . . . the next term is

A 1 **B** 4 **C** 5 **D** 13 **E** 16

2 Which of the following is an arithmetic sequence?

A 2, 4, 8, . . . **B** 2, 6, 18, . . . **C** 2, 4, 6, . . . **D** 2, 3, 5, . . . **E** 2, 4, 7, . . .

3 In the sequence: 27, 19, 11, 3, . . . the value of the common difference, D, is

A −24 **B** −8 **C** −5 **D** 3 **E** 8

4 In the sequence given by the recurrence relation: $t_0 = 63$, $t_{n+1} = t_n - 7$; what is the value of t_{15}?

A −42 **B** −35 **C** 15 **D** 161 **E** 168

5 A new boat is purchased for $110 000 with a loan. The loan is to be repaid with simple interest of 4.6% per annum. The total amount to be repaid after three years is

A $5060 **B** $15 180 **C** $110 000 **D** $120 120 **E** $125 180

6 Which of the following is a geometric sequence?

A 2, 4, 8, . . . **B** 2, 6, 10, . . . **C** 2, 4, 6, . . . **D** 2, 3, 5, . . . **E** 2, 4, 7, . . .

7 In the sequence: 27, 9, 3, 1, . . . the value of the common ratio, R, is

A −18 **B** −3 **C** 3 **D** $\frac{1}{3}$ **E** 18

8 In the sequence given by the recurrence relation: $t_0 = 10\,000$, $t_{n+1} = 0.2t_n$; what is the value of t_8?

A 0.0256 **B** 0.128 **C** 2000 **D** 20 000 **E** 2 560 000

9 The following recurrence relation can be used to model a compound-interest investment of $12 000 at the rate of 6% per annum. $V_0 = 12\,000$, $V_{n+1} = 1.06 \times V_n$. The value of the investment after 3 years, to the nearest cent, is

A $2.59 **B** $2160 **C** $2292.19 **D** $14 292.19 **E** $14 292.20

10 A new car is purchased for $135 000 with a loan. The loan is to be repaid with compound interest of 12.5% per annum. The total interest paid in the first three years, to the nearest cent, is

A $16 875.00 **B** $21 357.42 **C** $59 625.00 **D** $57 216.80 **E** $192 216.80

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Short-answer questions

1 Describe how terms are generated in each number sequence and give the next two terms.

a 2, 5, 8, 11, . . . **b** 47, 56, 65, 74, . . . **c** 16, 60, 16, 60, . . .
d 2, 6, 18, 54, . . . **e** 1000, 500, 250, 125, . . .

2 Given that the initial term in the sequence is t_0, for each sequence, state the value of the named terms

i t_0 **ii** t_3 **iii** t_5

a 12, 18, 24, 30, . . . **b** 20, 18, 16, 14, . . .
c 2, 10, 50, 250, . . .

3 Find t_{20} in the sequence: 7, 11, 15, 19, ...

4 The first term in an arithmetic sequence is 32 and the common difference is 11. Find t_{100}.

5 Generate and graph the first four terms in the sequence defined by

$$V_0 = 23, \qquad V_{n+1} = V_n + 7$$

6 The following recurrence relation can be used to model a simple-interest investment of \$20 000, paying interest at the rate of 4% per year.

$$V_0 = 20\,000, \qquad V_{n+1} = V_n + 800$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.
b When will the investment reach more than \$30 000?

7 Let V_n be the value (in dollars) of a printer after n years.
A recurrence relation that models the depreciating value of this printer over time is:

$$V_0 = 2000, \qquad V_{n+1} = V_n - 250$$

a What was the value of the printer when it was new?
b By how much (in dollars) did the printer depreciate each year?
c What was the percentage flat rate of depreciation?
d After how many years will the value of the printer be \$600?
e When will the printer devalue to half of its new price?

8 Generate and graph the first five terms of the geometric sequence defined by the recurrence relation

$$t_0 = 3, \qquad t_{n+1} = 2t_n.$$

9 Consider the recurrence relation given by

$$V_0 = 1000, \qquad V_{n+1} = 0.1V_n.$$

a Generate the first five terms of the geometric sequence defined by the recurrence relation.
b Graph the first five terms.
c Calculate the value of V_{10}.

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10 The first term in a geometric sequence is 8 and there is a common ratio of 2. Let t_n be the term after n applications of the rule. Find t_{10}.

11 The following recurrence relation can be used to model a compound-interest investment of \$20 000, paying interest at the rate of 4% per year.

$$V_0 = 20\,000, \qquad V_{n+1} = 1.04V_n$$

where V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach more than \$30 000?

12 Let V_n be the value (in dollars) of a work-station after n years.
A recurrence relation that models the depreciating value of this work-station over time is:

$$V_0 = 2000, \qquad V_{n+1} = 0.9V_n$$

a What was the value of the work-station when it was new?

b By how much (in dollars) did the work-station depreciate in the first year?

c What was the percentage flat rate depreciation?

d After how many years will the value of the work-station first be less than \$600?

e When will the work-station first devalue to less than half of its new price?

13 A given geometric sequence has a common ratio of 4. If $V_5 = 2048$, find the recurrence relation for the sequence.

Written-response questions

1 Chao wishes to buy a motor scooter which costs \$4100 when new. Its value depreciates at a flat rate of 8% per year.

a How much does the value of the motor scooter decline each year?

b How much is the motor scooter worth after one year?

c How much is the motor scooter worth after two years?

d Complete the following equations, where V_t is the value of the motor scooter at time t:
$V_1 = V_0 - \ldots$ and $V_2 = \ldots - \ldots$.

e Write the recurrence relation that models the depreciation of the motor scooter over time.

f After how many years will the value of the scooter first be less than \$1000?

2 Consider the recurrence relation:

$$t_0 = 5, \qquad t_{n+1} = t_n + 4.$$

a Construct a table showing the term number, n, and its value, t_n, for the first four terms.

b Use the table to plot the graph.

c Describe the graph and hence the sequence.

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3 Let V_n be the value (in dollars) of a delivery van after n years.
A recurrence relation that models the flat rate depreciating value of the van over time is:

$$V_0 = 64\ 000, \qquad V_{n+1} = V_n - 3200$$

a What was the value of the van when it was new?

b By how much (in dollars) did the van depreciate each year?

c What was the percentage flat rate depreciation that was applied to the van?

d After how many years will the value of the van first be less than \$40 000?

e When will the van devalue to half its original price?

4 A machine is purchased for \$30 000 and depreciates by \$0.15 for each unit that it produces.
Let V_n be the value (in dollars) of the machine after n units are produced.
A recurrence relation that models the unit cost depreciating value of the machine is:

$$V_0 = 30\ 000, \qquad V_{n+1} = V_n - 0.15$$

a How much is the machine worth after it produces 1000, 2000 and 3000 units?

b If the machine produces 20 000 units each year, find the value of the machine after 3 years.

c How many units will the machine need to produce before it is devalued to half of its original price?

5 The following recurrence relation can be used to model a compound-interest investment.

$$V_0 = 6200, \qquad V_{n+1} = 1.08V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a How much was initially invested?

b What was the interest rate for the investment?

c What was the value of the investment after 1, 2 and 3 years, to the nearest cent?

d How much interest did the investment earn in each of the first three years? Give your answer to the nearest cent.

6 Hugh bought a motorbike for \$9200 brand new. The bike depreciates at a reducing-balance rate of 15% per annum. Let V_n be the value of Hugh's bike after n years.

a How much does the value of the motorbike decline in the first year?

b Write down a recurrence relation to model the value of Hugh's motorbike after n years.

c Correct to the nearest dollar, how much is the motorbike worth after five years?

d After how many years will the value of the motorbike first be less than \$5000?

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7 Renee has $20 000 to invest for five years and must choose which investment offers the best deal. The bank offers the following investment options:

A Renee can invest her money and receive 5.5% simple interest per annum.

B Renee can invest her money and receive 5% interest per annum, compounding annually.

C Renee can invest her money and receive 4.5% interest per annum, compounding monthly.

Considering these options:

a How much would each option pay in interest for the first year, to the nearest dollar?

b Which option gives Renee the highest value at the end of five years?

c How much interest, in total, will Renee earn from this option? Give your answer to the nearest dollar.

8 The following recurrence relation can be used to model reducing-balance depreciation of a car:

$$V_0 = 22\ 500, \qquad V_{n+1} = 0.85V_n$$

where V_n is the value of the car after n years.

a State the initial value of the car.

b What was the annual depreciation rate of the car?

c Write down a rule for the value of the car after n years.

d Use the rule to find V_5, giving your answer to the nearest dollar.

e After how many years will the value of the car be less than half of its starting value?

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Chapter 4

Matrices

Chapter questions

- What is a matrix?
- How is the order of a matrix defined?
- How are the positions of the elements of a matrix specified?
- What are the rules for adding and subtracting matrices?
- How do we multiply a matrix by a scalar?
- What is the method for multiplying a matrix by another matrix?
- What are transition matrices and how do we apply them?
- What are inverse matrices and how can they be used to solve equations?
- How can your CAS calculator be used to do matrix operations?

A **matrix** is a rectangular group of numbers, set out in rows and columns. A matrix (plural matrices) can be used to store information, solve sets of simultaneous equations or be used in various applications. We will explore some of these applications while learning the basic theory of matrices.

4A The basics of a matrix

Learning intentions

- To be able to state the order of a given matrix.
- To be able to describe the location of an element in a matrix.

Matrices can be used to store and display information, for example, the sales of a market stall that operates on Friday and Saturday are recorded in matrix A.

$$A = \begin{array}{c} \\ \textit{Friday} \\ \textit{Saturday} \\ \\ \end{array} \begin{array}{c} \textit{Shirts} \quad \textit{Jeans} \quad \textit{Belts} \\ \begin{bmatrix} 6 & 8 & 4 \\ 3 & 7 & 1 \end{bmatrix} \\ \textit{column 1} \quad \textit{column 2} \quad \textit{column 3} \end{array} \begin{array}{l} \\ \textit{row 1} \\ \textit{row 2} \\ \\ \end{array}$$

Rows	Columns
Friday's sales are listed in row 1. Saturday's sales are listed in row 2.	The number of shirts sold is listed in column 1. The number of pairs of jeans sold is listed in column 2. The number of belts sold is listed in column 3.

We can read the following information from the matrix:

- on Friday, 8 pairs of jeans were sold
- on Saturday, 1 belt was sold
- the total number of items sold on Friday was $6 + 8 + 4 = 18$
- the total number of belts sold was $4 + 1 = 5$.

Order of a matrix

The **order** (or size) **of a matrix** is written as: number of rows × number of columns.

The number of rows is always given first, then the number of columns. For example, the order of matrix A in the market stall example above is 2×3; that is, 2 rows × 3 columns. It is called a 'two by three' matrix.

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Elements of a matrix

The numbers within a matrix are called its **elements**.

Matrices are usually named using capital letters, such as A, B, C. The corresponding lowercase letter with subscripts is used to denote an element of a matrix.

An element of a matrix

a_{ij} is the element in *row i*, *column j* of matrix A.

For example, in the matrix:

$$A = \begin{bmatrix} 6 & 8 & 4 \\ 3 & 7 & 1 \end{bmatrix}$$

- element a_{13} is in row 1, column 3, and its value is 4
- element a_{22} is in row 2, column 2, and its value is 7.

Example 1 Interpreting the elements of a matrix

Matrix B shows the number of boys and girls in Years 10 to 12 at a particular school.

$$B = \begin{matrix} \\ \text{Year 10} \\ \text{Year 11} \\ \text{Year 12} \end{matrix} \begin{matrix} \text{Boys} & \text{Girls} \\ \left[\begin{matrix} 57 \\ 48 \\ 39 \end{matrix}\right. & \left.\begin{matrix} 63 \\ 54 \\ 45 \end{matrix}\right] \end{matrix}$$

a Give the order of matrix B.

b What information is given by the element b_{12}?

c Which element gives the number of girls in Year 12?

d How many boys are there in total?

e How many students are in Year 11?

Explanation	Solution
a Count the rows, count the columns.	The order of matrix B is 3×2.
b The element b_{12} is in row 1 and column 2. This is where the Year 10 row meets the Girls column.	There are 63 girls in Year 10.
c Year 12 is row 3. Girls are column 2.	b_{32} gives the number of Year 12 girls.
d The sum of the Boys column gives the total number of boys.	The total number of boys is 144.
e The sum of the Year 11 row gives the total number of students in Year 11.	There are 102 students in Year 11.

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Now try this 1 **Interpreting the elements of a matrix (Example 1)**

Matrix C shows the number of men and women working in three different departments of an organisation.

$$C = \begin{matrix} & \textit{Men} & \textit{Women} \\ \textit{Finance} & 21 & 32 \\ \textit{Legal} & 11 & 18 \\ \textit{Sales} & 62 & 41 \end{matrix}$$

a Give the order of matrix C.

b What information is given by the element c_{21}?

c Which element gives the number of men working in Sales?

d How many women work in the three departments in total?

Hint 1 Determine the number of rows and columns to find the order.

Hint 2 Remember that rows come before columns.

Hint 3 To find the totals, add up all the relevant numbers from a row or a column.

Row matrices

A **row matrix** has a *single row* of elements.

In matrix A, the Friday sales from the market stall can be represented as a 1×3 *row* matrix.

$$\begin{matrix} & \textit{Shirts} & \textit{Jeans} & \textit{Belts} \\ \textit{Friday} & 6 & 8 & 4 \end{matrix}$$

Column matrices

A **column matrix** has a *single column* of elements.

In matrix A, the sales of jeans from the market stall can be represented as a 2×1 *column* matrix.

$$\begin{matrix} & \textit{Jeans} \\ \textit{Friday} & 8 \\ \textit{Saturday} & 7 \end{matrix}$$

Square matrices

In **square matrices**, the number of *rows* equals the number of *columns*.

Here are three examples.

$$[9] \quad \begin{bmatrix} 5 & 4 \\ 6 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 4 & 3 \\ 4 & 1 & 6 \\ 3 & 6 & 7 \end{bmatrix}$$

1×1 $\quad 2 \times 2$ $\quad 3 \times 3$

A square matrix is **symmetric** if all elements in the matrix can be swapped, as follows: $a_{ij} = a_{ji}$. The third square matrix above is an example of a symmetric matrix.

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How to enter a matrix using the TI-Nspire CAS

Enter the matrix $B = \begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$ into the TI-Nspire CAS. Display the element $b_{2\,1}$.

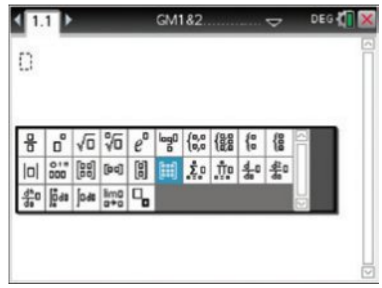
Steps

1 Press [⌂ on]>**New Document>Add Calculator**.

2 Press [template key] and use the cursor ▼ ▶ arrows to highlight the matrix template shown. Press [enter].

Note: Math Templates can also be accessed by pressing [ctrl] + [menu] **>Math Templates**.

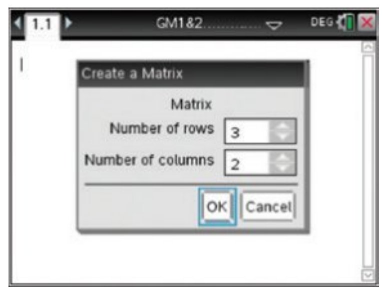
Note: You can also press [menu]> [7]> [1]> [1]



3 Press ◀ then ▲ or ▼ to select the **Number of rows** required (the number of rows in this example is 3).

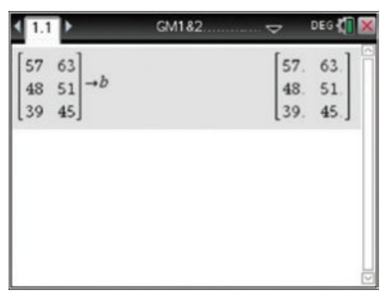
Press [tab] to move to the next entry and repeat for the **Number of columns** (the number of columns in this example is 2).

Press [tab] to highlight **OK** and press [enter].



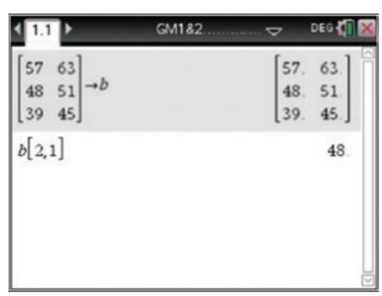
4 Type the values into the matrix template. Use [tab] or the arrow keys to move to the required position in the matrix to enter each value.

When the matrix has been completed, press [tab] or ▶ to move outside the matrix, and press [ctrl]+[var] followed by [B]. This will store the matrix as the variable **b**. Press [enter].



5 When you type B (or b) in the CAS calculator, it will paste in the matrix $\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$.

6 To display element $b_{2\,1}$ (the element in position Row 2, Column 1), type in **b[2,1]** and press [enter].



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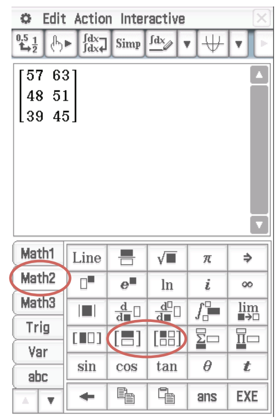
How to enter a matrix using the ClassPad

Enter the matrix $B = \begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$ into the ClassPad calculator.

Steps

1 Open the soft Keyboard in the **Main** application Main $\sqrt{\alpha}$.



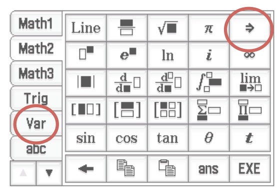
2 Select the Math2 keyboard.

3 Tap the 2×2 matrix icon followed by the 2×1 matrix icon. This will add a third row and create a 3×2 matrix.

4 Enter the values of $\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$.

Note: Tap in each new position to enter the new value or use the cursor key on the hard keyboard to navigate to a new position.

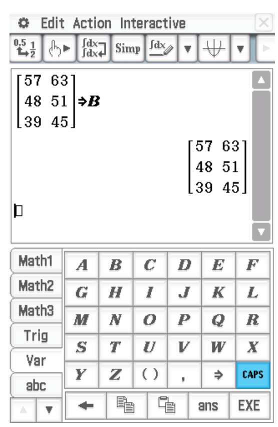
5 To assign the matrix the variable name B:



a Move the cursor to the very right-hand side of the matrix.

b From the keyboard, tap the variable assignment key ⇒, followed by the var, then CAPS (for uppercase letters) and ***B***. Press EXE to confirm your choice.

Note: Until it is reassigned, ***B*** will represent the matrix as defined above.



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Section Summary

- A **matrix** is a rectangular array of numbers, set out in rows and columns within square brackets. The rows are horizontal; the columns are vertical.
- The **order** (size) of a matrix is the number of rows × the number of columns.
- The element a_{ij} is the number in row i and column j of the matrix.
- A **row** matrix has a single row of elements.
- A **column** matrix has a single column of elements.
- A **square** matrix has an equal number of rows and columns.

Exercise 4A

Building understanding

1 Consider the following matrix:

$$A = \begin{bmatrix} 3 & 4 & 12 & 1 \\ 0 & 7 & 9 & 10 \end{bmatrix}$$

a State the number of rows in matrix A.

b State the number of columns in matrix A.

c State the order of matrix A.

d State the number of elements in matrix A.

2 Students in Year 11 were surveyed about whether they played a musical instrument or sport. The results were recorded in matrix S.

$$S = \begin{matrix} \\ \textit{Girls} \\ \textit{Boys} \end{matrix} \begin{matrix} \textit{Music} & \textit{Sport} \\ \left[\begin{matrix} 38 \\ 45 \end{matrix}\right. & \left.\begin{matrix} 53 \\ 36 \end{matrix}\right] \end{matrix}$$

a How many girls play sport?

b How many boys play sport?

c How many students from Year 11 play a musical instrument?

Developing understanding

Example 1

3 Matrix C is shown on the right.

$$C = \begin{bmatrix} 2 & 4 & 16 & 7 \\ 6 & 8 & 9 & 3 \\ 5 & 6 & 10 & 1 \end{bmatrix}$$

a Write down the order of matrix C.

b State the value of:

i c_{13} **ii** c_{24} **iii** c_{31}

c Write down the sum of the elements in row 3.

d Write down the sum of the elements in column 2.

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4 For each of the following matrices:

i state the order **ii** write down the values of the required elements.

a $A = \begin{bmatrix} 5 & 6 & 8 \\ 4 & 7 & 9 \end{bmatrix}$ Find a_{12} and a_{22}

b $B = \begin{bmatrix} 6 & 8 & 2 \end{bmatrix}$ Find b_{13} and b_{11}

c $C = \begin{bmatrix} 4 & 5 \\ 3 & 1 \\ 8 & -4 \end{bmatrix}$ Find c_{32} and c_{12}

d $D = \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix}$ Find d_{31} and d_{11}

e $E = \begin{bmatrix} 10 & 12 \\ 15 & 13 \end{bmatrix}$ Find e_{21} and e_{12}

f $F = \begin{bmatrix} 8 & 11 & 2 & 6 \\ 4 & 1 & 5 & 7 \\ 6 & 14 & 17 & 20 \end{bmatrix}$ Find f_{34} and f_{23}

5 Name which of the matrices in Question **4** are:

a row matrices **b** column matrices **c** square matrices.

6 Which three of these matrices is symmetric?

A $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

7 For matrix D, state the values of the following elements.

$$D = \begin{bmatrix} 3 & 4 & 6 & 11 & 2 \\ 5 & 1 & 9 & 10 & 4 \\ 8 & 7 & 2 & 0 & 1 \\ 6 & 8 & 5 & 8 & 2 \end{bmatrix}$$

a d_{23} **b** d_{45}

c d_{11} **d** d_{24}

e d_{42}

8 Consider the following matrices.

$$A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 5 & 3 \\ -3 & 4 & 8 \\ 7 & 6 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ -5 \end{bmatrix} \quad C = \begin{bmatrix} 8 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 4 & -3 & 0 & 1 & 9 \\ 6 & 11 & 2 & 7 & 5 \end{bmatrix}$$

a Write down the order of each matrix A, B, C and D.

b Identify the elements: a_{32}, b_{21}, c_{11} and d_{24} of matrices A, B, C and D respectively.

9 Some students were asked which of four sports they preferred to play, and the results were entered in the following matrix.

$$S = \begin{array}{c} \\ \textit{Year 10} \\ \textit{Year 11} \\ \textit{Year 12} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Tennis} & \textit{Basketball} & \textit{Football} & \textit{Hockey} \end{array} \\ \begin{bmatrix} 19 & 18 & 31 & 14 \\ 16 & 32 & 22 & 12 \\ 21 & 25 & 5 & 7 \end{bmatrix} \end{array}$$

a How many Year 11 students preferred basketball?

b Write down the order of matrix S.

c What information is given by s_{23}?

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10 Matrix F shows the number of hectares of land used for different purposes on two farms, X and Y. Row 1 represents Farm X and row 2 represents Farm Y. Columns 1, 2 and 3 show the amount of land used for wheat, cattle and sheep (W, C, S) respectively, in hectares.

$$F = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} W \quad C \quad S \\ \begin{bmatrix} 150 & 300 & 75 \\ 200 & 0 & 350 \end{bmatrix} \end{array} \begin{array}{c} \\ X \\ Y \end{array}$$

a How many hectares are used on:

i Farm X for sheep? **ii** Farm X for cattle?

iii Farm Y for wheat?

b Calculate the total number of hectares used on both farms for wheat.

c Write down the information that is given by:

i f_{22} **ii** f_{13} **iii** f_{11}

d Which element of matrix F gives the number of hectares used:

i on Farm Y for sheep? **ii** on Farm X for cattle?

iii on Farm Y for wheat?

e State the order of matrix F.

Testing understanding

11 Given the information provided, construct a matrix, B, to show the number of pies and sausage rolls sold by two bakeries last Saturday.

- Row 1 represents Bakery 1 and row 2 represents Bakery 2.
- Column 1 shows the number of pies, and column 2 shows the number of sausage rolls.
- Bakery 2 sold 165 pies and 181 sausage rolls.
- Bakery 1 sold 30 more pies than Bakery 2.
- Bakery 1 sold 40 fewer sausage rolls than Bakery 2.

12 A group of Year 11 students were surveyed about their preferred activity on a school trip. The results were entered into the following matrix, but unfortunately, some of the data was missing.

$$S = \begin{array}{r} \\ Boys \\ Girls \end{array} \begin{array}{c} \begin{array}{cccc} Ski & Mountain\ bike & Hike & Kayak \end{array} \\ \begin{bmatrix} 28 & \dots & 29 & \dots \\ \dots & 83 & \dots & 31 \end{bmatrix} \end{array}$$

Use the following information to complete the matrix.

- A total of 137 students wanted to go mountain bike riding.
- s_{23} is half of s_{11}.
- The total number of boys surveyed was 146.
- The number of girls who wanted to go skiing was 5 more than the number of girls who wanted to go kayaking.

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4B Adding and subtracting matrices

Learning intentions

- To be able to add and subtract matrices.
- To be able to identify a zero matrix.

Matrices of the same order can be added or subtracted by following a simple rule.

Rules for adding and subtracting matrices

1. Matrices are added by adding the elements that are in the same positions.
2. Matrices are subtracted by subtracting the elements that are in the same positions.
3. **Matrix addition and subtraction** can only be done if the two matrices have the *same order*.

Example 2 Addition and subtraction of matrices

Complete the following addition and subtraction of matrices.

a $\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 9 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 7 & 3 \\ 2 & 8 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -1 & 9 \\ 3 & 7 \end{bmatrix}$

Explanation	Solution
a 1 Write the addition.	$\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 9 & -1 \end{bmatrix}$
2 Add the elements that are in the same positions.	$= \begin{bmatrix} 2+9 & 4+8 \\ 5+9 & 1+(-1) \end{bmatrix}$
3 Evaluate each element.	$= \begin{bmatrix} 11 & 12 \\ 14 & 0 \end{bmatrix}$
b 1 Write the subtraction.	$\begin{bmatrix} 7 & 3 \\ 2 & 8 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -1 & 9 \\ 3 & 7 \end{bmatrix}$
2 Subtract the elements that are in the same positions, and evaluate each element.	$= \begin{bmatrix} 7-4 & 3-2 \\ 2-(-1) & 8-9 \\ 1-3 & 0-7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & -1 \\ -2 & -7 \end{bmatrix}$

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Now try this 2 Addition and subtraction of matrices (Example 2)

Complete the following addition and subtraction of matrices.

a $\begin{bmatrix} 3 & -2 & 6 \\ 8 & 4 & 12 \end{bmatrix} + \begin{bmatrix} 11 & 5 & 7 \\ 1 & -3 & -5 \end{bmatrix}$ **b** $\begin{bmatrix} 10 & -1 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ -3 & 10 \end{bmatrix}$

Hint 1 Write the addition or subtraction.

Hint 2 Add (or subtract) the elements that are in the same position.

Hint 3 Evaluate each element.

The zero matrix, 0

In a **zero matrix**, every element is zero. The zero matrix is sometimes called the null matrix.

The following are examples of zero matrices.

$$\begin{bmatrix} 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Just as in standard arithmetic, adding or subtracting a zero matrix does not make any change to the original matrix. For example:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Subtracting any matrix from itself gives a zero matrix. For example:

$$\begin{bmatrix} 9 & 4 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Section Summary

- Matrices of the same order can be **added** by adding elements in the same position.
- Matrices of the same order can be **subtracted** by subtracting the elements in the same position.
- A **zero matrix** is any matrix where every element is zero.

Exercise 4B

Building understanding

1 Which of the following matrices can be added together?

$$A = \begin{bmatrix} 9 & -5 \\ -3 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 14 \\ -3 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 6 & 4 & -2 \\ 8 & 1 & 9 \end{bmatrix}$$

$$E = \begin{bmatrix} 5 & 6 \\ 12 & 2 \\ 7 & -2 \end{bmatrix} \quad F = \begin{bmatrix} 5 & 13 & -7 \\ 8 & -2 & 6 \end{bmatrix} \quad G = \begin{bmatrix} -4 & 8 \\ 10 & 2 \end{bmatrix} \quad H = \begin{bmatrix} 9 & 6 \\ -8 & 3 \\ 2 & 1 \end{bmatrix}$$

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2 True or false?

$$\begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 5 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ -1 & 6 \end{bmatrix}$$

3 Calculate:

$$\begin{bmatrix} -3 & 3 \\ 9 & 12 \end{bmatrix} - \begin{bmatrix} -3 & 3 \\ 9 & 12 \end{bmatrix}$$

Developing understanding

Example 2

4 Complete the following addition and subtraction of matrices.

a $\begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 6 & 1 \end{bmatrix}$

b $\begin{bmatrix} 8 & 6 \\ 9 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix}$

c $\begin{bmatrix} 3 & 5 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d $\begin{bmatrix} 9 \\ 8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e $\begin{bmatrix} 8 & 6 \\ 2 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$

f $\begin{bmatrix} 7 & 4 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 2 & -8 \end{bmatrix}$

g $\begin{bmatrix} 4 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 5 \end{bmatrix}$

h $\begin{bmatrix} 7 & -5 \end{bmatrix} - \begin{bmatrix} 7 & -5 \end{bmatrix}$

i $\begin{bmatrix} 4 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 3 \end{bmatrix}$

5 Using the matrices given:

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 7 \\ 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 2 \\ 1 & 0 \\ 3 & -8 \end{bmatrix} \quad D = \begin{bmatrix} -3 & 5 \\ 4 & -2 \\ 1 & 7 \end{bmatrix} \quad E = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

find, where possible:

a $A + B$ **b** $B + A$ **c** $A - B$ **d** $B - A$

e $B + E$ **f** $C + D$ **g** $B + C$ **h** $D - C$

6 Two people conducted a telephone poll, surveying voters about their voting intentions. The results for each person's survey are given in matrix form.

Sample 1:

	Liberal	*Labor*	*Independent*	*Greens*
Men	19	21	7	3
Women	18	17	11	4

Sample 2:

	Liberal	*Labor*	*Independent*	*Greens*
Men	24	21	3	2
Women	19	20	6	5

Write a matrix showing the overall result of the survey.

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7 Three sports stores do a stocktake at the start of each month. The results for January and February are recorded in matrix form.

January:

$$\begin{array}{r} \\ \textit{Store 1} \\ \textit{Store 2} \\ \textit{Store 3} \end{array}\begin{array}{c} \begin{array}{cccc} \textit{Basketballs} & \textit{Netballs} & \textit{Cricket balls} & \textit{Footballs} \end{array} \\ \begin{bmatrix} 32 & 10 & 82 & 41 \\ 29 & 17 & 75 & 44 \\ 22 & 12 & 103 & 61 \end{bmatrix} \end{array}$$

February:

$$\begin{array}{r} \\ \textit{Store 1} \\ \textit{Store 2} \\ \textit{Store 3} \end{array}\begin{array}{c} \begin{array}{cccc} \textit{Basketballs} & \textit{Netballs} & \textit{Cricket balls} & \textit{Footballs} \end{array} \\ \begin{bmatrix} 26 & 10 & 25 & 26 \\ 12 & 12 & 21 & 31 \\ 22 & 5 & 30 & 18 \end{bmatrix} \end{array}$$

Write a matrix showing the overall change in the stock levels from January to February.

8 The weights and heights of four people were recorded and then checked again one year later.

2024 results:

$$\begin{array}{r} \\ \textit{Weight (kg)} \\ \textit{Height (cm)} \end{array}\begin{array}{c} \begin{array}{cccc} \textit{Arlo} & \textit{Beni} & \textit{Cal} & \textit{Dane} \end{array} \\ \begin{bmatrix} 32 & 44 & 59 & 56 \\ 145 & 155 & 160 & 164 \end{bmatrix} \end{array}$$

2025 results:

$$\begin{array}{r} \\ \textit{Weight (kg)} \\ \textit{Height (cm)} \end{array}\begin{array}{c} \begin{array}{cccc} \textit{Arlo} & \textit{Beni} & \textit{Cal} & \textit{Dane} \end{array} \\ \begin{bmatrix} 38 & 52 & 57 & 63 \\ 150 & 163 & 167 & 170 \end{bmatrix} \end{array}$$

a Write the matrix that gives the changes in each person's weight and height after one year.

b Who gained the most weight?

c Which person had the greatest height increase?

Testing understanding

9 Find the values of a, b, c and d in the following matrix addition.

$$\begin{bmatrix} 3 & 11 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 2 & 8 \end{bmatrix}$$

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10 Find the values of a, b, c and d in the following matrix subtraction.

$$\begin{bmatrix} 8 & 4 \\ -2 & 12 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ -9 & 0 \end{bmatrix}$$

11 The number of kicks, goals and handballs for three footballers was recorded for the first three rounds of the AFL football season. Unfortunately, some numbers were missing. Use the information below to complete the matrices.

Round 1:

$$\begin{array}{r} \\ Jack \\ Nick \\ Mykola \end{array} \begin{array}{c} \begin{array}{ccc} Kicks & Goals & Handballs \end{array} \\ \begin{bmatrix} 10 & 4 & 2 \\ \ldots & 0 & 26 \\ 18 & 1 & 12 \end{bmatrix} \end{array}$$

Round 2:

$$\begin{array}{r} \\ Jack \\ Nick \\ Mykola \end{array} \begin{array}{c} \begin{array}{ccc} Kicks & Goals & Handballs \end{array} \\ \begin{bmatrix} 7 & \ldots & 8 \\ 13 & 2 & 12 \\ 9 & \ldots & \ldots \end{bmatrix} \end{array}$$

Round 3:

$$\begin{array}{r} \\ Jack \\ Nick \\ Mykola \end{array} \begin{array}{c} \begin{array}{ccc} Kicks & Goals & Handballs \end{array} \\ \begin{bmatrix} \ldots & 3 & \ldots \\ 20 & 1 & 19 \\ \ldots & 4 & 11 \end{bmatrix} \end{array}$$

Totals:

$$\begin{array}{r} \\ Jack \\ Nick \\ Mykola \end{array} \begin{array}{c} \begin{array}{ccc} Kicks & Goals & Handballs \end{array} \\ \begin{bmatrix} 23 & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & 7 & 34 \end{bmatrix} \end{array}$$

- Jack kicked a total of 9 goals.
- Mykola had 6 more kicks in Round 1 than Round 3.
- Nick had a total of 43 kicks.
- In the third round, Jack had twice as many handballs than the first two rounds combined.

4C Scalar multiplication

Learning intentions

- ▶ To be able to perform scalar multiplication.
- ▶ To be able to apply scalar multiplication with addition and subtraction of matrices.

A *scalar* is just a number. Multiplying a matrix by a number is called **scalar multiplication**.

Multiplying a matrix by a scalar

Scalar multiplication is the process of multiplying a matrix by a number (a scalar).

In scalar multiplication, each element is multiplied by that scalar (number).

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Example 3 Scalar multiplication

If $A = \begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$, find $3A$.

Explanation

1 If $A = \begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$, then $3A = 3\begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$.

2 Multiply each number in the matrix by 3.

3 Evaluate each element.

Solution

$$3A = 3\begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 5 & 3 \times 1 \\ 3 \times -3 & 3 \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 3 \\ -9 & 0 \end{bmatrix}$$

Now try this 3 Scalar multiplication (Example 3)

If $B = \begin{bmatrix} -7 & 2 \\ 6 & -3 \end{bmatrix}$, find $5B$.

Hint 1 Write down the scalar multiplication.

Hint 2 Multiply each number in the matrix by the scalar.

Hint 3 Evaluate each element.

Scalar multiplication can also be used in conjunction with addition and subtraction of matrices.

Example 4 Scalar multiplication and subtraction of matrices

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find the matrix equal to $2A - 3B$.

Explanation

1 Write $2A - 3B$ in expanded matrix form.

2 Multiply the elements in A by 2 and the elements in B by 3.

3 Subtract the elements in corresponding positions and evaluate.

Solution

$$2A - 3B = 2 \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 3 \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0 & 2-3 \\ 0-3 & 2-0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

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Now try this 4 Scalar multiplication and subtraction of matrices (Example 4)

If $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, find the matrix equal to $3A - 2B$.

Hint 1 Write $3A - 2B$ in expanded matrix form.

Hint 2 Multiply the elements in A by 3 and the elements in B by 2.

Hint 3 Subtract the elements in corresponding positions.

Scalar multiplication has many practical applications. It is particularly useful in scaling up the elements of a matrix, for example, adding the GST to the cost of the prices of all items in a shop by multiplying a matrix of prices by 1.1 (adding GST of 10% is the same as multiplying by 1.1).

Example 5 Application of scalar multiplication

A gymnasium has the enrolments in courses shown in this matrix.

	Body building	*Aerobics*	*Fitness*
Men	70	20	80
Women	10	50	60

The manager wishes to double the enrolments in each course. Construct a matrix showing the new enrolments for men and women in each course.

Explanation

1 Each element in the matrix is multiplied by 2.

Solution

$$2 \times \begin{bmatrix} 70 & 20 & 80 \\ 10 & 50 & 60 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 70 & 2 \times 20 & 2 \times 80 \\ 2 \times 10 & 2 \times 50 & 2 \times 60 \end{bmatrix}$$

2 Evaluate each element.

=

	Body building	*Aerobics*	*Fitness*
Men	140	40	160
Women	20	100	120

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Now try this 5 Application of scalar multiplication (Example 5)

A popular burger chain has the following prices for its products, as shown in this matrix.

	Hamburger	Chips	Can of Softdrink
Sydney	13	4.50	3
Melbourne	12	5	2.50

The manager wishes to know how much to charge after 10% GST is added, which she knows is the same as multiplying the matrix by 1.1. Show this in a matrix.

Hint 1 Write down the scalar multiplication.

Hint 2 Multiply each number in the matrix by the scalar.

Hint 3 Evaluate each element.

How to add, subtract and scalar multiply matrices using the TI-Nspire CAS

If $A = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 1 & -2 \end{bmatrix}$, find:

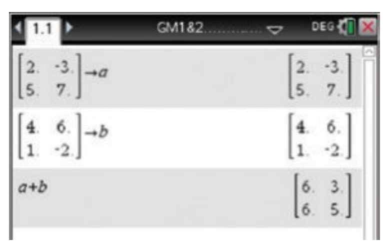
a $A + B$ **b** $A - B$ **c** $9A$ **d** $15A - 11B$

Steps

1 Press [home on]**>New Document>Add Calculator**.

2 Enter the matrices A and B into your calculator.

a To calculate $A + B$, type $A + B$ and then press [enter] to evaluate.



$$A + B = \begin{bmatrix} 6 & 3 \\ 6 & 5 \end{bmatrix}$$

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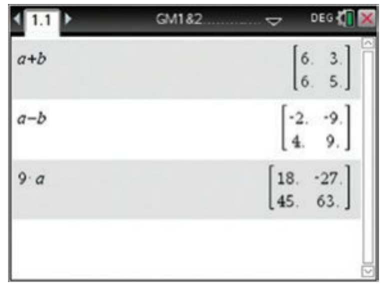
b To calculate $A - B$, type A – B and then press [enter] to evaluate.

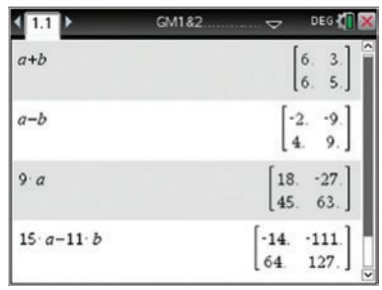
c To calculate $9A$, type $9A$ and then press [enter] to evaluate.



$$A - B = \begin{bmatrix} -2 & -9 \\ 4 & 9 \end{bmatrix}$$

$$9A = \begin{bmatrix} 18 & -27 \\ 45 & 63 \end{bmatrix}$$

d To calculate $15A - 11B$, type $15A - 11B$ and then press [enter] to evaluate.



$$15A - 11B = \begin{bmatrix} -14 & -111 \\ 64 & 127 \end{bmatrix}$$

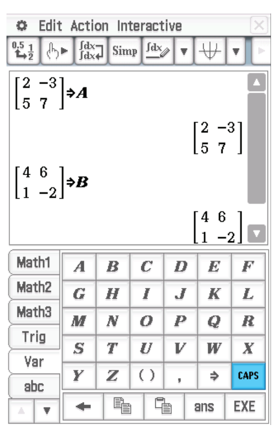
How to add, subtract and scalar multiply matrices using the ClassPad

If $A = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 1 & -2 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $9A$ **d** $15A - 11B$

Steps

1 Enter the matrices A and B into your calculator.



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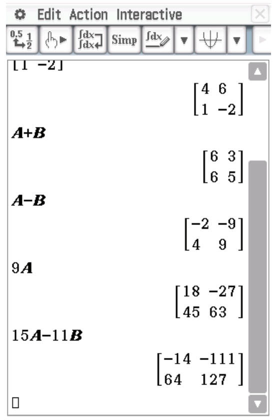
a To calculate $A + B$, type $\boldsymbol{A} + \boldsymbol{B}$ and then press EXE to evaluate.

b To calculate $A - B$, type $\boldsymbol{A} - \boldsymbol{B}$ and then press EXE to evaluate.

c To calculate $9A$, type $\boldsymbol{9A}$ and then press EXE to evaluate.

d To calculate $15A - 11B$, type $\boldsymbol{15A - 11B}$ and then press EXE to evaluate.



$$A + B = \begin{bmatrix} 6 & 3 \\ 6 & 5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -2 & -9 \\ 4 & 9 \end{bmatrix}$$

$$9A = \begin{bmatrix} 18 & -27 \\ 45 & 63 \end{bmatrix}$$

$$15A - 11B = \begin{bmatrix} -14 & -111 \\ 64 & 127 \end{bmatrix}$$

Section Summary

- **Scalar multiplication** is the multiplication of a matrix by a number (the scalar). When multiplying by a scalar, each element of the matrix is multiplied by the scalar (number).

Skill-sheet

Exercise 4C

Building understanding

1 Evaluate:

$$\begin{bmatrix} 3 \times 2 & 3 \times -1 \\ 3 \times 8 & 3 \times 7 \end{bmatrix}$$

2 Complete the following:

$$5\begin{bmatrix} -6 & 7 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 5 \times -6 & \ldots \times \ldots \\ 5 \times 3 & \ldots \times \ldots \end{bmatrix}$$

3 Complete the following:

$$2\begin{bmatrix} -2 & 5 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 \times -2 + 3 \times \ldots & \ldots \times \ldots + \ldots \times \ldots \\ 2 \times 1 + 3 \times \ldots & \ldots \times \ldots + \ldots \times \ldots \end{bmatrix}$$

Developing understanding

Example 3

4 Calculate the values of the following.

a $2\begin{bmatrix} 7 & -1 \\ 4 & 9 \end{bmatrix}$ **b** $5\begin{bmatrix} 0 & -2 \\ 5 & 7 \end{bmatrix}$ **c** $-4\begin{bmatrix} 16 & -3 \\ 1.5 & 3.5 \end{bmatrix}$ **d** $1.5\begin{bmatrix} 1.5 & 0 \\ -2 & 5 \end{bmatrix}$

e $3\begin{bmatrix} 6 & 7 \end{bmatrix}$ **f** $6\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ **g** $\frac{1}{2}\begin{bmatrix} 4 & 6 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ **h** $-1\begin{bmatrix} 3 & 6 & -8 \end{bmatrix}$

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Example 4

5 Given the matrices:

$$A = \begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 6 \\ 1 & -4 \end{bmatrix} \quad C = \begin{bmatrix} -3 & 4 \\ -2 & -5 \end{bmatrix} \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

find:

a $3A$ **b** $2B + 4C$ **c** $5A - 2B$ **d** $2O$ **e** $3B + O$

6 Enter the matrices $A = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}$ into your CAS calculator and evaluate:

a $17A - 14B$ **b** $29B - 21A$ **c** $9A + 7B$ **d** $3(5A - 4B)$

7 For the matrices:

$$A = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$

find the matrix for:

a $3A + 4B$ **b** $5C - 2D$ **c** $2(3A + 4B)$ **d** $3(5C - 2D)$

Example 5

8 A chemical plant uses four different chemicals: A, B, C and D, in particular amounts to make 1 litre each of Product 1 and Product 2. The amount of each chemical (in millilitres) to be used is given in the matrix below.

	A	*B*	*C*	*D*
Product 1	300	150	125	425
Product 2	250	170	260	320

The chemical plant needs to make 3 litres of each product for a customer.

a Perform scalar multiplication to find the total amount of each chemical required to make 3 litres of each product.

b Determine the total amount of chemical B required to make 3 litres of each product.

9 The expenses arising from costs and wages for each section of three department stores: A, B and C, are shown in the Costs matrix. The Sales matrix shows the money from the sale of goods in each section of the three stores. Figures represent the nearest million dollars.

Costs:

	Clothing	*Furniture*	*Electronics*
A	12	10	15
B	11	8	17
C	15	14	7

Sales:

	Clothing	*Furniture*	*Electronics*
A	18	12	24
B	16	9	26
C	19	13	12

a Write a matrix showing the profits in each section of each store.

b If 30% tax must be paid on profits, show the amount of tax that must be paid by each section of each store. No tax needs to be paid for a section that has made a loss.

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10 Zoe competed in the gymnastics rings and parallel bars events in a three-day gymnastics tournament. A win was recorded as 1 and a loss as 0. The three column matrices show the results for Saturday, Sunday and Monday.

$$\begin{array}{r} \\ \textit{Gymnastics rings} \\ \textit{Parallel bars} \end{array} \begin{array}{ccc} \textit{Sat} & \textit{Sun} & \textit{Mon} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}$$

a Create a 2×1 column matrix which records her total wins for each of the two types of events.

b Zoe received $50 for each win. Create a 2×1 matrix which records her total prize money for each of the two types of events.

Testing understanding

11 Consider the following matrices:

$$A = \begin{bmatrix} 6 & 3 \\ -2 & 14 \end{bmatrix} \quad B = \begin{bmatrix} 15 & -7 \\ 8 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -3 & 15 \\ 14 & -1 \\ -1 & 8 \end{bmatrix} \quad D = \begin{bmatrix} -18 \\ 2 \end{bmatrix}$$

Determine if the following are defined, and if so, evaluate the answer.

a $5A - 2B$ **b** $C - 2D$ **c** $2B - D$

12 Find the values of the pronumerals in the following matrix equations.

a

$$3\begin{bmatrix} 5 & a \\ -2 & 6 \end{bmatrix} + 2\begin{bmatrix} b & 3 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 19 & 3 \\ c & d \end{bmatrix}$$

b

$$5\begin{bmatrix} -7 & 10 \\ 9 & a \end{bmatrix} - 3\begin{bmatrix} b & -4 \\ d & 6 \end{bmatrix} = \begin{bmatrix} -47 & 2d \\ c & 2 \end{bmatrix}$$

4D Matrix multiplication

Learning intentions

- ▶ To be able to determine if matrix multiplication is possible for two matrices.
- ▶ To be able to determine the order of the resulting matrix formed under matrix multiplication.
- ▶ To be able to perform matrix multiplication by hand.
- ▶ To be able to perform matrix multiplication using a CAS calculator.

Matrix multiplication is the multiplication of a matrix by another matrix.

The matrix multiplication of two matrices, A and B, can be written as $A \times B$ or just AB. Matrix multiplication is not the simple multiplication of the numbers but instead requires the use of an algorithm involving the sum of pairs of numbers that have been multiplied.

The method of matrix multiplication can be demonstrated by using a practical example. The numbers of Books and Puzzles sold by Fatima and Gaia are recorded in matrix N. The selling prices of the Books and Puzzles are shown in matrix P.

$$N = \begin{array}{c} \\ \textit{Fatima} \\ \textit{Gaia} \end{array}\begin{array}{c} \begin{array}{cc} \textit{Books} & \textit{Puzzles} \end{array} \\ \begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \\ \textit{Books} \\ \textit{Puzzles} \end{array}\begin{array}{c} \$ \\ \begin{bmatrix} 20 \\ 30 \end{bmatrix} \end{array}$$

We want to make a matrix, S, that shows the value of the sales made by each person.

Fatima sold: **7** Books at **\$20** + **4** Puzzles at **\$30**.

Gaia sold: **5** Books at **\$20** + **6** Puzzles at **\$30**.

$$S = \begin{array}{c} \\ \textit{Fatima} \\ \textit{Gaia} \end{array}\begin{array}{c} \$ \\ \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 6 \times 30 \end{bmatrix} \end{array}$$

The steps used in this example follow the algorithm for the matrix multiplication of $N \times P$.

As we move across the *first row* of matrix N, we move down the *column* of matrix P, adding the products of the pairs of numbers as we go.

Then we move across the *second row* of matrix N and down the *column* of matrix P, adding the products of the pairs of numbers as we go.

$$\begin{array}{cc} N & \times \ P \end{array}$$

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ \square \quad \square \end{bmatrix}$$

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 6 \times 30 \end{bmatrix}$$

$$= \begin{bmatrix} 140 + 120 \\ 100 + 180 \end{bmatrix}$$

$$= \begin{array}{c} \$ \\ \begin{bmatrix} 260 \\ 280 \end{bmatrix} \end{array}\begin{array}{c} \\ \textit{Fatima} \\ \textit{Gaia} \end{array}$$

Rules for matrix multiplication

Given the way the products are formed, the number of columns in the first matrix must equal the number of rows in the second matrix. Otherwise, matrix multiplication is not defined.

Matrix multiplication

For matrix multiplication to be defined:

order of 1st matrix $m \times n$ — order of 2nd matrix $n \times p$

↑ must be the same ↑

For the earlier example of the books and puzzles sales:

order of 1st matrix 2×2 — order of 2nd matrix 2×1

↑ the same ↑

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Order of the product matrix

The order of the product matrix is given by:

order of 1st matrix $m \times n$ — order of 2nd matrix $n \times p$

order of answer: $m \times p$

For the earlier example of the books and puzzles sales:

order of 1st matrix 2×2 — order of 2nd matrix 2×1

order of answer: 2×1

Example 6 Rules for matrix multiplication

For the following matrices: $A = \begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 4 & 7 \end{bmatrix}$ $D = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

decide whether the matrix multiplication in each question below is defined. If matrix multiplication is defined, give the order of the answer matrix.

a AB **b** BA **c** CD

Explanation	Solution
a AB	
1 Write the order of each matrix.	$\underset{3 \times 2}{A}$ $\underset{2 \times 1}{B}$
2 The inside numbers are the same.	Matrix multiplication is defined for $A \times B$.
3 The outside numbers give the order of $A \times B$.	The order of the product, AB, is 3×1.
b BA	
1 Write the order of each matrix.	$\underset{2 \times 1}{B}$ $\underset{3 \times 2}{A}$
2 The inside numbers are not the same.	Multiplication is not defined for $B \times A$.
c CD	
1 Write the order of each matrix.	$\underset{1 \times 3}{C}$ $\underset{3 \times 1}{D}$
2 The inside numbers are the same.	Multiplication is defined for $C \times D$.
3 The outside numbers give the order of $C \times D$.	The order of the product, CD, is 1×1.

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Now try this 6 Rules for matrix multiplication (Example 6)

For the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ -5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -7 & -1 & 5 \\ 1 & 4 & 10 \end{bmatrix}$$

decide if matrix multiplication, AB, is defined, and if so, give the order of the answer matrix.

Hint 1 Write the order of each matrix.

Hint 2 Are the inside numbers the same?

Hint 3 If so, write down the order of the answer matrix.

Some people think of the matrix multiplication of $A \times B$ using a *run and dive* description.

Matrix multiplication of $A \times B$

The *run and dive* description of matrix multiplication is to add the products of the pairs made as you:

- *run* along the first row of A and *dive* down the first column of B,
- repeat running along the first row of A and diving down the next column of B until all columns of B have been used,
- now start running along the next row of A and repeat diving down each column of B, entering your results in a new row,
- repeat this routine until all rows of A have been used.

This procedure can be very tedious and error prone, so we will only do simple cases by hand. A CAS calculator will generally be used to do matrix multiplication.

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Example 7 **Matrix multiplication**

For the following matrices: $A = \begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 4 & 7 \end{bmatrix}$ $D = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

determine the following matrix multiplications by hand:

a AB **b** CD

Explanation	Solution
a AB	
1 Move across the first row of A and down the column of B, adding the products of the pairs.	$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ \\ \\ \end{bmatrix}$
2 Move across the second row of A and down the column of B, adding the products of the pairs.	$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \\ \\ \end{bmatrix}$
3 Move across the third row of A and down the column of B, adding the products of the pairs.	$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \\ 1 \times 8 + 3 \times 9 \end{bmatrix}$
4 Tidy up by doing some arithmetic.	$= \begin{bmatrix} 40 + 18 \\ 32 + 54 \\ 8 + 27 \end{bmatrix}$
5 Write your answer.	So $A \times B = \begin{bmatrix} 58 \\ 86 \\ 35 \end{bmatrix}$
b CD	
1 Move across the row of C and down the column of D, adding the products of the pairs.	$\begin{bmatrix} 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \times 8 + 4 \times 6 + 7 \times 5 \end{bmatrix}$
2 Tidy up by doing some arithmetic.	$= [16 + 24 + 35]$
3 Write your answer.	So $C \times D = [75]$

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Now try this 7 **Matrix multiplication (Example 7)**

For the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ -5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -7 & -1 & 5 \\ 1 & 4 & 10 \end{bmatrix}$$

perform matrix multiplication to find: $C = AB$.

Hint 1 Move across the first row of A and down the first column of B, adding the products of pairs to find the entry for c_{11}.

Hint 2 Move across the first row of A and down the second column of B, adding the products of pairs to find the entry for c_{12}.

Hint 3 Continue to find the other entries in the same way.

Hint 4 Tidy up by doing some arithmetic.

Hint 5 State the final answer.

In the previous example, $AB \neq BA$. Usually, when we reverse the order of the matrices in matrix multiplication, we get a different answer. This differs from ordinary arithmetic, where multiplication gives the same answer when the terms are reversed, for example, $3 \times 4 = 4 \times 3$.

Matrix multiplication

Remember: In general, matrix multiplication is not commutative. That is: $AB \neq BA$.

Matrix powers

Now that we can multiply matrices, we can also determine the **power of a matrix**. This is an important tool that will be useful when we meet communication and dominance matrices later in the next section and transition matrices later in the chapter.

Just as we define:

2^2 as 2×2,

2^3 as $2 \times 2 \times 2$,

2^4 as $2 \times 2 \times 2 \times 2$ and so on,

we define the various powers of matrices as:

A^2 as $A \times A$,

A^3 as $A \times A \times A$,

A^4 as $A \times A \times A \times A$ and so on.

Only square matrices can be raised to a power.

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Example 8 Evaluating matrix expressions involving powers

If $A = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$, determine:

a A^2 **b** AB^2

Explanation

1 Write down the matrices.

$A = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$

2 Enter the matrices A and B into your calculator.

$\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \to a$ $\qquad \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$

$\begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix} \to b$ $\qquad \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$

3 Type in each of the expressions as written, and press enter to evaluate. Write down your answer.

a^2 $\qquad \begin{bmatrix} 13 & 2 \\ 8 & 5 \end{bmatrix}$

$a \cdot b^2$ $\qquad \begin{bmatrix} -6 & -11 \\ -29 & -52 \end{bmatrix}$

Solution

a $A^2 = \begin{bmatrix} 13 & 2 \\ 8 & 5 \end{bmatrix}$

b $AB^2 = \begin{bmatrix} -6 & -11 \\ -29 & -52 \end{bmatrix}$

Note: For CAS calculators, you must use a multiplication sign between a and b^2 in the last example, otherwise it will be read as variable $(ab)^2$.

Now try this 8 Evaluating matrix expressions involving powers (Example 8)

For the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ -5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -7 & -1 & 5 \\ 1 & 4 & 10 \end{bmatrix}$$

Calculate A^2B.

Hint 1 Enter each matrix in your calculator.

Hint 2 Remember to use the multiplication symbol between a^2 and b on your calculator.

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Identity matrix

A special type of square matrix is called the **identity matrix**.

Identity matrix

The identity matrix (which is known by the letter I) is a square matrix of any size with 1s along the *leading diagonal* and 0s in all the other positions. For example,

$$[1] \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When the identity matrix, I, is multiplied by a matrix, for example, matrix A, the answer is A. For matrix A, $AI = A = IA$, meaning that the identity matrix multiplied by another matrix acts in the same way as the number 1 multiplied by another number. Thus, the identity matrix is a *multiplicative identity element*.

Example 9 The identity matrix

Consider the following matrices. $A = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a Find AI. **b** Find IA.

Explanation

a **1** Write the order of each matrix. The inside numbers are the same, so matrix multiplication is defined. The outside numbers tell us that the answer is a 2×2 matrix.

2 Do the matrix multiplication by hand or using your calculator.

b **1** Write the order of each matrix. The inside numbers are the same, so matrix multiplication is defined. The outside numbers tell us that the answer is a 2×2 matrix.

2 Do the matrix multiplication by hand or using your calculator.

Solution

a

$$\underset{2 \times 2}{A} \quad \underset{2 \times 2}{I}$$

$$A \times I = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \times 1 + 2 \times 0 & 5 \times 0 + 2 \times 1 \\ 8 \times 1 + 3 \times 0 & 8 \times 0 + 3 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

b

$$\underset{2 \times 2}{I} \quad \underset{2 \times 2}{A}$$

$$I \times A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 5 + 0 \times 8 & 1 \times 2 + 0 \times 3 \\ 0 \times 5 + 1 \times 8 & 0 \times 2 + 1 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

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Now try this 9 **The identity matrix (Example 9)**

$$A = \begin{bmatrix} -3 & 9 \\ 2 & -7 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a Find AI.

b Find IA.

Hint 1 Write the order of each matrix.

Hint 2 Perform the matrix multiplication by hand.

How to multiply two matrices using the TI-Nspire CAS

If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, find the matrix CD.

Steps

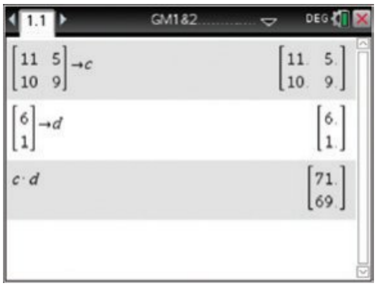
1 [⌂ on]>**New Document>Add Calculator**.

2 Enter the matrices C and D into your calculator.

3 To calculate matrix CD, type in $c \times d$. Press [enter] to evaluate.

Note: You must put a multiplication sign between the c and d.

Check: Since C has dimension 2×2 and D has dimension 2×1, matrix CD should be a 2×1 matrix.



4 Write your answer.

$$CD = \begin{bmatrix} 71 \\ 69 \end{bmatrix}$$

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How to multiply two matrices using the ClassPad

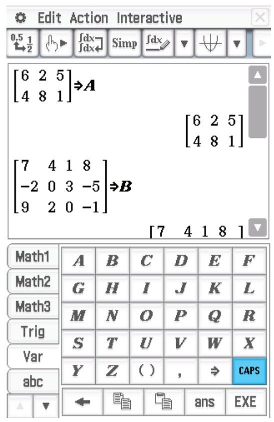
Find $A \times B$: $A = \begin{bmatrix} 6 & 2 & 5 \\ 4 & 8 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 7 & 4 & 1 & 8 \\ -2 & 0 & 3 & -5 \\ 9 & 2 & 0 & -1 \end{bmatrix}$

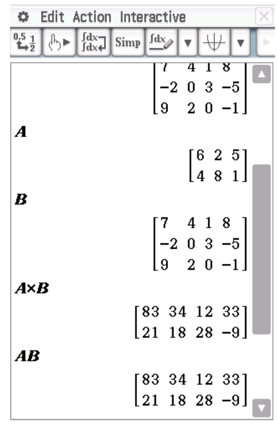
Steps

1 Enter the matrices A and B into your calculator.

2 To calculate $A \times B$, type $\boldsymbol{A} \times \boldsymbol{B}$ or $\boldsymbol{AB}$ and then press EXE to evaluate.

3 *Check:* Since A has dimensions 2×3 and B has dimensions 3×4, matrix AB should be a 2×4 matrix.





4 Write your answer.

$$AB = \begin{bmatrix} 83 & 34 & 12 & 33 \\ 21 & 18 & 28 & -9 \end{bmatrix}$$

Section Summary

- **Matrix multiplication** is the process of multiplying a matrix by another matrix. The entry for b_{ij} is found by adding the products of the pairs from row i and column j.
  For a 2×2 matrix:
  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$
  For example,
  $$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 \times 4 + 2 \times 0 & 3 \times 7 + 2 \times 6 \\ -1 \times 4 + 5 \times 0 & -1 \times 7 + 5 \times 6 \end{bmatrix}$$
- An **identity** matrix is a square matrix with 1s along the leading diagonal and 0s in all other positions. The identity matrix behaves like the number 1 in arithmetic. Any matrix multiplied by I remains unchanged. That is, for matrix A, $AI = A = IA$.

Skill-sheet

Exercise 4D

Building understanding

1 Complete the following:

$$\begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \times \cdots + (-5) \times \ldots \\ 3 \times \cdots + 1 \times \ldots \end{bmatrix}$$

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2 Determine the order of each matrix, and decide if the two matrices can be multiplied.

$$A = \begin{bmatrix} -3 & -7 & 2 \\ 5 & 9 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -6 & 4 \\ 11 & 8 \\ -1 & -3 \end{bmatrix}$$

3 Determine the order of the resulting matrix when matrix A and matrix B are multiplied together to form AB.

$$A = \begin{bmatrix} -2 & 6 & 1 \\ 8 & -12 & 6 \\ 0 & 5 & 6 \\ 1 & 4 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 20 & -6 & 2 & 1 \\ 0 & 5 & 8 & -1 \\ 7 & -9 & 3 & 2 \end{bmatrix}$$

4 Write down the 3×3 identity matrix.

Developing understanding

Example 6

5 For the following matrices: $A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 \\ 0 & 8 \\ 2 & -5 \end{bmatrix}$

- decide whether the matrix multiplication in each question below is defined.
- If matrix multiplication is defined, give the order of the answer matrix.

a AB **b** BA **c** CB **d** BC
e AA **f** BB **g** AC **h** CA

6 Write the orders of each pair of matrices and decide if matrix multiplication is defined.

a $\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

d $\begin{bmatrix} 8 & -2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Example 7

7 Perform the following matrix multiplications by hand, using the rule for multiplying matrices.

a $\begin{bmatrix} 4 & 2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 4 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -2 & -3 \end{bmatrix}$

d $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} 7 & 4 \\ 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **f** $\begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 9 \end{bmatrix}$

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g $\begin{bmatrix} 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$ h $\begin{bmatrix} 6 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ i $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$

8 Consider the following matrices: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

a Find AB. b Find BA. c Does $AB = BA$?

9 Perform the following matrix multiplications using your CAS calculator.

a $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$ b $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 6 \\ 7 & 8 \end{bmatrix}$ c $\begin{bmatrix} 5 & 8 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$ e $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ f $\begin{bmatrix} 2 & 5 \\ 4 & 1 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 3 \end{bmatrix}$

Example 8
Example 9

10 Noting that $A^2 = A \times A$, $A^3 = A \times A \times A$, etc., calculate:

i A^2 ii A^3 iii A^4

for each of the following matrices.

a $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ b $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ c $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Testing understanding

11 Consider the following list of matrices.

$$A = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix} B = \begin{bmatrix} 8 \\ -5 \end{bmatrix} C = \begin{bmatrix} -1 & 0 & 6 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$$

Identify which matrices can be multiplied together, and determine the order of the resulting matrix.

12 Perform the following matrix multiplication:

$$\begin{bmatrix} 1 & 7 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 6 & 5 & -1 \\ 3 & 0 & 2 \end{bmatrix}$$

13 Two matrices were multiplied together to give a third matrix, as shown below.

$$\begin{bmatrix} -6 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -27 & -30 \\ -37 & -18 \end{bmatrix}$$

Find the values of a, b, c and d to identify the second matrix.

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4E Inverse matrices and solving simultaneous equations using matrices

Learning intentions

- ► To be able to find the inverse of a matrix.
- ► To be able to solve simultaneous equations using matrices.

The inverse of a matrix

Written as A^{-1}, the **inverse** of matrix A is a matrix that multiplies A to make the identity matrix, I.

$$A \times A^{-1} = I = A^{-1} \times A$$

Only square matrices have inverses, but not all square matrices have inverses.

Finding the inverse of a matrix is best done using your CAS calculator. If the inverse does not exist, your CAS will give you an error message.

How to find the inverse of a matrix using a CAS calculator

Find the inverse of the matrix $A = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$.

Steps

1. Enter matrix A into your calculator.
2. Type in A ^ −1 and evaluate.
3. Form the product AA^{-1}. It should give you the identity matrix, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix} \rightarrow a$ $\quad \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$

a^{-1} $\quad \begin{bmatrix} 0.5 & -1 \\ -0.75 & 2 \end{bmatrix}$

$a \cdot a^{-1}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. Write your answer.

The inverse of A is

$$A^{-1} = \begin{bmatrix} 0.5 & -1 \\ -0.75 & 2 \end{bmatrix}$$

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How to find the inverse of a matrix using the ClassPad

Find the inverse of matrix $A = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$.

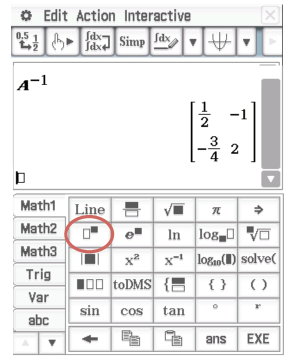
Steps

1 To calculate the inverse matrix, A^{-1}:

- Type in **A** ^ –1 or Type **A** □■ –1.
- Press EXE to evaluate.

2 Write your answer.

The inverse of A is: $A^{-1} = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{3}{4} & 2 \end{bmatrix}$.



The determinant of a matrix

The determinant is used in finding the inverse of a matrix, and in fact, determining if an inverse exists.

The determinant of a 2 × 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of A is given by:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$$

Example 10 Finding the determinant of a 2 × 2 matrix

Find the determinant of each of the following matrices:

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 7 \\ 2 & -2 \end{bmatrix}.$$

Explanation

1 Write down the matrix and use the rule: $\det(A) = a \times d - b \times c$.

2 Evaluate.

Solution

$\det(A) = 5 \times 6 - 2 \times 3 = 24$

$\det(B) = (-3) \times (-2) - 7 \times 2 = -8$

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Now try this 10 Finding the determinant of a 2 × 2 matrix (Example 10)

Find the determinant of the following matrix.

$$G = \begin{bmatrix} -3 & -9 \\ 2 & -7 \end{bmatrix}$$

Hint 1 Use the rule: $\det(G) = a \times d - b \times c$.

Hint 2 Be careful when multiplying negatives.

While we would typically use a calculator to find the inverse of a matrix, we can find the inverse of a 2 × 2 matrix by hand, using the determinant.

The rule for finding the inverse of a 2 × 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse, A^{-1}, is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $\frac{1}{a \times d - b \times c} \neq 0$. That is, provided the determinant is not equal to 0.

For example, if $A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$ then the inverse, $A^{-1} = \frac{1}{24} \begin{bmatrix} 6 & -2 \\ -3 & 5 \end{bmatrix}$

The rule shows that the inverse of a matrix cannot be found if the determinant of the matrix is equal to zero.

Solving simultaneous equations using matrices

Matrices can be used to solve simultaneous equations. If we have two equations in terms of x and y, we can write them out in matrix form and then use a CAS calculator to solve them.

Example 11 Solving simultaneous equations

Express the following simultaneous equations as matrices.

$$8x + 2y = 46$$
$$5x - 3y = 19$$

Explanation

1 Write each side of the equation as a matrix.

2 Write the left-hand side of the matrix equation as the product of two matrices.

Solution

$$\begin{bmatrix} 8x + 2y \\ 5x - 3y \end{bmatrix} = \begin{bmatrix} 46 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 46 \\ 19 \end{bmatrix}$$

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Now try this 11 **Solving simultaneous equations (Example 11)**

Express the following simultaneous equations as matrices.

$$6x - y = 15$$
$$-2x + 5y = -19$$

Hint 1 Write each side of the equation as a matrix.

Hint 2 Write the matrix on the left-side of the equation as a product of two matrices

How to solve simultaneous equations using a CAS calculator

Solve to find x and y:

$$5x + 2y = 21$$
$$7x + 3y = 29$$

Steps

1 The two simultaneous equations can be represented by the matrix equation shown.

$$\begin{bmatrix} 5x+2y \\ 7x+3y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$

2 The left-hand side of the matrix equation in step **1** can be written as the product of two matrices.

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$

3 Name the matrices as shown. Matrix X contains the solutions to the simultaneous equations.

$$A \times X = C$$

4 Enter matrix A and matrix C.

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \rightarrow a \qquad \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 21 \\ 29 \end{bmatrix} \rightarrow c \qquad \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$

5 We want to find the values of matrix X.

$X = A^{-1} \times C$

Since: $A \times X = C$

$$A^{-1} \times A \times X = A^{-1} \times C$$
$$I \times X = A^{-1} \times C$$
$$X = A^{-1} \times C$$

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6 Enter the matrix product $A^{-1} \times C$.

Note: Order is critical here: $X = A^{-1}C$, *not* CA^{-1}.

$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \to a$ $\quad\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

$\begin{bmatrix} 21 \\ 29 \end{bmatrix} \to c$ $\quad\begin{bmatrix} 21 \\ 29 \end{bmatrix}$

$a^{-1}.c$ $\quad\begin{bmatrix} 5 \\ -2 \end{bmatrix}$

7 Write matrix X.

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

8 Write the solutions to the equations.

So $x = 5$ and $y = -2$.

9 Substitute the values for x and y into the equations to check if they are correct.

Section Summary

- The **inverse matrix**, A^{-1}, of matrix A is defined such that $A \times A^{-1} = I$.
- **Simultaneous equations** can be solved by writing the equations in matrix form, $A \times X = C$, and then solved to find X: $X = A^{-1}C$. Use a CAS calculator for this step.

Skill-sheet

Exercise 4E

Building understanding

1 True or false: If $A \times B = I$, then B is the inverse of matrix A.

2 Confirm that D is the inverse of C by using your CAS calculator to find CD for:

$$C = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & -1 \\ -2.5 & 1.5 \end{bmatrix}$$

3 Convert the following simultaneous equations into matrix form by completing the following:

a $3x + 7y = 27$
$5x + 6y = 28$
$\begin{bmatrix} 3 & \ldots \\ \ldots & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27 \\ \ldots \end{bmatrix}$

b $2x + 8y = -2$
$3x + 20y = 5$
$\begin{bmatrix} \ldots & 8 \\ \ldots & \ldots \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ \ldots \end{bmatrix}$

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Developing understanding

4 Use your CAS calculator to find the inverse of each matrix. Check by showing that $AA^{-1} = I$ for each matrix.

a $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}$

e $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **f** $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \\ 2 & 0 & 2 \end{bmatrix}$ **g** $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 3 & 8 \\ 1 & 2 & 0 \end{bmatrix}$ **h** $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

5 Use your CAS calculator and the matrices shown below to find the following:

$$A = \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -7 & 2 \\ 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -4 & 9 \\ 6 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 8 \\ 4 & -1 \end{bmatrix}$$

a $(A + B)^{-1}$ **b** $(3C - D)^{-1}$ **c** $(CD)^{-1}$ **d** $AB + D^{-1}$

Example 10

6 Calculate the determinant for each of the matrices in the previous question.

7 Use your CAS calculator and the matrices shown below to find the following:

$$A = \begin{bmatrix} -1 & 7 \\ 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ -3 & 2 \end{bmatrix}$$

a A^{-1} **b** B^{-1} **c** $B^{-1}A^{-1}$ **d** $(AB)^{-1}$

Example 11

8 Express the following simultaneous equations in matrix form.

a $5x + y = 13$
$3x + 2y = 12$

b $x + 2y = 10$
$4x + y = 5$

c $7x - 2y = -31$
$-3x + 2y = -1$

d $6x + 5y = 38$
$9x + 3y = 66$

9 Use matrix methods on your CAS calculator to solve the following simultaneous equations.

a $3x + 2y = 12$
$5x + y = 13$

b $4x + 3y = 10$
$x + 2y = 5$

c $4x - 3y = 10$
$3x + y = 1$

d $8x + 3y = 50$
$5x + 2y = 32$

e $6x + 7y = 68$
$4x + 5y = 46$

f $6x - 5y = -27$
$7x + 4y = -2$

Testing understanding

10 Evaluate $A \times (AA^{-1}) \times (A^{-1}A) \times A^{-1}$ for

$$A = \begin{bmatrix} 6 & 2 \\ 5 & -1 \end{bmatrix}$$

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11 Express the following as two simultaneous equations.

$$\begin{bmatrix} 3 & -2 \\ 7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 23 \\ 55 \end{bmatrix}$$

12 Use matrix multiplication to find the inverse matrices below, by first finding the values of a, b, c and d.

a $$\begin{bmatrix} -1 & 7 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & -6 \\ 11 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4F Using matrices to model road and communication networks

Learning intentions

- ▶ To be able to summarise relationships in a network in a matrix.
- ▶ To be able to represent a social network in a communication matrix.

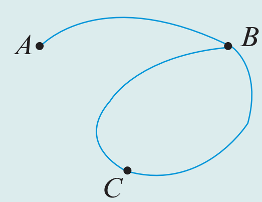
A **network** is a set of objects which are connected together. Examples of networks include towns which may be connected by roads, or people who may be connected by knowing one another. Networks can be illustrated by network diagrams where the objects are points (vertices) and the connections are lines (edges).

Example 12 Using a matrix to represent connections

The network diagram on the right shows road connections between three towns, A, B and C.

a Use a matrix to represent the road connections. Each element should describe the number of ways to travel *directly* from one town to another.

b What information is given by the sum of the elements in column B?



Explanation

a As there are three towns: A, B and C, use a 3×3 matrix to show the direct connections.

There are 0 roads directly connecting any town to itself. So enter 0 in row A, column A, and so on.

Note: If there were a road directly connecting town A to itself, it would be a loop from A back to A, and 1 would be added to that element in the matrix.

Solution

$$\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

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There is one road directly connecting B to A (or A to B). So enter 1 in row B, column A, and in row A, column B.

$$\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

There are no direct roads between C and A. So enter 0 in row C, column A, and row A, column C.

$$\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \\ 0 & & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

There are 2 roads between C and B. Enter 2 in row C, column B, and row B, column C.

$$\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

b The second column, B, shows the number of roads directly connected to town B.

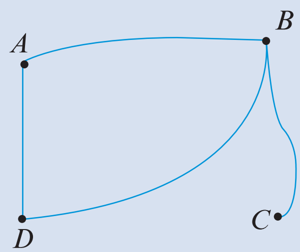
The sum of the second column, B, is the total number of roads directly connected to town B.
$1 + 0 + 2 = 3$

Now try this 12 Using a matrix to represent connections (Example 12)

The network diagram on the right shows the road connections between four towns, A, B, C and D.

a Use a matrix to represent the road connections. Each element should describe the number of ways to travel *directly* from one town to another.

b What information is given by the sum of the elements in column D?



Hint 1 Determine how many rows and columns should be in the matrix.

Hint 2 If there is no road between two towns, place a 0 in the corresponding entry. If there is one road, place a 1.

Hint 3 Sum up the numbers in the second column. Think about what each entry represents so that you can understand what the sum of the column represents.

A particular type of network is a social network where the objects are individuals, and the connections between them indicate that they communicate with each other.

Diagrams can be used to show the communication links between people in a social network. A line between two people in a diagram indicates that there is a direct communication link between the people in the network. This is represented by a '1' in the associated matrix.

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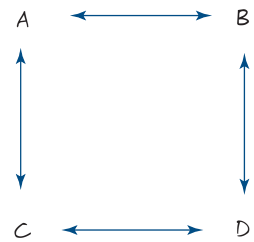
The absence of a line between two people in a network indicates that they cannot communicate with each other directly. This is shown by a '0' in the associated matrix.

When people communicate directly with each other, we say that there is a **'one step' communication link**. When people can communicate indirectly with each other via another person, we say that there is a **'two-step' communication link**.

Consider the communication matrix, S, representing the network shown below.

$$S = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} \\ A \\ B \\ C \\ D \end{matrix}$$



The matrix S can be multiplied by itself, S^2, to determine the number of ways that pairs of people in a network can communicate with each other via a third person.

$$S^2 = \begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \end{array} \begin{matrix} \\ A \\ B \\ C \\ D \end{matrix}$$

For example, this tells us that A can communicate with D through two different people. Looking at the network above, we can see that A can communicate with D through B (ABD) or through C (ACD). We can also see that A can communicate with themselves through two different people: B (ABA) or with C (ACA). It also tells us that A cannot communicate with B through one other person.

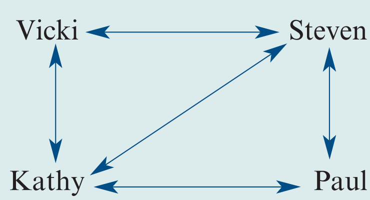
The matrix S^3 would give the three-step communications between people, which is the number of ways of communicating with someone via two people.

The matrix method of investigating communications can be applied to friendships, travel between towns and other types of two-way connections.

Example 13 Using matrices to model the communication links in a social network

The diagram shows the communications within a group of friends. In this diagram:

a Use a matrix, N, to record the presence or absence of direct communication links between the people in the network. Use the first letter of each name to label the columns and rows. Explain how the matrix can be read, using the labels.



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b Why is the matrix symmetric?

c What information is given by the sum of column K?

d N^2 gives the number of two-step communications between people. Namely, how many ways one person can communicate with someone via another person. Find the matrix N^2, the square of matrix N.

e Use the matrix N^2 to find the number of two-step ways Kathy can communicate with Steven, and write the connections.

f In the N^2 matrix, there is a 3 where the S column meets the S row. This indicates that there are three two-step communications which Steven can have with himself. Explain how this can be given a sensible interpretation.

Solution

a

$$N = \begin{array}{c} \begin{matrix} V & S & K & P \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} V \\ S \\ K \\ P \end{matrix}$$

Reading from column S and row K, a 1 indicates that Steven communicates with Kathy. The number 0 is used where there is no communication, for example, between Vicki and Paul.

b The symmetry occurs because the communication is two-way. For example, Vicki communicates with Steven and Steven communicates with Vicki.

c The sum of a column gives the total number of people that a given person can communicate with.

For example, Kathy can communicate with: $1 + 1 + 0 + 1 = 3$ people.

d

$$N^2 = \begin{array}{c} \begin{matrix} V & S & K & P \end{matrix} \\ \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix} \end{array} \begin{matrix} V \\ S \\ K \\ P \end{matrix}$$

e Reading down the K column to the S row, there are 2 two-step communications between Kathy and Steven. These can be found in the arrows diagram.

Kathy → Vicki → Steven

Kathy → Paul → Steven

f There are three ways Steven can communicate with himself via another person.

Steven → Vicki → Steven

Steven → Kathy → Steven

Steven → Paul → Steven

For example, using the first case above, Steven might ring Vicki and ask her to ring him back later to remind him of an appointment.

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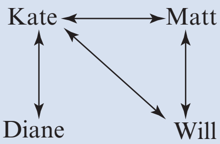
Now try this 13 **Using matrices to model the communication links in a social network (Example 13)**

The diagram shows the communications within a group of friends.



a Record the social links in a matrix, A, using the first letter of each name to label the columns and rows. Explain how the matrix should be read.

b Explain why there is a symmetry about the leading diagonals of the matrix.

c What information is given by the sum of a column or a row?

d Find the matrix A^2, the square of A.

e Using the matrix A^2, find how many ways that Matt can communicate with Kate, and write the connections.

Hint 1 Determine how many rows and columns should be in the matrix.

Hint 2 If there is no communication between individuals, then place a zero. If two individuals communicate directly, then place a 1.

Hint 3 To square a matrix, multiply the matrix by itself.

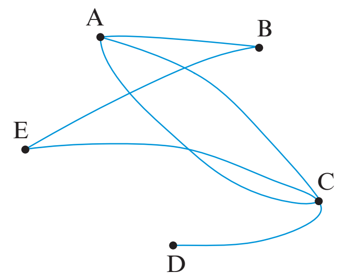
Section Summary

- A matrix can model communication or connection between two or more towns or people. Connection matrices are always square matrices.
- Matrices and matrix techniques can be used to model and analyse the properties of road, communication and other real-world networks.

Exercise 4F

Building understanding

Consider the following network of roads between five towns.



1 List the towns that are directly connected to B.

2 Which two towns are directly connected to each other by two roads?

3 Complete the matrix to represent this network:

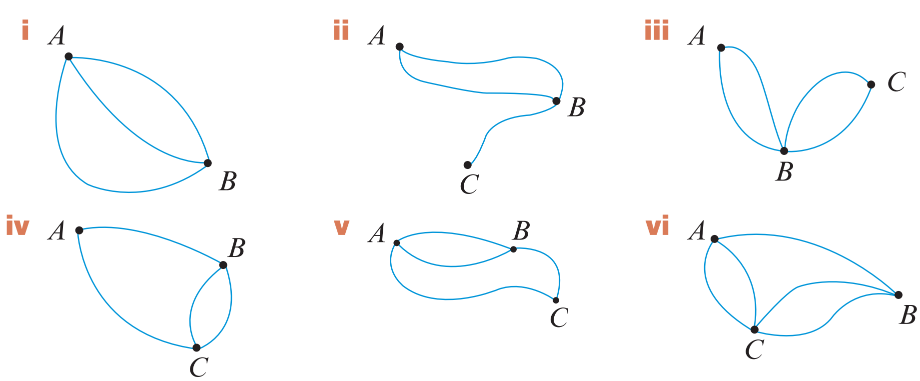
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ \ldots & 0 & 0 & \ldots & 1 \\ \ldots & 0 & 0 & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & 0 \end{bmatrix} \end{array}$$

Developing understanding

Example 12

4 The road networks below show roads connecting towns.

a In each case, use a matrix to record the number of ways of travelling *directly* from one town to another.



b What does the sum of the second column of each matrix represent?

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5 The matrices shown below record the number of ways of going directly from one town to another.

a In each case, represent each matrix with a road network between towns A, B and C.

i
$$\begin{array}{c} \begin{matrix} \text{A} & \text{B} & \text{C} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix}$$

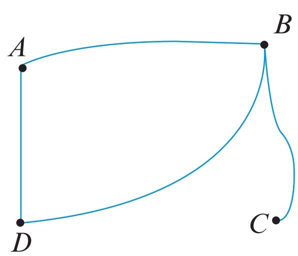
ii
$$\begin{array}{c} \begin{matrix} \text{A} & \text{B} & \text{C} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix}$$

iii
$$\begin{array}{c} \begin{matrix} \text{A} & \text{B} & \text{C} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix}$$

iv
$$\begin{array}{c} \begin{matrix} \text{A} & \text{B} & \text{C} \end{matrix} \\ \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix}$$

b State the information that is given by the sum of the first column in the matrices of part **a**.

6 The network diagram opposite has lines showing which people from the four people, A, B, C and D, have met.



a Represent the network using a matrix. Use 0 when two people have *not* met and 1 when they have met.

b How can the matrix be used to tell who has met the most people?

c Who has met the most number of people?

d Who has met the fewest number of people?

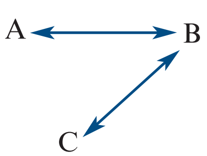
Example 13

7 Consider the following communication matrix where the letters represent the names of people, a '1' indicates that there is direct communication between the people represented in the respective row and column, and a '0' indicates that there is no direct communication. Given that communication is a two-way process, find the error in the communication matrix.

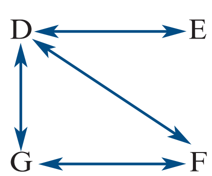
$$\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

8 Write the matrix for each communication diagram. Use the number 1 when direct communication between two people exists and 0 for no direct communication.

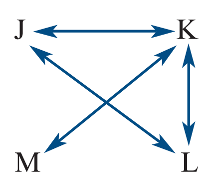
a



b



c



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9 Road connections between towns are recorded in the matrices below. The letters represent towns. The number 1 indicates that there is a road directly connecting the two towns. The number 0 indicates that there is no road directly connecting the two towns.

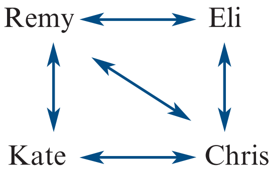
Draw a diagram corresponding to each matrix showing the roads connecting the towns.

a $\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \end{matrix}$

b $\begin{array}{c} \begin{matrix} P & Q & R & S \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} P \\ Q \\ R \\ S \end{matrix}$

c $\begin{array}{c} \begin{matrix} T & U & V & W \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} T \\ U \\ V \\ W \end{matrix}$

10 Communication connections between Chris, Eli, Kate and Remy are shown in the diagram.



a Write a matrix, Q, to represent the connections. Label the columns and rows in alphabetical order using the first letter of each name. Enter 1 to indicate that two people communicate directly or 0 if they do not.

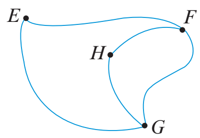
b What information is given by the sum of column R?

c **i** Find Q^2.

ii Using the matrix Q^2, find the total number of ways that Eli can communicate with a person via another person.

iii Write the chain of connections for each way that Eli can communicate to a person via another person.

11 Roads connecting the towns Easton, Fields, Hillsville and Gorges are shown in the diagram. The first letter of each town is used.



a Use a matrix, R, to represent the road connections. Label the columns and rows in alphabetical order using the first letter of each town's name. Write 1 when two towns are directly connected by a road and write 0 if they are not connected.

b What does the sum of column F reveal about the town Fields?

c **i** Find R^2.

ii How many ways are there to travel from Fields to a town via another town? Include ways of starting and ending at Fields.

iii List the possible ways of part **ii**.

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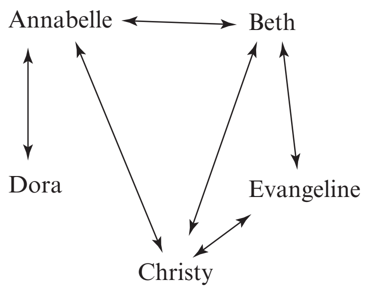
Testing understanding

12 Communication between five people is shown below:



a Construct a matrix, F, to represent the connections between the five people. Label each row and column with the first letter of their name. Enter 1 if the two individuals *directly* communicate and 0 if they do not.

b What is the minimum number of steps required to ensure that all individuals can communicate with everyone? Hint: Calculate F^2, F^3 etc.

c List all the possible paths of communication between Evangeline and Dora in 3-step communication.

13 The Hampden football league is made up of ten teams: North Warrnambool (N), South Warrnambool (S), Warrnambool (W), Hamilton (H), Camperdown (Ca), Cobden (Co), Portland (Pl), Port Fairy (PF), Terang (T) and Koroit (K).
The first three rounds were as follows:

1 N vs. S, W vs. H, Ca vs. Co, Pl vs. PF, T vs. K

2 N vs. W, S vs. Ca, Co vs. Pl, PF vs. T, K vs. H

3 N vs H, S vs. Co, W vs. Pl, PF vs. K, T vs. Ca

Given this information:

a Construct a matrix, labelling each row and column with each team. Place a 1 in the matrix if the two teams play each other in the first three rounds and a 0 if they do not play each other.

b State the sum of each row and each column.

c What would the sum of row 6 be after 8 rounds of football?

d What should the matrix look like at the end of round 9?

e What happens to the matrix in round 10?

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4G Introduction to transition matrices

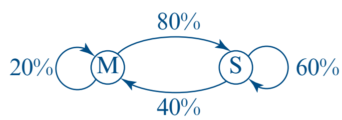
Learning intentions

- To be able to set up a transition matrix.
- To be able to interpret a transition matrix.

Suppose that in a certain town you have two choices of activity each Saturday night: going to the movies (M) or staying at home (S). Suppose further that it has been established that:

- 20% of people who go to the movies this Saturday will go again next Saturday (and hence, 80% of people who go to the movies this Saturday will stay home next Saturday).
- 40% of people who stay home this Saturday will go to the movies next Saturday (and hence, 60% of people who stay home this Saturday will stay home again next Saturday.)

The diagram below describes this information.



The information can also be summarised in a matrix as shown below. Note that the percentages have been written as decimals.

$$\begin{array}{c} \text{this Saturday} \\ \begin{array}{cc} M & S \end{array} \\ \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix} \begin{array}{c} M \\ S \end{array} \text{ next Saturday} \end{array}$$

This matrix is an example of a **transition matrix**.

Properties of a transition matrix

A transition matrix is always a square matrix.

An important feature of a transition matrix is that each column total of the proportions (or percentages) must equal 1 (100%).

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Setting up a transition matrix

Example 14 **Setting up a transition matrix**

An amusement park has two rides for pre-school children. These are the Ferris wheel (F) and the Merry-go-round (M). After each ride, children can choose which ride they want to go on next. The park observes the following:

- 70% of children on the Ferris wheel will go on the Ferris wheel again.
- 40% of children on the Merry-go-round will go on the Merry-go-round again.

a Find the percentage of children on the Ferris wheel who will go on the Merry-go-round next time.

b Find the percentage of children on the Merry-go-round who will go on the Ferris wheel next time.

c Construct a transition matrix that describes the transition from one step to the next for this situation.

Explanation

a Children on the Ferris wheel either go on the Ferris wheel again or the Merry-go-round. Thus, the percentages must add up to 100%.

Solution

30% of children will go on the Merry-go-round next time.

b Children on the Merry-go-round either go on the Merry-go-round again or the Ferris wheel. Thus, the percentages must add up to 100%.

Solution

60% of children will go on the Ferris wheel next time.

c Set up the transition matrix with F and M representing the Ferris wheel and Merry-go-round.

The first column represents the percentages (written as decimals) for children currently on the Ferris wheel, so place 0.7 and 0.3 in these positions.

The second column represents the percentages for children currently on the Merry-go-round, so place 0.6 and 0.4 in these positions.

Solution

this ride

$$\begin{matrix} & F & M & \\ & \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix} & \begin{matrix} F \\ M \end{matrix} & \text{next ride} \end{matrix}$$

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Now try this 14 **Setting up a transition matrix (Example 14)**

An observant sales assistant has noted whether customers buy hot food (H) or cold food (C) in the store for lunch. They notice that:

- 80% of customers who buy hot food for lunch will buy hot food again the next day.
- 60% of customers who buy cold food for lunch will buy cold food again the next day.

Construct a transition matrix that describes the percentages of customers choosing hot or cold food for lunch from one day to the next.

Hint 1 Calculate the percentage of people buying cold food tomorrow who purchased hot food today.

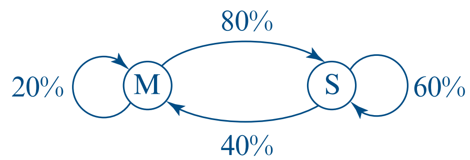
Hint 2 Calculate the percentage of people buying hot food tomorrow who purchased cold food today.

Hint 3 Set up a transition matrix with labels H and C for the rows and columns. Fill in the percentages - remember that 80% should be written as 0.8.

Applying a transition matrix

Considering our earlier example of going to the movies or staying home, we saw that the transition matrix, T, and its transition diagram could be used to describe the weekly pattern of behaviour.

$$T = \begin{array}{c} \text{this week} \\ \begin{array}{cc} M & S \end{array} \\ \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} M \\ S \end{array} \text{ next week}$$



Using this information alone, a number of predictions can be made.
For example, if 200 people go to the movies this week, the transition matrix predicts that:

- 20% or 40 of these people will go to the movies next week ($0.2 \times 200 = 40$)
- 80% or 160 of these people will stay at home next week ($0.8 \times 200 = 160$)

Further, if 150 people stayed at home this week, the transition matrix predicts that:

- 40% or 60 of these people will go to the movies next week ($0.4 \times 150 = 60$)
- 60% or 90 of these people will stay at home next week ($0.6 \times 150 = 90$)

Thus, we can predict that if 200 people went to the movies this week, and 150 stayed home, then the number of people going to the movies next week is $0.2 \times 200 + 0.4 \times 150 = 100$, and the number of people who stay home next week is $0.8 \times 200 + 0.6 \times 150 = 250$.

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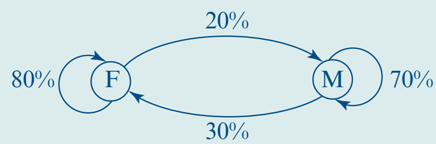
Example 15 Interpreting a transition matrix

The following transition matrix, T, and its transition diagram can be used to describe the movement of children between a Ferris wheel and a Merry-go-round.

$$T = \begin{array}{c} \text{this ride} \\ \begin{matrix} F & M \end{matrix} \\ \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \end{array} \begin{matrix} F \\ M \end{matrix} \text{ next ride}$$



a What percentage of children on the Ferris wheel are predicted to go on the:

i Ferris wheel next time? **ii** Merry-go-round next time?

b 140 children went on the Ferris wheel. How many of these children do you expect to go on the:

i Ferris wheel next time? **ii** Merry-go-round next time?

c If 140 children went on the Ferris wheel and 80 children went on the Merry-go-round, how many children do you predict will go on the Ferris wheel next time?

Explanation	Solution
a You should read from the first column of the transition matrix.	
i Consider the first row.	80%
ii Consider the second row.	20%
b You should read from the first column of the transition matrix.	
i Since 80% of children on the Ferris wheel are expected to go on the Ferris wheel next time, then we multiply 0.8 by the number of children on the Ferris wheel.	$0.8 \times 140 = 112$ We expect 112 children from the Ferris wheel to go on the Ferris wheel next time.
ii Since 20% of children on the Ferris wheel are expected to go on the Merry-go-round next time, then we multiply 0.2 by the number of children on the Ferris wheel.	$0.2 \times 140 = 28$ We expect 28 children from the Ferris wheel to go on the Merry-go-round next time.
c Since 30% of children on the Merry-go-round are expected to go on the Ferris wheel next time, we multiply 0.3 by the number of children on the Merry-go-round. This is added to our answer from above.	$0.3 \times 80 = 24$ $112 + 24 = 136$ We predict that 136 students will go on the Ferris wheel next time.

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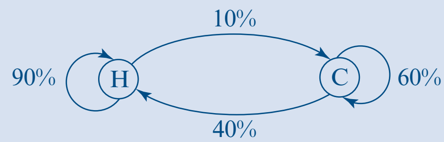
Now try this 15 **Interpreting a transition matrix (Example 15)**

The following transition matrix, T, and its transition diagram can be used to describe customers choosing between a hot and a cold meal, given their previous choice.

$$\begin{array}{c} \text{today} \\ \begin{array}{cc} H & C \end{array} \end{array}$$
$$T = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix} \begin{array}{c} H \\ C \end{array} \text{ tomorrow}$$



a Of the customers who purchased hot food today, what percentage of people do you expect to buy hot meals and cold meals tomorrow?

b If the store sold 400 hot meals on one day, how many of those who purchased hot meals do you expect to purchase hot meals the next day?

c If the store sold 400 hot meals and 500 cold meals on one day, how many hot meals in total do you expect them to sell the next day?

Section Summary

- A transition matrix is always a square matrix.
- An important feature of a transition matrix is that each column total of the proportions (or percentages) must equal 1 (100%).

Exercise 4G

Building understanding

1 Consider the following matrix.

$$\begin{array}{c} \text{this time} \\ \begin{array}{cc} A & B \end{array} \end{array}$$
$$T = \begin{bmatrix} 0.8 & 0.3 \\ \square & \square \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ next time}$$

Fill in the blanks so that the matrix is a transition matrix.

2 An amusement park has two rides: a Ferris wheel and a Merry-go-round. Experience has shown that:

- 60% of children on the Ferris wheel go on the Ferris wheel again.
- 50% of children on the Merry-go-round go on the Merry-go-round again.

a What percentage of children on the Ferris wheel go on the Merry-go-round next time?

b What percentage of children on the Merry-go-round go on the Ferris wheel next time?

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3 Consider the following transition matrix.

$$T = \begin{array}{c} \text{this time} \\ \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{next time} \end{array}$$

Using the transition matrix above, if the initial population at two different locations, A and B, is 50 people each:

a How many people at location A are predicted to stay at location A next period?

b How many people at location B are predicted to move to location A next period?

c How many people are predicted to be at location A at the start of the next period?

Developing understanding

Example 14

4 The local football club kiosk sells pies and dim sims. They observe that:

- 60% of customers who buy a pie, choose to buy a pie next week.
- 80% of customers who buy dim sims, choose to buy dim sims next week.

Find the percentage of customers who:

a bought a pie this week who are predicted to buy dim sims next week.

b bought dim sims this week who are predicted to buy a pie next week.

c Construct a transition matrix that describes the transition between the two foods.

5 A new resident to an area is trying to work out the weather patterns. They observe:

- it rains 70% of the time tomorrow if it rains today.
- it rains 60% of the time tomorrow if it does not rain today.

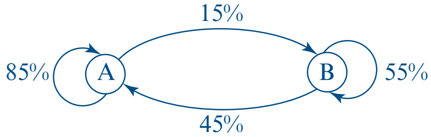
a Find the percentage of the time that it will:

i not rain tomorrow if it rained today.

ii not rain tomorrow if it did not rain today.

b Construct a transition matrix that describes the transition between raining and not raining.

6 The following diagram describes the transition between two states, A and B.



Construct a transition matrix that can be used to represent the diagram.

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Example 15

7 The following matrix, T, is used to describe the daily pattern of morning coffee orders at a cafe, where L represents a latte and F represents a flat white. On Monday, the cafe sells 160 lattes and 100 flat whites.

$$T = \begin{array}{c} \text{Monday} \\ \begin{array}{cc} L & F \end{array} \\ \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ L \\ F \end{array} \text{Tuesday}$$

a What percentage of customers who buy a latte on Monday are predicted to buy a:

i latte on Tuesday?

ii flat white on Tuesday?

b If the cafe sells 160 lattes on Monday, how many of the people who buy a latte on Monday are predicted to buy a latte on Tuesday?

c If the cafe sells 160 lattes on Monday, how many of the people who buy a latte on Monday are predicted to buy a flat white on Tuesday?

d If the cafe sells 160 lattes and 100 flat whites on Monday, in total, how many people are predicted to buy a latte on Tuesday?

e If the cafe sells 160 lattes and 100 flat whites on Monday, in total, how many people are predicted to buy a flat white on Tuesday?

8 The following matrix, Q, is used to describe whether students arrive on time to school each morning, where T represents being on time and L represents being late. On Monday, 170 students were on time and 30 students were late.

$$Q = \begin{array}{c} \text{today} \\ \begin{array}{cc} T & L \end{array} \\ \begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ T \\ L \end{array} \text{tomorrow}$$

a How many students who were on time on Monday are predicted to be on time on Tuesday?

b How many students who were late on Monday are predicted to be on time on Tuesday?

c In total, how many students do you expect to be on time on Tuesday?

d In total, how many students do you expect to be late on Tuesday?

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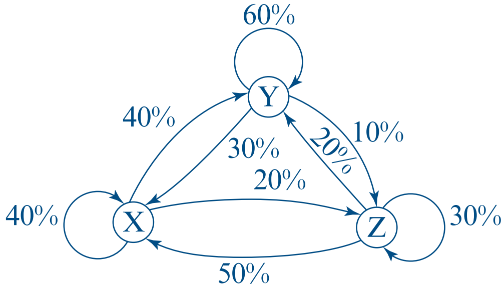
Testing understanding

9 The following diagram describes the transition between three states: X, Y and Z. Complete the following transition matrix to represent this diagram.



$$\begin{array}{c} \text{this time} \\ \begin{array}{ccc} X & Y & Z \end{array} \end{array}$$
$$\begin{bmatrix} 0.4 & \square & 0.5 \\ \square & 0.6 & \square \\ 0.2 & \square & \square \end{bmatrix} \begin{array}{l} X \\ Y \text{ next time} \\ Z \end{array}$$

4H Using recursion to answer questions that require multiple applications of a transition matrix

Learning intentions

- ▶ To be able to use recursion to generate a sequence of **state matrices**.
- ▶ To be able to determine what happens to the sequence of state matrices in the long term.

Consider the earlier example of going to the movies or staying home. We can now ask questions about the number of people we expect to be at the movies after 1 week, 2 weeks and so on.

This type of problem is similar to those considered in financial modelling. For example, if \$3000 is invested at 5% per annum, how much will we have after 1 year, 2 years, 3 years etc?

In the same way that financial problems were solved using a **recurrence relation** to model the change in growth of value in an investment or loan over time, a recurrence relation can be formed using matrices which can model the change in the number of people choosing to go to the movies from week to week.

The starting point for this example is a matrix that tells us how many people initially go to the movies and how many stay at home. This is called the **initial state matrix** and is denoted S_0.

$$S_0 = \begin{bmatrix} 200 \\ 150 \end{bmatrix}$$

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To find the number of people who go to the movies after 1 week, we use the transition matrix, T, to generate the next **state matrix** in the sequence, S_1, as follows:

$$S_1 = TS_0$$
$$= \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix}\begin{bmatrix} 200 \\ 150 \end{bmatrix} = \begin{bmatrix} 0.2 \times 200 + 0.4 \times 150 \\ 0.8 \times 200 + 0.6 \times 150 \end{bmatrix}$$
$$= \begin{bmatrix} 100 \\ 250 \end{bmatrix}$$

Thus, after one week, we predict that 100 people will go to the movies and 250 people will stay home.

A similar calculation can be performed to find out how many people we predict will go to the movies and will stay home after two weeks.

$$S_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix}\begin{bmatrix} 100 \\ 250 \end{bmatrix} = \begin{bmatrix} 0.2 \times 100 + 0.4 \times 250 \\ 0.8 \times 100 + 0.6 \times 250 \end{bmatrix} = \begin{bmatrix} 120 \\ 230 \end{bmatrix}$$

It is clear that there is a pattern. So far, we have seen that:

$S_1 = T \times S_0$

$S_2 = T \times S_1$

Continuing this pattern we have:

$S_3 = T \times S_2$

$S_4 = T \times S_3$

$S_5 = T \times S_4$

or or more generally, after n weeks (or n applications of the transition matrix, T),

$$S_{n+1} = T \times S_n.$$

Matrix recurrence relations

A matrix recurrence relation for generating a sequence of state matrices associated with a transition matrix is given by:

$$S_0 = \text{ initial state matrix}, \qquad S_{n+1} = T \times S_n$$

where T is the transition matrix and S_n is the state matrix after n applications of the transition matrix, T, or in practical situations, n time intervals.

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Example 16 Using a recursion relation to generate successive state matrices

Children at a fair can either go on the Ferris wheel or the Merry-go-round. At the start, 140 children went on the Ferris wheel and 80 children went on the Merry-go-round. Use the recurrence relation:

$$S_0 = \text{ initial value,} \qquad S_{n+1} = TS_n$$

where:

$$S_0 = \begin{bmatrix} 140 \\ 80 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

to predict the number of children on the Ferris wheel and Merry-go-round after 1 turn and after 3 turns.

Explanation

1 Use the rule: $S_{n+1} = TS_n$, to predict the number of children on the Ferris wheel and Merry-go-round after one turn, by forming the product $S_1 = TS_0$ and evaluating.

2 To predict the number of children on the Ferris wheel and Merry-go-round after three turns, first calculate the number after two turns and then three.

Solution

$$S_1 = TS_0 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 140 \\ 80 \end{bmatrix} = \begin{bmatrix} 136 \\ 84 \end{bmatrix}$$

Thus, we predict that 136 children will go on the Ferris wheel and 84 will go on the Merry-go-round.

$$S_2 = TS_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 136 \\ 84 \end{bmatrix} = \begin{bmatrix} 134 \\ 86 \end{bmatrix}$$

$$S_3 = TS_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 134 \\ 86 \end{bmatrix} = \begin{bmatrix} 133 \\ 87 \end{bmatrix}$$

Now try this 16 Using a recursion relation to generate successive state matrices (Example 16)

A cafe keeps track of the percentage of people who purchase hot food and cold food each day. On the first day, 400 hot meals and 500 cold meals are chosen.

Use the recurrence relation:

$$S_0 = \text{ initial value} \qquad S_{n+1} = TS_n$$

where:

$$S_0 = \begin{bmatrix} 400 \\ 500 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}$$

to predict how many people chose a hot or cold meal after 1 day or after 3 days.

Hint 1 Use $S_1 = TS_0$, to find the number of hot and cold meals after 1 day.

Hint 2 Use the recurrence relation to find the number of hot and cold meals after 2 days, so you can then find the number after 3 days.

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An explicit rule for determining the state matrix after n applications of the recursion rule: $S_{n+1} = TS_n$

While we can use the recurrence relation:

$$S_0 = \text{initial state matrix} \qquad S_{n+1} = TS_n$$

to generate a sequence of state matrices step-by-step, there is a more efficient way when we need to determine the state matrix, S_n, for a large value of n. For example, consider the following matrices, defining the initial population (10 000 and 35 000) of two regional towns, and a transition matrix that defines the proportion of each population that moves between the two towns each year.

$$S_0 = \begin{bmatrix} 10\,000 \\ 35\,000 \end{bmatrix} \qquad T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

Using the recurrence relation, the population state matrix for the two towns after one year is:

$$S_1 = TS_0 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \begin{bmatrix} 10\,000 \\ 35\,000 \end{bmatrix} = \begin{bmatrix} 10\,250 \\ 34\,750 \end{bmatrix}$$

After two years:

$$S_2 = TS_1 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \begin{bmatrix} 10\,250 \\ 34\,750 \end{bmatrix} = \begin{bmatrix} 10\,450 \\ 34\,550 \end{bmatrix}$$

Thus, $S_2 = T \times (TS_0) = T^2 S_0$, so the rule for finding the population after n years is: $S_n = T^n S_0$.

An explicit rule for determining S_n after n transitions (or time intervals)

If the recurrence relation for determining matrices is:

$$S_0 = \text{initial state matrix}, \qquad S_{n+1} = TS_n$$

then an explicit rule for finding S_n is:

$$S_n = T^n \times S_0$$

where T is the transition matrix and S_0 is the initial state matrix.

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Example 17 Using matrices to model and analyse population growth and decay

A study was conducted to investigate the change in the populations of Geelong (G) and Ballarat (B) due to movement of people between the two cities.

At the start of the study, the populations of Geelong and Ballarat were respectively 250 000 and 110 000 people. Using a matrix approach to model future changes, the initial state matrix is:

$$S_0 = \begin{bmatrix} 250\,000 \\ 110\,000 \end{bmatrix}$$

The transition matrix that can be used to chart the population movements between the two cities from year to year is:

$$T = \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix}$$

a Using the recursion rule: $S_{n+1} = TS_n$, predict the population of Geelong after one year.

b Using the recursion rule: $S_{n+1} = TS_n$, predict the population of Geelong after two, three and four years.

c Using the rule: $S_n = T^n S_0$, predict Geelong's population after 20, 30 and 40 years.

d What do you notice about the predicted population of Geelong? What do you think will happen beyond 40 years?

Explanation	Solution
a Calculate: $S_1 = TS_0$, and read off the top line.	$S_1 = \begin{bmatrix} 241\,500 \\ 118\,500 \end{bmatrix}$ The predicted population after one year in Geelong is 241 500.
b Use the rule: $S_{n+1} = TS_n$, and then read off the top line.	After two years: 235 125 After three years: 230 344 After four years: 226 758
c Use the rule: $S_n = T^n S_0$, and then read off the top line.	After twenty years: 216 108 After thirty years: 216 006 After forty years: 216 000
d Consider what is happening to the population over time - is it increasing or decreasing? Is it approaching a particular value?	Over time, the predicted population in Geelong is declining. Initially, the population is falling rapidly from 250 000 to 235 125 in one year, but then appears to stabilise at around 216 000 people.

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Now try this 17 **Using matrices to model and analyse population growth and decay (Example 17)**

A study of the relative growth in duck populations at two nesting sites, A and B, is to be analysed using matrix methods.

At the start of the study, there were 8000 ducks at site A and 10 000 ducks at site B. For these populations, the initial population state matrix is:

$$S_0 = \begin{bmatrix} 8000 \\ 10\,000 \end{bmatrix}$$

Over time, ducks move from one site to the other site. The transition matrix that can be used to describe this movement is:

$$T = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

a What is the predicted population in each location after one, two and three years?

b What is the predicted population in each site after twenty, thirty and forty years?

c What do you notice about the predicted population, and what do you think will happen beyond forty years?

Hint 1 Use the rule: $S_n = T^n S_0$, to find the population each year.

Hint 2 Consider what is happening to the population over time - is it increasing or decreasing? Is it approaching a particular value?

From the Geelong and Ballarat population example above, the long-run populations of the two cities become stable and do not change from one year to the next. This does not mean that nobody is moving between the two cities, rather that the number of people moving from Geelong to Ballarat and Ballarat to Geelong is equal. We can see this by calculating the expected number of people moving between each city when $n = 41$(meaning 41 years). In the example, the population of Geelong when $n = 40$ was 216 000, and the population of Ballarat was 144 000. When $n = 41$, the number of people moving from Geelong to Ballarat will be:

$$0.1 \times 216\,000 = 21\,600$$

The number of people moving from Ballarat to Geelong will be:

$$0.15 \times 144\,000 = 21\,600$$

Since these two numbers are equal, we refer to this as the **steady state** or the **equilibrium state**.

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Section Summary

- A **transition matrix**, T, is a square matrix which represents the transition (movement) from one state in a sequence to the next state in a sequence.
- A **state matrix**, S_n, is a column matrix which lists the numbers in each state after n time periods.
- The **recurrence relation**:

$$S_0 = \text{ initial state matrix} \qquad S_{n+1} = T \times S_n$$

can be used to generate a sequence of state matrices, $S_0, S_1, S_2, \ldots$, where successive states only depend on their immediate predecessor.
- An explicit rule for finding S_n is: $S_n = T^n S_0$, where T is the transition matrix and S_0 is the initial state matrix.

Skill-sheet

Exercise 4H

Building understanding

1 A factory has a large number of machines. The machines can be in one of two states: operating (O) and broken (B). Broken machines can be repaired and go back into operation, and operating machines can break down. Construct an initial state matrix if there are initially 100 machines in operation and 25 broken machines.

Example 16

2 Consider the following matrices defining the number of students in a year level who get above a C on their maths test and those who get a C or lower, and the transition matrix for the next period. Assume getting above a C is listed first.

$$S_0 = \begin{bmatrix} 60 \\ 140 \end{bmatrix} \qquad T = \begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix}$$

How many students do you expect to get above a C on the next maths test?

3 For the initial state matrix, $S_0 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ and the transition matrix, $T = \begin{bmatrix} 0.1 & 0.8 \\ 0.9 & 0.2 \end{bmatrix}$:

a use the recursion relation: S_0 = initial state matrix, $S_{n+1} = TS_n$, to determine: S_1, S_2 and S_3

b determine the value of T^2

c use the rule: $S_n = T^n S_0$, to determine S_7.

Developing understanding

Example 17

4 A small town has two fish and chip shops (Bill's Barnacles and Sally's Seafood), and local residents choose which shop to use each week. Consider the following matrices defining the initial customers and a transition matrix that describes the movement between the two shops each week.

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Assume that Bill is listed first.

$$S_0 = \begin{bmatrix} 200 \\ 150 \end{bmatrix} \qquad T = \begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix}$$

a How many people are predicted to go to Bill's Barnacles next week?

b How many people are predicted to go to Bill's Barnacles after four weeks? What about Sally's Seafood?

c How many people are predicted to go to Bill's Barnacles after twelve weeks? What about Sally's Seafood?

d In the long term, how many customers do you expect to go to Bill's Barnacles and Sally's Seafood each week?

5 In a football club, players can either be available to play or are injured. 90% of players who are available in one week are available the next week, while 40% of injured players are available to play the next week.

a Construct a fully labelled transition matrix to describe this situation, where available players are listed first. Call the matrix T.

b The club has a total of 50 players on their list, all of whom are initially available. Construct a column matrix, S_0, that describes this situation.

c Using T and S_0, how many players do you expect will be available after one week?

d In the long run, how many players do you expect the club to have on its injury list each week?

6 A local town produces a weekly newspaper. Residents can either read the newspaper (R) or ignore the newspaper (I).
In a given week:

- 80% of people who read the paper will read it next week.
- 20% of people who read the paper will ignore it next week.
- 25% of people who ignore the paper will read it next week.
- 75% of people who ignore the paper will ignore it next week.

a Construct a fully labelled transition matrix to describe the situation, where reading the paper is listed first. Call the matrix T.

b Initially, 4000 residents read the paper and 2000 residents ignore it. Write down a column matrix to describe the situation. Call the matrix S_0.

c How many of the residents who read the paper this week do we expect to read it next week? How many of those do we expect to ignore it next week?

d How many of the residents who ignore the paper this week do we expect to read it next week? How many of those do we expect to ignore it next week?

e How many residents do we expect to read the paper after 6 weeks?

f In the long term, how many residents of the town do we expect to read the paper each week? How many residents do we expect to ignore the paper each week?

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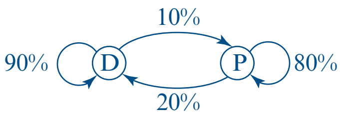
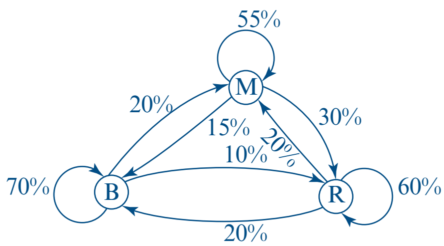
7 Residents in a large apartment block in the city have the option of either driving to work (D) or catching public transport (P).
Consider the diagram opposite that shows the way that people change between the two options on each day.



a Construct a fully labelled transition matrix to describe the situation, where driving to work is listed first. Call the matrix T.
b Initially, 40 residents drive to work and 160 residents get public transport. Write down a column matrix to describe the situation. Call the matrix S_0.
c How many of the residents who drive to work today do you expect will drive to work tomorrow? How many of those residents do you expect will get public transport tomorrow?
d How many of the residents who get public transport today do you expect will get public transport tomorrow? How many of those residents do you expect will drive to work tomorrow?
e How many residents do you expect to get public transport after 5 days?
f In the long term, how many residents do you expect to get public transport? How many residents do you expect to drive to work?

Testing understanding

8 A juice bar sells three types of juices: Banana Bonanza, Mango Magic and Radical Raspberry. The patterns of daily customers are recorded as shown in the diagram.



a Construct an appropriately labelled 3 by 3 transition matrix to describe this situation. Call the matrix T.
b On Monday, 300 people buy Banana Bonanza, 200 buy Mango Magic and 50 buy Radical Raspberry. Construct a column matrix, S_0, that describes this situation.
c Using matrix multiplication, calculate the number of each type of juice that you expect the store to sell on Tuesday.
d The store is concerned about stock levels. Calculate the total number of each type of drink that you expect the store to sell in the first week from Monday (based on S_0) through to the following Sunday.
e In the long run, how many of each juice do you expect the store will sell each day?

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4I Applications of matrices

Learning intentions

- To be able to use matrix multiplication to solve application problems.
- To be able to use row and column matrices to extract information from matrices.

Data represented in matrix form can be multiplied to produce new useful information.

Example 18 Business application of matrices

Freddy and George's store has a sales promotion. One free cinema ticket is given with each DVD purchased. Two cinema tickets are given with the purchase of each game.

The number of DVDs and games sold by Freddy and George is given in matrix S.

The selling price of a DVD and a game, together with the number of free tickets, is given by matrix P.

$$S = \begin{matrix} & \text{DVDs} & \text{Games} \\ \text{Freddy} & 7 & 4 \\ \text{George} & 5 & 6 \end{matrix} \qquad P = \begin{matrix} & \$ & \text{Tickets} \\ \text{DVDs} & 20 & 1 \\ \text{Games} & 30 & 2 \end{matrix}$$

Find the matrix product, $S \times P$, and interpret.

Explanation

Complete the matrix multiplication, $S \times P$.

Solution

$$\begin{matrix} & \text{DVDs} & \text{Games} \\ \text{Freddy} & 7 & 4 \\ \text{George} & 5 & 6 \end{matrix} \times \begin{matrix} & \$ & \text{Tickets} \\ \text{DVDs} & 20 & 1 \\ \text{Games} & 30 & 2 \end{matrix}$$

$$= \begin{matrix} & \$ & \text{Tickets} \\ \text{Freddy} & 7 \times 20 + 4 \times 30 & 7 \times 1 + 4 \times 2 \\ \text{George} & 5 \times 20 + 6 \times 30 & 5 \times 1 + 6 \times 2 \end{matrix}$$

$$= \begin{matrix} & \$ & \text{Tickets} \\ \text{Freddy} & 260 & 15 \\ \text{George} & 280 & 17 \end{matrix}$$

Explanation

Interpret the matrix.

Solution

Freddy had sales of \$260 and gave out 15 tickets.
George had sales of \$280 and gave out 17 tickets.

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Now try this 18 **Business application of matrices (Example 18)**

Jacky and Peter's store has a special sales promotion. One free prize ticket is given with each drink purchased. Two prize tickets are given with the purchase of each hamburger.

The number of drinks and hamburgers sold by Jacky and Peter are given in matrix S.

The selling price of a drink and hamburger, together with the number of free prize tickets, is given by matrix P.

$$S = \begin{array}{r} \\ \textit{Jacky} \\ \textit{Peter} \end{array}\begin{array}{c} \begin{array}{cc} \textit{Drinks} & \textit{Hamburgers} \end{array} \\ \begin{bmatrix} 3 & 10 \\ 6 & 8 \end{bmatrix} \end{array} \qquad P = \begin{array}{r} \\ \textit{Drinks} \\ \textit{Hamburgers} \end{array}\begin{array}{c} \begin{array}{cc} \$ & \textit{Tickets} \end{array} \\ \begin{bmatrix} 4 & 1 \\ 12 & 2 \end{bmatrix} \end{array}$$

Find the matrix product, $S \times P$, and interpret.

Hint 1 Write down the matrices to be multiplied.
Hint 2 Carry out the matrix multiplication using the algorithm.
Hint 3 Determine what each element in the matrix is telling you.

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Properties of row and column matrices

Row and column matrices provide efficient ways of extracting information from data stored in large matrices. Matrices of a convenient size will be used to explore some of the surprising and useful properties of row and column matrices.

Example 19 **Using row and column matrices to extract information**

Three rangers completed their monthly park surveys of feral animal sightings, as shown in Matrix S.

$$S = \begin{matrix} \\ Aaron \\ Barry \\ Chloe \end{matrix} \begin{matrix} \begin{matrix} cats & dogs & foxes & rabbits \end{matrix} \\ \begin{bmatrix} 27 & 9 & 34 & 59 \\ 18 & 15 & 10 & 89 \\ 35 & 6 & 46 & 29 \end{bmatrix} \end{matrix} \quad A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

a Evaluate $S \times B$.

b What information about matrix S is given in the product $S \times B$?

c Evaluate $A \times S$.

d What information about matrix S is given in the product $A \times S$?

Explanation

Solution

a Matrix multiplication of a 3×4 and a 4×1 matrix produces a 3×1 matrix.

$$S \times B = \begin{bmatrix} 27 & 9 & 34 & 59 \\ 18 & 15 & 10 & 89 \\ 35 & 6 & 46 & 29 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 27 + 9 + 34 + 59 \\ 18 + 15 + 10 + 89 \\ 35 + 6 + 46 + 29 \end{bmatrix} = \begin{bmatrix} 129 \\ 132 \\ 116 \end{bmatrix}$$

b Look at the second last step in the working of $S \times B$.

Each row of SB gives the sum of the rows in S. Namely, the total sightings made by each ranger.

c Matrix multiplication of a 1×3 and a 3×4 matrix produces a 1×4 matrix.

$$A \times S = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 27 & 9 & 34 & 59 \\ 18 & 15 & 10 & 89 \\ 35 & 6 & 46 & 29 \end{bmatrix}$$

$$A \times S = \begin{bmatrix} 27+18+35 & 9+15+6 & 34+10+46 & 59+89+29 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 30 & 90 & 177 \end{bmatrix}$$

d In the second last step of part **c**, we see that each element is the sum of the sightings for each type of animal.

Each column of AS gives the sum of the columns in S, which gives the sum of the sightings of each type of animal.

Now try this 19 **Using row and column matrices to extract information (Example 19)**

Four students recorded the number of minutes they spent watching television each weekday and recorded their times in matrix T.

$$T = \begin{array}{c} \\ Beth \\ Zara \\ Aria \\ Wanda \end{array} \begin{array}{c} \begin{array}{ccccc} Mon & Tue & Wed & Thur & Fri \end{array} \\ \begin{bmatrix} 45 & 25 & 80 & 0 & 140 \\ 0 & 20 & 0 & 20 & 120 \\ 30 & 30 & 0 & 30 & 110 \\ 40 & 0 & 0 & 20 & 120 \end{bmatrix} \end{array} \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

a Evaluate $T \times B$.

b What information about matrix T is given in the product, $T \times B$?

c Evaluate $A \times T$.

d What information about matrix T is given in the product, $A \times T$?

Hint 1 Write down the matrices to be multiplied.

Hint 2 Carry out the matrix multiplication using the algorithm.

Hint 3 Determine what each element in the matrix is telling you.

Section Summary

- Matrices can be used to store information. Matrix multiplication can allow key information from matrices to be extracted.

Exercise 4I

Building understanding

1 Joe and Stephen sell cans of softdrink and bottles of water at the senior school fair. The number of cans and bottles are given in matrix D, and the prices of cans and bottles are given in matrix P.

$$D = \begin{array}{c} \\ Joe \\ Stephen \end{array} \begin{array}{c} \begin{array}{cc} Cans & Bottles \end{array} \\ \begin{bmatrix} 214 & 103 \\ 162 & 189 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \\ Cans \\ Bottles \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{array}$$

a True or false: Matrix P is a column matrix.

b Complete the following matrix multiplication:

$$\begin{bmatrix} 214 & 103 \\ 162 & 189 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 214 \times 3 + 103 \times \ldots \\ \cdots \times 3 + \cdots \times 2 \end{bmatrix}$$

c What does the top line of the resulting matrix tell you?

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Developing understanding

Example 18

2 The matrix below shows the number of milkshakes and sandwiches that Helen had for lunch one week. The number of kilojoules (kJ) present in each food is given in the second matrix.

$$\begin{array}{c} \\ Helen \end{array}\begin{array}{c} \begin{array}{cc} Milkshakes & Sandwiches \end{array} \\ \begin{bmatrix} 2 & 3 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ Milkshakes \\ Sandwiches \end{array}\begin{array}{c} kJ \\ \begin{bmatrix} 1400 \\ 1000 \end{bmatrix} \end{array}$$

Use matrix multiplication to calculate how many kilojoules Helen had for lunch over the week.

3 Consider the following two matrices, where the first matrix shows the number of cars and bicycles owned by two families and the second matrix records the wheels and seats for cars and bicycles.

$$\begin{array}{c} \\ Smith \\ Jones \end{array}\begin{array}{c} \begin{array}{cc} Cars & Bicycles \end{array} \\ \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ Car \\ Bicycle \end{array}\begin{array}{c} \begin{array}{cc} Wheels & Seats \end{array} \\ \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix} \end{array}$$

Use matrix multiplication to find a matrix that gives the numbers of wheels and seats owned by each family.

4 Eve played a game of darts. The parts of the dartboard that she hit during one game are recorded in matrix H. The bull's eye is a small area in the centre of the dartboard. The points scored for hitting different regions of the dartboard are shown in matrix P.

$$H = \begin{array}{c} \\ Hits \end{array}\begin{array}{c} \begin{array}{ccc} Bull's\ eye & Inner\ region & Outer\ region \end{array} \\ \begin{bmatrix} 2 & 13 & 5 \end{bmatrix} \end{array}$$

$$P = \begin{array}{c} Points \\ \begin{bmatrix} 20 \\ 5 \\ 1 \end{bmatrix} \end{array}\begin{array}{l} \\ Bull's\ eye \\ Inner\ region \\ Outer\ region \end{array}$$

Use matrix multiplication to find a matrix giving her score for the game.

5 On a Saturday morning, Michael's cafe sold 18 quiches, 12 soups and 64 coffees. A quiche costs \$5, soup costs \$8 and a coffee costs \$3.

a Construct a row matrix to record the number of each type of item sold.

b Construct a column matrix to record the cost of each item.

c Use matrix multiplication of the matrices from parts **a** and **b** to find the total value of the morning sales.

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6 Han's stall at the football made the sales shown in the table.

Tubs of chips	Pasties	Pies	Sausage rolls
90	84	112	73

The selling prices were: chips \$4, pastie \$5, pie \$5 and a sausage roll \$3.

a Record the numbers of each product sold in a row matrix.

b Write the selling prices in a column matrix.

c Find the total value of the sales by using matrix multiplication of the row and column matrices found in parts **a** and **b**.

Example 19

7 The number of study hours completed by three students over four days is shown in matrix H.

$$H = \begin{matrix} \\ Issie \\ Jack \\ Kaiya \end{matrix} \begin{matrix} \begin{matrix} Mon & Tues & Wed & Thur \end{matrix} \\ \begin{bmatrix} 2 & 3 & 2 & 3 \\ 1 & 4 & 0 & 2 \\ 3 & 4 & 3 & 2 \end{bmatrix} \end{matrix}$$

Using matrix multiplication with a suitable row or column matrix, complete the following.

a Produce a matrix showing the total study hours for each student.

b Hence, find a matrix with the average hours of study for each student.

c Obtain a matrix with the total number of hours studied on each night of the week.

d Hence, find a matrix with the average number of hours studied each night. Round your answers to one decimal place.

8 Matrix R records four students' results in five tests.

$$R = \begin{matrix} \\ Ellie \\ Felix \\ Gavin \\ Hannah \end{matrix} \begin{matrix} \begin{matrix} T1 & T2 & T3 & T4 & T5 \end{matrix} \\ \begin{bmatrix} 87 & 91 & 94 & 86 & 88 \\ 93 & 76 & 89 & 62 & 95 \\ 73 & 61 & 58 & 54 & 83 \\ 66 & 79 & 83 & 90 & 91 \end{bmatrix} \end{matrix}$$

Choose an appropriate row or column matrix, and use matrix multiplication to complete the following.

a Obtain a matrix with the sum of each student's results.

b Hence, give a matrix with each student's average test score.

c Derive a matrix with the sum of the scores for each test.

d Hence, give a matrix with the average score on each test.

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Testing understanding

9 Scalar multiplication occurs when a number multiplies a matrix. For a 2×2 matrix it has the general form:

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Suggest why the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ is called a *scalar matrix*.

10 The mobile phone bills of Anna, Boyd and Charlie for the four quarters of 2022 are recorded in matrix P.

$$P = \begin{matrix} & Q1 & Q2 & Q3 & Q4 \\ Anna & 47 & 43 & 52 & 61 \\ Boyd & 56 & 50 & 64 & 49 \\ Charlie & 39 & 41 & 44 & 51 \end{matrix} \qquad E = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

a Find $P \times E$, and comment on the result of that matrix multiplication.

b State the matrix, F, needed to extract the second quarter, $Q2$, costs.

To find the four quarterly costs on Charlie's phone bill, another matrix is required.

c What will be the order of the matrix that displays Charlie's quarterly costs?

d State the order of a matrix, G, that when it multiplies a 3×4, gives a 1×4 matrix as the result? Should matrix G pre-multiply (GP) or post-multiply (PG) matrix P?

e Suggest a suitable matrix G that will multiply matrix P and produce a matrix of Charlie's quarterly phone bills. Check that it works.

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Review

Key ideas and chapter summary

Assign-ment

Matrix A **matrix** is a rectangular array of numbers set out in rows and columns within square brackets. The rows are horizontal; the columns are vertical.

Order of a matrix The **order (size) of a matrix** is the number of rows × number of columns. The number of rows is always given first.

Elements of a matrix The **elements of a matrix** are the numbers within it. The position of an element is given by its row and column in the matrix. Element a_{ij} is in row i and column j. The row is always given first.

Connections A matrix can be used to record various types of connections, such as social communications and roads directly connecting towns.

Equal matrices Two matrices are equal when they have the same numbers in the same positions. They need to have the same order.

Adding matrices Matrices of the same order can be *added* by adding numbers in the same positions.

Subtracting matrices Matrices of the same order can be *subtracted* by subtracting numbers in the same positions.

Zero matrix, *O* A **zero matrix** is any matrix with zeroes in every position.

Scalar multiplication **Scalar multiplication** is the multiplication of a matrix by *a number*, where each element of the matrix is multiplied by the scalar.

Matrix multiplication **Matrix multiplication** is the process of multiplying a matrix by another matrix. The entry for b_{ij} is found by adding the products of the pairs from row i and column j. For a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

For example,

$$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 3\times4+2\times0 & 3\times7+2\times6 \\ -1\times4+5\times0 & -1\times7+5\times6 \end{bmatrix}$$

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Identity matrix, I

An **identity matrix**, I, behaves like the number 1 in arithmetic. Any matrix multiplied by I remains unchanged. For 2×2 matrices,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $AI = A = IA$.

Matrix multiplication by the identity matrix is commutative.

Transition matrix

A **transition matrix**, T, is a matrix which describes the transition (movement) from one step to the next of a sequence.

Inverse matrix, A^{-1}

When any matrix, A, is multiplied by its **inverse matrix**, A^{-1}, the answer is I, the identity matrix. That is:

$$A \times A^{-1} = I$$

Solving simultaneous equations

Two **simultaneous equations**, for example:

$$5x + 2y = 21$$

$$7x + 3y = 29$$

can be written in matrix form as:

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$

$$A \times X = C$$

The equations can be solved (for x and y) by finding the values in matrix X, as:

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \times C$$

Order is critical: $X = A^{-1}C$, *not* CA^{-1}.
This is best done using a CAS calculator.

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Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

4A **1 I can state the order of a given matrix.** ☐

e.g. State the order of the following matrix:

$$\begin{bmatrix} 7 & -2 & 5 \\ 10 & 0 & 4 \\ -8 & 12 & 3 \end{bmatrix}$$

4A **2 I can describe the location of an element in a matrix.** ☐

e.g. Considering the following matrix, B, state the value of the element in b_{23}.

$$B = \begin{bmatrix} 57 & 63 & 19 \\ 48 & 54 & 6 \\ 39 & 45 & 22 \\ 37 & 59 & 31 \\ 42 & 22 & 19 \end{bmatrix}$$

4B **3 I can add and subtract matrices.** ☐

e.g. Complete the following matrix addition.

$$\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 9 & -1 \end{bmatrix}$$

4B **4 I can identify a zero matrix.** ☐

e.g. Identify the zero matrix from the matrices below.

$$\begin{bmatrix} 7 & 3 \\ 2 & 8 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4C **5 I can perform scalar multiplication.** ☐

e.g. If $A = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$, then find $5A$.

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4C **6** I can apply scalar multiplication with addition and subtraction of matrices. ☐

e.g. If $A = \begin{bmatrix} 5 & 3 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, then find $2A + 3B$.

4D **7** I can determine if matrix multiplication is possible for two matrices. ☐

e.g. State whether it is possible to find AB or BA for the following matrices:

$$A = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 5 \\ -3 & 9 \end{bmatrix}$$

4D **8** I can determine the order of the resulting matrix formed under matrix multiplication. ☐

e.g. Determine the order of the answer matrix for the following matrix multiplication:

$$\begin{bmatrix} 9 & -11 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 7 & -6 & 2 & -2 \\ 4 & -3 & 1 & 5 \end{bmatrix}$$

4D **9** I can perform matrix multiplication by hand. ☐

e.g. Calculate the following matrix multiplication by hand:

$$\begin{bmatrix} 5 & 2 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 9 \\ -4 & -5 & 0 \end{bmatrix}$$

4D **10** I can perform matrix multiplication using a CAS calculator. ☐

e.g. Calculate the following matrix multiplication using a CAS calculator:

$$\begin{bmatrix} 6 & -11 & 3 & 7 \\ 0 & 8 & -2 & 8 \end{bmatrix}\begin{bmatrix} 12 & -3 \\ 1 & 5 \\ 4 & -6 \\ 10 & -2 \end{bmatrix}$$

4D **11** I can identify and construct an identity matrix of a given size. ☐

e.g. Construct an identity matrix of size 3×3.

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4E **12** I can find the inverse of a matrix. ☐

e.g. Find the inverse of the matrix $\begin{bmatrix} 7 & 2 \\ -1 & 3 \end{bmatrix}$

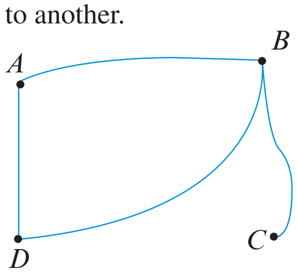
4E **13** I can solve simultaneous equations using matrices. ☐

e.g. Solve the following pair of simultaneous equations using matrices and a CAS calculator.

$$8x + 2y = 46 \qquad 5x - 3y = 19$$

4F **14** I can represent a road network in a matrix. ☐

e.g. Use a matrix to record the number of ways of travelling directly from one town to another.



4F **15** I can summarise relationships in a network in a matrix. ☐

e.g. Draw a network to show the direct connections between towns *A*, *B* and *C* given in the matrix below.

$$\begin{matrix} & \text{A} & \text{B} & \text{C} & \\ & \left[\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right. & \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} & \left.\begin{matrix} 1 \\ 1 \\ 0 \end{matrix}\right] & \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \end{matrix} \end{matrix}$$

4G **16** I can set up a transition matrix. ☐

e.g. A charity sends out a monthly letter asking for donations. They notice that 60% of patrons who donated last month will donate this month, while 30% of patrons who did not donate last month will donate this month. Construct a transition matrix that describes how the behaviour of patrons who donate (D) or do not donate (N) changes from month to month.

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Review

4G **17** I can interpret a transition matrix. ☐

e.g. The following transition matrix, T, can be used to describe whether customers bring their shopping bags to the supermarket (S) or leave them at home (H), based on their behaviour last week.

$$T = \begin{matrix} & \text{This week} & \\ & \begin{matrix} S & H \end{matrix} & \\ & \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} & \begin{matrix} S \\ H \end{matrix} \text{ Next week} \end{matrix}$$

If 150 people took their bags for their weekly shop, and 90 people left their bags at home this week, how many people do you expect to bring their own shopping bags next week?

4H **18** I can use recursion to generate a sequence state matrix. ☐

e.g. Consider the following matrices: one defining the initial populations in Geelong and Ballarat and the other, a transition matrix defining the proportion of the population moving between the two cities each year. Assume Geelong is listed first.

$$S_0 = \begin{bmatrix} 250\,000 \\ 110\,000 \end{bmatrix} \qquad T = \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix}$$

How many people are expected to be in Geelong after five years?

4H **19** I can determine what happens to a sequence of state matrices in the long term. ☐

e.g. In the example of Geelong and Ballarat above, how many people are expected to be in Geelong in the long term?

4I **20** I can use matrix multiplication to solve application problems. ☐

e.g. The first matrix gives the hours for which Tom (row 1) and Louise (row 2) agreed to chop firewood (column 1) and mow the lawns (column 2). The second matrix gives the hourly rate of pay for chopping firewood (row 1) and mowing the laws (row 2).

$$\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Use matrix multiplication to generate a matrix giving the total earnings for Tom and Louise.

4I **21** I can use row and column matrices to extract specific information from matrices. ☐

e.g. On Saturday, Jack's coffee van sold 98 cups of coffee and 25 cups of tea. On Sunday, he sold 85 cups of coffee and 31 cups of tea. The price of coffee is $4 and the price of a cup of tea is $3. Construct a matrix for his sales. Using matrix multiplication with a suitable row or column matrix, produce a matrix showing the total number of drinks he sold each day.

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Multiple-choice questions

Use the matrix, F, in Questions 1 and 2.

$$F = \begin{bmatrix} 4 & 8 & 6 \\ 5 & 1 & 7 \end{bmatrix}$$

1 The order of matrix F is

A 6 **B** 2×3 **C** 3×2 **D** $2 + 3$ **E** $3 + 2$

2 The element f_{21} is:

A 3 **B** 2 **C** 8 **D** 1 **E** 5

3 Three students were asked the number of electronic devices their family owned. The results are shown in this matrix.

	TVs	*DVD Players*	*Laptops*
Caroline	4	3	2
Delia	1	0	5
Emir	2	1	3

The number of laptops owned by Emir's family is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

4 The matrix below gives the numbers of roads directly connecting one town to another. The total number of roads directly connecting town E to other towns is

	D	*E*	*F*
D	0	2	1
E	2	0	3
F	1	3	0

A 0 **B** 2 **C** 3 **D** 5 **E** 12

5 For these two matrices to be equal, the required value of x is

$$\begin{bmatrix} 4 & 3x \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

A 2 **B** 3 **C** 4 **D** 6 **E** 18

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Use matrices M and N in Questions 6 to 10.

$$M = \begin{bmatrix} 7 & 6 \\ 4 & 3 \end{bmatrix} \quad N = \begin{bmatrix} 5 & -2 \\ 1 & 0 \end{bmatrix}$$

6 The matrix $M + N$ is

A $\begin{bmatrix} 12 & 8 \\ 5 & 3 \end{bmatrix}$ **B** $\begin{bmatrix} 12 & 4 \\ 4 & 3 \end{bmatrix}$ **C** $\begin{bmatrix} 12 & 4 \\ 4 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 12 & 4 \\ 5 & 3 \end{bmatrix}$ **E** $\begin{bmatrix} 12 & 4 \\ 5 & 0 \end{bmatrix}$

7 The matrix $M - N$ is

A $\begin{bmatrix} 2 & 8 \\ 3 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 8 \\ 3 & 3 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 8 \\ 4 & 3 \end{bmatrix}$

8 The matrix $N - N$ is

A 0 **B** $\begin{bmatrix} 0 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 5 & -2 \\ 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} -5 & 2 \\ -1 & 0 \end{bmatrix}$

9 The matrix $2N$ is

A $\begin{bmatrix} 10 & -4 \\ 2 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 7 & 0 \\ 3 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 10 & -4 \\ 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 10 & -2 \\ 2 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 10 & -4 \\ 2 & 2 \end{bmatrix}$

10 The matrix $2M + N$ is

A $\begin{bmatrix} 14 & 10 \\ 7 & 5 \end{bmatrix}$ **B** $\begin{bmatrix} 14 & 6 \\ 7 & 5 \end{bmatrix}$ **C** $\begin{bmatrix} 24 & 8 \\ 10 & 6 \end{bmatrix}$ **D** $\begin{bmatrix} 19 & 14 \\ 9 & 6 \end{bmatrix}$ **E** $\begin{bmatrix} 19 & 10 \\ 9 & 6 \end{bmatrix}$

Use the matrices P, Q, R and S in Questions 11 to 14.

$$P = \begin{bmatrix} 5 & 4 & 1 \\ 7 & 6 & 8 \end{bmatrix} \quad Q = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \quad R = \begin{bmatrix} 4 & 7 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

11 Matrix multiplication is not defined for

A PQ **B** SS **C** SP **D** PS **E** RS

12 The order of matrix QR is

A 1×1 **B** 3×2 **C** 2×3 **D** 6 **E** 5

13 Which of the following matrix multiplications gives a 1×3 matrix?

A QQ **B** RQ **C** PR **D** QR **E** RP

14 The matrix multiplication PQ gives the matrix

A $\begin{bmatrix} 34 \\ 50 \end{bmatrix}$ **B** $\begin{bmatrix} 10 & 24 & 0 \\ 14 & 36 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 10 & 14 \\ 24 & 0 \\ 36 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 5 & 4 & 1 \\ 7 & 6 & 8 \\ 2 & 6 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 34 & 50 \end{bmatrix}$

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15 The identity matrix for 2 × 2 matrices is

A $\begin{bmatrix} 1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 The inverse of the matrix, $A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ is

A $\begin{bmatrix} -4 & -5 \\ -2 & -3 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ **C** $\frac{1}{2}\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$ **E** $\frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$

Short-answer questions

Use matrix A in Questions 1 to 4.

$$A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 3 & 4 & 7 & 9 \end{bmatrix}$$

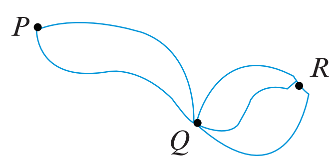
1 State the order of matrix A.

2 Identify the element $a_{1\,3}$.

3 If $C = [5 \quad 6]$, find CA.

4 If the order of a matrix, B, was 4×1, what would be the order of the matrix resulting from AB?

5 Roads are shown joining towns P, Q and R. Use a matrix to record the number of roads directly connecting one town to another town.



6 Use the matrices below:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 5 \\ 7 & 6 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

to find

a $3A$ **b** $A + B$ **c** $B - A$ **d** $2A + B$

e $A - A$ **f** AB **g** BA **h** A^{-1}

i A^2 **j** AI **k** AA^{-1}

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Written-response questions

1 Farms A and B have their livestock numbers recorded in the matrix shown.

$$\begin{array}{c} \\ \textit{Farm A} \\ \textit{Farm B} \end{array}\begin{array}{c} \begin{array}{ccc} \textit{Cattle} & \textit{Pigs} & \textit{Sheep} \end{array} \\ \begin{bmatrix} 420 & 50 & 100 \\ 300 & 40 & 220 \end{bmatrix} \end{array}$$

a How many pigs are on Farm B?

b What is the total number of sheep on both farms?

c Which farm has the largest total number of livestock?

2 A bakery recorded the sales for Shop A and Shop B of cakes, pies and rolls in a Sales matrix, S. The prices were recorded in the Prices matrix, P.

$$S = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{ccc} \textit{Cakes} & \textit{Pies} & \textit{Rolls} \end{array} \\ \begin{bmatrix} 12 & 25 & 18 \\ 15 & 21 & 16 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \\ \textit{Cakes} \\ \textit{Pies} \\ \textit{Rolls} \end{array}\begin{array}{c} \$ \\ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \end{array}$$

a How many pies were sold by Shop B?

b What is the selling price of pies?

c Calculate the matrix product SP.

d What information is contained in matrix SP?

e Which shop had the largest income from its sales? How much were its takings?

3 Patsy and Geoff decided to participate in a charity fun run.

a Patsy plans to walk for 4 hours and jog for 1 hour. Geoff plans to walk for 3 hours and jog for 2 hours. Construct a matrix showing how many hours Patsy and Geoff spend walking and jogging.

b Walking raises \$2 per hour and consumes 1500 kJ/h (kilojoules per hour). Jogging raises \$3 per hour and consumes 2500 kJ/h. Construct a matrix showing how much is earned and how many kilojoules are consumed by walking and jogging.

c Use matrix multiplication to find a matrix that shows the money raised and the kilojoules consumed by Patsy and Geoff.

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4 An insurance company wants to be able to predict the number of accidents their customers will have in future years. Historically, they know that 60% of their customers who had an accident (A) in the last year will have another accident in the next year. They also know that 5% of customers who did not have an accident (N) last year will have an accident next year.

a What percentage of customers who had an accident last year are expected not to have an accident next year?

b Construct a transition matrix to describe this situation. Call the matrix, T.

c Last year, within one region, 4000 of the insurance company's customers had an accident while 26 000 did not have an accident. The company decided to use this as the starting point for making their predictions. Write down a column matrix, S_0, that describes this situation.

d Using T and S_0, how many people do we expect to have an accident in the next year?

e How many people do we expect to have an accident in five years' time?

f In the long term, how many people do we expect to have an accident each year and how many do we expect to not have an accident?

5 Supermarkets sell eggs in cartons of 12, apples in packets of 8 and yoghurt tubs in sets of 4. This is represented by matrix A. The cost for each type of packet is given by matrix B.

$$A = \begin{array}{r} \\ \textit{Items per packet} \end{array}\begin{array}{c} \begin{array}{ccc} \textit{Eggs} & \textit{Apples} & \textit{Yoghurt} \end{array} \\ \begin{bmatrix} 12 & 8 & 4 \end{bmatrix} \end{array} \qquad B = \begin{array}{r} \textit{Eggs} \\ \textit{Apples} \\ \textit{Yoghurt} \end{array}\overset{\$}{\begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}}$$

The sales of each type of packet are given by matrix C as a column matrix and by matrix D as a row matrix.

$$C = \begin{array}{r} \textit{Eggs} \\ \textit{Apples} \\ \textit{Yoghurt} \end{array}\overset{\textit{Packets}}{\begin{bmatrix} 100 \\ 50 \\ 30 \end{bmatrix}} \qquad D = \begin{array}{r} \\ \textit{Packets} \end{array}\begin{array}{c} \begin{array}{ccc} \textit{Eggs} & \textit{Apples} & \textit{Yoghurt} \end{array} \\ \begin{bmatrix} 100 & 50 & 30 \end{bmatrix} \end{array}$$

Choose the appropriate matrices and use matrix multiplication to find:

a the total number of items sold (counting each egg, apple or yoghurt tub as an item)

b the total value of all sales.

6 We are told that 2 apples and 3 bananas cost \$6. This can be represented by the equation $2x + 3y = 6$, where x represents the cost of an apple and y the cost of a banana.

a Write an equation for: 6 apples and 5 bananas cost \$14.

b With your equation and the equation given, use matrix methods on your CAS calculator to find the cost of an apple and the cost of a banana.

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Chapter 5

Linear relations and modelling

Chapter questions

- How do we use a formula?
- How do we create a table of values with and without a CAS calculator?
- How do we solve linear equations?
- How do we develop a formula?
- How do we determine the slope of a straight-line graph?
- How do we find the equation of a straight-line graph?
- How do we sketch a straight-line graph from its equation?
- How do we use straight-line graphs to model practical situations?
- How do we find the intersection of two linear graphs?
- What are simultaneous equations and how do we solve them?
- How can we use simultaneous equations to solve practical problems?
- What are piecewise linear graphs?
- What are step graphs?

Linear relations and equations connect two or more variables, and they produce a straight line when graphed. Many everyday situations can be described and investigated using a linear graph and its equation. Examples occur in technology, science and business, such as the depreciating value of a newly purchased car or the short-term growth of a newly planted tree. In this chapter, the properties of linear graphs and their equations will be applied to modelling linear growth and decay in the real world.

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5A Substitution of values into a formula and constructing a table of values

Learning intentions

- To be able to solve practical problems by substituting values into formulas.
- To be able to construct a table of values.

Substituting values into a formula

A **formula** is a mathematical relationship connecting two or more variables.

For example:

- $C = 45t + 150$ is a formula for relating the cost, C dollars, of hiring a plumber for t hours. C and t are the variables.
- $P = 4L$ is a formula for finding the perimeter of a square, where P is the perimeter and L is the side length of the square. P and L are the variables.

By substituting all known variables into a formula, we are able to find the value of an unknown variable.

Example 1 Using a formula

The cost of hiring a windsurfer is given by the rule:

$$C = 10 + 40t$$

where C is the cost, in dollars, and t is the time, in hours. How much will it cost to hire a windsurfer for 2 hours?

Explanation	Solution
1 Write the formula.	$C = 10 + 40t$
2 To determine the cost of hiring a windsurfer for 2 hours, substitute $t = 2$ into the formula.	$C = 10 + 40 \times 2$
3 Evaluate.	$C = 90$
4 Write your answer.	It will cost \$90 for a 2-hour hire.

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Now try this 1 Using a formula (Example 1)

The cost of hiring a bobcat is:

$$C = 330 + 80t$$

where C is the cost, in dollars, and t is the time, in hours. How much will it cost to hire a bobcat for 10 hours?

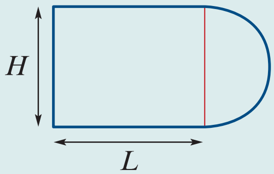
Hint 1 Substitute the value for t into the formula.

Hint 2 Remember that the answer needs to be given in $.

Example 2 Using a formula

The perimeter of this shape can be given by the formula:

$$P = 2L + H\left(1 + \frac{\pi}{2}\right)$$



In this formula, L is the length of the rectangle and H is the height. Find the perimeter, to one decimal place, if $L = 16.1$ cm and $H = 3.2$ cm.

Note: π is the ratio of the circumference of any circle to its diameter. It is an *irrational* number. An approximate value of π is 3.14159, to 5 decimal places. Calculators have a special key for π.

Explanation	Solution
1 Write the formula.	$P = 2L + H\left(1 + \frac{\pi}{2}\right)$
2 Substitute values for L and H into the formula.	$P = 2 \times 16.1 + 3.2\left(1 + \frac{\pi}{2}\right)$
3 Evaluate.	$P = 40.4$ (to one decimal place)
4 Give your answer with correct units.	The perimeter is 40.4 cm.

Now try this 2 Using a formula (Example 2)

The perimeter of a rectangle can be given by the formula:

$$\mathrm{P} = 2L + 2W$$

In this formula, L is the length of the rectangle and W is the width. Find the perimeter if $L = 26.5$ cm and $W = 14.8$ cm.

Hint 1 Substitute the values for L and W into the formula.

Hint 2 Remember to write the answer with correct units.

Constructing a table of values

We can use a formula to construct a **table of values**. This can be done by substitution (by hand) or using your TI-Nspire or ClassPad.

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Example 3 Constructing a table of values (by hand)

The formula for converting degrees Celsius to degrees Fahrenheit is given by:

$$F = \frac{9}{5}C + 32$$

Use this formula to construct a table of values for F, using values of C in intervals of 10 between $C = 0$ and $C = 100$.

Explanation

Draw up a table of values for:

$F = \frac{9}{5}C + 32$, and then substitute values of $C = 0, 10, 20, 30, \ldots$ into the formula to find F.

Solution

If $C = 0$, $F = \frac{9}{5}(0) + 32 = 32$

If $C = 10$, $F = \frac{9}{5}(10) + 32 = 50$

and so on.

The table would then look as follows:

C	0	10	20	30	40	50	60	70	80	90	100
F	32	50	68	86	104	122	140	158	176	194	212

Now try this 3 Constructing a table of values (by hand) (Example 3)

The perimeter of a square of side length L is given by the formula:

$$P = 4L$$

Use this formula to construct a table of values for P, using values of L in intervals of 10 between $L = 0$ and $L = 100$.

How to construct a table of values using the TI-Nspire

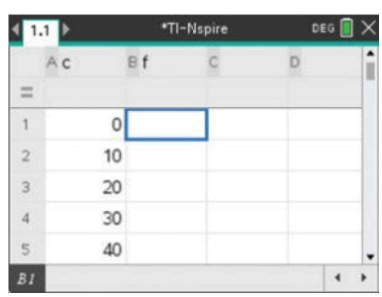
The formula for converting degrees Celsius to degrees Fahrenheit is given by:

$$F = \frac{9}{5}C + 32$$

Use this formula to construct a table of values for F, using values of C in intervals of 10 between $C = 0$ and $C = 100$.

Steps

1. Start a new document: **Press** ctrl + N.
2. Select **Add Lists & Spreadsheet**. **Name** the lists c (for Celsius) and f (for Fahrenheit). Enter the data 0–100 in intervals of 10 into a list named c, as shown.



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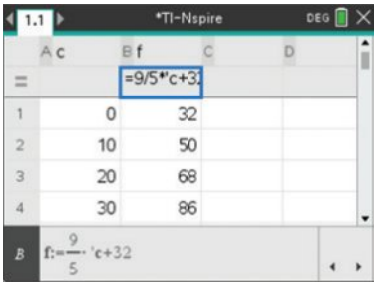
3 Place the cursor in the formula cell of column B (i.e. list f) and type in: **= 9 ÷ 5 × c + 32**.

Hint: If you typed in c you will need to select **Variable Reference** when prompted. This prompt occurs because c can also be a column name. Alternatively, pressing the var key and selecting **c** from the variable list will avoid this issue.

Press enter to display the values given.

Use the ▼ arrow to move down through the table.



How to construct a table of values using the ClassPad

The formula for converting degrees Celsius to degrees Fahrenheit is given by:

$$F = \frac{9}{5}C + 32$$

Use this formula to construct a table of values for F, using values of C in intervals of 10 between $C = 0$ and $C = 100$.

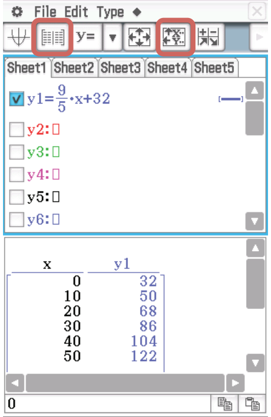
Steps

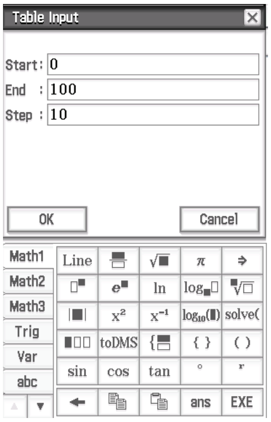
1 Enter the data into your calculator using the **Graph & Table** application. From the application menu screen, locate the built-in **Graph & Table** application, . Tap to open. Tapping Menu from the icon panel (just below the touch screen) will display the application menu if it is not already visible.

2 a Adjacent to **y1=** type in the formula $\frac{9}{5}x + 32$. Then press EXE.

b Tap the **Table Input** () icon to set the table entries as shown and tap OK.

c Tap the icon to display the required table of values. Scrolling down will show more values in the table.





Note: $y1$ is used to represent the variable F and x is used to represent the variable C.

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Section Summary

- A **formula** is a mathematical relationship connecting two or more variables.
- A **table of values** can be found by substitution (by hand) or using a TI-Nspire or ClassPad.

Exercise 5A

Building understanding

Example 1

1 The cost of hiring a removalist has a fixed fee of $300 and is given by the rule:

$$C = 300 + 120t$$

where C is the total cost in dollars and t is the number of hours for which the removalist is hired.

a Substitute $t = 1$ into the rule to find the cost of hiring the removalist for one hour.

b Substitute $t = 2$ into the rule to find the cost of hiring the removalist for two hours.

Example 2

2 The perimeter of a rectangle is given by the formula:

$$P = 2L + 2W$$

In this formula, L is the length of the rectangle and W is the width.

a Substitute $L = 4$ and $W = 2$ to find the perimeter of a rectangle with length 4 cm and width 2 cm.

b Substitute $L = 5.8$ and $W = 3.5$ to find the perimeter of a rectangle with length 5.8 cm and width 3.5 cm.

c What values for L and W would you substitute in to find the perimeter of a rectangle with length 7.9 cm and width 2.7 cm?

Example 3

3 A football club wishes to purchase a number of pies at a cost of $3 each. How much does it cost for:

a 43 pies? **b** 44 pies?

c If C is the cost ($) and x is the number of pies, complete the table by putting your answers to part **a** and **b** into the table. Then find the unknown values, showing the amount of money needed to purchase from 40 to 45 pies.

x	40	41	42	43	44	45
C($)						

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Developing understanding

4 The cost of hiring a dance hall is given by the rule:

$$C = 1200 + 50t$$

where C is the total cost, in dollars, and t is the number of hours for which the hall is hired.
Find the cost of hiring the hall for:

a 4 hours **b** 6 hours **c** 4.5 hours

5 The distance, d km, travelled by a car in t hours at an average speed of v km/h is given by the formula: $d = v \times t$.
Find the distance travelled by a car travelling at a speed of 95 km/h for 4 hours.

6 Taxi fares are calculated using the formula:

$$F = 4 + 1.3K$$

where K is the distance travelled, in kilometres, and F is the cost of the fare in dollars.
Find the costs of the following trips.

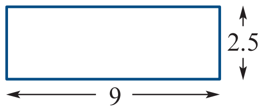
a 5 km **b** 8 km **c** 20 km

7 The circumference, C, of a circle with a radius, r, is given by $C = 2\pi r$.
Find, to two decimal places, the circumferences of the circles with the following radii.

a An earring with $r = 3$ mm **b** A circular garden bed with $r = 7.2$ m

8 If the area of a rectangle is given by $A = L \times W$, find the value of A for the following rectangles.

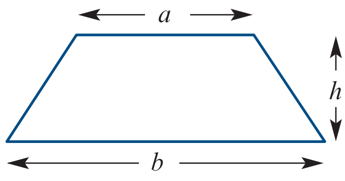
a $L = 3$ and $W = 4$ **b** $L = 15$ and $W = 8$ **c**



9 The area of a trapezium, as shown, is $A = \frac{1}{2}h(a + b)$.
Find A if:

a $h = 1, a = 3, b = 5$

b $h = 5, a = 2.5, b = 3.2$



10 The formula used to convert temperature from degrees Fahrenheit to degrees Celsius is:

$$C = \frac{5}{9}(F - 32)$$

Use this formula to convert the following temperatures to degrees Celsius. Give your answers to one decimal place.

a 50°F **b** 0°F **c** 212°F **d** 92°F

11 The formula for calculating simple interest is:

$$I = \frac{PRT}{100}$$

where P is the principal (amount invested or borrowed), R is the interest rate per annum and T is the time (in years). In the following questions, give your answers to the nearest cent (to two decimal places).

a Frank borrows \$5000 at 12% for 4 years. How much interest will he pay?

b Henry invests \$8500 for 3 years with an interest rate of 7.9%. How much interest will he earn?

12 In Australian football, a goal is worth 6 points and a behind is worth 1 point. The total number of points, P, is given by:

$$P = 6 \times \text{number of goals} + \text{number of behinds}$$

a Find the number of points if:

i 2 goals and 3 behinds are kicked **ii** 8 goals and 20 behinds are kicked.

b In a match, Redteam scores 4 goals and 2 behinds and Greenteam scores 3 goals and 10 behinds. Which team wins the match?

13 The circumference of a circle is given by:

$$C = 2\pi r$$

where r is the radius. Complete the table of values to show the circumferences of circles with radii from 0 to 1 cm in intervals of 0.1 cm. Give your answers to three decimal places.

r (cm)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
C (cm)	0	0.628	1.257	1.885							

14 A phone bill is calculated using the formula:

$$C = 40 + 0.18n$$

where C is the total cost and n represents the number of calls made. Complete the table of values to show the cost for 50, 60, 70, . . . 130 calls.

n	50	60	70	80	90	100	110	120	130
C(\$)	49	50.80	52.60						

15 The amount of energy (E) in kilojoules expended by an adult male of mass (M), at rest, can be estimated using the formula:

$$E = 110 + 9M$$

Complete the table of values in intervals of 5 kg for males of mass 60–120 kg to show the corresponding values of E.

M (kg)	60	65	70	75	80	85	90	95	100	105	110	115	120
E (kJ)	650	695											

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16 Anita has \$10 000 that she wishes to invest at a rate of 4.5% per annum. She wants to know how much interest she will earn after 1, 2, 3, ...10 years. Using the formula:

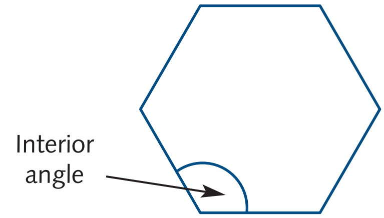
$$I = \frac{PRT}{100}$$

where P is the principal and R is the interest rate (%), construct a table of values, and use a calculator to find how much interest, I, she will have after $T = 1, 2, \ldots 10$ years.

17 The sum, S, of the interior angles of a polygon with n sides is given by the formula:

$$S = 90(2n - 4)$$

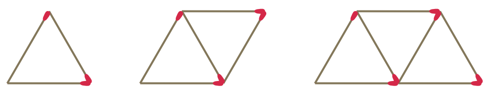
Construct a table of values showing the sum of the interior angles of polygons with 3 to 10 sides.



n	3	4	5					
S	180°	360°						

Testing understanding

18 The number of matchsticks used for each shape below follows the pattern 3, 5, 7, . . .



The rule for finding the number of matches used in this sequence is:

number of matches $= a + (n - 1)d$

where a is the number of matches in the first shape ($a = 3$), d is the number of extra matches used for each shape ($d = 2$) and n is the shape number.
Find the number of matches in the:

a 6th shape (so, $n = 6$) **b** 11th shape **c** 50th shape

19 Suggested cooking times for roasting x kilograms of different types of meat are given in the table.

Meat type	Minutes/kilogram
Chicken (well done)	45 min/kg + 20 mins
Lamb (medium)	55 min/kg + 25 mins
Lamb (well done)	65 min/kg + 30 mins
Beef (medium)	55 min/kg + 20 mins
Beef (well done)	65 min/kg + 30 mins

a How long, to the nearest minute, will it take to cook:

i a 2 kg chicken? **ii** 2.25 kg of beef (well done)?
iii a piece of lamb weighing 2.4 kg (well done)? **iv** 2.5 kg of beef (medium)?

b At what time should you put a 2 kg leg of lamb into the oven to have served medium at 7:30 p.m.?

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5B Solving linear equations and developing formulas

Learning intentions

- To be able to solve linear equations with one unknown.
- To be able to set up linear equations.

Solving linear equations

Practical applications of mathematics often involve the need to be able to solve **linear equations**. An equation is a mathematical statement that says that two things are equal. For example, these are all equations:

$x - 3 = 5$ $\qquad$ $2w - 5 = 17$ $\qquad$ $3m = 24$

Linear equations come in many different forms in mathematics but are easy to recognise because the powers on the unknown values are always 1. For example:

- $m - 4 = 8$ is a linear equation, with unknown value m
- $3x = 18$ is a linear equation, with unknown value x
- $4y - 3 = 17$ is a linear equation, with unknown value y
- $a + b = 0$ is a linear equation, with two unknown values, a and b
- $x^2 + 3 = 9$ is *not* a linear equation (the power of x is 2, not 1), with unknown value x
- $c = 16 - d^2$ is *not* a linear equation (the power of d is 2), with two unknowns, c and d.

The process of finding the unknown value is called solving the equation. When solving an equation, opposite (or inverse) operations are used so that the unknown value to be solved is the only term remaining on one side of the equation. Opposite operations are indicated in the table below.

Operation	+	−	×	÷
Opposite operation	−	+	÷	×

Remember: The equation must remain balanced. To balance an equation, add or subtract the same number to or from both sides of the equation or multiply or divide both sides of the equation by the same number.

Example 4 Solving a linear equation by hand

Solve the following linear equations:

a $x + 6 = 10$ $\qquad$ **b** $3y = 18$ $\qquad$ **c** $4(x - 3) = 24$

Explanation	Solution
a The equation needs to be 'undone,' leaving the unknown value by itself on one side of the equation.	
1 Write the equation.	$x + 6 = 10$

2 Subtract 6 from both sides of the equation. This is the opposite process to adding 6.

$$x + 6 - 6 = 10 - 6$$
$$\therefore x = 4$$

3 Check your answer by substituting the found value for x into the original equation. If each side gives the same value, the solution is correct.

$$\begin{aligned} \text{LHS} &= x + 6 \\ &= 4 + 6 \\ &= 10 \\ &= \text{RHS} \end{aligned}$$

$\therefore$ Solution is correct.

b **1** Write the equation.

$$3y = 18$$

2 The opposite process of multiplying by 3 is dividing by 3. Divide both sides of the equation by 3.

$$\frac{3y}{3} = \frac{18}{3}$$
$$\therefore y = 6$$

3 Check that the solution is correct by substituting $y = 6$ into the original equation.

$$\begin{aligned} \text{LHS} &= 3y \\ &= 3 \times 6 \\ &= 18 \\ &= \text{RHS} \end{aligned}$$

$\therefore$ Solution is correct.

c **Method 1**

1 Write the equation.

$$4(x - 3) = 24$$

2 Expand the brackets.

$$4x - 12 = 24$$

3 Add 12 to both sides of the equation.

$$4x - 12 + 12 = 24 + 12$$
$$4x = 36$$

4 Divide both sides by 4.

$$\frac{4x}{4} = \frac{36}{4}$$
$$\therefore x = 9$$

5 Check that the solution is correct by substituting $x = 9$ into the original equation (see 4 below).

Method 2

1 Write the equation.

$$4(x - 3) = 24$$

2 Divide both sides by 4.

$$\frac{4(x - 3)}{4} = \frac{24}{4}$$
$$x - 3 = 6$$

3 Add 3 to both sides of the equation.

$$x - 3 + 3 = 6 + 3$$
$$\therefore x = 9$$

4 Check that the solution is correct by substituting $x = 9$ into the orginal equation.

$$\begin{aligned} \text{LHS} &= 4(x - 3) \\ &= 4(9 - 3) \\ &= 4 \times 6 = 24 = \text{RHS} \end{aligned}$$

$\therefore$ Solution is correct.

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Now try this 4 Solving a linear equation by hand (Example 4)

Solve the equation $2x - 6 = 10$.

Hint 1 Add 6 to both sides of the equation.

Hint 2 Divide by 2.

Note: All of the above linear equations can be solved using the **solve(** command on a CAS calculator.

Example 5 Solving a linear equation using a CAS calculator

Solve the equation $-4 - 5b = 8$.

Explanation

1 Use the **solve(** command on your CAS calculator to solve for b, as shown opposite.

Note: 1. Set the mode of your calculator to Approximate (or press (ctrl + enter)) (TI-Nspire) or Decimal (ClassPad) before using **solve(**.
2. Ensure that you use the variable b on CAS.

Solution

solve $(-4 - 5b = 8, b)$ $\quad b = -2.4$

Now try this 5 Solving a linear equation using a CAS calculator (Example 5)

Solve the equation $-5 - 2x = 15$.

Hint 1 Use the **solve(** function on a CAS calculator.

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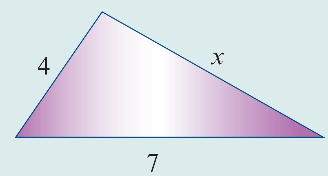
Developing a formula: setting up linear equations with one unknown

In many practical problems, we often need to set up a linear equation before finding the solution to a problem. Some practical examples are given below, showing how a linear equation is set up and then solved.

Example 6 Setting up a linear equation

Find an equation for the perimeter of this triangle.

Note: Perimeter is the distance around the outside of a shape.



Explanation

1. Choose a variable to represent the perimeter.
2. Add up all the sides of the triangle and let them equal the perimeter, P.
3. Write your answer.

Solution

Let P be the perimeter.

$P = 4 + 7 + x$

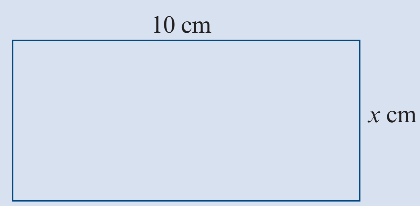
$P = 11 + x$

The required equation is:

$P = 11 + x$

Now try this 6 Setting up a linear equation (Example 6)

Find an equation for the perimeter of this rectangle.



Hint 1 Choose a variable to represent the perimeter.

Hint 2 Add up all the sides of the rectangle and let them equal the perimeter.

Hint 3 $x + x = 2x$

Example 7 Setting up and solving a linear equation

If 11 is added to a certain number, the result is 25. Find the number.

Explanation

1. Choose a variable to represent the number.
2. Using the information, write an equation.
3. Solve the equation by subtracting 11 from both sides of the equation.
4. Write your answer.

Solution

Let n be the number.

$n + 11 = 25$

$n + 11 - 11 = 25 - 11$

$\therefore n = 14$

The required number is 14.

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Now try this 7 Setting up and solving a linear equation (Example 7)

If 7 is added to a certain number, the result is 49. Find the number.

Hint 1 Choose a variable to represent the number.
Hint 2 Use the information to write an equation.
Hint 3 Solve the equation.

Example 8 Setting up and solving a linear equation

A car rental company has a fixed charge of \$110 plus \$84 per day for the hire of a car. The Brown family have budgeted \$650 for the hire of a car during their family holiday. For how many days can they hire a car?

Explanation	Solution
1 Choose a variable (d) for the number of days that the car is hired for. Use the information to write an equation.	Let d be the number of days that the car is hired for. $110 + 84d = 650$
2 Solve the equation. First, subtract 110 from both sides of the equation. Then divide both sides of the equation by 84.	$110 + 84d - 110 = 650 - 110$ $84d = 540$ $\frac{84d}{84} = \frac{540}{84}$ $\therefore d = 6.428$
3 Write your answer in terms of complete days. Car hire works on a daily rate, so 6.428 days is not an option. We therefore round down to 6 days to ensure that the Brown family stays within their budget of \$650.	The Brown family could hire a car for 6 days.

Now try this 8 Setting up and solving a linear equation (Example 8)

A 12-seater van can be hired for a fixed charge of \$56 plus \$80 per day. Chris and his friends have budgeted \$500 for the hire of a 12-seater van during their holiday. For how many days can they hire a 12-seater van?

Hint 1 Choose a variable for the number of days that the van is hired for.
Hint 2 Write an equation for the cost, using the given information.
Hint 3 Solve the equation.
Hint 4 Give your answer in whole days.

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Section Summary

- A **linear equation** is an equation whose unknown values are always to the power of one.
- To **solve** a linear equation means to find the value of the unknown variable.
- **Linear equations** can be solved by hand or by using a CAS calculator.
- **Linear equations** can be used to solve practical problems.

Skill-sheet

Exercise 5B

Building understanding

Example 4

1 Solve the following linear equations.

a $x + 6 = 15$ **b** $y + 11 = 26$ **c** $m - 5 = 1$

d $m - 5 = -7$ **e** $6 + e = 9$ **f** $-n + 5 = 1$

2 Solve the following linear equations.

a $5x = 15$ **b** $3g = 27$ **c** $6j = -24$

d $\frac{r}{3} = 4$ **e** $\frac{t}{-2} = 6$ **f** $\frac{h}{-8} = -5$

3 For the linear equation, $2a + 15 = 27$:

a What is the first step in solving this equation? **b** What is the second step?

c What is the solution to this equation?

4 For the linear equation, $\frac{y}{4} - 10 = 0$:

a What is the first step in solving this equation? **b** What is the second step?

c What is the solution to this equation?

Developing understanding

5 Solve the following linear equations.

a $v + 7 = 2$ **b** $9 - k = 2$ **c** $3 - a = -5$

d $-5b = -25$ **e** $13 = 3r - 11$ **f** $\frac{x+1}{3} = 2$

6 Solve the following linear equations. (You do not necessarily have to multiply brackets out first.)

a $2(y - 1) = 6$ **b** $8(x - 4) = 56$ **c** $3(g + 2) = 12$

d $3(4x - 5) = 21$ **e** $8(2x + 1) = 16$ **f** $3(5m - 2) = 12$

g $\frac{2(a-3)}{5} = 6$ **h** $\frac{4(r+2)}{6} = 10$

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7 Solve these equations by first ensuring that the unknown variable is on one side of the equation.

a $2x = x + 5$ **b** $2a + 1 = a + 4$ **c** $4b - 10 = 2b + 8$
d $7 - 5y = 3y - 17$ **e** $3(x + 5) - 4 = x + 11$ **f** $6(c + 2) = 2(c - 2)$
g $2f + 3 = 2 - 3(f + 3)$ **h** $5(1 - 3y) - 2(10 - y) = -10y$

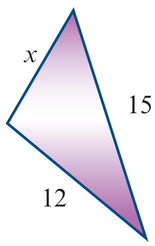
Example 5

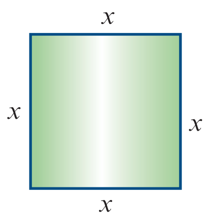
8 Solve the following linear equations using a CAS calculator. Give answers to one decimal place where appropriate.

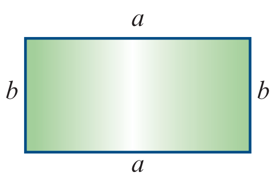
a $3a + 5 = 11$ **b** $4b + 3 = 27$ **c** $2w + 5 = 9$
d $7c - 2 = 12$ **e** $3y - 5 = 16$ **f** $4f - 1 = 7$
g $3 + 2h = 13$ **h** $2 + 3k = 6$ **i** $-4(g - 4) = -18$
j $\frac{2(s-6)}{7} = 4$ **k** $\frac{5(t+1)}{2} = 8$ **l** $\frac{-4(y-5)}{5} = 2.4$
m $2(x - 3) + 4(x + 7) = 10$ **n** $5(g + 4) - 6(g - 7) = 25$ **o** $5(p + 4) = 25 + (7 - p)$

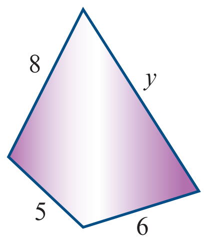
Example 6

9 Find an expression for the perimeter, P, for each of the following shapes.

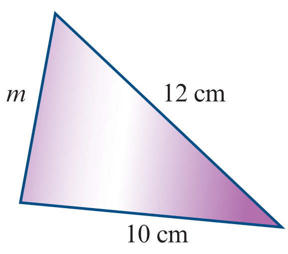
a


b


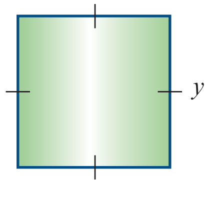
c


d


10 **a** Write an expression for the perimeter of this triangle.
b If the perimeter, P, of the triangle is 30 cm, solve the equation to find the value of m.



11 **a** Write an equation for the perimeter of this square.
b If the perimeter is 52 cm, what is the length of one side?



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Example 7 **12** Seven is added to a number and the result is 15.

a Write an equation, using n to represent the number.

b Solve the equation for n.

13 Five is added to twice a number and the result is 17. What is the number?

14 When a number is doubled and 15 is subtracted, the result is 103. Find the number.

15 A bicycle hire company charges a fixed cost of \$20 plus \$15 per hour of hire.

a Define a variable to use for the time. Remember to specify the units.

b Write a formula showing the cost, C, of hiring a bicycle, using your variable for time from part **a**.

c Find the cost of hiring a bicycle for:

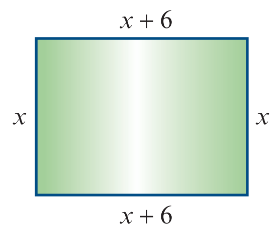
i 2 hours **ii** 5 hours

16 The perimeter of a rectangle is 84 cm. The length of the rectangle is 6 cm longer than the width, as shown in the diagram.

a Write an expression for the perimeter, P, of the rectangle.

b Find the value of x.

c Find the lengths of the sides of the rectangle.



Example 8 **17** Year 11 students want to run a social. The cost of hiring a band is \$820 and they are selling tickets at \$15 per person. The profit, P, is found by subtracting the band hire cost from the money raised from selling tickets. The students would like to make a profit of \$350. Use the information to write an equation, and then solve the equation to find how many tickets they need to sell.

Testing understanding

18 The cost for printing cards at the Stamping Printing Company is \$60 plus \$2.50 per card. Kate paid \$122.50 to print invitations for her party. How many invitations were printed?

19 A raffle prize of \$1000 is divided between Anne and Barry, so that Anne receives 3 times as much as Barry. How much does each person receive?

20 Bruce cycles x kilometres then walks half as far as he cycles. If the total distance covered is 45 km, find the value of x.

21 Amy and Ben live 17.2 km apart. They cycle to meet each other. Ben travels at 12 km/h and Amy travels at 10 km/h.

a How long (to the nearest minute) until they meet each other?

b What distances, to one decimal place, have they both travelled?

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5C Developing a formula: setting up and solving an equation in two unknowns

Learning intentions

- To be able to set up and solve linear equations with two unknowns.

It is often necessary to develop formulas so that problems can be solved. Constructing a formula is similar to developing an equation from a description.

Example 9 Setting up and solving a linear equation in two unknowns

Sausage rolls cost $1.30 each and party pies cost 75 cents each.

a Construct a formula for finding the cost, C dollars, of buying x sausage rolls and y party pies.

b Find the cost of 12 sausage rolls and 24 party pies.

Explanation	Solution
a 1 Work out a formula using x. One sausage roll costs \$1.30. Two sausage rolls cost $2 \times \$1.30 = \$2.60$. Three sausage rolls cost $3 \times \$1.30 = \$3.90$ *etc*. Write a formula using x.	x sausage rolls cost $x \times 1.30 = 1.3x$
2 Work out a formula using y. One party pie costs \$0.75. Two party pies cost $2 \times \$0.75 = \$1.50$ Three party pies cost $3 \times \$0.75 = \$2.25$ *etc*. Write a formula using y.	y party pies cost $y \times 0.75 = 0.75y$
3 Combine to get a formula for total cost, C.	$C = 1.3x + 0.75y$
b 1 Write the formula for C.	$C = 1.3x + 0.75y$
2 Substitute $x = 12$ and $y = 24$ into the formula.	$C = 1.3 \times 12 + 0.75 \times 24$
3 Evaluate.	$C = 33.6$
4 Give your answer in dollars and cents.	The total cost for 12 sausage rolls and 24 party pies is \$33.60.

Now try this 9 Setting up and solving a linear equation in two unknowns (Example 9)

Lemon tarts cost $3.50 each and apple crumble tarts cost $4.75 each.

a Construct a formula for finding the cost, C dollars, of buying x lemon tarts and y apple crumble tarts.

b Find the cost of 10 lemon tarts and 15 apple crumble tarts.

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Hint 1 Write out a rule for the cost of x lemon tarts.
Hint 2 Write out a rule for the cost of y apple crumble tarts.
Hint 3 Combine the two rules to give the total cost.
Hint 4 Substitute values for x and y in the formula.

Section Summary

- Practical problems can be solved by developing **formulas**.

Skill-sheet

Exercise 5C

Building understanding

Example 9

1 At a French patisserie, a baguette costs \$4.50 and a large family quiche costs \$14.30.
 a State the cost of buying x baguettes.
 b State the cost of buying y family quiches.
 c Construct a formula for the cost, \$$C$, of x baguettes and y family quiches.

2 The cost, \$$C$, of buying x muffins and y cookies is $C = 4x + 2.5y$.
 a How much does it cost to buy a single muffin?
 b How much does it cost to buy one cookie?
 c How much does it cost to buy 8 muffins and 4 cookies?

Developing understanding

3 Balloons cost 50 cents each and streamers costs 20 cents each.
 a Construct a formula for the cost, \$$C$, of x balloons and y streamers.
 b Find the cost of 25 balloons and 20 streamers.

4 Tickets to a concert cost \$40 for adults and \$25 for children.
 a Construct a formula for the total amount, \$$C$, paid by x adults and y children.
 b How much money altogether was paid by 150 adults and 315 children?

5 At the football canteen, chocolate bars cost \$1.60 and muesli bars cost \$1.40.
 a Construct a formula to show the total money, \$$C$, made by selling x chocolate bars and y muesli bars.
 b How much money would be made if 55 chocolate bars and 38 muesli bars were sold?

6 At the bread shop, custard tarts cost \$1.75 and iced doughnuts \$0.70 cents.
 a Construct a formula to show the total cost, \$$C$, if x custard tarts and y iced doughnuts are purchased.
 b On Monday morning, Mary bought 25 custard tarts and 12 iced doughnuts. How much did it cost her?

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7 At the beach cafe, Marion takes orders for coffee and milkshakes. A cup of coffee costs \$3.50 and a milkshake costs \$5.00.

a Let x = number of coffees ordered and y = number of milkshakes ordered. Using x (coffee) and y (milkshakes), write a formula showing the cost, \$$C$, of the number of coffees and milkshakes ordered.

b Marion took orders for 52 cups of coffee and 26 milkshakes. How much money did this make?

8 Joe sells budgerigars for \$30 and parrots for \$60.

a Write a formula showing the money, \$$C$, made by selling x budgerigars and y parrots.

b Joe sold 28 parrots and 60 budgerigars. How much money did he make?

Testing understanding

9 James has been saving fifty-cent and twenty-cent pieces.

a If James has x fifty-cent pieces and y twenty-cent pieces, write a formula to show the number, N, of coins that James has.

b Write a formula to show the value, V dollars, of James's collection.

c When James counts his coins, he has 45 fifty-cent pieces and 77 twenty-cent pieces. How much money does he have in total?

5D Drawing straight-line graphs and finding their slope

Learning intentions

- To be able to draw a straight-line graph.
- To be able to find the slope of a straight line.

Plotting straight-line graphs

Relations defined by equations such as:

$$y = 1 + 2x \qquad y = 3x - 2 \qquad y = 10 - 5x \qquad y = 6x$$

are linear relations, and they generate straight-line graphs.

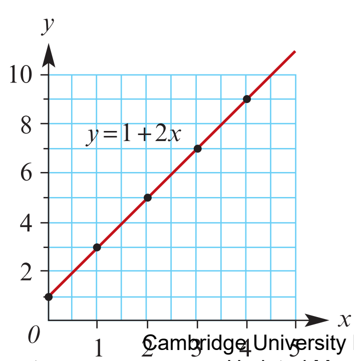
For example, consider the relation $y = 1 + 2x$. To plot this graph, we can form a table.

x	0	1	2	3	4
y	1	3	5	7	9

We can then plot the values from the table on a set of axes, as shown opposite.

The points appear to lie on a straight line.

A ruler can then be used to draw in this straight line to give the graph of $y = 1 + 2x$.



Example 10 Constructing a graph from a table of values

Plot the graph of $y = 8 - 2x$ by forming a table of values of y, using $x = 0, 1, 2, 3, 4$.

Explanation

1 Set up the table of values.
When $x = 0$, $y = 8 - 2 \times 0 = 8$.
When $x = 1$, $y = 8 - 2 \times 1 = 6$,
and so on.

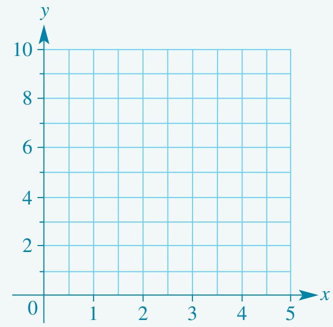
2 Draw, label and scale a set of axes to cover all values.

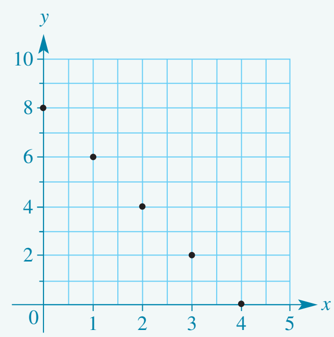
3 Plot the values in the table on the graph by marking with a dot (●). The first point is (0, 8). The second point is (1, 6), and so on.

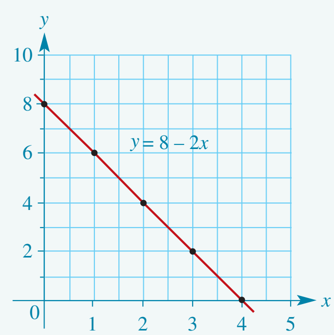
4 The points appear to lie on a straight line. Use a ruler to draw in the straight line. Label the line $y = 8 - 2x$.

Solution

x	0	1	2	3	4
y	8	6	4	2	0







Now try this 10 Constructing a graph from a table of values (Example 10)

Plot the graph of $y = 9 - 3x$ by forming a table of values of y, using $x = 0, 1, 2, 3$.

Hint 1 Draw up a table of values.

Hint 2 Plot points on a scaled set of axes and connect with a ruler.

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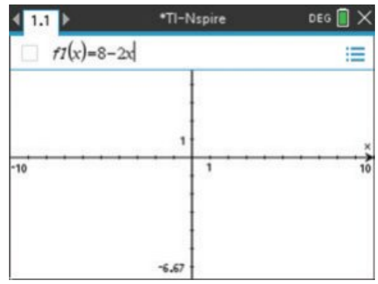
How to draw a straight-line graph and show a table of values using the TI-Nspire

Use a CAS calculator to draw the graph $y = 8 - 2x$ and show a table of values.

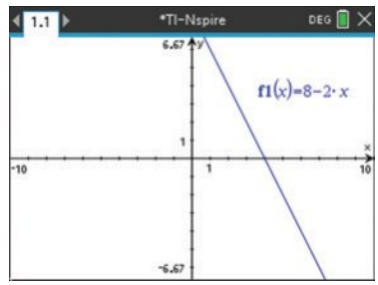
Steps

1 Start a new document ([ctrl] + [N]) and select **Add Graphs**.

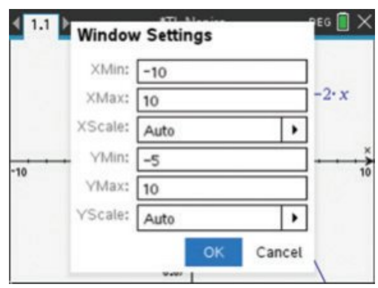


2 Type in the equation as shown. Note that f1(x) represents the y. Press [enter] to obtain the graph on the right.

Hint: If the function entry line is not visible, press [tab].

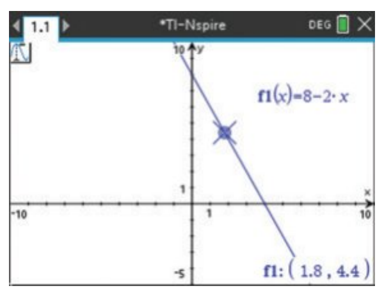


3 Change the window setting to see the key features of the graph. Use [menu]>**Window/Zoom**>**Window Settings** and edit as shown. Use the [tab] key to move between the entry lines. Press [enter] when finished editing the settings. The re-scaled graph is shown on the right.

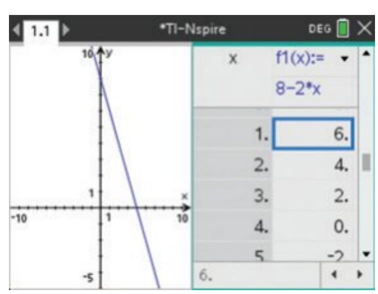


4 To show values on the graph, use [menu]>**Trace**>**Graph Trace** and then use the ◄► arrows to move along the graph.

Note: Press [esc] to exit the **GraphTrace** tool.



5 To show a table of values, press [ctrl] + [T]. Use the ▲▼ arrows to scroll through the values in the table.



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How to draw a straight-line graph and show a table of values using the ClassPad

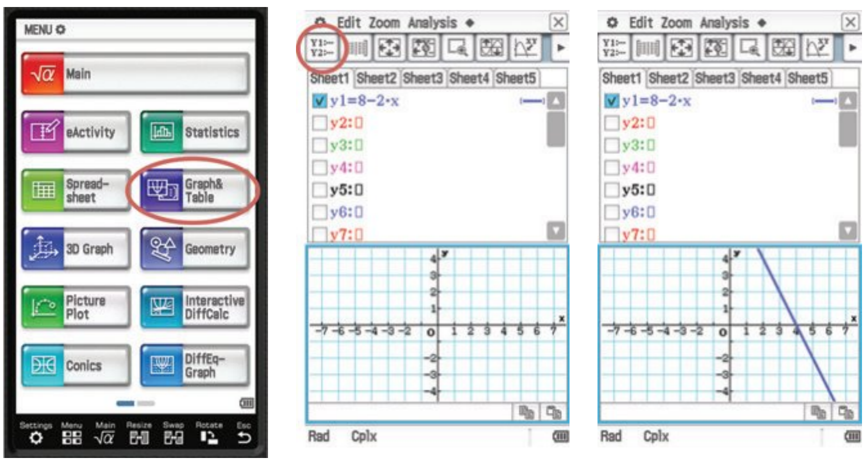
Use a CAS calculator to draw the graph of $y = 8 - 2x$ and show a table of values.

Steps

1 Open the **Graphs and Table** application.

2 Enter the equation into the graph editor window by typing **8 – 2*x***. Tick the box and press EXE.

3 Tap the icon to plot the graph.

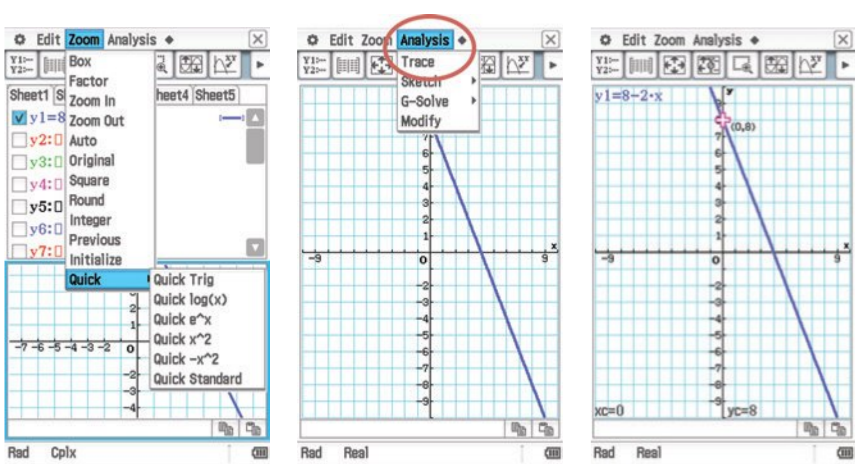


4 To adjust the graph screen, go to **Zoom > Quick > Quick Standard**. Quick Standard changes the window settings to [-10, 10] in the x and y directions.

5 Tap resize from the toolbar to fill the screen with the graph window.

6 Select **Analysis>Trace** to place a cursor on the graph. The coordinates of the point will be displayed at the location of the cursor. E.g. (0, 8).

7 Use the cursor key to move the cursor along the line.



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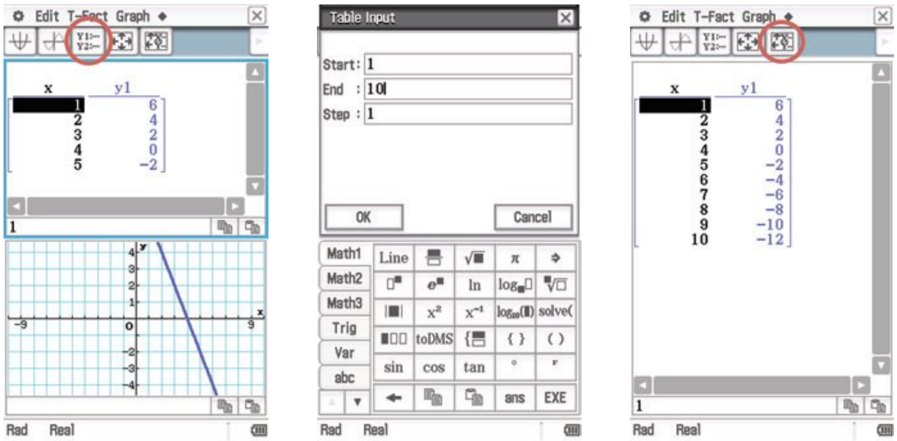
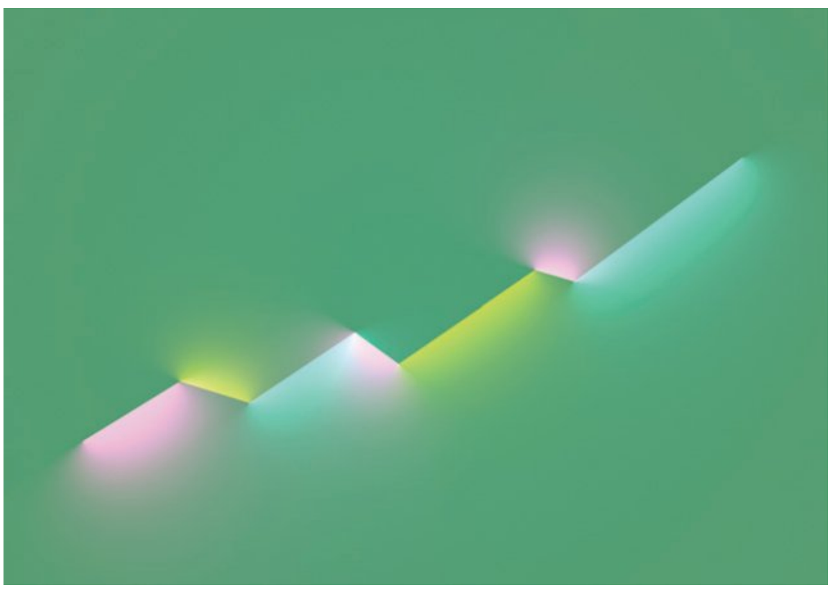
8 Tap the icon from the toolbar to display a table of values.

9 Tap the icon from the toolbar to open the **Table Input** dialog box. The values displayed in the table can be adjusted by changing the values in this window.



10 Tap resize from the toolbar to fill the screen with the table window.



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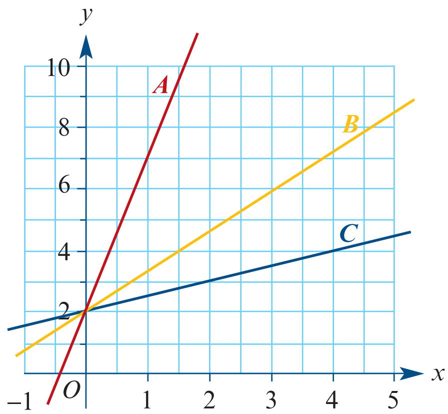
Positive and negative slopes of a straight line

One thing that makes one straight-line graph look different from another is its steepness or **slope**. Another name for slope is **gradient**[1].

For example, the three straight lines on the graph below all cut the y-axis at $y = 2$, but they have quite different slopes.



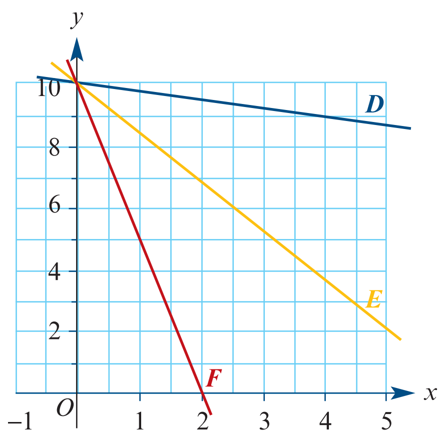
Line A has the steepest slope while Line C has the gentlest slope. Line B has a slope somewhere in between.

In all cases, the lines have **positive slopes**; that is, they rise from left to right.

Similarly, the three straight lines on the graph opposite all cut the y-axis at $y = 10$, but they have quite different slopes.

In this case, Line D has the gentlest slope while Line F has the steepest slope. Line E has a slope somewhere in between.

In all cases, the lines have **negative slopes**; that is, they fall from left to right.



[1] Note: For linear graphs, the terms slope and gradient mean the same thing. However, when dealing with practical applications of linear graphs and, most particularly, in statistics applications, the word 'slope' is preferred. For this reason, and to be consistent with the VCE General Mathematics curricula, this is the term used throughout this book.

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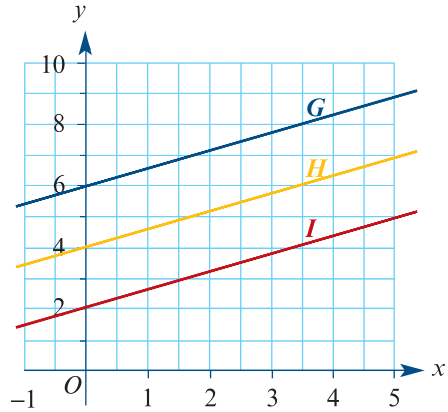
By contrast, the three straight lines, *G*, *H* and *I*, on the graph opposite, cut the *y*-axis at different points, but they all have the *same* slope.



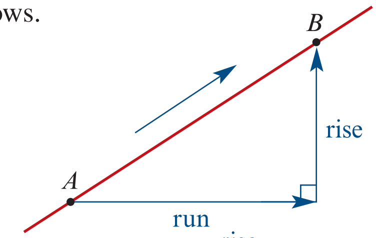
Calculating the slope

When talking about the **slope of a straight line**, we want to be able to do more than say that it has a gentle positive slope. We would like to be able to give the slope a value that reflects this fact. We do this by defining the slope of a line as follows.

First, two points, *A* and *B*, on the line are chosen. As we go from *A* to *B* along the line, we move:

- up by a distance called the **rise**
- and across by a distance called the **run**.

The slope is found by dividing the rise by the run.



$$\textbf{slope} = \frac{\textbf{rise}}{\textbf{run}}$$

Example 11 Finding the slope of a line from a graph: positive slope

Find the slope of the line through the points (1, 4) and (4, 8).

Explanation

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

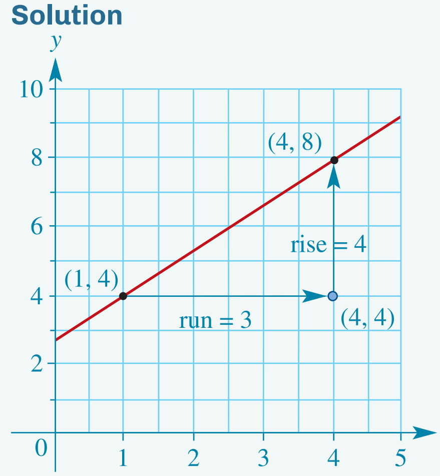
$$\text{rise} = 8 - 4 = 4$$

$$\text{run} = 4 - 1 = 3$$

$$\therefore \text{ slope} = \frac{4}{3} = 1.33 \text{ (to 2 decimal places)}$$

Note: To find the 'rise,' look at the *y*-coordinates.
To find the 'run,' look at the *x*-coordinates.

Solution



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Now try this 11 **Finding the slope of a line from a graph: positive slope (Example 11)**

Find the slope of the line through the points (2, 6) and (3, 9).

Hint 1 By sketching a graph, find the values for the rise and the run.

Hint 2 Divide the rise by the run.

Example 12 **Finding the slope of a line from a graph: negative slope**

Find the slope of the line through the points (0, 10) and (4, 2).

Explanation

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

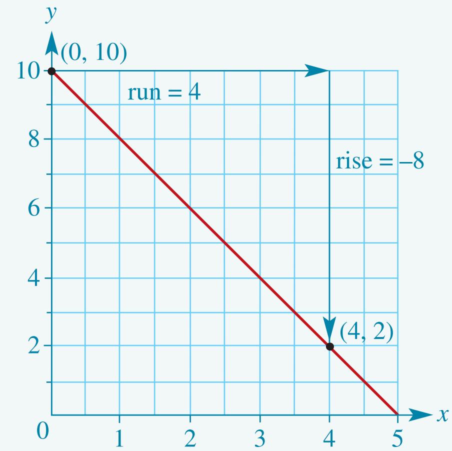
$$\text{rise} = 2 - 10 = -8$$

$$\text{run} = 4 - 0 = 4$$

$$\therefore \text{slope} = \frac{-8}{4} = -2$$

Note: In this example, we have a negative 'rise' which represents a 'fall'.

Solution



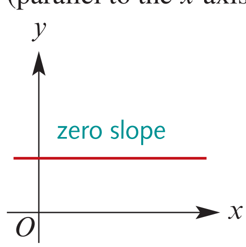
Now try this 12 **Finding the slope of a line from a graph: negative slope (Example 12)**

Find the slope of the line through the points (1, 4) and (5, 2).

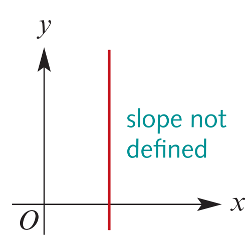
Hint 1 By sketching a graph, find the values for the rise and the run.

Hint 2 Divide the rise by the run.

A straight-line graph that is horizontal (parallel to the x-axis) has a slope of **zero**.



A straight-line graph that is vertical (parallel to the y-axis) has a slope that is **undefined**.



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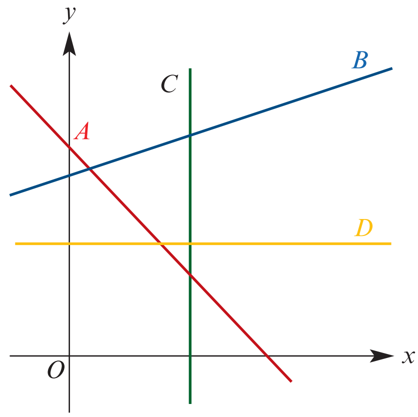
Section Summary

- The graph of a linear equation is a straight line.
- The **slope** of a straight-line graph tells us the steepness of the graph.
- The **slope** of a graph can be **positive**, **negative**, **zero** or **undefined**.
- A **positive slope** rises from left to right.
- A **negative slope** falls from left to right.
- The slope of a straight-line graph that is horizontal (parallel to the x-axis) is **zero**.
- The slope of a straight-line graph that is vertical (parallel to the y-axis) is **undefined**.

Exercise 5D

Building understanding

1 Identify the slope of each of the straight-line graphs A, B, C and D as: positive, negative, zero, or undefined.



2 Complete the table below for the equation $y = 3 + 2x$.

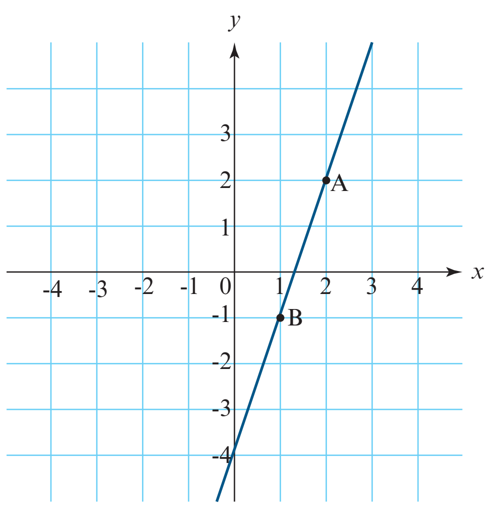
x	−2	−1	0	1	2	3
y	−1				7	

3 Consider the following graph.

a State the rise from point B to point A.

b State the run from point B to point A.

c Using your answers from part **a** and **b**, state the slope of the line.



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Developing understanding

Example 10

4 Plot the graph of the linear equations below by first completing a table of values for y when $x = 0, 1, 2, 3, 4$.

a $y = 2 + x$

b $y = 1 + 2x$

c $y = 10 - x$

d $y = 9 - 2x$

5 Use your CAS calculator to plot a graph and generate a table of values of y for $x = 0, 1, 2, 3, 4$.

a $y = 4 + x$

b $y = 2 + 3x$

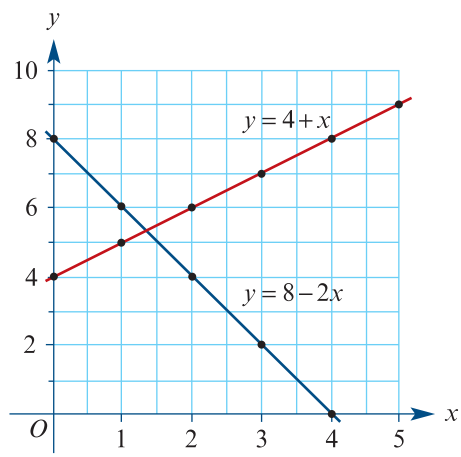
c $y = 10 + 5x$

d $y = 100 - 5x$

6 Two straight-line graphs, $y = 4 + x$ and $y = 8 - 2x$, are plotted as shown opposite.

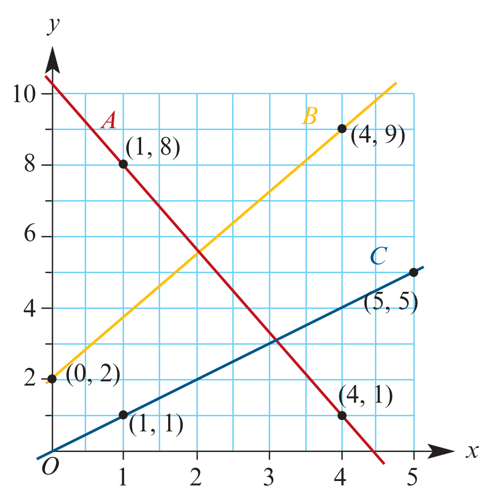
a Reading from the graph of $y = 4 + x$, determine the missing coordinates: (0, ?), (2, ?), (?, 7), (?, 9).

b Reading from the graph of $y = 8 - 2x$, determine the missing coordinates: (0, ?), (1, ?), (?, 4), (?, 2).



Example 11

7 Find the slope, to two decimal places, of each of the lines (A, B, C) shown on the graph below.



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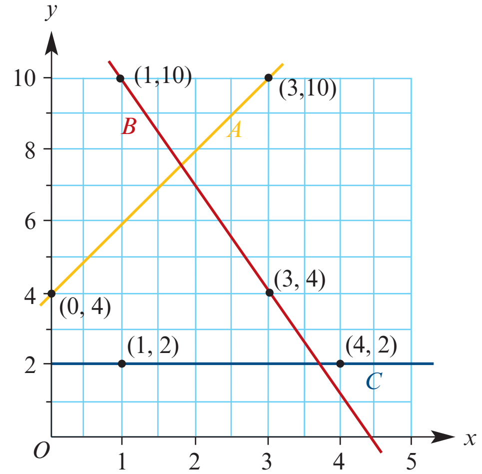
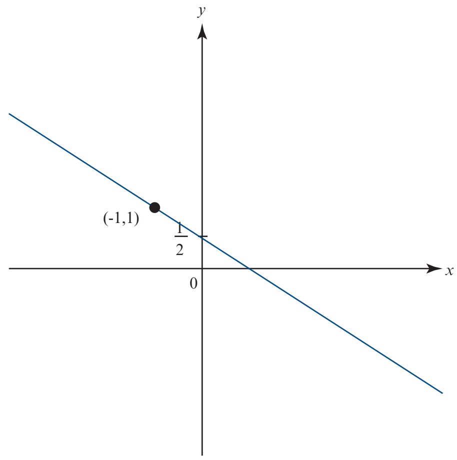
Example 12 **8** Find the slope of each of the lines (A, B, C) shown on the graph below.



Testing understanding

9 Find the slope of the graph.



10 Consider the following rules:

- $y = 3x$
- $y = 3x + 1$
- $y = 3x + 2$

a Sketch each line.

b Find the slope of each line.

c What do you notice about the lines and your answer to part **b**?

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5E Equations of straight lines

Learning intentions

- To be able to use the formula for finding the slope of a straight line.
- To be able to find the y-intercept and slope from a straight-line equation.

A formula for finding the slope of a line

While the 'rise/run' method for finding the slope of a line will always work, some people prefer to use a formula for calculating the slope. The formula is derived as follows.

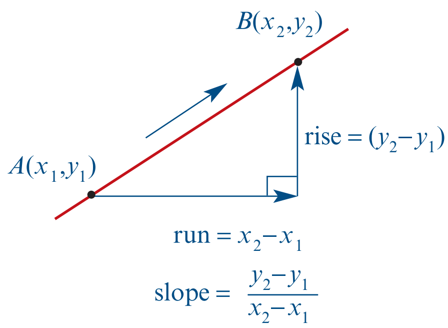
Label the coordinates of point A: (x_1, y_1).

Label the coordinates of point B: (x_2, y_2).

By definition: slope $= \dfrac{\text{rise}}{\text{run}}$.

From the diagram:

$$\text{rise} = y_2 - y_1$$
$$\text{run} = x_2 - x_1$$



The slope of a line can be found by substituting in the formula:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 13 Finding the slope of a line using the formula for the slope

Find the slope of the line that passes through the points (1, 7) and (4, 2) using the formula for the slope of a line. Give your answer to two decimal places.

Explanation

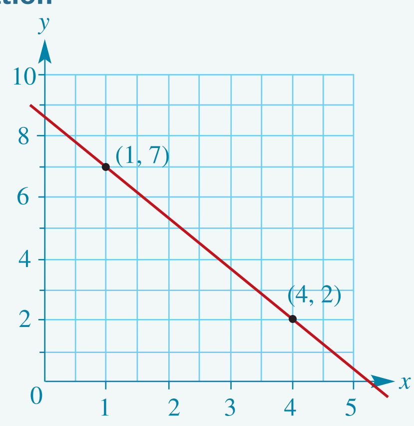
Use

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Let $(x_1, y_1) = (1, 7)$ and $(x_2, y_2) = (4, 2)$.

Note: To use this formula it does not matter which point you call (x_1, y_1) and which point you call (x_2, y_2). The rule still works.

Solution



$$\text{slope} = \frac{2-7}{4-1}$$
$$= -1.67 \text{ (to 2 d.p.)}$$

Now try this 13 **Finding the slope of a line using the formula for the slope (Example 13)**

Find the slope of the line through the points (2, 8) and (6, 3) using the formula for the slope of a line. Give your answer to two decimal places.

Hint 1 Write down the formula for the slope.

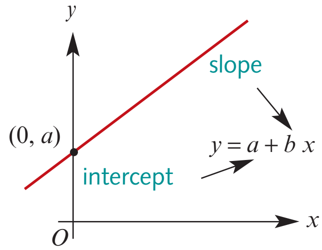
Hint 2 Substitute your values into the formula.

The intercept-slope form of the equation of a straight line

We can write the equation* of a straight line in the form: $y = a + bx$.

We call $y = a + bx$ the **intercept-slope form** of the equation of a straight line because:

- a = the ***y*-intercept** of the graph (i.e, when $x = 0$)
- b = the **slope** of the graph.



The intercept–slope form of the equation of a straight line is useful in modelling relationships in many practical situations. It is also the form used in **bivariate** (two-variable) statistics.

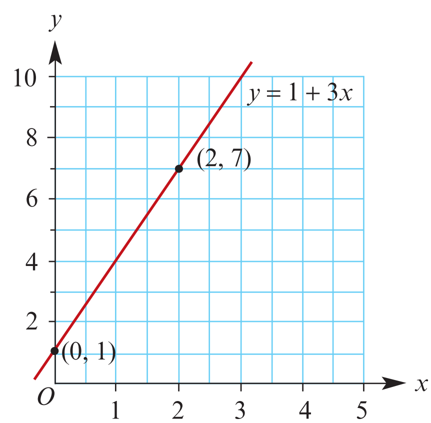
An example of the equation of a straight line written in intercept–slope form is $y = 1 + 3x$.

Its graph is shown opposite.

From the graph we see that the:

$$y\text{-intercept} = 1$$

$$\text{slope} = \frac{7-1}{2-0} = \frac{6}{2} = 3$$



That is:

- the y-intercept corresponds to the (*constant*) term in the equation (intercept = 1)
- the slope is given by the *coefficient of x* in the equation (slope = 3).

* Note: You may be used to writing the equation of a straight line as $y = mx+c$. However, when we are using a straight-line graph to model (represent) real world phenomena, we tend to reverse the order of the terms and use 'a' for the intercept and 'b' for the slope (rather than 'c' and 'm') and write the equation as $y = a + bx$. This is particularly true in performing statistical computations where your calculator will use 'a' for slope and 'b' for the y-intercept. You will see this later in the book, so it is worth making the change now. This is also the language used in the VCE written examinations.

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Intercept–slope form of the equation of a straight line

When we write the equation of a straight line in the form:

$$y = a + bx$$

we are using the **intercept-slope form** of the equation of a straight line.

a = the y-intercept of the graph (where the graph cuts the y-axis)

b = the slope of the graph

Example 14 Finding the *y*- intercept and slope of a line from its equation

Write down the y-intercept and slope of each of the straight-line graphs defined by the following equations.

a $y = -6 + 9x$ **b** $y = 10 - 5x$ **c** $y = -2x$ **d** $y - 4x = 5$

Explanation

For each equation:

1 Write the equation. If it is not in intercept–slope form, rearrange the equation.

2 Write down the y-intercept and slope. When the equation is in intercept–slope form, $y = a + bx$, the value of:

a = the y-intercept (the constant term)

b = the slope (the coefficient of x).

Solution

a $y = -6 + 9x$

y-intercept $= -6$ slope $= 9$

b $y = 10 - 5x$

y-intercept $= 10$ slope $= -5$

c $y = -2x$ or $y = 0 - 2x$

y-intercept $= 0$ slope $= -2$

d $y - 4x = 5$ or $y = 5 + 4x$

y-intercept $= 5$ slope $= 4$

Now try this 14 Finding the *y*-intercept and slope of a line from its equation (Example 14)

Write down the y-intercept and slope of the following straight-line graphs.

a $y = 4 - 2x$ **b** $y = 3x$ **c** $y - 3x = 6$

Hint 1 Rearrange equations, if necessary, into intercept-slope form.

Hint 2 Remember in the intercept-slope form, $y = a + bx$, a is the y-intercept and b is the slope.

Example 15 Writing the equation of a straight line given its *y*-intercept and slope

Write down the equations of the straight lines with the following y-intercepts and slopes.

a y-intercept $= 9$ slope $= 6$ **b** y-intercept $= 2$ slope $= -5$

c y-intercept $= -3$ slope $= 2$

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Explanation

The equation of a straight line is $y = a + bx$. In this equation, a = y-intercept and b = slope.

Form an equation by inserting the given values of the y-intercept and the slope for a and b in the standard equation $y = a + bx$.

Solution

a y-intercept = 9 slope = 6

equation : $y = 9 + 6x$

b y-intercept = 2 slope = −5

equation : $y = 2 + (-5)x$

or $y = 2 - 5x$

c y-intercept = −3 slope = 2

equation : $y = -3 + 2x$

or $y = 2x - 3$

Now try this 15 **Writing the equation of a straight line given its y-intercept and slope (Example 15)**

Write down the equations of the straight lines with the following y-intercepts and slopes.

a y-intercept = 5 slope = 3

b y-intercept = 6 slope = −2

c y-intercept = −4 slope = 5

Hint 1 Write down the equation of a straight line in **intercept-slope** form.

Hint 2 Substitute in values for the y-intercept and the slope.

Sketching straight-line graphs

Since only two points are needed to draw a straight line, all we need to do is find two points on the graph and then draw a line passing through these two points. When the equation of a straight line is written in intercept–slope form, one point on the graph is immediately available: the y-intercept. A second point can then be quickly calculated by substituting a suitable value of x into the equation.

When we draw a graph in this manner, we call it a **sketch graph**.

Example 16 **Sketching a straight-line graph from its equation**

Sketch the graph of $y = 8 + 2x$.

Explanation

1 Write the equation of the line.

2 As the equation is in intercept–slope form, the y-intercept is given by the constant term. Write it down.

3 Find a second point on the graph. Choose an x-value (not 0) that makes the calculation easy: $x = 5$ would be suitable.

Solution

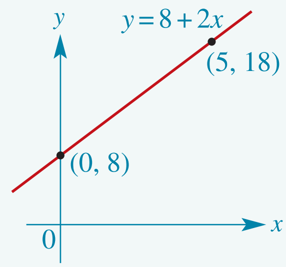
$y = 8 + 2x$

y-intercept = 8

When $x = 5$, $y = 8 + 2(5) = 18$

$\therefore$ (5, 18) is a point on the line.

4 To sketch the graph:
- draw a set of labelled axes
- mark in the two points with coordinates
- draw a straight line through the points
- label the line with its equation.



Now try this 16 Sketching a straight-line graph from its equation (Example 16)

Sketch the graph of $y = 5 + 3x$.

Hint 1 Find the y-intercept.

Hint 2 Choose a value for x to find another point on the graph.

Section Summary

- The formula for finding the **slope** of a line is $\dfrac{y_2 - y_1}{x_2 - x_1}$.
- The **intercept-slope form** of a straight-line equation is $y = a + bx$ where a is the **y-intercept** and b is the **slope**.

Skill-sheet

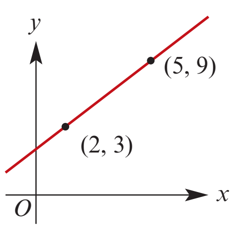
Exercise 5E

Building understanding

Example 13

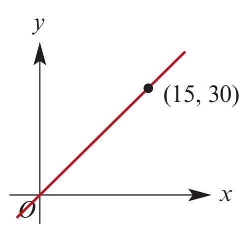
1 Use the formula, slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$ to find the slope of each of the lines shown below. An uncompleted formula is under each graph to assist you.

a



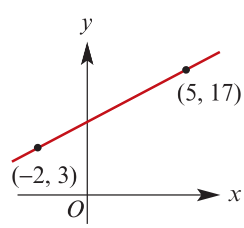
slope $= \dfrac{9 - \square}{5 - \square}$

b



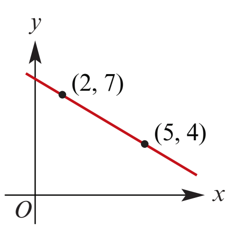
slope $= \dfrac{30 - \square}{\square - 0}$

c



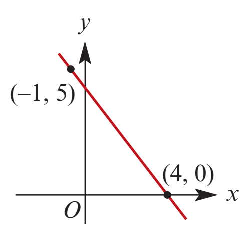
slope $= \dfrac{\square - \square}{\square - \square}$

d



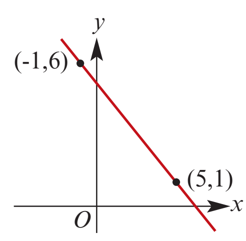
slope $= \dfrac{7 - \square}{2 - \square}$

e



slope $= \dfrac{5 - \square}{\square - 4}$

f



slope $= \dfrac{\square - \square}{\square - \square}$

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Developing understanding

Example 14

2 Write down the y-intercepts and slopes of the straight lines with the following equations.

a $y = 5 + 2x$ **b** $y = 6 - 3x$ **c** $y = 15 - 5x$ **d** $y = 3x$

3 Find the y-intercept and the slope by first rearranging the equations to make y the subject.

a $y + 3x = 10$ **b** $4y + 8x = -20$ **c** $x = y - 4$ **d** $x = 2y - 6$
e $2x - y = 5$ **f** $y - 5x = 10$ **g** $2.5x + 2.5y = 25$ **h** $y - 2x = 0$
i $y + 3x - 6 = 0$ **j** $y = 3$ **k** $4x - 5y - 8 = 7$ **l** $2y - 8 = 2(3x - 6)$

Example 15

4 Write down the equation of the straight line that has:

a y-intercept = 2, slope = 5 **b** y-intercept = 5, slope = 10
c y-intercept = −2, slope = 4 **d** y-intercept = 0, slope = −3
e y-intercept = −2, slope = 0 **f** y-intercept = 1.8, slope = −0.4
g y-intercept = 2.9, slope = −2 **h** y-intercept = −1.5, slope = −0.5

Example 16

5 Sketch the graphs of the straight lines with the following equations, clearly showing the y-intercepts and the coordinates of one other point.

a $y = 5 + 2x$ **b** $y = 5 + 5x$ **c** $y = 20 - 2x$
d $y = -10 + 10x$ **e** $y = 4x$ **f** $y = 16 - 2x$

Testing understanding

6 Sketch the graphs of the following straight lines.

a $y - x = 3$ **b** $y + 2x = 1$ **c** $y - 3x - 4 = 0$

5F Finding the equation of a straight-line graph

Learning intentions

- ▶ To be able to use the y-intercept and the slope to find the equation of a straight line.
- ▶ To be able to use two points on a graph to find the equation of a straight line.

Using the intercept and slope to find the equation of a straight line

We have learned how to construct a straight-line graph from its equation. We can also determine the equation from a graph. In particular, if the graph shows the y-intercept, it is a relatively straightforward procedure.

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Finding the equation of a straight-line graph from its intercept and slope

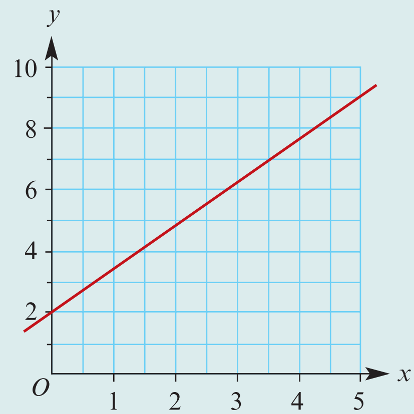
To find the equation of a straight line in intercept–slope form ($y = a + bx$) from its graph:

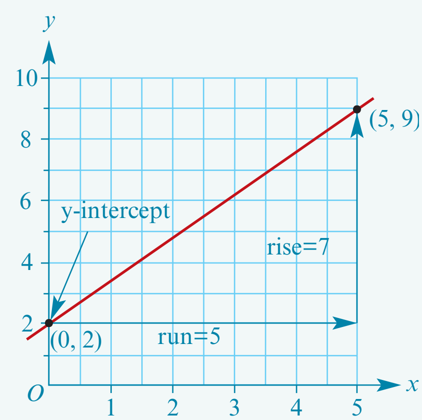
1. identify the y-intercept (a)
2. use two points on the graph to find the slope (b)
3. substitute these two values into the standard equation $y = a + bx$.

Note: This method *only works* when the graph *scale includes* $x = 0$.

Example 17 Finding the equation of a line: intercept–slope method

Determine the equation of the straight-line graph shown opposite.



Explanation	Solution
1 Write the general equation of a line in intercept–slope form.	$y = a + bx$
2 Read the y-intercept from the graph. **3** Find the slope using two well-defined points on the line, for example, (0, 2) and (5, 9).	 y-intercept = 2 so $a = 2$ $\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{7}{5} = 1.4$ so $b = 1.4$
4 Substitute the values of a and b into the equation.	$y = 2 + 1.4x$
5 Write your answer.	$y = 2 + 1.4x$ is the equation of the line.

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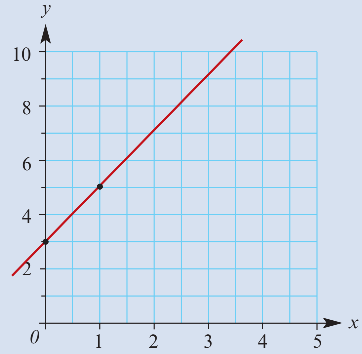
Now try this 17 **Finding the equation of a line: intercept–slope method (Example 17)**

Determine the equation of this straight-line graph.



Hint 1 Find the y-intercept.

Hint 2 Find the slope using two points on the line.

Hint 3 Substitute the values for a and b into the straight-line equation $y = a + bx$.

Using two points on a graph to find the equation of a straight line

Unfortunately, not all straight-line graphs show the y-intercept. When this happens, we have to use the two-point method for finding the equation of the line.

Finding the equation of a straight-line graph using two points

The general equation of a straight-line graph is $y = a + bx$.

1. Use the coordinates of two points on the line to determine the slope (b).
2. Substitute this value for the slope into the equation. There is now only one unknown, a.
3. Substitute the coordinates of one of the two points on the line into this new equation and solve for the unknown (a).
4. Substitute the values of a and b into the general equation, $y = a + bx$, to obtain the equation of the straight line.

Note: This method works in *all* circumstances.

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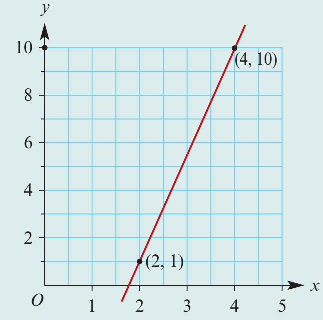
Example 18 Finding the equation of a straight line using two points on the graph

Find the equation of the line that passes through the points (2, 1) and (4, 10).



Explanation

1. Write down the general equation of a straight-line graph.

Solution: $y = a + bx$

2. Use the coordinates of the two points on the line to find the slope (b).

Solution: $\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{10 - 1}{4 - 2} = \frac{9}{2} = 4.5$

so $b = 4.5$

3. Substitute the value of b into the general equation.

Solution: $y = a + 4.5x$

4. To find the value of a, substitute the coordinates of one of the points on the line (either will do) and solve for a.

Solution: Using the point (2, 1):

$1 = a + 4.5 \times 2$

$1 = a + 9$

$a = -8$

5. Substitute the values of a and b into the general equation, $y = a + bx$, to find the equation of the line.

Solution: Thus, the equation of the line is:

$y = -8 + 4.5x$

Now try this 18 Finding the equation of a straight line using two points on the graph (Example 18)

Find the equation of the line that passes through the points (2, 4) and (3, 10).

Hint 1 Write down the general equation of a straight line.

Hint 2 Find the slope of the line (b-value).

Hint 3 Substitute b and either of the given coordinates into the general equation.

Hint 4 Solve for a.

Finding the equation of a straight-line graph from two points using a CAS calculator

While the intercept–slope method of finding the equation of a line from its graph is relatively quick and easy to apply, using the two-point method to find the equation of a line can be time consuming.

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An alternative to using either of these methods is to use the line-fitting facility of your CAS calculator. You will meet this method again when you study the topic 'Investigating relationships between two numerical variables' later in the year.

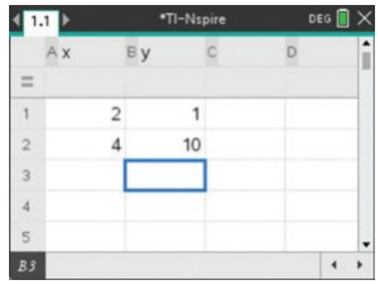
How to find the equation of a line from two points using the TI-Nspire

Find the equation of the line that passes through the two points (2, 1) and (4, 10).

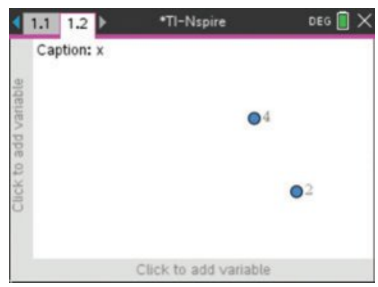
Steps

1 Write the coordinates of the two points. Label one point *A*, the other *B*.

2 Start a new document (ctrl + doc▼ and select **Add Lists & Spreadsheet**.
Enter the coordinate values into lists named ***x*** and ***y***.



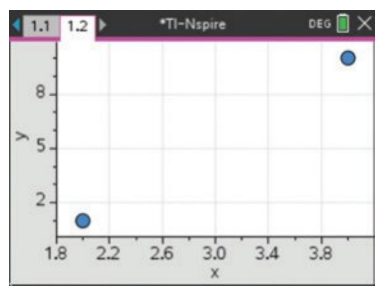
3 Plot the two points on a scatterplot. Press ctrl + N and select **Add Data & Statistics**.
(or press on and arrow to and press enter)
Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.



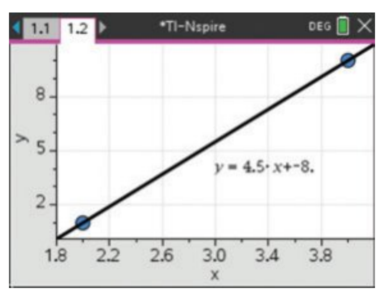
4 To construct a scatterplot:

a Press tab and select the variable ***x*** from the list. Press enter to paste the variable ***x*** to the ***x***-axis.

b Press tab again and select the variable ***y*** from the list. Press enter to paste the variable ***y*** to the ***y***-axis to generate the required scatter plot.



5 Use the **Regression** command to plot a line through the two points and determine its equation.
Press menu**>Analyze>Regression>Show Linear (a+bx)** and enter to complete the task.
Correct to one decimal place, the equation of the line is: $y = -8.0 + 4.5x$.



6 Write your answer. The equation of the line is $y = -8 + 4.5x$.

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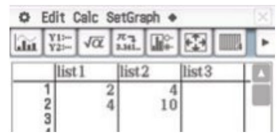
How to find the equation of a line from two points using the ClassPad

Find the equation of the line that passes through the two points (2, 4) and (4, 10).

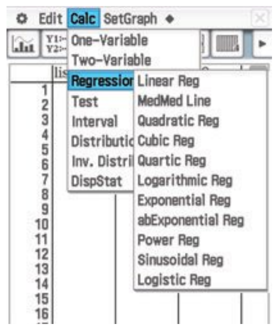
Steps

1 Open the **Statistics** application and enter the coordinate values into the lists as shown.

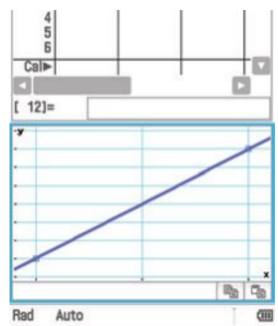


2 To find the equation of the line $y = ax + b$ that passes through the two points:

- Select **Calc** from the menu bar
- Select **Regression** and **Linear Reg**
- Ensure that the **Set Calculation** dialog box is set as shown
- Press OK.



3 The results are given in a **Stat Calculation** dialog box.



The equation of the line is $y = -2 + 3x$.

Note: Tapping **OK** will automatically display the graph window with the line drawn through the two points. This confirms that the line passes through the two points.

Section Summary

- The equation of a straight line can be found using the y-intercept and the slope.
- The equation of a straight line can be found using two points on the line.
- The equation of a straight line can be found using a CAS calculator.

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Skill-sheet

Exercise 5F

Building understanding

The graphs for Questions 1-3 are shown below Question 3.

1 For Graph 1 below:

a Find the y-intercept.

b Find the slope of the line using the formula for the slope.

c State the equation of the straight line in the form $y = a + bx$.

2 For Graph 2 below:

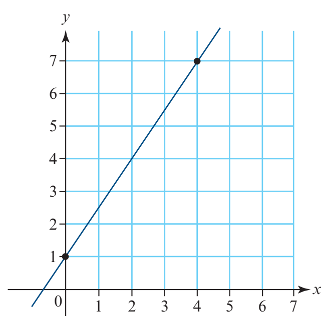
a Find the slope of the line using the formula for the slope.

b Find the y-intercept (a) by substituting the slope for b and the point (3, 0) into $y = a + bx$.

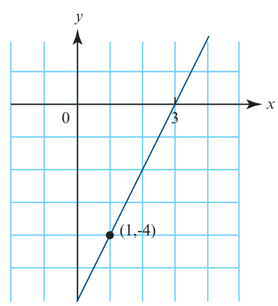
3 For Graph 3 below:

a Find the slope of the line using the formula.

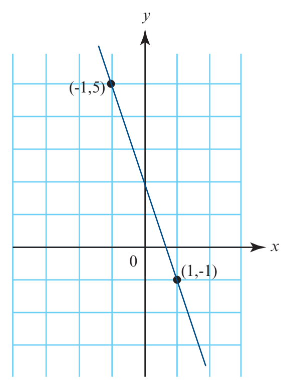
b Find the y-intercept by substituting the slope for b and the point $(-1, 5)$ into $y = a + bx$.



Graph 1



Graph 2

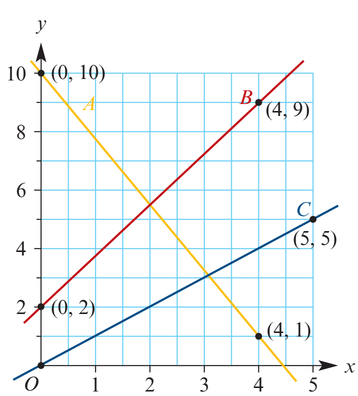


Graph 3

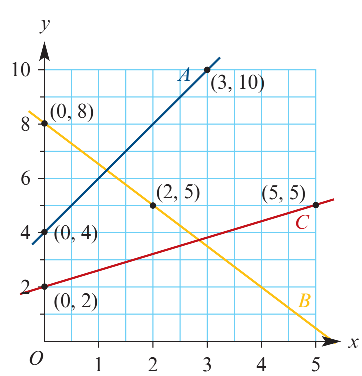
Developing understanding

Example 17

4 Find the equation of the lines (A, B, C) shown on the graph below. Write your answers in the form $y = a + bx$.



5 Find the equation of the lines (A, B, C) shown on the graph below. Write your answers in the form $y = a + bx$.



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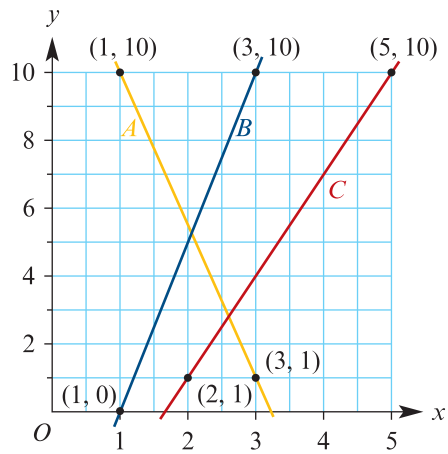
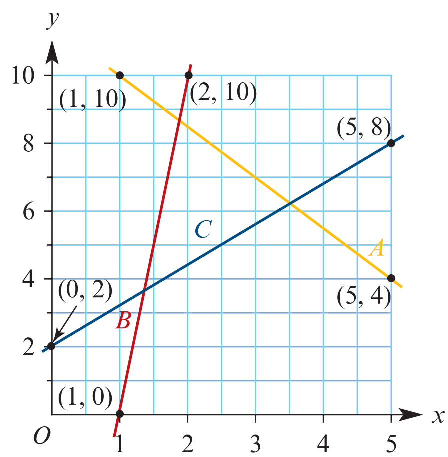
Example 18

6 Find the equation of the lines (A, B, C) on the graph below. Write your answers in the form $y = a + bx$.

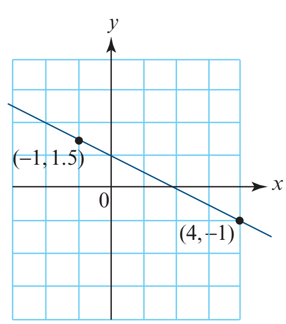


7 Use a CAS calculator to find the equation of each of the lines (A, B, C) on the graph below. Write your answers in the form $y = a + bx$.



Testing understanding

8 Find the equation of the straight-line graph shown.



9 A line passes through (0, 2) and (5, 9). Find the equation of the line in the form $y = a + bx$.

10 A line passes through (2, −4) and (−4, 8). Find the equation of the line in the form $y = a + bx$.

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5G Linear modelling

Learning intentions

- To be able to construct linear models.
- To be able to interpret and analyse linear models.

Many real-life relationships between two variables can be described mathematically by linear (straight-line) equations. This is called **linear modelling**.

These linear models can then be used to solve problems, such as finding the time taken to fill a partially filled swimming pool with water, estimating the depreciating value of a car over time or describing the growth of a plant over time.

Modelling plant growth with a linear equation

Some plants, such as tomato plants, grow remarkably quickly.

When first planted, the height of this plant was 5 cm.

The plant then grows at a constant rate of 6 cm per week for the next 10 weeks.

From this information, we can now construct a mathematical model that can be used to chart the growth of the plant over the following weeks and predict its height at any time during the first 10 weeks after planting.

Start After 1 week After 2 weeks

Constructing a linear model

Let h be the height of the plant (in cm).

Let t be the time (in weeks) after it was planted.

For a linear growth model we can write:

$$h = a + bt$$

where:

- a is the initial height of the plant; in this case, 5 cm (in graphical terms, the y-intercept)
- b is the constant rate at which the plant's height increases each week; in this case, 6 cm per week (in graphical terms, the slope of the line).

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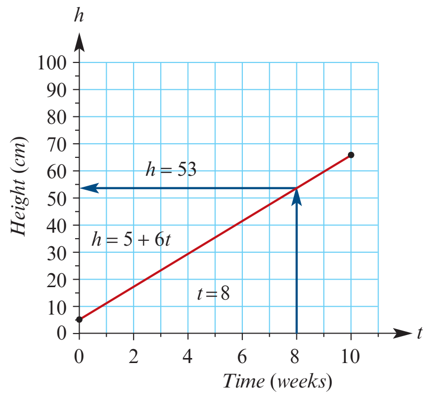
Thus we can write: $h = 5 + 6t$ for $0 \leq t \leq 10$ (see note below). The graph for this model is plotted here.



Three important features of the linear model $h = 5 + 6t$ for $0 \leq t \leq 10$ should be noted:

- The h-intercept gives the height of the plant at the start; that is, its height when $t = 0$. The plant was 5 cm tall when it was first planted.
- The *slope* of the graph gives the growth rate of the plant. The plant grows at a rate of 6 cm per week; that is, each week the height of the plant increases by 6 cm.
- The graph is only plotted for $0 \leq t \leq 10$. This is because the model is only valid for the time (10 weeks) when the plant is growing at the constant rate of 6 cm a week.

Note: The expression $0 \leq t \leq 10$ is included to indicate the range of the number of weeks for which the model is valid. In more formal language this would be called the **domain** of the model.

Using a linear model to make predictions

To use the mathematical model to make predictions, we simply substitute a value of t into the model and evaluate.

For example, after eight weeks' growth ($t = 8$), the model predicts the height of the plant to be:

$$h = 5 + 6(8) = 53 \text{ cm}$$

This value could also be read directly from the graph, as shown above.

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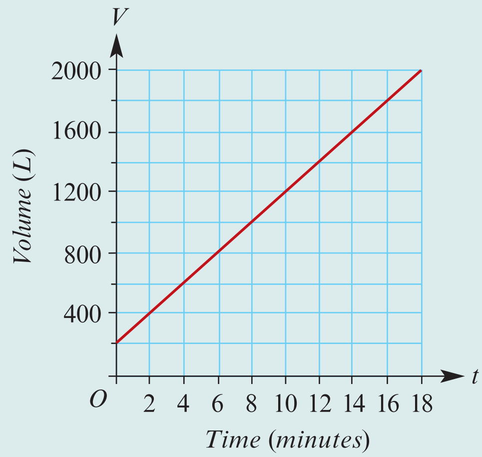
Interpreting and analysing the graphs of linear models

Example 19 Graphs of linear models with a positive slope

Water is pumped into a partially full tank. The graph gives the volume of water, V, (in litres) after t minutes.



a How much water is in the tank at the start ($t = 0$)?

b How much water is in the tank after 10 minutes ($t = 10$)?

c The tank holds 2000 L. How long does it take to fill?

d Find the equation of the line in terms of V and t.

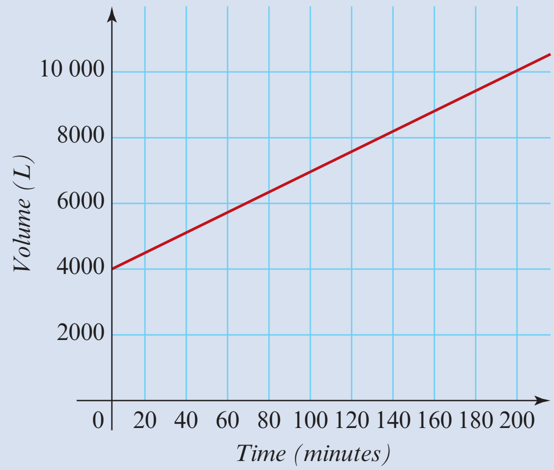
e Use the equation to calculate the volume of water in the tank after 15 minutes.

f At what rate is the water pumped into the tank; that is, how many litres are pumped into the tank each minute?

Explanation	Solution
a Read from the graph (when $t = 0$, $V = 200$).	200 L of water are in the tank.
b Read from the graph (when $t = 10$, $V = 1200$).	1200 L of water are in the tank.
c Read from the graph (when $V = 2000$, $t = 18$).	It takes 18 minutes to fill the tank.
d The equation of the line is $V = a + bt$. a is the V-intercept. Read from the graph. b is the slope. Calculate using two points on the graph, say (0, 200) and (18, 2000). **Note:** You can use your calculator to find the equation of the line if you wish.	$V = a + bt$ $a = 200$ $b = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{2000 - 200}{18 - 0}$ $= 100$ $\therefore\ V = 200 + 100t\ (t \geq 0)$
e Substitute $t = 15$ into the equation. Evaluate.	$V = 200 + 100(15) = 1700$ L
f The rate at which water is pumped into the tank is given by the slope of the graph, 100 (from part **d**).	The rate is 100 L/min.

Now try this 19 Graphs of linear models with a positive slope (Example 19)

Petrol is pumped into a partially full storage tank. The graph shows the volume of petrol, V, (in litres) after t minutes.



a How much petrol is in the tank at the start ($t = 0$)?

b How much petrol is in the tank after 100 minutes ($t = 100$)?

c The tank holds 10 000 L. How long does it take to fill?

d Find the equation of the line in terms of V and t.

e Use the equation to calculate the volume of water in the tank after 150 minutes.

f At what rate is the water pumped into the tank; that is, how many litres are pumped into the tank each minute?

Hint 1 Read from the graph when $t = 0$.

Hint 2 Read from the graph when $t = 100$.

Hint 3 Read from the graph when $V = 10\,000$.

Hint 4 Find the slope of the graph and then write an equation for the line ($V = a + bt$) using the slope and V-intercept.

Hint 5 Substitute in $t = 150$.

Hint 6 The slope will tell you the rate.

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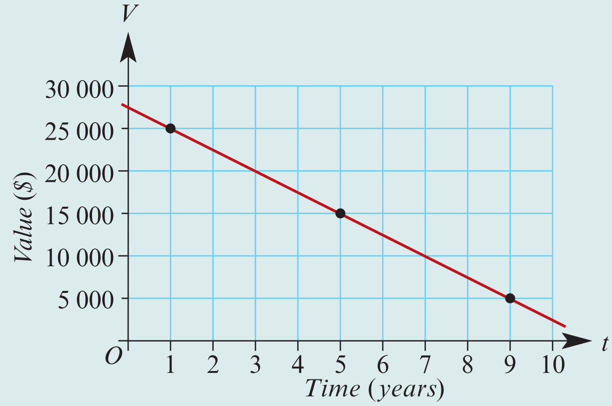
Example 20 Graphs of linear models with a negative slope

The value of new cars depreciates with time. The graph shows how the value, V, (in dollars) of a new car depreciates with time, t, (in years).



a What was the value of the car when it was new?

b What was the value of the car when it was 5 years old?

c Find the equation of the line in terms of V and t.

d At what rate does the value of the car depreciate with time; that is, by how much does its value decrease each year?

e When does the equation predict the car will have no (zero) value?

Explanation	Solution
a Read from the graph (when $t = 0, V = 27\,500$).	\$27 500 when the car was new
b Read from the graph (when $t = 5, V = 15\,000$).	\$15 000 when the car was 5 years old
c The equation of the line is $V = a + bt$. ■ a is the V-intercept. Read from the graph. ■ b is the slope. Calculate using two points on the graph, say (1, 25 000) and (9, 5000). Note: You can use your calculator to find the equation of the line if you wish.	$V = a + bt$ $a = 27\,500$ $b = \text{slope} = \dfrac{25\,000 - 5000}{1 - 9}$ $= -2500$ $\therefore\ V = 27\,500 - 2500t$ for $t \geq 0$
d The slope of the line is –2500, so the car depreciates in value by \$2500 per year.	\$2500 per year
e Substitute into the equation and solve for t.	$0 = 27\,500 - 2500t$ $2500t = 27\,500$ $\therefore t = \dfrac{27\,500}{2500} = 11$ years

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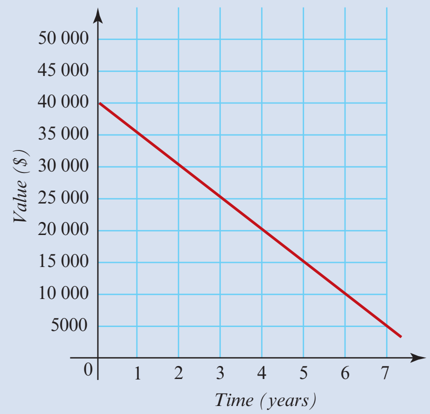
Now try this 20 Graphs of linear models with a negative slope (Example 20)

A car's value depreciates with time. Its value, V, over time, t, (in years) is shown on the graph.



a What was the value of the car when it was new?

b What was the value of the car when it was 3 years old?

c Find the equation of the line in terms of V and t.

d At what rate does the value of the car depreciate with time?

e When does the equation predict the car will have no (zero) value?

Hint 1 Look at the V-intercept.

Hint 2 Find the slope of the graph.

Hint 3 Write an equation for the line in the form $V = a + bt$.

Hint 4 The slope tells you the rate.

Section Summary

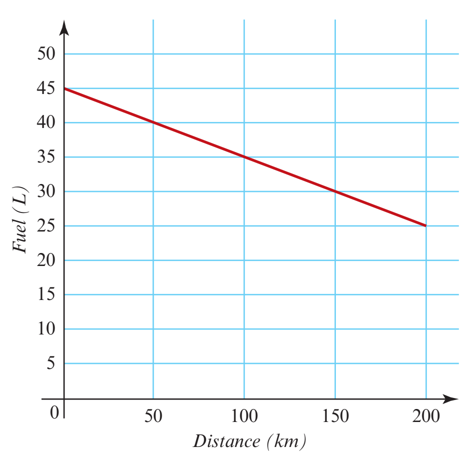
- **Linear modelling** uses linear (straight-line) equations to describe relationships between variables.
- A linear model can be used to make predictions.

Exercise 5G

Building understanding

Example 19

1 The graph shows the amount of fuel in the tank of a car after driving for a distance, d, kilometres.



a State the amount of fuel in the tank initially, when $d = 0$.

b State the amount of fuel in the tank after travelling $d = 50$ kilometres.

c State the amount of fuel in the tank when $d = 200$ kilometres.

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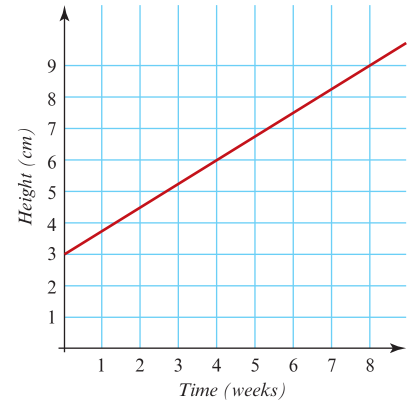
2 The graph shows the height, h, in cm, for the first 8 weeks' growth of a plant.



a What was the height of the plant initially, (at $t = 0$)?
b What was the height of the plant after four weeks, (at $t = 4$)?
c What was the height of the plant after 8 weeks?
d Find the slope of the straight line.
e Write down the linear model in terms of h and t.

Developing understanding

3 An empty 20-litre cylindrical beer keg is to be filled with beer at a constant rate of 5 litres per minute.
Let V be the volume of beer in the keg after t minutes.

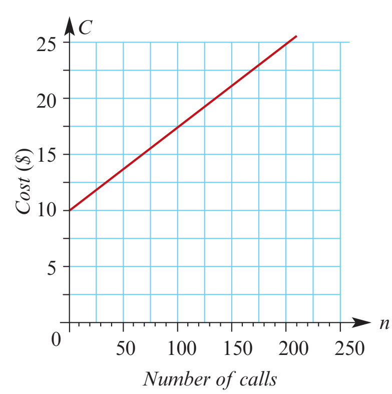
a Write down a linear model in terms of V and t to represent this situation.
The beer keg is filled in 4 minutes.
b Sketch the graph showing the coordinates of the intercept and its end point.
c Use the model to predict the volume of beer in the keg after 3.2 minutes.

4 A home waste removal service charges \$80 to come to your property. It then charges \$120 for each cubic metre of waste it removes. The maximum amount of waste that can be removed in one visit is 8 cubic metres.
Let C be the total charge for removing w cubic metres of waste.

a Write down a linear model in terms of C and w to represent this situation.
b Sketch the graph showing the coordinates of the intercept and its end point.
c Use the model to predict the cost of removing 5 cubic metres of waste.

Example 19
Example 20

5 A phone company charges a monthly service fee plus the cost of calls. The graph shown gives the total monthly charge, C dollars, for making n calls. This includes the service fee.



a How much is the monthly service fee ($n = 0$)?
b How much does the company charge if you make 100 calls a month?
c Find the equation of the line in terms of C and n.
d Use the equation to calculate the cost of making 300 calls in a month.
e How much does the company charge per call?

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6 A motorist fills the tank of her car with unleaded petrol, which costs \$1.57 per litre. Her tank can hold a maximum of 60 litres of petrol. When she started filling her tank, there was already 7 litres in her tank.
Let C be the cost of adding v litres of petrol to the tank.

a Write down a linear model in terms of C and v to represent this situation.

b Sketch the graph of C against v, showing the coordinates of the intercept and its end point.

c Use the model to predict the cost of filling the tank of her car with petrol.

7 The graph opposite, shows the volume of saline solution, V mL, remaining in the reservoir of a saline drip after t minutes.

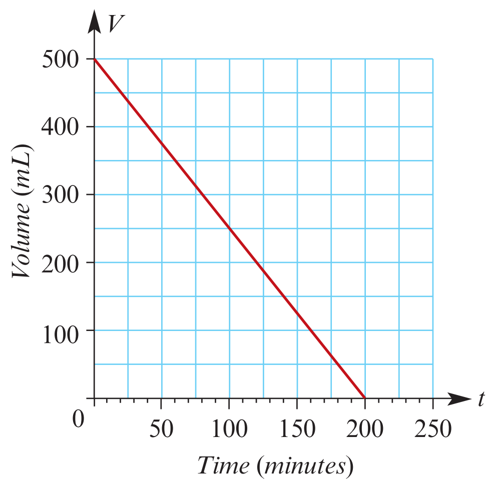
a How much saline solution was in the reservoir at the start?

b How much saline solution remains in the reservoir after 40 minutes? Read the result from the graph.

c How long does it take for the reservoir to empty?

d Find the equation of the line in terms of V and t.

e Use the equation to calculate the amount of saline solution in the reservoir after 115 minutes.



f At what rate (in mL/minute) is the saline solution flowing out of the drip?

Testing understanding

8 A swimming pool when full contains 10 000 litres of water. Due to a leak, it loses on average 200 litres of water per day.
Let V be the volume of water remaining in the pool after t days.

a Write down a linear model in terms of V and t to represent this situation.

b The pool continues to leak. How long will it take to empty the pool?

c Sketch the graph showing the coordinates of the intercept and its end point.

d Use the model to predict the volume of water left in the pool after 30 days.

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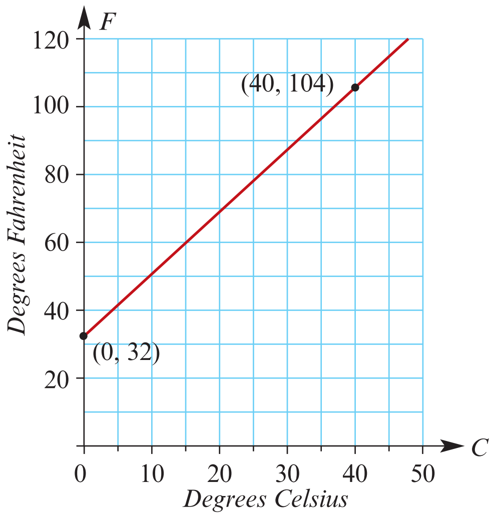
9 The graph, opposite, can be used to convert temperatures in degrees Celsius (C) to temperatures in degrees Fahrenheit (F).

a Find the equation of the line in terms of F and C.

b Use the equation to predict the temperature in degrees Fahrenheit when the temperature in degrees Celsius is:

i 50°C **ii** 150°C **iii** −40°C



c Complete the following sentence by filling in the box.
When the temperature in Celsius increases by 1 degree, the temperature in Fahrenheit increases by ☐ degrees.

5H Solving simultaneous equations

Learning intentions

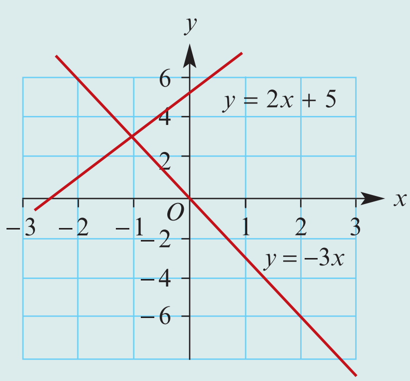
- To be able to find the intersection point of two lines.
- To be able to solve simultaneous equations.

Finding the point of intersection of two linear graphs

Two straight lines will always intersect unless they are parallel. The point at which two straight lines intersect can be found by sketching the two graphs on the one set of axes and then reading off the coordinates at the point of intersection. When we find the *point of intersection*, we are said to be **solving the equations simultaneously**.

Example 21 Finding the point of intersection of two linear graphs

The graphs of $y = 2x + 5$ and $y = -3x$ are shown. Write their point of intersection.



Solution

From the graph, it can be seen that the point of intersection is $(-1, 3)$.

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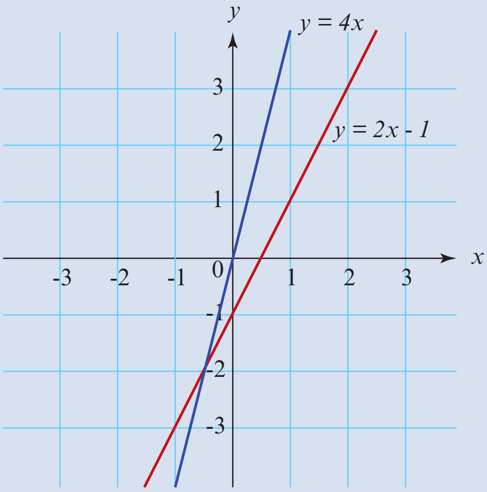
Now try this 21 Finding the point of intersection of two linear graphs (Example 21)

The graphs of $y = 2x - 1$ and $y = 4x$ are shown. Write their point of intersection.



A CAS calculator can also be used to find the point of intersection.

How to find the point of intersection of two linear graphs using the TI-Nspire

Use a CAS calculator to find the point of intersection of the simultaneous equations $y = 2x + 6$ and $y = -2x + 3$.

Steps

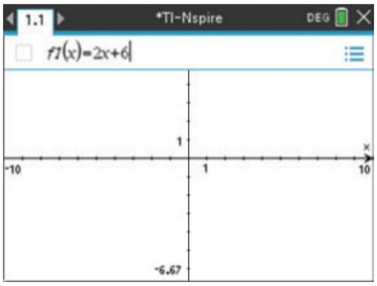
1 Start a new document (ctrl + N) and select **Add Graphs**.

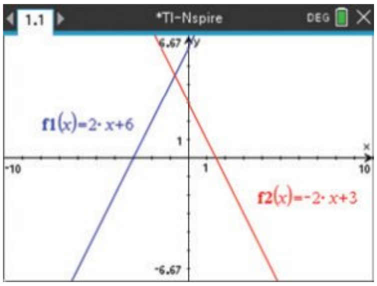
2 Type in the first equation as shown. Note that $f1(x)$ represents the y. Press ▼ and the edit line will change to $f2(x)$ and the first graph will be plotted. Type in the second equation and press enter to plot the second graph.

Hint: If the entry line is not visible, press tab.

Hint: To see all entered equations, move the cursor onto the ☰ and press [click] del.

Note: To change window settings, press menu **>Window/Zoom>Window Settings** and change to suit. Press enter when finished.





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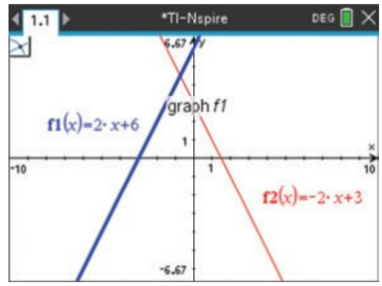
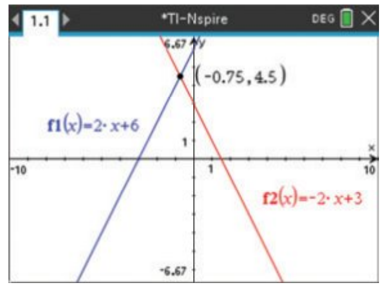
3 To find the point of intersection, press [menu]**>Geometry>Points & Lines>Intersection Point(s)**. Move the cursor to one of the graphs until it flashes, press [click], then move to the other graph and press [click]. The solution will appear. Alternatively, use [menu]**>Analyze Graph>Intersection**.



4 Press [enter] to display the solution on the screen. The coordinates of the point of intersection are $x = -0.75$ and $y = 4.5$.

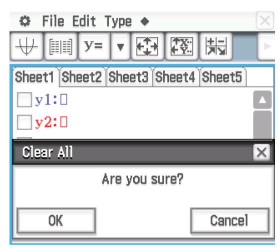


How to find the point of intersection of two linear graphs using the ClassPad

Use a CAS calculator to find the point of intersection of the simultaneous equations $y = 2x + 6$ and $y = -2x + 3$.

Steps

1 Open the built-in **Graphs and Tables** application. Tapping Menu from the icon panel (just below the touch screen) will display the Application menu if it is not already visible.



2 If there are any equations from previous questions, go to **Edit Clear all** and tap **OK**.

3 Enter the equations into the graph editor window. Tick the boxes. Tap the [graph] icon to plot the graphs.

4 To adjust the graph window, tap **Zoom, Quick Standard**. Alternatively, tap the [view window] icon and complete the **View Window** dialog box.

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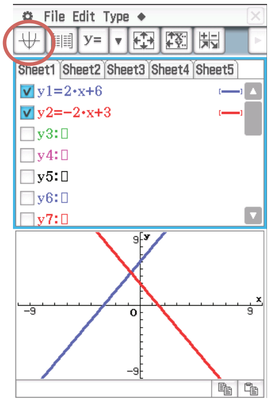
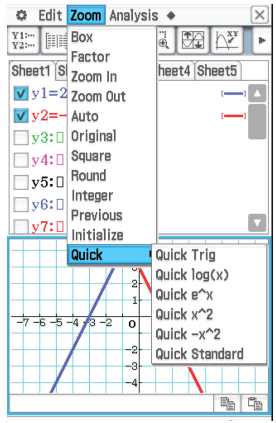
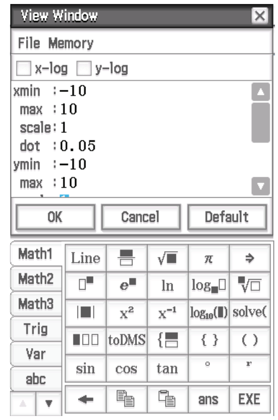
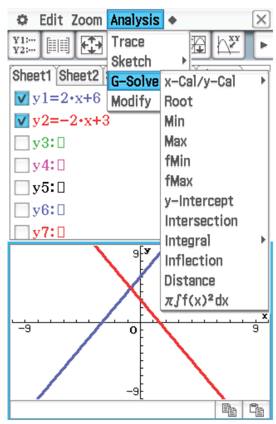
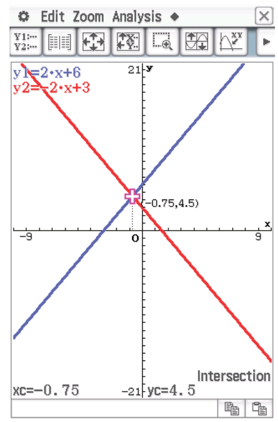






5 Solve by finding the point of intersection. Select **Analysis** > **G-solve**> **Intersection**. Tap Resize to view graph only. The required solution is displayed on the screen: $x = -0.75$ and $y = 4.5$.





Solving simultaneous linear equations using a CAS calculator

Another way of solving simultaneous equations, rather than drawing the graphs on a CAS calculator, is to use the **solve simultaneous functions** on the CAS calculator.

How to solve a pair of simultaneous linear equations algebraically using the TI-Nspire

Solve the following pair of simultaneous equations:

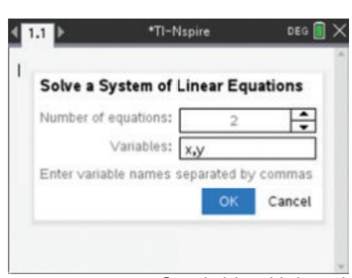
$$24x + 12y = 36$$

$$45x + 30y = 90$$

Steps

1 Start a new document and select **Add Calculator**.

2 Press menu>**Algebra>Solve System of Equations>Solve System of Linear Equations**. Complete the pop-up screen as shown.



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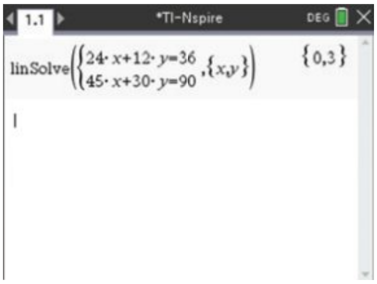
(The default settings are for two equations with variables x & y). A simultaneous equation template will be pasted to the screen.

3 Enter the equations as shown into the template. Use the [tab] key to move between entry boxes.

4 Press [enter] to display the solution, which is interpreted as $x = 0$ and $y = 3$.



5 Write your answer. $x = 0, y = 3$

How to solve a pair of simultaneous linear equations algebraically using the ClassPad

Solve the following pair of simultaneous equations:

$$24x + 12y = 36$$

$$45x + 30y = 90$$

Steps

1 Open the built-in **Main** application.

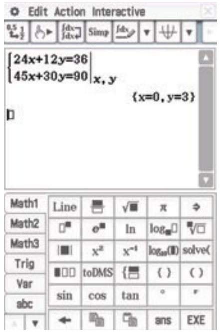
a Press [Keyboard] on the front of the calculator to display the built-in keyboard.

b Tap the simultaneous equations icon.

c Enter the information

$$\begin{cases} 24x + 12y = 36 \\ 45x + 30y = 90 \end{cases}\Bigg|_{x,\, y}$$

2 Press [EXE] to display the solution, $x = 0$ and $y = 3$.



3 Write your answer. $x = 0, y = 3$

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Section Summary

- When two straight lines intersect and the point of intersection is found, we are solving the equations **simultaneously**.
- **Simultaneous equations** can be solved graphically or by using a CAS calculator.

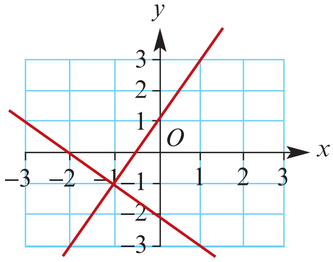
Exercise 5H

Building understanding

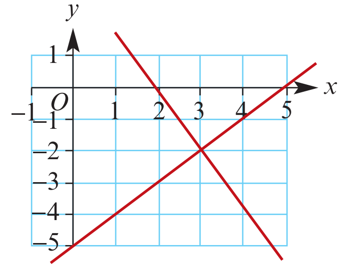
Example 21

1 For each of the following graphs, state the value of x where the two straight lines intersect.

a

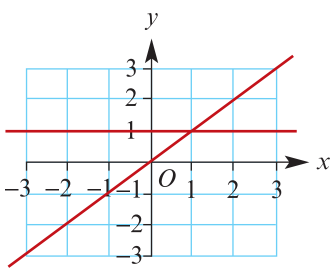


b

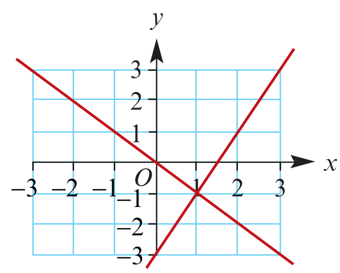


2 State the point of intersection for each of these pairs of straight lines.

a



b



Developing understanding

3 Using a CAS calculator, find the point of intersection of each of these pairs of lines.

a $y = x - 6$ and $y = -2x$

b $y = x + 5$ and $y = -x - 1$

c $y = 3x - 2$ and $y = 4 - x$

d $x - y = 5$ and $y = 2$

e $x + 2y = 6$ and $y = 3 - x$

f $2x + y = 7$ and $y - 3x = 2$

g $3x + 2y = -4$ and $y = x - 3$

h $y = 4x - 3$ and $y = 3x + 4$

Testing understanding

4 Solve the simultaneous equations, to one decimal place, using a CAS calculator.

a $2x + 5y = 3$
$x + y = 3$

b $3x + 2y = 5.5$
$2x - y = -1$

c $3x - 8y = 13$
$-2x - 3y = 8$

d $2m - n = 1$
$2n + m = 8$

e $15x - 4y = 6$
$-2y + 9x = 5$

f $2.9x - 0.6y = 4.8$
$4.8x + 3.1y = 5.6$

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5I Practical applications of simultaneous equations

Learning intentions

- To solve practical problems using simultaneous equations.

Simultaneous equations can be used to solve problems in real situations. It is important to define the unknown quantities with appropriate variables before setting up the equations.

Example 22 Using simultaneous equations to solve a practical problem

Mark buys 3 roses and 2 gardenias for \$15.50. Peter buys 5 roses and 3 gardenias for \$24.50. How much did each type of flower cost?

Explanation

Strategy: Using the information given, set up a pair of simultaneous equations to solve.

1 Choose appropriate variables to represent the cost of roses and the cost of gardenias.

Solution: Let r be the cost of a rose and g be the cost of a gardenia.

2 Write equations using the information given in the question.

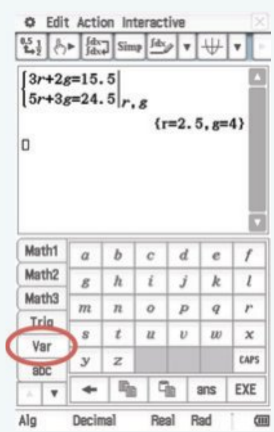
Solution:

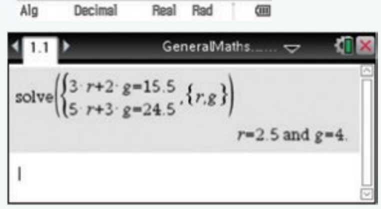
$3r + 2g = 15.5 \quad (1)$

$5r + 3g = 24.5 \quad (2)$

3 Use your CAS calculator to solve the two simultaneous equations.

Solution:





4 Write down the solutions.

Solution: $r = 2.50$ and $g = 4$

5 Check by substituting $r = 2.5$ and $g = 4$ into equation (2).

Solution:

$\text{LHS} = 5(2.5) + 3(4)$

$= 12.5 + 12 = 24.5 = \text{RHS}$

6 Write your answer with the correct units.

Solution: Roses cost \$2.50 each and gardenias cost \$4 each.

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Now try this 22 **Using simultaneous equations to solve a practical problem (Example 22)**

Ian bought 6 plain croissants and 6 chocolate croissants for \$46.20. Anne bought 5 plain croissants and 3 chocolate croissants for \$30.10. How much was each type of croissant?

Hint 1 Choose appropriate variables to represent the cost of plain croissants and chocolate croissants.

Hint 2 Write 2 equations using information from the question.

Hint 3 Use a CAS calculator to solve the simultaneous equations.

Example 23 **Using simultaneous equations to solve a practical problem**

The perimeter of a rectangle is 48 cm. If the length of the rectangle is three times the width, determine its dimensions.

Explanation	Solution
Strategy: Using the information given, set up a pair of simultaneous equations to solve.	
1 Choose appropriate variables to represent the dimensions of width and length.	Let w = width and l = length
2 Write two equations from the information given in the question. Label the equations as (1) and (2). Remember: The perimeter of a rectangle is the distance around the outside and can be found using $2w + 2l$. Note: If the length, l, of a rectangle is three times its width, w, then this can be written as $l = 3w$.	$2w + 2l = 48 \quad (1)$ $l = 3w \quad (2)$
3 Use your CAS calculator to solve the two simultaneous equations.	See Example 22.
4 Give your answer in the correct units.	The dimensions of the rectangle are width 6 cm and length 18 cm.

Now try this 23 **Using simultaneous equations to solve a practical problem (Example 23)**

The perimeter of a rectangle is 64 cm. If the length of one side of the rectangle is four times its width, determine its dimensions.

Hint 1 Choose appropriate variables to represent the length and the width of the rectangle.

Hint 2 Write two equations using information from the question.

Hint 3 Use a CAS calculator to solve the simultaneous equations.

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Example 24 Using simultaneous equations to solve a practical problem

Tickets for a movie cost \$19.50 for adults and \$14.50 for children. Two hundred tickets were sold, giving a total of \$3265. How many children's tickets were sold?

Explanation	Solution
Strategy: Using the information given, set up a pair of simultaneous equations to solve.	
1 Choose appropriate variables to represent the number of adult tickets sold and the number of children's tickets sold.	Let a be the number of adult tickets sold and c be the number of children's tickets sold.
2 Write two equations using the information given in the question. Note: The total number of adult and child tickets is 200, which means that $a + c = 200$.	$19.5a + 14.5c = 3265 \quad (1)$ $a + c = 200 \quad (2)$
3 Use your CAS calculator to solve the two simultaneous equations.	See Example 22.
4 Write down your solution.	127 children's tickets were sold.

Now try this 24 Using simultaneous equations to solve a practical problem (Example 24)

Tickets to a football game cost \$26.95 for adults and \$6.00 for children. In one morning, 498 tickets were sold, giving a total of \$9692. How many adult tickets were sold?

Hint 1 Choose appropriate variables to represent the cost of an adult ticket and the cost of a child's ticket.

Hint 2 Write two equations using information from the question, and solve with CAS.

Section Summary

- **Simultaneous equations** can be used to solve practical problems.

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Exercise 5I

Building understanding

Example 22

1 Jessica bought 5 crayons and 6 pencils for \$12.75, and Tom bought 7 crayons and 3 pencils for \$13.80.
Using c for crayon and p for pencil, complete the following to form a pair of simultaneous equations to solve.

$\square c + \square p = 12.75$

$\square c + \square p = 13.80$

2 Peter buys 50 litres of petrol and 5 litres of motor oil for \$109. His brother Anthony buys 75 litres of petrol and 5 litres of motor oil for \$146.

a Using p for petrol and m for motor oil, complete the following to form a pair of simultaneous equations to solve.

$\square p + \square m = 109$

$\square p + \square m = 146$

b How much does a litre of petrol cost?

c How much does a litre of motor oil cost?

3 Six oranges and ten bananas cost \$7.10. Three oranges and eight bananas cost \$4.60.

a Write down a pair of simultaneous equations to solve.

b How much does an orange cost?

c How much does a banana cost?

Developing understanding

4 The weight of a box of nails and a box of screws is 2.5 kg. Four boxes of nails and a box of screws weigh 7 kg. Determine the weight of each.

5 An enclosure at a wildlife sanctuary contains wombats and emus. If the number of heads totals 28 and the number of legs totals 88, determine the number of each species present.

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Example 23 **6** The perimeter of a rectangle is 36 cm. If the length of the rectangle is twice its width, determine its dimensions.

7 The sum of two numbers x and y is 52. The difference between the two numbers is 8. Find the values of x and y.

8 The sum of two numbers is 35 and their difference is 19. Find the numbers.

9 Bruce is 4 years older than Michelle. If their combined age is 70, determine their individual ages.

10 A boy is 6 years older than his sister. In three years' time he will be twice her age. What are their present ages?

11 A chocolate thickshake costs \$2 more than a fruit smoothie. Jack pays \$27 for 3 chocolate thickshakes and 4 fruit smoothies. What is the cost of

a a chocolate thickshake?

b a fruit smoothie?

12 In 4 years' time, a mother will be three times as old as her son. Four years ago she was five times as old as her son. Find their present ages.

Example 24 **13** The registration fees for a mathematics competition are \$1.20 for students aged 8–12 years and \$2 for students 13 years and over. One hundred and twenty-five students have already registered and an amount of \$188.40 has been collected in fees. How many students between the ages of 8 and 12 have registered for the competition?

14 A computer company produces 2 laptop models: standard and deluxe. The standard laptop requires 3 hours to manufacture and 2 hours to assemble. The deluxe model requires $5\frac{1}{2}$ hours to manufacture and $1\frac{1}{2}$ hours to assemble. The company allows 250 hours for manufacturing and 80 hours for assembly over a limited period. How many of each model can be made in the time available?

15 A chemical manufacturer wishes to obtain 700 litres of a 24% acid solution by mixing a 40% solution with a 15% solution. How many litres of each solution should be used?

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Testing understanding

16 In a hockey club there are 5% more boys than there are girls. If there is a total of 246 members in the club, what is the number of boys and the number of girls?

17 The owner of a service station sells unleaded petrol for $1.42 per litre and diesel fuel for $1.54 per litre. In five days he sold a total of 10 000 litres and made $14 495. How many litres of each type of petrol did he sell? Give your answer to the nearest litre.

18 James had $30 000 to invest. He chose to invest part of it at 5% and the other part at 8%. Overall he earned $2100 in interest. How much did he invest at each rate?

19 The perimeter of a rectangle is 120 metres. The length is one and a half times the width. Calculate the width and length.

5J Piecewise linear graphs

Learning intentions

- To be able to construct and analyse piecewise linear graphs.
- To be able to construct and analyse step graphs.

Sometimes a situation requires two linear graphs to obtain a suitable model. The graphs we use to model such situations are called **piecewise linear graphs**.

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Example 25 Constructing a piecewise linear graph model

The amount, C dollars, charged to supply and deliver x m^3 of crushed rock is given by the equations:

$$C = 50 + 40x \qquad (0 \le x < 3)$$
$$C = 80 + 30x \qquad (3 \le x \le 8)$$

a Use the appropriate equation to determine the cost to supply and deliver the following amounts of crushed rock.

i 2.5 m^3 **ii** 3 m^3 **iii** 6 m^3

b Use the equations to construct a piecewise linear graph for $0 \le x \le 8$.

Explanation

a **1** Write the equations.

2 Then, in each case:
- choose the appropriate equation.
- substitute the value of x and evaluate.
- write down your answer.

b The graph has two line segments.

1 Determine the coordinates of the end points of both lines.

2 Draw a set of labelled axes and mark in the points with their coordinates.

3 Join up the end points of each line segment with a straight line.

4 Label each line segment with its equation.

Solution

$C = 50 + 40x \qquad (0 \le x < 3)$

$C = 80 + 30x \qquad (3 \le x \le 8)$

i When $x = 2.5$
$C = 50 + 40(2.5) = 150$
Cost for 2.5 m^3 of crushed rock is \$150.

ii When $x = 3$
$C = 80 + 30(3) = 170$
Cost for 3 m^3 of crushed rock is \$170.

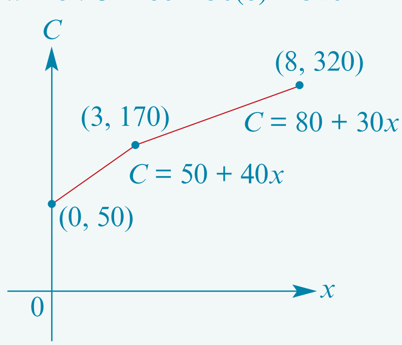
iii When $x = 6$,
$C = 80 + 30(6) = 260$
Cost for 6 m^3 of crushed rock is \$260.

b $x = 0 : C = 50 + 40(0) = 50$

$x = 3 : C = 50 + 40(3) = 170$

$x = 3 : C = 80 + 30(3) = 170$

$x = 8 : C = 80 + 30(8) = 320$



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Now try this 25 **Constructing a piecewise linear graph model (Example 25)**

The amount, C dollars, charged to supply and deliver x m^3 of sand is given by the equations:

$$C = 20 + 30x \qquad (0 \le x < 4)$$

$$C = 60 + 20x \qquad (4 \le x \le 10)$$

a Use the appropriate equation to determine the cost to supply and deliver the following amounts of sand.

i 3.5 m^3 **ii** 4 m^3 **iii** 8 m^3

b Use the equations to construct a piecewise linear graph for $0 \le x \le 10$.

Hint 1 Decide whether 3.5, 4 and 8 are in the $(0 \le x < 4)$ or $(4 \le x \le 10)$ interval. This will tell you in which equation to substitute $x = 3.5$, $x = 4$ and $x = 8$.

Hint 2 Sketch the two graphs on the same axes. They will join up at $x = 4$.

Hint 3 Label each line segment.

Step graphs

A **step graph** is a type of piecewise graph that uses line segments where each line segment is horizontal.

Example 26 **Constructing a step graph**

Sophie invests \$10 000, which earns compound interest of 12% per annum. The interest is calculated at the end of each year and added to the amount invested; i.e. \$10 000 is invested at 12% per annum compound interest.

The amount of money she has in the account for the first 5 years is shown in the table. Sketch the graph of the amount in the account against the year.

Time period (years)	Amount of interest earned (to the nearest dollar)	Total amount
0–1	0	10 000
1–2	1200	11 200
2–3	1344	12 544
3–4	1505	14 049
4–5	1686	15 735

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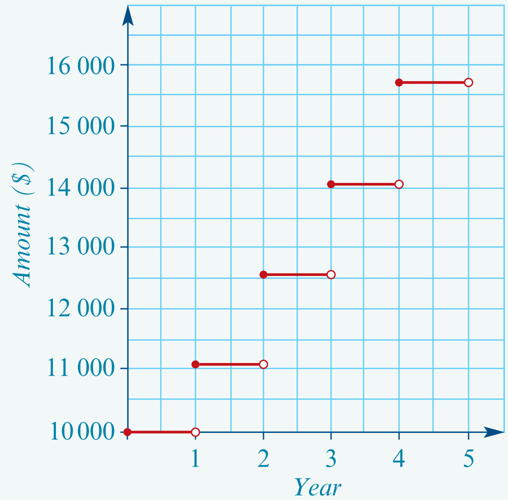
Explanation

For each time interval, draw the appropriate line segment. In this case, ●——○ indicates that change takes place at the end of the year.

Solution



Section Summary

- A piecewise linear graph uses two or more linear graphs.
- A step graph uses line segments where each line segment is horizontal.

Exercise 5J

Building understanding

1 Consider the piecewise linear equation:

$C = 90 + 10x \quad (0 \le x < 4)$

$C = 50 + 20x \quad (4 \le x < 8)$

Which equation would you use to find C if:

a $x = 2$

b $x = 6$

2 Consider the piecewise linear equation:

$D = 45 - 5t \quad (0 \le t < 3)$

$D = 90 - 20t \quad (3 \le t < 8)$

Which equation would you use to find D if:

a $t = 2$

b $t = 3$

c $t = 7$

3 Sketch the following piecewise equation.

$y = 1 + 3x \quad (0 \le x < 3)$

$y = 4 + 2x \quad (3 \le x \le 6)$

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Developing understanding

Example 25

4 An empty tank is being filled from a mountain spring. For the first 30 minutes, the equation giving the volume, V, of water in the tank (in litres) at time, t minutes, is:

$$V = 15t \quad (0 \le t \le 30)$$

After 30 minutes, the flow from the spring slows down. For the next 70 minutes, the equation giving the volume of water in the tank at time, t, is given by the equation:

$$V = 150 + 10t \quad (30 < t \le 100)$$

a Use the appropriate equation to determine the volume of water in the tank after:

i 20 minutes **ii** 30 minutes **iii** 60 minutes **iv** 100 minutes.

b Use the equations to construct a piecewise linear graph for $0 \le t \le 100$.

5 A multistorey car park has tariffs as shown. Sketch a step graph showing this information.

First 2 hours	\$5.00
2–3 hours (more than 2, less than or equal to 3)	\$7.50
3–4 hours (more than 3, less than or equal to 4)	\$11.00
4–8 hours (more than 4, less than or equal to 8)	\$22.00

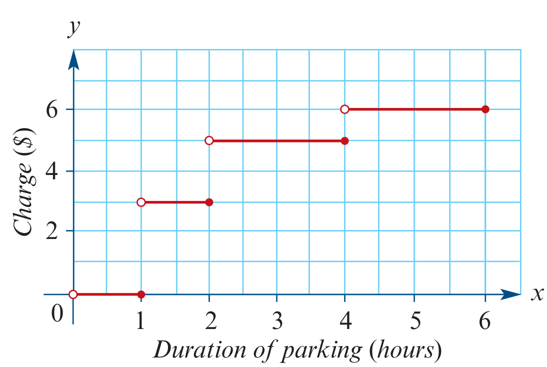
Example 26

6 This step graph shows the charges for a market car park.

a How much does it cost to park for 40 minutes?

b How much does it cost to park for 2 hours?

c How much does it cost to park for 3 hours?



Testing understanding

7 A National Park has an entrance fee of \$20 and charges \$10 per person for a guided tour with a group of 1 to 6 people. For a group of 7-10 people, the cost remains constant at \$80.

a Complete the following piecewise linear equation where C is the cost (\$) and n is the number of people.

$C =$ ☐, $(1 \le n \le$ ☐$)$

$C =$ ☐, $($☐ $\le n \le$ ☐$)$

b Sketch a graph of the piecewise linear equation.

c Find the cost for

i a family of 4 **ii** 10 people

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Key ideas and chapter summary

Assign-ment

Formula

A **formula** is a mathematical relationship connecting two or more variables.

Linear equation

A **linear equation** is an equation whose unknown values are always to the power of 1.

Slope of a straight-line graph

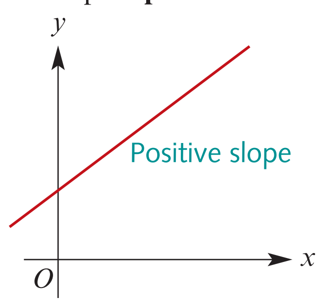
The **slope** of a straight-line graph is defined to be:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

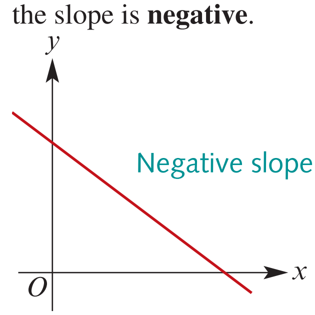
where (x_1, y_1) and (x_2, y_2) are two points on the line.

Positive and negative slope

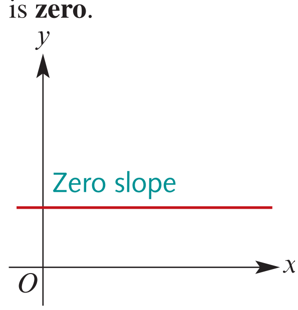
If the line rises to the right, the slope is **positive**.



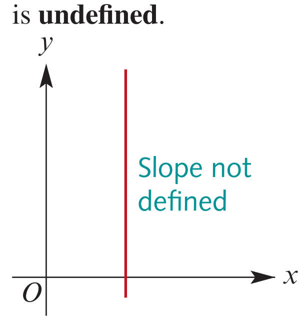
If the line falls to the right, the slope is **negative**.



If the line is horizontal, the slope is **zero**.



If the line is vertical, the slope is **undefined**.



Equation of a straight-line graph: the intercept–slope form

The equation of a straight line can take several forms.

The **intercept–slope form** is:

$y = a + bx$

where a is the **y-intercept** and b is the **slope** of the line.

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Linear model

A **linear model** has a linear equation or relation of the form:

$$y = a + bx \quad \text{where} \quad c \leq x \leq d$$

where a, b, c and d are constants.

Simultaneous equations

Two straight lines will always intersect, unless they are parallel. At the point of intersection the two lines will have the same coordinates. When we find the point of intersection, we are solving the equations simultaneously. **Simultaneous equations** can be solved graphically, algebraically or by using a CAS calculator.

Example:

$$3x + 2y = 6$$

$$4x - 5y = 12$$

are a pair of simultaneous equations.

Piecewise linear graphs

Piecewise linear graphs are used in practical situations where more than one linear equation is needed to model the relationship between two variables.

Step graphs

A **step graph** is a particular type of piecewise linear graph made of horizontal intervals or 'steps'.

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Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

5A **1** I can substitute values into linear equations and formulas. ☐

e.g. The cost, C, of hiring a trailer is $C = 10t + 50$, where t is the time in hours. How much will it cost to hire the trailer for 6 hours?

5A **2** I can construct tables of values from given formulas. ☐

e.g. The weekly wage, W, of a car salesperson is given by $W = 790 + 40n$, where n is the number of cars sold. Construct a table of values to show how much their weekly wage will be if they sell 5 to10 cars.

5B **3** I can solve linear equations. ☐

e.g. Solve the linear equation $2x - 5 = 25$ for x.

5B **4** I can use linear equations to solve practical problems. ☐

e.g. A plumber charges $100 up front and $60 for each hour, h, that they work. How much do they earn if they work for 5 hours?

5C **5** I can develop formulas from descriptions. ☐

e.g. A sausage roll costs $2.90 and a pie costs $2.50. Write a formula showing the cost, C, of x sausage rolls and y pies.

5D **6** I can draw a straight-line graph using a CAS calculator or from constructing a table of values. ☐

e.g. Draw the graph of $y = -2x + 5$.

5E **7** I can find the slope of a straight line given two points on the line. ☐

e.g. Find the slope of the line that goes through the points $(1, 8)$ and $(4, 2)$.

5E **8** I can find the intercept and slope of a straight-line graph from its equation and from its graph. ☐

e.g. What is the y-intercept and the slope of the straight line $y = -7 + 2x$?

5F **9** I can find the equation of a straight line given the slope and the y-intercept. ☐

e.g. Find the equation of the line with a slope of 3, going through the point $(0, 5)$.

5F **10** I can find the equation of a straight line given two points on the graph. ☐

e.g. Find the equation of the line that goes through the points $(3, -1)$ and $(-2, 7)$.

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5G **11** I can construct a linear model to represent a practical situation using a linear equation or a straight-line graph. ☐

e.g. The height, h, of a plant is 70 cm tall when it is first planted. For the next 6 months, it grows 3 cm every month. Write down a linear model for the situation.

5H **12** I can solve simultaneous equations. ☐

e.g. Solve the simultaneous equations $5x - 2y = 17$ and $3x + 4y = 20$. Give your answer to two decimal places.

5I **13** I can use simultaneous equations to solve practical problems. ☐

e.g. Six croissants and four chocolate eclairs cost \$45.20 and five croissants and eight chocolate eclairs cost \$62.40. What is the price of a croissant?

5J **14** I can use piecewise linear graphs that model a practical situation. ☐

e.g. A car park is open for 10 hours a day. The cost, C, for the time, t, is given by the following piecewise equation.

$$C = 12t \qquad (0 \le t < 4)$$

$$C = 8 + 10t \qquad (4 \le t \le 10)$$

Find the cost for parking your car for 8 hours.

5J **15** I can draw and interpret step graphs. ☐

e.g. Draw a step graph to show the cost of hiring a chain saw. The cost is defined by:

One hour or less	\$40
More than 1 hour but less than or equal to 2 hours	\$65
More than 2 hours but less than or equal to 3 hours	\$90
More than 3 hours but less than or equal to 4 hours	\$115

Multiple-choice questions

1 If $a = 4$, then $3a + 5 =$

A 12 **B** 17 **C** 27 **D** 34 **E** 39

2 If $b = 1$, then $2b - 9 =$

A -11 **B** -7 **C** 12 **D** 13 **E** 21

3 If $C = 50t + 14$ and $t = 8$, then $C =$

A 72 **B** 414 **C** 512 **D** 522 **E** 1100

4 If $P = 2L + 2W$, then the value of P when $L = 6$ and $W = 2$ is:

A 12 **B** 14 **C** 16 **D** 30 **E** 48

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5 If $x = -2, y = 3$ and $z = 7$, then $\frac{z - x}{y} =$

A -3 **B** $-\frac{5}{3}$ **C** $\frac{5}{3}$ **D** 3 **E** 9

6 If $a = 2, b = 5, c = 6$ and $d = 10$, then $bd - ac =$

A 7 **B** 24 **C** 38 **D** 39 **E** 484

7 The solution to $4x = 24$ is:

A $x = 2$ **B** $x = 6$ **C** $x = 8$ **D** $x = 20$ **E** $x = 96$

8 The solution to $\frac{x}{3} = -8$ is:

A $x = -38$ **B** $x = -24$ **C** $x = -\frac{8}{3}$ **D** $x = \frac{8}{3}$ **E** $x = 24$

9 The solution to $2v + 5 = 11$ is:

A $v = 3$ **B** $v = 6$ **C** $v = 8$ **D** $v = 16$ **E** $v = 17$

10 The solution to $3k - 5 = -14$ is:

A $k = -6.3$ **B** $k = -3$ **C** $k = 3$ **D** $k = 19$ **E** $k = 115.67$

11 The cost of hiring a car for a day is \$60 plus 0.25c per kilometre. Michelle travels 750 kilometres. Her total cost is:

A \$187.50 **B** \$188.10 **C** \$247.50 **D** \$810 **E** \$18810

12 Given $v = u + at$ and $v = 11.6$ when $u = 6.5$ and $a = 3.7$, the value of t, to two decimal places, is:

A 1.37 **B** 1.378 **C** 1.38 **D** 4.89 **E** 9.84

13 The solution to the pair of simultaneous equations:

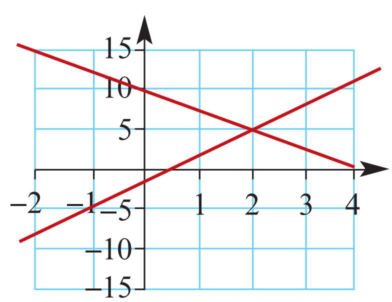
$y = 5x$

$y = 2x + 6$ is:

A $(-2, 0)$ **B** $(-1, -5)$ **C** $(3, 0)$ **D** $(2, 10)$ **E** $(5, 2)$

14 The point of intersection of the lines shown in the diagram is:

A $(5, 2)$ **B** $(0, 0)$ **C** $(0, 9)$
D $(2, 5)$ **E** $(4, 10)$



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15 The solution to the pair of simultaneous equations:

$2x + 3y = -6$

$x + 3y = 0$ is:

A $(-6, -2)$ **B** $(6, 2)$ **C** $(2, 3)$ **D** $(-2, 6)$ **E** $(-6, 2)$

16 The equation of a straight line is $y = 4 + 3x$. When $x = 2$, y is:

A 2 **B** 3 **C** 4 **D** 6 **E** 10

17 The equation of a straight line is $y = 5 + 4x$. The y-intercept is:

A 2 **B** 3 **C** 4 **D** 5 **E** 20

18 The equation of a straight line is $y = 10 - 3x$. The slope is:

A -3 **B** 0 **C** 3 **D** 7 **E** 10

19 The equation of a straight line is $y - 2x = 3$. The slope is:

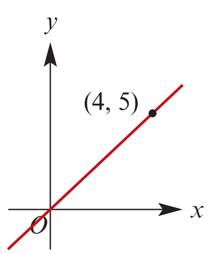
A -3 **B** -2 **C** 0 **D** 2 **E** 3

20 The slope of the line passing through the points (5, 8) and (9, 5) is:

A -1.3 **B** -1 **C** -0.75 **D** 0.75 **E** 1.3

21 The graph of $y = 4 + 5x$ is:

A



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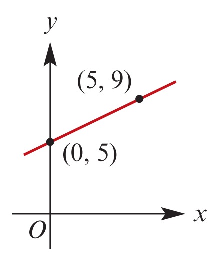

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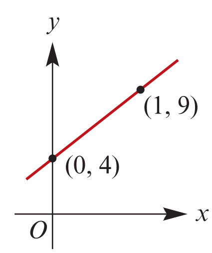

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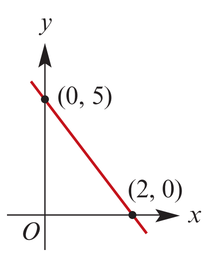
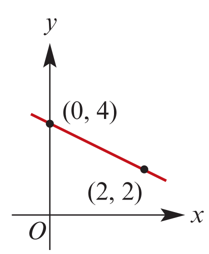


E



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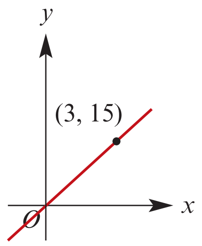
22 The graph of $y = 15 - 3x$ is:

A



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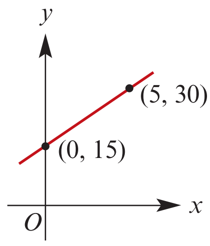

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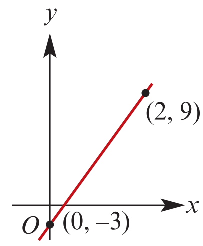

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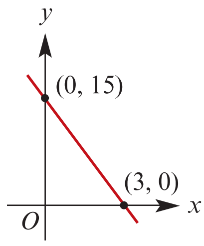

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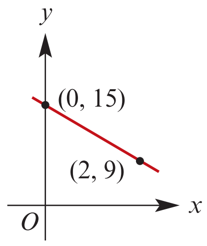
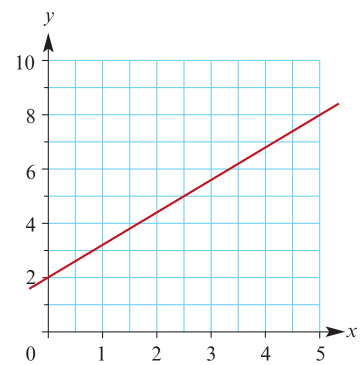


Questions 23 and 24 relate to the following graph.



23 The y-intercept is:

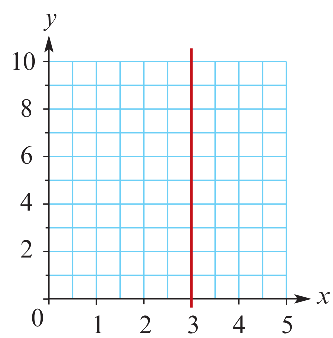
A -2 **B** 0
C 2 **D** 5
E 8

24 The slope is:

A 1.6 **B** 1.2
C 2 **D** 5
E 8

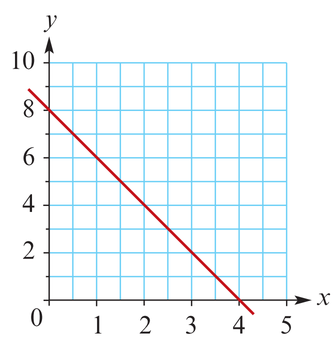
25 The slope of the line in the graph, shown opposite, is:

A negative
B zero
C positive
D three
E undefined



26 The equation of the graph, shown opposite, is:

A $y = -2 + 8x$
B $y = 4 - 2x$
C $y = 8 - 2x$
D $y = 4 + 2x$
E $y = 8 + 2x$



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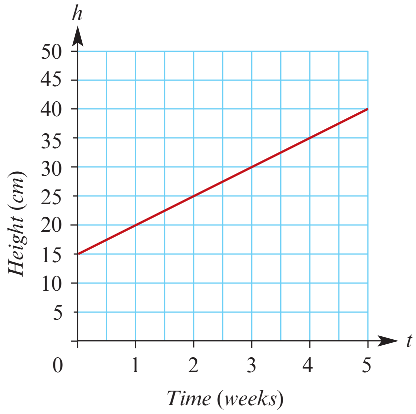
27 Which of the following points lies on the line $y = -5 + 10x$?

A $(1, -5)$ **B** $(1, 5)$ **C** $(1, 15)$ **D** $(2, 20)$ **E** $(2, 23)$

28 The graph opposite shows the height of a small tree, h, as it increases with time, t. Its growth rate is closest to:

A 1 cm/week
B 3 cm/week
C 5 cm/week
D 8 cm/week
E 15 cm/week



Short-answer questions

1 Solve the following equations for x.

a $x + 5 = 15$ **b** $x - 7 = 4$ **c** $16 + x = 24$
d $9 - x = 3$ **e** $2x + 8 = 10$ **f** $3x - 4 = 17$
g $x + 4 = -2$ **h** $3 - x = -8$ **i** $6x + 8 = 26$
j $3x - 4 = 5$ **k** $\frac{x}{5} = 3$ **l** $\frac{x}{-2} = 12$

2 If $P = 2l + 2b$, find P if:

a $l = 12$ and $b = 8$ **b** $l = 40$ and $b = 25$.

3 If $A = \frac{1}{2}bh$, find A if:

a $b = 6$ and $h = 10$ **b** $b = 12$ and $h = 9$.

4 The formula for finding the circumference of a circle is given by $C = 2\pi r$, where r is the radius. Find the circumference of a circle with radius 15 cm, to two decimal places.

5 For the equation $y = 33x - 56$, construct a table of values for x in intervals of 5 from -20 to 25.

a For what value of x is $y = 274$?
b When $y = -221$, what value is x?

6 I think of a number, double it and add 4. If the result is 6, what is the original number?

7 Four less than three times a number is 11. What is the number?

8 Find the point of intersection of the following pairs of lines.

a $y = x + 2$ and $y = 6 - 3x$ **b** $y = x - 3$ and $2x - y = 7$
c $x + y = 6$ and $2x - y = 9$

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9 Solve the following pairs of simultaneous equations.

a $y = 5x - 2$ and $2x + y = 12$

b $x + 2y = 8$ and $3x - 2y = 4$

c $2p - q = 12$ and $p + q = 3$

d $3p + 5q = 25$ and $2p - q = 8$

e $3p + 2q = 8$ and $p - 2q = 0$

10 Plot the graphs of these linear relations by hand.

a $y = 2 + 5x$

b $y = 12 - x$

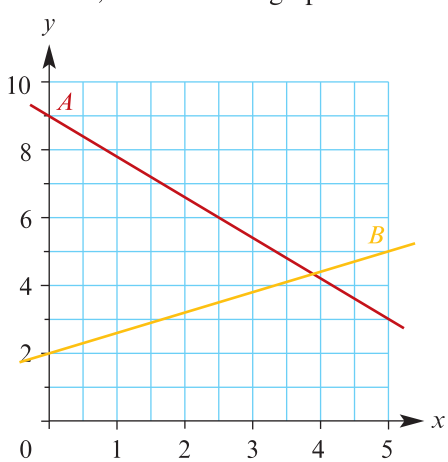
c $y = -2 + 4x$

11 A linear model for the amount C, in dollars, charged to deliver w cubic metres of builders' sand is given by $C = 95 + 110w$, for $0 \le w \le 7$.

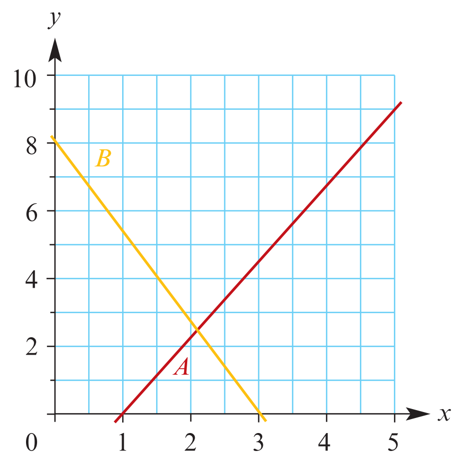
a Use the model to determine the total cost of delivering 6 cubic metres of sand.

b When the initial cost of \$95 is paid, what is the cost for each additional cubic metre of builders' sand?

12 Find the slope of each of the lines A and B, shown on the graph below.



13 Find the slope of each of the lines A and B (to two decimal places) shown on the graph below.



Written-response questions

1 The cost, C, of hiring a boat is given by $C = 25 + 8h$, where h represents hours.

a What is the cost if the boat is hired for 4 hours?

b For how many hours was the boat hired if the cost was \$81?

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2 A phone bill is calculated using the formula $C = 25 + 0.50n$, where n is the number of calls made.

a Complete the table of values below for values of n from 60 to 160.

n	60	70	80	90	100	110	120	130	140	150	160
C											

b What is the cost of making 160 phone calls?

3 An electrician charges \$80 up front and \$45 for each hour, h, that he works.

a Write a linear equation for the total charge, C, of any job.

b How much would a 3-hour job cost?

4 Two families went to the theatre. The first family bought tickets for 3 adults and 5 children and paid \$73.50. The second family bought tickets for 2 adults and 3 children and paid \$46.50.

a Write down two simultaneous equations that could be used to solve the problem.

b What was the cost of an adult's ticket?

c What was the cost of a child's ticket?

5 The perimeter of a rectangle is 10 times the width. The length is 9 metres more than the width. Find the width of the rectangle.

6 A secondary school offers three languages: French, Indonesian and Japanese. There are 105 students in Year 9. Each student studies one language. The Indonesian class has two-thirds the number of students that the French class has, and the Japanese class has five-sixths the number of students of the French class. How many students study each language?

7 A new piece of machinery is purchased by a business for \$300 000. Its value is then depreciated each month using the graph below.

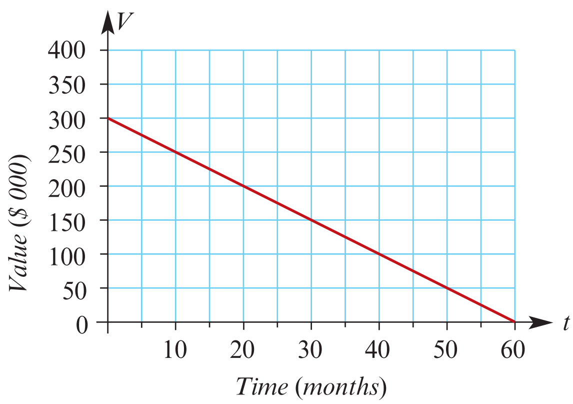
a What is the value of the machine after 20 months?

b When does the line predict that the machine has no value?

c Find the equation of the line in terms of value, V, (in thousands) and time, t.

d Use the equation to predict the value of the machine after 3 years.

e By how much does the machine depreciate in value each month?



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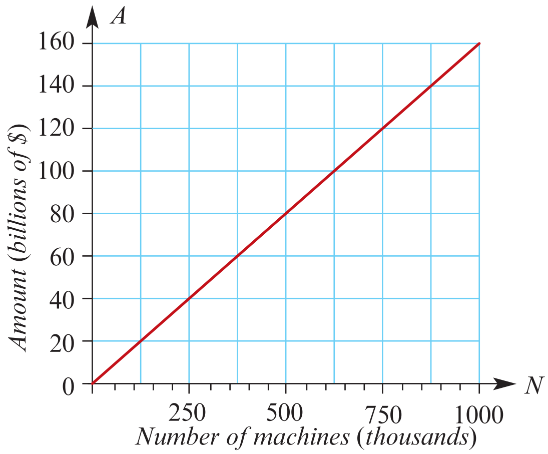
Review

8 The amount of money transacted through ATMs has increased with the number of ATMs available. The graph charts this increase.

a What was the amount of money transacted through ATMs when there were 500 000 machines?

b Find the equation of the line in terms of amount of money transacted, A, and number of ATMs, N. (Leave A in billions and N in thousands).

c Use the equation to predict the amount transacted when there were 600 000 machines.



d If the same rule applies, how much money is predicted to be transacted through ATM machines when there are 1 500 000 machines?

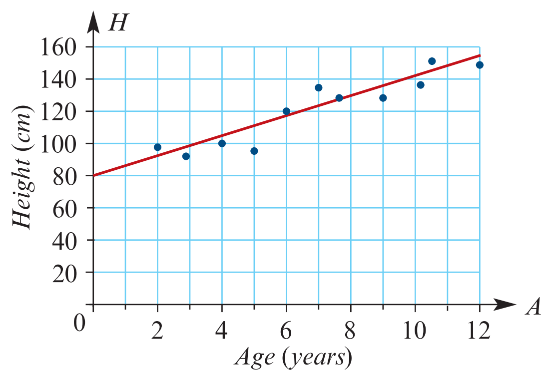
e By how much does the amount of money transacted through ATMs increase with each 1000 extra ATMs?

9 The heights, H, of a number of children are shown, plotted against age, A. Also shown is a line of good fit.

a Find the equation of the line of good fit in terms of H and A.

b Use the equation to predict the height of a child, aged 3 (to the nearest cm).

c Complete the following sentence by filling in the box.
The equation of the line of good fit tells us that, each year, children's heights increase by ☐ cm.



10 To conserve water, one charging system increases the amount people pay as the amount of water used increases. The charging system is modelled by:

$$C = 5 + 0.4x \ (0 \le x < 30) \qquad C = -31 + 1.6x \ (x \ge 30)$$

C is the charge, in dollars, and x is the amount of water used, in kilolitres (kL).

a Use the appropriate equation to determine the charge for using:

i 20 kL **ii** 30 kL **iii** 50 kL

b How much does a kilolitre of water cost when you use:

i less than 30 kL? **ii** more than 30 kL?

c Use the equations to construct a piecewise graph for $0 \le x \le 50$.

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Chapter 6

Revision of Unit 1

6A Multiple-choice questions

Chapter 2: Investigating and comparing data distributions

1 (VCAA-style question) In an experiment, researchers were interested in the effect of water on the growth of plants. They exposed groups of plants to three levels of water each day (5 – 10 mL, 15 – 20 mL, 25 – 30 mL), and then at the end of the experiment, classified the plants by size (as small, medium, large). The variables *levels of water* and *size* are:

A both ordinal variables

B a numerical variable and an ordinal variable respectively

C an ordinal variable and a numerical variable respectively

D an ordinal variable and a nominal variable respectively

E both numerical

2 (VCAA-style question) For which of the following variables is a bar chart an appropriate display?

A Petrol consumption (km/litre)

B Distance between towns (km)

C Time to complete a puzzle (seconds)

D Price of houses ($)

E Coffee size (small, medium, large, jumbo)

3 Which of the following displays are appropriate for numerical data:

A Dot plot and bar chart

B Dot plot and percentage bar chart

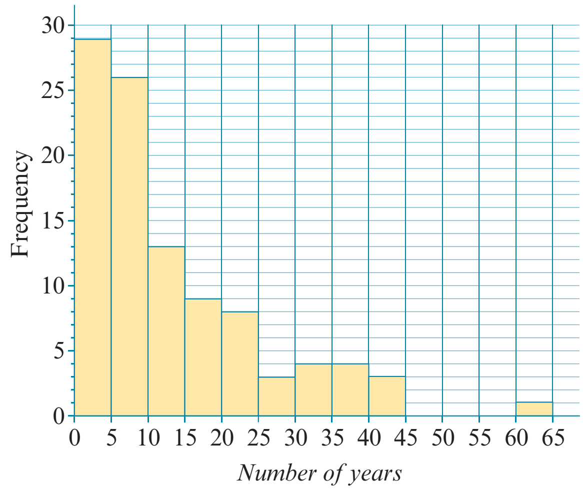
C Histogram and bar chart

D Histogram, dot plot and stem plot

E Stem plot, bar chart and percentage bar chart

The following information relates to Questions 4 -7.

A group of 100 people were asked how many years they have been doing their current type of work. Their responses are summarised in the following histogram.



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4 The percentage of employees who have worked from 5 to less than 10 years is closest to:

A 29% **B** 26% **C** 55% **D** 13% **E** 39%

5 (VCAA-style question) The median number of years employees have worked for their current employer is in the interval:

A 0 to less than 5 years **B** 5 to less than 10 years **C** 10 to less than 15 years
D 15 to less than 20 years **E** 20 to less than 25 years

6 The modal interval for number of years worked for their current employer is:

A 0 to less than 5 years **B** 5 to less than 10 years **C** 10 to less than 15 years
D 15 to less than 20 years **E** 20 to less than 25 years

7 (VCAA-style question) The shape of the histogram is:

A Positively skewed **B** Negatively skewed
C Symmetric **D** Bimodal
E Random

8 (VCAA-style question) Ten people were asked how many cups of coffee they each drink a week, giving the following data:

7 5 10 2 5 7 5 14 21 0

The mean and standard deviation of this data are closest to:

A $\bar{x} = 7.6$ and $s = 5.8$ **B** $\bar{x} = 7.6$ and $s = 6.1$ **C** $\bar{x} = 5.8$ and $s = 5.5$
D $\bar{x} = 8.4$ and $s = 5.8$ **E** $\bar{x} = 8.4$ and $s = 5.5$

9 A group of 500 people were asked how many hours per week they watched television. The five-number summary for the data collected is:

Min = 0 $Q_1 = 5$ $M = 18$ $Q_3 = 35$ Max = 92

The values of the lower and upper fences are:

A lower fence = 5 and upper fence = 35
B lower fence = 0 and upper fence = 92
C lower fence = −25 and upper fence = 65
D lower fence = −40 and upper fence = 80
E lower fence = 5 and upper fence = 80

10 (VCAA-style question) Suppose the distribution of heart rates is approximately symmetric and bell shaped, with a mean of 80 beats per minute and a standard deviation of 10. Approximately what percentage of people have heart rates between 60 and 100?

A 50% **B** 68% **C** 95% **D** 99.7% **E** 100%

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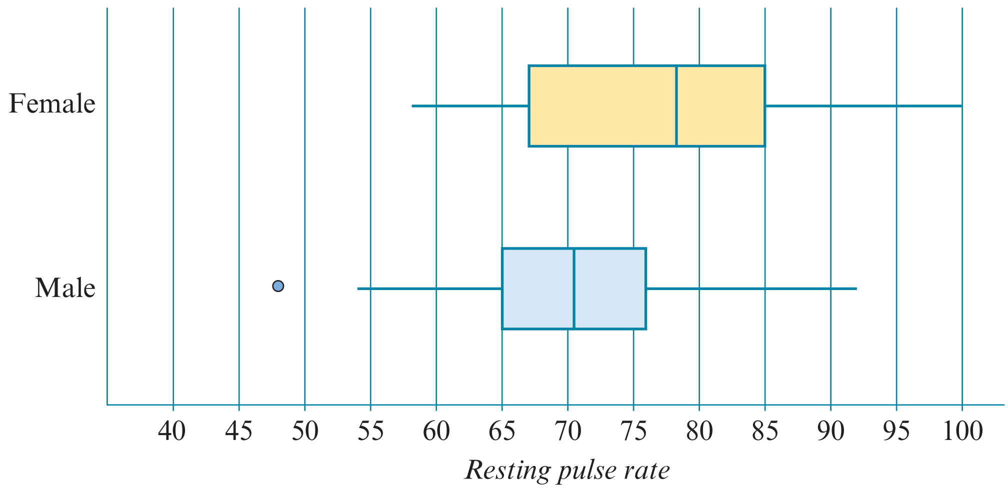
The following information relates to Questions 11 and 12.

The resting pulse rates in beats/min for a group of male students and a group of female students are summarised in the following boxplots.



11 (VCAA-style question) Which of the following statements is not true?

A About 50% of male students have pulse rates greater than 70 beats/min.

B About 25% of female students have pulse rates more than 85 beats/min.

C More than 75% of female students have pulse rates higher than, at most, 25% of male students.

D The lowest pulse rate for males is about 54 beats/min.

E The interquartile range of the pulse rates for males is about 11.

12 Using the median as a measure of the average pulse rate, which of the following can be concluded from the boxplots?

A The pulse rates of female students are lower on average and more variable than the pulse rates of male students.

B The pulse rates of female students are lower on average and less variable than the pulse rates of male students.

C The pulse rates of female students are higher on average and more variable than the pulse rates of male students.

D The pulse rates of female students are higher on average and less variable than the pulse rates of male students.

E The pulse rates of female students are higher on average and about the same in variability as the pulse rates of male students.

13 (VCAA-style question) The distribution of the length (in mm) of a species of earth worm is bell shaped, with a mean of 75.5 mm and a standard deviation of 1.5 mm.

The percentage of the worms with a length less than 72.5 mm is closest to:

A 2.5% **B** 5% **C** 16% **D** 84% **E** 95%

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Chapter 3: Sequences and finance

14 (VCAA-style question) The sequence generated by the recurrence relation $V_0 = 6,\ V_{n+1} = V_n + 3$ is

A 6, 9, 46, 136, 406, . . . **B** 6, 9, 12, 15, 18, . . . **C** 6, 2, −2, −6, −10, . . .
D 6, 14, 44, 134, 401, . . . **E** 6, −14, 44, −134, . . .

15 (VCAA-style question) The sequence generated by the recurrence relation $V_0 = 6,\ V_{n+1} = V_n - 4$ is

A 6, 16, 46, 136, 406, . . . **B** 6, 9, 12, 15, 18, . . . **C** 6, 2, −2, −6, −10, . . .
D 6, 14, 44, 134, 401, . . . **E** 6, −14, 44, −134, 404, . . .

16 (VCAA-style question) Which of the following could be the first five terms, t_0, t_1, t_2, t_3, t_4, of an arithmetic sequence?

A 2, 4, 2, 4, 2 **B** 1, 10, 100, 1000, 10000 **C** −189, −89, 11, 111, 211
D 1, 4, 9, 16, 25 **E** 4, 4, 6, 6, 8

17 Which of the following is not an arithmetic sequence?

A 11, 2, −8, −19, . . . **B** 4, 7, 10, 13, . . . **C** 57, 51, 45, 39, . . .
D −3, −5, −7, −9, . . . **E** 1, 2, 3, 4, . . .

18 (VCAA-style question) Brian initially has two camelias in his garden. Every week, he will plant three more camelias.
A recurrence model for the number of camelias in Brian's backyard after n weeks is

A $T_0 = 2,\ T_{n+1} = 3T_n$ **B** $T_0 = 2,\ T_{n+1} = 3T_n + 3$ **C** $T_0 = 2,\ T_{n+1} = T_n + 3$
D $T_0 = 2,\ T_{n+1} = T_n - 3$ **E** $T_0 = 2,\ T_{n+1} = 3T_n - 3$

19 (VCAA-style question) Lee invests \$40 000 with a bank. She will be paid simple interest at the rate of 5.1% per annum. If V_n is the value of Lee's investment after n years, a recurrence model for Lee's investment is

A $V_0 = 40\,000,\ V_{n+1} = V_n + 5.1$ **B** $V_0 = 40\,000,\ V_{n+1} = 5.1V_n$
C $V_0 = 40\,000,\ V_{n+1} = 0.051V_n + 102$ **D** $V_0 = 40\,000,\ V_{n+1} = V_n + 2040$
E $V_0 = 40\,000,\ V_{n+1} = 5.1V_n + 2000$

20 A sequence is generated from the recurrence relation $V_0 = 50,\ V_{n+1} = V_n - 20$.
The rule for the value of the term V_n is

A $V_n = 50n - 20$ **B** $V_n = 50 - 20n$ **C** $V_n = 50n$
D $V_n = 50 + 20n$ **E** $V_n = 50n - 20$

21 (VCAA-style question) A computer is depreciated using a flat-rate depreciation method. It was purchased for \$3200 and depreciates at the rate of 15% per annum.
The value of the computer after 4 years is

A \$1280 **B** \$1920 **C** \$1760 **D** \$2720 **E** \$5120

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22 For the arithmetic sequence with $t_0 = 39, t_1 = 36, t_2 = 33, \ldots$, the term t_8 is equal to

A 14 **B** 15 **C** 16 **D** 24 **E** 26

23 For an arithmetic sequence with terms $t_0, t_1, t_2, \ldots$, it is known that $t_1 = 26$ and $t_5 = 10$. The term t_3 is equal to:

A 14 **B** 18 **C** 32 **D** 40 **E** 52

24 (VCAA-style question) A recurrence relation is defined by:
$t_{n+1} - t_n = -7$ with $t_0 = 3$. The term t_n is equal to

A $3 + 7n$ **B** $3 + 6n$ **C** $-7 + 3n$ **D** $-3n + 7$ **E** $3 - 7n$

25 Which one of the following is not a geometric sequence?

A $2, 30, 450, 6750 \ldots$
B $-3, 9, -27, 81, -243, 729, -2187 \ldots$
C $7, 21, 55, 165, \ldots$
D $1, 3, 9, 27, 81 \ldots$
E $60, 12, 2.4, 0.48, 0.096, \ldots$

26 (VCAA-style question) A car purchased on 1 June 2022 loses value at a depreciation rate of 20% per year. The original purchase price was $25 000. The value of the car, in dollars, on 1 June 2028 will be closest to

A $7000 **B** $8000 **C** $9000 **D** $10 000 **E** $11 000

27 (VCAA-style question) The following is a geometric sequence:
2000, 500, 125, 31.25, . . . The common ratio, R, is equal to

A −500 **B** 0.25 **C** 0.5 **D** 0.75 **E** −0.5

28 (VCAA-style question) The present value of an investment (to the nearest dollar) is $72 000. It was reached by investing $$P$ at a compound interest rate of 3% for 4 years. The value of P was?

A $63 741 **B** $63 971 **C** $64 008 **D** $65 392 **E** $81 036

Chapter 4: Matrices

Use matrices A and B in Questions 29 to 32.

$$A = \begin{bmatrix} 3 & -5 \\ 1 & 8 \\ 2 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & -2 \\ -1 & 0 \\ 3 & 7 \end{bmatrix}$$

29 The order of matrix A is

A 6 **B** 2×3 **C** 3×2 **D** 2, 3 **E** 3, 2

30 The element a_{12} is

A 3 **B** −5 **C** 1 **D** 8 **E** 2

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31 The matrix $A + B$ is

A $\begin{bmatrix} 8 & -3 \\ 2 & 8 \\ 5 & 3 \end{bmatrix}$ **B** $\begin{bmatrix} 8 & -3 \\ 2 & 8 \\ 5 & 11 \end{bmatrix}$ **C** $\begin{bmatrix} 8 & -7 \\ 2 & 8 \\ 5 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 8 & -7 \\ 0 & 8 \\ 5 & 3 \end{bmatrix}$ **E** $\begin{bmatrix} 8 & -3 \\ 0 & 8 \\ 5 & 3 \end{bmatrix}$

32 The matrix $2A - B$ is

A $\begin{bmatrix} 1 & -8 \\ 1 & 16 \\ 1 & -15 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & -6 \\ 1 & 16 \\ 1 & -15 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & -6 \\ 3 & 16 \\ 1 & -15 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & -8 \\ 3 & 16 \\ 1 & -15 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & -7 \\ 2 & 8 \\ -1 & -11 \end{bmatrix}$

Use the matrices E, F, G, H in Questions 33 to 35.

$$E = \begin{bmatrix} 4 & 5 & 2 \\ 0 & 9 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} \quad G = \begin{bmatrix} -2 & 8 \end{bmatrix} \quad H = \begin{bmatrix} 3 & 2 \\ -1 & 7 \end{bmatrix}$$

33 Matrix multiplication is not defined for

A EF **B** GH **C** GE **D** HE **E** EH

34 The order of matrix FG is

A 1×1 **B** 1×2 **C** 3×2 **D** 6 **E** 5

35 The inverse of matrix H is

A $\begin{bmatrix} 7 & 2 \\ 1 & 3 \end{bmatrix}$ **B** $\frac{1}{23}\begin{bmatrix} 7 & -2 \\ 1 & 3 \end{bmatrix}$ **C** $\begin{bmatrix} 7 & -2 \\ 1 & 3 \end{bmatrix}$ **D** $\frac{1}{23}\begin{bmatrix} 7 & 2 \\ 1 & 3 \end{bmatrix}$ **E** $23\begin{bmatrix} 7 & -2 \\ 1 & 3 \end{bmatrix}$

36 Three students were asked the number of pets they each had. The following matrix shows the number of pets owned by the three friends.

	Dogs	*Cats*	*Fish*
Daniel	2	1	0
Eloise	1	0	4
Freddy	1	3	0

The number of pets owned by Freddy's family is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

37 The matrix opposite shows the number of different social media applications used by three friends: Manesh (M), Nikita (N) and Paolo (P).

The total number of ways that Manesh directly connects to each of his friends is:

	M	*N*	*P*
M	0	3	2
N	3	0	4
P	2	4	0

A 1 **B** 2 **C** 3 **D** 4 **E** 5

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38 (VCAA-style question) The matrix below shows how five people, Anna (A), Ben (B), Charis (C), Damien (D) and Emma (E), can communicate with each other. A '1' in the matrix indicates that the sender (row) can send a message directly to the receiver (column).

$$\text{Sender}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \text{Receiver} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Damien wishes to send a message to Anna. Which one of the following shows the order through which the message could be sent?

A $D-C-A$

B $D-E-A$

C $D-C-E-B-A$

D $D-E-B-A$

E $D-C-E-A$

39 (VCAA-style question)

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 3 \\ 8 \end{bmatrix}$$

Which of the following systems of simultaneous linear equations best represents the matrix equation above?

A

$$\begin{aligned} v + 2x + y &= 5 \\ 2v + w &= 13 \\ w + 2x + y &= 3 \\ v + x + 2y &= 8 \end{aligned}$$

B

$$\begin{aligned} v + 2w + y &= 5 \\ w + x &= 13 \\ 2v + 2x + y &= 3 \\ v + x + 2y &= 8 \end{aligned}$$

C

$$\begin{aligned} v &= 5 \\ w &= 13 \\ x &= 3 \\ y &= 8 \end{aligned}$$

D

$$\begin{aligned} v + 2w + x &= 5 \\ 2v + w &= 13 \\ v + 2w + x &= 3 \\ v + w + 2x &= 8 \end{aligned}$$

E

$$\begin{aligned} v &= 5 \\ w &= 3 \\ x &= -1 \\ y &= 12 \end{aligned}$$

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Chapter 5: Linear relations and modelling

40 If $a = 2$, $b = 3$ and $c = 5$, then $ab - c$ is:

A 0 **B** 1 **C** 18 **D** 27 **E** 28

41 The solution to $2x + 3 = 12$ is:

A 2.4 **B** 4.5 **C** 7.5 **D** 9 **E** 18

42 The equation of a straight line is $y = 3x - 7$. When $x = 5$, y is:

A -7 **B** 5 **C** 8 **D** 15 **E** 28

43 A music band charges a fixed fee of \$400, plus \$150 per hour of music.
The equation that represents the total amount, \$$C$, charged for t hours of music is:

A $C = 150t$ **B** $C = 400t$ **C** $C = 550t$
D $C = 400 + 150t$ **E** $C = 150 + 400t$

44 The slope and the y-intercept of the straight line $y = 3 - 2x$ are:

A slope = 1, y-intercept = 3 **B** slope = 2, y-intercept = 3
C slope = -2, y-intercept = 3 **D** slope = 3 y-intercept = 2
E slope = 3 y-intercept = -2

45 The solution to the simultaneous equations:

$$2x - 3y = 5$$
$$7x + 8y = 3$$

is:

A $(-0.78, 1.32)$ **B** $(1, -1)$ **C** $(5, 3)$ **D** $(1.32, -0.78)$ **E** $(2.5, 2.7)$

46 If 5 times a number added to 71 gives 101, then the number is:

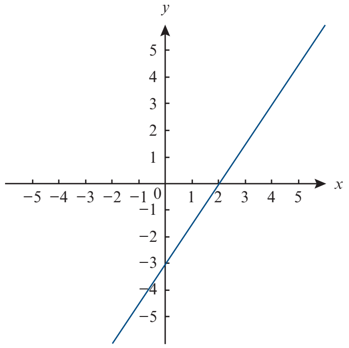
A 3 **B** 6 **C** 25 **D** $34\frac{2}{5}$ **E** 53

47 The slope of the line passing through the points $(3, -1)$ and $(4, 1)$ is:

A -2 **B** 0 **C** $\frac{1}{2}$ **D** 2 **E** $\frac{7}{2}$

48 The equation of the graph shown is:

A $y = -3 - 2x$
B $y = -2 - 3x$
C $y = -3 + 2x$
D $y = 1.5x - 3$
E $y = 3 - 2x$



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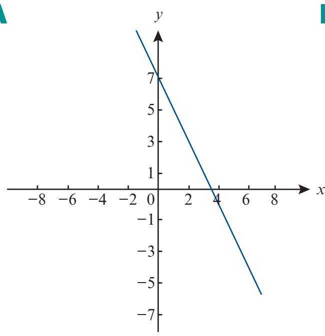
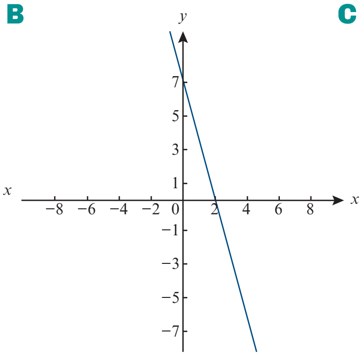
49 The graph of $y = 7 - 2x$ is:

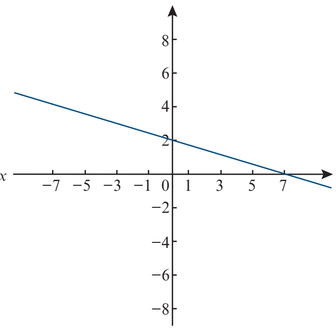
A



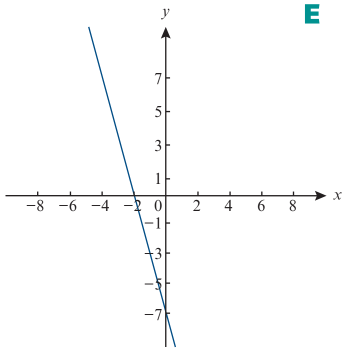
B



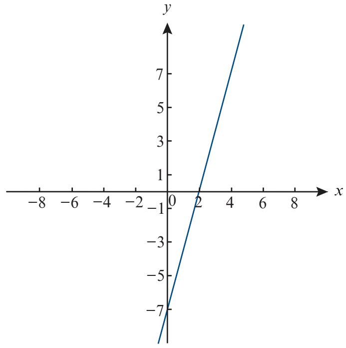
C



D



E



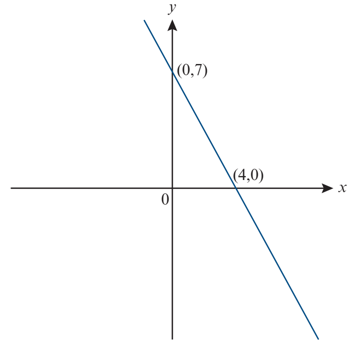
50 The solution to the following simultaneous equations is:

$$3x + 6y = 60$$
$$x + 9y = 69$$

A $x = 6, y = 6$ **B** $x = 6, y = 7$ **C** $x = 7, y = 6$ **D** $x = 3, y = 6$ **E** $x = 1, y = 9$

51 The graph shows a line intersecting the x-axis at (4, 0) and the y-axis at (0, 7). The gradient of the line is:

A $-\frac{4}{7}$ **B** $-\frac{7}{4}$ **C** $\frac{4}{7}$ **D** $\frac{7}{4}$ **E** 3



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52 (VCAA-style question) A sports store is having a sale. Each soccer ball costs \$15 and each basketball costs \$30. Chloe plans to restock the sporting equipment at her school. She buys 8 balls for a total of \$165. The number of basketballs that Chloe buys is:

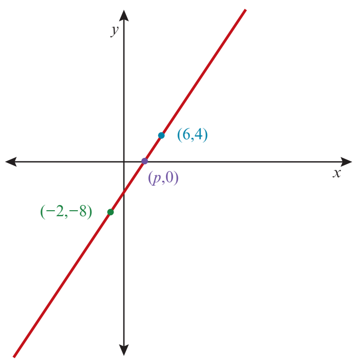
A 1 **B** 2 **C** 3 **D** 4 **E** 5

53 (VCAA-style question) Sebastian makes and sells postcards. Each postcard costs 70 cents to make and sells for \$3.50. Sebastian also has a fixed cost for making a batch of postcards. On Saturday, Sebastian sold 62 postcards for a profit of \$153.60. Sebastian's fixed cost for making a batch of postcards is:

A \$ 20 **B** \$ 100 **C** \$ 173.60 **D** \$ 4123 **E** \$ 4276.60

Use the following information to answer Questions 54 and 55.

The graph below shows a straight line that passes through the points (6,4), (−2, −8) and (p,0).



54 (VCAA-style question) The coordinates of another point on the line are:

A (4,0) **B** (5,3) **C** (7,5) **D** (12,13) **E** (15,15)

55 (VCAA-style question) The value of P is:

A 1 **B** 2 **C** 5/2 **D** 3 **E** 10/3

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6B Written-response questions

Chapter 2: Investigating and comparing data distributions

1 A group of 200 people was asked to describe their general happiness by choosing one of the responses: very happy, pretty happy, and not too happy. The data is summarised in the following frequency table:

General happiness	Frequency	
	Number	%
Very happy	53	26.5
Pretty happy		59.0
Not too happy	29	
Total	200	100.0

a What type of data has been collected: categorical or numerical?

b Complete the frequency table and the percentage frequency table.

c Construct a percentage bar chart.

d Write a report summarising the responses, quoting appropriate percentages to support your conclusion.

2 The following data gives daily rainfall, in mm, for a city for the 31 days of May:

0	4	0	1	2	0	0	6	1	9	10	0
0	1	2	3	2	1	0	1	1	0	0	0
2	8	9	1	1	7	3					

a Construct a dot plot of the data.

b Determine the values of:

i the range

ii the median

iii the interquartile range.

c In what percentage of the days was there no rain?

d Construct a histogram of the rainfall data, starting at 0 and with interval widths of 2.

3 Annual salaries data for a group of employees in a company were collected in 2016 and again in 2020.

a A five-number summary for the salaries in the Marketing team of the company in 2016 was as follows:

Min = \$29 000 Q_1 = \$49 000 M = \$63 000 Q_3 = \$92 000 Max = \$133 000

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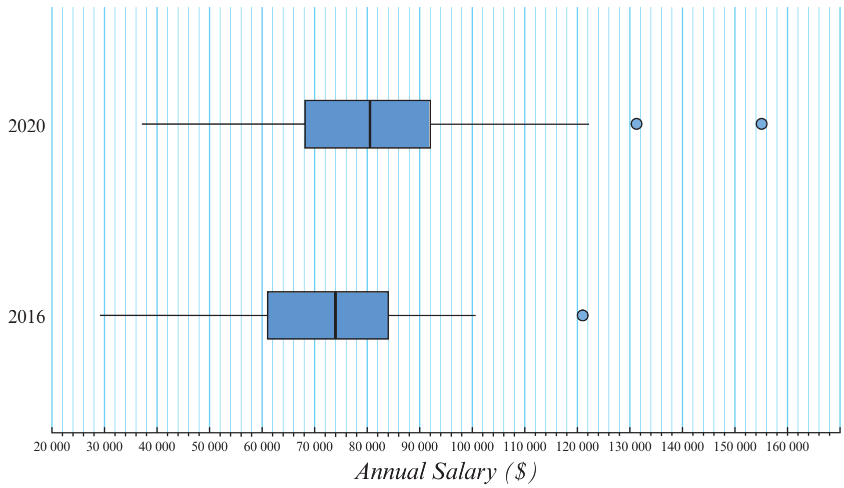
i Use the five-number summary to construct a boxplot.

ii Determine the values of the upper and lower fences, and use these to confirm that there are outliers.

iii What percentage of Marketing team employees earned more than $49 000?

b The boxplots below display the distribution of the salaries of all employees in 2016 and 2020:



i What is the median salary for each year?

ii Use the information from the boxplots to write a short report comparing the salaries at the company in 2016 and in 2020.

4 The number of goals scored by each member of two basketball teams in the finals is as follows:

Team A

20 19 16 14 11 10 9 8 8 7 6 4 2 0 0

Team B

31 30 15 13 13 12 10 6 6 4 2 2 2 2 0

a Which of the variables, *team* or *score*, is a categorical variable?

b Construct back-to-back stem plots of this data, using the following stems:

c Determine the five-number summary for each team.

d Identify any outliers in the scores for each team.

e How do you think the mean score for Team A would compare to the mean score for Team B? Give a reason for your answer.

Score

Team A		Team B
	0	
	0	
	1	
	1	
	2	
	2	
	3	

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Chapter 3: Sequences and finance

5 The following recurrence relation can be used to model a simple interest investment of \$2000, paying interest at the rate of 3.8% per annum.

$$V_0 = 2000, \qquad V_{n+1} = V_n + 76$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b Use your calculator to determine how many years it takes for the value of the investment to first be more than \$3000.

c Write down a recurrence relation model if \$1500 was invested at the rate of 6.0% per annum.

6 Jane and Kate each inherit \$50 000 from a long-lost relative.

a Jane decides to loan her money to a friend, who agrees to pay her \$750/month for 10 years.

i How much does the friend repay in total, including monthly repayments and the principal.

ii What is the equivalent simple interest rate per annum of this investment?

b Kate decides to invest her money in an investment account, which pays 6.75% compound interest per annum, compounding monthly.

i How much will Kate have in her investment account after 10 years?

ii Find the lowest annual interest rate, correct to 1 decimal place, that would make Kate's investment worth at least as much as Jane's investment after 10 years if interest compounds monthly.

7 The following recurrence relation can be used to model the depreciation of a computer with purchase price \$2500 and annual depreciation of \$400.

$$V_0 = 2500, \quad V_{n+1} = V_n - 400$$

In the recurrence relation, V_n is the value of the computer after n years.

a Use the recurrence relation to find the value of the computer after 1, 2 and 3 years.

b Use your calculator to determine how many years it takes for the value of the computer to be less than \$1000.

c Write down a recurrence relation model if the computer was purchased for \$1800 and depreciated at \$350 per annum.

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8 A factory has a tall chimney stack from which a pollutant gas is emitted at the rate of 1500 kilograms per day. New technology has been developed that enables the emissions to be reduced in stages to a minimum of 200 kilograms per day.

a Using one method of installation, the emissions will be reduced by a constant amount each day until the minimum emission of 200 kilograms per day is reached. Consider the case where the emissions are reduced by 130 kilograms each day. The installation will be completed after 12 days, and from the 13th day, the emissions will be 200 kilograms per day. Let E_i be the emissions after i days of the installation process.

i Use this information to complete the table below.

Day(i)	0	1	2	3	4	5	6	7	8	9	10
E_i	1500	1370	1240								

ii Write the recurrence relation in terms of E_i to describe the reduction in emissions for $0 \le i \le 12$.

iii Write down the rule for E_n in terms of n, where $0 \le n \le 12$.

b Now suppose that the installation is to be completed by the end of the fifth day (K_5), so that from the sixth day, the emissions will be 200 kilograms per day. (K_i is the emissions after i days of the new installation process) .

i By what constant amount must the emissions be reduced each day during the installation period?

ii Draw a graph for the sequence for $0 \le i \le 5$.

iii Write the recurrence relation in terms of K_i to describe the reduction in emissions for $0 \le i \le 5$.

iv Write down the rule for K_n in terms of n, where $0 \le n \le 5$.

9 The number of rabbits on a remote island is causing concern. Scientists sent to investigate the problem find that, on their arrival, there are 360 000 rabbits on the island.

a If the rabbit population on the island increases at a constant rate of 12 000 rabbits per week:

i How many rabbits will be on the island one week after the scientists' arrival?

ii Write down a rule for the number of rabbits on the island, recorded at n weeks after the arrival of the scientists.

iii How many weeks would it take for this rabbit population to grow to 600 000?

iv Calculate the constant weekly increase that would lead to a rabbit population of one million on this island, recorded at 40 weeks after the scientists' arrival.

Continued

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b To help reduce the rabbit population, the scientists, upon their arrival on the island, introduced a viral infection into the rabbit population. The scientists kept a record of the number of rabbits killed each week by the virus, with the following results for the first three weeks:

Week number (n)	1	2	3
Number of rabbits killed by virus	512	768	1152

The scientists found that the number of rabbits killed each week by the virus followed a geometric sequence

i Show that the common ratio, R, for the sequence is 1.5.

ii Calculate the number of rabbits that die from the virus in Week 5.

iii Calculate the total number of rabbits that die from the virus during the first six weeks of the scientists' stay on the island.

Chapter 4: Matrices

10 Mary and Derek are fitness enthusiasts. Each week, they record how far they run, cycle and swim in km. The information for last week is recorded in matrix F.

$$F = \begin{array}{c} \\ Mary \\ Derek \end{array} \begin{array}{c} \begin{array}{ccc} Run & Cycle & Swim \end{array} \\ \begin{bmatrix} 38 & 85 & 4 \\ 12 & 130 & 0 \end{bmatrix} \end{array}$$

a How many kilometres did Mary do in total last week?

b What is the total distance that Mary and Derek ran last week?

c Mary cycles to and from work each day (Monday to Friday) and then goes for a longer ride on the weekends. If her workplace is 7 km from her home, how far does she cycle on the weekend?

11 A sushi shop records the sales for two stores (A and B) of Handrolls, Sashimi packs and Maki packs in matrix S. The prices were recorded in the prices matrix, P.

$$S = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} Handrolls & Sashimi & Maki \end{array} \\ \begin{bmatrix} 60 & 15 & 8 \\ 45 & 21 & 10 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \\ Handrolls \\ Sashimi \\ Maki \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 4 \\ 15 \\ 10 \end{bmatrix} \end{array}$$

a How many Handrolls were sold by Shop B?

b What is the selling price of the Sashimi Packs?

c Calculate the matrix product SP.

d What information is contained in matrix SP?

e Which shop had the larger income from its sales? How much were its takings?

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Chapter 5: Linear relations and modelling

12 A shop sells fruit in two types of gift boxes: standard and deluxe. Each standard box contains 1 kg of peaches and 2 kg of apples, and each deluxe box contains 2 kg of peaches and 1.5 kg of apples. On one particular day, the shop sold 12 kg of peaches and 14 kg of apples in gift boxes. How many of each kind of box were sold on the day?

13 A gardener charges a fixed fee of \$50 plus \$15 for each hour of work.

a Using t for time (in hours), write a formula for the total cost, C, of t hours of work.

b Sketch a graph of the cost, C.

c Martha paid the gardener \$76.25. How many hours did the gardener work?

14 A person's total body water - the amount of water in their body - is dependent on their gender, age, height and weight.
Total body water (TBW) for males and females can be found by using the following formulas:

$$\underset{\text{(litres)}}{Male\ TBW} = 2.447 - 0.09516 \times \underset{\text{(years)}}{age} + 0.1074 \times \underset{\text{(cm)}}{height} + 0.3362 \times \underset{\text{(kg)}}{weight}$$

$$\underset{\text{(litres)}}{Female\ TBW} = -2.097 + 0.1069 \times \underset{\text{(cm)}}{height} + 0.2466 \times \underset{\text{(kg)}}{weight}$$

a What is the TBW for a female of height 175 cm and weight 62 kg? Give your answer correct to two decimal places.

b Calculate the TBW, correct to two decimal places, for a 45-year-old male of height 184 cm and weight 87 kg.

c A healthy 27-year-old female has a TBW of 32 and weighs 62 kg. What is her height, to the nearest cm?

d What would be the TBW, correct to two decimal places, for a 78-year-old man of height 174 cm and weight 80 kg?

e Over a period of a week, the 78-year-old man's weight rapidly increases to 95 kg. What is his new TBW, correct to two decimal places?

f Construct a table showing the TBW for a 22-year-old male of height 185 cm, with weights in increments of 5 kg from 60–120 kg. Give your answers correct to two decimal places.

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6C Investigations

Statistics investigation

1 To investigate the age of parents at the birth of their first child, a hospital recorded the ages of the mothers and fathers for the first 40 babies born in the hospital for each of the years 1970, 1990 and 2010. The data is given below:

1970 Mother									
23	22	33	19	19	26	20	15	26	17
18	31	24	20	29	28	25	45	28	22
1970 Father									
29	15	39	29	22	35	32	26	37	29
25	31	20	34	28	22	33	25	34	46
1990 Mother									
28	14	38	28	21	34	31	25	36	28
24	30	19	33	27	21	32	24	33	45
1990 Father									
31	27	46	31	26	28	30	27	43	37
39	22	27	35	31	29	32	27	38	35
2010 Mother									
30	26	45	32	25	27	29	26	42	36
38	21	26	34	37	28	28	37	37	34
2010 Father									
37	31	39	36	21	34	34	23	17	37
23	33	31	32	24	39	45	30	35	34

a **i** Construct parallel boxplots of the mothers' and fathers' ages in 1970.

ii Determine the median and interquartile range for the ages of the mothers and fathers in 1970.

iii Based on the analyses in part **ai** and part **aii**, write a report comparing the ages of mothers and fathers in 1970.

b **i** Construct parallel boxplots of the mothers' and fathers' ages in 1990.

ii Determine the median and interquartile range for the ages of the mothers and fathers in 1990.

iii Based on the analyses in part **bi** and part **bii**, write a report comparing the ages of mothers and fathers in 1990.

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c **i** Construct parallel boxplots of the mothers' and fathers' ages in 2010.

ii Determine the median and interquartile range for the ages of the mothers and fathers in 2010.

iii Based on the analyses in part **ci** and part **cii**, write a report comparing the ages of mothers and fathers in 2010.

d Use your previous analyses to describe the changes in mothers' ages and fathers' ages over time.

2 Does Year 12 require more study time than Year 11? Carry out a statistical investigation to answer this question. The statistical investigation process is the principal means of problem-solving and modelling in statistics, and it encompasses the following steps which you should follow:

- Pose the question – Decide what variables would allow you to address the problem. Options could include the number of hours each week students spend on their homework, or the number of homework sessions per week for each year level.
- Data – Collect the data from a sample of students at each of the year levels, or obtain it if it already exists. The sample size should be large enough to analyse: at least 20 students at each year level would be desirable.
- Analyse – Summarise and display the data to answer the question posed. Here, the analysis methods would include summary statistics and boxplots.
- Conclusion – Interpret the results and communicate what has been learned in a written report.

Matrices investigation - Encoding and decoding

3 In the past, commonly used codes replaced each letter of the alphabet with a randomly chosen number or symbol. The intended recipient could use a list of the changes to change each number back to a letter. The weakness in this type of code is that, in the English language, E is the most frequently occurring letter, followed by T and then A. A table of frequencies for letters can be used to replace numbers occurring with about the same frequency, and hence, to break the code.

Matrices can be used to encode words so that each letter does *not* have the same number throughout the coded message. This makes the message extremely difficult to decode, without knowing the secret encoding matrix. We will use this simple table with the letters numbered in order, but use a matrix to encode the final message.

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	space
15	16	17	18	19	20	21	22	23	24	25	26	27

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Revision

Example 1

Encode the message: 'Meet me,' then show how to decode the encoded message.
Use the encoding matrix, C, where:

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Explanation | **Solution**

Encoding the message

a Write the letters in a 2×4 matrix, M.

$$M = \begin{bmatrix} \text{M E E T} \\ \text{M E} \end{bmatrix}$$

b Use the table on the opposite page to replace each letter with its number.
Do all of the following steps using your graphics calculator.

$$M = \begin{bmatrix} 13 & 5 & 5 & 20 \\ 27 & 13 & 5 & 27 \end{bmatrix}$$

c Multiply the message matrix, M, by the secret encoding matrix, C.
Notice that the three Es now have three different numbers in this encoded message matrix, CM.

$$CM = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 13 & 5 & 5 & 20 \\ 27 & 13 & 5 & 27 \end{bmatrix}$$

$$= \begin{bmatrix} 107 & 49 & 25 & 121 \\ 67 & 31 & 15 & 74 \end{bmatrix}$$

Decoding the message

a Work out the *inverse* of the secret encoding matrix. Use a calculator.

The inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ is $C^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

b Multiply the encoded message matrix, CM, by the inverse of the encoding matrix, C^{-1}, to return to the message matrix, M.

$$C^{-1} \times CM = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 107 & 49 & 25 & 121 \\ 67 & 31 & 15 & 74 \end{bmatrix}$$

$$M = \begin{bmatrix} 13 & 5 & 5 & 20 \\ 27 & 13 & 5 & 27 \end{bmatrix}$$

c Use the table above to replace each number with its letter.

$$M = \begin{bmatrix} \text{M E E T} \\ \text{M E} \end{bmatrix}$$

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A 2 × 2 matrix can be used to encode any information written as a matrix with two rows. Credit card numbers consisting of 16 digits can be written into a 2 × 8 matrix and encoded using a 2 × 2 matrix.
Choose a 2 × 2 encoding matrix with small positive numbers so that the numbers in the encoded message do not get too large.

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	space
15	16	17	18	19	20	21	22	23	24	25	26	27

a Use the table for swapping letters with numbers, and use matrix C to *encode* each of the following messages.

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

i $\begin{bmatrix} F & B & I & \\ K & N & O & W \end{bmatrix}$

ii $\begin{bmatrix} M & A & P & \\ L & O & S & T \end{bmatrix}$

iii $\begin{bmatrix} A & P & E & \\ F & A & C & E \end{bmatrix}$

iv $\begin{bmatrix} F & I & N & D \\ & T & O & M \end{bmatrix}$

v $\begin{bmatrix} N & O & & G & U & A & R & D \\ & T & O & N & I & G & H & T \end{bmatrix}$

vi $\begin{bmatrix} M & E & E & T & & A & N & N \\ A & T & & J & O & H & N & S \end{bmatrix}$

b Use the letters-to-numbers table and the *inverse* of matrix S to *decode* the following messages.

$$S = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

i $\begin{bmatrix} 10 & 27 & 33 & 62 \\ 19 & 45 & 53 & 97 \end{bmatrix}$

ii $\begin{bmatrix} 28 & 41 & 52 & 59 \\ 47 & 62 & 85 & 91 \end{bmatrix}$

iii $\begin{bmatrix} 20 & 39 & 63 & 59 \\ 34 & 66 & 101 & 91 \end{bmatrix}$

iv $\begin{bmatrix} 37 & 65 & 29 & 79 \\ 56 & 109 & 44 & 131 \end{bmatrix}$

v $\begin{bmatrix} 22 & 45 & 48 & 67 & 73 & 15 & 36 & 59 \\ 40 & 75 & 82 & 114 & 119 & 23 & 57 & 91 \end{bmatrix}$

vi $\begin{bmatrix} 26 & 51 & 30 & 35 & 58 & 23 & 39 & 58 \\ 46 & 84 & 55 & 66 & 89 & 37 & 59 & 89 \end{bmatrix}$

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c Use the encoding matrix, B, to do the following.

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

i Encode the credit card number written into this 2×8 matrix:

$$\begin{bmatrix} 3 & 1 & 4 & 7 & 2 & 3 & 8 & 1 \\ 6 & 0 & 5 & 8 & 9 & 3 & 0 & 7 \end{bmatrix}$$

ii Decode the encoded credit card number received in this matrix:

$$\begin{bmatrix} 7 & 10 & 8 & 13 & 12 & 0 & 12 & 12 \\ 8 & 19 & 9 & 19 & 20 & 0 & 16 & 21 \end{bmatrix}$$

d Make up an encoded message to fit within a 2×8 matrix. Give a classmate the 2×2 encoding matrix, and see whether they can decode it.

e Work with a matrix with 3 rows, and use a 3×3 matrix to encode.

Financial mathematics investigation

4 One of the first big purchases you are likely to make is buying your first car. How much can you afford to pay for that car, and how will you go about financing the purchase?

In this investigation, you are going to use available resources to determine the best strategy. You will need to investigate each of the following questions.

a What can you afford?
Assuming that you will need to finance the car, what can you afford to repay each week or fortnight? You will need to consider your likely salary, as well as your other living costs. Some of the major banks will give advice regarding this amount and include 'affordability' calculators on their websites.

b How should you finance the car?
Compare some different forms of finance (such as variable interest personal loans, fixed interest personal loans, credit cards), some different financial institutions, as well as some of the financing options offered directly by the car dealerships to determine your best finance option.

c What car should you buy?
Cars often depreciate in value very quickly, especially if they are purchased new. In the worst-case scenario, you can end up owing more money on a car than its current market value! Compare the depreciation of two or three different brands of car when purchased new, and at various stages over the period of time for which you have decided to finance it. How much will the car be worth when it is finally paid for?

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5 Leonne is repaying $250 000 at $200 a month. She also decides to pay back $2500 at the end of the first year, $5000 at the end of the second, $7500 at the end of the third and so on.

a Use a spreadsheet to give a table of the amount owing each year until there is zero owing. The first three years and the spreadsheet rules are given here.

n	An
0	250000
1	245100
2	237700
3	227800

n	An
0	250000
1	=B2-2400-2500*A3
=A3+1	=B3-2400-2500*A4
=A4+1	=B4-2400-2500*A5

b Now consider what happens when Leonne is also paying 1.5% per annum interest on the amount owing each year. Use your spreadsheet to calculate the amount owing each year. How much longer does it take to repay the loan?

c Experiment to find the maximum interest which is payable, and for her to be able to pay the $250 000 back in less than 15 years if she uses the same repayment method.

Linear relations and modelling investigation

6 As part of its urban renewal strategy, Camtown Council makes $\frac{1}{4}$ hectare of land available for building middle-income homes. The project manager decides to build 10 houses on blocks of varying sizes.

There are five small blocks, three medium-sized blocks and two large blocks. The medium-sized blocks are 100 m^2 larger than the small blocks, and the large blocks are 200 m^2 larger than the medium-sized ones.

What are the sizes of the blocks?

Note: $1\text{hectare} = 10\,000\,\text{m}^2$

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Chapter 7

Investigating relationships between two numerical variables

Chapter questions

- What is bivariate data?
- What are response and explanatory variables?
- What is a scatterplot, how is it constructed and what does it tell us?
- What do we mean when we describe the association between two numerical variables in terms of direction, form and strength?
- What is Pearson's correlation coefficient, how is it calculated and what does it tell us?
- What is the difference between association and causation?
- How do we fit a line of good fit to a scatterplot by eye?
- How do we fit a line of good fit to a scatterplot using the least squares method?
- How do we interpret the intercept and slope of a line fitted to a scatterplot in the context of the data?
- How do we use a line fitted to a scatterplot to make predictions?
- What is the difference between interpolation and extrapolation?

In this chapter, we begin our study of **bivariate data**; data which is recorded on two variables from the same subject. Measuring the height and the weight of a particular person would be an example of bivariate data.

Bivariate data arises when we consider questions such as: Is the new treatment for a cold more effective than the old treatment? Do younger people spend more time using social media than older people?

Each of these questions is concerned with understanding the association between the two variables. To investigate such relationships will require us to develop some new statistical tools. In this chapter, we will only consider analysis of bivariate data where **both of the variables are classified as numerical**.

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7A Scatterplots

Learning intentions

- To be able to define **explanatory** and **response** variables.
- To be able to identify which of the two variables in the data may be the **explanatory variable** and which may be the **response variable**.
- To be able to construct a **scatterplot** by hand and by using a CAS calculator.

Response and explanatory variables

When we analyse **bivariate data**, we try to answer questions such as: 'Is there an association between these two variables?' More specifically, we want to answer the question: 'Does knowing the value of one of the variables tell us anything about the value of the other variable?'

For example, let us take as our two variables the *mark* a student obtained on a test and the amount of *time* they spent studying for that test. It seems reasonable that the more time one spends studying, the better mark you will achieve. That is, the amount of *time* spent studying may help to **explain** the *mark* obtained. For this reason we call *time* the **explanatory variable (EV)**. And, since the *mark* may go up or down in response to the amount of *time* spent studying, we call *mark* the **response variable (RV)**. In general, we anticipate that the value of the explanatory variable will have some effect on the value of the response variable.

Response and explanatory variables

When investigating associations (relationships) between two variables, the explanatory variable (EV) is the variable we expect to explain or predict the value of the response variable (RV).

Note: The explanatory variable is also sometimes called the independent variable (IV), and the response variable, the dependent variable (DV).

Identifying response and explanatory variables

It is important to be able to identify the explanatory and response variables before starting to explore the association between two numerical variables. Consider the following examples.

Example 1 **Identifying the response and explanatory variables**

We wish to investigate the question: 'Do older people sleep less?' The variables here are *age* and *time spent sleeping*. Which is the response variable (RV), and which is the explanatory variable (EV)?

Explanation	Solution
When looking to see if the length of time people spent sleeping is explained by their age, *age* is the EV and *time spent sleeping* is the RV.	EV: age RV: time spent sleeping

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Example 2 **Identifying the response and explanatory variables**

We wish to investigate the association between kilojoule consumption and weight loss. The variables in the investigation are *kilojoule consumption* and *weight loss.* Which is the response variable (RV), and which is the explanatory variable (EV)?

Explanation

Since we are looking to see if the weight loss can be explained by the amount people eat, *kilojoule consumption* is the EV and *weight loss* is the RV.

Solution

EV: kilojoule consumption
RV: weight loss

Example 3 **Identifying the response and explanatory variables**

Can we predict a person's height from their wrist circumference? The variables in this investigation are *height* and *wrist circumference*. Which is the response variable (RV), and which is the explanatory variable (EV)?

Explanation

Since we wish to predict height from wrist circumference, *wrist circumference* is the EV. *Height* is then the RV.

Solution

EV: wrist circumference
RV: height

It is important to note that, in Example 3, we could have asked the question the other way around, that is 'Can we predict a person's wrist circumference from their height?' In that case, *height* would be the EV and *wrist circumference* would be the RV. The way we ask our statistical question is an important factor when there is no obvious EV and RV.

Now try this 3 **Identifying response and explanatory variables (Example 3)**

A teacher is concerned that her students are tired in class. She suspects it is because they spend too much time on social media when they should be sleeping. The variables in the investigation are *amount of sleep* and *time on social media*. Which is the response variable (RV), and which is the explanatory variable (EV)?

Hint 1 Consider the way the teacher has posed the question. Which variable does the teacher think the students should reduce? That is the explanatory variable here.

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Constructing a scatterplot manually

The first step in investigating an association between two numerical variables is to construct a visual display of the data, which we call a **scatterplot**.

The scatterplot

- A scatterplot is a plot which enables us to display bivariate data when **both of the variables are numerical**.
- In a scatterplot, each point represents a single case.
- When constructing a scatterplot, it is conventional to use the **vertical** or **y-axis** for the response variable (RV) and the **horizontal** or **x-axis** for the explanatory variable (EV).

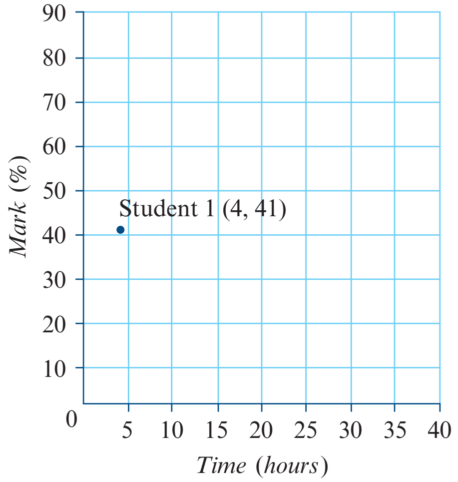
We will illustrate the process by constructing a scatterplot of the marks students obtained on an examination (the RV) and the times they spent studying for the examination (the EV).

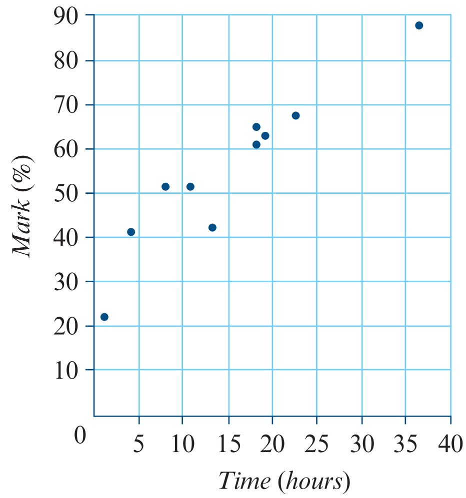
Student	1	2	3	4	5	6	7	8	9	10
Time (hours)	4	36	23	19	1	11	18	13	18	8
Mark (%)	41	87	67	62	23	52	61	43	65	52

In this scatterplot, each point will represent an individual student, and:

- The horizontal or x-coordinate of the point represents the time spent studying.
- The vertical or y-coordinate of the point represents the mark obtained.

The following scatterplots show how a scatterplot is constructed. The scatterplot on the left shows the point for Student 1, who studied 4 hours for the examination and obtained a mark of 41. The completed scatterplot on the right shows the data plotted for all students (one point for each student).





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Example 4 Understanding a scatterplot

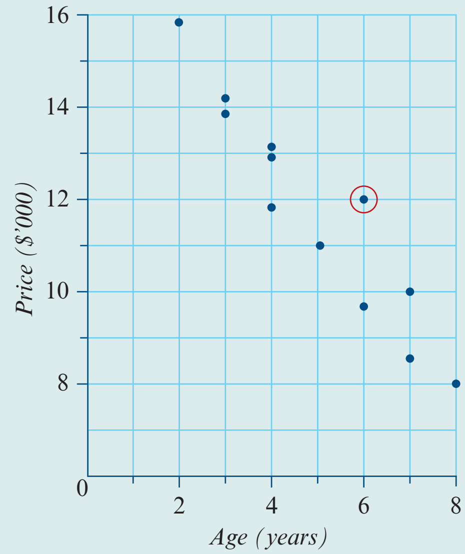
The scatterplot shown has been constructed from data collected to investigate the association between the *price* of a second-hand car and its *age*.

Use the scatterplot to answer the following questions.

1 Which is the explanatory variable and which is the response variable?

2 How many cars are in the data set?

3 How old is the car circled? What is its price?



Explanation

1 The EV will be on the horizontal axis and the RV on the vertical axis.

2 The number of cars is equal to the number of points on the scatterplot.

3 The x-coordinate of the point will be the car's age and the y-coordinate is its price.

Solution

EV: Age
RV: Price

12 Cars

The car is 6 years old, and its price is $12 000.

Now try this 4 Understanding a scatterplot (Example 4)

Construct a scatterplot of the maximum temperature on a summer day (the EV) and the number of bottles of water a student drank over the course of that day (the RV) over a 10-day period.

Day	1	2	3	4	5	6	7	8	9	10
Temperature	24	26	22	25	29	32	29	35	33	28
Number of bottles of water	2	3	2	3	4	5	4	6	6	4

Hint 1 Label the horizontal axis: *Temperature*. You can start the scale at 0, but it is acceptable to use a scale from 20 to 35 (for example), as long as it covers the values of the data.

Hint 2 Label the vertical axis: *Number of bottles of water*, with a scale from 0 to 6.

Hint 3 Note that the completed plot shows only 9 dots, as two points in the data set are identical.

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Using a CAS calculator to construct a scatterplot

While you need to understand the principles of constructing a scatterplot and maybe need to construct one by hand for a few points, in practice you will use a CAS calculator to complete this task.

How to construct a scatterplot using the TI-Nspire CAS

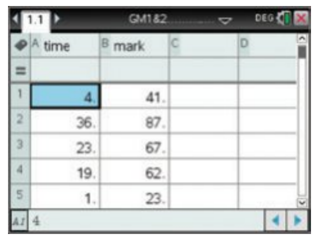
The data below shows the marks that 10 students obtained on an examination and the time they spent studying for the examination.

Time (hours)	4	36	23	19	1	11	18	13	18	8
Mark (%)	41	87	67	62	23	52	61	43	65	52

Use a calculator to construct a scatterplot. Use *time* as the explanatory variable.

Steps

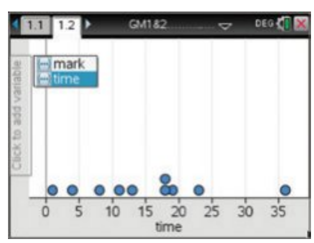
1 Start a new document (ctrl + N) and select **Add Lists & Spreadsheet**.
Enter the data into lists named *time* and *mark*.



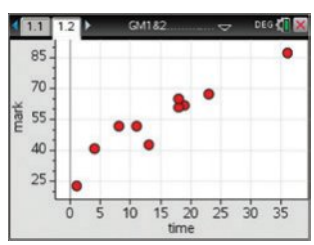
2 Statistical graphing is done through the **Data & Statistics** application.
Press ctrl + I and select **Add Data & Statistics**
(or press on and arrow to [icon] and press enter).
Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.

3 To construct a scatterplot.

a Press tab and select the variable *time* from the list. Press enter to paste the variable, *time*, to the x-axis.



b Press tab again and select the variable *mark* from the list. Press enter to paste the variable, *mark*, to the y-axis to generate the required scatterplot. The plot is automatically scaled.
Note: To change colour, move the cursor over the plot and press ctrl + menu**>Colour>Fill Colour.**



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How to construct a scatterplot using the ClassPad

The data below shows the marks that students obtained on an examination and the times they spent studying for the examination.

Time (hours)	4	36	23	19	1	11	18	13	18	8
Mark (%)	41	87	67	62	23	52	61	43	65	52

Use a calculator to construct a scatterplot. Use *time* as the explanatory variable.

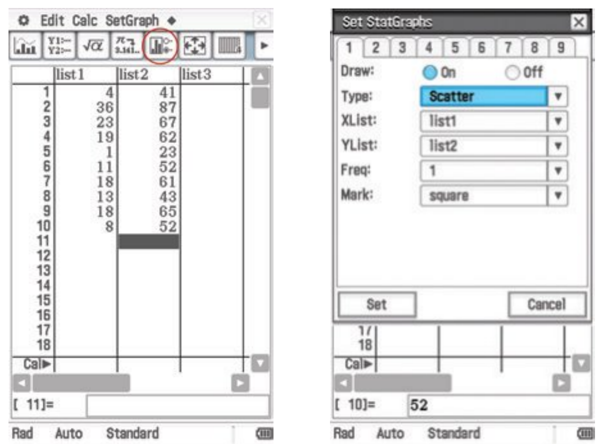
Steps

1 Open the **Statistics** application.

2 Enter the values into lists, with *time* in list1 and *mark* in list2.

3 Tap to open the **Set StatGraphs** dialog box.

4 Complete the dialog box as shown and tap SET.

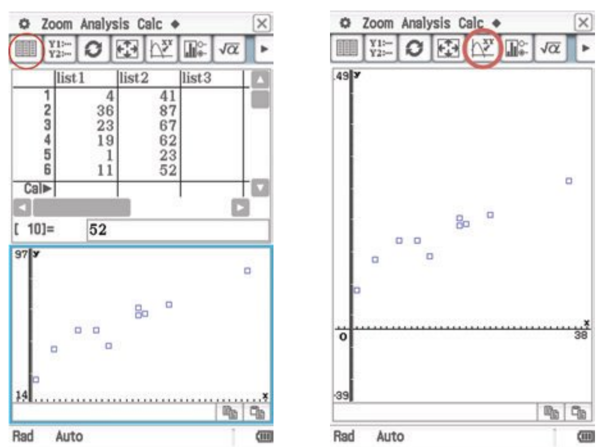


5 Tap to plot a scaled graph in the lower half of the screen.

6 Tap Resize to give a full-screen sized graph. Tap to return to a half-screen.

7 Tap to place a marker on the first data point: $(x_c = 4, y_c = 41)$.

8 Use the horizontal cursor arrow to move from point to point.



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Section Summary

- **Bivariate Data** arises when data on two different variables are collected for each individual or case.
- Usually, one of the two variables can be identified as the **explanatory** variable and the other as the **response** variable.
- The explanatory variable (EV) is the variable we expect to explain or predict the value of the response variable (RV).
- When both variables are numerical, bivariate data can be displayed in a **scatterplot**.
- When constructing a scatterplot, the EV is plotted on the horizontal axis, and the RV is plotted on the vertical axis.

Exercise 7A

Building understanding

Example 1–3

1 Identify the EV and the RV in each of the following situations.

a We wish to predict the *diameter* of a certain type of tree from its *age*.

b The association between *weight loss* and the number of *weeks* a person is on a diet is to be studied.

c Data is collected to investigate the association between *age* of a second-hand textbook and its *selling price*.

d The association between the number of *hours* a gas heating system is used and the *amount* of gas used is to be investigated.

e A study is to be made of the association between the number of *runs* a cricketer scores and the number of *balls bowled* to them.

2 For which of the following pairs of variables would it be appropriate to construct a scatterplot to investigate a possible association?

a Car *colour* (blue, green, black, . . .) and its *size* (small, medium, large)

b A food's *taste* (sweet, sour, bitter) and its *sugar content* (in grams)

c The *weight* (in kg) of 12 males and the *weight* (in kg) of 12 females

d The *time* people spend exercising each day, in minutes, and their *resting pulse rate* in beats per minute

e The *arm span* (in centimetres) and *gender* of a group of students

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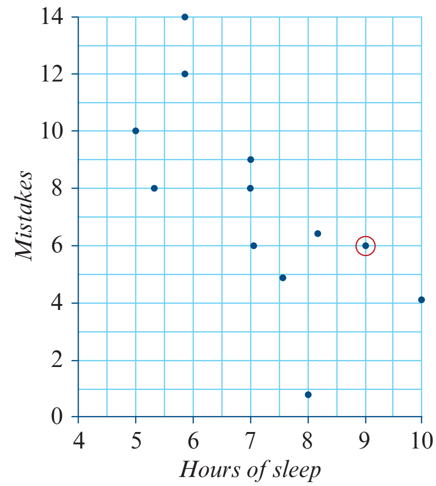
Example 4

3 This scatterplot has been constructed from data collected to investigate the association between the amount of sleep a person has the night before a test and the number of mistakes they make on the test. Use the scatterplot to answer the following questions.

a Which is the EV and which is the RV?

b How many people are in the data set?

c How much sleep did the individual circled have, and how many mistakes did they make?



Developing understanding

4 The table below shows the heights and weights of eight people.

Height (cm)	190	183	176	178	185	165	185	163
Weight (kg)	77	73	70	65	65	65	74	54

Use your calculator to construct a scatterplot with the variable *height* as the explanatory variable and the variable *weight* as the response variable.

5 The table below shows the ages of 11 couples when they got married.

Age of wife	26	29	27	21	23	31	27	20	22	17	22
Age of husband	29	43	33	22	27	36	26	25	26	21	24

Use your calculator to construct a scatterplot with the variable *wife* (age of wife) as the explanatory variable and the variable *husband* (age of husband) as the response variable.

6 The table below shows the number of seats and airspeeds (in km/h) of eight aircraft.

Airspeed	830	797	774	736	757	765	760	718
Seats	405	296	288	258	240	193	188	148

Use your calculator to construct a scatterplot with the variable *seats* as the explanatory variable and the variable *airspeed* as the response variable.

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7 The table below shows the response times of 10 patients (in minutes) given a pain relief drug and the drug dosages (in milligrams).

a Which variable is the explanatory variable?

b Use your calculator to construct an appropriate scatterplot.

Drug dosage	0.5	1.2	4.0	5.3	2.6	3.7	5.1	1.7	0.3	0.6
Response time	65	35	15	10	22	16	10	18	70	50

8 The table below shows the number of people in a cinema at 5-minute intervals after the advertisements started.

Number in cinema	87	102	118	123	135	137
Time	0	5	10	15	20	25

a Which is the explanatory variable?

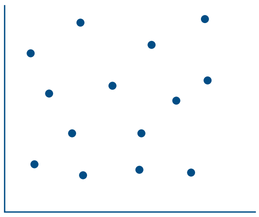
b Use your calculator to construct an appropriate scatterplot.

7B How to interpret a scatterplot

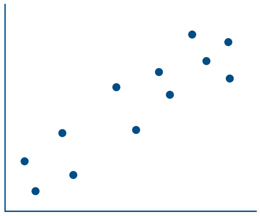
Learning intentions

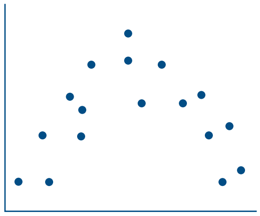
- To be able to use a scatterplot to identify an association between two variables.
- To be able to use the scatterplot to classify an association according to:
  - **Direction**, which may be **positive** or **negative**
  - **Form**, which may be **linear** or **non-linear**
  - **Strength**, which may be **weak**, **moderate** or **strong**.

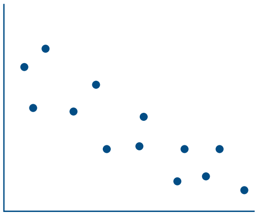
What features do we look for in a scatterplot to help us identify and describe any associations present? First, we look to see if there is a clear pattern in the scatterplot. In the scatterplot opposite, there is no clear pattern in the points. The points are randomly scattered across the plot, so we conclude that there is no association.



For the three examples below, there is a clear (but different) pattern in each set of points, so we conclude that there is an association in each case.







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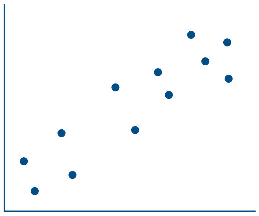
Having found a pattern, we need to be able to describe these associations clearly, as they are obviously quite different. The three features we look for in the pattern of points are **direction**, **form** and **strength.**

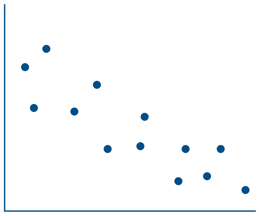
Direction of an association

We begin by looking at the overall pattern in the scatterplot.

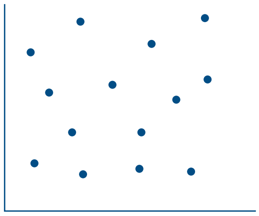
- If the points in the scatterplot trend upwards as we go from left to right, we say there is a **positive association** between the variables. That is, the values of the explanatory variable and the response variable tend to increase together.



- If the points in the scatterplot trend downwards as we go from left to right, we say there is a **negative association** between the variables. That is, as the values of the explanatory variable increase, the values of the response variable tend to decrease.



- If there is no pattern in the scatterplot, that is, the points just seem to randomly scatter across the plot, we say there is **no association** between the variables.



In general terms, we can classify the direction of an association as follows.

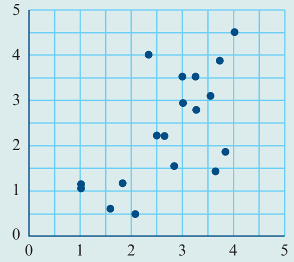
Direction of an association

- Two variables have a **positive association** when the value of the response variable tends to increase as the value of the explanatory variable increases.
- Two variables have a **negative association** when the value of the response variable tends to decrease as the value of the explanatory variable increases.
- Two variables have **no association** when there is no consistent change in the value of the response variable when the value of the explanatory variable increases.

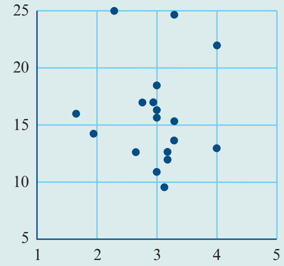
Example 5 Direction of an association

Classify each of the following scatterplots as exhibiting positive, negative or no association.

a



b



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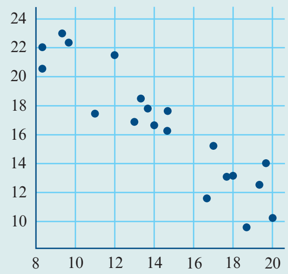
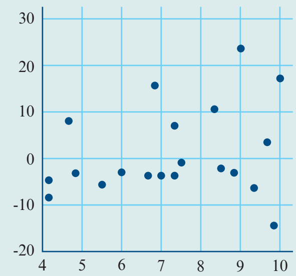
c



d

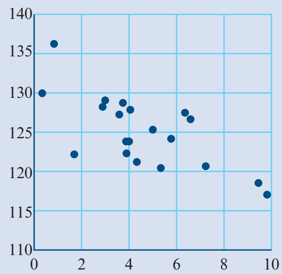


Explanation	Solution
a The points trend *upwards* from left to right.	The direction of the association is **positive**.
b There is *no clear pattern* in the scatterplot.	The scatterplot shows **no** association.
c The points trend *downwards* from left to right.	The direction of the association is **negative**.
d There is *no clear pattern* in the scatterplot.	The scatterplot shows **no** association.

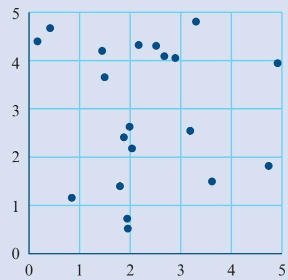
Now try this 5 Direction of an association (Example 5)

Describe the association in each of the following scatterplots.

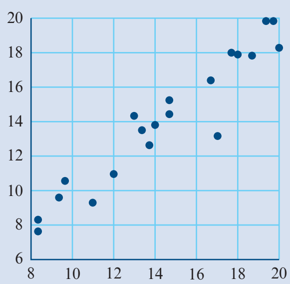
a



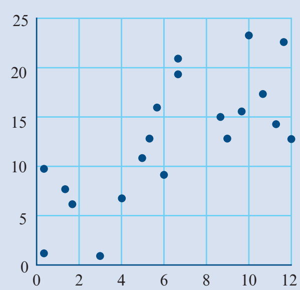
b



c



d



Hint 1 Use the scatterplots in Example 5 as a guide.

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Once we have identified the direction of an association, we can interpret this specifically in terms of the variables under investigation. So, for example:

- If there is a positive association between *height* and *weight*, then we can say that those individuals who are taller also tend to be heavier.
- If there is a negative association between *hours of sleep* and *reaction time*, then we can say that those individuals who have slept fewer hours tend to have slower reaction times.
- If there is no association between *height* and *reaction time*, then we are saying that the height of an individual does not seem to relate to their reaction time.

Example 6 **Interpreting the direction of an association**

Write a sentence interpreting each of the following associations:

a There is a positive association between *study time* and *score on the exam*.

b There is a negative association between *study time* and *time spent watching TV*.

Solution

a Those people who spend more time studying tend to score higher marks on the exam.

b Those people who spend more time studying tend to spend less time watching TV.

Now try this 6 **Interpreting the direction of an association (Example 6)**

Write a sentence interpreting each of the following associations:

a There is a negative association between *height above sea level* and *temperature*.

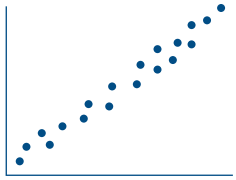
b There is a positive association between *number of years spent studying* and *salary*.

Hint 1 Use the wording of the answers in Example 6 as a model for your answers.

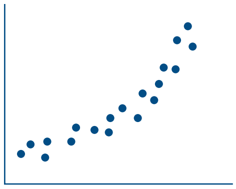
Form of an association

The next feature that interests us in an association is its general form. Do the points in a scatterplot tend to follow a linear pattern or a curved pattern?
For example:

- the association shown in the scatterplot opposite is **linear**. We can imagine the points in the scatterplot to be scattered around some **straight line**.



- the association shown in the scatterplot opposite is **non-linear**. We can imagine the points in the scatterplot to be scattered around a **curved line** rather than a straight line.



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In general terms, we can describe the **form of an association** as follows.

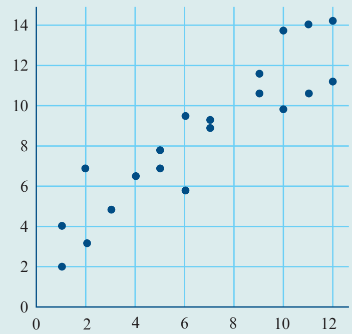
Form

A scatterplot is said to have a **linear form** when the points tend to follow a straight line.
A scatterplot is said to have a **non-linear form** when the points tend to follow a curved line.

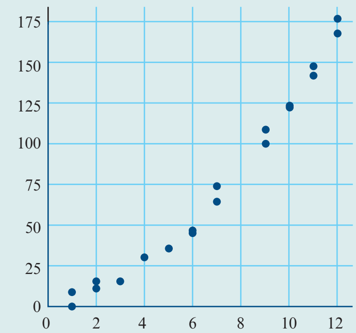
Example 7 Form of an association

Classify the **form** of the association in each of the following scatterplots as linear or non-linear.

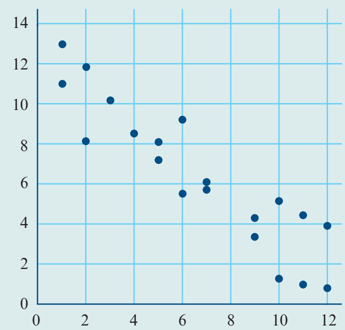
a



b



c



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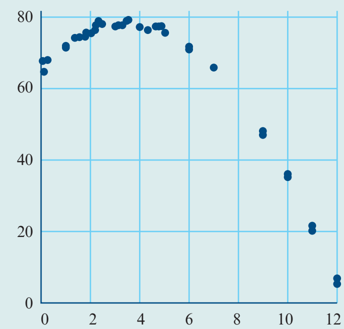

Explanation	Solution
a There is a clear straight-line pattern.	The association is linear.
b There is a clear curved pattern.	The association is non-linear.
c There is a clear straight-line pattern.	The association is linear.
d There is a clear curved pattern.	The association is non-linear.

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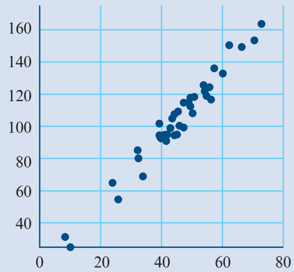
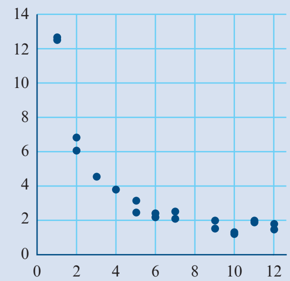
Now try this 7 Form of an association (Example 7)

Classify the *form* of the association in each of the following scatterplots.

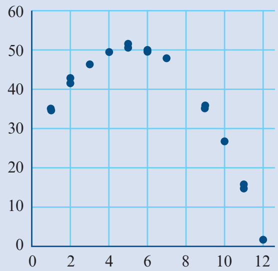
a



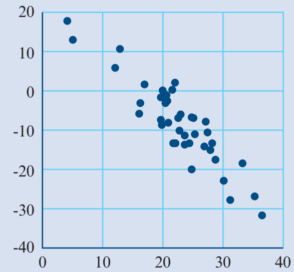
b



c



d

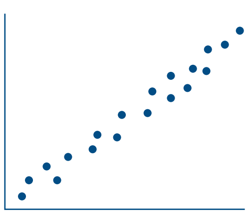


Hint 1 Use the scatterplots in Example 7 as a guide.

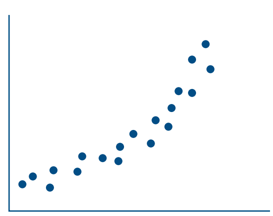
Strength of an association

The **strength of an association** is a measure of how much scatter there is in the scatterplot.

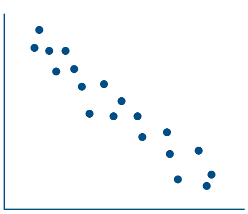
When there is a **strong association** between the variables, there is only a small amount of scatter in the plot, and a pattern is clearly seen.



Strong positive association



Strong positive association



Strong negative association

As the amount of scatter in the plot increases, the pattern becomes less clear. This indicates that the association is less strong. In the examples on the following page, we might say that there is a **moderate association** between the variables.

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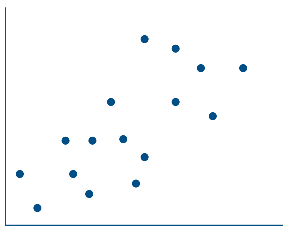
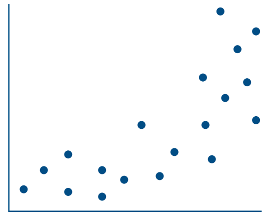
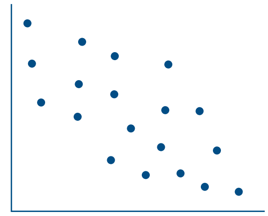


Moderate positive association

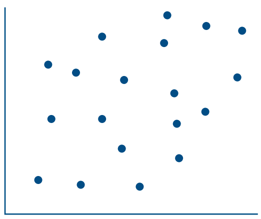


Moderate positive association

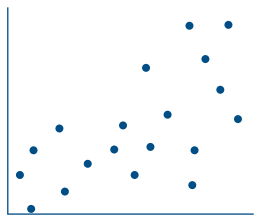


Moderate negative association

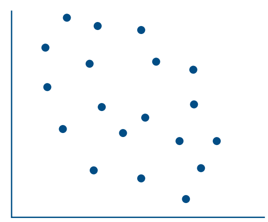
As the amount of scatter increases further, the pattern becomes even less clear. This indicates that any association between the variables is weak. The scatterplots below are examples of **weak associations** between the variables.



Weak positive association

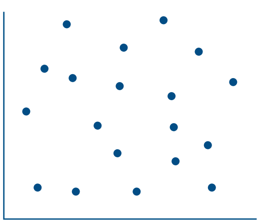


Weak positive association

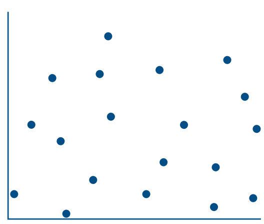


Weak negative association

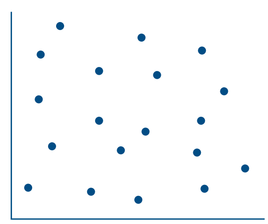
Finally, when all we have is scatter, as seen in the scatterplots below, no pattern can be seen. In this situation, we say that there is **no association** between the variables.



No association



No association



No association

In general terms, we can describe the **strength of an association** as follows.

Strength

An association is classified as:

- **Strong** if the points on the scatterplot tend to be tightly clustered about a trend line.
- **Moderate** if the points on the scatterplot tend to be moderately clustered about a trend line.
- **Weak** if the points on the scatterplot tend to be loosely clustered about a trend line.

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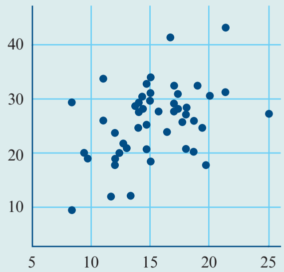
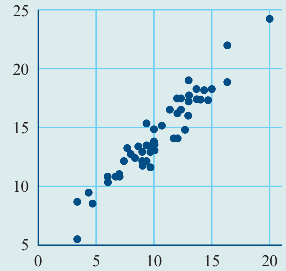
Example 8 Strength of an association

Classify the **strength** of the association in each of the following scatterplots as strong, moderate or weak.

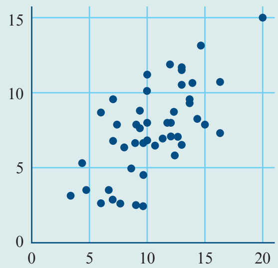
a



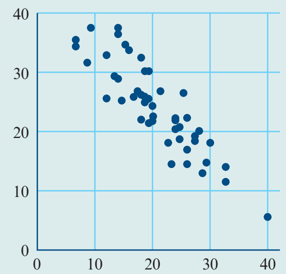
b



c



d



Explanation

a The points are loosely clustered.

b The points are tightly clustered.

c The points are moderately clustered.

d The points are tightly clustered.

Solution

The association is weak.

The association is strong.

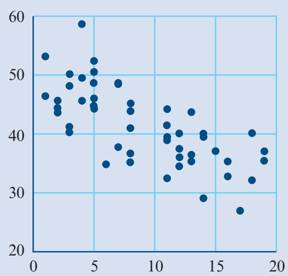
The association is moderate.

The association is strong.

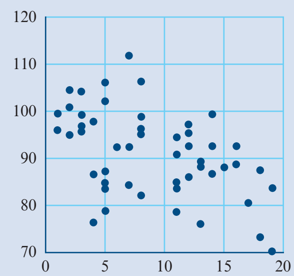
Now try this 8 Strength of an association (Example 8)

Classify the *strength* of the association in each of the following scatterplots as strong, moderate or weak.

a



b



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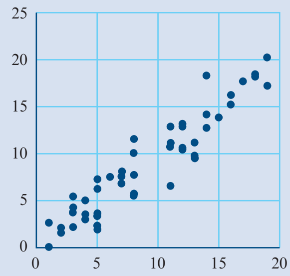
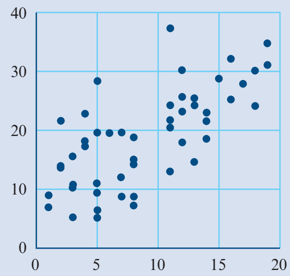
c



d



Hint 1 Use the scatterplots in Example 8 as a guide.

At the moment, you only need to be able to estimate the strength of an association, as strong, moderate, weak or none, by comparing it with the standard scatterplots given. However, the previous examples will have shown you that it is sometimes quite difficult to judge the difference between the strength of these associations by merely looking at the scatterplot. In the next section, you will learn about a statistic, the **correlation coefficient**, which can be used to give a value to the strength of linear association from the data values.

Section Summary

From a scatterplot, we can describe key features of a bivariate association.

- **Direction**. The two variables in the scatterplot have:
  - a positive association when the value of the response variable tends to increase as the value of the explanatory variable increases,
  - a negative association when the value of the response variable tends to decrease as the value of the explanatory variable increases,
  - no association when there is no consistent change in the value of the response variable when the values of the explanatory variable increase.
- **Form**. An association is classified as:
  - linear when the points tend to follow a straight line,
  - non-linear when the points tend to follow a curved line.
- **Strength**. An association is classified as:
  - Strong if the points on the scatterplot tend to be tightly clustered about a trend line,
  - Moderate if the points on the scatterplot tend to be moderately clustered about a trend line,
  - Weak if the points on the scatterplot tend to be loosely clustered about a trend line.

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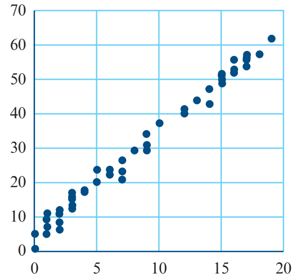
Exercise 7B

Building understanding

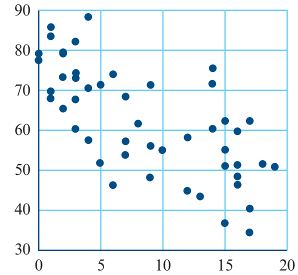
Example 5

1 Classify each of the following scatterplots according to **direction** (positive or negative).

a



b



Example 7

2 Classify each of the scatterplots in Question 1 according to **form** (linear or non-linear).

Example 8

3 Classify each of the scatterplots in Question 1 according to **strength** (weak, moderate or strong).

Developing understanding

Example 6

4 Write a sentence interpreting each of the following associations:

a There is a positive association between *fitness level* and *amount* of daily exercise.

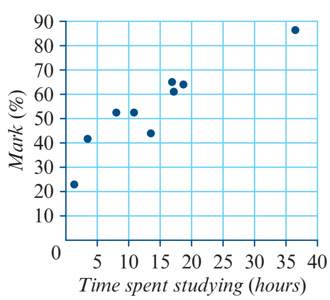
b There is a negative association between *time* taken to run a marathon and *speed* of the runner.

5 The variables in each of the following scatterplots are associated. In each case:

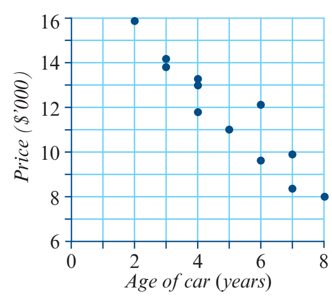
i Describe the association in terms of its direction (positive/negative), form (linear/non-linear) and strength (strong/moderate/weak).

ii Write a sentence describing the direction of the association in terms of the variables in the scatterplot.

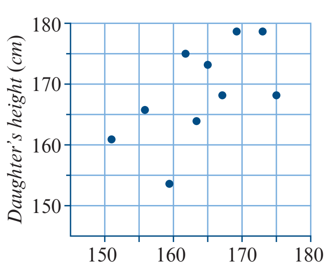
a



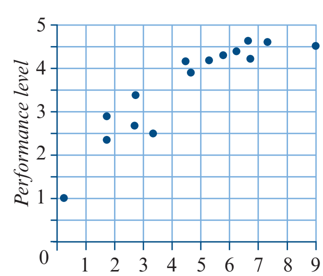
b



c



d



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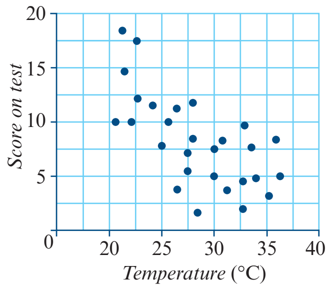
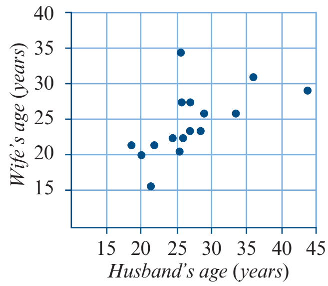
e



f



Testing understanding

6 In a mathematics class, a group of students were asked to draw circles on squared paper. They measured the diameter of the circles they had drawn, and then estimated the areas of the circles by counting the squares. Their results are given in the following table:

diameter (cm)	3.5	6.2	5.4	3.7	7.3	8.6	3.7	2.9	2.1	9.7	3.7
area (cm^2)	9.5	30.0	22.7	10.2	42.6	57.7	10.5	5.7	2.7	74.4	11.0

a Use your calculator to construct a scatterplot of the data, with the variable *diameter* as the explanatory variable and the variable *area* as the response variable.

b Describe the scatterplot in terms of direction, form and strength.

c Create a new column of data in your calculator by squaring values of the diameter (that is, diameter × diameter). Construct another scatterplot of the data, this time with the variable $diameter^2$ as the explanatory variable and *area* as the response variable.

d Describe the second scatterplot in terms of direction, form and strength.

e What has been the effect on the scatterplot of using $diameter^2$ rather than *diameter* as the explanatory variable?

7C Pearson's correlation coefficient (r)

Learning intentions

- ▶ To be able to understand Pearson's correlation coefficient, r, as a measure of the strength of a linear association between two variables.
- ▶ To be able to use technology to find the value of Pearson's correlation coefficient, r.
- ▶ To be able to classify the strength of a linear association as weak, moderate or strong, based on the value of Pearson's correlation coefficient, r.
- ▶ To be able to define and differentiate the concepts of association and causation.

When an association is linear, the most commonly used measure of strength of the association is Pearson's correlation coefficient, r. It gives a numerical measure of the degree to which the points in the scatterplot tend to cluster around a straight line.

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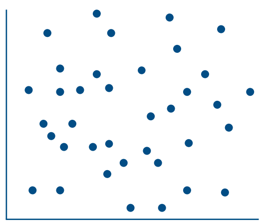
There are two key assumptions when using Pearson's correlation coefficient, r. These are:

- the data from both variables are numerical
- the association is linear.

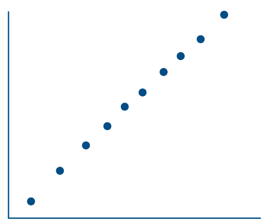
Properties of Pearson's correlation coefficient (r)

Pearson's correlation coefficient has the following properties:

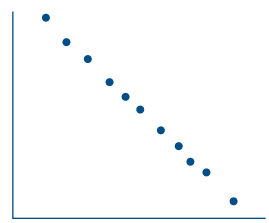
- no linear association, $r = 0$
- a perfect positive linear association, $r = +1$
- a perfect negative linear association, $r = -1$.



$r = 0$

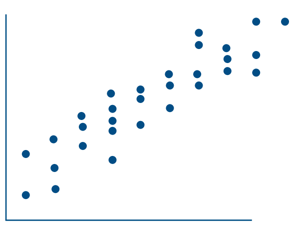


$r = +1$

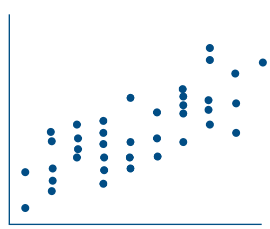


$r = -1$

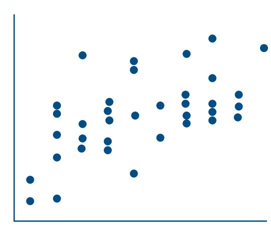
In practice, the value of r will be somewhere between $+1$ and -1 and rarely, exactly zero, as shown in the selection of scatterplots below.



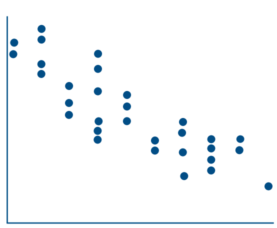
$r = 0.915$



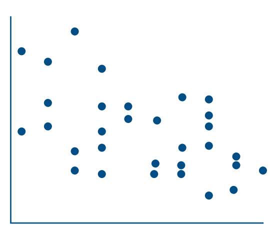
$r = 0.767$



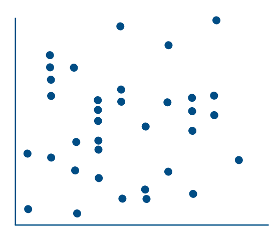
$r = 0.551$



$r = -0.874$



$r = -0.501$



$r = 0.150$

These scatterplots illustrate an important point – the stronger the association, the larger the magnitude of Pearson's correlation coefficient.

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Pearson's correlation coefficient

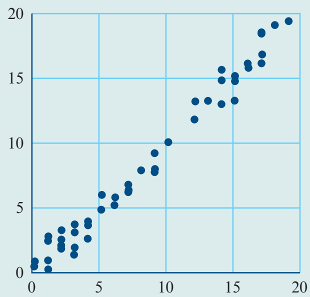
The Pearson's correlation coefficient, r:

- measures the **strength** of a **linear association**, with larger values indicating stronger relationships
- has a value between −1 and +1
- is positive if the direction of the linear association is positive
- is negative if the direction of the linear association is negative
- is close to zero if there is no association.

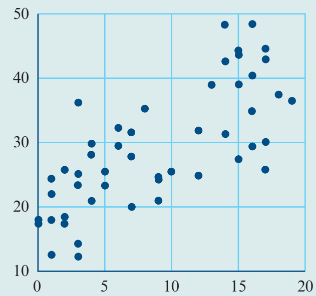
Example 9 Estimating the correlation coefficient from a scatterplot

Estimate the value of the correlation coefficient, r, in each of the following plots, using the plots on page 420 as a guide.

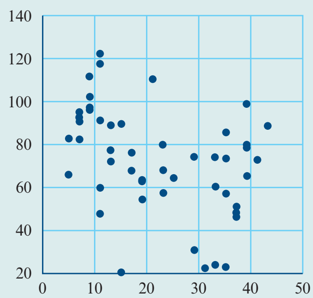
a



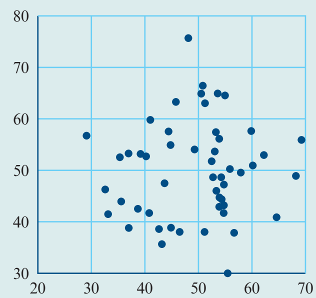
b



c



d



Explanation	Solution
a Points are tightly clustered, similar to Plot 1, and the direction is positive.	Estimate: $r \approx 0.9$
b Points look to be more loosely clustered than Plot 2 but not as loose as Plot 3, and the direction is positive.	Estimate: $r \approx 0.7$
c Points look to be slightly more loosely clustered than Plot 5, and the direction is negative.	Estimate: $r \approx -0.4$
d The points look random as in Plot 6.	Estimate: $r \approx 0$

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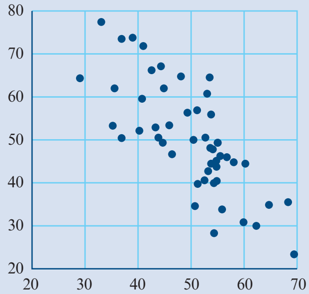
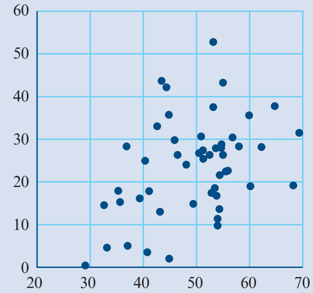
Now try this 9 **Estimating the correlation coefficient from a scatterplot (Example 9)**

Estimate the value of the correlation coefficient, r, in each of the following plots, using the plots on page 420 as a guide.

a



b



Hint 1 Firstly decide whether the value of r is positive or negative.

Hint 2 Then use the plots on page 420 to estimate the value of r.

Determining the value of Pearson's correlation coefficient, *r*

The formula for calculating r is:

$$r = \frac{1}{n-1}\sum\left(\frac{x-\bar{x}}{s_x}\right)\left(\frac{y-\bar{y}}{s_y}\right)$$

In this formula, $\bar{x}$ and s_x are the mean and standard deviation of the x scores, and $\bar{y}$ and s_y are the mean and standard deviation of the y scores.

After the mean and standard deviation, Pearson's correlation coefficient is one of the most frequently computed descriptive statistics. The presence of a linear association should always be confirmed with a scatterplot before Pearson's correlation coefficient is calculated. And, like the mean and the standard deviation, Pearson's correlation coefficient can be very sensitive to the presence of outliers, particularly for small data sets.

Pearson's correlation coefficient, r, is rather tedious to calculate by hand and is usually evaluated with the aid of technology.

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How to calculate Pearson's correlation coefficient, *r*, using the TI-Nspire CAS

The following data shows the per capita income (in $'000) and the carbon dioxide emissions (in tonnes) of 11 countries.
Determine the value of Pearson's correlation coefficient, r, for these data.

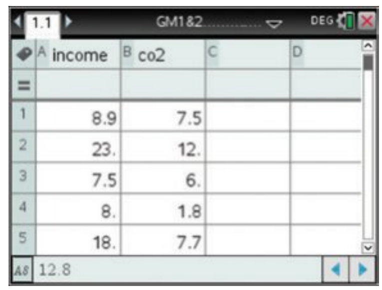
Income ($'000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 *(tonnes)*	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

Steps

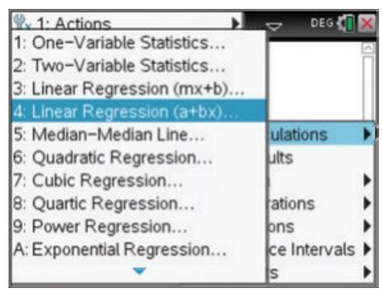
1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**.
Enter the data into lists named *income* and *co2*.

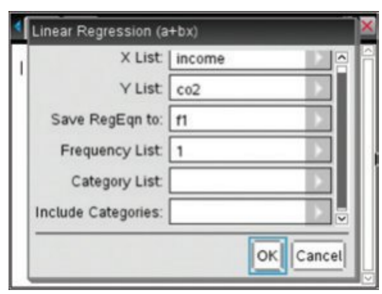
3 Statistical calculations can be done in the Calculator application. Press ctrl + I and select **Calculator**.



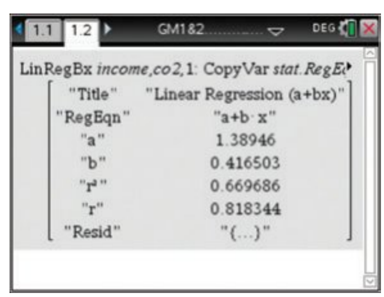
4 Press menu **>Statistics>Stat Calculations> Linear Regression (a + bx)** to generate the screen opposite.



5 Press menu to generate the pop-up screen as shown. To select the variable for the X List entry, use ▶ and enter to select and paste in the list name, *income*. Press tab to move to the Y List entry, use ▶ ▼ and enter to select and paste in the list name, *co2*.



6 Press enter to exit the pop-up screen, and generate the results shown in the screen opposite.



7 The value of the correlation coefficient is $r = 0.818344\ldots$ or 0.818, rounded to three decimal places.

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How to calculate Pearson's correlation coefficient, *r*, using the ClassPad

The following data shows the per capita income (in $'000) and the carbon dioxide emissions (in tonnes) of 11 countries.
Determine the value of Pearson's correlation coefficient, *r*, for the given data.

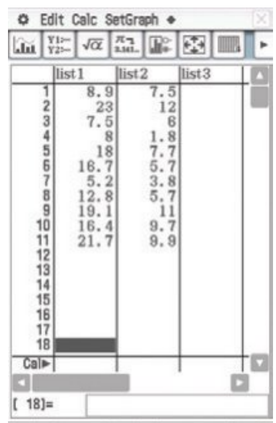
Income ($'000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 *(tonnes)*	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

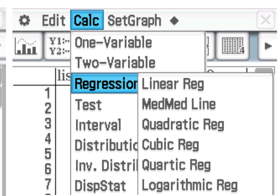
Steps

1 Open the **Statistics** application.

2 Enter the data into the columns.
- *Income* in List1
- CO_2 in List2

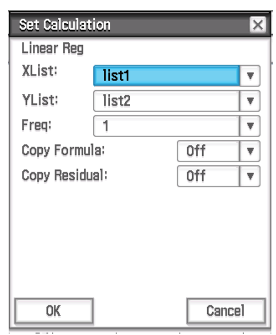
3 Select **Calc>Regression>Linear Reg** from the menu bar.

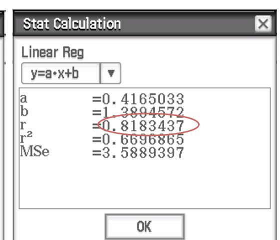




4 Press EXE.
This opens the **Set Calculation** dialog box, as shown to the right.

5 Tap **OK** to confirm your selections.





6 The value of the correlation coefficient is $r = 0.818344\ldots$ or 0.818, rounded to three decimal places.

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Example 10 Calculating the correlation coefficient using a calculator

Scores in two tests for a group of ten students are given in the following table. Determine the value of the correlation coefficient, r, for these data, rounded to four decimal places.

Score test 1 (30)	14	17	26	17	15	13	29	25	17	30
Score test 2 (20)	9	11	15	13	10	9	16	14	12	19

Explanation

1 Enter the data into lists named *test1* and *test2*.

2 Determine the value of r following the instructions for your calculator.

Solution

$r = 0.9499$

Now try this 10 Calculating the correlation coefficient using a calculator (Example 10)

The hours spent studying for each of two tests by a group of students are given in the following table. Determine the value of the correlation coefficient, r, for these data, rounded to four decimal places.

Hours studying for test 1	9	13	7	2	8	7	6	3	10	6
Hours studying for test 2	7	12	6	2	8	7	5	6	11	8

Hint 1 Always begin by entering the data into named lists.

Hint 2 Carefully follow the instructions for your calculator.

Guidelines for classifying the strength of a linear association

Pearson's correlation coefficient, r, can be used to classify the strength of a linear association as follows:

$0.75 \le r \le 1$	**strong** positive association
$0.5 \le r < 0.75$	**moderate** positive association
$0.25 \le r < 0.5$	**weak** positive association
$-0.25 < r < 0.25$	**no** association
$-0.5 < r \le -0.25$	**weak** negative association
$-0.75 < r \le -0.5$	**moderate** negative association
$-1 \le r \le -0.75$	**strong** negative association

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Example 11 Classifying the strength of a linear association

Classify the strength of each of the following linear associations using the previous table.

a $r = 0.35$ **b** $r = -0.507$

c $r = 0.992$ **d** $r = -0.159$

Explanation	Solution
a The value 0.35 is more than 0.25 and less than 0.5. That is, $0.25 \le r < 0.5$.	weak, positive
b The value −0.507 is more than −0.75 and less than −0.5. That is, $-0.75 < r \le -0.5$.	moderate, negative
c The value 0.992 is more than 0.75 and less than 1. That is, $0.75 \le r \le 1$.	strong, positive
d The value −0.159 is more than −0.25 and less than 0.25. That is, $-0.25 < r < 0.25$.	no association

Now try this 11 Classifying the strength of a linear association (Example 11)

Classify the strength of each of the following linear associations.

a $r = 0.807$ **b** $r = -0.818$

c $r = 0.224$ **d** $r = -0.667$

Hint 1 In each part, compare the value of r to the interval given in the table on the previous page.

Hint 2 Remember: the sign of the correlation coefficient tells you the direction of the association, and the value tells you the strength.

Correlation and causation

A strong correlation between two variables means that they vary together, both increasing together if the correlation is positive, or one decreasing as the other increases if the correlation is negative. The existence of even a strong correlation between two variables is not, in itself, sufficient to imply that altering one variable **causes a change** in the other. It only implies that this **may** be the explanation.

Suppose, for example, we were to find a high correlation between the smoking rate and the incidence of heart disease across a group of countries. We cannot conclude from this correlation coefficient alone that smoking **causes** heart disease. Another possible explanation is that people who smoke neglect lifestyle factors such as exercise and diet. It could well be that people who smoke also tend not to exercise regularly, and it is the lack of exercise which causes heart disease.

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- A **correct** interpretation of a high correlation between smoking rate and heart disease across a group of countries would be: "Those countries which have higher rates of smoking also tend to have higher incidence of heart disease".
- An **incorrect** interpretation of a high correlation between smoking rate and heart disease across a group of countries would be: "As the smoking rate increases then the incidence of heart disease will also increase". Also **incorrect** would be to state: "Reducing the smoking rate would also reduce the incidence of heart disease".

It is for this reason that we need to be very careful when interpreting the correlation coefficient.

Example 12 Correlation and causation

The correlation coefficient, r, between the per capita income (in \$'000) and the carbon dioxide emissions (in tonnes) of 11 countries is equal to 0.818. Does this mean that reducing the per capita income would result in decreased carbon dioxide emissions?

Explanation

We cannot **infer** (which means deduce or conclude) causation, even when there is a strong correlation.

Solution

No, we can only conclude that those countries with higher per capita income also tend to have higher carbon dioxide emissions.

Now try this 12 Correlation and causation (Example 12)

If the heights and the scores obtained on a test of mathematical ability by a group of primary school students in Year Prep to Year 6 were recorded, a strong correlation would be found. Can it be inferred from this that taller people are better at mathematics? Give a possible non-causal explanation.

Section Summary

- **Pearson's correlation coefficient**, r, is a measure of the strength of a linear association.
- Assumptions when using Pearson's correlation coefficient, r, are:
  - the data from both variables are **numerical**
  - the association is **linear**.
- Pearson's correlation coefficient, r:
  - has a value between –1 and +1, with larger values indicating stronger associations
  - is close to zero if there is **no** association
  - is **positive** if the direction of the linear association is positive
  - is **negative** if the direction of the linear association is negative.

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Section Summary

- The strength of the correlation coefficient is classified according to the following table:

$0.75 \leq r \leq 1$	**strong** positive association
$0.5 \leq r < 0.75$	**moderate** positive association
$0.25 \leq r < 0.5$	**weak** positive association
$-0.25 < r < 0.25$	**no** association
$-0.5 < r \leq -0.25$	**weak** negative association
$-0.75 < r \leq -0.5$	**moderate** negative association
$-1 \leq r \leq -0.75$	**strong** negative association

- The existence of even a strong correlation between two variables is not sufficient to conclude that there is a causal association between them.

Skill-sheet

Exercise 7C

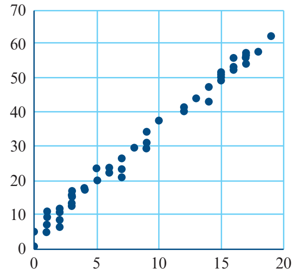
Building understanding

1 What are the two key assumptions justifying the use of Pearson's correlation coefficient to quantify the strength of the association between two variables?

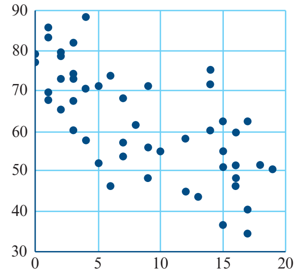
Example 9

2 Estimate the value of the correlation coefficient, r, in each of the following plots, using the plots on page 420 as a guide.

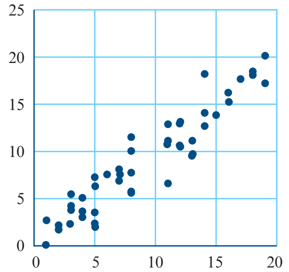
a



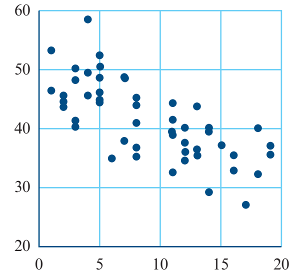
b



c



d



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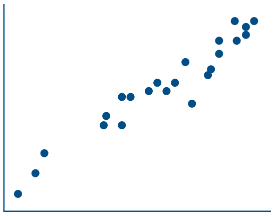
3 Estimate the value of the correlation coefficient, r, in each of the following plots, using the plots on page 420 as a guide.

a



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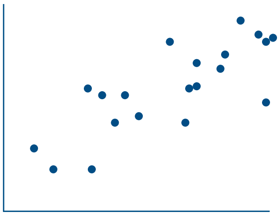
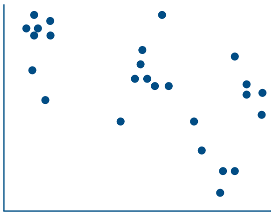
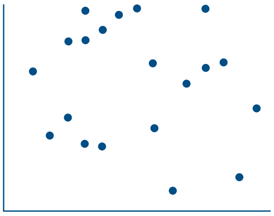


c



d



Developing understanding

Example 10

4 Determine the value of Pearson's correlation coefficient, r, for the data in the following table. Give your answer rounded to three decimal places.

X	7	3	7	12	8	17	6	3	10	6
Y	10	2	6	12	8	7	8	6	11	8

5 The table below shows the *weight* (in kg) and blood *glucose* level (in mg/100 mL) of eight adults.

Weight	82.1	70.1	76.6	82.1	83.9	73.2	66.0	77.5
Glucose	101	89	98	100	108	104	94	89

Use your calculator to determine the value of Pearson's correlation coefficient for this data set. Give your answer rounded to three decimal places.

6 The table below shows the scores which a group of nine students obtained on two class tests, *Test 1* and *Test 2*, as part of their school-based assessment.

Test 1	33	45	27	42	50	38	17	35	29
Test 2	43	46	36	34	48	34	29	41	28

Use your calculator to determine the value of Pearson's correlation coefficient for this data set. Write your answer rounded to three decimal places.

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7 The table below shows the carbohydrate content (*carbs*) and the fat content (*fat*) in 100 g of nine breakfast cereals.

Carbs (g)	88.7	67.0	77.5	61.7	86.8	32.4	72.4	77.1	86.5
Fat (g)	0.3	1.3	2.8	7.6	1.2	5.7	9.4	10.0	0.7

Use your calculator to determine the value of Pearson's correlation coefficient for this data set. Give your answer rounded to three decimal places.

Example 11

8 Use the guidelines on page 428 to classify the strength of a linear association for which Pearson's correlation coefficient is calculated to be:

a $r = 0.205$ **b** $r = -0.303$ **c** $r = -0.851$ **d** $r = 0.333$
e $r = 0.952$ **f** $r = -0.740$ **g** $r = 0.659$ **h** $r = -0.240$
i $r = -0.484$ **j** $r = 0.292$ **k** $r = 1$ **l** $r = -1$

Example 12

9 There is a strong positive correlation between the number of bars and the number of school teachers in cities around the world. Can we conclude from this that school teachers spend a lot of time in bars? Give a possible non-causal explanation.

10 There is a strong negative correlation between birth rate and life expectancy in a country. Can we conclude that decreasing the birth rate in a country will help increase the life expectancy of its citizens? Give a possible non-causal explanation.

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11 In a survey of nine problem gamblers, the respondents were asked the *amount* (in dollars) they had spent on gambling and the *number of hours* they had spent gambling in the past week. The data collected is recorded in the table below.

Hours	10	11	12	15	20	21	25	35	40
Amount	500	530	300	750	1000	1200	2000	2300	5000

a The aim is to predict the amount of money spent on gambling from the time spent gambling. Which is the explanatory variable and which is the response variable?

b Construct a scatterplot of these data.

c Determine the value of the correlation coefficient, r, to three decimal places.

d Describe the association between the variables *amount* and *hours* in terms of strength, direction and form.

12 The following data was recorded through the National Health Survey:

Region	*Percentage with eye disease*	
	Male (%)	Female (%)
Australia	40.7	49.1
Other Oceania countries	46.1	66.2
United Kingdom	74.5	75.0
North-West Europe	71.2	71.5
Southern & Eastern Europe	71.6	74.6
North Africa & the Middle East	52.2	57.5
South-East Asia	47.7	54.8
All other countries	56.0	62.0

a Which is the explanatory variable and which is the response variable?

b Construct a scatterplot of these data, with *percentage of males* on the horizontal axis and *percentage of females* on the vertical axis.

c Determine the value of the correlation coefficient, r, to three decimal places.

d Describe the association between the male and female eye disease percentages for these countries in terms of strength, direction and form.

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Testing understanding

13 The following table gives the educational level (*education*), the number of years the person has worked for the company (*years*) and their current salary to the nearest thousand dollars (*salary*) for a group of current employees of a particular company.

education	*years*	*salary*	*education*	*years*	*salary*
Secondary	2	52	Tertiary	2	62
Secondary	3	64	Tertiary	3	69
Secondary	2	56	Tertiary	4	75
Secondary	4	63	Tertiary	5	76
Secondary	7	65	Tertiary	4	72
Secondary	6	64	Tertiary	4	68
Secondary	7	52	Tertiary	1	63
Secondary	10	65	Tertiary	8	85
Secondary	5	59	Tertiary	3	67
Secondary	5	62	Tertiary	6	77

a **i** Use your calculator to construct a scatterplot of the data for all 20 employees, with the variable *salary* as the response variable and the variable *years* as the explanatory variable.

ii Determine the value of the correlation coefficient, r, to three decimal places.

b **i** Use your calculator to construct a scatterplot of *salary* against *years* for those employees who have Secondary educational level.

ii Determine the value of the correlation coefficient, r, for these employees to three decimal places.

c **i** Use your calculator to construct a scatterplot of *salary* against *years* for those employees who have Tertiary educational level.

ii Determine the value of the correlation coefficient, r, for these employees to three decimal places.

d Based on the values of the three correlation coefficients determined in parts **a, b** and **c**, how would you describe the association between *salary* and *years* for the employees of this company?

7D Fitting a linear model to the data

Learning intentions

- To be able to fit a linear model to the data **by eye**.
- To be able to determine the equation of the line fitted by eye from the graph.
- To be able to use the method of **least squares** for fitting a linear model to the data.

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- To be able to calculate the intercept and slope of the least squares line from the correlation coefficient and summary statistics.
- To be able to use a CAS calculator to determine the intercept and slope of the least squares line from the bivariate data.

Once we identify a linear association between two numerical variables, we can go one step further by fitting a linear model to the data and finding its equation. This model gives us a better understanding of the association between the two variables and allows us to make predictions. The process of modelling an association with a straight line is known as **linear regression**, and the resulting line is often called the **regression line** or line of good fit.

As discussed in Chapter 5, the equation of the regression line is given by the rule:

$$y = a + bx$$

where a is the y-intercept and b is the slope.

Fitting a line 'by eye'

The simplest method of fitting a line is to use a ruler to draw a line on the scatterplot that seems to balance out the points around the line. This is called **fitting a line 'by eye'**. Once the line has been drawn on the scatterplot, then its equation can be determined from the plot, using the skills that you developed in Chapter 5F, as shown in the following examples.

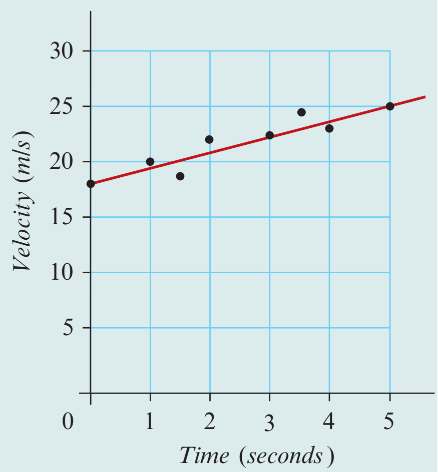
Example 13 Fitting a line by eye using the intercept and slope

A straight line has been fitted by eye to a set of data that records the velocity, v, (in m/s) of an accelerating car at time, t, seconds.

a What is the car's velocity when *time* = 0?

b What is the slope of the line?

c Write down the equation of the line in terms of *velocity* and *time*.



Explanation

a Where *time* = 0, the y-intercept can be read from the graph.

Solution

y-intercept $a \approx 18\ m/s$

b Calculate the slope by using two points on the graph, say (0, 18) and (5, 25). Any two points can be used.

$$\text{slope } b = \frac{\text{rise}}{\text{run}} = \frac{25 - 18}{5 - 0} = 1.4$$

c The general equation of the line is of the form $y = a + bx$.

$$velocity = 18 + 1.4 \times time$$

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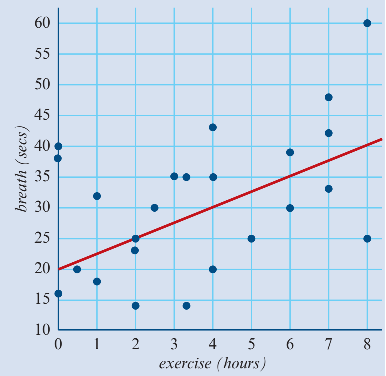
Now try this 13 Fitting a line by eye using the intercept and slope (Example 13)

A straight line has been fitted by eye to a scatterplot showing how long a group of students could hold their breath for, in seconds, (*breath*) and the number of hours they spend each week in exercise (*exercise*).



a What is the value of *breath* when *exercise* = 0?

b What is the slope of the line?

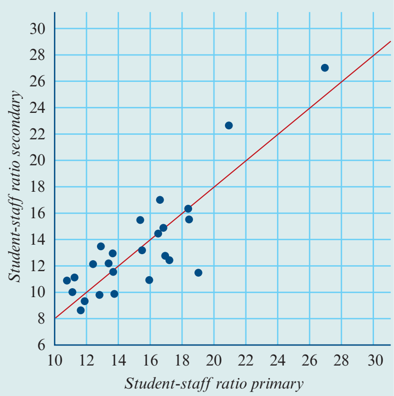
c Write down the equation of the line in terms of *exercise* and *breath*.

Hint 1 Remember to label the intercept as a and the slope as b.

Example 14 Fitting a line by eye using two points on the graph

A straight line has been fitted by eye to a scatterplot of the *student-staff ratio for secondary* against the *student-staff ratio for primary* for a group of countries.

Determine the equation of this line in terms of the variables in the question.

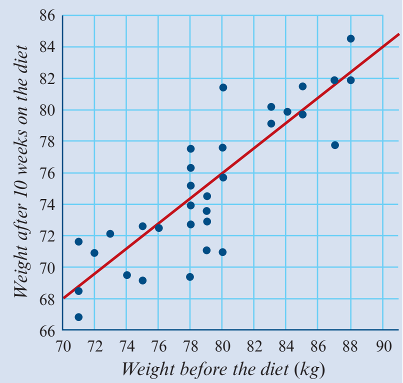


Explanation	Solution
1 Find two points on the line where the coordinates of the points can be read easily from the graph. There may be a few, any two points will do.	Suitable points are (14,12) and (30, 28).
2 Calculate the slope (b) by using the coordinates of the two points.	$b = \frac{\text{rise}}{\text{run}} = \frac{28 - 12}{30 - 14} = 1.0$
3 To find the value of a, substitute the coordinates of one of the points on the line (either will do) into the general rule $y = a + bx$ and solve for a.	Using the point (14,12) $12 = a + 1 \times 14$ $\therefore a = 12 - 14 = -2$
4 Write down the equation of the line in terms of the variables in the question.	*student-staff ratio secondary* = $-2 + 1\times$ *student-staff ratio primary*

Now try this 14 Fitting a line by eye using two points on the graph (Example 14)

A straight line has been fitted by eye to a scatterplot showing the weights for a group of males before and after they spent 10 weeks on a weight reduction diet.

Determine the equation of this line in terms of the variables in the question.



Hint 1 Find two points on the line where the coordinates are clear.

Hint 2 Find the slope, b, first.

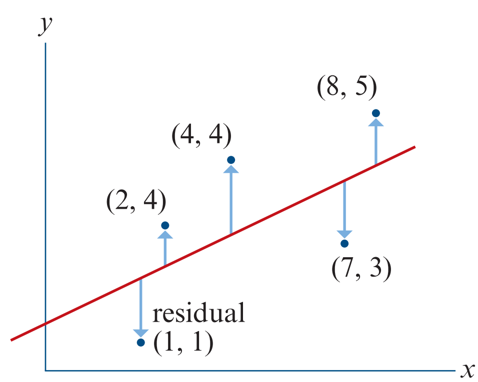
Hint 3 Use either point to substitute into the general rule to find the value of a.

Using the least squares line to model a linear association

A better approach to fitting a line is to use a mathematical strategy known as the **least squares method**.

It is very unlikely that all of the points in the scatterplot will lie exactly on a straight line, so our goal is to find a line that best fits the data in some way. To do this, we start by considering the vertical distances between the line and the actual data points. These are called **residuals** and are shown by the blue arrows in the scatterplot below.

One way of fitting the line is to find the line that has the smallest value for the sum of the residuals. However, because some residuals will be positive (because the point is *above* the line) and some will be negative (because the point is *below* the line) then these residuals will tend to cancel out.



We have met this problem before when calculating standard deviation, and we solve it the same way, by squaring the residuals to make them all positive, and then they can be added together.

The least squares line is the equation of the line that minimises the sum of these squared residuals (hence the name of the method: least squares). The exact solution for these values can be found mathematically, using the techniques of calculus; however, this is beyond the mathematics required for General Mathematics.

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The equation of the least squares regression line

The equation of the least squares regression line is given by, $y = a + bx$, where:

the *slope* (b) is given by: $b = \dfrac{rs_y}{s_x}$

and

the *intercept* (a) is then given by : $a = \bar{y} - b\bar{x}$

- r is the correlation coefficient
- s_x and s_y are the standard deviations of x and y
- $\bar{x}$ and $\bar{y}$ are the mean values of x and y.

The assumptions made in using the least squares method to model a linear association are the same as those for Pearson's correlation coefficient. These are:

- the variables are numerical
- the association is linear.

Example 15 Using the formula to find the intercept and slope of the least squares line

Find the equation of the least squares regression line, $y = a + bx$, when:

$r = 0.600$ $s_x = 9.80$ $s_y = 5.40$ $\bar{x} = 165.0$ $\bar{y} = 62.3$

Give the values of the intercept and slope, rounded to three **significant figures**.

Note: To revise significant figures and how to calculate them, see Chapter 10, section 10A.

Explanation	Solution
1 First, substitute in the formula for b to find the slope.	$b = \dfrac{rs_y}{s_x} = \dfrac{0.600 \times 5.40}{9.80} = 0.3306$
2 Next, substitute in the formula for a to find the intercept.	$a = \bar{y} - b\bar{x} = 62.3 - 0.3306 \times 165.0$ $= 7.751$
3 Substitute a and b in the formula for a straight line, giving values rounded to three significant figures.	$y = 7.75 + 0.331x$

Now try this 15 Using the formula to find the intercept and slope of the least squares line (Example 15)

Find the equation of the least squares regression line, $y = a + bx$, when:

$r = 0.700$ $s_x = 2.30$ $s_y = 3.40$ $\bar{x} = 15.2$ $\bar{y} = 24.5$

Give the values of the intercept and slope, rounded to three significant figures.

Hint 1 You must always start by finding the value of b (since this is required in the formula for a).

Hint 2 Work with more significant figures than required, and only round to the number asked for in the answer.

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Your CAS calculator can also be used to find the equation for the least squares regression line from the data, as in the following examples.

How to determine and graph the least squares regression line using the TI-Nspire CAS

The following data shows the per capita income (*income*) and the carbon dioxide emissions (CO_2) of 11 countries.

Income ($'000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 (tonnes)	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

a Construct a scatterplot to display these data, with *income* as the explanatory variable (EV).

b Fit a least squares regression line to the scatterplot and determine its equation.

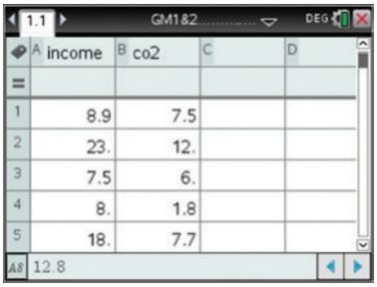
c Write the equation of the regression line in terms of the variables *income* and CO_2, with the coefficients given, rounded to three significant figures.

d Determine and write down the value of the correlation coefficient, r, rounded to three significant figures.

Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**. Enter the data into lists named *income* and co_2.



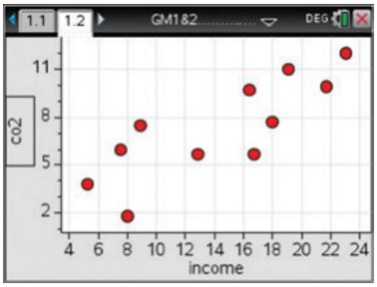
3 Identify the explanatory variable (EV) and the response variable (RV).

EV: *income*

RV: co_2

Note: In saying that we want to predict **co_2** from **income**, we are implying that income is the EV.

4 Press ctrl + I, select **Data & Statistics** and construct a scatterplot with the *income* (EV) on the horizontal (or x-) axis and co_2 (RV) on the vertical (or y-) axis.



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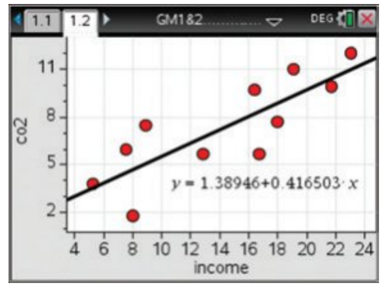
5 Press [menu] **>Analyze>Regression>Show Linear (a+bx)** to display the least squares regression line on the scatterplot. Simultaneously, the equation of the regression line is shown, written using the variables y and x:

$y = 1.389\ldots + 0.416\ldots$ or

$y = 1.39 + 0.417x$ to 3 significant figures

Note: The calculator assumes that the variable on the x-axis is the EV.



6 Write down the equation of the least squares regression line in terms of the variables *income* and co_2. Write the coefficients, rounded to three significant figures.

$CO_2 = 1.39 + 0.417 \times income$

7 **a** Press [ctrl] + [I] and select Calculator to open the Calculator application.

b Now press [var], locate then select stat.r and press [enter] to display the value of r.

8 Write down the value of the correlation coefficient, rounded to three significant figures.

$r = 0.818$

How to determine and graph the least squares regression line using the ClassPad

The following data shows the per capita income (in \$'000) and the carbon dioxide emissions (in tonnes) of 11 countries.

Income (\$'000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 (tonnes)	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

a Determine and graph the equation of the least squares regression line that will enable CO_2 emissions to be predicted from income.

b Write the equation in terms of the variables *income* and co_2, with the coefficients given, rounded to three significant figures.

c Determine and write down the value of the correlation coefficient, r, to three significant figures.

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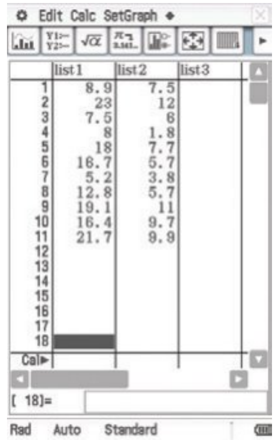
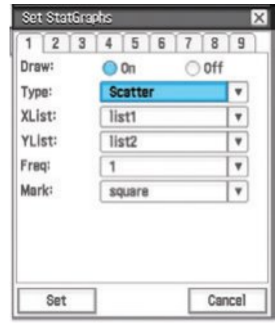
Steps

1 Open the **Statistics** application.

2 Enter the data into columns:

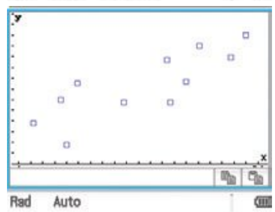
- Income in List1
- CO_2 in List2

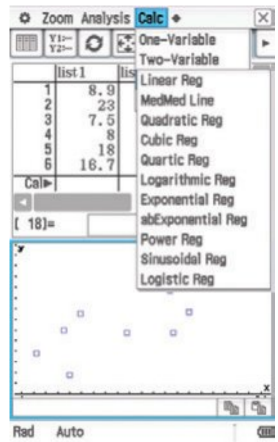




3 Tap to open the **Set StatGraphs** dialog box and complete as shown.
Tap Set to confirm your selections.

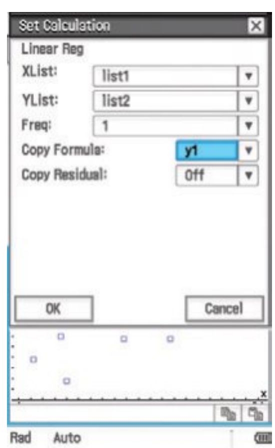
4 Tap in the toolbar at the top of the screen to plot the scatterplot in the bottom half of the screen.

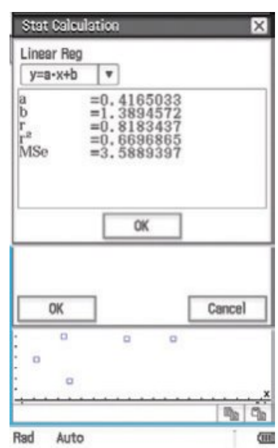




5 To calculate the equation of the least squares regression line:

- Tap **Calc>Regression>Linreg** from the menu bar.
- Complete the **Set Calculations** dialog box as shown.
- Tap **OK** to confirm your selections in the **Set Calculations** dialog box.
- This generates the regression results in **Stat Calculation**, shown opposite.





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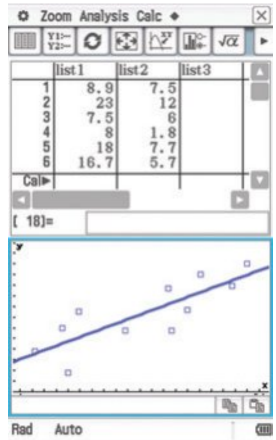
6 Tapping **OK** a second time automatically plots and displays the regression line on the plot.

7 Write down the equation of the least squares line in terms of the variables *income* and CO_2 and the value of the correlation coefficient, to three decimal places.



$CO_2 = 1.39 + 0.417 \times income$

$r = 0.818$

Section Summary

- The simplest method of finding the equation of a linear model is to draw a line **by eye** on the scatterplot.
- The vertical distance between a point on a bivariate plot and a line fitted to the data is called a **residual**.
- The method of **least squares** is a method for fitting a straight line to a scatterplot, based on minimising the sum of the squared residuals.
- The equation of the least squares line is given by $y = a + bx$, where:

  the slope (b) is given by: $b = \dfrac{rs_y}{s_x}$

  and

  the intercept (a) is then given by : $a = \bar{y} - b\bar{x}$

  Here:

  - r is the correlation coefficient
  - s_x and s_y are the standard deviations of x and y
  - $\bar{x}$ and $\bar{y}$ are the mean values of x and y.

  A CAS calculator can be used to determine the intercept and slope of the least squares line from the bivariate data.

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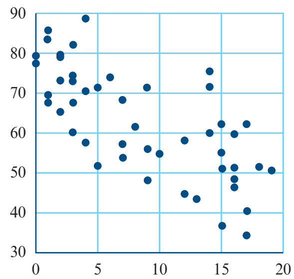
Skill-sheet

Exercise 7D

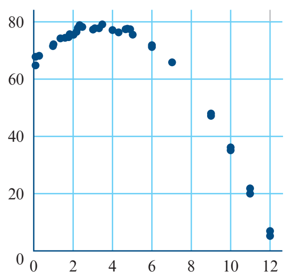
Building understanding

1 For each of the following plots, indicate whether it would be appropriate or not to fit a least squares regression line to the data.

a

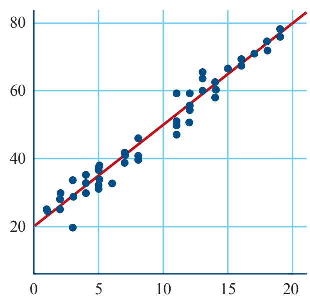


b

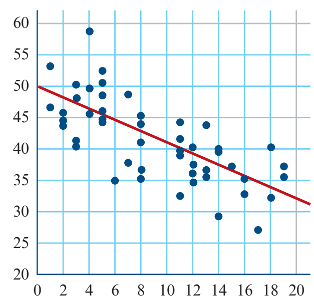


2 For each of the following scatterplots, determine the y-intercept (a), the slope (b) and hence the equation of the line $y = a + bx$. Round your answers to 3 significant figures.

a

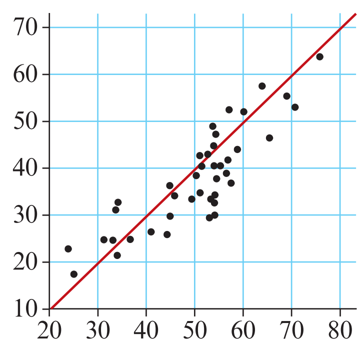


b



Example 13

3 For the following scatterplot, determine the equation of the line shown on the scatterplot. Round your answer to two significant figures.



4 Use your calculator to find the equation of the least squares regression line, $y = a + bx$, which fits this data. Give your answers to three significant figures.

x	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
y	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

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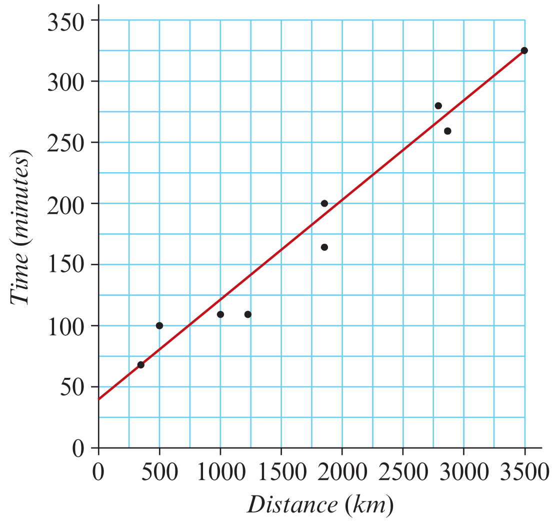
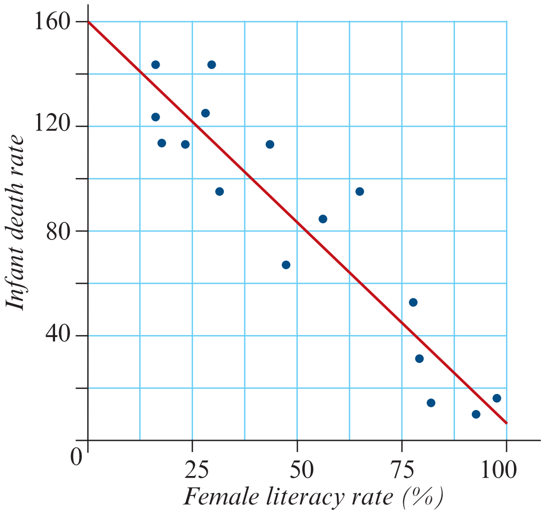
Developing understanding

Example 14

5 A straight line has been fitted by eye to a plot of travelling time, in minutes, (*Time*) against distance travelled, in km, (*Distance*) for nine plane trips between nine different cities. Determine the equation of the line in terms of *Time* and *Distance*.



6 A straight line has been fitted by eye to a plot of *Infant death rate* (per 100 000 people) against *Female literacy rate (%)* for a number of countries. Determine the equation of the line in terms of these variables. Give answers correct to one decimal place.



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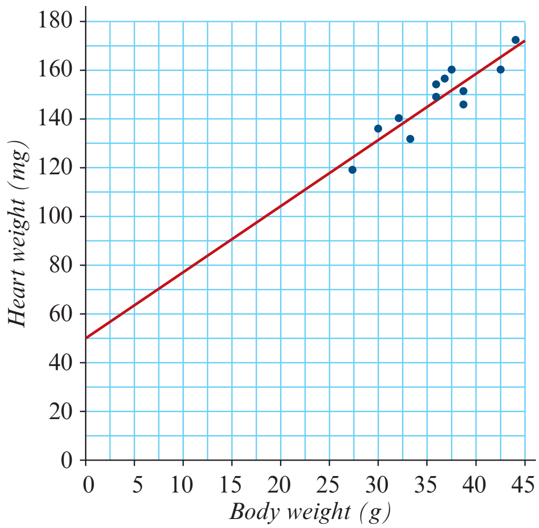
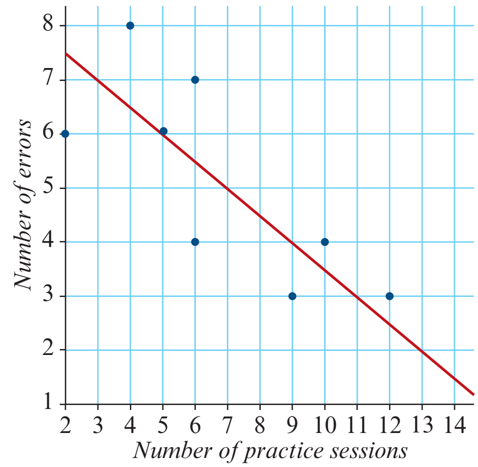
7 A straight line has been fitted by eye to a plot of *heart weight (mg)* against *body weight (g)* for twelve laboratory mice.
Determine the equation of the line in terms of these variables. Give answers correct to one decimal place.

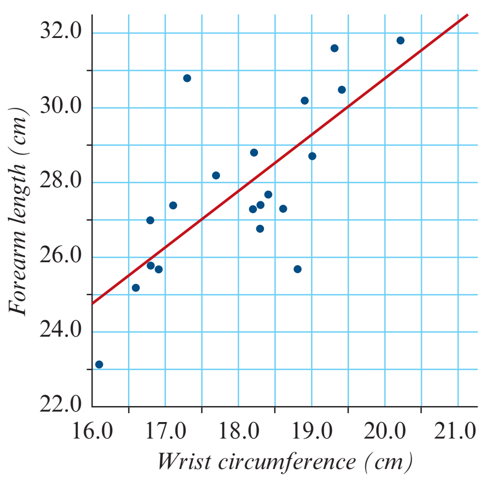


8 A straight line has been fitted by eye to a plot of *number of errors* made on a final assessment against *number of practice sessions* attended before the assessment for a group of university students.

Determine the equation of the line shown on the scatterplot in terms of these variables.



9 The graph opposite shows the wrist circumference and forearm length, both in centimetres, of 20 people. A least squares line has been fitted to the scatterplot, with wrist circumference as the explanatory variable. Use the scatterplot to find the equation of the line in terms of the variables in the question.



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10 Find the equation of the least squares regression line, $y = a + bx$, when:

$r = 0.8 \quad s_x = 6.8 \quad s_y = 20.5 \quad \bar{x} = 115.0 \quad \bar{y} = 123.5$

Give the values of the intercept and slope, rounded to two decimal places.

11 Find the equation of the least squares regression line, $y = a + bx$, when:

$r = -0.600 \quad s_x = 2.50 \quad s_y = 1.70 \quad \bar{x} = 18.0 \quad \bar{y} = 15.7$

Give the values of the intercept and slope, rounded to three significant figures.

Example 15

12 The table below shows the weight (in kg) and blood glucose level (in mg/100 mL) of eight adults.

Weight (kg)	82.1	70.1	76.6	82.1	83.9	73.2	66.0	77.5
Glucose (mg/100 mL)	101	89	98	100	108	104	94	89

a Construct a scatterplot to display the data, with *weight* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *glucose* and *weight* with the coefficients given, rounded to three significant figures.

d Determine the correlation coefficient to three significant figures.

13 The table below shows the scores which a group of nine students obtained on two class tests, Test 1 and Test 2, as part of their school-based assessment.

Test 1	33	45	27	42	50	38	17	35	29
Test 2	43	46	36	34	48	34	29	41	28

a Construct a scatterplot to display these data, with *Test 1* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *Test 2* and *Test 1* with the coefficients given, rounded to three significant figures.

d Determine the correlation coefficient to three significant figures.

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14 The table below shows the carbohydrate content, in grams, (*carbs*) and the fat content, in grams, (*fat*) in 100 grams of nine breakfast cereals.

Carbs	88.7	67.0	77.5	61.7	86.8	32.4	72.4	77.1	86.5
Fat	0.3	1.3	2.8	7.6	1.2	5.7	9.4	10.0	0.7

a Construct a scatterplot to display these data, with *carbs* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *fat* and *carbs* with the coefficients given, rounded to three significant figures.

d Determine the correlation coefficient to three significant figures.

15 The table below shows the *age* and *height* of six young children.

Age (months)	36	40	44	52	56	60
Height (cm)	84	87	90	92	94	96

a Construct a scatterplot to display these data, with *age* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *height* and *age* with the coefficients given, to three significant figures.

d Determine the correlation coefficient to three significant figures.

16 The following table gives the *height* and *arm span*, both in centimetres, for a group of eight people.

Height (cm)	162	170	164	153	171	166	170	163
Arm span (cm)	163	168	165	154	165	164	170	165

a Construct a scatterplot to display these data, with *height* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *arm span* and *height* with the coefficients given, to three significant figures.

d Determine the correlation coefficient to three significant figures.

Testing understanding

17 The statistical analysis of a set of bivariate data involving variables x and y resulted in the information displayed in the table below.

Mean	$\bar{x} = 8.97$	$\bar{y} = 4.42$
Standard deviation	$s_x = 4.29$	$s_y = 1.69$
Equation of the least squares line	$y = 1.62 + 0.312x$	

Use this information to determine the value of the correlation coefficient, r.

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18 In a mathematics class, a group of students were asked to draw circles on squared paper. They measured the diameters of the circles they had drawn and then estimated the areas of the circles by counting the squares. Their results are given in the following table. Their teacher suggested they investigate the association between the area of each circle and the square of its diameter.

diameter (cm)	3.5	6.2	5.4	3.7	7.3	8.6	3.7	2.9	2.1	9.7	3.7
area (cm^2)	9.5	30.0	22.7	10.2	42.6	57.7	10.5	5.7	2.7	74.4	11.0

a Construct a scatterplot of the data, with the variable $diameter^2$ as the explanatory variable and the variable *area* as the response variable.

b Use your calculator to find the intercept and slope of the least squares line for this scatterplot, rounded to three significant figures.

c Complete this equation: $area = \square + \square \times diameter^2$

d Compare this equation with what you know to be the exact association between the diameter and area of a circle.

7E Interpreting and predicting from a linear model

Learning intentions

- ▶ To be able to interpret the slope and intercept of a linear model in the context of the data.
- ▶ To be able to use the equation of a linear model to predict the value of the response variable, based on the value of the explanatory variable.
- ▶ To be able to understand the difference between **interpolation** and **extrapolation** when making predictions.

Interpreting the slope and intercept of a model in the context of the data

Whatever method is used to determine the equation of a straight line, the intercept and slope of the line can be interpreted in the context of the data in the question. This interpretation gives us further insights into the association between the variables.

Interpreting the slope and intercept of a linear model

For the regression line $y = a + bx$:

- the slope (b) tells us on average the change in the response variable (y) for each one-unit increase or decrease in the explanatory variable (x)
- the intercept (a) tells us on average the value of the response variable (y) when the explanatory variable (x) equals 0.

Note: The interpretation of the y-intercept in a data context may not be sensible when $x = 0$ is not within the range of observed x values.

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Example 16 Interpreting the slope and intercept of a linear model

A regression line is used to model the association between the *time*, in hours, a group of students spent studying for an examination and their *mark (%)*. The equation of the regression line is:

$mark = 30.8 + 1.62 \times time$

a **i** Write down the value of the intercept.

ii Interpret the intercept in the context of these variables.

b **i** Write down the value of the slope.

ii Interpret the slope in the context of these variables.

Explanation	Solution
a **i** In a linear equation of the form $y = a + bx$; the intercept is a.	intercept = 30.8
ii The intercept ($a = 30.8$) gives the average *mark* (y) when the study *time* (x) equals 0.	On average, students who spend no time studying for the examination will obtain a mark of 30.8.
b **i** In a linear equation of the form $y = a + bx$; the slope is b.	slope = 1.62
ii The slope ($b = 1.62$) gives the average change in the *mark* (y) associated with a one-unit increase in the variable *time* (x).	On average, students' marks increase by 1.62 for each extra hour of study.

Now try this 16 Interpreting the slope and intercept of a linear model (Example 16)

A regression line is used to model the association between the time, in hours, students spend doing household chores *(chores)* and the hours they spend in part-time work *(work)*, which ranged from 0 to 8 hours for this group of students. The equation of the regression line is:

$chores = 8.0 - 0.30 \times work$

a **i** Write down the value of the intercept.

ii Interpret the intercept in the context of these variables.

b **i** Write down the value of the slope.

ii Interpret the slope in the context of these variables.

Hint 1 Use the wording in Example 16 as a model for your answers.

Hint 2 The sign of the slope tells you if the association between the variables is positive or negative. This is an important consideration when interpreting the slope.

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Using the model to make predictions: interpolation and extrapolation

The aim of linear regression is to model the association between two numerical variables by using the equation of a straight line. This equation can then be used to make predictions.

The data below shows the times that 10 students spent studying for an exam and the marks they subsequently obtained.

Time (*hours*)	4	36	23	19	1	11	18	13	18	8
Mark (%)	41	87	67	62	23	52	61	43	65	52

If we fitted a linear model to this data using the least squares method, we would have an equation close to:

$$mark = 30.8 + 1.62 \times time$$

Using this equation and rounding off to the nearest whole number, we would predict that a student who spent:

- 0 hours studying would obtain a mark of 31% (mark = 30.8 + 1.62 × 0 = 31%)
- 8 hours studying would obtain a mark of 44% (mark = 30.8 + 1.62 × 8 = 44%)
- 12 hours studying would obtain a mark of 50% (mark = 30.8 + 1.62 × 12 = 50%)
- 30 hours studying would obtain a mark of 79% (mark = 30.8 + 1.62 × 30 = 79%)
- 80 hours studying would obtain a mark of 160% (mark = 30.8 + 1.62 × 80 = 160%)

This last result, 160%, points to one of the limitations of substituting values into a regression equation without thinking carefully. Using this regression equation, we predict that a student who studies for 80 hours will obtain a mark of more than 100%, which is impossible. Something is wrong!

The problem is that we are using the regression equation to make predictions well outside the range of values used to calculate this equation. The maximum time any student spent studying for this exam was 36 hours; yet, we are using the equation we calculated to try to predict the exam mark for someone who studies for 80 hours. Without knowing that the model works equally well for someone who spends 80 hours studying, which we don't, we are venturing into unknown territory and can have little faith in our predictions.

As a general rule, a regression equation only applies to the range of values of the explanatory variables used to determine the equation. Thus, we are reasonably safe using the line to make predictions that lie roughly within this data range, say from 1 to 36 hours. The process of making a prediction within the range of values of the explanatory variable used to derive the regression equation is called **interpolation**, and we can have some faith in these predictions.

However, we must be extremely careful about how much faith we put into predictions made outside the range of values of the explanatory variable. Making predictions outside the data range is called **extrapolation**.

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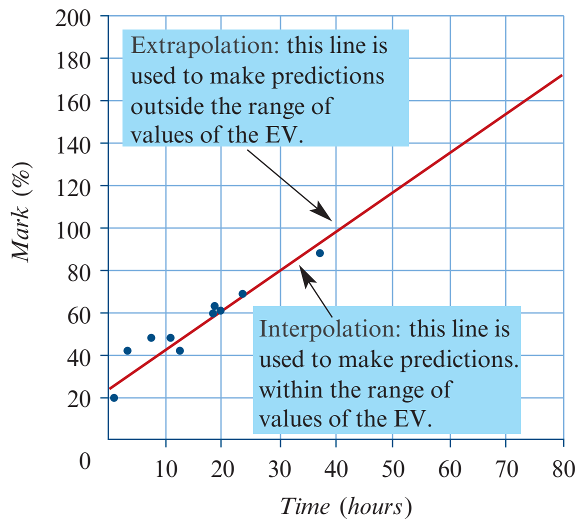
Interpolation and extrapolation

Predicting within the range of values of the explanatory variable is called **interpolation**. Interpolation is generally considered to give a **reliable** prediction.

Predicting outside the range of values of the explanatory variable is called **extrapolation**. Extrapolation is generally considered to give an **unreliable** prediction.

For example, if we use the regression line to predict the examination mark for 30 hours of studying time, we would be interpolating. However, if we use the regression line to predict the examination mark for 50 hours of studying time, we would be extrapolating. Extrapolation is a less reliable process than interpolation because we are going beyond the range of values of the explanatory variable, and it is quite possible that the association may no longer be linear.



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Example 17 Using the linear model to make predictions

The equation relating the weights, in kg, and heights, in cm, of a group of students whose heights ranged from 163 cm to 190 cm, is:

$weight = -40 + 0.60 \times height$

Use this equation to predict the weight of students with the following heights.
Are you interpolating or extrapolating?

a 170 cm **b** 65 cm

Explanation	Solution
a Substitute 170 into the equation and evaluate.	The weight of a person of height 170 cm is predicted to be: $weight = -40 + 0.60 \times 170 = 62$ kg Interpolating: predicting within the range of values of the EV.
b Substitute 65 into the equation and evaluate.	The weight of a person of height 65 cm is predicted to be: $weight = -40 + 0.60 \times 65 = -1.0$ kg which is not possible. Extrapolating: we are predicting well outside the range of values of the EV.

Now try this 17 Using the linear model to make predictions (Example 17)

A regression line is used to model the association between the time, in hours, which students spend doing household chores *(chores)* and the hours they spend in part-time work *(work)*, which ranged from 0 to 8 hours for this group of students.
The equation of the regression line is:

$chores = 8.00 - 0.30 \times work$

Use this equation to predict the hours spent doing chores by a student who works the following hours per week in their part-time job. Are you interpolating or extrapolating?

a 2 hours **b** 10 hours

Hint 1 Substitute the value for *work* in the equation given.

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Section Summary

- For the regression line, $y = a + bx$:
  - on average, the response variable (y) changes by b units for each one-unit increase in the explanatory variable (x).
  - on average, the intercept (a) predicts the value of the response variable (y) when the explanatory variable (x) equals 0.
- The linear model, $y = a + bx$, can be used to predict the value of the response variable (y) for a particular value of the explanatory variable (x).
- Predicting within the range of the values of the explanatory variable used to fit the linear model is called **interpolation**; predicting outside the range of values of the explanatory variable used to fit the linear model is called **extrapolation**.
- Interpolation is generally considered to give a **reliable** prediction.
- Extrapolation is generally considered to give an **unreliable** prediction.

Skill-sheet

Exercise 7E

Building understanding

1 The following linear model which allows *height*, in centimetres, to be predicted from *age*, in months, was determined from data collected from a group of children aged from 12 to 36 months.

$$height = 69 + 0.50 \times age$$

a Which is the EV and which is the RV?

b Write down the values of the intercept and slope.

c Complete the following sentences:

i The slope tells us, on average, that *height* increases by ☐ cm for each ☐ month increase in age.

ii The intercept tell us, that, on average, students aged ☐ months will be ☐ cm tall.

2 Complete the following sentences.
Using a linear model to make a prediction:

a within the range of data is called ________.

b outside the range of data is called ________.

3 The linear model relating students' marks (%) on an oral French test (*mark*) to the number of hours they spent practising speaking French in the week before the test (*practice*) has the equation:

$$mark = 48.1 + 2.20 \times practice$$

The linear model predicts that:

a a student who practised for 5 hours will score 48.1 + 2.20 × ____ = ____ per cent.

b a student who practised for 8 hours will score 48.1 + 2.20 × ____ = ____ per cent.

Developing understanding

Example 16

4 The equation, *price* = 37 650 – 4200 × *age*, can be used to predict the *price* of a used car (in dollars) from its *age* (in years).

a **i** For this regression equation, write down the value of the intercept.

ii Interpret the intercept in the context of the variables in the equation.

b **i** For this regression equation, write down the value of the slope.

ii Interpret the slope in the context of the variables in the equation.

5 The following regression equation can be used to predict the flavour rating of yoghurt from its percentage fat content, (*calories*).

$$flavour\ rating = 40 + 2.0 \times calories$$

a **i** For this regression equation, write down the value of the intercept.

ii Interpret the intercept in the context of the variables in the equation.

b **i** For this regression equation, write down the value of the slope.

ii Interpret the slope in the context of the variables in the equation.

Example 17

6 For children between the ages of 36 and 60 months, the equation relating their *height* (in cm) to their *age* (in months) is:

$$height = 72 + 0.40 \times age$$

Use this equation to predict the height (to the nearest cm) of a child with the following age. Are you interpolating or extrapolating?

a 40 months old **b** 55 months old **c** 70 months old

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7 When preparing between 25 and 100 *meals*, a cafeteria's *cost* (in dollars) is given by the equation:

$$cost = 175 + 5.80 \times meals$$

Use this equation to predict the cost (to the nearest dollar) of preparing the following meals. Are you interpolating or extrapolating?

a no meals

b 60 meals

c 89 meals

8 For women of heights from 150 cm to 180 cm, the equation relating a *daughter's height* (in cm) to her *mother's height* (in cm) is:

$$daughter\text{'s } height = 18.3 + 0.910 \times mother\text{'s } height$$

Use this equation to predict (to the nearest cm) the adult height of women whose mothers are the following heights. Are you interpolating or extrapolating?

a 168 cm tall

b 196 cm tall

c 155 cm tall

9 Students sit for two exams, two weeks apart. The following regression equation can be used to predict the students' marks on exam 2 from the marks they obtained on exam 1.

$$mark\ on\ exam\ 2 = 15.7 + 0.650 \times mark\ on\ exam\ 1$$

a i For this regression equation, write down the value of the intercept.

ii Interpret the intercept in the context of the variables in the equation.

b i For this regression equation, write down the value of the slope.

ii Interpret the slope in the context of the variables in the equation.

c Use the equation to predict the mark on exam 2 for a student who obtains a mark of 20 on exam 1. Give your answer to the nearest mark.

10 It has been suggested that the blood *glucose* level (in mg/100 mL) of adults can be predicted from their *weight* (in kg).

$$glucose = 51 + 0.62 \times weight$$

a i For this regression equation, write down the value of the intercept.

ii Interpret the intercept in the context of the variables in the equation.

b i For this regression equation, write down the value of the slope.

ii Interpret the slope in the context of the variables in the equation.

c Use the equation to predict the blood glucose level of a person who weighs 75 kg. Give your answer rounded to one decimal place.

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Testing understanding

11 A study of the association between the average score in an examination in each of 25 schools and the student:staff ratio in that school, resulted in the information below.

Variable	Mean	Stand dev
student:staff ratio	13.404	4.128
score	71.669	12.013
Correlation coefficient	$r = -0.651$	

Use this information to predict the value of average examination scores in a school with a student:staff ratio of 15. Give your answer rounded to one decimal place.

12 The following table gives the educational level (*education*), the number of years the person has worked for the company (*years*) and their current salary to the nearest thousand dollars (*salary*) for a group of current employees of a particular company.

education	*years*	*salary*	*education*	*years*	*salary*
Secondary	2	52	Tertiary	2	62
Secondary	3	64	Tertiary	3	69
Secondary	2	56	Tertiary	4	75
Secondary	4	63	Tertiary	5	76
Secondary	7	65	Tertiary	4	72
Secondary	6	64	Tertiary	4	68
Secondary	7	52	Tertiary	1	63
Secondary	10	65	Tertiary	8	85
Secondary	5	59	Tertiary	3	67
Secondary	5	62	Tertiary	6	77

a **i** Find the equation of a linear model which allows *salary* to be predicted from *years* for those with Secondary education, rounding the values of the intercept and slope to three decimal places.

ii Interpret the intercept and slope in the context of the variables in this equation.

iii Use the equation to predict the *salary* for an employee with Secondary education who has worked for the company for 5 years. Round your answer to the nearest hundred dollars.

b **i** Find the equation of a linear model which allows *salary* to be predicted from *years* for those with Tertiary education, rounding the values of the intercept and slope to three decimal places.

ii Interpret the intercept and slope in the context of the variables in this equation.

iii Use the equation to predict the *salary* for an employee with Tertiary education who has worked for the company for 5 years. Round your answer to the nearest hundred dollars.

c Are the predictions in parts **a iii** and **b iii** reliable? Explain your reasoning.

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Review

Key ideas and chapter summary

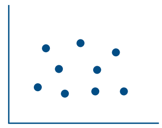
Assign-ment

Explanatory and response variables The **explanatory variable** is used to explain or predict the value of the **response variable**.

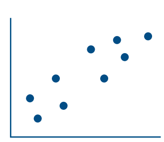
Scatterplot A two-dimensional data plot where each point represents the value of two related variables in a bivariate data set. In a scatterplot, the **response variable (RV)** is plotted on the vertical axis and the **explanatory variable (EV)** on the horizontal axis.

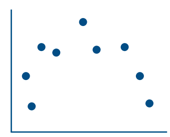
A scatterplot is used to help identify and describe the association between two **numerical** variables.

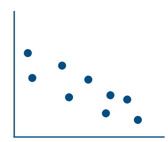
Identifying associations (relationships) between two numerical variables A random cluster of points (no clear pattern) indicates that there is **no association** between the variables.



A clear pattern in the scatterplot indicates that there is an **association between the variables**.







Describing associations in scatterplots Associations are described in terms of:

- **direction** (positive or negative)
- **form** (linear or non-linear)
- **strength** (strong, moderate, weak or none).

Pearson's correlation coefficient (r) **Pearson's correlation coefficient** (r) is a statistic that measures the direction and strength of a linear association between a pair of numerical variables. The strength of the linear association can be classified as follows:

$0.75 \le r \le 1$	**strong** positive association
$0.5 \le r < 0.75$	**moderate** positive association
$0.25 \le r < 0.5$	**weak** positive association
$-0.25 < r < 0.25$	**no** association
$-0.5 < r \le -0.25$	**weak** negative association
$-0.75 < r \le -0.5$	**moderate** negative association
$-1 \le r \le -0.75$	**strong** negative association

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Review

Correlation and causation An association (correlation) between two variables does not automatically imply that the observed association between the variables is **causal**.

Linear regression A straight line can be used to model a linear association between two numerical variables. The association can then be described by a rule of the form, $y = a + bx$.

In this equation:

- ***y*** is the **response variable**
- ***x*** is the **explanatory variable**
- ***a*** is the ***y*-intercept**
- ***b*** is the **slope of the line**.

Fitting a line by eye The simplest method of fitting a linear model to a scatterplot is to draw in a line **by eye** which follows the trend of the data.

The least squares method The **least squares method** for fitting a line to a scatterplot minimises the sum of the squares of the residuals.

Interpreting the intercept and slope For the regression line, $y = a + bx$:

- the slope (b) tells us, on average, the change in the response variable (y) for each one-unit increase or decrease in the explanatory variable (x)
- the intercept (a) tells us, on average, the value of the response variable (y) when the explanatory variable (x) equals 0.

Making predictions The **regression line**, $y = a + bx$, enables the value of y to be predicted for a given value of x, by substitution into the equation.

Interpolation and extrapolation Predicting within the range of the values of the explanatory variable is called **interpolation**, and will give a **reliable** prediction. Predicting outside the range of the values of the explanatory variable is called **extrapolation**, and will give an **unreliable** prediction.

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Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

7A **1** Where appropriate, I can identify the EV and RV in bivariate data. ☐

e.g. I wish to predict the length of a person's foot (*foot length*) from their height (*height*), both measured in centimetres. Which is the explanatory variable (EV), and which is the response variable (RV)?

7A **2** Having identified the EV and RV, I can construct an appropriate scatterplot. ☐

e.g. In order to predict the length of a person's foot from their height, a group of 16 students collected the following data set, which gives height (*height*) and the length of their foot (*foot length*) in centimetres. Construct an appropriate scatterplot.

height	*foot length*	*height*	*foot length*
172	24	182	32
172	22	160	26
177	28	153	22
165	20	177	28
176	28	172	26
155	25	173	25
166	22	185	40
166	30	159	24

7B **3** I can use a scatterplot to describe an observed association between two numerical variables in terms of direction, form and strength, and the meaning of the association, within the context of the data. ☐

e.g. Describe the association in the previous scatterplot between *height* and *foot length* in terms of direction, form and strength.

7C **4** I can estimate the value of the correlation coefficient, r, from a scatterplot and calculate its value from data using technology. ☐

e.g. Estimate the value of the correlation coefficient, r, between *height* and *foot length* by comparing the scatterplot to those on page 420, and then use your calculator to find its exact value. Give your answer rounded to three decimal places.

7C **5** I can classify the value of the correlation coefficient, r, as weak, moderate or strong. ☐

e.g. Use the table on page 428 to classify the value of the correlation coefficient, r, between *height* and *foot length* as weak, moderate or strong.

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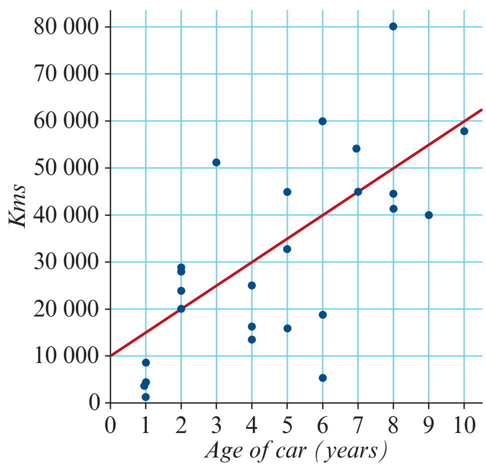
Review

7C **6** I can recognise that an association (correlation) between two variables does not automatically imply that the observed association between the variables is causal. ☐

e.g. There is a strong positive correlation between sales of umbrellas and traffic accidents. Can we conclude that decreasing the sales of umbrellas will help decrease the number of traffic accidents?

7D **7** I can fit a linear model by eye to a scatterplot and find the equation of the line. ☐

e.g. Determine the equation of the line fitted by eye to the scatterplot, below, which shows the association between the age of a group of cars, in years, (*age of car*) and the number of kilometres they have travelled (*kms*).



7D **8** I can determine the equation of the least squares line fitted to the data to model an observed linear association. ☐

e.g. Find the equation of the least squares line which would enable *foot length* to be predicted from *height*. Give the values of the intercept and slope, rounded to two decimal places.

7E **9** I can interpret the slope and intercept of the linear model in the context of data. ☐

e.g. Interpret the intercept and slope of the regression line relating *height* and *foot length* in terms of these variables.

7E **10** I can use the model to make predictions, differentiating between interpolation and extrapolation. ☐

e.g. Use the linear model line relating *height* and *foot length* to predict the foot length of people with the following heights, indicating in each whether you are interpolating or extrapolating. Give your answers rounded to one decimal place.

a 160 cm tall **b** 100 cm tall

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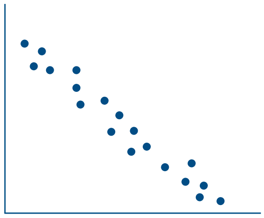
Multiple-choice questions

1 For which one of the following pairs of variables would it be appropriate to construct a scatterplot?

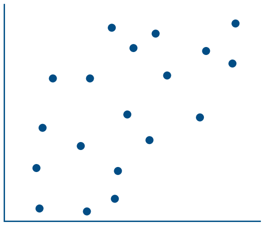
A *eye colour* (blue, green, brown, other) and *hair colour* (black, brown, blonde, other)
B *test score* and *sex* (male, female)
C *political party preference* (Labor, Liberal, Other) and *age* in years
D *age* in years and *blood pressure* in mmHg
E *height* in cm and *sex* (male, female)

2 For the scatterplot shown, the association between the variables is best described as:



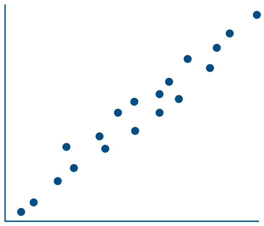
A weak linear negative
B strong linear negative
C no association
D weak linear positive
E strong linear positive

3 For the scatterplot shown, the association between the variables is best described as:



A weak linear negative
B weak non-linear negative
C no association
D weak linear positive
E strong non-linear positive

4 For the scatterplot shown, the association between the variables is best described as:



A weak linear positive
B strong linear positive
C no association
D moderate linear positive
E strong non-linear positive

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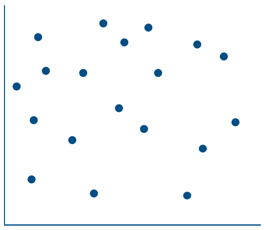
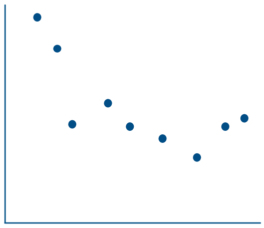
5 For the scatterplot shown, the association between the variables is best described as:



A weak non-linear negative
B strong linear negative
C no association
D weak non-linear positive
E weak linear positive

6 For the scatterplot shown, the association between the variables is best described as:



A weak negative linear
B strong negative linear
C no association
D weak non-linear
E strong non-linear

7 The association between birth weight and infant mortality rate is negative. Given this information, it can be concluded that:

A birth weight and infant mortality rate are not related.
B infant mortality tends to increase as birth weight increases.
C infant mortality tends to decrease as birth weight decreases.
D infant mortality tends to decrease as birth weight increases.
E the values of infant mortality are, in general, less than the corresponding values of birth weight.

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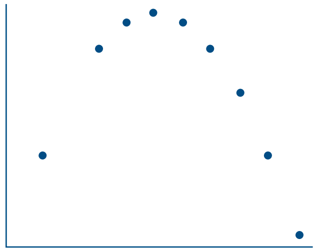
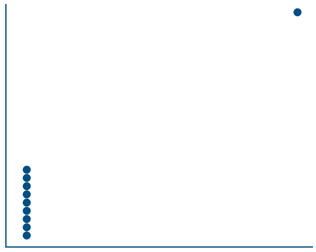
8 For which of the following scatterplots would it make sense to calculate the correlation coefficient (r) to indicate the strength of the association between the variables?

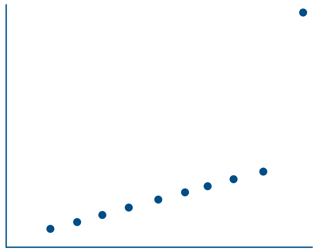
A



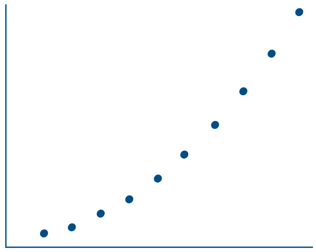
B



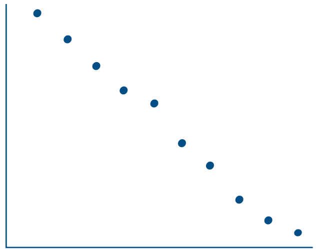
C



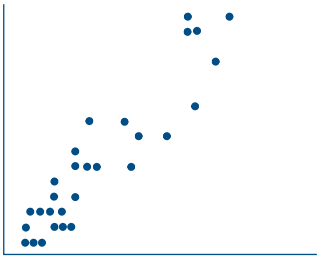
D



E

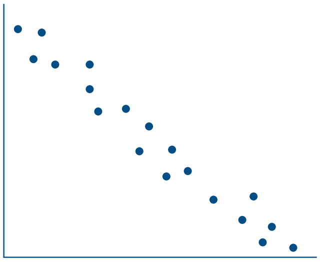


9 For the scatterplot shown, the value of Pearson's correlation coefficient, r, is closest to:



A 0.28 **B** 0.41 **C** 0.63
D 0.86 **E** 0.99

10 For the scatterplot shown, the value of Pearson's correlation coefficient, r, is closest to:



A −0.90 **B** −0.64 **C** −0.23
D 0.64 **E** 0.90

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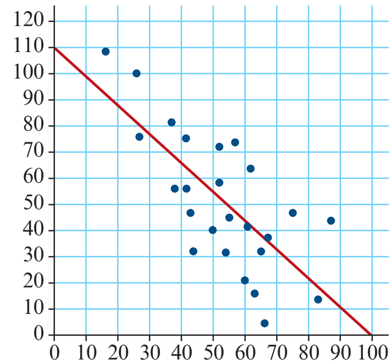
11 A correlation coefficient of $r = -0.32$ would classify a linear association as:

A weak positive **B** weak negative **C** moderately positive
D close to zero **E** moderately weak

12 A line is fitted by eye to the scatterplot, as shown. The equation of this line is:



A $y = 110 - 1.10x$ **B** $y = -110 + 1.10x$ **C** $y = -110 - 1.10x$
D $y = 1.10 - 110x$ **E** $y = 1.10 + 1.10x$

13 The equation of the least squares regression line, $y = a + bx$, when:

$r = 0.500 \quad s_x = 2.40 \quad s_y = 12.5 \quad \bar{x} = 8.00 \quad \bar{y} = 34.5$

is given by:

A $y = 13.7 - 2.60x$ **B** $y = 2.60 + 13.7x$ **C** $y = 33.6 + 0.096x$
D $y = 13.7 + 2.60x$ **E** $y = -13.7 + 2.60x$

The following information relates to Questions 14 and 15.

The weekly *income* and weekly *expenditure* on food for a group of 10 university students is given in the following table.

Income ($/week)	150	250	300	600	300	380	950	450	850	1000
Expenditure ($/week)	40	60	70	120	130	150	200	260	460	600

14 The value of the Pearson correlation coefficient, r, for these data is closest to:

A 0.2 **B** 0.4 **C** 0.6 **D** 0.7 **E** 0.8

15 The least squares regression line that enables weekly *expenditure* (in dollars) on food to be predicted from weekly *income* (in dollars) is closest to:

A *expenditure on food* = 0.482 + 42.9 × *income*
B *expenditure on food* = 0.482 − 42.9 × *income*
C *expenditure on food* = −42.9 + 0.482 × *income*
D *expenditure on food* = 239 + 1.36 × *income*
E *expenditure on food* = 1.36 + 239 × *income*

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The following information relates to Questions 16 and 17.

The equation of a regression line that enables the weekly *amount* spent on entertainment (in dollars) to be predicted from weekly *income* is given by:

$$amount = 40 + 0.10 \times income$$

16 Using this equation, the amount spent on entertainment by an individual with a weekly income of $600 is predicted to be:

A $40 **B** $46 **C** $100 **D** $240 **E** $24 060

17 From the equation of the regression line it can be concluded that, on average:

A the weekly *amount* spent on entertainment increases by 40 cents a week for each extra dollar of weekly income.

B the weekly *amount* spent on entertainment increases by 10 cents a week for each extra dollar of weekly income.

C the weekly *income* increases by 10 cents for each dollar increase in the amount spent on entertainment each week.

D $40 is spent on entertainment each week.

E $40.10 is spent on entertainment each week.

Short-answer questions

1 The following table gives the *number* of times the ball was inside the team's 50-metre line in an AFL football game and the team's final score (in points) in that game.

Number	64	57	34	61	51	52	53	51	64	55	58	71
Score (points)	90	134	76	92	93	45	120	66	105	108	88	133

a Which variable is the RV?

b Construct a scatterplot of *score* against *number*.

c Use the scatterplot to describe the association in terms of direction, form and strength.

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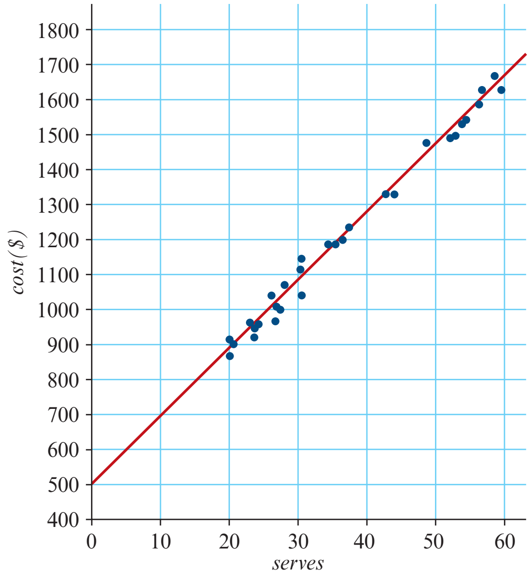
2 The following scatterplot shows the costs of catering a meal, in dollars, (*cost*) with the number of meals served (*serves*). A line fitted by eye is shown on the following scatterplot.



a Which variable is the RV and which is the EV?

b Find the equation of the line shown on the plot in terms of *cost* and *serves*. Round your answer to three significant figures.

3 The *distance* travelled to work and the *time* taken for ten company employees are given in the opposite table. *Distance* is the response variable.

a Determine the value of the Pearson correlation coefficient, r, for this set of data. Round your answer to three decimal places.

b Determine the equation of the least squares line for this data, and write the equation in terms of the variables *distance* and *time*. Round your answer to three significant figures.

Distance (km)	Time (min)
12	15
50	75
40	50
25	50
45	80
20	50
10	10
3	5
10	10
30	35

4 From a data set relating *height (cm)* and *weight (kg)* for a group of students it was determined that the correlation coefficient was $r = 0.75$. It was also found that the mean height for the group was 174.5 cm, with a standard deviation of 9.3 cm, and that the mean weight was 65.9 kg, with a standard deviation of 10.8 kg.

a Find the slope of the least squares regression line which would enable *weight* to be predicted from *height*. Round your answer to three significant figures.

b Find the intercept for this line. Round your answer to three significant figures.

c Hence, write down the equation of the least squares regression line in terms of *weight* and *height*.

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5 The regression equation:

$$taste\ score = -22 + 7.3 \times magnesium\ content$$

can be used to predict the *taste score* of a country town's drinking water from its *magnesium content* (in mg/litre).

a Which variable is the explanatory variable?

b Write down and interpret the slope of the regression line.

c Use the regression line to predict the taste score of a country town's drinking water whose magnesium content is 16 milligrams/litre, rounded to one decimal place.

6 The *time* taken to complete a task and the number of *errors* on the task were recorded for a sample of 10 primary school children.

Time (s)	22.6	21.7	21.7	21.3	19.3	17.6	17.0	14.6	14.0	8.8
Errors	2	3	3	4	5	5	7	7	9	9

a Determine the equation of the least squares regression line that fits this data, with *errors* as the response variable. Round your answer to three significant figures.

b Determine the value of Pearson's correlation coefficient to two decimal places.

7 Researchers found a strong positive correlation between students' exam results in Mathematics and in French. Can we conclude that encouraging students to study harder in French would improve their scores in Mathematics?

Written-response questions

1 A marketing firm wanted to investigate the association between the number of times a song was played on the radio (*played*) and the number of downloads sold the following week (*weekly sales*).

The following data was collected for a random sample of ten songs.

Played	47	34	40	34	33	50	28	53	25	46
Weekly sales	3950	2500	3700	2800	2900	3750	2300	4400	2200	3400

a Which is the explanatory variable and which is the response variable?

b Construct a scatterplot of this data.

c Determine the value of the Pearson correlation coefficient, r, for this data. Round your answer to four decimal places.

d Describe the association between *weekly sales* and *played* in terms of direction, form and strength.

e Determine the equation for the least squares regression line and write it down in terms of the variables *weekly sales* and *played*. Round the values of coefficients to three significant figures.

f Interpret the slope and intercept of the regression line in the context of the problem.

g Use your equation to predict the number of downloads of a song when it was played on the radio 100 times in the previous week.

h In making this prediction, are you interpolating or extrapolating?

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Review

2 To test the effect of driving instruction on driving skill, 10 randomly selected learner drivers were given a *score* on a driving skills test. The number of *hours* of instruction for each learner was also recorded. The results are displayed in the table below.

Hours	19	2	5	9	16	4	19	26	14	8
Score	32	12	17	19	23	16	28	36	30	23

a Which is the explanatory variable and which is the response variable?

b Construct a scatterplot of these data.

c Determine the correlation coefficient, r, and round your answer to 4 decimal places.

d Describe the association between *score* and *hours* in terms of direction, strength and form (and outliers, if any).

e Determine the equation for the least squares regression line and write it down in terms of the variables *score* and *hours*. Give coefficients rounded to three significant figures.

f Interpret the slope and the intercept (if appropriate) of the regression line.

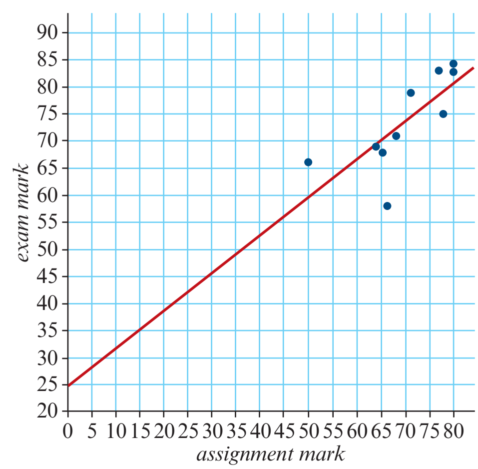
g Predict the score after 10 hours of instruction to the nearest whole number.

3 To investigate the association between marks, in percentages, on an assignment and the final examination mark, data was collected from 10 students.

a The following scatterplot shows this data, with the least squares regression line added.

Use the scatterplot to determine the equation of the least squares regression line, and write it down in terms of the variables, final *exam mark* and *assignment mark*. Write the values in the equation, rounded to two significant figures.

b Interpret the intercept and slope of the least squares regression line in terms of the variables in the study.



c Use your regression equation to predict the final exam mark for a student who scored 50% on the assignment. Give your answer rounded to the nearest mark.

d How reliable is the prediction made in part **c**?

e The scatterplot shows that there is a strong positive linear association between the *assignment mark* and the final *exam mark*. The correlation coefficient is $r = 0.76$. Given this information, a student wrote: 'Good final exam marks are the result of good assignment marks'. Comment on this statement.

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Chapter 8

Graphs and networks

Chapter questions

- What is a graph?
- How do we identify the features of a graph?
- How do we draw a graph?
- How do we apply graphs in practical situations?
- How do we construct an adjacency matrix from a graph?
- How do we define and draw a planar graph?
- How do we identify the type of walk on a graph?
- How do we find the shortest path between two vertices of a graph?
- How do we find the minimum distance required to connect all vertices of a graph?

In this chapter, we show how **graphs** and **networks** can model and analyse everyday networks, maps and many kinds of organisational charts and diagrams.

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8A What is a graph?

Learning intentions

- ▶ To be able to define and identify a graph, vertex, edge and loop.
- ▶ To be able to find the degree of a vertex.
- ▶ To be able to find the sum of degrees.

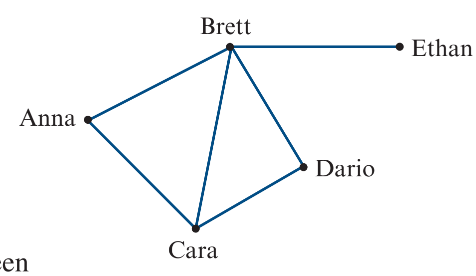
There are many situations in everyday life that involve connections between people or objects. Towns are connected by roads, computers are connected to the internet and people connect to each other through being friends on social media. A diagram that shows these connections is called a **graph**.

Graph elements: definitions

Graphs, vertices, edges

A **graph** is a diagram that consists of a set of points, called **vertices** (plural of vertex), and a set of lines, called **edges**.

A graph can be used to show how five people - Anna, Brett, Cara, Dario and Ethan - are connected on a social media website.



In the graph:

- the 5 vertices represent the 5 people
- the 6 edges represent the 6 connections between the people on the social media website
- Anna is a friend of Brett and Cara
- Brett is a friend of Anna, Cara, Dario and Ethan
- Cara is a friend of Anna, Brett and Dario
- Dario is a friend of Brett and Cara
- Ethan is a friend of Brett.

We will now explore the properties of graphs in more detail.

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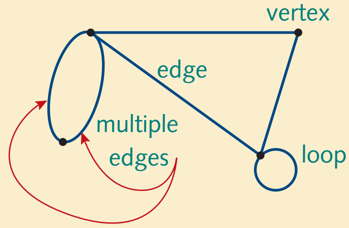
Graph elements: definitions

A **graph** is a diagram that consists of a set of points, called **vertices**, that are joined by a set of lines, called **edges**.

- A **loop** is an edge in a graph that joins a **vertex** in a graph to itself.
- Two or more edges that connect the same vertices are called **multiple edges**.



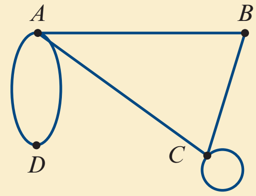
The **degree of a vertex** is the number of edges attached to the vertex.

The degree of a vertex, A, is written as: **deg(A)**.

For example, in the graph opposite,

deg(A) = 4, deg(B) = 2, deg(C) = 4 and deg(D) = 2.

Note: A loop contributes *two degrees* to a vertex because a loop is attached to its vertex at both ends.

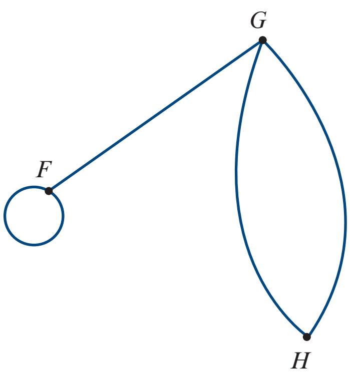


- In any graph, the **sum of degrees** of the vertices is equal to *twice the number of edges*.

  Note: A loop contributes *one edge* to the graph.
- We say a vertex is **even** if the degree of the vertex is even, and we say a vertex is **odd** if the degree of the vertex is odd.

For example, consider the graph opposite.

- There are 3 vertices
- There are 4 edges
- The vertex F has a loop
- There are multiple edges between vertices G and H
- The degree of vertex F is 3, so we write deg(F) = 3
- The degree of vertex G is 3, so we write deg(G) = 3
- The degree of vertex H is 2, so we write deg(H) = 2
- The sum of degrees for this graph is 3 + 3 + 2 = 8, which is *twice* the number of edges. (In this case there are 4 edges, and $2 \times 4 = 8$.)



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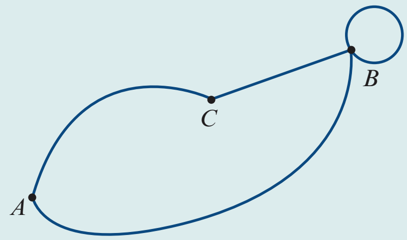
Example 1 Properties of a graph

Consider the graph shown.



a How many vertices does this graph have?

b How many edges does this graph have?

c What is the degree of vertex A?

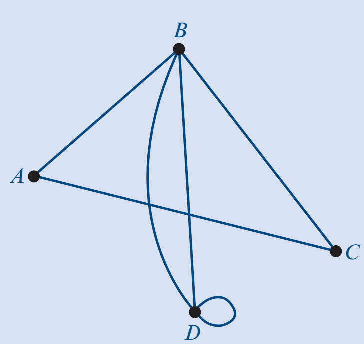
d What is the degree of vertex B?

e What is the *sum of degrees* for this graph?

Explanation	Solution
a Vertices are the set of points that make up a graph. There are three points: A, B and C.	There are 3 vertices.
b Edges are the lines that connect the points. There are four lines. Note: A loop is considered one edge that connects a vertex to itself.	There are 4 edges.
c The degree of a vertex is the number of edges attached to it. There are two edges attached to vertex A.	$\deg(A) = 2$
d Vertex B has one edge connected to vertex A, one edge connected to vertex C and a loop. Note: A loop contributes two degrees to a vertex.	$\deg(B) = 4$
e This graph has four edges. The *sum of degrees* is equal to *twice the number of edges*. Note: $\deg(A) = 2$, $\deg(B) = 4$, $\deg(C) = 2$, therefore $2 + 4 + 2 = 8$	Sum of degrees $= 2 \times 4 = 8$

Now try this 1 Properties of a graph (Example 1)

Consider the graph shown.



a How many vertices does this graph have?

b How many edges does this graph have?

c What is the degree of vertex A?

d What is the degree of vertex D?

e What is the *sum of degrees* for this graph?

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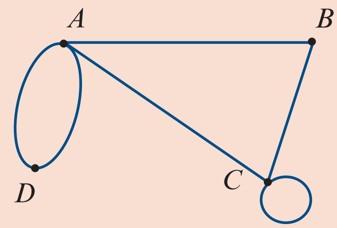
Section Summary

- A **graph** is a diagram that consists of a set of points, called **vertices**, and a set of lines, called **edges**.
- A **loop** is an edge that connects a vertex to itself.
- In this graph, there are 4 **vertices**: *A, B, C,* and *D*.
- In this graph, there are 6 **edges**: *AB, AC, AD (twice), BC* and the loop at *C*.



- The **degree of vertex** *A*, written deg(*A*), is the *number of edges attached to the vertex*. Loops contribute two degrees to a vertex. For example, in the graph above: deg(B) = 2 and deg(C) = 4.
- The **sum of degrees** is equal to *twice the number of edges*.
- We say a vertex is **even** if the degree of the vertex is even, and we say a vertex is **odd** if the degree of the vertex is odd.

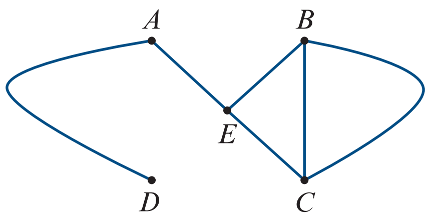
Exercise 8A

Building understanding

Example 1

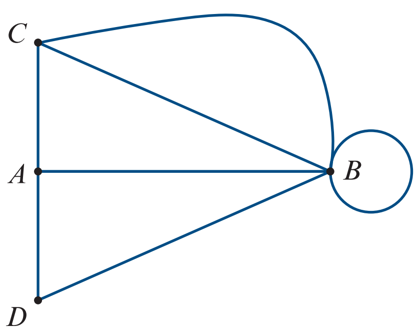
1 For each graph shown, copy and complete the statements by filling in the boxes.

a



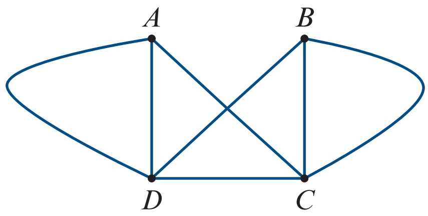
i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv deg(A) = ☐
v deg(E) = ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

b



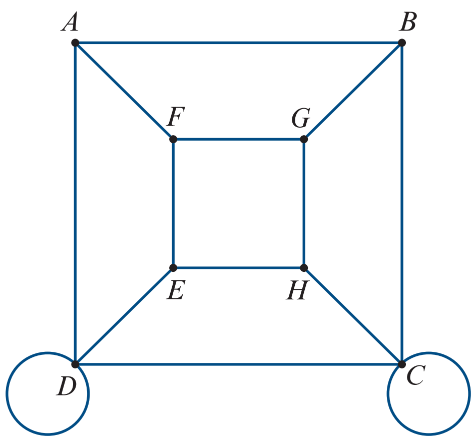
i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv deg(B) = ☐
v deg(D) = ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

c



i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv deg(A) = ☐
v deg(C) = ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

d



i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv deg(C) = ☐
v deg(F) = ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

2 What is the sum of the degrees of the vertices of a graph with:

a five edges? **b** three edges? **c** one edge?

In each case, draw an example of the graph and check your answer.

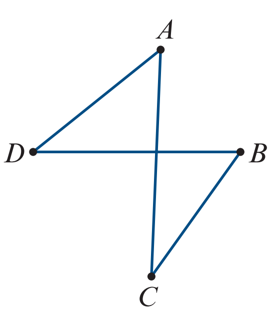
Developing understanding

3 Will the sum of the vertex degrees of a graph always equal twice the number of edges? Explain your reasoning.

4 Consider the graph opposite.
A loop is added at vertex A:

a how will this change the degree of vertex A?

b how many edges are added to the graph?



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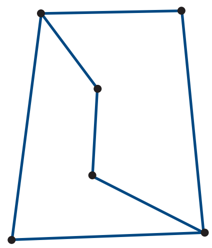
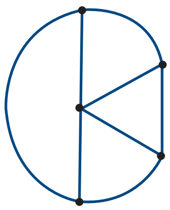
5 In the graph shown below, the sum of the degrees of the vertices is:

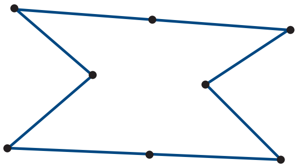


A 6 **B** 7 **C** 12 **D** 13 **E** 14

6 Two graphs, labelled Graph 1 and Graph 2, are shown below.



Graph 1

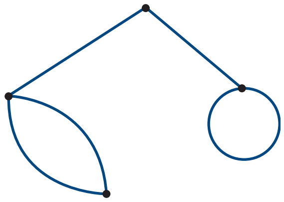


Graph 2

The sum of degrees of the vertices of Graph 1 is:

A two less than the sum of the degrees of the vertices of Graph 2.
B one less than the sum of the degrees of the vertices of Graph 2.
C equal to the sum of the degrees of the vertices of Graph 2.
D one more than the sum of the degrees of the vertices of Graph 2.
E two more than the sum of the degrees of the vertices of Graph 2.

7 The graph below has four vertices and five edges.



How many of the vertices in this graph have an odd degree?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

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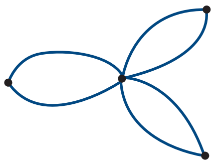
8 Consider the following graph.



Which one of the following statements is **not** true for this graph?

A There are four vertices.

B There are three loops.

C All vertices have an even degree.

D Three of the vertices have the same degree.

E The sum of the degrees of the vertices is twelve.

The Königsberg bridge problem

The problem that began the scientific study of graphs and networks is known as the *Königsberg bridge problem*. The problem began as follows.

The centre of the old German city of Königsberg was located on an island in the middle of the Pregel River. The island was connected to the banks of the river and to another island by five bridges. Two other bridges connected the second island to the banks of the river, as shown.

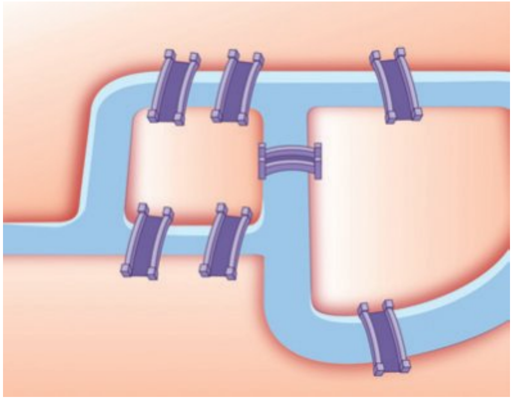
The seven bridges of Königsberg.

The problem simplified

A simplified view of the situation is shown in the drawing below.

Can a continuous walk be planned so that all bridges are crossed only once?

Whenever someone tried to walk the route, they either ended up missing a bridge or crossing one of the bridges more than once. Two such walks are marked on the diagrams that follow. See if you can trace out a walk on the diagram that crosses every bridge, but only once.



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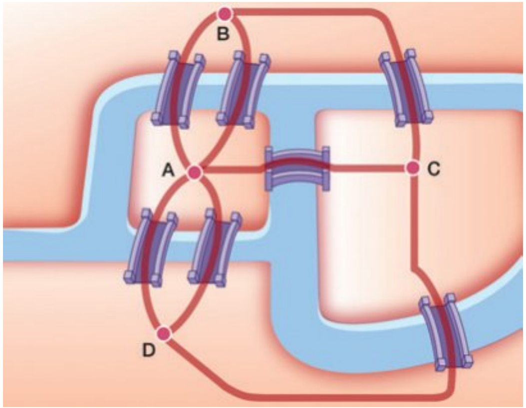
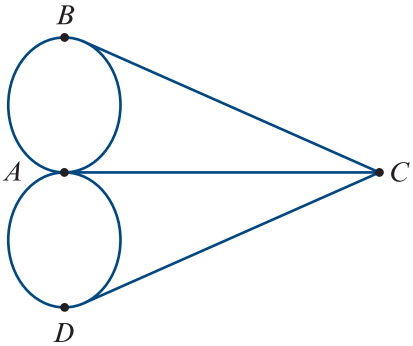
Enter the mathematician

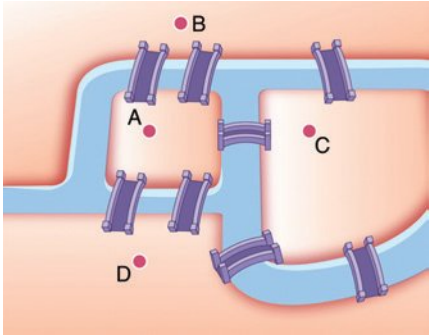
The Königsberg bridge problem was well-known in 18th century Europe and attracted the attention of the Swiss mathematician Euler (pronounced 'Oil-er'). He started analysing the problem by drawing a simplified diagram to represent the situation, as shown below. We now call this type of simplified diagram a *graph*.





Euler's diagram

9 **a** The picture opposite shows a situation in which an eighth bridge has been added.



i With a pencil or the tip of your finger, see whether you can trace out a continuous walk that crosses each of the bridges only once. Such a walk exists.

ii Construct a graph to represent this new situation with eight bridges. Labelled dots have been placed on the picture to help you draw your graph.

iii Your graph should have only two odd vertices. Which vertices have an odd degree?

iv As you will learn later, when the graph has only two odd vertices, you can only complete the task if you start at the places represented by the odd vertices. You will then finish at the place represented by the other odd vertex. Check to see.

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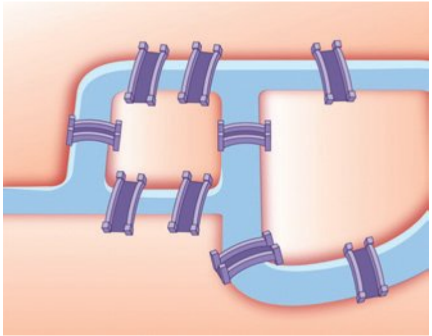
b A ninth bridge has been added, as shown opposite.



i With a pencil or the tip of your finger, see whether you can trace out a continuous walk that crosses each of the bridges only once. Such a walk exists.

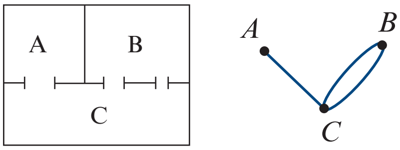
ii Construct a graph to represent this situation.

iii Your graph should *not* have any odd vertices; that is, they should all be even. Check to see.

iv As you will learn later, when the graph has only even vertices, you can start your walk from any island or any riverbank and still complete the task. Check to see.

Testing understanding

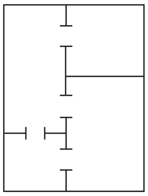
A graph is used to represent the floor plan of a house. The vertices A, B and C represent the rooms, and the edges represent the doorways connecting the rooms. See the diagram of a floor plan and a graph that represents it below.

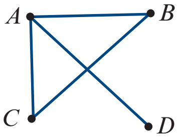


From the graph, we can see that:

- The single edge in the graph between vertices A and C shows that there is one doorway connecting room A to room C.
- The two edges between vertices B and C show that there are two doorways connecting room C to room B.
- There is no edge between vertices A and B, showing that there is no doorway connecting room A to room B.

10 A graph is used to represent the floor plan of a second house.
In the graph, the vertices A, B, C and D represent the rooms, and the edges represent the doorways connecting the rooms. The room labels are missing from the floor plan.
Use the information in the graph to correctly label the rooms in the plan.





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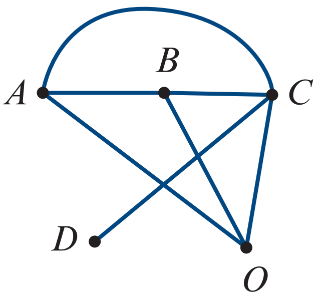
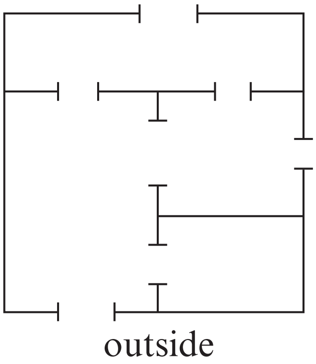
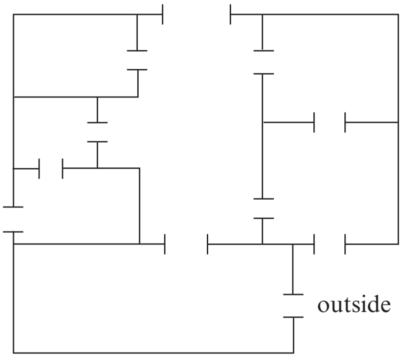
11 A graph is used to represent the floor plan of a third house. In this house, the doorways enable people to move between the rooms of the house, and for some of the rooms, between the room and the outside. In the graph, the vertices labelled A, B, C and D represent the rooms of the house, and the vertex labelled O represents the outdoor area surrounding the house.

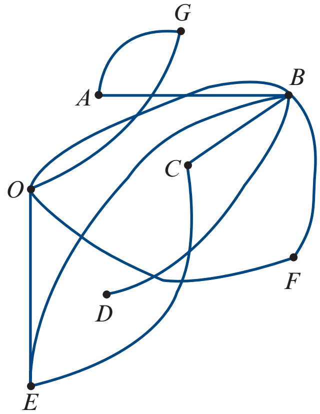




The room labels are missing from the floor plan. Use the information in the graph to correctly label the rooms in the floor plan.

12 A graph is used to represent the floor plan of a fourth house.
In the graph, the vertices A, B, C, D, E, F and G represent the rooms, and edges represent the doorways connecting the rooms. The label, O, is used to label the outside area of the house which can be accessed via a doorway from some rooms. The room labels are missing from the plan.
Use the information in the graph to correctly label the rooms in the plan.





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8B Isomorphic (equivalent) connected graphs and adjacency matrices

Learning intentions

- To be able to identify connected graphs and bridges.
- To be able to identify isomorphic graphs.
- To be able to use an adjacency matrix to represent a graph.

Connected graphs and bridges

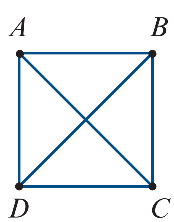
Connected graphs

So far, all the graphs we have encountered have been **connected**.

Connected graphs

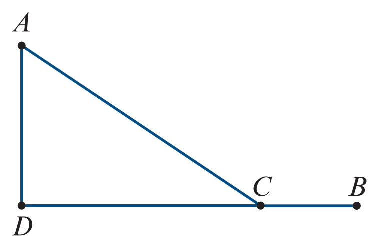
In a **connected graph**, every vertex is connected to every other vertex, either directly or indirectly. That is, every vertex in the graph can be reached from every other vertex in the graph.

For example, the three graphs shown below are all connected.



Graph 1

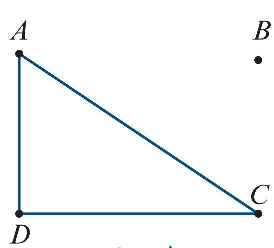
Graph 2



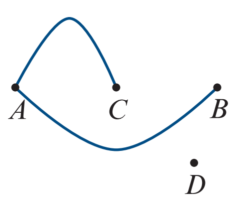
Graph 3

The graphs are connected because, starting at any vertex, say A, you can always find a path along the edges of the graph to take you to every other vertex.

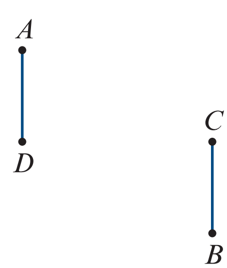
However, the three graphs below are *not connected*, because there is not a path along the edges that connects vertex A (for example) to every other vertex in the graph.



Graph 1



Graph 2



Graph 3

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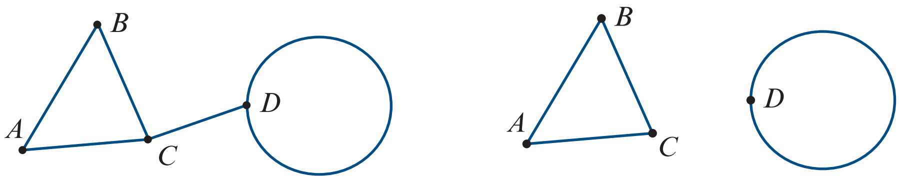
Bridges

Connected graphs have applications in a range of problems, such as planning airline routes, communication systems and computer networks, where a single missing connection can lead to an inoperable system. Such critical connections are called *bridges*.

Bridge

A **bridge** is an edge in a connected graph that, if removed, leaves the graph disconnected.

In Graph 1 below, edge CD is a bridge because removing CD from the graph leaves it disconnected, as shown in Graph 2.



Graph 1 Graph 2

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Isomorphic graphs

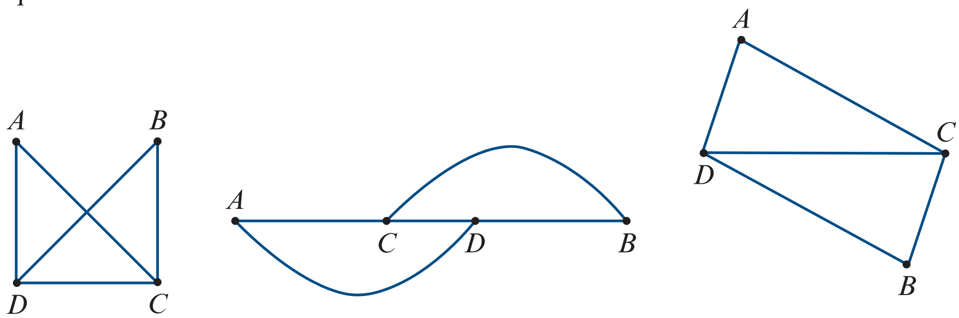
Different-looking graphs can contain the same information.

Isomorphic (equivalent) graphs

Isomorphic graphs are graphs that contain identical information. They have the

- same number of vertices
- same number of edges
- and the same connections between vertices.

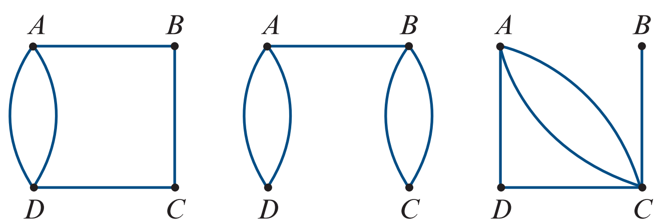
For example, the three graphs below look quite different, but in graphical terms, they are equivalent.



Graph 1 Graph 2 Graph 3

This is because they contain the same information. Each graph has the same number of edges (5) and vertices (4), corresponding vertices have the same degree (e.g. deg(A) = 2 for each graph) and the edges join the vertices in the same way (A to C, A to D, B to C, B to D, and D to C).

However, the three graphs below, although having the same numbers of edges and vertices, are not isomorphic. This is because corresponding vertices do *not* have the same degree and the edges do *not* connect the same vertices.



Graph 1 Graph 2 Graph 3

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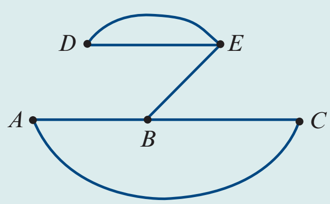
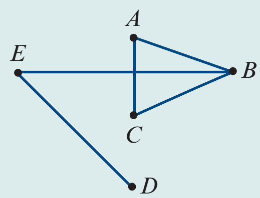
Example 2 Identifying bridges and isomorphic graphs

Consider the following three graphs.

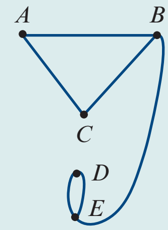
Graph 1



Graph 2



Graph 3



a Which of the three graphs above are connected?

b Graph 2 contains two bridges. Identify the bridges.

c Two of the graphs are isomorphic. Identify the isomorphic graphs.

Explanation

a In each of the three graphs, every vertex is connected to every other vertex, directly or indirectly, through another vertex.

b **1** One bridge is *DE*. If you remove the edge between the vertices *D* and *E*, the graph will now be disconnected *(the vertex D will no longer be connected to any vertices).*

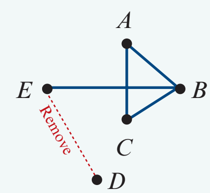
2 Likewise, if you remove the edge between the vertices *E* and *B*, the graph will also be disconnected *(the vertices D and E will no longer be connected to the vertices A, B and C).*

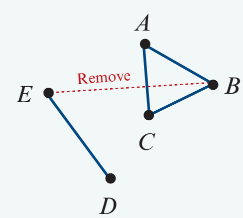
c **1** First, check if all three graphs contain the same number of vertices and edges. All three graphs have five vertices, *however*, Graphs 1 and 3 have six edges, whereas Graph 2 only has five edges.

2 Next, check that all edges connect to the same vertices. All edges in Graphs 1 and 3 connect to the same vertices *(e.g. DE twice, BE, BC, AB and AC).*

Solution

All three graphs are connected.





The two bridges are *DE* and *EB*.

The two isomorphic graphs are Graph 1 and Graph 3.

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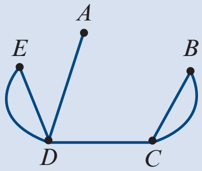
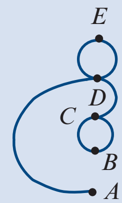
Now try this 2 **Identifying bridges and isomorphic graphs (Example 2)**

Consider the following three graphs.

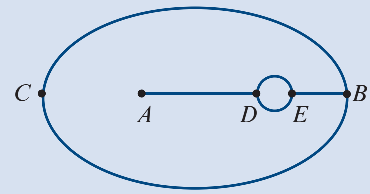
Graph 1



Graph 2



Graph 3



a Which of the graphs above are connected?

b Graph 1 contains two bridges. Identify the bridges.

c Two of the graphs are isomorphic. Identify the isomorphic graphs.

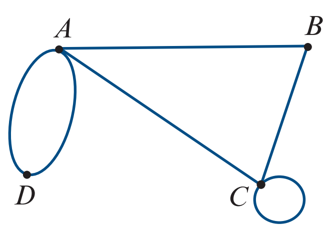
Adjacency matrices

Matrices are a compact way of communicating the information in a graph. There are various types of matrices that can be used to represent the information in a graph. We will only consider one type, the **adjacency matrix**.

Adjacency matrix

An **adjacency matrix** is a square matrix that uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph.

Adjacency matrices are useful when the information in a graph needs to be entered into a computer. For example:



	A	B	C	D
A	0	1	1	2
B	1	0	1	0
C	1	1	1	0
D	2	0	0	0

On the left, we have a graph with four vertices, A, B, C and D.
On the right, we have a 4×4 adjacency matrix (four rows and four columns).
The rows and columns are labelled A to D, as shown, to match the vertices in the graph.
The numbers in the matrix refer to the number of edges joining the corresponding vertices.
For example, in this matrix:

- the '0' in row A, column A, indicates that no edges connect vertex A to vertex A.
- the '1' in row B, column A, indicates that one edge connects vertex A to vertex B.
- the '2' in row D, column A, indicates that two edges connect vertex A to vertex D.
- the '1' in row C, column C, indicates that one edge connects vertex C to vertex C (a loop) and so on until the matrix is complete.

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Example 3 Drawing a graph from an adjacency matrix

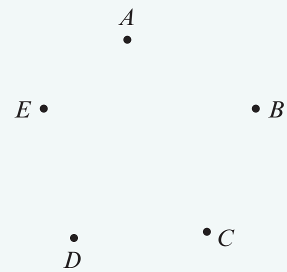
Draw the graph that is represented by the following adjacency matrix.

	A	B	C	D	E
A	0	0	2	0	1
B	0	0	2	1	0
C	2	2	0	1	0
D	0	1	1	1	0
E	1	0	0	0	0

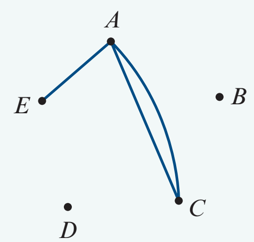
Explanation

1 Draw a dot for each vertex, and label *A* to *E*.

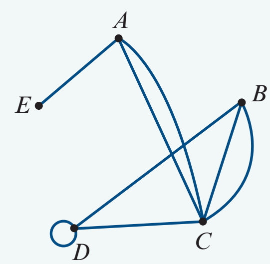
Solution



2 Starting with row *A*, there is a ‘2’ in column *C* and a ‘1’ in column *E*. This means vertex *A* has three edges (two connecting to vertex *C* and one connecting to vertex *E*). Draw these edges.



3 Continue this process, row by row, until all edges are drawn. Note the ‘1’ in row *D*, column *D*, refers to a *loop*, because it’s an edge that connects a vertex to itself.



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Now try this 3 Drawing a graph from an adjacency matrix (Example 3)

Draw the graph that is represented by the following adjacency matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Section Summary

- A graph is **connected** if every vertex in the graph is accessible from every other vertex in the graph along a series of adjacent edges.
- A **bridge** is a single edge in a connected graph that, if removed, leaves the graph disconnected. A graph can have more than one bridge.
- Graphs are said to be **isomorphic** if:
  - they have the same numbers of edges and vertices
  - corresponding vertices have the same degree, and the edges connect the same vertices.
- An **adjacency matrix** summarises the information in a graph. There is one row and one column for each vertex. It uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph. A loop (an edge that connects a vertex to itself) is counted as one edge in the matrix.

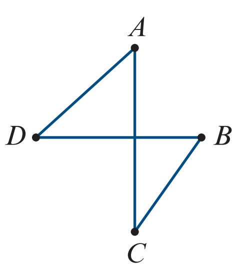
Exercise 8B

Building understanding

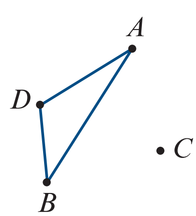
Example 2

1 Which of the following graphs are connected?

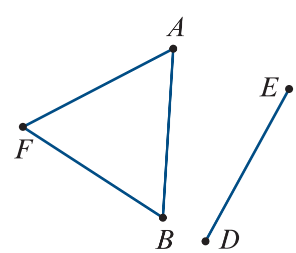
A



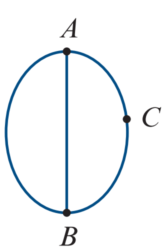
B



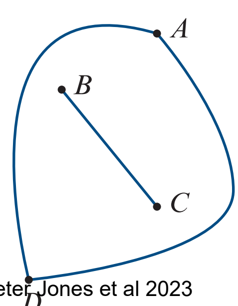
C



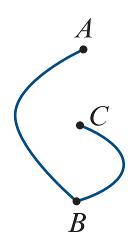
D



E



F



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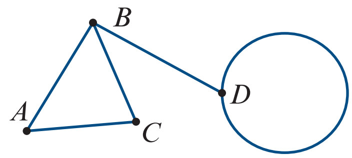
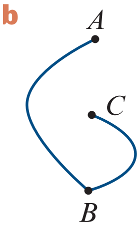
2 Identify the bridge (or bridges) in the graphs below.

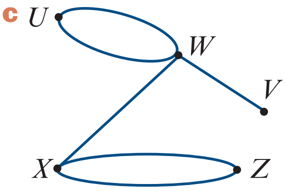
a



b

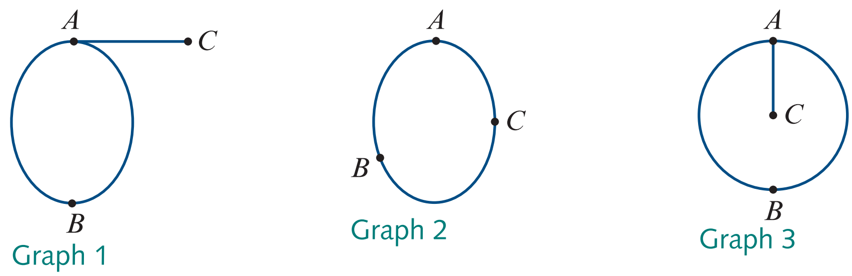


c

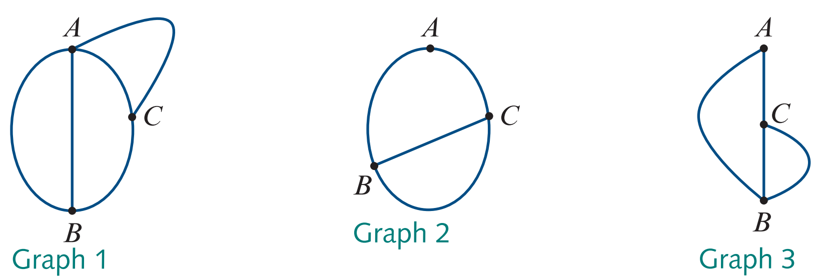


3 In each of the following sets of three graphs, two of the graphs are isomorphic. In each case, identify the isomorphic graphs.

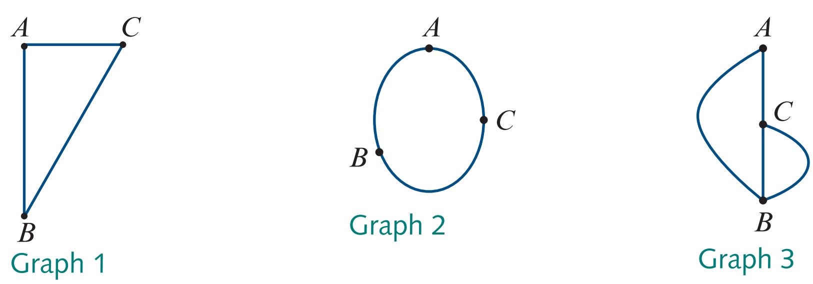
a



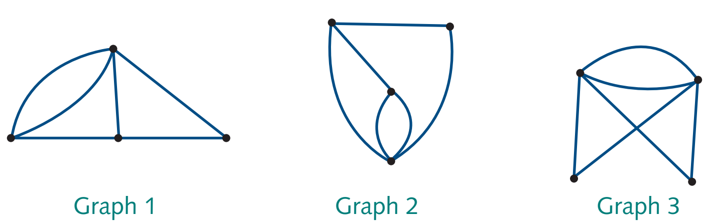
b



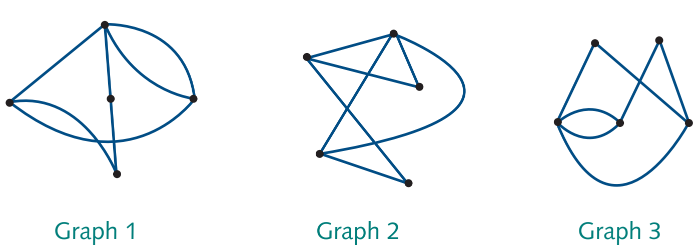
c



d



e



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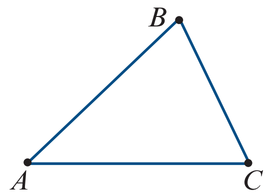
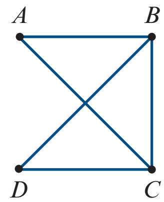
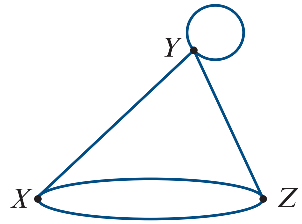
4 Construct an adjacency matrix for each of the following graphs.

a


b


c


Developing understanding

5 Draw a connected graph with:

a three vertices and three edges
b three vertices and five edges
c four vertices and six edges
d five vertices and five edges

6 Draw a graph that is *not* connected with:

a three vertices and two edges
b four vertices and three edges
c four vertices and four edges
d five vertices and three edges

7 What is the smallest number of edges that can form a connected graph with four vertices?

8 Draw a graph with four vertices in which every edge is a bridge.

Example 3

9 Construct a graph for each of the following adjacency matrices.

a
$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

c
$$\begin{array}{c} \\ W \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

d
$$\begin{array}{c} \\ F \\ G \\ H \\ I \\ J \end{array}\begin{array}{c} \begin{array}{ccccc} F & G & H & I & J \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

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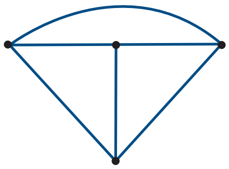
Testing understanding

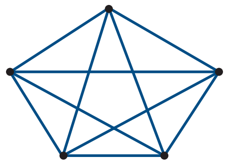
10 A graph has six vertices and only one bridge. What is the minimum number of edges that this graph must have if it is a connected graph?

11 The following graphs are connected. How many edges in each graph can be removed so that the graph will have the minimum number of edges to remain connected?

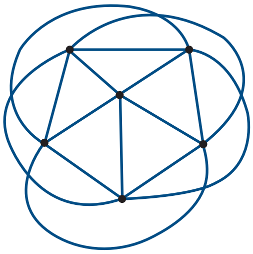
a



b

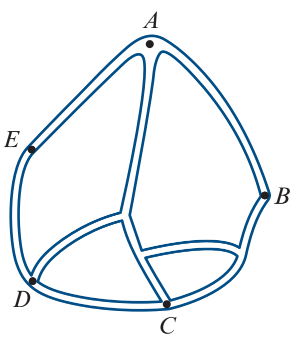


c

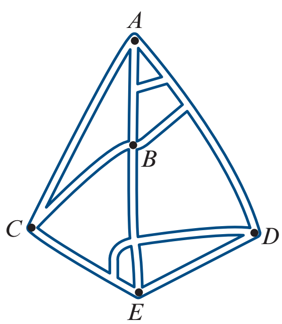


12 The following maps below show all the road connections between five towns, A, B, C, D and E. Construct an adjacency matrix for each of the maps below.

a



b



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8C Planar graphs and Euler's formula

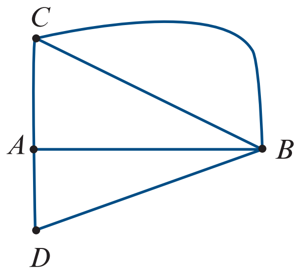
Learning intentions

- To be able to classify a graph as a **planar graph**.
- To be able to redraw a graph in an equivalent planar form.
- To be able to identify the number of faces of a graph.
- To be able to verify Euler's formula.
- To be able to apply Euler's formula to find unknown properties of a planar graph.

Planar graphs

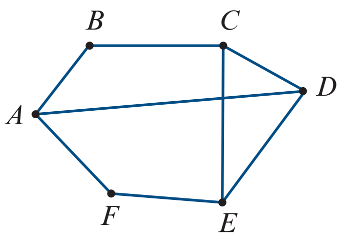
Planar graphs

A **planar graph** can be drawn on a plane (page surface) so that no edges intersect (cross), except at the vertices.

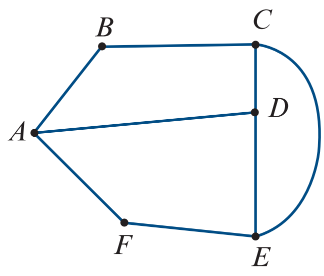


A planar graph: no intersecting edges

Some graphs do not initially appear to be planar; for example, Graph 1, shown below left. However, Graph 2 (below right) is equivalent (isomorphic) to Graph 1. Graph 2 is clearly planar, so Graph 1 is also planar.



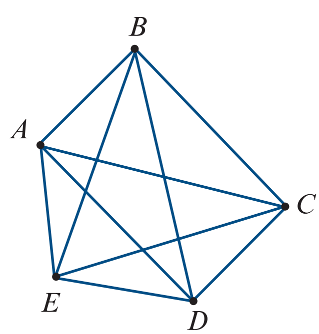
Graph 1



Graph 2

Not all graphs are planar.

For example, the graph opposite cannot be redrawn in an equivalent planar form, no matter how hard you try.



Non-planar graph

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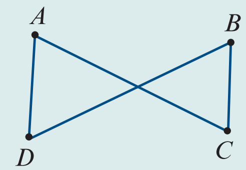
Example 4 Redrawing a graph in planar form

Redraw the graph shown opposite in planar form.



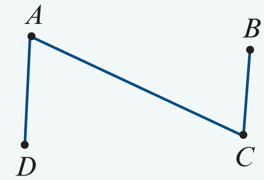
Explanation

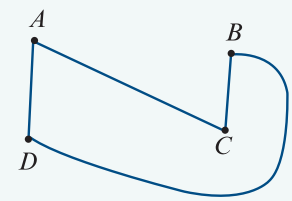
1 Redraw the graph with edge DB removed.

Note: We have removed edge DB because it intersects edge AC.

2 Replace edge DB as a curved line that avoids intersecting with the other three edges. The graph is now in an equivalent planar form: no edges intersect, except at vertices.

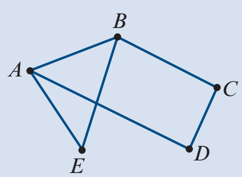
Solution (there are others)





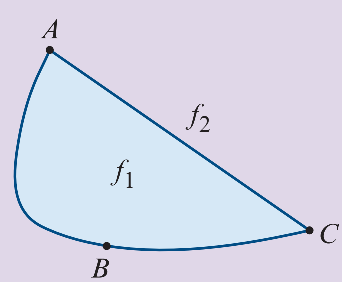
Now try this 4 Redrawing a graph in planar form (Example 4)

Redraw the graph below in planar form.



Faces of a graph

The graph opposite can be regarded as dividing the paper it is drawn on into two regions. These regions are called **faces**.



Faces

A **face** is a region of space. The space may be enclosed within the connected edges of a graph or the space surrounding the graph.

One face, f_1, is bounded by the graph.

The other face, f_2, is the region surrounding the graph. This 'outside' face is infinite.

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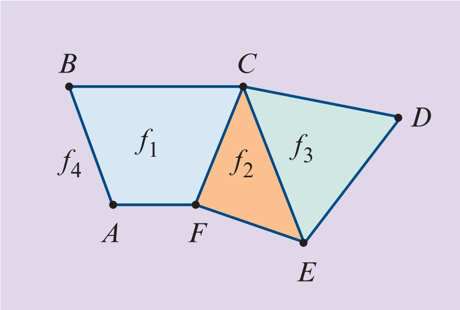
The graph opposite divides the paper into four regions, so we say that it has four faces: f_1, f_2, f_3 and f_4. Here f_4 is an infinite face.



Euler's formula

Euler discovered that, for connected planar graphs, there is a relationship between the *number of vertices*, v, the *number of edges*, e, and the *number of faces*, f.

Euler's formula

For a connected planar graph:

number of vertices + number of faces = number of edges + 2

or

$$v + f = e + 2$$

where v = number of vertices, e = number of edges and f = number of faces.

For example, for the graph opposite:

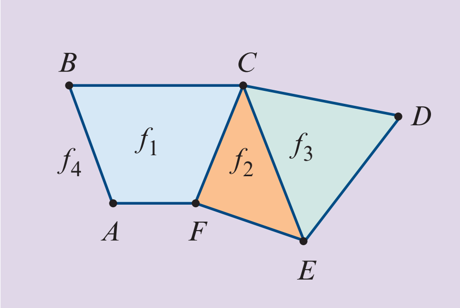
$v = 6$, $f = 4$ and $e = 8$.

$v + f = e + 2$

$6 + 4 = 8 + 2$

$10 = 10$

confirming Euler's formula.

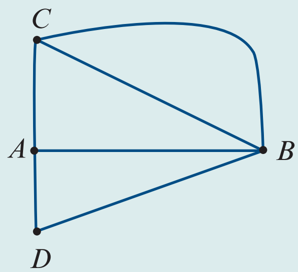


Example 5 Verifying Euler's formula

Consider the connected planar graph shown.

a Write down the number of vertices, v, the number of edges, e, and the number of faces, f.

b Verify Euler's formula.



Explanation	Solution
a 1 There are four vertices: A, B, C, D, so $v = 4$.	Number of vertices: $v = 4$

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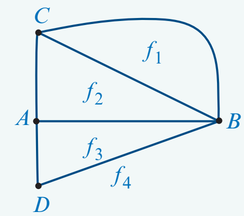
2 There are six edges: *AB, AC, AD, BC* (×2) and *BD*, so $e = 6$.

3 There are four faces, so $f = 4$.

Tip: Mark the faces on the diagram. Do not forget the infinite face, f_4, that surrounds the graph.

Number of edges: $e = 6$



Number of faces: $f = 4$

b **1** Write down Euler's formula.

2 Substitute the values of v, e, and f. Evaluate.

3 Write your conclusion.

Euler's formula:

$v + f = e + 2$

$4 + 4 = 6 + 2$

$8 = 8$

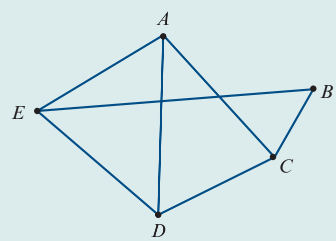
∴ Euler's formula is verified.

Example 6 Verifying Euler's formula

Consider the connected planar graph shown.

a Write down the number of vertices, v, the number of edges, e, and the number of faces, f.

b Verify Euler's formula.



Explanation

a **1** There are five vertices: *A, B, C, D, E*, so $v = 5$.

2 There are seven edges: *AC, AD, AE, BC, BE, CD, DE*, so $e = 7$.

3 The original graph had two edges crossing, therefore it must be redrawn in planar form to determine how many faces. There are four faces, so $f = 4$.

Tip: Mark the faces on the diagram. Do not forget the face, f_4, that surrounds the graph.

b **1** Write down Euler's formula.

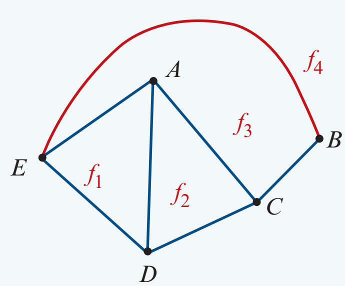
2 Substitute the values of v, e, and f. Evaluate.

3 Write your conclusion.

Solution

Number of vertices: $v = 5$

Number of edges: $e = 7$



Number of faces: $f = 4$

Euler's formula: $v + f = e + 2$

$5 + 4 = 7 + 2$

$9 = 9$

∴ Euler's formula is verified.

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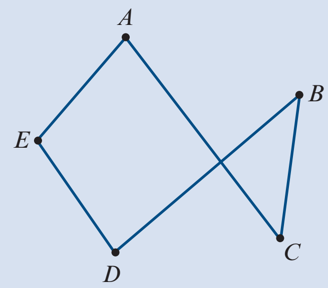
Now try this 6 Verifying Euler's formula (Examples 5 and 6)

Consider the connected graph opposite.
Verify Euler's formula.



Example 7 Using Euler's formula to find unknown characteristics of a graph

For a planar connected graph, find:

a the number of edges, given it has four vertices and six faces.

b the number of faces, given it has four vertices and five edges.

Explanation	Solution
a **1** Write down v and f.	$v = 4$, $f = 6$
2 Write down Euler's formula.	Euler's formula:
3 Substitute the values of v and f.	$v + f = e + 2$
4 Solve for e.	$4 + 6 = e + 2$
	$10 = e + 2$
	$e = 8$
5 Write your answer.	Therefore this graph has 8 edges.
b **1** Write down v and e.	$v = 4$, $e = 5$
2 Write down Euler's formula.	Euler's formula:
3 Substitute the values of v and e.	$v + f = e + 2$
4 Solve for f.	$4 + f = 5 + 2$
	$4 + f = 7$
	$f = 7 - 4$
	$f = 3$
5 Write your answer.	Therefore this graph has 3 faces.

Now try this 7 Using Euler's formula to find unknown characteristics of a graph (Example 7)

For a planar connected graph, find the number of vertices given it has 5 faces and 8 edges.

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Section Summary

- **Planar graphs** are graphs that can be drawn so that no two edges cross or intersect, except at the vertices.
- A **face** is an area in a graph that is enclosed by the edges. The space that surrounds a graph is also always counted as a face.
- The number of faces can only be determined when the graph is drawn in planar form (no crossing edges).
- **Euler's formula**: $v + f = e + 2$
  where:
  v = number of vertices,
  f = number of faces (including the surrounding space)
  and e = number of edges.
- Euler's formula only applies to planar graphs.

Skill-sheet

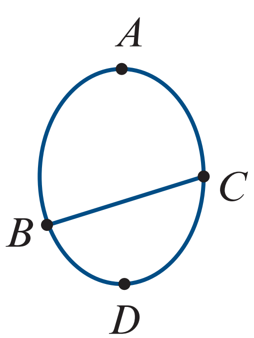
Exercise 8C

Building understanding

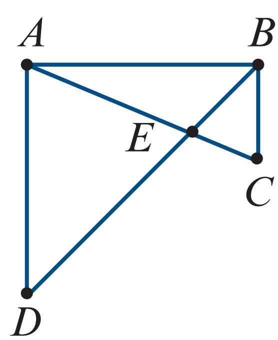
Example 4

1 Which of the following graphs are drawn in planar form?

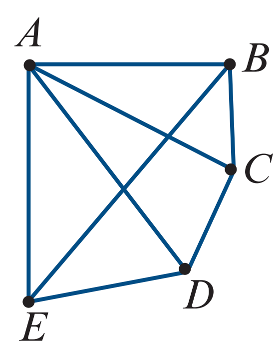
A



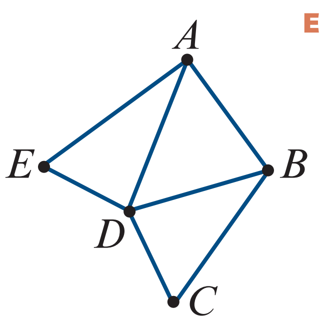
B



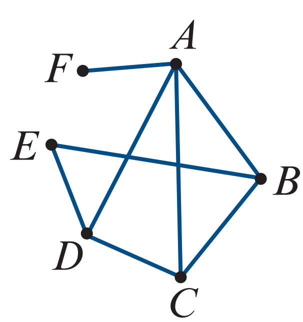
C



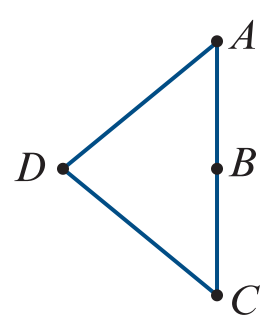
D



E



F



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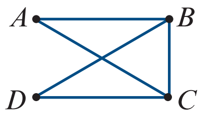
2 Redraw each graph in an equivalent planar form.

a



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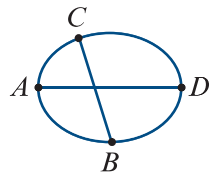
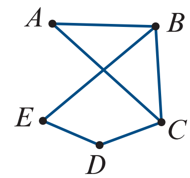
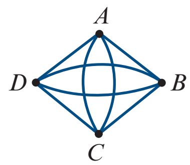


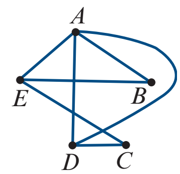
c



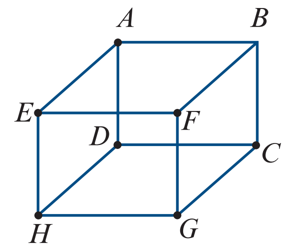
d



e



f



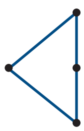
Developing understanding

Example 5
Example 6

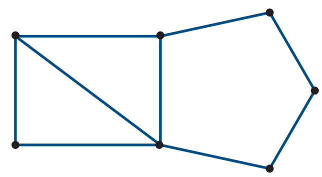
3 For each of the following graphs:

i state the values of v, e and f **ii** verify Euler's formula.

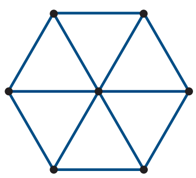
a



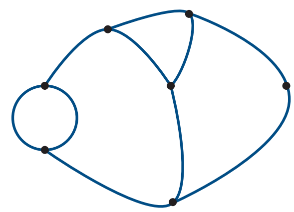
b



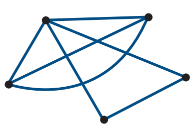
c



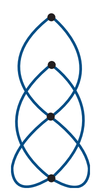
d



e



f



Example 7

4 For a planar connected graph, find:

a f given $v = 4$ and $e = 4$

b v given $e = 3$ and $f = 2$

c e given $v = 3$ and $f = 3$

d v given $e = 6$ and $f = 4$

e f given $v = 4$ and $e = 6$

f f given $v = 6$ and $e = 11$

g e given $v = 10$ and $f = 11$

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5 The following adjacency matrices represent planar graphs. Find the number of faces for each graph.

a
$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

c
$$\begin{array}{c} \\ V \\ W \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{ccccc} V & W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

d
$$\begin{array}{c} \\ F \\ G \\ H \\ I \\ J \end{array}\begin{array}{c} \begin{array}{ccccc} F & G & H & I & J \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Testing understanding

6 A connected planar graph has six vertices and nine edges. A further three edges were added to the graph. The number of faces increased by:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

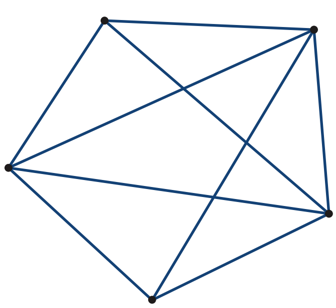
7 A planar graph has four faces. This graph could have:

A Seven vertices and seven edges
B Seven vertices and four edges
C Seven vertices and five edges
D Four vertices and seven edges
E Five vertices and seven edges

8 A planar graph has five vertices. Three vertices have degree four and two have degree three. The number of faces of this graph is:

A 6 **B** 9 **C** 11 **D** 12 **E** 18

9 Consider the graph shown opposite.
Which one of the following statements is not true for the graph?



A The graph has even and odd vertices.
B The graph has five faces.
C Euler's formula is verified for this graph.
D The graph is planar with five vertices.
E The graph is connected.

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10 A graph is connected and has five vertices and four edges. Consider the following four statements.

- The graph is planar.
- The graph can have multiple edges between two vertices.
- All the vertices of the graph can have an even degree.
- Two vertices of the graph can have an odd degree.

The number of these statements which are true for this graph is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

11 A connected planar graph has an equal number of vertices and edges. The number of faces in this graph is equal to:

A 1 **B** 2 **C** 3 **D** 4

E Cannot determine, more information is required.

12 A connected planar graph has an equal number of vertices and faces. The number of edges in this graph is equal to:

A the number of faces

B twice the number of faces

C half the number of faces

D two less than twice the number of faces

E two more than half the number of faces

8D Walks, trails, paths, circuits and cycles

Learning intentions

- ► To be able to identify a walk as a trail, path, circuit or cycle.

Many practical problems that can be modelled by graphs involve moving around a graph, for example, designing a postal delivery route or solving the Königsberg bridge problem. To solve such problems, you will need to know about a number of concepts that we use to describe the different ways we can move around a graph. We will use the following graph to explore these ideas.

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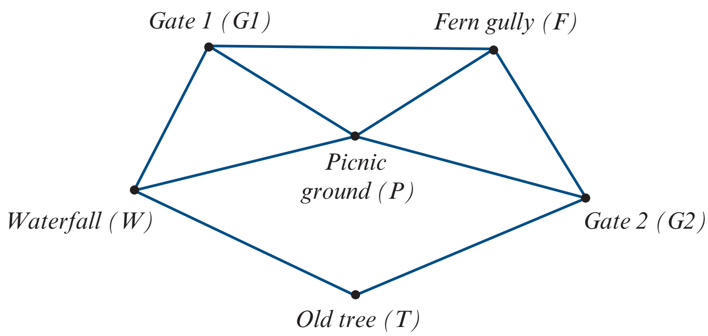


The graph above represents a series of tracks in Sherbrooke Forest that connect a picnic ground (vertex P), a waterfall (vertex W), a very old tree (vertex T) and a fern gully (vertex F).

People can enter and leave the forest through either Gate 1 (vertex $G1$) or Gate 2 (vertex $G2$). The edges in the graph represent tracks that connect these places, for example, the edge WP represents the track between the waterfall and the picnic ground.

Walks, trails and paths

Informally, a walk is any route through a graph that moves from one vertex to another along the joining edges. When there is no ambiguity, a walk in a graph can be specified by listing the vertices visited on the walk.

Walk

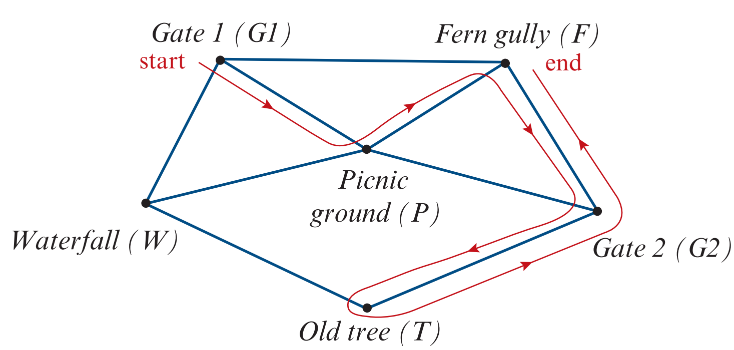
A **walk** is a sequence of edges linking successive vertices, that connects two different vertices in a graph.

A walk in the forest

Using the forest track graph (shown in blue), an example of a *walk* is:

G1-P-F-G2-T-G2-F

The red arrows on the graph trace out a walk.



Note1: The double red arrows on the graph indicate that this track is walked along in both directions.

Note2: A walk does not require all of its edges or vertices to be different.

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Trail

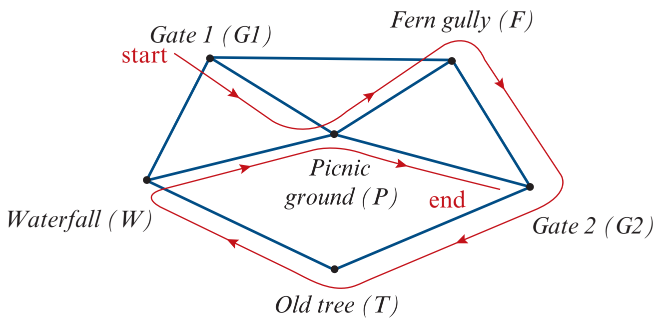
A **trail** is a walk with no repeated edges.

A forest trail

Using the forest track graph, an example of a *trail* is:

G1-P-F-G2-T-W-P-G2

The red arrows on the graph trace out this trail. It has *no repeated edges*. However, there are two repeated vertices, *P* and *G*2. This is permitted on a trail.



Path

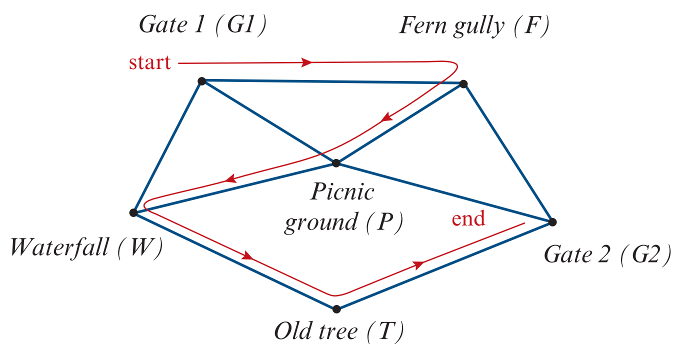
A **path** is a walk with no repeated edges and no repeated vertices.

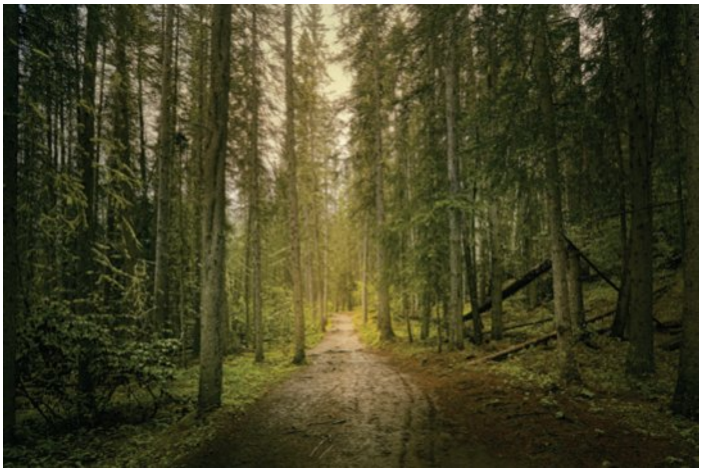
A path in the forest

Using the forest track graph, an example of a *path* is:

G1-F-P-W-T-G2

The red arrows on the graph trace out this path. There are *no repeated edges* or *vertices*.





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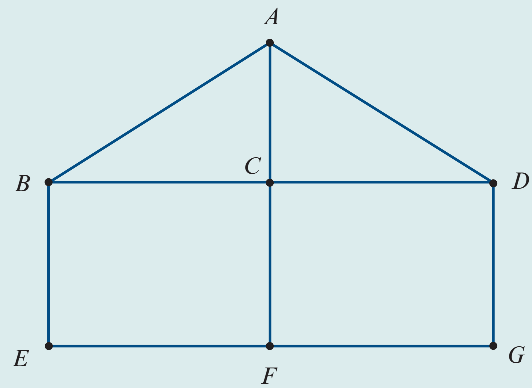
Example 8 Identifying types of walks

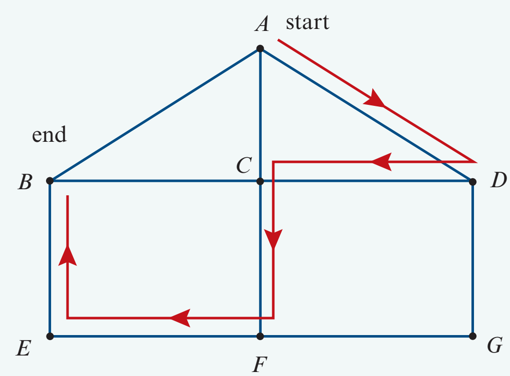
The graph opposite shows how seven vertices are connected. One walk is described as *A*-*D*-*C*-*F*-*E*-*B*. Does this walk represent a trail or a path?



Explanation

This walk starts at one vertex, ends at a different vertex, has no repeated edges and no repeated vertices, so it must be a path.

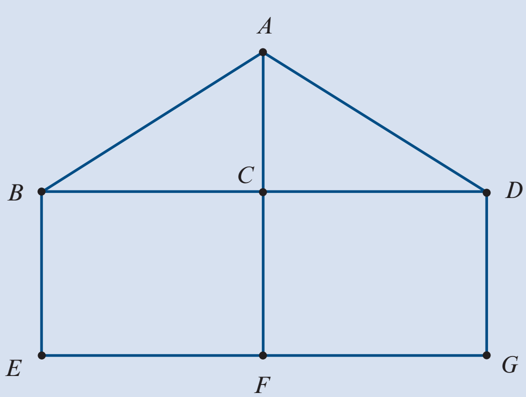
Solution



This walk is a path.

Now try this 8 Identifying types of walks (Example 8)

The graph below shows how seven vertices are connected. One walk is described as *B*-*E*-*F*-*C*-*B*-*A*. Does this walk represent a trail or a path?



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Circuits and cycles

There is nothing to stop a walk, trail or path starting and ending at the same vertex. When this happens, we say that the walk, trail or path is closed. Because *closed trails* and *closed paths* are so important in practice, we give them special names. We call them *circuits* and *cycles*.

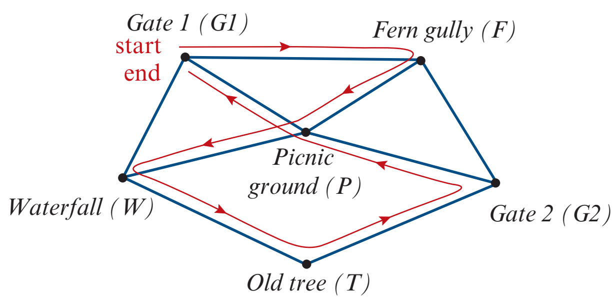
Circuit

A **circuit** is a walk that starts and ends at the same vertex and has no repeated edges.

A circuit in the forest

Using the forest track graph, an example of a *circuit* is:

G1-F-P-W-T-G2-P-G1



This circuit *starts* and *ends* at the *same* vertex (*G1*). The red arrows on the graph trace out this circuit.

There are *no repeated edges*.
However, the circuit passes through vertex *P* twice.

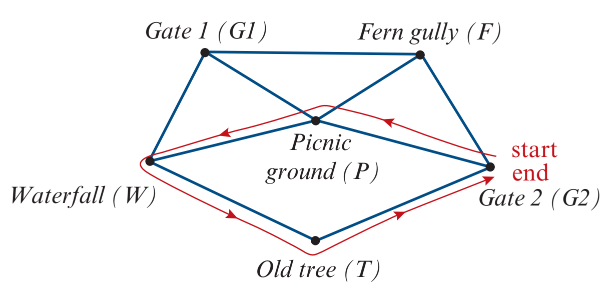
Cycle

A **cycle** is a walk that starts and ends at the same vertex, has no repeated edges and has no repeated vertices.

A cycle in the forest

Using the forest track graph, an example of a *cycle* is:

G2-P-W-T-G2



The red arrows on the graph trace out this cycle. Except for the first vertex (and last) there are no repeated edges or vertices in a cycle.

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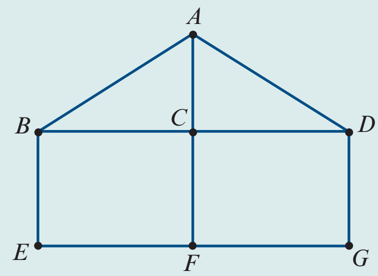
Example 9 Identifying types of walks

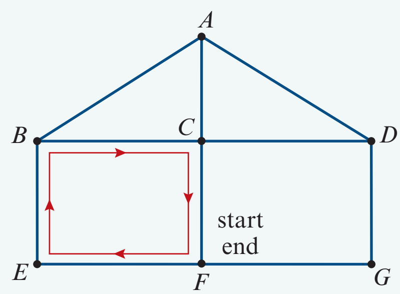
The graph opposite shows how seven vertices are connected. One walk is described as *F*-*E*-*B*-*C*-*F*. Does this walk represent a circuit or a cycle?



Explanation

This walk starts and ends at the same vertex, has no repeated edges and no repeated vertices, so it must be a cycle.

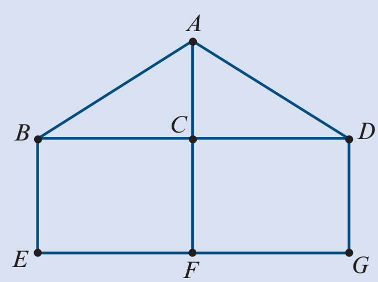
Solution



This walk is a cycle.

Now try this 9 Identifying types of walks (Example 9)

The graph opposite shows how seven vertices are connected. One walk is described as: *A*-*C*-*F*-*G*-*D*-*C*-*B*-*A*. Does this walk represent a circuit or a cycle?



Section Summary

- A **walk** is a sequence of edges linking successive vertices, that connects at least two different vertices in a graph.
- A **trail** is a walk with no repeated edges.
- A **path** is a walk with no repeated edges and no repeated vertices.
- A **circuit** is a walk that starts and ends at the same vertex and has no repeated edges.
- A **cycle** is a walk that starts and ends at the same vertex, has no repeated edges and no repeated vertices (except for the first/last vertex).

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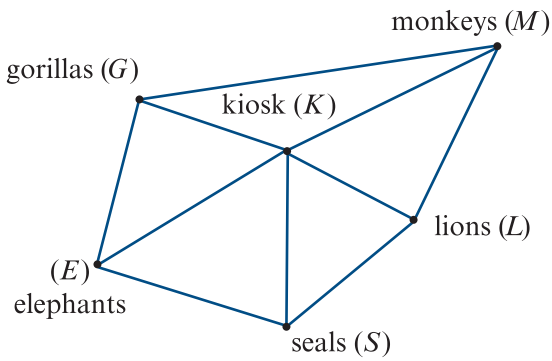
Skill-sheet

Exercise 8D

Building understanding

1 The graph below shows the pathways linking five animal enclosures in a zoo to each other and to the kiosk.



a Which of the following represents a trail in the graph?

- **i** S-L-K-M-K
- **ii** G-K-L-S-E-K-M
- **iii** E-K-L-K

b Which of the following represents a path in the graph?

i K-E-G-M-L **ii** E-K-L-M **iii** K-S-E-K-G-M

c Which of the following represents a circuit in the graph?

i K-E-G-M-K-L-K **ii** E-S-K-L-M-K-E **iii** K-S-E-K-G-K

d Which of the following represents a cycle in the graph?

i K-E-G-K **ii** G-K-M-L-K-G **iii** L-S-E-K-L

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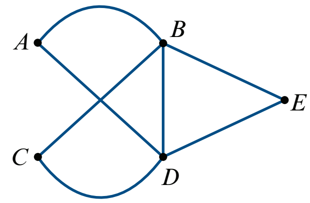
Developing understanding

Example 8

2 Using the graph opposite, identify the walks below as a trail, path, circuit, cycle or walk only.



a *A-D-B-A*

b *A-D-E-B-D-C-B-A*

c *C-B-E-D-B*

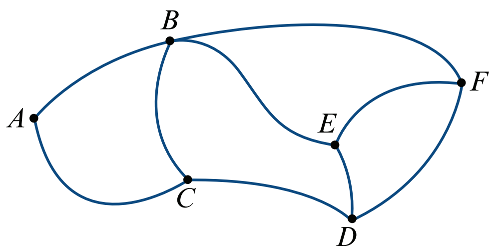
d *C-D-E-B-A*

e *D-C-B-D-E-B-A-D*

f *E-B-D-C*

Example 9

3 Using the graph opposite, identify the walks below as a trail, path, circuit, cycle or walk only.



a *A-B-E-F-B*

b *B-C-D-E-B*

c *C-D-E-F-B-A*

d *A-B-E-F-B-E-D*

e *E-F-D-C-B-E*

f *B-C-A-B-F-E-B*

Testing understanding

4 The road connections between different towns are represented below. For each:

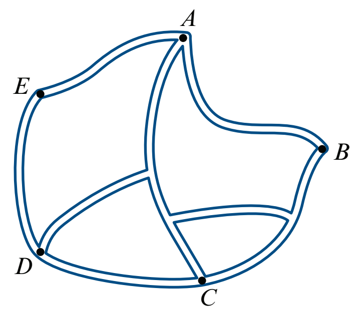
i How many different paths are there from Town *A* to Town *D*?

ii How many different cycles are there, starting at Town *A*?

a

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b



8E Eulerian trails and circuits (extension)

Learning intentions

- To be able to identify a walk as an Eulerian trail.
- To be able to identify a walk as an Eulerian circuit.
- To be able to use the degrees of the vertices to identify when an Eulerian trail or circuit is possible.

Trails and circuits that follow every edge of a graph without duplicating any edge are called **Eulerian trails** and **Eulerian circuits**; named after the pioneering work done by Euler.

Eulerian trail

An **Eulerian trail** follows every edge of a graph with no repeated edges.
An Eulerian trail will exist if the graph:

- is connected
- has exactly *zero or two* vertices that have an *odd degree*.

If there are zero odd vertices, the Eulerian trail can start at any vertex in the graph. If there are two odd vertices, the Eulerian trail will start at one of the odd vertices and finish at the other.

Eulerian circuit

An **Eulerian circuit** is an Eulerian trail that starts and finishes at the same vertex.
An Eulerian circuit will exist if the graph:

- is connected
- has vertices that *all* have an *even degree*.

An Eulerian circuit can start at *any* of the vertices.

The reason we are interested in Eulerian trails and circuits is because of their practical significance. If, for example, a graph shows towns as vertices and roads as edges, then being able to identify a route through the graph that follows every road can be important for mail delivery or for checking the conditions of the roads.

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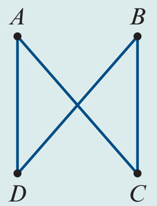
Example 10 **Identifying Eulerian trails and circuits**

For each of the following graphs:

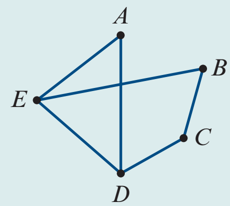
i Determine whether the graph has an Eulerian trail, an Eulerian circuit, both or neither, and state why.

ii If the graph has an Eulerian trail or an Eulerian circuit, show one example.

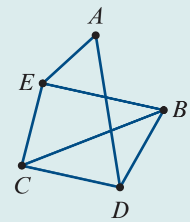
a



b



c



Solution

a Both: zero odd vertices.

A-C-B-D-A

b Eulerian trail: two odd vertices, the rest even.

E-A-D-E-B-C-D

c Neither: more than two odd vertices.

Note: In both cases, more than one solution is possible.

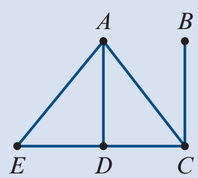
Now try this 10 **Identifying Eulerian trails and circuits (Example 10)**

For each of the following graphs:

i Determine whether the graph has an Eulerian trail, an Eulerian circuit, both or neither, and state why.

ii If the graph has an Eulerian trail or an Eulerian circuit, show one example.

a



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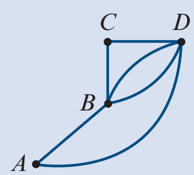
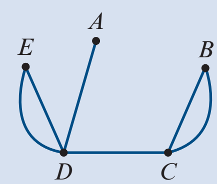


c



Applications of Eulerian trails and circuits

In everyday life, Eulerian trails and circuits relate to situations like the following:

- A postie wants to deliver mail without travelling along any street more than once.
- A visitor to a tourist park wants to minimise the distance they walk to see all of the attractions by not having to retrace their steps at any stage.
- A road inspector wants to inspect the roads linking several country towns without having to travel along each road more than once.

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Section Summary

Eulerian trail: a walk that follows every edge of a graph with no repeated edges.
An Eulerian trail will exist if the graph:

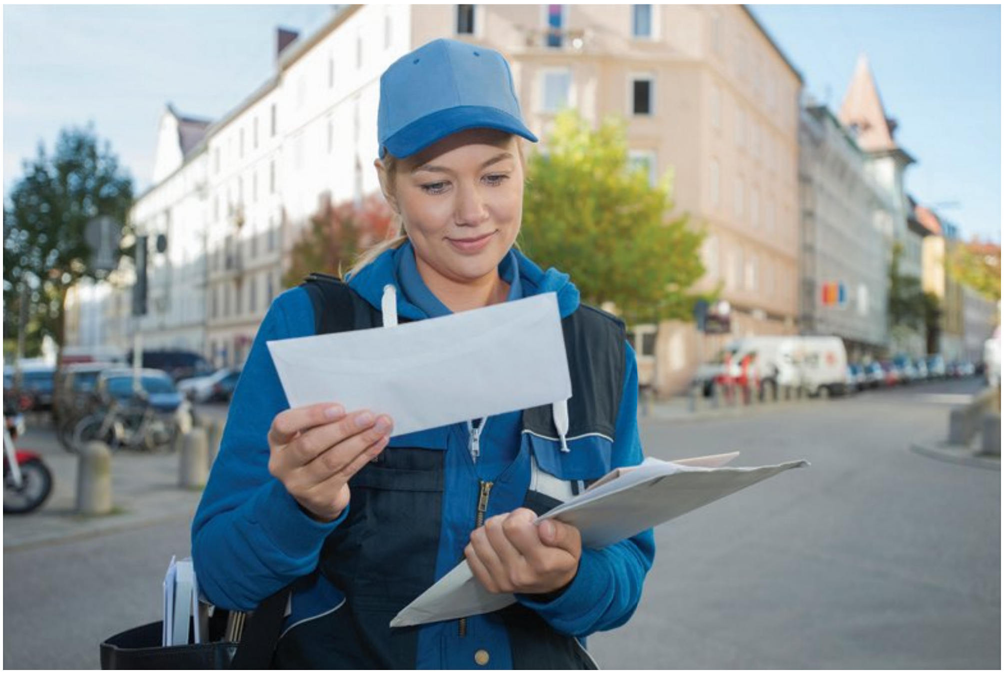
- is connected
- has exactly *zero or two* vertices that have an *odd degree*.

If there are zero odd vertices, the Eulerian trail can start at any vertex in the graph. If there are two odd vertices, the Eulerian trail will start at one of the odd vertices and finish at the other.

Eulerian circuit: an Eulerian trail that starts and finishes at the same vertex.
An Eulerian circuit will exist if the graph:

- is connected
- has vertices that *all* have an *even degree*.

An Eulerian circuit can start at *any* of the vertices.

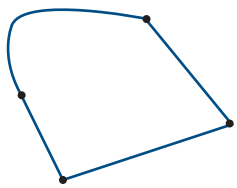


Exercise 8E

Building understanding

Example 10

1 For each of the following graphs, determine whether the graph has an Eulerian trail, an Eulerian circuit, both or neither, and state why.

a


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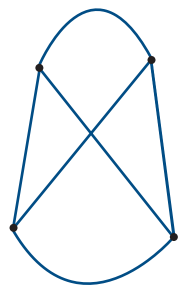

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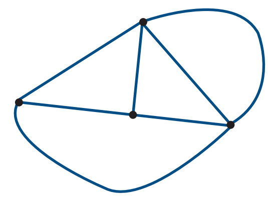
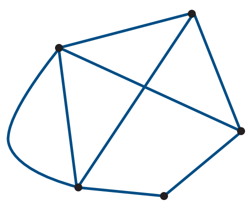


d


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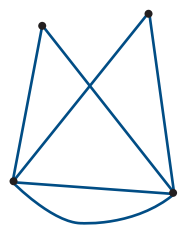

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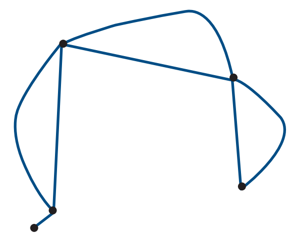

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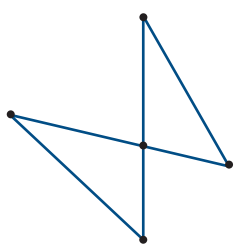

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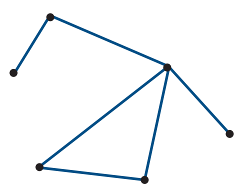
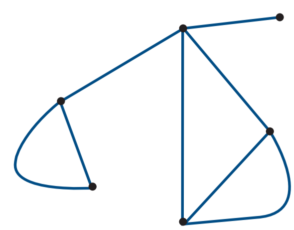
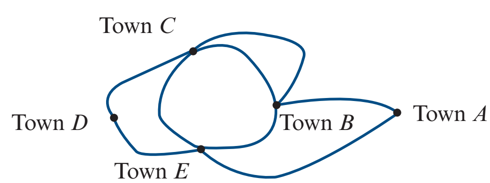


i


Developing understanding

2 A road inspector lives in Town *A* and is required to inspect all roads connecting the neighbouring towns, *B*, *C*, *D* and *E*. The network of roads is shown on the right.



a Is it possible for the inspector to set out from Town *A*, carry out their inspection by travelling over every road linking the five towns only once, and return to Town *A*? Explain.

b Show one possible route she can follow.

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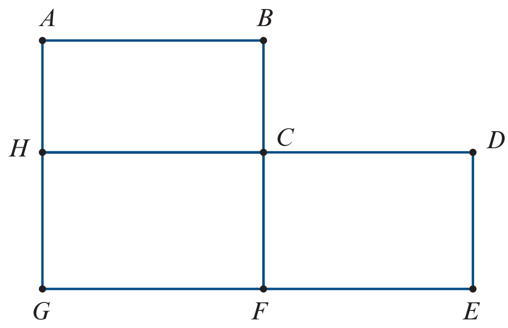
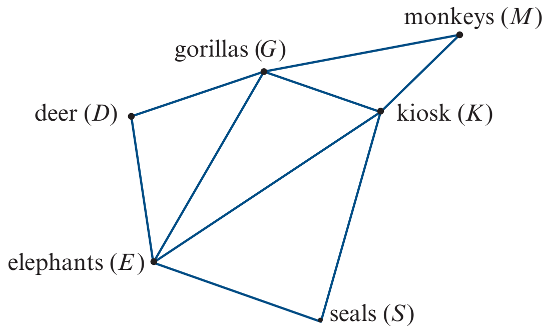
3 A postie has to deliver letters to the houses located on the graph of streets shown on the right.



a Is it possible for the postie to start and finish her deliveries at the same point in the graph without retracing her steps at some stage? If not, why not?

b It is possible for the postie to start and finish her deliveries at different points in the graph without retracing her steps at some stage. Identify one such route.

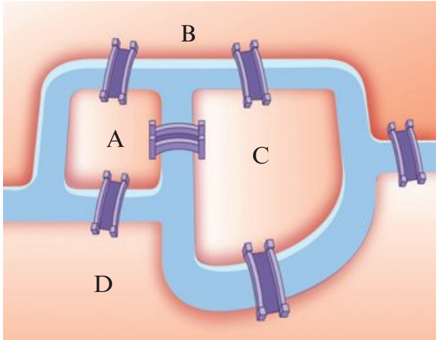
4 The graph below models the pathways linking five animal enclosures in a zoo to the kiosk and to each other.



a Is it possible for the zoo's street sweeper to follow a route that enables its operator to start and finish at the kiosk without travelling down any one pathway more than once? If so, explain why.

b If so, write down one such route.

5 Two islands are connected to the banks of a river by six bridges. See opposite.



a Draw a graph to represent this situation. Label the vertices: A, B, C and D to represent the riverbanks and the two islands. Use the edges of the graph to represent the bridges.

b It is not possible to plan a walking route that passes over each bridge once only. Why not?

c **i** Show where another bridge could be added to make such a walk possible.

ii Draw a graph to represent this situation.

iii Explain why it is now possible to find a walking route that passes over each bridge once only. Mark one such route on your graph.

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Testing understanding

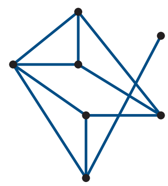
6 A graph has five vertices: A, B, C, D and E. The adjacency matrix for this graph is shown below.

a Does an Eulerian trail or Eulerian circuit exist? Give a reason for your answer.

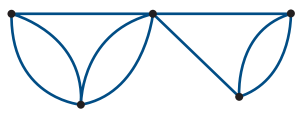
b If the element in row A, column C, changed to a zero, and the element in row C, column A, was also changed to a zero, would your answer to part **a** change? Justify your reasoning with a diagram and with reference to the degrees of the vertices.

	A	B	C	D	E
A	0	0	1	1	0
B	0	0	1	0	3
C	1	1	0	1	1
D	1	0	1	0	0
E	0	3	1	0	0

7 Consider the graph opposite. What is the minimum number of edges that must be added for an Eulerian circuit to exist?



8 An Eulerian trail for the graph opposite will be possible if only one edge is removed. In how many different ways could this be done?



8F Hamiltonian paths and cycles (extension)

Learning intentions

- To be able to identify a walk as a Hamiltonian path.
- To be able to identify a walk as a Hamiltonian cycle.

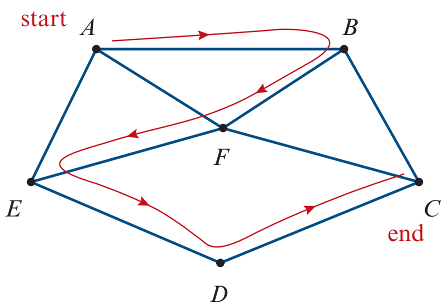
Eulerian trails and circuits focus on edges.

Hamiltonian paths and cycles focus on vertices.

Hamiltonian path

Hamiltonian path

A **Hamiltonian path** visits every vertex of a graph, with no repeated vertices.



For example, in the graph opposite, *A-B-F-E-D-C* is a Hamiltonian path. It starts at vertex A and ends at vertex C, visiting every vertex of the graph. (Follow the arrows.)

Note: A Hamiltonian path does not have to involve all edges.

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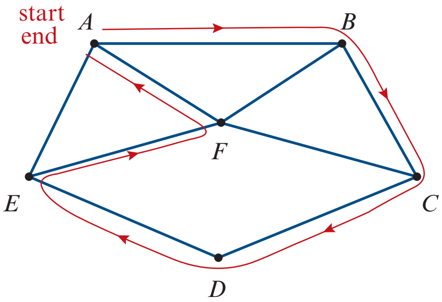
Hamiltonian cycle



Hamiltonian cycle

A **Hamiltonian cycle** visits every vertex of a graph with no repeated vertices, except for starting and finishing at the same vertex.

For example, in the second graph, *A-B-C-D-E-F-A* is a Hamiltonian cycle. It starts and finishes at vertex *A*, visiting every vertex of the graph. (Follow the arrows.)

Note: A Hamiltonian cycle does not have to involve all edges.

Unfortunately, unlike Eulerian trails and circuits, there are *no simple rules* for determining whether a network contains a Hamiltonian path or cycle. It is just a matter of 'trial and error'.

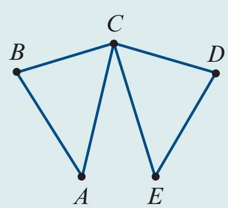
Example 11 Identifying Hamiltonian paths and cycles

For each of the following graphs:

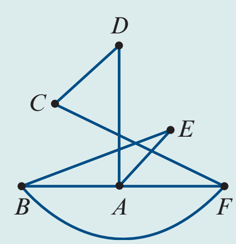
i Identify a Hamiltonian path, starting at vertex *A*.

ii State whether a Hamiltonian cycle is possible or not. If possible, identify one. If not possible, explain why not.

a



b



Solution

a **i** Hamiltonian path:

$A - B - C - D - E$

ii A Hamiltonian cycle is not possible. Whichever vertex you start at, you cannot return to the same vertex without repeating a vertex.

b **i** Hamiltonian path:

$A - E - B - F - C - D$

ii Yes, a Hamiltonian cycle is possible. One example:

$A - E - B - F - C - D - A$

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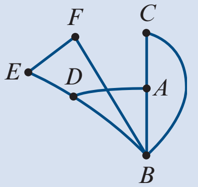
Now try this 11 **Identifying Hamiltonian paths and cycles (Example 11)**

For each of the following graphs:

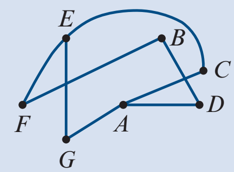
i Identify a Hamiltonian path, if possible, starting at vertex A.

ii State whether a Hamiltonian cycle is possible or not. If possible, identify one.

a



b



Applications of Hamiltonian paths and cycles

Hamiltonian paths and cycles have many practical applications. In everyday life, a Hamiltonian path would apply to situations like the following:

- You plan a trip from Melbourne to Mildura, with visits to Bendigo, Halls Gap, Horsham, Stawell and Ouyen on the way, but do not want to visit any town more than once.

Hamiltonian cycles relate to situations like the following:

- A courier leaves her depot to make a succession of deliveries to a variety of locations before returning to her depot. She does not like to go past each location more than once.
- A tourist plans to visit all of the historic sites in a city without visiting each more than once.
- You are planning a trip from Melbourne to visit Shepparton, Wodonga, Bendigo, Swan Hill, Natimuk, Warrnambool and Geelong before returning to Melbourne. You don't want to visit any town more than once.

In all these situations, there would be several suitable routes. However other factors, such as time taken or distance travelled, may need to be taken into account in order to determine the best route. This is an issue addressed in the next section: weighted graphs and networks.

Section Summary

- **Hamiltonian path**: a walk that involves every vertex of a graph with no repeated vertices.
- **Hamiltonian cycle**: a walk that involves every vertex of a graph with no repeated vertices, except for starting and ending at the same vertex.
- Both Hamiltonian paths and cycles can only be identified by inspection.

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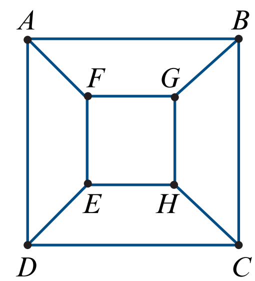
Exercise 8F

Building understanding

1 List a Hamiltonian path for the graph shown.

a Starting at A and finishing at D

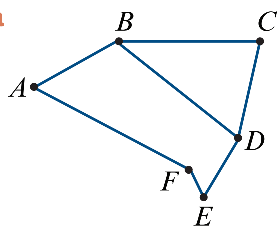
b Starting at F and finishing at G



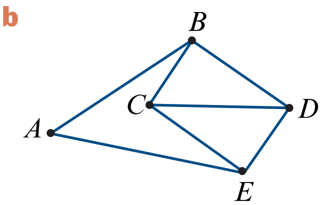
Example 11

2 Identify a Hamiltonian cycle in each of the following graphs (if possible), starting at A each time.

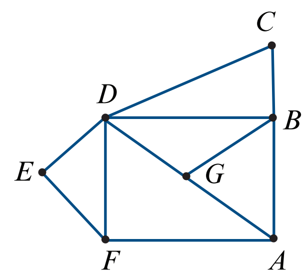
a



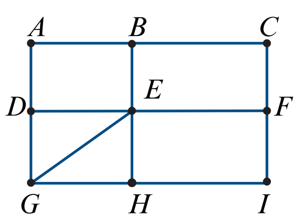
b



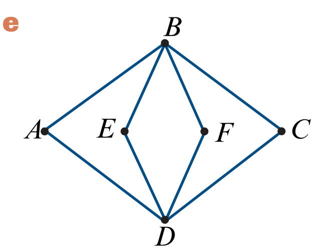
c



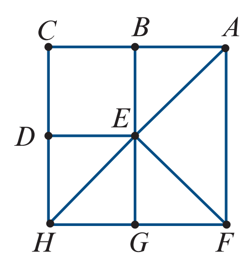
d



e

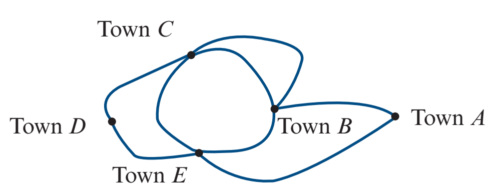


f



Developing understanding

3 A tourist wants to visit a second-hand bookshop in each of five different towns: Apsley (A), Berrigama (B), Cleverland (C), Donsley (D) and Everton (E).



The graph of roads connecting the towns is shown above.
Can a tourist start a tour that visits each town only once by starting at:

a Cleverland and finishing at Everton? If so, identify one possible route and give its mathematical name.

b Cleverland and finishing at Apsley? If so, identify one possible route and give its mathematical name.

c Everton and finishing at Everton? If so, identify one possible route and give its mathematical name.

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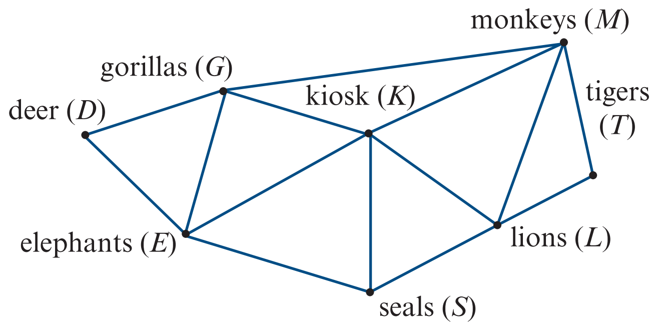
4 The graph opposite models the pathways linking seven animal enclosures in a zoo to the kiosk and to each other.



a Is it possible for a visitor to the zoo to start their visit at the kiosk and see all of the animals without visiting any one animal enclosure more than once? If so, identify a possible route, and give this route its mathematical name.

b Is it possible for a visitor to the zoo to start their visit at the deer enclosure and finish at the kiosk without visiting the kiosk or any enclosure more than once? If so, identify a possible route and give this route its mathematical name.

Testing understanding

5 The opposite adjacency matrix represents a graph.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \end{array}$$

a How many faces does this graph have?

b State the two vertices connected by the bridge in this graph.

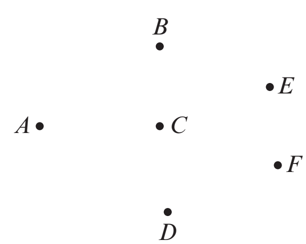
c Does this graph contain an Eulerian trail, Eulerian circuit or neither?

d Does this graph contain a Hamiltonian path, Hamiltonian cycle or neither?

6 What is the minimum number of edges that must be added to the following set of vertices for:

a a Hamiltonian path to exist

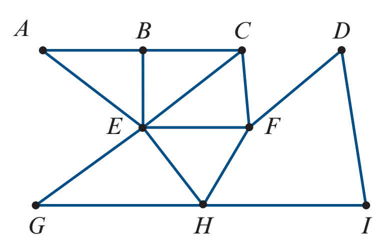
b a Hamiltonian cycle to exist.



7 What is the maximum number of edges that can be removed from the graph opposite for:

a a Hamiltonian path to still exist

b a Hamiltonian cycle to still exist.



8 **a** Do all connected graphs contain a Hamiltonian path? Explain your reasoning.

b Do all connected graphs contain a Hamiltonian cycle? Justify your reasoning with reference to the degree of vertices.

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8G Weighted graphs, networks and the shortest path problem

Learning intentions

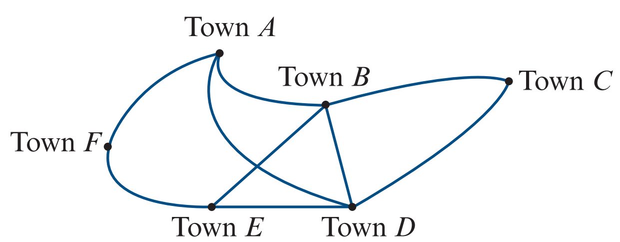
- To be able to analyse a weighted graph.
- To be able to represent a real-world situation using a network.
- To be able to find the shortest path between two vertices for a network.

Weighted graphs and networks

The graph opposite shows how six towns are connected by road.

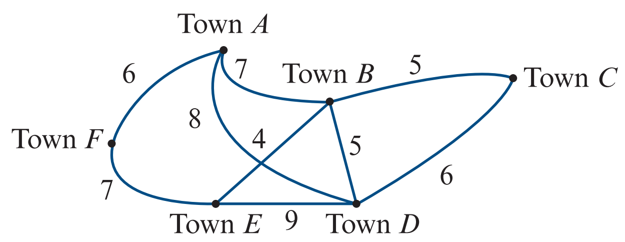
The towns are represented by the vertices of the graph.

The roads between towns are represented by the edges.



We can give more information about the situation we are representing with the graph by adding numbers to the edges.

The weighted graph opposite shows the distances between the towns (in kilometres).



Weighted graph

A **weighted graph** is a graph that has a number associated with each edge.

The weighted graph above could be called a network.

Network

A **network** is a weighted graph in which the weights are physical quantities, for example, distance, time or cost.

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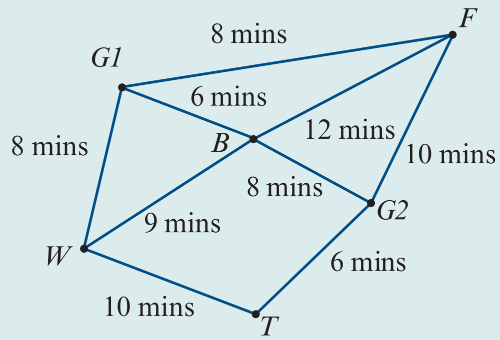
Example 12 Interpreting a network

The network opposite is used to model the tracks in a forest connecting a suspension bridge (B), a waterfall (W), a very old tree (T) and a fern gully (F).

Walkers can enter or leave the forest through either Gate 1 ($G1$) or Gate 2 ($G2$).

The numbers on the edges represent the times (in minutes) taken to walk directly between these places.



a How long does it take to walk from the bridge directly to the fern gully?

b How long does it take to walk from the old tree to the fern gully via the waterfall and the bridge?

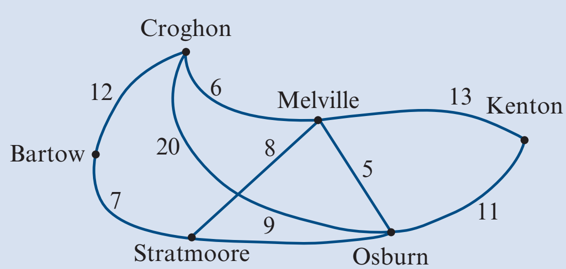
Explanation	Solution
a Identify the edge that directly links the bridge with the fern gully, and read off the time.	The edge is B-F. The time taken is 12 minutes.
b Identify the path that links the old tree to the fern gully, visiting the waterfall and the bridge on the way. Add up the times.	The path is T-W-B-F. The time taken is $10 + 9 + 12 = 31$ minutes.

Now try this 12 Interpreting a network (Example 12)

The weighted graph in the diagram below shows towns, represented by vertices, and the roads between those towns, represented by edges. The numbers represent the lengths of each road, in kilometres.

a How far is it from Stratmoore directly to Osburn?

b How far is it from Stratmoore to Kenton, via Melville then Osburn?



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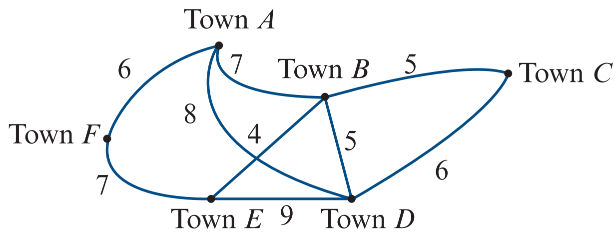
The shortest path problem

Another question we might have when presented with a road network like the one shown is, 'What is the shortest distance between certain towns?'

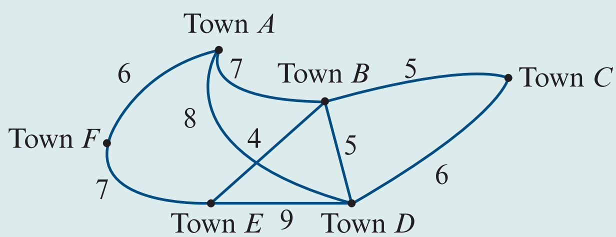


While this question is easily answered if all of the towns are directly connected by a road, for example, Town A and Town B, the answer is not so obvious if we have to travel through other towns to get there, for example, Town F and Town C.

While there are sophisticated techniques for solving the shortest path problem (which are presented in Units 3 and 4), we will identify and compare the lengths of the likely candidates for the shortest path by inspection.

Example 13 **Finding the shortest path by inspection**

Find the shortest route between Town C and Town F in the network shown opposite.



Explanation	Solution
1 Identify all of the likely shortest routes between Town C and Town F and calculate their lengths.	C-D-E-F: The distance is $6 + 9 + 7 = 22$ km. C-B-E-F: The distance is $5 + 4 + 7 = 16$ km. C-B-A-F: The distance is $5 + 7 + 6 = 18$ km.

Note: In theory, when using the 'by inspection' method to solve this problem, we need to list all possible routes between Town C and Town F and determine their lengths. However, we can save time by eliminating any route that passes through any town more than once or uses any road more than once. We can also eliminate any route that 'takes the long way around' rather than using the direct route, for example, when travelling from Town B to Town D, we can ignore the route that goes via Town A because it is longer.

2 Compare the different path lengths to identify the shortest path and write your answer. Note: You should compare the lengths of all likely paths because there can be more than one shortest path in a network.	The shortest path is C-B-E-F.

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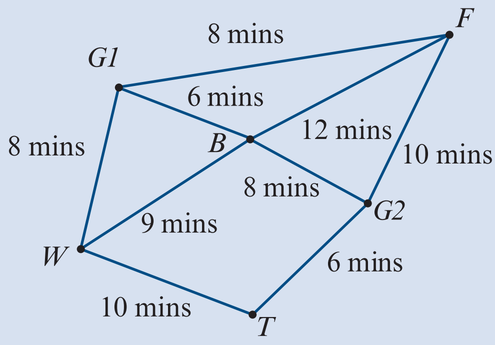
Now try this 13 Finding the shortest path by inspection (Example 13)

Find the shortest route between the waterfall (W) and the fern gully (F) in the network shown on the right.



Section Summary

- A **weighted graph** is a graph where a number is associated with each edge. These numbers are called weights.
- A **network** is a weighted graph in which the weights are physical quantities, for example, distance, time and cost.
- Determining the shortest distance or time or the least cost to move around a network is called the **shortest path problem**.

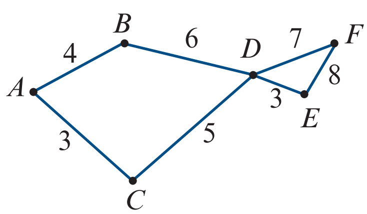
Skill-sheet

Exercise 8G

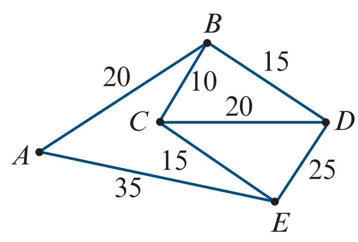
Building understanding

Example 13

1 Find the shortest path from vertex A to vertex E in this network. The numbers represent time, in hours.



2 Find the shortest path from vertex A to vertex D in this network. The numbers represent lengths, in metres.



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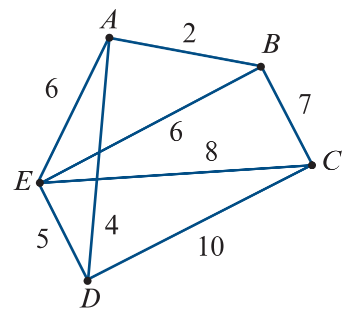
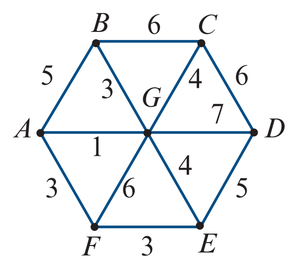
3 Find the shortest path from vertex B to vertex D in this network. The numbers represent cost, in dollars.



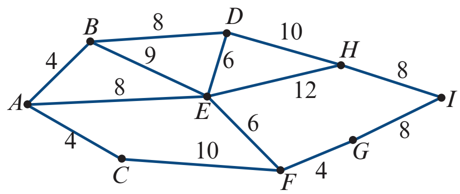
4 Find the shortest path from vertex B to vertex F in this network. The numbers represent time, in minutes.



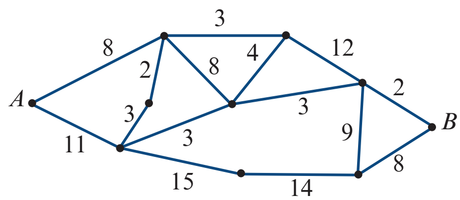
Developing understanding

Example 12

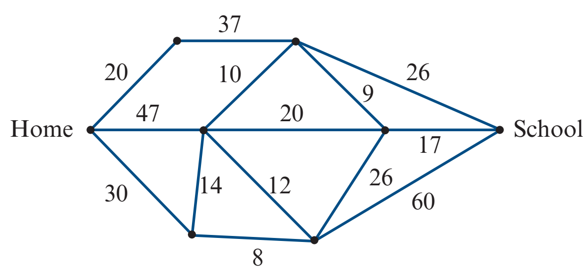
5 The network opposite shows the distance, in kilometres, along walkways that connect the landmarks A, B, C, D, E, F, G, H and I in a national park. Find the shortest path from A to I.



6 In the network opposite, the vertices represent small towns and the edges represent roads. The weights on the edges indicate the distance, in km, between towns. Determine the length of the shortest path between towns A and B.



7 Victoria rides her bike to school each day. The edges of the network opposite represent the roads that Victoria can use to ride to school. The numbers on the edges give the time taken, in minutes, to travel along each road. What is the shortest time that Victoria can ride between home and school?



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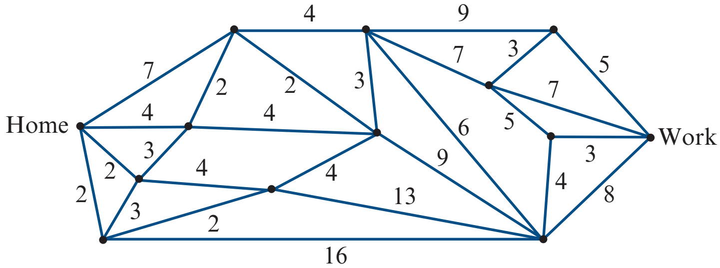
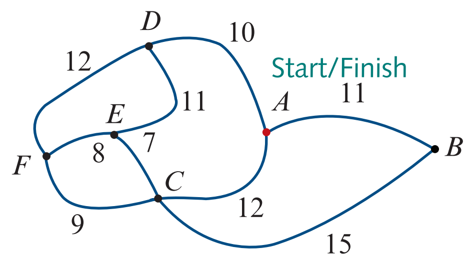
8 Renee drives to work each day. The edges of the network below represent the roads that Renee can use to drive to work. The numbers on the edges give the time, in minutes, to travel along each road. What is the shortest time that Renee can drive between home and work?



9 The graph opposite shows a mountain bike rally course. Competitors must pass through each of the checkpoints, A, B, C, D, E and F. The average times, in minutes, taken to ride between the checkpoints are shown on the edges of the graph.

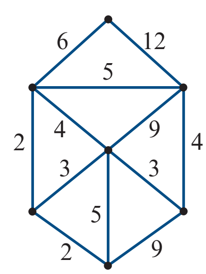


Competitors must start and finish at checkpoint A but can pass through the other checkpoints in any order they wish.
Which route would have the shortest average completion time?

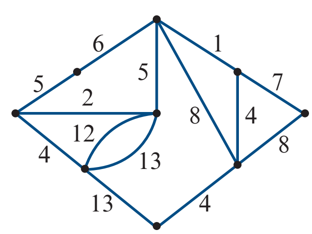
Testing understanding

10 For each of these graphs, determine the length of the shortest Hamiltonian path.

a



b



11 Is an Eulerian trail possible for each of the graphs above? Why is it unnecessary to determine the length of the shortest Eulerian trail?

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8H Minimum spanning trees and greedy algorithms

Learning intentions

- To be able to identify a tree.
- To be able to find a spanning tree for a graph.
- To be able to find the minimum spanning tree for a weighted graph using greedy algorithms such as Prim's or Kruskal's.
- To be able to apply a greedy algorithm to find the shortest path in a network from one vertex to another.

In the previous applications of networks, the weights on the edges of the graph were used to determine a minimum weight pathway through the graph from one vertex to another. In other applications, it is more important to minimise the number and weights of the edges in order to keep all vertices connected to the graph. For example, a number of towns might need to be connected to a water supply. The cost of connecting the towns can be minimised by connecting each town into a network of water pipes only once, rather than connecting each town to every other town. Problems of this type can be solved with the use of a **tree**.

Trees

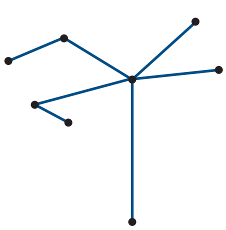
Tree

A **tree** is a connected graph that contains no cycles, multiple edges or loops.

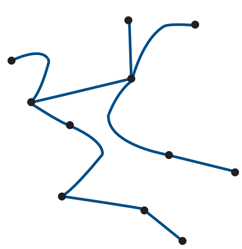
A tree may be part of a larger graph.

If a tree has n vertices, it will have $n - 1$ edges.

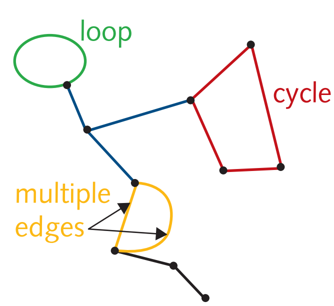
For example, Graphs 1 and 2 below are examples of trees. Graph 3 is *not* a tree.



Graph 1: a tree



Graph 2: a tree



Graph 3: not a tree

Graphs 1 and 2 are trees: they are connected and have no cycles, multiple edges or loops.

Graph 3 is *not* a tree because it has several cycles (loops and multiple edges count as cycles).

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For trees, there is a relationship between the number of vertices and the number of edges.

- In Graph 1, the tree has 8 vertices and 7 edges.
- In Graph 2, the tree has 11 vertices and 10 edges.

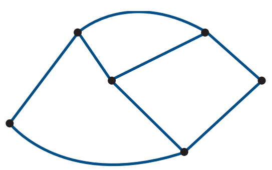
An inspection of other trees would show that, in general, the number of edges in a tree is one less than the number of vertices. A tree with n vertices has $n - 1$ edges.

Spanning trees

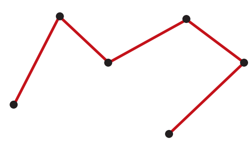
Spanning tree

A **spanning tree** is a tree that connects *all* of the vertices in a connected graph.

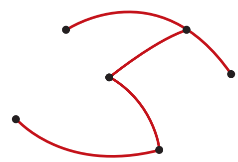
For example, Graphs 2 and 3 below are different spanning trees of Graph 1. Note that the spanning trees connect all of the vertices from Graph 1; each spanning tree has 6 vertices and 5 edges.



Graph 1



Graph 2



Graph 3

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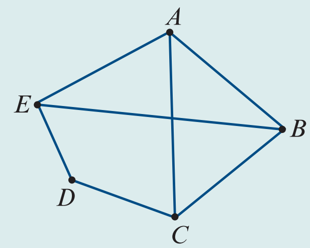
Example 14 Finding a spanning tree in a graph

Find two spanning trees for the graph shown opposite.



Explanation

1 The graph has five vertices and seven edges. A spanning tree will have five (n) vertices and four ($n - 1$) edges.

2 To form a spanning tree, remove any *three* edges, provided that:

- all the vertices remain connected
- there are no multiple edges or loops.

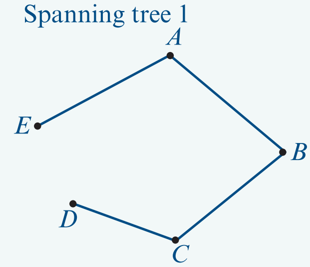
Spanning tree 1 is formed by removing edges *EB, ED* and *CA*.

Spanning tree 2 is formed by removing edges *EA, AC* and *CD*.

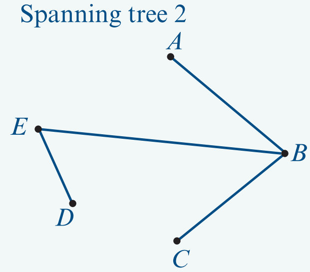
Note: Several other possibilities exist.

Solution

Spanning tree 1

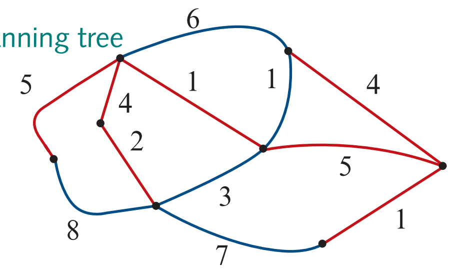


Spanning tree 2



Minimum spanning trees

For weighted graphs or networks, it is possible to determine the 'length' of each spanning tree by adding up the weights of the edges in the tree.



For the spanning tree (highlighted in red) opposite:

$$\text{Length} = 5 + 4 + 2 + 1 + 5 + 4 + 1$$
$$= 22 \text{ units}$$

Minimum spanning tree

A **minimum spanning tree** is a spanning tree of minimum length. There may be more than one.

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A minimum spanning tree may represent the minimum distance, minimum time, minimum cost, etc. There may be more than one minimum spanning tree in a weighted graph.

Minimum spanning trees have many real-world applications, such as planning the layout of a computer network or a water supply system for a new housing estate. In these situations, we might want to minimise the amount of cable or water pipe needed for the job. Alternatively, we might want to minimise the time needed to complete the job or its cost.

Algorithmic methods for determining minimum spanning trees

To date, we have used inspection to identify minimum spanning trees in a weighted graph. While this is practical for simple weighted graphs, it is less appropriate when solving practical problems which are likely to involve more complex weighted graphs. A more systematic or algorithmic approach is needed.

An algorithm is a set of instructions or rules to follow for solving a problem or performing a complex task. In the everyday world, a recipe for baking a cake is an algorithm. In arithmetic, it could be the series of steps we can follow to reliably perform a long division between two multi-digit numbers by hand.

In the world of graphs and networks, an algorithm is a series of steps you can follow to enable you to evaluate some property of a graph or network. In this topic, we are interested in algorithms that will enable us to identify a minimum spanning tree in a network and determine its length in a systematic and routine manner.

There are several algorithms that have been developed for this purpose. We will consider two: Prim's algorithm and Kruskal's algorithm, which are the most used algorithms for this task.

Prim's algorithm

Prim's algorithm for finding a minimum spanning tree

Prim's algorithm is a set of rules to determine a minimum spanning tree for a graph.

1. Choose a starting vertex (any will do).
2. Inspect the edges, starting from this vertex, and choose the one with the lowest weight. (If there are two edges that have the same weight, it does not matter which one you choose.) The starting vertex, the edge and the vertex it connects to form the beginning of the minimum spanning tree.
3. Next, inspect all the edges, starting from both of the vertices you have in the tree so far. Choose the edge with the lowest weight, ignoring edges that would connect the tree back to itself. The vertices and edges you already have, plus the extra edge and vertex it connects to, form the minimum spanning tree so far.
4. Repeat the process until all the vertices are connected.

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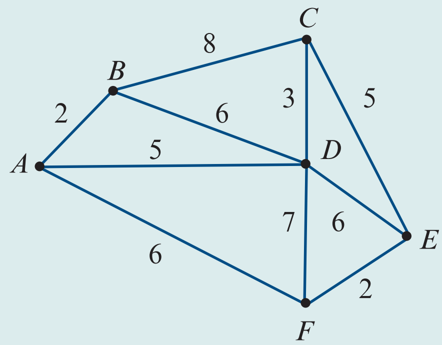
Example 15 Finding the minimum spanning tree by applying Prim's algorithm

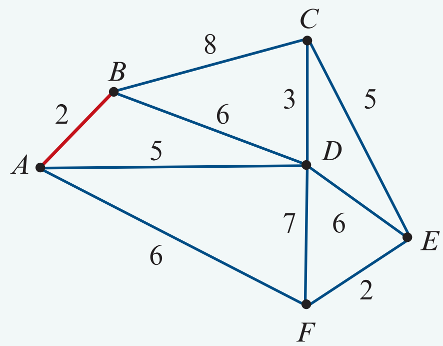
Apply Prim's algorithm to find the minimum spanning tree for the graph shown on the right. Write down the total weight of the minimum spanning tree.



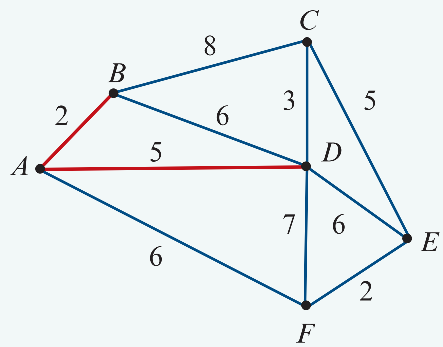
Explanation

1 Start with vertex A.
The smallest weighted edge from vertex A is to B, with weight 2.

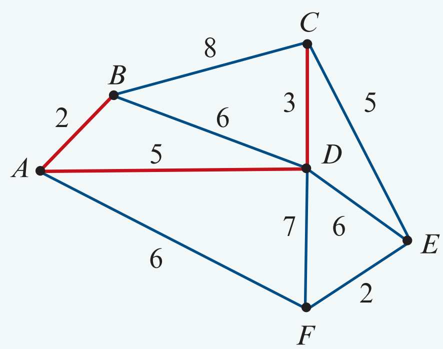
Solution



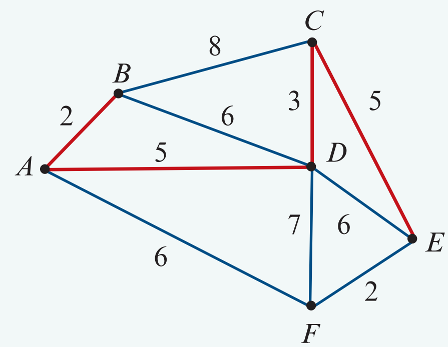
2 Look at vertices A and B. The smallest weighted edge from either vertex A or vertex B is from A to D, with weight 5.



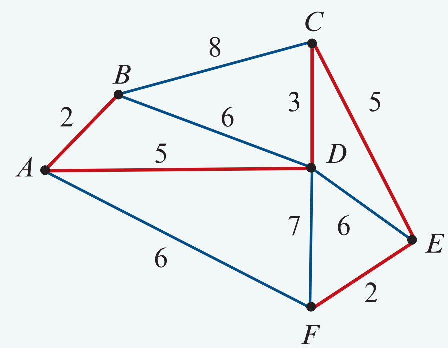
3 Look at vertices A, B and D. The smallest weighted edge from vertix A, B or D is from D to C, with weight 3.



4 Look at vertices A, B, D and C. The smallest weighted edge from vertex A, B, D or C is from C to E, with weight 5.



5 Look at vertices A, B, D, C and E. The smallest weighted edge from either vertex A, B, D, C or E is from E to F, with weight 2.
All vertices have been included in the graph. This is the minimum spanning tree.

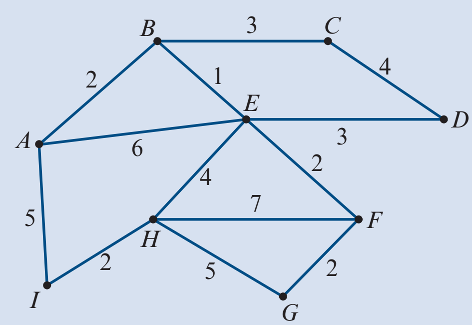


6 Add the weights to find the total weight of the minimum spanning tree.

The total weight of the minimum spanning tree = 2 + 5 + 3 + 5 + 2 = 17.

Now try this 15 **Finding the minimum spanning tree by applying Prim's algorithm (Examples 14 and 15)**

Apply Prim's algorithm to obtain a minimum spanning tree for the graph shown, and calculate its length.



Kruskal's algorithm

Kruskal's algorithm is another algorithm that can be used to determine a minimum spanning tree for a network.
Prim's algorithm starts from a vertex and builds up the spanning tree step-by-step from one vertex to an adjacent vertex, to eventually form a minimum spanning tree for the network.
By contrast, Kruskal's algorithm sorts all the edges from lowest to highest weight and keeps adding the edges of the same or next lowest weight (that do not form cycles) until all the vertices have been covered.

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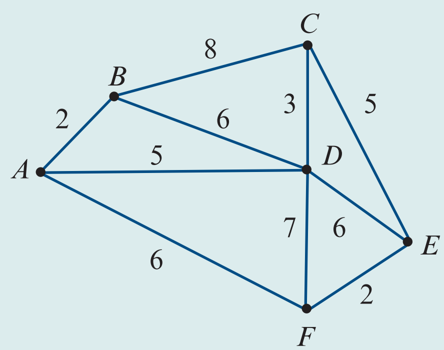
Kruskal's algorithm for finding a minimum spanning tree

1. Choose the edge with the least weight as the starting edge. If there is more than one least-weight edge, any will do.
2. Next, from the remaining edges, choose an edge of least weight which does not form a cycle. If there is more than one least-weight edge, any will do.
3. Repeat the process until all vertices are connected. The result is a minimum spanning tree.
4. Determine the length of the spanning tree by summing the weights of the chosen vertices.

Example 16 Kruskal's algorithm for determining a minimum spanning tree

Apply Kruskal's algorithm to obtain a minimum spanning tree for the graph shown, and calculate its length.

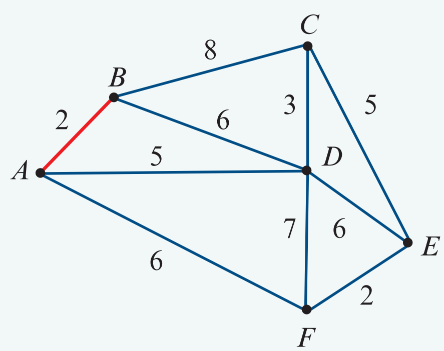


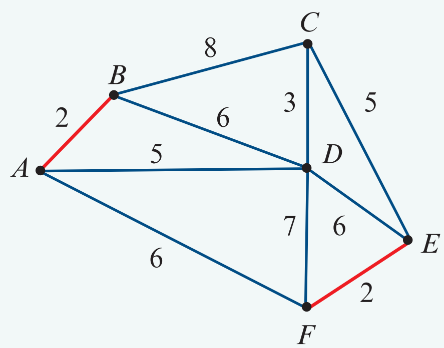
Explanation

1. Choose the edge with the least weight. If there is more than one, it doesn't matter which you choose. There are two edges of weight 2. We will choose edge AB. Draw it in as shown in red, opposite.

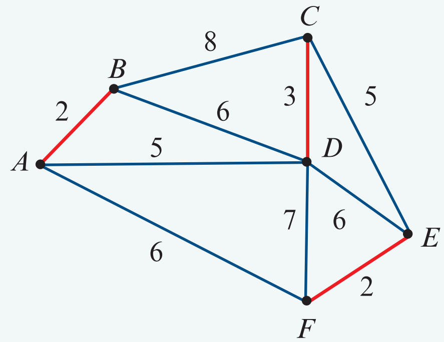
2. From the remaining edges, choose the edge with the least weight. There is one, edge FE, with weight 2. Draw FE in, as shown opposite.

Solution

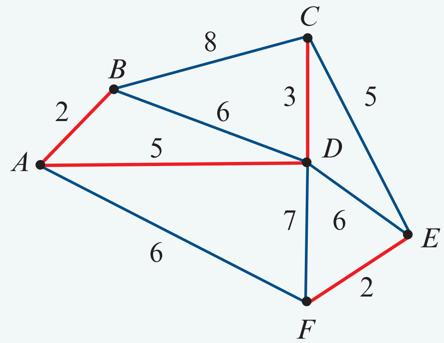




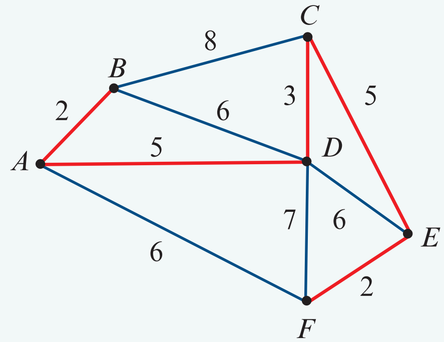
3 Continuing the process, the next edge we choose is CD, with a weight of 3. Draw CD in, as shown opposite.



4 Continuing the process, there are two edges of weight 5. Neither form a cycle so it does not matter which of these we choose. We will choose edge AD. Draw AD in, as shown opposite.



5 We can now add in the remaining edge of weight 5, CE. Draw CE in, as shown opposite.
All vertices have now been joined and a minimum spanning tree has been determined.

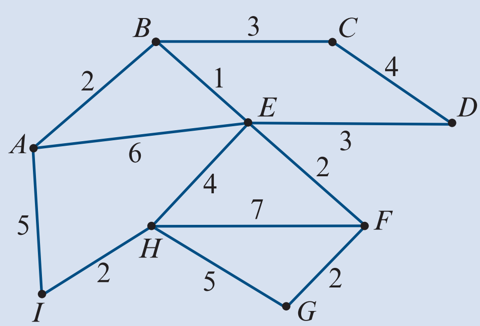


6 Find the length of the minimum spanning tree by adding the weights of the chosen edges.

Minimum spanning tree
Length = $2 + 5 + 3 + 5 + 2 = 17$

Now try this 16 **Kruskal's algorithm for determining a minimum spanning tree (Example 16)**

Apply Kruskal's algorithm to obtain a minimum spanning tree for the graph shown, and calculate its length



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Greedy algorithms

Prim's and Kruskal's algorithms are examples of greedy algorithms.

What is a greedy algorithm?

A **greedy algorithm** is a simple, intuitive set of rules that can be used to solve optimisation problems. It breaks up the solution of the optimisation problem into a series of simple steps that find the optimum solution at each step in the process. The expectation is that finding the optimum solution at each step in the solution will lead to the optimum solution for the entire problem.

Because a greedy algorithm must consider all the directly available information at each step in the process, it can be very time consuming compared to some alternative but less intuitive methods that do not have this requirement. This is particularly true when dealing with large and complex networks, but these situations are beyond the scope of this course.

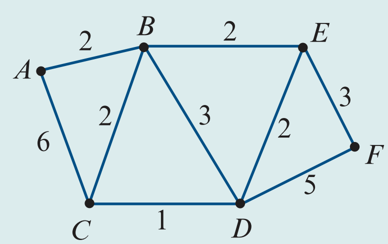
Another problem with using greedy algorithms to solve optimisation problems is that the methodology does not always lead to the optimal solution for all optimisation problems. However, this is not the case for Prim's and Kruskal's algorithms. A greedy algorithm can be used to identify the shortest path from one vertex to another.

Dijkstra's greedy algorithm for finding the shortest path (Extension)

Next year, you will learn to use Dijkstra's algorithm to find the shortest path between two points in a network. It is also a greedy algorithm.

Example 17 **Dijkstra's algorithm for determining the shortest path**

Find the shortest path from A to F in the weighted graph shown on the right.

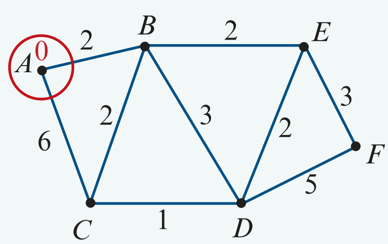


Explanation

1 Assign the starting vertex a zero, and circle the vertex and its new value of zero.

Solution

A is the starting vertex; it is assigned zero and it is circled.



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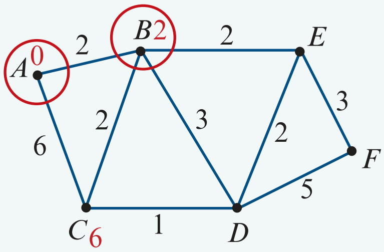
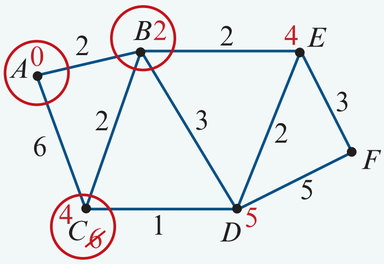
2 Assign a value to each vertex connected to the starting vertex. The value assigned is the length of the edge connecting it to the starting vertex. Circle the vertex with the lowest assigned value.

The starting vertex, *A*, is connected to vertices *B* and *C*. The vertex *B* is assigned 2 and the vertex *C* is assigned 6. Vertex *B* is circled because it has the lowest value.



3 From the newly circled vertex, assign a value to each vertex connected to it by *adding* the value of each connecting edge to the newly circled vertex's value. If a connecting vertex already has a value assigned to it, and the new value is less than it, replace it with the new value. If a vertex is circled, it cannot have its value changed. Consider all uncircled vertices and circle the one with the lowest value.

The newly circled vertex, *B*, is connected to three vertices; *C, D* and *E*. Starting with vertex *B*'s value of 2, *E* is assigned 4 (adding 2 from the connecting edge) and *D* is assigned 5 (adding 3 from the connecting edge). The vertex *C* will be re-assigned 4 (adding 2 from the connecting edge) because it is lower than 6. Now there are two uncircled vertices with the lowest assigned value of 4, vertices *C* and *E*; it **does not** matter which one you circle.



Continued

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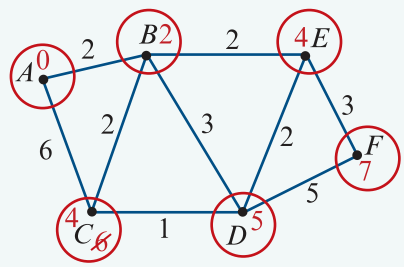
4 Repeat Step 3 until the destination vertex and its assigned value are circled. The length of the shortest path will be the assigned value of the destination vertex. The shortest path is found by backtracking. Starting at the destination vertex, move to the circled vertex whose value is equal to the destination vertex's assigned value minus the connecting edge value. Continue to minus the connecting edge value from one circled vertex to the next until you reach the starting vertex.

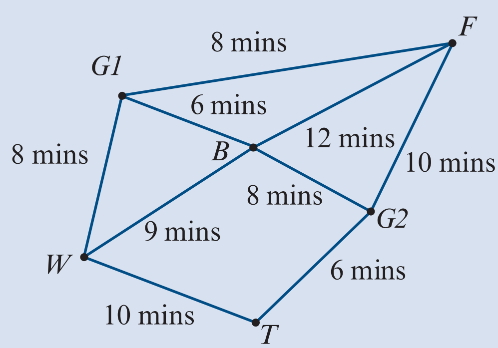
Note: Once a vertex is assigned a value, it cannot be assigned a larger value, even if it has not been circled yet. You do not need to circle all vertices. Stop when the destination vertex is circled.

Vertex F is the destination vertex, assigned a value of 7. Therefore the shortest path from A to F has a length of 7. To find the shortest path, start at F and consider the two connecting edges to it. The edge of length 3 is correct because 7 minus 3 equals 4, the value of vertex E. Likewise, minus the connecting edge of 2 to vertex B to equal 2, then minus the 2 to the connecting edge with A to equal zero. Therefore, the shortest path from A to F is: A-B-E-F, with a length of 7.



Now try this 17 Dijkstra's algorithm for determining the shortest path (Example 17)

Find the shortest route between the waterfall *(W)* and the fern gully *(F)* in the network shown on the right.



Section Summary

- A **tree** is a connected graph that contains no cycles, multiple edges or loops. A tree may be part of a larger graph.
- A tree with n vertices has $n - 1$ edges.
- A **spanning tree** is a tree that connects *all* of the vertices in a connected graph.
- A **minimum spanning tree** is the spanning tree of *minimum* length.
- **Prim's and Kruskal's algorithms** are different types of **greedy algorithms.** They give a set of rules to determine a minimum spanning tree for a graph.

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Skill-sheet

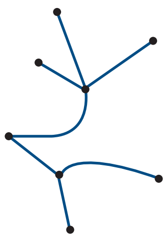
Exercise 8H

Building understanding

1 **a** How many edges are there in a tree with 15 vertices?
b How many vertices are there in a tree with 5 edges?
c Draw two different trees with four vertices.
d Draw three different trees with five vertices.

2 A connected graph has eight vertices and ten edges. Its spanning tree has ______ vertices and ______ edges.

3 Which of the following graphs are trees?

A

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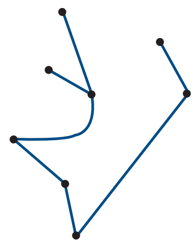
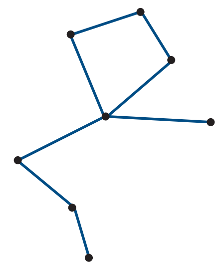
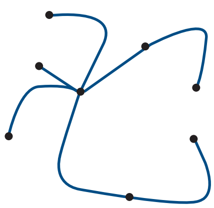
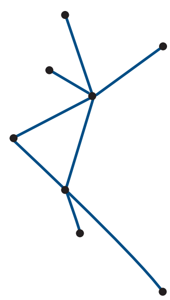
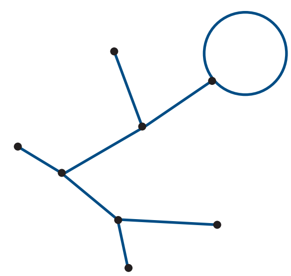
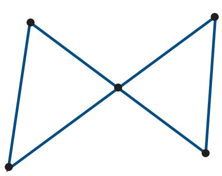

C

D

E

F


Example 14

4 For each of the following graphs, draw three different spanning trees.

a

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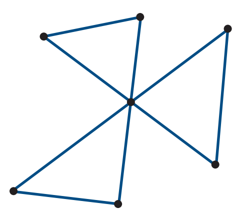
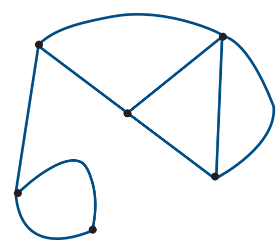

c


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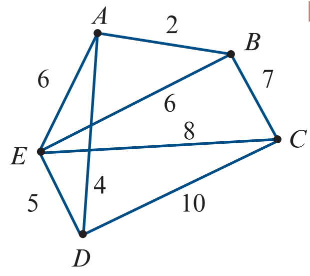
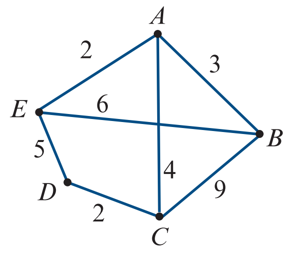
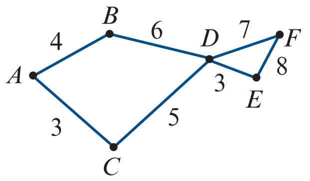
Developing understanding

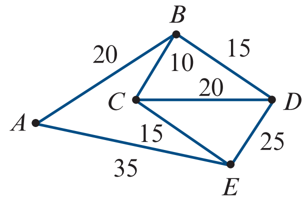
Example 15 **5** For each of the following connected graphs, use Prim's algorithm to determine the minimum spanning tree and its length.

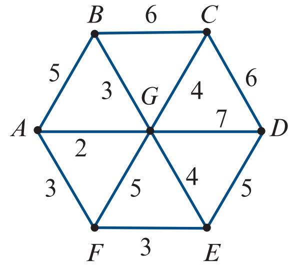
a


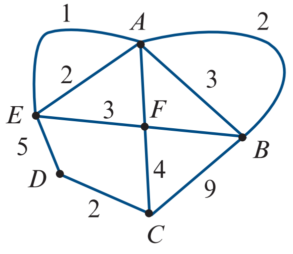
b


c


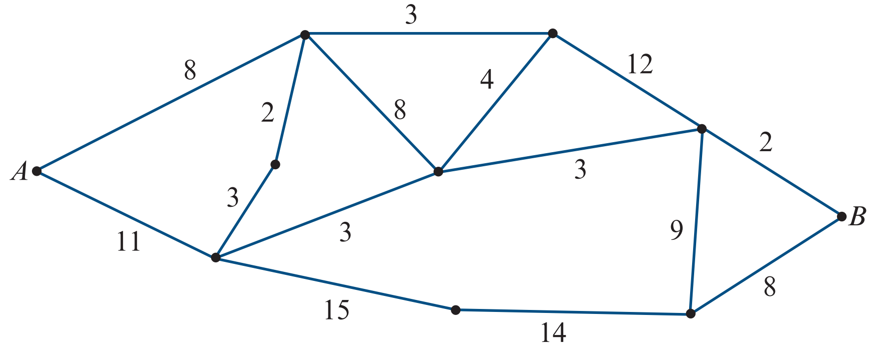
Example 16 **6** For each of the following connected graphs, use Kruskal's algorithm to determine the minimum spanning tree and its length.

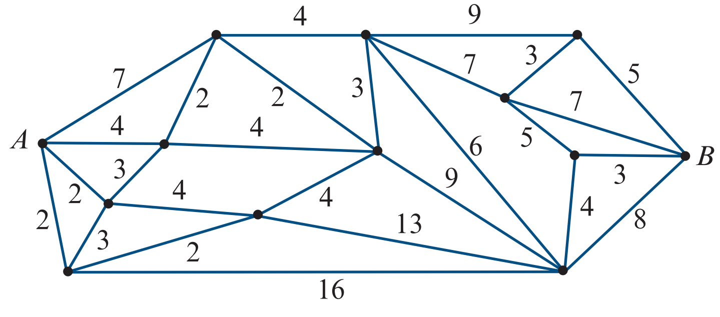
a


b


c


Example 17 **7** Using a greedy algorithm, find the shortest path from A to B in each of the following.

a


b


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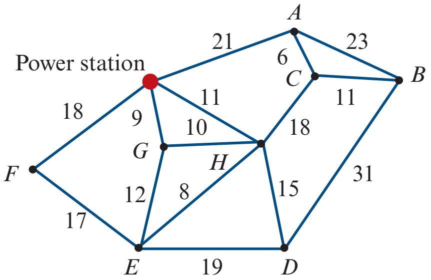
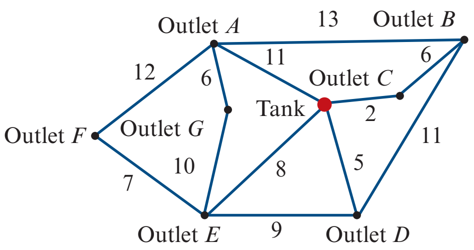
8 Power is to be connected by cable from a power station to eight substations (A to H). The distances (in kilometres) of the substations from the power station and from each other are shown in the network opposite. Determine the minimum length of cable needed.



9 Water is to be piped from a water tank to seven outlets on a property. The distances (in metres) of the outlets from the tank and from each other are shown in the network below.



a Starting at the tank, determine the minimum length of pipe needed.

b Due to high demand, the pipe connecting outlet A to outlet F and the pipe connecting outlet A to outlet B must always be in use. Given these new conditions, how much longer will the new minimum length of pipe be?

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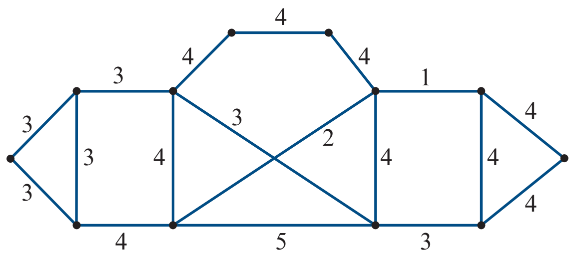
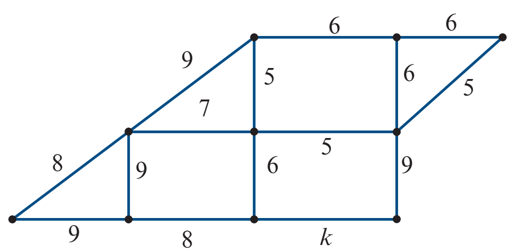
Testing understanding

10 A minimum spanning tree is to be drawn for the weighted graph below.



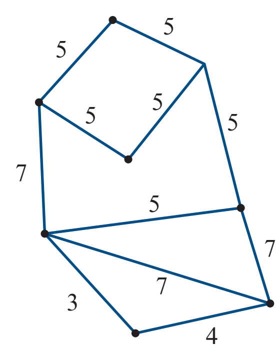
How many edges with weight 4 will **not** be included in the minimum spanning tree?

11 The minimum spanning tree for the graph below includes the edge with weight labelled k.



The total weight of all edges for the minimum spanning tree is 58. Find the value of k.

12 Consider the weighted graph below. How many different minimum spanning trees are possible?



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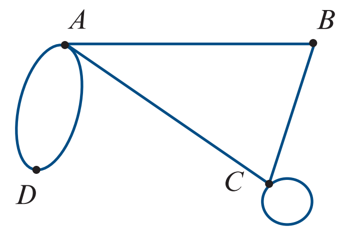
Key ideas and chapter summary

Assign-ment

Graph or network

A **graph** is a diagram that consists of a set of points called **vertices** and a set of lines called **edges**. C has one edge which links C to itself. This edge is called a **loop**.



Vertices and edges

In the graph above, A, B, C, and D are **vertices** and the lines AB, AD, AC, and BC are **edges**.

Degree of a vertex

The **degree of vertex** A, written deg(A), is the *number of edges attached to the vertex*. A loop will contribute to the degree of a vertex twice.

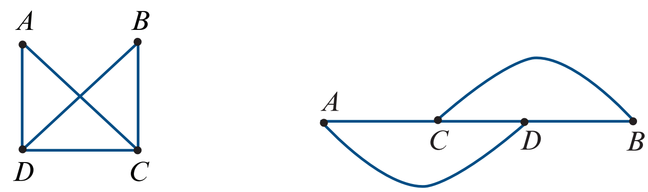
For example, in the graph above: deg(B) = 2 and deg(C) = 4.

Isomorphic graphs

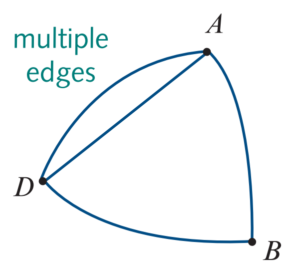
Two graphs are said to be **isomorphic** (equivalent) if:

- they both have the same number of edges and vertices
- corresponding vertices have the same degree and the edges connect to the same corresponding vertices.

For example, the two graphs below are isomorphic (or equivalent).



Multiple edges



The graph above is said to have **multiple edges**, as there are two edges joining A and D.

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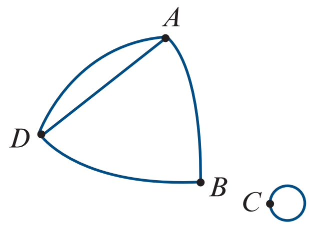
Adjacency matrix

An **adjacency matrix** is a square matrix that uses a zero or positive integer to record the number of edges connecting each pair of vertices in the graph.

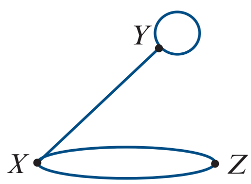
An example of a graph and its adjacency matrix is shown below.



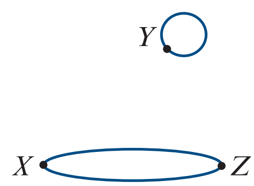
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Connected graph and bridges

A **graph** is **connected** if there is a path between each pair of vertices.
A **bridge** is a single edge in a connected graph that, if removed, leaves the graph disconnected. A graph can have more than one bridge.
Graph 1 is a connected graph.
Graph 2 is not a connected graph.
Edge XY in Graph 1 is a bridge because removing it leaves Graph 1 disconnected.



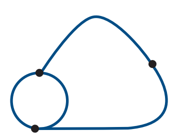
Graph 1



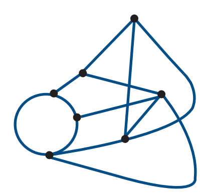
Graph 2

Planar graph

A graph that can be drawn in such a way that no two edges intersect, except at the vertices, is called a **planar graph**.



planar graph



non-planar graph

Euler's formula

For any **connected planar** graph, **Euler's formula** states:

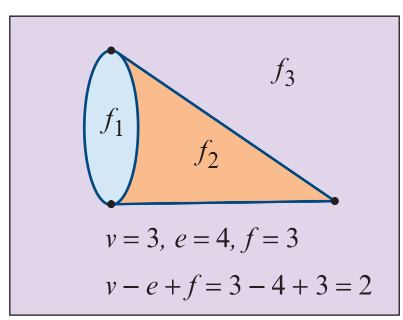
$v + f = e + 2$

v = the number of vertices

e = the number of edges

f = the number of faces.

and also, $v - e + f = 2$



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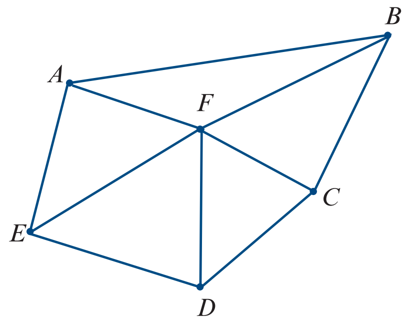
Walk, trail, path, circuit and cycle



A **walk** is a sequence of edges linking successive vertices in a graph.
In the graph above, *E-A-F-D-C-F-E-A* is a walk.
A **trail** is a walk with no repeated edges.
In the graph, *A-F-D-E-F-C* is a trail.
A **circuit** is a walk that has no repeated edges that starts and ends at the same vertex.
In the graph, *A-F-D-E-F-B-A* is a circuit.
A **path** is a walk with no repeated vertices.
In the graph, *F-A-B-C-D* is a path.
A **cycle** is a walk with no repeated vertices that starts and ends at the same vertex.
In the graph, *B-F-C-B* is a cycle.

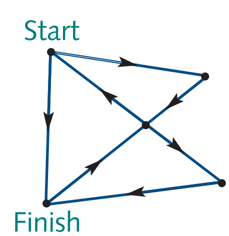
Eulerian trail

A trail that includes every edge just once (but does not start and finish at the same vertex) is called an **Eulerian trail**.

Condition for an Eulerian trail

To have an Eulerian trail (but not an Eulerian circuit), a graph must be connected and have exactly zero or two odd vertices, with the remaining vertices being even.

For example, the graph opposite is connected.
It has two odd vertices and three even vertices.
It has an Eulerian trail that starts at one of the odd vertices and finishes at the other.



Eulerian circuit

An Eulerian trail that starts and finishes at the same vertex is called an **Eulerian circuit**.

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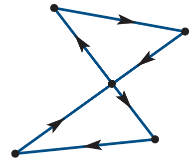
Review

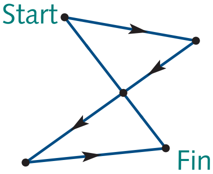
Condition for an Eulerian circuit

To have an **Eulerian circuit**, a graph must be connected, and all vertices must be even.
In the network shown, all vertices are even. It has an Eulerian circuit.
The circuit starts and finishes at the same vertex.



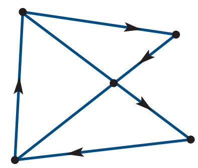
Hamiltonian path

A **Hamiltonian path** is a path through a graph that passes through each vertex exactly once but does not necessarily start and finish at the same vertex.



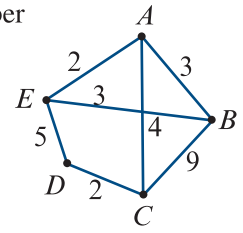
Hamiltonian cycle

A **Hamiltonian cycle** is a Hamiltonian path that starts and finishes at the same vertex.



Weighted graphs and networks

A **weighted graph** is one where a number is associated with each edge. These numbers are called weights. When the weights are physical quantities, for example, distance, time or cost, a weighted graph is often called a **network**.

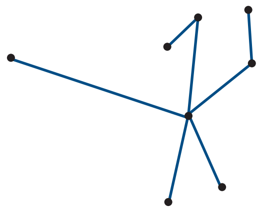


The shortest path problem

Determining the shortest distance or time or the least cost to move around a network is called the **shortest path problem**.

Tree

A **tree** is a connected graph that contains no cycles, multiple edges or loops.
A tree with n vertices has $n - 1$ edges.
The tree (right) has 8 vertices and 7 edges.



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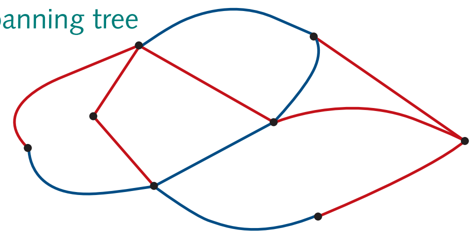
Spanning tree

A **spanning tree** of a graph is a subgraph that contains all the vertices of a connected graph, without multiple edges, cycles or loops.



Minimum spanning tree

A **minimum spanning tree** is a spanning tree for which the sum of the weights of the edges is as small as possible.

Greedy algorithm

A **greedy algorithm** is a simple, intuitive set of rules that can be used to solve optimisation problems. The expectation is that finding the optimum solution at each step in the solution will lead to the optimum solution for the entire problem.

Prim's algorithm

Prim's algorithm is a systematic method for determining a minimum spanning tree in a connected graph. It can start at any vertex. Prim's algorithm is a greedy algorithm.

Kruskal's algorithm

Kruskal's algorithm is a systematic method for determining a minimum spanning tree in a connected graph. It starts at the edge with the least weight. Kruskal's algorithm is a greedy algorithm.

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Review

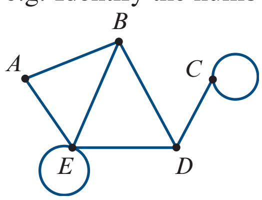
Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

8A **1 I can identify the number of vertices, edges and loops of a graph.** ☐

e.g. Identify the number of vertices, edges and loops in the graph below.



8A **2 I can determine the degree of a vertex and the sum of degrees of a graph.** ☐

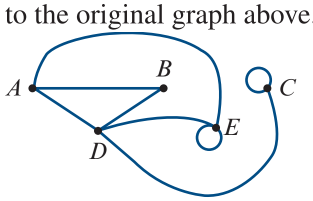
e.g. Determine the degree of each vertex in the graph above and the sum of degrees of the graph.

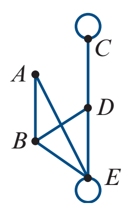
8B **3 I can identify a connected graph and a bridge.** ☐

e.g. Give a reason why the graph above would be considered connected.
Are there any bridges present? If yes, identify the two vertices the bridge(s) exist between.

8B **4 I can identify isomorphic graphs.** ☐

e.g. Identify whether both, one or none of the following graphs are isomorphic to the original graph above.





8B **5 I can use an adjacency matrix to represent a graph.** ☐

e.g. Complete the following matrix to represent the original graph above.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} \quad & \quad & \quad & \quad & \quad \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right] \end{array}$$

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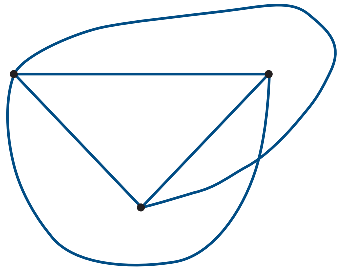
8C **6** I can classify a graph as planar and use isomorphic graphs to help identify them. ☐

e.g. Is the following graph planar? Justify your reasoning.



8C **7** I can identify the number of faces of a graph. ☐

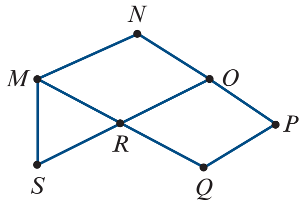
e.g. Determine the number of faces for the graph above.

8C **8** I can verify Euler's formula. ☐

e.g. Verify Euler's formula for the graph above.

8D **9** I can classify a walk as either a trail, path, circuit or cycle. ☐

e.g. For the graph below, is the walk $S\text{-}R\text{-}Q\text{-}P\text{-}O\text{-}N\text{-}M\text{-}S$ considered a trail, path, circuit or cycle?



8E **10** I can identify an Eulerian trail. ☐

e.g. For the graph in Question **9**, find an Eulerian trail.

8E **11** I can identify an Eulerian circuit. ☐

e.g. For the graph in Question **9**, an Eulerian circuit does not exist. Give a reason why this graph does not contain one, referring to the degrees of vertices.

8F **12** I can identify a Hamiltonian path. ☐

e.g. For the graph in Question **9**, find a Hamiltonian path.

8F **13** I can identify a Hamiltonian cycle. ☐

e.g. For the graph in Question **9**, find a Hamiltonian cycle.

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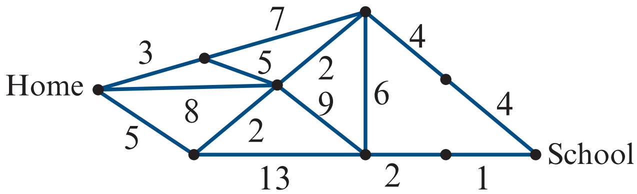
8G **14** I can find the shortest path of a weighted network. ☐

e.g. For the graph below, find the length of the shortest path from Home to School.

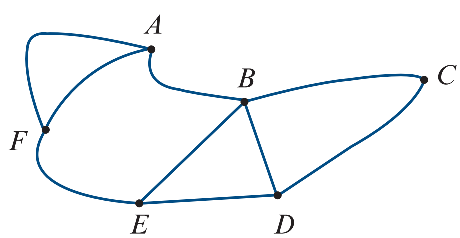


8H **15** I can apply a greedy algorithm such as Prim's or Kruskal's algorithm to find a minimum spanning tree in a network. ☐

e.g. For the graph above, find the minimum spanning tree and state its length.

Multiple-choice questions

The following graph relates to Questions 1 to 4.



1 The number of vertices in the graph above is:

A 3 **B** 5 **C** 6 **D** 7 **E** 9

2 The number of edges in the graph above is:

A 3 **B** 5 **C** 6 **D** 7 **E** 9

3 The degree of vertex B in the graph above is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

4 The number of even vertices in the graph above is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

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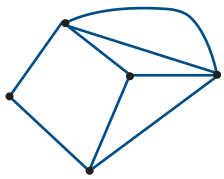
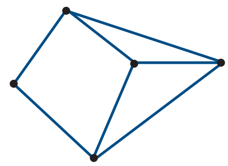
5 For which graph below is the sum of the degrees of the vertices equal to 14?

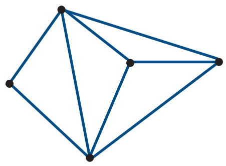
A



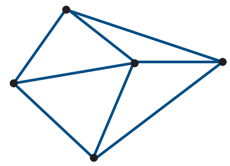
B



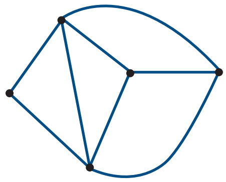
C



D

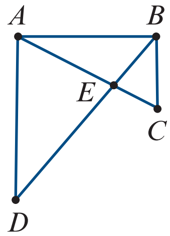


E

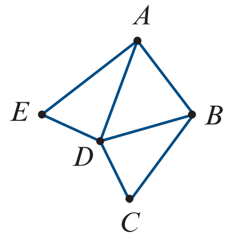


6 The graph that matches the matrix $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$ is:

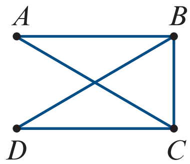
A



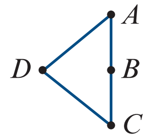
B



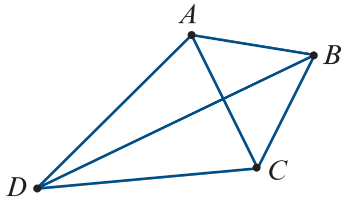
C



D



E



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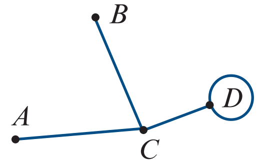
7 The adjacency matrix that matches the graph shown is:



A
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

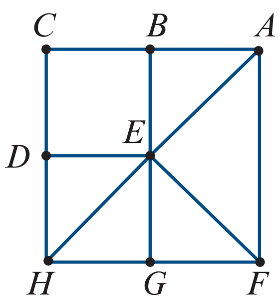
B
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

C
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \end{array}$$

D
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

E
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

The graph below is to be used when answering Questions 8 to 11.



8 The sequence of vertices *C-B-E-A-E-G* represents:

- **A** a walk only
- **B** a trail
- **C** a path
- **D** a circuit
- **E** a cycle

9 The sequence of vertices *D-E-H-G-E-A-B-C-D* represents:

- **A** a walk only
- **B** a trail but not a circuit
- **C** a path but not a cycle
- **D** a circuit
- **E** a cycle

10 The sequence of vertices *C-B-E-A-F-E-G-H* represents:

- **A** a walk only
- **B** a trail but not a circuit
- **C** a path but not a cycle
- **D** a circuit
- **E** a cycle

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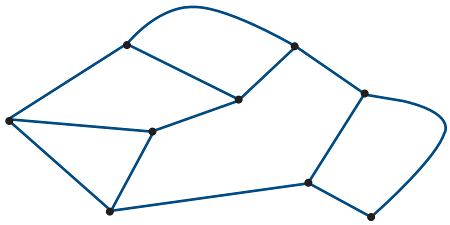
11 The sequence of vertices D-E-A-F-G-H-D represents:

A a walk only
B a trail but not a circuit
C a path but not a cycle
D a cycle
E none of these

The graph below is to be used when answering Questions 12 and 13.



12 The sum of the degrees of the vertices is:

A 22 **B** 23 **C** 24 **D** 25 **E** 26

13 For this graph:

A $v = 9, e = 13, f = 5$
B $v = 9, e = 13, f = 6$
C $v = 10, e = 13, f = 5$
D $v = 9, e = 14, f = 6$
E $v = 11, e = 13, f = 5$

14 Euler's formula for a planar graph is:

A $v - e = f + 2$
B $v - e + f = 2$
C $v + e + f = 2$
D $e - v + f = 2$
E $v - e = f - 2$

15 A connected graph with 10 vertices divides the plane into five faces. The number of edges connecting the vertices in this graph will be:

A 5
B 7
C 10
D 13
E 15

16 For a connected graph to have an Eulerian trail but not an Eulerian circuit:

A all vertices must be odd
B all vertices must be even
C there must be exactly two odd vertices and the rest even
D there must be exactly two even vertices and the rest odd
E an odd vertex must be followed by an even vertex

17 For a connected graph to have an Eulerian circuit:

A all vertices must be odd
B all vertices must be even
C there must be exactly two odd vertices and the rest even
D there must be exactly two even vertices and the rest odd
E an odd vertex must be followed by an even vertex

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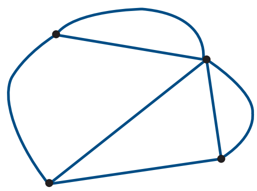
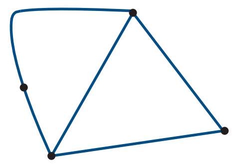
Review

The graphs below are to be used when answering Questions 18 and 19.

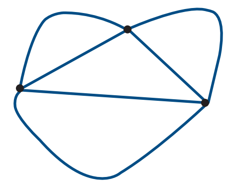
A



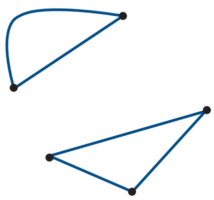
B



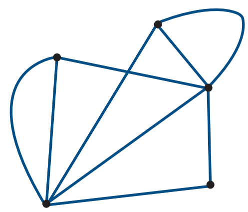
C



D



E

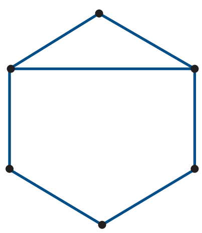


18 Which one of the graphs above has an Eulerian trail but not an Eulerian circuit?

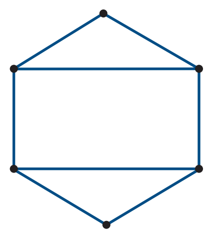
19 Which one of the graphs above has an Eulerian circuit?

20 Which one of the following graphs has an Eulerian circuit?

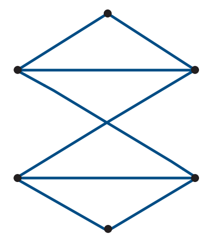
A



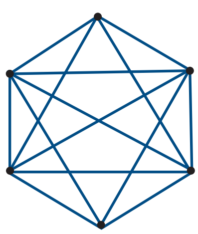
B



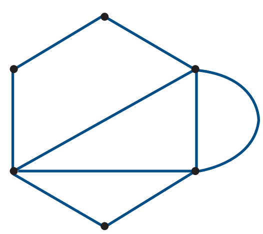
C



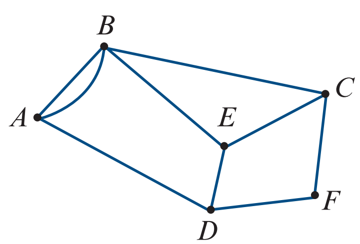
D



E

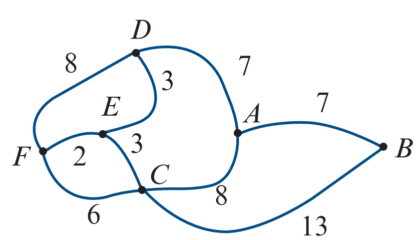


21 For the graph shown, which additional edge could be added to the graph so that the graph formed would contain an Eulerian trail?



A *AF* **B** *AD*
C *AB* **D** *CF*
E *BF*

22 The length of the shortest path from *F* to *B* in the graph shown is:



A 17 **B** 18
C 19 **D** 20
E 21

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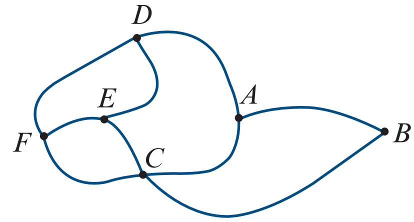
23 Which one of the following paths is a Hamiltonian cycle for the graph shown?



A *F-E-D-F*

B *F-E-D-A-B-C-E-F*

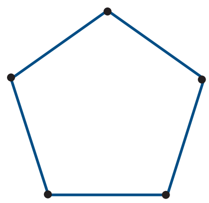
C *F-E-D-A-B-C-F*

D *F-C-A-B-D-E-F*

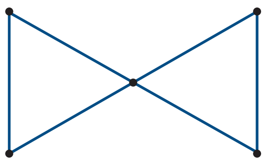
E *F-D-E-C-A-B-C-F*

24 Of the following, which graph has both an Eulerian circuit and Hamiltonian cycle?

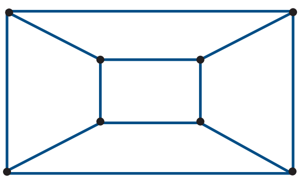
A



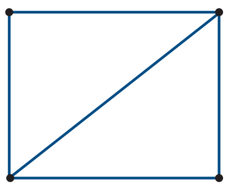
B



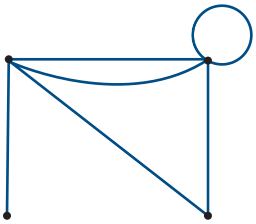
C



D

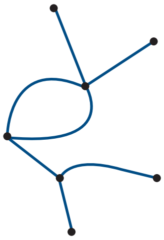


E

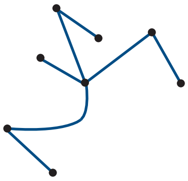


25 Which one of the following graphs is a tree?

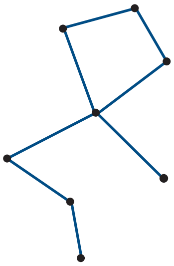
A



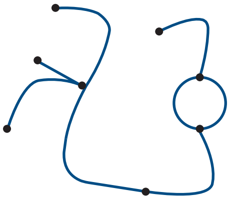
B



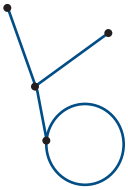
C



D



E



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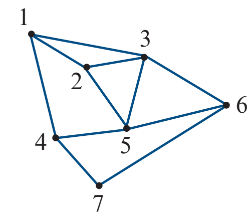
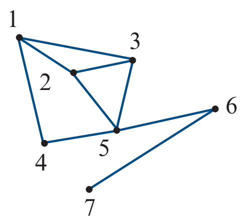
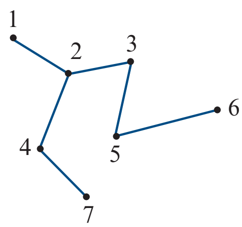
26 Which one of the following graphs is a spanning tree for the graph shown?



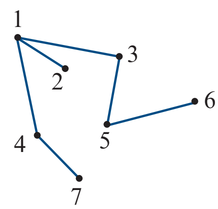
A



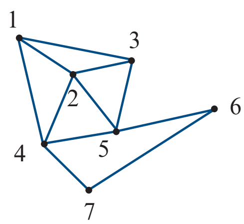
B



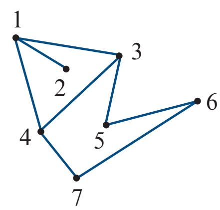
C



D



E



The campsite information below is to be used when answering Questions 27 to 29.

27 A park ranger wants to check all of the campsites in a national park, starting at and returning to the park office. The campsites are all interconnected with walking tracks. She would like to check the campsites without having to visit each campsite more than once. If possible, the route she should follow is:

A an Eulerian trail but not a circuit
B an Eulerian circuit
C a Hamiltonian path but not a cycle
D a Hamiltonian cycle
E a minimum spanning tree

28 The park authorities plan to pipe water to each of the campsites from a spring located in the park. They want to use the least amount of water pipe possible. If possible, the water pipes should follow:

A an Eulerian trail but not a circuit
B an Eulerian circuit
C a Hamiltonian path but not a cycle
D a Hamiltonian cycle
E a minimum spanning tree

29 Each week, a garbage collection route starts at the tip and collects the rubbish left at each of the campsites before returning to the tip to dump the rubbish collected. The plan is to visit each campsite only once. If possible, the garbage collection route should follow:

A an Eulerian trail but not a circuit
B an Eulerian circuit
C a Hamiltonian path but not a cycle
D a Hamiltonian cycle
E a minimum spanning tree

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Short-answer questions

1 Draw a connected graph with:

a four vertices, four edges and two faces

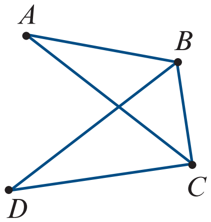
b four vertices, five edges and three faces

c five vertices, eight edges and five faces

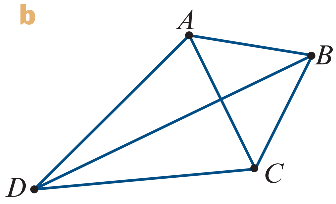
d four vertices, five edges, three faces and two bridges

2 Redraw each of the following graphs in planar form.

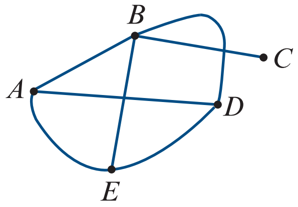
a



b



c

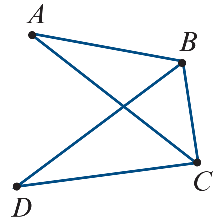


3 For the graph shown, write down:

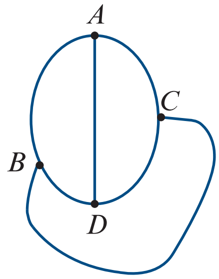
a the degree of vertex C

b the numbers of odd and even vertices

c the route followed by an Eulerian trail, starting at vertex B.



4 Construct an adjacency matrix for the graph below.

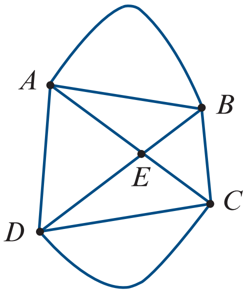


5 For the graph shown, write down:

a the degree of vertex C

b the number of odd and even vertices

c the route followed by an Eulerian circuit, starting at vertex A.



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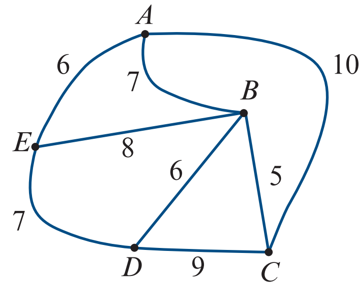
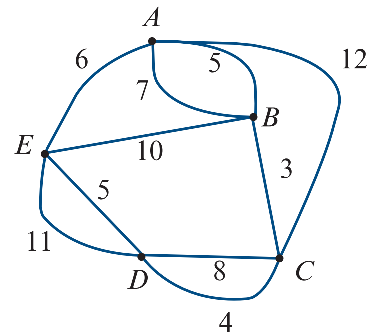
6 For the weighted graph shown, determine the length of the minimum spanning tree.



7 In the network opposite, the numbers on the edges represent distances in kilometres.
Determine the length of:

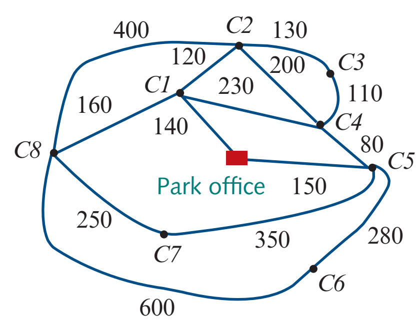
a the shortest path between vertex A and vertex D

b the length of the minimum spanning tree.



Written-response questions

1 The diagram opposite shows the network of walking tracks in a small national park. These tracks connect the campsites to each other and to the park office. The lengths of the tracks (in metres) are also shown.



a The network of tracks is planar. Explain what this means.

b Verify Euler's formula for this network.

c A ranger at campsite $C8$ plans to visit campsites $C1$, $C2$, $C3$, $C4$ and $C5$ on her way back to the park office. What is the shortest distance she will have to travel?

d How many even and how many odd vertices are there in the network?

e Each day, the ranger on duty has to inspect each of the tracks to make sure that they are all passable.

i Is it possible for her to do this, starting and finishing at the park office and travelling only once on each track? Explain why.

ii Identify one route that she could take.

f Following a track inspection after wet weather, the Head Ranger decides that it is necessary to put gravel on some walking tracks to make them weatherproof. What is the minimum length of track that will need to be gravelled to ensure that all campsites and the park office are accessible along a gravelled track?

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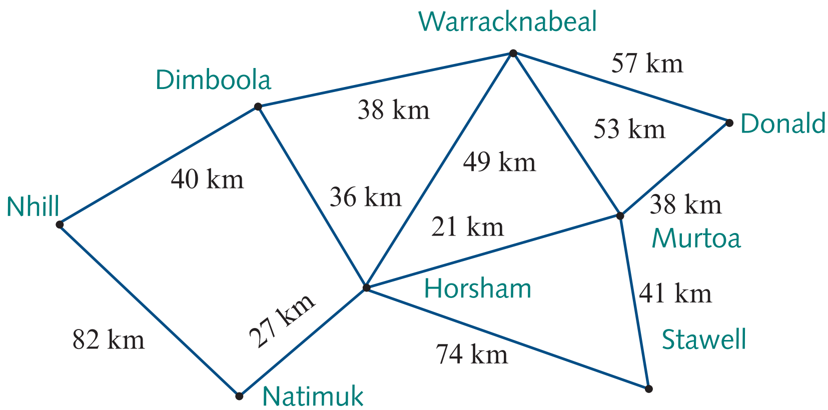
g A ranger wants to inspect each of the campsites but not pass through any campsite more than once on her inspection tour. She wants to start and finish her inspection tour at the park office.

  i What is the technical name for the route she wants to take?

  ii With the present layout of tracks, she cannot inspect all the tracks without passing through at least one campsite twice. Suggest where an additional track could be added to solve this problem.

  iii With this new track, write down a route she could follow.

2 The network below shows the major roads connecting eight towns in Victoria and the distances between them, in kilometres.



a Find the shortest distance between Nhill and Donald using these roads.

b Verify Euler's formula for this network.

c An engineer based in Horsham needs to inspect each road in the network without travelling along any of the roads more than once. He would also like to finish his inspection at Horsham.

  i Explain why this cannot be done.

  ii The engineer can inspect each road in the network without travelling along any of the roads more than once if he starts at Horsham but finishes at a different town. Which town is that? How far will he have to travel in total?

  iii Identify one route, starting at Horsham, that the engineer can take to complete his inspection without travelling along any of the roads more than once.

d A telecommunications company wants to connect all of the towns to a central computer system located in Horsham. What is the minimum length of cable that they will need to complete this task?

e The engineer can complete his inspection in Horsham by only travelling along one of the roads twice. Which road is that?

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Chapter 9

Variation

Chapter questions

- How do we recognise relationships involving **direct variation**?
- How do we determine the **constant of variation** in cases involving direct variation?
- How do we solve problems involving direct variation?
- How do we recognise relationships involving **inverse variation**?
- How do we determine the **constant of variation** in cases involving inverse variation?
- How do we solve problems involving inverse variation?
- How do we establish the relationship that exists between two variables from given data?
- How do we use data transformations to linearise a relationship?
- How do we use and interpret **log scales**?

In Chapter 7, we looked at how we might better understand an association between two variables for which we have data from the same subject. The techniques that we developed in that chapter were based on data collected on the two variables, and to a large extent, could only be used if we could establish that a linear association existed between the two variables.

In this chapter, we again look at the relationship between two variables, but this time we take a more mathematical approach, exploring situations where an exact formula relating the two variables can be found. We use the ideas of direct and indirect variation to establish the formulae which connect the two variables. We also look in more detail at some special cases where a non-linear relationship exists and develop some new techniques which enable us to linearise such relationships using data transformation.

9A Direct variation

Learning intentions

- To be able to recognise direct variation.
- To be able to find the constant of variation for direct variation.
- To be able to solve practical problems involving direct variation.

For the variables x and y, if $y = kx$ where k is a positive constant, we say that 'y varies **directly** as x'. Sometimes we use the following phrase, which has exactly the same meaning: 'y is directly proportional to x'. The positive constant k is called the **constant of variation**.

Using symbols, we write 'y is directly proportional to x' as: $\boldsymbol{y \propto x}$.

The symbol $\propto$ means '**is proportional to**' or '**varies as**'.

We note that, as x increases, y increases.

For example, if $y = 3x$, then $y \propto x$ and 3 is the constant of variation.

An example of direct variation is with the variables *distance* and *time* when driving at a constant speed.

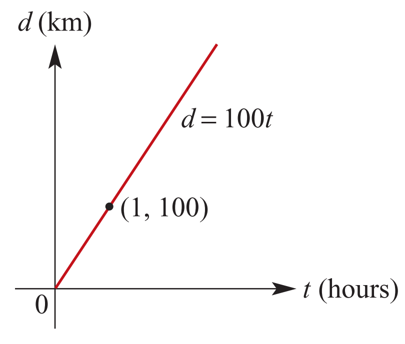
For example, Emily drives from her home in Appleton to visit her friend, Kim, who lives 600 km away in Brownsville. She drives at a constant speed of 100 km/h, and each hour, she notes how far she has travelled.

Time (t hours)	1	2	3	4	5	6
Distance (d km)	100	200	300	400	500	600

It can be seen that, as t increases, d also increases. The rule relating time and distance is: $d = 100t$.

This is an example of **direct variation**, and in this case, 100 is called the **constant of variation**.

We can say that the distance travelled **varies directly** as the time taken, or that d is **proportional** to t ($\boldsymbol{d \propto t}$). The graph of $\boldsymbol{d}$ **against** $\boldsymbol{t}$ is a straight line, passing through the origin.



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- The variable, y, is said to **vary directly** as x, if $y = kx$, for some positive constant, k.
- The constant, k, is called the **constant of variation**.
- In direct variation, $k = \dfrac{y \text{ value}}{\text{corresponding value of } x}$
- The statement 'y varies directly as x' is written symbolically as $\boldsymbol{y \propto x}$.
- If $y \propto x$, then the graph of y against x is a straight line, passing through the origin.

Determining the constant of variation

If y varies directly as x, then we can write $y = kx$, where k is the constant of variation.

The constant of variation can be found if we know just one value of x and the corresponding value of y.

Example 1 Finding the constant of variation

Use the table of values to determine the constant of variation, k, and hence complete the table:

$y \propto x$

x	3	5	7	
y	21		49	63

Explanation

1 Rewrite the variation expression as an equation, with k as the constant of variation.

2 Substitute corresponding values for x and y, and solve for k.

3 Substitute $k = 7$ in $y = kx$.

4 Substitute the value for x to find the corresponding y value.

5 Substitute the value of y to find the corresponding x value.

6 Complete the table.

Solution

$y \propto x$
$y = kx$

When $x = 3$, $y = 21$
$21 = 3k$
$\therefore\ k = 7$
$y = 7x$

When $x = 5$
$y = 7(5)$
$\therefore\ y = 35$

When $y = 63$
$63 = 7(x)$
$\therefore\ x = 9$

x	3	5	7	9
y	21	35	49	63

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Now try this 1 Finding the constant of variation (Example 1)

Use the table of values to find the constant of variation and complete the table.

$y \propto x$

x	2	4	6	
y	10		30	100

Hint 1 Rewrite the variation expression as an equation with constant of variation, k.
Hint 2 Substitute known corresponding values for x and y to find k.
Hint 3 Solve for unknown y by substituting relevant x value in the equation.
Hint 4 Solve for unknown x by substituting relevant y value in the equation.

Variation involving powers

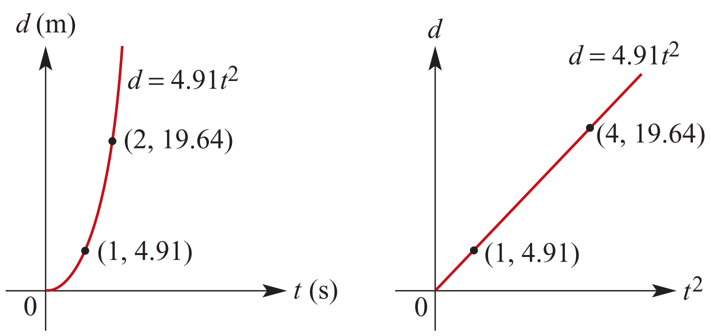
Sometimes, one of the variables may be raised to a power.
For example, a metal ball is dropped from the top of a tall building, and the distance it has fallen, in metres, is recorded each second.

Time (t s)	0	1	2	3	4	5
Distance (d m)	0	4.91	19.64	44.19	78.56	122.75

As t increases, d also increases. The rule relating time and distance is $d = 4.91t^2$.

This is another example of direct variation. In this case, we say that the distance fallen varies directly as the *square* of the time taken, or that d is proportional to t^2. We write $d \propto t^2$.

The graph of d against t is a parabola. However, the graph of d against t^2 is a straight line passing through the origin.



- If $y \propto x^n$, then $y = kx^n$, where k is a **constant of variation**.
- If $y \propto x^n$, then the graph of y against x^n is a straight line passing through the origin.

For all examples of direct variation, where one variable increases the other will also increase. It should be noted, however, that not all increasing trends are examples of direct variation.

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Example 2 Finding the constant of variation involving powers

Given that $y \propto x^2$, use the table of values to determine the constant of variation, k, and hence complete the table.

x	2	4	6	
y	12		108	192

Explanation

1 Rewrite the variation expression as an equation, with k as the constant of variation.

2 Substitute corresponding values for x and y and solve for k.

3 Rewrite equation for y, substituting k.

4 Check with other values that the correct value for k has been found.

5 Substitute values for x to find corresponding y values.

6 Substitute the value of y to find the corresponding x value.

7 Complete the table.

Solution

$y \propto x^2$
$y = kx^2$

When $x = 2$, $y = 12$
$12 = (2^2)k$
$\therefore \quad k = 3$

$y = 3x^2$

When $x = 6$, $y = 3(6^2) = 108$

When $x = 4$, $y = 3(4^2)$
$\therefore \quad y = 48$

When $y = 192$
$192 = 3(x^2)$
$\therefore \quad x = 8$

x	2	4	6	8
y	12	48	108	192

Now try this 2 Finding the constant of variation involving powers (Example 2)

Given that $y \propto x^2$, use the table of values to find the constant of variation and then complete the table.

x	2	4	6	
y		32	72	288

Hint 1 Rewrite the variation expression as an equation, with constant of variation, k.
Hint 2 Substitute known corresponding values for x and y into the equation, remembering to square each x value.
Hint 3 Solve equation for unknown y value by multiplying x^2 value by k.
Hint 4 Solve for x by substituting known y value into equation.

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Example 3 Solving a direct variation practical problem

In an electrical wire, the resistance (R ohms) varies directly as the length (L m) of the wire.

a If a 6 m wire has a resistance of 5 ohms, what is the resistance of a 4.5 m wire?

b How long is a wire for which the resistance is 3.8 ohms?

Explanation	Solution
a	
1 Write down the variation expression.	$R \propto L$
2 Rewrite expression as an equation with k as the constant of variation.	$R = kL$
3 Find the constant of variation by substituting given values for R and L.	When $L = 6, R = 5$ $5 = k(6)$ $\therefore \quad k = \frac{5}{6}$
4 Substitute value for k and write down the equation.	$R = \frac{5L}{6}$
5 Substitute $L = 4.5$ and solve to find R.	$R = \frac{5 \times 4.5}{6}$ $= 3.75$
6 Write your answer.	A wire of length 4.5 m has a resistance of 3.75 ohms.
b	
1 Write down the equation.	$R = \frac{5L}{6}$
2 Substitute $R = 3.8$ and solve to find L.	$3.8 = \frac{5 \times L}{6}$ $\therefore \quad L = 4.56$
3 Write your answer.	A wire of resistance 3.8 ohms has a length of 4.56 m.

Now try this 3 Solving a direct variation practical problem (Example 3)

A car is travelling at a constant speed. The distance travelled (d) varies directly as the time taken (t).

a If it takes 2 hours to travel 190 km, how far has the car travelled in 3 hours?

b How long does it take to travel 500 km? Give your answer in hours and minutes, rounded to the nearest whole minute.

Hint 1 Write down the variation expression and then rewrite as an equation with constant of variation, k.

Hint 2 Substitute known values to find k.

Hint 3 Use equation to solve for unknown values.

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Section Summary

Direct variation

- The variable y **varies directly** as x if $y = kx$, for some positive constant k.
  We can also say that y is **proportional** to x, and we can write $\boldsymbol{y \propto x}$.
- The constant, k, is called the **constant of variation** or **constant of proportionality**.
- If y is proportional to x, then the graph of y against x is a straight line through the origin. The slope of the line is the constant of variation.
- If $y \propto x$, then for any two non-zero values x_1 and x_2 and the corresponding values y_1 and y_2,

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = k$$

Skill-sheet

Exercise 9A

Building understanding

Example 1

1 Complete the following tables for the given equations.

a $y = 3x$

x	2	4	6	
y	6	12		24

b $y = 4x$

x	0	1	2	
y	0			12

2 Rewrite the following expressions of variation as equations, using k as the constant of variation.

a $y \propto x$ **b** $y \propto x^2$ **c** $y \propto x^5$
d $a \propto b$ **e** $z \propto w$ **f** $y \propto \sqrt{x}$

3 Write expressions of variation for the following statements.

a m varies directly as n
b y varies directly as the square of x
c y varies directly as the square root of x
d s varies directly as the square of t

Developing understanding

Example 2

4 For each of the following, determine the constant of variation, k, and hence complete the table of values:

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a $y \propto x$

x	3	4		12
y	21		49	84

b $y \propto x$

x	4	9	14	
y	2	4.5		10

c $y \propto x^2$

x	2	4	6	
y	8	32		128

5 If $y \propto x$ and $y = 42$ when $x = 7$, find:

a y when $x = 9$

b x when $y = 102$

6 If $M \propto n^2$, and $M = 48$ when $n = 4$, find:

a M when $n = 10$

b n when $M = 90$, correct to one decimal place.

Example 3

7 The area (A) of a triangle of fixed base length varies directly as its perpendicular height (h). If the area of the triangle is 60 cm^2 when its height is 10 cm, find:

a the area when its height is 12 cm

b the height when its area is 120 cm^2

8 The cost of potatoes varies directly with their weight. If potatotes weighing 4 kg cost $15.60, find:

a the cost of potatoes weighing 6 kg

b how many kgs of potatoes you could buy for $25, correct to one decimal place.

9 The distance (d) that a car travels at a constant speed varies directly with the time (t) taken. If the car travels 330 kms in 3 hours, find:

a the time taken to travel 500 kms

b the distance travelled after

i 5 hours **ii** 90 minutes

Testing understanding

10 The weight (W) of a square sheet of lead varies directly with the square of its side length (L). If a sheet of side length 20 cm weighs 18 kg, find the weight of a sheet that has an area of 225 cm^2.

11 The distance (d) to the visible horizon varies directly with the square root of the height (h) of the observer above sea level. An observer 1.8 m tall can see 4.8 km out to sea when standing on the shoreline. How far could the person see if they climbed a 4-m tower? Give your answer to two decimal places.

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9B Inverse variation

Learning intentions

- To be able to recognise inverse variation.
- To be able to find the constant of variation for inverse variation.
- To be able to solve practical problems involving inverse variation.

For the variables x and y, if $y = \frac{k}{x}$ where k is a positive constant, we say that 'y varies **inversely** as x'. Sometimes we use the following phrase, which has exactly the same meaning: 'y is **inversely proportional** to x'. The positive constant, k, is called the **constant of variation**.

Using symbols, we write it as $y \propto \frac{1}{x}$.

We note that as x increases, y decreases.

For example, if $y = \frac{3}{x}$, then $y \propto \frac{1}{x}$ and 3 is the constant of variation.

For example, a builder employs a number of bricklayers to build a brick wall. Three bricklayers will complete the wall in 8 hours. However, if the builder employs six bricklayers, the wall will be completed in half the time. The more bricklayers employed, the shorter the time taken to complete the wall. The time taken (t) decreases as the number of bricklayers (b) increases. This is an example of **inverse variation**.

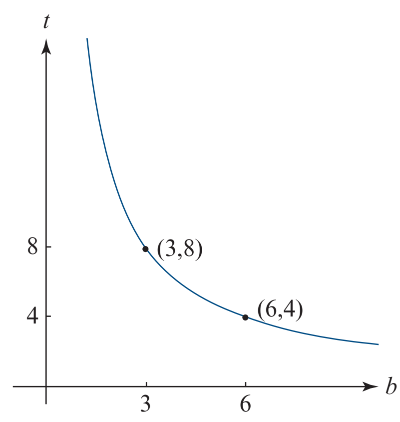
In this case we say:

t **varies inversely** as b

or t is **inversely proportional** to b

and we write $t \propto \frac{1}{b}$.

The graph of this is shown here.



If y **varies inversely** as x, then $\boldsymbol{y \propto \frac{1}{x}}$ and $y = \frac{k}{x}$, where k is the constant of variation.

In the table below, $y \propto \frac{1}{x}$ and the constant of variation is 6. We can say that $y = \frac{6}{x}$.

x	$\frac{1}{3}$	$\frac{1}{2}$	1	2
y	18	12	6	3

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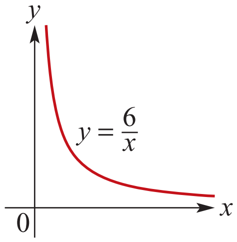
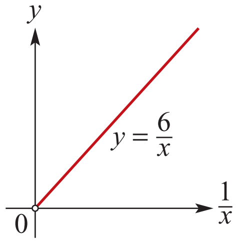
Note: $y \propto \frac{1}{x}$ is equivalent to $xy = k$, for a positive constant k. That is, the product is constant. This is often a useful way to check for inverse variation when given data in table form.

When we plot the graph of y against x, we obtain a curved shape called a **hyperbola**. However, the graph of y against $\frac{1}{x}$ is a straight line which does not have a value at the origin (as $\frac{1}{x}$ is undefined).





For all examples of inverse variation, as one variable increases, the other will decrease, and vice versa. The graph of y against x will show a downward trend. It should be noted, however, that not all decreasing trends are examples of inverse variation.

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Example 4 Determining the constant of variation for inverse variation

Use the table of values to find the constant of variation, k, and hence complete the table.

$y \propto \frac{1}{x}$

x	1	2	3	4	
y	3		1	0.75	0.6

Explanation	Solution
1 Write down the variation expression.	$y \propto \frac{1}{x}$
2 Rewrite the expression as an equation with constant of variation, k.	$y = \frac{k}{x}$
3 Substitute known values for x and y.	When $x = 1, y = 3$ $3 = \frac{k}{1}$
4 Solve for k.	$k = 3$
5 Rewrite equation, substituting k.	$y = \frac{3}{x}$
6 Check with other values that the correct value for k has been found.	When $x = 3, y = \frac{3}{3} = 1$
7 Substitute value for x to find corresponding y value.	When $x = 2$ $y = \frac{3}{2} = 1.5$
8 Substitute value for y to find corresponding x value.	When $y = 0.6$ $0.6 = \frac{3}{x}$ $x = \frac{3}{0.6}$ $\therefore\ x = 5$
9 Complete the table.	see table below

x	1	2	3	4	5
y	3	1.5	1	0.75	0.6

Now try this 4 Determining the constant of variation for inverse variation (Example 4)

Use the table of values to find the constant of variation, k, and hence complete the table.

$y \propto \frac{1}{x}$

x	2	4	5	10	
y	1	0.5	0.4		0.1

Hint 1 Rewrite the variation expression as an equation with constant of variation, k.

Hint 2 Substitute known values for x and y and solve for k.

Hint 3 Substitute value for x to find corresponding unknown y value.

Hint 4 Substitute value for y to find corresponding unknown x value.

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Example 5 Solving an inverse variation practical problem

For a cylinder of fixed volume, the height (h cm) is inversely proportional to the square of the radius (r cm).
If a cylinder of height 15 cm has a base radius of 4.2 cm, how high would a cylinder of equivalent volume be if its radius was 3.5 cm?

Explanation	Solution
1 Write down the variation expression, and then rewrite it as an equation with constant of variation, k.	$h \propto \frac{1}{r^2}$ $h = \frac{k}{r^2}$
2 Substitute known values for h and r, and solve for k.	When $h = 15, r = 4.2$ $15 = \frac{k}{(4.2)^2}$ $k = 15(4.2)^2$ $\therefore\ k = 264.6$
3 Write the equation.	$h = \frac{264.6}{r^2}$
4 To find the height when the radius is 3.5 cm, substitute $r = 3.5$ into equation and solve for h.	$h = \frac{264.6}{(3.5)^2}$ $h = 21.6$
5 Write your answer with correct units.	A cylinder with radius 3.5 cm has a height of 21.6 cm.

Now try this 5 Solving an inverse variation practical problem (Example 5)

The time taken (t hours) to empty a tank is inversely proportional to the rate, r, of pumping. If it takes two hours for a pump to empty a tank at a rate of 1200 litres per minute, how long will it take to empty a tank at a rate of 2000 litres per minute?

Hint 1 Write the variation expression and then rewrite as an equation with constant of variation, k.

Hint 2 Substitute values for t and r and solve for k.

Hint 3 Write down the equation, substitute value for r and solve for t.

Section Summary

Inverse variation

- The variable y **varies inversely** as x if $y = \frac{k}{x}$, for some positive constant, k.
  We can also say that y is **inversely proportional** to x, and we can write $y \propto \frac{1}{x}$.
- If y varies inversely as x, then the graph of y against $\frac{1}{x}$ is a straight line (not defined at the origin) and the gradient is the constant of variation.
- If $y \propto \frac{1}{x}$, then $x_1y_1 = x_2y_2 = k$, for any two values x_1 and x_2 and the corresponding values y_1 and y_2.

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Skill-sheet

Exercise 9B

Building understanding

1 Complete the following tables for the given equations.

a $y = \frac{20}{x}$

x	2	4	5	
y	10	5		2

b $y = \frac{5}{x}$

x	1	2	4	
y	5			1

2 Rewrite the following expressions of variation as equations, using k as the constant of variation.

a $y \propto \frac{1}{x}$
b $y \propto \frac{1}{x^2}$
c $y \propto \frac{1}{x^3}$
d $m \propto \frac{1}{n}$
e $z \propto \frac{1}{w}$
f $y \propto \frac{1}{\sqrt{x}}$

3 Write expressions of variation for the following statements.

a A varies inversely as r
b y varies inversely as the square of x
c y varies inversely as the square root of x
d m varies inversely as the cube of n
e s varies inversely as t

Developing understanding

Example 4

4 For each of the following, determine the constant of variation, k, and hence complete the table of values:

a $y \propto \frac{1}{x}$

x	1	2	4	
y	10	5		1

b $y \propto \frac{1}{x}$

x	2	4	10	
y	1	$\frac{1}{2}$		$\frac{1}{15}$

c $y \propto \frac{1}{x}$

x	1	2	4	
y	1	$\frac{1}{2}$		$\frac{1}{5}$

d $y \propto \frac{1}{x}$

x	0.5	1		5
y	1	0.5	0.25	

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5 If $y \propto \frac{1}{x}$ and $y = 20$ when $x = 5$, find:

a y when $x = 10$ **b** x when $y = 50$

6 If $a \propto \frac{1}{b}$ and $a = 1$ when $b = 2$, find:

a a when $b = 4$ **b** b when $a = \frac{1}{8}$

7 If $a \propto \frac{1}{b}$ and $a = 5$ when $b = 2$, find:

a a when $b = 4$ **b** b when $a = 1$

Example 5

8 The current (I amperes) that flows in an electrical appliance varies inversely with the resistance (R ohms). If the current is 3 amperes when the resistance is 80 ohms, find the current when the resistance is 100 ohms.

9 The time taken (t) to check-in passengers on a flight varies inversely with the number of people (n) working at the check-in desks.

a Write a variation expression to show this.

b When 5 people are working at the check-in desks, it takes 40 minutes to check-in passengers. How long will it take if there are 8 people working?

10 The length of a string on a musical instrument varies inversely to its frequency vibration. A 32.5 cm string on a violin has a frequency vibration of 400 cycles per second. What would be the frequency vibration of a 24.2 cm string? Give your answer to the nearest whole number.

Testing understanding

11 The gas in a cylindrical canister occupies a volume of 22.5 cm^3 and exerts a pressure of 1.9 kg/cm^2. If the volume (V) varies inversely with the pressure (P), find the pressure if the volume is reduced to 15 cm^3.

9C Data transformations

Learning intentions

- To be able to use the squared transformation.
- To be able to use the reciprocal transformation.

When the data from two variables are plotted on a graph, we may see that there is a clear relationship between the variables, but that this relationship is not linear. By changing the scale of one of the variables, it is sometimes possible to change the relationship to linear form, which is easier to analyse using techniques we have already developed.

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We can change the scale of a variable by applying a mathematical function to the values of one of the variables. This is a strategy that we used in the previous sections, changing the scale from x to x^2 in Example 2.

Changing the scale of a variable is called transforming the data.

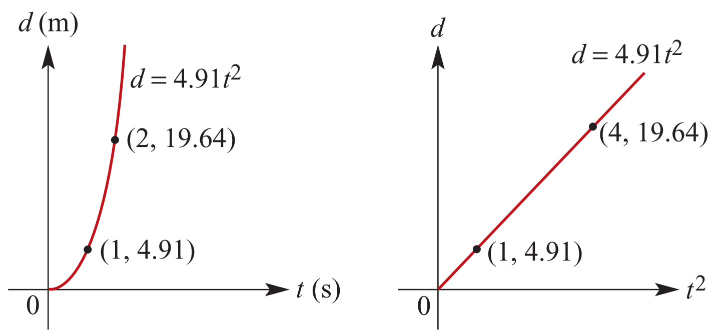
The process of changing the scale so that the relationship is linear is called **linearisation.**

Knowing which transformation to apply to linearise a particular relationship is complex and will be studied in further detail in General Mathematics Units 3 and 4. In this section, we will look at two very useful transformations: the squared transformation and the reciprocal transformation.

The squared transformation: x^2

In this transformation, we change the scale on the horizontal axis from $\boldsymbol{x}$ to $\boldsymbol{x^2}$.

Referring back to the example on direct variation involving powers (page 555) where a metal ball was dropped from a tall building and the distance and time were recorded, we noticed that the graph of $\boldsymbol{d}$ against $\boldsymbol{t}$ was a parabola (curved graph).
However, when we plotted $\boldsymbol{d}$ against $\boldsymbol{t^2}$ we obtained a straight-line graph, passing through the origin.



This is an example of using the squared transformation, and it has the effect of linearising the graph (changing the data to a straight line), which is easier to analyse. The slope of this line is the constant of variation.

Example 6 **Transforming data using the x^2 transformation**

Plot the points for the given table of values, and then use an x^2 transformation to check if this gives a straight line.

x	0	1	2	3	4
y	1	2	5	10	17

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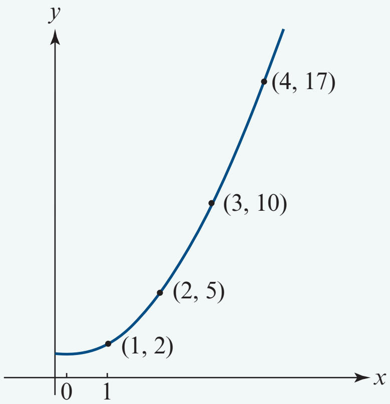
Explanation

1 Plot corresponding x and y values on a graph. This is clearly non-linear.

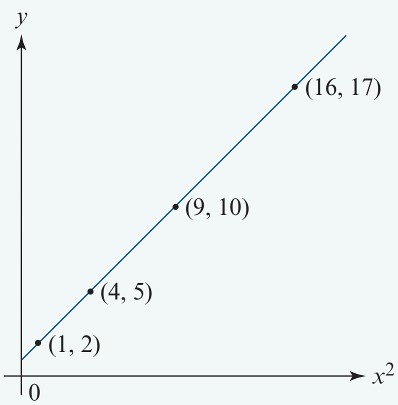
Solution



2 Square all x values.

x	0	1	2	3	4
x^2	0	1	4	9	16
y	1	2	5	10	17

3 Plot corresponding x^2 and y values on a graph. **Remember** to label the horizontal axis as x^2.



4 Check to see if the graph has become linear.

Graph is a straight line, so the graph has been linearised with the x^2 transformation.

Now try this 6 **Transforming data using the x^2 transformation (Example 6)**

Plot the points for the given table of values and then use an x^2 transformation to check if this gives a straight line.

x	0	1	2	3	4
y	3	4	7	12	19

Hint 1 Square each x value.

Hint 2 Plot each corresponding x^2 value and y value, and check to see if this gives a straight line.

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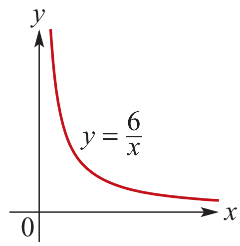
The reciprocal transformation: $\frac{1}{x}$

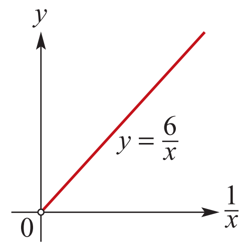
In this transformation we change the scale on the horizontal axis from $\boldsymbol{x}$ to $\boldsymbol{\frac{1}{x}}$.

Using our earlier example (page 561) of $y \propto \frac{1}{x}$ with a constant of variation of 6, the graph of y against x gives a curved line, called a hyperbola. However, the graph of y against $\frac{1}{x}$ gives a straight line with a slope of 6. The graph will not have a point at the origin (indicated on the graph by an open circle) because $\frac{1}{0}$ is undefined.

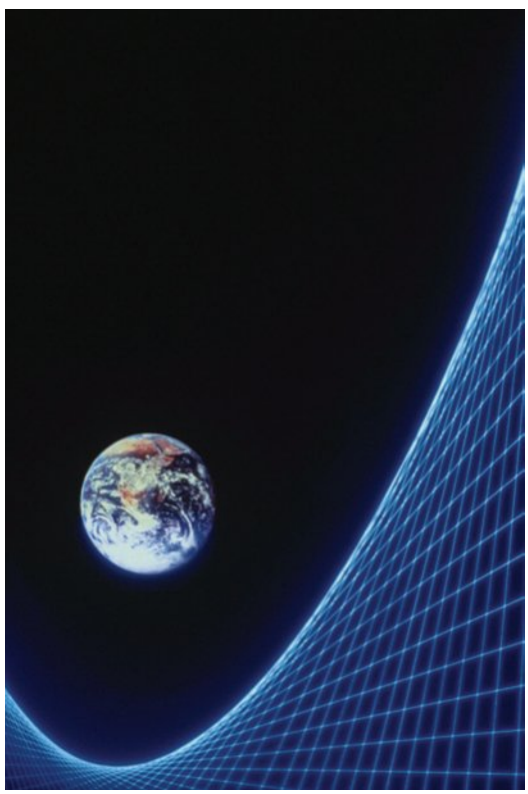
The table of values has been rewritten here with the values of $\frac{1}{x}$ included. Next to the table are the graphs showing $\boldsymbol{y}$ plotted against $\boldsymbol{x}$, and then $\boldsymbol{y}$ plotted against $\boldsymbol{\frac{1}{x}}$.

x	$\frac{1}{3}$	$\frac{1}{2}$	1	2
$\frac{1}{x}$	3	2	1	$\frac{1}{2}$
y	18	12	6	3





We can see that by changing the horizontal axis from x to $\frac{1}{x}$ we obtain a linear graph.



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Example 7 Transforming data using the $\frac{1}{x}$ transformation

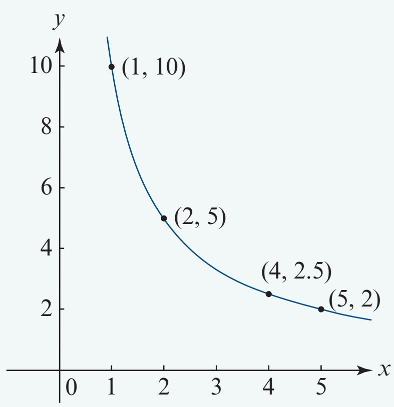
Plot the points for the given table of values, and then use the $\frac{1}{x}$ transformation to check if this gives a straight line.

x	1	2	4	5
y	10	5	2.5	2

Explanation

1 Plot corresponding x and y values on a graph.

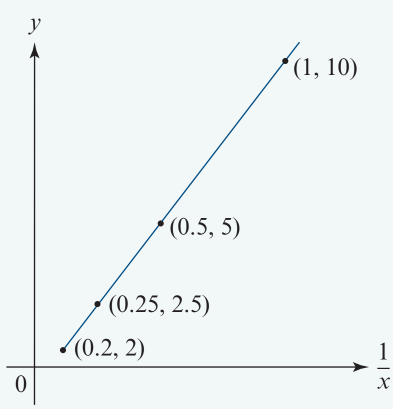
Solution



2 Find the reciprocal $(\frac{1}{x})$ of all x values.

x	1	2	4	5
$\frac{1}{x}$	1	0.5	0.25	0.2
y	10	5	2.5	2

3 Plot corresponding $\frac{1}{x}$ and y values on a graph. **Remember** to label horizontal axis as $\frac{1}{x}$.



4 Check to see if the graph has become linear.

Graph is a straight line, so the graph has been linearised with the $\frac{1}{x}$ transformation.

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Now try this 7 Transforming data using the $\frac{1}{x}$ transformation (Example 7)

Plot the points for the given table of values, and then use a $\frac{1}{x}$ transformation to check if this gives a straight line.

x	1	2	4	8
y	4	2	1	0.5

Hint 1 Divide 1 by each x value to find $\frac{1}{x}$.

Hint 2 Plot each corresponding $\frac{1}{x}$ value and y value, and check to see if this gives a straight line.

The CAS calculator can be used to perform the x^2 and the $\frac{1}{x}$ transformations.

Example 8 Using a CAS calculator to perform the x^2 transformation

For the given table of values, use the x^2 transformation and plot the graph of y against x^2 to check if this transformation gives a linear graph.

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

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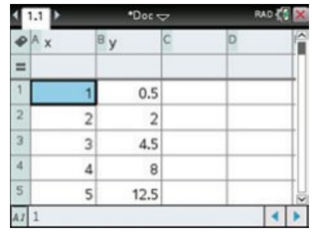
Using the TI-Nspire CAS to perform a squared transformation

Steps

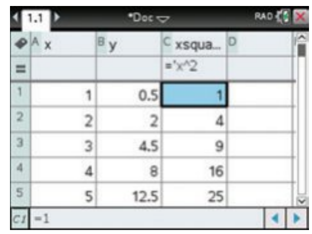
1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**.
Enter the data into lists named *x* and *y*, as shown.

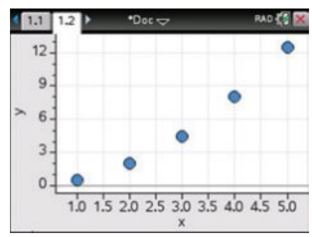


3 Name column C as *xsquared*.

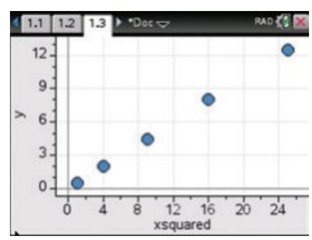
4 Move the cursor to the grey cell below *xsquared*. Enter the expression = x^2 by pressing =, then typing x^2. Pressing enter calculates and displays the values of x^2.



5 Press ctrl + I and select **Add Data & Statistics**.
Construct a scatterplot of *y* against *x*. The plot is clearly non-linear.



6 Press ctrl + I and select **Add Data & Statistics**.
Construct a scatterplot of *y* against x^2.
The plot is now linear.



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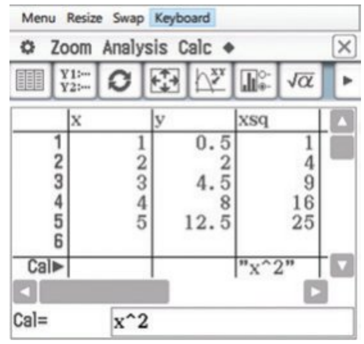
Using the CAS Classpad to perform a squared transformation

Steps

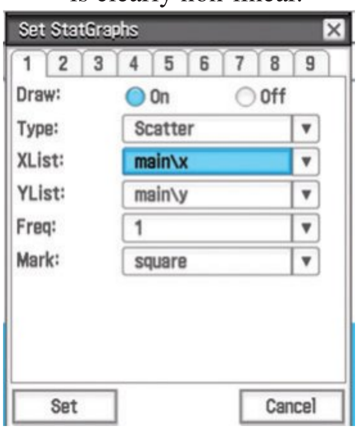
1 In the Statistics application, enter the data into lists named *x* and *y*. Name the third list *xsq* (for x^2).

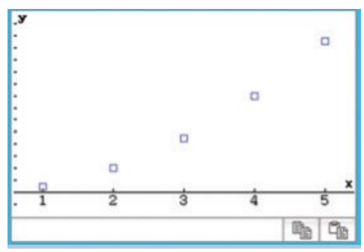
2 Place the cursor in the calculation cell at the bottom of the third column and type x^2. This will calculate the values of x^2.



3 Construct a scatterplot of *y* against *x*.

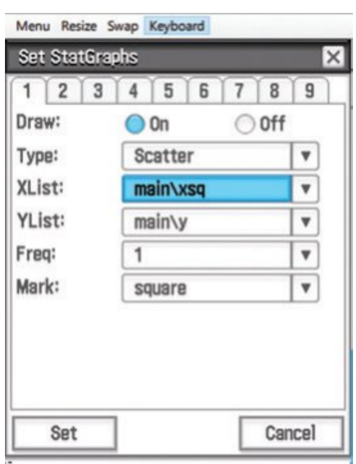
- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot. The scatterplot is clearly non-linear.

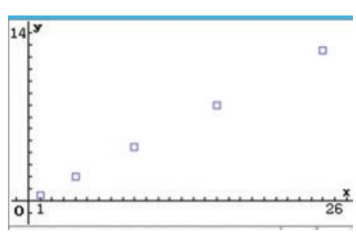




4 Construct a scatterplot of *y* against x^2.

- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot. The plot is now clearly linear.





Now try this 8 Using a CAS calculator to perform the x^2 transformation (Example 8)

For this table of values, use the x^2 transformation and plot the graph of y against x^2 to check if this transformation gives a linear graph.

x	1	2	3	4
y	3	12	27	48

Example 9 Using a CAS calculator to perform the $\frac{1}{x}$ transformation

For the given table of values, use the $\frac{1}{x}$ transformation, and plot the graph of y against $\frac{1}{x}$ to check if this transformation gives a linear graph.

x	1	2	3	4	5
y	4	2	1.3	1	0.8

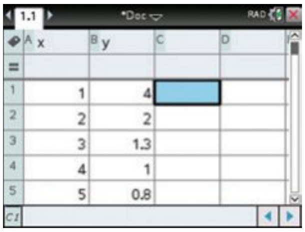
Using the TI-Nspire CAS to perform a $\frac{1}{x}$ transformation

Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**.

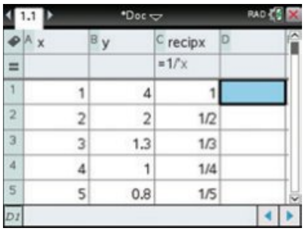
Enter the data into lists named x and y, as shown.



3 Name column C as *recipx*.

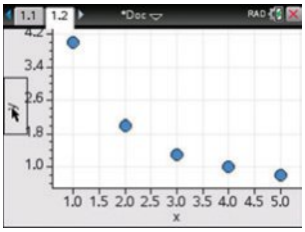
4 Move the cursor to the grey cell below *recipx*. Enter the expression $= \frac{1}{x}$ by pressing =, then typing **1/x**.

Pressing enter calculates and displays the values of $\frac{1}{x}$.



5 Press ctrl+I and select **Add Data & Statistics**.

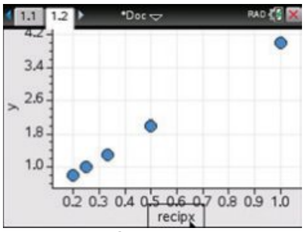
Construct a scatterplot of y against x. The plot is clearly non-linear.



6 Press ctrl+I and select **Add Data & Statistics**.

Construct a scatterplot of y against $\frac{1}{x}$.

The plot is now linear.



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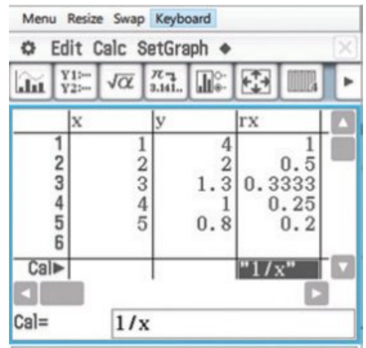
Using the CAS Classpad to perform a $\frac{1}{x}$ transformation

Steps

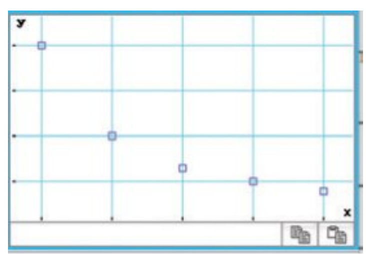
1 In the Statistics application, enter the data into lists named *x* and *y*. Name the third list *rx* (for $\frac{1}{x}$, the reciprocal of x).



2 Place the cursor in the calculation cell at the bottom of the third column and type $\frac{1}{x}$. This will calculate the values of $\frac{1}{x}$.

3 Construct a scatterplot of y against x.

- Tap [icon] and complete the **Set StatGraphs** dialog box as shown.
- Tap [icon] to view the scatterplot. The scatterplot is clearly non-linear.



4 Construct a scatterplot of y against $\frac{1}{x}$.

- Tap [icon] and complete the **Set StatGraphs** dialog box as shown.
- Tap [icon] to view the scatterplot. The plot is now clearly linear.

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Now try this 9 **Using a CAS calculator to perform the $\frac{1}{x}$ transformation (Example 9)**

For the given table of values, use the $\frac{1}{x}$ transformation and plot the graph of y against $\frac{1}{x}$ to check if this transformation gives a linear graph.

x	1	2	4	5
y	0.5	0.25	0.125	0.1

Hint 1 Follow instructions for the relevant CAS calculator above.

Section Summary

- A graph or a table of values may be used to help decide between direct and inverse variation.
- A set of data can be transformed using a squared (x^2) or a reciprocal ($\frac{1}{x}$) transformation.
- Direct variation If $y \propto x^2$, then the graph of y against x^2 will be a straight line through the origin. The slope of this line will be the constant of variation, k.
- Inverse variation If $y \propto \frac{1}{x}$, then the graph of y against $\frac{1}{x}$ will be a straight line, not defined at the origin. The slope of this line will be the constant of variation, k.

Exercise 9C

Building understanding

1 Copy and complete the following:

a In direct variation, if the values of x increase, the values of y ☐.

b Direct variation graphs are ☐ lines that pass through the ☐.

c In inverse variation, as the values of x increase, the values of y ☐.

2 Do the following tables show direct or inverse variation?

a

x	3	6	9	12
y	4	2	$\frac{4}{3}$	1

b

x	3	6	9	12
y	18	72	162	288

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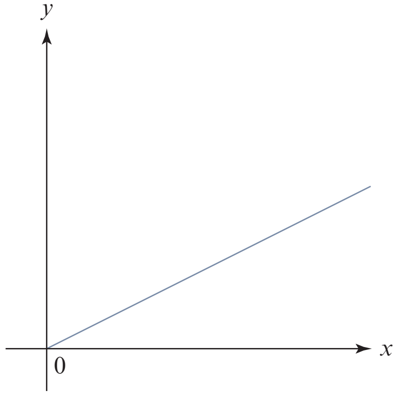
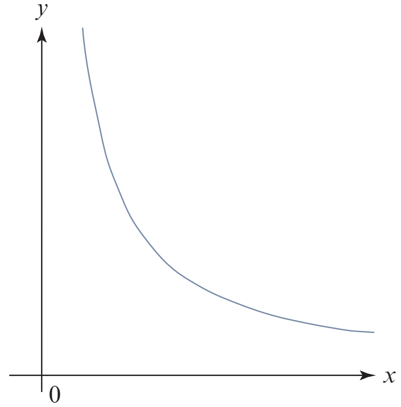
3 Do the following graphs show direct or inverse variation?

a



b



4 Complete the following tables to give x^2 and $\frac{1}{x}$ values.

a

x	2	4	6	8
x^2	4			64
$\frac{1}{x}$	$\frac{1}{2}$		$\frac{1}{6}$	

b

x	10	20	30	40
x^2			900	
$\frac{1}{x}$		$\frac{1}{20}$		

Developing understanding

5 The following tables show different types of variation. Which one shows:

a direct, $y \propto x$ **b** inverse, $y \propto \frac{1}{x}$ **c** direct, $y \propto x^2$

i

x	1	2	3	4	5
y	4	16	36	64	100

ii

x	0	3	6	9	12
y	0	2	4	6	8

iii

x	1	5	10	15	20
y	5	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

Example 6

6 Use the x^2 transformation to check if plotting the values of x^2 and y is linear.

x	1	2	3	4	5
y	7.5	9	11.5	15	19.5

7 Use the x^2 transformation to check if plotting the values of x^2 and y is linear.

x	2	2.5	3	3.5	4
y	6.3	15	21.6	27	30.5

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Example 7

8 Use the $\frac{1}{x}$ transformation to check if plotting the values of $\frac{1}{x}$ and y is linear.

x	0.2	0.4	0.5	1
y	50	20	14	2

Example 8

9 Use a CAS calculator to check if y plotted against x^2 is linear.

x	2	4	6	8	10
y	9	21	41	69	105

Example 9

10 Use a CAS calculator to check if y plotted against $\frac{1}{x}$ is linear.

x	2	3	4	8	12
y	12	8	6	3	2

Testing understanding

11 Given the following data:

x	1	2	3	4
y	8	19	34	55

a would you use the x^2 or the $\frac{1}{x}$ transformation to obtain a straight-line graph?

b Perform the transformation that you selected in part **a**, and check to see that this gives a straight-line graph.

12 The time taken to mow a lawn is inversely proportional to the number of mowers. It takes 2 hours to mow a public area of grass with one ride-on mower. If we have 2 ride-on mowers, it will take half the time. This information is represented in the table below, where the time taken, t, is in minutes, and the number of ride-on mowers is represented by n.

n	1	2	3	4	5	6
t	120	60	40			

a Complete the table.

b What transformation would you use to linearise this data?

c Plot the resulting graph.

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9D Logarithms

Learning intentions

- To be able to understand orders of magnitude.
- To be able to use the log transformation ($\log_{10} x$).

What are logarithms?

Possibly the most commonly used data transformation when working with real data is the logarithmic transformation. Before we see how useful this transformation can be, we need to develop our understanding of logarithms. Consider the numbers:

1, 10, 100, 1000, 10 000, 100 000, 1 000 000

Such numbers can also be written as:

$10^0, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6$

In fact, if we make it clear we are only talking about powers of 10, we can simply write down the powers:

0, 1, 2, 3, 4, 5, 6

These powers are called the **logarithms** of the numbers, or **logs** for short. When we use logarithms to write numbers as powers of 10, we say we are working with logarithms to the base 10.

We write $\log_{10} 100$ to mean log to the base 10 of 100. Often we leave the base 10 out and simply write log 100. (We only do this if the base is 10).

Knowing the powers of 10 is important when using logarithms to the base 10.

Example 10 Evaluating a logarithm

Write the number 100 as a power of 10, and then write down its logarithm.

Explanation	Solution
1 Write 100 as a power of 10.	$100 = 10^2$
2 Write down the logarithm.	$\log(100) = \log(10^2)$ $= 2$

Now try this 10 Evaluating a logarithm (Example 10)

Write the number 10 000 as a power of 10, and then write down its logarithm.

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Example 11 Using a CAS calculator to find logarithms

Find the log of 45 to one decimal place.

Explanation	Solution
1 Open a calculator screen, enter log(45) from the keyboard, and press ENTER (TI-Nspire) or EXE (Casio).	$\log_{10}(45)$ 1.65321
2 Write the answer to one decimal place.	$\log(45) = 1.65\ldots$ $= 1.7$ to one decimal place

Now try this 11 Using a CAS calculator to find logarithms (Example 11)

Find the logarithm of 245 to one decimal place.

Hint 1 Use your calculator.

Hint 2 Correct to one decimal place means that there will be only one number after the decimal point.

Hint 3 Round up if the digit after the first decimal place is 5 or more.

Example 12 Using a CAS calculator to evaluate a number if logarithm is known

Find the number whose logarithm is 3.1876 to one decimal place.

Explanation	Solution
1 If the logarithm of a number is 3.1876, then the number is $10^{3.1876}$.	
2 Enter the expression and press ENTER (TI-Nspire) or EXE (Casio).	$10^{3.1876} = 1540.281\ldots$
3 Write the answer to one decimal place.	$= 1540.3$ to one decimal place

Now try this 12 Using a CAS calculator to evaluate a number if logarithm is known (Example 12)

Find the number whose log is 2.8517 to one decimal place.

Hint 1 Use your calculator to evaluate $10^{2.8517}$.

Hint 2 Correct to one decimal place means that there will be only one number after the decimal point.

Hint 3 Round up if the digit after the first decimal place is 5 or more.

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Order of magnitude

Increasing an object by an order of magnitude of 1 means that the object is ten times larger.

An increase by order of magnitude	Increase in size
1	10^1 = 10 times larger
2	10^2 = 100 times larger
3	10^3 = 1000 times larger
6	10^6 = 1 000 000 times larger

Decreasing an object by an order of magnitude of 1 means that the object is ten times smaller.

An increase by order of magnitude

In general, an increase by n orders of magnitude is the equivalent of multiplying a quantity by 10^n.

A decrease by order of magnitude

In general, a decrease by n orders of magnitude is the equivalent of dividing a quantity by 10^n.

It is easy to see the order of magnitude of various numbers when they are written in standard form (e.g. 500 in standard form is $5 \times 100 = 5 \times 10^2$, so the order of magnitude of 500 is 2).

Example 13 Finding the order of magnitude of a number written in standard form

What is the order of magnitude of 1200?

Explanation

1 Write 1200 in standard form.

2 Look at the power of 10 to find the order of magnitude. Write your answer.

Note: The order of magnitude of 1.2 is 0.

Solution

$1200 = 1.2 \times 10^3$

The power of 10 is 3, so the order of magnitude of 1200 is 3.

Now try this 13 Finding the order of magnitude of a number written in standard form (Example 13)

What is the order of magnitude of 35 500?

Hint 1 Write 35 500 in standard form.

Hint 2 Look at the power of 10 to find the order of magnitude.

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Using the logarithmic transformation: $\log_{10}(x)$

As well as using the squared and the reciprocal transformation, we can also use a log transformation to linearise data. In this case, we change the scale on the horizontal axis from x to $\log_{10}(x)$.

Example 14 Transforming data using the $\log_{10}(x)$ transformation

For the given table of values, perform a $\log_{10}(x)$ transformation to check if this gives a straight line, as the graph of y against x is not a straight line.

x	5	10	150	500
y	3.1	4.0	7.5	9.1

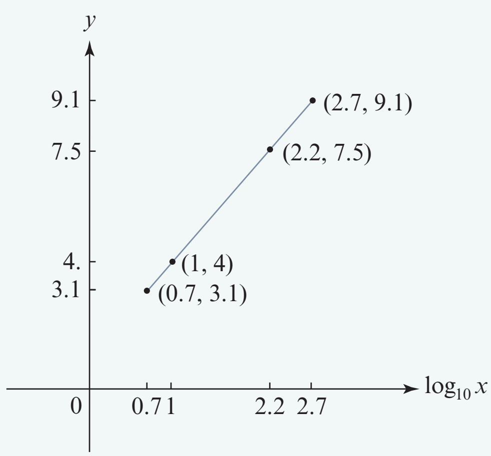
Explanation

1 Find $\log_{10}(x)$ for all x values.

Solution

x	5	10	150	500
$\log_{10}(x)$	0.7	1	2.2	2.7
y	3.1	4.0	7.5	9.1

2 Plot corresponding $\log_{10}(x)$ and y values on a graph. **Remember** to label horizontal axis as $\log_{10}(x)$.



3 Check to see if the graph has become linear.

Graph has been linearised with the $\log_{10}(x)$ transformation.

Now try this 14 Transforming data using the $\log_{10}(x)$ transformation (Example 14)

Plot the points for the given table of values, and then use a $\log_{10}(x)$ transformation to check if this gives a straight line.

x	2	4	6	8
y	1.6	2.2	2.6	2.8

Hint 1 Find $\log_{10}(x)$ for each x value.

Hint 2 Plot each corresponding $\log_{10}(x)$ value and y value, and check to see if this gives a straight line.

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A CAS calculator can also be used to perform the $\log_{10}(x)$ transformation.

Example 15 Using a CAS calculator to perform the $\log_{10}(x)$ transformation

For the given table of values, use the $\log_{10}(x)$ transformation to check if this transformation gives a linear graph.

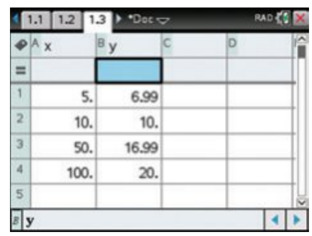
x	5	10	50	100
y	6.99	10	16.99	20

Using the TI-Nspire CAS to perform a $\log_{10}(x)$ transformation

Steps

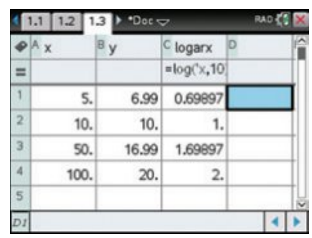
1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**.
Enter the data into lists named x and y, as shown.

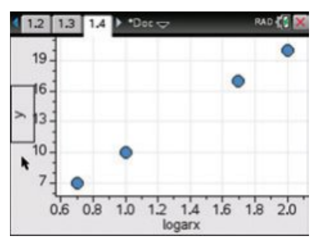


3 Name column C as *logarx* .

4 Move the cursor to the grey cell below *logarx*. Enter the expression = $\log_{10} x$ by pressing =, then entering $\log_{10} x$. Pressing enter calculates and displays the values of $\log_{10} x$.



5 Press ctrl+I and select **Add Data & Statistics**.
Construct a scatterplot of y against $\log_{10} x$.

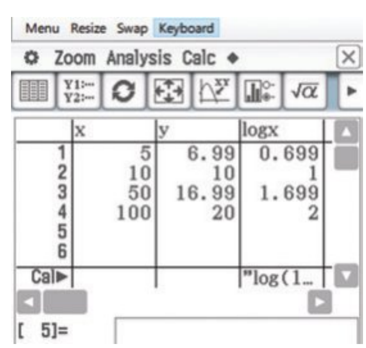


Using the CASIO ClassPad to perform a $\log_{10}(x)$ transformation

Steps

1 In the Statistics application, enter the data into lists named **x** and **y**. Name the third list: $\log x$ (for $\log_{10}(x)$) .

2 Place the cursor in the calculation cell at the bottom of the third column, and type $\log_{10}(x)$. This will calculate the values of $\log_{10}(x)$.



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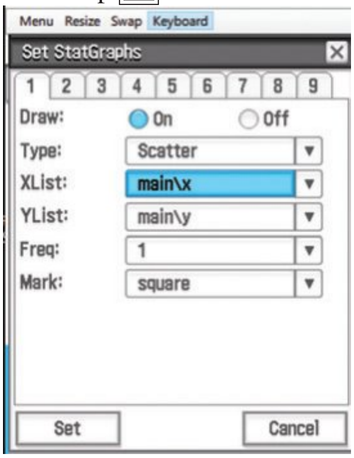
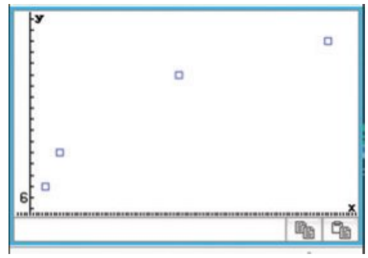
3 Construct a scatterplot of y against x.

- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot. The scatterplot is clearly non-linear.





4 Construct a scatterplot of y against $\log_{10}(x)$.

- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot. The plot is now clearly linear.

Now try this 15 **Using a CAS calculator to perform the $\log_{10}(x)$ transformation (Example 15)**

For the given table of values, use the $\log_{10}(x)$ transformation to check if this transformation gives a linear graph.

x	10	100	500	1000
y	5	8	10.1	11

Hint 1 Follow instructions for the relevant CAS calculator above.

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Example 16 Applying the logarithmic transformation

Plot the heartbeat/minute of mammals against the logarithm of their body weight.

Mammal	Body weight (g)	Heartbeat/minute
Shrew	2.5	1400
Chick	50	400
Rabbit	1000	205
Monkey	5000	190
Tree kangaroo	8000	192
Giraffe	900 000	65
Elephant	5 000 000	30
Blue whale	170 000 000	16

Note 1: If we plot the heartbeat/minute of mammals against their body weight, we will be starting from a very small weight value of 2.5 grams for a shrew to 170 tonne = 170 000 kilograms = 170 000 000 grams for a blue whale.

Plotting the actual body weight values on a horizontal axis is difficult because of the large range of values for the body weight of mammals.

However, if the body weight values are written more compactly as logarithms (powers) of 10, then these logarithms can be placed on a logarithmic scale graph.

We have seen that $\log_{10} 100 = 2$. This can also be expressed as $\log(100) = 2$.

Note 2: $\log_{10} x$ is often written as $\log(x)$.

Explanation	Solution
1 Convert each mammal's body weight to logarithms.	
Weight of shrew is 2.5 grams. Find logarithm (log) of 2.5.	$\log(2.5) = 0.40$ (to two decimal places)
Weight of tree kangaroo is 8000 grams. Use a calculator to find log(8000).	$\log(8000) = 3.90$ (to two decimal places)
Weight of giraffe is 900 000 grams. Use a calculator to find log(900 000).	$\log(900\,000) = 5.95$ (to two decimal places)
Weight of blue whale is 170 000 000 grams. Use a calculator to find log(170 000 000).	$\log(170\,000\,000) = 8.23$ (to two decimal places)

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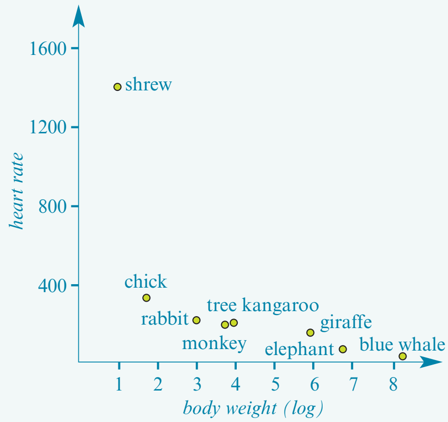
2 Use a calculator to find the logarithms (logs) of the body weight of the other mammals.
Record your results.

Mammal	Body weight (g)	log(*weight*)
Shrew	2.5	0.40
Chick	50	1.70
Rabbit	1000	3.00
Monkey	5000	3.70
Tree kangaroo	8000	3.90
Giraffe	900 000	5.95
Elephant	5 000 000	6.70
Blue whale	170 000 000	8.23

3 Plot the logarithms of the animals' body weights on the horizontal axis of the graph, and the heart rate on the vertical axis.



Now try this 16 Applying the logarithmic transformation (Example 16)

Plot the heart beat/minute of the animals below against the logarithm of their life span.

Animal	Life span (days)	Heartbeat/minute
Mayfly	1	1260
Fly	28	250
Mouse	365	600
Rabbit	3 650	205
Horse	10 950	40
Elephant	31 390	30
Galapagos Tortoise	69 350	6

Hint 1 Use a calculator to find the logarithm of each animal's life span.
Hint 2 Place the logarithms of the animal's life span on the horizontal axis.
Hint 3 The vertical axis will represent the heart rate.
Hint 4 Label axes and points clearly.

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On a log scale:

- In moving from 1 to 2 we are increasing by a factor of 10.
- In moving from 2 to 3 we are increasing by a factor of 10.
- In moving from 2 to 5 we are increasing by a factor of 1000 ($10^3 = 10 \times 10 \times 10$).

Example 17 Applying logarithms

Using the logarithmic values from the table in Example 16, how many times heavier than a rabbit is a giraffe? Give your answer to the nearest hundred.

Explanation	Solution
1 Subtract the logarithm of the weight of a rabbit from the logarithm of the weight of a giraffe.	$5.95 - 3.00$ $= 2.95$
2 The difference between these logarithms means that a giraffe is $10^{2.95}$ times heavier than a rabbit.	
3 Evaluate $10^{2.95}$	$10^{2.95} = 891.25$
4 Write your answer to the nearest hundred.	A giraffe is 900 times heavier than a rabbit.

Now try this 17 Applying logarithms (Example 17)

Using the logarithmic values from the table in Example 16, how many times heavier than a tree kangaroo is an elephant? Give your answer to the nearest whole number.

Hint 1 Subtract the logarithm of a tree kangaroo from the logarithm of an elephant.
Hint 2 Evaluate 10 to the power of this logarithm.

Section Summary

- A logarithm is the power to which a number must be raised in order to get another number.
- The order of magnitude of a physical quantity is its magnitude in powers of 10.
- A logarithmic transformation, $\log_{10}(x)$, can be used to linearise data.

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Exercise 9D

Building understanding

1 Write these numbers as powers of 10.

a 100 **b** 1000 **c** 10 **d** 1 **e** 10 000

2 Write down the logarithm (log) of the following:

a 10^5 **b** 10^8 **c** 10^0 **d** 10^9 **e** $10^{4.5}$

3 Use your calculator to evaluate to 2 decimal places:

a $10^{1.5}$ **b** $10^{2.2}$ **c** $10^{3.8}$ **d** $10^{0.7}$ **e** $10^{6.9}$

4 What is the order of magnitude of:

a 10^4 **b** 10^{33} **c** 10^{100}

Developing understanding

Example 10

5 Write the number as a power of 10, and then write down its logarithm.

a 1000 **b** 1 000 000 **c** 10 000 000 **d** 1 **e** 10

Example 11

6 Use your calculator to evaluate to three decimal places.

a log(300) **b** log(5946) **c** log(10 390) **d** log(7.25)

Example 12

7 Find the numbers, to two decimal places, with logarithms of:

a 2.5 **b** 1.5 **c** 0.5 **d** 0

Example 13

8 What is the order of magnitude of the following numbers?

a 46 000 **b** 559 **c** 3 000 000 000
d 4.21×10^{12} **e** 600 000 000 000

9 A city has two TAFE colleges with 4000 students each. What is the order of magnitude of the total number of TAFE students in the city?

10 At the football stadium, 35 000 people attend a football match each week. What is the order of magnitude of the number of people who attend 8 weeks of games?

11 A builder buys 9 boxes, each containing 1000 screws, to build a deck.

a What is the order of magnitude of the total number of screws?

Once the deck is completed, the number of screws left is 90.

b What is the order of magnitude of the number of screws that are left?

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Example 14

12 Use the $\log_{10}(x)$ transformation to check if plotting the values of $\log_{10}(x)$ and y is linear.

x	20	125	250	500	1000
y	8.6	10.2	10.8	11.4	12

Example 15

13 Use a CAS calculator and the $\log_{10}(x)$ transformation to check if plotting the values of $\log_{10}(x)$ and y is linear.

x	5	50	500	5000
y	3.7	4.7	5.7	6.7

Example 16
Example 17

14 Use the logarithmic values for the animals' weights in Example 16 to find how much heavier a tree kangaroo is than a shrew, to the nearest thousand.

Testing understanding

15 The radius of the planet Jupiter is 69 910 km. What is the order of magnitude of Jupiter's diameter?

16 If the logarithm of a gorilla's body weight (in kg) is 2.3, what is the gorilla's actual weight? Give your answer to two decimal places.

9E Further modelling of non-linear data

Learning intentions

- ▶ To be able to model non-linear data with $kx^2 + c$.
- ▶ To be able to model non-linear data with $\frac{k}{x} + c$.
- ▶ To be able to model non-linear data with $k\log(x) + c$.

Modelling non-linear data

Once we are able to transform data so that we have a straight line, we can then use our knowledge of straight-line graphs to model the non-linear data. We will investigate three different ways of modelling our data. They are, using:

- $y = kx^2 + c$
- $y = \dfrac{k}{x} + c$ where $k > 0$
- $y = k\log(x) + c$ where $k > 0$.

Instead of plotting y against x, we will plot y against x^2, $\dfrac{1}{x}$ or $\log_{10}(x)$. For each of these, k is the slope and c is the y-intercept.

In Chapter 5, we used the straight-line equation: $y = a + bx$, where a is the y-intercept and b is the slope.

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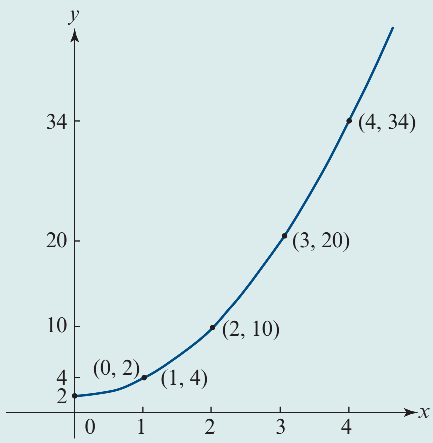
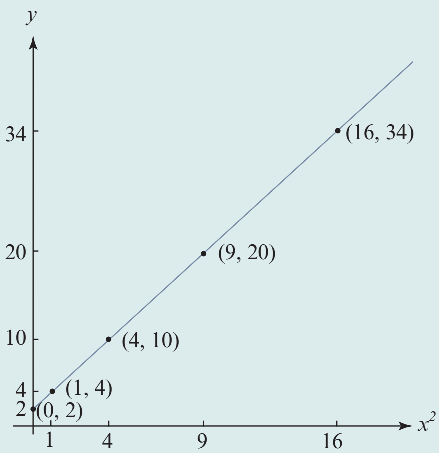
Example 18 Using the model $y = kx^2 + c$

The data in the given table has undergone an x^2 transformation, and the graph of both the original data and the transformed data is shown. Find a rule connecting the variables x and y.

x	0	1	2	3	4
x^2	0	1	4	9	16
y	2	4	10	20	34





Explanation

Since the graph of y against x^2 gives a straight line, we can find the equation of this straight line.

1 Find k by finding the slope of the straight-line graph. Select any two points on the line to find the slope.

Solution: $k = \text{slope} = \dfrac{\text{rise}}{\text{run}}$

Use (4, 10) and (9, 20)

$$k = \frac{20 - 10}{9 - 4} = \frac{10}{5} = 2$$

2 Find the y-intercept (c). This is where the horizontal axis is 0 ($x^2 = 0$).

Alternatively, we can substitute k and a known value for x and y into the equation $y = kx^2 + c$ and solve for c.

Solution: The graph crosses the y-axis at 2.

So $c = 2$.

or

$y = kx^2 + c$ Use (2, 10)

$10 = 2(2^2) + c$

$c = 2$

3 Substitute k and c values into the equation $y = kx^2 + c$.

Solution: $y = 2x^2 + 2$

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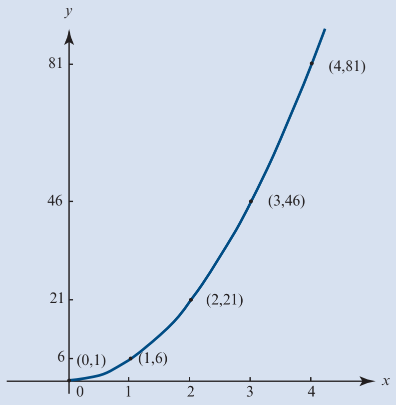
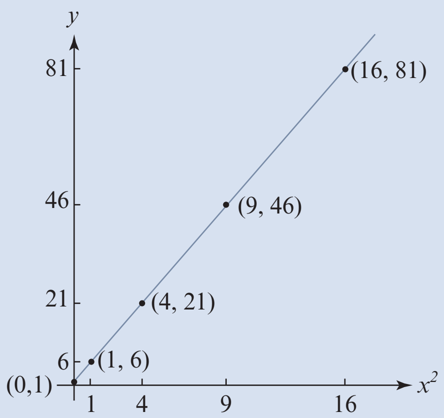
Now try this 18 Using the model $y = kx^2 + c$ (Example 18)

The data in the given table has undergone an x^2 transformation, and the graph of both the original data and the transformed data is shown. Find a rule connecting the variables x and y.

x	0	1	2	3	4
x^2	0	1	4	9	16
y	1	6	21	46	81





Hint 1 Find k, the slope of the straight-line graph.

Hint 2 Find c, the y-intercept.

Hint 3 Substitute k and c in the equation $y = kx^2 + c$.

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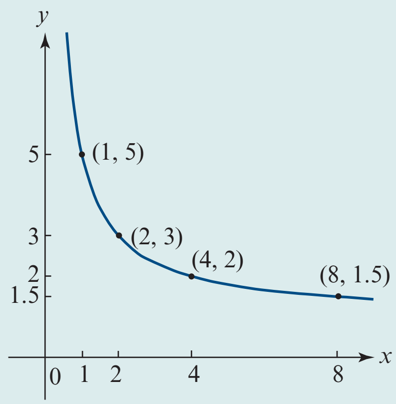
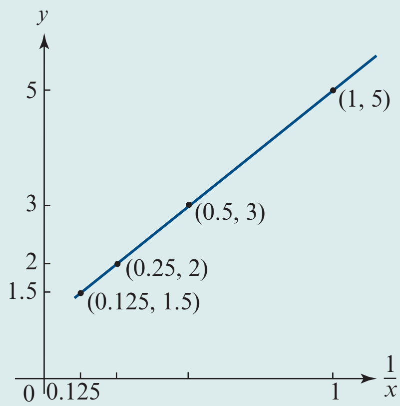
Example 19 Using the model $y = \frac{k}{x} + c$

The data in the given table has undergone a $\frac{1}{x}$ transformation, and the graph of both the original data and the transformed data is shown. Find a rule connecting the variables x and y.

x	1	2	4	8
$\frac{1}{x}$	1	0.5	0.25	0.125
y	5	3	2	1.5





Explanation

Since the graph of y against $\frac{1}{x}$ gives a straight line, we can find the equation of this straight line.

1 The k value is given by the value of the slope. Find k by selecting any two points on the line. e.g. (0.5, 3) and (1, 5).

2 Substitute k and a known value for x and y into the equation $y = \frac{k}{x} + c$ and solve for c.

3 Substitute values for k and c in the equation $y = \frac{k}{x} + c$.

Solution

$k = \text{slope} = \frac{\text{rise}}{\text{run}}$

$k = \frac{5-3}{1-0.5} = \frac{2}{0.5} = 4$

$y = \frac{4}{x} + c$ Use (2, 3)

$3 = \frac{4}{2} + c$

$\therefore c = 1$

$y = \frac{4}{x} + 1$

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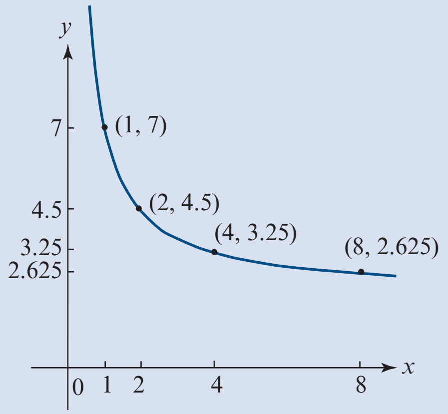
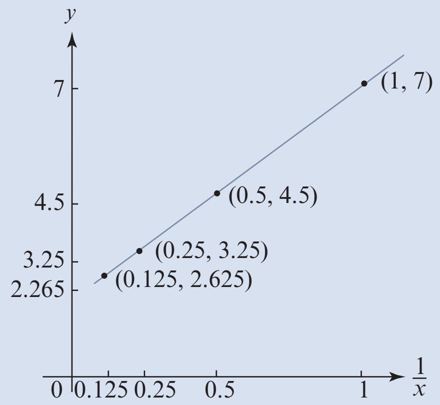
Now try this 19 Using the model $y = \frac{k}{x} + c$ (Example 19)

The data in the given table has undergone a $\frac{1}{x}$ transformation, and the graph of both the original data and the transformed data is shown. Find a rule connecting the variables x and y.

x	1	2	4	8
$\frac{1}{x}$	1	0.5	0.25	0.125
y	7	4.5	3.25	2.625

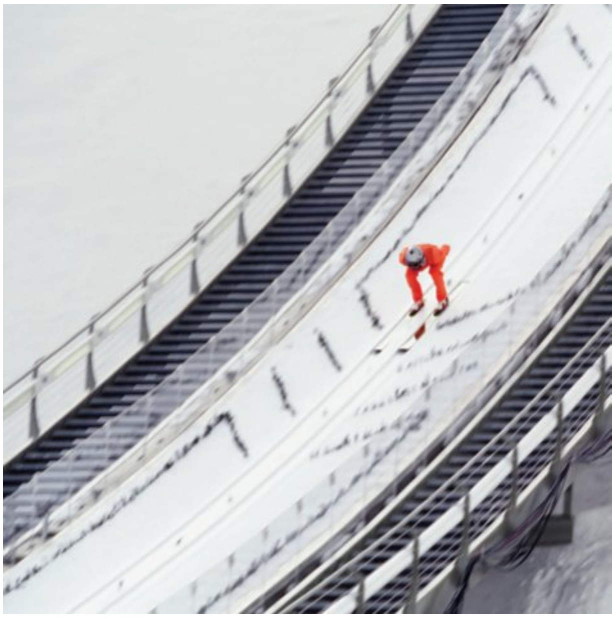




Hint 1 Find k, the slope of the straight-line graph.

Hint 2 Find c, the y-intercept.

Hint 3 Substitute k and c in the equation $y = \frac{k}{x} + c$.



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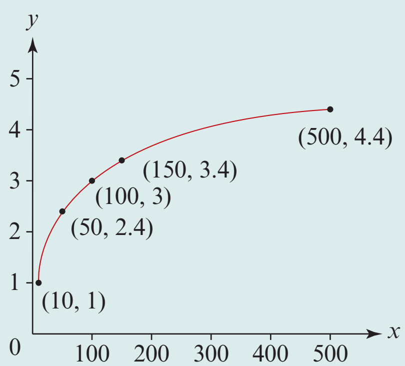
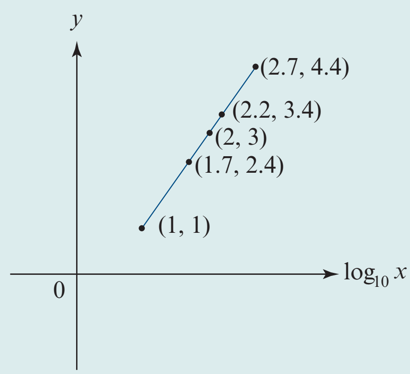
If the data can be linearised with a $\log_{10}(x)$ transformation, we can use the model $y = k\log_{10}(x) + c$.

Example 20 Using the model $y = k\log_{10}(x) + c$

The data in the given table has undergone a $\log_{10}(x)$ transformation, and the graph of the transformed data is shown. Find a rule connecting the variables x and y.

x	10	50	100	150	500
$\log_{10}(x)$	1	1.7	2	2.2	2.7
y	1	2.4	3	3.4	4.4





Explanation

Since the graph of y against $\log_{10}(x)$ gives a straight line, we can find the equation of this straight line.

1 The k value is given by the value of the slope.
Find k by selecting any two points on the line, e.g. (1.7, 2.4) and (2, 3).

2 Find the y-intercept (c).
Substitute k and a known value for x and y into the equation $y = k\log_{10}(x) + c$ and solve for c.

3 Substitute k and c values into the equation $y = k\log_{10}(x) + c$.

Solution

$k = \text{slope} = \dfrac{\text{rise}}{\text{run}}$

$k = \dfrac{3 - 2.4}{2 - 1.7} = \dfrac{0.6}{0.3} = 2$

$y = k\log_{10}(x) + c$ Use (10, 1)

$1 = 2\log_{10}(10) + c$

$\therefore c = -1$

$y = 2\log_{10}(x) - 1$

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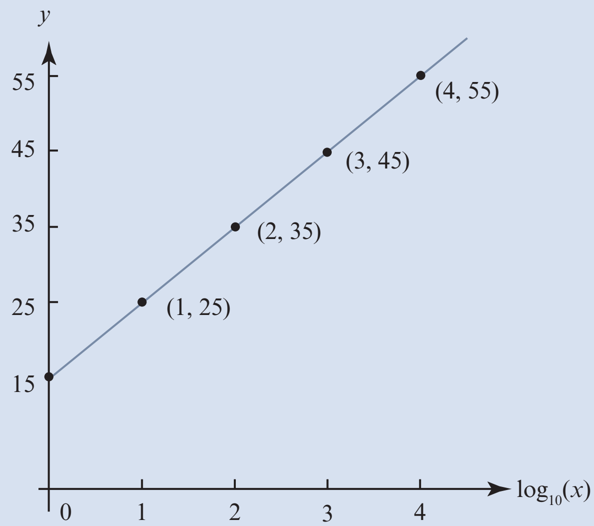
Now try this 20 Using the model $y = k\log_{10}(x) + c$ (Example 20)

The data in the given table has undergone a $\log_{10}(x)$ transformation, and the graph of the transformed data is shown. Find a rule connecting the variables x and y.

x	1	10	100	1000	10 000
$\log_{10}(x)$	0	1	2	3	4
y	15	25	35	45	55



Hint 1 Find the slope of the straight-line graph to give k.

Hint 2 c is the y-intercept.

Hint 3 Substitute k and c into the equation $y = k\log_{10}(x) + c$.

Section Summary

- Non-linear data can be modelled using:

1 $kx^2 + c$ **2** $\dfrac{k}{x} + c$ **3** $k\log_{10}(x) + c$

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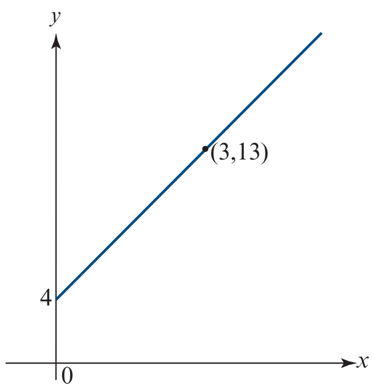
Skill-sheet

Exercise 9E

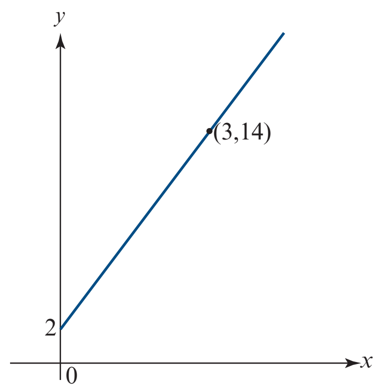
Building understanding

1 For the graphs shown below, give the slope and the y-intercept.

a



b



2 If y is plotted against x^2 for the equation $y = 3x^2 + 7$, what is the value of:

a the slope **b** the y-intercept?

3 If y is plotted against $\frac{1}{x}$ for the equation $y = \frac{2}{x} - 5$, what is the value of:

a the slope **b** the y-intercept?

4 If y is plotted against $\log_{10}(x)$ for the equation $y = 4\log_{10}(x) + 3$, what is the value of:

a the slope **b** the y-intercept?

Developing understanding

Example 18

5 The following data has undergone an x^2 transformation. Complete the table, sketch the graph of y against x^2, and find a rule connecting variables x and y.

x	1	2	3	4
x^2		4		16
y	5	11	21	35

Example 19

6 The following data has undergone a $\frac{1}{x}$ transformation. Complete the table, sketch the graph of y against $\frac{1}{x}$, and find a rule connecting variables x and y.

x	0.1	0.2	0.5	1
$\frac{1}{x}$	10		2	
y	102	52	22	12

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Example 20

7 The following data has undergone a $\log_{10}(x)$ transformation. Complete the table, sketch the graph of y against $\log_{10}(x)$, and find the relationship between x and y.

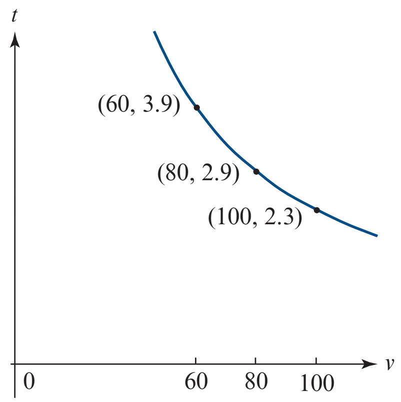
x	10	100	1000	10 000
$\log_{10}(x)$	1			4
y	11	17	23	29

Testing understanding

8 During a mice plague, the population, P, of mice increases as shown in the table, where t is the time in days. Perform a squared transformation on the variable time, t, and find a rule which allows population, P, to be predicted from time, t.

t	1	5	10	20
P	15	135	510	2010

9 The time taken to travel from Santiago to Valparaiso, in Chile, and the average speed travelled by a car is shown in the graph below.



a Find a rule describing the relationship between the time taken to travel from Santiago to Valparaiso, t hours, and the average speed, v km/hr.
Hint: Use $\frac{1}{v}$ transformation.

b If it takes Pablo two hours and six minutes to travel from Santiago to Valparaiso, what was his average speed? Give your answer to the nearest kilometre/hour.

10 The table below gives the number of people, N, infected by a virus after t days. Apply a $\log_{10}(t)$ transformation to find the relationship between N and t.

t	1	10	100	500	1000
N	15	115	215	285	315

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Key ideas and chapter summary

Assignment

Direct variation

y **varies directly** as x is written as $\boldsymbol{y \propto x}$

As x increases, y will also increase.

If $\boldsymbol{y \propto x}$, then $y = kx$, where k is the constant of proportionality.

The graph of y against x is a straight line through the origin.

Inverse variation

y **varies inversely** as x is written as $\boldsymbol{y \propto \frac{1}{x}}$

As x increases, y will decrease.

If $\boldsymbol{y \propto \frac{1}{x}}$ then $y = \frac{k}{x}$, where k is the constant of proportionality.

Order of magnitude

The order of magnitude of a physical quantity is its magnitude in powers of 10.

Transformations to linearity

Relationships between variables can be established by transforming data to linearity. This process is called **linearisation**.

The **squared** transformation involves changing the $\boldsymbol{x}$ values to $\boldsymbol{x^2}$.

The **reciprocal** transformation involves changing the $\boldsymbol{x}$ values to $\boldsymbol{\frac{1}{x}}$.

The **logarithmic** transformation involves changing the $\boldsymbol{x}$ values to $\boldsymbol{\log_{10}(x)}$.

Modelling non-linear data

We can model non-linear data using either:

$$y = kx^2 + c,$$

$$y = \frac{k}{x} \quad \text{or}$$

$$y = k\log_{10}(x) + c \quad \text{where } k > 0.$$

Skills checklist

Checklist

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

9A **1** I can solve for the constant of variation, k, in direct variation. ☐

e.g. The distance (d) travelled by a vehicle is directly proportional to the time (t) taken. Using the information in the following table, find the constant of variation.

Time (t)	1	2	3
Distance (d)	95	190	285

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9A

2 I can solve for the constant of variation, k, in inverse variation. ☐

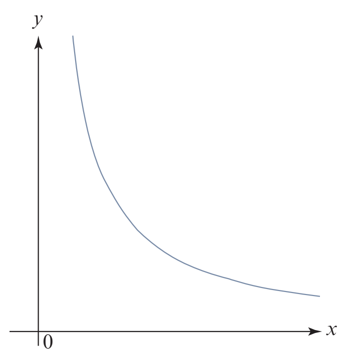
e.g. If F is inversely proportional to d, use the following table to find the constant of variation.

d	10	20	50
F	10	5	2

9B

3 I can determine direct or inverse variation from a graph. ☐

e.g. Does the following graph show direct or inverse variation?



9C

4 I can use the x^2 transformation (the squared transformation). ☐

e.g. Complete the following table giving an x^2 transformation, and check that this linearises the data.

x	1	2	5
x^2			
y	12	27	132

9C

5 I can use the $\frac{1}{x}$ transformation (the reciprocal transformation). ☐

e.g. Complete the following table giving a $\frac{1}{x}$ transformation, and check that this linearises the data.

x	0.5	0.2	0.1
$\frac{1}{x}$			
y	5	35	85

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9D **6** I can evaluate logarithms. ☐

e.g. Find the log of 592 to two decimal places.

9D **7** I can state the order of magnitude of a number. ☐

e.g. What is the order of magnitude of 478 000?

9D **8** I can use and interpret log scales when used to represent quantities that range over multiple orders of magnitude. ☐

e.g. Using the logarithmic values for the mammals' weights in Example 16, find how much heavier than a monkey is an elephant.

9D **9** I can use a calculator to evaluate a number if the logarithm is known. ☐

e.g. Find the number whose logarithm is 2.1567 to one decimal place.

9D **10** I can use the $\log_{10}(x)$ transformation (log transformation). ☐

e.g. Complete the following table giving a $\log_{10}(x)$ transformation, and check that this linearises the data.

x	10	100	1000
$\log_{10}(x)$			
y	3	6	9

9E **11** I can model non-linear data using either: ☐

$$y = kx^2 + c, \quad y = \frac{k}{x} + c \quad \text{or} \quad y = k\log_{10}(x) + c.$$

e.g. Find a rule connecting the variables x and y if the data in the given table has undergone an x^2 transformation.

x	1	5	10	15
x^2	1	25	100	225
y	23	95	320	695

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Multiple-choice questions

1 For the values in the table shown, it is known that $y \propto x$. The value of k, the constant of variation, is equal to:

x	0	3	9	18
y	0	12	36	72

A 1 **B** 3 **C** 4 **D** 9 **E** 12

2 For the values in the table shown, it is known that $y \propto x$. The value of k, the constant of variation, is equal to:

x	2	8	16
y	20	80	160

A 2 **B** 4 **C** 10 **D** 20 **E** 60

3 For the values in the table shown, it is known that $y \propto \frac{1}{x}$. The value of k, the constant of variation, is equal to:

x	1	2	3	4
y	3	1.5	1	0.75

A $\frac{1}{3}$ **B** $\frac{1}{2}$ **C** 1 **D** 2 **E** 3

4 For the values in the table shown, it is known that $y \propto x^2$. The value of k, the constant of variation, is equal to:

x	2	3	6
y	$\frac{4}{3}$	3	12

A 3 **B** 9 **C** $\frac{1}{3}$ **D** 2 **E** $\frac{4}{3}$

5 For the values in the table shown, it is known that $y \propto \frac{1}{x}$. The value of k, the constant of variation, is equal to:

x	2	4	8
y	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

A $\frac{1}{2}$ **B** 1 **C** 4 **D** 2 **E** $\frac{1}{4}$

6 Assume that $a \propto b$, and that $a = 18$ when $b = 3$. If $b = 7$, then $a =$

A 4 **B** 6 **C** 15 **D** 22 **E** 42

7 Assume that $a \propto b^2$ and that $a = 32$ when $b = 2$. If $b = 4$, then $a =$

A 4 **B** 16 **C** 32 **D** 64 **E** 128

8 Assume that $p \propto \frac{1}{q}$ and that $p = \frac{1}{3}$ when $q = 3$. If $p = 1$, then $q =$

A -3 **B** $\frac{1}{3}$ **C** $\sqrt{3}$ **D** 1 **E** -3

9 The given table of values follows the rule: $y = kx^2 + c$.

x	1	2	3	4
y	4	22	52	94

The values of k and c respectively are:

A 6 and 2 **B** 6 and -2 **C** 2 and -1 **D** -2 and 6 **E** 2 and 1

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10 The given table of values follows the rule: $y = \frac{k}{x} + c$.

x	1	2	4	5
y	6	3.5	2.25	2

The values of k and c respectively are:

A 5 and 1 **B** 2 and 3 **C** 2 and 4 **D** 4 and 2 **E** 4 and 1

11 The following data can be modelled by $y = k\log_{10}(x) + c$.

x	1	10	100	1000
y	50	350	650	950

The values of k and c respectively are:

A 1 and 50 **B** 1 and 300 **C** 50 and 0 **D** 50 and −150 **E** 300 and 50

Short-answer questions

1 If $a \propto b$ and $a = 8$ when $b = 2$,

a find a when $b = 50$

b find b when $a = 60$

2 If $y \propto x^2$ and $y = 108$ when $x = 6$,

a find y when $x = 4$

b find x when $y = 90$. Give your answer to two decimal places.

3 If $y \propto \frac{1}{x}$ and $y = 3$ when $x = 2$,

a find y when $x = \frac{1}{2}$

b find x when $y = \frac{2}{3}$

4 The distance, d metres, which an object falls, varies directly as the square of the time, t seconds, for which it has been falling. If an object falls 78.56 m in 4 s, find (to two decimal places):

a the constant of variation

b the formula connecting d and t

c the distance fallen in 10 s

d the time taken to fall 19.64 m (to the nearest second).

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5 The time taken for a journey is inversely proportional to the average speed of travel. If it takes 4 hours travelling at 30 km/h, how long will it take travelling at 50 km/h?

6 For a constant resistance, the voltage (v volts) of an electrical circuit varies directly as the current (I amps). If the voltage is 24 volts when the current is 6 amps, find the current when the voltage is 72 volts.

7 The number of square tiles needed to surface the floor of a hall varies inversely as the square of the side length of the tile used. If 2016 tiles of side length 0.4 m would be needed to surface the floor of a certain hall, how many tiles of side length 0.3 m would be required?

8 The time taken to fill a tank with water varies inversely as the volume of water poured in per minute. It takes 45 minutes to fill a tank using a pipe with a flow rate of 22 litres per minute. What flow rate is being used if it takes 30 minutes to fill the tank?

9 How many times stronger is a magnitude 6 earthquake compared to a magnitude 3 earthquake?

Written-response questions

1 A certain type of hollow sphere is designed in such a way that the mass varies directly as the square of the diameter. Three spheres of this type are made. The first has mass 5 kg and diameter 10 cm, the second has diameter 14 cm and the third has mass 6 kg. Find, to two decimal places:

a the mass of the second sphere

b the diameter of the third sphere.

2 The time taken, t, to paint a building varies inversely as the number, n, of painters working. It takes 6 painters 20 days to paint a building.

a How long will it take 4 painters?

b The building needs to be painted in 15 days. How many painters should be employed?

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3 a The air in a tube occupies 43.5 cm^3 and the pressure is 2.8 kg/cm^2. If the volume (V) varies inversely as the pressure (P), find the formula connecting V and P.

b Calculate the pressure when the volume is decreased to 12.7 cm^3 (to two decimal places).

4 The weight (w kg) which a beam supported at each end will carry without breaking varies inversely as the distance (d m) between supports. A beam which measures 6 m between supports will just carry a load of 500 kg.

a Find the formula connecting w and d.

b What weight could the beam carry if the distance between the supports were 5 m?

c What weight could the beam carry if the distance between the supports were 9 m? Give your answer to two decimal places.

5 The table shows the relationship between the pressure and the volume of a fixed mass of gas when the temperature is constant.

Pressure (p)	12	16	18
Volume (v)	12	9	8

a What is a possible equation relating p and v?

b Using this equation, find:

i the volume when the pressure is 72 units

ii the pressure when the volume is 3 units.

c Sketch the graph relating v and $\frac{1}{p}$.

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Chapter 10

Measurement, scale and similarity

Chapter questions

- ▶ How do we make useful approximate values for measurements?
- ▶ What is Pythagoras' theorem?
- ▶ How do we use Pythagoras' theorem?
- ▶ How do we find the perimeter of a shape?
- ▶ How do we find the area of a shape?
- ▶ What is a composite shape?
- ▶ How do we find the volume of an object?
- ▶ How do we find the surface area of a shape?
- ▶ What does it mean when we say that two figures are similar?
- ▶ What are the tests for similarity for triangles?
- ▶ How do we know whether two solids are similar?
- ▶ How can we use a scale factor to make a similar but larger object?

Measurement explores length, area, volume and capacity. Approximate and more concise ways of writing numbers are reviewed. Pythagoras' theorem is used to find lengths within two-dimensional and three-dimensional objects. Perimeters and areas of various shapes are investigated, including similar shapes, and we explore using scale factor to compare objects of similar shapes but different sizes.

10A Approximations, decimal places and significant figures

Learning intentions

- To be able to round numbers to the required accuracy.
- To be able to express numbers in scientific notation.
- To be able to round numbers to the required significant figures.

Approximations are useful when it is not practical to give exact numerical values. Some numbers are too long (e.g. 0.573 128 9 or 107 000 000 000) to work with, and they are rounded to make calculations easier. Some situations do not require an exact answer, and a stated degree of accuracy is often sufficient.

Rules for rounding

Rules for rounding

1. Look at the value of the digit to the right of the specified digit.
2. If the value is 5, 6, 7, 8 or 9, *round the specified digit up*.
3. If the value is 0, 1, 2, 3 or 4, *leave the specified digit unchanged*.

Example 1 Rounding to the nearest thousand

Round 34 867 to the nearest thousand.

Explanation	Solution
1 Look at the first digit to the right of the thousands. It is an 8.	34 ⇓8 67
2 As it is 5 or more, increase the thousands digit by one. So the 4 becomes a 5. The digits to the right all become zero. Write your answer. Note: 34 867 is closer to 35 000 than 34 000.	35 000

Now try this 1 Rounding to the nearest thousand (Example 1)

Round 57 642 to the nearest thousand.

Decimal places

23.798 is a decimal number with three digits after the decimal point. The first digit after the decimal point (7) is the first (or one) decimal place. Depending on the required accuracy, we round to one decimal place, two decimal places, etc.

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Example 2 Rounding to a number of decimal places

Round 94.738 295 to two decimal places.

Explanation	Solution
1 For two decimal places, count two places to the right of the decimal point and look at the digit to the right (8).	94.738 295
2 As 8 is '5 or more', increase the digit of the second decimal place by one. (3 becomes 4) Write your answer.	= 94.74 (to two decimal places)

Now try this 2 Rounding to a number of decimal places (Example 2)

Round 43.632 697 to two decimal places.

Scientific notation (standard form)

When we work with very large or very small numbers, we often use *scientific notation*, also called *standard form*.

To write a number in scientific notation, we express it as a number between 1 and 10, multiplied by a power of 10. More precisely, we use a number greater than or equal to 1, and less than 10.
For example, 134.7 written in scientific notation is 1.347×10^2.
Similarly, 0.0823 written in scientific notation is 8.23×10^{-2}.

Example 3 Writing a number in scientific notation

Write the following numbers in scientific notation.
a 7 800 000 **b** 0.000 000 5

Explanation	Solution
a **1** Place a decimal point to the right of the first non-zero digit.	7.800 000
2 Count the number of places the decimal point needs to move and whether it is to the left or right.	 Decimal point needs to move 6 places to the right from 7.8 to make 7 800 000.
3 To move the decimal point 6 places to the right, we need to multiply by 10^6. Write your answer.	$7\,800\,000 = 7.8 \times 10^6$

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b **1** Place a decimal point after the first non-zero digit.	5.0
2 Count the number of places the decimal point needs to move and whether it is to the left or right.	7 places 0.0 0 0 0 0 0 5 Decimal point needs to move 7 places to the left from 5.0 to make 0.000 000 5
3 To move the decimal point 7 places to the left, we need to multiply by 10^{-7}. Write your answer.	$0.000\,000\,5 = 5.0 \times 10^{-7}$

Now try this 3 Writing a number in scientific notation (Example 3)

Write the following numbers in scientific notation.

a 670 000 **b** 0.000 006

Example 4 Writing a scientific notation number as a basic numeral

Write the following scientific notation numbers as basic numerals.

a 3.576×10^{7} **b** 7.9×10^{-5}

Explanation	Solution
a **1** Multiplying 3.576 by 10^{7} means that the decimal point needs to be moved 7 places to the right.	3.576×10^{7} 7 places 3.5 7 6 0 0 0 0 × 10^{7}
2 Move the decimal place 7 places to the right and write your answer. Zeroes will need to be added as placeholders.	= 35 760 000
b **1** Multiplying 7.9 by 10^{-5} means that the decimal point needs to be moved 5 places to the left.	7.9×10^{-5} 5 places 0.0 0 0 0 7 9 × 10^{-5}
2 Move the decimal place 5 places to the left and write your answer.	= 0.000 079

Now try this 4 Writing a scientific notation number as a basic numeral (Example 4)

Write the following scientific notation numbers as basic numerals.

a 4.231×10^{6} **b** 8.2×10^{-4}

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All digits that appear in scientific notation are regarded as **significant figures**.

Significant figures

When a football match attendance is reported to be 64 000 people, it means that when measured in thousands, the crowd was estimated to be closer to 64 000 than 63 000 or 65 000. The figures 6 and 4 are called significant because the two figures make a claim to being accurate to the nearest thousand. We say 64 000 is accurate to two *significant figures*. Whereas 64 208 makes a claim of accuracy involving the thousands, hundreds, tens and units to describe the crowd size and is said to be accurate to 5 significant figures: 6, 4, 2, 0 and 8.

When asked to write a number to a required number of significant figures, the many zeroes in numbers such as 26 508 000 000 and 0.000 000 076 can cause confusion. As a first step, write the number in scientific notation. Then round to the required significant figures.

$26\,508\,000\,000 = 2.6508 \times 10^{10}$ is 5 significant figures.

2.651×10^{10} is 4 significant figures.

2.65×10^{10} is 3 significant figures.

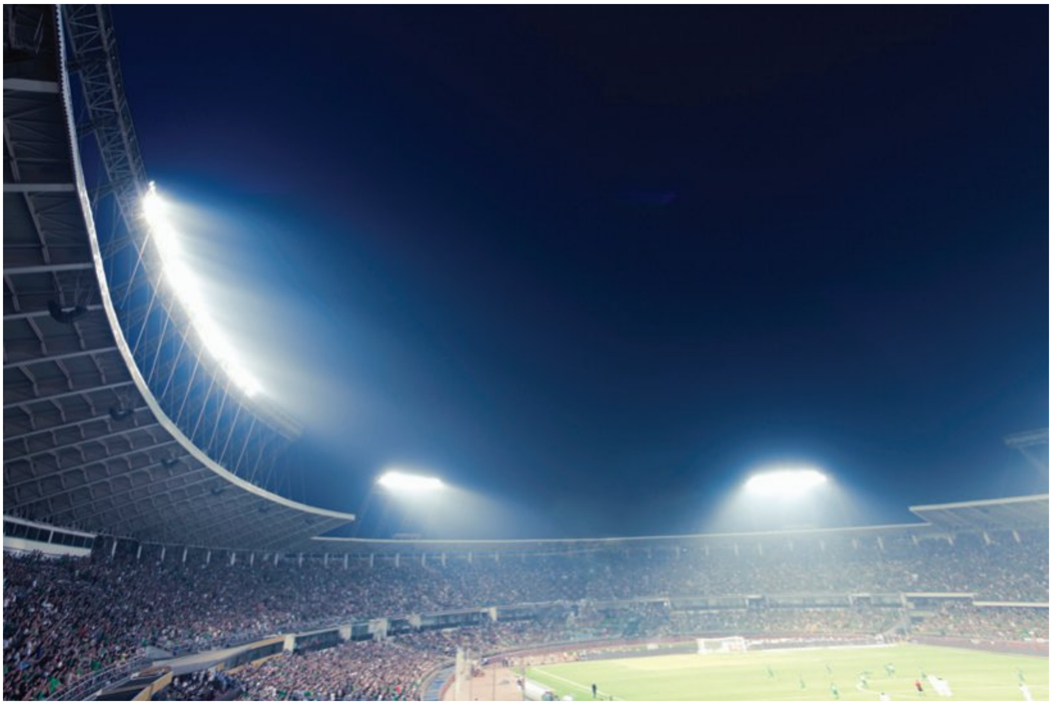
2.7×10^{10} is 2 significant figures.

3×10^{10} is 1 significant figure.

Similarly,

$0.000\,000\,076 = 7.6 \times 10^{-8}$ is 2 significant figures.

8×10^{-8} is 1 significant figure.



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Example 5 Rounding to a certain number of significant figures

Write each number in scientific notation, then round to two significant figures.

a 9764.809 4 **b** 0.000 004 716 8

Explanation	Solution
a **1** To write in scientific notation, put a decimal point after the first non-zero digit, and multiply by the required power of 10.	$9.764\,809\,4 \times 10^3$
2 To round to the second significant figure, check if the third digit is greater than 5.	The third digit (6) is greater than 5, so the second digit must be increased by 1 (so changes from 7 to 8).
3 Round to two significant figures. Write your answer.	$9.8 \times 10^3 = 9\,800$
b **1** To write in scientific notation, put a decimal point after the first non-zero digit, and multiply by the required power of 10.	$4.716\,8 \times 10^{-6}$
2 To round to the second significant figure, check if the third digit is greater than 5.	The third digit is not greater than 5, so the second digit is unchanged.
3 Round to two significant figures. Write your answer.	$4.7 \times 10^{-6} = 0.0000047$

Now try this 5 Rounding to a certain number of significant figures (Example 5)

Round to 3 significant figures.

a 57.892 607 **b** 0.000 471 68

Section Summary

- When rounding a number, look at the value of the digit to the right of the specified digit. Round the specified digit up if the following digit is 5 or more, and leave the specified digit unchanged if the following digit is less than 5.
- To determine the number of **decimal places** in a number, count the digits after the decimal point.
- A number written in **scientific notation** is expressed as a number between 1 and 10, multiplied by a power of 10.
- To write a number to the required number of **significant figures**, write the number in scientific notation then round to the required number of significant figures.

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Exercise 10A

Building understanding

1 By counting the number of places after the decimal point, state how many decimal places the following numbers have?

a 3.473 **b** 40.15 **c** 678.098
d 6.02 **e** 0.0005

2 Round the following to the nearest whole number.

a 3.8 **b** 12.1 **c** 67.02 **d** 556.73

3 How many places to the right does the decimal point need to move to change the following scientific notation numbers to basic numerals?

a 6.43×10^3 **b** 4.01×10^5 **c** 7.1×10^4

4 How many places to the left does the decimal point need to move to change the following scientific notation numbers to basic numerals?

a 8.14×10^{-3} **b** 5.01×10^{-1} **c** 6.2×10^{-2}

Developing understanding

Example 1

5 Round to the nearest hundred.

a 482 **b** 46 770 **c** 79 399 **d** 313.4

6 Round to the nearest dollar.

a $ 689.79 **b** $20.45 **c** $927.58 **d** $13.50

Example 2

7 Use a calculator to find answers to the following. Give each answer to the number of decimal places indicated in the brackets.

a 3.185×0.49 (2) **b** $0.064 \div 2.536$ (3)
c 0.474×0.0693 (2) **d** $12.943 \div 6.876$ (4)

8 Calculate the following to two decimal places.

a $\sqrt{7^2 + 14^2}$ **b** $\sqrt{3.9^2 + 2.6^2}$ **c** $\sqrt{48.71^2 - 29^2}$

Example 3

9 Write these numbers in scientific notation.

a 792 000 **b** 14 600 000 **c** 500 000 000 000 **d** 0.000 009 8
e 0.145 697 **f** 0.000 000 000 06 **g** 2 679 886 **h** 0.0087

Example 4

10 Write these scientific notation numbers as basic numerals.

a 5.3467×10^4 **b** 3.8×10^6 **c** 7.89×10^5 **d** 9.21×10^{-3}
e 1.03×10^{-7} **f** 2.907×10^6 **g** 3.8×10^{-12} **h** 2.1×10^{10}

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11 Express the following approximate numbers using scientific notation.

a The mass of the Earth is 6 000 000 000 000 000 000 000 000 kilograms.

b The circumference of the Earth is 40 000 000 metres.

c The diameter of an atom is 0.000 000 000 1 metre.

d The radius of the Earth's orbit around the Sun is 150 000 000 kilometres.

Example 5

12 Write the following to the number of significant figures indicated in each of the brackets.

a 4.8736 (2) **b** 0.078 74 (3)

c 1506.862 (5) **d** 5.523 (1)

13 Calculate the following and give your answer to the number of significant figures indicated in each of the brackets.

a $4.3968 \times 0.000\,743\,8$ (2) **b** $0.611\,35 \div 4.1119$ (5)

c $3.4572 \div 0.0109$ (3) **d** $50\,042 \times 0.0067$ (3)

Testing understanding

14 Round 35.8997 to three decimal places.

15 Which is the smallest number?

A 7.87×10^{-1} **B** 2.0×10^{0} **C** 0.5×10^{-1}

D 3.21×10^{-3} **E** 0.00067×10^{3}

16 Write the following numbers:

i to the number of significant figures indicated in the brackets.

ii to the number of decimal places indicated in the brackets.

a 421.389 (2) **b** 64.031 (3) **c** 5090.0493 (3)

d 70.549 (2) **e** 0.4573 (2) **f** 0.405 (2)

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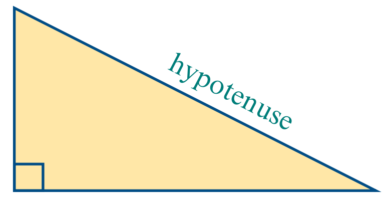
10B Pythagoras' theorem

Learning intentions

- To be able to find the length of an unknown side in a right-angled triangle using Pythagoras' theorem.
- To be able to find the length of an unknown side in a three-dimensional diagram.

Pythagoras' theorem is a relationship connecting the side lengths of a right-angled triangle. In a right-angled triangle, the side *opposite* the **right angle** is called the **hypotenuse**. The hypotenuse is always the longest side of a right-angled triangle.

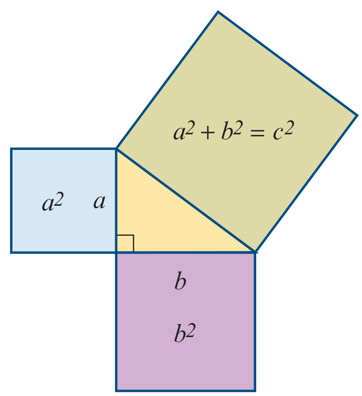


Pythagoras' theorem

Pythagoras' theorem states that, for any right-angled triangle, the sum of the areas of the squares of the two shorter sides (a and b) equals the area of the square of the hypotenuse (c).

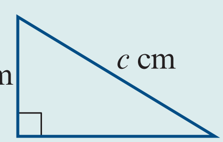
$$a^2 + b^2 = c^2$$

Pythagoras' theorem can be used to find the length of a side in a right-angled triangle when the lengths of the other two sides are known.



Example 6 Using Pythagoras' theorem to calculate the length of the hypotenuse

Calculate the length of the hypotenuse in the triangle opposite to two decimal places.



Explanation	Solution
1 Write Pythagoras' theorem.	$a^2 + b^2 = c^2$
2 Substitute known values.	$4^2 + 10^2 = c^2$
3 Take the square root of both sides, then evaluate.	$c = \sqrt{4^2 + 10^2}$ $= 10.770\ldots$
4 Write your answer to two decimal places, with correct units.	The length of the hypotenuse is 10.77 cm.

Hint: To ensure that you get a decimal answer, set your calculator to approximate or decimal mode. (See the Appendix, which is available through the Interactive Textbook.)

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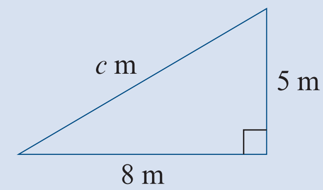
Now try this 6 **Using Pythagoras' theorem to calculate the length of the hypotenuse (Example 6)**

Find the length of the hypotenuse to two decimal places.



Hint 1 The hypotenuse is opposite the right angle.

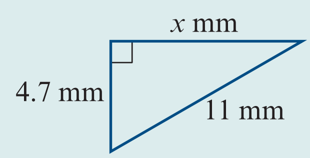
Hint 2 Work to at least three decimal places, then round your answer to two decimal places.

Hint 3 Answer using the correct units.

Pythagoras' theorem can also be rearranged to find sides other than the hypotenuse.

Example 7 **Using Pythagoras' theorem to calculate the length of an unknown side in a right-angled triangle**

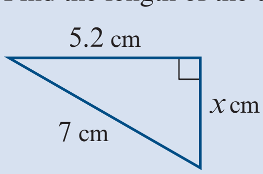
Calculate the length of the unknown side, x, in the triangle opposite to one decimal place.



Explanation	Solution
1 Write Pythagoras' theorem.	$a^2 + b^2 = c^2$
2 Substitute known values and the given variable.	$x^2 + 4.7^2 = 11^2$
3 Rearrange the formula to make x the subject, then evaluate.	$x = \sqrt{11^2 - 4.7^2}$ $= 9.945\ldots$
4 Write your answer to one decimal place, with correct units.	The length of x is 9.9 mm to one decimal place.

Now try this 7 **Using Pythagoras' theorem to calculate the length of an unknown side in a right-angled triangle (Example 7)**

Find the length of the unknown side to one decimal place.



Hint 1 Notice that the unknown is not the hypotenuse in this question.

Hint 2 Work to at least two decimal places, then round your answer to one decimal place.

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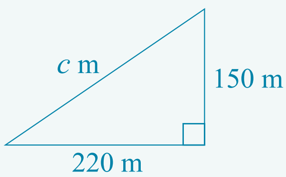
Pythagoras' theorem can be used to solve many practical problems.

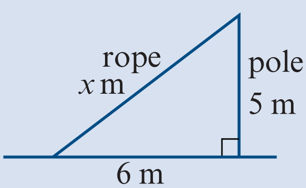
Example 8 **Using Pythagoras' theorem to solve a practical problem**

A helicopter hovers at a height of 150 m above the ground and is a horizontal distance of 220 m from a landing pad. Find the direct distance of the helicopter from the landing pad to two decimal places.

Explanation	Solution
1 Draw a diagram to show which distance is to be found.	
2 Write Pythagoras' theorem.	$c^2 = a^2 + b^2$
3 Substitute known values.	$c^2 = 150^2 + 220^2$
4 Take the square root of both sides, then evaluate.	$c = \sqrt{150^2 + 220^2}$ $= 266.270\ldots$
5 Write your answer to two decimal places, with correct units.	The helicopter is 266.27 m from the landing pad.

Now try this 8 **Using Pythagoras' theorem to solve a practical problem (Example 8)**

A rope tied to the top of a 5 m pole is secured to the ground by a peg, 6 m from the base of the pole. What is the length of the rope to one decimal place?



Pythagoras' theorem in three dimensions

When solving three-dimensional problems, it is essential to carefully draw diagrams. In general, to find lengths in solid figures, we must first identify the correct right-angled triangle in the plane containing the unknown side. Remember, a plane is a flat surface, such as the cover of a book or a tabletop.

Once it has been identified, the right-angled triangle should be drawn separately from the solid figure, displaying as much information as possible.

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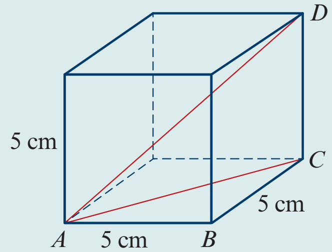
Example 9 Using Pythagoras' theorem in three dimensions

The cube in the diagram on the right has side lengths of 5 cm.

Find the length:

a AC to one decimal place.

b AD to one decimal place.



Explanation

a **1** Locate the relevant right-angled triangle in the diagram.
Draw the right-angled triangle ABC that contains AC, and then mark in the known side lengths.

2 Using Pythagoras' theorem, calculate the length AC.
Retain the lengthy decimal in your calculator for use in part **b**.

3 Write your answer with correct units and to one decimal place.

b **1** Locate the relevant right-angled triangle in the diagram.

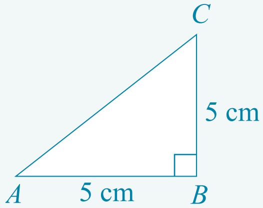
2 Draw the right-angled triangle ACD that contains AD, and mark in the known side lengths. From part **a**, put $AC = 7.07$ cm, working with at least one more decimal place than the answer requires.

3 Using Pythagoras' theorem, calculate the length AD.

4 Write your answer with correct units and to one decimal place.

Solution

a



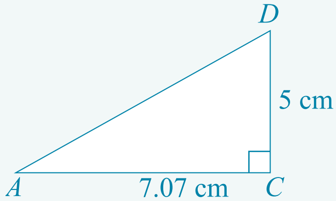
$$(AC)^2 = (AB)^2 + (BC)^2$$

$$\therefore AC = \sqrt{5^2 + 5^2}$$

$$= 7.071\ldots$$

The length AC is 7.1 cm to one decimal place.

b



$$(AD)^2 = (AC)^2 + (CD)^2$$

$$\therefore AD = \sqrt{7.07^2 + 5^2}$$

$$= 8.659\ldots$$

The length AD is 8.7 cm to one decimal place.

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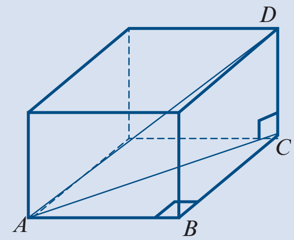
Now try this 9 Using Pythagoras' theorem in three dimensions (Example 9)

In the box shown, $AB = 8$ cm, $BC = 9$ cm and $CD = 6$ cm. Find to one decimal place:

a The length AC.

b The length AD.



Hint 1 In three-dimensional diagrams, right angles often appear distorted.

Hint 2 Identify the right-angled triangles, and draw them separately as they would appear, flat on the page.

Hint 3 Check: The diagonal to the opposite corner through a box with sides a, b and c is given by $\sqrt{a^2 + b^2 + c^2}$.

Section Summary

- In a right-angled triangle, the **hypotenuse** is the longest side and is opposite the right angle.
- Pythagoras' theorem states that, in any right-angled triangle:

$$c^2 = a^2 + b^2$$

  where c is the length of the hypotenuse and the other two sides have lengths of a and b.
- Given the lengths of any two sides in a right-angled triangle, Pythagoras' theorem can be used to find the length of the third side.
- Pythagoras' theorem can be used to find a distance in a three-dimensional figure. Draw a separate diagram for each triangle with a required unknown side.

Skill-sheet

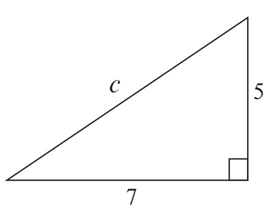
Exercise 10B

Building understanding

1 **a** Write Pythagoras' theorem for a right-angled triangle with sides a, b and c, where c is the hypotenuse.

b Complete: Let $a = 5$ and $b = ...$

c Substitute the values of a and b into the rule for Pythagoras' theorem from part **a**.



d Find the value of $a^2 + b^2$, then take the square root to find the value of c to one decimal place.

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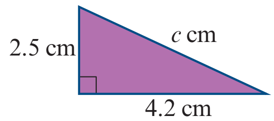
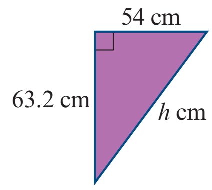
2 Find the length of the unknown side in each of these triangles to one decimal place.

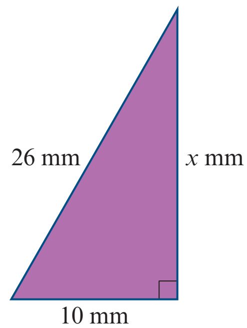
a



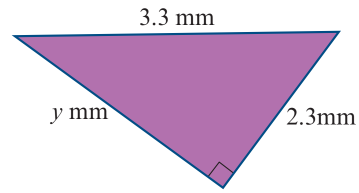
b



c



d

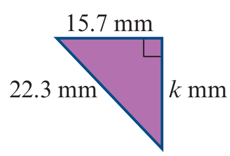


Developing understanding

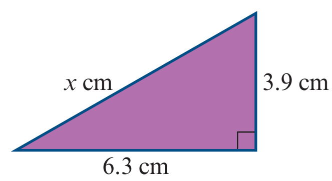
Example 6
Example 7

3 Determine the length of the unknown side in each of these triangles to one decimal place.

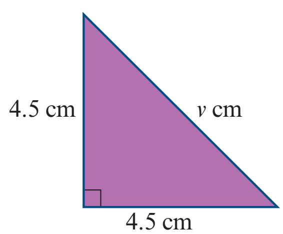
a



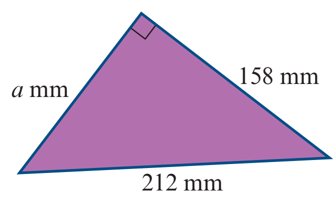
b



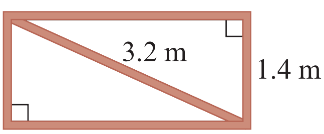
c



d



4 A farm gate that is 1.4 m high is supported by a diagonal bar of length 3.2 m. Find the width of the gate to one decimal place.



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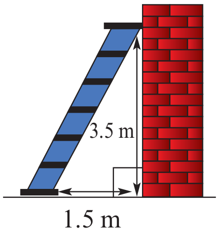
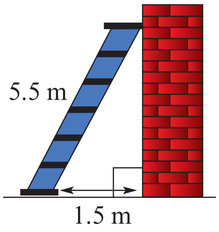
Example 8

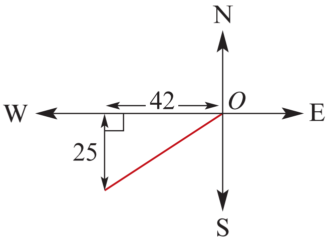
5 A ladder rests against a brick wall, as shown in the diagram on the right. The base of the ladder is 1.5 m from the wall, and the top reaches 3.5 m up the wall. Find the length of the ladder to one decimal place.



6 The base of a ladder leaning against a wall is 1.5 m from the base of the wall. If the ladder is 5.5 m long, find how high the top of the ladder is from the ground to one decimal place.

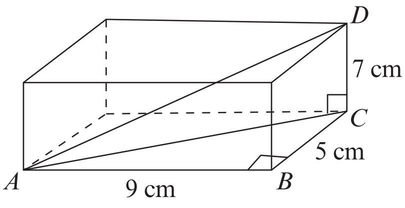


7 A ship sails 42 km due west and then 25 km due south. How far is the ship from its starting point? Answer to two decimal places.



8 A flying fox on a school camp starts from a tower 25 m high and finishes on the ground, 100 metres from the base of the tower. What is the distance travelled along the flying fox to the nearest metre?

9 Follow the steps below to find the length of the diagonal AD in the given diagram to one decimal place.



a The right-angled triangle ABC has sides AC, AB and BC. Draw the right-angled triangle ABC, then write Pythagoras' theorem for the sides AC, AB and BC.

b Substitute the values for sides AB and BC into your equation from part **a**.

c Calculate the value of $AB^2 + BC^2$, then take the square root to find the length AC. Keep the value in your calculator to use in part **e**.

The right-angled triangle ACD has sides AC, CD and AD.

d Draw the right-angled triangle ACD, then write Pythagoras' theorem for the sides AD, AC and CD.

e Substitute the values for sides AC and CD into your equation from part **c**.

f Calculate the value of $AC^2 + CD^2$, then take the square root to find the length AD to one decimal place.

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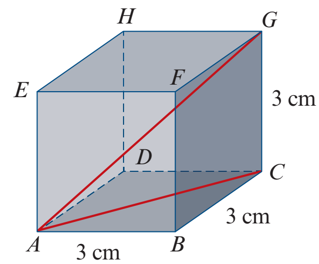
Example 9

10 The cube shown in the diagram has sides of 3 cm. Find the length of:

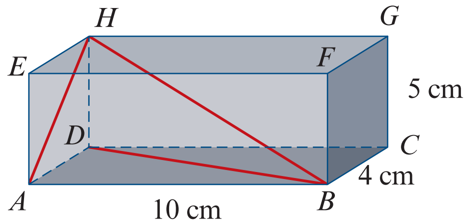
a AC to three decimal places

b AG to two decimal places.



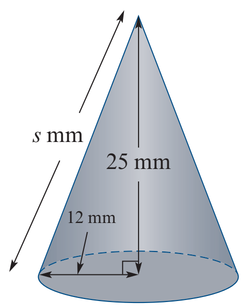
11 For this cuboid, calculate, to two decimal places, the lengths:

a DB **b** BH **c** AH

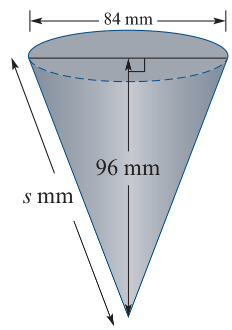


12 Find the sloping height, s, of each of the following cones to two decimal places.

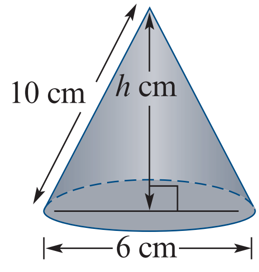
a



b



13 The slant height of this circular cone is 10 cm and the diameter of its base is 6 cm. Calculate the height of the cone to two decimal places.

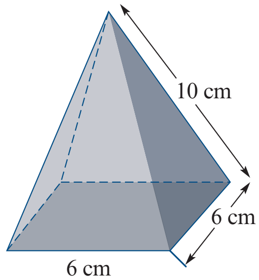


14 For each of the following square-based pyramids, find to one decimal place:

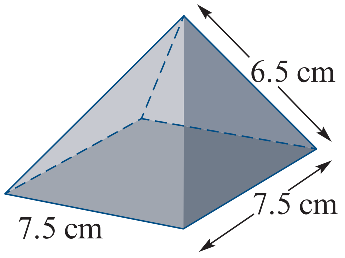
i the length of the diagonal on the base

ii the height of the pyramid.

a



b



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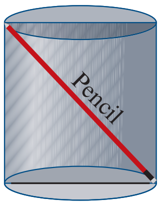
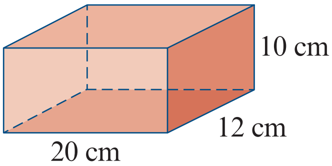
15 Find the length of the longest pencil that will fit inside a cylinder with height 15 cm and a diameter of 8 cm.



16 Chris wants to use a rectangular pencil box. What is the length of the longest pencil that would fit inside the box shown on the right? (Answer to the nearest centimetre.)

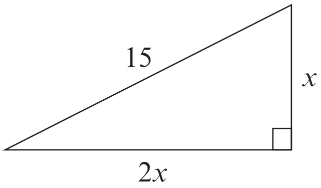


17 A broomstick is 145 cm long. Would it be able to fit in a cupboard measuring 45 cm by 50 cm and height 140 cm?

18 In the primate enclosure at the zoo, a rope is to be attached from the bottom corner of the enclosure to the opposite top corner for the monkeys to swing and climb on. If the enclosure measures 8 m by 10 m by 12 m, what is the length of the rope? Give your answer to two decimal places.

Testing understanding

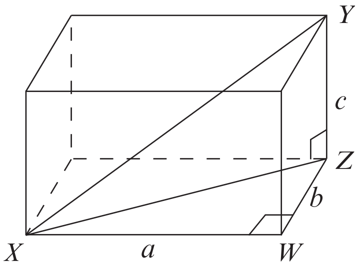
19 Find the value of x to one decimal place.



20 **a** In right-angled triangle WXZ, find an expression for $(XZ)^2$ in terms of a and b.

b In right-angled triangle XYZ, find an expression for $(XY)^2$ in terms of $(XZ)^2$ and $(YZ)^2$.

c Hence, find an expression for the length XY in terms of a, b and c.



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10C Perimeter and area

Learning intentions

- To be able to determine the perimeters and areas of regular shapes, such as rectangles, parallelograms, trapeziums and triangles.
- To be able to find the perimeter and area of composite shapes.
- To be able to find the circumference and area of a circle when given its radius.

Mensuration is a part of mathematics that looks at the measurement of length, area and volume. It comes from the Latin word *mensura*, which means 'measure'.

Perimeters of regular shapes

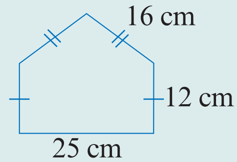
Perimeter

The **perimeter** of a two-dimensional shape is the total distance around its edge.

Example 10 Finding the perimeter of a shape

Find the perimeter of the shape shown.

The same dashes indicate sides of the same length.



Explanation

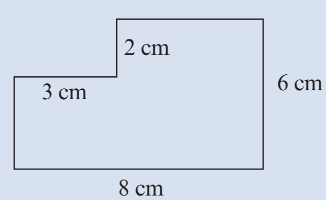
To find the perimeter, add up all the side lengths of the shape.

Solution

$$\begin{aligned}\text{Perimeter} &= 25 + 12 + 12 + 16 + 16\\ &= 81 \text{ cm}\end{aligned}$$

Now try this 10 Finding the perimeter of a shape (Example 10)

By first finding the unknown lengths, find the perimeter of the given shape.



Hint 1 Where a length is not shown, look at the opposite side of the rectangle.

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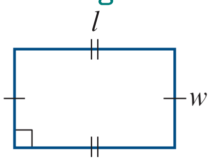
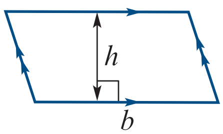
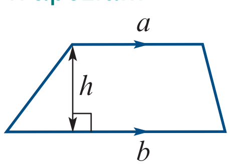
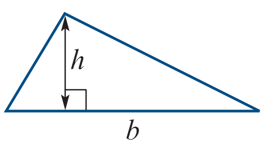
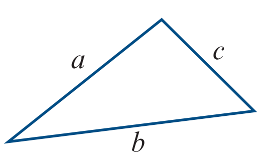
Areas of regular shapes

Area

The **area** of a shape is a measure of the region enclosed by its boundaries.

When calculating area, the answer will be in *square units*, i.e. $\text{mm}^2, \text{cm}^2, \text{m}^2, \text{km}^2$.

Shape	Area	Perimeter
Rectangle 	$A = lw$	$P = 2l + 2w$ or $P = 2(l + w)$
Parallelogram 	$A = bh$	Sum of four sides
Trapezium 	$A = \frac{1}{2}(a + b)h$	Sum of four sides
Triangle 	$A = \frac{1}{2}bh$	Sum of three sides
Heron's formula for finding the area of a triangle with three side lengths known. 	$A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$ (s is the semi-perimeter)	$P = a + b + c$

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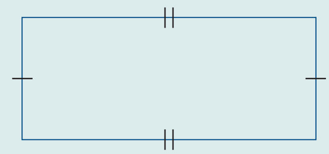
Example 11 Finding the perimeter of a rectangle

Find the perimeter of the rectangle shown.



Explanation

1 Since the shape is a rectangle, use the formula $P = 2l + 2w$.

2 Substitute length and width values into the formula. Evaluate.

3 Give your answer with correct units.

Solution

$$\begin{aligned} P &= 2l + 2w \\ &= 2 \times 12 + 2 \times 5 \\ &= 34\text{ cm} \end{aligned}$$

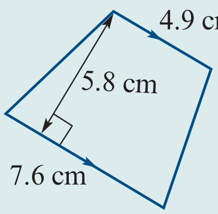
The perimeter is 34 cm.

Now try this 11 Finding the perimeter of a square (Example 11)

Find the perimeter of a square with sides of 3 m.

Example 12 Finding the area of a shape

Find the area of the given shape.



Explanation

1 Since the shape is a trapezium, use the formula $A = \frac{1}{2}(a + b)h$.

2 Substitute the values for a, b and h.

3 Evaluate.

4 Give your answer with correct units.

Solution

$$\begin{aligned} A &= \frac{1}{2}(a + b)h \\ &= \frac{1}{2}(4.9 + 7.6)5.8 \\ &= 36.25\text{ cm}^2 \end{aligned}$$

The area of the shape is 36.25 cm^2.

Now try this 12 Finding the area of a shape (Example 12)

Find the area of a trapezium with a distance of 8 cm between the two parallel sides that have lengths of 10 cm and 14 cm.

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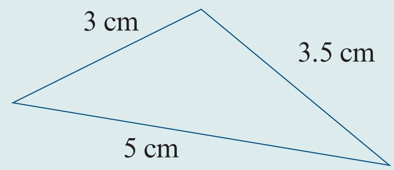
Example 13 **Finding the area of a triangle using Heron's formula**

Find the area of the following triangle. Give your answer to two decimal places.



Explanation	Solution
1 Since the height of the triangle is not given, you need to use Heron's formula as the three side lengths are known.	
2 Write down Heron's formula.	$A = \sqrt{s(s-a)(s-b)(s-c)}$
3 Find the perimeter of the triangle by adding the three side lengths.	$P = 3 + 3.5 + 5$ $= 11.5$
4 Divide the perimeter by 2 to find s, the semi-perimeter.	$s = \frac{11.5}{2}$ $= 5.75$
5 Substitute the value for s into Heron's formula to find the area of the triangle.	$A = \sqrt{5.75(5.75-3)(5.75-3.5)(5.75-5)}$ $= 5.16562\ldots$
6 Give your answer to two decimal places and with correct units.	The area of the triangle is 5.17 cm^2 to two decimal places.

Now try this 13 **Finding the area of a triangle using Heron's formula (Example 13)**

Find the area of a triangle with sides of 4 m, 5 m and 6 m to one decimal place.

The formulas for area and perimeter can be applied to many practical situations.

Example 14 **Finding the area and perimeter in a practical problem**

A display board for a classroom measures 150 cm by 90 cm.

a If ribbon costs \$0.55 per metre, how much will it cost to add a ribbon border around the display board?

b The display board is to be covered with yellow paper. What is the area to be covered? Give you answer in m^2 to two decimal places.

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Explanation	Solution
a **1** To find the length of ribbon required, we need to work out the perimeter of the display board. The display board is a rectangle, so use the formula: $P = 2l + 2w$	$P = 2l + 2w$
2 Substitute $l = 150$ and $w = 90$ and evaluate.	$P = 2(150) + 2(90)$ $= 480$ cm of ribbon.
3 To convert from centimetres to metres, divide by 100.	$P = 480 \div 100$ $= 4.8$ m
4 To find the cost of the ribbon, multiply the length of the ribbon by \$0.55. Write your answer.	$4.8 \times 0.55 = \$2.64$ The cost of ribbon is \$2.64.
b **1** To find the area, use $A = lw$.	$A = lw$
2 Substitute $l = 150$ and $w = 90$ and evaluate.	$= 150 \times 90$ $= 13\,500 \text{ cm}^2$
3 Convert your answer to m^2 by dividing by $(100 \times 100 = 10\,000)$.	$A = 13\,500 \div 10\,000$ $= 1.35$
4 Write your answer with correct units.	Area to be covered is 1.35 m^2.

Now try this 14 **Finding the area and perimeter in a practical problem (Example 14)**

The paint in a 2-litre can covers 6 m^2. How many litres will be needed to paint a wall which is 3 m high and 9 m long?

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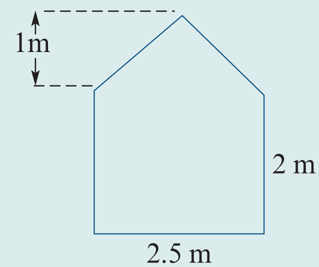
Composite shapes

Composite shapes

A composite shape is a shape that is made up of two or more basic shapes.

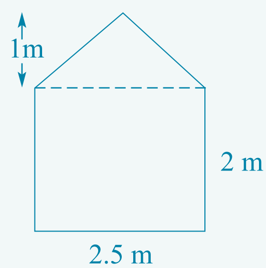
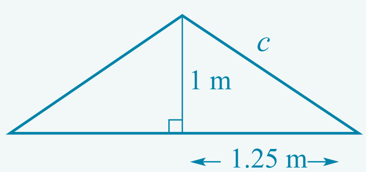
Example 15 Finding the perimeter and area of a composite shape in a practical problem

A gable window at a reception venue is to have LED lights around its perimeter (but not along the bottom of the window). The window is 2.5 m wide and the height of the room is 2 m. The height of the gable is 1 m, as shown in the diagram.



a Calculate the length of LED lights needed to two decimal places.

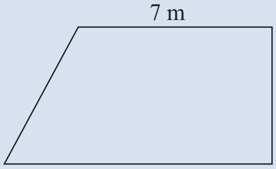
b The glass in the window needs to be replaced. Find the total area of the window to two decimal places.

Explanation	Solution
a The window is made of two shapes: a rectangle and a triangle.	
1 First find the length of the slant edge of the triangle. Draw a diagram and label the slant edge as c.	
Use Pythagoras' theorem to find c. Note: The length of the base of the triangle is 1.25 m ($\frac{1}{2}$ of 2.5 m).	$c^2 = 1^2 + 1.25^2$ $\therefore c = \sqrt{1^2 + 1.25^2}$ $c = 1.6007\ldots$ $c = 1.600$ m
2 Add all the outside edges of the window, but do not include the bottom length.	$2 + 2 + 1.600 + 1.600 =$ 7.20 to two decimal places.
3 Write your answer with correct units.	Require 7.20 m of LED lights.

b **1** To find the total area of the window, first find the area of the rectangle by using the formula $A = bh$.	$A = bh$
2 Substitute the values for b and h.	$= 2.5 \times 2$
3 Evaluate and write your answer with correct units.	$= 5\text{ m}^2$
4 Find the area of the triangle by using the formula $A = \frac{1}{2}bh$.	$A = \frac{1}{2}bh$
5 Substitute the values for b and h.	$= \frac{1}{2} \times 2.5 \times 1$
6 Evaluate and write your answer with correct units.	$= 1.25\text{ m}^2$
7 To find the total area of the window, add the area of the rectangle and the area of the triangle.	Total area = area of rectangle + area of triangle $= 6.25\text{ m}^2$
8 Give your answer with correct units to two decimal places.	Total area of window is 6.25 m^2 to two decimal places.

Now try this 15 Finding the perimeter and area of a composite shape in a practical problem (Example 15)

A courtyard is being renovated and the landscape gardener needs to order the materials required.



a Find the area of grass that will be needed for the courtyard.

b What will be the total length of wooden border required?

Hint 1 To find the length of the sloping edge, consider it as part of a right-angled triangle.

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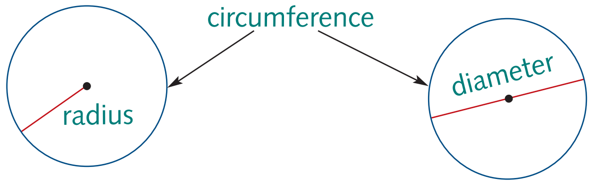
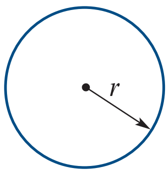
The circumference and area of a circle

The perimeter of a circle is also known as the **circumference** (C) of the circle.



The area and the circumference of a circle are given by the following formulas.

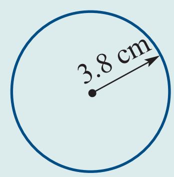
	Area	Circumference
Circle  	$A = \pi r^2$ where r is the **radius**	$C = 2\pi r$ or $C = \pi d$ where d is the **diameter**

Example 16 Finding the circumference and area of a circle

For the circle shown, find:

a the circumference to one decimal place

b the area to one decimal place.



Explanation	Solution
a 1 For the circumference, use the formula $C = 2\pi r$.	$C = 2\pi r$
2 Substitute $r = 3.8$ and evaluate.	$= 2\pi \times 3.8$ $= 23.876\ldots$
3 Give your answer to one decimal place and with correct units.	The circumference of the circle is 23.9 cm to one decimal place.
b 1 To find the area of the circle, use the formula $A = \pi r^2$.	$A = \pi r^2$
2 Substitute $r = 3.8$ and evaluate.	$= \pi \times 3.8^2$ $= 45.364\ldots$
3 Give your answer to one decimal place and with correct units.	The area of the circle is 45.4 cm^2 to one decimal place.

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Now try this 16 **Finding the circumference and area of a circle (Example 16)**

A circle has a radius of 12 metres. Find to one decimal place:

a the circumference of the circle

b the area of the circle.

Hint 1 Take care to give the correct units for the area. This will depend on the units for the radius that were used in the question.

Section Summary

- The perimeter of a two-dimensional shape is the total distance around its edge.
- The areas of common shapes can be found using the formulas:

Rectangle	Area $= lw$	$l =$ length, $w =$ width
Parallelogram	Area $= bh$	$b =$ base, $h =$ height
Trapezium	Area $= \frac{1}{2}(a+b)h$	a, b are the parallel sides, $h =$ height
Triangle	Area $= \frac{1}{2}bh$	$b =$ base, $h =$ height

When the three sides a, b and c are known, use Heron's formula:

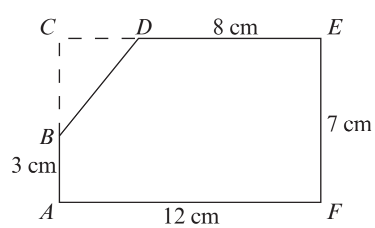
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{1}{2}(a+b+c)$$

- The circumference of a circle is given by $C = 2\pi r$, where r is the radius and π is a constant.
- The area of a circle can be calculated using $A = \pi r^2$, where r is the radius.

Exercise 10C

Building understanding

1 Follow the steps to find the area of the shape $ABDEF$ to one decimal place.



a Find the area of the rectangle $ACEF$.

b Find the length of the side BC and the side CD.

c Find the area of the triangle BCD.

d Find the area $ABDEF$ by subtracting the answer to part **c** from the answer to part **a**.

2 A circle has a radius of 7 cm. Substitute $r = 7$ and complete the following:

a Find, to two decimal places, the circumference.

Use $C = 2\pi r$
$= 2\pi(...)$
$= ...$ cm

b Find, to two decimal places, the area of the circle.

Use $A = \pi r^2$
$= \pi(...)^2$
$= ...\ \text{cm}^2$

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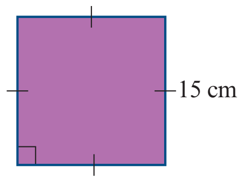
Developing understanding

Example 10
Example 11

3 For each of the following shapes, find to one decimal place:

i the perimeter **ii** the area

a



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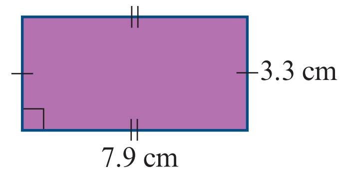

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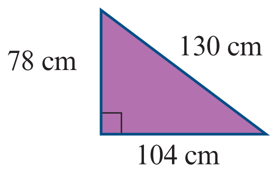

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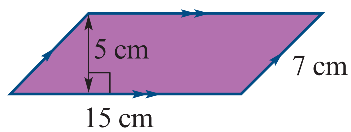
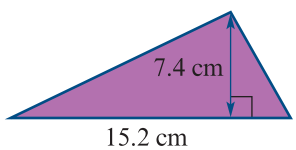


Example 12
Example 13

4 Find the areas of the given shapes to one decimal place where appropriate.

a



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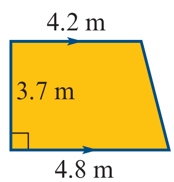

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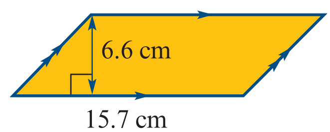

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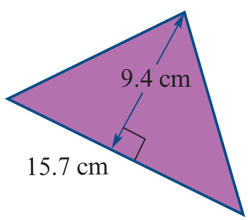

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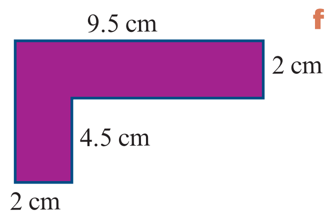

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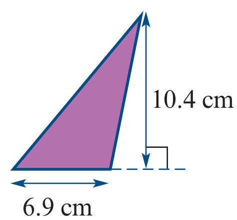

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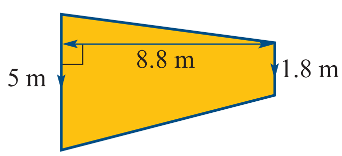

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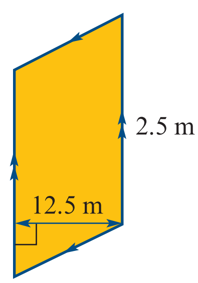

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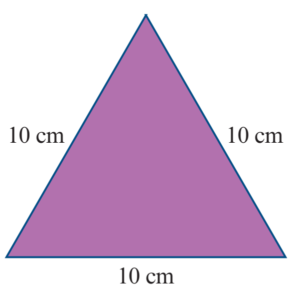

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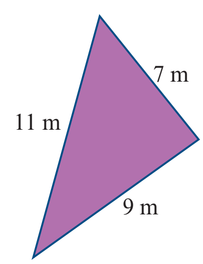

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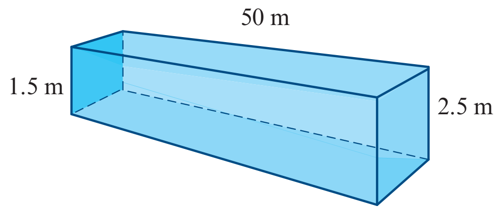
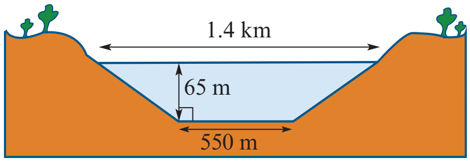
5 A 50 m swimming pool increases in depth from 1.5 m at the shallow end to 2.5 m at the deep end, as shown in the diagram (*not* to scale). Calculate the area of a side wall of the pool.



6 A dam wall is built across a valley that is 550 m wide at its base and 1.4 km wide at its top, as shown in the diagram (*not* to scale). The wall is 65 m deep. Calculate the area of the dam wall.



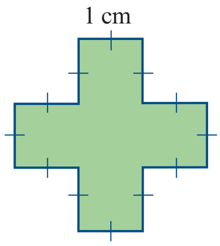
Example 14

7 Ray wants to tile a rectangular area measuring 1.6 m by 4 m outside her holiday house. The tiles that she wishes to use are 40 cm by 40 cm. How many tiles will she need?

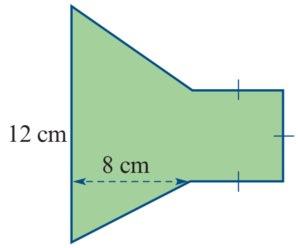
8 One litre of paint covers 9 m^2. How much paint is needed to paint a wall measuring 3 m by 12 m?

9 Find the area of the following composite shapes.

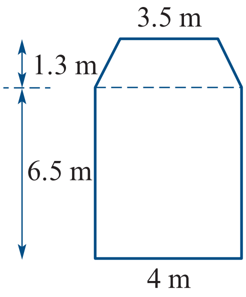
a



b



10 A driveway, as shown in the diagram, is to be paved. What is the area of the driveway to two decimal places?

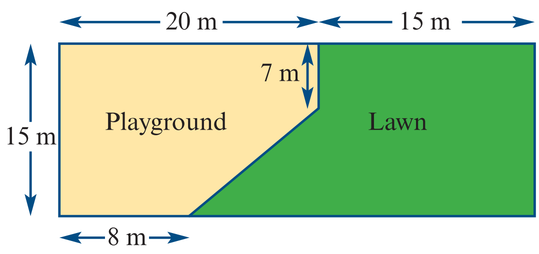


Example 15

11 A local council plans to fence a rectangular piece of land to make a children's playground and a lawn, as shown. (Not drawn to scale.)

a What is the area of the children's playground?

b What is the area of the lawn?



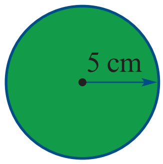
Example 16

12 For each of the following circles, find:

i the circumference to one decimal place

ii the area to one decimal place.

a



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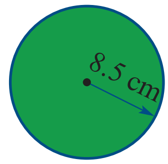
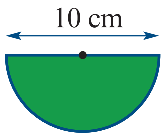


13 For each of the following shapes, find:

i the perimeter to two decimal places

ii the area to two decimal places.

a



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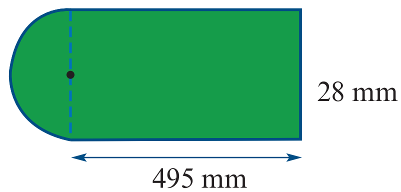

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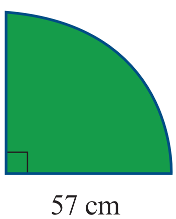

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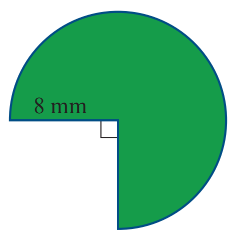
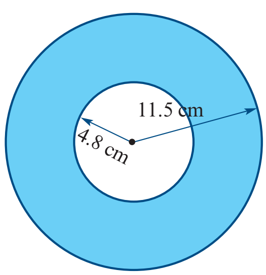


14 Find the shaded areas in the following diagrams to one decimal place.

a



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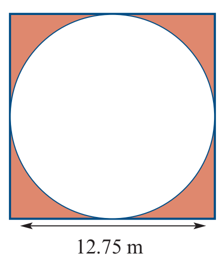

15 A fence needs to be built around a sports field that has two parallel straight sides, 400 m long, and semicircular ends with a diameter of 80 m.

a What length of fencing, to two decimal places, is required?

b What area will be enclosed by the fencing to two decimal places?

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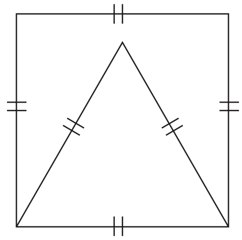
16 Three juggling rings cut from a thin sheet are to be painted. The diameter of the outer circle of the ring is 25 cm and the diameter of the inside circle is 20 cm. If both sides of the three rings are to be painted, what is the total area to be painted? (Ignore the inside and outside edges.) Round your answer to the nearest cm^2.

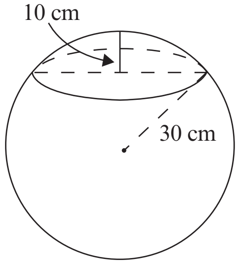
17 A path 1.2 m wide surrounds a circular garden bed whose diameter is 7 metres. What is the area of the path? Give the answer to two decimal places.

Testing understanding

18 An equilateral triangle with sides of 12 cm fits inside a square which also has sides of 12 cm.
What fraction of the square does the equilateral triangle occupy? Answer to three decimal places.



19 A sphere with a radius of 30 cm is sliced horizontally, 10 cm below the top of the sphere.
What is the circumference of the circle formed?
Answer to one decimal place.



10D Length of an arc and area of a sector

Learning intentions

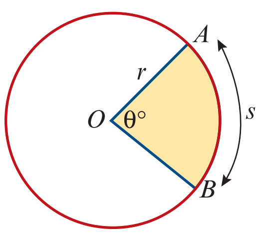
- To be able to find the length of an arc.
- To be able to find the area of a sector.

Length of an arc

Recall that the circumference, C, of a circle of radius r is given by $C = 2\pi r$. The fraction of the circumference will be $\frac{\theta}{360}$.

Therefore, the length, s, of an arc that subtends an angle of $\theta°$ at the centre is:

$$s = \frac{\theta}{360} \times 2\pi r$$



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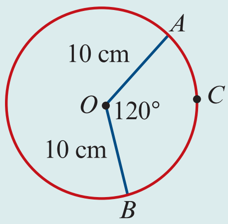
Length of an arc

The length, s, of an arc of a circle with a radius, r, that subtends an angle of $\theta°$ at the centre is given by:

$$s = \frac{\pi r\theta}{180}$$

Example 17 Calculating the length of an arc

In this circle with a centre at point O and a radius length of 10 cm, the angle subtended at O by arc ACB has a magnitude of 120°. Find the length of the arc, ACB, to one decimal place.



Explanation	Solution
1 Write down the formula.	$s = \frac{\pi r\theta}{180}$
2 Substitute $\theta = 120°$ and $r = 10$.	$s = \frac{\pi \times 10 \times 120}{180}$ $= \frac{20\pi}{3}$ ≈ 20.9 cm (to one decimal place)

Now try this 17 Calculating the length of an arc (Example 17)

Find the length of an arc that subtends an angle of 52° at the centre of a circle with a radius of 18 cm. Answer to one decimal place.

Area of a sector

If $\angle AOB = \theta°$, the area of the sector is a fraction of the area of the circle.

The area, A, of a circle of radius r is given by $A = \pi r^2$.

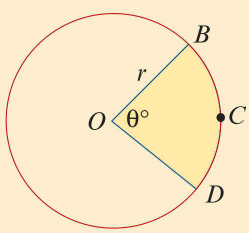
Minor and major sectors

In the diagram below with circle centre O, the yellow region is a **minor sector** and the unshaded region is a **major sector**.

Area of a sector

The area, A, of a sector of a circle with a radius, r, where the arc (BCD) of the sector subtends an angle of $\theta°$ at the centre is given by:

$$A = \frac{\pi r^2\theta}{360}$$



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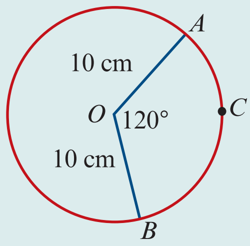
Example 18 Calculating the area of a sector

In this circle with a centre at point O and a radius length of 10 cm, the angle subtended at O by arc ACB has magnitude 120°.



Find:

a the area of the minor sector AOB

b the area of the major sector AOB.

Give your answer to two decimal places.

Explanation

a **1** Write down the formula.

$$A = \frac{\pi r^2 \theta}{360}$$

2 Substitute $\theta = 120°$ and $r = 10$.

$$A = \frac{\pi \times 100 \times 120}{360} = \frac{100\pi}{3} \approx 104.72 \text{ cm}^2$$

b **1** Write down the formula.

$$A = \frac{\pi r^2 \theta}{360}$$

2 Substitute $\theta = 240°$ and $r = 10$.

Note: The angle is 240°, which is 360° minus 120°.

Solution

$$A = \frac{\pi \times 100 \times 240}{360} = \frac{200\pi}{3} \approx 209.44 \text{ cm}^2$$

Now try this 18 Calculating the area of a sector (Example 18)

Find the area of a sector that subtends an angle of 73° at the centre of a circle with a radius of 34 cm. Answer to one decimal place.

Hint 1 Take care to give the correct units for the area. This will depend on the units for the radius that were used in the question.

Section Summary

- The length, s, of an arc of a circle with a radius, r, that subtends an angle of $\theta°$ at the centre is given by:

$$s = \frac{\pi r \theta}{180}$$

- The area, A, of a sector of a circle with a radius, r, where the arc of the sector subtends an angle of $\theta°$ at the centre is given by:

$$A = \frac{\pi r^2 \theta}{360}$$

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Skill-sheet

Exercise 10D

Building understanding

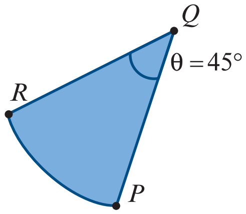
1 What fraction of a circle is each sector?

a Angle at the centre is 90°.
b Angle at the centre is 270°.
c Angle at the centre is 30°.
d Angle at the centre is 120°.
e Angle at the centre is 60°.
f Angle at the centre is 150°.

Developing understanding

2 Find the arc length of each sector. The radius is 10 cm. Answer to two decimal places.

a



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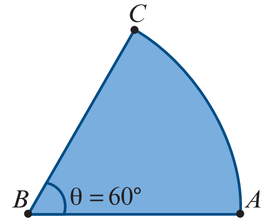

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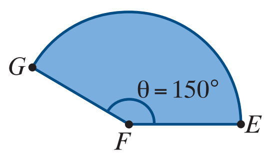

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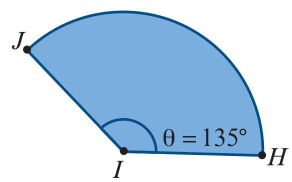


Example 17

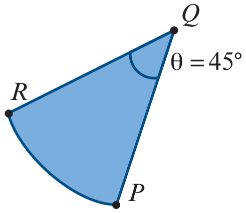
3 Findthe arc length of the following arcs where the radius of the circle (given in cm) and the angle subtended at the centre are given. Give your answer to two decimal places.

a $r = 15,\ \theta = 50°$
b $r = 20,\ \theta = 15°$
c $r = 30,\ \theta = 150°$
d $r = 16,\ \theta = 135°$
e $r = 40,\ \theta = 175°$
f $r = 30,\ \theta = 210°$

4 Find the perimeter of a sector that makes an angle of 58° in a circle with a radius of 3 metres. Answer to one decimal place.

5 Find the area of each of the following sectors, to one decimal place. The radius of the circle is 10 cm.

a



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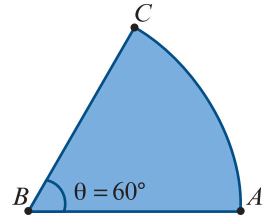

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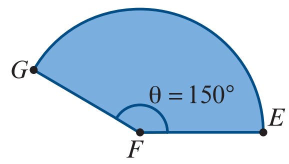

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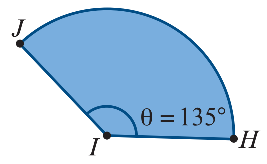

Example 18

6 Find the area of each sector where the radius of the circle (given in cm) and the angle subtended at the centre are given. Give your answer to two decimal places.

a $r = 10, \ \theta = 150°$ **b** $r = 40, \ \theta = 35°$ **c** $r = 45, \ \theta = 150°$

d $r = 16, \ \theta = 300°$ **e** $r = 50, \ \theta = 108°$ **f** $r = 30, \ \theta = 210°$

7 Find, to two decimal places, the size of the angle subtended at the centre of a circle of radius length 30 cm, by:

a an arc of length 50 cm **b** an arc of length 25 cm.

Testing understanding

8 Find the area of the region between an equilateral triangle of side length 10 cm and the circle that passes through the three vertices of the triangle.

9 A person stands on level ground, 60 m from the nearest point of a cylindrical tank of radius length 20 m. Calculate to two decimal places:

a the circumference of the tank

b the percentage of the circumference that is visible to the person when viewed from ground level. Draw a clearly labelled diagram and use trigonometry to find the angles involved (see page 696).

10 The minute hand of a large clock is 4 m long.

a How far does the tip of the minute hand move between 12:10 p.m. and 12:35 p.m?

b What is the area covered by the minute hand between 12:10 p.m. and 12:35 p.m?

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10E Volume

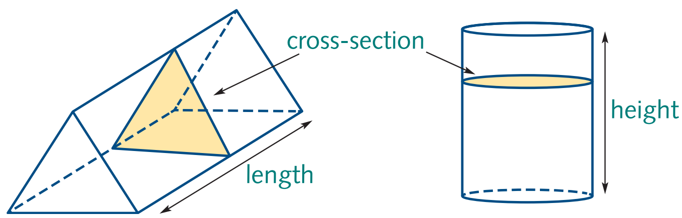
Learning intentions

- To be able to determine the volumes of rectangular prisms, triangular prisms, square prisms, cylinders and cones.
- To be able to find the capacity of three-dimensional containers.

Volume

Volume is the amount of space occupied by a three-dimensional object.

Prisms and cylinders are three-dimensional objects that have a uniform cross-section along their entire length. The volume of a prism or cylinder is found by using its **cross-sectional area**.

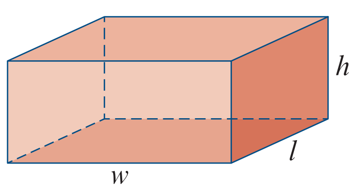
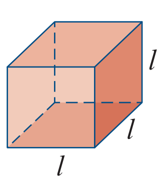


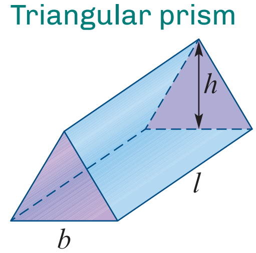
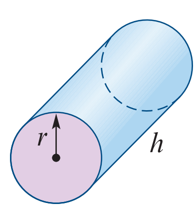
For prisms and cylinders:

volume = area of cross-section × height (or length)

When calculating volume, the answer will be in *cubic units*, i.e. mm^3, cm^3, m^3.

The formulas for the volumes of regular prisms and a cylinder are given in the table below.

Shape	Volume
Rectangular prism (cuboid) 	$V = lwh$
Square prism (cube) 	$V = l^3$

Shape	Volume
Triangular prism 	$V = \frac{1}{2}bhl$
Cylinder 	$V = \pi r^2 h$

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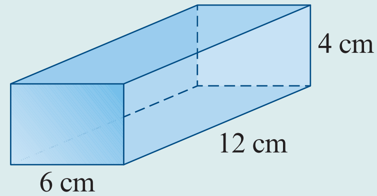
Example 19 Finding the volume of a cuboid

Find the volume of the cuboid shown.



Explanation

1 Use the formula $V = lwh$.

2 Substitute in $l = 12$, $w = 6$ and $h = 4$.

3 Evaluate.

4 Give your answer with correct units.

Solution

$V = lwh$

$= 12 \times 6 \times 4$

$= 288\ \text{cm}^3$

The volume of the cuboid is $288\ \text{cm}^3$.

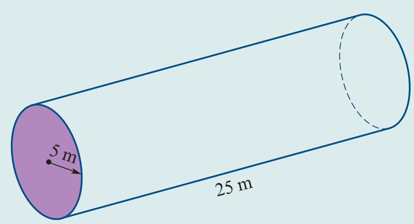
Now try this 19 Finding the volume of a cuboid (Example 19)

A cuboid has a length of 4 metres, a width of 3 metres and a height of 2 metres. Find its volume.

Hint 1 Take care to give the correct units for volume. The units for volume will depend on the units for length that were used.

Example 20 Finding the volume of a cylinder

Find the volume of this cylinder in cubic metres. Give your answer to two decimal places.



Explanation

1 Use the formula $V = \pi r^2 h$.

2 Substitute in $r = 5$ and $h = 25$ and evaluate.

3 Write your answer to two decimal places and with correct units.

Solution

$V = \pi r^2 h.$

$= \pi \times 5^2 \times 25$

$= 1963.495\ldots$

The volume of the cylinder is $1963.50\ \text{m}^3$, to two decimal places.

Now try this 20 Finding the volume of a cylinder (Example 20)

A cylinder has a radius of 18 cm and a height of 24 cm. Determine its volume to one decimal place.

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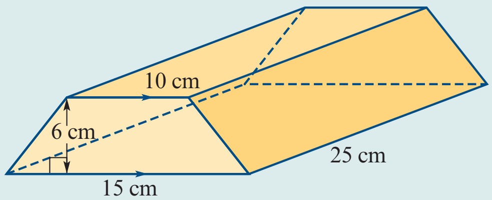
Example 21 Finding the volume of a three-dimensional object

Find the volume of the three-dimensional object shown.



Explanation

Strategy: To find the volume, find the area of the light-orange shaded cross-section and multiply it by the length of the shape.

1 Find the area of the cross-section, which is a trapezium. Use the formula $A = \frac{1}{2}(a + b)h$.

Substitute in $a = 10, b = 15$ and $h = 6$ and evaluate.

Solution

$$\text{Area of trapezium} = \frac{1}{2}(a + b)h = \frac{1}{2}(10 + 15)6 = 75 \text{ cm}^2$$

2 To find the volume, multiply the area of the cross-section by the length of the shape (25 cm).

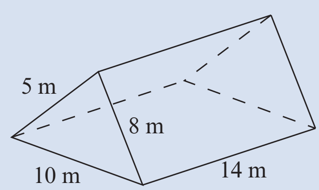
$$V = \text{area of cross-section} \times \text{length} = 75 \times 25 = 1875 \text{ cm}^3$$

3 Give your answer with correct units.

The volume of the shape is 1875 cm^3.

Now try this 21 Finding the volume of a three-dimensional object (Example 21)

Find the volume of the triangular prism shown.
Give your answer to one decimal place.



Hint 1 The volume of a prism is the cross-section area times the length.

Hint 2 What rule is used to find the area of a triangle when you know the length of the three sides?

Hint 3 Work to at least two decimal places and round your answer to one decimal place.

Capacity

Capacity

Capacity is the amount of substance that an object can hold.

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The difference between volume and capacity is that volume is the space available whilst capacity is the amount of substance that fills the volume.

For example:

- a cube that measures 1 metre on each side has a volume of one cubic metre (m^3) and is able to hold 1000 litres (L) (capacity)
- a bucket of volume 7000 cm^3 can hold 7000 mL (or 7 L) of water.

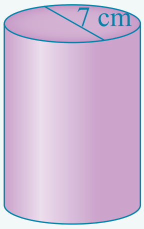
Volume to capacity conversion

The following conversions are useful to remember.

$$1 \text{ m}^3 = 1000 \text{ litres (L)}$$
$$1 \text{ cm}^3 = 1 \text{ millilitre (mL)}$$
$$1000 \text{ cm}^3 = 1 \text{ litre (L)}$$

Example 22 Finding the capacity of a cylinder

A drink container is in the shape of a cylinder. How many litres of water can it hold if the height of the cylinder is 20 cm and the diameter is 7 cm? Give your answer to two decimal places.

Explanation	Solution
1 Draw a diagram of the cylinder.	
2 Use the formula for finding the volume of a cylinder: $V = \pi r^2 h$.	$V = \pi r^2 h$
3 Since the diameter is 7 cm then the radius is 3.5 cm. Substitute $h = 20$ and $r = 3.5$.	$V = \pi \times 3.5^2 \times 20$
4 Evaluate to find the volume of the cylinder.	$V = 769.6902\ldots$ The volume of the cylinder is 769.69 cm^3
5 As there are 1000 cm^3 in a litre, divide the volume by 1000 to convert to litres.	$\frac{769.69}{1000} = 0.76969\ldots$
6 Give your answer to two decimal places and with correct units.	Cylinder has capacity of 0.77 litres to two decimal places.

Now try this 22 Finding the capacity of a cylinder (Example 22)

A cylindrical fuel can has a radius of 9 cm and a height of 30 cm. How many litres of fuel could it hold? Give your answer to one decimal place.

Hint 1 First find the volume of the cylinder.
Hint 2 Convert cubic centimetres into litres.

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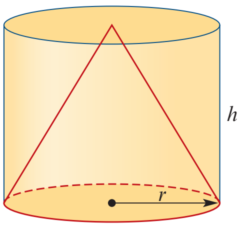
Volume of a cone

Calculating the volume of a cone is not required by the Study Design; however, it is often considered when studying the volume of solids.

A cone can fit inside a cylinder, as shown in the diagram. The cone occupies one-third of the volume of the cylinder containing it. Therefore, the formula for finding the volume of a cone is:

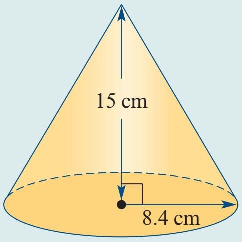
$$\text{volume of cone} = \tfrac{1}{3} \times \text{volume of its cylinder}$$
$$\text{volume of cone} = \tfrac{1}{3} \times \text{ area of base} \times \text{height}$$
$$V = \tfrac{1}{3}\pi r^2 h$$



The cone in the diagram is called a right circular cone because a line drawn from the centre of the circular base to the vertex at the top of the cone is perpendicular to the base.

Example 23 Finding the volume of a cone

Find the volume of this right circular cone.
Give your answer to two decimal places.



Explanation	Solution
1 Use the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.	$V = \dfrac{1}{3}\pi r^2 h$
2 Substitute $r = 8.4$ and $h = 15$ and evaluate.	$= \dfrac{1}{3}\pi(8.4)^2 \times 15$ $= 1108.353\ldots$
3 Give your answer to two decimal places and with correct units.	The volume of the cone is 1108.35 cm^3 to two decimal places.

Now try this 23 Finding the volume of a cone (Example 23)

Find the volume of a cone with a height of 3 m and a base with a radius of 2.5 m.
Give your answer to one decimal place.

Hint 1 Work to at least two decimal places then round your answer to one decimal place.

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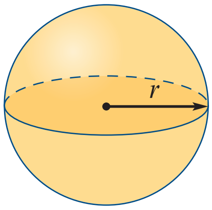
Volume of a sphere

The volume of a sphere of radius r can be found by using the formula:

$$V = \frac{4}{3}\pi r^3$$



Example 24 Finding the volume of a sphere

Find the volume of this sphere, giving your answer to two decimal places.

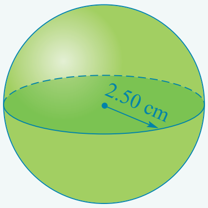
Explanation

1 Use the formula: $V = \frac{4}{3}\pi r^3$.

2 Substitute $r = 2.5$ and evaluate.

3 Give your answer to two decimal places and with correct units.

Solution



$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi \times 2.5^3$$
$$= 65.449\ldots$$

The volume of the sphere is 65.45 cm^3.

Now try this 24 Finding the volume of a sphere (Example 24)

Find the volume of a sphere with a radius of 45 cm. Answer to one decimal place.

Hint 1 Work to at least two decimal places and round your answer so it is to one decimal place.

Section Summary

- Volume is the amount of space occupied by a three-dimensional object.
- The volumes of common objects can be found using the formulas:

Rectangular prism	$V = lwh$	l = length, w = width, h = height
Triangular prism	$V = \frac{1}{2}bhl$	b = base, h = height, l = length
Cylinder	$V = \pi r^2 h$	r = radius, h = height
Cone	$V = \frac{1}{3}\pi r^2 h$	r = radius of the base, h = height
Sphere	$V = \frac{4}{3}\pi r^3$	r = radius.

- Capacity is the amount of substance an object can hold.

  Volume can be converted into capacity using:

$$1\ m^3 = 1000 \text{ litres (L)}$$
$$1\ cm^3 = 1 \text{ millilitre (mL)}$$
$$1000\ cm^3 = 1 \text{ litre (L)}$$

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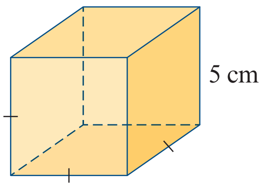
Exercise 10E

Building understanding

Example 19–21

1 Find the volumes of the following solids. Give your answers to one decimal place where appropriate.

a



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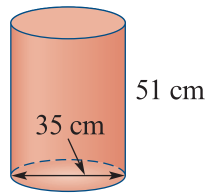

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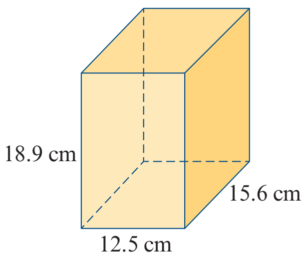

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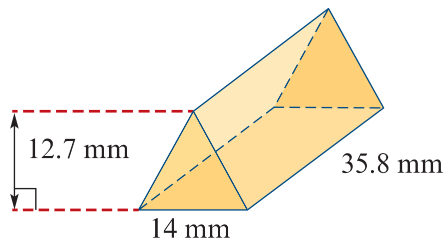

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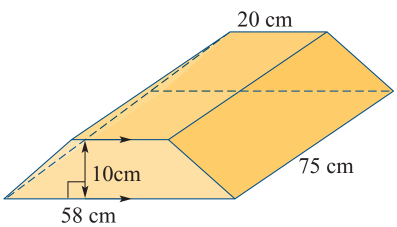

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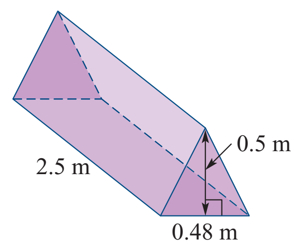

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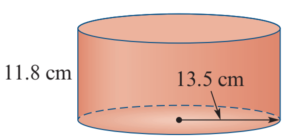

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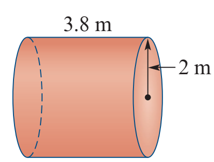
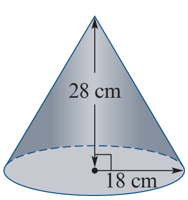


Example 23

i



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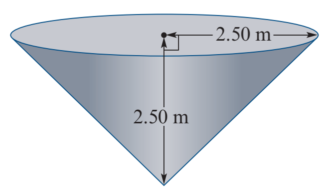
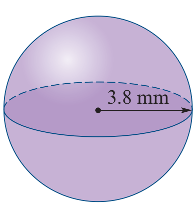


Example 24

k



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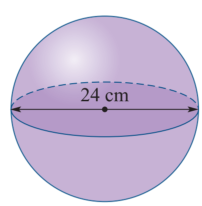

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Developing understanding

2 A cylindrical plastic container is 15 cm high, and its circular end surfaces each have a radius of 3 cm. What is its volume, to the nearest cm^3?

3 What is the volume, to the nearest cm^3, of a rectangular box with dimensions 5.5 cm by 7.5 cm by 12.5 cm?

Example 23

4 Find the volume, to two decimal places, of the cones with the following dimensions.

- **a** Base radius 3.50 cm, height 12 cm
- **b** Base radius 7.90 m, height 10.80 m
- **c** Base diameter 6.60 cm, height 9.03 cm
- **d** Base diameter 13.52 cm, height 30.98 cm

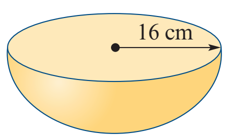
Example 24

5 Find the volumes, to two decimal places, of the following balls.

- **a** Tennis ball, radius 3.5 cm
- **b** Basketball, radius 14 cm
- **c** Golf ball, radius, 2 cm

6 Find the volumes, to two decimal places, of the following hemispheres.

a



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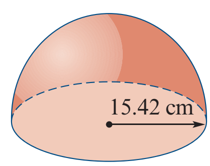


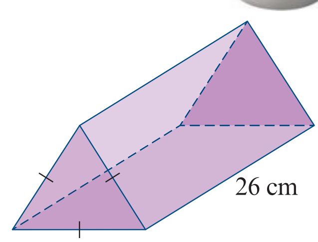
Example 22

7 How many litres of water does a fish tank with dimensions 50 cm by 20 cm by 24 cm hold when full?

8 **a** What is the volume, to two decimal places, of a cylindrical paint tin with height 33 cm and diameter 28 cm?

b How many litres of paint would fill this paint tin? Give your answer to the nearest litre.

9 A chocolate bar is made in the shape of an equilateral triangular prism. What is the volume of the chocolate bar if the length is 26 cm and the side length of the triangle is 4.5 cm? Give your answer to the nearest cm^3.



10 What volume of crushed ice will fill a snow cone level to the top if the snow cone has a top radius of 5 cm and a height of 15 cm? Give your answer to the nearest cm^3.

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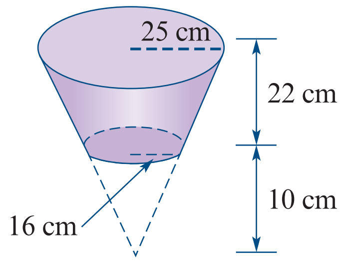
11 What is the volume, to two decimal places, of a cone with height 2.6 m and diameter of 3.4 m?

12 How many litres of water can be poured into a cone with a diameter of 2.8 cm and a height of 10 cm?

13 A solid figure is *truncated* when a portion of the bottom is cut and removed. Find the volume, to two decimal places, of the truncated cone shown in the diagram.



14 A plastic cone of height 30 cm and diameter 10 cm is truncated to make a rain gauge. The rain gauge has a height of 25 cm. What is its capacity? Give your answer to the nearest millilitre.

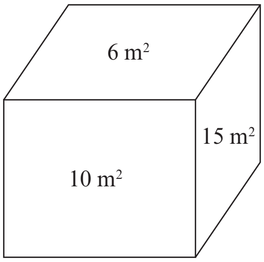
15 Len wants to serve punch at Christmas time in his new hemispherical punch bowl with diameter of 38 cm. How many litres of punch could be served, given that 1 millilitre (mL) is the amount of fluid that fills 1 cm^3? Answer to the nearest litre.

Testing understanding

16 The area of each face of the rectangular prism is shown.

a Find the volume of the rectangular prism.

b If the area of each face shown was X m^2, Y m^2 and Z m^2, what would be the volume of the rectangular prism?



17 A flat-bottomed silo for grain storage was constructed as a cylinder with a cone on top. The cylinder has a circumference of 53.4 m and a height of 10.8 m. The total height of the silo is 15.3 m. What is the volume of the silo? Give your answer to the nearest m^3.

18 A cylindrical water tank with a base of 250 cm^2 had enough water in it so that when a sphere with a volume of 500 cm^3 was lowered into the tank it was completely submerged in the water. How much did the water level rise?

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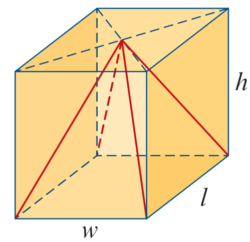
10F Volume of a pyramid

Learning intentions

- To be able to calculate the volume of a square pyramid given its height and the width of its base.
- To be able to calculate the volume of a pyramid given its height and the area of its base.

A square pyramid can fit inside a prism, as shown in the diagram. The pyramid occupies one third of the volume of the prism containing it. The formula for finding the volume of a pyramid is therefore:

$$\text{volume of pyramid} = \tfrac{1}{3} \times \text{volume of its prism}$$
$$\text{volume of pyramid} = \tfrac{1}{3} \times \text{area of base} \times \text{height}$$
$$V = \tfrac{1}{3}lwh$$

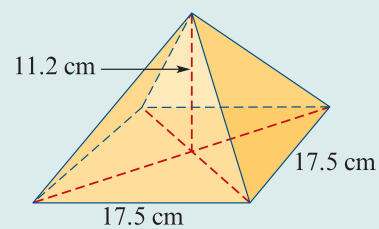


For the volume of pyramids with various-shaped bases:

$$V = \tfrac{1}{3} \times \text{area of base} \times \text{height}$$

Example 25 Finding the volume of a square pyramid

Find the volume of a square pyramid of height 11.2 cm and base length of 17.5 cm. Give your answer to two decimal places.



Explanation	Solution
1 Use the formula: $V = \frac{1}{3} \times \text{area of base} \times \text{height}$	$V = \frac{1}{3} \times \text{area of base} \times \text{height}$
2 Substitute the values for the area of the base (in this example, the base is a square) and height of the pyramid, and evaluate.	$= \frac{1}{3} \times 17.5^2 \times 11.2$ $= 1143.333\ldots$
3 Give your answer to two decimal places and with correct units.	The volume of the pyramid is $1143.33\ \text{cm}^3$.

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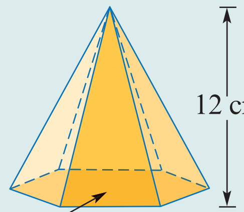
Now try this 25 Finding the volume of a square pyramid (Example 25)

Determine the volume of a square pyramid that has a height of 54 m and the sides of its base are 38 m long.

Example 26 Finding the volume of a hexagonal pyramid

Find the volume of this hexagonal pyramid that has a base of area 122 cm^2 and a height of 12 cm.



Explanation	Solution
1 Use the formula: $V = \frac{1}{3} \times \text{area of base} \times \text{height}$	$V = \frac{1}{3} \times \text{area of base} \times \text{height}$
2 Substitute the values for the area of the base (122 cm^2) and height (12 cm) and evaluate.	$= \frac{1}{3} \times 122 \times 12$ $= 488\ cm^3$
3 Give your answer with correct units.	The volume is 488 cm^3.

Now try this 26 Finding the volume of a hexagonal pyramid (Example 26)

A hexagonal pyramid has a base area of 320 m^2 and a height of 24 m. Find its volume.

Section Summary

- The volume of a pyramid is given by the formula:

$$V = \tfrac{1}{3} \times \text{area of base} \times \text{height}$$

- For a square or rectangular-based pyramid, the formula becomes:

$V = \frac{1}{3}lwh$ $\quad$ l = side length, w = width, h = height

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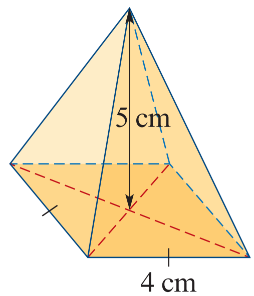
Exercise 10F

Building understanding

Example 25
Example 26

1 Find the volumes of the following right pyramids to two decimal places.

a


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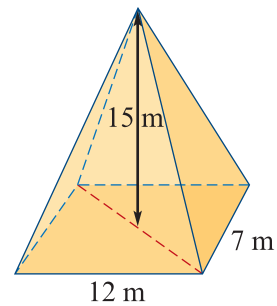
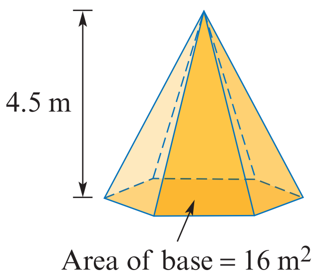


c


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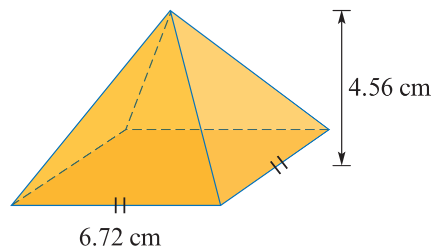
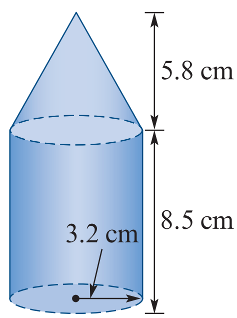


Developing understanding

2 The first true pyramid in Egypt is known as the Red Pyramid. It has a square base approximately 220 m long and is about 105 m high. What is its volume?

3 Find the volumes of these composite objects to one decimal place.

a


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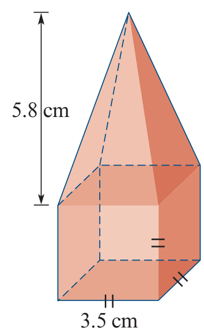
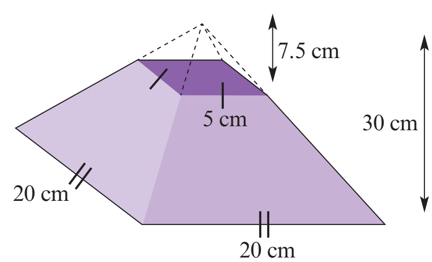


Testing understanding

4 Calculate the volume of the following truncated pyramid to one decimal place.



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10G Surface area

Learning intentions

- To be able to find the surface area of objects with plane surfaces, such as prisms, cuboids and pyramids.
- To be able to find the surface area of objects with curved surfaces, such as cylinders, cones and spheres.

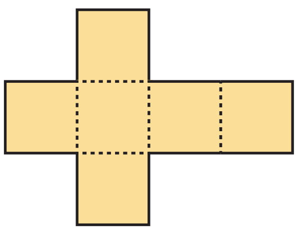
To find the **surface area (*SA*)** of a solid, you need to find the area of each of the surfaces of the solid and then add them all together.

Solids with plane faces (prisms and pyramids)

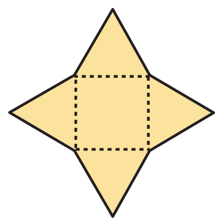
It is often useful to draw the **net** of a solid to ensure that all sides have been added.

A **net** is a flat diagram consisting of the plane faces of a polyhedron, arranged so that the diagram may be folded to form the solid.

For example: The net of a cube and of a square pyramid are shown below.



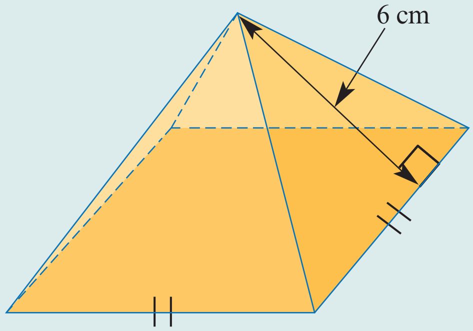
Net of a cube



Net of a square pyramid

Example 27 Finding the surface area of a pyramid

Find the surface area of this square-based pyramid.



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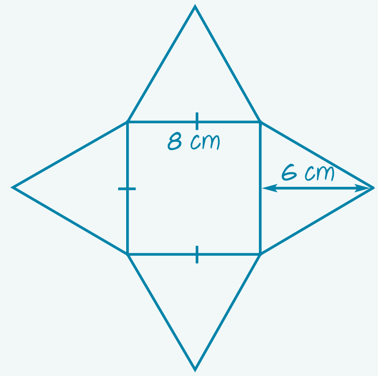
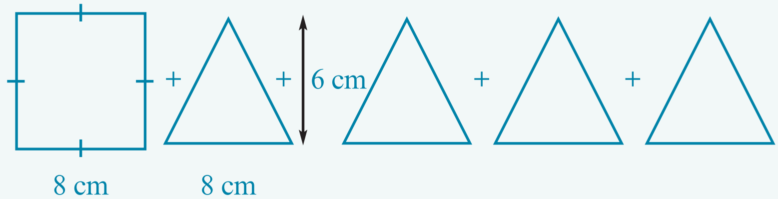
Explanation | **Solution**

1 Draw a net of the square pyramid. Note that the net is made up of one square and four identical triangles, as shown right.





2 Write down the formula for the surface area, using the net as a guide, and evaluate.

Surface area = area of  + 4 

$= 8 \times 8 + 4 \times (\frac{1}{2} \times 8 \times 6)$

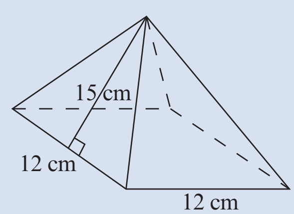
$= 160$

The surface area of the square pyramid is $160\ \text{cm}^2$.

Note: To find the area of the square, multiply the length by the width (8×8). To find the area of the triangles, use $A = \frac{1}{2}bh$, where b is 8 and h is 6.

Now try this 27 Finding the surface area of a pyramid (Example 27)

Find the surface area of the square-based pyramid shown.



Hint 1 Use the units of length given to decide what are the correct units of volume.

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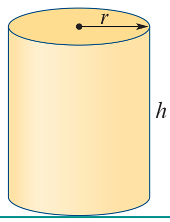
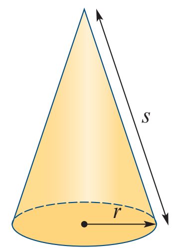
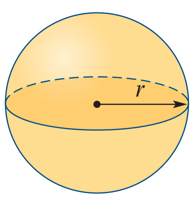
Solids with curved surfaces (cylinder, cone, sphere)

Calculating the surface area of a cone is not required by the Study Design; however, it is often considered when studying the surface area of solids.

For some special objects, such as the cylinder, cone and sphere, formulas to calculate the surface area can be developed. The formulas for the surface area of a cylinder, cone and sphere are given below.

Shape	Surface area
Cylinder 	$SA = 2\pi r^2 + 2\pi rh$

Shape	Surface area
Cone 	$SA = \pi r^2 + \pi rs$

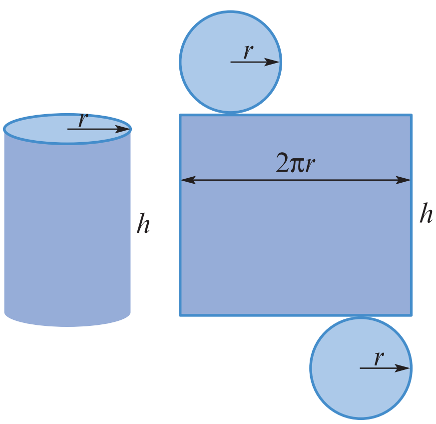
Shape	Surface area
Sphere 	$SA = 4\pi r^2$

To develop the formula for the surface area of a cylinder, we first draw a net, as shown.

The **total surface area** of a cylinder can therefore be found using:

$$\begin{aligned} SA &= \text{area of ends} + \text{area of curved surface} \\ &= \text{area of 2 circles} + \text{area of rectangle} \\ &= 2\pi r^2 + 2\pi rh \end{aligned}$$

Note: $2\pi r$ is the circumference of the circle, and this is the side length of the rectangle.



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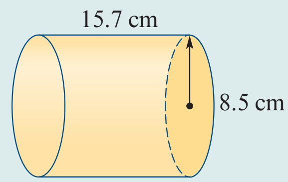
Example 28 Finding the surface area of a cylinder

Find the surface area of this cylinder to one decimal place.



Explanation

1. Use the formula for the surface area of a cylinder.
2. Substitute $r = 8.5$ and $h = 15.7$ and evaluate.
3. Give your answer to one decimal place and with correct units.

Solution

$SA = 2\pi r^2 + 2\pi rh$

$= 2\pi(8.5)^2 + 2\pi \times 8.5 \times 15.7$

$= 1292.451\ldots$

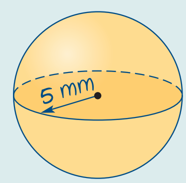
The surface area of the cylinder is $1292.5\ cm^2$ to one decimal place.

Now try this 28 Finding the surface area of a cylinder (Example 28)

Find the surface area of a cylinder with a height of 45 m and a radius of 30 m to one decimal place.

Example 29 Finding the surface area of a sphere

Find the surface area of a sphere with radius 5 mm to two decimal places.



Explanation

1. Use the formula $SA = 4\pi r^2$.
2. Substitute $r = 5$ and evaluate.
3. Give your answer to two decimal places and with correct units.

Solution

$SA = 4\pi r^2$

$= 4\pi \times 5^2$

$= 314.159\ldots$

The surface area of the sphere is $314.16\ mm^2$.

Now try this 29 Finding the surface area of a sphere (Example 29)

Find the surface area of a sphere with a radius of 20 cm to one decimal place.

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Section Summary

- Solids with plane faces, such as prisms, cuboids and pyramids, can be represented by their net as a way of calculating the total surface area.
- The surface areas of solids with curved surfaces can be found using the formulas:

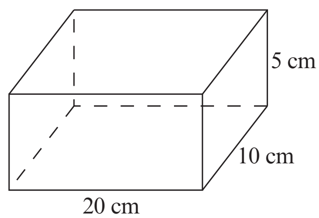
Cylinder	$SA = 2\pi r^2 + 2\pi rh$	r = radius, h = height
Cone	$SA = \pi r^2 + \pi rs$	r = radius, s = sloping (or slant) edge
Sphere	$SA = 4\pi r^2$	r = radius

Skill-sheet

Exercise 10G

Building understanding

1 To find the surface area of a cuboid (box), its net could be drawn as a useful aid. Alternatively, you could directly calculate the different faces.



a Find the total area of the top and bottom faces.

b What is the total area of the two side faces?

c Find the total area of the front and back faces.

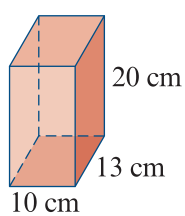
d Calculate the total surface area.

Developing understanding

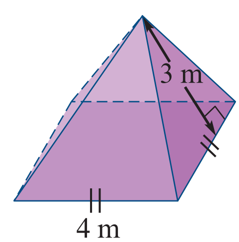
Example 27

2 Find the surface areas of these prisms and pyramids to one decimal place.

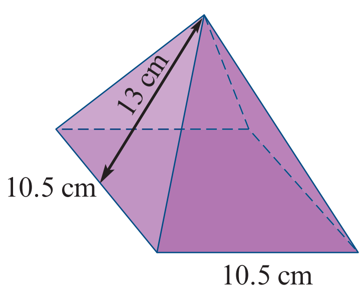
a



b



c



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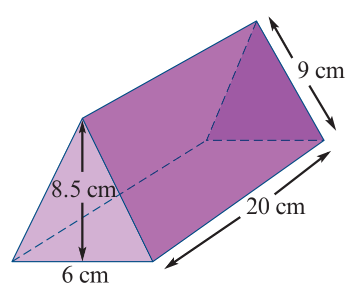

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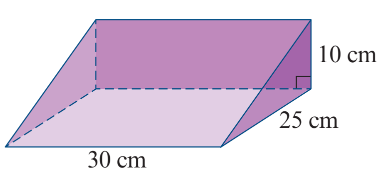

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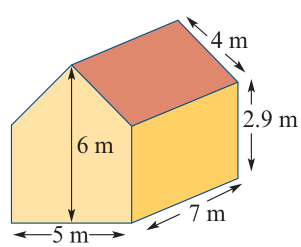

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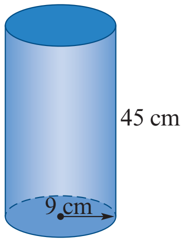
Example 28
Example 29

3 Find the surface area of each of these solids with curved surfaces to two decimal places.

a



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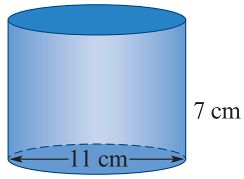

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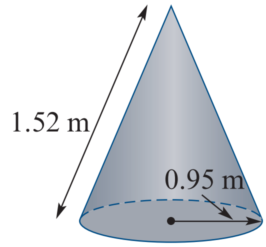

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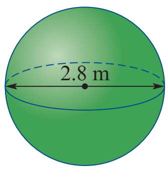

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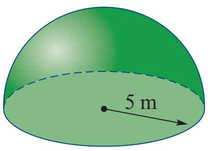

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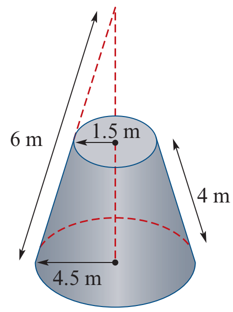


4 A tennis ball has a radius of 3.5 cm. A manufacturer wants to provide sufficient material to cover 100 tennis balls. What area of material is required? Give your answer to the nearest cm^2.

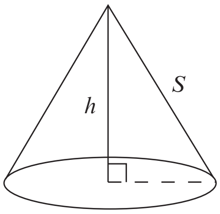
5 A set of 10 conical paper hats are covered with material. The height of a hat is 35 cm and the diameter is 19 cm.

a What amount of material, in m^2, will be needed? Give your answer to two decimal places.

b Tinsel is to be placed around the base of the hats. How much tinsel, to the nearest metre, is required?

Testing understanding

6 When the curved surface of a cone was cut along the sloping edge, S, and laid flat on the table, it formed a semicircle with a radius of 20 cm.
Find the height h of the cone to one decimal place.



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10H Similar figures

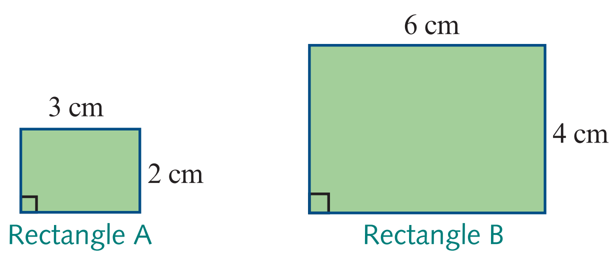
Learning intentions

- To be able to determine when shapes are similar and find their scale factor.
- To be able to find the scale factor for the areas of similar shapes.
- To be able to use similar triangles to find an unknown value.

Shapes that are similar have the same shape but are different sizes. The three frogs below are **similar figures**.

Polygons (closed plane figures with straight sides), like the rectangles in the diagram below, are similar if:

- corresponding angles are equal
- corresponding sides are proportional (which means that each pair of corresponding side lengths are in the same *ratio*).



For example, the two rectangles above are similar because their corresponding angles are equal and their side lengths are in the same ratio.

Ratio of side lengths = 6 : 3 which simplifies to 2 : 1

Ratio of side lengths = 4 : 2 which simplifies to 2 : 1

The ratios could be written as 3 : 6 and 2 : 4, simplifying to 1 : 2, as long as the measurements are read in the same order from one diagram to the next.
Ratios are sometimes written as fractions, such as $\frac{1}{2}$ for 1 : 2.

When we enlarge or reduce a shape by a **scale factor**, the *original* and the *image* are similar.

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Scale factor

When determining the **scale factor** of similar shapes, the numerator is a length of the second shape and the denominator is the corresponding length from the first (or original) shape.

In the previous diagram,

$$\textbf{scale factor} = \frac{\textit{a length of the second shape}}{\textit{corresponding length of the first shape}} = \frac{6}{3} = 2$$

Rectangle A has been enlarged by a scale factor, $k = 2$, to give rectangle B.

We say that rectangle A has been *scaled up* to give rectangle B.

We can also compare the rectangles' areas.

Area of rectangle A = 6 cm^2

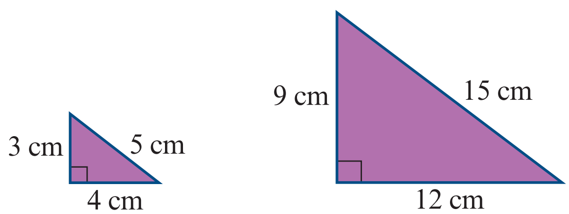
Area of rectangle B = 24 cm^2

The area of rectangle A has been enlarged by a scale factor, $k^2 = 4$, to give rectangle B.

We notice that, as the lengths are enlarged by a scale factor of 2, the area is enlarged by a scale factor of $2^2 = 4$.

Area scale factor

When all the lengths are multiplied by a scale factor of k, the area is multiplied by a scale factor of k^2.



The two triangles above are similar because their corresponding side lengths are in the same ratio.

Ratio of side lengths = 3 : 9 or 1 : 3

Ratio of side lengths = 4 : 12 or 1 : 3

Ratio of side lengths = 5 : 15 or 1 : 3

$$\text{Scale factor, } k = \frac{\textit{a length of the second shape}}{\textit{corresponding length of the first shape}} = \frac{15}{5} = 3$$

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We would expect the area scale factor, k^2, or the ratio of the triangles' areas, to be 9 ($= 3^2$).

$$\text{Area of small triangle} = 6\text{ cm}^2$$

$$\text{Area of large triangle} = 54\text{ cm}^2$$

$$\text{Area scale factor, } k^2 = \text{Ratio of areas} = \frac{54}{6} = 9.$$

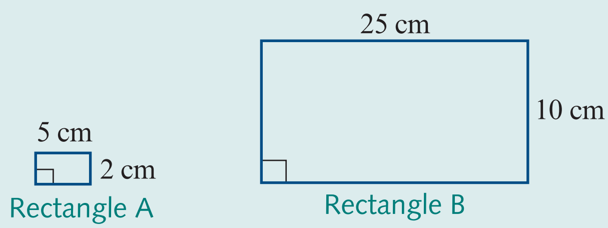
Shapes can be scaled up or scaled down. When a shape is made larger, it is *scaled up*, and when it is made smaller, it is *scaled down*.

Example 30 Finding the scale factor of length and area

The rectangles shown are similar.

a Find the scale factor of their side lengths.

b Find the scale factor of their areas.



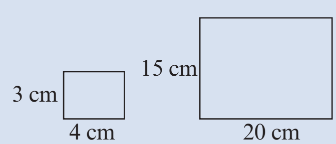
Explanation	Solution
a 1 To find the scale factor, put a length of the second shape as the numerator and the corresponding length from the first (or original) shape as the denominator.	$\frac{25}{5} = 5$
2 The small rectangle has been scaled up by a scale factor of 5. Write your answer.	The scale factor of the side lengths is 5.
b 1 Since the lengths are multiplied by a scale factor of 5, the area will be multiplied by a scale factor of 5^2.	$5^2 = 25$
2 The area of the small rectangle has been scaled up by a scale factor of 25. Write your answer.	The scale factor of the areas is 25.

Now try this 30 Finding the scale factor of length and area (Example 30)

For the similar rectangles shown:

a Find the scale factor of their sides.

b Find the scale factor of their areas.



Hint 1 The scale factor of their sides is found by dividing the length of a side in the second shape by the length of the corresponding side in the first shape.

Hint 2 The area scale factor is the length scale factor squared.

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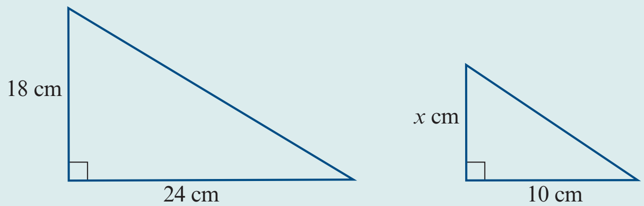
When scaling down from a larger figure to a smaller figure, the scale factor will always be less than one. This is because the numerator is always a length of the second shape, which in this case is smaller than the denominator (the corresponding length of the first shape).

Example 31 Using similar triangles to find unknown values

The following two triangles are similar. Find the value of x.

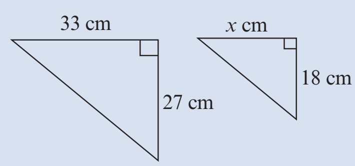


Explanation	**Solution**
1 Since the triangles are similar, their corresponding side lengths are in the same ratio.	$\frac{x}{18} = \frac{10}{24}$
2 Solve for x. Multiply both sides by 18.	$\frac{x}{18} \times 18 = \frac{10}{24} \times 18$
3 Write your answer using correct units.	$x = 7.5$ cm

Now try this 31 Using similar triangles to find unknown values (Example 31)

Two similar triangles are shown.

Find the value of x.



Hint 1 Since the triangles are similar, the ratio of their corresponding side lengths will be equal. So x over 33 is equal to ... over ... .

Hint 2 Now solve for x.

Section Summary

- Similar figures have the same shape but different sizes.
- Similar figures have:
  - corresponding angles that are equal
  - corresponding pairs of sides that are in the same proportion.
- The *scale factor* is the length of a side in the second shape divided by the length of the corresponding side in the first shape.
- The scale factor can be used to find an unknown length in similar triangles.
- The scale factor for area is the ratio of the areas of the similar shapes.
- A scale factor of k for lengths implies a scale factor of k^2 for areas.

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Exercise 10H

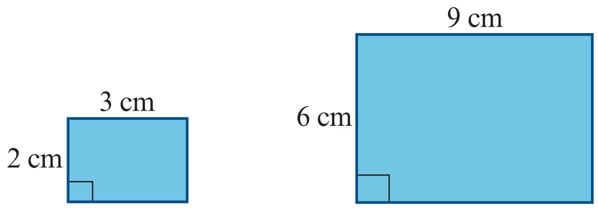
Building understanding

Example 30

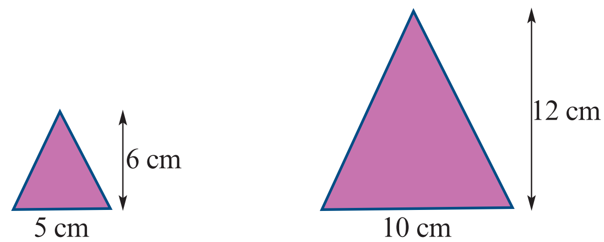
1 The following pairs of figures are similar. For each pair, find:

i the scale factor of their side lengths **ii** the scale factor of their areas.

a



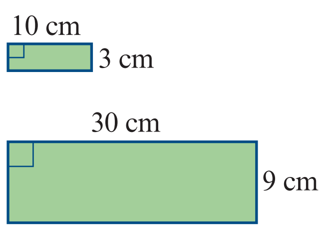
b



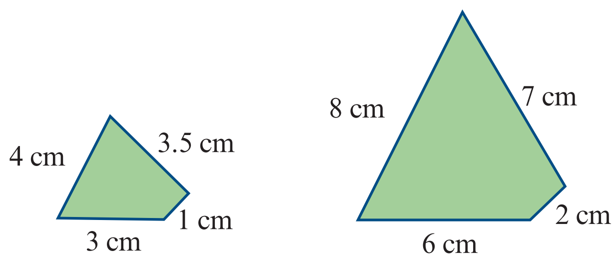
Developing understanding

2 Which of the following pairs of figures are similar? For those that are similar, find the scale factor of their sides.

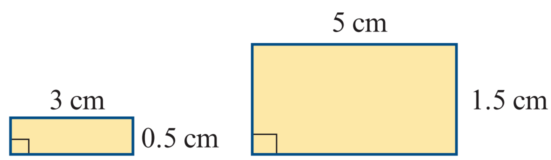
a



b



c



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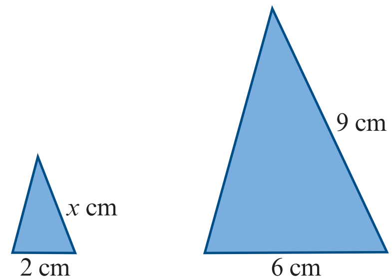
Example 31

3 The following triangles are similar.

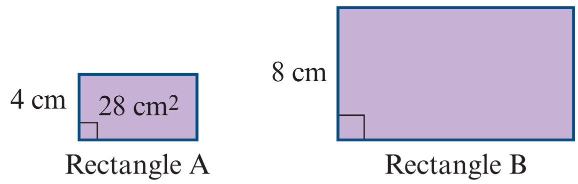
a Find the scale factor.

b Find the value of x.

c Find the ratio of their areas.



4 The two rectangles shown right are similar. The area of rectangle A is 28 cm^2. Find the area of rectangle B.



5 A photo is 12 cm by 8 cm. It is to be enlarged and then framed. If the dimensions are tripled, what will be the area of the new photo?

6 What is the scale factor if a photo has been enlarged from 15 cm by 9 cm, to 25 cm by 15 cm? Give your answer to two decimal places.

Testing understanding

7 A scale on a map is 1 : 500 000.

a What is the actual distance between two towns if the distance on the map is 7.2 cm? Give your answer in kilometres.

b If the actual distance between two landmarks is 15 km, find the distance represented on the map in cm.

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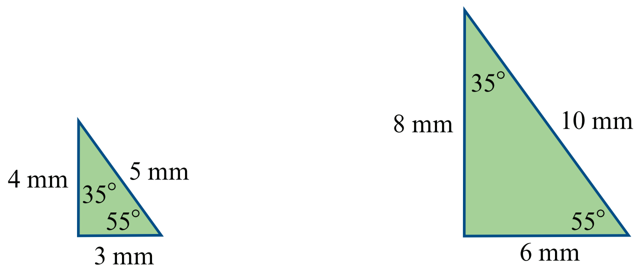
10I Similar triangles

Learning intentions

- To be able to use the tests for similar triangles to determine if triangles are similar.

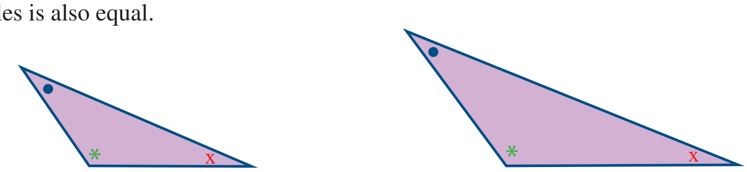
In mathematics, two **triangles** are said to be **similar** if they have the same shape. As in the previous section, this means that corresponding angles are equal and the lengths of corresponding sides are in the same ratio.

For example, these two triangles are similar.

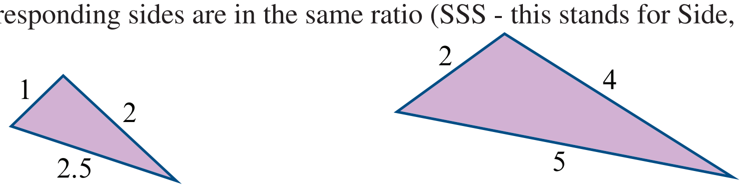


Two triangles can be tested for similarity by considering the following necessary conditions.

- Corresponding angles are equal (AA - this stands for Angle, Angle).
  Remember: If two pairs of corresponding angles are equal, then the third pair of corresponding angles is also equal.

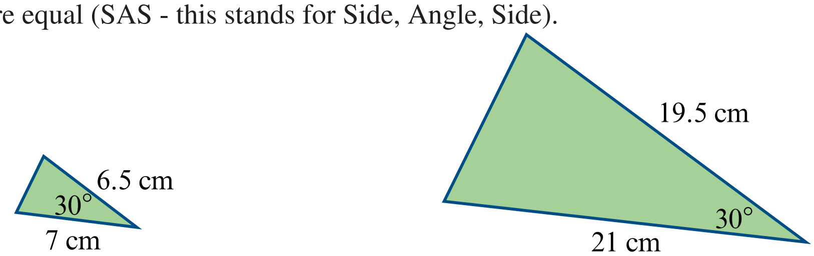


- Corresponding sides are in the same ratio (SSS - this stands for Side, Side, Side).



$$\frac{5}{2.5} = \frac{4}{2} = \frac{2}{1} = 2$$

- Two pairs of corresponding sides are in the same ratio and the included corresponding angles are equal (SAS - this stands for Side, Angle, Side).



$$\frac{19.5}{6.5} = \frac{21}{7} = 3$$

Both triangles have an included corresponding angle of 30°.

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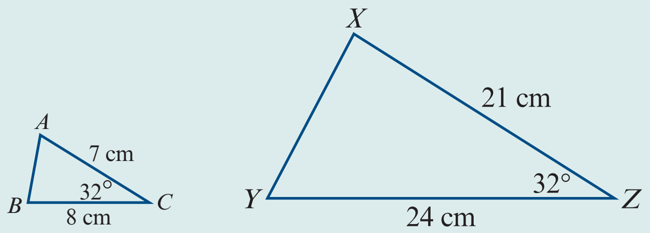
Example 32 Checking if triangles are similar

Explain why triangle ABC is similar to triangle XYZ.



Explanation

1. Compare corresponding side ratios:
   AC and XZ
   BC and YZ.
2. Triangles ABC and XYZ have an included corresponding angle.
3. Write an explanation as to why the two triangles are similar.

Solution

$\frac{XZ}{AC} = \frac{21}{7} = \frac{3}{1}$

$\frac{YZ}{BC} = \frac{24}{8} = \frac{3}{1}$

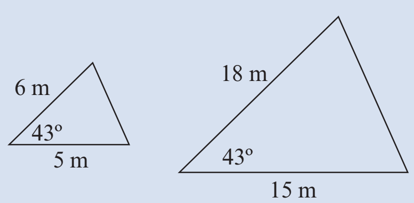
The angle 32° is included and corresponding.

They have two pairs of corresponding sides in the same ratio and the included corresponding angles are equal. (SAS)

Now try this 32 Checking if triangles are similar (Example 32)

Show that the smaller triangle is similar to the larger triangle.

State the rule which was used to decide that they are similar.



Hint 1 Find the side length ratios.

Section Summary

Two triangles are similar if:

- Corresponding angles are equal, AA OR
- Corresponding sides are in the same ratio, SSS OR
- Two pairs of corresponding sides are in the same ratio and the included angles are equal, SAS.

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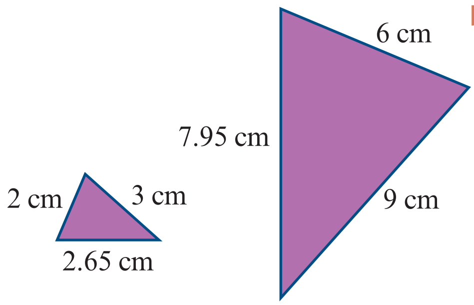
Exercise 10I

Building understanding

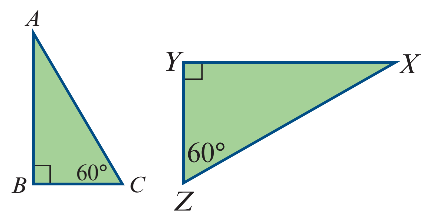
Example 32

1 Three pairs of similar triangles are shown below. Explain why each pair of triangles are similar.

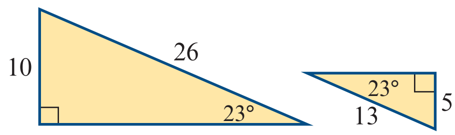
a



b



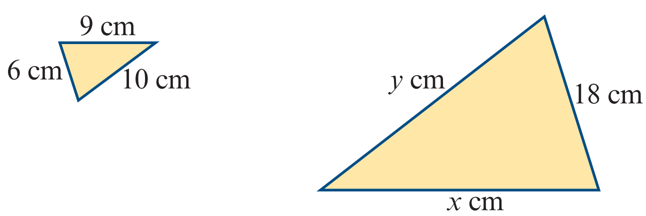
c



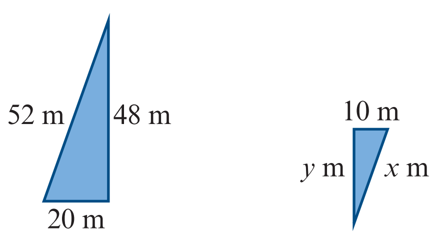
Developing understanding

2 Calculate the missing dimensions, marked x and y, in these pairs of similar triangles by first finding the scale factor.

a



b



3 A triangle with sides 5 cm, 4 cm and 8 cm is similar to a larger triangle with a longest side of 56 cm.

a Find the scale factor.

b Find the lengths of the larger triangle's other two sides.

c Find the perimeter of the larger triangle.

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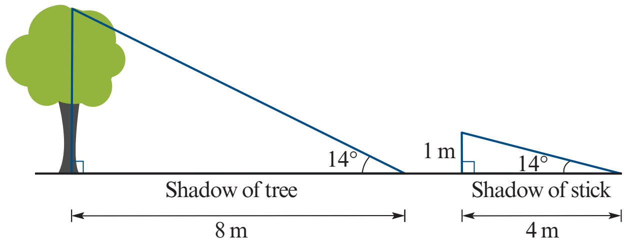
4 A tree and a one metre vertical stick cast their shadows at a particular time in the day. The shadow lengths are shown in the diagram below (*not* drawn to scale).

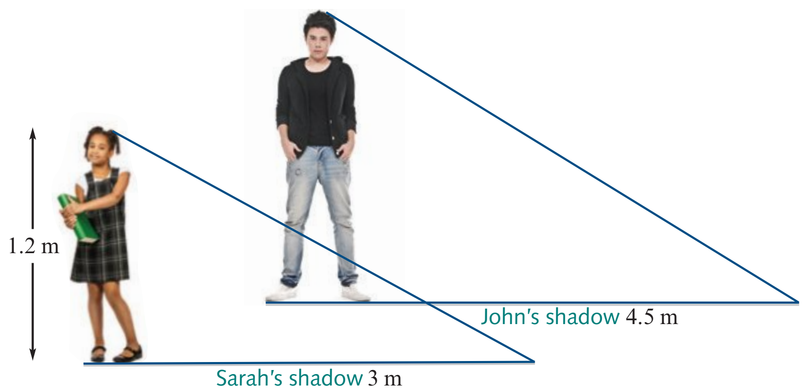
a Give reasons why the two triangles shown are similar.

b Find the scale factor for the side lengths of the triangles.

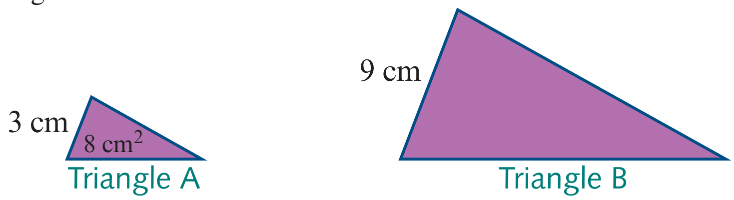
c Find the height of the tree.



5 John and his younger sister, Sarah, are standing side by side. Sarah is 1.2 m tall and casts a shadow 3 m long. How tall is John if his shadow is 4.5 m long?

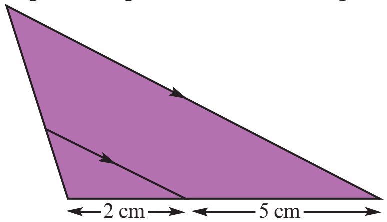


6 The area of triangle A is 8 cm^2. Triangle B is similar to triangle A. What is the area of triangle B?



Testing understanding

7 Given that the area of the small triangle in the following diagram is 2.4 cm^2, find the area of the larger triangle to one decimal place.



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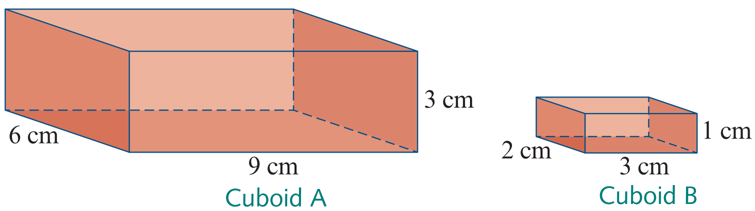
10J Similar solids

Learning intentions

- To be able to determine when two solids are similar, and to calculate their scale factor.

Two solids are similar if they have the same shape, and the ratios of their corresponding linear dimensions are equal.

Cuboids



The two cuboids are similar because:

- they are the same shape (both are cuboids)
- the ratios of the corresponding dimensions are the same.

$$\frac{\text{length of cuboid B}}{\text{length of cuboid A}} = \frac{\text{width of cuboid B}}{\text{width of cuboid A}} = \frac{\text{height of cuboid B}}{\text{height of cuboid A}}$$

$$\frac{2}{6} = \frac{3}{9} = \frac{1}{3} = \frac{1}{3}$$

$$\text{Volume scale factor, } k^3 = \text{Ratio of volumes} = \frac{2 \times 3 \times 1}{6 \times 9 \times 3} = \frac{6}{162} = \frac{1}{27} = \frac{1}{3^3}$$

As the length dimensions are reduced by a scale factor of $\frac{1}{3}$, the volume is reduced by a scale factor of $\frac{1}{3^3} = \frac{1}{27}$.

Scaling volumes

When all the dimensions are multiplied by a scale factor of k, the volume is multiplied by a scale factor of k^3.

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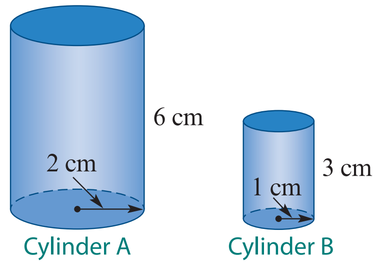
Cylinders

These two cylinders are similar because:

- they are the same shape (both are cylinders)
- the ratios of the corresponding dimensions are the same.

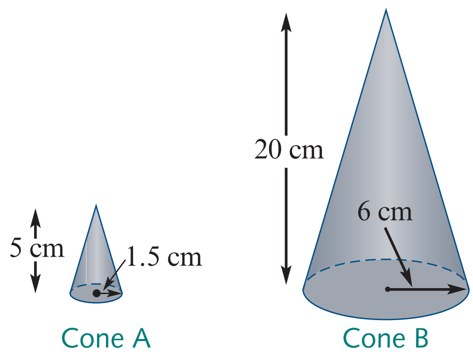
$$\frac{3}{6} = \frac{1}{2}$$

$$\frac{\text{height of cylinder B}}{\text{height of cylinder A}} = \frac{\text{radius of cylinder B}}{\text{radius of cylinder A}}$$



$$\text{Volume scale factor, } k^3 = \text{Ratio of volumes} = \frac{\pi \times 1^2 \times 3}{\pi \times 2^2 \times 6} = \frac{3}{24} = \frac{1}{8} = \frac{1}{2^3}$$

Cones



These two cones are similar because:

- they are the same shape (both are cones)
- the ratios of the corresponding dimensions are the same.

$$\frac{20}{5} = \frac{6}{1.5} = 4$$

$$k = \frac{\text{height of cone B}}{\text{height of cone A}} = \frac{\text{radius of cone B}}{\text{radius of cone A}} = 4$$

$$\text{Volume scale factor, } k^3 = \text{Ratio of volumes} = \frac{\frac{1}{3} \times \pi \times 6^2 \times 20}{\frac{1}{3} \times \pi \times 1.5^2 \times 5}$$

$$= 64 = 4^3$$

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Example 33 Comparing volumes of similar solids

Two solids are similar such that the larger one has all of its dimensions three times that of the smaller solid. How many times larger is the larger solid's volume?

Explanation	Solution
1 Since all of the larger solid's dimensions are 3 times those of the smaller solid, the volume will be 3^3 times larger. Evaluate 3^3.	$3^3 = 27$
2 Write your answer.	The larger solid's volume is 27 times the volume of the smaller solid.

Now try this 33 Comparing volumes of similar solids (Example 33)

A solid has dimensions seven times those of a smaller similar solid. How many times is the volume of the larger solid greater than the volume of the smaller solid?

Hint 1 A scale factor of k for lengths implies a scale factor of k^3 for volumes.

Section Summary

- Similar solids have the same shape and a constant ratio for their corresponding sides.
- Lengths with a scale factor of k imply that there is a scale factor of k^3 for the volumes.

Skill-sheet

Exercise 10J

Building understanding

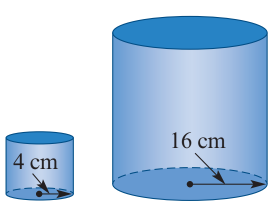
Example 33

1 Two cylindrical water tanks are similar such that the height of the larger tank is 3 times the height of the smaller tank. How many times larger is the volume of the larger tank compared to the volume of the smaller tank?

2 Two cylinders are similar and have radii of 4 cm and 16 cm, respectively.

a What is the ratio of their heights?

b What is the ratio of their volumes?



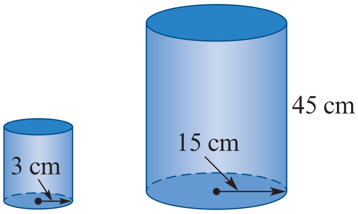
Developing understanding

3 Find the ratio of the volumes of two cuboids whose sides are in the ratio $\frac{3}{1}$.

4 The radii of the bases of two similar cylinders are in the ratio $\frac{5}{1}$. The height of the larger cylinder is 45 cm. Calculate:

a the height of the smaller cylinder

b the ratio of the volumes of the two cylinders.

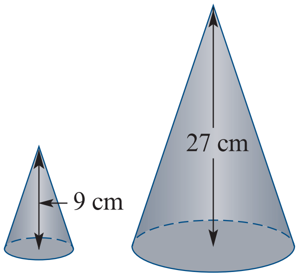


5 Two similar cones are shown at right. The ratio of their heights is $\frac{3}{1}$.

a Determine whether the smaller cone has been scaled up or down to give the larger cone.

b What is the volume scale factor?

c The volume of the smaller cone is 120 cm^3. Find the volume of the larger cone.



6 The radii of the bases of two similar cylinders are in the ratio 3 : 4. The height of the larger cylinder is 8 cm. Calculate:

a the height of the smaller cylinder

b the ratios of the volumes of the two cylinders.

Testing understanding

7 A pyramid has a square base of side 4 cm and a volume of 16 cm^3. Calculate:

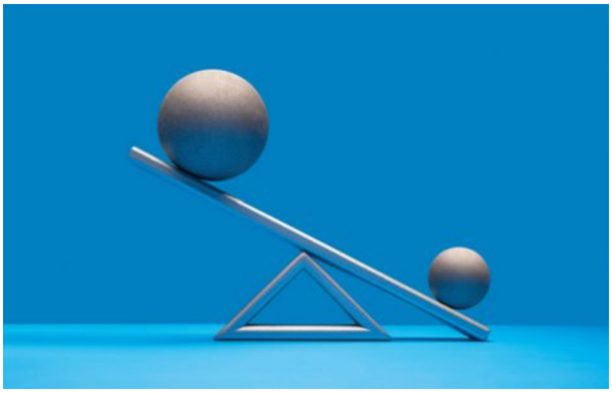
a the height of the pyramid

b the height and the length of the side of the base of a similar pyramid with a volume of 1024 cm^3.

8 Two spheres have diameters of 12 cm and 6 cm respectively. Calculate:

a the ratio of their surface areas

b the ratio of their volumes.



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Key ideas and chapter summary

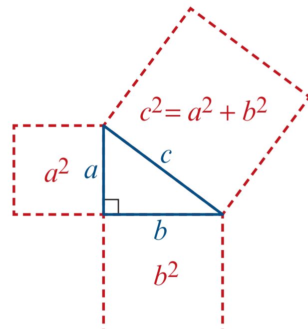
Assign-
ment

Scientific notation
To write a number in scientific notation, express it as a number between 1 and 10, multiplied by a power of 10.

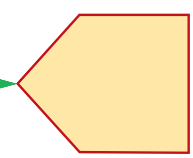
Rounding
If the number following the specified digit is 5 or more, than round the specified digit up. If the following number is less than 5, then leave the specified digit unchanged.
e.g. 5.417 rounded to two decimal places is 5.42 (number after the 1 is 7, so round up).

Significant figures
To write a number to the required number of **significant figures**, write the number in scientific notation, then round to the required number of significant figures.

Pythagoras' theorem
Pythagoras' theorem states that:
For any right-angled triangle, the sum of the areas of the squares of the two shorter sides (a and b) equals the area of the square of the hypotenuse (c): $c^2 = a^2 + b^2$



Perimeter (P)
Perimeter is the distance around the edge of a two-dimensional shape.

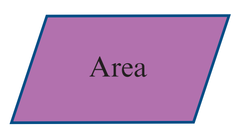


Perimeter of rectangle
$P = 2l + 2w$

Circumference (C)
Circumference is the perimeter of a circle: $C = 2\pi r$.

Length of an arc
The length, s, of an arc of a circle with a radius, r, that subtends an angle of $\theta°$ at the centre is given by:
$$s = \frac{\pi r \theta}{180}$$

Area (A)
Area is the measure of the region enclosed by the boundaries of a two-dimensional shape.



Area of a sector
The area, A, of a sector of a circle with a radius, r, where the arc of the sector subtends an angle of $\theta°$ at the centre is given by:
$$A = \frac{\pi r^2 \theta}{360}$$

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Area formulas

Area of rectangle $= lw$ Area of parallelogram $= bh$

Area of triangle $= \frac{1}{2}bh$ Area of trapezium $= \frac{1}{2}(a+b)h$

Area of circle $= \pi \times r^2$

Heron's formula

Area of triangle $= A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$ and a, b and c are the sides of the triangle.

Volume (V)

Volume is the amount of space occupied by a 3-dimensional object.

- For prisms and cylinders,
  Volume = area of cross-section × height
- For pyramids and cones,
  Volume $= \frac{1}{3} \times$ area of base × height

Volume formulas

Volume of cube $= l^3$ Volume of cuboid $= lwh$

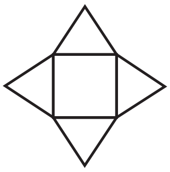
Volume of triangular prism $= \frac{1}{2}bhl$ Volume of cylinder $= \pi r^2 h$

Volume of cone $= \frac{1}{3}\pi r^2 h$ Volume of pyramid $= \frac{1}{3}lwh$

Volume of sphere $= \frac{4}{3}\pi r^3$

Surface area (SA)

Surface area is the total of the areas of all the surfaces of a solid. When finding surface area, it is often useful to draw the net of the shape.



Surface area formulas

Surface area of cylinder $= 2\pi r^2 + 2\pi rh$

Surface area of cone $= \pi r^2 + \pi rs$

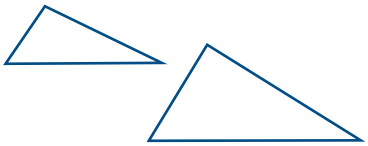
Surface area of sphere $= 4\pi r^2$

Similar figures or solids

Similar figures or solids are the same shape but different sizes. Corresponding sides are in the same ratio.

Similar triangles

Triangles are shown to be **similar** if:



- corresponding angles are similar (AA)
- corresponding sides are in the same ratio (SSS)
- two pairs of corresponding sides are in the same ratio and the included corresponding angles are equal (SAS).

Ratios of area and volume for similar shapes

When all the dimensions of similar shapes are multiplied by a scale factor of k, the areas are multiplied by a scale factor of k^2 and the volumes are multiplied by a scale factor of k^3.

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Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

10A **1** I can round numbers to a specific number of decimal places. ☐

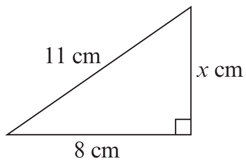
e.g. Round 307.509 to 2 decimal places.

10A **2** I can round numbers to a specific number of significant figures. ☐

e.g. Round 307.509 to 2 significant figures.

10B **3** I can find the length of an unknown side in a right-angled triangle. ☐

e.g. Find the length of the unknown side, x, to one decimal place.



10B **4** I can find the length of an unknown side in a three-dimensional object using Pythagoras' theorem. ☐

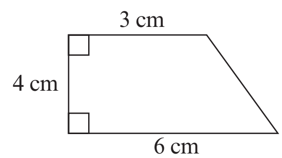
e.g. A cube has sides of 10 cm. Find the length of the diagonal from one corner, through the centre of the cube, to the opposite corner. Give your answer to one decimal place.

10C **5** I can determine the perimeters and areas of regular shapes, such as rectangles, parallelograms, trapeziums and triangles. ☐

e.g. Find the perimeter and area of a triangle with sides of 6 cm, 7 cm and 8 cm to the nearest whole number.

10C **6** I can find the perimeter and area of composite shapes. ☐

e.g. Find the perimeter and area of the trapezium shown.



10C **7** I can find the circumference and area of a circle when given its radius. ☐

e.g. Find the circumference and area of a circle with a radius of 7 cm, to one decimal place.

10D **8** I can find the length of an arc. ☐

e.g. An arc on a circle with a radius of 10 m, subtends an angle of 76° at the centre of the circle. Find the length of the arc to one decimal place.

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10D **9** I can find the area of a sector. ☐

e.g. A sector of a circle with a radius of 10 m, subtends an angle of 76° at the centre of the circle. Find the area of the sector to one decimal place.

10E **10** I can find the volumes of rectangular prisms, triangular prisms, square prisms and cylinders. ☐

e.g. A crystal has a triangular cross-section with a base length of 44 mm and a height of 28 mm.The crystal is 52 mm long. Find its volume.

10E **11** I can find the capacity of three-dimensional containers. ☐

e.g. A rectangular can has a base area of 600 cm^2 and a height of 45 cm. How many litres of petrol could it hold?

10E **12** I can calculate the volume of a cone given its height and radius. ☐

e.g. Find the volume of a cone, 30 cm high and with a base radius of 18 cm, to one decimal place.

10E **13** I can calculate the volumes of spheres and hemispheres when given their radii. ☐

e.g. Find the volume of a hemisphere with a radius of 15 cm to one decimal place.

10F **14** I can calculate the volume of a square or a hexagonal pyramid given its height and the width (or area) of its base. ☐

e.g. Determine the volume of a pyramid with a height of 48 m and a square base with sides of 58 m.

10F **15** I can determine the surface area of objects with plane surfaces, such as prisms, cuboids and pyramids. ☐

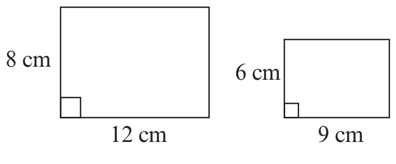
e.g. Determine the surface area of a rectangular prism with sides of 2, 3 and 4 metres.

10G **16** I can find the surface area of objects with curved surfaces, such as cylinders, cones and spheres. ☐

e.g. Find the surface area of a cone with a radius of 26cm and a sloping edge of 39 cm to one decimal place.

10H **17** I can determine when shapes are similar and find their scale factor. ☐

e.g. Show that the shapes are similar and find the scale factor.



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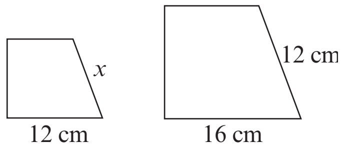
Review

10H **18** I can find the scale factor for the areas of similar shapes. ☐

e.g. Give the scale factor for the areas in the previous question.

10H **19** I can use the scale factor to find unknown values. ☐

e.g. Find the value of x for these similar shapes.



10I **20** I can use the appropriate tests to determine if triangles are similar. ☐

e.g. Two triangles have sides of 3, 5, 6 and 6, 10, 12. Determine if they are similar, giving a reason.

10J **21** I can determine when two solids are similar and calculate their scale factor. ☐

e.g. One cuboid has sides of 6, 9 and 12, while the other has sides of 10, 15 and 20. Determine if they are similar, and if so, calculate their scale factor.

Multiple-choice questions

1 3.895 rounded to two decimal places is:

A 3.8 **B** 3.89 **C** 3.9 **D** 3.99 **E** 4.0

2 4679 rounded to the nearest hundred is:

A 4600 **B** 4670 **C** 4680 **D** 4700 **E** 5000

3 5.21×10^5 is the same as:

A 0.000 052 1 **B** 260.50 **C** 52 105 **D** 521 000 **E** 52 100 000

4 0.0048 written in scientific notation is:

A 4.8×10^{-4} **B** 4.8×10^{-3} **C** 48×10^{-3} **D** 48×10^{-2} **E** 4.8×10^{3}

5 28 037.2 rounded to two significant figures is:

A 7.2 **B** 20 000.2 **C** 20 007 **D** 28 000 **E** 28 000.2

6 0.030 69 rounded to two significant figures is:

A 0.000 69 **B** 0.03 **C** 0.0306 **D** 0.0307 **E** 0.031

7 Which one of these numbers does *not* have exactly three significant figures?

A 0.0572 **B** 12.0 **C** 60.40 **D** 30 700 **E** 333 000

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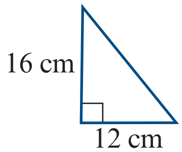
8 The three side measurements of five different triangles are given below. Which one is a right-angled triangle?

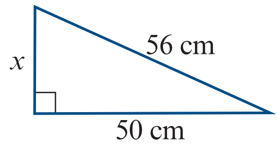
A 1, 2, 3 **B** 4, 5, 12 **C** 9, 11, 15 **D** 10, 10, 15 **E** 15, 20, 25

9 The length of the hypotenuse for the triangle shown is:



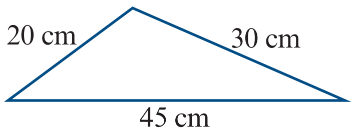
A 7.46 cm **B** 10.58 cm
C 20 cm **D** 28 cm
E 400 cm

10 The value of x in the triangle shown is:



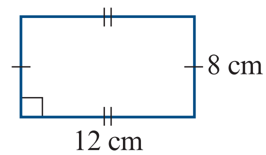
A 6 cm **B** 25.22 cm
C 75.07 cm **D** 116 cm
E 636 cm

11 The perimeter of the triangle shown is:



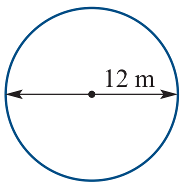
A 50 cm **B** 90 cm
C 95 cm **D** 95 cm^2
E 450 cm

12 The perimeter of the rectangle shown is:



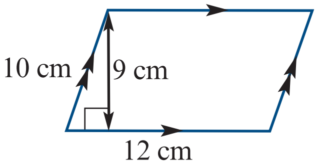
A 20 cm **B** 28 cm
C 32 cm **D** 40 cm
E 96 cm

13 The circumference of a circle with diameter 12 m is closest to:



A 18.85 m **B** 37.70 m
C 113.10 m **D** 118.44 m
E 453.29 m

14 The area of the shape shown is:



A 44 cm^2 **B** 90 cm^2
C 108 cm^2 **D** 120 cm^2
E 180 cm^2

15 The area of a circle with radius 3 cm is closest to:

A 9.42 cm^2 **B** 18.85 cm^2 **C** 28.27 cm^2 **D** 31.42 cm^2 **E** 113.10 cm^2

16 The length of an arc that subtends an angle of 49° on a circle of radius 5.4 m is closest to:

A 0.46 m **B** 2.3 m **C** 4.6 m **D** 9.2 m **E** 23 m

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17 The area of a sector that subtends an angle of 110° on a circle of radius 34 cm is closest to:

A 1109 cm^2 **B** 1110 cm^2 **C** 1119 cm^2 **D** 1900 cm^2 **E** 1901 cm^2

18 The volume of a cube with side length 5 cm is:

A 30 cm^3 **B** 60 cm^3 **C** 125 cm^3 **D** 150 cm^3 **E** 625 cm^3

19 The volume of a box with length 11 cm, width 5 cm and height 6 cm is:

A 22 cm^3 **B** 44 cm^3 **C** 302 cm^3 **D** 330 cm^3 **E** 1650 cm^3

20 The volume of a sphere with radius 16 mm is closest to:

A 67.02 mm^3 **B** 268.08 mm^3 **C** 1072.33 mm^3
D 3217 mm^3 **E** 17 157.28 mm^3

21 The volume of a cone with base diameter 12 cm and height 8 cm is closest to:

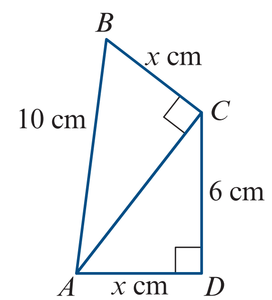
A 301.59 cm^3 **B** 904.78 cm^3 **C** 1206.37 cm^3
D 1809.56 cm^3 **E** 3619.11 cm^3

22 The volume of a cylinder with radius 3 m and height 4 m is closest to:

A 12 m^3 **B** 12.57 m^3 **C** 37.70 m^3
D 113.10 m^3 **E** 452.39 m^3

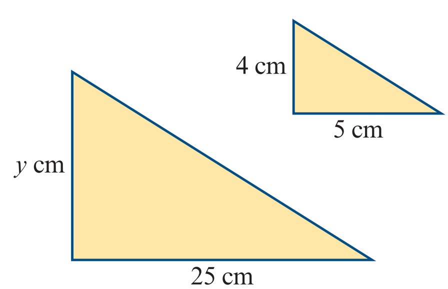
23 In the diagram, angles *ACB* and *ADC* are right angles. If *BC* and *AD* each have a length of *x* cm, then *x* is closest to:

A 4 **B** 5 **C** 5.66
D 7.07 **E** 8.25



24 The two triangles shown are similar.
The value of *y* is:

A 9 cm **B** 16 cm **C** 20 cm
D 21 cm **E** 24 cm



25 The diameter of a large sphere is 4 times the diameter of a smaller sphere. It follows that the ratio of the volume of the larger sphere to the volume of the smaller sphere is:

A 4 : 1 **B** 8 : 1 **C** 16 : 1 **D** 32 : 1 **E** 64 : 1

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Short-answer questions

1 Write each of the following in scientific notation.

a 2945 **b** 0.057 **c** 369 000 **d** 850.9

2 Write the basic numeral for each of the following.

a 7.5×10^{3} **b** 1.07×10^{-3} **c** 4.56×10^{-1}

3 Write the following to the number of significant figures indicated in the brackets.

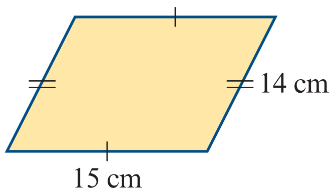
a 8.916 (2) **b** 0.0589 (2) **c** 809 (1)

4 Write the following to the number of decimal places indicated in the brackets.

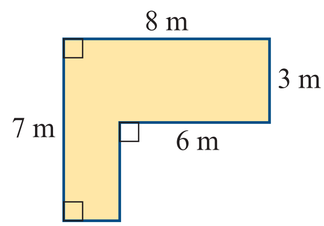
a 7.145 (2) **b** 598.241 (1) **c** 4.0789 (3)

5 Find the perimeters of these shapes.

a



b

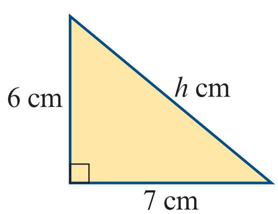


6 Find the perimeter of a square with side length 9 m.

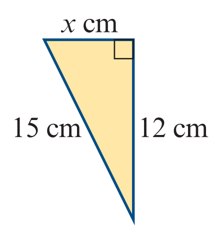
7 Find the perimeter of a rectangle with length 24 cm and width 10 cm.

8 Find the lengths of the unknown sides, to two decimal places, in the following triangles.

a

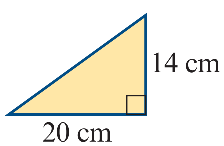


b

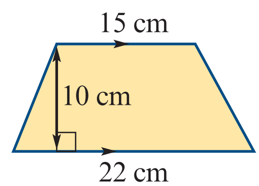


9 Find the areas of the following shapes.

a



b



10 Find the surface area of a cube with side length 2.5 m.

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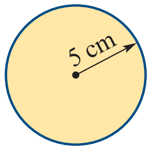
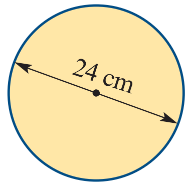
11 Find the circumferences of the following circles to two decimal places.

a



b



12 Find the areas of the circles in Question **11** to two decimal places.

13 A soup can has a diameter of 7 cm and a height of 13.5 cm.

- **a** How much metal, to two decimal places, is needed to make the can?
- **b** A paper label is made for the outside cylindrical shape of the can. How much paper, in m^2, is needed for 100 cans? Give your answer to two decimal places.
- **c** What is the capacity of one can, in litres, to two decimal places?

14 A circular swimming pool has a diameter of 4.5 m and a depth of 2 m. How much water will the pool hold to the nearest litre?

15 The radius of the Earth is approximately 6400 km. Calculate:

- **a** the surface area in square kilometres
- **b** the volume to four significant figures.

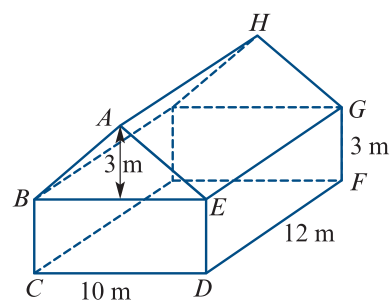
16 The diameter of the base of an oilcan in the shape of a cone is 12 cm and its height is 10 cm. Find:

- **a** its volume in square centimetres to two decimal places
- **b** its capacity to the nearest millilitre.

17 A pyramid with a square base of side length 8 m has a height of 3 m. Find the length of a sloping edge to one decimal place.

18 For the solid shown on the right, find to two decimal places:

- **a** the area of rectangle *BCDE*
- **b** the area of triangle *ABE*
- **c** the length *AE*
- **d** the area of rectangle *AEGH*
- **e** the surface area.



19 Find the volume of a rectangular prism with length 3.5 m, width 3.4 m and height 2.8 m.

20 You are given a circle of radius r. The radius increases by a scale factor of 2. By what factor does the area of the circle increase?

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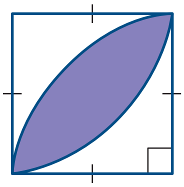
21 You are given a circle of diameter d. The diameter decreases by a scale factor of $\frac{1}{2}$. By how much does the area of the circle decrease?

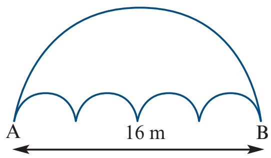
22 The radius of each arc equals the length of the side of the square. For the shaded region, find to two decimal places:

a the perimeter

b the area.



23 Which is the shorter path from A to B? Is it along the four semicircles or along the larger semicircle? Give reasons for your answer.

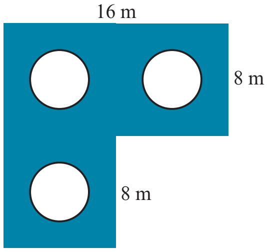


Written-response questions

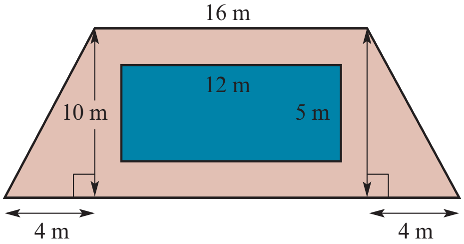
1 A lawn has three circular flowerbeds in it, as shown in the diagram. Each flowerbed has a radius of 2 m. A gardener has to mow the lawn and use a whipper-snipper to trim all the edges. Calculate:

a the area to be mown

b the length of the edges to be trimmed. Give your answer to two decimal places.



2 Chris and Gayle decide to build a swimming pool on their new housing block. The pool will measure 12 m by 5 m and it will be surrounded by timber decking in a trapezium shape. A safety fence will surround the decking. The design layout of the pool and surrounding area is shown in the diagram.



a What length of fencing is required? Give your answer to two decimal places.

b What area of timber decking is required?

c The pool has a constant depth of 2 m. What is the volume of the pool?

d The interior of the pool is to be painted white. What surface area is to be painted?

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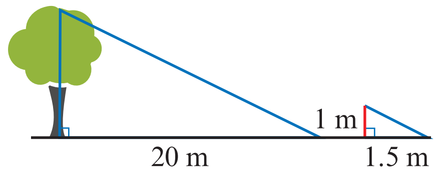
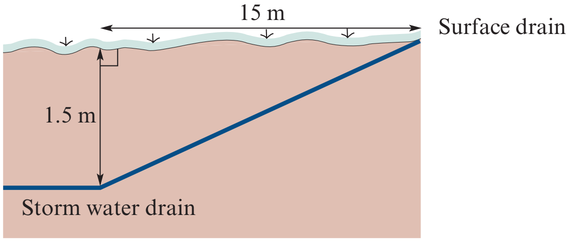
3 A biologist studying gum trees wanted to calculate the height of a particular tree. She placed a one metre ruler on the ground which cast a shadow on the ground measuring 1.5 m. The gum tree cast a shadow of 20 m, as shown in the diagram below (*not* to scale). Calculate the height of the tree. Give your answer to two decimal places.



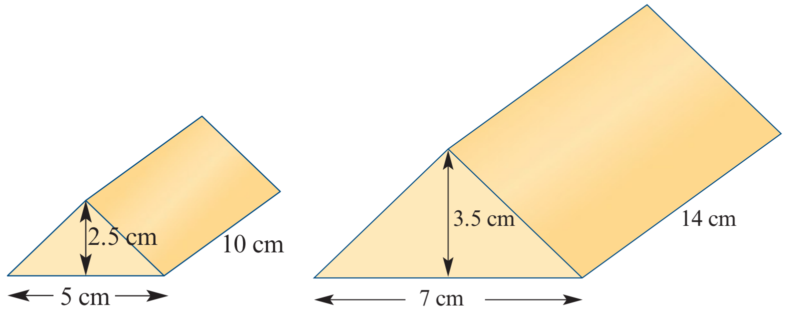
4 A builder is digging a trench for a cylindrical water pipe. From a drain at ground level, the water pipe goes 1.5 m deep where it joins a storm water drain. The horizontal distance from the surface drain to the storm water drain is 15 m, as indicated in the diagram below (*not* to scale).



a Calculate the length of water pipe required to connect the surface drain to the storm water drain to two decimal places.

b If the radius of the water pipe is 20 cm, what is the volume of the water pipe? Give your answer to two decimal places.

5 Two similar triangular prisms are shown below.



a Find the ratio of their surface areas.

b Find the ratio of their volumes.

c What is the volume of the smaller prism to the nearest cm^3?

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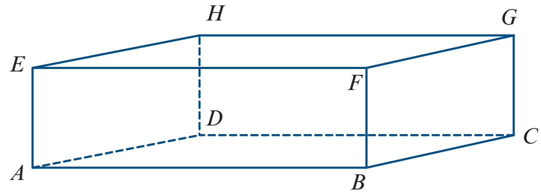
6 The length of a rectangular prism is eight times its height. The width is four times the height. The length of the diagonal between two opposite vertices (*A* and *G*) is 36 cm.
Find the volume of the prism.



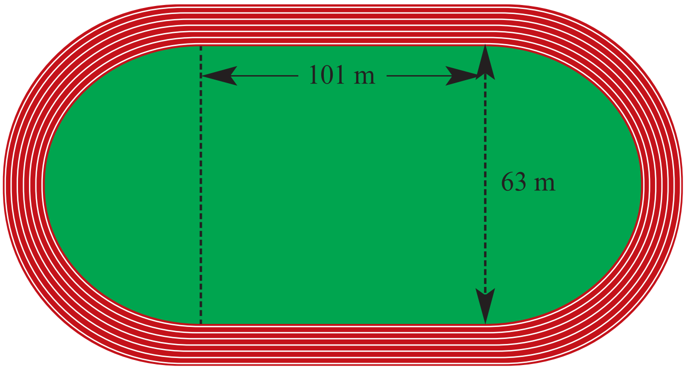
7 The volume of a cone of height 28.4 cm is 420 cm^3. Find the height of a similar cone whose volume is 120 cm^3 to two decimal places.

8 An athletics track is made up of a straight stretch of 101 m and two semicircles on the ends, as shown in the diagram. There are 6 lanes, each one metre wide.

a What is the total distance, to the nearest metre, around the inside lane?

b If 6 athletes run around the track keeping to their own lane, how far, to the nearest metre, would each athlete run?

c Draw a diagram and indicate at which point each runner should start so that they all run the same distance.



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Chapter 11

Applications of trigonometry

Chapter questions

- How are sin θ, cos θ and tan θ defined using a right-angled triangle?
- How can the trigonometric ratios be used to find the side lengths or angles in right-angled triangles?
- What is meant by an angle of elevation or an angle of depression?
- How are three-figure bearings measured?
- How can the sine and cosine rules be used to solve triangles which are not right-angled?
- What are the three rules that are used to find the area of a triangle?

Trigonometry can be used to solve many practical problems. How high is that tree? What is the height of the mountain we can see in the distance? What is the exact location of the fire that has just been seen by fire spotters? How wide is the lake? What is the area of this irregular-shaped paddock?

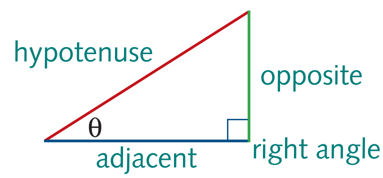
11A Trigonometry basics

Learning intentions

- To be able to name the sides of a right-angled triangle.
- To be able to know the definitions of the trigonometric ratios.
- To be able to use a CAS calculator to find the value of a trigonometric ratio for a given angle.

Although you are likely to have studied some trigonometry, it may be helpful to review a few basic ideas.

Naming the sides of a right-angled triangle

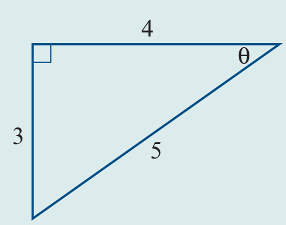


- The **hypotenuse** is the longest side of the right-angled triangle and is always opposite the right angle (90°).
- The **opposite** side is directly opposite the angle θ.
- The **adjacent** side is beside the angle θ, but it is not the hypotenuse. It runs from θ to the right angle.

The opposite and adjacent sides are located in relation to the position of angle θ. If θ was in the other corner, the sides would have to swap their labels.
The letter θ is the Greek letter *theta*. It is commonly used to label an angle.

Example 1 Identifying the sides of a right-angled triangle

Give the lengths of the hypotenuse, the opposite side and the adjacent side in the triangle shown.



Explanation	Solution
The hypotenuse is opposite the right angle.	The hypotenuse: $h = 5$
The opposite side is opposite the angle θ.	The opposite side: $o = 3$
The adjacent side is between θ and the right-angle.	The adjacent side: $a = 4$

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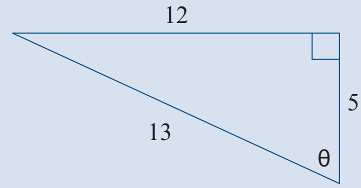
Now try this 1 **Identifying the sides of a right-angled triangle (Example 1)**

Refer to the diagram to answer the questions below.



a What is the name of the side that is 12 units long?

b Name the side that is 5 units long.

c Give the name of the side that is 13 units long.

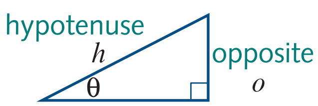
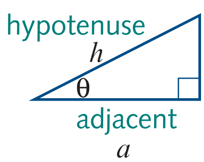
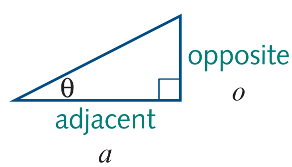
Hint 1 It is opposite the angle θ.

Hint 2 It is between the angle θ and the right angle.

Hint 3 It is opposite the right angle.

The trigonometric ratios

The **trigonometric ratios**, **sin** θ, **cos** θ and **tan** θ, can be defined in terms of the sides of a right-angled triangle.

		
$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\sin\theta = \dfrac{o}{h}$	$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\cos\theta = \dfrac{a}{h}$	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ $\tan\theta = \dfrac{o}{a}$

"SOH — CAH — TOA"

This mnemonic, **SOH-CAH-TOA**, is often used by students to help them remember the rule for each trigonometric ratio.

In this mnemonic:

- **SOH** reminds us that **S**ine equals **O**pposite over **H**ypotenuse
- **CAH** reminds us that **C**osine equals **A**djacent over **H**ypotenuse
- **TOA** reminds us that **T**an equals **O**pposite over **A**djacent.

Or you may prefer:

'**S**ir **O**liver's **H**orse **C**ame **A**mbling **H**ome **T**o **O**liver's **A**rms'

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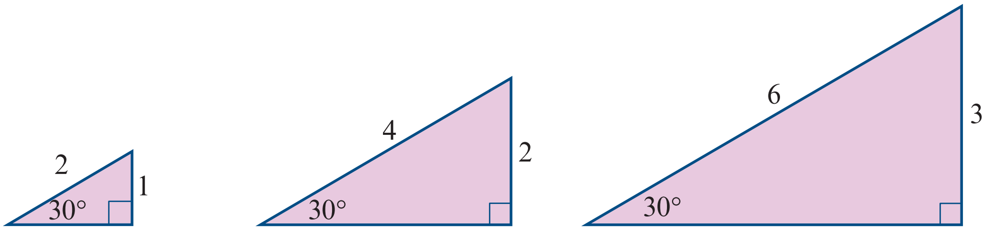
The meaning of the trigonometric ratios

Using a calculator we find, for example, that $\sin 30° = 0.5$. This means that in *all* right-angled triangles with an angle of 30°, the length of the side opposite the 30° divided by the length of the hypotenuse is always 0.5.



$\dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{1}{2} = 0.5$ $\dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{2}{4} = 0.5$ $\dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{3}{6} = 0.5$

Try drawing any right-angled triangle with an angle of 30° and check that the ratio:

$$\frac{\text{opposite}}{\text{hypotenuse}} = 0.5$$

Similarly, for *any* right-angled triangle with an angle of 30°, the ratios cos 30° and tan 30° always have the same values:

$$\cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} \text{ is always } \frac{\sqrt{3}}{2} = 0.8660 \text{ (to four decimal places)}$$

$$\tan 30° = \frac{\text{opposite}}{\text{adjacent}} \text{ is always } \frac{1}{\sqrt{3}} = 0.5774 \text{ (to four decimal places).}$$

A calculator gives the value of each trigonometric ratio for any angle entered.

Using your CAS calculator to evaluate trigonometric ratios

Warning!

Make sure that your calculator is set in DEGREE mode before attempting the example on the following page.

See the Appendix, which can be accessed online through the Interactive Textbook.

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Example 2 Finding the values of trigonometric ratios

Use your graphics calculator to find, to four decimal places, the value of:

a sin 49° **b** cos 16° **c** tan 27.3°

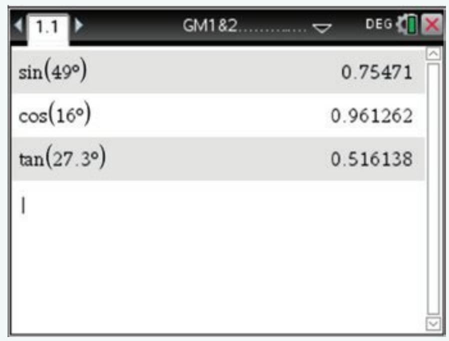
Explanation / Solution

1 For the **TI-Nspire CAS**, ensure that the mode is set in **Degree** and **Approximate (Decimal)**. Refer to Appendix to set mode.

2 In a Calculator page, press [trig], select **sin** and type 49.

3 Repeat for **b** and **c** as shown on the calculator screen.

Optional: you can add a degree symbol from the [αβ°] palette if desired. This will override any mode settings.



4 Write your answer to four decimal places.

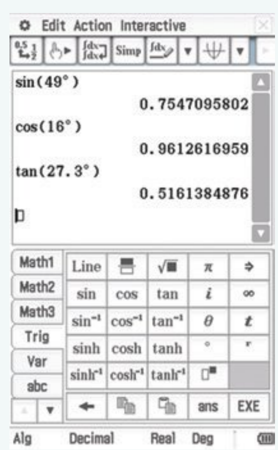
a sin (49°) = 0.7547
b cos (16°) = 0.9613
c tan (27.3°) = 0.5161

5 For **ClassPad**, in the Main application ensure that the status bar is set to **Decimal** and **Degree** mode.

6 To enter and evaluate the expression:

- Display the **keyboard**
- In the Trig palette select [sin]
- Type [4][9][°][)]
- Press [EXE]

7 Repeat for **b** and **c** as shown on the calculator screen.



8 Write your answer to four decimal places.

a sin (49°) = 0.7547
b cos (16°) = 0.9613
c tan (27.3°) = 0.5161

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Now try this 2 Finding the values of trigonometric ratios (Example 2)

Find the values of the trigonometric ratios to three decimal places:

a tan 28° **b** cos 43° **c** sin 62.8°

Hint 1 Make sure your calculator is in Degree mode.

Hint 2 Choose the required trigonometric button on your CAS calculator.

Hint 3 Type the required angle and press **enter** or **EXE**.

Hint 4 Look at the fourth decimal place. If it is 5 or larger, increase the third decimal place by 1.

Section Summary

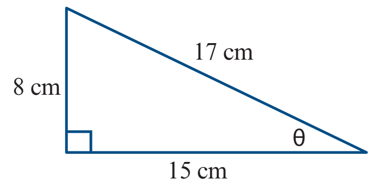
- In a right-angled triangle:
  The hypotenuse is the longest side and is opposite the right-angle
  The opposite side is directly opposite the angle θ
  The adjacent side runs from θ to the right-angle.
- The trigonometric ratios are:
  $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

Skill-sheet

Exercise 11A

Building understanding

1 Use the position of the angle θ to name each side.

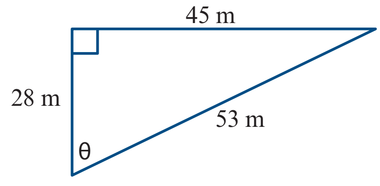


a Name the side that is 8 cm long.

b Write the name of the side that is 15 cm long.

c What is the name of the side that is 17 cm long?

2 Name each side by using the position of the angle θ.

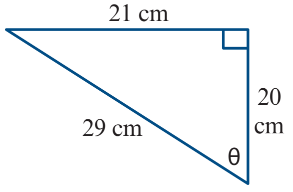


a Write the name of the side that is 45 m long.

b What is the name of the side that is 28 m long?

c Name the side that is 53 m long.

3 Using the sides given in the diagram, write the trigonometric ratio for each of the following:



a sin θ

b cos θ

c tan θ

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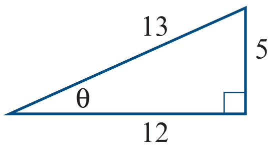
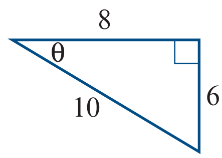
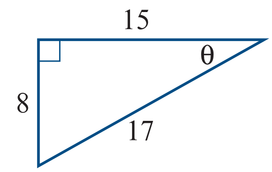
Developing understanding

Example 1

4 State the values of the hypotenuse, the opposite side and the adjacent side in each triangle.

a


b


c


Example 2

5 Write the ratios for sin θ, cos θ and tan θ for each triangle in Question **4**.

6 Find the values of the following trigonometric ratios to four decimal places.

a sin 27° **b** cos 43° **c** tan 62° **d** cos 79°
e tan 14° **f** sin 81° **g** cos 17° **h** tan 48°

Testing understanding

Use Pythagoras' theorem to find the answers as fractions.

7 Given $\cos\theta = \frac{20}{29}$, find $\sin\theta$.

8 Use $\sin\theta = \frac{9}{41}$ to find $\tan\theta$.

11B Finding an unknown side in a right-angled triangle

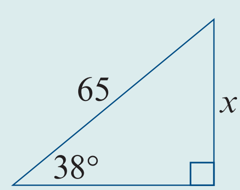
Learning intentions

- ▶ To be able to choose the required trigonometric ratio when finding an unknown side of a right-angled triangle.
- ▶ To be able to substitute in values and solve the required equation to find the length of the unknown side.

The trigonometric ratios can be used to find unknown sides in a right-angled triangle, given an angle and one side. When the unknown side is in the numerator (top) of the trigonometric ratio, proceed as follows.

Example 3 Finding an unknown side in a right-angled triangle

Find the length of the unknown side, x, in the triangle shown to two decimal places.



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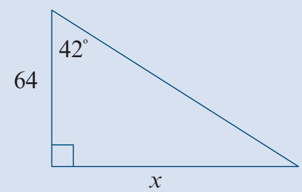
Explanation	Solution
1 The sides involved are the opposite and the hypotenuse, so use sin θ.	$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
2 Substitute in the known values.	$\sin 38^\circ = \dfrac{x}{65}$
3 Multiply both sides of the equation by 65 to obtain an expression for x. Use a calculator to evaluate.	$65 \times \sin 38^\circ = x$ $x = 65 \times \sin 38^\circ$ $= 40.017\ldots$
4 Write answer to 2 decimal places.	$x = 40.02$

Now try this 3 **Finding an unknown side in a right-angled triangle (Example 3)**

Find the length of the unknown side, x, in the triangle shown to one decimal place.



Hint 1 What is the position of the unknown side, looking from the given angle?

Hint 2 What position name should be given to the side that is 64 units long?

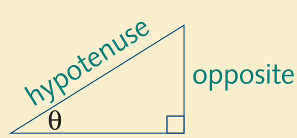
Hint 3 What trigonometric ratio uses the position names of the ‘x’ and the ‘64’ side?

Finding an unknown side in a right-angled triangle

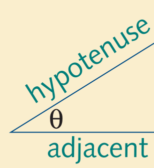
1 Draw the triangle and write in the given angle and side. Label the unknown side as x.

2 Use the trigonometric ratio that includes the given side and the unknown side.

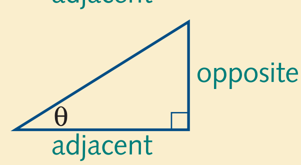
a For the opposite and the hypotenuse, use $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$



b For the adjacent and the hypotenuse, use $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$



c For the opposite and the adjacent, use $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$



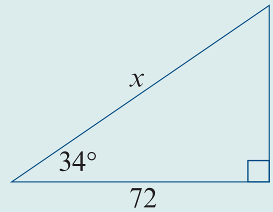
3 Rearrange the equation to make x the subject.

4 Use your calculator to find the value of x to the required number of decimal places.

An extra step is needed when the unknown side is in the denominator (at the bottom) of the trigonometric ratio, as in the example on the following page.

Example 4 Finding an unknown side which is in the denominator of the trig ratio

Find the value of x in the triangle shown to two decimal places.



Explanation

1 The sides involved are the adjacent and the hypotenuse, so use cos θ.

2 Substitute in the known values.

3 Multiply both sides by x.

4 Divide both sides by cos 34° to obtain an expression for x. Use a calculator to evaluate.

5 Write your answer to two decimal places.

Solution

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 34° = \frac{72}{x}$$

$$x\cos 34° = 72$$

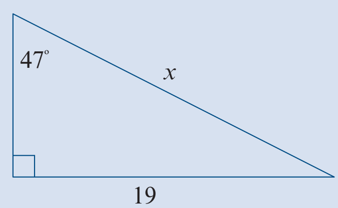
$$x = \frac{72}{\cos 34°}$$

$$= 86.847\ldots$$

$$x = 86.85$$

Now try this 4 Finding an unknown side which is in the denominator of the trig ratio (Example 4)

Find the length of the unknown side, x, in the triangle shown to one decimal place.



Hint 1 What are the position names of side x and the side that is 19 units long?

Hint 2 Choose the trigonometric ratio that uses the position names of the two sides involved.

Hint 3 Write the trigonometric equation for the given angle and sides.

Hint 4 Multiply both sides of the equation by x. Now what do you need to divide both sides by?

Section Summary

To find an unknown side in a right-angled triangle:

- Use the position of the given angle to name the given side and the required side.
- Write an equation using the trigonometric ratio that uses the given and required sides.
- Substitute the given values and solve the equation to find the unknown side to the required decimal places.

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Skill-sheet

Exercise 11B

Building understanding

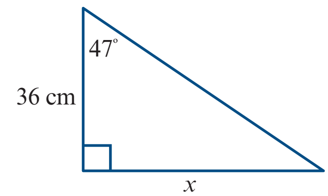
1 **a** State the name of the side that is 36 cm long.

b What is the name of the unknown side, x?

c Write the trigonometric ratio rule that uses the names of the sides in parts **a** and **b**.

d Substitute the value of the angle and the known side into the rule. Call the unknown side x.

e Solve the equation by multiplying both sides by the denominator and using your CAS calculator to find the value of x to two decimal places.



2 **a** Give the name of the side that is 19 cm long.

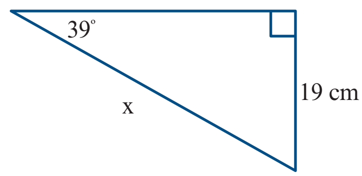
b State the name of the unknown side, x.

c Write the trigonometric ratio rule that uses the names of the sides in parts **a** and **b**.

d Substitute the value of the angle and the known side into the rule. Call the unknown side x.

e Because the unknown, x, is the denominator, two steps are needed. Multiply both sides by x, then divide both sides by $\sin 39°$, to make x the subject.

f Use your CAS calculator to find x to one decimal place.



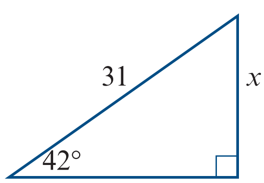
Developing understanding

Example 3

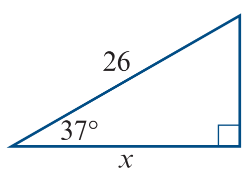
3 In each right-angled triangle below:

- decide whether the $\sin \theta$, $\cos \theta$ or $\tan \theta$ ratio should be used
- then find the unknown side, x, to two decimal places.

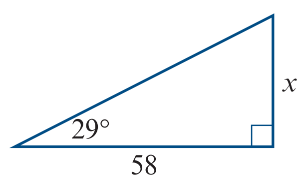
a



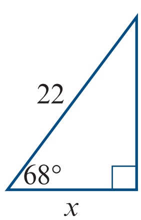
b



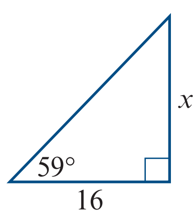
c



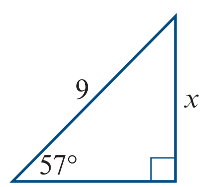
d



e



f



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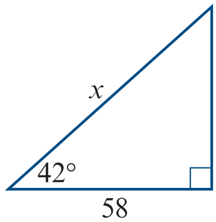
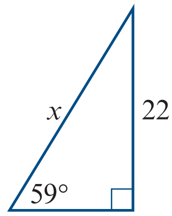
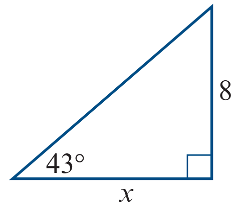
Example 4

4 Find the unknown side, x, in each right-angled triangle below to two decimal places.

a


b


c


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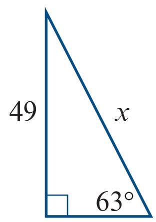

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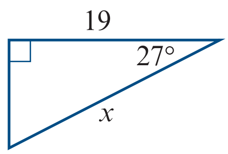

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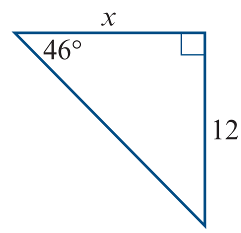
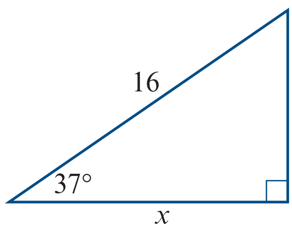


5 Find the length of the unknown side shown in each triangle to one decimal place.

a


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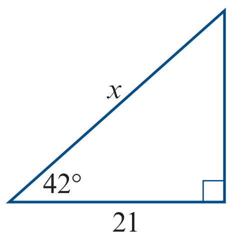

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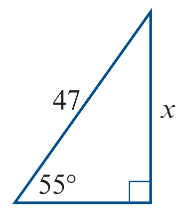

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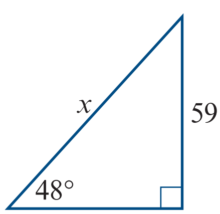

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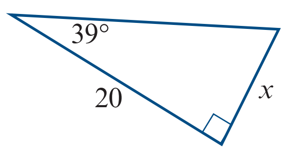

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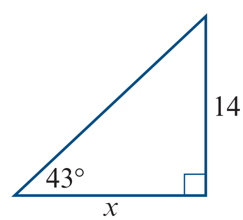
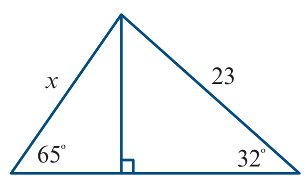
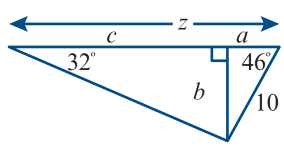


Testing understanding

6 Find the length of the unknown side, x, in the diagram shown.
Give your answer to one decimal place.



7 Find the length, z, to one decimal place.



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11C Finding an angle in a right-angled triangle

Learning intentions

- To be able to use a CAS calculator to find an angle when given the value of its trigonometric ratio.
- To be able to find the required angle in a right-angled triangle when given two sides of the triangle.

Finding an angle from a trigonometric ratio value

Before we look at how to find an unknown angle in a right-angled triangle, it will be useful to see how to find the angle when we know the value of the trigonometric ratio.

Suppose a friend told you that they found the sine value of a particular angle to be 0.8480, and challenged you to find out the mystery angle that had been used.

This is equivalent to saying:

$\sin\theta = 0.8480$, find the value of angle θ.

To do this, you need to work backwards from 0.8480 by undoing the sine operation to get back to the angle used. It is as if we have to find reverse gear to undo the effect of the sine function.

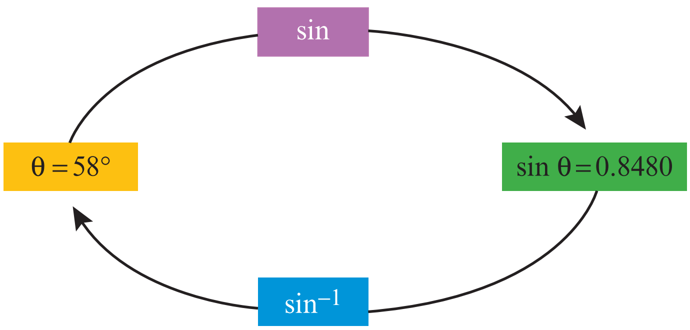
The reverse gear for sine is called the inverse of sine, written $\sin^{-1}$. The superscript, -1, is not a power. It is just saying, let us undo, or take one step backwards from using, the sine function.

The step to find θ when $\sin\theta = 0.8480$ can be written as:

$$\sin^{-1}(0.8480) = \theta$$

This process is summarised in the following diagram.

- The top arrow in the diagram corresponds to: given θ, find $\sin\theta$. We use the sine function on our calculator to do this by entering $\sin 58°$ into a calculator to obtain the answer, 0.8480.
- The bottom arrow in the diagram corresponds to: given $\sin\theta = 0.8480$, find θ. We use the $\sin^{-1}$ function on our calculator to do this by entering $\sin^{-1}(0.8480)$ to obtain the answer, 58°.



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Similarly:

- The inverse of cosine, written as $\cos^{-1}$, is used to find θ when, for example, $\cos\theta = 0.5$.
- The inverse of tangent, written as $\tan^{-1}$, is used to find θ when, for example, $\tan\theta = 1.67$.

You will learn how to use the $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ functions of your calculator in the following example.

Example 5 Finding an angle from a trigonometric ratio

Find the angle, θ, to one decimal place, given:

a $\sin\theta = 0.8480$ **b** $\cos\theta = 0.5$ **c** $\tan\theta = 1.67$

Explanation	Solution
a We need to find $\sin^{-1}(0.8480)$.	
1 For **TI-Nspire CAS**, press [trig], select $\sin^{-1}$, then press [0] [.] [8] [4] [8] [0] [enter].	$\sin^{-1}(0.848)$ 57.9948
2 For **ClassPad**, tap [$\sin^{-1}$] [0] [.] [8] [4] [8] [0] [)] [EXE].	
3 Write your answer to one decimal place.	θ = 58.0°
b We need to find $\cos^{-1}(0.5)$.	
1 For **TI-Nspire CAS**, press [trig], select $\cos^{-1}$, then press [0] [.] [5] [enter].	$\cos^{-1}(0.5)$ 60
2 For **ClassPad**, tap [$\cos^{-1}$] [0] [.] [5] [)] [EXE].	
3 Write your answer to one decimal place.	θ = 60°
c We need to find $\tan^{-1}(1.67)$.	
1 For **TI-Nspire CAS**, press [trig], select $\tan^{-1}$, then press [1] [.] [6] [7] [enter].	$\tan^{-1}(1.67)$ 59.0867
2 For **ClassPad**, tap [$\tan^{-1}$] [1] [.] [6] [7] [)] [EXE].	
3 Write your answer to one decimal place.	θ = 59.1°

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Now try this 5 Finding an angle from a trigonometric ratio (Example 5)

Find the angle, θ, to two decimal places, given:

a $\cos\theta = 0.6847$ **b** $\tan\theta = 7.5509$ **c** $\sin\theta = 0.2169$

Hint 1 In each question, select the required inverse function: $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ on your CAS calculator.

Hint 2 Then enter the given decimal value of the trigonometric function, and press **enter** or **EXE** to find the required angle.

Getting the language right

The language we use when finding an angle from a trig ratio is difficult when you first meet it. The samples below are based on the results of Example **5**.

- When you see:

  $\sin(58°) = 0.8480$

  think: 'the sine of the angle 58° equals 0.8480'.

- When you see:

  $\sin^{-1}(0.8480) = 58°$

  think: 'the angle whose sine is 0.8480 equals 58°'.

- When you see:

  $\cos(60°) = 0.5$

  think: 'the cosine of the angle 60° equals 0.5'.

- When you see:

  $\cos^{-1}(0.5) = 60°$

  think: 'the angle whose cosine is 0.5 equals 60°'.

- When you see:

  $\tan(59.1°) = 1.67$

  think: 'the tan of the angle 59.1° equals 1.67'.

- When you see:

  $\tan^{-1}(1.67) = 59.1°$

  think: 'the angle whose tan is 1.67 equals 59.1°'.

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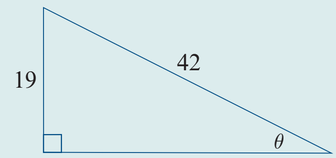
Finding an angle given two sides

Example 6 Find an angle, given 2 sides in a right-angled triangle

Find the angle, θ, in the right-angled triangle shown to one decimal place.



Explanation

1 The sides involved are the opposite and the hypotenuse, so use sin θ.

2 Substitute in the known values.

3 Write the equation to find an expression for θ. Use a calculator to evaluate.

4 Write your answer to one decimal place.

Solution

$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\sin\theta = \dfrac{19}{42}$

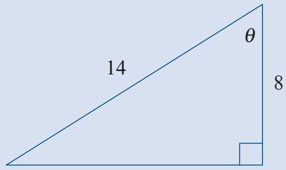
$\theta = \sin^{-1}\left(\dfrac{19}{42}\right) = 26.896\ldots$

$\theta = 26.9°$

The three angles in a triangle add to 180°. As the right angle is 90°, the other two angles must add to make up the remaining 90°. When one angle has been found, just subtract it from 90° to find the other angle. In Example **6**, the other angle must be 90° − 26.9° = 63.1°.

Now try this 6 Find an angle, given 2 sides in a right-angled triangle (Example 6)

Find the angle, θ, in the triangle shown to two decimal places.



Hint 1 What are the position names of the sides that are 8 and 14 units long?

Hint 2 Which of: $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ can use the sides from Hint 1 to find the angle θ?

Finding an angle in a right-angled triangle

1 Draw the triangle with the given sides shown. Label the unknown angle as θ.

2 Use the trigonometric ratio that includes the two known sides.

- If given the opposite and hypotenuse, use $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
- If given the adjacent and hypotenuse, use $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
- If given the opposite and adjacent, use $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

3 Divide the side lengths to find the value of the trigonometric ratio.

4 Use the appropriate inverse function key to find the angle, θ.

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Section Summary

To find an unknown angle:

- Use the position of the required angle to name the two given sides.
- Write an equation using the trigonometric ratio with the required angle and the two given sides.
- Substitute the given values and solve the equation to find the unknown angle, to the required number of decimal places.

Skill-sheet

Exercise 11C

Building understanding

1 Give answers to one decimal place.

a Given $\cos\theta = 0.4867$, use $\cos^{-1}(0.4867)$ to find the value of θ.

b Given $\tan\theta = 0.6384$, use $\tan^{-1}(0.6348)$ to find the value of θ.

c Given $\sin\theta = 0.3928$, use $\sin^{-1}(0.3928)$ to find the value of θ.

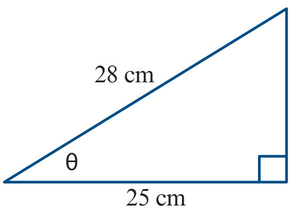
2 To find the value of θ, answer the following questions. Give θ to one decimal place.

a State the name of the side that is 28 cm long.

b Write the name of the side that is 25 cm long.

c Write the trigonometric ratio rule that uses the names of the two sides in parts **a** and **b**.

d Substitute the values of the sides into the rule.



e To find θ, use the inverse cosine, $\cos^{-1}$, feature on your CAS calculator to evaluate $\cos^{-1}(\frac{25}{28})$.

Developing understanding

Example 5

3 Find the unknown angle, θ, to one decimal place.

a $\sin\theta = 0.4817$

b $\cos\theta = 0.6275$

c $\tan\theta = 0.8666$

d $\sin\theta = 0.5000$

e $\tan\theta = 1.0000$

f $\cos\theta = 0.7071$

g $\sin\theta = 0.8660$

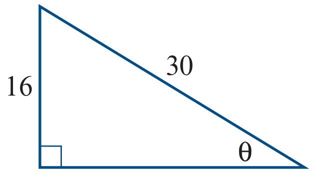
h $\tan\theta = 2.500$

i $\cos\theta = 0.8383$

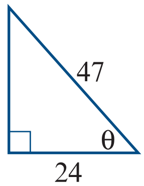
Example 6

4 Find the unknown angle, θ, in each triangle to one decimal place.

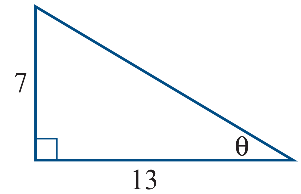
a Use $\sin^{-1}$ for this triangle.



b Use $\cos^{-1}$ for this triangle.



c Use $\tan^{-1}$ for this triangle.



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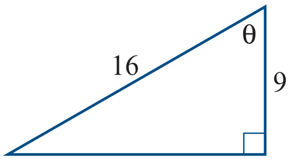

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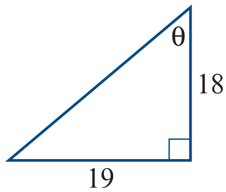

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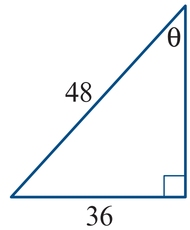

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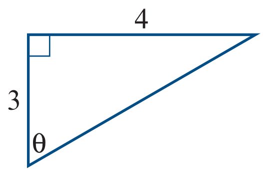

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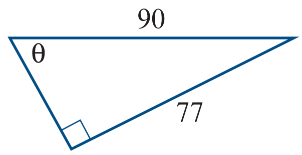

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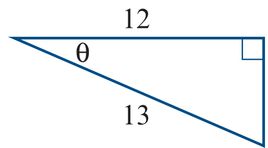
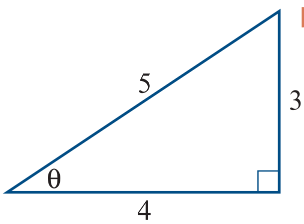
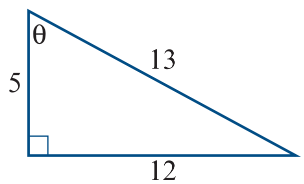


5 Find the value of θ in each triangle to one decimal place.

a



b



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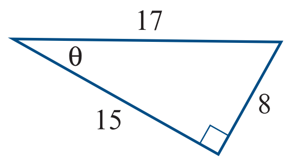
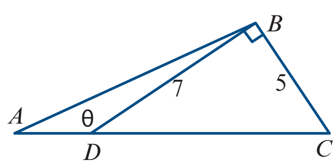
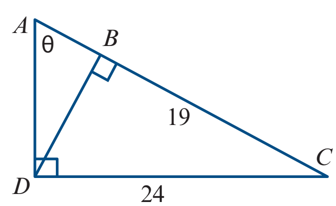


Testing understanding

6 Find the unknown angle, θ, to one decimal place.



7 Find the unknown angle, θ, to one decimal place.



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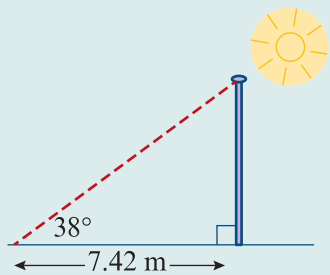
11D Applications of right-angled triangles

Learning intentions

- To be able to draw clearly labelled diagrams of practical situations, showing the given sides and angles.
- To be able to set up and solve equations to find unknown sides and angles.

Example 7 Application requiring a length

A flagpole casts a shadow 7.42 m long. The sun's rays make an angle of 38° with the level ground. Find the height of the flagpole to two decimal places.



Explanation

1 Draw a diagram showing the right-angled triangle. Include all the known details and label the unknown side as x.

2 The opposite and adjacent sides are involved, so use tan θ.

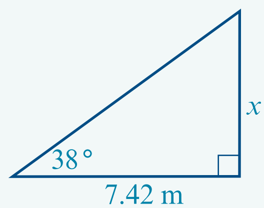
3 Substitute in the known values.

4 Multiply both sides by 7.42.

5 Use your calculator to find the value of x.

6 Write your answer to two decimal places.

Solution



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 38^\circ = \frac{x}{7.42}$$

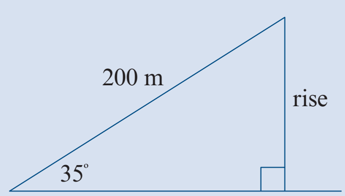
$$7.42 \times \tan 38^\circ = x$$

$$x = 5.797\ldots$$

The height of the flagpole is 5.80 m.

Now try this 7 Application requiring a length (Example 7)

A person walked 200 m up a slope of 35°. How much did they rise vertically? Answer to the nearest metre.



Hint 1 What are the position names of the sloping side and the rise?

Hint 2 Now use the trigonometric ratio that uses those position names.

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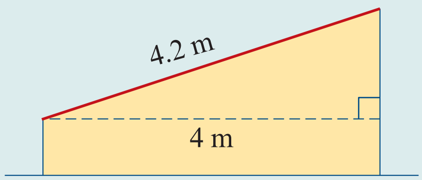
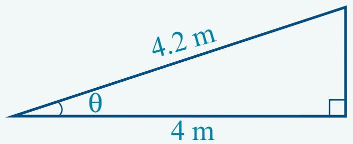
Example 8 Application requiring an angle

A sloping roof uses sheets of corrugated iron, 4.2 m long, on a shed, 4 m wide. There is no overlap of the roof past the sides of the walls. Find the angle that the roof makes with the horizontal to one decimal place.

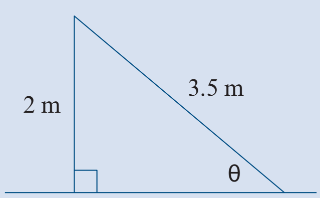


Explanation	Solution
1 Draw a diagram showing the right-angled triangle. Include all known details and label the required angle.	
2 The adjacent and hypotenuse are involved, so use cos θ.	$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
3 Substitute in the known values.	$\cos\theta = \dfrac{4}{4.2}$
4 Write the equation to find θ.	$\theta = \cos^{-1}\left(\dfrac{4}{4.2}\right)$
5 Use your calculator to find the value of θ.	$\cos^{-1}\left(\dfrac{4}{4.2}\right)$ 17.7528
6 Write your answer to one decimal place.	The roof makes an angle of 17.8° with the horizontal.

Remember: Always evaluate a mathematical expression as a whole, rather than breaking it into several smaller calculations. Rounding-off errors accumulate as more approximate answers are fed into the calculations.

Now try this 8 Application requiring an angle (Example 8)

A vertical ladder of a children's slide is 2 m in height. The length of the slide is 3.5 m. Find the unknown angle, θ, that the slide makes with the level ground to the nearest degree.



Hint 1 What are the position names of the given lengths?

Hint 2 Which of: $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ uses the sides named in Hint 1?

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Section Summary

To solve applications questions:

- Draw a clearly labelled diagram showing the given values and a symbol for the required value.
- Use the position of the angle to choose the required trigonometric equation.
- Substitute the given values and solve the equation to the required number of decimal places.

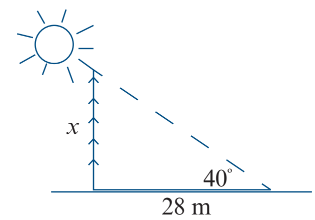
Skill-sheet

Exercise 11D

Building understanding

Answer to one decimal place.

1 A tree casts a shadow that is 28 m long, making an angle of 40° with the horizontal ground. To find the height, x, of the tree, answer the questions below.



a Name the 28 m side.

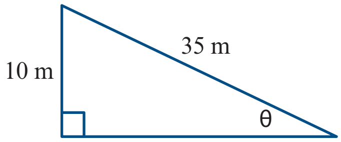
b Write the name of the side x.

c Write the trigonometric ratio rule that uses the sides named in parts **a** and **b**.

d Substitute the known values of the angle and a side into the equation. Call the unknown side x.

e Multiply both sides of the equation by the value of the denominator to get x by itself. Use your CAS calculator to find the value of x.

2 The starting ramp for a ski race is 35 m long and 10 m high. Find the angle the ramp makes with the horizontal by answering the questions below.



a State the name of the side that is 35 m long.

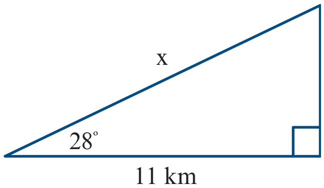
b Give the name of the side that is 10 m long?

c Write the trigonometric ratio rule that uses the sides named in parts **a** and **b**?

d Substitute the values of the side lengths into the equation.

e To find θ, use the inverse sine, $\sin^{-1}$, feature on your CAS calculator to evaluate $\sin^{-1}(\frac{10}{35})$.

3 A map showed a bushwalker had moved a horizontal distance of 11 km as she walked up a slope of 28°. To find the distance she had walked up the slope, answer the questions on the following page.



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a Give the name of the 11 km side and side x.

b State the trigonometric ratio rule that uses the sides in part **a**.

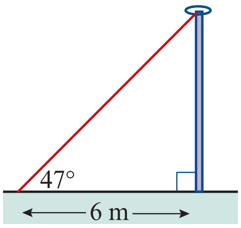
c Substitute the known value for the side and the angle into the equation.

d Multiply both sides by x. Divide both sides by $\cos 28°$. Use your calculator to find the value of x.

Developing understanding

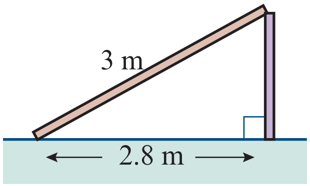
Example 7

4 A pole is supported by a wire that runs from the top of the pole to a point on the level ground, 6 m from the base of the pole. The wire makes an angle of 47° with the ground. Find the height of the pole to two decimal places.

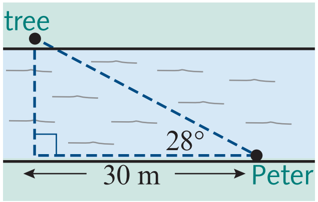


Example 8

5 A 3-metre log rests with one end on the top of a post and the other end on the level ground, 2.8 m from the base of the post. Find the angle the log makes with the ground to one decimal place.



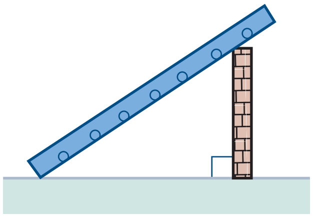
6 Peter noticed that a tree was directly opposite him on the far bank of the river. After he walked 30 m along his side of the river, he found that his line of sight to the tree made an angle of 28° with the riverbank. Find the width of the river, to the nearest metre.



7 A ladder rests on a wall that is 2 m high. The foot of the ladder is 3 m from the base of the wall on level ground.

a Copy the diagram and include the given information. Label as θ the angle the ladder makes with the ground.

b Find the angle the ladder makes with the ground to one decimal place.



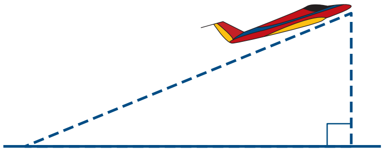
8 An aeroplane maintains a flight path of 17° with the horizontal after it takes off. It travels for 2 km along that flight path.

a Show the given and required information on a copy of the diagram.

b Find to two decimal places:

i the horizontal distance of the aeroplane from its take-off point

ii the height of the aeroplane above ground level.



9 A 3 m ladder rests against an internal wall. The foot of the ladder is 1 m from the base of the wall. Find the angle the ladder makes with the floor to one decimal place.

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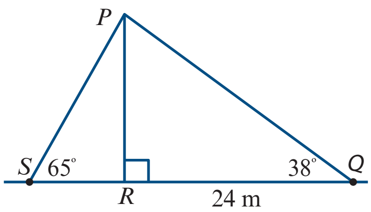
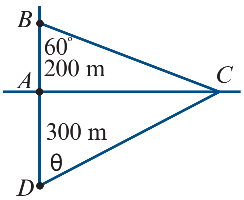
10 The entrance to a horizontal mining tunnel has collapsed, trapping the miners inside. The rescue team decide to drill a vertical escape shaft from a position 200 m further up the hill. If the hill slopes at 23° from the horizontal, how deep does the rescue shaft need to be to meet the horizontal tunnel? Answer to one decimal place.

Testing understanding

11 A cable, PQ, is secured 24 m from the base of the pole PR.
A second cable, SP, is required to make an angle of 65° with the horizontal ground.
Find the length of the cable, SP, needed to one decimal place.

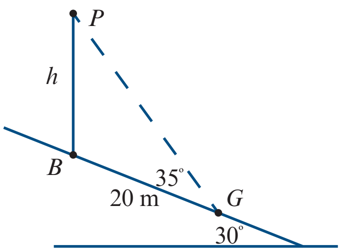


12 Two spotlights, B and D, are located at ground level, 200 m north and 300 m south of a highway, respectively. The spotlight at B directed its beam at an angle of 60° to BA, so that it shone on a parked car, C, on the highway.



At what angle, θ, to one decimal place, should the spotlight at D direct its beam to shine on the car?

13 A vertical pole on a slope of 30° is secured 20 m down the slope by a cable, PG, that makes an angle of 35° with the slope. Find the height of the pole to one decimal place.

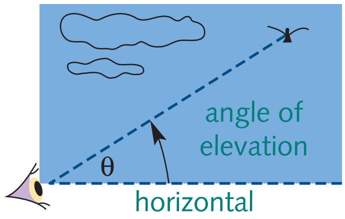


11E Angles of elevation and depression

Learning intentions

- To be able to identify and label the angles of elevation and depression in diagrams of practical situations.
- To be able to choose the appropriate trigonometric ratios and solve equations to find unknown sides and angles.

The **angle of elevation** is the angle through which you *raise* your line of sight from the horizontal when you are looking *up* at something.



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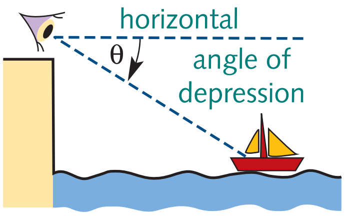
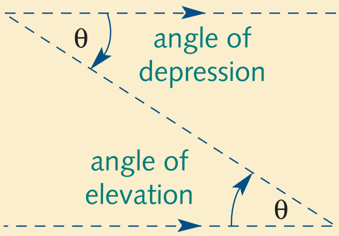
The **angle of depression** is the angle through which you *lower* your line of sight from the horizontal when you are looking *down* at something.



Angles of elevation and depression

angle of elevation = angle of depression

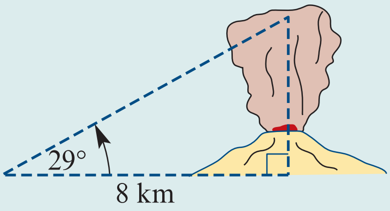
The diagram shows that the angle of elevation and the angle of depression are alternate angles ('Z' angles), so they are equal.

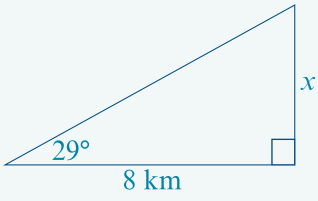


Applications of angles of elevation and depression

Example 9 Angle of elevation

A park ranger measured the top of a plume of volcanic ash to be at an angle of elevation of 29°. From her map she noted that the volcano was 8 km away. She calculated the height of the plume to be 4.4 km. Show how she might have done this. Give your answer to one decimal place.



Explanation	Solution
1 Draw a right-angled triangle showing the given information. Label the required height, x.	
2 The opposite and adjacent sides are involved so use tan θ.	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
3 Substitute in the known values.	$\tan 29° = \dfrac{x}{8}$
4 Multiply both sides by 8.	$8 \times \tan 29° = x$
5 Use your calculator to find the value of x.	$x = 4.434\ldots$
6 Write your answer to one decimal place.	The height of the ash plume was 4.4 km.

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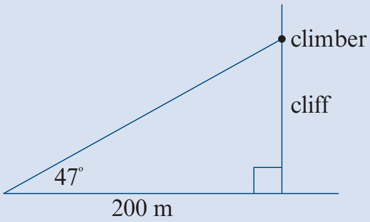
Now try this 9 Angle of elevation (Example 9)

The angle of elevation of a rock climber scaling a vertical cliff was 47°. The horizontal distance to the base of the cliff was 200 m. How high was the climber up the face of the cliff? Answer to the nearest metre.



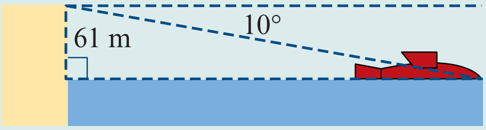
Hint 1 State the position names of the given length and the required length?

Hint 2 Write the trigonometric equation that uses the given angle and length and the required length.

Hint 3 Solve the equation to find the required length to the nearest metre.

Example 10 Angle of depression

From the top of a cliff that was 61 m above sea level, Chen saw a capsized yacht. He estimated the angle of depression to be about 10°. How far was the yacht from the base of the cliff, to the nearest metre?



Explanation

1 Draw a diagram showing the given information. Label the required distance, x.

2 Mark in the angle at the yacht corner of the triangle. This is also 10° because it and the angle of depression are alternate (or 'Z') angles.

Note: The angle between the cliff face and the line of sight is *not* 10°.

3 The opposite and adjacent sides are involved, so use tan θ.

4 Substitute in the known values.

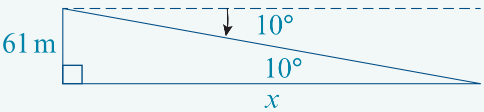
5 Multiply both sides by x.

6 Divide both sides by tan 10°.

7 Do the division using your calculator.

8 Write your answer to the nearest metre.

Solution



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 10° = \frac{61}{x}$$

$$x \times \tan 10° = 61$$

$$x = \frac{61}{\tan 10°}$$

$$x = 345.948\ldots$$

The yacht was 346 m from the base of the cliff.

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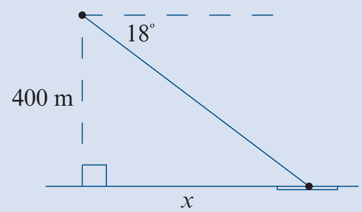
Now try this 10 Angle of depression (Example 10)

From a helicopter flying at a height of 400 m, the navigator sighted a landing platform at an angle of depression of 18°. Find the horizontal distance to the landing platform to the nearest metre.



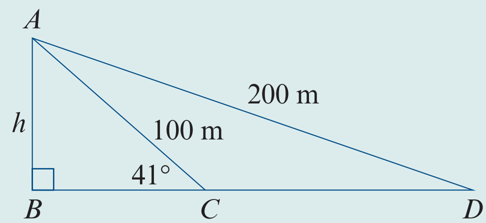
Hint 1 Find an angle inside the right-angled triangle.
Hint 2 What are the position names of the given and required distances?
Hint 3 Choose the relevant trigonometric ratio.
Hint 4 Write the equation with the given values and symbol for the unknown value.
Hint 5 Solve the equations to find the distance to the nearest metre.

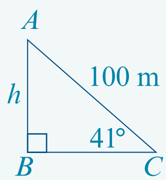
Example 11 Application with two right-angled triangles

A cable 100 m long makes an angle of elevation of 41° with the top of a tower.

a Find the height, h, of the tower, to the nearest metre.

b Find the angle of elevation, to the nearest degree, that a cable that is 200 m long would make with the top of the tower.



Explanation	Solution
Strategy: Find h in triangle ABC, then use it to find the angle in triangle ABD.	
a **1** Draw triangle ABC, showing the given and required information.	
2 The opposite and hypotenuse are involved, so use sin θ.	$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
3 Substitute in the known values.	$\sin 41^\circ = \dfrac{h}{100}$
4 Multiply both sides by 100.	$h = 100 \times \sin 41^\circ$
5 Evaluate 100 sin(41°) using your calculator, and store the answer as the value of the variable h for later use.	$100 \cdot \sin(41^\circ) \rightarrow h$ 65.6059
6 Write your answer to the nearest metre.	The height of the tower is 66 m.

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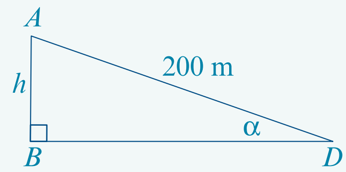
b **1** Draw triangle ABD, showing the given and required information.



2 The opposite and hypotenuse are involved, so use sin α.

$$\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}}$$

3 Substitute in the known values. In part **a** we stored the height of the tower as h.

$$\sin \alpha = \frac{h}{200}$$

4 Write the equation to find α.

$$\alpha = \sin^{-1}\left(\frac{h}{200}\right)$$

5 Use your calculator to evaluate α.

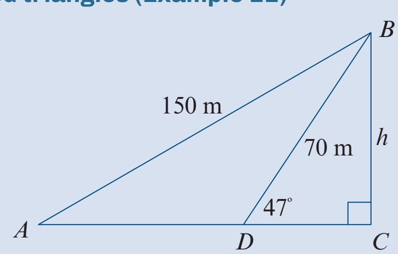
$100 \cdot \sin(41°) \rightarrow h$	65.6059
$\sin^{-1}\left(\frac{h}{200}\right)$	19.1492

6 Write your answer to the nearest degree.

The 200 m cable would have an angle of elevation of 19°.

Now try this 11 Application with two right-angled triangles (Example 11)

A wire that is 70 m long has secured the top, B, of a transmission tower to the point D on level ground. The angle of elevation looking from the point D to the top of the tower was 47°.



a Find the height of the tower, to the nearest metre.

Hint 1 What are the position names of the sides in triangle BCD?
Hint 2 Write and solve the equation for the appropriate trigonometric ratio.

b Find the angle of elevation, to the nearest degree, that a wire 150 m from point A to B, makes with the ground.

Hint 1 Write your answer from part **a** for the length BC onto triangle ABC, using at least two decimal places.
Hint 2 Choose the relevant trigonometric ratio and write the equation.
Hint 3 Use the appropriate inverse from: $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ to find θ.

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Section Summary

- The **angle of elevation** is the angle from the horizontal through which you raise your line of sight to view the object.
- The **angle of depression** is the angle from the horizontal through which you lower your line of sight to view the object.

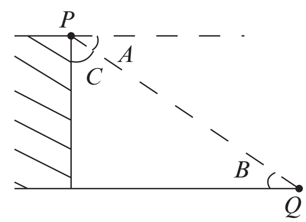
Skill-sheet

Exercise 11E

Building understanding

Answer to one decimal place.

1 The angle of depression, when looking at point Q from P, is 38°.



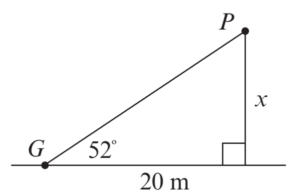
a State the value of angle A.

b Give angle B.

c Find angle C.

d What is the angle of elevation, looking from point Q to point P?

2 The angle of elevation of the top of a pole, P, viewed 20 m from the base of the pole, at G, is 52°.



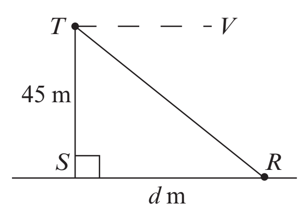
a Give the names for the side x and the 20 m distance.

b Write the trigonometric ratio rule which uses the names in part **a**.

c Substitute the known values of the angle and a side into the equation. Call the unknown side x.

d Solve the equation to one decimal place.

3 A spotlight at the top of a 45-m tower, T, makes an angle of depression of 40° as it shines on a rabbit, R. The rabbit is d metres from the base of the tower. Copy the diagram and write the 40° angle into its correct position.



a Is the angle of depression $\angle STR$, $\angle TRS$ or $\angle RTV$?

b State the trigonometric rule which uses the 45 m and d m sides.

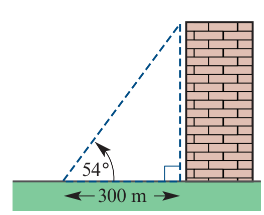
c Substitute the known values and unknown side, d, into the rule.

d Solve the equation, giving the value of d to one decimal place.

Developing understanding

Example 9

4 After walking 300 m away from the base of a tall building on level ground, Elise measured the angle of elevation to the top of the building to be 54°. Find the height of the building to the nearest metre.



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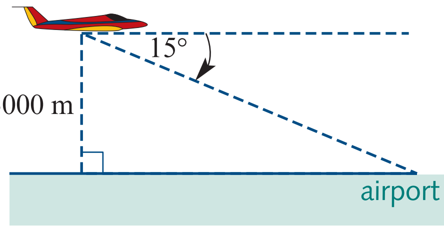
Example 10

5 The pilot of an aeroplane saw an airport at sea level at an angle of depression of 15°. His altimeter showed that the aeroplane was at a height of 3000 m. Find the horizontal distance of the plane from the airport to the nearest metre.

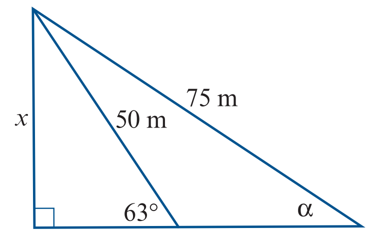


6 What would be the angle of elevation to the top of a radio transmitting tower that is 100 m tall and 400 m from the observer? Answer to the nearest degree.

Example 11

7 **a** Find the unknown length, x, to one decimal place.

b Find the unknown angle, α, to the nearest degree.

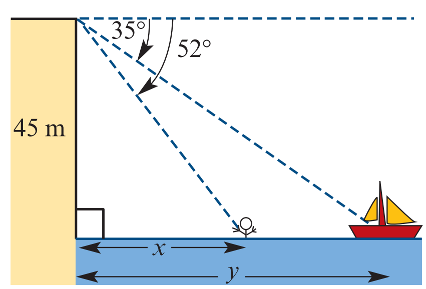


8 From the top of a cliff that is 45 m high, an observer looking along an angle of depression of 52° could see a man swimming in the sea. The observer could also see a boat at an angle of depression of 35°. Calculate to the nearest metre:

a the distance, x, of the man from the base of the cliff

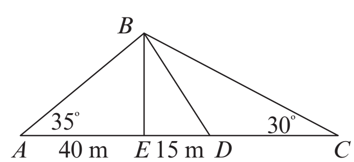
b the distance, y, of the boat from the base of the cliff

c the distance from the man to the boat.



Testing understanding

9 The angle of elevation of the top of the pole BE is 35° when read from point A.
The point A is 40 m from the base of the pole.



a Find the angle of elevation of the top of the pole when measured from point D, 15 m from the base of the pole.

b At what distance from D should a cable be secured so that it makes an angle of 30° with the horizontal ground?

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11F Bearings and navigation

Learning intentions

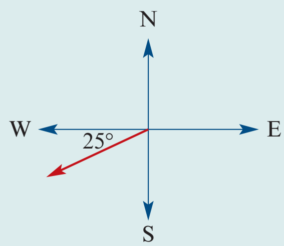
- To be able to use three-figure bearings to draw navigation and surveying diagrams.
- To be able to solve the appropriate equations to find unknown bearings and distances.

True bearings or three-figure bearings

A **true bearing** is the angle measured clockwise from north around to the required direction. True bearings are also called **three-figure bearings** because they are written using three numbers or figures. For example, 090° is the direction measured 90° clockwise from north, better known as east!

Example 12 Determining three-figure bearings

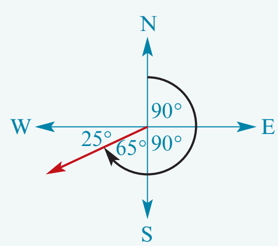
Give the three-figure bearing for the direction shown.



Explanation

1 Calculate the total angles swept out clockwise from north.
There is an angle of 90° between each of the four points of the compass.

Solution

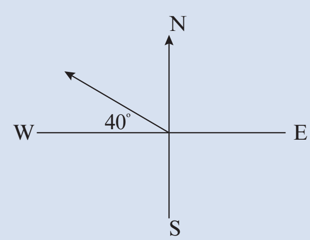


2 Write your answer.

The angle from north
$= 90° + 90° + 65° = 245°$
or $270° - 25° = 245°$
The three-figure bearing is 245°.

Now try this 12 Determining three-figure bearings (Example 12)

Give the three-figure bearing for the direction shown.



Hint 1 From the north direction, sweep clockwise to the required direction.

Hint 2 The angle swept out is three right angles plus 40°.

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Navigation problems

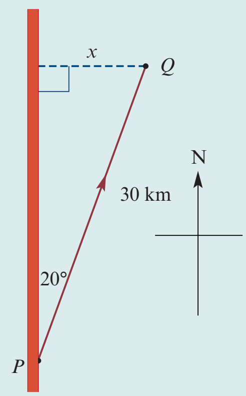
Navigation problems usually involve a consideration of not only the *direction* of travel, given as a bearing, but also the *distance* travelled.

Example 13 Navigating using a three-figure bearing

A group of bushwalkers leave point P, which is on a road that runs north–south, and walk for 30 km on a bearing 020° to reach point Q.

a What is the shortest distance, x, from Q back to the road to one decimal place?

b Looking from point Q, what would be the three-figure bearing of their starting point?



Explanation

a **1** Show the given and required information in a right-angled triangle.

2 The opposite and hypotenuse are involved, so use sin θ.

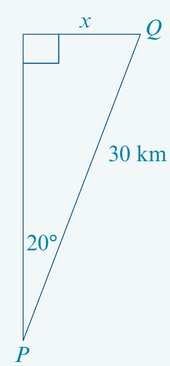
3 Substitute in the known values.

4 Multiply both sides by 30.

5 Find the value of x using your calculator.

6 Write your answer to one decimal place.

Solution



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 20^\circ = \frac{x}{30}$$

$$30 \times \sin 20^\circ = x$$

$$x = 10.260\ldots$$

The shortest distance to the road is 10.3 km.

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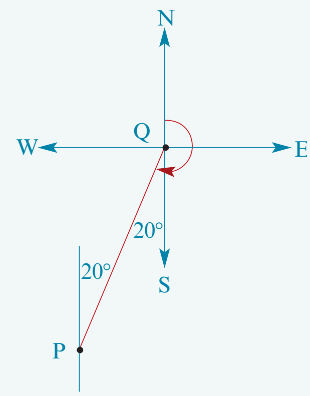
b **1** Draw the compass points at Q.
Enter the alternate angle 20°.

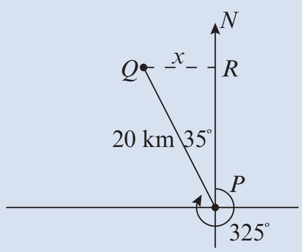


2 Standing at Q, add all the angles when facing north and then turning clockwise to look at P. This gives the three-figure bearing of P when looking from Q.

The angle from north is
$180° + 20° = 200°$
The three-figure bearing is 200°.

Now try this 13 Navigating using a three-figure bearing (Example 13)

A car was travelling north along a highway but then left the highway at the point P and travelled across the desert, heading on a bearing of 325°. The car broke down at point Q after travelling 20 km.



a Find the shortest distance, x, for the driver to walk to the highway at R to one decimal place.

Hint 1 Use the position names of the two sides involved to decide which trigonometric ratio to use.

Hint 2 Write the appropriate equation and solve to find x.

b From the point Q, what three-figure bearing should the driver take if, instead, he decided to walk back to P?

Hint 1 Draw the points of the compass at Q and use the alternate angles rule.

Hint 2 Turning clockwise from north to face P, what angle was swept out?

Section Summary

- The **three-figure bearing** is the angle swept clockwise from north to the required direction.

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Skill-sheet

Exercise 11F

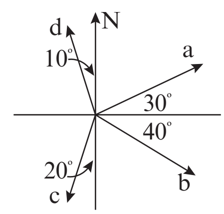
Building understanding

1 Give the three-figure bearings for the compass directions.

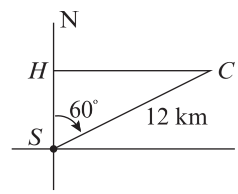
a North
b East
c South
d West

2 Give the three-figure bearings for each of the directions shown.

a Direction **a**
b Direction **b**
c Direction **c**
d Direction **d**



3 A car stopped at S on a highway that pointed in a North-South direction. The car then travelled for 12 km along a dirt road with a bearing of 060° and broke down at C. The driver needed to know the distance, x, he would have to walk to reach the highway at H.



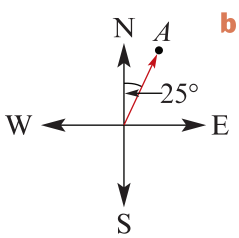
a In triangle SHC, give the names for the 12 km side and side HC.
b Write the trigonometric ratio rule that uses the names of the sides in part **a**.
c Substitute the value of the angle and the known side into the rule. Call the unknown side x.
d Solve the equation by multiplying both sides by the value of the denominator. Use your CAS calculator to find the value of x to one decimal place. As this is a practical problem, answer by responding in the context of the question.

Developing understanding

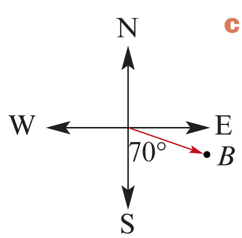
Example 12

4 State the three-figure bearing of each of the points A, B, C and D.

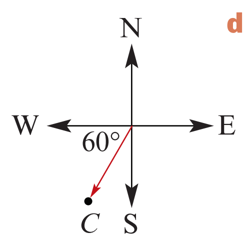
a



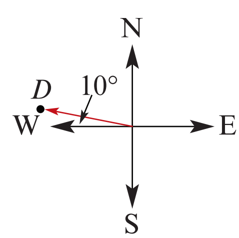
b



c



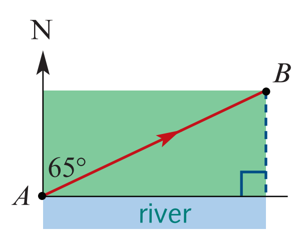
d



Example 13

5 Kirra camped overnight at point A beside a river that ran east–west. She walked on a bearing of 065° for 18 km to point B.

a What angle did her direction make with the river?
b What is the shortest distance from B to the river to two decimal places?



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6 A ship sailed 3 km west, then 2 km south.

a Give its three-figure bearings from an observer who stayed at its starting point to the nearest degree.

b For a person on the ship, what would be the three-figure bearings, looking back to the starting point?

7 A ship left port, P, and sailed 20 km on a bearing of 230°. It then sailed north for 30 km to reach point C. Give the following distances to one decimal place and directions to the nearest degree.

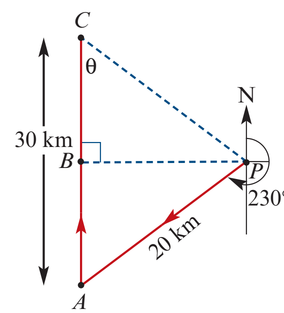
a Find the distance AB.

b Find the distance BP.

c Find the distance BC.

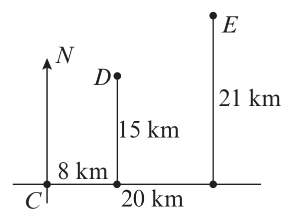
d Find the angle θ at point C.

e State the three-figure bearing and distance of the port, P, from the ship at C.



Testing understanding

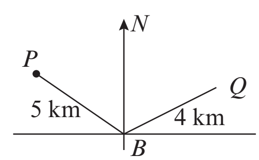
8 A bushwalker left the campsite, C, and walked 8 km east, then 15 km north to reach the point D. A friend walked 20 km east, then 21 km north to the point E. Find the bearing of point E from D to the nearest degree.



9 A surveyor walked 5 km on a bearing of 310° from a base camp, B, to reach point P. She then returned to camp B and walked 4 km on a bearing of 060° to the point Q.

a Find the bearing she had to walk from P to B.

b What bearing would be needed to return to B from Q?



11G The sine rule

Learning intentions

In a non-right-angled triangle

- To be able to use the sine rule to find an unknown angle, given two sides and an opposite angle.
- To be able to use the sine rule to find an unknown side, given two angles and a side.
- To be able to find the required angles and sides when the given information fits two possible triangles.

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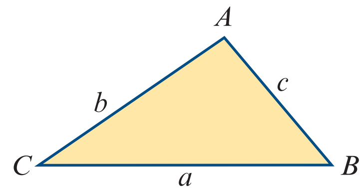
Standard triangle notation

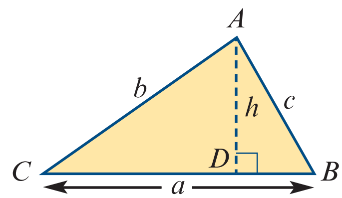
The convention for labelling a non-right-angled triangle is to use the upper case letters, A, B, and C, for the angles at each corner. The sides are named using lower case letters so that side a is opposite angle A, and so on.



This notation is used for the sine rule and cosine rule. Both rules can be used to find angles and sides in triangles that do not have a right angle.

How to derive the sine rule

In triangle ABC, show the height, h, of the triangle by drawing a perpendicular line from D on the base of the triangle to A.



In triangle ADC,	$\sin C = \frac{h}{b}$
So	$h = b \times \sin C$
In triangle ABD,	$\sin B = \frac{h}{c}$
So	$h = c \times \sin B$
We can make the two rules for h equal to each other.	$b \times \sin C = c \times \sin B$
Divide both sides by sin C.	$b = \frac{c \times \sin B}{\sin C}$
Divide both sides by sin B.	$\frac{b}{\sin B} = \frac{c}{\sin C}$

If the triangle was redrawn with side c as the base, then using similar steps we would get: $\frac{a}{\sin A} = \frac{b}{\sin B}$

We can combine the two rules as shown in the following box.

The sine rule

In any triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The **sine rule** is used to find the sides and angles in a non-right-angled triangle when given:

- two sides and an angle opposite one of the given sides
- two angles and one side.

Note: If neither of the two given angles is opposite the given side, find the third angle using $A + B + C = 180°$.

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The sine rule can take the form of any of these three possible equations:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \frac{a}{\sin A} = \frac{c}{\sin C}$$

Each equation has two sides and two angles opposite those sides. If we know three of the parts, we can find the fourth.

So if we know two angles and a side opposite one of the angles, we can find the side opposite the other angle. Similarly, if we know two sides and an angle opposite one of those sides, we can find the angle opposite the other side.

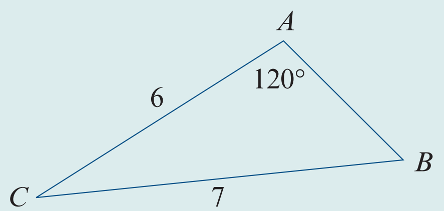
Although we have expressed the sine rule using a triangle *ABC*, for any triangle, such as *PQR*, the pattern of fractions consisting of 'side / sine of angle' pairs would appear as:

$$\frac{p}{\sin P} = \frac{q}{\sin Q} \qquad \frac{q}{\sin Q} = \frac{r}{\sin R} \qquad \frac{p}{\sin P} = \frac{r}{\sin R}$$

Using the sine rule

Example 14 Using the sine rule, given two sides and an opposite angle

Find angle *B* in the triangle shown to one decimal place.



Explanation	Solution
1 We have the pairs $a = 7$ and $A = 120°$, $b = 6$ and $B = ?$ with only *B* unknown. So use $\frac{a}{\sin A} = \frac{b}{\sin B}$.	$\frac{a}{\sin A} = \frac{b}{\sin B}$
2 Substitute in the known values.	$\frac{7}{\sin 120°} = \frac{6}{\sin B}$
3 Cross-multiply.	$7 \times \sin B = 6 \times \sin 120°$
4 Divide both sides by 7.	$\sin B = \frac{6 \times \sin 120°}{7}$
5 Write the equation to find angle *B*.	$B = \sin^{-1}\left(\frac{6 \times \sin 120°}{7}\right)$
6 Use your calculator to evaluate the expression for *B*.	$B = 47.928\ldots°$
7 Write your answer to one decimal place.	Angle *B* is 47.9°.

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When an angle, such as B, is unknown, the fractions on each side of the sine rule can be flipped as the first step.
For example:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Then just multiply both sides by b to find $\sin B$, and solve.

In Example **14**, now that we know that $A = 120°$ and $B = 47.9°$, we can use the fact that the angles in a triangle add to 180° to find C.

$$\begin{aligned} A + B + C &= 180° \\ 120° + 47.9° + C &= 180° \\ 167.9° + C &= 180° \\ C &= 180° - 167.9° = 12.1° \end{aligned}$$

As we now know that $A = 120°$, $a = 7$ and $C = 12.1°$, we can find side c using:

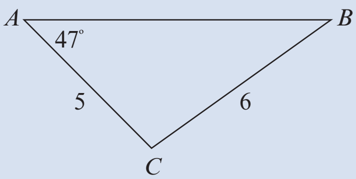
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

The steps are similar to those in the example.

Finding all the angles and sides of a triangle is called solving the triangle.

Now try this 14 **Using the sine rule, given two sides and an opposite angle (Example 14)**

Find angle B to one decimal place.

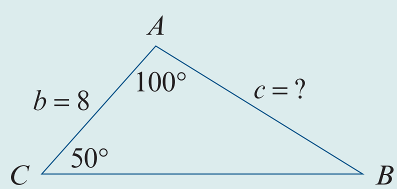


Hint 1 Write the sine rule using the letter names for the given information and the required angle.

Hint 2 Substitute in the known values and solve to find angle B.

Example 15 **Using the sine rule, given two angles and one side**

Find side c in the triangle shown to one decimal place.



Explanation	Solution
1 Find the angle opposite the given side by using $A + B + C = 180°$	$A + B + C = 180°$ $100° + B + 50° = 180°$ $B + 150° = 180°$ $B = 30°$

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2 We have the pairs $b = 8$ and $B = 30°, c = ?$ and $C = 50°$ with only c unknown. So use $\frac{b}{\sin B} = \frac{c}{\sin C}$.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

3 Substitute in the known values.

$$\frac{8}{\sin 30°} = \frac{c}{\sin 50°}$$

4 Multiply both sides by $\sin 50°$.

$$c = \frac{8 \times \sin 50°}{\sin 30°}$$

5 Use your calculator to find c.

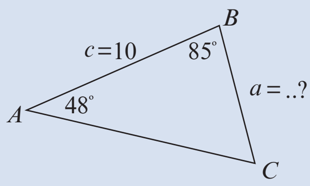
$$c = 12.256\ldots$$

6 Write your answer to one decimal place.

Side c is 12.3 units long.

Now try this 15 Using the sine rule, given two angles and one side (Example 15)

Find side a to one decimal place.



Hint 1 Find angle C, using $A + B + C = 180°$.

Hint 2 Write the sine rule using the letter names for the known information and the required side.

Hint 3 Substitute in the known values and solve to find side a.

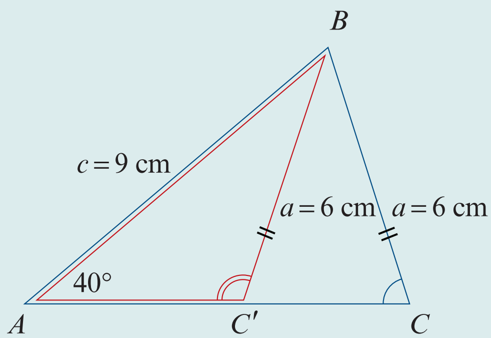
Ambiguous case

Sometimes, two triangles can be drawn to fit the given information. This can happen when you are given two sides and an angle *not* between the two given sides. The solution strategy uses the sine rule and the fact that the angles at the base of an isosceles triangle are equal.

Example 16 Ambiguous case using the sine rule

In triangle ABC, $A = 40°$, $c = 9$ cm and $a = 6$ cm.

Side c is drawn for 9 cm at 40° to the base. From vertex B, side a must be 6 cm long when it meets the base of the triangle. When side a is measured out with a compass, it can cross the base in two possible places - C and C'.



There are two possible triangles. ABC drawn in blue and ABC' drawn in red.

Find the two possible values for angle C, shown as $\angle BCA$ and $\angle BC'A$ in the diagram. Give the answers to two decimal places.

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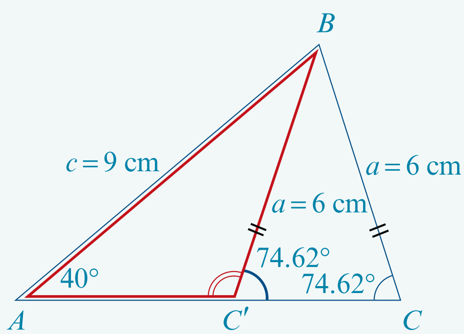
Explanation	Solution
1 Using the sine rule, we need two angle-side pairs with only one unknown. The unknown is angle C. Both sides of the sine rule were flipped to make $\sin C$ a numerator.	$\frac{\sin C}{c} = \frac{\sin A}{a}$ $\frac{\sin C}{9} = \frac{\sin 40^\circ}{6}$ $\sin C = \frac{9 \times \sin 40^\circ}{6}$ $C = \sin^{-1}\left(\frac{9 \times \sin 40^\circ}{6}\right)$ $C = 74.62^\circ$
2 Clearly $\angle BC'A$ is greater than 90°, so it is not the value of angle C just calculated.	So $\angle BCA = 74.62^\circ$

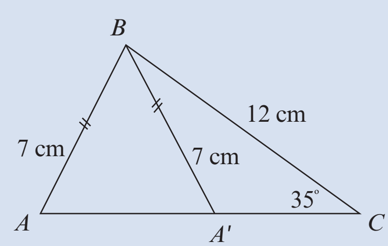


Explanation	Solution
3 The two angles at the base of the isosceles triangle $C'BC$ are equal.	$\angle BC'C = \angle BCC'$ So $\angle BC'C = 74.62^\circ$
4 Two angles on a straight line add to 180°.	$\angle BC'A + 74.62^\circ = 180^\circ$ $\angle BC'A = 180^\circ - 74.62^\circ = 105.38^\circ$ The possible values for angle C are 74.62° and 105.38°

Now try this 16 Ambiguous case using the sine rule (Example 16)

In triangle ABC, $C = 35^\circ$, $a = 12$ cm and $c = 7$ cm.

Side a was drawn for 12 cm at 35° to the base. From the vertex B, side c must be 7 cm long when it meets the base of the triangle. When side c is measured out with a compass, it can cross the base in two possible places, A and A'.



There are two possible triangles: CBA and CBA'.

Find the two possible values for angle A, seen as $\angle BAC$ and $\angle BA'C$ in the diagram. Answer to two decimal places.

Hint 1 In triangle ABC, use the sine rule to find angle A, known as $\angle BAC$.

Hint 2 Find $\angle BA'A$ and hence $\angle BA'C$, which is angle A'.

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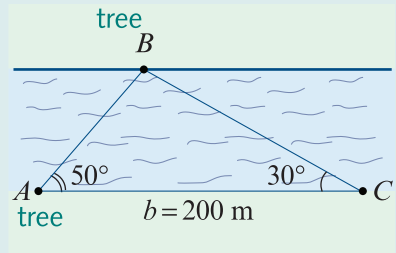
Example 17 Application of the sine rule

Leo wants to tie a rope from a tree at point A to a tree at point B on the other side of the river. He needs to know the length of rope required.

When he stood at A, he saw the tree at B at an angle of 50° with the riverbank. After walking 200 metres east to C, the tree was seen at an angle of 30° with the riverbank.



Find the length of rope required to reach from A to B to two decimal places.

Explanation	Solution
1 To use the sine rule, we need two angle-side pairs with only one item unknown. The unknown is the length of the rope, side c. Angle $C = 30°$ is given.	So one part of the sine rule equation will be: $\dfrac{c}{\sin C}$
2 We know side $b = 200$ and need to find angle B to use the sine rule equation:	$\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$
3 Use $A + B + C = 180°$ to find angle B.	$A + B + C = 180°$ $50° + B + 30° = 180°$ $B = 100°$
4 We have the pairs: $b = 200$ and $B = 100°$ $c = ?$ and $C = 30°$ with only c unknown. So use $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$.	$\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$
5 Substitute in the known values.	$\dfrac{c}{\sin 30°} = \dfrac{200}{\sin 100°}$
6 Multiply both sides by sin 30° .	$c = \dfrac{200 \times \sin 30°}{\sin 100°}$
7 Use your calculator to find c.	$c = 101.542...$
8 Write your answer to two decimal places.	The rope must be 101.54 m long.

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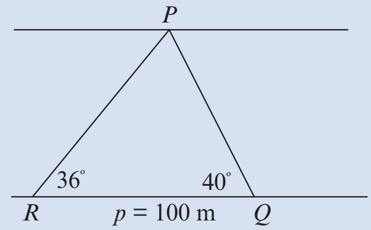
Now try this 17 **Application of the sine rule (Example 17)**

Engineers needed to construct a bridge across a canyon from P on the edge of one side to Q on the edge of the other side. The diagram is the view looking down on the parallel sides of the canyon and the proposed position of the bridge, PQ.

From point R on one side of the canyon, a surveyor sighted post P on the other side at an angle of 36° to the edge of the canyon. Moving 100 m to Q, she sighted point P at an angle of 40° to the edge.

Find the required length of the bridge to two decimal places.



Hint 1 Find angle P.

Hint 2 Use the sine rule to find PQ $(=r)$.

Tips for solving trigonometry problems

- Always make a rough sketch in pencil as you read the details of a problem. You may need to make changes as you read more, but it is very helpful to have a sketch to guide your understanding.
- In any triangle, the longest side is opposite the largest angle. The shortest side is opposite the smallest angle.
- When labelling the sides and angles of a triangle, make sure the name of a side is opposite the angle with the same letter. For example, side c is opposite angle C.
- When you have found a solution, re-read the question and check that your answer fits well with the given information and your diagram.
- Round answers for each part to the required decimal places. Keep more decimal places when the results are used in further calculations. Otherwise, rounding off errors accumulate.

Section Summary

- The sine rule can take the form of any of these three possible equations:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \frac{a}{\sin A} = \frac{c}{\sin C}$$

- Each equation consists of two sides and two angles opposite those sides. If three parts are known, the sine rule can be used to find the fourth unknown part.
- In the ambiguous case, two possible triangles can be drawn from the given information. Draw the two triangles within one diagram, and use the fact that the base angles of an isosceles triangle are equal.

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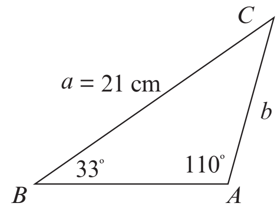
Skill-sheet

Exercise 11G

Building understanding

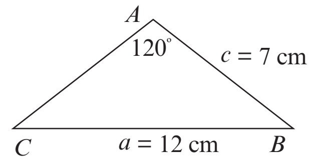
1 **a** For a triangle, ABC, write the three possible sine rule equations.
b For a triangle, PQR, write the three possible sine rule equations.

2 In triangle ABC, $A = 110°$, $a = 21$ cm and $B = 33°$.



a To find side b, which form of the sine rule should be used?
b Substitute the known values into the equation.
c Solve the equation to find side b to one decimal place.

3 In triangle ABC, $A = 120°$, $a = 12$ cm and $c = 7$ cm.



a To find angle C, which form of the sine rule should be used?
b Substitute the known values into the equation.
c When the unknown, such as $\sin C$, is in the denominator, the equation is easier to solve if each fraction is flipped. Flip each fraction.
d Solve the equation to find angle C to one decimal place.

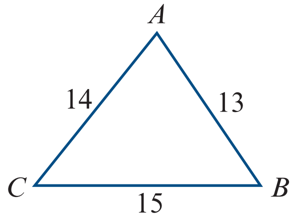
Developing understanding

In this exercise, calculate lengths correct to two decimal places and angles to one decimal place, where necessary.

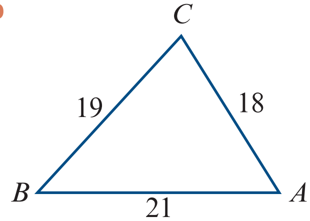
Basic principles

4 In each triangle, state the lengths of sides a, b and c.

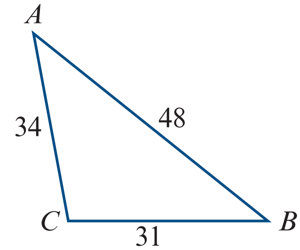
a



b

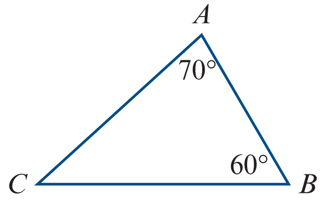


c

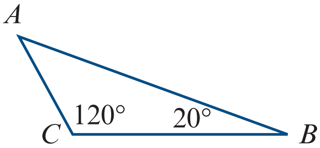


5 Find the value of the unknown angle in each triangle. Use $A + B + C = 180°$.

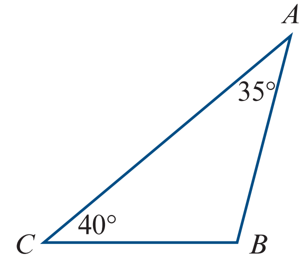
a



b



c



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6 In each of the following, a student was using the sine rule to find an unknown part of a triangle but was unable to complete the final steps of the solution. Find the unknown value by completing each problem. For ambiguous cases, find one possible value.

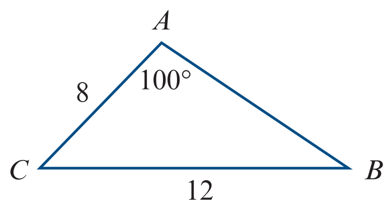
a $\dfrac{a}{\sin 40^\circ} = \dfrac{8}{\sin 60^\circ}$ **b** $\dfrac{b}{\sin 50^\circ} = \dfrac{15}{\sin 72^\circ}$ **c** $\dfrac{c}{\sin 110^\circ} = \dfrac{24}{\sin 30^\circ}$

d $\dfrac{17}{\sin A} = \dfrac{16}{\sin 70^\circ}$ **e** $\dfrac{26}{\sin B} = \dfrac{37}{\sin 95^\circ}$ **f** $\dfrac{21}{\sin C} = \dfrac{47}{\sin 115^\circ}$

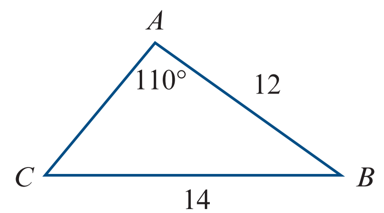
Using the sine rule to find angles

Example 14

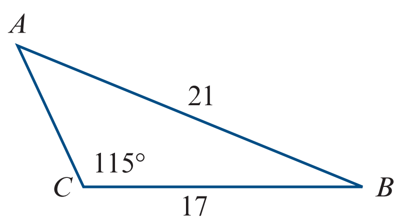
7 a Find angle B.



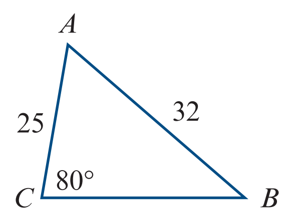
b Find angle C.



c Find angle A.



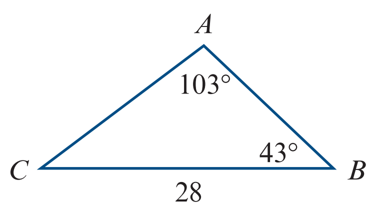
d Find angle B.



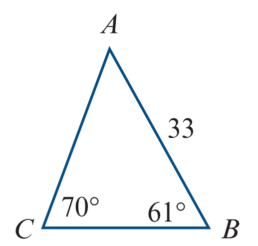
Using the sine rule to find sides

Example 15

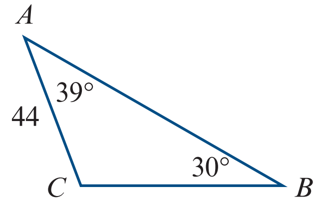
8 a Find side b.



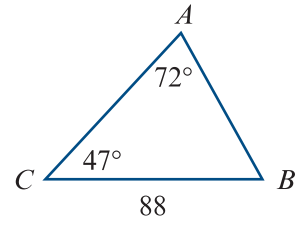
b Find side b.



c Find side a.



d Find side c.



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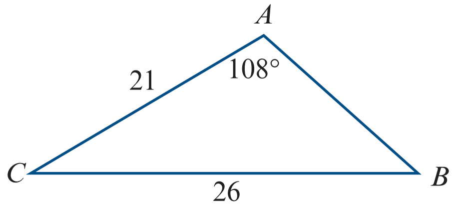
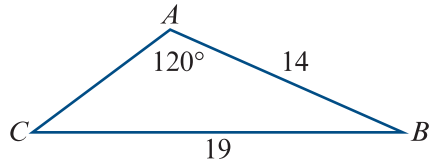
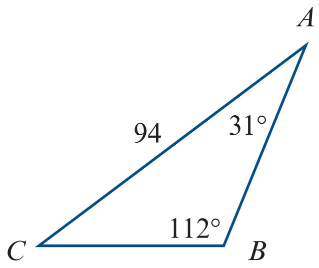
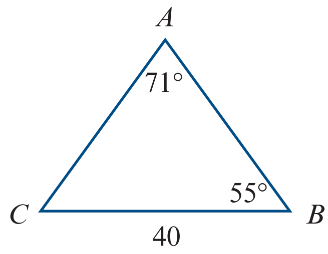
Solving triangles using the sine rule

9 Solve (find all the unknown sides and angles of) the following triangles.

a


b


c


d


10 In the triangle ABC, $A = 105°$, $B = 39°$ and $a = 60$. Find side b.

11 In the triangle ABC, $A = 112°$, $a = 65$ and $c = 48$. Find angle C.

12 In the triangle PQR, $Q = 50°$, $R = 45°$ and $p = 70$. Find side r.

13 In the triangle ABC, $B = 59°$, $C = 74°$ and $c = 41$. Find sides a and b and angle A.

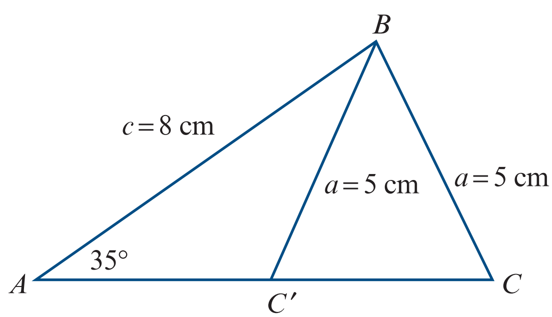
14 In the triangle ABC, $a = 60$, $b = 100$ and $B = 130°$. Find angles A and C and side c.

15 In the triangle PQR, $P = 130°$, $Q = 30°$ and $r = 69$. Find sides p and q and angle R.

The ambiguous case of the sine rule

Example 16

16 In triangle ABC, $A = 35°$, $c = 8$ cm and $a = 5$ cm.
Two triangles, ABC and ABC', can be drawn using the given information.
Give the angles to two decimal places.



a Use the sine rule to find $\angle BCA$ in triangle ABC.

b Use isosceles triangle $C'BC$ to find $\angle BC'C$.

c Find $\angle AC'B$ by using the rule for two angles on a straight line.

d Give the possible values for C and C', the angles opposite side c.

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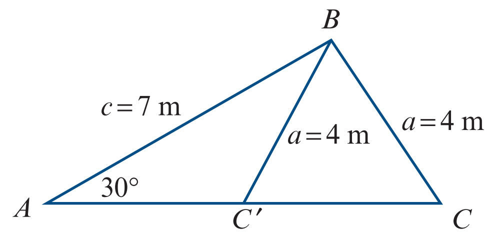
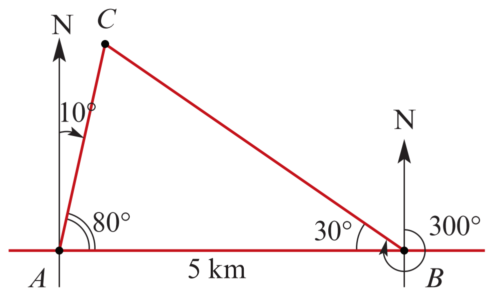
17 In triangle ABC, $A = 30°$, $c = 7$ m and $a = 4$ m.
Find the two possible values for angle C, shown as $\angle BCA$ and $\angle BC'A$ in the diagram. Give the angles to two decimal places.



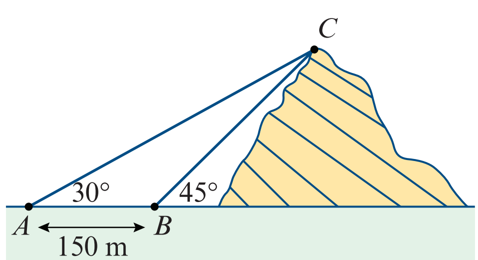
Applications

Example 17

18 A fire-spotter, located in a tower at A, saw a fire in the direction 010°. Five kilometres to the east of A, another fire-spotter at B saw the fire in the direction 300°.
Find the distance of the fire from each tower.



19 A surveyor standing at point A measured the angle of elevation to the top of the mountain as 30°. She moved 150 m closer to the mountain and, at point B, measured the angle of elevation to the top of the mountain as 45°.
There is a proposal to have a strong cable from point A to the top of the mountain to carry tourists in a cable car. What is the length of the required cable?



20 A naval officer sighted the smoke of a volcanic island on a bearing of 044°. A navigator on another ship 25 km due east of the first ship saw the smoke on a bearing of 342°.

a Find the distance of each ship from the volcano.

b If the ship closest to the volcano can travel at 15 km/h, how long will it take to reach the volcano?

21 An air-traffic controller at airport A received a distress call from an aeroplane low on fuel. The bearing of the aeroplane from A was 070°. From airport B, 80 km north of airport A, the bearing of the aeroplane was 120°.

a Which airport was closest for the aeroplane?

b Find the distance to the closest airport.

c The co-pilot estimates fuel consumption to be 1525 litres per 100 km. The fuel gauge reads 1400 litres. Is there enough fuel to reach the destination?

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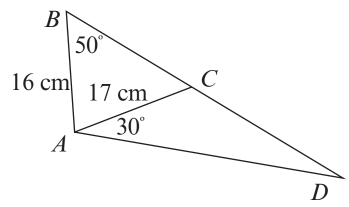
Testing understanding

22 Find the length AD to one decimal place.



23 Decide which of the descriptions given is for:

i a possible triangle **ii** an impossible triangle **iii** an ambiguous case

a $A = 30°$, $a = 4.5$ and $c = 10$ **b** $C = 40°$, $b = 12$ and $c = 8.5$

c $B = 50°$, $a = 8$ and $b = 9$

11H The cosine rule

Learning intentions

In a non-right-angled triangle

- To be able to use the cosine rule to find the unknown side when given two sides and the angle between them.
- To be able to use the cosine rule to find an angle in a triangle when given the three sides.
- To be able to identify when the sine rule or the cosine rule should be used.

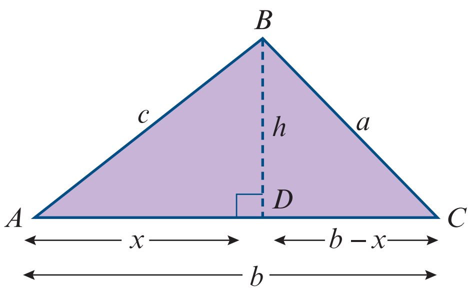
The **cosine rule** can be used to find the length of a side in any non-right-angled triangle when two sides and the angle between them are known. When you know the three sides of a triangle, the cosine rule can be used to find any angle.

How to derive the cosine rule

In the triangle ABC, show the height, h, of the triangle by drawing a line perpendicular from D on the base of the triangle to B.

Let $AD = x$

As $AC = b$, then $DC = b - x$.



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In triangle *ABD*, $\cos A = \dfrac{x}{c}$

Multiply both sides by *c*. $x = c\cos A$ ①

Using Pythagoras' theorem in triangle *ABD*. $x^2 + h^2 = c^2$ ②

Using Pythagoras' theorem in triangle *CBD*. $(b - x)^2 + h^2 = a^2$

Expand (multiply out) the squared bracket. $b^2 - 2bx + x^2 + h^2 = a^2$

Use ① to replace *x* with *c* cos *A*. $b^2 - 2bc\cos A + x^2 + h^2 = a^2$

Use ② to replace $x^2 + h^2$ with c^2. $b^2 - 2bc\cos A + c^2 = a^2$

Reverse and rearrange the equation. $a^2 = b^2 + c^2 - 2bc\cos A$

Repeating these steps with side *c* as the base, we get: $b^2 = a^2 + c^2 - 2ac\cos B$

Repeating these steps with side *a* as the base, we get: $c^2 = a^2 + b^2 - 2ab\cos C$

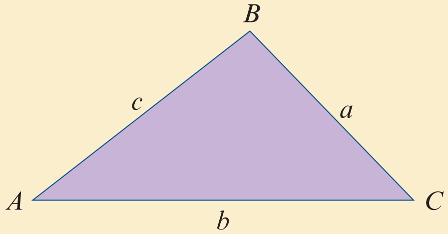
The three versions of the cosine rule can be rearranged to give rules for cos *A*, cos *B*, and cos *C*.

The cosine rule

The cosine rule in any triangle, *ABC*:

- when given two sides and the angle between them, the third side can be found using one of the equations:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



- when given three sides, any angle can be found using one of the following rearrangements of the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

In triangles using different letters, the cosine rule follows the same pattern. For example, in triangle *PQR*:

$$p^2 = q^2 + r^2 - 2qr\cos P$$
$$q^2 = p^2 + r^2 - 2pr\cos Q$$
$$r^2 = p^2 + q^2 - 2pq\cos R$$

"The square of one side equals the sum of the squares of the other sides, minus twice their product, times the cosine of the angle between them."

If the angle is 90°, a right-angled triangle is formed and Pythagoras' theorem results.

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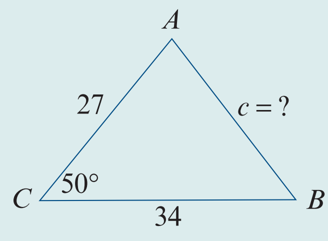
Using the cosine rule

Example 18 **Using the cosine rule, given two sides and the angle between them**

Find side c, to two decimal places, in the triangle shown.



Explanation

1 Write down the given values and the required unknown value.

2 We are given two sides and the angle between them. To find side c, use $c^2 = a^2 + b^2 - 2ab\cos C$

3 Substitute the given values into the rule.

4 Take the square root of both sides.

5 Use your calculator to find c.

6 Write your answer to two decimal places.

Solution

$a = 34, b = 27, c = ?, C = 50°$

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 34^2 + 27^2 - 2 \times 34 \times 27 \times \cos 50°$

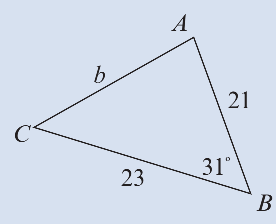
$c = \sqrt{(34^2 + 27^2 - 2 \times 34 \times 27 \times \cos 50°)}$

$c = 26.548...$

The length of side c is 26.55 units.

Now try this 18 **Using the cosine rule, given two sides and the angle between them (Example 18)**

Find side b to two decimal places.



Hint 1 Write the letter names for the given values and the required unknown.

Hint 2 Choose a form of the cosine rule for b^2.

Hint 3 Substitute in the known values and solve to find side b to two decimal places.

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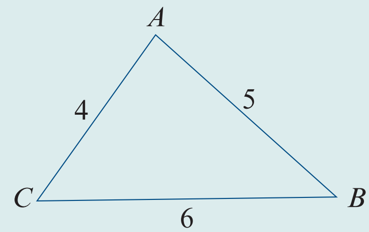
Example 19 Using the cosine rule to find an angle, given three sides

Find the largest angle, to one decimal place, in the triangle shown.



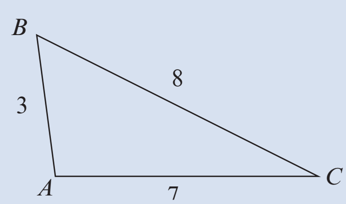
Explanation	Solution
1 Write down the given values.	$a = 6, b = 4, c = 5$
2 The largest angle is always opposite the largest side, so find angle A.	$A = ?$
3 We are given three sides. To find angle A use: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
4 Substitute the given values into the rule.	$\cos A = \dfrac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5}$
5 Write the equation to find angle A.	$A = \cos^{-1}\left(\dfrac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5}\right)$
6 Use your calculator to evaluate the expression for A. Make sure that your calculator is in DEGREE mode. Tip: Wrap all the terms in the numerator (top) within brackets. Also put brackets around all of the terms in the denominator (bottom).	$A = 82.819\ldots°$
7 Write your answer to one decimal place.	The largest angle is 82.8°.

When finding an angle, such as A, a negative value for $\cos A$ indicates that:

$$90° < A < 180°$$

Now try this 19 Using the cosine rule to find an angle, given three sides (Example 19)

Find the smallest angle to one decimal place.



Hint 1 The smallest angle is opposite the smallest side.

Hint 2 Write a form of the cosine rule with $\cos C$.

Hint 3 Substitute in the known values and solve to find angle C to one decimal place.

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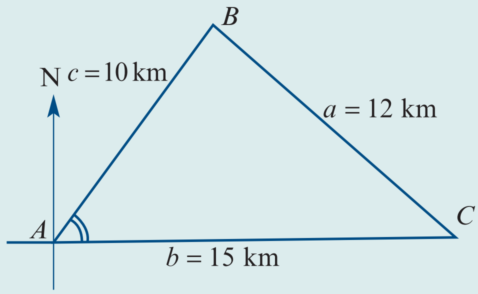
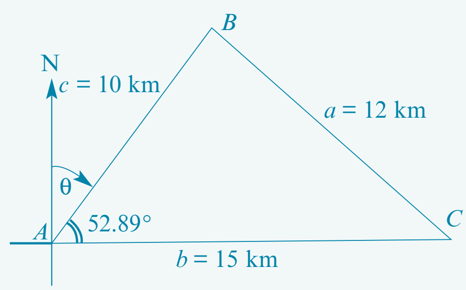
Example 20 Application of the cosine rule: finding an angle and a bearing

A yacht left point A and sailed 15 km east to point C. Another yacht also started at point A and sailed 10 km to point B, as shown in the diagram. The distance between points B and C is 12 km.



a What was the angle between their directions as they left point A? Give the angle to two decimal places.

b Find the bearing of point B from the starting point, A, to the nearest degree.

Explanation	Solution
a 1 Write the given values.	$a = 12, b = 15, c = 10$
2 Write the form of the cosine rule for the required angle, A.	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
3 Substitute the given values into the rule.	$\cos A = \dfrac{15^2 + 10^2 - 12^2}{2 \times 15 \times 10}$
4 Write the equation to find angle A.	$A = \cos^{-1}\left(\dfrac{15^2 + 10^2 - 12^2}{2 \times 15 \times 10}\right)$
5 Use your calculator to evaluate the expression for A.	$A = 52.891°$
6 Give the answer to two decimal places.	The angle was 52.89°.
b 1 The bearing, θ, of point B from the starting point, A, is measured clockwise from north.	
2 Consider the angles in the right angle at point A.	$\theta + 52.89° = 90°$
3 Find the value of θ.	$\theta = 90° - 52.89°$ $= 37.11°$
4 Write your answer.	The bearing of point B from point A is 037°.

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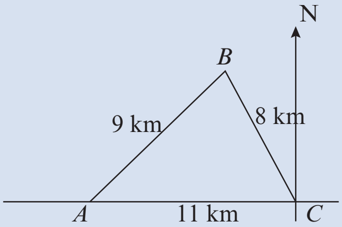
Now try this 20 **Application of the cosine rule: finding an angle and a bearing (Example 20)**

A bushwalker left their camp at point C and walked 8 km to point B, as shown in the diagram. A friend walked 11 km to point A, a distance of 9 km from B.



a What was the angle between their directions as they left C? Answer to one decimal place.

Hint 1 Write the form of the cosine rule with $\cos C$, and substitute in the known values.

Hint 2 Solve to find angle C.

b What was the bearing of point B from their starting point, C, to the nearest degree?

Hint 1 Add the angles swept out as you sweep clockwise from north until you face point B.

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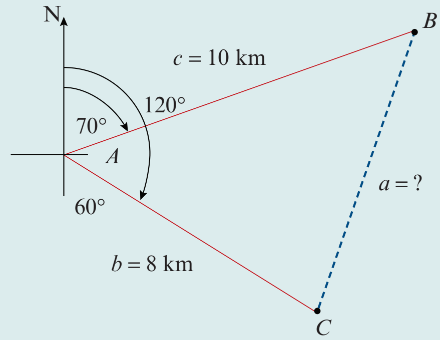
Example 21 Application of the cosine rule involving bearings

A bushwalker left his base camp and walked 10 km in the direction 070°.

His friend also left the base camp but walked 8 km in the direction 120°.

a Find the angle between their paths.

b How far apart were they when they stopped walking? Give your answer to two decimal places.

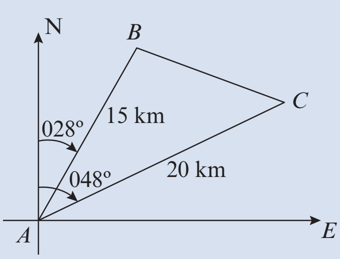


Explanation	Solution
a 1 Angles lying on a straight line add to 180°.	$60^\circ + A + 70^\circ = 180^\circ$ $A + 130^\circ = 180^\circ$ $A = 50^\circ$
2 Write your answer.	The angle between their paths was 50°.
b 1 Write down the known values and the required unknown value.	$a = ?\ b = 8,\ c = 10,\ A = 50$
2 We have two sides and the angle between them. To find side a, use $a^2 = b^2 + c^2 - 2bc\cos A$.	$a^2 = b^2 + c^2 - 2bc\cos A$
3 Substitute in the known values.	$a^2 = 8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 50^\circ$
4 Take the square root of both sides.	$a = \sqrt{(8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 50^\circ)}$
5 Use a calculator to find the value of a.	$a = 7.820\ldots$
6 Answer to two decimal places.	Distance between them was 7.82 km.

Now try this 21 Application of the cosine rule involving bearings (Example 21)

A sailor sailed 15 km in a bearing of 028° from the port at A and stopped at B.

Her friend sailed 20 km from port A on a bearing of 048° and stopped at the point C.



a Find the angle between their courses.

Hint 1 What was the difference in the angles swept out from north?

b How far apart were they when they stopped? Answer to two decimal places.

Hint 1 Use the form of the cosine rule for a^2.

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Section Summary

The cosine rule can be used in a triangle, ABC, to find unknown sides.

- To find an unknown side when given two sides and an included angle, use the equation for the unknown side:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

- To find an unknown angle when given three sides, use the equation for the unknown angle:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

- In a triangle with other lettering, use the pattern: The square of one side equals the sum of the squares of the other two sides, minus twice their product, times the cosine of the angle between them.

Skill-sheet

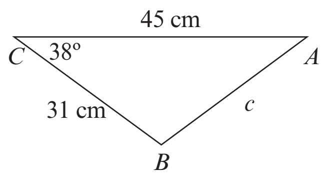
Exercise 11H

Building understanding

1 **a** For the triangle ABC, write the three possible forms of the cosine rule.

b Write the three possible forms of the cosine rule for the triangle XYZ.

2 In triangle ABC, $C = 38°$, $a = 31$ cm and $b = 45$ cm.

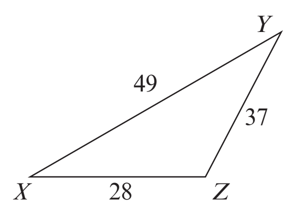


a To find side c, which form of the cosine rule should be used?

b Substitute the known values into the equation.

c Solve the equation to find side c to one decimal place.

3 In triangle XYZ, $x = 37$, $y = 28$ and $z = 49$.



a To find angle Y, which form of the cosine rule should be used?

b Substitute the known values into the equation.

c Tidy up the equation after evaluating the squares and calculating the product.

d Add $3626\cos Y$ to both sides. Subtract 784 from both sides. Divide both sides by 3626 to get an equation for $\cos Y$.

e Use the inverse cosine, $\cos^{-1}$, feature on your CAS calculator to find angle Y to one decimal place. Alternatively, after part **b**, use the Solve command on your CAS calculator to find angle Y.

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Developing understanding

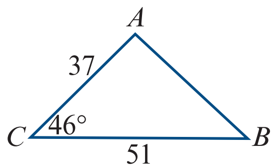
In this exercise, calculate lengths to two decimal places and angles to one decimal place.

Using the cosine rule to find sides

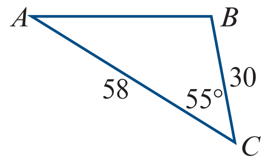
Example 18

4 Find the unknown side in each triangle.

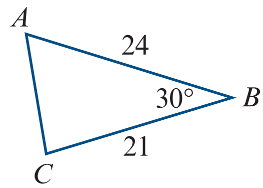
a



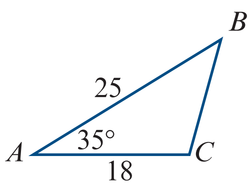
b



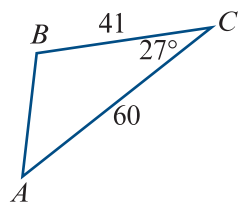
c



d



e



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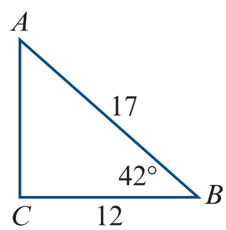


5 In the triangle ABC, $a = 27, b = 22$ and $C = 40°$. Find side c.

6 In the triangle ABC, $a = 18, c = 15$ and $B = 110°$. Find side b.

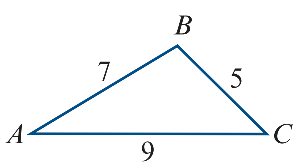
7 In the triangle ABC, $b = 42, c = 38$ and $A = 80°$. Find side a.

Using the cosine rule to find angles

Example 19

8 Find angle A in each triangle.

a



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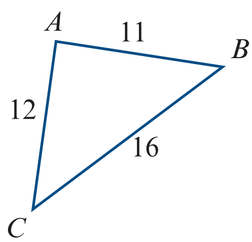

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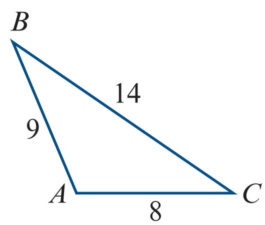

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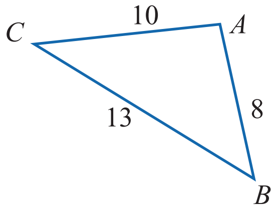

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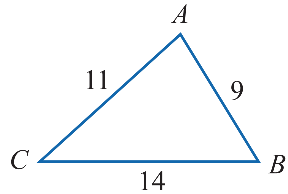

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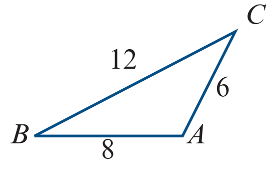

9 In the triangle ABC, $a = 31$, $b = 47$ and $c = 52$. Find angle B.

10 In the triangle RST, $r = 66$, $s = 29$ and $t = 48$. Find angle T.

11 Find the smallest angle in the triangle ABC, with $a = 120, b = 90$ and $c = 105$.

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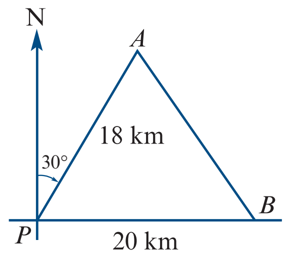
Applications

Example 20

12 A farm has a triangular shape with fences of 5 km, 7 km and 9 km in length. Find the size of the smallest angle between the fences. The smallest angle is always opposite the smallest side.

Example 21

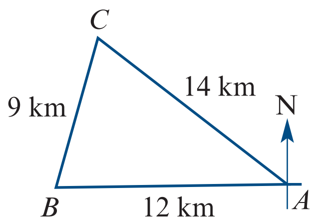
13 A ship left the port, P, and sailed 18 km on a bearing of 030° to point A. Another ship left port P and sailed 20 km east to point B. Find the distance from A to B to one decimal place.



14 A bushwalker walked 12 km west, from point A to point B. Her friend walked 14 km, from point A to point C, as shown in the diagram. The distance from B to C is 9 km.

a Find the angle at A, between the paths taken by the bushwalkers, to one decimal place.

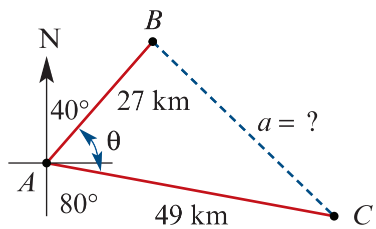
b What is the bearing of point C from A? Give the bearing to the nearest degree.



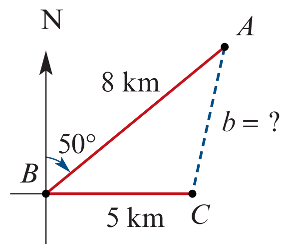
15 A ship left port A and travelled 27 km on a bearing of 040° to reach point B. Another ship left the same port and travelled 49 km on a bearing of 100° to arrive at point C.

a Find the unknown angle, θ, between the directions of the two ships.

b How far apart were the two ships when they stopped?

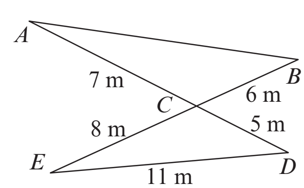


16 A battleship, B, detected a submarine, A, on a bearing of 050° and at a distance of 8 km. A cargo ship, C, was 5 km due east of the battleship. How far was the submarine from the cargo ship?



Testing understanding

17 Find distance AB to one decimal place.



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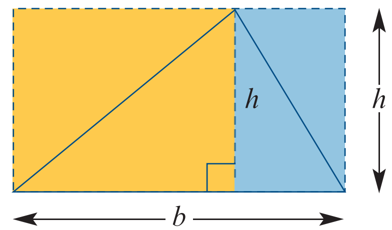
11I The area of a triangle

Learning intentions

- To be able to identify from the given information which of the three area rules should be used to find the area of the triangle.

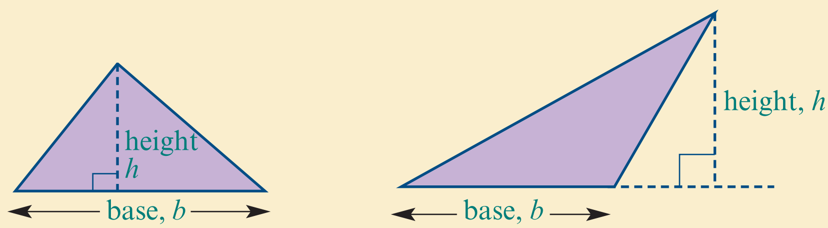
Area of a triangle = $\frac{1}{2}$ × base × height

From the diagram, we see that the area of a triangle with a base, b, and height, h, is equal to half the area of the rectangle, $b \times h$, that it fits within.



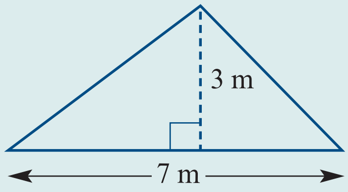
Area of a triangle

$$\text{Area of a triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times b \times h$$



Example 22 **Finding the area of a triangle using $\frac{1}{2}$ × base × height**

Find the area of the triangle shown to one decimal place.



Explanation	**Solution**
1 As we are given values for the base and height of the triangle, use Area = $\frac{1}{2} \times \text{base} \times \text{height}$	Base, $b = 7$ Height, $h = 3$ Area of triangle = $\frac{1}{2} \times b \times h$

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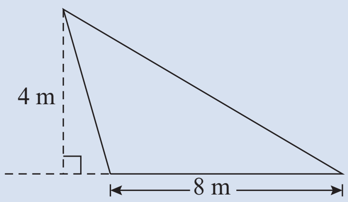
2 Substitute the given values.	$= \frac{1}{2} \times 7 \times 3$
3 Evaluate.	$= 10.5\ \text{m}^2$
4 Write your answer.	The area of the triangle is $10.5\ \text{m}^2$

Now try this 22 **Finding the area of a triangle using $\frac{1}{2}$ × base × height (Example 22)**

Find the area of the triangle shown to one decimal place.



Hint 1 Use: area of a triangle = $\frac{1}{2} \times$ base $\times$ height.

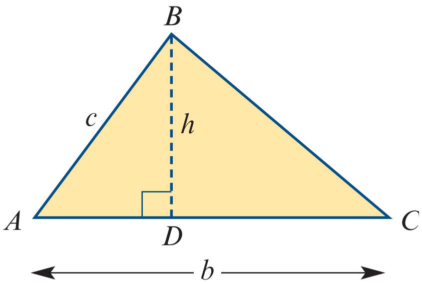
Hint 2 Round the answer to one decimal place and give the correct units for area.

Area of a triangle = $\frac{1}{2} bc \sin A$

In triangle ABD,

$$\sin A = \frac{h}{c}$$

$$h = c \times \sin A$$



So we can replace h with $c \times \sin A$ in the rule:

$$\text{Area of a triangle} = \frac{1}{2} \times b \times h$$

$$\text{Area of a triangle} = \frac{1}{2} \times b \times c \times \sin A$$

Similarly, using side c or a for the base, we can make a complete set of three rules:

Area of a triangle

$$\text{Area of a triangle} = \tfrac{1}{2}\, bc \sin A$$

$$\text{Area of a triangle} = \tfrac{1}{2}\, ac \sin B$$

$$\text{Area of a triangle} = \tfrac{1}{2}\, ab \sin C$$

Notice that each version of the rule follows the pattern:

$$\text{Area of a triangle} = \tfrac{1}{2} \times (\text{product of two sides}) \times \mathit{sin}(\text{angle between those two sides})$$

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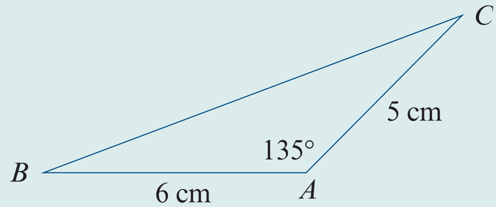
Example 23 Finding the area of a triangle using $\frac{1}{2}bc \sin A$

Find the area of the triangle shown to one decimal place.



Explanation

1. We are given two sides, b and c, and the angle, A, between them, so use:
   Area of a triangle $= \frac{1}{2}bc \sin A$
2. Substitute values for b, c and A into the rule.
3. Use your calculator to find the area.
4. Write your answer to one decimal place.

Solution

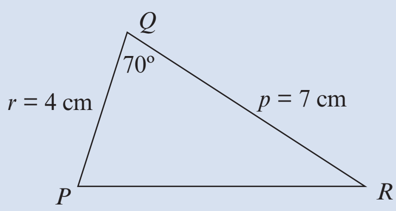
$b = 5, c = 6, A = 135°$

$$\text{Area of triangle} = \frac{1}{2}bc \sin A$$
$$= \frac{1}{2} \times 5 \times 6 \times \sin 135°$$
$$= 10.606\ldots$$

The area of the triangle is 10.6 cm^2.

Now try this 23 Finding the area of a triangle using $\frac{1}{2}bc \sin A$ (Example 23)

Find the area of the triangle shown to one decimal place.



Hint 1 The pattern of the area rule needed is:

Area of a triangle $= \frac{1}{2} \times$ (product of two sides) $\times$ sin(angle between the two sides)

Heron's rule for the area of a triangle

Heron's rule can be used to find the area of any triangle when we know the lengths of the three sides.

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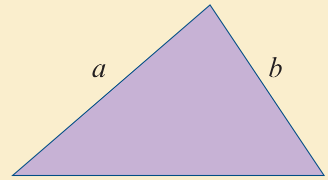
Heron's rule for the area of a triangle

Area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

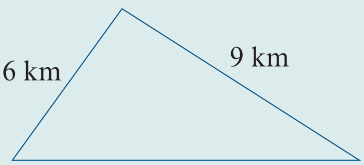
where $s = \frac{1}{2}(a+b+c)$

The variable s is called the *semi-perimeter* because it is equal to half the sum of the sides.



Example 24 Finding the area of a triangle using Heron's formula

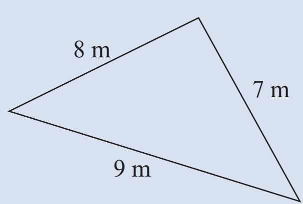
The boundary fences of a farm are shown in the diagram. Find the area of the farm to the nearest square kilometre.



Explanation	Solution
1 As we are given the three sides of the triangle, use Heron's formula. Start by finding s, the semi-perimeter.	Let $a = 6, b = 9, c = 11$ $s = \frac{1}{2}(a+b+c)$ $= \frac{1}{2}(6+9+11) = 13$
2 Write Heron's formula.	Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$
3 Substitute the values of s, a, b and c into Heron's formula.	$= \sqrt{13(13-6)(13-9)(13-11)}$ $= \sqrt{13 \times 7 \times 4 \times 2}$
4 Use your calculator to find the area.	$= 26.981\ldots$
5 Write your answer.	The area of the farm, to the nearest square kilometre, is 27 km^2.

Now try this 24 Finding the area of a triangle using Heron's formula (Example 24)

Find the area of the triangle shown to one decimal place.



Hint 1 Calculate the semi-perimeter using

$$s = \frac{1}{2}(a+b+c).$$

Hint 2 Write Heron's formula and substitute in the values for s, a, b and c.

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Section Summary

There are three rules for finding the area of a triangle, ABC:

- When given the length of the base and the perpendicular height, use:

$$Area = \tfrac{1}{2}\ base \times height$$

- When given two sides and the angle between those sides, use the given information to choose from:

$$Area = \tfrac{1}{2}\ bc \sin A$$
$$Area = \tfrac{1}{2}\ ac \sin B$$
$$Area = \tfrac{1}{2}\ ab \sin C$$

- When given the three sides of the triangle, use Heron's formula:

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$
$$where\ s = \tfrac{1}{2}(a+b+c)$$

Skill-sheet

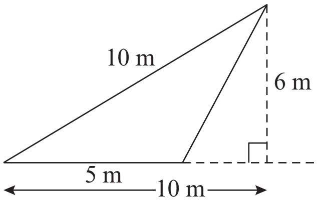
Exercise 11I

In this exercise, calculate areas to one decimal place, where necessary.

Building understanding

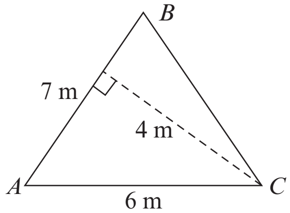
1 Finding the area of a triangle using:
Area $= \dfrac{1}{2}$ base $\times$ height.

a Give the perpendicular height.
b What is the length of the base?
c Find the area of the triangle.



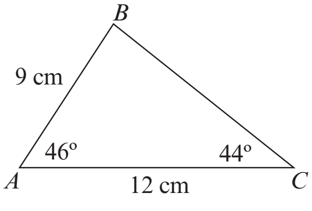
2 Find the area of triangle ABC. The base is not always the horizontal line. In this case, the perpendicular height is measured from the base, AB.

a State the perpendicular height.
b What is the length of the base?
c Find the area.



3 Find the area of the triangle using:
Area $= \dfrac{1}{2}\ bc \times \sin\theta$
where θ is the angle between the two given sides.

a Which angle should be used? Why?
b Find the area to one decimal place.



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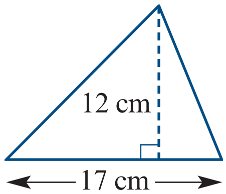
Developing understanding

Finding areas using $\frac{1}{2} \times$ base $\times$ height

Example 22

4 Find the area of each triangle.

a



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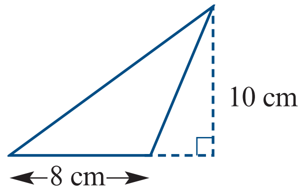

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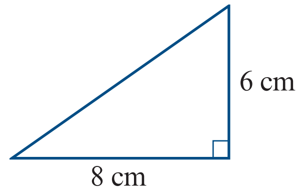
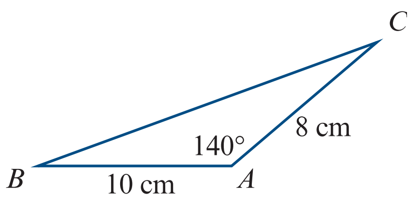


Finding areas using $\frac{1}{2}$ bc sin A

Example 23

5 Find the areas of the triangles shown.

a



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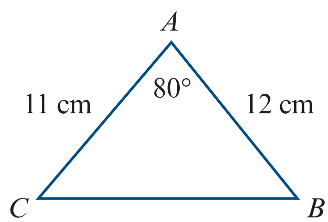

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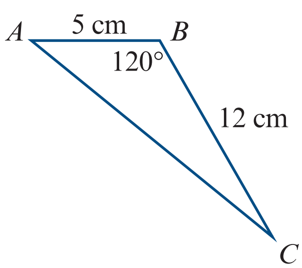

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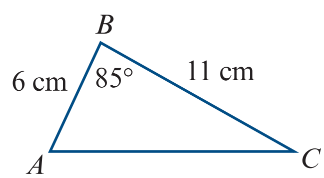
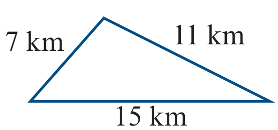


Finding areas using Heron's formula

Example 24

6 Find the area of each triangle.

a



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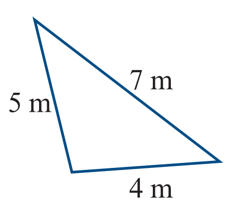

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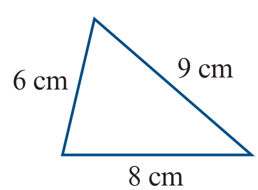

Mixed problems

7 For each triangle on the following page, choose the rule for finding its area from:

i $\frac{1}{2}$ *base* $\times$ *height*

ii $\frac{1}{2}\, bc\ \sin A$

iii $\frac{1}{2}\, ac\ \sin B$

iv $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a + b + c)$

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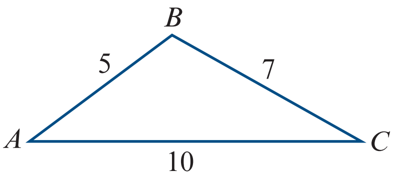
a



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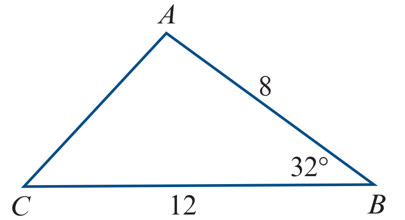

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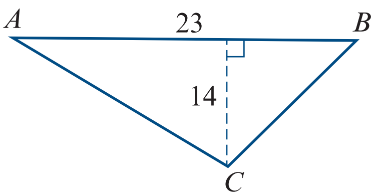

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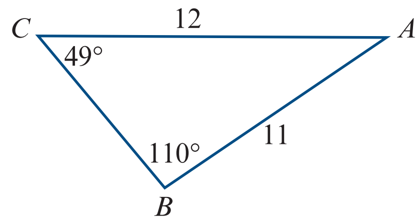
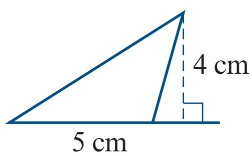


8 Find the area of each triangle shown.

a



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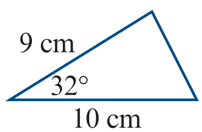

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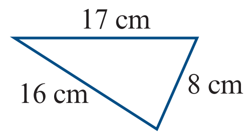

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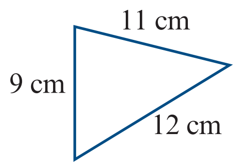

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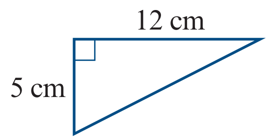

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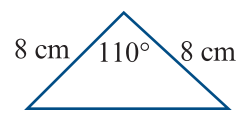


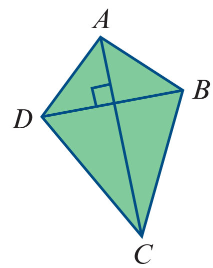
9 Find the area of a triangle with a base of 28 cm and a height of 16 cm.

10 Find the area of triangle RST with side r (42 cm), side s (57 cm) and angle T (70°).

11 Find the area of a triangle with sides of 16 km, 19 km and 23 km.

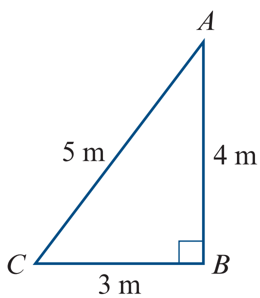
Applications

12 The kite shown is made using two sticks, AC and DB. The length of AC is 100 cm and the length of DB is 70 cm. Find the area of the kite.

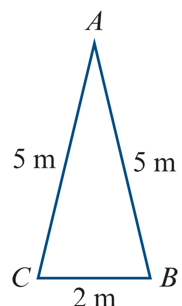


13 Three students, A, B and C, stretched a rope loop that was 12 m long into different triangular shapes. Find the area of each shape.

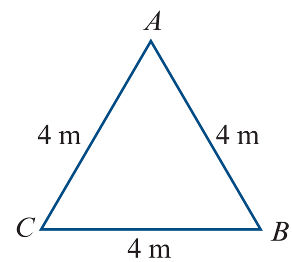
a



b



c



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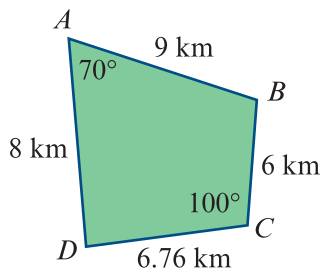
14 A farmer needs to know the area of her property with the boundary fences as shown. Give answers to two decimal places.

Hint: Draw a line from B to D to divide the property into two triangles.

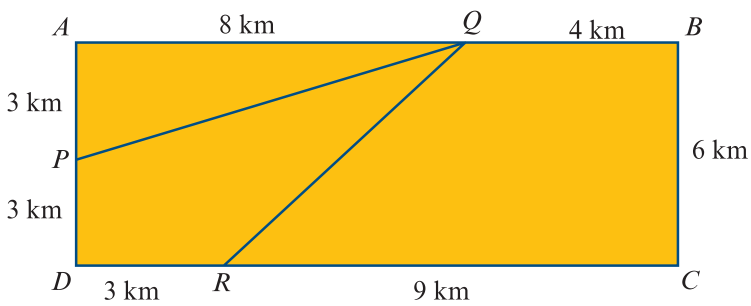


a Find the area of triangle ABD.

b Find the area of triangle BCD.

c State the total area of the property.

15 A large rectangular area of land, $ABCD$ in the diagram, has been subdivided into three regions as shown.



a Find the area of:

i region PAQ **ii** region $QBCR$ **iii** region $PQRD$.

b Find the size of angle PQR to one decimal place.

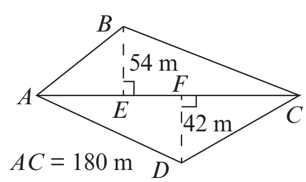
Testing understanding

16 A technique that surveyors use to find the area of an irregular four-sided shape is to measure the length, AC, joining opposite corners, then the lengths of lines perpendicular to AC to the other corners.

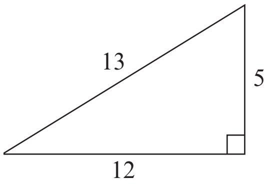
The measurements for the lengths were:

$AC = 180$ m, $BE = 54$ m and $DF = 42$ m.

Find the area $ABCD$ to one decimal place.

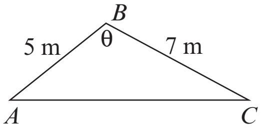


17 Show how the three different area rules can be used to find the area of the triangle shown.



18 Triangle ABC has sides of 5 m and 7 m. Angle θ is between the two given sides.

Find the angle, θ, that would give the maximum area for triangle ABC.



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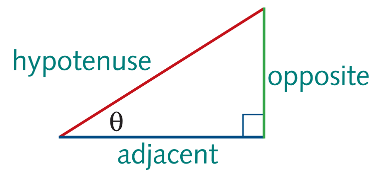
Key ideas and chapter summary

Assignment

Naming the sides of a right-angled triangle

The **hypotenuse** is the longest side and is always opposite the right angle (90°).
The **opposite** side is directly opposite the angle θ (the angle being considered).
The **adjacent** side is beside angle θ and runs from θ to the right angle.



Trigonometric ratios

The **trigonometric ratios** are sin θ, cos θ and tan θ:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

SOH-CAH-TOA

This helps you to remember the trigonometric ratio rules.

Degree mode

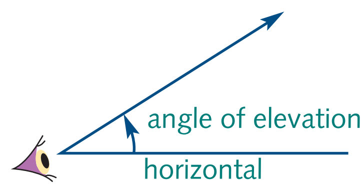
Make sure your calculator is in DEGREE mode when doing calculations with trigonometric ratios.

Applications of right-angled triangles

Always draw well-labelled diagrams, showing all known sides and angles. Also label any sides or angles that need to be found.

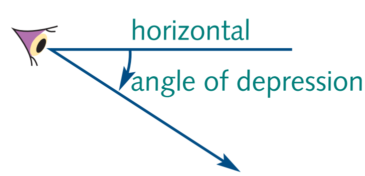
Angle of elevation

The **angle of elevation** is the angle through which you *raise* your line of sight from the horizontal, looking *up* at something.



Angle of depression

The **angle of depression** is the angle through which you *lower* your line of sight from the horizontal, looking *down* at something.

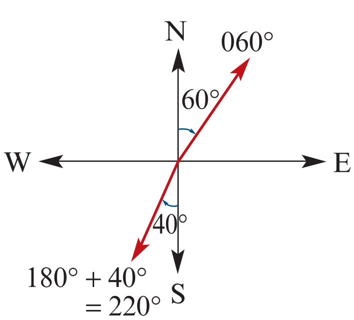


Angle of elevation = angle of depression

The angles of elevation and depression are alternate ('Z') angles, so they are equal.

Three-figure bearings

Three-figure bearings are measured clockwise from North and always have three digits, e.g. 060°, 220°.



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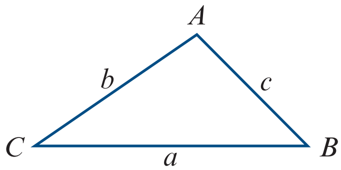
Distance, speed and time

Navigation problems often involve distance, speed and time, as well as direction.

Distance travelled = time taken × speed

Labelling a non-right-angled triangle

Side a is always opposite angle A, and so on.



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use the **sine rule** when given:

- two sides and an angle opposite one of those sides
- two angles and one side.

If neither angle is opposite the given side, find the third angle using $A + B + C = 180°$.

Ambiguous case of the sine rule

The ambiguous case of the sine rule occurs when it is possible to draw two different triangles that both fit the given information.

Cosine rule

The **cosine rule** has three versions. When given two sides and the angle between them, use the rule that starts with the required side:

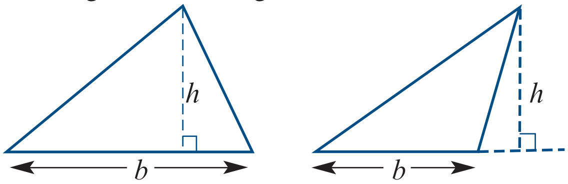
$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

- Use the formula: area of triangle $= \frac{1}{2} \times b \times h$, if the base and height of the triangle are known:



- Use the formula: area of triangle $= \frac{1}{2} \times bc\sin A$, if two sides and the angle between them are known.
- Use Heron's formula if the lengths a, b and c, of the three sides of the triangle are known.

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{where } s = \tfrac{1}{2}(a+b+c)$$

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Review

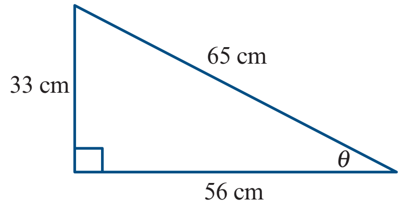
Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

11A **1** I can name the sides of a right-angled triangle. ☐

e.g. Name the sides 33 cm, 56 cm and 65 cm long.



11A **2** I can use the definitions of trigonometric ratios. ☐

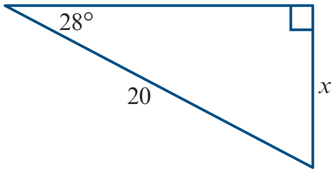
e.g. Using the diagram in the previous question, state the values of the trigonometric ratios for: $\cos\theta$, $\sin\theta$ and $\tan\theta$.

11A **3** I can use a CAS calculator to find the value of a trigonometric ratio for a given angle. ☐

e.g. Find $\cos 27°$, $\sin 58°$ and $\tan 73°$ to four decimal places.

11A **4** I can choose the required trigonometric ratio rule when finding an unknown side in a right-angled triangle. ☐

e.g. Name the trigonometric ratio needed to find x.



11B **5** I can substitute in values and solve the required equation to find the unknown side. ☐

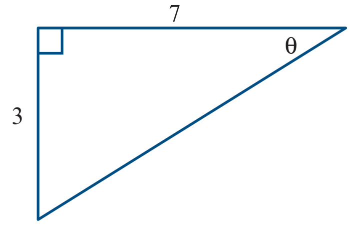
e.g. Use the triangle in the previous question to find side x to one decimal place.

11C **6** I can use a CAS calculator to find the required angle when given the value of its trigonometric ratio. ☐

Find θ, to one decimal place, when $\cos\theta = 0.7431$.

11C **7** I can find the required angle in a right-angled triangle given two sides of the triangle. ☐

e.g. Find angle θ to one decimal place.



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11D **8** I can draw clearly labelled diagrams of practical situations, showing the given sides and angles. ☐

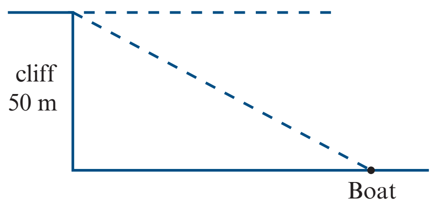
e.g. A tree casts a shadow that is 23 metres long when the sun's rays make an angle of 34° with the horizontal ground. Draw a clearly labelled diagram.

11D **9** I can set up and solve equations to find unknown sides and angles. ☐

e.g. Find the height of the tree described in the previous question to one decimal place.

11E **10** I can identify and label the angles of elevation and depression in diagrams of practical situations. ☐

e.g. From the height of a 50 m cliff, a boat is seen at an angle of depression of 28°. Write the information into the diagram shown.



11E **11** I can choose the appropriate trigonometric ratios and solve equations to find unknown sides and angles. ☐

e.g. In the situation described above, find the distance of the boat from the base of the cliff to the nearest metre.

11F **12** I can use three-figure bearings to draw navigation and surveying diagrams. ☐

e.g. A surveyor stopped on a highway pointing North and then walked on a bearing of 050° for 3 km. Show this in a clearly labelled diagram.

11F **13** I can solve the appropriate equations to find unknown bearings and distances. ☐

e.g. In the situation described above, what is the shortest distance the surveyor needs to walk to reach the highway to one decimal place.

11G **14** I can use the sine rule to find an unknown angle, given two sides and an opposite angle. ☐

e.g. In triangle ABC, $B = 115°$, $b = 27$ m and $c = 24$ m. Find angle C to one decimal place.

11G **15** I can use the sine rule to find an unknown side, given 2 angles and a side. ☐

e.g. In triangle ABC, $A = 30°$, $C = 110°$ and $c = 49$ km. Find side a to nearest km.

11G **16** I can find the required angles and sides when given information that fits two possible triangles. ☐

e.g. In triangle ABC, $A = 35°$, $a = 15$ and $c = 20$. Find the two possible values for angle C to one decimal place.

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Review

11H **17** I can identify when the sine rule or the cosine rule should be used. ☐

e.g. In triangle ABC, $a = 14$, $b = 17$ and $c = 16$. Which rule should be used to find angle A?

11H **18** I can use the cosine rule to find the unknown side when given two sides and the angle between them. ☐

e.g. In triangle ABC, $B = 70°$, $a = 8$ and $c = 10$. Find side b to one decimal place.

11H **19** I can find the required angle in a triangle when given the three sides. ☐

e.g. In triangle ABC, $a = 21$, $b = 23$ and $c = 26$. Find angle C to one decimal place.

11I **20** I can use the given information to decide which area rule should be used. ☐

e.g. In triangle ABC, $a = 20$, $b = 23$ and $c = 27$. Name the rule that should be used to find the area.

11I **21** I can use the appropriate rule to find the area of a given triangle. ☐

e.g. In triangle ABC, $A = 47°$, $b = 29$ cm and $c = 31$ cm. Find the area of triangle ABC to one decimal place.

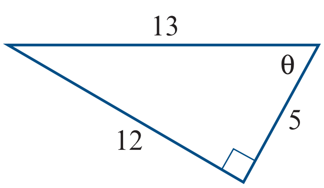
Multiple-choice questions

1 In the triangle shown, $\sin\theta$ equals:

A $\frac{5}{13}$ **B** $\frac{5}{12}$

C $\frac{12}{13}$ **D** $\frac{13}{12}$

E $\frac{12}{5}$

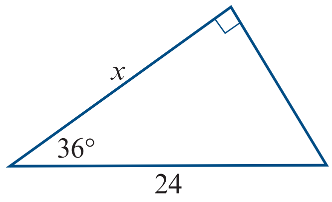


2 The unknown length, x, is given by:

A $24 \sin 36°$ **B** $24 \tan 36°$

C $24 \cos 36°$ **D** $\frac{\sin 36°}{24}$

E $\frac{\cos 36°}{24}$

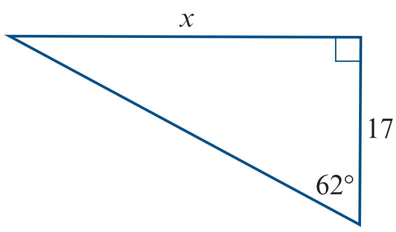


3 To find length x we should use:

A $17 \sin 62°$ **B** $17 \tan 62°$

C $17 \cos 62°$ **D** $\frac{\tan 62°}{17}$

E $\frac{\sin 62°}{17}$



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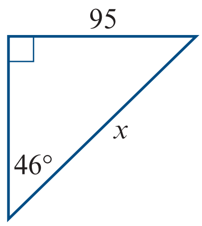
4 The unknown side, x, is given by:

A $95 \tan 46°$ **B** $\dfrac{95}{\cos 46°}$ **C** $\dfrac{\sin 46°}{96}$

D $95 \sin 46°$ **E** $\dfrac{95}{\sin 46°}$

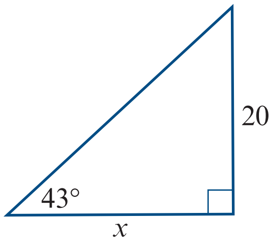


5 To find side x we need to calculate:

A $\dfrac{\tan 43°}{20}$ **B** $\dfrac{20}{\tan 43°}$

C $20 \tan 43°$ **D** $20 \cos 43°$

E $20 \sin 43°$

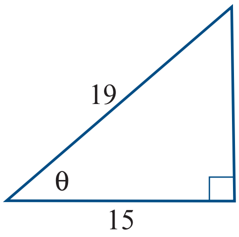


6 To find angle θ we need to use:

A $\cos^{-1}\left(\dfrac{15}{19}\right)$ **B** $\cos\left(\dfrac{15}{19}\right)$

C $\sin^{-1}\left(\dfrac{15}{19}\right)$ **D** $15 \sin(19)$

E $19 \cos(15)$

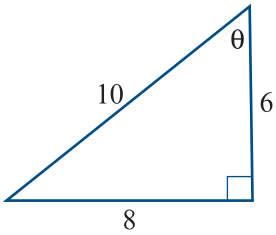


7 The unknown angle, θ, to one decimal place, is:

A 36.9° **B** 38.7°

C 51.3° **D** 53.1°

E 53.3°

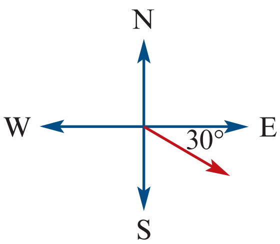


8 The direction shown has the three-figure bearing:

A 030° **B** 060°

C 120° **D** 210°

E 330°

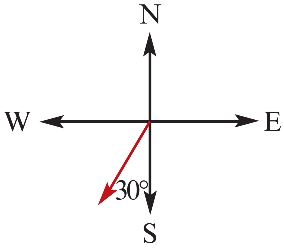


9 The direction shown could be described as the three-figure bearing:

A −030° **B** 030°

C 060° **D** 120°

E 210°



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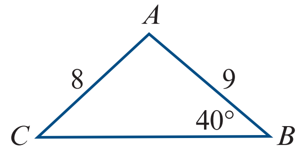
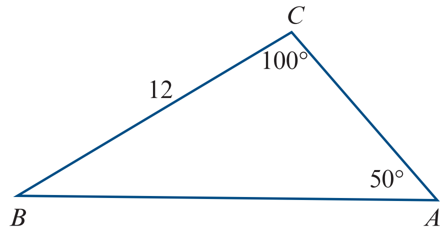
10 In this triangle, angle C equals:



A 34.8° **B** 46.3°
C 53.9° **D** 55.2°
E 86.1°

11 To find length c in triangle ABC we should use:

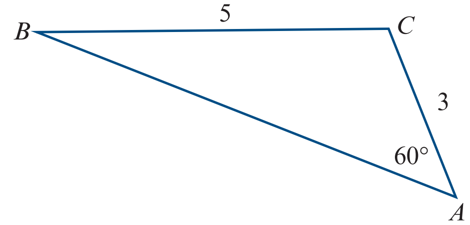


A $\dfrac{12\sin 100^\circ}{\sin 30^\circ}$ **B** $\dfrac{12\sin 50^\circ}{\sin 100^\circ}$

C $\dfrac{\sin 50^\circ}{12\sin 100^\circ}$ **D** $\dfrac{12\sin 100^\circ}{\sin 50^\circ}$

E $\dfrac{\sin 100^\circ}{12\sin 50^\circ}$

12 In triangle ABC, $\sin B$ equals:

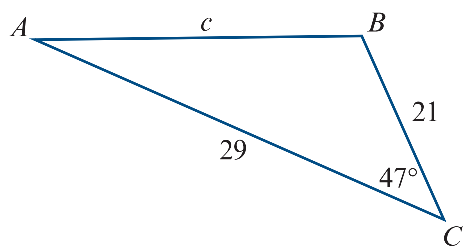


A $\dfrac{3}{5}$ **B** $\dfrac{3\sin 60^\circ}{5}$

C $\dfrac{3}{5\sin 60^\circ}$ **D** $\dfrac{5\sin 60^\circ}{3}$

E $\dfrac{5}{3\sin 60^\circ}$

13 Which expression should be used to find length c in triangle ABC?



A $\dfrac{1}{2}(21)(29)\cos 47^\circ$

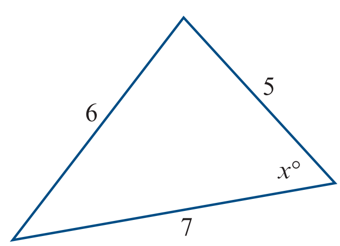
B $\cos^{-1}\left(\dfrac{21}{29}\right)$

C $\sqrt{21^2 + 29^2}$

D $21^2 + 29^2 - 2(21)(29)\cos 47^\circ$

E $\sqrt{21^2 + 29^2 - 2(21)(29)\cos 47^\circ}$

14 For the given triangle, the value of $\cos x$ is given by:



A $\dfrac{6^2 - 7^2 - 5^2}{2(7)(5)}$ **B** $\dfrac{7^2 + 5^2 - 6^2}{2(7)(5)}$

C $\dfrac{5}{7}$ **D** $\dfrac{7^2 - 5^2 - 6^2}{2(5)(6)}$

E $\dfrac{5^2 - 6^2 - 7^2}{2(5)(6)}$

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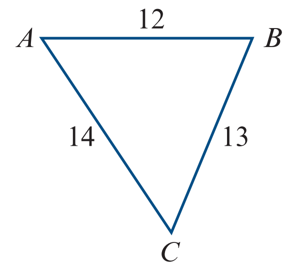
15 To find angle C we should use the rule:



A $\cos C = \dfrac{adjacent}{hypotenuse}$

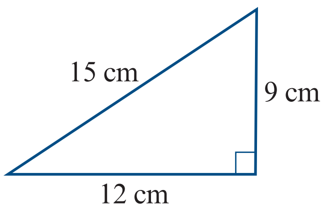
B $\sin C = \dfrac{opposite}{hypotenuse}$

C $\cos C = \dfrac{a^2 + c^2 - b^2}{2ac}$

D $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

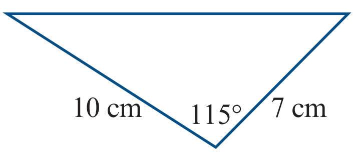
E $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

16 The area of the triangle shown is:



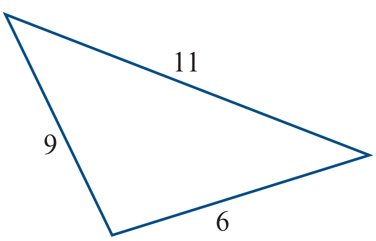
A 36 cm^2 **B** 54 cm^2
C 67.5 cm^2 **D** 90 cm^2
E 108 cm^2

17 The area of the triangle shown, to two decimal places, is:



A 14.79 cm^2 **B** 31.72 cm^2
C 33.09 cm^2 **D** 35.00 cm^2
E 70.00 cm^2

18 The area of the triangle shown is given by:



A $26(26 - 6)(26 - 9)(26 - 11)$

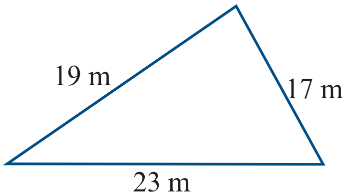
B $\sqrt{26(26 - 6)(26 - 9)(26 - 11)}$

C $\sqrt{13(13 - 6)(13 - 9)(13 - 11)}$

D $\sqrt{6^2 + 9^2 + 11^2}$

E $13(13 - 6)(13 - 9)(13 - 11)$

19 The area of the triangle shown, to one decimal place, is:



A 29.5 m^2 **B** 158.6 m^2 **C** 161.5 m^2
D 195.5 m^2 **E** 218.5 m^2

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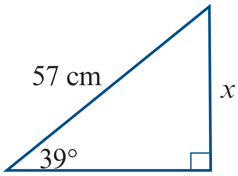
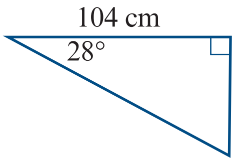
Short-answer questions

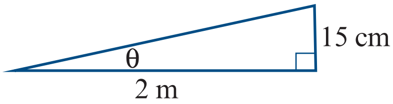
1 Find the length of x to two decimal places.



2 Find the length of the hypotenuse to two decimal places.

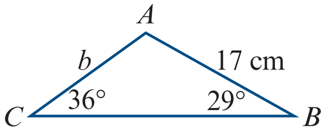


3 A road rises 15 cm for every 2 m travelled horizontally.
Find the angle of slope θ to the nearest degree.

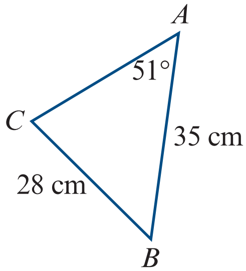


4 **a** Find the sides of a right-angled triangle for which $\cos\theta = \frac{72}{97}$ and $\tan\theta = \frac{65}{72}$.
b Hence, find $\sin\theta$.

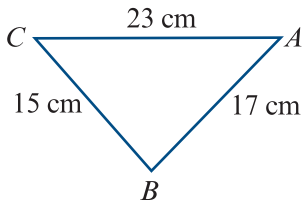
5 Find the length of side b to two decimal places.



6 Find two possible values for angle C to one decimal place.



7 Find the smallest angle in the triangle shown to one decimal place.



8 A car travelled 30 km east, then travelled 25 km on a bearing of 070°. How far was the car from its starting point? Answer to two decimal places.

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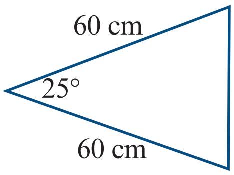
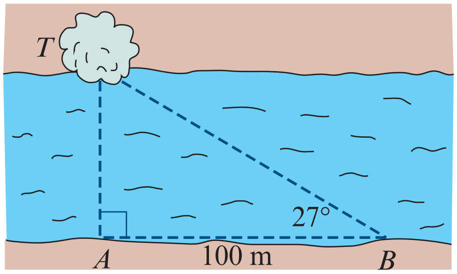
9 A pennant flag is to have the dimensions shown. What area of cloth will be needed for the flag? Answer to one decimal place.



10 Find the area of an equilateral triangle with sides of 8 m to one decimal place.

Written-response questions

1 Tim was standing at point A when he saw a tree, T, directly opposite him on the far bank of the river. He walked 100 m along the riverbank to point B and noticed that his line of sight to the tree made an angle of 27° with the riverbank. Answer the following to two decimal places.



a How wide was the river?

b What is the distance from point B to the tree?

Standing at B, Tim measured the angle of elevation to the top of the tree to be 18°.

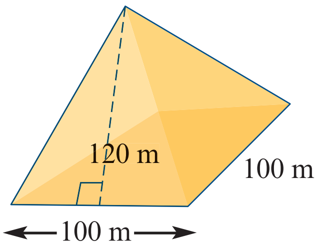
c Make a clearly labelled diagram showing distance TB, the height of the tree and the angle of elevation, then find the height of the tree.

2 A yacht, P, left port and sailed 45 km on a bearing of 290°. Another yacht, Q, left the same port but sailed for 54 km on a bearing of 040°.

a What was the angle between their directions?

b How far apart were they at that stage (to two decimal places)?

3 The pyramid shown has a square base with sides of 100 m. The line down the middle of each side is 120 m long.



a Find the total surface area of the pyramid. (As the pyramid rests on the ground, the area of its base is not part of its surface area.)

b If 1 kg of gold can be rolled flat to cover 0.5 m^2 of surface area, how much gold would be needed to cover the surface of the pyramid?

c At today's prices, 1 kg of gold costs $62 500. How much would it cost to cover the pyramid with gold?

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Chapter 12

Revision of Unit 2

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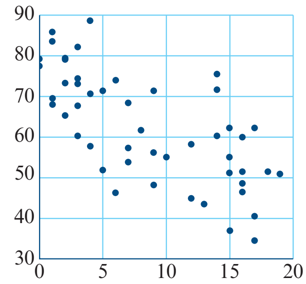
12A Multiple-choice questions

Chapter 7: Investigating relationships between two numerical variables

1 For which one of the following pairs of variables would it be appropriate to construct a scatterplot?

A *computer preference* (apple, PC) and *phone preference* (iPhone, android)
B *test score* and *year level* (Year11, Year 12)
C *interest in politics* (very interested, interested, not interested) and *age* in years
D *time spent on social media*, in minutes, and *age group* (less than 18, 18-25, over 25)
E *age*, in years, and *reaction time*, in seconds

2 For the scatterplot shown, the association between the variables is best described as:

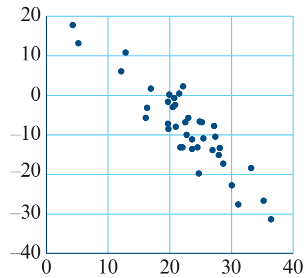


A weak linear negative
B strong linear negative
C no association
D weak linear positive
E strong linear positive

3 (VCAA-type question) A designer of luxury handbags discovered that there was a positive association between the price of their handbags and demand for the handbags. Given this information, it can be concluded that:

A there is a linear association between the price of the handbags and demand.
B demand for the handbags is not related to their price.
C demand for the handbags tends to increase as the price of the handbags increases.
D demand for the handbags tends to decrease as the price of the handbags increases.
E demand for the handbags tends to increase as the price of the handbags decreases.

4 For the scatterplot shown, the value of the Pearson's correlation coefficient, r, is closest to:



A -0.82 **B** -0.51 **C** 0.15
D 0.53 **E** 0.77

5 A correlation coefficient of $r = 0.64$ would classify a linear relationship as:

A weak, positive **B** weak, negative **C** moderate, positive
D moderate, negative **E** moderately weak

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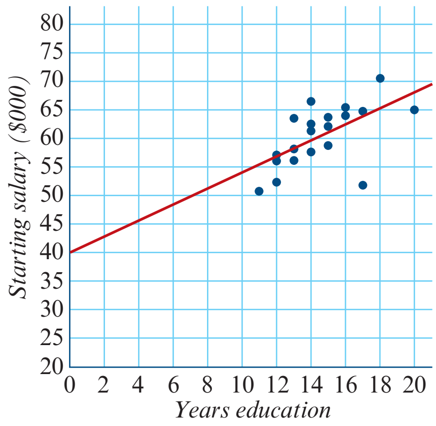
6 (VCAA-type question) The following scatterplot shows the association between the number of years employees in a company had spent in formal education (*years education*) and their *starting salary*, in $000s. The least squares regression line is also shown on the scatterplot.



The equation of this line is:

A *years education* = 40 + 1.4 × *starting salary*

B *starting salary* = 1.4 + 40 × *years education*

C *years education* = 40 + 0.714 × *starting salary*

D *expenditure* = 0.714 − 40 × *years education*

E *starting salary* = 40 + 1.4 × *years education*

7 (VCAA-type question) The equation of the least squares regression line, $y = a + bx$, when:

$r = -0.600$ $s_x = 3.20$ $s_y = 6.40$ $\bar{x} = 48.7$ $\bar{y} = 63.3$

is given by:

A $y = 4.86 - 1.20x$ **B** $y = 77.9 - 0.3x$ **C** $y = 1.2 + 122x$

D $y = 122 - 1.20x$ **E** $y = 0.3 - 77.9x$

The following information relates to Questions 8 and 9.

The *age*, in years, and percentage *body fat* for a group of 8 people is given in the following table.

age	58	72	67	43	51	52	25	35
body fat (%)	20.1	26.1	25.8	19.5	14.1	27.0	6.1	4.1

8 The value of Pearson's correlation coefficient, r, for these data is closest to:

A 0.4 **B** 0.6 **C** 0.7 **D** 0.8 **E** 0.9

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9 The least squares regression line that enables *body fat* to be predicted from *age* is closest to:

A $age = 23.9 + 1.49 \times body\ fat$

B $body\ fat = -6.40 + 0.481 \times body\ fat$

C $age = 6.40 + 0.481 \times body\ fat$

D $body\ fat = 23.9 + 1.49 \times age$

E $body\ fat = -6.40 + 0.846 \times body\ fat$

The following information relates to Questions 10 and 11.

Data from a large group of males was used to determine the equation of a regression line that enables *weight*, in kilograms, to be predicted from percentage *body fat* as follows:

$$weight = 61.32 + 1.057 \times bodyfat$$

10 Using this equation, the weight of an individual with a *body fat* of 20% is predicted to be:

A 61.5 kg **B** 78.4 kg **C** 82.5 kg **D** 125.1 kg **E** 1251.1 kg

11 From the equation of the regression line, it can be concluded that on average there is:

A an increase of 1 kg in *weight* for each 1.057 percentage increase in *body fat*.

B a decrease of 1 kg in *weight* for each 1 percentage increase in *body fat*.

C a decrease of 1 kg in *weight* for each 1 percentage increase in *body fat*.

D an increase of 61.32 kg in *weight* for each 1 percentage point increase in *body fat*.

E a decrease of 1.057 kg in *weight* for each 1 percentage decrease in *body fat*.

12 In an investigation of the association between hearing test scores and age, the equation of the least squares line fitted to the data was:

$$hearing\ test\ score = 48.9034 - 0.0428681 \times age$$

When the coefficients in the equation are rounded to four significant figures, the equation becomes:

A $hearing\ test\ score = 48.9034 - 0.0429 \times age$

B $hearing\ test\ score = 48.9 - 0.0429 \times age$

C $hearing\ test\ score = 48.90 - 0.04287 \times age$

D $hearing\ test\ score = 48.903 - 0.0429 \times age$

E $hearing\ test\ score = 48.900 - 0.04287 \times age$

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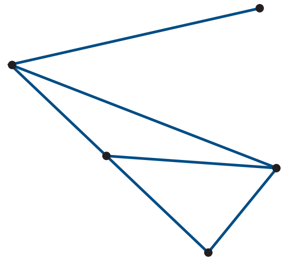
Chapter 8: Graphs and networks

13 (VCAA-type question) A graph has 6 vertices. The minimum number of edges required for this graph to be connected is:

A 3 **B** 4 **C** 5 **D** 6 **E** 7

14 (VCAA-type question) The sum of the degrees of the graph shown is:

A 8
B 9
C 10
D 11
E 12

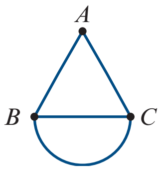


15 (VCAA-type question) A tree with 10 vertices will have:

A 8 edges **B** 9 edges **C** 10 edges **D** 11 edges **E** 12 edges

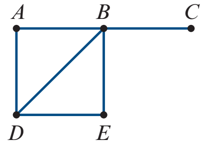
16 For the graph shown, the number of paths of length 3 from vertex A to vertex A is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4



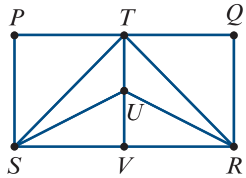
17 The graph shown does not have an Eulerian circuit.
The addition of which edge will ensure that the graph has an Eulerian circuit?

A AB **B** AC **C** AE
D DE **E** DC



18 (VCAA-type question)
For the graph shown opposite, which of the following is a Hamiltonian cycle.

A $T-P-S-V-R-U-Q-T$
B $T-Q-R-T-P-S-V-U-T$
C $R-V-S-P-T-Q-R-U$
D $U-V-S-P-T-Q-R$
E $U-V-S-P-T-Q-R-U$



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19 A graph has four vertices, A, B, C and D, and the adjacency matrix of this graph is:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 3 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

This graph has an Eulerian trail that begins and ends at which of the following pairs of vertices?

A A and B **B** A and C **C** A and D

D B and C **E** B and D

20 (VCAA-type question) A connected planar graph has 8 vertices and 8 faces. The number of edges for this graph is:

A 10 **B** 11 **C** 12 **D** 13 **E** 14

21 A connected planar graph that has 6 vertices could have:

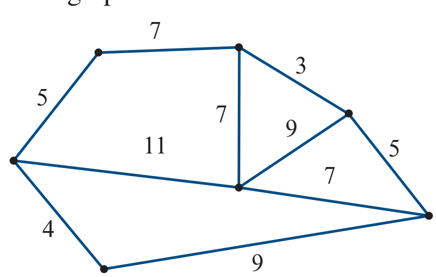
A 4 edges and 4 faces **B** 7 edges and 5 faces **C** 11 edges and 6 faces

D 10 edges and 6 faces **E** 8 edges and 3 faces

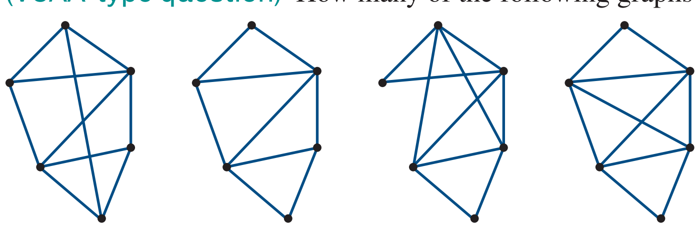
22 The total weight on the minimum spanning tree for the graph is:

A 25 **B** 27 **C** 31

D 33 **E** 34



23 (VCAA-type question) How many of the following graphs are non-planar?



A 0 **B** 1 **C** 2 **D** 3 **E** 4

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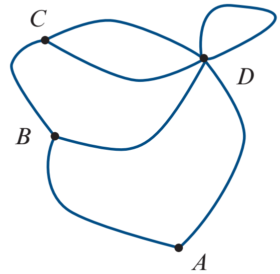
24 (VCAA-type question) Consider the following graph.



An adjacency matrix that could be used to represent the graph is:

A $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \end{array}$

B $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$

C $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \end{array}$

D $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \end{array}$

E $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \end{array}$

Chapter 9: Variation

25 If $m \propto n$ and $m = 9$ when $n = 4$, then the constant of variation, k, equals:

A $\frac{9}{4}$ **B** 13 **C** 36 **D** $\frac{4}{9}$ **E** 5

26 If $y \propto \frac{1}{x}$ and $y = 14$ when $x = 2$, the value of y when $x = 7$ is:

A $\frac{1}{4}$ **B** 4 **C** 9 **D** 19 **E** 49

27 If $x \propto \frac{1}{y}$ and y is multiplied by 5, then x will be:

A decreased by 5 **B** increased by 5 **C** multiplied by 5
D divided by 5 **E** none of these

28 The area of a triangle varies directly as the base length, provided the height of the triangle is constant. If the area equals 14 when the base length is 2.4, then the base length (correct to three decimal places) when the area is 18 will equal:

A 3.086 **B** 5.000 **C** 6.400 **D** 9.600 **E** 0.324

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29 (VCAA-type question) The area of ground that can be covered by a bag of mulch is inversely proportional to the depth at which the mulch is spread. If a bag of mulch covers 1.25 m^2 at a depth of 2 cm, then approximately how much area could be covered at a depth of 3 cm?

A 0.42 m^2 **B** 0.63 m^2 **C** 0.83 m^2 **D** 1.88 m^2 **E** 3.75 m^2

30 (VCAA-type question) The given table of values follows the rule: $y = kx^2 + c$.

x	1	2	3	4
y	4.5	9	16.5	27

The values of k and c respectively are:

A 1 and 5 **B** 1.5 and 3 **C** 2 and 11 **D** 3 and 6 **E** 5 and 1.5

31 The given table of values follows the rule: $y = \frac{k}{x} + c$.

x	1	2	4	5
y	7.5	4.5	3	2.7

The values of k and c respectively are:

A 1 and 6.5 **B** 2 and 3 **C** 2 and 5 **D** 6 and 1.5 **E** 8 and 2.5

32 (VCAA-type question) The following data can be modelled by $y = k\log_{10}(x) + c$.

x	1	10	100	1000
y	5	105	205	305

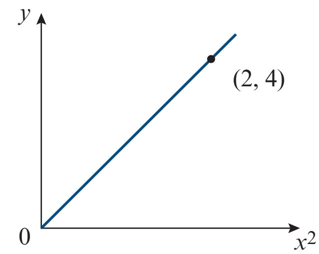
The values of k and c respectively are:

A 1 and 5 **B** 1 and 105 **C** 10 and 15 **D** 100 and 5 **E** 300 and 50

33 (VCAA-type question) The rule connecting y and x as shown in the graph is:

A $y = 2x$ **B** $y = 2x^2$ **C** $y = 2\sqrt{x}$

D $y = \frac{1}{2}x$ **E** $y = x^2 + 2$



34 The pressure, P, of a given quantity of gas is inversely proportional to its volume, V, as long as the temperature remains constant. If the pressure is 80 when the volume is 60, the value of P when $V = 80$ is:

A 60 **B** 4800 **C** 20 **D** 100 **E** $106\frac{2}{3}$

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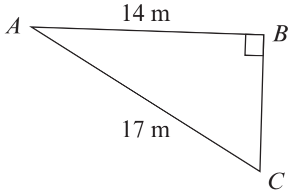
Chapter 10: Measurement, scale and similarity

35 (VCAA-type question) When rounded to 3 significant figures, 3978.6249 is:

A 3.97×10^2 **B** 3.98×10^2 **C** 3978.624
D 3978.625 **E** 3.98×10^3

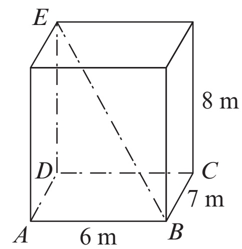
36 (VCAA-type question) A right-angled triangle, *ABC*, has side lengths *AB* = 14 m and *AC* = 17 m, as shown to the right. The length of *BC*, in metres, is closest to:

A 3.0 **B** 9.0 **C** 9.6
D 22.0 **E** 31.0



37 The distance, *BE*, from the corner, *B*, of the cuboid shown, directly to the far corner, *E*, is closest to:

A 9 m **B** 11 m **C** 12 m
D 17 m **E** 21 m

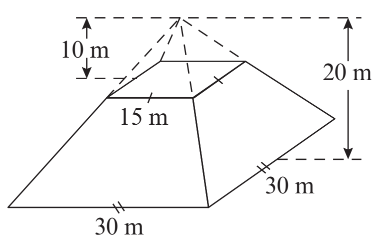


38 (VCAA-type question) The capacity of a petrol can that is 45 cm high with a diameter of 35 cm is closest to:

A 5 litres **B** 43 litres **C** 43 000 cm^3
D 173 litres **E** 173 000 cm^3

39 (VCAA-type question) A pyramid with an original height of 20 m has a small pyramid, 10 m high, removed from its peak, as shown in the diagram . The square base of the original pyramid has a length of 30 m, while the small pyramid has a base with a length of 15 m.
The volume remaining, in m^3, is closest to:

A 2250 **B** 5250 **C** 6000
D 7875 **E** 15 750



40 The volume of a hemisphere with a diameter of 50 cm is closest to:

A 7854 cm^3 **B** 31 416 cm^3 **C** 32 725 cm^3 **D** 65 450 cm^3 **E** 261 799 cm^3

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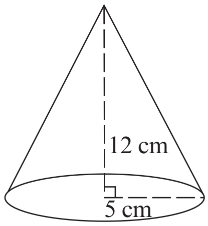
41 (VCAA-type question)

A cone with a base radius of 5 cm and a height of 12 cm is shown in the diagram to the right.

Its surface area, in square centimetres, is closest to:

A 267.0 **B** 282.7 **C** 314.2

D 340.3 **E** 534.1

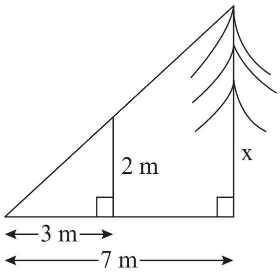


42 (VCAA-type question)

The tree in the diagram to the right cast a shadow that was 7 metres long. A person 2 m tall cast a shadow 3 m long.

The height, x m, of the tree was closest to:

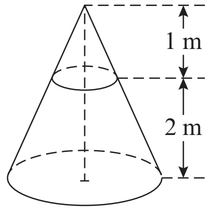
A 4.7 **B** 6.0 **C** 10.5 **D** 14.0 **E** 21.0



43 A cone that was one third the height of a large cone was sliced off the top of the large cone. What fraction of the large cone's volume remained?

A $\frac{8}{27}$ **B** $\frac{2}{3}$ **C** $\frac{19}{27}$

D $\frac{8}{9}$ **E** $\frac{26}{27}$



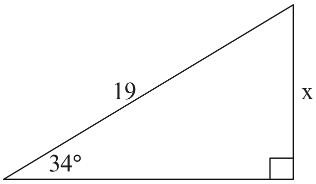
Chapter 11: Applications of trigonometry

Choose an answer, correct to one decimal place, unless otherwise asked.

44 The length of the unknown side, x, is:

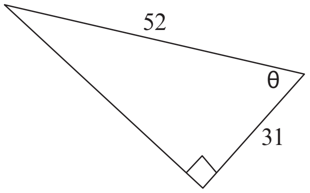
A 10.6 **B** 12.8 **C** 15.8

D 19.6 **E** 34.0



45 The angle, θ, is:

A 30.8° **B** 36.6° **C** 54.3° **D** 53.4° **E** 59.2°



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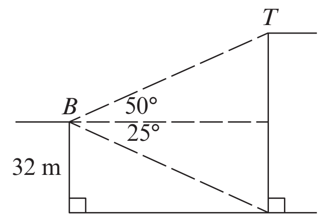
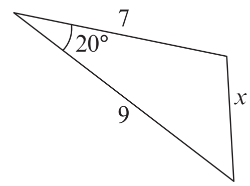
46 From the top of a building, B, that is 32 m tall, the angle of elevation of the top of a nearby building, T, was 50°. The angle of depression of the base of the building was 25°. The height of the nearby building, to the nearest metre, was:



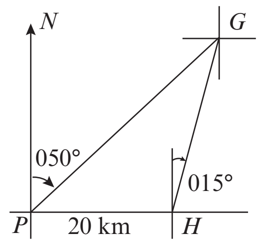
A 77 m **B** 80 m **C** 82 m **D** 101 m **E** 114 m

47 In the triangle shown, the value of x is:



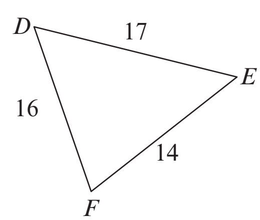
A 3.4 **B** 5.6 **C** 5.7 **D** 11.0 **E** 15.8

48 (VCAA-type question)
From a port, P, a ship, G, was sighted on a bearing of 050°.
A ship, H, 20 km east of port P, reported ship G on a bearing of 015°.
The distance, to the nearest kilometre, between the two ships was:



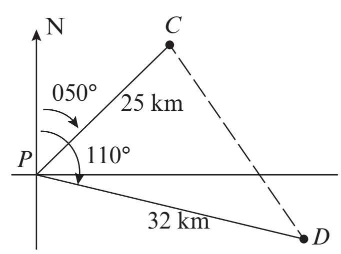
A 13 km **B** 17 km **C** 18 km **D** 22 km **E** 24 km

49 (VCAA-type question) In the triangle shown, angle DEF is:



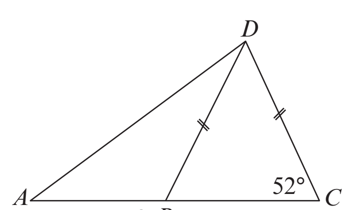
A 34.6° **B** 48.8° **C** 61.0° **D** 61.2° **E** 72.0°

50 (VCAA-type question) Ship C travelled 25 km on a bearing of 050° from port P. Meanwhile, ship D sailed 32 km from port P on a bearing of 110°. The distance, to the nearest kilometre, between the two ships was:



A 20 km **B** 27 km **C** 29 km **D** 31 km **E** 49 km

51 (VCAA-type question) In the given triangle, the angle ABD is:



A 38° **B** 52° **C** 76° **D** 128° **E** 138°

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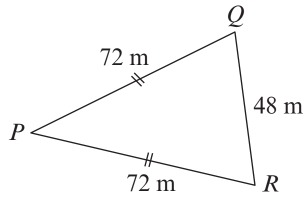
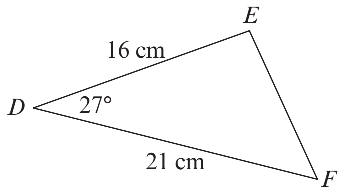
52 (VCAA-type question) The area of the triangle shown is closest to:

A 432 m^2 **B** 748 m^2 **C** 864 m^2
D 1629 m^2 **E** 1728 m^2



53 (VCAA-type question) In triangle DEF, side DE is 16 cm long and side DF is 21 cm. The angle, EDF, is 27°. The area of the triangle is closest to:

A 76 cm^2 **B** 150 cm^2 **C** 153 cm^2
D 168 cm^2 **E** 336 cm^2



12B Written-response questions

Chapter 7: Investigating relationships between two numerical variables

1 The table shows the number of times a student revised in the two-week period before the mathematics exam (*revised*) and their exam score (*exam score*) for a group of Year 11 students.

revised	10	9	9	8	7	7	6	5	4	3	0
exam score	85	85	83	84	82	80	75	60	72	64	60

a Which is the explanatory variable and which is the response variable?
b Construct a scatterplot of this data.
c Determine the value of the Pearson correlation coefficient, r, for this data.
d Describe the relationship between *exam score* and *revised* in terms of direction, form and strength.
e Determine the equation for the least squares regression line, and write it down in terms of the variables *exam score* and *revised*.
f Interpret the slope and intercept of the regression line in the context of the problem.
g Use your equation to predict the exam score for a student who revised 12 times in the two weeks before the exam.
h In making this prediction, are you interpolating or extrapolating?

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2 A teacher is concerned that students who spend a lot of time playing video games do not spend enough time reading. The following table shows the data she collected from a group of 10 of her students, who recorded the number of hours they spent reading (*reading*) and the number of hours they spent playing computer games (*games*) in one week.

reading	10	4	7	8	6	3	4	1	10	8
games	7	15	13	15	8	20	10	21	0	2

a Construct a scatterplot of these data, with *games* as the explanatory variable and *reading* as the response variable.

b Determine the correlation coefficient, r, and give your answer to four decimal places.

c Describe the association between *reading* and *games* in terms of direction, form and strength.

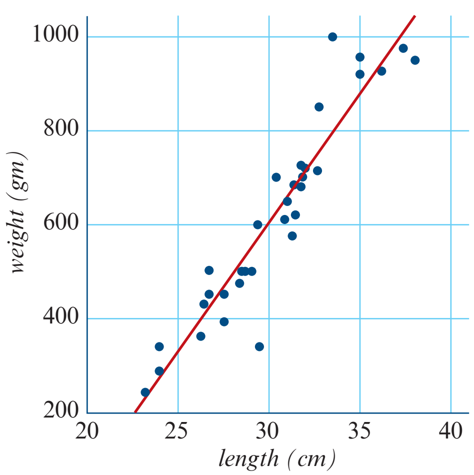
d Determine the equation for the least squares regression line and write it down in terms of the variables *reading* and *games*. Give coefficients to three significant figures.

e Interpret the slope and the intercept (if appropriate) of the regression line.

f Predict the number of hours a student who plays 10 hours of computer games would spend reading.

g How reliable is the prediction made in part **f**.

3 To investigate the association between the weight of a certain species of fish, in grams, (*weight*) and its length, in cm, (*length*), data was collected from a sample of 35 fish. The following scatterplot shows this data with the least squares line added.



a Which is the explanatory variable and which is the response variable?

b The value of the correlation coefficient, r, is 0.937. Describe the relationship between *weight* and *length* in terms of direction, form and strength.

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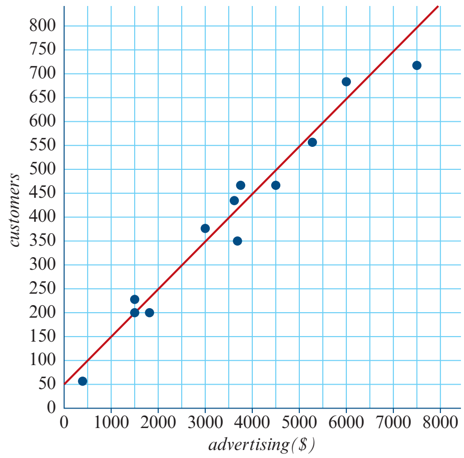
c It was determined from the data that the mean length of the sample of fish is 30.306 cm with a standard deviation of 3.594 cm, and the mean weight of the sample of fish is 617.829 grams with a standard deviation of 209.206 grams. Use this information to determine the equation of the least squares regression line, and write it down in terms of the variables *weight* and *length*. Write the values in the equation to two decimal places.

d Interpret the intercept and slope of the least squares regression line in terms of the variables in the study.

e Use your regression equation to predict the weight of a fish which is 50 cm long. Give your answer to the nearest gram.

f How reliable is the prediction made in part **e**?

4 Data was collected to investigate the association between the number of customers served at a coffee shop each week (*customers*) and the amount of money the shop owner spent in advertising (*advertising*), in dollars, over the previous week. The following scatterplot shows this data, with a line fitted by eye added.



a Which is the explanatory variable and which is the response variable?

b Describe the relationship between *customers* and *advertising* in terms of direction, form and strength.

c Use the scatterplot to determine the equation of the line shown, and write it down in terms of the variables *customers* and *advertising*. Write the values in the equation to two decimal places.

d How many customers would the coffee shop expect if they spend nothing on advertising the previous week?

e How much on average does the coffee shop need to spend on advertising to attract one additional customer?

f The coffee shop owner finds a new advertising option which promised to cost only $2 to attract each additional customer. Write down the equation which would summarise this relationship between *customers* and *advertising*.

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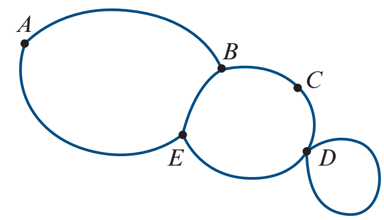
Chapter 8: Graphs and networks

5 The tracks in a park are shown in the diagram opposite.



a What is the degree of vertex D?

b What is the mathematical name for a walk that follows every edge without repeating an edge and returns to the starting point?

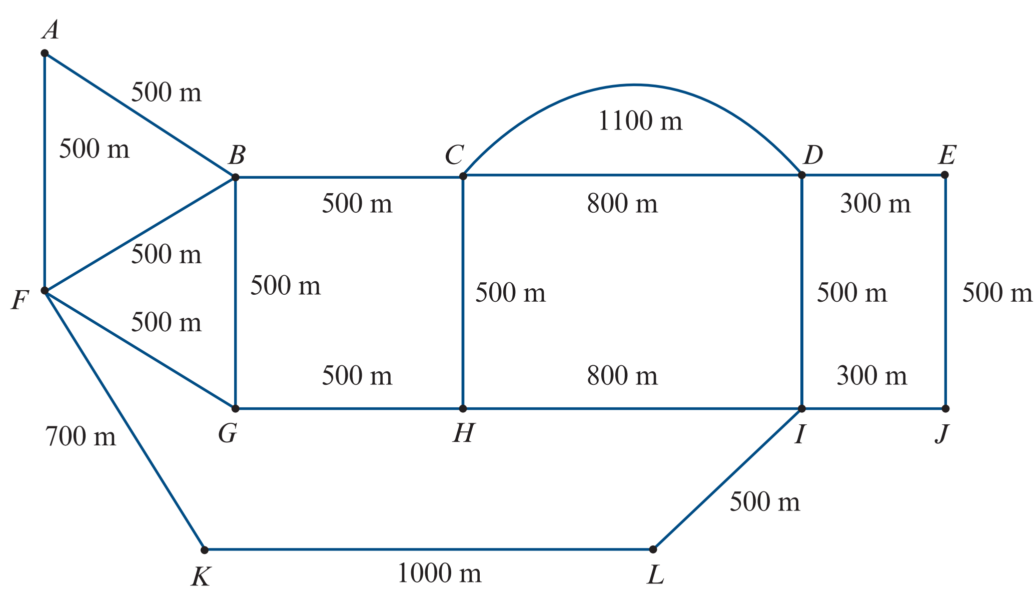
c How do you know the walk described in part **b** is not possible in the park?

d Which section of the track could be repeated to make it possible to return to the starting point and only walking on each of the other tracks once?

e An incomplete adjacency matrix for the network is shown opposite. Copy the adjacency matrix and enter the five missing elements in the spaces provided.

	A	B	C	D	E
A			0	0	1
B		0	1	0	1
C	0	1	0	1	0
D	0	0	1		1
E	1	1	0		0

6 The network below shows the road layout in a new housing estate. The vertices represent the road intersections and the weighted edges represent roads and their lengths.



a Cars can enter the estate through gates located at either A or J. What is the shortest distance a car can travel if it enters the estate at gate A and leaves at gate J?

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b The post office is located at intersection G. The postie must travel along each road in the estate at least once.

i Can the postie start and finish their delivery round (which includes every road in the housing estate) at intersection G, without travelling along the same road more than once? If not, why not?

ii Can the postie start their delivery round at intersection G and finish at some other intersection in this network of roads, without travelling along the same road more than once? If so, where would they end the delivery round?

iii If the postie must start and finish their delivery round at the post office at G, what is the shortest distance they will cover and still travel along each road at least once?

c Broadband is to be provided to the estate by connecting cables from the 'exchange,' located at intersection L, to distribution nodes, located at each of the intersections in the estate. What is the shortest length of cable needed to complete this task?

Chapter 9: Variation

7 A stone falls from rest down a mine shaft. It falls d metres in t seconds, where d varies directly as the square of t. It falls 20 metres in the first 2 seconds.

a Find the rule for d in terms of t.

b If it takes 5 seconds to reach the bottom of the shaft:

i Find the depth of the shaft.

ii Find the time taken to reach halfway down the shaft.

c Sketch the graph of d against t, for t between 0 and 5 inclusive.

8 The effectiveness of a pain-killing drug is being tested by varying the dosage given (d mL) and recording both the time (t min) for the drug to take effect and the time (T min) before the effect of the drug wears off. The following data was recorded.

Dosage (d mL)	10	20	40
Time to take effect (t min)	60	30	15
Time to wear off (T min)	120	480	1920

a Establish the relationship between d and t. (Assume that t is inversely proportional to d.)

b Establish the relationship between d and T. (Assume that T is directly proportional to d^2.)

c A patient is given a dose of 30 mL. How long will it take for the drug to take effect, and how long before it wears off?

d If a second patient takes an amount of the drug which takes effect in 40 minutes, calculate the dosage taken.

e How long would it take before the effect of the dosage in part **d** wears off?

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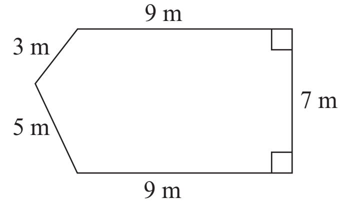
Chapter 10: Measurement, scale and similarity

Answer to one decimal place unless asked otherwise.

9 The diagram shows a plan for a children's sandpit.



a Find the total area of the sandpit.

b The area is to be filled with sand to a depth of 20 cm. Find the volume of sand needed.

c Calculate the cost of the sand if sand costs \$60 per cubic metre.

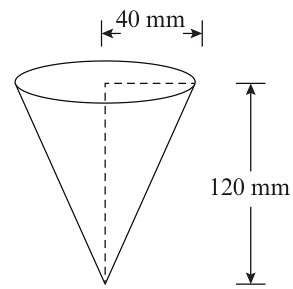
d Find the cost of timber for the border if it costs \$9 per metre.

10 A landscape gardener has a 12 m and a 30 m length of flexible garden border edging. She intends to use the 12 m length for the inner circle of a garden bed border and the 30 m length for the outer circular border.

a What would be the radius of each circle?

b Find the area of the garden bed enclosed by the two circular borders.

11 An inverted cone is used to collect rainwater. The cone is 120 mm high with a base radius of 40 mm.



a If 12 mm of rain fell overnight, what volume of water would have been collected in the rain gauge? [Hint: The volume of water collected depends on the area of the mouth of the cone × rainfall.]

b The gauge was emptied and after the next day's rainfall it was found to be filled to half the depth of the cone. Calculate the volume of water in the rain gauge.

c What was the overnight rainfall?

12 A proposal for a rectangular park measures 8 cm by 10 cm on a map with a scale of 1:5000.

a Find the actual area proposed for the park in hectares. One hectare equals 10 000 m^2.

b What diameter should be used on the map to draw a circular pond with a diameter of 50 metres?

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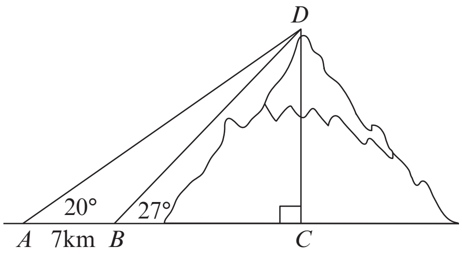
Chapter 11: Applications of trigonometry

Answer to one decimal place unless asked otherwise.

13 Suppose in your attempt to measure the height of Mount Everest from points A and B which are 7 km apart, you measured the angle of elevation of the summit to be 20° and 27°, respectively.



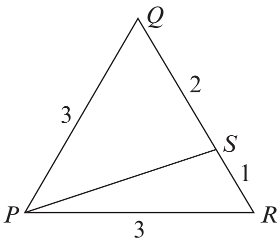
a Find the angle ABD to the nearest degree.

b Determine the angle ADB to the nearest degree.

c Calculate the distance AD, in metres, to two decimal places.

d Hence, determine the height CD to the nearest metre.

14 In equilateral triangle PQR, a line is drawn from P to S as shown.

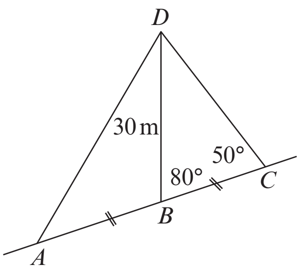


a Find the length PS.

b Find the angle QPS.

c Show how two different rules can be used to find the area of triangle PQS.

15 A vertical transmission tower, BD, is 30 m high and makes an angle of 80° with the sloping hillside. A cable, DC, makes an angle of 50° with the hillside.



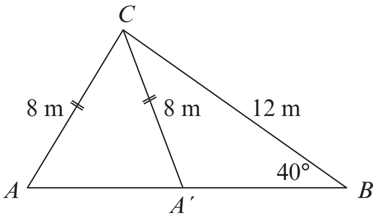
a Find the length of the cable, DC.

b What is the distance, BC, from the base of the tower to the point C where the cable is anchored to the ground?

c Find the length of cable AD, required to secure the cable at point A, where $AB = BC$.

d What acute angle will the cable, AD, make with the hillside?

16 In triangle ABC, we are given $B = 40°$, $b = 8$ m and $a = 12$ m. It is possible to draw two triangles, ABC and $A'BC$, using the given information.



a In triangle ABC, find angle CAB.

b Find angle $CA'B$.

c Give the values of the three angles in triangle ABC.

d State the values of the angles in triangle $A'BC$.

e Find the length of AB in triangle ABC.

f Find the length of $A'B$ in triangle $A'BC$.

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12C Investigations

1 Cricket Captains The table gives details of Australian Test Cricket Captains, based on all tests from 1930 until 2021.

Name	Years	Played	Won	Name	Years	Played	Won
W M Woodfull	1930–34	25	14	W M Lawry	1967–71	25	9
V Y Richardson	1935–36	5	4	B N Jarman	1968	1	0
D G Bradman	1936–48	24	15	I M Chappell	1970–75	30	15
W A Brown	1945–46	1	1	G S Chappell	1975–83	48	21
A L Hassett	1949–53	24	14	G N Yallop	1978–79	7	1
A R Morris	1951–55	2	0	K J Hughes	1978–85	28	4
I W Johnson	1954–57	17	7	A R Border	1984–94	93	32
R R Lindwall	1956–57	1	0	M A Taylor	1994–1999	50	26
I D Craig	1957–58	5	3	S R Waugh	1999–2004	57	41
R Benaud	1958–64	28	12	R Ponting	2004–2010	77	48
R N Harvey	1961	1	1	M Clarke	2010–2014	47	24
R B Simpson	1963–78	39	12	S Smith	2014–2018	34	18
B C Booth	1965–66	2	0	T Paine	2018–2021	23	11

a **i** Construct a stem plot and a boxplot for the number of tests played by the captains.

ii Describe the distribution of the number of tests captained in terms of shape, centre, spread and outliers, quoting the values of appropriate statistics.

iii Who is Australia's longest serving captain on the basis of this data?

b **i** Construct a scatterplot of matches won (*won*) against the matches played (*played*) by each captain. *Played* is the EV. Estimate the value of the correlation coefficient, r.

ii Describe the association between matches played and matches won in terms of strength, direction, form and outliers if any.

iii Determine the correlation coefficient, r, and compare it to your earlier estimate.

iv Fit a least squares regression line to the data and write its equation in terms of the variables *played* and *won*.

v Write down the slope of the least squares regression line and interpret it in terms of the variables *played* and *won*.

vi Based on your analyses, who would you suggest was Australia's most successful cricket captain?

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2 **Global warming** The table gives details of the average maximum yearly temperature (C°) (*maximum*) and the average minimum yearly temperature (C°) (*minimum*) in Melbourne for the years 1995 to 2020.

Year	Av Max Temp	Av Min Temp	Year	Av Max Temp	Av Min Temp
1995	18.5	9.4	2008	20.2	9.6
1996	18.6	9.0	2009	20.9	10.0
1997	20.1	9.6	2010	20.0	10.0
1998	19.7	9.2	2011	19.9	9.8
1999	20.3	9.9	2012	20.2	9.7
2000	20.2	10.0	2013	20.9	10.2
2001	20.1	9.9	2014	21.2	10.4
2002	20.4	9.5	2015	20.9	9.7
2003	20.1	9.7	2016	20.5	10.4
2004	19.6	9.6	2017	21.1	9.8
2005	20.6	9.8	2018	21.1	9.8
2006	20.2	9.2	2019	21.3	9.9
2007	21.3	10.4	2020	19.8	9.5

a **i** Construct parallel boxplots of average minimum and average maximum temperatures.

ii Use the boxplots to write a report comparing distributions in terms of shape, centre and spread.

b **i** Construct a scatterplot using *maximum* as the RV and *minimum* as the EV.

ii Describe the association between *maximum* and *minimum* in terms of direction, form and strength.

iii Determine the value of the correlation coefficent, r, between *maximum* and *minimum* temperatures.

iv Fit a least squares regression line which could be used to predict *maximum* from *minimum*.

v Interpret the intercept and slope of this regression line in terms of *maximum* and *minimum* temperatures.

c **i** Construct a scatterplot of *maximum* against *year*.

ii Describe the association between *maximum* and *year* in terms of direction, form and strength.

iii Fit a least squares regression line which could be used to predict *maximum* temperature from the *year*.

iv Interpret the intercept and slope of this regression line in terms of *maximum* temperature and *year*.

v Use the regression line to predict the average yearly max temp in 2025.

vi Are you interpolating or extrapolating when making this prediction?

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d **i** Construct a scatterplot of *minimum* against *year*.

ii Describe the association between *minimum* and *year* in terms of direction, form and strength.

iii Fit a least squares regression line which could be used to predict *minimum* temperature from the *year*.

iv Interpret the intercept and slope of this regression line in terms of *minimum* temperature and *year*.

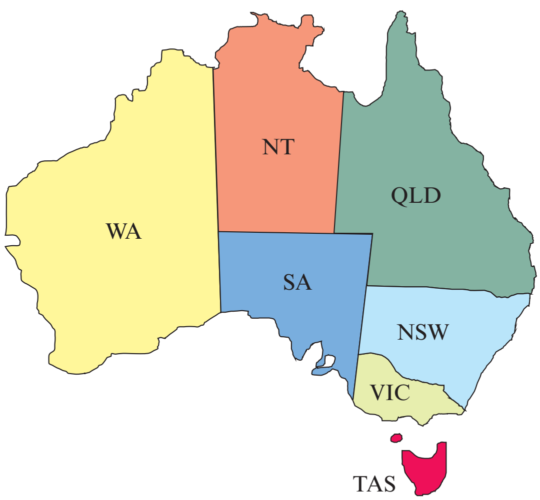
v Use the regression line to predict the average yearly minimum temperature in 2025.

vi Are you interpolating or extrapolating when making this prediction?

e Use the results of parts **c** and **d** to write a paragraph describing how the temperature in Melbourne has changed since 1995. Include a suggestion of other variables which could be included to extend the study.

3 Road Trip

Nick and Maria live in Melbourne and will soon have family visiting them from Greece. They would like to plan a trip around Australia, focusing on the major cities in each of the states and territories.



a On a previous trip, Nick and Maria travelled from Melbourne to Sydney to Brisbane, back to Sydney, across to Canberra and finally returning to Melbourne.

i Using vertices to represent each capital city and an edge for each direct route between a capital city, construct a network to show Nick and Maria's complete route.

ii Show that Euler's formula is verified for this network.

iii What is the sum of degrees of this network?

iv Are there any bridges? If yes, indicate the vertices they exist between. If no, give a reason for why there are none.

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v What type of walk can Nick and Maria's previous trip be described as? Justify your reasoning with reference to the vertices and edges of your network.

vi How many different spanning trees are possible for your network?

The following table includes the distances travelled and flight times for their previous trip.

From	To	Distance (km)	Time
Brisbane	Sydney	750	1 hour 30 minutes
Canberra	Melbourne	469	1 hour 5 minutes
Canberra	Sydney	236	45 minutes
Melbourne	Sydney	705	1 hour 20 minutes

vii Calculate the total distance travelled by Nick and Maria on their previous trip.

viii Calculate the total time Nick and Maria spent flying.

The following adjacency matrix summarises all domestic flights between the capital cities of Australia. A '1' is used to represent a direct flight between the cities and '0' to represent no direct flight. The first letter of each capital city is used to represent each city, e.g. M = Melbourne, A = Adelaide, B = Brisbane etc.

$$\begin{array}{c} \begin{matrix} A & B & C & D & H & M & P & S \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \\ H \\ M \\ P \\ S \end{matrix} \end{array}$$

b **i** Plan three potential journeys Nick and Maria could take their relatives on if they intend to visit all capital cities of Australia, starting and ending their route in Melbourne.

ii Create a network that illustrates all three potential routes.

iii Is your network planar? Justify your reasoning with a calculation.

iv Draw three different spanning trees for this network.

v If they could start their journey in any city, which ones could they start in if they must end in the same city they start? Write the sequence of some potential journeys.

Because Nick and Maria can work remotely, they do not need to end their journey in their home city, Melbourne. They do not intend on travelling to every city anymore and plan to end their journey in Perth.

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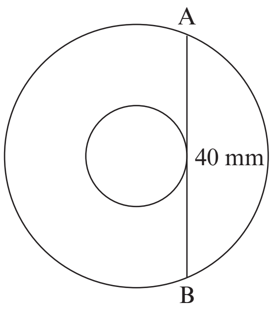
c **i** If this new journey must include three other cities between Melbourne and Perth, suggest three new routes they could take, and create a network that includes all three routes between Melbourne and Perth. Use the matrix above to help identify flight paths between the capital cities.

ii Research the distance or time of flights between cities and calculate the shortest or cheapest journey they could take.

4 Gold Medal You have won a gold medal in the form of an annulus. You can design your medal with any sizes for the inner and outer circle. The only condition is that the chord, AB, that is a tangent to the inner circle, must be exactly 40 mm long.

a Design your own gold annulus. Provide a clearly labelled diagram. Describe the steps you used when designing an annulus that satisfied the required condition, and show all of the calculations involved.



b State the inner radius, r, and the outer radius, R, of the annulus you have designed, showing the calculations used.

c Calculate the area of the gold annulus you have designed.

d Design another annulus that meets the requirement that the chord which is a tangent to the inner circle is 40 mm long, but with different values for the inner and outer radii. Provide a clearly labelled diagram and working.

e Calculate the area of the gold annulus you designed in part **d**.

f Comment on the results of your investigation so far. What conclusion are you inclined to make? Provide further evidence to support or confirm your conclusion.

g What would be the area of an annulus if the chord was c mm long? Does this result confirm the results of your earlier calculations?

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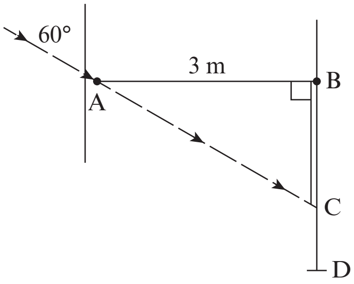
5 **Awning angles** A three-metre awning, AB, is intended to provide some shade, BC, for the clothes in a display window, BD.
Give answers to one decimal place.

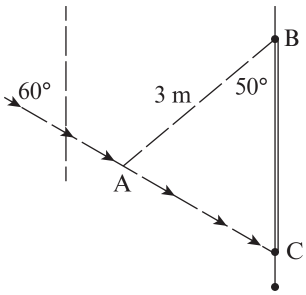
a At the time of interest, the morning sun makes an angle of 60° from the vertical.

i Find the length of the shade, BC.



The awning is hinged at point B.

ii What is the length of shade, BC, when the awning is adjusted so the angle, ABC, is 50°?



b Assume the angle the sun makes with the vertical is 60°. For simplicity, use θ for the angle ABC. Your goal is to find the value of θ that would give the maximum length of shade.
Several methods are possible. Support your answer with clear and logical working.

c What is the rule connecting the angle of the sun with the value of θ for maximum shade?

Revision

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Glossary

A

Adjacency matrix [p. 482] A square matrix that uses a zero or a positive integer to record the number of edges connecting each pair of vertices in the graph.

Ambiguous case (trigonometry) [p. 718] Occurs when it is possible to draw two different triangles that both fit the given information.

Angle of depression [p. 704] The angle between the horizontal and a direction *below* the horizontal.

Angle of elevation [p. 703] The angle between the horizontal and a direction *above* the horizontal.

Arc [p. 633] The part of a circle between two given points on the circle. The length of the arc is given by

$$S = \frac{\pi r \theta}{180}$$

where r is the radius of the circle and θ is the angle subtended by the arc of the centre of the circle.

Area [p. 622] The area of a shape is a measure of the region enclosed by its boundaries, measured in square units.

Area formulas [p. 622] Formulas used to calculate the areas of regular shapes, including rectangles, triangles, parallelograms, trapeziums and circles.

Area of a triangle rules [p. 622]

- $Area = \frac{1}{2} \times base \times height$
- $Area = \frac{1}{2} bc \sin A$
- Heron's rule: $Area = \sqrt{s(s-a)(s-b)(s-c)}$ and $s = \frac{1}{2}(a+b+c)$ where a, b and c are side lengths.

Arithmetic sequence [p. 144] A sequence in which each new term is made by adding a constant amount (D), called the common difference, to the current term. Given the value of the first term in an arithmetic sequence (a) and the common difference, there are rules for finding the nth term.

B

Back-to-back stem plot [p. 113] A back-to-back stem plot is used to compare two related numerical data sets, constructed with a single stem and two sets of leaves (one for each group).

Bar chart [p. 40] A statistical graph used to display the frequency distribution of categorical data, using vertical bars.

Bearing [p. 710] *See* Three-figure bearing.

Bivariate data [p. 401] Data which is recorded on two variables from the same subject.

BODMAS An aid for remembering the order of operations: **B**rackets first; **O**rders (powers, square roots) and fractions **O**f numbers; **D**ivision and **M**ultiplication, working left to right; **A**ddition and **S**ubtraction, working left to right.

Boxplot [p. 100] A plot of numerical data constructed using the 5-number summary, and showing outliers if present.

Bridge [p. 479] A single edge in a connected graph that, if removed, leaves the graph disconnected. A graph can have more than one bridge.

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C

Categorical data [p. 35] Data generated by a categorical variable, and where the data values are the names of categories.

Categorical variable [p. 37] Variables that classify or name a quality or attribute, and which generate data values which are the names of categories. Categorical variables are either nominal or ordinal.

Causation [p. 426] When the change in the value of one variable causes a change in the value of a second variable. A high correlation between two variables cannot be assumed to mean they are causally related.

Circuit [p. 500] A walk with no repeated edges that starts and ends at the same vertex. *See also* Walk.

Circumference [p. 628] The circumference of a circle is the length of its boundary. The circumference, C, of a circle with radius, r, is given by $C = 2\pi r$.

Column matrix [p. 221] A matrix with only one column.

Common difference [p. 144] The fixed amount (D) that is added to make each new term in an arithmetic sequence.

Common ratio [p. 168] The fixed number (R) that multiplies the current term to make the next term in a geometric sequence.

Compound interest [p. 183] Under compound interest, the interest paid on a loan or investment is credited or debited to the account at the end of each period. The interest earned for the next period is based on the principal plus previous interest earned. The amount of interest earned increases each year.

Connected graph [p. 478] A graph is connected if there is a path between each pair of vertices.

Continuous data [p. 36] Numerical data that can take any value, sometimes in an interval, and often arises from measuring.

Correlation coefficient, *r* [p. 420] A statistical measure of the strength of the linear association between two numerical variables.

Cosine ratio (cos θ) [p. 684] In right-angled triangles, the ratio of the side adjacent to a given angle (θ) to the hypotenuse.

Cosine rule [p. 726] In non-right-angled triangles, a rule used to find:

- the third side, given two sides and an included angle
- an angle, given three sides.

For triangle ABC and side a, the rule is:

$$a^2 = b^2 + c^2 - 2bc\ \cos A$$

or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Similar rules exist for sides b and c.

Cycle [p. 500] A walk with no repeated vertices that starts and ends at the same vertex. *See also* Walk.

D

Data [p. 35] Information collected about a variable.

Degree of a vertex (deg(*A*)) [p. 469] The number of edges that are attached to the vertex. The degree of a loop is two. The degree of vertex A is written as deg(A).

Depreciation [p. 159] Depreciation occurs when the value of an asset decreases over time.

Direction of association [p. 410] The direction of an association is positive if the values of the response variable tend to increase as the values of the explanatory variable increase, and negative if the values of the response variable tend to decrease as the values of the explanatory variable increase.

Direct variation [p. 553] Exists between any two variables when one quantity is directly dependent on the other. An increase in one quantity leads to an increase in the other whilst a decrease in one quantity leads to a decrease in the other quantity. The ratio of the two quantities remains the same, i.e. = k.

Discrete data [p. 36] Data which only take particular numerical values, usually whole numbers, and often arises from counting.

Discrete variable [p. 37] A variable which generates discrete numerical data, such as the number of people in a queue.

Distribution [p. 47] The pattern in a set of data values. It reflects how frequently different data values occur.

Dot plot [p. 69] A plot used for small sets of numerical data, where individual data values are plotted as dots on a number line.

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E

Edge [p. 468] A line joining one vertex in a graph or network to another vertex or to itself (a loop).

Elements of a matrix [p. 220] The numbers or symbols displayed in a matrix.

Eulerian circuit [p. 504] An Eulerian trail that starts and finishes at the same vertex. To have an Eulerian circuit, the graph must have all even vertices.

Eulerian trail [p. 504] A trail in a connected graph that includes every edge. To have an Eulerian trail (but not an Eulerian circuit), the graph must have *exactly two* odd vertices. The trail starts at one odd vertex and finishes at the other.

Euler's formula [p. 490] The formula $v - e + f = 2$, which relates the number of vertices (v), edges (e) and faces (f) in a connected planar graph.

Explanatory variable [p. 401] When investigating associations in bivariate data, the explanatory variable (EV) is the variable used to explain or predict the value of the response variable (RV).

Extrapolation [p. 449] Using an equation to predict the value of the response variable using a value of the explanatory variable which is outside the range of the values used to determine that equation.

F

Face [p. 489] An area in a graph or network that can only be reached by crossing an edge. One face is always the infinite area surrounding a graph.

Fitting a line by eye [p. 433] A line drawn on a scatterplot with a ruler that aims to capture the linear trend of the data points.

Five-number summary [p. 100] A list of the five key points in a data distribution: the minimum value (Min), the first quartile (Q_1), the median (M), the third quartile (Q_3) and the maximum value (Max).

Flat-rate depreciation [p. 159] Flat-rate depreciation is an example of linear decay where a constant amount that is subtracted from the value of an item at regular intervals of time.

Form of association [p. 412] The description of the association between variables as linear or non-linear.

Formula [p. 300] A mathematical relation connecting two or more variables, for example, $C = 5t + 20, P = 2L + 2W, A = \pi r^2$.

Frequency [p. 39] The number of times a value or a group of values occurs in a data set. Sometimes known as the count.

Frequency table [p. 39] A listing of the values that a variable takes in a data set along with how often (frequently) each value occurs. Frequency can also be recorded as a percentage.

G

Geometric sequence [p. 168] A sequence in which each new term is made by multiplying the current term by a constant called the common ratio (R).

Gradient of a straight line [p. 323] *See* Slope of a straight line.

Graph [p. 468] In a specific mathematical sense, as opposed to its common usage, a graph is a diagram that consists of a set of points called vertices that are joined by a set of lines called edges. Each edge joins two vertices.

Greedy algorithm [p. 528] A greedy algorithm is a simple, intuitive set of rules that can be used to solve optimisation problems.

Grouped data [p. 52] Where there are many different values of a numerical variable, the data is generally grouped into intervals such as 0-9, 10-19,...

GST [p. 16] GST (goods and services tax) is a tax, currently at the rate of 10%, that is added to most purchases of goods and services.

H

Hamiltonian cycle [p. 510] A Hamiltonian path that starts and finishes at the same vertex.

Hamiltonian path [p. 509] A path in a connected graph that passes through every vertex in the graph once only. It may or may not start and finish at the same vertex.

Heron's rule (Heron's formula) [p. 622] A rule for calculating the area of a triangle from its three sides.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ and is called the semi-perimeter.

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Histogram [p. 56] A statistical graph used to display the frequency distribution of a numerical variable; suitable for medium-sized to large-sized data sets.

Hypotenuse [p. 612] The longest side in a right-angled triangle.

I

Identity matrix (I) [p. 245] A matrix that behaves like the number one in arithmetic, represented by the symbol I. Any matrix multiplied by an identity matrix remains unchanged.

Inequality [p. 78] A mathematical relation involving the use of $<, \leq, >, \geq$ or $\neq$, for example, $2x \leq 7,\ 3x + 5y > 9,\ -1 \leq x < 2$.

Inflation [p. 199] The tendency of prices to increase with time, resulting in the loss of purchasing power.

Intercept–slope form of the equation of a straight line [p. 330] A linear equation written in the form, $y = a + bx$, where a and b are constants. In this equation, a represents the y-intercept and b represents the slope. For example, $y = 5 - 2x$ is the equation of a straight line with y-intercept of 5 and the slope of -2.

Interest [p. 157] An amount of money paid (earned) for borrowing (lending) money over a period of time. It may be calculated on a simple or compound basis.

Interest rate [p. 157] The rate at which interest is charged or paid. It is usually expressed as a percentage of the money owed or lent.

Interpolation [p. 448] Using an equation to predict the value of the response variable using a value of the explanatory variable which is within the range of the values used to determine that equation.

Interquartile range (IQR) [p. 86] A summary statistic that measures the spread of the middle 50% of values in a data distribution. It is defined as $\text{IQR} = Q_3 - Q_1$.

Inverse matrix (A^{-1}) [p. 250] A matrix that, when multiplied by the original matrix, gives the identity matrix (I). For a matrix A, the inverse matrix is written as A^{-1} and has the property that $A^{-1}A = AA^{-1} = I$.

Inverse trigonometric ratios ($\sin^{-1}, \cos^{-1}, \tan^{-1}$) [p. 693] Used to find an angle θ when given the value of sin θ, cos θ, tan θ.

Inverse variation [p. 560] If one variable increases, the other will decrease, whilst if one variable decreases, the other increases. The product of the corresponding values of the variables is constant, (i.e. the constant of proportionality).

Isomorphic graphs [p. 480] Equivalent graphs or networks; graphs that have the same number of edges and vertices, identically connected.

K

Kruskal's algorithm [p. 526] An algorithm (procedure) for determining a minimum spanning tree in a connected graph or network.

L

Least squares method [p. 435] A method of fitting a line to a scatterplot, based on minimising the sum of the squares of the residuals.

Linear decay [p. 163] When a recurrence rule involves subtracting a fixed amount, the terms in the resulting sequence are said to decay linearly.

Linear equation [p. 308] An equation that has a straight line as its graph. In linear equations, the unknown values are always to the power of 1, for example, $y = 2x - 3, y + 3 = 7, 3x = 8$.

Linear growth [p. 163] When a recurrence rule involves adding a fixed amount, the terms in the resulting sequence are said to grow linearly.

Linearisation [p. 566] The process of transforming data to linearity. The squared, reciprocal and logarithmic transformations may be used.

Linear regression [p. 433] The process of fitting a straight line to bivariate data.

Line of good fit [p. 433] A line used to approximately model the linear relationship between two variables. It is needed when the data values do not lie exactly on a straight line. Also known as a regression line.

Logarithm ($\log_{10} x$) [p. 578] A way of writing numbers as a power of ten. For example, if $x = 100 = 10^2$, then $\log_{10} x = 2$.

Loop [p. 469] An edge in a graph or network that joins a vertex to itself.

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M

Matrix [p. 219] *A* rectangular array of numbers or symbols set out in rows and columns within square brackets. (Plural – matrices.)

Matrix multiplication [p. 239] The process of multiplying a matrix by a matrix.

Maximum (Max) [p. 67] The largest value in a set of numerical data.

Mean ($\bar{x}$) [p. 77] A summary statistic that can be used to locate the centre of a symmetric distribution. The mean is given by $\bar{x} = \frac{\Sigma x}{n}$, where Σx is the sum of the data values and n is the number of data values.

Median (*M*) [p. 78] The midpoint of an ordered data set that has been divided into two equal parts, each with 50% of the data values. It is equal to the middle value (for an odd number of data values) or the average of the two middle values (for an even number of data values). It is a measure of the centre of the distribution.

Minimum (Min) [p. 67] The smallest value in a set of numerical data.

Minimum spanning tree [p. 522] The spanning tree of minimum length in a connected weighted graph or network. A graph may have more than one.

Modal category or interval [p. 42] The category or data interval that occurs most frequently in a data set.

Mode [p. 42] The most frequently occurring value in a data set. There may be more than one.

Multiple edges [p. 469] Two or more edges that connect the same two vertices in a graph or network.

N

Negative association (bivariate data) [p. 410] An association where the values of the explanatory variable tend to decrease as the values of the response variable increase.

Negatively skewed distribution [p. 68] A data distribution that has a long tail to the left. In negatively skewed distributions, the majority of data values fall to the right of the mean.

Negative slope [p. 323] A straight-line graph with a negative slope represents a decreasing y-value as the x-value increases. For the graph of a straight line with a negative slope, y decreases at a constant rate with respect to x.

Network [p. 514] A set of points called vertices and connecting lines called edges, enclosing and surrounded by areas called faces.

No association [p. 68] A state of no consistent change in the value of the response variable when the values of the explanatory variable change.

Nominal data [p. 35] A type of categorical data where data values are the names of groups.

Nominal variable [p. 37] A type of categorical variable where the values of the variable are the names of groups.

Non-linear equation [p. 588] An equation with a graph that is *not* a straight line. In non-linear equations, the unknown values are not all to the power of 1, for example, $y = x^2 + 5, 3y^2 = 6, b^3 = 27$.

Numerical data [p. 36] Data obtained by measuring or counting some quantity. Numerical data can be discrete (for example, the *number* of people waiting in a queue) or continuous (for example, the *amount of time* people spent waiting in a queue).

O

Order of a matrix [p. 219] An indication of the size and shape of a matrix, written as $m \times n$, where m is the number of rows and n is the number of columns.

Order of magnitude [p. 580] The quantity of powers of 10 in a number.

Ordinal data [p. 36] A type of categorical data where the values of the variable are the names of groups, and there is an inherent order in the categories.

Ordinal variable [p. 37] A type of categorical variable where when the values of the variable are the names of groups, and there is an inherent order in the categories.

Outliers [p. 102] Data values that appear to stand out from the main body of a data set. Using box plots, possible outliers are defined as data values greater than $Q_3 + 1.5 \times \text{IQR}$ or less than $Q_1 - 1.5 \times \text{IQR}$.

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P

Parallel boxplot [p. 114] A statistical graph in which two or more boxplots are drawn side by side. Used to compare distributions in terms of shape, centre and spread.

Path [p. 498] A walk with no repeated vertices or edges. *See also* Walk.

Pearson's correlation coefficient [p. 420] *See* Correlation coefficient.

Percentage [p. 2] The number as a proportion of one hundred, indicated by the symbol %. For example, 12% means 12 per one hundred.

Percentage change [p. 8] The amount of the increase or decrease of a quantity expressed as a percentage of the original value.

Percentage frequency [p. 39] Frequency of a value or group of values, expressed as a percentage of the total frequency.

Perimeter [pp. 300, 621] The distance around the edge of a two-dimensional shape.

Piecewise linear graph [p. 361] A graph made up of two or more parts of different straight-line graphs, each representing different intervals on the x-axis. Sometimes called a segmented linear graph.

Planar graph [p. 488] A graph or network that can be drawn in such a way that no two edges intersect, except at the vertices.

Positive association (bivariate data) [p. 410] An association where the values of the explanatory variable and the response variable tend to increase together.

Positively skewed distribution [p. 68] A data distribution that has a long tail to the right. In positively skewed distributions, the majority of data values fall to the left of the mean.

Positive slope [p. 323] A positive slope represents an increasing y-value with increase in x-value. For the graph of a straight line with a positive slope, y increases at a constant rate with respect to x.

Prim's algorithm [p. 523] An algorithm (procedure) for determining a minimum spanning tree in a connected graph or network.

Principal (P) [p. 157] The initial amount of money borrowed, lent or invested.

Pronumeral [p. 244] A symbol (usually a letter) that stands for a numerical quantity or variable.

Pythagoras' theorem [p. 612] A rule for calculating the third side of a right-angled triangle given the length of the other two sides. In triangle ABC, the rule is: $a^2 = b^2 + c^2$, where a is the length of the hypotenuse.

Q

Quartiles (Q_1, Q_2, Q_3) [p. 86] Summary statistics that divide an ordered data set into four equal-sized groups, each containing 25% of the scores.

R

Radius [p. 628] The distance from the centre of a circle (or sphere) to any point on its circumference (or surface); equal to half the diameter.

Range (R) [p. 85] The difference between the smallest (minimum) and the largest (maximum) values in a data set: a measure of spread.

Recurrence relation [p. 141] A rule that enables the next term in a sequence to be generated using one or more previous terms. For example, 'Starting with 3, each new term is made by adding 5 to the current term'. Written as: $t_0 = 3$ and $t_{n+1} = t_n + 5$. Gives: 3, 8, 13, 18, . . .

Regression line [p. 433] *See* Line of good fit.

Response variable [p. 401] When investigating associations (relationships) between two variables, the response variable (RV) is the variable we are trying to explain or predict, using values of the explanatory variable.

Right angle [p. 612] An angle equal to 90°.

Rise [p. 323] *See* Slope of a straight line.

Row matrix [p. 221] A matrix with only one row.

Run [p. 323] *See* Slope of a straight line.

S

(s) [p. 622] *See* Heron's rule.

Scalar multiplication [p. 231] Multiplying a matrix by a number.

Scale factor (areas) [p. 657] The scale factor, k^2, by which the area of a two-dimensional shape is scaled (increased or decreased) when its linear dimensions are scaled by a factor of k.

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Scale factor (lengths) [p. 657] The scale factor, k, by which lengths are scaled (increased or decreased) to find corresponding lengths in a similar shape.

Scale factor (volumes) [p. 666] The scale factor, k^3, by which the volume of a solid shape is scaled (increased or decreased) when its linear dimensions are scaled by a factor of k.

Scatterplot [p. 403] A statistical graph used for displaying bivariate data. Data pairs are represented by points on a coordinate plane, with the *EV* plotted on the horizontal axis and the *RV* plotted on the vertical axis.

Scientific notation (standard form) [p. 606] Numbers are expressed as a value between 1 and 10 multiplied by a power of 10. Namely as, a *times* 10^n, where $1 \leq a < 10$ and n is an integer. For example, 23.45180 written in scientific notation is 2.345180×10^1.

Sector [p. 633] The area bounded by two radii and an arc of a circle. The area, A, of a sector of a circle with a radius r, where the arc of the sector subtends an angle of θ at the centre, is given by

$$A = \frac{\pi r^2 \theta}{360}$$

Sequence [p. 133] A list of numbers or symbols written down in succession, for example, 5, 15, 25.

Shortest path [p. 516] The path through a graph or network with minimum length.

Significant figures [p. 608] Are used as an expression of the accuracy claimed for a measurement. Write the number in scientific notation, then round to the required number of significant figures.

Similar figures [p. 656] Figures that have exactly the same shape but differ in size.

Similar triangles [p. 662] Different sized triangles in which the corresponding angles are equal. The ratios of the corresponding pairs of sides are always the same.

Simple interest [p. 157] Interest that is calculated for an agreed period and paid only on the original amount invested or borrowed. Also called flat-rate interest.

Simultaneous linear equations [p. 350] Two or more linear equations in two or more variables, for values that are common solutions to all equations. For example, $3x - y = 7$ and $x + y = 5$ are a pair of simultaneous linear equations in x and y, with the common solution $x = 3$ and $y = 2$.

Sine ratio (sin θ) [p. 684] In right-angled triangles, the ratio of the side opposite a given angle (θ) to the hypotenuse.

Sine rule [p. 715] In non-right-angled triangles, a rule used to find:

- an unknown side, given the angle opposite and another side and its opposite angle
- an unknown angle, given the side opposite and another side and its opposite angle.

For triangle ABC the rule is:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Skewness [p. 68] Lack of symmetry in a data distribution. It may be positive or negative.

Slope of a straight line [p. 323] The ratio of the increase in the dependent variable (y) to the increase in the independent variable (x) in a linear equation. Also known as the gradient.

$$\text{slope} = \frac{\text{rise}}{\text{run}}.$$

SOH–CAH –TOA [p. 684] A memory aid for remembering the trigonometric ratio rules.

Solution [p. 24] A value that can replace a variable and make an equation or inequality true.

Spanning tree [p. 521] A subgraph of a connected graph or network that contains all the vertices of the original graph, but without any multiple edges, circuits or loops.

Spread of a distribution [p. 85] A measure of the degree to which data values are clustered around some central point in the distribution. Measures of spread include standard deviation (s), interquartile range (IQR) and range (R).

Square matrix [p. 221] A matrix with the same number of rows as columns.

Standard deviation (*s*) [p. 88] A summary statistic that measures the spread of the data values around the mean. It is given by:

$$s = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}}.$$

Stem plot (stem-and-leaf plot) [p. 70] A display of a numerical data set, formed by splitting the actual data values into two parts, a stem and a leaf; suitable for small to medium-sized data sets.

Strength of an association (relationship) [p. 414] The degree of association between two variables, classified as weak, moderate or strong. It is determined by observing the degree of scatter in a scatterplot or calculating a correlation coefficient.

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Summary statistics [p. 77] Statistics that give numerical values to special features of a data distribution, such as centre and spread. Summary statistics include the mean, median, range, standard deviation and IQR.

Surface area [p. 650] The total of the areas of each of the surfaces of a solid.

Symmetric distribution [p. 67] A data distribution in which the data values are evenly distributed around the mean. In a symmetric distribution, the mean and the median are equal.

T

Tangent ratio (tan θ) [p. 684] In right-angled triangles, the ratio of the side opposite a given angle θ to the side that is adjacent to the angle.

Term [p. 133] One value in a sequence or series; or one value in an algebraic expression.

Three-figure bearing [p. 710] An angular direction, measured clockwise from north and written with three digits, for example, 060°, 324°. Also called a true bearing.

Total surface area (TSA) [p. 650] The total surface area (TSA) of a solid is the sum of the surface areas of all of its faces.

Trail [p. 498] A walk with no repeated edges. *See also* Walk.

Tree [p. 520] A connected graph with no circuits, multiple edges or loops.

Trigonometric ratios [p. 684] In right-angled triangles, the ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \ \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}},$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

True bearing [p. 710] *See* Three-figure bearing.

U

Undefined [p. 325] Has no meaning; has no value. The slope, or gradient, of a vertical line is undefined because $\frac{rise}{run}$ gives a zero denominator.

Unit-cost depreciation [p. 161] Unit-cost depreciation is an example of linear decay that is calculated based on units of use rather than by time. The value of the asset declines by a constant amount for each unit of use (e.g. per 100 kilometres).

V

Variable [p. 300] A quantity that can have many different values in a given situation. Symbols such as x, y and z are commonly used to represent variables.

Variation [p. 553] A relationship between two or more variables.

Vertex [p. 468] The points in a graph or network. (Plural – vertices).

Volume [p. 638] The volume of a solid is a measure of the amount of space enclosed within it, measured in cubic units.

Volume formulas [p. 638] Formulas used to calculate the volumes of solids, including cubes, cuboids, prisms, pyramids, cylinders, cones and spheres.

W

Walk [p. 497] A sequence of edges, linking successive vertices, that connects two different vertices in a graph.

Weighted graph [p. 514] A graph in which a number is associated with each edge. These numbers are called weights. When the numbers represent the size of some quantity (such as distance or time), a weighted graph is often called a network.

Y

***y*-intercept** [p. 329] The point at which a graph cuts the y-axis.

Z

Zero matrix (*O*) [p. 228] A matrix that behaves like zero in arithmetic, represented by the symbol O. Any matrix with zeros in every position is a zero matrix.

Zero slope [p. 325] A horizontal line has zero slope. The equation of this line has the form $y = c$ where c is any constant.

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Answers

Chapter 1

Section 1A

Now try this

1 20% **2** 25% **3** $\frac{39}{50}$ **4** 0.45
5 $27 **6** 40% **7** 3%

Exercise 1A

1 a $\frac{17}{100}$ **b** $\frac{94}{100}$ **c** $\frac{71}{100}$
2 a $\frac{7}{100}$ **b** $\frac{13}{100}$ **c** $\frac{1}{2}$
d $\frac{1}{10}$ **e** $\frac{1}{5}$
3 a 11% **b** 23% **c** 79%
4 a 50% **b** 40% **c** 25%
5 a 78% **b** 37% **c** 56.1%
6 a 25% **b** 80% **c** 15% **d** 70%
e 19% **f** 79% **g** 215% **h** 3957%
i 7.3% **j** 100%
7 a i $\frac{1}{4}$ **ii** 0.25
b i $\frac{1}{2}$ **ii** 0.5
c i $\frac{3}{4}$ **ii** 0.75
d i $\frac{17}{25}$ **ii** 0.68
e i $\frac{23}{400}$ **ii** 0.0575
f i $\frac{34}{125}$ **ii** 0.272
g i $\frac{9}{2000}$ **ii** 0.0045
h i $\frac{3}{10000}$ **ii** 0.0003
i i $\frac{13}{200000}$ **ii** 0.000 065
j i 1 **ii** 1
8 a $114 **b** $110 **c** 25.5 m **d** $1350
e 1.59 cm **f** 2.64 **g** 0.161 **h** $4570
i $77 700 **j** $19 800
9 80% **10** 37.5%
11 95.6% **12** 83.33%
13 20% **14** 37.5%
15 65.08% **16** 150%
17 168 **18** 56 **19** 50

Section 1B

Now try this

8 $1045 **9** 11.04 km **10** $120
11 19.18% **12** 12%

Exercise 1B

1 a $38 **b** $152
2 a $3 **b** $33
3 a $1.25 **b** $23.75
4 a $37 **b** $148
5 a $4.50; $85.49 **b** $18.90; $170.10
c $74.85; $424.15 **d** $49.80; $199.20
e $17.99; $61.96 **f** $5.74; $17.21
g $165; $435 **h** $19.05; $44.45
i $330; $670
6 a $1050 **b** $1225 **c** $215.25
d $656.25
7 a $12.95 **b** $202.95
8 14 840 **9** 21.95%
10 26 880 km
11 a 13% **b** 26% **c** 6%
d 24% **e** 18% **f** 23%
12 7.08%

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13 **a** 19% **b** 33% **c** 45% **d** 20% **e** 33% **f** 16%

14 **a** 25% **b** 40% **c** 7.5%

15 Decreasing $60 by 8%

16 3.23%

Section 1C

Now try this

13 $85 **14** $1540
15 $670 **16** $105

Exercise 1C

1 **a** $90 **b** $76 **c** $59.90 **d** $6.50 **e** $257.20 **f** $4875.50

2 **a** $12.13 **b** $36.75 **c** $108.55 **d** $39.50

3 **a** $152.90 **b** $2945.80 **c** $10 835 **d** $1534.50

4 $2180.91

5 $3635.45

6 **a** $99.80 **b** $9.98

7 **a** $5209.09 **b** $52 090.91

8 $145.36

9 Buy the second mower to save $0.50.

Section 1D

Now try this

18 8 : 11 : 1 **19** 2 : 7
20 13 : 20 **21** 32

Exercise 1D

1 **a** 17 : 9 **b** 9 : 17 **c** 17 : 26

2 **a** 1 : 3 **b** 2 : 5 **c** 16 : 25

3 35 : 15

4 **a** 80 : 40 **b** 70 : 9 **c** 80 : 120 **d** 40 : 4 **e** 40 : 4 : 80

5 **a** 4 : 5 **b** 2 : 9 **c** 2 : 5 : 3 **d** 1 : 3 **e** 3 : 1 **f** 20 : 3 **g** 9 : 4

6 **a** 12 : 5 **b** 1 : 20 **c** 3 : 8 **d** 25 : 3 **e** 3 : 100 : 600 **f** 100 000 : 100 : 1 **g** 4 : 65 **h** 50 : 10 : 2 : 1

7 **a** 5 **b** 72 **c** 120 **d** 5000 **e** 24

8 **a** False **b** False 3 : 4 = 15 : 20 **c** True **d** False 60 : 12 = 15 : 3 = 5 : 1 **e** False The girl would be 8. **f** True

9 **a** 100 : 60 : 175 : 125 : 125
b 20 : 12 : 35 : 25 : 25
c 300 g rolled oats, 180 g coconut, 525 g flour, 375 g brown sugar, 375 g butter, 9 tbsp water, 6 tbsp golden syrup, 3 tsp bicarb soda

Section 1E

Now try this

22 $6.00, $12.00 and $18.00

Exercise 1E

1 **a** 48,48 **b** 64,32 **c** 60,36 **d** 76, 20

2 **a** 32 m and 8 m **b** 5 m and 35 m **c** 30 m and 10 m **d** 20 m and 20 m

3 **a** $300 and $200 **b** $50, $200 and $250 **c** $50, $400 and $50 **d** $160, $180 and $160 **e** $250, $125, $100 and $25

4 **a** 50 bananas **b** 5 mangos **c** 75 pieces of fruit

5 **a** 1.5 litres **b** 6 litres

6 3 kilometres

7 $1 440 000, $1 080 000, $1 080 000

Section 1F

Now try this

23 $60

Exercise 1F

1 **a** **i** $3 **ii** $33 **iii** 6
b **i** 7 **ii** 42 **iii** 56

2 **a** $15.60 **b** 84 seconds **c** $885 **d** 10 kilometres

3 7 red, 28 yellow

4 **a** 270 km **b** 225 km **c** 30 km **d** 105 km **e** 330 km **f** 67.5 km

5 73 g cone for $2

6 Brand A

7 51 eggs

8 **a** 550 kilometres **b** 17 litres

9 $300

10 250 Australian dollars

Chapter 1 Review

Skills Checklist answers

1 75% **2** 67% **3** $\frac{2}{5}$ **4** 0.85
5 $15 **6** $120 **7** 56% **8** $6000
9 $998.18 **10** $145 **11** 10 : 20 **12** 2 : 3
13 64, 192, 768 **14** $500

Multiple-choice questions

1 D **2** C **3** A **4** E **5** C
6 B **7** A

Short-answer questions

1 **a** 0.75 **b** 0.4 **c** 0.275

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2 a $\frac{1}{10}$ **b** $\frac{1}{5}$ **c** $\frac{11}{50}$

3 a 24 **b** \$10.50 **c** \$13.25

4 a \$51.90 **b** \$986.10

5 \$862.50

6 20%

7 46%

8 a \$27.90 **b** \$44.50

9 a \$18 **b** \$190

10 a False **b** False

11 a \$320 and \$480 **b** \$160 and \$640
c \$160, \$240 and \$400
d \$200, \$200 and \$400

12 18 cups

13 a 27 m **b** 140 m

14 \$20

15 301 km

Chapter 2

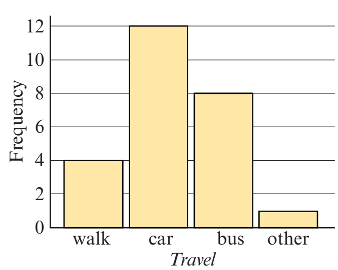
Section 2A

Now try this

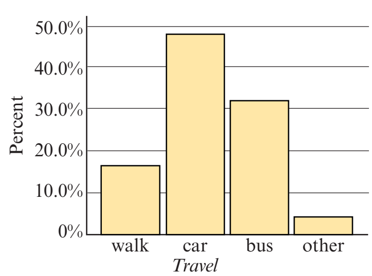
5

	Frequency	
Travel	Number	%
walk	4	16.0
car	12	48.0
bus	8	32.0
other	1	4.0
Total	25	100.0

6 a



b



Exercise 2A

1 a Categorical **b** Numerical
c Categorical **d** Numerical
e Categorical

2 a Nominal **b** Ordinal
c Ordinal **d** Nominal

3 a Discrete **b** Discrete
c Continuous **d** Continuous
e Discrete

4 a Nominal **b** Ordinal
c Numerical(discrete) **d** Ordinal

5 a Nominal

b

	Frequency	
Sex	Number	%
female	5	33.3
male	10	66.7
Total	15	100.0

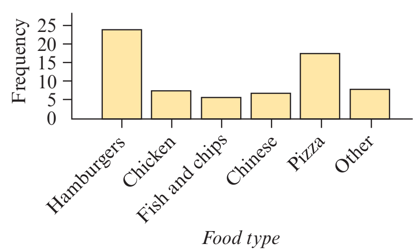
6 a Ordinal

b

	Frequency	
Shoe size	Number	%
7	3	15
8	7	35
9	4	20
10	3	15
11	2	10
12	1	5
Total	20	100

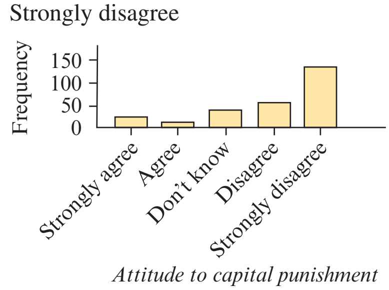
7 a 69; 8.7%, 26.1% **b** Nominal
c 7 students **d** 10.1%
e Hamburger

f



8 a 53; 16.4%, 20.7% **b** Ordinal
c 21 people **d** 50.4%
e Strongly disagree

f



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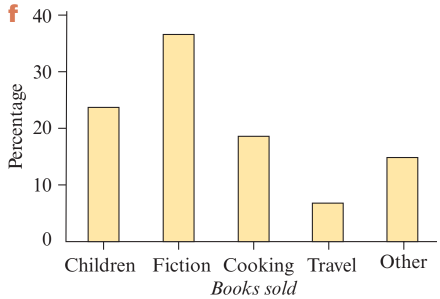
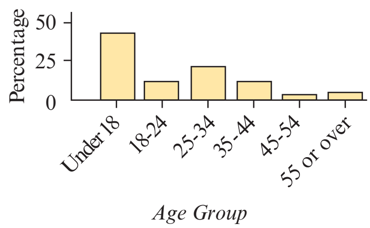
9 **a** 38.4%, 6.5%, 100.0% **b** Nominal
c 89 books **d** 22.8%
e 232 books
f



10 **a** 200 club members **b** Ordinal
c 12% **d** Under 18
e



11 **a** *level of activity*, ordinal **b** 80
c 18 **d** moderate, 68.75%

12 **a** Ordinal **b** 80 employees

Section 2B

Now try this

7 A group of 25 children were asked to choose and activity from painting, story time or playdough. The most popular activity was playdough, chosen by 14 of the children. The next most popular was painting, chosen by 8 children. The least popular activity was story time, chosen by only 3 of the children.

8 A sample of 66 people were asked to respond to the question "How much of the time have you felt happy in the last week?". The response chosen most often was "most of the time", chosen by 47.0% of the sample, followed by "almost all of the time" chosen by 27.3%, and then "some of the time" chosen by 22.7%. Very few people (only 3.0%) responded that they felt happy "none of the time".

Exercise 2B

1 69, hamburgers, 26.1%

2 Strongly disagreed, 20.7%, 16.4%

3 A group of 200 students were asked how they prefer to spend their leisure time. The most popular response was using the internet and digital games (42%), followed by listening to music (23%), reading (13%) and watching TV or going to a movie (12%) and phoning friends (4%). The remaining 6% said 'other'. Internet and digital games for this group of students was clearly the most popular leisure time activity.

4 A group of 600 employees from a large company were asked about the importance of salary in how they felt about the job. The majority of employees said that it was important (55%), or very important (30%). Only a small number of employees said that it was somewhat important (10%) with even fewer saying that it was not at all important (5%). Salary was clearly important to almost all of the employees in this company.

Section 2C

Now try this

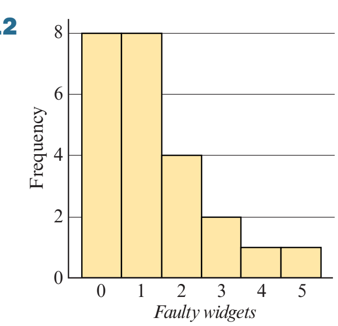
9

Number of widgets	Frequency	
	Number	%
0	8	33.3
1	8	33.3
2	4	16.7
3	2	8.3
4	1	4.2
5	1	4.2
Total	30	100.0

10

	Frequency	
Dined in restaurant	Number	%
0–9	17	56.7
10–19	5	16.7
20–29	3	10.0
30–39	0	0
40–49	1	3.3
50–59	3	10
60–69	1	3.3
Total	30	100

12



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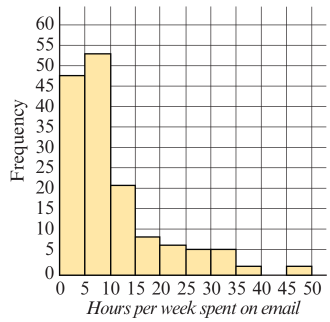
13

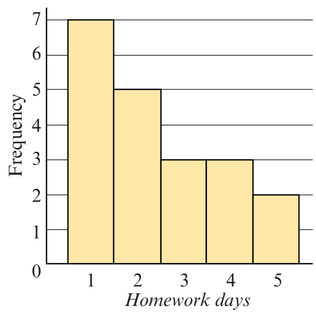


Exercise 2C

1 a

Days	Frequency
1	7
2	5
3	3
4	3
5	2
Total	20

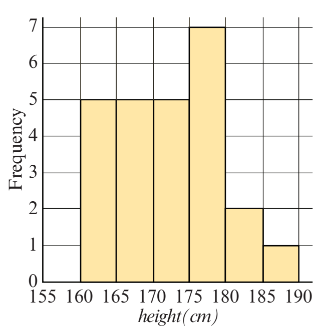
b



2 a

Height (cm)	Frequency
160–164	5
165–169	5
170–174	5
175–180	7
180–184	2
185–190	1
Total	25

b



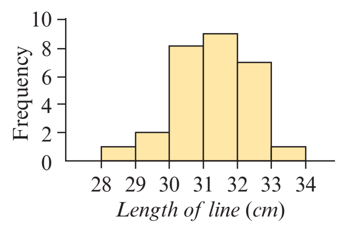
3

Number of magazines	Frequency	
	Number	Percent
0	4	26.7
1	4	26.7
2	3	20.0
3	2	13.3
4	1	6.7
5	1	6.7
Total	15	100.1

4

Amount of money ($)	Frequency	
	Number	Percent
0.00–4.99	13	65
5.00–9.99	3	15
10.0 –14.99	2	10
15.00–19.99	1	5
20.00–24.99	1	5
Total	20	100

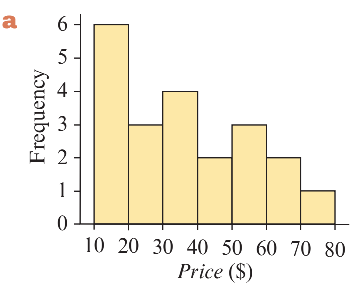
5 a i 2 students **ii** 3 students
iii 8 students
b i 32.1% **ii** 39.3% **iii** 89.3%
c



6 a 4 students **b** 2 children
c 5 students **d** 28 students

7 a 0 students **b** 48 students
c 60–69 marks **d** 33 students
e 31% failed

8 a



b i $30 to < $40 **ii** 4 books
iii $10 to < $20

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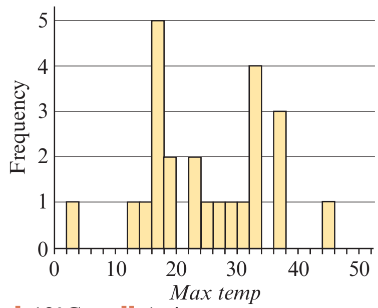
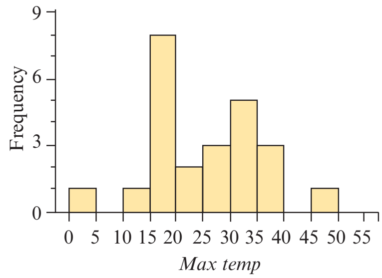
9 a



b **i** 12°C **ii** 1 city

c

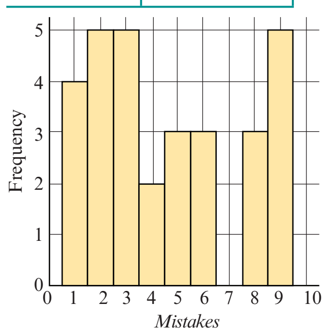


d **i** 2 cities **ii** 15°C to < 20°C

10 a

Mistakes	Frequency
1	4
2	5
3	5
4	2
5	3
6	3
7	0
8	3
9	5
Total	30

b

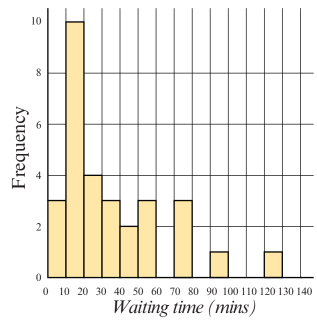


c No unique mode. **d** 46.7%

11 a

Waiting time (mins)	Frequency
0–9	3
10–19	10
20–29	4
30–39	3
40–49	2
50–59	3
60–69	0
70–79	3
80–89	0
90–99	1
100–109	0
110–119	0
120–129	1
Total	30

b

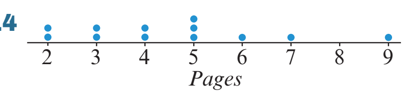


c 10 - 19 minutes and 33.3%

Section 2D

Now try this

14



15

Weight 1 | 5 means 15 kg

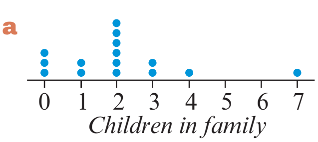
1	9
2	0 4 5 8 8 9
3	0 3 4 7 8
4	1 2 3 4 5 5
5	5
6	0 8
7	0

Exercise 2D

1 a Positively skewed **b** Negatively skewed
c Symmetric

2 a Centre **b** Neither **c** Both

3 a



b 2 children

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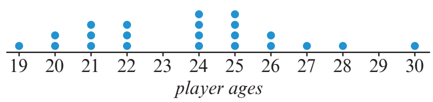
4 a

b 7 people

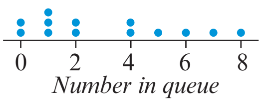
5 a negatively skewed
b symmetric
c positively skewed

6 a



b No unique mode
c Approximately symmetric **d** 59%

7 a



b Around 12:25 p.m.

8 a *English marks*

1	7
2	3 3 6 8
3	2 5 5 8 9
4	3 4 6 6 9
5	0 2 8 9
6	1 4 5 6 9
7	5 8 9 9
8	3 3 4 9
9	2 3 4 7

5|0 represents 50 marks

b 58.3% **c** 17 marks

9 a symmetric **b** negatively skewed
c positively skewed

10 a 40 people **b** Symmetric
c 21 people

11 a *Battery time (hours)*

0	4
1	7 9
2	0 1 2 4 5 6 6 7 7 8
3	0 0 1 1 3 3 4 7
4	0 1 6

2|5 represents 25 hours

b 9 batteries

12 a *Homework time (minutes)*

0	0
1	0 0 4 5 5 6 9
2	0 0 1 3 7 8 9
3	3 7 9
4	6
5	6
6	3
7	0

4|6 represents 46 minutes.

b 2 students
c Positively skewed

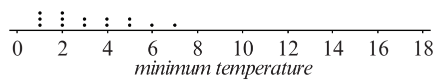
13 a *Price ($)*

2	5 8
3	5 6 9
4	5 6 9
5	2
6	8
7	5 5 6 8 9
8	2 4
9	5
10	
11	
12	
13	
14	9
15	

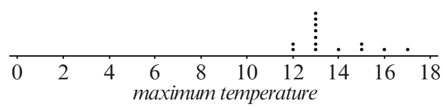
16|4 represents $164

b Approximately symmetric with an outlier ($149)

14 a



b



c The distribution of minimum temperature is centred much lower than the distribution of maximum temperature, as expected. It also looks to be a little more variable.

15 a *Body fat %* 1 | 2 means 12%

0	4
0	5 5 7 9 9
1	0 0 1 2 3 3 4
1	7 7 7 7 7 8 9
2	0 0 1 1 4
2	5 6 8 9
3	0

b The distribution would have exactly the same spread, but the centre would be 2% lower.

16 a Symmetric. We would expect the number of both very short and very tall people to reduce similarly as we get further away from the centre of the distribution.
b Positively skewed. There would be a long positive tail as there are many expensive houses.
c Negatively skewed, as many babies are born before full term, and not many after.

Section 2E

Now try this

18 a 5 **b** 3.5

Exercise 2E

1 a 26 **b** 2.36

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2 a 11 12 17 19 21 24 32 34 35 53 62 63 95
b 32

3 a mean = 5 **b** mean = 5
c mean = 15 **d** mean = 101
e mean = 2.8

4 a $M = 9$ **b** $M = 6.5$ **c** $M = 27$
d $M = 106.5$ **e** $M = 1.2$

5 a $M = 57$ mm
b $M = 27.5$ hours

6 a $\bar{x} = 12.5$ ha, $M = 7.4$ ha
b The median. Since the distribution is positively skewed, the median is typical of more suburbs.

7 a 55.42 **b** 28

8 a $\bar{x} = \$393\,386$, $M = \$340\,000$
b The median. Since the distribution is positively skewed, the median is typical of more apartment prices.

9 $M = 17, \bar{x} = 16.95$

10 35.3

11 69.3

12 a 16.05 **b** 15.0

13 a 2.5 **b** 3.3

14 a The median is the average of the 24th and 25 values when the data is ordered. The 24th value is in the interval 50 - 59, and the 25th value is in the interval 60 - 69. Thus we could infer that the median is about 60, although we cannot give it an exact value. **b** We cannot determine the exact value of the mean from a histogram where the data have been grouped. However, as the distribution is approximately symmetric we could infer that it would be similar in value to the median.

15 a The median would be in the interval where the cumulative percentage frequency is 50%. Thus it is in the interval 180.0 - 184.9, although we cannot give it an exact value. **b** We cannot determine the exact value of the mean from a histogram where the data have been grouped. However, as the distribution is approximately symmetric we could infer that it would be similar in value to the median.

Section 2F

Now try this

19 range = 5.9 kg

20 IQR = 2.3 kg

21

	x	$(x-\bar{x})$	$(x-\bar{x})^2$
	0	-2.25	5.0625
	1	-1.25	1.5625
	3	0.75	0.5625
	5	2.75	7.5625
Sum	9	0	14.75

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} = \sqrt{\frac{14.75}{4-1}} = 2.217$$

Exercise 2F

1 a 0 1 1 1 2 2 2 3 6 7 **b** R = 7 **c** $Q_1 = 1$
d $Q_3 = 3$ **e** IQR = 2

2 a $\bar{x} = 3$ **b** s = 2.160

3 a IQR = 10.5, $R = 21$ **b** IQR = 8, $R = 11$
c IQR = 7, $R = 12$ **d** IQR = 4.5, $R = 8$
e IQR = 1.1, $R = 2.7$

4 a IQR = 9.5 mm
b IQR = 11 hours

5 $M = 3, Q_1 = 1, Q_3 = 5.5$

6 $\bar{x} = 365.8$, $s = 8.4$, $M = 366.5$, IQR = 12.5, $R = 31$

7 $\bar{x} = 214.1$, $s = 36.5$, $M = 207.5$, IQR = 39, $R = 145$

8 a R = 9 **b** IQR = 2
c s = 2.164

9 $\bar{x} = 3.5$ kg, $s = 0.6$ kg, $M = 3.5$ kg, IQR = 1 kg, $R = 2.4$ kg

10 a **i** $\bar{x} = 6.71$, $M = 6.2$
ii IQR = 1.45, $s = 0.90$
b **i** $\bar{x} = 12.71$, $M = 7.3$
ii IQR = 1.85, $s = 17.75$
c The error does not affect the median or interquartile range very much. It doubles the mean and increases the standard deviation by a factor of 20.

Section 2G

Exercise 2G

1 a (13.5, 18.1) **b** (11.2, 20.4)
c (8.9, 22.7)

2 a (382.3, 488.9) **b** (329.0, 542.2)
c (275.7, 595.5)

3 a 95% **b** 99.7%

4 a 99.7% **b** 95%

5 a 68% **b** 99.7%

6 a 68% **b** 95%

7 a = 996 mL, b = 1004 mL

8 a $a = \$38\,000, b = \$144\,000$

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b $a = \$11\,500, b = \$170\,500$

9 a $a = 3$ hours, $b = 9.4$ hours

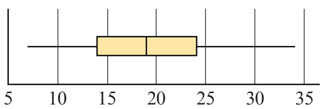
b $c = 1.4$ hours, $d = 11$ hours

c $e = 6.2$ hours **d** 13.5%

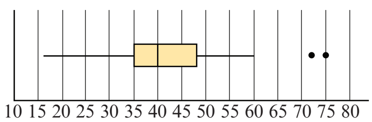
Section 2H

Now try this

24



25

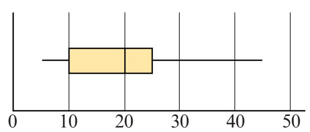


26 a 75% **b** 0% **c** 25% **d** 50%

e 75%

Exercise 2H

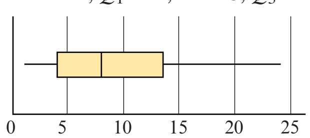
1



2

a Min = 1, $Q_1 = 4$, $M = 8$, $Q_3 = 13.5$, Max = 24

b



3 lower fence = 4, upper fence = 28

4 a lower fence = 15, upper fence = 95

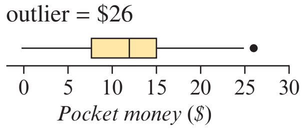
b outliers are 14 and 99

5

a lower fence = -\$3.125, upper fence = \$25.875

b outlier = \$26

c



6 a 50%

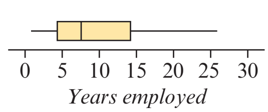
b approx 0%, although there is one outlier

c 75% **d** 25%

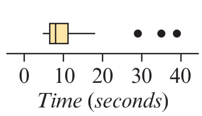
7 a 75% **b** 25% **c** 50% **d** 0%

e 75% **f** 25%

8

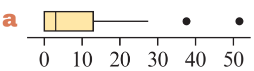


9 a



b 6 seconds

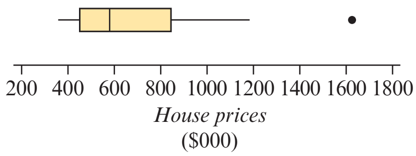
10 a



b There are two possible outliers; the people who borrowed 38 and 52 books respectively.

c 13 or more

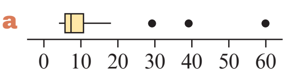
11 a



b There was one outlier, the unit which sold for \$1 625 000.

c From \$579 000 to \$1 625 000.

12 a



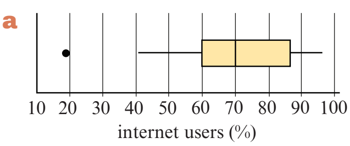
b There are three possible outliers, the three children who took 29, 39, and 60 seconds respectively to complete the puzzle.

c 11 seconds

13 a \$800 **b** \$1200 **c** \$1600

d Upper fence = \$3400

14 a



b There is one possible outlier, Afghanistan, that recorded extremely low percentages of internet users (18.8%).

c 59.9

Section 2I

Now try this

27 The median number of hours spent relaxing by this group of males was higher (M = 30 hours) and the median for this group of females (M = 24.5 hours). The spread of number of hours spent relaxing was lower for males (IQR = 14.5 hours) than for the females (IQR = 19 hours). In conclusion, the males spent more time relaxing that the females, and this amount of time was less variable for the males than the females.

28 The median amount of time spent in an exam was slightly higher for the females in the group ($M \approx 122$ mins) than for the males ($M \approx 116$ mins). The spread for the two groups was almost the same, (females: $IQR \approx 45$ mins, males: $IQR \approx 44$ mins.) In conclusion, females tended to spend longer in

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the exam than males, with similar variability. There was one female who spent an unusually short time in the exam, only 40 minutes.

Exercise 2I

Note: The written reports should only be regarded as sample reports. There are many ways of writing the same thing.

1 **a** Test A: M = 57, Test B: M = 73

b lower

c Test A: IQR = 16, Test B: IQR = 12

d more

2 **a** The foot length for males (median = 26 cm) is longer than the foot length for females (median = 24 cm).

b The foot length for males (IQR = 3.5cm) is similar in variability to the foot length for females (IQR = 3.5 cm).

3 **a** Females: $M = 34$ years, IQR = 28 years
Males: $M = 25.5$ years, IQR = 13 years

b Report: The median age of the females ($M = 34$ years) was higher than the median age of males ($M = 25.5$ years). The spread of ages of the females (IQR = 28 years) was greater than the spread of ages of the males (IQR = 13 years). In conclusion, the median age of the females admitted to the hospital on that day was higher than the males. Their ages were also more variable.

4 **a** Class A: 6; Class B: 2

b Class A: $M = 76.5$ marks, IQR = 30.5 marks;
Class B : $M = 78$ marks, IQR = 12 marks

c Report: The median mark for Class A ($M = 76.5$) was lower than the median mark for Class B ($M = 78$). The spread of marks for Class A (IQR = 30.5) was greater than the spread of marks of Class B (IQR = 12). In conclusion, Class B had a higher median mark than Class A and their marks were less variable.

5 **a**

Japan		Australia
3	0	2 3 3 4 4
9 7 6 5 5	0	5 5 6 7 7 8 9
4 4 2	1	1 1 4
7 5	1	5 7
3 3 2 2	2	1 3
9 8 6	2	
2	3	3
	3	
4	4	

6|2 represents 26   1|5 represents 15

b Japanese : $M = 16$ nights,
IQR = 16.5 nights;
Australians : $M = 7.5$ nights,
IQR = 10 nights

c Report: The median time spent away from home by the Japanese tourists ($M = 16$ nights) was much higher than the median time spent away from home by the Australian tourists ($M = 7.5$ nights). The spread in the spread in the time spent away from home by the Japanese (IQR = 16.5 nights) was also greater than the spread in the time spent away from home by the Australians (IQR = 10). In conclusion, the time spent away from home by the Japanese tourists was longer and more variable than for the Australian tourists.

6 **a** Year 12: $M = 5.5$ hours, IQR = 4.5 hours;
Year 8: $M = 3$ hours, IQR = 2.5 hours
(values can vary a little)

b Report: The median homework time for the year 12 students ($M = 5.5$ hours/week) was higher than the median homework time for year 8 students ($M = 3$ hours/week). The spread in the homework time for the year 12 students (IQR = 4.5 hours/week) was also greater than for the year 8 students (IQR = 2.5 hours/week). In conclusion, the median homework time for the year 12 students was higher than for the year 8 students and the homework time was more variable.

7 **a** Males: $M = 22\%$, IQR = 12.3%;
females : $M = 10\%$, IQR = 11.5% (values can vary a little)

b Report: The median smoking rate for males ($M = 22\%$) was much higher than for females ($M = 10\%$). The spread in smoking rates for males (IQR = 12.3%) was similar to females (IQR = 11.5%). In conclusion, median smoking rates were higher for males than for females but the variability in smoking rates was similar.

8 **a** Before: $M = 26$, IQR = 7.5, outlier = 46;
After : $M = 30$, IQR = 10.5,
outliers = 50 and 54 (values can vary a little)

b Report: The median number of sit-ups before the fitness class ($M = 26$) was lower than after the fitness class ($M = 30$). The spread in number of sit-ups before the fitness class (IQR = 7.5) was less than after the fitness class (IQR = 10.5). There was one outlier before the fitness class, the person who did 44 sit-ups. There were two

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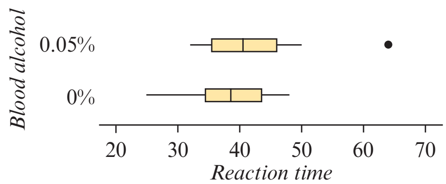
outliers after the fitness class, the person who did 50 sit-ups and the person who did 55 sit-ups. In conclusion, the median number of sit-ups increased after taking the fitness class and there was an increase in the variability of the number of sit-ups people could do.

9 a



b Report: The median time is slightly higher for the 0.05% blood alcohol group (M = 40.5) than for the 0% blood alcohol group (M = 38.5). The spread in time is also slightly higher for the 0.05% blood alcohol group (IQR = 9.5) than for 0% blood alcohol (IQR = 9.0). There was one outlier, the person with 0.05% blood alcohol who had a very long time of 64 seconds.

In conclusion, the median time was longer for the 0.05% blood alcohol group than for the 0% blood alcohol group but the variability in times was similar.

10 a People who were not thinking of changing to a different kind of work tended to work in occupations with higher median occupational prestige score (M = 48), than those who sometimes think of changing (M = 44), who in turn tended to work in occupations with higher median occupational prestige scores than those who were thinking of changing (M = 39). The variability in occupational prestige scores was lowest for those thinking of changing work (IQR = 17), and similar for those not thinking of changing (IQR = 20) and maybe thinking of changing (IQR = 21). **b** Yes, there appears to be an association between the variables, with people who are in occupations with higher occupational prestige scores tending to be less likely to want to change to a different kind of work.

Chapter 2 Review

Skills Checklist answers

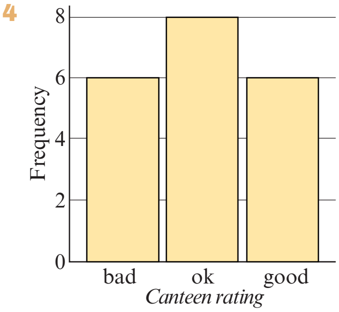
1 **a** continuous **b** discrete **c** nominal **d** ordinal

2

	Frequency	
Canteen rating	Number	%
bad	6	30
ok	8	40
good	6	30
Total	20	100

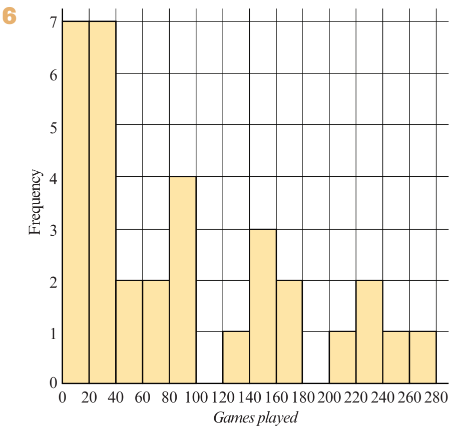
3 mode = ok

4



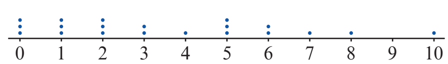
5 A group of 20 children were asked rate their canteen as bad, ok or good. The most popular response was ok, chosen by 40% of the students. The responses bad and good received the same percentage of responses, 30% each.

6



7 positively skewed

8



9

Age 2 | 2 means 22

1	7 9 9 9
2	0 0 1 1 2 3 6 7 8 9
3	4 4 7
4	5
5	6
6	6

10 M = 24.5, $\bar{x}$ = 29.15. The median is preferable as the distribution is positively skewed.

11 Q_1 = 20, Q_3 = 34, IQR = 14

12 s = 13.180

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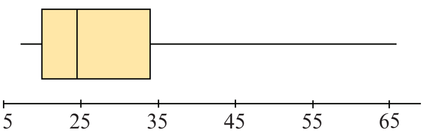
13 99.7%

14 min = 17, Q_1 = 20, M = 24.5, Q_3 = 34, max = 66



15 lower fence = -1, upper fence = 55, thus 56 and 66 are outliers.

Multiple-choice questions

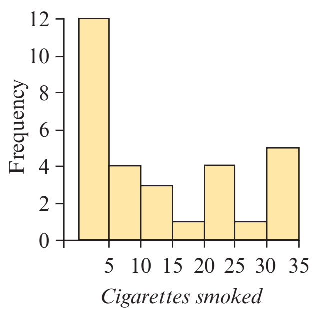
1 D	**2** D	**3** C	**4** C	**5** B
6 D	**7** D	**8** D	**9** D	**10** D
11 D	**12** A	**13** C	**14** C	**15** E
16 B	**17** B	**18** B	**19** C	**20** D
21 C	**22** A	**23** C		

Short-answer questions

1 a Discrete **b** Ordinal

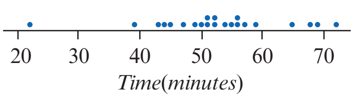
2 a Categorical **b** 7.5% **c** 35

3 a



b positively skewed **c** 5-<10

4 a



b *Time (minutes)*

2	2
3	9
4	3 4 5 7 9
5	0 1 1 2 2 4 5 6 6 7 9
6	5 8 9
7	2

4|7 represents 47 minutes

c M = 52 minutes, Q_1 = 47 minutes, Q_3 = 57 minutes

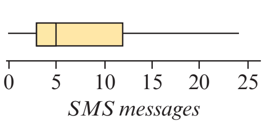
d lower fence = 32, upper fence = 72, thus 22 is an outlier

5 $\bar{x}$ = \$283.57,

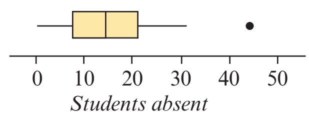
s = \$122.72, M = \$267.50, IQR = \$90,

R = \$495

6



7 a



b 14.5 students **c** 27.8%

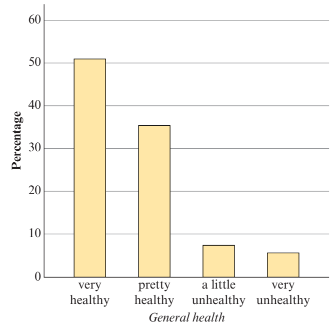
Written-response questions

1 a Ordinal

b

General health	Number	%
very healthy	255	51.0
pretty healthy	178	35.6
a little unhealthy	38	7.6
very unhealthy	29	5.8
Total	500	100.0

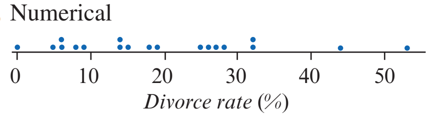
c



d Five hundred people were asked about their level of general health. The vast majority of the respondents indicated they were healthy, with 50.1% choosing very healthy and 35.6% choosing pretty healthy. Only 7.6% responded that they were a little unhealthy, and even fewer that they were very unhealthy (5.8%).

2 a Numerical

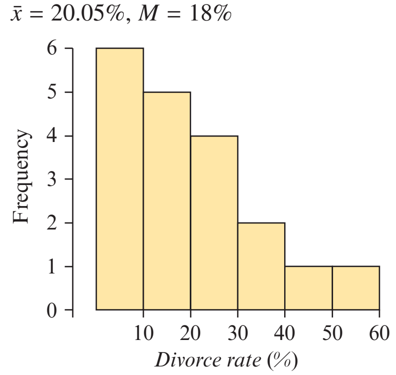
b



c Positively skewed

d 21.05%

e $\bar{x}$ = 20.05%, M = 18%



f i Positively skewed **ii** 5 countries

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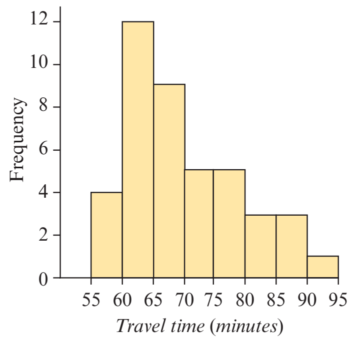
3 a

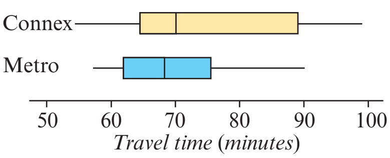


i 9 days **ii** Positively skewed

b $\bar{x} = 69.60$ minutes, $s = 9.26$ minutes, Min $= 57$ minutes, $Q_1 = 62$ minutes, $M = 68$ minutes, $Q_3 = 76$ minutes, Max $= 90$ minutes

c **i** 69.60 **ii** 68 **iii** 33, 14 **iv** 76 **v** 9.26

d



e The median travel times for Connex ($M = 70$ minutes) tend to be longer than the median travel times for Metro ($M = 68$ minutes). The spread of times is also longer for Connex (IQR = 24 minutes) compared to Metro (IQR = 14 minutes). Both median travel times and variability in travel times was less for Metro than for Connex.

Chapter 3

Section 3A

Now try this

1 a Oscillating **b** Increasing **c** Limiting to 1

2 a 12 **b** 8 **c** 18

3 7, 12, 17, 22, 27

Exercise 3A

1 a 8, 14 **b** 27

2 a Starting value: 10, Rule: Subtract 2
b Starting value: *a*, Rule: Add 3 letters

3 a 5 **b** Add 6 **c** 35, 41, 47

4 a Constant **b** Increasing
c Oscillating
d Oscillating, limiting to zero
e Decreasing **f** Increasing

5 a 19 **b** 6 **c** 1 **d** 3 **e** 48 **f** 2

6 a 64 **b** 3 **c** 81 **d** 3 **e** 62.5 **f** 162

7 a March **b** *i* **c** ♦ **d** ⇒ **e** Wednesday **f** ↑

8 a Add 3: 17, 20 **b** Add 9: 55, 64
c Subtract 4: 22, 18 **d** Subtract 8: 34, 26
e Multiply by 2: 48, 96
f Multiply by 3: 324, 972
g Divide by 2: 8, 4
h Multiply by −2: 48, −96
i Add previous two terms: 8, 13

9 3, 5, 7, 9, 11

10 90, 84, 78, 72, 66

11 5, 11, 23, 47, 95

12 a 125 **b** 120 **c** −720

13 a Multiply by 2, 3, 4, . . . : 240, 1440
b Square integers: 36, 49
c Odd squares: 81, 121

14 −32, 64, −128, 256, −512

Section 3B

Now try this

4 a 46 **b** 34 **c** 26

5 12, 17, 22, 27, 32

Exercise 3B

1 20

2 a 8, 4, 3, 11, 14 **b** t_0, t_1, t_2, t_3, t_4

3 a 9 **b** 2

4 a 6 **b** 21 **c** 16 **d** 26 **e** 31 **f** 36

5 a i) $t_0 = 6$, ii) $t_3 = 18$, iii) $t_1 = 10$
b i) $t_0 = 2$, ii) $t_3 = 128$, iii) $t_1 = 8$
c i) $t_0 = 29$, ii) $t_3 = 8$, iii) $t_1 = 22$
d i) $t_0 = 96$, ii) $t_3 = 12$, iii) $t_1 = 48$

6 a 20 **b** 16 **c** 8 **d** 24 **e** 28 **f** 40

7 a $t_0 = 14, t_3 = 32, t_6 = 50$
b $t_0 = 2, t_3 = 54, t_6 = 1458$
c $t_0 = 40, t_3 = 16, t_6 = -8$
d $t_0 = 8000, t_3 = 1000, t_6 = 125$

8 a 1, 3, 5, 7, 9 **b** 100, 90, 80, 70, 60
c 52, 64, 76, 88, 100

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9 a $V_0 = 3, V_{n+1} = V_n + 7$
b $V_0 = 9, V_{n+1} = V_n + 4$
c $V_0 = 16, V_{n+1} = V_n - 3$

10 a 3, 10, 17, 24, 31 **b** 9, 13, 17, 21, 25
c 16, 13, 10, 7, 4

11 a $V_0 = 11, V_{n+1} = V_n + 4$
b $V_0 = 43, V_{n+1} = V_n - 4$
c $V_0 = 3, V_{n+1} = V_n - 4$

12 a T_1 **b** T_4 **c** T_8

13 a 4
b The next term is larger so the sequence is not obtained by subtraction. The difference between t_0 and t_1 is 5 but the difference between t_1 and t_2 is 10 so the difference is not obtained by adding.
c The next term is larger so the sequence is not obtained by dividing by a number greater than 1. Since $t_1 = t_0 \times \frac{9}{4}$ but $t_2 = t_1 \times \frac{19}{9}$, the sequence cannot be generated by multiplying.
d The starting value is 4 and the rule is to add 5 times the number of iterations that have been applied.
e $V_0 = 4, V_{n+1} = V_n + 5(n+1)$.

Section 3C

Now try this

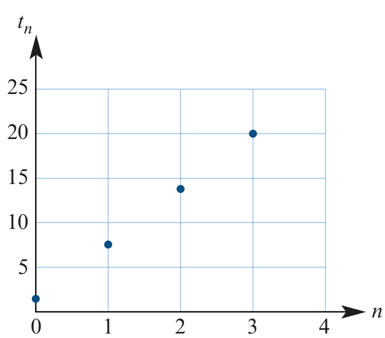
6 Common difference: −4, Next term: 7

7 Sequence 2

8 a

n	0	1	2	3
t_n	2	8	14	20

b

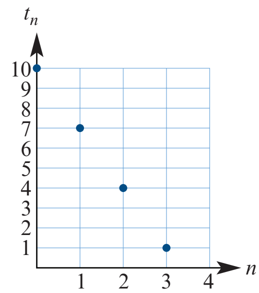


c The points lie on a straight line with a positive slope.

9 a

n	0	1	2	3
t_n	10	7	4	1

b



c The points lie on a straight line with a negative slope.

Exercise 3C

1 a 8 **b** 16 **c** 42

2 a 4, 4 **b** −2, −2

3 a

n	4	11	18	25	32
t_n	t_0	t_1	t_2	t_3	t_4

b 25
c 7, 7, 7, 7. Arithmetic
d $t_5 = 39$ and $t_6 = 46$

4 a $D = 6, t_3 = 23$ **b** $D = -4, t_3 = 5$
c $D = 4, t_3 = 23$ **d** $D = -4, t_3 = -4$
e $D = -5, t_3 = 20$ **f** $D = 0.5, t_3 = 3$

5 a 41, 47 **b** 2, −1 **c** 0, −0.5 **d** 59, 67
e −15, −27 **f** 2, 2.3

6 a Arithmetic $D = 3$ **b** Not arithmetic
c Arithmetic $D = -4$ **d** Arithmetic $D = -3$
e Not arithmetic **f** Arithmetic $D = 0$

7 a 3, 8, 13, 18, 23 **b** 16, 9, 2, −5, −12
c 1.6, 3.9, 6.2, 8.5, 10.8
d 8.7, 5.6, 2.5, −0.6, −3.7
e 293, 226, 159, 92, 25

8 a $t_6 = 31$ **b** $t_{12} = 21$ **c** $t_{10} = 5$ **d** $t_{15} = 45$

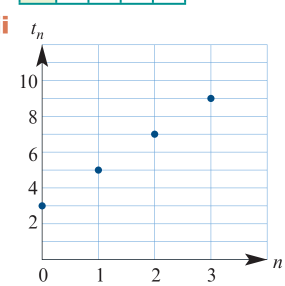
9 8 weeks

10 a 107, 114, 121, 128 **b** 184

11 a **i** 9
ii

n	0	1	2	3
t_n	3	5	7	9

iii

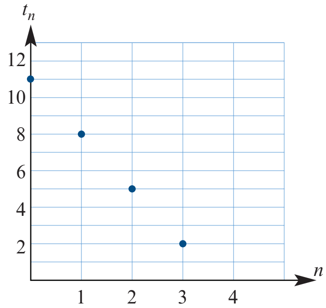


iv The straight line has a positive slope
b **i** 2

ii

n	0	1	2	3
t_n	11	8	5	2

iii



iv The straight line has a negative slope

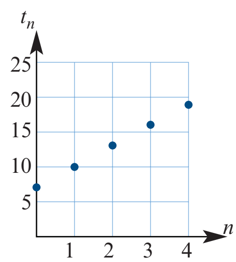
12 a 28, 33 **b** −10, −16 **c** 24, 33
d 13, 8 **e** 11, 19 **f** 13, 21 **g** 29, 18
h 29, 15 **i** 23, 39, 55

13 a $D = -3$ **b** Subtract 300
c $t_{100} = -302$ **d** $t_{200} = -602$

Section 3D

Now try this

10 7, 10, 13, 16, 19



11 404

Exercise 3D

1 a $a = 7, D = 4$ **b** $a = 8, D = -3$
c $a = 14, D = 9$ **d** $a = 62, D = -27$
e $a = -9, D = 5$ **f** $a = -13, D = -6$

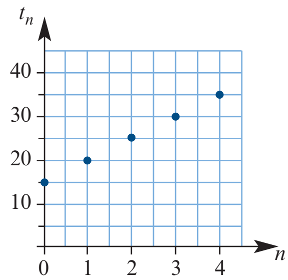
2 a 27 **b** −7 **c** 59 **d** −73
e 16 **f** −43

3 $a = 12, D = 10, n = 20, t_{20} = 212$

4

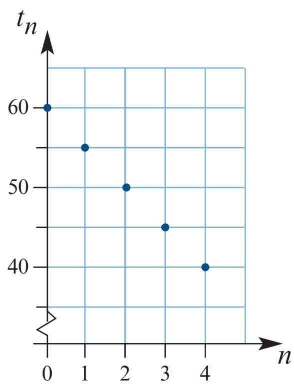
$$t_1 = t_0 + 1 \times 3 = 7 + 1 \times 3 = 10$$
$$t_2 = t_0 + 2 \times 3 = 7 + 2 \times 3 = 13$$
$$t_3 = t_0 + 3 \times 3 = 7 + 3 \times 3 = 16$$
$$t_{20} = t_0 + 20 \times 3 = 7 + 20 \times 3 = 67$$

5 a 15, 20, 25, 30, 35



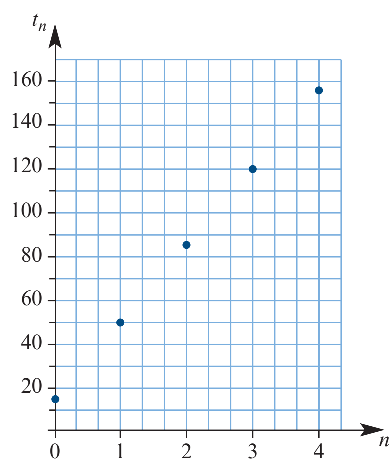
b 235

6 a 60, 55, 50, 45, 40



b 10

7 a 15, 50, 85, 120, 155



b 540

8 a 123 **b** 314 **c** −454 **d** 451
e −123 **f** 26 **g** −212 **h** −8

9 331

10 1927

11 −40

12 15 metres

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13 a $10 = a + 4 \times D$ **b** $18 = a + 8 \times D$
c $a = 2, D = 2$ **d** $t_5 = 12, t_9 = 20$
e 2, 4, 6

14 10 terms

Section 3E

Now try this

12 \$234.60

13 a \$3156, \$3312, \$3468
b After 7 years

14 a \$3441, \$3182, \$2923
b After 11 years

15 a $V_0 = 9000, V_{n+1} = V_n - 15$
b \$9185, \$9170, \$9155
c 1 400 000 sheets

16 a \$576 **b** \$17 760 **c** After 14 years

Exercise 3E

1 \$200

2 \$360

3 $V_n = V_0 + n \times D$

4 a \$730 **b** \$627 **c** \$11 588.40

5 a 10 450, 10 900, 11 350
b After 10 years

6 a 8400, 8800, 9200 **b** 10 years
c \$8000 **d** 5%

7 a \$63 000 **b** \$45 000 **c** 10 years
d $V_0 = 95\,000, V_{n+1} = V_n - 11\,400$

8 a \$2400 **b** \$300 **c** 12.5% **d** 6 years
e 4 years

9 a \$17 000 **b** \$14 000 **c** 90 000
d $V_0 = 20\,000, V_{n+1} = V_n - 250$

10 a \$26 500 **b** \$70 **c** 190,000 uses

11 a \$800 **b** \$44 000 **c** After 10 years

12 a \$500 **b** \$2000 **c** 8 years

13 \$8500

Section 3F

Now try this

17 a $R = 5$ **b** $R = \frac{1}{3}$

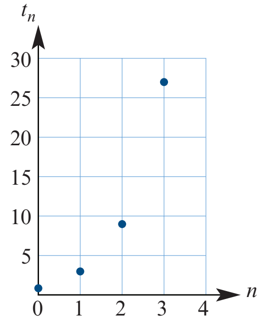
18 Sequence 2

19 a 27

b

n	0	1	2	3
t_n	1	3	9	27

c



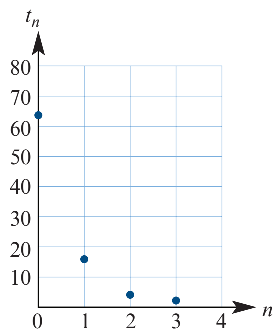
d The values lie along a curve and they are increasing.

20 a 1

b

n	0	1	2	3
t_n	64	16	4	1

c



d The values lie along a curve and they are decreasing.

Exercise 3F

1 a $\frac{1}{2}$ **b** 27 **c** 3 **d** $\frac{1}{16}$

2 a 2 **b** 3 **c** $\frac{1}{2}$ **d** $\frac{1}{2}$

3 a Arithmetic **b** Geometric
c Geometric **d** Arithmetic

4 a 2 **b** $\frac{1}{4}$ **c** 5 **d** 4
e $\frac{1}{2}$ **f** 6 **g** 10 **h** 7

5 a Geometric $R = 2$ **b** Geometric $R = 3$
c Not geometric **d** Geometric $R = 3$
e Geometric $R = 0.5$ **f** Not geometric
g Not geometric **h** Geometric $R = \frac{1}{3}$
i Geometric $R = 2$

6 a 56, 112 **b** 375, 1875 **c** 36, 108
d 5, 10 **e** 8, 512 **f** 9, 81

7 a 109 375 **b** 139 968 **c** 1.5
d 470 596 **e** 2.5 **f** 5 845 851

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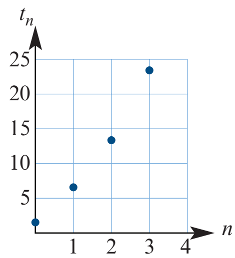
8 a i 24

ii

n	0	1	2	3
t_n	3	6	12	24

iii



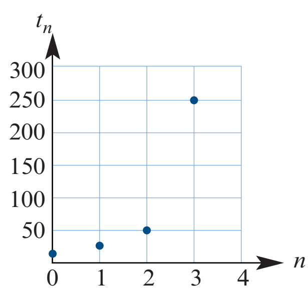
iv The graph is a curve with values increasing

b i 250

ii

n	0	1	2	3
t_n	21	10	50	250

iii



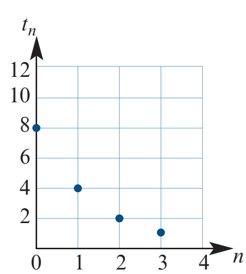
iv The graph is a curve with values increasing

9 a i 1

ii

n	0	1	2	3
t_n	8	4	2	1

iii



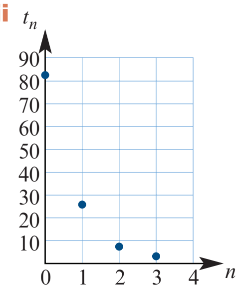
iv The graph is a curve with values decreasing and approaching zero.

b i 3

ii

n	0	1	2	3
t_n	81	27	9	3

iii



iv The graph is a curve with values decreasing and approaching zero.

10 a 1.1 **b** 22, 24.2, 26.62

11 a 0.9 **b** 90, 81, 72.9

12 a 1.2 **b** 1.2, 1.2 **c** 20% **d** 20 736

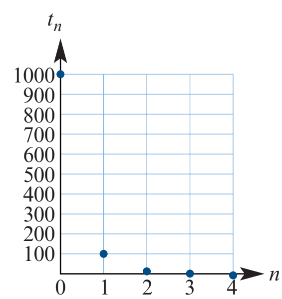
13 a 500, 195, 73, 24.2, 4.68 **b** Neither

c Never reach 800 since decreasing and starts at 500. **d** 5 iterations

Section 3G

Now try this

21 1000, 100, 10, 1, 0.1



22 5 242 880

Exercise 3G

1 a $a = 2, R = 3$ **b** $a = 5, R = 4$

c $a = 5, R = 2$ **d** $a = 3, R = 4$

2 a 486 **b** 5120 **c** 160 **d** 3072

3

$$t_1 = R^1 \times t_0 = 2^1 \times 6 = 2 \times 6 = 12$$

$$t_2 = R^2 \times t_0 = 2^2 \times 6 = 4 \times 6 = 24$$

$$t_3 = R^3 \times t_0 = 2^3 \times 6 = 8 \times 6 = 48$$

$$t_{20} = R^{20} \times t_0 = 2^{20} \times 6 = 1\,048\,576 \times 6$$

$$= 6\,291\,456$$

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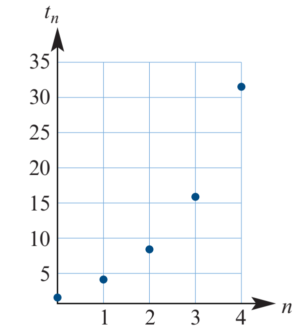
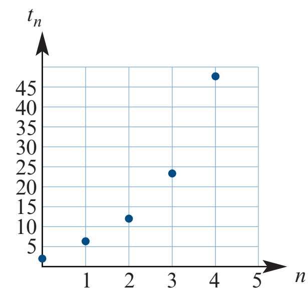
4 a 2, 4, 8, 16, 32



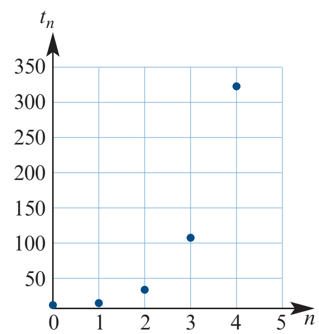
b $t_{10} = 2048$

5 a 3, 6, 12, 24, 48



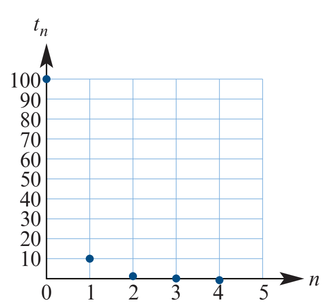
b $t_{12} = 12\ 288$

6 a 4, 12, 36, 108, 324



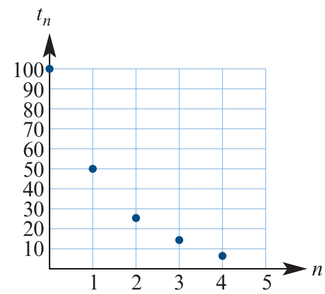
b $t_{10} = 236\ 196$

7 a 100, 10, 1, 0.1, 0.01



b $t_{10} = \dfrac{1}{100\ 000\ 000}$

8 a 100, 50, 25, 12.5, 6.25



b $t_{15} = 0.00305$

9 a 80 **b** 320 **c** 40 960 **d** 327 680
e 10 485 760 **f** 335 544 320

10 a 6 442 450 944
b 823 564 528 378 596
c 0.00000186 **d** 0.00000931

11 4 194 304

12 5120

13 0.004768

14 0.0078125m^2

15 a $12 = R^2 \times t_0$ **b** $96 = R^5 \times t_0$
c $R = 2, t_0 = 3$ **d** $V_0 = 3, V_{n+1} = 2V_n$
e $t_3 = 24, t_4 = 48$ **f** 3, 6, 12, 24, 48, 96

Section 3H

Now try this

23 a \$3120, \$3245, \$3375
b Five years

24 a $V_0 = 25\ 000, V_{n+1} = 1.048V_n$
b $V_0 = 25\ 000, V_{n+1} = 1.012V_n$
c $V_0 = 25\ 000, V_{n+1} = 1.004V_n$

25 a \$49 496, \$45 536.32, \$41 893.41
b After 8 years

Exercise 3H

1 a Geometric growth **b** Geometric decay
c Neither **d** Neither **e** Geometric decay
f Geometric growth

2 a \$5000 **b** 5% **c** \$5250

3 a \$40 000 **b** 10% **c** \$36 000

4 a 1.125% **b** 0.375%

5 a \$10 450, \$10 920.25, \$11 411.66
b After 5 years

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6 a \$210 400, \$221 340.80, \$232 850.52
b After six years

7 a \$8000 **b** After 4 years **c** 7.5%

8 a \$106 300, \$112 996.90, \$120 115.70
b After 12 years

9 a 0.5% per month
b $V_0 = 80\,000, V_{n+1} = 1.005V_n$
c \$80 400, \$80 802, \$81 206.01

10 a \$76 500, \$65 025, \$55 271.25, \$46 980.56, \$39 933.48
b \$24 524.15

11 a \$1200 **b** After 4 years
c After 2 years **d** 44%

12 a \$84 000 **b** \$81 060
c $V_0 = 84\,000, V_{n+1} = 0.965V_n$ **d** \$81 060
e \$84 000, \$81 060, \$78 222.90, \$75 485.10, \$72 843.12
f \$13 706.39

13 Option B is best, earning \$7401.73 interest

Section 3I

Now try this

26 a \$16 036 **b** \$7383

27 a \$53 756.66 **b** \$13 756.66
c \$1565.73
d $V_n = 1.0025^n \times 40\,000$
e \$53 974.14

28 \$45.80

29 \$5382.07

30 a \$3000 **b** \$3420 **c** \$3370.80
d \$3124.28 **e** \$3308
f Cheapest is credit card

31 \$4.24

32 a \$17.14 **b** \$40.72

33 \$48 266

Exercise 3I

1 a 15 **b** 135 **c** 1215 **d** 10 935

2 a 405 **b** 160 **c** 0.0625 **d** 0.78125

3 a 6700.48 **b** 8144.47 **c** 657 506.29

4 a \$10 000 **b** 10% **c** $V_n = 1.1^n \times 10\,000$
d \$16 105
e The investment is worth \$16 105 after 5 years

5 a \$12 000 **b** 8% **c** $V_n = 1.08^n \times 12\,000$
d \$16 326

6 a \$18 500 **b** 10% **c** $V_n = 0.9^n \times 18\,500$
d \$10 924
e The car is worth \$10 924 after 5 years

7 a \$9500 **b** 5% **c** $V_n = 0.95^n \times 9500$
d \$5688
e The boat is worth \$5688 after 10 years.

8 a \$520 000 **b** 0.35%
c $V_n = 0.9965^n \times 520\,000$
d \$509 175
e The balance of the loan is \$509 175 after 6 months

9 a \$12 461.82 **b** \$2461.82
c \$536.63 **d** $I_n = 1.00375^n \times 10\,000$
e \$12 517.96

10 a \$710 209.10 **b** \$410 209.10
c \$58 641.10
d $I_n = 1.0075^n \times 300\,000$
e \$735 407.12

11 a $V_n = 0.905^n \times 24\,000$
b \$14 569.82 **c** \$9430.18

12 a \$5900 **b** \$5871.21
c \$7648.63 **d** \$5338
e Buy now, pay later

13 a \$54.57 **b** \$110.29 **c** \$452.00 **d** \$45.99

14 a no interest on either card - choose either
b A: no interest, B: \$20.93. Choose A
c A: \$36.48, B: \$52.72. Choose A
d A:\$229.19, B: \$219.38. Choose B

15 a \$3.59 **b** \$3.72

16 a \$2.62 **b** \$7.10

17 a \$148 818.78 **b** \$58 917.67

18 5 years

19 30 years

Chapter 3 Review

Skills Checklist answers

1 Decreasing **2** 23 **3** 3, 5, 7, 9, 11

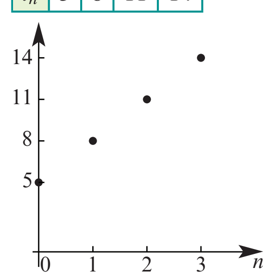
4 $t_0 = 4, t_3 = 16$ **5** 12, 19, 26, 33, 40

6 $d = 3$ and next term = 31

7 No it is not an arithmetic sequence.

8

n	0	1	2	3
t_n	5	8	11	14



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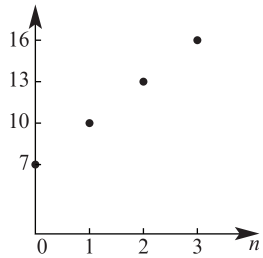
9 $t_0 = 7, t_1 = 10, t_2 = 13, t_3 = 16$



10 −20 **11** \$110 **12** 1680, 1760, 1840

13 \$15 300, \$12 600, \$9900

14 21 800, 21 600, 21 400

15 $t_0 = 80, t_{n+1} = t_n + 10\,500$, \$426 000

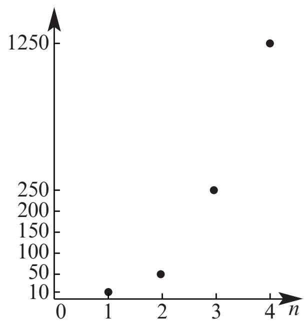
16 2

17 Yes it is a geometric sequence.

18 4, 8, 16, 32, 64

19

n	0	1	2	3	4
t_n	2	10	50	250	1250



20 8, 12, 18, 27, 40.5 **21** $t_4 = 160$

22 \$5300, \$5618, \$5955.08

23 $t_0 = 2000, t_{n+1} = 1.0105t_n$

24 \$24 750, \$22 275, \$20 047.50

25 \$599 140 **26** \$4016.33

27 \$48.32 **28** \$2768.75

29 Compounding at 6.5% **30** \$2.53

31 \$4.01 **32** \$58 383.89

Multiple-choice questions

1 E **2** C **3** B **4** A **5** E
6 A **7** D **8** A **9** D **10** D

Short-answer questions

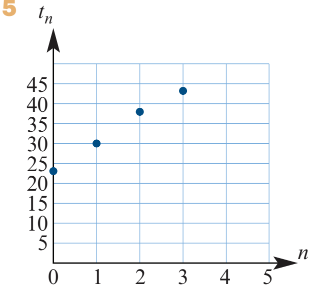
1 a Add 3: 14, 17 **b** Add 9: 83, 92
c Oscillate: 16, 60
d Multiply by 3: 162, 486
e Multiply by $\frac{1}{2}$: 62.5 and 31.25

2 a i 12 **ii** 30 **iii** 42
b i 20 **ii** 14 **iii** 10
c i 2 **ii** 250 **iii** 6250

3 87

4 1132

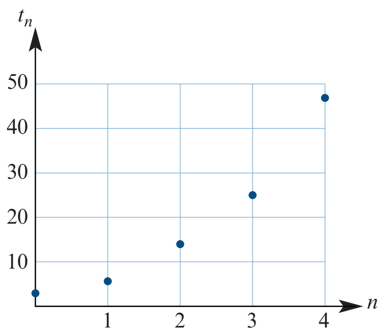
5



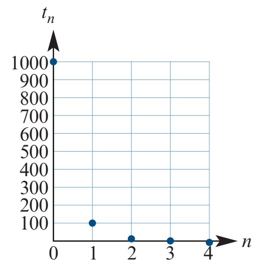
6 a 20 800, 21 600, 22 400
b Thirteen years

7 a \$2000 **b** \$250 **c** 12.5% **d** 6 years
e 4 years

8 3, 6, 12, 24, 48



9 a 1000, 100, 10, 1, 0.1
b



c 0.0000001

10 8192

11 a \$20 800, \$21 632, \$22 497.28
b Eleven years

12 a \$2000 **b** \$200 **c** 10% **d** 12 years
e 7 years

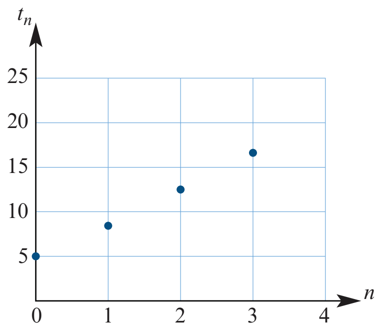
13 $V_0 = 2, V_{n+1} = 4^n \times V_0$

Written-response questions

1 a \$328 **b** \$3772 **c** \$3444
d $V_1 = V_0 - 328$ $V_2 = V_1 - 328$
e $V_{n+1} = 4100 - 328n$ **f** 10 years

2 a

n	0	1	2	3
t_n	5	9	13	17

b


c The points lie on a straight line with a positive slope. Thus, the sequence is arithmetic.

3 a $64 000 b $3200 c 5% d 8 years
e 10 years

4 a $29 500, $29 700, $29 550 b $21 000
c 100 000

5 a $6200 b 8%
c $6696, $7231.68, $7810.21
d $496, $535.68, $578.53

6 a $1380 b $V_0 = 9200, V_{n+1} = 0.85 \times V_n$
c $4082 d Four years

7 a A: $1100, B: $1000, C: $919
b A: $5500, B: $5526, C: $5036, so Option B gives the highest value.
c $5526

8 a $22 500 b 15% c $V_n = 22\,500 \times 0.85^n$
d $9983 e Five years

Chapter 4

Section 4A

Now try this

1 a 3×2 b Number of men in legal
c c_{31} d 91

Exercise 4A

1 a 2 b 4 c 2×4 d 8

2 a 53 b 36 c 83

3 a 3×4 b i 16 ii 3 iii 5
c 22 d 18

4 a i 2×3 ii 6, 7
b i 1×3 ii 2, 6
c i 3×2 ii −4, 5
d i 3×1 ii 9, 8
e i 2×2 ii 15, 12
f i 3×4 ii 20, 5

5 a B b D c E

6 *B, C, D*

7 a 9 b 2 c 3 d 10
e 8

8 a A: 4×3, B: 2×1, C: 1×2 D: 2×5
b $a_{32} = 4, b_{21} = -5, c_{11} = 8, d_{24} = 7$

9 a 32 b 3×4
c 22 Year 11 students prefer football

10 a i 75 ha ii 300 ha iii 200 ha
b 350 ha
c i Farm *Y* uses 0 ha for cattle.
ii Farm *X* uses 75 ha for sheep.
iii Farm *X* uses 150 ha for wheat.
d i f_{23} ii f_{12} iii f_{21}
e 2×3

11

$$\begin{array}{cc} & \begin{array}{cc} \textit{Pies} & \textit{Sausage Rolls} \end{array} \\ B = & \begin{bmatrix} 195 & 141 \\ 165 & 181 \end{bmatrix} \begin{array}{l} \textit{Bakery } 1 \\ \textit{Bakery } 2 \end{array} \end{array}$$

12

$$\begin{array}{cc} & \begin{array}{cccc} \textit{Ski} & \textit{Mountain bike} & \textit{Hike} & \textit{Kayak} \end{array} \\ S = & \begin{array}{l} \textit{Boys} \\ \textit{Girls} \end{array} \begin{bmatrix} 28 & 54 & 29 & 35 \\ 36 & 83 & 14 & 31 \end{bmatrix} \end{array}$$

Section 4B

Now try this

2 a $\begin{bmatrix} 14 & 3 & 13 \\ 9 & 1 & 7 \end{bmatrix}$

b $\begin{bmatrix} 4 & -6 \\ 7 & -3 \end{bmatrix}$

Exercise 4B

1 A, C, G or D, F or E, H

2 False

3 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4 a $\begin{bmatrix} 9 & 10 \\ 6 & 3 \end{bmatrix}$ b $\begin{bmatrix} 7 & 8 \\ 13 & 3 \end{bmatrix}$

c $\begin{bmatrix} 3 & 5 \\ 7 & 0 \end{bmatrix}$ d $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$

e $\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$ f $\begin{bmatrix} 4 & -2 \\ 3 & 9 \end{bmatrix}$

g $\begin{bmatrix} 12 & 7 \end{bmatrix}$ h $\begin{bmatrix} 0 & 0 \end{bmatrix}$

i $\begin{bmatrix} 0 & 0 \end{bmatrix}$

5 a $\begin{bmatrix} 8 & 5 \\ 3 & 7 \end{bmatrix}$ b $\begin{bmatrix} 8 & 5 \\ 3 & 7 \end{bmatrix}$

c $\begin{bmatrix} -2 & -9 \\ 1 & 1 \end{bmatrix}$ d $\begin{bmatrix} 2 & 9 \\ -1 & -1 \end{bmatrix}$

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e Not possible

f $\begin{bmatrix} 3 & 7 \\ 5 & -2 \\ 4 & -1 \end{bmatrix}$

g Not possible

h $\begin{bmatrix} -9 & 3 \\ 3 & -2 \\ -2 & 15 \end{bmatrix}$

6

	Liberal	*Labor*	*Independent*	*Green*
Men	43	42	10	5
Women	37	37	17	9

7

	Basketballs	Netballs	Cricket balls	Footballs
Store 1	−6	0	−57	−15
Store 2	−17	−5	−54	−13
Store 3	0	−7	−73	−43

8 a

	Arlo	*Beni*	*Cal*	*Dane*
Weight (kg)	6	8	−2	7
Height (cm)	5	8	7	6

b Beni

c Beni

9 $a = 4, b = 0, c = -3, d = 9$

10 $a = 1, b = -6, c = 7, d = 12$

11 Round 1:

	Kicks	*Goals*	*Handballs*
Jack	10	4	2
Nick	10	0	26
Mykola	18	1	12

Round 2:

	Kicks	*Goals*	*Handballs*
Jack	7	2	8
Nick	13	2	12
Mykola	9	2	11

Round 3:

	Kicks	*Goals*	*Handballs*
Jack	6	3	20
Nick	20	1	19
Mykola	12	4	11

Total:

	Kicks	*Goals*	*Handballs*
Jack	23	9	30
Nick	43	3	57
Gary	39	7	34

Section 4C

Now try this

3 $\begin{bmatrix} -35 & 10 \\ 30 & -15 \end{bmatrix}$

4 $\begin{bmatrix} 1 & -4 \\ 6 & 1 \end{bmatrix}$

5

	Hamburger	*Chips*	*Can of Softdrink*
Sydney	14.30	4.95	3.30
Melbourne	13.20	5.50	2.75

Exercise 4C

1 $\begin{bmatrix} 6 & -3 \\ 24 & 21 \end{bmatrix}$

2 $\begin{bmatrix} 5 \times -6 & 5 \times 7 \\ 5 \times 3 & 5 \times -2 \end{bmatrix}$

3 $\begin{bmatrix} 2 \times -2 + 3 \times 4 & 2 \times 5 + 3 \times -1 \\ 2 \times 1 + 3 \times 0 & 2 \times 0 + 3 \times 3 \end{bmatrix}$

4 a $\begin{bmatrix} 14 & -2 \\ 8 & 18 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -10 \\ 25 & 35 \end{bmatrix}$

c $\begin{bmatrix} -64 & 12 \\ -6 & -14 \end{bmatrix}$ **d** $\begin{bmatrix} 2.25 & 0 \\ -3 & 7.5 \end{bmatrix}$

e $\begin{bmatrix} 18 & 21 \end{bmatrix}$ **f** $\begin{bmatrix} -12 \\ 30 \end{bmatrix}$

g $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ **h** $\begin{bmatrix} -3 & -6 & 8 \end{bmatrix}$

5 a $\begin{bmatrix} 9 & -12 \\ 6 & 15 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 28 \\ -6 & -28 \end{bmatrix}$

c $\begin{bmatrix} 1 & -32 \\ 8 & 33 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

e $\begin{bmatrix} 21 & 18 \\ 3 & -12 \end{bmatrix}$

6 a $\begin{bmatrix} 79 & -31 \\ 68 & -36 \end{bmatrix}$ **b** $\begin{bmatrix} -121 & 50 \\ -84 & 103 \end{bmatrix}$

c $\begin{bmatrix} 13 & -2 \\ 36 & 53 \end{bmatrix}$ **d** $\begin{bmatrix} 69 & -27 \\ 60 & -30 \end{bmatrix}$

7 a $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 5 & 3 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 6 \\ 14 \\ 8 \end{bmatrix}$

d $\begin{bmatrix} 0 & 15 & 9 & 9 \end{bmatrix}$

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8 a

$$\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 1650 & 960 & 1155 & 2235 \end{bmatrix} \end{array}$$

b 960 mL

9 a

$$\begin{array}{l} \\ \text{Store } A \\ \text{Store } B \\ \text{Store } C \end{array} \begin{array}{c} \begin{matrix} \text{Clothing} & \text{Furniture} & \text{Electronics} \end{matrix} \\ \begin{bmatrix} 6 & 2 & 9 \\ 5 & 1 & 9 \\ 4 & -1 & 5 \end{bmatrix} \end{array}$$

b

$$\begin{array}{l} \\ \text{Store } A \\ \text{Store } B \\ \text{Store } C \end{array} \begin{array}{c} \begin{matrix} \text{Clothing} & \text{Furniture} & \text{Electronics} \end{matrix} \\ \begin{bmatrix} 1.8 & 0.6 & 2.7 \\ 1.5 & 0.3 & 2.7 \\ 1.2 & 0 & 1.5 \end{bmatrix} \end{array}$$

10 a

$$\begin{array}{r} \\ \text{Gymnastics rings} \\ \text{Parallel bars} \end{array} \begin{array}{c} \text{Wins} \\ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{array}$$

b

$$\begin{array}{r} \\ \text{Gymnastics rings} \\ \text{Parallel bars} \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 150 \\ 100 \end{bmatrix} \end{array}$$

11 a $\begin{bmatrix} 0 & 29 \\ -26 & 70 \end{bmatrix}$

b undefined

c undefined

12 a $a = -1, b = 2, c = -8, d = 32$

b $a = 4, b = 4, c = -48, d = 31$

Section 4D

Now try this

6 2×3

7 $\begin{bmatrix} -20 & 1 & 25 \\ 37 & 13 & -5 \end{bmatrix}$

8 $\begin{bmatrix} -23 & 16 & 70 \\ 174 & 21 & -135 \end{bmatrix}$

9 a $\begin{bmatrix} -3 & 9 \\ 2 & -7 \end{bmatrix}$

b $\begin{bmatrix} -3 & 9 \\ 2 & -7 \end{bmatrix}$

Exercise 4D

1 $\begin{bmatrix} 2 \times 6 + (-5) \times 7 \\ 3 \times 6 + 1 \times 7 \end{bmatrix}$

2 A:2×3, B:3×2. Yes, they can be multiplied to give AB

3 4×4

4 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5 a Defined: 2×1 **b** Not defined

c Defined: 3×1 **d** Not defined

e Defined: 2×2 **f** Not defined

g Not defined **h** Defined: 3×2

6 a 1×2 and 2×1, defined

b 1×2 and 3×1, not defined

c 1×3 and 3×1, defined

d 1×3 and 2×1, not defined

e 1×4 and 4×1, defined

f 1×4 and 3×1, not defined

7 a $\begin{bmatrix} 22 \\ 33 \end{bmatrix}$ **b** $\begin{bmatrix} 64 \\ 53 \end{bmatrix}$ **c** $\begin{bmatrix} -4 & -3 \\ -14 & -20 \end{bmatrix}$

d $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ **e** $\begin{bmatrix} 11 \\ 1 \\ 7 \end{bmatrix}$ **f** $\begin{bmatrix} 21 \end{bmatrix}$ **g** $\begin{bmatrix} 8 \end{bmatrix}$

h $\begin{bmatrix} 30 \end{bmatrix}$ **i** $\begin{bmatrix} 36 \end{bmatrix}$

8 a $\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$ **b** $\begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$

c No

9 a $\begin{bmatrix} 0 & -8 \\ 4 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 16 & 14 \\ 16 & 14 \end{bmatrix}$

c $\begin{bmatrix} 83 \end{bmatrix}$ **d** $\begin{bmatrix} 4 \end{bmatrix}$ **e** $\begin{bmatrix} 3 & 3 \end{bmatrix}$ **f** $\begin{bmatrix} 31 \\ 35 \\ 21 \end{bmatrix}$

10 a i $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **iii** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b i $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c i $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **ii** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

11 AC: 3×3, CA: 1×1, BC: 2×3 DB: 2×1

12 $\begin{bmatrix} 27 & 5 & 13 \\ 18 & 15 & -3 \end{bmatrix}$

13 $\begin{bmatrix} 1 & 4 \\ -7 & -2 \end{bmatrix}$

Section 4E

Now try this

10 39

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11 $\begin{bmatrix} 6 & -1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ -19 \end{bmatrix}$

Exercise 4E

1 True

2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3 a $\begin{bmatrix} 3 & 7 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27 \\ 28 \end{bmatrix}$

b $\begin{bmatrix} 2 & 8 \\ 3 & 20 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

4 a $\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & -2 \\ -2 & 4.5 \end{bmatrix}$

c $\begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}$ **d** $\begin{bmatrix} -1.5 & 3.5 \\ 1 & -2 \end{bmatrix}$

e $\begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}$

f $\begin{bmatrix} 0.1 & -0.2 & 0.35 \\ 0.4 & 0.2 & -0.6 \\ -0.1 & 0.2 & 0.15 \end{bmatrix}$

g $\begin{bmatrix} 16 & 10 & -15 \\ -8 & -5 & 8 \\ 3 & 2 & -3 \end{bmatrix}$

h $\begin{bmatrix} -0.25 & 0.125 & 0.5 \\ 0 & 0.5 & 0 \\ 0.5 & -0.25 & 0 \end{bmatrix}$

5 a $\begin{bmatrix} -\frac{6}{31} & \frac{1}{31} \\ \frac{7}{31} & \frac{4}{31} \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{25}{566} & \frac{19}{566} \\ \frac{7}{283} & \frac{6}{283} \end{bmatrix}$

c $\begin{bmatrix} \frac{5}{344} & \frac{41}{2752} \\ -\frac{1}{86} & \frac{9}{688} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{767}{32} & \frac{21}{4} \\ -\frac{103}{8} & 13 \end{bmatrix}$

6 a 19 **b** −13 **c** −86 **d** −32

7 a $\begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{7} & \frac{1}{28} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{1}{14} & -\frac{3}{14} \\ \frac{3}{28} & \frac{5}{28} \end{bmatrix}$

c $\begin{bmatrix} -\frac{3}{98} & \frac{1}{98} \\ \frac{5}{196} & \frac{13}{392} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{3}{98} & \frac{1}{98} \\ \frac{5}{196} & \frac{13}{392} \end{bmatrix}$

8 a $\begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 12 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

c $\begin{bmatrix} 7 & -2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -31 \\ -1 \end{bmatrix}$

d $\begin{bmatrix} 6 & 5 \\ 9 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 38 \\ 66 \end{bmatrix}$

9 a $x = 2, y = 3$ **b** $x = 1, y = 2$
c $x = 1, y = -2$ **d** $x = 4, y = 6$
e $x = 9, y = 2$ **f** $x = -2, y = 3$

10 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

11

$3x - 2y = 23$

$7x - 5y = 55$

12 a $a = -\frac{8}{29}, b = \frac{7}{29}, c = \frac{3}{29}, d = \frac{1}{29}$

b $a = \frac{2}{43}, b = \frac{3}{43}, c = -\frac{11}{86}, d = \frac{5}{86}$

Section 4F

Now try this

12 a

$$\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

b The sum of the elements in column D is the total number of roads directly connected to Town D.

13 a

$$\begin{array}{c} \begin{matrix} K & M & D & W \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} K \\ M \\ D \\ W \end{matrix}$$

A 1 represents communication and a 0 represents no direct communication.

b The symmetry occurs because communication is two-way.

c The sum of a column or row gives the total number of people that a given person can communicate with.

d

$$\begin{array}{c} \begin{matrix} K & M & D & W \end{matrix} \\ \begin{bmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array} \begin{matrix} K \\ M \\ D \\ W \end{matrix}$$

e One way. (Matt-Will-Kate)

Exercise 4F

1 A, E

2 A, C

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3
$$\begin{array}{cc} \begin{matrix} A & B & C & D & E \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \end{array}$$

4 a i
$$\begin{array}{cc} \begin{matrix} A & B \end{matrix} & \\ \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} & \begin{matrix} A \\ B \end{matrix} \end{array}$$

ii
$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

iii
$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

iv
$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

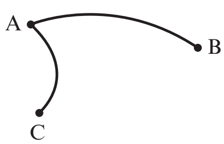
v
$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

vi
$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

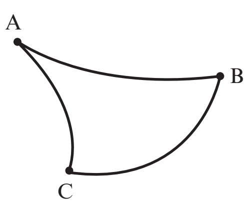
b The number of roads directly connected to B.

5 a Many answers are possible. Examples:

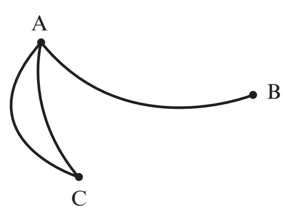
i



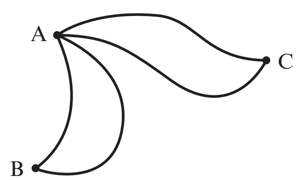
ii



iii



iv



b The number of roads directly connected to town A.

6 a
$$\begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b Compare the sums of the rows (or columns). The person with the highest total has met the most people.

c Person B

d Person C

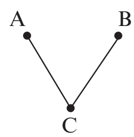
7 D communicates with A, yet A does not communicate with D.

8 a
$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

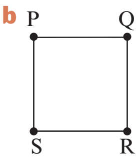
b
$$\begin{array}{cc} \begin{matrix} D & E & F & G \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{matrix} D \\ E \\ F \\ G \end{matrix} \end{array}$$

c
$$\begin{array}{cc} \begin{matrix} J & K & L & M \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \begin{matrix} J \\ K \\ L \\ M \end{matrix} \end{array}$$

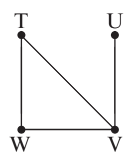
9 a



b



c



10 a
$$Q = \begin{array}{cc} \begin{matrix} C & E & K & R \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} & \begin{matrix} C \\ E \\ K \\ R \end{matrix} \end{array}$$

b Remy communicates with 3 people.

c i
$$Q^2 = \begin{array}{cc} \begin{matrix} C & E & K & R \end{matrix} & \\ \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix} & \begin{matrix} C \\ E \\ K \\ R \end{matrix} \end{array}$$

ii Add column E to get 6 ways.

iii $E \to R \to C$
$E \to R \to E$
$E \to C \to E$
$E \to R \to K$
$E \to C \to K$
$E \to C \to R$

11 a
$$R = \begin{array}{cc} \begin{matrix} E & F & G & H \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{matrix} E \\ F \\ G \\ H \end{matrix} \end{array}$$

b Three roads directly connected to Fields.

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c i

$$R^2 = \begin{array}{c} \\ \\ \end{array}\begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}\begin{matrix} E \\ F \\ G \\ H \end{matrix} \quad \text{(columns } E\ F\ G\ H\text{)}$$

ii 7

iii

$F \to G \to E$

$F \to E \to F$

$F \to G \to F$

$F \to H \to F$

$F \to E \to G$

$F \to H \to G$

$F \to G \to H$

12 a

$$\begin{matrix} & A & B & C & D & E & \\ \end{matrix}$$
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}\begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}$$

b Three

c $E - C - A - D$ or $E - B - A - D$

13 a

$$\begin{matrix} N & S & W & H & Ca & Co & Pl & PF & T & K \end{matrix}$$
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{matrix} N \\ S \\ W \\ H \\ Ca \\ Co \\ Pl \\ PF \\ T \\ K \end{matrix}$$

b 3

c 8

d All 1s with 0s on the diagonal

e One 2 in each row and column, with 0s on the diagonal and 1s in each of the other cells.

Section 4G

Now try this

14

$$B = \begin{matrix} & H & C \\ \end{matrix}$$
$$B = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}\begin{matrix} H \\ C \end{matrix} \quad \text{(columns } H\ C\text{)}$$

15 a H: 90%, C: 10% **b** 360 **c** 560

Exercise 4G

1

$$T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{matrix} A \\ B \end{matrix} \quad \text{(columns } A\ B\text{)}$$

2 a 40% **b** 50%

3 a 45 **b** 15 **c** 60

4 a 40%

b 20%

c This week

$$T = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{matrix} P \\ D \end{matrix} \text{ Next week} \quad \text{(columns } P\ D\text{)}$$

5 a i 30%

ii 40%

b Today

$$T = \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}\begin{matrix} R \\ N \end{matrix} \text{ Tomorrow} \quad \text{(columns } R\ N\text{)}$$

6 This time

$$T = \begin{bmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{bmatrix}\begin{matrix} A \\ B \end{matrix} \text{ Next time} \quad \text{(columns } A\ B\text{)}$$

7 a i 75%

ii 25%

b 120

c 40

d 160

e 100

8 a 153 **b** 21 **c** 174 **d** 26

9 This time

$$\begin{bmatrix} 0.4 & 0.3 & 0.5 \\ 0.4 & 0.6 & 0.2 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}\begin{matrix} X \\ Y \\ Z \end{matrix} \text{ Next time} \quad \text{(columns } X\ Y\ Z\text{)}$$

Section 4H

Now try this

16 One day: H - 560, C - 340, Three days: H - 680, C - 220

17 a 1 year: 7600, 10 400, 2 years: 7400, 10 600, 3 years: 7300, 10 700

b 20 years: 7200, 10 800, 30 years: 7200, 10 800, 40 years: 7200, 10 800

c Over time, the predicted population in site one declines and settles at 7200 while the population at site B increases and settles at 10 800.

Exercise 4H

1

$$\begin{bmatrix} 100 \\ 25 \end{bmatrix}\begin{matrix} O \\ B \end{matrix}$$

2 65

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3 a

$$S_1 = \begin{bmatrix} 100 \\ 200 \end{bmatrix} \quad S_2 = \begin{bmatrix} 170 \\ 130 \end{bmatrix} \quad S_3 = \begin{bmatrix} 121 \\ 179 \end{bmatrix}$$

b

$$\begin{bmatrix} 0.73 & 0.24 \\ 0.27 & 0.76 \end{bmatrix}$$

c

$$S_7 = \begin{bmatrix} 136.3321 \\ 163.6679 \end{bmatrix}$$

4 a 198 **b** 195, 155 **c** 194, 156 **d** 194, 156

5 a

$$T = \overset{\text{This week}}{\begin{matrix} A & I \end{matrix}} \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix} \begin{matrix} A \\ I \end{matrix} \text{ Next week}$$

b

$$S_0 = \begin{bmatrix} 50 \\ 0 \end{bmatrix}$$

c 45

d 10

6 a

$$T = \overset{\text{This week}}{\begin{matrix} R & I \end{matrix}} \begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \begin{matrix} R \\ I \end{matrix} \text{ Next week}$$

b

$$S_0 = \begin{bmatrix} 4000 \\ 2000 \end{bmatrix}$$

c 3200, 800

d 500, 1500

e 3352

f 3333, 2667

7 a

$$T = \overset{\text{Today}}{\begin{matrix} D & P \end{matrix}} \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{matrix} D \\ P \end{matrix} \text{ Tomorrow}$$

b

$$S_0 = \begin{bmatrix} 40 \\ 160 \end{bmatrix}$$

c 36, 4

d 128, 32

e 82

f 67, 133

8 a

$$T = \overset{\text{This week}}{\begin{matrix} B & M & R \end{matrix}} \begin{bmatrix} 0.7 & 0.15 & 0.2 \\ 0.2 & 0.55 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \begin{matrix} B \\ M \\ R \end{matrix} \text{ Next week}$$

b

$$S_0 = \begin{bmatrix} 300 \\ 200 \\ 50 \end{bmatrix}$$

c B: 250, M: 180, R: 120

d B: 1609, M: 1232, R: 1009

e B: 203, M: 169, R: 178

Section 4I

Now try this

18

$$\begin{bmatrix} 132 & 23 \\ 120 & 22 \end{bmatrix}$$

Jacky has sales of \$132 and gives 23 tickets.
Peter has sales of \$120 and gives 22 tickets.

19 a

$$\begin{bmatrix} 290 \\ 160 \\ 200 \\ 180 \end{bmatrix}$$

b Total time spent watching television by each student.

c

$$\begin{bmatrix} 115 & 75 & 80 & 70 & 490 \end{bmatrix}$$

d Total time spent watching television each day.

Exercise 4I

1 a True

b

$$\begin{bmatrix} 214 \times 3 + 103 \times 2 \\ 162 \times 3 + 189 \times 2 \end{bmatrix}$$

c Top line tells you Joe's revenue from selling cans and bottles.

2 5800 kJ

3

$$\begin{matrix} \\ \textit{Smith} \\ \textit{Jones} \end{matrix} \overset{\begin{matrix} \textit{Wheels} & \textit{Seats} \end{matrix}}{\begin{bmatrix} 14 & 13 \\ 12 & 9 \end{bmatrix}}$$

4 110

5 a

$$\overset{\begin{matrix} \textit{Quiche} & \textit{Soup} & \textit{Coffee} \end{matrix}}{\begin{bmatrix} 18 & 12 & 64 \end{bmatrix}}$$

b

$$\begin{matrix} \textit{Quiche} \\ \textit{Soup} \\ \textit{Coffee} \end{matrix} \overset{\$}{\begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}}$$

c \$378

6 a

$$\overset{\begin{matrix} \textit{Chips} & \textit{Pastie} & \textit{Pie} & \textit{Sausage roll} \end{matrix}}{\begin{bmatrix} 90 & 84 & 112 & 73 \end{bmatrix}}$$

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b
$$\begin{array}{r} \\ Chips \\ Pastie \\ Pie \\ Sausage\ roll \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 4 \\ 5 \\ 5 \\ 3 \end{bmatrix} \end{array}$$

c $1559

7 a
$$\begin{array}{c} \\ I \\ J \\ K \end{array} \begin{array}{c} Hrs \\ \begin{bmatrix} 10 \\ 7 \\ 12 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ I \\ J \\ K \end{array} \begin{array}{c} Av.\ Hrs \\ \begin{bmatrix} 2.5 \\ 1.75 \\ 3 \end{bmatrix} \end{array}$$

c
$$Hrs \begin{array}{c} \begin{matrix} M & Tu & W & Th \end{matrix} \\ \begin{bmatrix} 6 & 11 & 5 & 7 \end{bmatrix} \end{array}$$

d
$$Av.\ Hrs \begin{array}{c} \begin{matrix} M & Tu & W & Th \end{matrix} \\ \begin{bmatrix} 2 & 3.7 & 1.7 & 2.3 \end{bmatrix} \end{array}$$

8 a
$$\begin{array}{c} \\ E \\ F \\ G \\ H \end{array} \begin{array}{c} Total\ score \\ \begin{bmatrix} 446 \\ 415 \\ 329 \\ 409 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ E \\ F \\ G \\ H \end{array} \begin{array}{c} Av.\ score \\ \begin{bmatrix} 89.2 \\ 83 \\ 65.8 \\ 81.8 \end{bmatrix} \end{array}$$

c
$$Test\ total \begin{array}{c} \begin{matrix} T1 & T2 & T3 & T4 & T5 \end{matrix} \\ \begin{bmatrix} 319 & 307 & 324 & 292 & 357 \end{bmatrix} \end{array}$$

d
$$Test\ av. \begin{array}{c} \begin{matrix} T1 & T2 & T3 & T4 & T5 \end{matrix} \\ \begin{bmatrix} 79.75 & 76.75 & 81 & 73 & 89.25 \end{bmatrix} \end{array}$$

9
$$k \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
The matrix multiplies each item in a second matrix by k.

10 a $\begin{bmatrix} 52 \\ 64 \\ 44 \end{bmatrix}$ Mobile phone bill for Q3 **b** $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

c 1×4 **d** 1×3 Pre-multiply

e $G = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

Chapter 4 Review

Skills Checklist answers

1 3×3 **2** $b_{23} = 6$ **3** $\begin{bmatrix} 11 & 12 \\ 14 & 0 \end{bmatrix}$

4 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **5** $\begin{bmatrix} 20 & -5 \\ 10 & 0 \end{bmatrix}$

6 $\begin{bmatrix} 16 & 3 \\ -4 & 3 \end{bmatrix}$

7 BA is possible. AB is not possible.

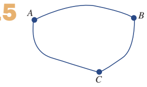
8 2×4 **9** $\begin{bmatrix} 7 & -5 & 45 \\ -3 & -1 & -9 \\ -13 & -19 & 9 \end{bmatrix}$

10 $\begin{bmatrix} 143 & -105 \\ 80 & 36 \end{bmatrix}$ **11** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12 $\frac{1}{23}\begin{bmatrix} 3 & -2 \\ 1 & 7 \end{bmatrix}$ **13** $x = \frac{88}{17}, y = \frac{39}{17}$

14
$$\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

15



16
$$\begin{array}{c} \begin{matrix} D & N \end{matrix} \\ \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \end{array} \begin{array}{c} \\ D \\ N \end{array}$$

17 147 **18** 224 068

19 216 000 **20** $\begin{bmatrix} 44 \\ 38 \end{bmatrix}$ **21** $\begin{bmatrix} 98 & 25 \\ 85 & 31 \end{bmatrix}\begin{bmatrix} 123 \\ 116 \end{bmatrix}$

Multiple-choice questions

1 B	**2** E	**3** C	**4** D	**5** A
6 D	**7** D	**8** C	**9** A	**10** E
11 D	**12** B	**13** E	**14** A	**15** D
16 E				

Short-answer questions

1 2×4

2 1

3 $\begin{bmatrix} 38 & 34 & 47 & 54 \end{bmatrix}$

4 2×1

5
$$\begin{array}{c} \begin{matrix} P & Q & R \end{matrix} \\ \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ P \\ Q \\ R \end{array}$$

6 a $\begin{bmatrix} 9 & 3 \\ 12 & 6 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 6 \\ 11 & 8 \end{bmatrix}$

c $\begin{bmatrix} -3 & 4 \\ 3 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} 6 & 7 \\ 15 & 10 \end{bmatrix}$

e $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **f** $\begin{bmatrix} 7 & 21 \\ 14 & 32 \end{bmatrix}$

g $\begin{bmatrix} 20 & 10 \\ 45 & 19 \end{bmatrix}$ **h** $\begin{bmatrix} 1 & -0.5 \\ -2 & 1.5 \end{bmatrix}$

i $\begin{bmatrix} 13 & 5 \\ 20 & 8 \end{bmatrix}$ **j** $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

k $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Written-response questions

1 a 40 pigs **b** 320 sheep **c** Farm A

2 a 21 pies **b** $2 **c** $\begin{bmatrix} 104 \\ 103 \end{bmatrix}$

d Value of sales for each shop

e Shop A, $104

3 a

$$\begin{array}{c c} & \begin{array}{cc} \textit{Hours walking} & \textit{Hours jogging} \end{array} \\ \begin{array}{c} Patsy \\ Geoff \end{array} & \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \end{array}$$

b

$$\begin{array}{c c} & \begin{array}{cc} \$ & kJ \end{array} \\ \begin{array}{c} Walking \\ Jogging \end{array} & \begin{bmatrix} 2 & 1500 \\ 3 & 2500 \end{bmatrix} \end{array}$$

c

$$\begin{array}{c c} & \begin{array}{cc} \$ & kJ \end{array} \\ \begin{array}{c} Patsy \\ Geoff \end{array} & \begin{bmatrix} 11 & 8500 \\ 12 & 9500 \end{bmatrix} \end{array}$$

4 a 40% **b**

$$T = \begin{array}{c} \begin{array}{cc} A & N \end{array} \\ \begin{bmatrix} 0.6 & 0.05 \\ 0.4 & 0.95 \end{bmatrix} \end{array} \begin{array}{c} A \\ N \end{array}$$

c $\begin{bmatrix} 4000 \\ 26\,000 \end{bmatrix}$ **d** 3700 **e** 3367

f 3333, 26 667

5 a 1720 **b** 1250

6 a $6x + 5y = 14$

b Apple $1.50, banana $1

Chapter 5

Section 5A

Now try this

1 $1130

2 82.6 cm

3

L	0	10	20	30	40	50	60	70	80	90	100
P	0	40	80	120	160	200	240	280	320	360	400

Exercise 5A

1 a $420 **b** $540

2 a 12 cm **b** 18.6 cm **c** $L = 7.9$, $W = 2.7$

3 a $129 **b** $132

c

x	40	41	42	43	44	45
C($)	120	123	126	129	132	135

4 a $1400 **b** $1500 **c** $1425

5 380 km

6 a $10.50 **b** $14.40 **c** $30

7 a 18.85 mm **b** 45.24 m

8 a $A = 12$ **b** $A = 120$ **c** $A = 22.5$

9 a $A = 4$ **b** $A = 14.25$

10 a 10°C **b** −17.8°C

c 100°C **d** 33.3°C

11 a $2400 **b** $2014.50

12 a i 15 points **ii** 68 points

b Greenteam

13

r	0	0.1	0.2	0.3	0.4	0.5
C	0	0.628	1.257	1.885	2.513	3.142

r	0.6	0.7	0.8	0.9	1.0
C	3.770	4.398	5.027	5.655	6.283

14

n	50	60	70	80	90	100
C($)	49	50.80	52.60	54.40	56.20	58.00

n	110	120	130
C($)	59.8	61.6	63.4

15

M(kg)	60	65	70	75	80	85	90
E(kJ)	650	695	740	785	830	875	920

M(kg)	95	100	105	110	115	120
E(kJ)	965	1010	1055	1100	1145	1190

16

T (years)	1	2	3	4	5
I($)	450	900	1350	1800	2250

T (years)	6	7	8	9	10
I($)	2700	3150	3600	4050	4500

17

n	3	4	5	6
S	180°	360°	540°	720°

n	7	8	9	10
S	900°	1080°	1260°	1440°

18 a 13 **b** 23 **c** 101

19 a i 1 h 50 min **ii** 2 h 56 min

iii 3 h 6 min **iv** 2 h 38 min

b 5:15 p.m.

Section 5B

Now try this

4 $x = 8$ **5** $x = -10$ **6** $P = 20 + 2x$

7 42 **8** 5 days

Exercise 5B

1 a $x = 9$ **b** $y = 15$ **c** $m = 6$ **d** $m = -2$

e $e = 3$ **f** $n = 4$

2 a $x = 3$ **b** $g = 9$ **c** $j = -4$ **d** $r = 12$

e $t = -12$ **f** $h = 40$

3 a Subtract 15 from both sides of the equation.

b Divide by 2. **c** $a = 6$

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4 a Add 10 to both sides of the equation.
b Multiply by 4. **c** $y = 40$

5 a $v = -5$ **b** $k = 7$ **c** $a = 8$ **d** $b = 5$
e $r = 8$ **f** $x = 5$

6 a $y = 4$ **b** $x = 11$ **c** $g = 2$ **d** $x = 3$
e $x = 0.5$ **f** $m = 1.2$ **g** $a = 18$ **h** $r = 13$

7 a $x = 5$ **b** $a = 3$ **c** $b = 9$ **d** $y = 3$
e $x = 0$ **f** $c = -4$ **g** $f = -2$ **h** $y = -5$

8 a $a = 2$ **b** $b = 6$ **c** $w = 2$ **d** $c = 2$
e $y = 7$ **f** $f = 2$ **g** $h = 5$ **h** $k = 1.3$
i $g = 8.5$ **j** $s = 20$ **k** $t = 2.2$ **l** $y = 2$
m $x = -2$ **n** $g = 37$ **o** $p = 2$

9 a $P = 27 + x$ **b** $P = 4x$
c $P = 2a + 2b$ **d** $P = 19 + y$

10 a $P = 22 + m$ **b** 8 cm

11 a $P = 4y$ **b** 13 cm

12 a $n + 7 = 15$ **b** 8

13 6

14 59

15 a t is time in hours
b $C = 20 + 15t$
c i \$50 **ii** \$95

16 a $P = 4x + 12$ **b** 18 cm
c 24 cm, 18 cm

17 78 tickets

18 25 invitations

19 Anne \$750, Barry \$250

20 30 km

21 a 47 min **b** Ben 9.4 km, Amy 7.8 km

Section 5C

Now try this

9 a $C = 3.5x + 4.75y$ **b** \$106.25

Exercise 5C

1 a \$4.50x **b** \$14.30y **c** $C = 4.5x + 14.3y$

2 a \$4 **b** \$2.50 **c** \$42

3 a $C = 0.5x + 0.2y$ **b** \$16.50

4 a $C = 40x + 25y$ **b** \$13 875

5 a $C = 1.6x + 1.4y$ **b** \$141.20

6 a $C = 1.75x + 0.7y$ **b** \$52.15

7 a $C = 3.5x + 5y$ **b** \$312

8 a $C = 30x + 60y$ **b** \$3480

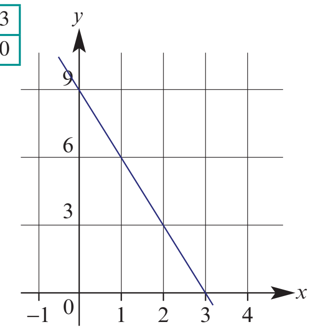
9 a $N = x + y$ **b** $V = 0.5x + 0.2y$
c \$37.90

Section 5D

Now try this

10

x	0	1	2	3
y	9	6	3	0



11 3 **12** −0.5

Exercise 5D

1 A negative, B positive, C undefined, D zero.

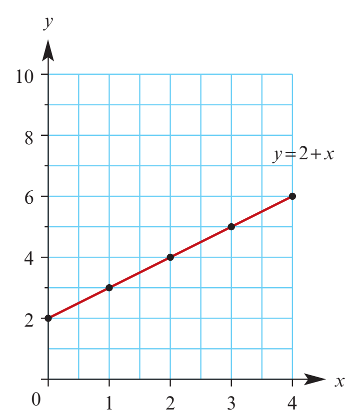
2

x	-2	-1	0	1	2	3
y	-1	1	3	5	7	9

3 a 3 **b** 1 **c** 3

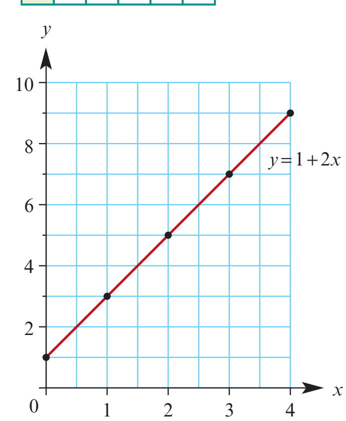
4 a

x	0	1	2	3	4
y	2	3	4	5	6



b

x	0	1	2	3	4
y	1	3	5	7	9



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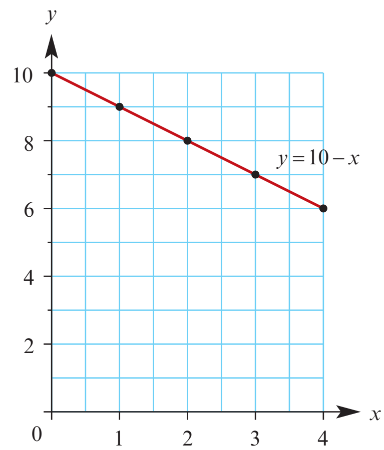
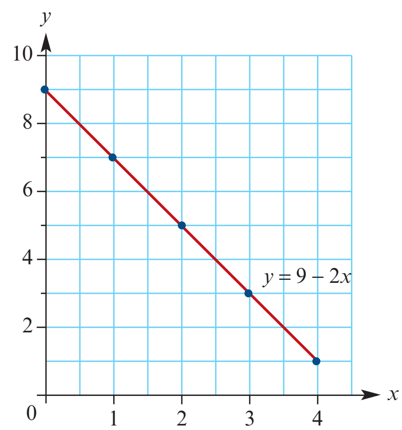
c

x	0	1	2	3	4
y	10	9	8	7	6

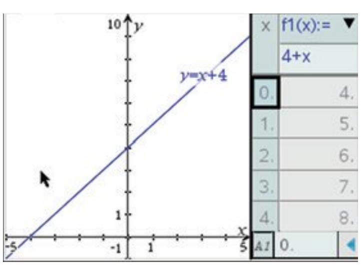


d

x	0	1	2	3	4
y	9	7	5	3	1



5 a



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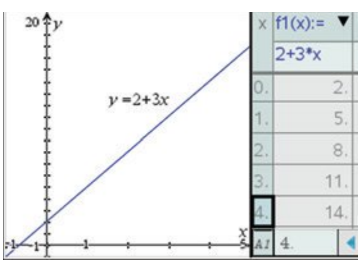
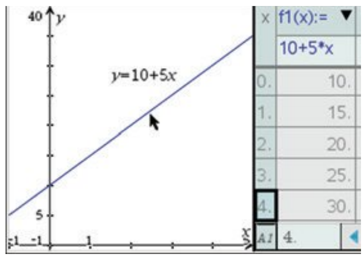
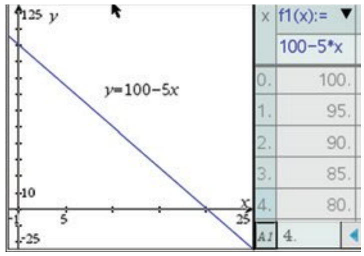


c



d



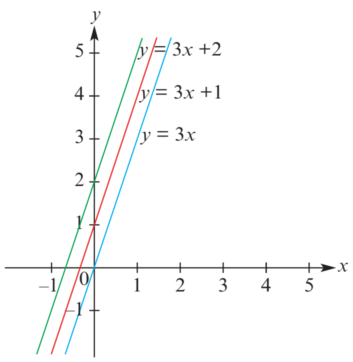
6 a (0, 4), (2, 6), (3, 7), (5, 9)
b (0, 8), (1, 6), (2, 4), (3, 2)

7 A − 2.33, B 1.75, C 1.00

8 A 2, B − 3, C 0

9 $-\frac{1}{2}$

10 a



b 3, 3, 3
c The lines are parallel. The slopes are all 3.

Section 5E

Now try this

13 −1.25

14 a y-intercept = 4, slope = -2
b y-intercept = 0, slope = 3
c y-intercept = 6, slope = 3

15 a $y = 5 + 3x$ **b** $y = 6 - 2x$
c $y = -4 + 5x$

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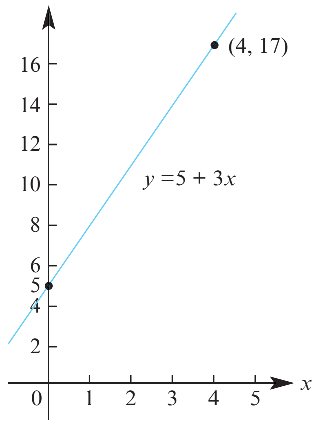
16



Exercise 5E

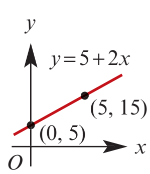
1 a 2 **b** 2 **c** 2 **d** -1
e -1 **f** $-\frac{5}{6}$

2 a y−intercept = 5, slope = 2
b y−intercept = 6, slope = -3
c y−intercept = 15, slope = -5
d y−intercept = 0, slope = 3

3 a y-intercept = 10, slope = −3
b y-intercept = −5, slope = −2
c y-intercept = 4, slope = 1
d y-intercept = 3, slope = 0.5
e y-intercept = −5, slope = 2
f y-intercept = 10, slope = 5
g y-intercept = 10, slope = −1
h y-intercept = 0, slope = 2
i y-intercept = 6, slope = −3
j y-intercept = 3, slope = 0
k y-intercept = −3, slope = 0.8
l y-intercept = −2, slope = 3

4 a $y = 2 + 5x$ **b** $y = 5 + 10x$
c $y = -2 + 4x$ **d** $y = -3x$ **e** $y = -2$
f $y = 1.8 - 0.4x$ **g** $y = 2.9 - 2x$
h $y = -1.5 - 0.5x$

5 a



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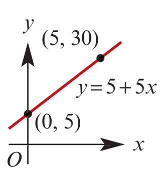

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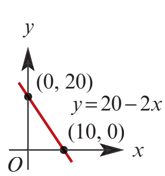
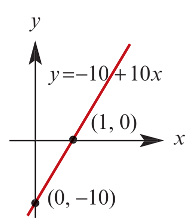
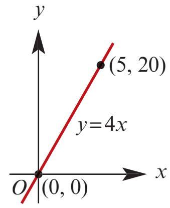


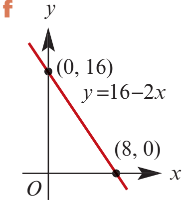
d



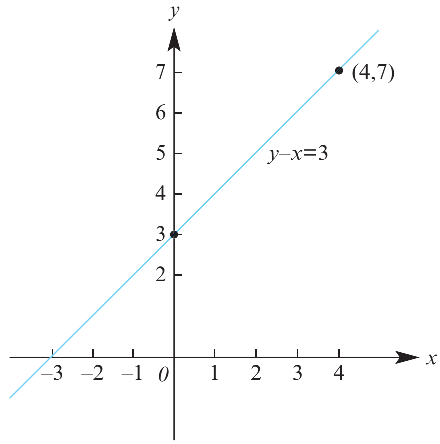
e



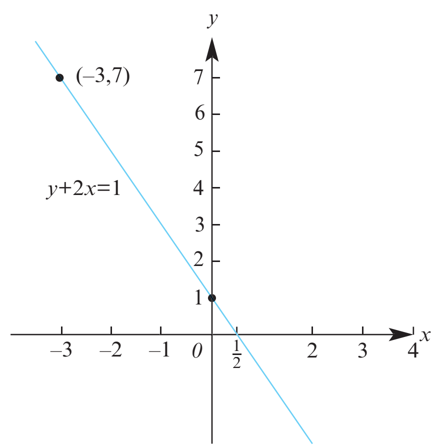
f



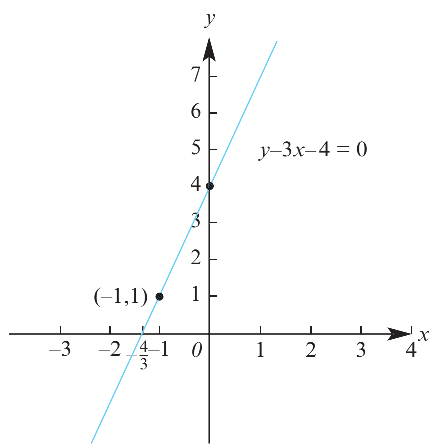
6 a



b



c



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Section 5F

Now try this

17 $y = 3 + 2x$ **18** $y = -8 + 6x$

Exercise 5F

1 a 1 **b** 1.5 **c** $y = 1 + 1.5x$

2 a 2 **b** -6

3 a -3 **b** 2

4 A: $y = 10 - 2.25x$ B: $y = 2 + 1.75x$
C: $y = x$

5 A: $y = 4 + 2x$ B: $y = 8 - 1.5x$
C: $y = 2 + 0.6x$

6 A: $y = 14.5 - 4.5x$ B: $y = -5 + 5x$
C: $y = -5 + 3x$

7 A: $y = 11.5 - 1.5x$ B: $y = -10 + 10x$
C: $y = 2 + 1.2x$

8 $y = 1 - 0.5x$

9 $y = 2 + 1.4x$

10 $y = -2x$

Section 5G

Now try this

19 a 4000 L **b** 7000 L
c 200 minutes or 3 hours 20 minutes
d $V = 4000 + 30t$ **e** 8 500 L
f 30 L/minute

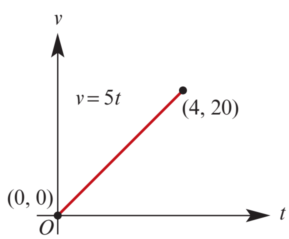
20 a \$40 000 **b** \$25 000 **c** $V = 40000 - 5000t$
d \$5 000 per year **e** 8 years

Exercise 5G

1 a 45 L **b** 40 L **c** 25 L

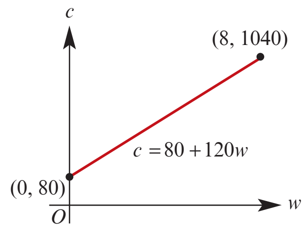
2 a 3 cm **b** 6 cm **c** 9 cm
d $\frac{3}{4} = 0.75$ **e** $h = 3 + 0.75t$

3 a $V = 5t$ for $0 \le t \le 4$
b



c 16 litres

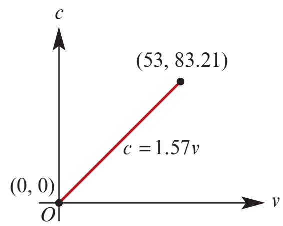
4 a $C = 80 + 120w$
b



c \$680

5 a \$10
b \$17.50
c $C = 10 + 0.075n$
d \$32.50
e \$0.075 = 8 cents

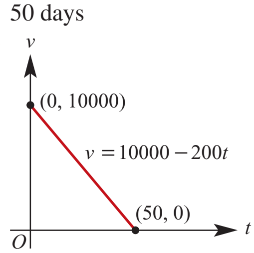
6 a $C = 1.57v$, for $0 \le v \le 53$
b



c \$83.21

7 a 500 mL **b** 400 mL **c** 200 minutes
d $V = 500 - 2.5t$ **e** 212.5 mL
f 2.5 mL/min

8 a $V = 10\,000 - 200t$
b 50 days
c



d 4000 litres

9 a $F = 32 + 1.8C$ (or as more commonly written: $F = \frac{9}{5}C + 32$)
b i 122°F **ii** 302°F **iii** −40°F
c 1.8

Section 5H

Now try this

21 $(-0.5, -2)$

Exercise 5H

1 a −1 **b** 3

2 a (1,1) **b** (1,−1)

3 a (2, −4) **b** (−3, 2) **c** (1.5, 2.5)
d (7, 2) **e** (0, 3) **f** (1, 5)
g (0.4, −2.6) **h** (7, 25)

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4 **a** $x = 4, y = -1$ **b** $x = 0.5, y = 2$
c $x = -1, y = -2$ **d** $m = 2, n = 3$
e $x = 1.3, y = 3.5$
f $x = 1.5, y = -0.6$, to 1 d.p.

Section 5I

Now try this

22 plain croissant \$3.50, chocolate croissant \$4.20

23 Width = 6.4 cm, Length = 25.6 cm

24 320 tickets

Exercise 5I

1 $5c + 6p = 12.75$
$7c + 3p = 13.80$

2 a $50p + 5m = 109$
$75p + 5m = 146$
b \$1.48 **c** \$7

3 a $6a + 10b = 7.10$
$3a + 8b = 4.60$
b 60c
c 35c

4 Nails 1.5 kg, screws 1 kg

5 12 emus, 16 wombats

6 length = 12 cm, width = 6 cm

7 22, 30 **8** 8, 27

9 Bruce 37, Michelle 33

10 Boy is 9, sister is 3

11 a \$5 **b** \$3

12 Mother 44, son 12

13 77 students

14 10 standard, 40 deluxe

15 252 litres (40%), 448 litres (15%)

16 126 boys, 120 girls

17 7542 litres unleaded, 2458 litres diesel

18 \$10 000 at 5%, \$20 000 at 8%

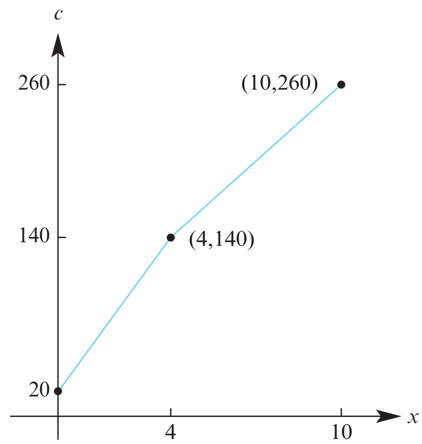
19 Width 24 m, length 36 m

Section 5J

Now try this

25 a i \$125 **ii** \$140 **iii** \$220

b

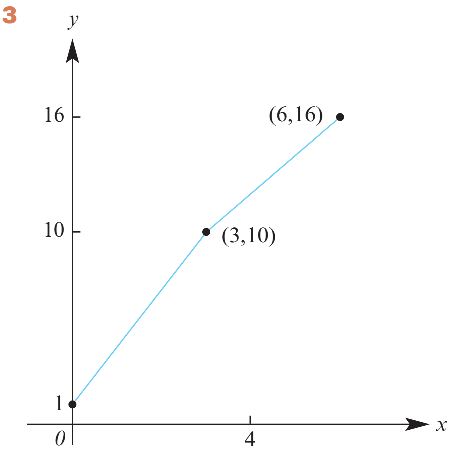


Exercise 5J

1 a $C = 90 + 10x$ **b** $C = 50 + 20x$

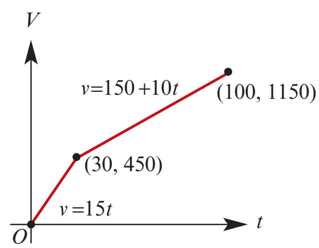
2 a $D = 45 - 5t$ **b** $D = 90 - 20t$
c $D = 90 - 20t$

3



4 a i 300 L **ii** 450 L **iii** 750 L
iv 1150 L

b



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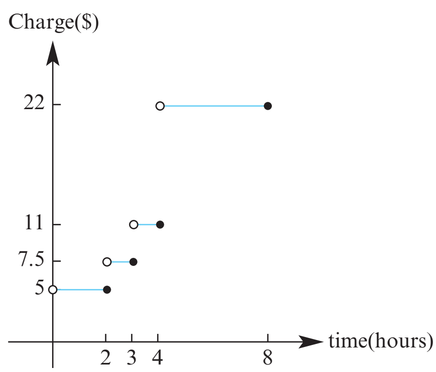
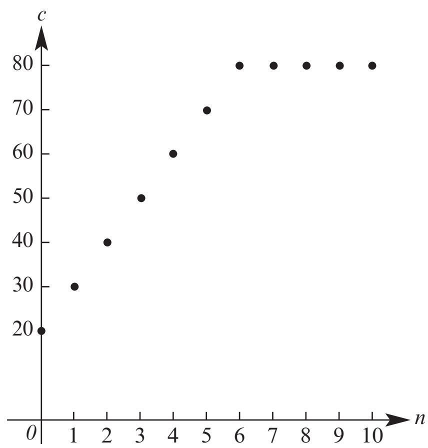


6 a \$0 **b** \$3 **c** \$5

7 a $C = 20 + 10n \quad (1 \le n \le 6)$

$C = 80 \quad (7 \le n \le 10)$

b



c i \$60 **ii** \$80

Chapter 5 Review

Skills Checklist answers

1 (5A) \$110

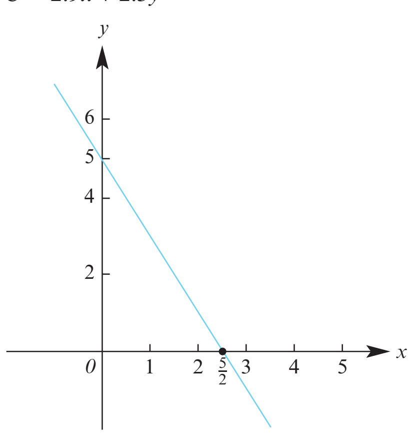
2 (5A)

n	5	6	7	8	9	10
W	990	1030	1070	1110	1150	1190

3 (5B) $x = 15$ **4** (5B) \$400

5 (5C) $C = 2.9x + 2.5y$

6 (5D)



7 (5E) −2 **8** (5E) y-intercept= −7, slope = 2

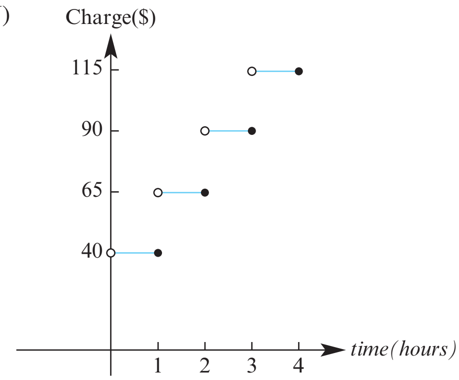
9 (5F) $y = 5 + 3x$ **10** (5F) $y = 3.8 - 1.6x$

11 (5G) $h = 70 + 3t \quad 0 \le t \le 6$ t is time in months

12 (5H) $x = 4.15, y = 1.88$ **13** (5I) \$4

14 (5J) \$88

15 (5J)



Multiple-choice questions

1 B	**2** B	**3** B	**4** C
5 D	**6** C	**7** B	**8** B
9 A	**10** B	**11** C	**12** C
13 D	**14** D	**15** E	**16** E
17 D	**18** A	**19** D	**20** C
21 C	**22** E	**23** C	**24** B
25 E	**26** C	**27** B	**28** C

Short-answer questions

1 a $x = 10$ **b** $x = 11$ **c** $x = 8$ **d** $x = 6$

e $x = 1$ **f** $x = 7$ **g** $x = -6$ **h** $x = 11$

i $x = 3$ **j** $x = 3$ **k** $x = 15$ **l** $x = -24$

2 a $P = 40$ **b** $P = 130$

3 a $A = 30$ **b** $A = 54$

4 94.25 cm

5

x	−20	−15	−10	−5	0
y	−716	−551	−386	−221	−56

x	5	10	15	20	25
y	109	274	439	604	769

a $x = 10$ **b** $x = -5$

6 1

7 5

8 a (1, 3) **b** (4, 1) **c** (5, 1)

9 a $x = 2, y = 8$ **b** $x = 3, y = 2.5$

c $p = 5, q = -2$ **d** $p = 5, q = 2$

e $p = 2, q = 1$

10 a

x	0	1	2	3	4
y	2	7	12	17	22

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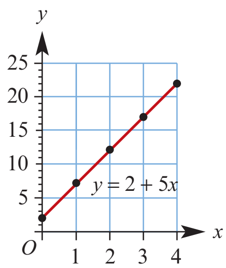
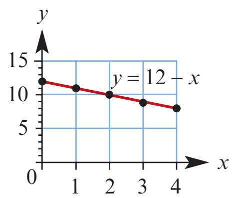


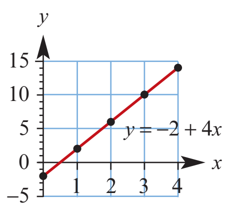
b

x	0	1	2	3	4
y	12	11	10	9	8



c

x	0	1	2	3	4
y	−2	2	6	10	14



11 a \$755 **b** \$110

12 A −1.2, B 0.6

13 A 2.25, B −2.67

Written-response questions

1 a \$57 **b** 7 hours

2 a

n	60	70	80	90	100	110
C	55	60	65	70	75	80

n	120	130	140	150	160
C	85	90	95	100	105

b \$105

3 a $C = 80 + 45h$ **b** \$215

4 a $3a + 5c = 73.5$ **b** \$12 **c** \$7.50

$2a + 3c = 46.5$

5 3 m

6 Indonesian 28; French 42; Japanese 35.

7 a \$200 000

b After 60 months (5 years)

c $V = 300 - 5t$ **d** \$120 000

e \$5000

8 a \$80 billion

b $A = 0.16N$ (with A in billions, N in thousands)

c \$96 billion **d** \$240 billion

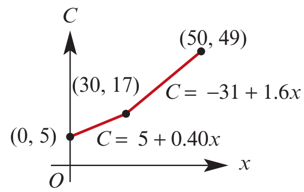
e \$0.16 billion

9 a $H = 80 + 6.67A$ **b** 100 cm **c** 6.67

10 a i \$13 **ii** \$17 **iii** \$49

b i \$0.40 (40 cents) **ii** \$1.60

c



Chapter 6

6A Multiple-choice questions

Chapter 2 Investigating and comparing data distributions

1 A **2** E **3** D **4** B **5** B
6 A **7** A **8** B **9** D **10** C
11 D **12** C **13** A

Chapter 3 Sequences and finance

14 B **15** C **16** C **17** A
18 C **19** D **20** B **21** A
22 B **23** B **24** E **25** C
26 A **27** B **28** B

Chapter 4 Matrices

29 C **30** B **31** D **32** D **33** E
34 C **35** B **36** D **37** E **38** A
39 A

Chapter 5 Linear relations and modelling

40 B **41** B **42** C **43** D **44** C
45 D **46** B **47** D **48** D **49** A
50 B **51** B **52** C **53** A **54** D
55 E

6B Written-response questions

Chapter 2 Investigating and comparing data distributions

1 a categorical

b

General happiness	Frequency	
	Number	%
Very happy	53	26.5
Pretty happy	118	59.0
Not too happy	29	14.5
Total	200	100.0

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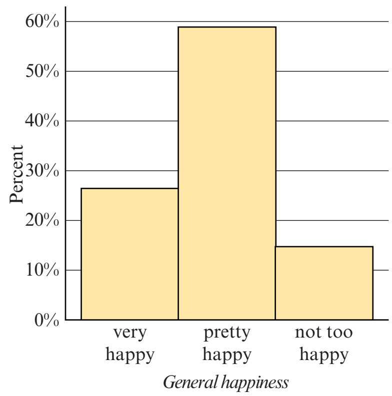
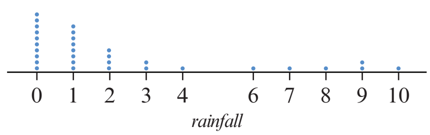
c



d A group of 200 people was asked to describe their general happiness by choosing one of the responses very happy, pretty happy, not too happy. The highest level of response was pretty happy, chosen by 59.0% of the people. Another 26.5% of people chose very happy. Only 14.5% of people said they were not too happy.

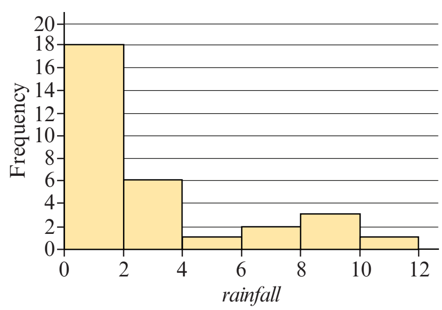
2 a



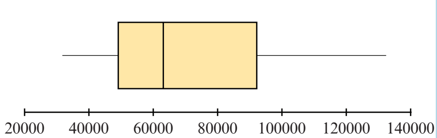
b **i** range = 10
ii median = 1
iii IQR = 3

c 32.3%

d



3 a **i**



ii lower fence = −\$15 500, upper fence = \$156 500
iii 75%

b **i** 2016 median = \$74 000, 2020 median = \$80 000
ii The median salary in 2020 (M = \$80 000) was higher than the median salary in 2016 (M = \$74 000). The spread of salaries in 2020 (IQR = \$24 000) was slightly higher than the spread in 2016 (IQR = \$23 000). There was one outlier in 2016, who had a salary of \$121,000 which was high compared to the rest of the employees. There were two outliers in 2020, who had high salaries of \$131 000 and \$155 000 respectively. In conclusion, the median salary has certainly increased over the four years from 2016 to 2020, while at the same time salaries have become more a little more variable.

4 a *Team*

b

Score

Team A		Team B
4 2 0 0	0	0 2 2 2 2 4
9 8 8 7 6	0	6 6
4 1 0	1	0 2 3 3
9 6	1	5
0	2	
	2	
	3	0 1

c Team A: Min = 0, Q_1 = 4, M = 8, Q_3 = 14, Max = 20. Team B: Min = 0, Q_1 = 2, M = 6, Q_3 = 13, Max = 31

d Team A: lower fence = −11, upper fence = 29, no outliers. Team B: lower fence = −14.5, upper fence = 29.5. Outliers 30 and 31.

e The mean for Team A is lower than the mean for Team B because of the influence of the outliers in Team B.

Chapter 3 Sequences and finance

5 a 2076, 2152, 2228
b After 14 years, \$3064
c $V_0 = 1500, V_{n+1} = V_n + 90$

6 a **i** \$140 000 **ii** 18%
b **i** 98 016.09 **ii** 10.3%

7 a \$2100, \$1700, \$1300 **b** 4 years
c $V_0 = \$1800, V_{n+1} = V_n - 350$

8 a **i**

Day(i)	0	1	2	3	4
E_i	1500	1370	1240	1110	980

Day(i)	5	6	7	8
E_i	850	720	590	460

ii $E_0 = 1500, E_{i+1} = E_i - 130$
iii $E_n = 1500 - 130n, \quad 0 \le n \le 12$

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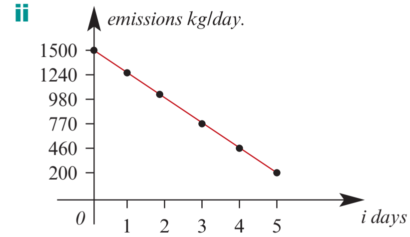
b **i** 260

ii



iii $K_0 = 1500, K_{i+1} = K_i - 260, \quad 0 \leq i \leq 5$

iv $K_n = 1500 - 260n$

9 **a** **i** 372 000 **ii** $R = 360\,000 + 12\,000n$

iii 20 weeks **iv** $K = 16\,000$

b **i** $R = 1.5$ **ii** 2592 **iii** 10640

Chapter 4 Matrices

10 **a** 127 km **b** 50 km **c** 15 km

11 **a** 45 **b** $15 **c** $\begin{bmatrix} 545 \\ 595 \end{bmatrix}$

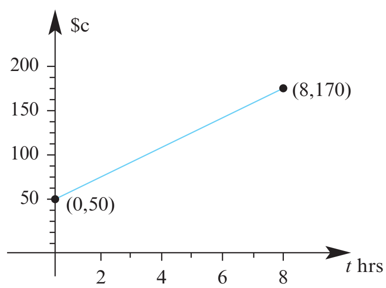
d Revenue at each shop **e** Shop B, $595

Chapter 5 Linear relations and modelling

12 4 standard, 4 deluxe

13 **a** $C = 50 + 15t$

b



c $t = 1.75$ hours

14 **a** 31.90 litres **b** 47.18 litres

c 176 cm **d** 40.61 litres

e 45.65 litres

f

Weight(kg)	60	65	70	75
TBW litres	40.39	42.08	43.76	45.44

Weight(kg)	80	85	90	95
TBW litres	47.12	48.80	50.48	52.16

Weight(kg)	100	110	115	120
TBW litres	55.52	57.20	58.89	60.57

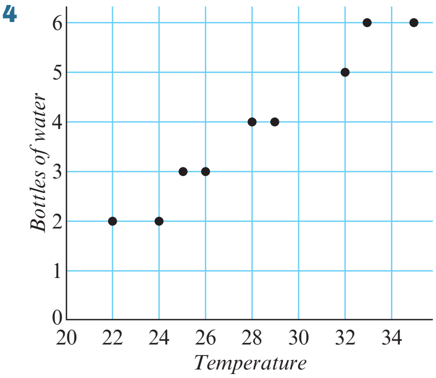
Chapter 7

Section 7A

Now try this

3 EV: *time on social media* RV: *amount of sleep*

4



Exercise 7A

1 **a** EV: *age* RV: *diameter*

b EV: *weeks* RV: *weight loss*

c EV: *age* RV: *price*

d EV: *hours* RV: *amount*

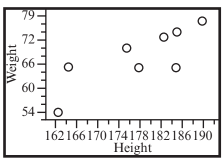
e EV: *balls bowled* RV: *runs*

2 D

3 **a** EV: *hours of sleep*, RV: *mistakes*

b 12 people **c** 9 hours, 6 mistakes

4



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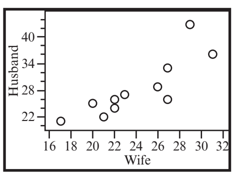

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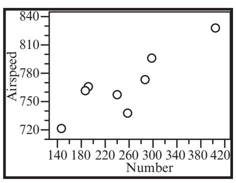
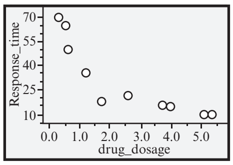


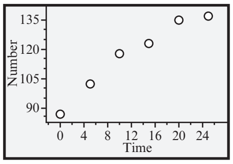
7 **a** *drug dosage*

b



8 **a** *time*

b



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Section 7B

Now try this

5 **a** negative **b** no association **c** positive **d** positive

6 **a** Temperature tends to be lower at higher altitudes. **b** Those people who spend more time studying tend to earn higher salaries.

7 **a** linear **b** non-linear **c** non-linear **d** linear

8 **a** moderate **b** weak **c** strong **d** moderate

Exercise 7B

Note: Estimates of strength can vary by one level and still be acceptable.

1 **a** positive **b** negative

2 **a** linear **b** non-linear

3 **a** strong **b** weak

4 **a** Those who have higher levels of fitness tend to do more daily exercise. **b** Those who take more time to run a marathon tend to run at slower speeds.

5 **a** Strong positive linear association. Those who spent more time studying for the exam tended to get higher marks.

b Strong negative linear association. The price of older cars tended to be lower as the age of the cars increased.

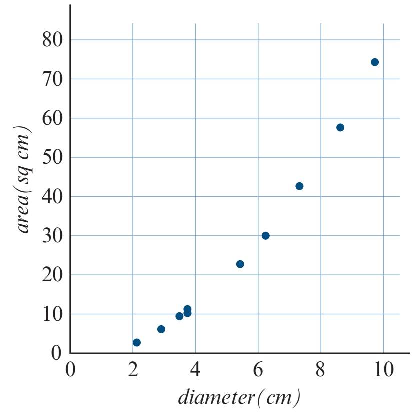
c Moderate positive linear association. Daughters from taller mothers also tended to be taller.

d Strong positive non-linear association. Performance level tended to increase with time spent practising for the first eight hours, after which performance appeared to level off.

e Moderate negative linear association. Students tended to score lower on a test when the temperature was higher.

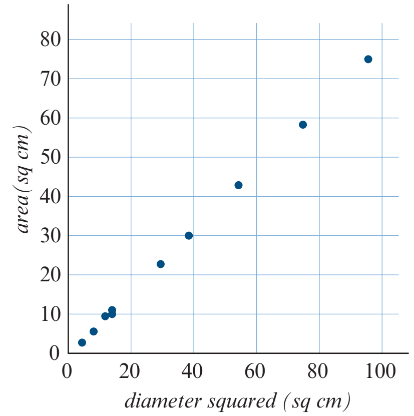
f Strong positive linear association. The older husbands also tended to have older wives.

6 **a**



b There is a strong, positive, non-linear association between the diameter of a circle and its area.

c



d There is a strong, positive, linear association between the square of the diameter of a circle and its area.

e The affect has been to change the form of the association from non-linear to linear.

Section 7C

Now try this

9 **a** $r \approx -0.7$ **b** $r \approx 0.4$

10 $r = 0.8804$

11 **a** strong, positive **b** strong, negative **c** no association **d** moderate, negative

12 No, we can only say that those children who are taller tend to be better at mathematics (which doesn't make much sense!).

Exercise 7C

1 Numerical data, linear association

2 Estimates could vary by ± 0.1

a 0.9 **b** −0.6 **c** 0.8 **d** −0.7

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3 Estimates could vary by ± 0.1

a 0.9 **b** 0.7 **c** −0.6 **d** 0

4 $r = 0.485$

5 $r = 0.570$

6 $r = 0.722$

7 $r = -0.396$

8 **a** None **b** Weak negative

c Strong negative **d** Weak positive

e Strong positive **f** Moderate negative

g Moderate positive **h** None

i Weak negative **j** Weak positive

k Perfect positive or strong positive

l Perfect negative or strong negative

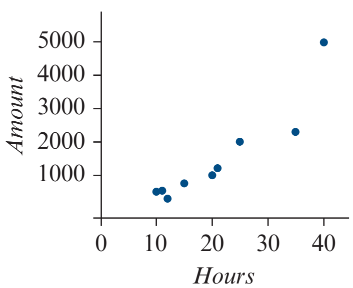
9 No, we can only conclude that those cities with a larger number of bars also tend to have a large number of teachers. Since the number of bars and the number of school teachers will both increase with the size of the city, then size of the city is likely to explain this correlation.

10 No, we can only conclude that those countries with a higher birth rate also tend to have a lower life expectancies. Possible variables include the general wealth of a country, expenditure on medical care, and educational level.

11 **a** EV: *hours* or the number of hours spent gambling

RV: *amount* or amount spent gambling

b

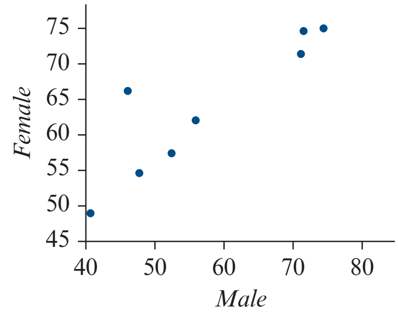


c $r = 0.922$

d Strong positive linear association: Those who gambled for longer tended to spend more on gambling.

12 **a** Either variable could be the EV.

b

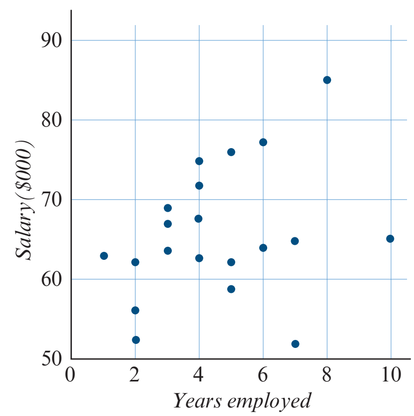


c $r = 0.894$

d Strong positive linear association with an outlier: Those countries with high percentages of males with eye disease also tended to have a high percentage of females with eye disease.

13 **a**

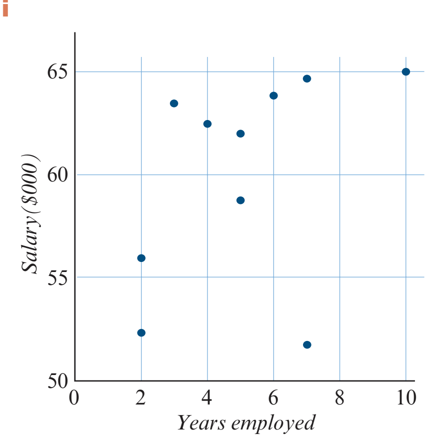
i



ii $r = 0.297$

b

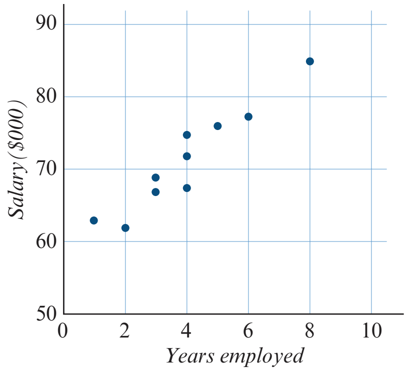
i



ii $r = 0.409$

c

i



ii $r = 0.954$

d When educational level is not taken into consideration there is only a weak correlation between salary and years of employment. However, when the two groups are considered separately there is a weak correlation between salary and years of employment for those with a secondary education, and a strong correlation between salary and years of employment for those with a tertiary education. For this group, those with higher years of employment did tend to earn higher salaries.

Section 7D

Now try this

13 a $a = 20$ **b** $b = 2.5$

c *breath* = 20 + 2.5 × *exercise*

14 *weight after 10 weeks on the diet* = 12 + 0.80× *weight before diet*

15 a = 8.77, b = 1.03

Exercise 7D

1 a yes, the variables are both numerical and the association is linear.

b no, the variables are both numerical but the association is non-linear.

2 a $y = 20.0 + 3.00x$

b $y = 50.0 - 0.900x$

3 $y = -10 + 1.0x$

4 $y = 1.39 + 0.417x$

5 *Time* = 40 + 0.08 × *Distance*

6 *Infant mortality rate* = 160.0 − 1.6 × *Female literacy rate*

7 *Heart weight* = 50 + 2.7 × *Body weight*

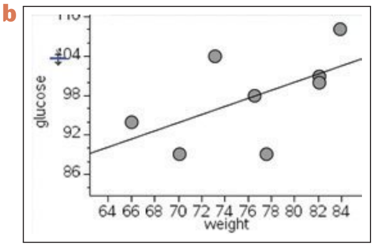
8 *number of errors* = 8.5 − 0.50 × *number of practice sessions*

9 *forearm length* = 0.75 + 1.5 × *wrist circumference*

10 a = −153.88, b = 2.41

11 a = 23.0, b = -0.408

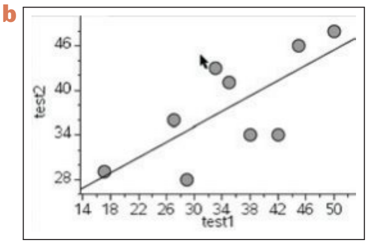
12 a & b



c *glucose* = 50.8 + 0.616 × *weight*

d $r = 0.570$

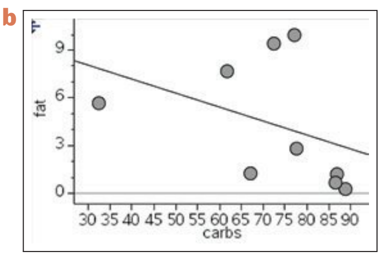
13 a & b



c *test* 2 = 19.6 + 0.516 × *test* 1

d $r = 0.722$

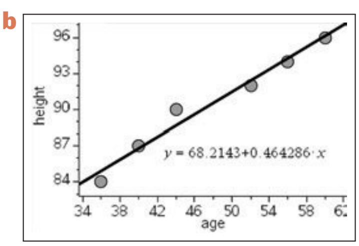
14 a & b



c *fat* = 10.7 − 0.0879 × *carbs*

d $r = -0.396$

15 a & b



c *height* = 68.2 + 0.464 × *age*

d $r = 0.985$

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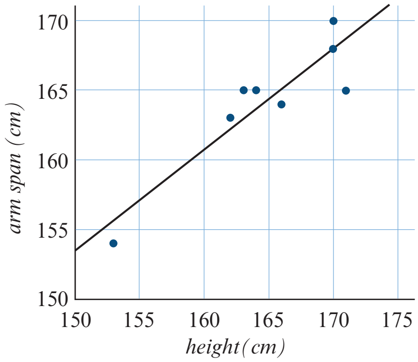
16 a & b

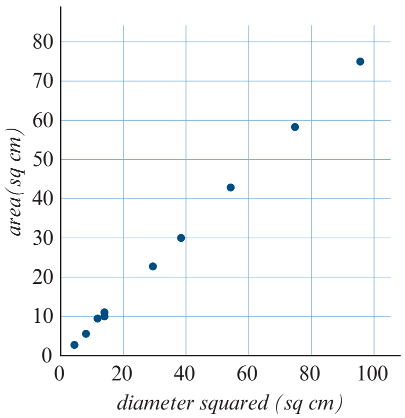


c *arm span* = 45.6 + 0.720 × *height*

d $r = 0.903$

17 $r = 0.792$

18 a



b a = −0.484, b = 0.795

c $area = -0.484 + 0.795 \times diameter^2$

d $area = \frac{\pi}{4} \times diameter^2 = 0.785 \times diameter^2$

Section 7E

Now try this

16 a i intercept = 8.00

ii On average, students who spend no time in part-time work will spend 8 hours doing household chores.

b i slope = −0.3

ii On average, the number of hours spent doing household chores decreases by 0.3 hours for each extra hour of spent in part time work.

17 a 7.4 hours, interpolating

b 5 hours, extrapolating

Exercise 7E

1 a EV = *age*. RV = *height*

b intercept = 69, slope = 0.5

c 0.5cm, 1 month **d** 0 month, 69cm

2 a Interpolation **b** Extrapolation

3 a 59% **b** 66%

4 a i 37 650

ii On average, the price of a new car of this type is $37 650.

b i −4200

ii On average, price of second-hand car of this type decreases by $4200 each year.

5 a i 40

ii On average, the flavour rating of yoghurt with zero fat content fat is 40.

b i 2.0

ii On average, the flavour rating of a yoghurt increases by 2.0 for each 1% increase in fat content.

6 a 88 cm, interpolation

b 94 cm, interpolation

c 100 cm, extrapolation

7 a $175, extrapolation **b** $523, interpolation

c $691, interpolation

8 a 171 cm, interpolation

b 197 cm, extrapolation

c 159 cm, interpolation

9 a i 15.7

ii On average, students obtaining a zero mark on exam 1 obtained a mark of 15.7 for exam 2.

b i 0.65

ii On average, marks in examination 2 increase by 0.65 marks for each additional mark obtained on examination 1.

c 29

10 a i 51

ii The intercept does not have a meaningful interpretation in this example.

b i 0.62

ii On average, the blood glucose level of a healthy adult increases by 0.62 mg/100 mL for each kilogram increase in weight.

c 97.5 mg/100 mL

11 68.7

12 a i salary = 55.916 + 0.840 × years

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ii Intercept: On average, new employees with secondary level education are paid \$55 916 per year. Slope: On average, the salary for employees with secondary level education increases by \$840 per year.

iii \$60 100

b i *salary* = 57.956 + 3.361 × *years*

ii Intercept: On average, new employees with tertiary level education are paid \$57 956 per year.
Slope: On average, the salary for employees with secondary level education increases by \$3361 per year.

iii \$74 800

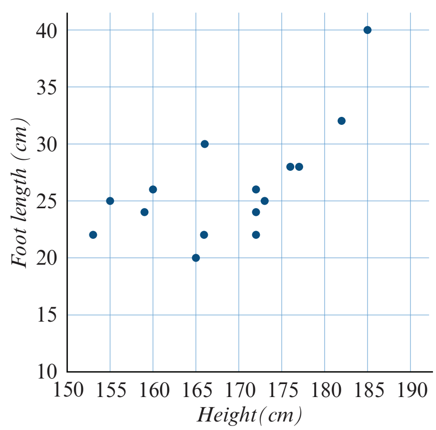
c Yes, both are interpolating.

Chapter 7 Review

Skills Checklist answers

1 EV: *height*, RV: *foot length*

2



3 There is a moderate, positive, linear relationship between a person's height and their foot length.

4 Estimate r = 0.7, exact value r = 0.657.

5 moderate

6 No, we can only say that increasing sales of umbrellas are associated with increasing number of traffic accidents.

7 *kms* = 10 000 + 5000 × *age of car*

8 *foot length* = −31.27 + 0.34 × *height*

9 Intercept: −31.27. This has no meaningful interpretation. Slope: On average, feet are is 0.34 cm longer for each additional 1 cm increase in height.

10 a 23.1 cm, interpolating

b 2.7 cm, extrapolating

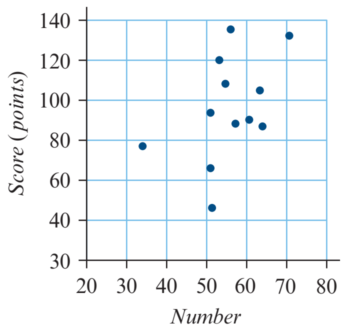
Multiple-choice questions

1 D	2 B	3 D	4 B	5 C
6 E	7 D	8 E	9 D	10 A
11 B	12 A	13 D	14 E	15 C
16 C	17 B			

Short-answer questions

1 a *Score*

b



c Weak linear positive association

2 a EV: *serves*, RV: *cost*

b *cost* = 500 + 20.0 × *serves*

3 a r = 0.927

b *distance* = 3.50 + 0.553 × *time*

4 a slope = 0.871

b intercept = -86.1

c *weight* = −86.1 + 0.871 × *height*

5 a *magnesium content*

b On average, taste scores increases by 7.3 points for each one mg/litre increase in magnesium content.

c 94.8 milligrams/litre

6 a *errors* = 14.9 – 0.533 × *time*

b r = −0.94

7 No, we can only say that those students who score highly in French also tend to score highly in Mathematics.

Written-response questions

1 a EV: *played*
RV: *weekly sales*

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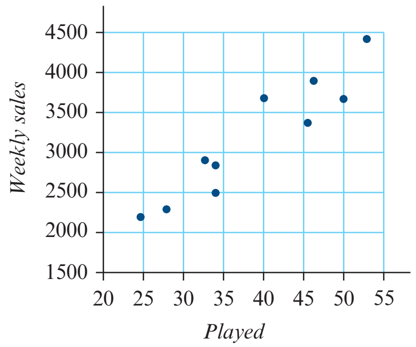
b



c $r = 0.9458$

d Strong, positive, linear relationship

e $\textit{weekly sales} = 293 + 74.3 \times \textit{played}$

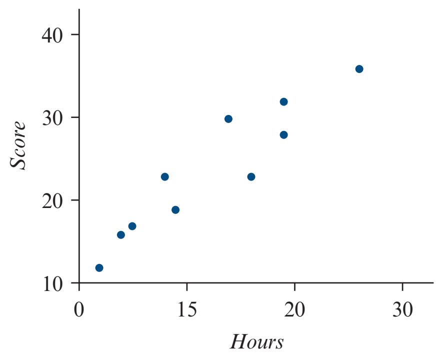
f Slope: on average, the number of down loads increases by 74.3 for each additional time the song is played on the radio in the previous week.
Intercept: predicts 293 downloads of the song if it is not played on radio in the previous week.

g 7723

h Extrapolating

2 a EV: *hours* RV: *score*

b



c $r = 0.9375$

d Positive strong, linear

e $\textit{score} = 12.3 + 0.930 \times \textit{hours}$

f Slope: on average, test scores of learner drivers increases by 0.93 marks when instruction time increases by one hour.
Intercept: on average, test scores of learner drivers who received no instruction prior to taking the test is 12.3 marks.

g 22

3 a $\textit{exam mark} = 25 + 0.69 \times \textit{assignment mark}$ (correct to 2 sig. figures)

b Intercept: on average those who score 0 on the assignment scored 25 marks on the final exam.
Slope: on average, exam marks increase by 0.69 for each additional mark obtained on the assignment.

c 60

d Reliable: The prediction made in part **d** falls well within the range of the values of the EV (interpolating).

e No, we can only say that those students who scored well on the assignment also tended to score well on the exam.

Chapter 8

Section 8A

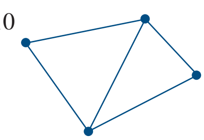
Now try this

1 a 4 **b** 6 **c** 2 **d** 4
e 12

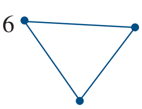
Exercise 8A

1 a i 5 **ii** 6 **iii** 0 **iv** 2
v 3 **vi** 4 **vii** 1
b i 4 **ii** 7 **iii** 1 **iv** 6
v 2 **vi** 2 **vii** 2
c i 4 **ii** 7 **iii** 0 **iv** 3
v 4 **vi** 2 **vii** 2
d i 8 **ii** 14 **iii** 2 **iv** 5
v 3 **vi** 8 **vii** 0

2 a 10

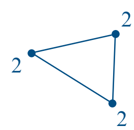


b 6

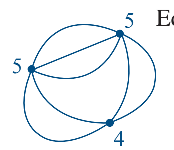


c 2

3 Yes. Each edge contributes two degrees overall to a graph, because it is connected to two vertices (or one vertex twice in the case of a loop); therefore the sum of degrees would be twice the number of edges.



Edges = 3. Sum of degrees = 6



Edges = 7. Sum of degrees = 14

4 a Increase by 2 **b** 1

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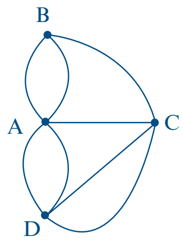
5 E **6** C **7** C **8** B

9 a i Yes. One option: *A-D-A-C-D-C-B-A-B*

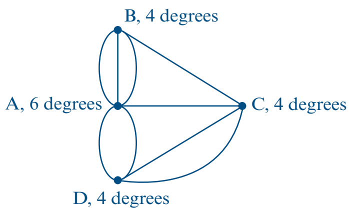
ii



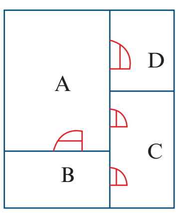
iii A and B

b i Yes. One option: *A-B-A-B-C-A-D-C-D-A*

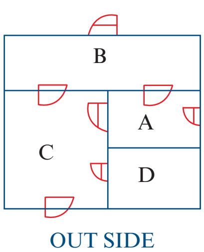
ii and iii All vertices are even.



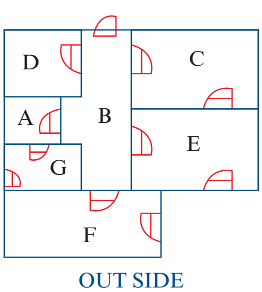
10 Note: B and C can be swapped.



11 Note: A and B can be swapped.



12



Section 8B

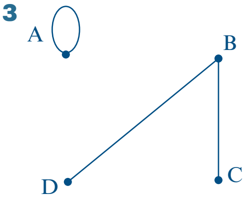
Now try this

2 a All graphs are connected

b AD and DC

c Graph 1 and Graph 2

3



Exercise 8B

1 A, D, F

2 a BD

b AB and AC

c XW and WV

3 a Graph 1 and Graph 3

b Graph 2 and Graph 3

c Graph 1 and Graph 2

d Graph 1 and Graph 2

e Graph 2 and Graph 3

4 a

$$\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

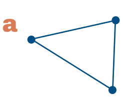
b

$$\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix} \end{array}$$

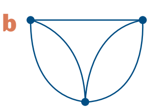
c

$$\begin{array}{c} \begin{matrix} X & Y & Z \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{matrix} X \\ Y \\ Z \end{matrix} \end{array}$$

5 a



b



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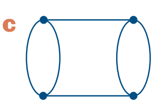
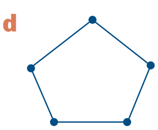
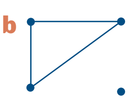


d

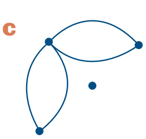


6 a

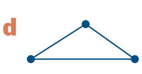
b



c



d



7 3

8

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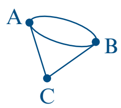
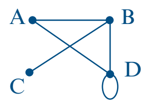
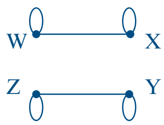
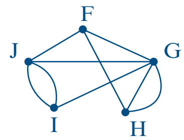
9 a 

b 

c 

d 

10 6

11 a 3 **b** 6 **c** 10

12 a

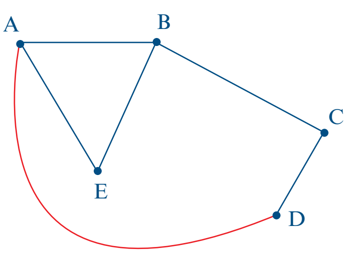
$$\begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 2 & 2 & 1 & 1 \\ 2 & 0 & 2 & 1 & 0 \\ 2 & 2 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}$$

b

$$\begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 1 & 4 & 1 & 2 & 0 \\ 4 & 1 & 2 & 3 & 2 \\ 1 & 2 & 0 & 1 & 2 \\ 2 & 3 & 1 & 0 & 3 \\ 0 & 2 & 2 & 3 & 1 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}$$

Section 8C

Now try this

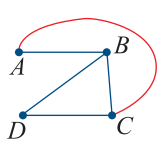
4 

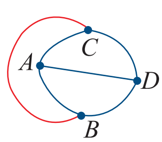
6 $v + f = e + 2;\quad 5 + 2 = 5 + 2;\quad 7 = 7$ **7** 5

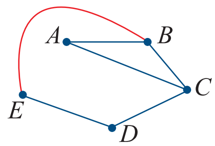
Exercise 8C

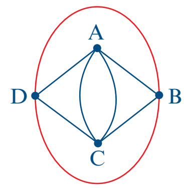
1 *A, B, D, F*

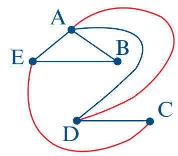
2 a Note: Other answers possible.

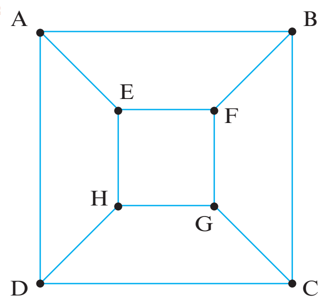


b 

c 

d 

e 

f 

3 a i $v = 4, e = 4, f = 2$
ii $v + f = e + 2;\quad 4 + 2 = 4 + 2;\quad 6 = 6$
b i $v = 7, e = 9, f = 4$
ii $v + f = e + 2;\quad 7 + 4 = 9 + 2;\quad 11 = 11$
c i $v = 7, e = 12, f = 7$
ii $v + f = e + 2;\quad 7 + 7 = 12 + 2;\quad 14 = 14$
d i $v = 7, e = 10, f = 5$
ii $v + f = e + 2; 7 + 5 = 10 + 2; 12 = 12$
e i $v = 5, f = 4, e = 7$
ii $5 + 4 = 7 + 2; 9 = 9$
f i $v = 4, f = 4, e = 6$
ii $4 + 4 = 6 + 2; 8 = 8$

4 a $f = 2$ **b** $v = 3$ **c** $e = 4$ **d** $v = 4$
e $f = 4$ **f** $f = 7$ **g** $e = 19$

5 a 4 **b** 5 **c** 4 **d** 4

6 D **7** E **8** A **9** B
10 C **11** B **12** D

Section 8D

Now try this

8 Trail **9** Circuit

Exercise 8D

1 a i No **ii** Yes **iii** No
b i Yes **ii** Yes **iii** No

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c **i** No **ii** Yes **iii** No
d **i** Yes **ii** No **iii** Yes

2 **a** Cycle **b** Circuit **c** Trail **d** Path
e Circuit **f** Path

3 **a** Trail **b** Cycle **c** Path **d** Walk
e Cycle **f** Circuit

4 **a** **i** 5 **ii** 12
b **i** 6 **ii** 23

Section 8E

Now try this

10 **a** **i** Neither
b **i** Both, zero odd vertices
ii *A-B-C-D-B-D-A*
c **i** Eulerian trail, only two odd vertices
ii *A-D-E-D-C-B-C*

Exercise 8E

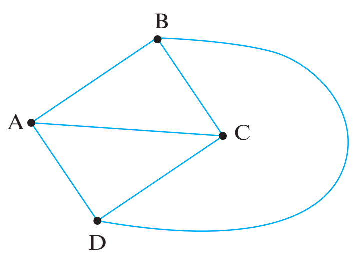
1 **a** Both, zero odd vertices
b Neither
c Eulerian trail, only two odd vertices
d Eulerian trail, only two odd vertices
e Both, zero odd vertices
f Eulerian trail, only two odd vertices
g Both, zero odd vertices
h Eulerian trail, only two odd vertices
i Neither

2 **a** Yes, an Eulerian circuit, all even vertices
b *A-E-D-C-E-B-C-B-A*

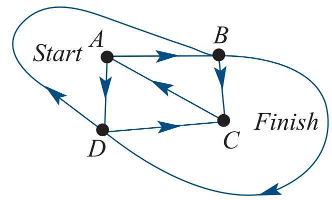
3 **a** No, not all vertices are even
b Yes, *H-A-B-C-H-G-F-C-D-E-F*

4 **a** Yes, an Eulerian circuit, all even vertices
b *K-S-E-K-G-E-D-G-M-K*

5 **a**



b No. More than two vertices have an odd degree.
c **i** One of the following: *AB, AC, AD, BC, BD* or *CD*
ii Possible solution:



iii The graph now has two odd vertices. Possible solutions:
$A-B-C-B-D-C-A-D$
or
$A-B-C-A-D-B-D-C$

6 **a** Eulerian circuit, all vertices are even
b Yes. Eulerian trail, only two odd vertices

7 3

8 5

Section 8F

Now try this

11 **a** **i** *A-D-E-F-B-C*
ii Yes, *A-D-E-F-B-C-A*
b **i** Not possible **ii** No

Exercise 8F

1 **a** *A-B-C-H-G-F-E-D*
b *F-E-H-C-D-A-B-G*

2 **a** *A-B-C-D-E-F-A*
b *A-B-D-C-E-A*
c *A-F-E-D-C-B-G-A*
d *A-B-C-F-I-H-E-G-D-A*
e Not possible
f *A-F-G-E-H-D-C-B-A*

3 **a** No
b Yes, Hamiltonian path *C-D-E-B-A*
c Yes, Hamiltonian cycle *E-D-C-B-A-E*

4 **a** Yes, Hamiltonian path *K-S-E-D-G-M-T-L*
b Yes, Hamiltonian path *D-G-E-S-L-T-M-K*

5 **a** 5
b *B* and *C*
c Eulerian trail
d Hamiltonian path

6 **a** 5
b Yes, 6

7 **a** 6
b 5

8 **a** No, although all vertices in a connected graph can be reached by another directly or indirectly, this may require the repeating of a vertex to achieve such a Hamiltonian path.

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b No, to achieve a Hamiltonian cycle, there must be at least one vertex with an even degree for the cycle to start and end at the same vertex with no repeated vertices or edges.

Section 8G

Now try this

12 a 9 km **b** 24 km

13 *W-G1-F*; 16 min

Exercise 8G

1 *A-C-D-E*; 11 hours **2** *A-B-D*; 35 metres

3 *B-A-D*; \$6 **4** *B-G-A-F*; 7 min

5 $A-E-F-G-I$ or $A-C-F-G-I$; 26 km

6 19 km **7** 80 min **8** 23 min

9 *A-B-C-E-F-D-A*

10 a 22 **b** 45

11 10a - No, 10b - Yes; all Eulerian trails in a graph will have the same total weighting because all edges are covered, so it is unnecessary to determine the length of the shortest one.

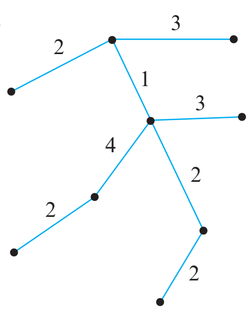
Section 8H

Now try this

15 19

16 19

17 *W-G1-F*; 16 minutes

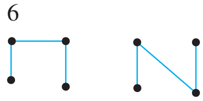


Exercise 8H

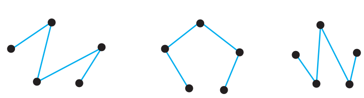
1 a 14

b 6

c



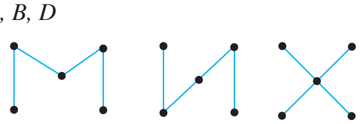
d



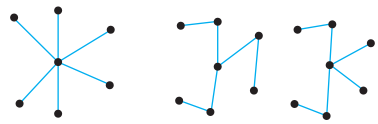
2 8, 7

3 *A, B, D*

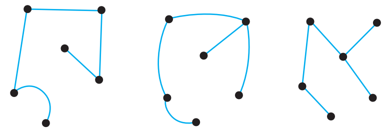
4 a



b



c



5 a 18 **b** 11 **c** 22

6 a 60 **b** 20 **c** 9

7 a 19 **b** 23

8 94 km

9 a 44 metres

b 7 metres longer

10 5

11 $k = 8$

12 4

Chapter 8 Review

Skills Checklist answers

1 5 vertices, 8 edges, 2 loops

2 $\deg(A) = 2$, $\deg(B) = 3$, $\deg(C) = 3$, $\deg(D) = 3$, $\deg(E) = 5$, sum of degrees = 16

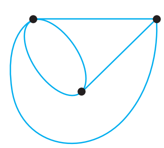
3 The graph is connected because there is a path between each pair of vertices.
Yes, the bridge is between the vertices *C* and *D*.

4 Second graph only.

5

	A	B	C	D	E
A	0	1	0	0	1
B	1	0	0	1	1
C	0	0	1	1	0
D	0	1	1	0	1
E	1	1	0	1	1

6 Yes, can redraw the graph with no edges crossing



7 4

8 $v = 3, e = 5, f = 4$; $v + f = e + 2$;
$3 + 4 = 5 + 2$; $7 = 7$ **9** Cycle

10 $M-S-R-M-N-O-R-Q-P-O$

11 Not all vertices have an even degree

12 $S-R-M-N-O-P-Q$

13 $S-R-Q-P-O-N-M-S$ **14** 17

15 23

Answers 8G → 8 Review

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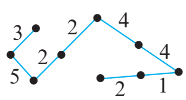

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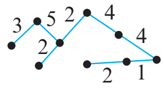


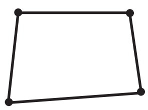
Multiple-choice questions

1 C **2** E **3** D **4** B **5** B
6 D **7** A **8** A **9** D **10** B
11 D **12** E **13** B **14** B **15** D
16 C **17** B **18** B **19** C **20** E
21 B **22** B **23** C **24** A **25** B
26 C **27** D **28** E **29** D

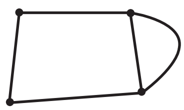
Short-answer questions

1 Many answers are possible. Examples:

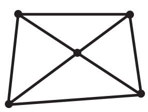
a



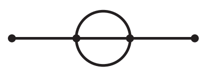
b



c

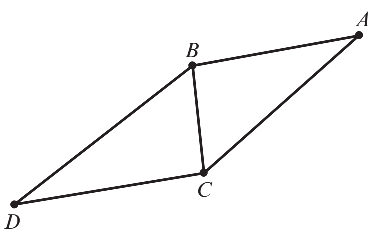


d

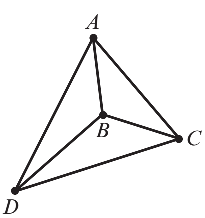


2 Other answers are possible in each case.

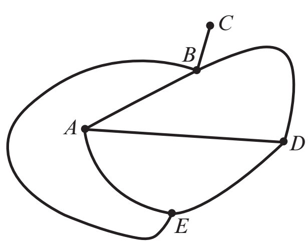
a



b



c



3 **a** deg(C) = 3
b 2 odd, 2 even
c Other answers are possible, ending at C and tracing each edge once only.
Example: $B - A - C - B - D - C$

4

	A	B	C	D
A	0	1	1	1
B	1	0	1	1
C	1	1	0	1
D	1	1	1	0

5 **a** deg(C) = 4
b No odd vertices, five even vertices
c Other answers are possible, ending at A and tracing each edge once only.
Example: $A - B - C - D - E - B - A - E - C - D - A$

6 24

7 **a** 11 km **b** 17 km

Written-response questions

1 **a** No edges intersect, except at vertices.
b $v = 9, e = 14, f = 7; 9 - 14 + 7 = 2$
c 750 m
d No odd, 9 even
e **i** Yes, all vertices are even.
ii Many answers are possible. Example:
$P - C1 - C8 - C2 - C1 - C4 - C2 - C3 - C4 - C5 - C7 - C8 - C6 - C5 - P$
f 1270 m
g **i** Hamiltonian cycle
ii $C7$ Park Office
iii $P - C1 - C2 - C3 - C4 - C5 - C6 - C8 - C7 - P$, or the same route in reverse

2 **a** 135 km (there are two shortest paths).
b $v = 8, e = 12, f = 6 : 8 - 12 + 6 = 2$
c **i** This network does not have an Eulerian circuit as it contains two odd vertices
ii Dimboola, 556 km
iii $H - S - M - H - W - Don - M - W - Dim - H - Nat - Nhill - Dim$
d 241 km
e The Dimboola/Horsham road

Chapter 9

Section 9A

Now try this

1 $k = 5$

x	2	4	6	20
y	10	20	30	100

2 $k = 2$

x	2	4	6	12
y	8	32	72	288

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3 a 285 km **b** 5 hrs 16 mins

Exercise 9A

1 a

x	2	4	6	8
y	6	12	18	24

b

x	0	1	2	3
y	0	4	8	12

2 a $y = kx$ **b** $y = kx^2$ **c** $y = kx^5$ **d** $a = kb$
e $z = kw$ **f** $y = k\sqrt{x}$

3 a $m \propto n$ **b** $y \propto x^2$ **c** $y \propto \sqrt{x}$ **d** $s \propto t^2$

4 a $k = 7$

x	3	4	7	12
y	21	28	49	84

b $k = \frac{1}{2}$

x	4	9	14	20
y	2	4.5	7	10

c $k = 2$

x	2	4	6	8
y	8	32	72	128

5 a 54 **b** 17

6 a 300 **b** 5.5

7 a 72 cm^2 **b** 20 cm

8 a \$23.40 **b** 6.4 kg

9 a 4.55 hours = 4hours 33 mins
b i 550 km **ii** 165 km

10 10.125 kg

11 8.62 km

Section 9B

Now try this

4 $k = 2$

x	2	4	5	10	20
y	1	0.5	0.4	0.2	0.1

5 1 hr 12 mins (1.2 hrs)

Exercise 9B

1 a

x	2	4	5	10
y	10	5	4	2

b

x	1	2	4	5
y	5	2.5	1.25	1

2 a $y = \frac{k}{x}$ **b** $y = \frac{k}{x^2}$ **c** $y = \frac{k}{x^3}$ **d** $m = \frac{k}{n}$
e $z = \frac{k}{w}$ **f** $y = \frac{k}{\sqrt{x}}$

3 a $A \propto \frac{1}{r}$ **b** $y \propto \frac{1}{x^2}$ **c** $y \propto \frac{1}{\sqrt{x}}$ **d** $m \propto \frac{1}{n^3}$
e $s \propto \frac{1}{t}$

4 a $k = 10$

x	1	2	4	10
y	10	5	2.5	1

b $k = 2$

x	2	4	10	30
y	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{15}$

c $k = 1$

x	1	2	4	5
y	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$

d $k = 0.5$

x	0.5	1	2	5
y	1	0.5	0.25	0.1

5 a 10 **b** 2

6 a $\frac{1}{2}$ **b** 16

7 a 2.5 **b** 10

8 2.4 amperes

9 a $t \propto \frac{1}{n}$ **b** 25 mins

10 537 cycles/sec

11 2.85 kg/cm^3

Section 9C

Now try this

6 Yes it does. **7** Yes it does.
8 Yes it does. **9** Yes it does.

Exercise 9C

1 a increase **b** straight, origin **c** decrease

2 a inverse **b** direct

3 a direct **b** inverse

4 a

x	2	4	6	8
x^2	4	16	36	64
$\frac{1}{x}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$

b

x	10	20	30	40
x^2	100	400	900	1600
$\frac{1}{x}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{40}$

5 a (ii) **b** (iii) **c** (i)

6 Yes it is linear.

7 No it is not linear.

8 Yes it is linear.

9 Yes it is linear.

10 Yes it is linear.

11 a x^2 transformation **b** Yes it does.

12 a

n	1	2	3	4	5	6
t	120	60	40	30	24	20

b $\frac{1}{x}$ transformation

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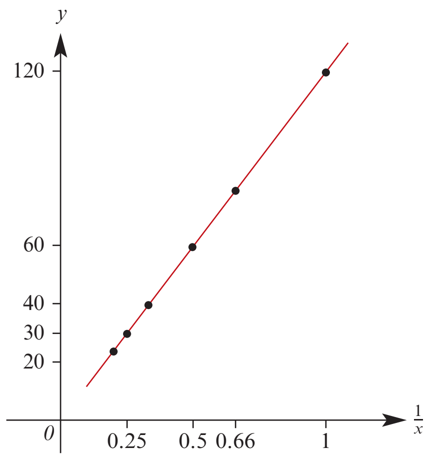
c



Section 9D

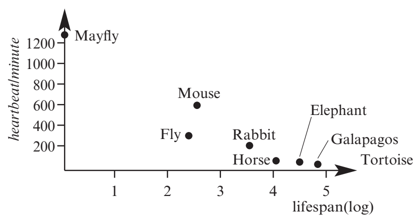
Now try this

10 $10^4, 4$ **11** 2.4 **12** 710.7 **13** 4

14 It gives a straight line.

15 Yes it does.

16



17 631 times heavier

Exercise 9D

1 **a** 10^2 **b** 10^3 **c** 10^1 **d** 10^0 **e** 10^4

2 **a** 5 **b** 8 **c** 0 **d** 9 **e** 4.5

3 **a** 31.62 **b** 158.49 **c** 6309.57 **d** 5.01 **e** 7 943 282.35

4 **a** 4 **b** 33 **c** 100

5 **a** $10^3, 3$ **b** $10^6, 6$ **c** $10^7, 7$ **d** $10^0, 0$ **e** $10^1, 1$

6 **a** 2.477 **b** 3.774 **c** 4.017 **d** 0.860

7 **a** 316.23 **b** 31.62 **c** 3.16 **d** 1

8 **a** 4 **b** 2 **c** 9 **d** 12 **e** 11

9 3

10 5

11 **a** 3 **b** 1

12 Yes it is.

13 Yes it is.

14 3000 times heavier

15 5

16 199.53 kg.

Section 9E

Now try this

18 $y = 5x^2 + 1$ **19** $y = \frac{5}{x} + 2$

20 $y = 10\log_{10}(t) + 15$.

Exercise 9E

1 **a** slope = 3, y-intercept = 4
b slope = 4 , y=intercept = 2

2 **a** 3 **b** 7

3 **a** 2 **b** −5

4 **a** 4 **b** 3

5

x	1	2	3	4
x^2	1	4	9	16
y	5	11	21	35

$y = 2x^2 + 3$

6

x	0.1	0.2	0.5	1
$\frac{1}{x}$	10	5	2	1
y	102	52	22	12

$y = \frac{10}{x} + 2$

7

x	10	100	1000	10000
$\log_{10}(x)$	1	2	3	4
y	11	17	23	29

$y = 6\log_{10}(x) + 5$

8 $P = 5t^2 + 10$

9 **a** $t = \frac{240}{v} - 0.1$ **b** 109 km/hr

10 $N = 100\log_{10}(x) + 15$

Chapter 9 Review

Skills Checklist answers

1 $k = 95$

2 $k = 100$

3 inverse variation

4

x	1	2	5
x^2	1	4	25
y	12	27	132

Yes it does.

5

x	0.5	0.2	0.1
$\frac{1}{x}$	2	5	10
y	5	35	85

Yes it does.

6 2.77

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7 5

8 1000 times

9 143.4

10

x	10	100	1000
$\log_{10}(x)$	1	2	3
y	3	6	9

Yes it does.

11 $y = 3x^2 + 20$

Multiple-choice questions

1 C **2** C **3** E **4** C
5 A **6** E **7** E **8** D
9 B **10** A **11** E

Short-answer questions

1 a 200 **b** 15

2 a 48 **b** 5.48

3 a 12 **b** 9

4 a 4.91 **b** $d = 4.91t^2$ **c** 491 m
d 2 seconds

5 2.4 hrs = 2 hrs 24 mins

6 18 amps

7 3584 tiles

8 33 litres/min

9 1000 times stronger

Written-response questions

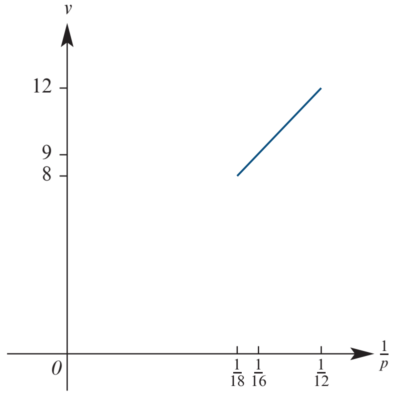
1 a 9.80 kg **b** 10.95 cm

2 a 30 days **b** 8 painters

3 a $V = \frac{121.8}{P}$ **b** 9.59 kg/cm^2

4 a $w = \frac{3000}{d}$ **b** 600 kg
c 333.33 kg

5 a $V = \frac{144}{P}$
b i 2 **ii** 48
c



Chapter 10

Section 10A

Now try this

1 58 000

2 43.63

3 a 6.7×10^5 **b** 6.0×10^{-6}

4 a 4 231 000 **b** 0.00082

5 a 57.9 **b** 0.000472

Exercise 10A

1 a 3 **b** 2 **c** 3 **d** 2
e 4

2 a 4 **b** 12 **c** 67 **d** 557

3 a 3 **b** 5 **c** 4

4 a 3 **b** 1 **c** 2

5 a 500 **b** 46 800
c 79 400 **d** 300

6 a \$ 690 **b** \$ 20
c \$ 928 **d** \$ 14

7 a 1.56 **b** 0.025 **c** 0.03
d 1.8823

8 a 15.65 **b** 4.69 **c** 39.14

9 a 7.92×10^5 **b** 1.46×10^7
c 5.0×10^{11} **d** 9.8×10^{-6}
e $1.456\,97 \times 10^{-1}$ **f** 6.0×10^{-11}
g $2.679\,886 \times 10^6$ **h** 8.7×10^{-3}

10 a 53 467 **b** 3 800 000
c 789 000 **d** 0.009 21
e 0.000 000 103 **f** 2 907 000
g 0.000 000 000 003 8 **h** 21 000 000 000

11 a 6×10^{24} **b** 4×10^7
c 1×10^{-10} **d** 1.5×10^8

12 a 4.9 **b** 0.0787 **c** 1506.9 **d** 6

13 a 0.0033 **b** 0.148 68
c 317 **d** 335

14 35.900

15 D

16 a i 420 **ii** 421.39
b i 64.0 **ii** 64.031
c i 5090 **ii** 5090.049
d i 71 **ii** 70.55
e i 0.46 **ii** 0.46
f i 0.41 **ii** 0.41

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Section 10B

Now try this

6 9.43 m
7 4.7 cm
8 7.8 m
9 a 12.0 cm b 13.5 cm

Exercise 10B

1 a $a^2 + b^2 = c^2$ b $b = 7$
c $5^2 + 7^2 = c^2$ d 8.6
2 a 4.9 cm b 83.1 cm c 24.0 mm
d 2.4 mm
3 a 15.8 mm b 7.4 cm c 6.4 cm
d 141.4 mm
4 2.9 m 5 3.8 m 6 5.3 m
7 48.88 km 8 103 m
9 a $AC^2 = AB^2 + BC^2$
b $AC^2 = 9^2 + 5^2$
c 10.2956... cm
d $AD^2 = AC^2 + CD^2$
e $AD^2 = 106 + 7^2$ f 12.4 cm
10 a 4.243 cm b 5.20 cm
11 a 10.77 cm b 11.87 cm c 6.40 cm
12 a 27.73 mm b 104.79 mm
13 9.54 cm
14 a i 8.5 cm ii 9.1 cm
b i 10.6 cm ii 3.8 cm
15 17 cm 16 25 cm 17 Yes it will fit
18 17.55 m 19 6.7
20 a $(XZ)^2 = a^2 + b^2$
b $(XY)^2 = (XZ)^2 + (YZ)^2$
c $XY = \sqrt{a^2 + b^2 + c^2}$

Section 10C

Now try this

10 28 cm 11 12 m 12 96 cm^2
13 9.9 m^2 14 9 litres
15 a 34 m^2 b 26 m
16 a 75.4 m b 452.4 m^2

Exercise 10C

1 a 84 cm^2 b 4 cm, 4cm c 8 cm^2
d 76 cm^2
2 a 43.98 cm b 153.94 cm^2
3 a i 60.0 cm ii 225.0 cm^2
b i 22.4 cm ii 26.1 cm^2
c i 312.0 cm ii 4056.0 cm^2
d i 44.0 cm ii 75.0 cm^2
4 a 56.2 m^2 b 16.7 m^2
c 103.6 cm^2 d 73.8 cm^2
e 28 cm^2 f 35.9 cm^2
g 29.9 m^2 h 31.3 m^2
i 43.3 cm^2 j 31.4 m^2
5 100 m^2 6 63 375 m^2
7 40 tiles 8 4 L
9 a 5 cm^2 b 125 cm^2
10 30.88 m^2
11 a 252 m^2 b 273 m^2
12 a i 31.4 cm ii 78.5 cm^2
b i 53.4 cm ii 227.0 cm^2
13 a i 25.71 cm ii 39.27 cm^2
b i 1061.98 mm ii 14 167.88 mm^2
c i 203.54 cm ii 2551.76 cm^2
d i 53.70 mm ii 150.80 mm^2
14 a 343.1 cm^2 b 34.9 m^2
15 a 1051.33 m b 37 026.55 m^2
16 1060 cm^2 17 30.91 m^2
18 0.433 19 140.5 cm

Section 10D

Now try this

17 16.3 cm 18 736.4 cm

Exercise 10D

1 a 1/4 b 3/4 c 1/12 d 1/3
e 1/6 f 5/12
2 a 7.85 cm b 10.47 cm
c 26.18 cm d 23.56 cm
3 a 13.09 cm b 5.24 cm
c 78.54 cm d 37.70 cm
e 122.17 cm f 109.96 cm
4 9.0 m
5 a 39.3 cm^2 b 52.4 cm^2
c 130.9 cm^2 d 117.8 cm^2
6 a 130.90 cm^2 b 488.69 cm^2
c 2650.72 cm^2 d 670.21 cm^2
e 2356.19 cm^2 f 1649.34 cm^2
7 a 95.50° b 47.75°
8 61.42 cm^2
9 a 125.66 m b 41.96%
10 a 10.47 m b 20.94 m^2

Section 10E

Now try this

19 24 m^3 20 24 429.0 cm^3 21 277.3 m^3
22 7.6 litres 23 19.6 m^3
24 381 703.5 cm^3

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Exercise 10E

1 a $125\ \text{cm}^3$ b $49\,067.8\ \text{cm}^3$ c $3685.5\ \text{cm}^3$ d $3182.6\ \text{mm}^3$ e $29\,250\ \text{cm}^3$ f $0.3\ \text{m}^3$ g $6756.2\ \text{cm}^3$ h $47.8\ \text{m}^3$ i $9500.2\ \text{cm}^3$ j $16.4\ \text{m}^3$ k $229.9\ \text{mm}^3$ l $7238.2\ \text{cm}^3$

2 $424\ \text{cm}^3$

3 $516\ \text{cm}^3$

4 a $153.94\ \text{cm}^3$ b $705.84\ \text{m}^3$ c $102.98\ \text{cm}^3$ d $1482.53\ \text{cm}^3$

5 a $179.59\ \text{cm}^3$ b $11\,494.04\ \text{cm}^3$ c $33.51\ \text{cm}^3$

6 a $8578.64\ \text{cm}^3$ b $7679.12\ \text{cm}^3$

7 24 L

8 a $20\,319.82\ \text{cm}^3$ b 20 L

9 $228\ \text{cm}^3$ 10 $393\ \text{cm}^3$ 11 $7.87\ \text{m}^3$

12 0.02 L 13 $18\,263.13\ \text{cm}^3$

14 782 mL 15 14 L

16 a $30\ \text{m}^3$ b $\sqrt{XYZ}$

17 $2791\ \text{m}^3$

18 2 cm

Section 10F

Now try this

25 $25\,992\ \text{m}^3$ 26 $2560\ \text{m}^3$

Exercise 10F

1 a $26.67\ \text{cm}^3$ b $420\ \text{m}^3$ c $24\ \text{m}^3$ d $68.64\ \text{cm}^3$

2 $1\,694\,000\ \text{m}^3$

3 a $335.6\ \text{cm}^3$ b $66.6\ \text{cm}^3$

4 $3937.5\ \text{cm}^3$

Section 10G

Now try this

27 $504\ \text{cm}^3$ 28 $14\,137.2\ \text{m}^2$

29 $5026.6\ \text{cm}^2$

Exercise 10G

1 a $400\ \text{cm}^2$ b $100\ \text{cm}^2$ c $200\ \text{cm}^2$ d $700\ \text{cm}^2$

2 a $1180\ \text{cm}^2$ b $40\ \text{m}^2$ c $383.3\ \text{cm}^2$ d $531\ \text{cm}^2$ e $2107.8\ \text{cm}^2$ f $176.1\ \text{m}^2$

3 a $3053.63\ \text{cm}^2$ b $431.97\ \text{cm}^2$ c $7.37\ \text{m}^2$ d $24.63\ \text{m}^2$ e $235.62\ \text{m}^2$ f $146.08\ \text{m}^2$

4 $15\,394\ \text{cm}^2$

5 a $1.08\ \text{m}^2$ b 6 m

6 17.3 cm

Section 10H

Now try this

30 a 5 b 25

31 22 cm

Exercise 10H

1 a i $k = 3$ ii $k^2 = 9$

b i $k = 2$ ii $k^2 = 4$

2 a similar, k=3 b similar, k=2 c not similar

3 a $k = 3$ b 3 cm c 1 : 9

4 $112\ \text{cm}^2$ 5 $864\ \text{cm}^2$

6 1.67 7 a 36 km b 3 cm

Section 10I

Now try this

32 Side Angle Side

Exercise 10I

1 a SSS b AA c SAS or SSS or AA

2 a $x = 27$ cm, $y = 30$ cm

b $x = 26$ m, $y = 24$ m

3 a 7 b 28 cm, 35 cm c 119 cm

4 a AA b $\frac{1}{2}$ c 2 m

5 1.8 m 6 $72\ \text{cm}^2$ 7 $29.4\ \text{cm}^2$

Section 10J

Now try this

33 343 times larger

Exercise 10J

1 27 times

2 a 1:4 b 1:64

3 $\frac{27}{1}$

4 a 9 cm b 1 : 125

5 a Scaled up b 27 c $3240\ \text{cm}^3$

6 a 6 cm b 27: 64

7 a 3 cm b Height = 12 cm, base = 16 cm

8 a $\frac{1}{4}$ b $\frac{1}{8}$

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Chapter 10 Review

Skills Checklist answers

1 307.51 **2** 310 **3** 7.5 cm
4 17.3 cm **5** 21 cm, 20 cm^2
6 18 m, 18 m^2 **7** 44.0 cm, 153.9 cm^2
8 13.3 m **9** 66.3 m^3 **10** 32 032 mm^3
11 27 litres **12** 10 178.8 cm^3
13 7068.6 cm^3 **14** 53 824 m^3
15 52 m^2 **16** 5309.3 cm^2
17 $\frac{6}{8} = \frac{9}{12} = \frac{3}{4}, k = \frac{3}{4}$ **18** $k^2 = \frac{9}{16}$ **19** 9
20 Yes, SSS **21** Yes, $k = \frac{5}{3}$

Multiple-choice questions

1 C **2** D **3** D **4** B **5** D
6 E **7** C **8** E **9** C **10** B
11 C **12** D **13** B **14** C **15** C
16 C **17** B **18** C **19** D **20** E
21 A **22** D **23** C **24** C **25** E

Short-answer questions

1 a 2.945×10^3 **b** 5.7×10^{-2}
c 3.69×10^5 **d** 8.509×10^2
2 a 7500 **b** 0.00107 **c** 0.456
3 a 8.9 **b** 0.059 **c** 800
4 a 7.15 **b** 598.2 **c** 4.079
5 a 58 cm **b** 30 m
6 36 m **7** 68 cm
8 a 9.22 cm **b** 9 cm
9 a 140 cm^2 **b** 185 cm^2
10 37.5 cm^2
11 a 31.42 cm **b** 75.40 cm
12 a 78.54 cm^2 **b** 452.39 cm^2
13 a 373.85 cm^2 **b** 2.97 m^2
c 0.52 litres
14 31 809 litres
15 a 514 718 540 km^2 **b** 1.098×10^{12} km^3
16 a 376.99 cm^3 **b** 377 mL
17 6.4 m
18 a 30 m^2 **b** 15 m^2 **c** 5.83 m
d 69.97 m^2 **e** 421.94 m^2
19 33.32 m^3
20 4 **21** $\frac{1}{4}$
22 a 50.27 cm **b** 146.12 cm^2
23 both equal 25.13 m

Written-response questions

1 a 154.30 m^2 **b** 101.70 m
2 a 61.54 m **b** 140 m^2 **c** 120 m^3 **d** 128 m^2
3 13.33 m
4 a 15.07 m **b** 1.89 m^3
5 a $\frac{1.96}{1}$ or 1 : 1.96 **b** $\frac{2.744}{1}$ or 1 : 2.744
c 63 cm^3
6 2048 cm^3
7 18.71 cm
8 a 400 m
b 400 m, 406 m, 412 m, 419 m, 425 m, 431m
c Starting from the inside lane each new starting point should be 6 m ahead, except for distance between 3rd and 4th runners which is 7 m.

Chapter 11

Section 11A

Now try this

1 a opposite **b** adjacent
c hypotenuse
2 a 0.532 **b** 0.731 **c** 0.889

Exercise 11A

1 a opposite **b** hypotenuse
c adjacent
2 a opposite **b** adjacent
c hypotenuse
3 a $\frac{21}{29}$ **b** $\frac{20}{29}$ **c** $\frac{21}{20}$
4 Answers are in order: hypotenuse, opposite, adjacent.
a 13, 5, 12 **b** 10, 6, 8
c 17, 8, 15
5 Answers are in order: $\sin\theta, \cos\theta, \tan\theta$.
a $\frac{5}{13}, \frac{12}{13}, \frac{5}{12}$ **b** $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}$
c $\frac{8}{17}, \frac{15}{17}, \frac{8}{15}$
6 a 0.4540 **b** 0.7314 **c** 1.8807 **d** 0.1908
e 0.2493 **f** 0.9877 **g** 0.9563 **h** 1.1106
7 $\frac{21}{29}$
8 $\frac{9}{40}$

Section 11B

Now try this

3 57.6 **4** 26.0

Exercise 11B

1 a adjacent **b** opposite
c $\tan\theta = \frac{opposite}{adjacent}$ **d** $\tan 47° = \frac{x}{36}$
e 38.61 cm

2 a opposite **b** hypotenuse

c $\sin\theta = \frac{opposite}{hypotenuse}$ **d** $\sin 39° = \frac{19}{x}$

e $x = \frac{19}{\sin 39°}$ **f** 30.2 cm

3 a $\sin\theta$, 20.74 **b** $\cos\theta$, 20.76

c $\tan\theta$, 32.15 **d** $\cos\theta$, 8.24

e $\tan\theta$, 26.63 **f** $\sin\theta$, 7.55

4 a 78.05 **b** 25.67 **c** 8.58 **d** 54.99

e 21.32 **f** 11.59

5 a 12.8 **b** 28.3 **c** 38.5 **d** 79.4

e 16.2 **f** 15.0

6 13.4

7 18.5

Section 11C

Now try this

5 a 46.79° **b** 82.46° **c** 12.53°

6 55.15°

Exercise 11C

1 a 60.9° **b** 32.4° **c** 23.1°

2 a hypotenuse **b** adjacent

c $\cos\theta = \frac{adjacent}{hypotenuse}$

d $\cos\theta = \frac{25}{28}$ **e** 26.8°

3 a 28.8° **b** 51.1° **c** 40.9° **d** 30.0°

e 45.0° **f** 45.0° **g** 60.0° **h** 68.2°

i 33.0°

4 a 32.2° **b** 59.3° **c** 28.3° **d** 55.8°

e 46.5° **f** 48.6° **g** 53.1° **h** 58.8°

i 22.6°

5 a 36.9° **b** 67.4° **c** 28.1°

6 144.5°

7 52.3°

Section 11D

Now try this

7 115 m **8** 35°

Exercise 11D

1 a adjacent **b** opposite

c $\tan\theta = \frac{opposite}{adjacent}$ **d** $\tan 40° = \frac{x}{28}$

e 23.5 m

2 a hypotenuse **b** opposite

c $\sin\theta = \frac{opposite}{hypotenuse}$ **d** $\sin\theta = \frac{10}{35}$

e 16.6°

3 a adjacent, hypotenuse

b $\cos\theta = \frac{adjacent}{hypotenuse}$

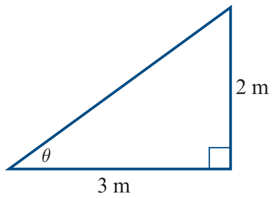
c $\cos 28° = \frac{11}{x}$ **d** 12.5 km

4 6.43 m

5 21.0°

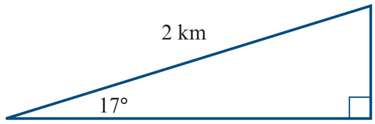
6 16 m

7 a



b 33.7°

8 a



b i Horizontal distance 1.91 km

ii Height 0.58 km

9 70.5° **10** 78.1 m **11** 20.7 m

12 49.1° **13** 27.1 m

Section 11E

Now try this

9 214 m

10 1231 m

11 a 51 m **b** 20°

Exercise 11E

1 a 38° **b** 38° **c** 52° **d** 38°

2 a opposite, adjacent **b** $\tan\theta = \frac{opposite}{adjacent}$

c $\tan 52° = \frac{x}{20}$ **d** 25.6 m

3 a $\angle RTV$ **b** $\tan\theta = \frac{opposite}{adjacent}$

c $\tan 40° = \frac{45}{d}$ **d** 53.6 m

4 413 m

5 11 196 m

6 14°

7 a 44.6 m **b** 36°

8 a 35 m **b** 64 m **c** 29 m

9 a 61.8° **b** 33.5 m

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Section 11F

Now try this

12 310°

13 a 11.5 km **b** 145°

Exercise 11F

1 a 000° **b** 090° **c** 180° **d** 270°

2 a 060° **b** 130° **c** 200° **d** 350°

3 a hypotenuse, opposite

b $\sin\theta = \dfrac{opposite}{hypotenuse}$ **c** $\sin 60° = \dfrac{x}{12}$

d The driver must walk 10.4 km

4 a 025° **b** 110° **c** 210° **d** 280°

5 a 25° **b** 7.61 km

6 a 236° **b** 056°

7 a 12.9 km **b** 15.3 km **c** 17.1 km

d 42° **e** 138°, 23.0 km

8 063°

9 a 130° **b** 240°

Section 11G

Now try this

14 37.6° **15** 10.2 **16** 79.51°, 100.49°

17 60.58 m

Exercise 11G

1

a $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}, \dfrac{a}{\sin A} = \dfrac{c}{\sin C}, \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

b $\dfrac{p}{\sin P} = \dfrac{q}{\sin Q}, \dfrac{p}{\sin P} = \dfrac{r}{\sin R}, \dfrac{r}{\sin R} = \dfrac{q}{\sin Q}$

2 a $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$ **b** $\dfrac{b}{\sin 33°} = \dfrac{21}{\sin 110°}$

c 12.2 cm

3 a $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$ **b** $\dfrac{7}{\sin C} = \dfrac{12}{\sin 120°}$

c $\dfrac{\sin C}{7} = \dfrac{\sin 120°}{12}$ **d** 30.3°

4 a $a = 15, b = 14, c = 13$

b $a = 19, b = 18, c = 21$

c $a = 31, b = 34, c = 48$

5 a $C = 50°$ **b** $A = 40°$ **c** $B = 105°$

6 a 5.94 **b** 12.08 **c** 45.11 **d** 86.8°

e 44.4° **f** 23.9°

7 a 41.0° **b** 53.7° **c** 47.2° **d** 50.3°

8 a 19.60 **b** 30.71 **c** 55.38 **d** 67.67

9 a $c = 10.16, B = 50.2°, C = 21.8°$

b $b = 7.63, B = 20.3°, C = 39.7°$

c $a = 52.22, c = 61.01, C = 37°$

d $b = 34.65, c = 34.23, C = 54°$

10 39.09 **11** 43.2° **12** 49.69

13 $a = 31.19, b = 36.56, A = 47°$

14 $A = 27.4°, C = 22.6°, c = 50.24$

15 $p = 154.54, q = 100.87, R = 20°$

16 a 66.60° **b** 66.60° **c** 113.40°

d 66.60°, 113.40°

17 61.04°, 118.96°

18 2.66 km from *A*, 5.24 km from *B*

19 409.81 m

20 a 26.93 km from naval ship, 20.37 km from other ship

b 1.36 h (1 h 22 min)

21 a Airport *A* **b** 90.44 km

c Yes

22 44.1 cm

23 a impossible **b** ambiguous

c only one possible triangle

Section 11H

Now try this

18 11.92

19 21.8°

20 a 53.8° **b** 324°

21 a 20° **b** 7.82 km

Exercise 11H

1 a $a^2 = b^2 + c^2 - 2bc\cos A$,
$b^2 = a^2 + c^2 - 2ac\cos B$,
$c^2 = a^2 + b^2 - 2ab\cos C$

b $x^2 = y^2 + z^2 - 2yz\cos X$,
$y^2 = x^2 + z^2 - 2xz\cos Y$,
$z^2 = x^2 + y^2 - 2xy\cos Z$

2 a $c^2 = a^2 + b^2 - 2ab\cos C$

b $c^2 = 31^2 + 45^2 - 2(31)(45)\cos 38°$

c 28.1 cm

3 a $y^2 = x^2 + z^2 - 2xz\cos Y$

b $28^2 = 37^2 + 49^2 - 2(37)(49)\cos Y$

c $784 = 3770 - 3626\cos Y$

d $\cos Y = 0.8235$ **e** 34.6°

4 a 36.72 **b** 47.62 **c** 12.00 **d** 14.55

e 29.95 **f** 11.39

5 17.41

6 27.09

7 51.51

8 a 33.6° **b** 88.0° **c** 110.7° **d** 91.8°

e 88.3° **f** 117.3°

9 63.2°

10 40.9°

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11 $B = 46.6°$

12 33.6°

13 19.1 km

14 a 39.6° **b** 310°

15 a 60.0° **b** 42.51 km

16 5.26 km

17 10.9 m

Section 11I

Now try this

22 16.0 m^2

23 13.2 cm^2

24 26.8 m^2

Exercise 11I

1 a 6 m **b** 5 m **c** 15 m^2

2 a 4 m **b** 7 m **c** 14 m^2

3 a 46° It's between the given lengths
b 38.8 cm^2

4 a 102 cm^2 **b** 40 cm^2 **c** 24 cm^2

5 a 25.7 cm^2 **b** 65.0 cm^2
c 26.0 cm^2 **d** 32.9 cm^2

6 a 36.0 km^2 **b** 9.8 m^2
c 23.5 cm^2

7 a iv **b** iii **c** i **d** ii

8 a 10 cm^2 **b** 23.8 cm^2 **c** 63.5 cm^2
d 47.3 m^2 **e** 30 m^2 **f** 30.1 m^2

9 224 cm^2

10 1124.8 cm^2

11 150.4 km^2

12 3500 cm^2

13 a 6 m^2 **b** 4.9 m^2 **c** 6.9 m^2

14 a 33.83 km^2 **b** 19.97 km^2
c 53.80 km^2

15 a i 12 km^2 **ii** 39 km^2 **iii** 21 km^2
b 29.6°

16 8640 m^2

17 30 square units

18 90°

Chapter 11 Review

Skills Checklist answers

1 opposite, adjacent, hypotenuse

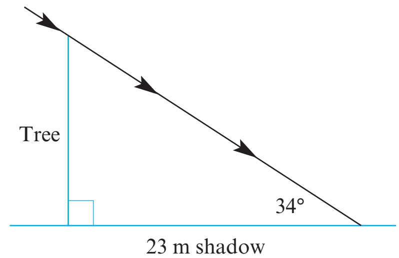
2 $\frac{56}{65}, \frac{33}{65}, \frac{33}{56}$

3 0.8910, 0.8480, 3.2709

4 $\sin\theta = \frac{opposite}{hypotenuse}$ **5** 9.4 **6** 42.0°

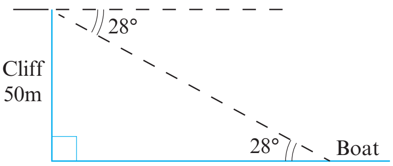
7 23.2°

8

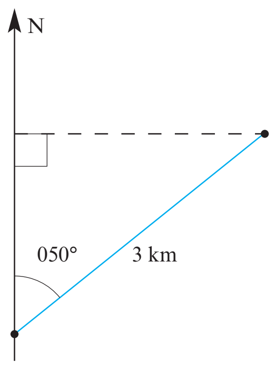


9 15.5 m

10



11 94 m **12**



13 2.3 km **14** 53.7° **15** 26 km

16 49.9°, 130.1° **17** cosine rule

18 10.5 **19** 72.3° **20** Heron's rule

21 328.7 cm^2

Multiple-choice questions

1 C **2** C **3** B **4** E **5** B

6 A **7** D **8** C **9** E **10** B

11 D **12** B **13** E **14** B **15** D

16 B **17** B **18** C **19** B

Short-answer questions

1 35.87 cm **2** 117.79 cm **3** 4°

4 a 65, 72, 97 or multiples of these **b** $\frac{65}{97}$

5 14.02 cm **6** 76.3°, 103.7°

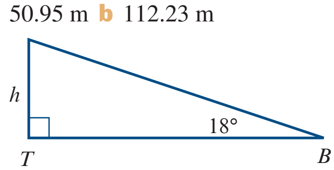
7 $A = 40.7°$ **8** 54.17 km **9** 760.7 cm^2

10 27.7 m^2

Written-response questions

1 a 50.95 m **b** 112.23 m

c



Height of tree = 36.47 m

2 a 110°

b 81.26 km

3 a 24 000 m^2 **b** 48 000 kg
c $3 000 000 000

Chapter 12

12A Multiple-choice questions

Chapter 7 Investigating relationships between two numerical variables

1 E **2** A **3** C **4** A
5 C **6** E **7** D **8** D
9 B **10** C **11** E **12** C

Chapter 8 Graphs and networks

13 C **14** E **15** B **16** E
17 E **18** E **19** C **20** E
21 D **22** C **23** E **24** A

Chapter 9 Variation

25 A **26** B **27** D **28** A
29 C **30** B **31** D **32** D
33 B **34** A

Chapter 10 Measurement, scale and similarity

35 E **36** C **37** C **38** B **39** B
40 C **41** B **42** A **43** E

Chapter 11 Applications of trigonometry

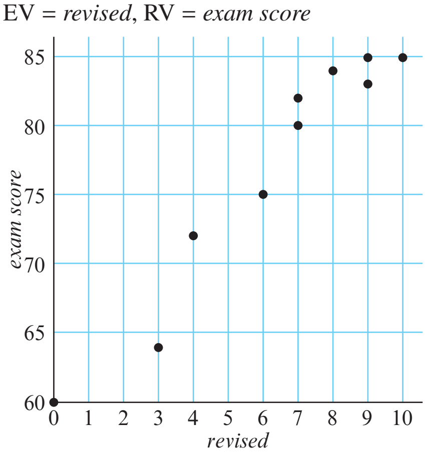
44 A **45** D **46** E **47** A **48** D
49 D **50** C **51** D **52** D **53** A

12B Written-response questions

Chapter 7 Investigating relationships between two numerical variables

1 a EV = *revised*, RV = *exam score*

b



c $r = 0.894$

d There is a strong, positive, linear relationship between exam score and the number of revision sessions.

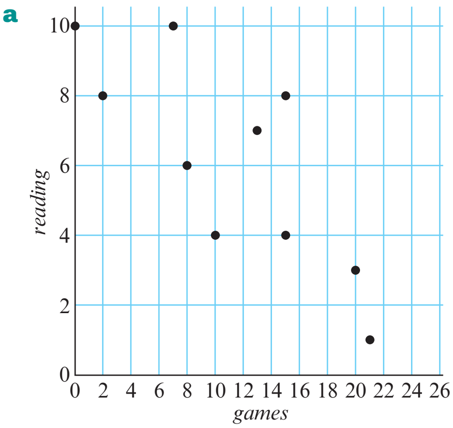
e *exam score* = 57.03 + 2.98 × *revised*

f Intercept: On average, students who do no revision sessions will score 57 on the examination. Slope: On average, a student's score on the examination will increase by 3 marks for each additional revision session.

g 93

h unreliable as we are extrapolating.

2 a



b There is a strong, negative association between the time spent playing games and the time spent reading.

c *reading* = 9.87 − 0.340 × *games*

d $r = 0.787$

e Intercept: On average, students who spend no time playing games will read for 9.87 hours. Slope: On average, students will read for 0.34 hours less for each additional hour they spend playing games.

f 6.47 hours

g reliable as we are interpolating.

3 a EV: *length*, RV: *weight*

b There is a strong, positive, linear relationship between length and weight.

c *weight* = −1053.15 + 54.54 × *length*

d Intercept: no meaningful interpretation, Slope: On average, fish will weigh 54.54 grams more for each additional cm in length.

e 1692 gm

f unreliable as we are extrapolating.

4 a EV: *advertising*, RV: *customers*

b There is a strong, positive linear relationship between the amount spent on advertising as the number of customers.

c *customers* = 50.0 + 0.1 × *advertising*

d 50

e $10

f *customers* = 50.0 + 0.5 × *advertising*

Chapter 8 Graphs and networks

5 a 4

b Eulerian trail

c Not possible

d BE

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e

	A	B	C	D	E
A	0	1	0	0	1
B	1	0	1	0	1
C	0	1	0	1	0
D	0	0	1	2	1
E	1	1	0	1	0

6 a 2600 m

b i No. Not all vertices are of even degree.

ii Yes.(Just two vertices of odd degree, G and H)

iii Edges total 10 500 m. 500 m from H to G. Total = 11 000 m

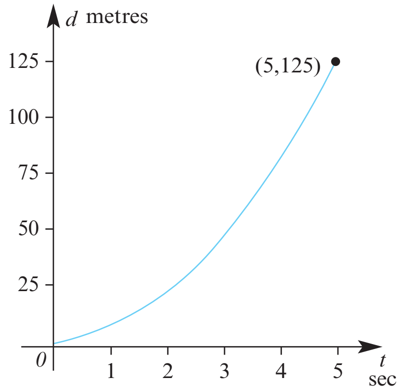
c 5600 m

Chapter 9 Variation

7 a $d = 5t^2$

b i 125 metres **ii** 3.54 seconds

c



8 a $t = \dfrac{600}{d}$ **b** $T = 1.20d^2$

c 20 minutes, 1080 minutes

d 15 mL **e** 270 minutes

Chapter 10 Measurement, scale and similarity

9 a 69.5 m^2 **b** 13.9 m^3 **c** \$833.90 **d** \$297.00

10 a 1.9 m, 4.8 m **b** 60.2 m^2

11 a 60 318.6 mm^3 **b** 25 132.7 mm^3

c 5 mm

12 a 20 hectares **b** 3 cm

Chapter 11 Applications of trigonometry

13 a 153° **b** 7° **c** 26 076.56 m

d 8919 m

14 a 2.6 units **b** 40.9°

c 2.6 sq units Using: $\frac{1}{2}s \times p \sin 60°$, Heron's rule

15 a 38.6 m **b** 30 m **c** 46.0 m **d** 40°

16 a 74.6° **b** 105.4°

c A = 74.6°, B = 40°, C = 65.4°

d A′= 105.4°, B = 40°, C = 34.6°

e 11.3 cm **f** 7.1 cm

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