Solutions to Exercise 1A

1 **a** *Time* in minutes is a measured quantity ⇒ numerical

b *Number of frogs* is a counted quantity⇒ numerical

c *Bank account numbers* are used only to name or identify a person's account. They have no numerical properties ⇒ categorical

d In this instance, a person's *height* is not measured (in cm, say), but recorded as falling into one of three categories (short, average, tall) ⇒ categorical

e *Time in hours* is a measured quantity ⇒ numerical

f *Number of people* is a counted quantity ⇒ numerical

g *Eye colour* recorded as (brown, blue, green), is a quality not a quantity ⇒ categorical

h *Postcode* is used only to name or identify a geographic area. It has no numerical properties ⇒ categorical

2 **a** *Colour* is a quality or characteristic that can only be used to name or identify but not order ⇒ nominal

b *Type of animal* is a quality or characteristic that can only be used to name or identify but not order ⇒ nominal

c *Floor level* a quality or characteristic that can be used to order ⇒ ordinal

d *Speed* a quality or characteristic that can be used to order ⇒ ordinal

e *Shoes size* is a quality or characteristic that can be used to order ⇒ ordinal

f *Family name* is a quality or characteristic that can only be used to name or identify but not order ⇒ nominal

3 **a** *Number of pages* is a counted quantity ⇒ discrete

b If *cost* is charged to the nearest dollar, eg. $24, $34, $67, etc then cost is discrete ⇒ discrete

c *Volume* is a measured quantity ⇒ continuous

d *Speed* is a measured quantity ⇒ continuous

e *Number of people* is a counted quantity ⇒ discrete

f *Temperature* is a measured quantity ⇒ continuous

4 The data collected is categorical, since the numbers do not represent quantities, but are naming categories. The data is ordinal, since there is an order to the categories ⇒ B

5 The variables *weight* and *age* are both ordinal, taking values which are categories that can be ordered ⇒ D

6 The variables *number of contact hours per week* and *number of subjects needed to complete degree* are both numerical variables, and both are discrete since their values are obtained by counting. The other variables are all categorical ⇒ B

Solutions to Exercise 1B

1 a Count the number of A's, B's and C's to find the frequencies, making sure they sum to 11.

Percentage of A's= $\frac{3}{11} \times 100 = 27.2727... = 27.3$ rounded to one decimal place.

Percentage of B's= $\frac{5}{11} \times 100 = 45.4545... = 45.5$ rounded to one decimal place.

Percentage of C's= $\frac{3}{11} \times 100 = 27.2727... = 27.3$ rounded to one decimal place.

	Frequency	
Grades	Count	%
A	3	27.3
B	5	45.5
C	3	27.3
Total	11	100.1

The percentages sum to 100.1, which is due to rounding.

b Count the number of 8's, 9's, 10's, 11's and 12's to find the frequencies, making sure they sum to 12.

Since the variable *shoe size* is ordinal, a shoe size of 11 has been included for completeness.

Percentage of 8's= $\frac{6}{12} \times 100 = 50.0$ rounded to one decimal place.

Percentage of 9's= $\frac{3}{12} \times 100 = 25.0$ rounded to one decimal place.

Percentage of 10's= $\frac{2}{12} \times 100 = 16.7$ rounded to one decimal place.

Percentage of 11's= $\frac{0}{12} \times 100 = 0$

Percentage of 12's= $\frac{1}{12} \times 100 = 8.3$ rounded to one decimal place.

	Frequency	
Shoe size	Count	%
8	6	50.0
9	3	25.0
10	2	16.7
11	0	0
12	1	8.3
Total	12	100.0

2 a *State of residence* is a quality or characteristic of a person ⇒ categorical.

b Count the number of A's, B's and C's to find the frequencies, making sure they sum to 11.

Percentage of Vic's= $\frac{6}{11} \times 100 = 54.5$ rounded to one decimal place.

Percentage of SA's.= $\frac{1}{11} \times 100 = 9.1$ rounded to one decimal place.

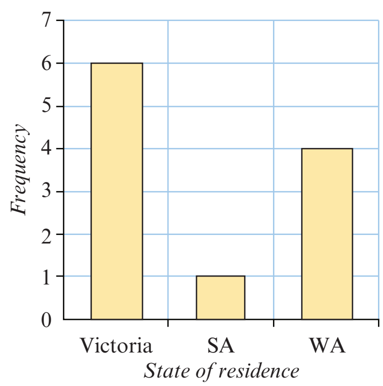
Percentage of WA's= $\frac{4}{11} \times 100 = 36.4$ rounded to one decimal place.

	Frequency	
State of residence	Count	%
Victoria	6	54.5
SA	1	9.1
WA	4	36.4
Total	11	100.0

c The bar chart is constructed from the frequency table.

The vertical axis should be labelled 'frequency'. The maximum frequency (count) is 6, so the vertical scale should be from 0 to 7, marked off in units of 1.

The horizontal axis should be labelled *State of residence*, and there will be a column for each value of the variable. There is no numerical scale on the horizontal axis, but the columns should be of equal width, and equidistant apart. They can be in any order.



3 a *Car size* recorded as 'small', 'medium' or 'large' is a quality or characteristic of a car ⇒ categorical.

b Count the number of A's, B's and C's to find the frequencies, making sure they sum to 20.

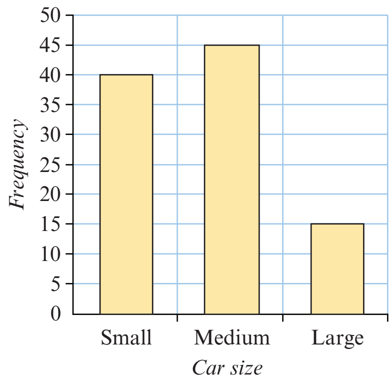
Percentage of small cars= $\frac{8}{20} \times 100 = 40$

Percentage of medium cars= $\frac{9}{20} \times 100 = 45$

Percentage of large cars= $\frac{3}{20} \times 100 = 15$

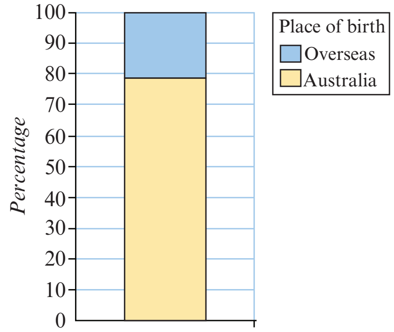
	Frequency	
Car size	Count	%
Small	8	40
Medium	9	45
Large	3	15
Total	20	100

c The percentage bar chart is constructed from the frequency table.
The vertical axis should be labelled ‘percentage’. The maximum percentage frequency is 45, so the vertical scale should be from 0 to 50, marked off in units of 10.
The horizontal axis should be labelled *Car size*, and there will be a column for each value of the variable. There is no numerical scale on the horizontal axis, but the columns should be of equal width, and equidistant apart. Since the variable car size is ordinal, the columns should be in size order, preferably from smallest to largest.



4 a Place of birth is a characteristic of a person that can be used to name or identify where a person was born. However, it has no ordering properties ⇒ nominal

b The vertical axis of a segmented bar chart is labelled ‘percentage’, and has a scale from 0 to 100, marked off in units of 10. To construct the segmented bar chart select the first value of the variable in the table (here it is Australia), and draw a bar from 0 to 80 (the percentage of the sample with that response). The next value of the variable in the table (Overseas) has a percentage frequency of 20. To add this segment start at 80 and add a bar of height 20 (ending at 100).
Colour each segment of the segmented bar chart with a different colour, and add a key indicating which value of variable is represented in each segment.



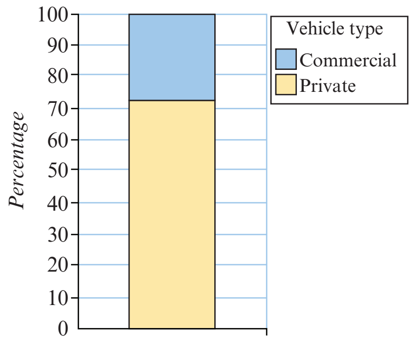
5 **a** *Type of car*, recorded as 'private' or 'commercial', can be used to identify a car. However, this information cannot be used to order cars in any meaningful way ⇒ nominal

b Sum the two count values to get the total count, which is 181 845.
To calculate the percentage frqeuencies divide each of the count values by the total count and multiply by 100.
Percentage of private vehicles= $\frac{132\,736}{181\,845} \times 100 = 73$
Percentage of commercial vehicles= $\frac{49\,109}{181\,845} \times 100 = 27$

Type of vehicle	Frequency Count	%
Private	132 736	73
Commercial	49 109	27
Total	181 845	100

c The vertical axis of a segmented bar chart is labelled 'percentage', and has a scale from 0 to 100, marked off in units of 10. To construct the segmented bar chart select the first value of the variable in the table (here it is Private), and draw a bar from 0 to 73 (the percentage of the sample with that response). The next value of the variable in the table (Private) has a percentage frequency of 27. To add this segment start at 73 and add a bar of height 27 (ending at 100).
Colour each segment of the segmented bar chart with a different colour, and add a

key indicating which value of variable is represented in each segment.



6 **a** The total number of schools sums to 20. Since the 3 percentages must sum to 100, and two percentages in the table sum to 45, the missing percentage must be 100 – 45 = 55.

b 5.

c There are 4 + 11 + 5 = 20 schools in total.

d 55% of schools are categorised as government.

e Report: 20 schools were classified according to school type. The majority of these schools, 55%, were found to be Government schools. Of the remaining schools, 25% were Independent while 20% were Catholic schools.'

7 **a** Taking the frequencies and percentages from the completed table:
Since the total number of students is 22, the two numbers in the table sum to 15, the number for 'rarely' = 22 – 15 = 7.
The total percentage must be 100%.
The sum of the percentages for 'regularly' and 'rarely' is equal to 54.5%. Thus the percentage for 'sometimes' equals 100 – 54.5 = 45.5%. ⇒ **B**

b Taking the frequencies and percentages from the completed table:
Report: When 22 students were asked the question, "How often do you play sport?", the dominant response was 'sometimes', given by 45.5% of the students. Of the remaining students, 31.8% of the students responded that they played sport 'rarely' while 22.7% said that they played sport 'regularly'.

8 Taking the frequencies and percentages from the table, and modelling on the previous reports:

The eye colours of 11 children were recorded. The majority, 54.5%, had brown eyes. Of the remaining children, 27.3% had blue eyes and 18.2% had hazel eyes.

9 The modal response is the response with the highest frequency. In this example, the highest frequency is 30, for the response 'most of time'. Adding the frequencies we see that there are a total fo 70 responses. Therefore the percentage who gave the modal response is $\frac{30}{70} \times 100 = 42.86\% \Rightarrow$ **B**

Solutions to Exercise 1C

1 **a** Count the number of 0's, 1's, 2's, 3's, 4's and 5's to find the frequencies, making sure they sum to 20. Percentage of 0's$= \frac{6}{20} \times 100 = 30$

Percentage of 1's$= \frac{4}{20} \times 100 = 20$

Percentage of 2's$= \frac{3}{20} \times 100 = 15$

Percentage of 3's$= \frac{3}{20} \times 100 = 15$

Percentage of 4's$= \frac{2}{20} \times 100 = 10$

Percentage of 5's$= \frac{2}{20} \times 100 = 10$

	Frequency	
Number	Count	%
0	6	30
1	4	20
2	3	15
3	3	15
4	2	10
5	2	10
Total	20	100

b The people who bought take-away food more than 3 times are those who bought it 4 times (10%) or 5 times (10%), which is a total of 20%

c The mode is the value with the highest frequency, which is 0 (frequency = 6).

2 **a** Count the number of 2's, 3's, 4's, 5's and 6's to find the frequencies, making sure they sum to 40.

Percentage of 2's$= \frac{1}{40} \times 100 = 2.5$

Percentage of 3's$= \frac{0}{40} \times 100 = 0$

Percentage of 4's$= \frac{17}{40} \times 100 = 42.5$

Percentage of 5's$= \frac{13}{40} \times 100 = 32.5$

Percentage of 6's$= \frac{9}{40} \times 100 = 22.5$

	Frequency	
Number	Count	%
2	1	2.5
3	0	0
4	17	42.5
5	13	32.5
6	9	22.5
Total	40	100.0

b The percentage of biscuits with three or less chocolates chips are those with 2 chocolate chips (2.5%) or three chocolate chips (0%), which is a total of 2.5%

c The mode is the value with the highest frequency, which is 4 (frequency = 17).

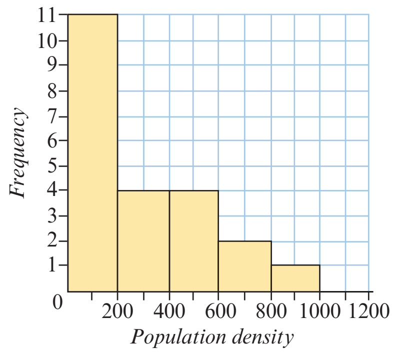
3 a Count the number of players with heights in in each interval.
For example, there are five players with heights from 160 cm to 164 cm, those with heights of 161 cm, 162 cm, 164 cm, 163 cm and 164 cm. Continue counting the numbers with the heights in each interval, making sure the total frequency is 25.

Height (cm)	Frequency
160−164	5
165−169	5
170−174	5
175−179	6
180−184	3
185−189	1
Total	25

b The interval with the highest frequency is 175−179, with a frequency of 6.

c There are 4 players with heights of 180 cm or more (1 + 3 = 4), which is a percentage of $\frac{4}{25} \times 100 = 16\%$

4 Label the vertical axis frequency. The highest frequency is 11, so mark in a vertical scale from 0 to 11.
Label the horizontal axis *Population density*.
Mark in equal size intervals, labelling only the beginning of each interval, ie 0, 200, 400, 600, 800, 1000.
Draw in the histogram bars to the appropriate height as given in the table.



5 **a** Remember that by convention the histogram columns include the beginning value of the interval but not the end value (which belongs to the next interval). So, the first column shows the percentage of sentences which have from 0 to less than 5 words. Since the number of words is discrete, then this would be from 0 to 4 words. Similarly, the second column shows the percentage of sentences with from 5 to 9 words, the third column the percentage of sentences with from 10 to 14 words, and so on. (Of course the idea of a sentence with 0 words is a nonsense, but the starting value of 0 and interval width of 5 have been chosen for convenience!).

i Reading from the graph: 17%

ii Reading from the graph: 13%

iii Reading from the graph: 46%

iv Summing the first three columns: 33%

b **i** Multiplying the percentage as read from the graph by 30 and dividing by 100% = 6

ii Multiplying the percentage as read from the graph by 30 and dividing by 100% = 4

c 15–19 is the highest bar and is therefore the mode in this case.

6 **a** Read the frequencies from the histogram and sum: 3 + 2 + 3 + 3 + 2 + 3 + 4 + 1 = 21

b Remember that by convention the histogram columns include the beginning value of the interval but not the end value (which belongs to the next interval). So, the first column shows the number of cricketers with a batting average from 0 to less than 5. Batting average will be a continuous variable, able to take any value on this interval. The second column shows the number of cricketers with batting averages from 5 to less than 10, the third column number of cricketers with batting averages from 10 to less than 15, and so on.

i The number of cricketers with batting averages of 20 or more $= 3+2+3+4+1 = 13$

ii The number of cricketers with batting averages less than 15 = 3 + 2 + 3 = 8

iii The number of cricketers with batting averages from 20 to less than 30 = 3 + 2 = 5

iv None of these cricketers had a batting average between 40 and 50.

c **i** There is only one player with a batting average of 50 or more, expressed as a percentage:

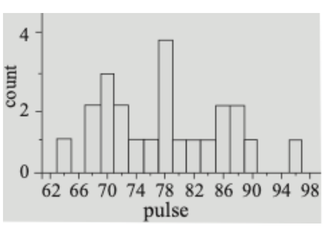
$$\frac{1}{21} \times 100 = 4.8\%$$

ii There are 3 cricketers with batting averages from 20 to less than 25, 2 with batting averages from 25 to less than 30, 3 with batting averages from 30 to less than 35, and 4 with batting averages from 35 to less than 40. That is a total of 2 + 3 + 3 + 4 = 12 with batting averages from 20 to less than 40, expressed as a percentage:

$$\frac{12}{21} \times 100 = 57.1\%$$

7 See calculator instructions page 17 (TI) or 19 (CASIO).

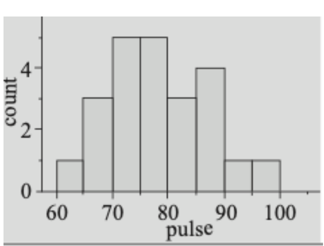
a



b **i** 69

ii 3; 69, 70, 70

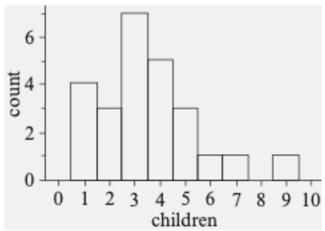
c



d 3

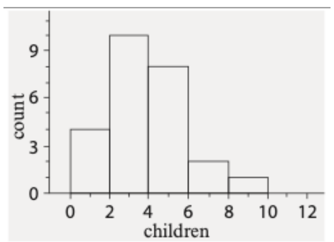
8 See calculator instructions page 17 (TI) or 19 (CASIO).

a



b 3.5, 5

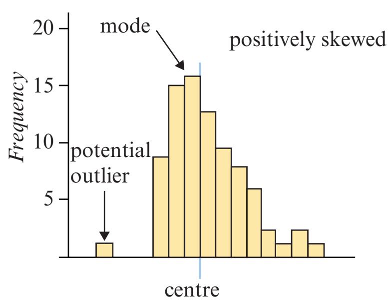
c



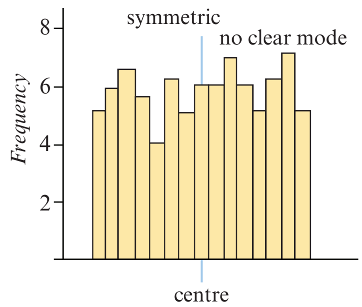
d **i** 2

ii 6 and 7

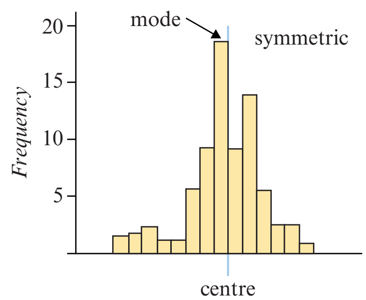
9 **a** The mode is the highest bar. There is one potential outlier and the histogram tails off to the right so is positively skewed.



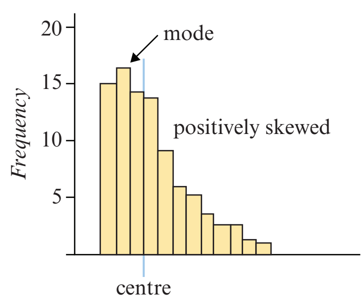
b The mode is not clear since two bars appear to be the same size. There are no outliers. The histogram is approximately symmetrically spread around its centre.



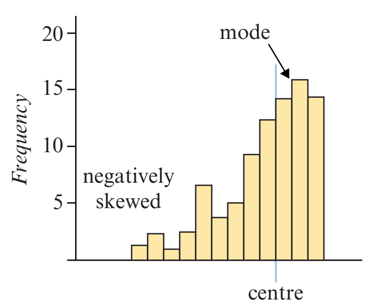
c The mode is the highest bar. There are no outliers and the histogram appears to be approximately symmetrically spread around its centre.



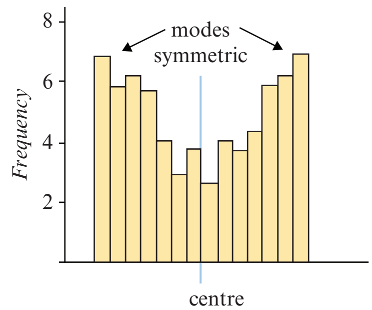
d The mode is the highest bar. There are no outliers and the histogram tails off to the right so is positively skewed.



e The mode is the highest bar. There are no potential outliers and the histogram tails off to the left so is negatively skewed



f There are two highest bars and thus two modes. There are no outliers and the histogram approximately symmetrically spread around its centre.



10 **a** All the distributions appear approximately symmetric around their respective centres.

b There are no clear outliers in any of the distributions.

c In A the central mark lies in the interval 8–10, in B it lies in the interval 24–26 and in C it lies in the interval 40–42. Note that the central interval in A and B is also the mode for those distributions.

d The spread is lowest in B since the range is only 8, compared to 14 for A and 18 for C.

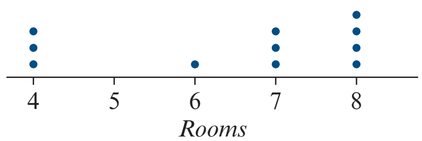
11 The interval width for this histogram is \$5000. Locate the interval from \$65 000 to less than \$70 000 on the horizontal axis. The height of this column is 30. Thus there are 30 people who earn from \$65 000 to less than \$70 000, which is option B.

12 The shape of this distribution is positively skewed, with a long upper tail (to the right). There are two values in the interval \$145 000 to \$150 000 which are possible outliers. The best answer is Option A.

Solutions to Exercise 1D

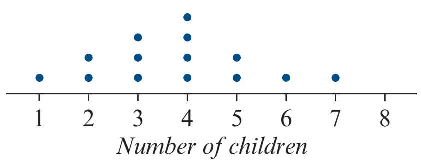
1 **a** discrete

b Label the horizontal axis *Rooms* and off a horizontal scale which includes from 4 to 8. The first value in the data is 4, mark in a dot at 4. The next value is 6, mark in a dot at 6, at the same height as the dot at 4. The next two values are both 7, mark in two dots at 7, one above the other, equally spaced. Continue until all of the data is included in the dot plot. Count the number of dots to ensure there are 11.



2 **a** discrete

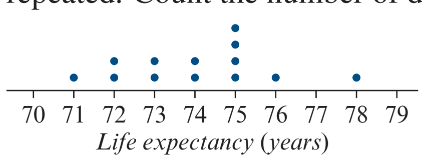
b Label the horizontal axis *Number of children* and mark off scale which includes from 1 to 7. The first value in the data is 1, mark in a dot at 1. The next value is 6, mark in a dot at 6, at the same height as the dot at 1. Continue until all of the data is included in the dot plot, stacking the dots on top of each other when a value is repeated. Count the number of dots to ensure there are 14.



c The mode is 4, which also has a frequency of 4. The mode is the most frequently occurring number of children for these families

3 **a** continuous

b Label the horizontal axis *Life expectancy (years)* and mark off scale which includes from 71 to 78. The first value in the data is 76, mark in a dot at 76. The next value is 75, mark in a dot at 75, at the same height as the dot at 71. Continue until all of the data is included in the dot plot, stacking the dots on top of each other when a value is repeated. Count the number of dots to ensure there are 13.

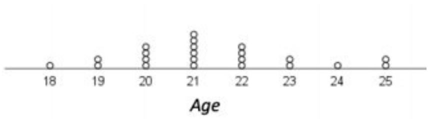


c The mode is 75, which has a frequency of 4; the mode is the most common life expectancy for these countries

4 a The shape of this distribution is negatively skewed, with a long lower tail (to the left).

b The shape of this distribution is positively skewed, with a long upper tail (to the right).

5 a Label the horizontal axis *Age* and mark off scale which includes from 18 to 25. The first value in the data is 22, mark in a dot at 22. The next value is 20, mark in a dot at 20, at the same height as the dot at 22. Continue until all of the data is included in the dot plot, stacking the dots on top of each other when a value is repeated. Count the number of dots to ensure there are 22.



b The mode is 21 years, which has a frequency of 6.

c The shape is approximately symmetric.

d There are 3 runners with ages less than 20 (1 at 18 and two at 19). The percentage is $\frac{3}{22} \times 100 = 13.63... = 14\%$ to the nearest whole percentage.

6 a The variable *urbanisation* is continuous: it is a percentage that can potentially take any value between 0 and 100, but has been rounded to the nearest percent.

b See Example 11 instructions on constructing a stem plot. The smallest data value is 3 and the largest is 80, so stems from 0 to 8 are suitable.

key: 1 | 6 = 16

0	33699
1	2267
2	0225789
3	15
4	46
5	
6	
7	99
8	0

7 a The variable *wrist circumference* is continuous because data values can take any value, but have been rounded to one decimal place.

b See page 31 for information on stem plots with split stems.

i 16|5 =16.5

16	5 7 9
17	0 1 2 3 6 6 7
18	2 4 5
19	3 9

ii 16|5 =16.5

16	5 7 9
17	0 1 2 3
17	6 6 7
18	2 4
18	5
19	3
19	9

8 **a** The shape of this distribution is positively skewed, with a longer tail to the higher values.

b The shape of this distribution is negatively skewed, with a longer tail to the lower values.

9 **a** Count the number of 'leaves' to find the number of data values (do not include the stems) = 40.

b The shape of the distribution is approximately symmetric.

c There are 2 people aged from 20-24, 7 aged from 25-29, 1 person ages 30 and 1 person aged 31, a total of 11 people aged less than 33 years.

10 The shape of the distribution is approximately symmetric, Option C.

11 Since there are 25 people, the slowest 20% are the slowest $\frac{20}{100} \times 25 = 5$ people. The slowest people are those which take the longest time. The five longest times are, 10.6 minutes, 9.5 minutes, 9.5 minutes, 8.9 minutes, and 8.5 minutes. So the slowest 20% of people took 8.5 minutes or more. Option B.

Solutions to Exercise 1E

1 **a** to **h**

To calculate the log of a number:

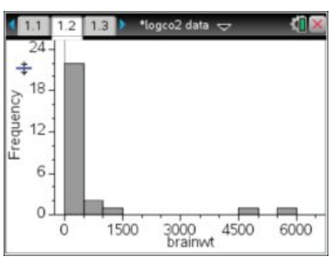
- Type **log(number)**, eg. log(2.5), and press **Enter** (or **exe**).
- Round the answer to one decimal place: eg. log(2.5) = 0.3979 . . . = 0.4 (to 1 d.p.)

2 **a** to **d**

To convert the log of a number to the number:

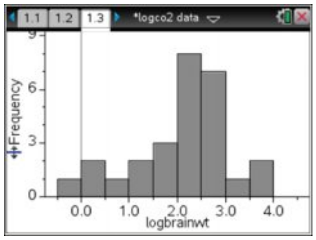
- Type **10^(number),** eg. 10^(−2.5), and press **Enter** (or **exe**).
- Round the answer to two significant figures as required: eg. log(−2.5) = 0.00316 . . . = 0.0032 (to 2 sig.figs.)

3 **a** To construct a histogram for the raw data, see instructions on page 17(TI) or page 19(CASIO).



The histogram tails off to the right indicating positive skew. The are also two columns well to the right of the main body of data which appear to be outliers.

b To construct a histogram with a log scale, see instructions on page 38(TI) or page 40(CASIO).



The histogram is slightly negatively skewed but much closer to symmetric.

4 **a** log(0.4) = −0.397 . . . = −0.4 to 1 d.p.

b log(5712) = 3.756 . . . = 3.76 to 3 sig. figs.

c $10^2 = 100$g

d $10^{-1} = 0.1$g

e **i** Since log(1000) = 3, 1000 g or more is equivalent to 3 or more on the log scale. Reading from the histogram, there are 5(= 3 + 2) animals with brain weights of 1000 g or more.

ii Since log(1) = 0, and log(100) = 2, from 1g to less than 100 g on the log scale is equivalent to from 0 to 2. Reading from the histogram, there are 12(= 4 + 8) animals with brain weights in this range.

iii 1g or more is equivalent to 0 or more on the log scale. Reading from the histogram, there are 24 (= 4 + 8 +7 + 3 + 2) animals with brain weights of 1g or more.

5 Since log(10 000) = 4, and log(100 000) = 5, from 10 000 to less than 100 000 on the log scale is equivalent to from 4 to 5. Reading from the histogram, there are 60 countries with carbon dioxide emissions in this range. The percentage of countries $= \frac{60}{239} \times 100 = 25.1$ Option B.

6 Since log(10 000 000) = 8, and log(100 000 000) = 9, from 10 000 000 to less than 100 000 000 on the log scale is equivalent to from 8 to 9. Reading from the histogram, there are 37 (= 12 + 35) countries with where tourism spend is in this range. Option D.

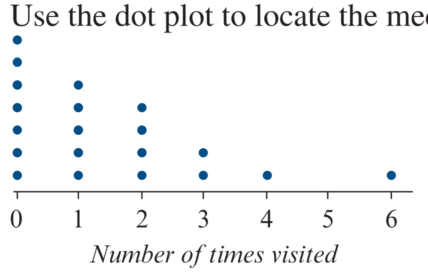
Solutions to Exercise 1F

1 **a** After ordering the 6 values, the two middle values are 4 and 6, median = (4 +6)/2 =5
To check that the median splits the data set into two equal parts, write out the date values in order, locate the median (see red line below)and count the number of data values each side of the median.
1 3 4|6 8 9

b After ordering the 5 values, middle value = median = 12
To check that the median splits the data set into two equal parts, write out the date values in order, locate the median (see red data value below)and count the number of data values each side of the median.
9 10 12 14 20

2 After ordering the 9 values, middle value = median = $850
430 500 650 750 850 1200 1790 2950 3500

3 Use the dot plot to locate the median.



There are 20 data values, so the median will be the mean of the 10th and 11th values. From the dot plot we can see that both of these values are equal to 1. Thus $M = 1$.

4

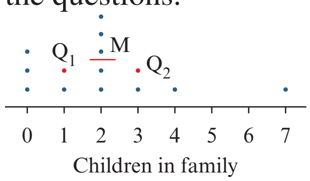
Time (minutes) key: 4|0 represents 4.0

4	2 6
5	1 3 6 8
6	0 1 5 6 7
7	1 3 4 5 7 8 9
8	0 2 5 9
9	5 5
10	6

a There are 25 values, so the median is the 13th value (from either end). That is, 7.3 minutes.

b The smallest value is 4.2 minutes, and the largest is 10.6 minutes. So the range $R = 10.6 - 4.2 = 6.4$ minutes.

5 Use the dot plot to locate the median and the quartiles and use these values to answer the questions.



a There are 14 values, so the median is the average of the 7th and 8th values, which are both 2. So $M = 2$

b There are 7 values in the each half, so Q_1 is the 4th value from the bottom, and Q_3 is the 4the value from the top. That is, $Q_1 = 1$ & $Q_3 = 3$.

c $IQR = Q_3 - Q_1 = 3 - 1 = 2$

d Range = max. – min. = 7 – 0 =7

e 0 0 0 1 1 2 2|2 2 2 3 3 4 7

Q_1 M Q_3

6 **a** There are 14 values, so the median is the average of the 7th and 8th values, which are 10 and 12 respectively. So $M = 11$.

b There are 7 values in the each half, so Q_1 is the 4th value from the bottom, and Q_3 is the 4the value from the top. That is, $Q_1 = 10$ & $Q_3 = 15$.

c $IQR = Q_3 - Q_1 = 15 - 10 = 5$ $Range = max - min = 25 - 7 = 18$.

7 **a** approximately symmetric, with no outliers.

b There are 20 values, so the median is the average of the 10th and 11th values, which are 25 and 27 respectively. So $M = 26$.

c There are 10 values in the each half. Q_1 is the average of the 5th and 6th values from the bottom, which are 16 and 19 respectively That is, $Q_1 = 17.5$. Q_3 is the average of the 5th and 6th values from the top, which are 30 and 31 respectively That is, $Q_3 = 30.5$.

d $IQR = Q_3 - Q_1 = 30.5 - 17.5 = 13$ $\quad$ $Range = max - min = 39 - 10 = 29.$

8 **a** Positively skewed with a possible outlier: The data rapidly tails of to the right and there is one value (6) that appears to be well separated from the main body of the data that could be an outlier.

b There are 23 values, so the median is the 12th value, so $M = 0$.

c There are 11 values in the each half, when the median is excluded. Q_1 is the 6th value from the bottom, so $Q_1 = 0$. Q_3 is the 6th values from the top so $Q_3 = 1$.

d $IQR = Q_3 - Q_1 = 1 - 0 = 1$

e $Range = max - min = 6 - 0 = 6.$

9 **a** Of the 17 values, so the median is the 9th value, so $M = 21$.

b median of the lower 50% (8 lowest values) $Q_1 = \dfrac{9 + 12}{2} = 10.5$. median of the upper 50% (8 highest values) $= Q_3 = \dfrac{26 + 30}{2} = 28$.

c $IQR = Q_3 - Q_1 = 28 - 10.5 = 17.5$

d $Range = max - min = 55 - 1 = 54.$

10 There are 44 data values, found by summing the frequencies (4+12+9+5+5+3+3+1+1+1 = 44). The median is the average of the 22nd and 23rd values, which are both in the interval 65 to less than 70. Thus the median will also be in this interval. There are 22 values in the each half. Q_1 is the average of the 11th and 12th values from the bottom, which are both in the interval 60 to less than 65. Thus Q_1 will also be in this interval. Q_3 is the average of the 11th and 12th values from the top, which are which are both in the interval 75 to less than 80. Thus, Q_3 will also be in this interval.

11 **a** There are 195 data values. The median is the 98th value, which is in the interval 5.0 to 9.9.

b There are 97 values in the each half (after median has been excluded). Q_1 is 49th value from the bottom, which is in the interval from 0.0 to 4.9. Q_3 is the 49th value from the top, which is in the interval 15.0 to 19.9. The maximum possible value for the IQR would be when Q_1 takes its minimum possible value (which is 0.0) and Q_3 takes its maximum possible value (which is 19.9). So the maximum possible value of the 1QR = 19.9 − 0.0 = 19.9.

12 **a** n = number of values = 4; $\sum x$ = sum of values = 12; $\bar{x} = \frac{\sum x}{n} = 3$

b n = number of values = 5; $\sum x$ = sum of values = 104; $\bar{x} = \frac{\sum x}{n} = 20.8$

c n = number of values = 7; $\sum x$ = sum of values = 21; $\bar{x} = \frac{\sum x}{n} = 3$

13 **a** After ordering and summing the 11 values, mean = $\frac{33}{11} = 3$; median = 6th value = 3; mode = most common value = 2

b After ordering and summing the 12 values, mean = $\frac{60}{12} = 5$; median = average of two middle values (5 and 5) = 5; mode = most common value = 5

14 After ordering and summing the 8 values,

a mean = $\frac{288.8}{8}$ = 36.1; median = average of two middle values (36.0 and 36.0) = 36.0

b Because the mean and median are very close to each other, the distribution can be assumed to be close to symmetric.

15 After ordering and summing the 7 values,

a mean = $\frac{\$25.55}{7}$ = \$3.65; median = middle value = \$1.70

b In this case, the median is a much better marker of a typical amount spent. This is because the mean has been positively skewed by the large positive outlier of \$16.55.

16 **a** Mean shouldn't be used, due to the distribution being strongly negatively skewed.

b No reason not to use the mean, the distribution is approximately symmetric.

c Mean shouldn't be used, due to outliers.

d Mean shouldn't be used, due to the distribution being very positively skewed.

e Mean shouldn't be used, due to the presence of outliers and the distribution being positively skewed.

f No reason not to use the mean.

17 a Since the distribution is approximately symmetric, either could be used, the distribution is approximately symmetric..

b mean = 82.55 kg (sum the data values and divide by 22); median = average of the 11th and 12th values = 82.5 kg.

18 There are only three measures of spread, and these are the features which differentiate them from each other.

a The *IQR*, by definition, will always incorporate 50% of the scores, specifically the middle 50% of scores.

b Since range = highest score – lowest score, the range only uses the smallest and largest scores.

c The standard deviation is the average amount by which the scores differ from the mean.

19 As all of the data values have the same, 7.1, the mean is 7.1 As data values do not vary, the standard deviation is 0.

20 It doesn't make sense to calculate a mean and standard deviation for *sex* (**b**), *post code* (**d**) and *weight* (underweight, etc.) (**f**), since all three are categorical variables and cannot be used to perform meaningful numerical calculations.

21 a Use you calculator. For instructions, see page xx (TI) or yy (CASIO).

b When the median and mean are similar in value, the distribution can be assumed to be close to symmetric.

22 Use you calculator. For instructions, see page xx (TI) or yy (CASIO)

23 There are 16 values, so the median is the average of the 8th and 9th values, which are 20 and 21 respectively. Thus M = 20.5, which is Option C.

24 There are 8 values in the each half. Q_1 is the average of the 4th and 5th values from the bottom, which are 9 and 12 respectively. That is, $Q_1 = 10.5$. Q_3 is the average of the 4th and 5th values from the top, which are both 26. That is, $Q_3 = 26$. Thus $IQR = Q_3 - Q_1 = 26 - 10.5 = 15.5$ which is Option C.

25 There are 159 values, so the median is the 80th value. Adding the frequencies starting from the lowest values we find that the 80th value is in the interval from 20 to less than 25 cm (2 + 18 + 24 + 33 = 77, 2 + 18 + 24 + 33 + 25 = 102). The only value given in that interval is 25.2 cm which is Option C.

26 There are 159 values, so there are 80 values in the lower half, and Q_1 is the average of the 40th and 41st values from the bottom, which are both in the interval from 15 to less than 20 cm, so Q_1 is in this interval. The only value given in that interval is 16.7 cm which is Option B.

Solutions to Exercise 1G

1 Locate Q_1, M and Q3 on the dot plot. Use these and the minimum and maximum values (read from the dot plot) to form the five-number summary
Five number summary: 4, 5, 6, 7, 9

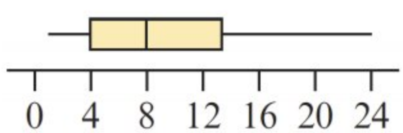
2 Locate Q_1, M and Q_3 on the stem plot.

key: 13 | 6						
13	6	7				
14	3	6	8	8	9	
15	2	5	8	8	8	
16	4	5	5	6	7	9
17	8	8	9			
18	2	9				

Use these and the minimum and maximum values (read from the stem plot) to form the five-number summary
Five number summary: 136, 148, 158, 169, 189

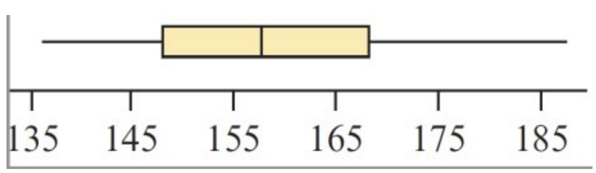
3 Draw an appropriately scaled number line. Mark in the each of the five numbers in the five number summary. Draw a box between Q_1 and Q_3, and locate the median in the box with a vertical line. Join the minimum to Q_1 with a horizontal line, join the maximum to Q_3 with a horizontal line.

a

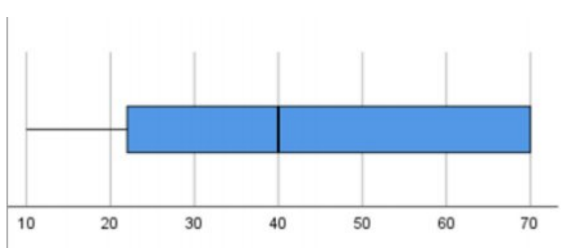


The scale here has been marked off in intervals of 4, other choices (such as 2 or 5) would also be correct.

b



4 a



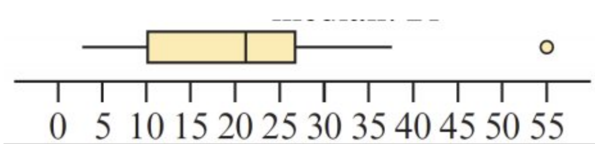
b There is no upper whisker because Q_3 and the maximum are equal, minimum = 3 and maximum = 55.

5 a The data are listed in order, so the minimum is 3 and the maximum is 55. There are 21 data values so the median is the 11th value, $M = 21$. There are 10 values in the bottom half, Q_1 is the average of the 5th and 6th values from the bottom which are 9 and 12, hence $Q_1 = 10.5$, and Q_3 is the average of the 5th and 6th values from the top which are 26 and 27, hence $Q3 = 26.5$.

b upper fence. $= Q_3 + 1.5 \times IQR = 26.5 + 1.5 \times 16 = 50.5$
$(IQR = Q_3 - Q_1 = 26.5 - 10.5 = 16)$

c The value 55 is an outlier as it is larger than the upper fence (50.5).

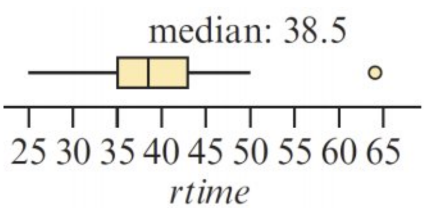
d



6 a Lower fence $= Q_1 - 1.5 \times IQR = 40 - 1.5 \times 22 = 7$
Upper fence $== Q_3 + 1.5 \times IQR = 62 + 1.5 \times 22 = 95$

b The data value 6 is less than the lower fence, so it is an outlier. The data value 99 is more than the upper fence, so it is an outlier.

7 a Follow the instructions for constructing a boxplot for a CAS calculator in the text.



b Five number summary: 25,35,38.5,43, possible outlier is at 64.

8 a i Median is middle vertical of coloured box = 10

ii Q_1 is left edge of the box = 5, Q_3 is right edge of the box = 21

iii $IQR = Q_3 - Q_1 = 16$

iv Minimum is left end of horizontal line = 0, maximum is right end of horizontal line = 45

v No outliers

b **i** Median is middle vertical of coloured box = 27

ii Q_1 is left edge of coloured box = 12, Q_3 is right edge of coloured box = 42

iii $IQR = Q_3 - Q_1 = 30$

iv Minimum is left end of horizontal line = 5, maximum is right end of horizontal line = 50

v No outliers

c **i** Median is middle vertical of coloured box = 38

ii Q_1 is left edge of coloured box = 32, Q_3 is right edge of coloured box = 42

iii $IQR = Q_3 - Q1 = 10$

iv Minimum is leftmost outlier = 5, maximum is right end of horizontal line = 50

v Outlier is 5

d **i** Median is middle vertical of coloured box = 16

ii Q_1 is left edge of coloured box = 14, Q_3 is right edge of coloured box = 21

iii $IQR = Q_3 - Q_1 = 7$

iv Minimum is leftmost outlier = 1.5, maximum is rightmost outlier = 50

v Outliers are 1.5, 3, 36, 40, 50

e **i** Median is middle vertical of coloured box = 34

ii Q_1 is left edge of coloured box = 30, Q_3 is right edge of coloured box = 39 (approximately)

iii $IQR = Q_3 - Q_1 = 9$ (approximately)

iv Since there is no tail, the minimum and Q_1 are equal. That is, minimum = 30. The maximum is upper outlier = 55

v Outlier is 55

9 **a** **i** Upper fence is right edge of coloured box + 1.5 times the $IQR = 40 + 1.5 \times 10 = 55$

ii Lower fence is left edge of coloured box – 1.5 times the $IQR = 30 - 1.5 \times 10 = 15$

b **i** Upper fence is right edge of coloured box + 1.5 times the $IQR = 70 + 1.5 \times 20 = 100$

ii Lower fence is left edge of coloured box – 1.5 times the IQR
$= 50 - 1.5 \times 20 = 20$

10 **a** Lower fence= $Q_1 - 1.5 \times IQR = 39 - 1.5 \times 6 = 30$
b no, 31 would still lie inside the lower fence

11 **a** 39 is the first quartile; 25% of values are less that the first quartile
b 45 is the third quartile; 75% of values are less than the first quartile
c 45 is the third quartile; 25% of values are greater than the first quartile
d 39 is the first quartile and 45 is the third quartile; 50% of values are between that the first and third quartiles
e 5 is less than the lowest data value and 45 is the third quartile; 75% of values are greater than 5 and less than the third quartile.

12 **a** 59 is the third quartile; 25% of values are above the third quartile
b 49.5 is the first quartile; 25% of values are below the first quartile
c 50% of values are between the first and third quartiles
d 57 is the median; 25% of values are between the median and the third quartile
e 75% of values are lower than the third quartile
f 70 is greater than any data value is the first quartile; 50% of values are between the median and any value greater than 67

13 Boxplot 1 matches histogram **B** because the histogram symmetric and has approximately equal length quartiles.
Box plot 2 is negatively skewed with a possible outlier at the upper end. Histogram **D** is negatively skewed distribution, with a high–value outlier.
Box plot 3 matches histogram **C** because the central body of data is approximately symmetric and there is both a high–value and low–value outlier.
Box plot 4 matches Histogram **A** because it is symmetric and the upper and lower quartiles are more spread out compared to the two middle quartiles.

14 **a** Because the median value of 41.5 is well towards right hand side of the box and the left hand whisker is longer than the right hand whisker we can say that the distribution is negatively skewed. There are no dots, so there are no outliers.
Reading values from the boxplot, we can say:
'The distribution is negatively skewed with no outliers. The distribution is centred at 41.5, the median value. The spread of the distribution, as measured by the *IQR,* is 15 and, as measured by the range, 47.'

b Because median value is towards the left hand side of the box and the right whisker is longer than the left whisker we can say that the distribution is postively skewed. There are no outliers.
Reading values from the boxplot, we can say:
'The distribution is positively skewed with no outliers. The distribution is centred at 800, the median value. The spread of the distribution, as measured by the *IQR,* is 1200 and, as measured by the range, 3200.'

15 **a** Because median value is well towards the right hand side of the box and the left whisker is longer than the right whisker we can say that the distribution is negatively skewed. The dot on the right indicates there is one outlier.
Reading values from the boxplot, we can say:
'The distribution is negatively skewed with an outlier. The distribution is centred at 39, the median value. The spread of the distribution, as measured by the *IQR,* is 10 and, as measured by the range, 45. There is an outlier at 5.'

b Because the median value of 16 is well towards to the left hand side of the box and the right hand whisker is longer than the left hand whisker we can say that the distribution is positively skewed. The five dots indicate that there are 5 outliers.
Reading values from the boxplot, we can say:
'The distribution is positively skewed with outliers. The distribution is centred at 16, the median value. The spread of the distribution, as measured by the *IQR,* is 6 and, as measured by the range, 35. The outliers are at 5, 8, 36, and 40.'

c Because the median value of 41 is approximately in the middle of the box and whiskers are similar in length, we can say that distribution is approximately symmetric. The four dots indicate that there are 4 outliers.
Reading values from the boxplot, we can say:
'The distribution is approximately symmetric and has 4 outliers. The distribution is centred at 41, the median value. The spread of the distribution, as measured by the *IQR,* is 7 and, as measured by the range, 36. The outliers are at 10, 15, 20 and 25.'

16 Because the median value of 70 is towards the left of the box, and the upper whisker is longer than the lower whisker, we can say that distribution is positively skewed. The two dots indicate that there are 2 outliers.

Reading values from the boxplot, and putting into the context of the question, we can say:

'The median time it takes Taj to travel to university is 70 minutes. The range of the distribution of travel time is 60 minutes, but the interquartile range is only 15 minutes. The distribution of travel times is positively skewed with two outliers, unusually long travel times of 110 minutes and 120 minute respectively.'

Note that there are many versions of the report that would be correct, this is just one example.

17 24 is the value of Q_3, so we can say about 25% of students scored more than 24 marks. Option B.

18 Reading from the boxplot Min = 1, Q_1 = 15, M = 18, Q_3 = 24, Max = 50. Option A.

19 Since the mean is quite a lot greater than the median we can say that the distribution is positively skewed. Option B.

20 The upper fence = $650 + 1.5 \times (650 - 120) = 1445$. Data values greater than this are 1550, 1600 and 1650. Option D.

Solutions to Exercise 1H

1 a 68% of values lie within 1 standard deviation of the mean.
134 + 20 = 154, 134 – 20 = 114, 68% of values lie between 114 and 154.

b 95% of values lie within 2 standard deviations of the mean.
134 + 40 = 174, 134 – 40 = 94, 95% of values lie between 94 and 174.

c 99.7% of values lie within 3 standard deviations of the mean.
134 + 60 = 194, 134 – 60 = 74, 99.7% of values lie between 74 and 194.

d 16% of values are greater than 1 standard deviation above the mean.
134 + 20 = 154, 16% of values are above 154.

e 2.5% of values are less than 2 standard deviations below the mean.
134 – 40 = 94, 2.5% of values are below 94.

f 0.15% of values are less than 3 standard deviations below the mean.
134 – 60 = 74, 2.5% of values are below 74.

g 50% of values are greater than the mean.
50% of values are greater than 134.

2 a 1.68 and 2.08 are both 1 standard deviation from the mean.
68% of values are between 1.68 and 2.08.

b 1.28 and 2.48 are both 3 standard deviations from the mean.
99.7% of values are between 1.28 and 2.48.

c 2.08 is 1 standard deviation above the mean.
16% of values are above 2.08.

d 2.28 is 2 standard deviations above the mean. 2.5% of values are above 2.28.

e 1.28 is 3 standard deviations below the mean.
0.15% of values are below 1.28.

f 1.88 is the mean.
50% of values are above 1.88.

3 a i 11 is one standard deviation below the mean. 84% of values are more than 11.

ii 14 is the mean. 50% of values are less than 14.

iii 20 is two standard deviations above the mean. 50% of values are greater than the 14 (the mean) 2.5% of values are greater than20 (2 SDs above the mean) 50–2.5 = 47.5 % of values are between 14 and 20.

b 8 is 2 standard deviations below the mean, which means that 2.5% of values are below 8.
Of the 1000 walkers we could expect approximately 2.5% of them, or
$\frac{2.5}{100} \times 1000 = 25$ walkers to complete the circuit in less than 8 minutes

4 a i 155 and 185 are both 3 standard deviations from the mean. 99.7% of values are between 155 and 185.

ii 180 is 2 standard deviation above the mean. 2.5% of values are above 180.

iii 160 is 2 standard deviations below the mean. 2.5% of values are below 160. 175 is 1 standard deviations above the mean. 16% of values are above 175. 100–2.5 –16 = 81.5% of values are between 160 and 175

b 175 is 1 standard deviation above the mean. 16% of values are above 175. In a sample of 5000 women we could expect approximately 16% of them, or
$\frac{16}{100} \times 5000 = 800$ of them to have heights greater than 175cm.

5 a i 66 is the mean. 50% of values are below 66.

ii 70 is 1 standard deviation above the mean. 16% of values are above 70.
66 is the mean.
50% of values are above 66
⇒ 50–16 = 34 % of values are between 160 and 175.

iii 62 is 1 standard deviation below the mean.
16% of values are below 62.
74 is 2 standard deviations above the mean.
2.5% of values are above 74.
⇒ 100–16 –2.5 = 81.5% of values are between 62 and 74

b 54 and 78 are both 3 standard deviations from the mean.
99.7% of values are between 54 and 78. In a sample of 2000 men we could expect approximately 99.7% of them, or
$\frac{99.7}{100} \times 2000 = 1994$ to have pulse rates between 54 and 78 beats/minute.

6 mean = 100, standard deviation = 20

a $z = \dfrac{120 - 100}{20} = 1$

b $z = \dfrac{140 - 100}{20} = 2$

c $z = \dfrac{80 - 100}{20} = -1$

d $z = \dfrac{100 - 100}{20} = 0$

e $z = \dfrac{40 - 100}{20} = -3$

f $z = \dfrac{110 - 100}{20} = 0.5$

7 mean = 30, standard deviation =7

a $z = \dfrac{37 - 30}{7} = 1.0$

b $z = \dfrac{23 - 30}{7} = -1.0$

c $z = \dfrac{40 - 30}{7} = 1.4$

d $z = \dfrac{20 - 30}{7} = -1.4$

8 **a** English: $z = \dfrac{69 - 60}{4} = 2.25$

Biology: $z = \dfrac{75 - 60}{5} = 3$

Chemistry: $z = \dfrac{55 - 55}{6} = 0$

Further Maths: $z = \dfrac{55 - 44}{10} = 1.1$

Psychology: $z = \dfrac{73 - 82}{4} = -2.25$

b *English*: A *z*–score of 2.25 is at least 2 standard deviations above the mean, so the student was within the top 2.5% of scores for English.
Biology: A *z*–score of 3 is 3 standard deviations above the mean, so the student was within the top 0.15% of scores for Biology.

Chemistry: A z–score of 0 is the mean, so the student was exactly average for Chemistry.
Further Maths: A z–score of 1.1 is at least 1 standard deviation above the mean, so the student was within the top 16% of scores for Further Maths.
Psychology: A z–score of –2.25 is at least 2 standard deviations below the mean, so the student was within the bottom 2.5% of scores for Psychology.

9 mean = 300, standard deviation = 2

a A z–score of 2 is 2 standard deviations above the mean, so approximately 2.5% of cans contain more soft drink than this can.

b Lower limit $z = \frac{302 - 300}{2} = 1$
Upper limit $z = \frac{306 - 300}{3} = 3$
Using the diagram on page 83 we see that between $z = 1$ and $z = 3$ lies 13.5% + 2.35% = 15.85% of the distribution. Thus we can say approximately 15.85% of the cans contain between 302 mL and 306 mL.

10 **a** A score of 49 is equivalent to a z–score of 2, so approximately 2.5% of people who sat the test will be eligible for the training program.

b People who are invited to resit the test are those with scores between 47 and 49. That is, those with z–scores between 1 and 2. Using the diagram on page 83 we see that this is 13.5% of people who sat the test.

11 **a** $x = 100 + 1 \times 20 = 120$

b $x = 100 + 0.8 \times 20 = 116$

c $x = 100 + 2.1 \times 20 = 142$

d $x = 100 + 0 \times 20 = 100$

e $x = 100 + (-1.4) \times 20 = 100 - 28 = 72$

f $x = 100 + (-2.5) \times 20 = 100 - 50 = 50$

12 Since 2.5% of retail assistants earn more than $30 per hour, this corresponds to a z–score of 2. The mean $\bar{x} = 27$. Substituting in the equation:
$x = \bar{x} + z \times s \quad \Rightarrow 30 = 27 + 2 \times s$

Hence $s = 1.50$

13 Since 16% of bananas weigh less than 96 gm, this corresponds to a z−score of -1. The standard deviation $s = 5$. Substituting in the equation:

$$x = \bar{x} + z \times s \qquad \Rightarrow 96 = \bar{x} - 1 \times 5$$

Thus $\bar{x} = 96 + 5 = 101$

14 Since 16% of babies weigh more than 4.0 gm, this corresponds to a z−score of 1. Substituting in the equation:

$$x = \bar{x} + z \times s \qquad \Rightarrow 4 = \bar{x} + 1 \times s \qquad \text{(equation 1)}$$

Since 0.15% of babies weigh more than 5.0 gm, this corresponds to a z−score of 3. Substituting in the equation:

$$x = \bar{x} + z \times s \qquad \Rightarrow 5 = \bar{x} + 3 \times s \qquad \text{(equation 2)}$$

Subtracting equation 1 from equation 2:

$1 = 2s \Rightarrow s = 0.5$

Substitute $s = 0.5$ in equation 1:

$4 = \bar{x} + 1 \times s \Rightarrow \bar{x} = 3.5$

Thus mean = 3.5 kg, stand dev = 0.5 kg

15 We know that 99.7% of students score between than 43 and 89 marks, which correspond to z−scores of -3 and 3. Substituting in the equation:

$$\Rightarrow 43 = \bar{x} - 3 \times s \qquad \text{(equation 1)}$$

$$\Rightarrow 89 = \bar{x} + 3 \times s \qquad \text{(equation 2)}$$

Subtracting equation 1 from equation 2:

$46 = 6s \Rightarrow s = 7.67$ (we need to keep two decimal places throughout the working)

Substitute $s = 7.67$ in equation 1:

$43 = \bar{x} - 3 \times 7.67 \Rightarrow \bar{x} = 66.0$

Thus mean = 66.0, stand dev = 7.7

16 **a** $z = \dfrac{56 - 54}{10} = 0.2$

b $x = 54 - 0.75 \times 10 = 46.5$ kg

c **i** A weight of 74 kg is 2 standard deviations above the mean so its standardised score is $z = 2$. Thus, using the 68-95-99.7% rule, 2.5% of these girls have weights greater than 74kg.

ii A weight of 54 kg is the mean weight of these girls so its standardised score is $z = 0$. A weight of 64 kg is one standard deviation above the mean so its standardised score is $z = 1$. Thus, using the 68-95-99.7% rule, 34% of these girls have weights between 54 and 74kg.

iii A standardised weight of $z = –1$ is one standard deviation below the mean. Thus, using the 68-95-99.7% rule, 16% of these girls have standardise weight less than –1.

iv A standardised weight of $z = –2$ is two standard deviation below the mean. Thus, using the 68-95-99.7% rule, 97.5% of these girls have standardised weights of more than –1.

17 **a** **i** A score of 115 is one standard deviation above the mean so its standardised score is $z = 1$. Thus, using the 68-95-99.7% rule, 16% of people have an IQ greater than 115.

ii A score of 70 is two standard deviation below the mean so its standardised score is $z = -2$. Thus, using the 68-95-99.7% rule, 2.5% of people have an IQ less than 70.

b The top 2.5% of the population have a $z-$score greater than 2. That is, an actual score greater than:

$$x = 100 + 2 \times 15 = 130$$

c $z - \text{score} = 2.2 \Rightarrow x = 100 + 2.2 \times 15 = 133$

18 **a** **i** A height of 152 is one standard deviation above the mean so its standardised score is $z = -1$. Using the diagram on page 83, 84% of of women are taller than 152 cm.

ii A height of 176 cm is two standard deviation above the mean so its standardised score is $z = 2$. Thus, Using the diagram on page 83, 97.5% of women are shorter than 176 cm.

b The tallest 0.15% of the women have a $z-$score greater than 3. That is, an actual height greater than:

$$x = 160 + 3 \times 8 = 184 \text{ cm.}$$

c The shortest 2.5% of the women have a $z-$score less than 2. That is, an actual height less than:

$$x = 160 - 2 \times 8 = 144 \text{ cm.}$$

d $z - \text{score} = -1.2 \Rightarrow x = 160 - 1.2 \times 8 = 150.4$ cm.

19 $z = \dfrac{71 - 50}{7} = 3$

Thus, using the 68-95-99.7% rule, 0.15% of students have scores greater than 71. ⇒ **A**

20 The top 2.5% of students have a z−score greater than 2. That is, an actual score greater than:

$x = 50 + 2 \times 7 = 64 \Rightarrow$ **D**

21 $z = \dfrac{57 - 50}{7} = 1$

A mark of 57 is 1 standard deviation above the mean. Thus, using the 68-95-99.7% rule, 16% of students have scores greater than 57.

The number of students who gained a mark of 57 or more is approximately $16000 \times \dfrac{16}{100} = 2560 \Rightarrow$ **C**

22 $z = \dfrac{43 - 50}{7} = -1$

$z = \dfrac{64 - 50}{7} = 2$

Using the diagram on page 83, 34% + 34% + 13.5% = 81.5% of students score between 43 and 64.

The number of students who score between 43 and 64 is approximately $16000 \times \dfrac{81.5}{100} = 13040 \Rightarrow$ **C**

23 Miller is in the fastest 2.5% of swimmers at his swimming club for a stroke if his z−score is less than -2. His z−scores are:

Butterfly: $z = \dfrac{38.8 - 46.2}{3.2} = -2.31$

Breaststroke: $z = \dfrac{51.4 - 55.1}{4.1} = -0.90$

Backstroke: $z = \dfrac{53.5 - 48.3}{2.5} = 2.08$

Freestyle: $z = \dfrac{33.3 - 38.2}{2.3} = -2.13$

Thus he is in the fastest 2.5% for two strokes, butterfly and freestyle. ⇒ **C**

Chapter Review: Solutions to Multiple-choice questions

1 *number of seats* is the only discrete numerical variable. ⇒ **A**

2 *age* is the only ordinal variable. ⇒ **B**

3 The brown hair segment (coloured blue) is approximately 36 percentage points in size. Since there are 200 students, this equates to $0.36 \times 200 = 72$ students with brown hair. ⇒ **D**

4 The brown hair segment is the largest, so brown hair is the most common hair colour. ⇒ **C**

5 The number of students who obtained scores less than 14 is found by adding the frequencies of the columns less than 14, that is $3 + 4 + 6 + 3 = 16$ ⇒ **D**

6 The main body of data looks to be positively skewed and there appears to an outlier. ⇒ **D**

7 There are 20 scores in total. The fist quartile is the average of the 5th and 6th values from the bottom, both of which lie on the interval 8 to < 10, so Q_1 is in this interval. The only option in this interval is 8.9 ⇒ **C**

8 $100 = 10^2$ so $\log(100) = 2$ ⇒ **C**

9 The number with log 2.314 is $10^{2.314} = 206$ to the nearest whole number. ⇒ **C**

10 $\log(16.8) = 1.22\ldots$ so on the log scale, Australia lies in the interval 1 – <1.5 ⇒ **D**

11 $\log(10) = 1$ so the percentage of countries with CO2 emissions under 10 tonnes is approximately $100 - (11 + 1) = 88\%$ ⇒ **E**

12 Minimum value = 22, Q_1 = 3rd value = 23, median = average of two middle values = 24.5, Q_3 = 8th value = 27, maximum value = 29 ⇒ **B**

13 There are 25 data points, so the median 13th value (from either end). Thus M = 28. ⇒ **C**

14 There are 25 data points, so there are 12 in each half (the median is omitted). Q_1 is the average of the 6th and 7th lowest values $= \dfrac{20 + 24}{2} = 22$
Q_3 is the average of the 6th and 7th highest values $= \dfrac{39 + 38}{2} = 39.5$
$IQR = Q_3 - Q_1 = 38.5 - 22 = 16.5$ ⇒ **B**

15 The main body of the data in the stem plot is roughly evenly spread around its centre, so the distribution is approximately symmetric. While there appears to be an outlier at 60, it is inside the lower fence which is at 63.25 (=38.5 + 1.5 ×16.5). ⇒ **A**

16 Reading from the midline in box plot A, the median is $M = 53$. ⇒ **B**

17 The length of the box in box plot B is 9 (= 75– 66), so the $IQR = 9$. ⇒ **A**

18 range = highest value – lowest value

In box plot C, the highest value is 80 (the end of the right hand whisker) and the lowest value is the outlier at 49, so range = 80 – 49 = 31 ⇒ **D**

19 In box plot A, the mid line is towards the left-hand side of the box and the left hand whisker is much shorter than the right hand whisker, so the distribution is positively skewed. ⇒ **D**

20 In box plot B, the mid line is centred in the box and the left hand whisker and the right hand whiskers are approximately equal in length, so the distribution is approximately symmetric. ⇒**A**

21 In box plot D, the mid line is just off centre the box and the left hand whisker and the right hand whiskers are approximately equal in length, so the main body of the data is approximately symmetric. There are also 4 outliers as indicated by the shaded circles. distribution is approximately symmetric with outliers. ⇒ **B**

22 In box plot D, the third quartile is 53, so 25% of the data values are great than 65. ⇒ **B**

23 $IQR = Q_3 - Q_1 = 65 - 60 = 5$;
$60 - 1.5 \times 5 = 52.5$ = lower fence,
$65 + 1.5 = 72.5$ = upper fence ⇒ **A**

24 Enter the data into your CAS calculator and calculate the mean and the standard deviation. ⇒ **D**

25 Since phone numbers are categorical variables and not numerical, calculating the mean and standard deviation a set of phone numbers will not give anything meaningful. ⇒ **B**

26 The mean is generally only an appropriate measure of centre when the distribution is symmetric and there are no outliers. The median is a much more robust measure of centre and would be preferred when the data is clearly skewed and/or there are outliers. ⇒ **D**

27 As every garden stake is reduced by the same amount, the mean value will decrease by this same amount, 5 cm, so the new mean = 180.5 − 5 = 175.5 cm.
Reducing the length of each stake by the same amount does not change the variability in length so the standard deviation stays the same, $s = 2.9$ cm ⇒ **C**

28 $z = \dfrac{50 - 55}{2.5} = -2.0$ ⇒ **B**

29 Because the trips are normally distributed, 50% of values lie above and below the mean. The mean is 78 mins.⇒ **C**

30 $z = \dfrac{70 - 78}{4} = -2$ $z = \dfrac{82 - 78}{4} = 1$
Using the diagram on page 83, 81.5% of trips have a z−score between -2 and 1. For 200 trips this is approximately $200 \times \dfrac{81.5}{100} = 163$ trips ⇒ **E**

31 $z = \dfrac{71 - 78}{4} = -1.75 \Rightarrow$ **A**

32 $x = 78 - 0.25 \times 4 = 77 \Rightarrow$ **A**

33 A standardised time of 2.1 is more than 2 standard deviations above the mean so it is very much above average. ⇒ **E**

34 Albie: $z = \dfrac{7.5 - 7.0}{1.2} = 0.42$

Lincoln: $z = \dfrac{4.9 - 7.0}{1.2} = 1.75$

Wendy: $z = \dfrac{8.0 - 7.0}{1.2} = 0.83$

Albie and Lincoln would both be able to join ⇒ **B**

35 Since 2.5% of bolts have diameter more than 4.94 mm, this corresponds to a z−score of 2. Substituting in the equation:

$x = \bar{x} + z \times s \Rightarrow 4.94 = \bar{x} + 2 \times s$ (equation 1)

Since 0.15% of bolts have diameter less than 4.84 mm, this corresponds to a z−score of -3. Substituting in the equation:

$x = \bar{x} + z \times s \Rightarrow 4.84 = \bar{x} - 3 \times s$ (equation 2)

Subtracting equation 2 from equation 1:

$0.1 = 5s \Rightarrow s = 0.02$

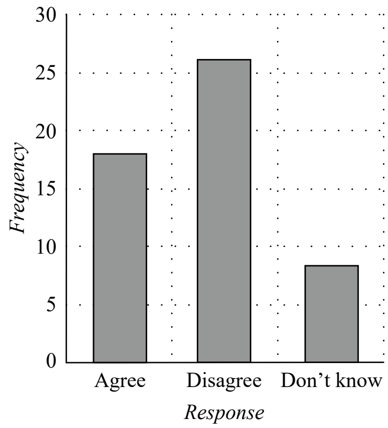
Substitute $s = 0.02$ in equation 1:

$4.94 = \bar{x} + 2 \times 0.02 \Rightarrow \bar{x} = 4.90$

Thus mean = 4.90 mm, stand dev = 0.02 mm. ⇒ **E**

Chapter Review: Extended-response questions

1 a

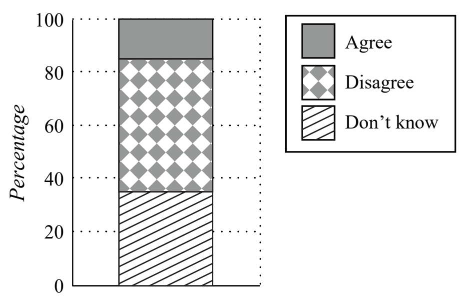


b Agree: $\frac{18}{52} = \frac{9}{26} \times 100\% = 34.6\%$

Disagree: $\frac{26}{52} = \frac{50}{100} \times 100\% = 50.0\%$

Don't know: $\frac{8}{52} = \frac{2}{13} \times 100\% = 15.4\%$

c



d Report: In response to the questions "Do you agree that the use of marijuana should be legalised" 50% of the 52 students disagreed. Of the remaining students, 34.6% agreed while 15.4% said that they didn't know.

2 a **i** Number of students surveyed = 3 + 5 + 5 + 9 + 6 + 8 + 8 + 2 + 2 + 2 = 50

ii 5 students spent from \$100 to less than \$105 per month.

b The modal interval is \$105–<\$110. This is the location of the tallest bar in the histogram.

c Number of students who spent \$110 or more = 6 + 8 + 8 + 2 + 2 + 2 = 28

d 3 + 5 = 8 students spent \$100 or less per month. $\frac{8}{50} = \frac{16}{100} \times 100\% = 16\%$

16% of students spent \$100 or less per month.

e **i** From the histogram, the distribution is approximately symmetric.

ii The median is the average of the 25th and 26th values. These are both in the interval \$100-<\$115, so the median will be in this interval.

iii The upper quartile will be the 13th value from the top of the distribution. Adding from right to left we see that the 13th value is in the interval \$120 - <\$125.

3 **a** The distribution is positively skewed.

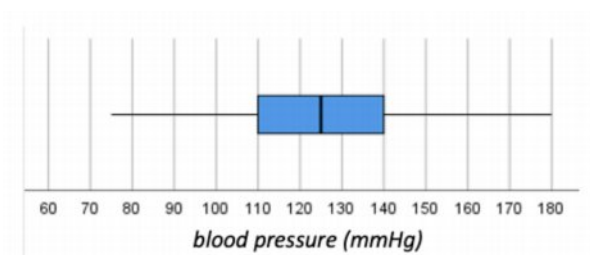
b There are 32 values, so the median is the average of the 16th and 17th values.
$M = \dfrac{2.6 + 2.7}{2} = 2.65$ kg.

c There are 16 values in each half of the data set. Q_1 is the average of the 8th and 9th smallest values.
$Q_1 = \dfrac{2.2 + 2.2}{2} = 2.20$ kg. Q_3 is the average of the 8th and 9th largest values.
$Q_3 = \dfrac{3.4 + 3.5}{2} = 3.45$ kg.
$IQR = Q_3 - Q_1 = 3.45 - 2.20 = 1.25$ kg.

d 5 people had a weight loss of more than 3.5 kg. This is $\dfrac{5}{32} \times 100 = 15.6\%$.

e Upper fence= $3.45 + 1.5 \times 1.25 = 5.33$ kg

4 **a**



b Lower fence= $110 - 1.5 \times 30 = 65$
Upper fence= $140 + 1.5 \times 30 = 185$
Since the minimum value is 75 (more than the lower fence) and the maximum value is 180 (less than the upper fence) there are no outliers.

c **i** $z = \dfrac{108 - 128}{20} = -1$ $z = \dfrac{148 - 128}{20} = 1$
The percentage of the distribution between $z-$scores of -1 and 1 is approximately 68%.

ii $x = 128 - 3 \times 20 = 68$ mmHg

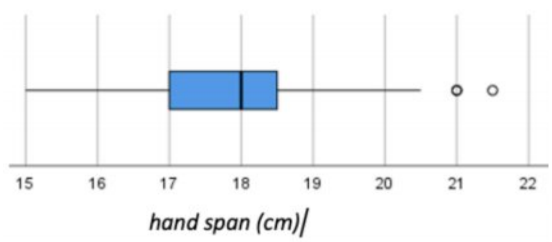
iii $z-$score = -3. Approximately 0.15% of people have a blood pressure 3 standard deviations below the mean. Of the 2000 people measured, we could expect approximately 3 to have blood pressure this low.

iv The lowest blood pressure in this group was 75, so there were no people with a blood pressure this low.

5 **a** The modal hand span (the one with the highest frequency) is 18.cm.

b mean−2× stand dev= $17.9 - 2 \times 1.1 = 15.7$ cm
Looking at the stem plot there are 4 values less 15.7
mean+2× stand dev= $17.9 + 2 \times 1.1 = 20.1$cm
Looking at the stem plot there are 7 values more than 20.1 cm
Total number of values more than 2 standard deviations below or above the mean = 4+7=11
Percentage of values more than 2 standard deviations below or above the mean $= \frac{11}{200} = 5.5\%$

c Lower fence= $17.0 - 1.5 \times 1.5 = 14.75$
There are no values less than the lower fence. Upper fence= $18.5 + 1.5 \times 1.5 = 20.75$
There are three values more than the upper fence, two at 21.0 and one at 21.5.



Solutions to Exercise 2A

1 In order to answer this question it is a good idea to make up names for the variables in the question, and then consider the values they may take. The actual names are not important, and you may use different ones from the ones used here.

a The two variables are *study load* which can take the values ‘full-time’ or ’part time’, and *travel mode* which can take the values ‘by car’ or ‘not by car’. These are both categorical variables.

b The two variables are *Year* which can take the values ‘Year 11’ or ’Year 12’, and *TV hours* which can take numerical values. This is one categorical variable and one numerical variable.

c The variable *income* will take numerical values, as will the variable *infant mortality*. These are both numerical variables.

d The two variables are *Attitude to gun control* which can take values such as ‘for’ or ‘against’, and *country of birth* which can take any number of values, which are all names of countries. These are both categorical variables.

2 **a** If we plan to predict (or explain) a fish’s toxicity from its colour, the variable *colour* is explanatory variable (EV) and *toxicity* is the response variable (RV).

b It makes more sense to explain a person’s *weight loss* in terms of a change in the *type of diet* they follow rather than the other way around. Thus *type of diet* is the explanatory variable and *weight loss* is the response variable.

c It makes more sense to explain the change in a second hand car’s *price* in terms of its *age* rather than the other way around. Thus *age* is the explanatory variable and *price* is the response variable.

d By suggesting that the cost of heating a house depends on the type of fuel used designates type of *fuel* as the explanatory variable. *Cost* is then the response variable.

e It makes more sense to explain the change in a house’s *price* in terms of its *location* rather than the other way around. Thus *location* is the explanatory variable and *price* is the response variable.

3 **a** It makes more sense to explain a change in a person's *exercise level* (categorical) in terms of their *age* (numerical) rather than the other way around. Thus *age* is the explanatory variable.

b It makes more sense to explain a change in a person's *salary level* (categorical) in terms of their *years of education* (numerical) rather than the other way around. Thus *years of education* is the explanatory variable.

c It makes more sense to explain the change in a person's *comfort level* (categorical) in terms of a change in *temperature* (numerical) rather than the other way around. Thus *temperature* is the explanatory variable.

d It makes more sense to explain the change in the *incidence of hay fever* (categorical) in terms of a change in the *time of year* (categorical) rather than the other way around. Thus *time of year* is the explanatory variable.

e It makes more sense to explain the difference in people's *musical taste* (categorical) in terms of their *age group* (categorical) rather than the other way around. Thus *age group* is the explanatory variable.

f It makes more sense to explain the differences in the *AFL team supported* (categorical) by people in terms of the differences in their *state of residence* (categorical) rather than the other way around. Thus *state of residence* is the explanatory variable.

4 The responses to the question "How concerned are you about climate change" are options which are numbers, but these number are not quantities, they label ordered categories. So the variable is ordinal. ⇒ **B**

5 The variables *weight* and *height* both have options which are ordered categories. So both of the variables are ordinal. ⇒ **B**

6 The researchers believe a change in *reaction time* may be explained by a change in *temperature*. Thus the response variable is *reaction time*, which is a numerical variable. ⇒ **C**

Solutions to Exercise 2B

1 **a** It is more likely that *gender* determines *intends to go to university* rather than the other way around, making *gender* the explanatory variable, and *intends to go to university* the response variable.

b Create the table, with the values of the explanatory variable labelling the columns, and the values of the response variable labelling the rows. Then go through the data, putting a tally mark in the appropriate cell for each person, as shown in the example in the text.

Intends to go to university	*Gender*	
	Male	Female
Yes	4	8
No	4	4
Total	8	12

2 **a** It is more likely that *age group* determines *reducing university fees* rather than the other way around, making *age group* the explanatory variable, and *reducing university fees* the response variable.

b Create the table, with the values of the explanatory variable labelling the columns, and the values of the response variable labelling the rows. Then go through the data, putting a tally mark in the appropriate cell for each person, as shown in the example in the text.

Reduce university fees?	*Age group*		
	17-18	19-25	26 or more
Yes	8	6	6
No	3	3	4
Total	11	9	10

c The column percentages are calculated by expressing each entry as a percentage of the column total. For example, in the column labelled 17-18 years:

Yes: $\frac{8}{11} \times 100 = 72.7\%$

No: $\frac{3}{11} \times 100 = 27.3\%$

Reduce university fees?	*Age group (%)*		
	17-18	19-25	26 or more
Yes	72.7	66.7	60.0
No	27.3	33.3	40.0
Total	100.0	100.0	100.0

3 **a** It is more likely that *enrolment status* determines *drinking behaviour* rather than the other way around, making *enrolment status* the explanatory variable.

b No: since the 'yes' and 'no' percentages were very similar for both full–time and part–time students, we can make the following statement: 'The percentages of full–time and part–time students who drank alcohol are similar, 80.5% to 81.8%. This indicates that drinking behaviour is not associated with enrolment status.'

4 **a** As *handedness* and *sex* are both natural attributes that cannot be manipulated in any way, the choice of the response variable is arbitrary. However, the way the table has been set up with *handedness* (left, right) defining the rows of the table, handedness is assumed to be the response variable.

b Sum the columns to find the total number males (22 + 222 = 242) and females (16 + 147 = 163) and use these values to convert the table to percentages by making the following computations: male-left: $\frac{22}{244} \times 100 = 9.0\%$

male right: $\frac{222}{244} \times 100 = 91.0\%$

female left: $\frac{16}{163} \times 100 = 9.8\%$

female right: $\frac{147}{163} \times 100 = 90.2\%$

c No: since the percentage of left handed males (9.0%) and females (9.8%) is similar, there is no evidence of an association between *handedness* and *sex*.

5 **a** It is possible that the *course* of a person may explain the how often they *exercised* but not the other way around, so *course* is the explanatory 'variable

b *Exercised* is as an ordinal variable because it's categories 'rarely', 'sometimes' and ‘regularly’ can be used to order the individuals involved in terms of how often they exercised.

c The percentage of Arts who exercised sometimes is given by the percentage in the ‘Arts’ column and the ‘sometimes’ row (54.9%)

d Yes: for example, since a greater percentage of Business students (18.6%) exercised regularly compared to Arts students (5.9%), we can conclude that the variable *exercised* and *course* are associated.

6 a Since we might expect a student’s *grade* to depend on which *class* they are in, *class* is the explanatory variable and *grade* is the response variable. The is how the table has been set up. Calculate column percentages, for example, in the column labelled Dr Evans:

Fail: $\frac{2}{18} = 11.1\%$

Pass: $\frac{11}{18} = 61.1\%$

Credit or above: $\frac{8}{18} = 27.8\%$

For Dr Smith: Fail: $\frac{3}{32} = 9.4\%$

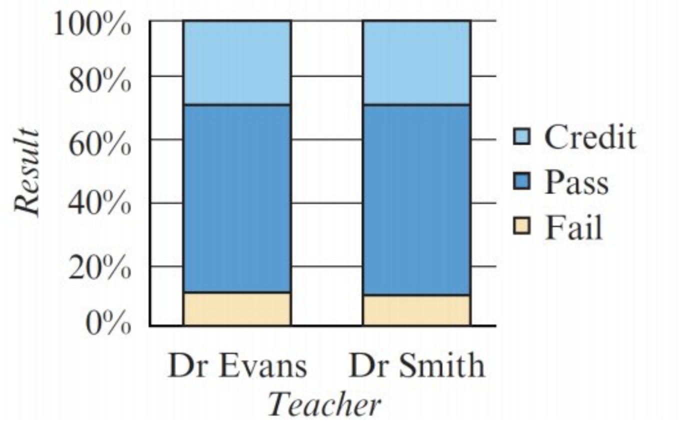
Pass: $\frac{11}{18} = 62.5\%$

Credit or above: $\frac{8}{18} = 28.1\%$

	Teacher (%)	
Result	Dr Evans	Dr Smith
Fail	11.1	9.4
Pass	61.1	62.5
Credit	27.8	28.1
Total	100.0	100.0

b The vertical axis is labelled percentage, and scaled appropriately (in 10’s or 20’s should be adequate).
There is a bar for each value of the explanatory variable, here that is one bar for Dr Evans, and one bar for Dr Smith, as shown below. Mark off each bar according to the values of the percentage frequencies, and then colour the segments in the bar which correspond to the same value of the response variable the same. For example, colour the segment which represents fail in each bar the same colour (here fail is coloured yellow). Finally, add a key indicates which colour represents each value of

the response variable.



c If we look across all rows, we can see that the heights of the segments, and thus the percentages, about the same for both teachers. An example of a report we could write is:
There is no evidence students of Dr Evans receive higher grades than students of Dr Smith. The percentage of students achieving each grade level almost the same for both classes (eg. 61.1% compared 62.5% for students who received a Pass).

7 If we look across all rows, we can see that the heights of the segments, and thus the percentages, about the different for each group. An example of a report we could write is:
The data supports the contention that people who are satisfied with their job are more likely to be satisfied with their life, with 70% of people who are satisfied with their job reporting that they are satisfied with their life, compared to only 50% of people who are dissatisfied with their job.

8 **a** Since we expect the *treatment outcome* to depend on the *type of treatment*, *type of treatment* is the explanatory variable and *treatment outcome* is the response variable.

b If we look across all rows, we can see that the heights of the segments, and thus the percentages, about the different for each group. An example of a report we could write is: The data supports the contention that the special pillow is more effective at treating snoring than the drug treatment, with just over 30% of people who used the special pillow reporting they were cured, compared to only 10% of people who used the drug reported they were cured.

9 **a** The percentage of widowed people who found life dull is given by the percentage in the ‘widowed’ column and the ‘dull’ row (11.9%).

b The percentage of people who were never married and found life exciting, is given by the percentage in the ‘never’ column and the ‘exciting’ row (52.3%)

c The way the table is set up with marital status defining the columns, marital status can be assumed to be the explanatory variable.

d *Attitude to life* is as an ordinal variable because its categories ‘exciting’, ‘pretty routine’ and ‘dull’ can be used to order the individuals involved in terms of their satisfaction with life.

e Yes: there are many ways this question can be answered. Several possibilities are given in the answers. In essence, it is sufficient to find two percentages in the same row (defined by the RV) but different columns (defined by the EV) that differ significantly to infer an association. For example, focusing on the ‘dull’ row, the fact that the percentage of ‘married’ people’ who found life ‘dull’ (3.7%) is very much less than the percentage of ‘widowed’ people who find life dull (11.9%) is sufficient to infer an association between *attitude to life* and *marital status*.

10 There are a total of 300 people in the study, 172 of whom do not have a tertiary qualification. That is: $\frac{172}{300} \times 100 = 57.3\% \Rightarrow$ **A**

11 There are a total of 172 people in the study do not have a tertiary qualification, 138 of who responded that they were happy with their lives. That is: $\frac{138}{172} \times 100 = 80.2\% \Rightarrow$ **B**

12 To answer this question we need to first determine the column percents, as shown below:

Happy with life?	*Tertiary qualification*	
	Yes	No
Yes	90.6%	80.2%
No	9.4%	19.8%
Total	100.0%	100.0%

To support the hypothesis we need to compare the percents across one of the rows, and support with a statement such as "90.6% of people with a tertiary qualification are happy, compared to 80.2% of those without a tertiary qualification" $\Rightarrow$ **C**

Solutions to Exercise 2C

1 **a** The variables are *country of origin* which is categorical, and *number of days away* which is numerical.

b To compare the distributions we need to determine the median and IQR for each group. Note that we don't usually compare the shapes of the distributions from a dot plots, since they are generally small data sets.
Japanese: Median= 17 $Q_1 = 8$ $Q_3 = 24.5$ so $IQR = 16.5$
Australia: Median= 7 $Q_1 = 4$ $Q_3 = 14.5$ so $IQR = 10.5$
The number of days these tourists spend away from home was associated with their country of origin. The median number of days spent away from home for Japanese tourists (M = 18 days) is considerably higher than for Australian tourists (M = 7 days). The variability for the number of days away is also higher for Japanese tourists (IQR = 16.5) compared to that for Australian tourists (IQR = 10.5).

2 **a** The variable *age* is numerical, and the variable *gender* is categorical.

b To compare the distributions we need to determine the median and IQR for each group.
Females: Median= 34 $Q_1 = 15$ $Q_3 = 43$ so $IQR = 28$
Males: Median= 25.5 $Q_1 = 21$ $Q_3 = 34$ so $IQR = 13$
From this information it can be concluded that the median age of the people admitted to the hospital during this week was associated with their age. The median age of the females (34 years) admitted to the hospital was considerably higher than the median age of males (25.5 years). The variability of the ages was also higher for the females (IQR = 28 years) compared that of the males (IQR = 13 years).

3 **a** The variable *number of hours spent online* is numerical, and the variable *year level* is categorical.

b To compare the distributions we need to determine the median and IQR for each group.
Year 10 (17 students): Median= 20 $Q_1 = 14$ $Q_3 = 23.5$ so $IQR = 9.5$
Year 11 (16 students): Median= 16.5 $Q_1 = 9.5$ $Q_3 = 22.5$ so $IQR = 13$
From this information it can be concluded that the median number of hours spent online was associated with year level. The median time spent online by the Year 10 students (20 hours) was higher than the median number of hours by the Year 11 students (16.5 hours). The variability of the hours spent online was lower for the Year 10 students (IQR = 9.5 hours) compared that of the Year 11 students (IQR = 13

hours).

4 **a** *Age at marriage* (in years) is numerical and *gender* is categorical.

b The values of the median and quartiles should be read from the boxplots and compared (the values quoted are only approximate). The shape can also be determined from the box plots.
For this data there is an association between age at marriage and gender. The age at marriage is higher for men (M = 23 years) than for women (M = 20.5 years). The variability is also greater for the men (IQR = 12 years) than for the women (IQR = 8.4 years). The distributions of age at marriage are positively skewed for both men and women. There are no outliers.

5 **a** *pulse rate*: numerical, *gender*: categorical

b The values of the median and quartiles should be read from the boxplots and compared (the values quoted are only approximate). The shape can also be determined from the box plots.
For this data there is an association between pulse rate and gender. The pulse rates for males (M = 73 beats/min) are lower than the pulse rates for women (M = 76 beats/min). The variability is also lower for the males (IQR = 8 beats/min) than for the women (IQR = 14 beats/min). Both distributions are approximately symmetric, with no outliers.

6 **a** Battery *lifetime* is measured in hours, so it is a numerical variable.
Battery *price* is classified as 'high', 'medium, and 'low' so it is a categorical variable.

b For this data there is an association between the life time of a battery and its price. The life time of the high price batteries (M = 51 hours) is longer than that of the medium price batteries (M = 35 hours), which is in turn slightly longer than that of the low price batteries (M = 32 hours). The variability in life time also increased with price, from IQR = 7 hours for the high price batteries, to IQR = 12 hours for the medium price batteries, and IQR = 17 hours for the low price batteries. All three distributions are approximately symmetric, with no outliers.

7 Consider each statement:
A 75% of babies born to non-smokers weigh more than the lightest 50% of babies born to smokers.

Since Q_1 for non- smokers = 7.0 lb, and the median M for smokers =7.0 lb, then this statement is TRUE.
B 50% of babies born to non-smokers weigh more than the heaviest 25% of babies born to smokers.
The median weight for non-smokers is about 7.4 lb, while Q_3 for smokers is about 7.8 lb, so this is FALSE
C 25% of the babies born to smokers weigh less than all the babies born to non-smokers.
Because Q_1 for the plot for smokers is greater than the minimum value for non-smokers this is FALSE. D All of the babies born to non-smokers weigh more than the heaviest 75% of the babies born to smokers.
Because Q_1 for the plot for smokers is greater than the minimum value for non-smokers this is FALSE. E The range of baby weights for smokers is less than the range of baby weights for non smokers.
The range of baby weights for smokers is from about 4.2 -10.0 lb, while the range of baby weights for non smokers is from about 5.7 - 10 lb, so this is FALSE.
$\Rightarrow$ **A**

8 Looking at the two boxplots, the median weight is 7.0 lb for Smokers which is less than the median weight of 7.4 lb for non-smokers, while the *IQR*'s for each are very similar (1.7 for Smokers,1.6 for non-smokers). $\Rightarrow$ **D**.

Solutions to Exercise 2D

1 **a** Plotting the variable number of seats on the horizontal axis of the scatterplot implies that in this investigation, *number of seats* is the explanatory variable.

b *Airspeed* values are recorded in km/h so the variable is numerical.

c 8 - count the number of data points (dots).

d around 800 km/h-Locate the data point that represents the aircraft that can seat 300 passengers and read off its airspeed.

2-5 Enter data into your calculator and follow the instructions on page 104 to generate the required plots.

Solutions to Exercise 2E

1 **a** As a general rule no association would be expected.

b A positive association would be expected, as the higher the level of education, the higher the level of skills usually required in high paying occupations.

c A positive association would be expected; tax paid should increase with salary.

d A positive association would be expected, the more frustrated people become with a difficult situation, the more they are likely they are to exhibit aggressive behaviour.

e A negative association would be expected. In a typical Australian city, population density tends because individual dwellings tend to occupy a greater area of land.

f A negative association would be expected, the more time spent using social media reduces the time available for studying.

2 **a** **i** The dots drift upwards to the right, so it is a positive association. he association is linear (no apparent curve), and of moderate strength, as the points are quite clearly scattered about a straight line. Thus we can say there is a moderate, positive, linear association between *smoking rate* and *lung cancer mortality rate*.

ii The dots drift downwards to the right, so it is a negative association. The association is linear (no apparent curve) but weak, as the points are only loosely scattered about s straight line. Thus we can say there is is weak, negative, linear association between *age* and *score on the aptitude test*.

iii The dots drift upwards to the right, so it is a positive association. The association is linear (no apparent curve), and of string strength, as the points are closely scattered about a straight line. Thus we can say there is a strong, positive, linear association between *Traffic volume* and *CO level*.

iv There is no clear pattern in the cloud of dots that make up the scatterplot indicating there is no association.

b **i** Those people who smoke more tend to have a higher lung cancer mortality rate.

ii Older children tended to score lower on the aptitude test.

iii Intersections with higher levels of traffic volume also tended to have higher CO levels.

iv There is no association so nothing to interpret.

Solutions to Exercise 2F

1 **a** Scatterplot A shows a strong, positive, non-linear association with no outliers. Scatterplot B shows a strong, negative, linear association with one outlier. Scatterplot C shows a weak (to moderate), negative, linear association with no outliers.

b It wouldn't be appropriate to use the correlation coefficient for scatterplot A (as the association is non-linear) or for scatterplot B (as the association has an outlier).

2 (Optional) Using the information given in the table, we can draw the following table:

x	$x - \bar{x}$	y	$y - \bar{y}$	$(x - \bar{x}) \times (y - \bar{y})$
2	−2	1	−4	8
3	−1	6	1	−1
6	2	5	0	0
3	−1	4	−1	1
6	2	9	4	8
Sum	**0**		**0**	**16**

$$\begin{aligned} r &= \frac{\sum(x - \bar{x})(y - \bar{y})}{(n-1)s_x s_y} \\ &= \frac{16}{4 \times 1.871 \times 2.915} \\ &= 0.73 \end{aligned}$$

3a, b & c Enter data into your calculator and follow the instructions on page 141 (TI) or page 142 (CASIO) to calculate the correlation coefficient.

4 **a** 0.818 > 0.75, strong positive

b 0.8782 > 0.75, strong positive

c −0.75 < −0.6727 < −0.5, moderate negative

Solutions to Exercise 2G

1 **a** Coefficient of determination:
$r^2 = (0.675)^2 = 0.456 = 45.6\%$

b Coefficient of determination:
$r^2 = (0.345)^2 = 0.119 = 11.9\%$

c Coefficient of determination:
$r^2 = (-0.567)^2 = 0.321 = 32.1\%$

d Coefficient of determination:
$r^2 = (-0.673)^2 = 0.453 = 45.3\%$

e Coefficient of determination:
$r^2 = (0.124)^2 = 0.015 = 1.5\%$

2 To help you answer questions **2a** to **e**, remember that:
the coefficient of determination (as a percentage) tells us the variation in the response variable (RV) explained by the variation in the explanatory variable (EV).

a The coefficient of determination =
$r^2 = (-0.611)^2 = 0.373$ or 37.3%,
which can be interpreted as '37.3% of the variation observed in hearing test scores (RV) can be explained by the variation in age (EV)'.

b The coefficient of determination = $r^2 = (0.716)^2 = 0.513$ or 51.3%, which can be interpreted as '51.3% of the variation observed in mortality rates can be explained by the variation in smoking rates'.

c The coefficient of determination = $r^2 = (-0.807)^2 = 0.651$ or 65.1%, which can be interpreted as '65.1% of the variation observed in life expectancy can can be explained by the variation in birth rates'.

d The coefficient of determination = $r^2 = (0.818)^2 = 0.669$ or 66.9%, which can be interpreted as '66.9% of the variation observed in daily maximum temperature can be explained by the variation in daily minimum temperatures'.

e The coefficient of determination = $r^2 = (0.8782)^2 = 0.771$ or 77.1%, which can be interpreted as '77.1% of the variation in the runs scored by a batsman can be explained by the variation in the number of balls they face.'

3 **a** $r^2 = 0.8215$,
$r = \sqrt{0.8215} = 0.906$ (from the scatterplot, the association is positive)

b $r^2 = 0.1243$,
$r = \sqrt{0.1243} = -0.353$ (from the scatterplot, the association is negative)

4 $r^2 = 0.3969$,
$r = \sqrt{0.3639} = -0.63$ (from the scatterplot, the association is positive) ⇒ **E**

5 The coefficient of determination $r^2 = 0.3969$ or 39.7% rounded to one decimal place. This can be interpreted as '39.7% of the variation in *time* can be explained by the variation in the *number of training sessions*'. ⇒ **A**

6 Since 39.7% of the variation in *time* can be explained by the variation in the *number of training sessions*, then 100% – 39.7% = 60.3% of the variation in *time* is NOT explained by the variation in the *number of training sessions* ⇒ **E**

7 Consider each of the alternatives:
A Older employees tend to have spent more years studying.
Since we do not have the correlation coefficient between *age* and *years spent studying* we do not know if this is true or not.
B The correlation between age and years spent studying is 0.32.
We cannot determine the value of the correlation coefficient between *age* and *years spent studying* from the information given.
C and D Age explains a higher percentage of the variation in income than years spent studying so we do not know if this is true or not.
The correlation between *years spent studying* and *income* = 0.73^2 = 53.3%. Thus *years spent studying* explains 53.5% of the variation in *income*.
The correlation between *age* and *income* = 0.45^2 = 20.3%. Thus *years spent studying* explains 20.3% of the variation in *income*.
Thus *years spent studying* explains a higher percentage of the variation in *income* than does age, which means C is FALSE and D is TRUE.
E Together age and years spent studying explain 100% of the variation in income.
We cannot add the values of the coefficient of determination in this way. FALSE
⇒ **D**

8 Consider each of the alternatives:
A The correlation coefficient between height (in centimetres) and weight (1 = light,

2 = medium, 3 = heavy) was found to be 0.68. Since *weight* here is an ordinal variable the correlation coefficient cannot be calculated. FALSE

B The correlation coefficient between height (in centimetres) and head circumference (in centimetres) was found to be 1.45.

Correlation coefficient cannot be more than 1. FALSE

C The correlation coefficient between blood pressure (in centimetres) and weight (in kg) was found to be -0.3, and the coefficient of determination was found to be $r^2 = -0.09$.

The coefficient of determination cannot be negative. FALSE

D The correlation coefficient between age (in years) and salary (in $000's) was found to be 0.68. This could be TRUE.

E The correlation coefficient between height (in centimetres) and head circumference (in centimetres) was found to be 0.49, and the coefficient of determination was found to be 70%.

If the correlation coefficient is equal to 0.48 then the coefficient of determination is equal to 24.0%. FALSE

⇒ **D**

Solutions to Exercise 2H

1 to 7 The solutions are integrated into the answers for these questions

8 . The first 3 options (A, B, C) are all causal statement, so all are incorrect. Because the correlation is positive, D is incorrect. The correct statement is E. $\Rightarrow$ **E**

Solutions to Exercise 2I

To help you answer these questions, consult the table on page 155.

1 **a** Two categorical variables ⇒ segmented bar chart

b Two numerical variables ⇒ scatterplot

c Numerical response variable (hours spent at the beach) and categorical explanatory variable (state of residence) with two or more sub categories ⇒ parallel box plots

d Two numerical variables ⇒ scatterplot

e Two numerical variables ⇒ scatterplot

f Two categorical variables ⇒ segmented bar chart

g Two categorical variables ⇒ segmented bar chart

h Numerical response variable (cigarettes smoked per day) and categorical explanatory variable (sex) ⇒ parallel box plots or a back-to-back stem plot because the variable sex only has two categories

2 The response variable is numerical and the explanatory variable is categorical but with only two categories so the correct response is **E** (back-to-back stem plot)

3 The response variable is numerical and the explanatory variable is categorical but with three categories so the correct response is **D** (parallel boxplots)

Chapter Review: Solutions to Multiple-choice questions

1 *Plays sport* and *gender* are both categorical variables. ⇒**A**

2 'Females–No' = 175 – 79 = 96 ⇒ **D**

3 Percentage 'Males–No' = $\frac{34}{102} \times 100\% = 33.3\%$ ⇒ **B**

4 When the table is percentaged, the percentage of males who play sport is much higher than the percentage of females who play sport indicating an association between playing sport and sex. ⇒**D**

5 The percentage of Year 12 students who do not read for leisure is given by the height of the blue segment of the Year 12 bar = 90% ⇒**E**

6 From the segmented bar chart we can construct the percentaged two-way frequency table as follows:

Read	*Year level*	
	Year 10	Year 12
Yes	25%	10%
No	75%	90%
Total	100.0%	100.0%

Determining the percentages for each table in turn we find the correct option is Table B. ⇒ **B**

7 To find an association we need to compare (and find a difference in) percentage across a row, and interpret correctly, in terms of which percentage is higher or lower. ⇒ **D**

8 To find an association we need to compare (and find a difference in) the medians or interquartile range.

The median for Group A is 19.5 minutes, while the median for Group B is 15 minutes. ⇒ **A**

9 Battery life is a numerical variable but brand is a categorical variable. ⇒**C**

10 The first statement supports the contention because the difference in median battery life shows one brand to be superior.
The second and third statements support the contention because they both show one brand to be more reliable than the other.
Therefore, all three statements support the contention. ⇒**E**

11 *Weight at age 21* and *weight at birth* are both measured in kg so they are numerical variables. ⇒**D**

12 The dots drift upwards towards the right of the graph and appear to be aggregated fairly closely around an imagined straight line fitted to the data. From this we conclude that there is a strong positive linear association. ⇒**E**

13 The percentage of variation in *weight at age 21* which is explained by the variation in *Weight at age 21* is given by the value of the coefficient of determination $r^2 = 0.58^2 = 33.6\%$. ⇒ **C**

14 If $r = -0.9$, this means the relationship is a strongly negative linear one. This means that as drug dosage increases, response time tends to decrease. ⇒**C**

15 Enter the data into you calculator and follow the instructions on page 141 (TI) or Page 142 (CASIO) to get: r = 0.7863 (to 4 decimal places) ⇒**C**

16 Coefficient of determination $= r^2 = (-0.7685)^2 = 0.5906$ ⇒**D**

17 Coefficient of determination $= r^2 = (0.765)^2 = 0.585 =$ 58.5%.
Since heart weight is the response variable, this result tells us that, for these mice, '58.5% of the variation in heart weight can be explained by the variation in body weights'. ⇒**A**

18 This association tells us that heavier mice tend to have heavier hearts (D) and lighter mice tend to have lighter hearts, but no more. Statement such as 'increasing the body weights of mice will increase their heart' (E) might also seem to be a reasonable option. However it is causal in nature and not justifiable conclusion to draw when all we know is that heart weight and body weight are strongly correlated. ⇒**E**

19 Since the response variable (*weight*) is numerical and the explanatory variable (*level of nutrition*) is categorical with more than two categories, we would use parallel box plots. ⇒**B**

20 Since the two variables are both categorical, we would use a segmented bar chart. ⇒**C**

21 This association tells us that people on higher salaries tend to recycle more garbage but we cannot assume that it is the higher salary alone that encourages a higher level of recycling of garbage. It may be that those on higher salaries have more education and thus are more aware of the environmental impact of recycling. ⇒**E**

22 As there is no logical reason to expect an association between marriage rate and falling out of a fishing boat and drowning, we conclude that the association is just coincidence. ⇒**C**

Chapter Review: Extended-response questions

1 a *number of accidents* and *age* are both categorical variables.

b We might expect that *age* could explain the *number of accidents* but not the other way around. Thus *age* is the explanatory variable and *number of accidents* the response variable.

c reading from the 'more than one accident' row and the 'age < 30' column we see that 470 drivers under the age of 30 had more than one accident.

d

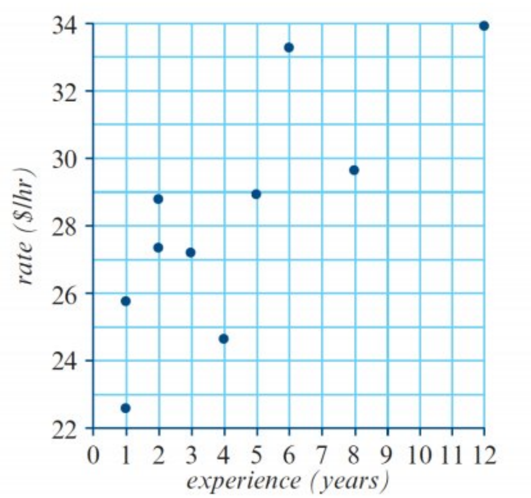
Number of accidents	*Age* < 30	*Age* ≥ 30
At most one accident	$\frac{130}{600} \times \frac{100}{1}\% \approx 21.7\%$	$\frac{170}{400} \times \frac{100}{1}\% \approx 42.5\%$
More than one accident	$\frac{470}{600} \times \frac{100}{1}\% \approx 78.3\%$	$\frac{230}{400} \times \frac{100}{1}\% \approx 57.5\%$
Total	100%	100%

2 a We might expect that *completed weeks of course* could explain the *converation test score* but not the other way around. Thus *completed weeks of course* is the explanatory variable and *converation test score* the response variable.

b For each of the distributions determine the median and interquartile range from the boxplots.
0 weeks: $M = 38$, $Q_1 = 32$, Q_3=44, so $IQR = 12$
6 weeks: $M = 42$, $Q_1 = 38$, Q_3=50 so $IQR = 12$
12 weeks: $M = 72$, $Q_1 = 64$, Q_3=78, so $IQR = 14$
Also note the shape of the distributions, and the values of any outliers. Compare each of these (centre, spread, shape and outliers) in a report such as the following: 'There is an association between the students' scores on the conversation test, and the number of weeks of the course they have completed. The median score at the beginning of the course (M=38) showed a little improvement after six weeks (M =42), followed by a very large improvement by the end of the 12 week course (M =72). The variability of the scores changed little over the course(IQR=12 at the beginning, IQR = 12 at 6 weeks, IQR = 14 at 12 weeks). The distributions of scores at 0 weeks is approximately symmetric with an outlier at 66, positively skewed with an outlier at 76 at 6 weeks, and approximately symmetric with no outliers at 12 weeks.'

3 a We might expect that *years* experience could explain the *rate* of pay, but not the other way around. Thus *years* is the explanatory variable and *rate* the response variable.

b



c The points trend upward as we go from left to right across the graph, indicating the association is positive. It is linear, as there are no distinct curves, and of moderate strength. Thus we can say: 'There is a moderate positive linear relationship; that is, people with more experience are generally being paid a higher hourly pay rate.'

d Enter the data into you calculator and follow the instructions on page 141 (TI) or Page 142 (CASIO) to get: $r = 0.786$ (to 3 decimal places)

e Use four decimal places for r when determining the value r^2. That is: $r^2 = 0.7864^2 = 0.6184 = 61.8\%$. So we can say that 61.8% of the variation in pay rate is explained by the variation in experience.

4 a The height of the yellow segment (which represents good) in the Before bar is 60%.

b Compare one value of the sleep quality rating and compare Before and After. In the following sample report the percentage for good are compared. 'For these people there is an association between the person's quality of sleep is and their participation in the course, with 85% of people rating their sleep quality as good after the course, compared to only 60% of people rating their sleep quality as good before the course.'

Solutions to Exercise 3A

1 A residual is the vertical difference between a value on a plot and the regression line drawn to fit the plot.

2 To find the least squares regression line, we must minimise the sum of the squares of the residual values. ⇒**C**

3 See page 170.

4 **a** In an equation of the form $y = a + bx$, x is the explanatory variable and y is the response variable.

b

$$\begin{aligned} b &= r \times \frac{s_y}{s_x} \\ &= 0.7818 \times \frac{6.619}{5.162} = 1.0025 \\ a &= \bar{y} - b\bar{x} \\ &= 19.91 - 1.002 \times 10.65 = 9.2333 \end{aligned}$$

Therefore,

$x = a + b \times y = 9.23 + 1.00 \times x$ (3 sig figures).

5 **a** Since *traffic volume* is to be used to predict *pollution level* so *traffic volume* is the explanatory variable (EV) and *pollution level* is the response variable (RV).

b

$$\begin{aligned} b &= r \times \frac{s_y}{s_x} \\ &= 0.940 \times \frac{97.9}{1.87} = 49.2118 \\ a &= \bar{y} - b\bar{x} \\ &= 231 - 49.2118 \times 11.4 = -330.0145 \end{aligned}$$

Therefore,

pollution level = $a + b \times$ *traffic volume* = $-330 + 49 \times$ *traffic volume* (2 sig. figs.)

6 **a** As stated in the questions, *birth rate* is to be used to predict *life expectancy*, so *birth rate* is the predictor or explanatory variable (EV) and *life expectancy* is the response variable (RV).

b

$$\begin{aligned} b &= r \times \frac{s_y}{s_x} \\ &= -0.810 \times \frac{9.99}{5.41} = -1.4957 \\ a &= \bar{y} - b\bar{x} \\ &= 55.1 + 1.4957 \times 34.8 = 107.15 \end{aligned}$$

Therefore,
life expectancy = $a + b \times x$ = 110 − 1.5 × *birth rate* (2 sig. figs.)

7 **a** In an equation of the form $y = a + bx$, x is the explanatory variable and y is the response variable.

b

$$\begin{aligned} r &= b \times \frac{s_x}{s_y} \\ &= -0.4847 \times \frac{4.796}{5.162} = -0.450 \end{aligned}$$

8 **a** As stated in the questions, *age* is to be used to predict *distance travelled*, so *age* is the predictor or explanatory variable (EV) and *distance travelled* is the response variable (RV).

b

$$\begin{aligned} r &= b \times \frac{s_x}{s_y} \\ &= 11.08 \times \frac{3.64}{42.6} = 0.947 \end{aligned}$$

9 **a** If the slope is negative, the correlation coefficient must also be negative. This follows from the rule: $b = r \times \frac{s_x}{s_y}$

b If the correlation coefficient is zero, this means the least squares regression line will be horizontal and thus will have a slope of zero. This also follows from the rule: $b = r \times \frac{s_x}{s_y}$

c The correlation coefficient being zero means the line will be horizontal and thus will have a constant y-value for its entire length. This y-value will thus be the average of all the y-values and will be the mean y-value, which is $\bar{y}$. This follows from the rule: $a = \bar{y} + b\bar{x}$ when $b = 0$.

10 Enter the data into your calculator and follow the instructions on page 173 (TI) or 174 (Casio).

11 Enter the data into your calculator and follow the instructions on page 173 (TI) or 174 (Casio).

12 **a** Enter the data into your calculator and follow the instructions on page 173 (TI) or 174 (Casio).

b *runs* = −2.6 + 0.73 × *balls faced*

13 **a** EV is *number of TVs*

b answers given in question

c *number of TVs* = 61.2 + 0.930 × *number of cars*

14 Only Option C is always true. There is no reason for either A or E to be true. The least squares line minimises the sum of the *squares* of the vertical distances from the line to each data point, not the sum of the vertical distances from the line to each data point, so Option B is not true. Option D has the response and explanatory variables reversed. ⇒ **C**

15

$$\begin{aligned} r &= b \times \frac{s_x}{s_y} \\ &= 1.45 \times \frac{3.42}{6.84} = 0.725 \end{aligned}$$

⇒ **A**

16 Enter the data into your calculator and follow the instructions on page 173 (TI) or 174 (Casio). ⇒ **C**

Solutions to Exercise 3B

1 Read values from the graph intercept ≈80

slope = $\dfrac{46 - 80}{8 - 0}$=4.25 or 4.3 rounded to 1 d.p.

Thus, *mark* = 80 − 4.3 × *days absent*

2 **a** From the equation of the least squares line: intercept = 2.9

This tells us that, on average, a person who is 0 cm in height has a hand span of 2.9 cm, which is clearly not sensible.

b From the equation of the least squares line: slope = 0.33

This tells us that, on average, on average a person's hand span increases by 0.33 cm for each 1 cm increase in height.

3 **a** From the equation of the least squares line: intercept = 575

This tells us that, on average the company will have $575 in sales when their online advertising expenditure is $0.

b From the equation of the least squares line: slope = 4.85

This tells us that, on average sales will increase by $4.85 for each additional $1.00 spent on online advertising.

4 *height* = 72 + 0.40 × *age*, for children between 36 months and 60 months.

a When *age* = 20 months, *height* = 72 + 0.40 × 20 = 80cm

Extrapolating because 20 is < 36.

b When *age* = 50 months, *height* = 72 + 0.40 × 50 = 92cm

Interpolating because 50 is between 36 and 60.

c When *age* = 65 months, *height* = 72 + 0.40 × 65 = 98cm

Extrapolating because 65 is > 60 .

5 *cost* = 487.50 + 6.70 × *meals*, for number of meals between 25 and 100.

a When *meals* = 0, *cost* = 487.50 + 6.70 × 0 = 487.50

Extrapolating because 0 is < 25.

b When *meals* = 80, *cost* = 487.50 + 6.70 × 80 = 1023.50

Interpolating because 80 is between 25 and 100.

item When $meals = 110$, $cost = 487.50 + 6.70 \times 110 = 1224.50$
Extrapolating because 110 is > 100 .

6 $son's\,height = 83.9 + 0.525 \times father's\,height$, for males from 150 cm to 190 cm tall.

a When $father's\,height = 170$ cm, $son's\,height = 83.9 + 0.525 \times 170 = 173$cm
Interpolating because 170 is between 150 and 190.

b When $father's\,height = 200$ cm, $son's\,height = 83.9 + 0.525 \times 200 = 189$cm
Extrapolating because 200 is more than 190.

c When $father's\,height = 155$ cm, $son's\,height = 83.9 + 0.525 \times 155 = 165$cm
Interpolating because 155 is between 150 and 190.

7 a Correlation between *IQ* and *exam score* = 0.45. Therefore $r^2 = 0.45^2 = 0.2025 =$ 20.3%.

b Correlation between *hours* and *exam score* = 0.65. Therefore $r^2 = 0.65^2 = 0.4225$ we can say 42.3% of the variation in *exam score* is explained by the variation in *hours*.

c Since only 20.3% of the variation in *exam score* is explained by the variation in *IQ*, compared to the 42.3% of the variation in *exam score* which is explained by the variation in *hours*, we can conclude that *hours* is more important.

8 a Substitute *height* = 160 into the equation of the regression line and evaluate:
$$hand\ span = 2.9 + 0.33 \times 160 = 55.7 \text{ cm}$$

b residual = actual − predicted
= 58.5 − 55.7 = 2.8 cm

9 a Substitute *weight* = 980 into the equation of the regression line and evaluate:
$fuel\,consumption = -0.1 + 0.01 \times 980 = 9.7$ litres/100 km

b residual = actual − predicted = 8.9 − 9.7 = −0.8 litres/100 km

10 Read the vertical distances between the points and the line from the graph.

a When $x = 1$, predicted $y = 8$, actual $y = 10$

residual = actual – predicted = 10 − 8 = 2

b When $x = 3$, predicted $y = 6$, actual $y = 5$
residual = actual – predicted = 5 − 6 = −1

c When $x = 8$, predicted $y = 2$, actual $y = 4$
residual = actual – predicted = 4 – 2 = 2

11 A: clear curved pattern in the residuals (not random), C: curved pattern in the residuals (not random).

12 **a** The equation of the least squares line is:
energy content = 27.8 + 14.7 × *fat content*
Reading values from this equation:
slope = 14.7
intercept = 27.8

b Intercept = 27.8. On average, standard bags of chips which contain 0 grams of fat have an energy content of 27.8 calories.

c Slope = 14.7. On average, the energy content of a standard bag of chips increases by 14.7 calories, for reach additional one gram of fat content.

d $(r^2) = 0.7569 = 75.7\%$, thus we can say that 75.7% of the variation in energy content can be explained by the variation in fat content.

e Substitute *fat content* = 8 into the equation of the regression line and evaluate:
energy content = 27.8 + 14.7 × 8 = 145.4 calories

f residual = actual – predicted = 132 – 145.4 = −13.4 calories

13 **a** Slope = −0.278. On average, the success rate decreases by 0.278% for each extra cm the golfer is from the hole.' Note: Since 0.278% per cm is the same as 27.8% per metre then this is an equivalent statement.

b Substitute *distance* = 90 into the equation of the regression line and evaluate:
success rate = 98.5 – 0.278 × 90 = 98.5 – 25.02 = 73.5%

c $0 = 98.5 - 0.278 \times distance$
$0.278 \times distance = 98.5$
$distance = \frac{98.5}{0.278} = 354.3$ cm = 3.54 metre

d $r = \pm\sqrt{0.497} = \pm 0.705$ to 3 d.p.
However, the association is negative because the slope of the regression line is negative, so $r = -0.705$.

e The coefficient of determination:
$r^2 = 0.497$.
Thus, since $0.497 \times 100\% = 49.7\%$, we can say that 49.7% of the variation in *success rate* is explained by the variation in distance of the golfer from the hole.

14 **a** Appropriate: the scatterplot shows that there is a clear linear association.

b Coefficient of determination $= r^2 = (0.967)^2 = 0.9351$

c In general terms, the coefficient of determination (r^2) gives the as the proportion (or percentage) of the variation in a response variable that can be explained by the explanatory variable.
Since $0.9351 \times 100\% = 93.5\%$, we can say that 93.5% of the variation in a person's *pay rate* is explained by the variation in their years of work *experience*.

d Replacing x and y by *experience* and *pay rate* respectively in the equation
$y = 18.56 + 0.289 \times x$
gives
pay rate $= 18.56 + 0.289 \times$ *experience*

e Intercept = \$18.56. On average the hourly pay rate for a worker with no experience is \$18.56.

f Slope = 0.289. On average, the *pay rate* increases \$0.29 per hour for each additional year of experience.

g **i** *pay rate* $= 8.56 + 0.289 \times$ *experience*
$= 18.56 + 0.289 \times 8 = 8.56 + 2.312$
$= \$20.87$

ii residual = actual – predicted
$= 21.20 - 20.87 = \$0.33$

h The absence of a clear pattern in the residual plot supports the assumption of linearity, and this is what is observed in the residual plot.

15 **a** $r = \pm\sqrt{0.370} = \pm 0.608$ to 3 d.p.
However, the association is negative because the slope of the regression line is

negative, so $r = -0.608$.

b Since 0.370 × 100% = 37.0%, we can say that 37.0% of the variation in the *hearing test scores* can be explained by the variation in *age*.

c Replacing x and y by *age* and hearing test *score* into the equation:
$y = 4.9 - 0.043 \times x$
gives *score* = 4.9 − 0.043 × *age*

d The slope of the line is −0.043. On average, hearing test scores decrease by 0.043 for each additional year of age

e **i** *score* = 4.9 − 0.043 × *age* = 4.9 − 0.043 × 20 = 4.04

ii residual = actual − predicted = 2.0 − 4.04 = −2.04

f **i** The plot point at 35 years is a vertical distance of about 0.3 above the regression line, so the residual is 0.3.

ii The plot point at 55 years is a vertical distance of about 0.4 below the regression line, so the residual is −0.4.

g The absence of a clear pattern in the residual plot supports the assumption of linearity which is observed in the residual plot.

16 **A: negative –** From the scatterplot, we can see that the association is negative since the points in the scatterplot drift downwards towards the right of the plot.
B: *drug dose* – This is the explanatory variable.
C: −0.9492 As read from the regression results screen.
D: 55.9 As read from the regression results screen.
E: −9.31 As read from the regression results screen.
E: decreases The slope of the regression line is negative.
H: 9.31 The slope of the regression line is −9.306
I: 55.9 The y-intercept is 55.89
J: 90.1 – Since $r^2 = 0.901$
K: *response time* and ***drug dose*** Drug dosage determines response time since drug dosage is the explanatory variable.
L: clear pattern – The residual graph can be seen to show a parabolic–like pattern.

17 Use the report in question 9 as a model.

18 Find two points on the graph through which the regression line passes. For example, (70, 60) and (120, 115).

Substitute in the equation $\frac{y - y_1}{x - x_1} == \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore \frac{y - 60}{x - 70} = \frac{115 - 60}{120 - 70}$

$\therefore \frac{y - 60}{x - 70} = 1.1$

$\therefore y - 60 = 1.1 \times (x - 70)$

$\therefore y = -17 + 1.1 \times x$

In terms of the variables in the question: *weight* = −17 + 1.1 × *waist* ⇒ **E**

19 The EV in this question is *life expectancy* and the RV is *health*. Enter the data into your calculator and construct a scatterplot of the data, and add the regression line (follow the instructions on page 173 (TI) or 174 (Casio)). To find out how many times the predicted value is greater than the actual value, count the number of points below the line, there are four. ⇒ **B**

20 The correct interpretation of the slope is "On average, the weight of the fish increases by 23.3 grams for reach centimetre increase in length." ⇒ **A**

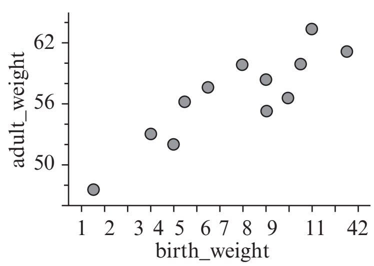
Solutions to Exercise 3C

1 Enter the data into your calculator and follow the instructions on page 173 (TI) or 174 (Casio).

2 Enter the data into your calculator and follow the instructions on page 173 (TI) or 174 (Casio).

3 **a** The aim is to predict their *adult weight* from a person's *birth weight*, which means that
RV: *adult weight*
EV: *birth weight*.
To provide the answers necessary to answer questions parts **b** to **i**, enter the data into your calculator and follow the instruction on page 13 (TI) and 174 (CASIO).

b



c **i** The points in the scatterplot tend to trend upwards to the right of the plot following a roughly a linear pattern with a relatively small amount of scatter. The is evidence of a strong positive, linear association. There are no apparent outliers.

ii Comparing the scatter with the standard plots in the text suggest an r of around 0.9

d Noting that the explanatory variable is *birth weight* and using the regression results generated by your calculator, the following the regression equation can be written down:
adult weight = 38.4 + 5.86 × *birth weight* (3 sig. figs.)

e Using the regression results generated by you calculator you will find that the coefficient of determination:
$r^2 = 0.7653\ldots$
Since $0.7653\ldots \times 100\% \approx 76.5\%$, we can say, using the standard interpretation, that 76.5% of the variation in *adult weight* is can be explained by the variation in birth weight.

f The slope of the regression line is 5.86 . . . ≈5.9. Using the standard interpretation, this tells us that, on average, adult weight increases by 5.9 kg for each additional kg of birth weight.

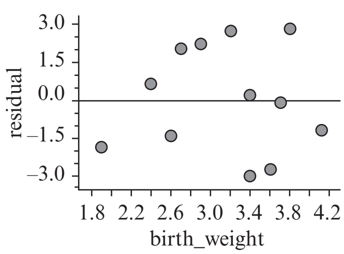
g **i** *adult weight* = 38.4 + 5.86 × *birth weight*
= 38.4 + 5.86 × 3.0
= 56.0 kg (to 1 d.p).

ii *adult weight* = 38.4 + 5.86 × *birth weight*
= 38.4 + 5.9 × 2.5 = 38.4 + 14.8
= 53.1 kg (to 1 d.p).

iii *adult weight* = 38.4 + 5.86 × *birth weight*
= 38.4 + 5.86 × 3.9
= 61.3 kg (to 1 d.p.)

h This contention is supported by the data, since the data suggests that 76.5% of the variation in a person's adult weight explained by the variation in their birth weight. Only 23.5% is explained by other factors

i The absence of a clear pattern in a residual plot shown supports the assumption of linearity which is observed in the residual plot below.



Chapter Review: Solutions to Multiple-choice questions

1 To use a straight line to model an association work, it is assumed that the variables being modelled are linearly related. ⇒**C**

2 The constant value = −1.2 = y-intercept;
the slope = 0.52. ⇒**D**

3 $r = \pm\sqrt{0.25} = \pm 0.5$
However, since the slope of the regression line is negative, r must be negative. Thus, $r = -0.5$. ⇒**A**

4 $r = b \times \frac{s_x}{s_y} = 1.328 \times \frac{1.871}{3.391} = 0.7327$
⇒ **C**

5 *time* is the EV, and *errors* is the RV.
$b = r \times \frac{s_y}{s_x} = -0.236 \times \frac{12.5}{2.40} = -1.229$
$a = \bar{y} - b \times \bar{x} = 34.5 + 8 \times 1.229 =$
44.33
∴ *errors* = 44.3 − 1.23 × *time* ⇒ **E**

6 *speed* is the EV, and *distance* is the RV.
$b = r \times \frac{s_y}{s_x} = -0.948 \times \frac{1.348}{1.124} = 1.138$
On average, for each additional km/hr of speed, the car takes an additional 1.14 metres to stop.
⇒ **C**

7 $y = 8 - 9x = 8 - 9 \times 5$
$= 8 - 45 = -37$ ⇒**B**

8 Enter the data into your calculator and follow instructions on page 173 (TI) or page 174 (CASIO).
⇒**B**

9 Enter the data into your calculator and follow instructions on page 173 (TI) or page 174 (CASIO).
⇒**D**

10 residual = actual – predicted
−5.4 = actual value – 78.6
actual value = 78.6 – 5.4
= 73.2 ⇒**A**

11 The y-intercept is just less than 9, say 8.8. The slope = $\frac{0 - 9}{10} = -0.9$ so that the equation is $y = 8.8 - 0.9x$ which is closest to option A: y = 8.7– 0.9x.
⇒**A**

12 The intercept cannot be determined directly from the graph because the y-axis in this graph is located at x = 20 not x = 0, so proceed as follows. The line passes through the points (20,2) and (30,10) so that the slope $\frac{10 - 2}{30 - 20} = \frac{8}{10} = 0.8$
Our equation is now $y = a + 0.8x$.
Substituting in the point (20,2), we get:
$2 = a + 0.8 \times 20$
$0 = a + 14$
$a = -14$
⇒**A**

13 All of the statements except for D are true. Reading from the equation of the least squares line, the intercept is −96, not 96, so option D is false.
⇒**D**

14 The slope the regression line is 0.95.

This tells us that, on average, *weight* increases by 0.95 kg for reach 1 cm increase in *height*.
⇒**E**

15 If $r = 0.79$, then $r^2 = 0.6241$. Since $0.6241 \times 100\% = 62\%$, we can say that 62% of the variation in *weight* is explained by the variation in *height*.
⇒**A**

16 $weight = -96 + 0.95 \times \text{height}$
$= -96 + 0.95 \times 179$
$= -96 + 170.05 = 74$
residual = actual − predicted
$= 82 - 74 = 8$ kg. ⇒**C**

17 There are 11 students, so the median is the 6th data value when they are in order. Consider only the student marks, which range from about 43 to 98. Counting from the lowest value, the 6th value looks to be about 67.
⇒ *C*

18 There are 11 students, so the median is the 6th data value when they are in order. Consider only the days absent, which range from 0 to 8, Counting from the lowest value, the 6th value is 2. ⇒ **A**

19 if $r^2 = 0.5$, then $r = \pm\sqrt{0.5} = \pm 0.707$

Since the slope of the regression line is negative, *r* must be negative. Thus, $r = -0.7$. ⇒**A**

20 Looking at the graph, when *days absent* = 2, there is one point about 10 marks above the line, and another about 10 marks below the line. ⇒ **D**

21 Looking at the graph, when *days absent* = 4, the predicted value is just over 60. The closest option is 62. ⇒ **C**

22 The EV in this question is *length* and the RV is *weight*. Enter the data into your calculator and construct a scatterplot of the data, and add the regression line (follow the instructions on page 173 (TI) or 174 (Casio)). To find out how many times the predicted value is greater than the actual value, count the number of points below the line, there are 5. ⇒ **C**

23 The percentage of the variation in fish *weight* which is explained by the variation length is equal to $0.965^2 = 0.931 = 93.1\%$. Therefore, the percentage of of the variation in fish weight which is not explained by the variation in length is $100\% - 93.1\% = 6.9\%$ ⇒ **D**

Chapter Review: Extended-response questions

1 **a** There are 8 data values, already in order for *age*. Median is the average of the 4th and 5th values $= \dfrac{4.9 + 5.1}{2} = 5.0$

b Enter the values of airspeed into your calculator to determine the mean and standard deviation of *airspeed*: mean= 767, stand dev= 35.

c Enter the values of *number of seats* into your calculator (the values for *airspeed* are already there). Use your calculator to fit a least squares line to the data.
$a = 673.45$, $b = 0.3717$. This the equation, with values of intercept and slope rounded to 3 significant figures is: $airspeed = 673 + 0.372 \times number\,of\ seats$

d $r^2 = 0.7409 = 74.1\%$

2 $hours\,of\ sunshine = 2850 - 6.88 \times days\,of\,rain$

a From the form of the equation, the EV is ***days of rain***.

b The slope is **−6.88** and the intercept is **2850**.

c Substitute *days of rain* = 120 in to the equation and evaluate.
hours of sunshine
$= 2850 - 6.88 \times 120 \approx$ **2024**

d The slope of the regression line is −6.88 so we can say that, on average, the hours of sunshine per year will **decrease** by **6.88** hours for each additional day of rain.

e $r^2 = 0.484$
$r = \pm\sqrt{0.4838}$
$= \pm 0.696$ (to 3 sig. figs.)
Because the slope of the line is negative r is negative, so
$\boldsymbol{r \approx -0.696}$

f $r^2 = 0.484$ or 48.4%
From this it follows that:
48.4% of the variation in sunshine hours can be explained by the variation in days of rain.

g **i** For 142 days of rain:
hours of sunshine $= 2850 - 6.88 \times 142 \approx$ **1873**

ii The residual value for this city = data value – predicted value

$= 1390 - 1873 = -480$

The residual value for this city is **−483** hours.

h Making predictions within the range of data used to determine the regression equation is called **interpolation**.

3 **a** In this situation, the RV is *cost*.

b From the graph we can see that there is a strong, positive, linear association between the cost of meals and the number of meals prepared.

c **i** *cost* = 222.48 + 4.039 × 21=$307.30

ii From the graph we can see line has been fitted to number of meals from 25 to 80. Since 21 is less than 25, we are making a prediction outside the range of data, so we are extrapolating.

d **i** $222.48: The intercept of the regression line tells us that,on average, when the *number of meals* = 0, the *cost* is $222.48. This represents the fixed costs of preparing meals.

ii $4.039: The slope of the regression line tells us that, on average, *cost* increases by $4.039 for each additional meal prepared.

d When the *number of meals*=50, predicted cost = 222.48 + 4.039 × 50=$424.43
Actual costs = $444. Therefore residual=actual – predicted= 444 − 424.43 = $19.57

4 **a** From the information given in the question, height is to be predicted from femur length. Thus *femur length* is the EV and *height* is the RV.

b Use the following rules to determine the slope and intercept of the least squares line.

$$b = r \cdot \frac{s_y}{s_x} \quad a = \bar{y} - b\bar{x}$$

$$b = r \cdot \frac{s_y}{s_x}$$

$$= 0.9939\left(\frac{10.086}{1.873}\right) = 5.352\ldots$$

$$a = \bar{y} - b\bar{x}$$

$$= 166.092 - 5.352 \times 24.246 = 36.32\ldots$$

Thus, correct to 3 sig. figs., the equation of the least squares line is:
height = 36.3 + 5.35 × *femur length*

c Given the slope = 5.35, it can be deduced that, on average, *height* increases by 5.35

cm for each additional cm increase in *femur length*.

d Given $r = 0.9939$,
$r^2 = 0.9939^2 \approx 0.988$ or 98.8%
From this it can be deduced that 98.8% of the variation in *height* can be explained by the variation in *femur length*.

e $r = b \times \frac{s_x}{s_y} = 0.926 \times \frac{10.761}{10.086} = 0.9879$
$r^2 = 0.9879^2 = 0.9761 = 97.6\%$

5 a From the information given in the question, height is to be predicted from age. Thus *age* is the EV and *height* is the RV.

b There is a strong, positive association between *age* and *height*. There are no outliers.

c Enter the data into your calculator and follow instructions on page 173 (TI) or page 174 (CASIO) and use the values of the statistics generated to arrive at the equation:
height = 76.64 + 6.366 × *age*

d *height* = 76.64 + 6.366 × 1 = 83.1 cm
As we are making a prediction outside the range of the data it is an extrapolation.

e Given the slope ≈ 6.4, it can be deduced that, on average, *height* increases by 6.4 cm for each extra year in *age*.

f Given $r = 0.9973$, $r^2 = 0.9973^2 \approx 0.995$ or 99.5%
From this it can be deduced that 99.5% of the variation in *height* can be explained by the variation in *age*.

g i When the age is 10,
height = 76.64 + 6.366 × 10
= 140.3 cm (to1 d.p.)
The height of a 10-year-old child is predicted to be 140.3 cm.

ii residual = actual – predicted
= 139.6 – 140.3 = −0.7 cm

h i use your calculator to generate the residual plot shown

ii There is a clear pattern in the residual plot which does not support the assumption of linearity.

6 a There is a moderate, positive linear association between *heart rate before exercise* and *heart rate after exercise*. There are no outliers.

b **i** *heart rate after exercise*= $85.671 + 0.561 \times 100 = 141.77 = 142$ to the nearest whole number.

ii From the graph *heart rate before exercise* ranges from about 64-96. We are extrapolating since $100 > 96$.

c *heart rate after exercise*= $85.671 + 0.561 \times 76 = 128.3$ to one decimal place. residual = actual – predicted= $122 - 128.3 = -6.3$

d **i** The residual plot is used to test the assumption of linearity.

ii The assumption of linearity is satisfied becuase there is no clear pattern in the plot, the points are randomly scattered.

Solutions to Exercise 4A

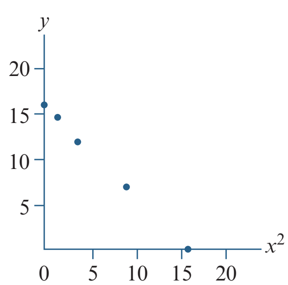
1 **a** $y = 7 + 8x^2$
When $x = 1.25$,
$= 7 + 8 \times 1.25^2 = 19.5$

b $y = 7 + 3x^2$
When $x = 1.25$,
$= 7 + 3 \times 1.25^2 = 11.7$ to 1 d.p.

c $y = 24.56 - 0.457x^2$
When $x = 1.23$,
$= 24.56 - 0.457 \times 1.23^2$
$= 23.8$ to 1 d.p.

d $y = -4.75 + 5.95x^2$
When $= 4.7$,
$= -4.75 + 5.95 \times 4.7^2$
$= 126.7$ to 1 d.p.

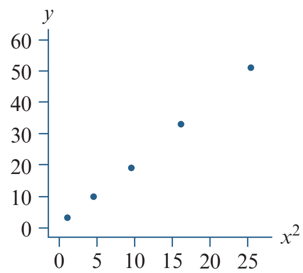
2 **a** Follow the calculator instructions on pages 214 (TI) and 216 (CASIO), enter the data into your calculator, apply an x^2 transformation and generate a scatterplot of y vs x^2 to show that the x^2 transformation has linearised the data.



b Fit a least squares line to the transformed to obtain the equation:
$y = 16 - x^2$

c Substitute $x = -2$ into the equation and evaluate.
$y = 16 - x^2$
when $x = -2$:
$y = 16 - (-2)^2$
$y = 12$

3 a Follow the calculator instructions on pages 196 (TI) and 170 (CASIO), enter the raw data into your calculator, apply an x^2 transformation and generate a scatterplot of y vs x^2 to show that the x^2 transformation has linearised the data.



b Fit a least squares line to the transformed to obtain the equation:

$y = 1 + 2x^2$

c Substitute $x = 6$ into the equation and evaluate.

$y = 1 + 2x^2$

when $x = 6$:

$y = 1 + 2(6)^2$

$y = 73$

4 a $y^2 = 16 + 4x$

When $x = 1.57$,

$y^2 = 16 + 4 \times 1.57$

or

$y = \pm\sqrt{(16 + 4 \times 1.57)}$

$y = \pm 4.7$ to 1 d.p.

b $y^2 = 1.7 - 3.4x$

When $x = 0.03$,

$y^2 = 1.7 - 3.4 \times 0.03$

or

$y = \pm\sqrt{(1.7 - 3.4 \times 0.030)}$

$= \pm 1.3$ to 1 d.p.

c $y^2 = 16 + 2x$

When $x = 10$,

$y = \pm\sqrt{(16 + 2 \times 10)} = \pm 6$

However, the question specifies that we only want the positive solution ($y > 0$), so:

$y = 6$

d $y^2 = 58 + 2x$

When $x = 3$,

$y = \pm\sqrt{(58 + 2 \times 3)} = \pm 8$
However, the question specifies that we only want the negative solution ($y > 0$),
so: $y = -8$

5 **a** Follow the calculator instructions on pages 169 (TI) and 170 (CASIO), enter the raw data into your calculator but, this time, and apply a y^2 transformation and generate a scatterplot of y^2 vs x to show that the y^2 transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:
$y^2 = 1.5 + 3.1x$ to 2 sig. figs.

c Substitute $x = 9$ into the equation and evaluate.
$y^2 = 1.5 + 3.1 \times 9$ or $y = \pm\sqrt{29.4} = \pm 5.42\ldots$
So, because y can only be positive in this situation (see the original scatterplot)
$y = 5.4$ to 1 d.p.

6 Follow the calculator instructions on pages 214 (TI) and 216 (CASIO), enter the raw data into your calculator but, this time, and apply a squared transformation to the variable *diameter* and generate a scatterplot of *number* vs $diameter^2$ to show that the $diameter^2$ transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:
$number = 0.0 + 4.1 \times diameter^2$ to 1 d.p.

b Substitute *diameter* = 1.3 into the equation and evaluate.
$number = 4 \times 1.3^2 = 6.76$
= 7 to the nearest person

7 Follow the calculator instructions on pages 169 (TI) and 170 (CASIO), enter the raw data into your calculator but, this time, and apply a squared transformation to the variable *time* and generate a scatterplot of $time^2$ vs *amount* to show that the $time^2$ transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:
$time^2 = 18 - 9.3 \times amount$ to 2 sig. figs.

b Substitute $x = 0.4$ into the equation and evaluate.
$time^2 = 18 - 9.3 \times 0.4 = 14.28$
or
$time = 3.8$ to 1 d.p. (as $time > 0$)

8 Follow the calculator instructions on pages 214 (TI) and 216 (CASIO), enter the raw data into your calculator but, this time, and apply a squared transformation to the x-variable. Fit a least squares regression line to the transformed data, giving $y = 262.8 - 3.256x \Rightarrow$ **D**

9 $y^2 = 79973 - 9533.4x$. When $x = 4$, $y^2 = 79973 - 9533.4 \times 4 = 41839.4$
$\therefore y = \sqrt{4}1839.4 = \pm 204.55$
From the graph we see that the value of y must be positive $\Rightarrow$ **A**

10 $y = 2370 + 0.238x^2$. When $x = 75$, $y = 2370 + 0.238^2 = 3708.75$
$\Rightarrow$ **B**

Solutions to Exercise 4B

1 **a** $y = 5.5 + 3.1 \log 2.3$
$y = 6.6$ to 1 d.p.

b $y = 0.34 + 5.2 \log 1.4$
$y = 1.1$ to 1 d.p.

c $y = -8.5 + 4.12 \log 20$
$y = -3.1$ to 1 d.p.

d $y = 196.1 - 23.2 \log 303$
$= 138.5$ to 1 d.p.

2 **a** Follow the calculator instructions on pages 224 (TI) and 225 (CASIO), enter the data into your calculator, apply an log x transformation and generate a scatterplot of y vs log x to show that the log x transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation: $y = 1 + 3 \log x$

c Substitute $x = 100$ into the equation and evaluate.

$$\begin{aligned} y &= 1 + 3 \log x \\ \text{when } x &= 100 \\ y &= 1 + 3 \log 100 = 7 \end{aligned}$$

3 **a** Follow the calculator instructions on pages 224 (TI) and 225(CASIO), enter the raw data into your calculator, apply an log x transformation and generate a scatterplot of y vs log x to show that the log x transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation: $y = 20 - \log x$

c Substitute $x = 1000$ into the equation and evaluate.
$y = 20 - 5 \log x$
when $x = 1000$
$y = 20 - 5 \log 1000$
$y = 5$

4 **a** $\log y = 2$
$y = 10^2 = 100$

b $\log y = 2.34$
$y = 10^{2.34} = 218.8$ to 1 d.p.

c $\log y = 3.5 + 2x$
When $x = 1.25$,
$\log y = 3.5 + 2 \times 1.25 = 6$
$y = 10^6 = 1\,000\,000$

d $\log y = -0.5 + 0.024x$
When $x = 17.3$,
$\log y = -0.5 + 0.024 \times 17.3$
$\log y = -0.0848\ldots$
$y \approx 10^{-0.0848} = 0.8$ to 1 d.p.

5 **a** Follow the calculator instructions on pages 224 (TI) and 225 (CASIO), enter the raw data into your calculator and apply a log y transformation. Generate a scatterplot of log y vs x to show that the log y transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:
$\log y = 1 + 2x$

c Substitute $x = 0.6$ into the equation and evaluate.
$\log y = 1 + 2x$
when $x = 0.6$
$\log y = 2.2$
or
$y = 10^{22} = 158.5$ to l d.p.

6 Follow the calculator instructions on pages 224 (TI) and 225 (CASIO), enter the raw data into your calculator, apply an log transformation to the variable *time*. Generate a scatterplot of *level* vs log *time* to show that the log transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:
level = $1.8 + 2.6 \times \log(\textit{time})$ (2 sig. figs.)

b Substitute *time* = 2.5 into the equation and evaluate.
level = $1.8 + 2.6 \times \log 2.5 = 2.8$ to 1 d.p.

7 Follow the calculator instructions on pages 224 (TI) and 225 (CASIO), enter the raw data into your calculator and apply a log transformation to the variable *number*. Generate a scatterplot of log *number* vs *month* to show that the log *number* transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:
$\log(number) = 1.314 + 0.08301 \times month$
(4 sig. figs.)

b Substitute *month* = 10 into the equation and evaluate.
$\log(number) = 1.314 + 0.08301 \times 10$
$= 2.1441\ldots$
So;
$number = 10^{2.1441} = 139.34\ldots$
$= 139$ (nearest whole number)

8 Follow the calculator instructions on pages 224 (TI) and 225 (CASIO), enter the data into your calculator and apply a log transformation to the *x*-variable. Fit a least squares regression line to the transformed data, giving $y = 2.78 + 0.973\log x \Rightarrow$ **C**

9 $time = 42.7 - 13.7 \times \log(horsepower)$
When *horsepower*= 180, $time = 42.7 - 13.7 \times \log(180) = 11.352. \Rightarrow$ **E**

10 $\log(shareprice) = 1.39 + 0.050 \times month$
When *month*=14, $\log(shareprice) = 1.39 + 0.050 \times 14 = 2.09$
$\therefore \quad shareprice = 10^{2.09} = \$123.03 \Rightarrow$ **A**

Solutions to Exercise 4C

1 a $y = 6 + \frac{22}{x}$

When $x = 3$,

$y = 6+ = \frac{22}{x} = 13.3$ to 1 d.p.

b $y = 4.9 - \frac{2.3}{x}$

When $x = 1.1$,

$y = 4.9 - \frac{2.3}{1.1} = 2.8$ to 1 d.p.

c $y = 8.97 - \frac{7.95}{x}$

When $x = 1.97$,

$y = 8.97 - \frac{7.95}{1.97} = 4.9$ to 1 d.p.

d $y = 102.6 + \frac{223.5}{x}$

When $x = 1.08$,

$y = 102.6 + \frac{223.5}{1.08} = 309.5$ to 1 d.p.

2 a Following the calculator instructions on pages 233 (TI) and 234 (CASIO), enter the raw data into your calculator, apply a reciprocal transformation and generate a scatterplot of y vs $1/x$ to show that the reciprocal $(1/x)$ transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:

$y = \frac{120}{x}$

c Substitute $x = 5$ into the equation and evaluate.

$y = \frac{120}{x}$

when $x = 5$:

$y = 24$

3 a $\frac{1}{y} = 3x$

When $x = 2$,

$\frac{1}{y} = 3 \times 2 = 6$

so

$y = \frac{1}{6} = 0.17$ to 2 d.p.

b $\frac{1}{y} = 6 + 2x$

When $x = 4$,

$\frac{1}{y} = 6 + 2 \times 4 = 14$

so

$y = \frac{1}{14} = 0.07$ to 2 d.p.

c $\frac{1}{y} = -4.5 + 2.4x$

When $x = 4.5$,

$\frac{1}{y} = -4.5 + 2.4 \times 4.5 = 6.3$

so

$y = \frac{1}{6.3} = 0.16$ to 2 d.p.

d $\frac{1}{y} = 14.7 + 0.23x$

When $x = 4.5$,

$\frac{1}{y} = 14.7 + 0.23 \times 4.5 = 15.735$

so

$y = \frac{1}{15.735} = 0.06$ to 2 d.p.

4 a Follow the calculator instructions on pages 233 (TI) and 234 (CASIO), enter the raw data into your calculator but, this time, and apply a *reciprocal* transformation and generate a scatterplot of $1/y$ vs x to show that the reciprocal ($1/y$) transformation has linearised the data.

b Fit a least squares line to the transformed data to obtain the equation:

$\frac{1}{y} = x$

c Substitute $x = 0.25$ into the equation and evaluate.

$$\frac{1}{y} = x$$

when $x = 0.25$

$y = 4$

5 Follow the calculator instructions on pages 233 (TI) and 234 (CASIO), enter the raw data into your calculator, apply a reciprocal transformation to the variable fuel *consumption* and generate a scatterplot of *horsepower* vs 1/*consumption* to show that the log transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:

$$horsepower = 22.1 + \frac{690}{consumption}$$

(3 sig. figs.)

b Substitute *time* = 2.5 into the equation and evaluate.

$$horsepower = 22.1 + \frac{690}{9} = 99 \text{ to the nearest whole number.}$$

6 Follow the calculator instructions on pages 233 (TI) and 234 (CASIO), enter the raw data into your calculator but, this time, and apply a reciprocal transformation to the variable *errors* and generate a scatterplot of 1/*errors* vs *times* to show that the reciprocal transformation has linearised the data.

a Fit a least squares line to the transformed data to obtain the equation:

$$\frac{1}{errors} = 0.050 \times times - 0.00024$$

(2 sig. figs.)

b Substitute *times* = 6 into the equation and evaluate.

$$\frac{1}{errors} = 0.050 \times 6$$

or

errors = 3.33… = 3 to the nearest whole number

Solutions to Exercise 4D

1 a Matching the shape of the scatterplot with the shapes shown on the circle of transformations, we see that the following transformations have the potential to linearise the plot:
$\log(x)$, $\frac{1}{x}$, $\log(y)$ and $\frac{1}{y}$

b The circle of transformations only applies to plots with a consistently increasing or decreasing trend.
Since the plot has a decreasing trend followed by an increasing a trend, the circle of transformations does not apply.

c Matching the shape of the scatterplot with the shapes shown on the circle of transformations, we see that the following transformations have the potential to linearise the plot:
$\log(y)$, $\frac{1}{y}$ and x^2

d Matching the shape of the scatterplot with the shapes shown on the circle of transformations, we see that the following transformations have the potential to linearise the plot:
y^2 and x^2.

2 a Follow the calculator instructions on pages 131, enter the raw data into your calculator and generate a scatterplot of *yield* vs *length*.

b Follow the calculator instructions on pages 195(TI) and 196 (CASIO), fit a least squares regression line to the data (and generate a residual plot)

$$yield = -620.0 + 80.23 \times length$$

c The residual plot shows a clear curved pattern, meaning a transformation might be required.

d Matching the shape of the scatterplot with the shapes shown on the circle of transformations, we see that the following transformations have the potential to linearise the plot: $\log(y)$, $\frac{1}{y}$ and x^2

e In this situation the most logical transformation is x^2, as it makes sense that the yield is related to area rather than length. Follow the calculator instructions on pages 214 (TI) and 216 (CASIO) and apply a squared transformation to the y-variable. Fit a least squares regression line to the transformed data, giving:

$yield = 3.983 + 2.030 \times length^2$

f $r^2 = 95.7\%$

3 a Follow the calculator instructions on pages 131, enter the raw data into your calculator and generate a scatterplot of *smoking* vs *cost*.

b Follow the calculator instructions on pages 195(TI) and 196 (CASIO), fit a least squares regression line to the data (and generate a residual plot)

$smoking = 22.294 - 9.501 \times cost$

c The residual plot shows a clear curved pattern, meaning a transformation might be required.

d Matching the shape of the scatterplot with the shapes shown on the circle of transformations, we see that the following transformations have the potential to linearise the plot: $\log(x)$, $\frac{1}{x}$

e Both transformations give good results. Follow the calculator instructions on pages 224 (TI) and 225 (CASIO) to apply a log transformation to the *x*-variable, or pages 233 (TI) and 234 (CASIO) to apply a reciprocal transformation to the *x*-variable. Fit a least squares regression line to the transformed data, giving:

$smoking = 3.420 + \frac{9.045}{cost}$ or

$smoking = 12.73 - 21.90 \times \log(cost)$

The $\frac{1}{x}$ transformation is more intuitive and easier to interpret.

f $\frac{1}{x} : r^2 = 99.3\%$ $\log x : r^2 = 99.6\%$

4 a Follow the calculator instructions on pages 131, enter the raw data into your calculator and generate a scatterplot of *density* vs *distance*.

b Follow the calculator instructions on pages 195(TI) and 196 (CASIO), fit a least squares regression line to the data (and generate a residual plot)

$density = 345.3 - 18.65 \times distance$

c The residual plot shows a clear curved pattern, meaning a transformation might be required.

d Matching the shape of the scatterplot with the shapes shown on the circle of transformations, we see that the following transformations have the potential to linearise the plot: x^2, y^2

e The x^2 transformation is easier to interpret. Follow the calculator instructions on pages 214 (TI) and 216 (CASIO) to apply a squared transformation to the x-variable. Fit a least squares regression line to the transformed data, giving:

$$density = 308.9 - 1.345 \times distance^2$$

f $r^2 = 99.1\%$

Chapter Review: Solutions to Multiple-choice questions

1 A square transformation has an expanding (stretching-out) effect upon large values in a data distribution. **A**

2 A log transformation has a compressing effect upon the high values in a data distribution. **D**

3 A reciprocal transformation has a compressing effect upon the high values in a data distribution. **D**

4 The scatterplot can be linearised by stretching out the high end of the y axis scale. A y^2 transformation can be used for this purpose. **B**

5 The scatterplot can be linearised by compressing the high end of the y axis scale ($\log y$ or $1/y$) or compressing the high end of the x axis scale ($\log x$ or $1/x$). **A**

6 The scatterplot can be linearised by stretching out the high end of the y axis scale. A y^2 transformation can be used for this purpose. **B**

7 Following the instructions on pages 224 (TI) and 225 (CASIO), enter the data into your calculator, perform a $\log x$ transformation and fit a least squares line to the transformed data to obtain the equation, $y = 7.04 + 3.86 \log x$. **E**

8 Following the instructions on pages 224 (TI) and 225 (CASIO), enter the data into your calculator, perform a $\log y$ transformation and fit a least squares line to the transformed data to obtain the equation $\log y = 2.88 + 0.256x$. **D**

9 For an x^2 transformation, we can write the general equation $y = a + b \times x^2$. y is the response variable = *weight*,
a is the y-intercept = 10,
b is the slope of the regression line = 5,
x is the explanatory variable = *width*.

Substituting all these values in gives us
$weight = 10 + 5 \times width^2$. **D**

10
$mark = 20 + 40 \times \log(hours)$

$mark = 20 + 40 \times \log(20)$

$mark = 72.04 \approx 72$ **D**

10
$1/y = 0.14 + 0.045x$

$= 0.14 + 0.045 \times 6$

$= 0.41$

$y = \dfrac{1}{0.41} = 2.439\ldots$

$y \approx 2.4$ **D**

Chapter Review: Extended-response questions

1 a Following the instructions on pages 233 (TI) and 234 (CASIO), enter the data into your calculator, perform a reciprocal y transformation and fit a least squares line to the transformed data to obtain the equation:

$$\frac{1}{age} = 2.606 - 1.053 \times length$$

b $\log(age) = -2.443 + 1.429 \times length$

$length = 2.00 \Rightarrow \log(age) = -2.443 + 1.429 \times 2 = 0.415$

$age = 10^{0.415} = 2.600$ years

2 a Following the instructions on pages 224(TI) and 225 (CASIO), enter the data into your calculator, perform a log x transformation and fit a least squares regression line to the transformed data giving:

$literacy\ rate = -44.2 + 33.3 \times \log GDP$.

b The residual plot shows no clear pattern.

c $literacy\ rate = -44.2 + 33.3 \times \log GDP$

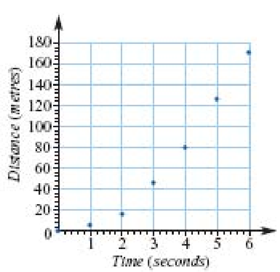
When $GDP = 10\,000 \Rightarrow literacy\ rate = -44.2 + 33.3 \times \log 10\,000$

$= -44.2 + 33.3 \times 4 = 89\%$

d When $GDP = 19\,860 \Rightarrow predicted\ literacy\ rate = -44.2 + 33.3 \times \log 19\,860 = 98.933\%$

actual literacy rate = 99% $\Rightarrow$ $residual = 99 - 98.933 = 0.077\%$

3 a Following the instructions on pages 173(TI) and 174 (CASIO), enter the data into your calculator, and construct a scatterplot with *distance* as the response variable as shown below.



Inspection of the scatterplot shows that the association between *distance* and *time* is strong, postive and non-linear.

b Perform an x^2 transformation to form a new variable $time^2$ and complete the table.

c Using the transformed data, fit a least squares line to the transformed data to obtain the equation:

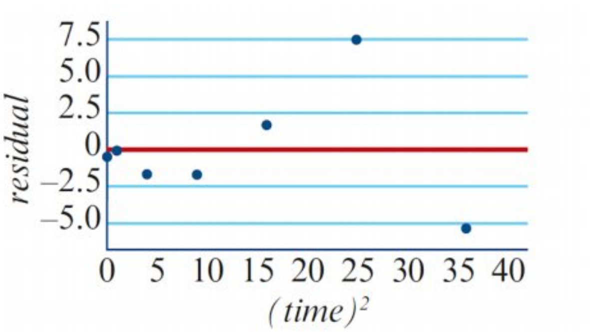
distance = 0.45 + 4.8 × *time*^{2}

c When *time* = 7 seconds,

distance = 0.45 + 4.8 ×7^2

= 235.65 ≈ 236 metres

f Follow the calculator instructions on page 214 (TI) or 216(CASIO) to construct a residual plot with *time*^{2} as the EV as shown below.



There is no clear pattern in the residual plot; it is essentially a random array of points, indicating that the assumption of linearity is justified.

4 a Following the instructions on pages 173 (TI) and 174 (CASIO), enter the data into your calculator, and construct a scatterplot with *mortality* as the response variable, and *doctors* as the explanatory variable. The plot is non-linear.

b Following the instructions on pages 233 (TI) and 234 (CASIO), enter the data into your calculator, and construct a scatterplot with *mortality* as the response variable, and $\frac{1}{doctors}$ as the response variable.

c Following the instructions on pages 233 (TI) and 234 (CASIO),fit a least squares regression line to the data (include a residual plot) with *mortality* as the response variable, and $\frac{1}{doctors}$ as the response variable, giving the equation

$$mortality = -1.194 + 3856 \times \frac{1}{doctors}$$

d The residual plots shows no clear structure, indicating the assumption of linearity is justified.

e r^2 = 82.8%

f $mortality = -1.194 + 3856 \times \frac{1}{doctors}$

When *doctors* = 100, *mortality* = −1.194 + 3856 × 0.01 = 37.366

g Since 100 in within the range of the data used for the regression we are interpolating, so the prediction should be reliable.

Solutions to Exercise 5A

1 Enter the data into your calculator and follow the instructions for creating a time series plot on page 255 for both TI and CASIO.

2 Enter the data into your calculator and follow the instructions for creating a time series plot on page 255 for both TI and CASIO.

3 Enter the data into your calculator and follow the instructions for creating a time series plot on page 255 for both TI and CASIO.

4 **Graph A:** There is a general tendency for the points to decrease in value as we go from left to right indicating a **decreasing trend.** The remaining variation appears to be no more than **irregular (random) fluctuations**.

Graph B: There is no general tendency for the points to decrease or increase in value as we go from left to right so there is **no** trend. The sole source of variation appears to be **irregular (random) fluctuations**.

Graph C: There is a general tendency for the points to increase in value as we go from left to right indicating a **increasing trend.** The remaining variation appears to be no more than **irregular (random) fluctuations**.

5 **Graph A:** There is a general tendency for the points to decrease in value as we go from left to right indicating a **decreasing trend.** There also appears to be **seasonality** as there are regularly spaced troughs and peaks with intervals of around one 1 year. The remaining variation appears to be no more than **irregular (random) fluctuations**.

Graph B: There is no apparent trend. However, there appears to be **cyclical variation**, since the troughs and the peaks are regularly spaced but at intervals of greater than 1 year. The remaining variation appears to be no more than **irregular (random) fluctuations**.

Graph C: There is a general tendency for the points to increase in value as we go from left to right indicating a **increasing trend.** There also appears to be **seasonality** as the troughs and peaks are roughly equally spaced with intervals of less than 1 year. The remaining variation appears to be no more than **irregular (random) fluctuations**.

6 **Graph A:** The plot shows a period of **decreasing trend** followed by a period of **increasing trend** which suggests there has been a **structural change**. The remaining variation appears to be no more than **irregular (random) fluctuations**.

Graph B: The values in the plot do not change in any way over time. Other than that, the plot has no defining features.

Graph C: There is no apparent trend but there appears to be **seasonality** as the troughs and peaks are roughly equally spaced with intervals of around one 1 year. The remaining variation appears to be no more than **irregular (random) fluctuations**.

7 **a** There is a clear **increasing trend**. The remaining variation appears to be no more than **irregular (random) fluctuations**.

b There is little change in the first fours years, then a drop in the fifth year, showing a **structural change**.The remaining variation appears to be no more than **irregular (random) fluctuations**.

c Ther is no apparent trend in the plot, but there is an unusually high value in 2018, indicating a possible **outlier**.The remaining variation appears to be no more than **irregular (random) fluctuations**.

8 The number of mobile phones per 100 people increases rapidly over the years 2000-2008. The number continues to increase from 2009 until 2019, but the increase in the number of phones is at much lower level than in the preceding years.

9 **a** Enter the data into your calculator and follow the instructions for creating a time series plot on page 255 for both TI and CASIO.

b The plot shows a steady increase in the population of Australia over the years 2012 - 2021.

10 **a** Enter the data into your calculator and follow the instructions for creating a time series plot on page 255 for both TI and CASIO.

b The plot shows a steady decline in the number of vehicle thefts over the years 2003 -2010, after which the number of vehicle thefts has remained reasonably steady, showing only irregular variation.

11 The number of cases of measles show an increasing trend between 1989 and 1992. In 1993-1994 there is a rapid increase in the number of measles cases, followed be a rapid decrease in 1995-1996. The number of cases continued to decrease until 2000, sine then have remained low, showing only irregular variation over the years 2001-2019.

12 The number of overseas arrivals (millions people per month) in Australia increased steadily from November 2011 until April 2020. The number of arrivals is clearly seasonal, with the peak time for arrivals in the January quarter each year. The number of arrivals dropped suddenly to almost zero in August 2020, and remained at this level until October 2021.

13 a i The plot for the men (red line) shows a general decrease in smoking rates for the period 1945 until 1992 which indicates the presence of a decreasing trend.
The plot for females (blue line) shows a general but slow increase in smoking rates for the period 1945 until 1975. This was followed by a general decrease in female smoking rates from 1975 to 1992 that that paralleled the decrease for men.

ii Since the two graphs have become closer and closer together between 1945 and 1992, we can say that the difference in smoking rates has decreased over that period.

b i Enter the data into your calculator and follow the instructions for creating a time series plot on page 255 for both TI and CASIO.

ii Whilst both plots show irregular fluctuation, overall the percentages of male and females who smoke have declined substantially over the years 2000-2018.

iii The difference in smoking rates between males and females has remained almost the same over these years.

14 E
There is a general tendency for the points to increase in value as we go from left to right indicating a **increasing trend.** There also appears to be **seasonality** as there are regularly spaced troughs and peaks within the years. The remaining variation appears to be no more than **irregular (random) fluctuations**.

15 D

There is no increasing or decreasing trend showing, but there appears to be an **outlier** in 2019. The remaining variation appears to be no more than **irregular (random) fluctuations**.

Solutions to Exercise 5B

1 a i $\dfrac{5+3+1}{3} = 3$

ii $\dfrac{1+0+2}{3} = 1$

iii $\dfrac{5+2+5}{3} = 4$

b i $\dfrac{5+2+5+3+1}{5} = 3.2$

ii $\dfrac{1+0+2+3+0}{5} = 1.2$

iii $\dfrac{2+5+3+1+0}{5} = 2.2$

c i $\dfrac{5+2+5+3+1+0+2}{7} = 2.6$

ii $\dfrac{5+3+1+0+2+3+0}{7} = 2$

d $\dfrac{5+2+5+3+1+0+2+3+0}{9} = 2.3$

2 a $\dfrac{33.5+21.6+18.1}{3} = 24.4$

b $\dfrac{21.6+18.1+16.2+17.9+26.4}{5} = 20.0$

c $\dfrac{28.9+33.5+21.6+18.1+16.2+17.9+26.4}{7} = 23.2$

3 $t = 2 \quad 3mm = \dfrac{10+12+8}{3} = 10$

$t = 3 \quad 3mm = \dfrac{12+8+4}{3} = 8 \quad 5mm = \dfrac{10+12+8+4+12}{5} = 9.2$

$t = 4 \quad 3mm = \dfrac{8+4+12}{3} = 8 \quad 5mm = \dfrac{12+8+4+12+8}{5} = 8.8$

$t = 5 \quad 3mm = \dfrac{4+12+8}{3} = 8 \quad 5mm = \dfrac{8+4+12+8+10}{5} = 8.4$

$t = 6 \quad 3mm = \dfrac{12+8+10}{3} = 10 \quad 5mm = \dfrac{4+12+8+10+18}{5} = 10.4$

$t = 7 \quad 3mm = \dfrac{8+10+18}{3} = 12 \quad 5mm = \dfrac{12+8+10+18+2}{5} = 10$

$t = 8 \quad 3mm = \dfrac{10 + 18 + 2}{3} = 10$

4 a Follow the calculator instructions given on page 255 for both TI and CASIO.

b Day 3: $3mm = \dfrac{27 + 28 + 40}{3} = 31.7$

Day 5: $3mm = \dfrac{40 + 22 + 23}{3} = 28.3$

Day 7: $3mm = \dfrac{23 + 22 + 21}{3} = 22$

Day 4: $5mm = \dfrac{27 + 28 + 40 + 22 + 23}{5} = 28$

Day 6: $5mm = \dfrac{40 + 22 + 23 + 22 + 21}{5} = 25.6$

Day 6: $5mm = \dfrac{22 + 23 + 22 + 21 + 25}{5} = 22.6$

c Follow the calculator instructions given on page 255 for both TI and CASIO.
Once constructed, you are expected to comment on the effect of smoothing on the time series plots.
The following comments are based on the plot below.
Noting that the smoothed graphs show a much smaller temperature variation from day to day, we can say:
'The smoothed plot show that the 'average' maximum temperature changes relatively slowly over the 10-day period (the 5-day average varies by only 5 degrees) when compared to the daily maximum, which can vary quite widely (for example, nearly 20 degrees between the 4th and 5th day) over the same period of time.'

5 a Follow the calculator instructions given on page 255 for both TI and CASIO.

b Day 4: $3mm = \dfrac{0.737 + 0.751 + 0.724}{3} = 0.737$

Day 6: $3mm = \dfrac{0.724 + 0.724 + 0.712}{3} = 0.720$

Day 7: $3mm = \dfrac{0.724 + 0.724 + 0.735}{3} = 0.724$

Day 4: $5mm = \dfrac{0.743 + 0.754 + 0.737 + 0.751 + 0.724}{5} = 0.742$

Day 6: $5mm = \dfrac{0.751 + 0.724 + 0.724 + 0.712 + 0.735}{5} = 0.729$

Day 6: $5mm = \dfrac{0.724 + 0.724 + 0.712 + 0.735 + 0.716}{5} = 0.722$

c Follow the calculator instructions given on page 255 for both TI and CASIO. The smoothed plots have removed the irregular variation, and making the downward trend over the 10-day period quite easy to see.

6 **a** t=3 $2mm = \dfrac{\frac{2+5}{2} + \frac{5+3}{2}}{2} = 3.8$

b t=8 $2mm = \dfrac{\frac{2+3}{2} + \frac{3+0}{2}}{2} = 2.0$

7 **a** t=3 $4mm = \dfrac{\frac{5+2+5+3}{4} + \frac{2+5+3+1}{4}}{2} = 3.3$

b t=6 $4mm = \dfrac{\frac{3+1+0+2}{4} + \frac{1+0+2+3}{4}}{2} = 1.5$

c t=4 $6mm = \dfrac{\frac{5+2+5+3+1+0}{6} + \frac{2+5+3+1+0+2}{6}}{2} = 2.4$

d t=6 $6mm = \dfrac{\frac{5+3+1+0+2+3}{6} + \frac{3+1+0+2+3+0}{6}}{2} = 1.9$

8 **a** Day 5 $2mm = \dfrac{\frac{14.1+12.5}{2} + \frac{12.5+13.3}{2}}{2} = 13.1$

b Day 5 $4mm = \dfrac{\frac{11.6+14.1+12.5+13.3}{2} + \frac{14.1+12.5+13.3+6.4}{4}}{2} = 12.2$

c Day 5 $6mm = \dfrac{\frac{3.5+11.6+14.1+12.5+13.3+6.4}{6} + \frac{11.6+14.1+12.5+13.3+6.4+8.5}{6}}{2} = 10.7$

9 **a** Follow the calculator instructions given on page 255 for both TI and CASIO.

b Sept $2mm = \dfrac{\frac{7+9}{2} + \frac{9+10}{2}}{2} = 8.8$

c Follow the calculator instructions given on page 255 for both TI and CASIO. When comparing the plots we see that the smoothing has not made a lot of difference. Apart from the irregular fluctuation in the data the most obvious feature of the plots is the large number of complaints made between April and July.

10 **a** Follow the calculator instructions given on page 255 for both TI and CASIO.

b August $4mm = \dfrac{\frac{52.3+42.1+58.9+79.9}{4} + \frac{42.1+58.9+79.9+81.5}{4}}{2} = 62.0$

c Follow the calculator instructions given on page 255 for both TI and CASIO. When comparing the plots we see that the smoothing had made clearer the steady increase in rainfall between June and October.

11 A

$$\text{5-mean smoothed value for Friday} = \frac{77 + Thur + Fri + Sat + 10}{5} = 39$$

$$77 + Thur + Fri + Sat + 10 = 5 \times 39 = 195$$

$$Thur + Fri + Sat = 195 - 77 - 10 = 108$$

$$\text{3-mean smoothed value for Friday} = \frac{Thur + Fri + Sat}{3} = \frac{108}{3} = 36$$

12 C

$$\text{Day 6} \quad 6mm = \frac{\frac{1.10+1.25+1.29+1.37+2.42+1.95}{6} + \frac{1.25+1.29+1.37+2.42+1.95+2.05}{6}}{2} = 1.64$$

13 E

$$\text{July: } 5mm = \frac{May + Jun + Jul + Aug + Sep}{5} = \frac{831}{5} = \$166.20$$

14 D

There are 12 data points. Seven mean smoothing starts at the 4th data point and ends at the 4th past data point ie we lose three points at each end. So there will be 6 left.

Solutions to Exercise 5C

1

a–d To locate the median of a set of points graphically:

- locate the median horizontally and mark with a vertical dotted line
- locate the median vertically and mark with a horizontal line.
- the median of the points is then given by the point of intersection of these two lines.

See page 278 for a step-by-step worked example.

2 See Example 12 for a step-by-step worked example of three median smoothing.

3 See Example 12 for a step-by-step worked example of three median smoothing. The smoothed plots shows the change to lower sales in 2017

4 See Example 13 for a step-by-step worked example of five median smoothing.

5 See Example 13 for a step-by-step worked example of five median smoothing. The smoothed plot the exchange rate decreasing over the 10 day period.

6 See Example 12 for a step-by-step worked example of three median smoothing. See Example 13 for a step-by-step worked example of five median smoothing.

7 See Example 12 for a step-by-step worked example of three median smoothing. See Example 13 for a step-by-step worked example of five median smoothing. The effect of median smoothing is to smooth out local irregular fluctuations in the plot to hopefully reveal any underlying feature. In this case smoothing revealed a dip in GDP growth between years 6 and 10.

8 **D**
See description on page 278 on how to find the median value. There are 12 points so the median is the average of the 6th and 7th points, when placed in order. Counting up form the lowest point we see the 6th value is in June (180) and the 7th value is in

August (200). The average of these two values is 190.

9 D

See Example 123 for a step-by-step worked example of five median smoothing.

10 D

For nine median smoothing the smoothed value is the median of the point and the four points in either side of it (making 9 points in total), so we are looking for the median of the points for April, May, June, July, August, September, October, November, December.

Solutions to Exercise 5D

1 **a** We know that the average seasonal index must be 1 by definition. Since there are 12 seasons (months) in this case, the total sum of seasonal indices must be 12. Since
$1.2 + 1.3 + 1.0 + 1.0 + 0.9 + 0.8 + 0.7 + 0.9 + 1.0 + 1.1 = 11.0$
the seasonal index for December is $12.0 - 11.0 = 1.0$.

b Seasonal index for Feb = 1.3. To interpret the seasonal index subtract 1, and then convert to a percentage.
$(1.3 - 1.0) = 0.3 = 30\%$ We can then say: Sales in Feb are 30% **higher** than in an average month.

c Seasonal index for Sept = 0.9. To interpret the seasonal index subtract 1, and then convert to a percentage.
$(0.9 - 1.0) = -0.1 = -10\%$ We can then say: Sales in Sept are 10% **lower** than in an average month.

2 deseasonalised value= $\dfrac{\text{actual value}}{\text{seasonal index}}$

a deseasonalised sales = $\dfrac{8.6}{1.1}$=7.8

b deseasonalised sales = $\dfrac{6.0}{0.9}$=6.7

3 actual value = deseasonalised value × seasonal index

a actual sales = $5.6 \times 0.7 = 3.9$

b actual sales = $6.9 \times 1.0 = 6.9$

4 deseasonalised sales= $\dfrac{\text{actual sales}}{\text{seasonal index}} = \dfrac{1}{\text{seasonal index}} \times \text{actual sales}$

a SI = 0.7 ⇒ deseasonalised sales= $\dfrac{1}{0.7} \times$ actual sales = 1.429 × actual sales
To find the percentage subtract 1, and then convert to a percentage.
$(1.429 - 1.0) = 0.429 = 42.9\%$. Sales should be increased by 42.9%.

b SI = 0.7 ⇒ deseasonalised sales= $\dfrac{1}{1.3} \times$ actual sales = 0.769 × actual sales
To find the percentage subtract 1, and then convert to a percentage.
$(0.769 - 1.0) = -0.231 = -23.1\%$. Sales should be decreased by 23.1%.

5 **a** Seasonal indices should add to 4. $0.8 + 1.0 + 1.3 = 2.8 \Rightarrow$ SI for Quarter 4= $4 - 2.8 = 1.2$

b deseasonalised sales = $\frac{1060}{0.7}$=1514.285.

c deseasonalised sales= $\frac{1868}{1.3} = 1436.923$

d actual sales = $1256 \times 0.8 = 1004.8$

6

	Summer	**Autumn**	**Winter**	**Spring**
Year 1	56	125	126	96
Deseasonalised	$\frac{56}{0.5} = 112$	$\frac{125}{1} = 125$	$\frac{126}{1.3} = 96.9...$	$\frac{96}{1.2} = 80$
Seasonal index	0.5	1	1.3	$4.0 - 0.5 - 1.0 - 1.3 = 1.2$

7

a, c

	Quarter 1	**Quarter 2**	**Quarter 3**	**Quarter 4**
Year 1	198	145	86	168
Deseasonalised	$\frac{198}{1.3} = 152.30\ldots$	$\frac{145}{1.02} = 142.15\ldots$	$\frac{86}{0.58} = 148.27\ldots$	$\frac{168}{1.10} = 152.72\ldots$
Seasonal index	1.30	$4 - (1.30 + 0.58 + 1.10) = 1.02$	0.58	1.10

b The seasonal index of the average month is 1.0
The seasonal index for Quarter 1 is 1.3 which is $(1 - 1.3) \times 100 = 30\%$ greater than the average quarter
The number of waiters employed in Quarter 1 is 30% more than the average quarter.

8 Seasonal average = $\frac{60 + 50 + 15 + 18}{4} = 67.25$

Q1	Q2	Q3	Q4
$\frac{60}{67.25} = 0.89$	$\frac{56}{67.25} = 0.83$	$\frac{75}{67.25} = 1.12$	$\frac{78}{67.25} = 1.16$

9 Seasonal average = 13.5. The indices in the table below are calculated by dividing the *Sales* figure for each month by 13.5 and rounding to 2 decimal places.

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Sales	12	13	14	17	18	15	9	10	8	11	15	20
Index	0.89	0.96	1.04	1.26	1.33	1.11	0.67	0.74	0.59	0.81	1.11	1.48

10 Seasonal average Year 1=$\frac{22+19+25+23+20+18+20+15+14+11+23+30}{12} = 20$

Seasonal average Year 2=$\frac{21+20+23+25+22+17+19+17+16+11+25+31}{12} = 20.58$

The indices in the table for year 1 below are calculated by dividing the *Sales* figure for each month by 20. Answers are shown to two decimal places but actual workings were to three decimal places, to ensure final answer was accurate to two decimal places.The indices in the table for year 2 below are calculated by dividing the *Sales* figure for each month by 20.583. Answers are shown to two decimal places but actual workings were to three decimal places, to ensure final answer was accurate to two decimal places.

The Average SI is the average of the SI for each month over the two years.

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Index Year 1	1.10	0.95	1.25	1.15	1.00	0.90	1.00	0.75	0.70	0.55	1.15	1.50
Index Year 2	1.02	0.97	1.12	1.21	1.07	0.83	0.92	0.83	0.78	0.53	1.21	1.51
Average SI	1.06	0.96	1.18	1.18	1.03	0.86	0.96	0.79	0.74	0.54	1.18	1.50

11 **a** The deseasonalised figures in the table below are calculated by dividing the *Sales* figure for each day by the appropriated Seasonal index.

Day	Mon	Tu	Wed	Thu	Fri	Sat	Sun
Week	$\frac{124}{0.8}$	$\frac{110}{0.7}$	$\frac{45}{0.3}$	$\frac{67}{0.5}$	$\frac{230}{1.5}$	$\frac{134}{1.0}$	$\frac{330}{2.2}$
Week 2	$\frac{120}{0.8}$	$\frac{108}{0.7}$	$\frac{57}{0.3}$	$\frac{74}{0.5}$	$\frac{215}{0.5}$	$\frac{150}{1.0}$	$\frac{345}{2.2}$

b Enter the data into your calculator and follow the instructions for creating a time series plot on page 255 for both TI and CASIO. Number the days from 1 to 14.

12 **a** Enter the data into your calculator and follow the instructions for creating a time series plot on page 255 for both TI and CASIO. Number the quarters from 1 to 12.

b Determine the quarterly average retail job vacancies for each year:

$$\text{Year 1} = \frac{212 + 194 + 196 + 227}{4} = 207.25$$

$$\text{Year 2} = \frac{220 + 197 + 196 + 239}{4} = 213$$

$$\text{Year 3} = \frac{231 + 205 + 203 + 245}{4} = 221$$

The indices in the table for year 1 below are calculated by dividing the *jobs* figure for each month by 207.25.
The indices in the table for year 2 below are calculated by dividing the *Sales* figure for each month by 213.
The indices in the table for year 3 below are calculated by dividing the *Sales* figure for each month by 221.
The final SI is obtained by averaging over the three years.

Jobs	*Q1*	*Q2*	*Q3*	*Q4*
Year 1	1.023	0.936	0.946	1.095
Year 2	1.033	0.925	0.920	1.122
Year 3	1.045	0.928	0.919	1.109
Average	1.03	0.93	0.92	1.11

c The deseasonalised figures in the table below are calculated by dividing the *jobs* figure for each day by the appropriate Average Seasonal index.

Year	*Q1*	*Q2*	*Q3*	*Q4*
1	206	209	211	205
2	214	212	211	215
3	224	220	218	221

d Enter the deseasonalised data into your calculator and follow the instructions for creating a time series plot on page 255 for both TI and CASIO. Number the quarters from 1 to 12.

13 E

SI = 0.93 ⇒ deseasonalised customers= $\frac{1}{0.93}$ × actual customers = 1.075 × actual customers

To find the percentage subtract 1, and then convert to a percentage.

(1.075 − 1.0) = 0.075 = 7.5%. Customers should be increased by 7.5%.

14 D

SI = 1.18 ⇒ deseasonalised sales= $\frac{1}{1.18}$ × actual sales = 0.847 × actual sales

To find the percentage subtract 1, and then convert to a percentage.

(0.847 − 1.0) = −0.153 = −15.3%. Sales should be decreased by 15.3%.

15 E

actual customers= 700 × 0.68 = 476

16 B

$$\text{Monthly average} = \frac{223 + 190 + \ldots + 307}{12} = 204.25$$

$$\text{SI}_{August} = \frac{153}{204.25} = 0.75$$

17 B

2020: $\text{SI}_{Winter} = \frac{68.4}{71.3} = 0.9593$

2021: $\text{SI}_{Winter} = \frac{67.7}{71.0} = 0.9535$

2020: $\text{SI}_{Winter} = \frac{68.3}{71.6} = 0.9539$

$$\text{Average} = \frac{0.959 + 0.954 + 0.954}{3} = 0.956$$

Solutions to Exercise 5E

1 **a** From the plot we can see a steady increasing trend in the data plots, meaning the number of Australian university students has increased steadily over the time period.

b Enter the data into your calculator and follow the instructions on page 173 (TI) or 174 (CASIO) to fit a straight line to the data with *year* as the EV.
$students(000s) = -37\,563 + +18.927 \times year$ (to 5 sig. figs).
slope = 18.927: this means that, on average, the number of university students in Australia has increased by 18 927 each year.

c In the year 2030 the predicted number of students is:
$students(000s) = -37\,563 + +18.927 \times 2030 = 858.81$
$\Rightarrow students = 858.81 \times 1000 = 858\,810 = 859\,000$ to the nearest 1000 students.

2 **a** Enter the data into your calculator and follow the instructions on page 255 for both TI and CASIO to construct a time series plot with *sales* as the EV.

b From the plot, a generally decreasing trend in the percentage of sales made in department stores is evident.

c Follow the instructions on page 173 (TI) or 174 (CASIO) to fit a straight line to the data with *sales* as the EV.
$sales(\%) = 12.5 - 0.258 \times year$
slope = 0.258: On average, the percentage of sales made in department stores decreased by 0.258% per year.

d In Year 15 the predicted percentage of total retail sales made in department stores is:
$sales(\%) = 12.5 - 0.258 \times 15 = 8.6\%$

3 **a** Follow the instructions on page 173 (TI) or 174 (CASIO) to fit a straight line to the data with *year* as the EV.
$age = -147 + 0.0882 \times year$ (to 3 sig figs)
slope = 0.0882: this means that, on average, the age of mothers having their first child in Australia increased by around 0.0882 years each year ($0.0882 \times 12 = 1.06$ months).

b In 2030 the predicted average age of mothers having their first child is:
$age = -147 + 0.0882 \times year = 32.0$ years.
This prediction is likely to be unreliable we are extrapolating well beyond the data.

4 a Follow the instructions on page 173 (TI) or 174 (CASIO) to fit a straight line to the data with *year* as the EV.

earnings = −83 280 + 42.07 × *year* (to 4 sig figs)

slope = 42.07: this means that, on average, the average weekly earnings in Australia increased by around $42.07 each year.

b In 2030 the predicted the average weekly earnings in Australia is:

earnings = −83 280 + 42.07 × 2030 = $2122.10

This prediction is likely to be unreliable we are extrapolating well beyond the data.

5 a Enter the deseasonalised sales data into your calculator and follow the instructions on page 173 (TI) or 174 (CASIO) to fit a straight line to the data with *quarter* as the EV.

deseasonalised number = 50.9 + 1.59 × *quarter* (to 3 sig. figs).

b The fourth quarter year 4 is quarter number 16. Substituting *quarter number* = 16 into the equation we get:

deseasonalised number = 50.9 + 1.59× 16 = 76.34

To reseasonalise this value we need to multiply by the seasonal index quarter 4:

actual number = 76.34 × 1.18 = 90.08 or 90 to the nearest whole number.

6 a Deseasonalised sales and rounded to the nearest whole number.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Year 1	$\frac{138}{1.13} = 122$	$\frac{60}{0.47} = 128$	$\frac{73}{0.62} = 118$	$\frac{230}{1.77} = 130$
Year 2	$\frac{283}{1.13} = 250$	$\frac{115}{0.47} = 245$	$\frac{163}{0.62} = 263$	$\frac{417}{1.77} = 236$

b Enter the data into your calculator and follow the instructions on page 255 for both TI and CASIO to construct a time series plot plot of the actual data and the deseasonalised data as shown below.

The deseasonalised plot suggests a general increase in sales over time indicating a positive trend.

c Follow the instructions on page 136 (TI) or 137 (CASIO) to fit a straight line to the deaseasonalised data with *quarter* as the EV.

deseasonalised sales = 80.8 + 23.5 × *quarter* (to 3 sig. figs.)

d For the first quarter of Year 4, *quarter* = 13

deseasonalised sales = 80.8 + 23.5 ×13 = 386.3

sales = *deseasonalised sales* × *seasonal index*

= 386.3 × 1.13

= 437 to the nearest whole number
The forecast for sales in the first quarter of Year 4 is 437.

7 C

April-June quarter in 2025 is Quarter 14. The predicted deseasonalised number of visitors in Quarter 14 is :

deseasonalised number of visitors = 38 345 + 286.5 ×14 = 42 356

number of visitors = deseasonalised number of visitors × *seasonal index*

= 42 356 × 0.91

= 38544 to the nearest whole number

8 E

November in 2025 is Month 59. The predicted deseasonalised sales in Month 59 is :

deseasonalised sales = 197 + 1.2 ×59 = 267.8

sales = deseasonalised sales × *seasonal index*

= 267.8 × 1.18

= 316 to the nearest whole number

Chapter Review: Multiple-choice questions

1 **E**

The time series plot is shows a slight downward trend, a clear seasonal pattern (quarterly) and irregular fluctuations.

2 **E**

The time series plot an increasing trend from the years 2000-2015, then a significant drop (structural change) before then continuing to increase.

3 **A**

$$\frac{2.4+3.4+4.4}{3} = 3.4$$

4 **B**

$$\frac{2.4+3.4+4.4+2.7+5.1}{5} = 3.6$$

5 **E**

$$2mm = \frac{\frac{2.7+5.1}{2} + \frac{5.1+3.7}{2}}{2} = 4.15$$

6 **C**

$$4mm = \frac{\frac{3.4+4.4+2.7+5.1}{4} + \frac{4.4+2.7+5.1+3.7}{4}}{2}$$

$= 3.9375$

7 **D**

Graphically locate the median of the three points centred on Quarter 2 and read the value off the vertical axis (M=65) .

8 **D**

Graphically locate the median of the five points centred on Quarter 3 and read value off the vertical axis (M=68)

9 **C**

We know that the average seasonal index must be 1 by definition. Since there are 12 seasons (months) in this case, the total sum of seasonal indices must be 12.

Since $12 - (1.0 + 1.1 + 0.9 + 1.0 + 1.0 + 1.2 + 1.1 + 1.1 + 1.1 + 1.0 + 0.7) = 0.8$ the seasonal index for Feb is 0.8.

10 **D**

$$6mm = \frac{\frac{2.80+...+4.32}{6} + \frac{2.78+...+3.95}{6}}{2}$$

$= \$2.99$

11 **B**

Five mean smoothing would give the first value at Day 3 and the last value at Day 8. The number of smoothes values is therefore 6.

12 **C**

Five mean smoothed value for March

$$= \frac{\text{Jan} + \text{Feb} + \text{Mar} + \text{Apr} + \text{May}}{5}$$

$$= \frac{427+230}{5} = \$131.40$$

13 **B**

To deseasonalise a data value, we divide the data value by its seasonal index: $\frac{432}{1.8} = 240$

14 **A**

actual value = deseasonalised value × seasonal index

actual sales Winter = $380 \times 0.3 = 114$

15 **E**

The seasonal index of the average month is 1.0.
The seasonal index for spring is 1.5 which is 50% greater than the average month.

16 **D**

The seasonal index for Autumn is 0.4, so

$\text{deseasonlised value} = \dfrac{\text{actual value}}{0.4}$

$= \dfrac{1}{0.4} \times \text{actual value}$

$= 2.50 \times \text{actual value}$

Percentage increase

$= (2.50 - 1) \times 100 = 150\%$

17 **D**

Seasonal average

$= \dfrac{1048 + 677 + 593 + 998}{4} = 829.$

Seasonal index for Autumn is the value for Autumn divided by the seasonal average $= \dfrac{677}{829} = 0.82.$

18 **A**

When *year* = 2026

predicted change in enrolments

$= -3480 \times 2026 = 24.98$

19 **B**

slope = 1.73: On average, the percentage change in enrolments has increased by 1.73% each year from 2012 to 2019.

20 **E**

Sunday March 20 is Day 15.

predicted deseasonalised price

$= 189.9 + 0.23 \times 15 = 193.35$

predicted seasonalised price

$= 193.35 \times 1.2 = \$232.02$

Chapter Review: Extended-response questions

1 **a** Enter the data into your calculator and follow the instructions on page 255 for both TI and CASIO to construct a time series plot plot of the data with *year* as the EV.

b Carbon dioxide emissions decreased between 2009 and 2014, then remained reasonable steady over the years 2014–2017, showing only irregular fluctuations. Between 2017 and 2018 there was a small decrease in carbon dioxide emissions.

c Follow the instructions on page 173 (TI) or 174 (CASIO) to fit a straight line to the data with *year* as the EV.
$CO_2\ emissions = 612.0 - 0.2958 \times year$

d $Year = 2026$
$predicted\ emissions = 612.0 - 0.2958 \times 2026 = 12.7$

e Unreliable as we are extrapolating 8 years beyond the period in which the data were collected.

2 **a** **i** Enter the data for China into your calculator and follow the instructions on page 173 (TI) or 174 (CASIO) to fit a straight line to the data with *year* as the EV.
$inflation = 332 - 0.164 \times year$

ii Follow the instructions on page 255 for both TI and CASIO to construct a time series plot of the data for China with *year* as the EV. Add the regression line to the plot following the instructions on page 173 (TI) or 174 (CASIO).

b **i** Enter the data for Australia into your calculator and follow the instructions on page 173 (TI) or 174 (CASIO) to fit a straight line to the data with *year* as the EV.
$inflation = 339 - 0.167 \times year$

ii Follow the instructions on page 255 for both TI and CASIO to construct a time series plot of the data for Australia with *year* as the EV. Add the regression line to the plot following the instructions on page 173 (TI) or 174 (CASIO).

c The trend lines are parallel. As such, they will never cross, so the inflation rate for China will remain higher than the inflation rate for Australia.

d $2mm = \dfrac{\frac{2.5+1.5}{2} + \frac{1.5+1.3}{2}}{2} = 1.7$

3 a 2020: quarterly average $= \dfrac{97 + 112 + 480 + 678}{4} = 341.75$

2021: quarterly average $= \dfrac{107 + 145 + 496 + 730}{4} = 369.5$

To find the SI's for each quarter we divide by the quarterly average for that year:

Year	Sum	Aut	Win	Spr
2020	$\frac{97}{341.75}$	$\frac{112}{341.75}$	$\frac{480}{341.75}$	$\frac{678}{341.75}$
2021	$\frac{107}{369.5}$	$\frac{145}{369.5}$	$\frac{496}{369.5}$	$\frac{730}{369.5}$

=

Year	Sum	Aut	Win	Spr
2020	0.284	0.328	1.405	1.984
2021	0.290	0.392	1.342	1.976

To find the final seasonal indices we average over the two years and round to two decimal places, giving:

	Sum	Aut	Win	Spr
SI	0.29	0.36	1.37	1.98

b To deseasonalise the data we divide the actual data by the appropriate seasonal index, giving:

	Sum	Aut	Win	Spr
Deseas. data	269	239	255	273

Solutions to Exercise 6A

1 The variables *campus* and *transport* are both nominal variables. The data values they generate that can be used to group individuals into separate categories (eg female, male) but there is no underlying order to these categories. ⇒ **A**

2 The variable *number* is the only discrete numerical variable. ⇒ **B**

3 Reading the table we see that there are two regional students (R) in the sample, but only one uses public transport (3). ⇒ **B**

4 Since the first quartile starts to the right of the value 30, we can safely say that less than a quarter of all observations are less than 30. ⇒ **B**

5 The modal response is the one with the highest percentage frequency. Reading from the graph this value is 29%. ⇒ **D**

6 Since the histogram tails off slowly to the left from the peak it is negatively skewed. ⇒ **B**

7 Negatively skewed distribution has a box plot with the median to the right hand side of the box, a short right hand whisker and a long left-hand whisker. ⇒ **D**

8 The histogram tells us that 11+10 = 21 students scores 50 or more.
We know that 63 students sat the test.
Since $\frac{21}{63} \times 100\% = 33\%$, we conclude that 33% of students scored 50 or more.
⇒ **E**

9 The histogram tells us that 4 students scored from 30 to less than 35, 7 students from 35 to less than 40, and 9 students from 40 to less than 45, so that 20 (= 4 + 7 + 9) students have scores between 30 and 45. ⇒ **E**

10 For grouped data, the mode (or modal interval) is the most commonly occurring group of values. In a histogram this group of values (50–55) is identified by the highest bar. ⇒ **E**

11 Since there are 63 values, the median value is the 32nd value.
Counting up from the left-hand side of the histogram, the 32nd value (the median) is in the 40–45 interval.⇒ **D**

12 $IQR = Q_3 - Q_1 = 60 - 50 = 10$.
Lower limit for outliers (upper fence)
$= Q_1 - 1.5 \times IQR$
$= 50 - 1.5 \times 10 = 35$.
Upper limit for outliers (lower fence)
$= Q_3 + 1.5 \times IQR$
$= 60 + 1.5 \times 10 = 75$.
Thus, outliers are defined as values lower than 35 or higher than 75.

$\Rightarrow$**A**

13 $n = 20$ so $M = \dfrac{10^{th} + 11^{th}}{2} = \dfrac{36 + 42}{2} = 39 \Rightarrow$ **D**

14 $n = 20$ so $Q1 = \dfrac{26 + 28}{2} = 27$

$Q1 = \dfrac{53 + 51}{2} = 52$

$IQR = 52 - 27 = 25 \Rightarrow$ **A**

15 The fastest 30% of people = the fastest $\dfrac{30}{100} \times 20 = 6$ people. Reading from the stem plot, 28 minutes or less. $\Rightarrow$ **A**

16 log(10 000 000) = 7

Number of countries with log(*Expenditure*) $\geq$ 7=14+16+8+12 = 50.

Percentage of countries=$\dfrac{50}{72} \times 100 = 69.4 \Rightarrow$ **E**

17 Adding the percentage frequencies left to right 4+12+26 = 42, 4+12+26 =34 = 76. The median is in the interval 180-182. $\Rightarrow$ **C**

18 Adding the percentage frequencies left to right 4+12=16, 4+12+26 = 42, so Q_1 is in the interval 178-180. Adding the percentage frequencies right to left 2+4=6, 2+4+18= 24, 2+4+18+32= 56 so Q_3 is in the interval 180-182. The maximum value of the $IQR = 182 - 178 = 4 \Rightarrow$ **B**

19 From the 68-95-99.7% rule, 95% of ants have lengths between:

mean – 2 *SD* and *mean* + 2 *SD*

or

$4.8 - 2 \times 1.2 = 2.4$ and $4.8 + 2 \times 1.2 = 7.2 \Rightarrow$ **B**

20 *actual length* = *mean* + $z \times$ *SD*

$= 4.8 + (-1.2) \times 1.2 = 3.36 \Rightarrow$ **B**

21 3.6 is one SD below the mean.

As a consequence of the 68-95-99.7% rule, 16% of ants have lengths less than 3.6 mm. $\Rightarrow$ **C**

22 6 is one SD above the mean.

As a consequence of the 68-95-99.7% rule, 16% of ants have lengths more than 6, (100 –16)% = 84% of ants have lengths less than 6 mm. $\Rightarrow$ **E**

23 3.6 is one SD below the mean.

7.2 is two SDs above the mean.

As a consequence of the 68-95-99.7% rule,

- 84% of ants have lengths more than 3.6 mm
- 2.5% of ants have lengths above 7.2 mm,

so 84–2.5 = 81.5% of ants have lengths between 3.6 and 7.2 mm. $\Rightarrow$ **D**

24 2.4 is two SDs below the mean.

4.8 is the mean.

As a consequence of 68-95-99.7% rule, 95% of ants have lengths within two standard deviations of the mean so half

of these, 47.5%, have lengths between 2.4 and 4.8 mm.
\$47.5% of 1000 = 475 ants. $\Rightarrow$ **C**

25 $2.5\% > 68 \Rightarrow \bar{x} + 2s = 68$
$0.15\% < 16 \Rightarrow \bar{x} - 3s = 16$
Subtracting the two equations gives:
$5s = 52 \Rightarrow s = \frac{52}{5} = 10.4$
Substituting in Equation 1:
$\bar{x} = 68 - 2 \times 10.4 = 47.2 \Rightarrow$ **A**

26 81 for Biology is equivalent to a standardised score
$z = (81–54)/15 = 1.8$
To obtain the same standardised score in Legal Studies, Sashi's actual score would need to be:
actual score = *mean* + $z \times$ *SD*
$= 78 + 1.8 \times 5$
$= 87$
$\Rightarrow$ **D**

Solutions to Exercise 6B

1 EV=*temperature* which is ordinal. ⇒ **E**

2 There are 222 participants aged 35 or more.
Percentage = $\frac{222}{400} \times 100 = 55.5\% \Rightarrow$ **A**

3 There are 178 participants aged under 35. Of these, 136 are satisfied with their career choice.
Percentage = $\frac{136}{178} \times 100 = 76.4\% \Rightarrow$ **E**

4 To investigate the association we need to calculate row percentages, and then compare across one of the rows. The only option that does this is C. ⇒ **C**

5 From the information given in the question it can be concluded that that,

- A is true, the boxwidths are the same.
- B is true, the median pay rates increase in 1980 is the lowest.
- C is not true, the IQR for 1980 is less than the others.
- D is true, Q_3 1980 is less than the medians of 1990 and 2000.
- E is true.

⇒ **C**

6 Reading from the graph = 20%. ⇒ **B**

7 We need to calculate row percentages for each of these tables to find the one that matches the segmented bar chart. Only the table in E does this:

	Dyslexia	
Dominant hand	Yes	No
Left	20%	10%
Right	80%	90%

⇒ **E**

8 To investigate the association we need to compare the correct row percentages across one of the rows. The only option that does this is B. ⇒ **B**

9 Both variable need to be numerical. ⇒ **B**

10 The points in the scatterplot trend down from left to right and can be imaged to closely follow a straight line with minimal scatter around the line. Of the options given, $r = -0.8$ is best option. ⇒ **E**

11 The points in the scatterplot trend down from left to right and can be imaged to closely follow a straight line with minimal scatter around the line. Of the options given, $r = -0.8$ is best option. ⇒ **E**

12 Moving the point at (1, 5) to the point (6, 5) reduces the total scatter in the plot which would not change the direction of the association but would strengthen the correlation. Thus r value would remain

negative and would be closer to −1.
⇒ **E**

13 $r^2 = 0.8464 \Rightarrow r = \pm\sqrt{8464} = 0.92$
Since people who own more cars also tend to own more computers we know that r is positive ⇒ **D**

14 $r^2 = 0.8484 \Rightarrow 84.64\%$ of the variation in computer ownership can be explained by the variation in car ownership. ⇒ **B**

15 Since 84.6% of the variation in computer ownership can be explained by the variation in car ownership, then it follows that 100 − 84.6 = 15.4% of the variation in computer ownership is not explained by the variation in car ownership. ⇒ **E**

16 A back–to–back stem plot is useful for displaying associations between a numerical variable and a categorical variable with **two** categories. *Height* in centimetres versus *sex* (male, female) fulfils these requirements. ⇒ **D**

17 Since both variables are categorical the best choice would be an appropriately percentaged table. ⇒ **A**

18 Since a numerical variable is being plotted against a categorical variable with **more than two** categories, parallel box plots would be the most appropriate choice. ⇒ **D**

Solutions to Exercise 6C

1 $b = \frac{rs_y}{s_x} = \frac{0.675 \times 4.983}{2.567} = 1.3 \Rightarrow$ **C**

2 $r = \frac{bs_x}{s_y} = \frac{0.475 \times 4.65}{4.78} = 10.462 \Rightarrow$ **C**

3 Using your calculator, the equation of the least squares line is:
number= 227.3 – 4.312× *temperature*
$\Rightarrow$ **A**

4 Coefficient of determination:
$r^2 = 0.89 \Rightarrow$ **D**

5 Given:
number of errors= 8.8 – 0.12× *study time*
When *study time* = 35,
number of errors = 8.8 – 0.12 × 35
= 4.6 $\Rightarrow$ **B**

6 residual value = actual – predicted
actual value = 6
predicted value = 8.8 – 0.12 × 10
= 7.6
residual value = 6 – 7.6 = –1.6
$\Rightarrow$ **B**

7 Option **E** is clearly **not** true, the *EV* is *study time* not *number of errors.*
All of the other options can be shown to be true. $\Rightarrow$ **E**

8 From the given regression equation, the slope = –0.12.
This means that, on average, the number of errors made decreases by 0.12 for each extra minute spent studying. $\Rightarrow$ **B**

9 Since $r^2 = 0.8198$ or 81.98 % ≈ 82%, we can say that 82% of the variation in the *number of errors* made can be explained by the variation in *study time.*
$\Rightarrow$ **D**

10 Use the points (0,210) and 14,50).
$a \approx 210$
$b = \frac{50 - 210}{14 - 0} = \frac{160}{14} = 11.4$
Hence, *average rainfall* = 210 – 11× *temperature range*.

$\Rightarrow$ **A**

11 From the given correlation coefficients:

- Option A is not true, we don't know anythings about the correlation between two tests.
- Option B is not true, we don't know anythings about the correlation between two tests.
- Option C is not true, we cant add values of r^2.
- Option D is not true, we don't know anythings about the correlation between two tests.
- Option E is **true**: a correlation coefficient of r =0.662 indicates a stronger association than a correlation coefficient of r = 0.462

$\Rightarrow$ **E**

12 Enter the the raw data into your calculator, apply an x^2 transformation to obtain the required equation:
$y = 131.6 - 1.64x^2$
$\Rightarrow$ **E**

13 $y^2 = 20076 - 2397.2x$
When $x = 2$
$y^2 = 20076 - 2397.2 \times 2 = 15281.6$
$\therefore x = \sqrt{15281.6} = 123.6 \Rightarrow$ **B**

14 Enter the the raw data into your calculator, apply a 1/y transformation to obtain the required equation:
$1/y = 0.08 + 0.16x$

$\Rightarrow$ **A**

13 $population = 58\,170 + 43.17 \times year^2$
When $year = 10$,
$population = 58\,170 + 43.17 \times 10^2$
$= 62\,487 \Rightarrow$ **E**

14 $weight^2 = 52 + 0.78 \times area$
When $area = 8.8$,
$weight^2 = 52 + 0.78 \times 8.8$
$= \sqrt{58.86\ldots} \approx 7.7$
$\Rightarrow$ **C**

15 $\log number = 1.31 + 0.083 \times month$
When $month = 6$,
$\log number = 1.31 + 0.083 \times 6 = 1.808$
or $number = 10^{1.808} = 64.2\ldots \approx 64 \Rightarrow$ **D**

Solutions to Exercise 6D

1 There is a clear upwards trend and random variation, but no variation that can be said to be clearly cyclical or seasonal. ⇒ **A**

2 This question is concerned with the **difference** in share price between the two companies. This distance is represented by the distance between the two time series lines which clearly **decreases** with time. ⇒ **A**

3 $\frac{4+5+4}{3} = 4.33\ldots$ ⇒ **A**

4 $\frac{4+4+8+6+91}{5} = 6.2 \Rightarrow$ **B**

5 $2mm = \frac{\frac{8+6}{2} + \frac{6+9}{2}}{2}$

$= \frac{7+7.5}{2} = 7.25 \Rightarrow \text{E}$

6 $4mm = \frac{\frac{4+5+4+4}{4} + \frac{5+4+4+8}{4}}{2}$

$= \frac{4.25+5.25}{2} = 4.75 \Rightarrow$ **B**

7 $5mm = \frac{25 + Wed + Thu + Fri + 84}{5}$

$= 38$

$\therefore 25 + Wed + Thu + Fri + 84 = 5 \times 38 = 190$

$\therefore Wed + Thu + Fri = 190 - 25 - 84 = 81$

$3mm = \frac{Wed + Thu + Fri}{3} = \frac{81}{3} = 27$

⇒ **A**

8 $6mm = \frac{\frac{1.15+1.10+1.25+1.29+1.37+2.42}{6} + \frac{1.10+1.25+1.29+1.37+2.42+1.95}{6}}{2}$

$= 1.50$

⇒ **B**

9 Monthly average

$= \frac{Jan + Feb + \cdots + Nov + Dec}{12}$

$= \frac{123 + 90 + \ldots + 85 + 107}{12}$

$= 114.7$

Sept SI$= \frac{95}{114.4} = 0.83$

⇒ **D**

10 actual cost $= \frac{1}{0.90} \times$ deseasonalised cost

$= 1.111 \times$deseasonalised cost

To find the actual cost we need to increase by $(1.111 - 1) \times 100 = 11.1\%$

⇒ **E**

11 deseasonalised sales $= \frac{800}{0.8} = 1000$

⇒ **D**

12 SI$= 4 - 1.1 - 0.9 - 0.8 = 1.2 \Rightarrow$ **E**

13 actual sales = 91 564 × 1.45

= \$132 767.8 ⇒ **E**

14 The seasonal index of an average quarter is 1.0.

The seasonal index for quarter 3 is 0.85 which is 15% less than the average quarterly.

⇒ **B**

15 To find the 3-median smoothed value for month 9 compare the number of calls for months 8, 9, 10 on the time series plot. The median of these three points is at month 8 ≈ 362. ⇒ **B**

16 To find the 5-median smoothed value for month 9 compare the number of calls for months 8, 9, 10,11, 12 on the time series plot. The median of these five points is at month 9 ≈ 375. ⇒ **D**

17 Choose the points (0,20) and (10,4)

$a = 20$

$$b = \frac{4 - 20}{10 - 0} = \frac{-16}{1} = -1.6$$

Hence, $y = 20 - 1.6t$

⇒ **A**

Chapter Review: Extended-response questions

1 **a** The numerical variables are *age* and *distance*.

b Using your calculator to find summary statistics $\bar{x} = 7.17$ km, $s = 3.46$ km

c $z = \dfrac{x - \bar{x}}{s} = \dfrac{13 = 7.17}{3.46} = 1.7$

d Tally the number of students in each category from the data in the table:

	Gender	
Study mode	Female	Male
On campus	3	3
Online	4	2
Total	7	5

e **i** Reading from the segmented bar chart =60%

ii We need to quote and compare online (or on-campus) percentages for each course. eg: Yes, there is an association between study mode preference and course. A higher percentage of students business chose to study online (60%), compared to only 36% for both students of Health and Social Science.

2 **a** Looking at both the histogram and boxplot we can see that the distribution of *distance* is positively skewed, with outliers at 17 km, 18 km, and 19 km.

b The histogram doesn't help here as there is not interval from 4-5. Looking at the boxplot we see that $Q_1 = 4$ and $M = 5$ so approximately 25% of the students travel from 4-5 km. Since there are 120 students then this is $120 \times \dfrac{25}{100} = 30$ students.

c **i** $IQR = Q_3 - Q_1 = 8 - 4 = 4$
Lower fence= $4 - 1.5 \times 4 = -2$
Upper fence = $8 + 1.5 \times 4 = 14$

ii A distance of 1 km is within the fences.

d **i** $M_{online} = 6$ km $\quad M_{on-campus} = 5$ km
$\therefore M_{online}$ is 1 km more than $M_{on-campus}$

ii $IQR_{online} = 9 - 4 = 5$ km $\quad IQR_{on-campus} = 7.5 - 4 = 3.5$ km
$\therefore IQR_{online}$ is 1.5 km more than $IQR_{on-campus}$

3 **a** Males $b = 0.815$. On average, height increases by 0.815 cm for each additional 1 cm increase in arm span.

b **i** Females: $r = \dfrac{bs_x}{s_y} = \dfrac{1.028 \times 6.319}{8.083} = 0.8037$

$r^2 = (0.8037)^2 = 64.6\%$

ii Males: $r = \dfrac{bs_x}{s_y} = \dfrac{0.815 \times 9.832}{9.583} = 0.8362$

$r^2 = (0.8363)^2 = 69.9\%$

iii Since the value of the coefficient of determination for males (69.9%) is higher than the value for females (64.6%), then we can say that arm span is a better predictor of height for males than for females.

c **i** *armspan* = 160

predicted female height = −4.199 + 1.028 × 160 = 160.3

predicted male height = 31.705 + 0.815 × 160 = 162.1

The models predict that when both have arm span measurements of 160 cm, a male will be 1.8 cm taller than a female.

ii *armspan* = 190

predicted female height = −4.199 + 1.028 × 190 = 191.1

predicted male height = 31.705 + 0.815 × 190 = 186.6

The models predict that when both arm span measurements of 190 cm, a female will be 4.6 cm taller than a male.

iii The differences predicted is not reliable for a height of 160 cm as this is value is outside the range of height data for males. The prediction is not reliable for a height of 190 cm as this value is outside the range of height data for females.

4 a There is a moderate strength, non-linear association between *expenditure* and *score*.

b Compare this scatterplot to the Circle of Transformations on page 239

$\Rightarrow y^2, \log x, \dfrac{1}{x}$

c i The linearity assumption.

ii No, there is a clear structure in the residual plot. If the linearity assumption had been met the residuals would have been randomly scattered around a horizontal line at $y = 0$.

d i $a = 12.99 \qquad b = 120.6$

score= 12.99 + 120.6× log(*expenditure*)

ii *score*= 12.99 + 120.6× log(10 000)=495

5 a i The five-moving median for month 10 is the median for months 8, 9, 10, 11, 12. From the time series plot the median is at month 9 = \$12 000

ii The seven-moving median for month 9 is the median for months 6, 7, 8, 9, 10, 11, 12.

From the time series plot the median is at month 8 =\$11 400 ≈ \$11 000

b $2mm = \dfrac{\frac{33\,543.77+58\,726.68}{2} + \frac{58\,726.68+57\,836.01}{2}}{2} = \$52\,208.29$

c i $b = 1525.80 \Rightarrow$ On average the value of bitcoin has increased by \$1525.80 each month.

ii January 2024 = month 37

predicted bitcoin= 36 382.73 + 1525.799 × 37 = \$92 837

iii Unreliable as we are extrapolating several years beyond the period in which the data were collected.

Solutions to Exercise 7A

1 a Starting value/First term: 2
Second term: $2 + 6 = 8$
Third term: $8 + 6 = 14$
Fourth term: $14 + 6 = 20$
Fifth term: $20 + 6 = 26$
Gives 2, 8, 14, 20, 26

b Starting value/First term: 5
Second term: $5 - 3 = 2$
Third term: $2 - 3 = -1$
Fourth term: $-1 - 3 = -4$
Fifth term: $-4 - 3 = -7$
Gives 5, 2, −1, −4, −7

c Starting value/First term: 1
Second term: $1 \times 4 = 4$
Third term: $4 \times 4 = 16$
Fourth term: $16 \times 4 = 64$
Fifth term: $64 \times 4 = 256$
Gives 1, 4, 16, 64, 256

d Starting value/First term: 64
Second term: $64 \div 2 = 32$
Third term: $32 \div 2 = 16$
Fourth term: $16 \div 2 = 8$
Fifth term: $8 \div 2 = 4$
Gives 64, 32, 16, 8, 4

2 a Starting value/First term: 6
Second term: $6 \times 2 + 2 = 14$
Third term: $14 \times 2 + 2 = 30$
Fourth term: $30 \times 2 + 2 = 62$
Fifth term: $62 \times 2 + 2 = 126$
Gives 6, 14, 30, 62, 126

b Starting value/First term: 24
Second term: $24 \times 0.5 + 4 = 16$
Third term: $16 \times 0.5 + 4 = 12$
Fourth term: $12 \times 0.5 + 4 = 10$
Fifth term: $10 \times 0.5 + 4 = 9$
Gives 24, 16, 12, 10, 9

c Starting value/First term: 1
Second term: $1 \times 3 - 1 = 2$
Third term: $2 \times 3 - 1 = 5$
Fourth term: $5 \times 3 - 1 = 14$
Fifth term: $14 \times 3 - 1 = 41$
Gives 1, 2, 5, 14, 41

d Starting value/First term: 124
Second term: $124 \times 0.5 - 2 = 60$
Third term: $60 \times 0.5 - 2 = 28$
Fourth term: $28 \times 0.5 - 2 = 12$
Fifth term: $12 \times 0.5 - 2 = 4$
Gives 124, 60, 28, 12, 4

3 a Using your CAS, enter 4, then press '+2 ENTER' 4 times to read off the first five terms.

4	4
4 + 2	6
6 + 2	8
8 + 2	10
10 + 2	12

Thus: 4, 6, 8, 10, 12

b Using your CAS, enter 24, then press '-4 ENTER' 4 times to read off the first five terms.

24	24
24 – 4	20
20 – 4	16
16 – 4	12
12 – 4	8

Thus: 24, 20, 16, 12, 8

c Using your CAS, enter 2, then press '×3 ENTER' 4 times to read off the first five terms.

2	2
$2 \cdot 3$	6
$6 \cdot 3$	18
$18 \cdot 3$	54
$54 \cdot 3$	162

Thus: 2, 6, 18, 54, 162

d Using your CAS, enter 50, then press '÷5 ENTER' 4 times to read off the first five terms.

50	50
$50 \div 5$	10
$10 \div 5$	2
$0.4 \div 5$	0.4
$0.08 \div 5$	0.08

Thus: 50, 10, 2, 0.4, 0.08

e Using your CAS, enter 5, then press '×2 + 3 ENTER' 4 times to read off the first five terms.

5	5
$5 \cdot 2 + 3$	13
$13 \cdot 2 + 3$	29
$29 \cdot 2 + 3$	61
$61 \cdot 2 + 3$	125

Thus: 5, 13, 29, 61, 125

f Using your CAS, enter 18, then press '×0.8 + 2 ENTER' 4 times to read off the first five terms.

18	18
$18 \cdot 0.8 + 2$	16.4
$16.4 \cdot 0.8 + 2$	15.12
$15.12 \cdot 0.8 + 2$	14.096
$14.096 \cdot 0.8 + 2$	13.2768

Thus: 18, 16.4, 15.12, 14.096

4 a $W_0 = 2$

$W_1 = W_0 + 3 = 2 + 3 = 5$

$W_2 = W_1 + 3 = 5 + 3 = 8$

$W_3 = W_2 + 3 = 8 + 3 = 11$

$W_4 = W_3 + 3 = 11 + 3 = 14$

Thus: 2, 5, 8, 11, 14

b $D_0 = 50$

$D_1 = D_0 - 5 = 50 - 5 = 45$

$D_2 = D_1 - 5 = 5 - 5 = 40$

$D_3 = D_2 - 5 = 8 - 5 = 35$

$D_4 = D_3 - 5 = 11 - 5 = 30$

Thus: 50, 45, 40, 35, 30

c $M_0 = 1$

$M_1 = 3 \times M_0 = 3 \times 1 = 3$

$M_2 = 3 \times M_1 = 3 \times 3 = 9$

$M_3 = 3 \times M_2 = 3 \times 9 = 27$

$M_4 = 3 \times M_3 = 3 \times 27 = 81$

Thus: 1, 3, 9, 27, 81

d $L_0 = 3$

$L_1 = -2 \times L_0 = -2 \times 3 = -6$

$L_2 = -2 \times L_1 = -2 \times -66 = 12$

$L_3 = -2 \times L_2 = -2 \times 12 = -24$

$L_4 = -2 \times L_3 = -2 \times -24 = 48$

Thus: 3, –6, 12, –24, 48

e $K_0 = 5$

$K_1 = 2 \times K_0 - 1 = 2 \times 5 - 1 = 9$

$K_2 = 2 \times K_1 - 1 = 2 \times 9 - 1 = 17$

$K_3 = 2 \times K_2 - 1 = 2 \times 17 - 1 = 33$

$K_4 = 2 \times K_3 - 1 = 2 \times 33 - 1 = 65$

Thus: 5, 9, 17, 33, 65

f $F_0 = 2$
$F_1 = 2 \times F_0 + 3 = 2 \times 2 + 3 = 7$
$F_2 = 2 \times F_1 + 3 = 2 \times 7 + 3 = 17$
$F_3 = 2 \times F_2 + 3 = 2 \times 17 + 3 = 37$
$F_4 = 2 \times F_3 + 3 = 2 \times 37 + 3 = 77$
Thus: 2, 7, 17, 37, 77

g $S_0 = -2$
$S_1 = 3 \times S_0 + 5 = 3 \times -2 + 5 = -1$
$S_2 = 3 \times S_1 + 5 = 3 \times -1 + 5 = 2$
$S_3 = 3 \times S_2 + 5 = 3 \times 2 + 5 = 11$
$S_4 = 3 \times S_3 + 5 = 3 \times 11 + 5 = 38$
Thus: $-2, -1, 2, 11, 38$

h $V_0 = -10$
$V_1 = -3 \times V_0 + 5 = -3 \times -10 + 3 = 35$
$V_2 = -3 \times V_1 + 5 = -3 \times 35 + 3 = -100$
$V_3 = -3 \times V_2 + 5 = -3 \times -100 + 3 = 305$
$V_4 = -3 \times V_3 + 5 = -3 \times 305 + 3 = -910$
Thus: $-10, 35, -100, 305, -910$

5 a Using your CAS, enter 12, then press '×6 – 15 ENTER' 4 times to read off the first five terms. Thus: 12, 57, 327, 1947, 11 667

b Using your CAS, enter 20, then press '×3 + 25 ENTER' 4 times to read off the first five terms. Thus: 20, 85, 280, 865, 2620

c Using your CAS, enter 2, then press '×4 + 3 ENTER' 4 times to read off the first five terms. Thus: 2, 11, 47, 191, 767

d Using your CAS, enter 64, then press '×0.25 – 1 ENTER' 4 times to read off the first five terms. Thus: 64, 15, 2.75, −0.3125, −1.078125

e Using your CAS, enter 48 000, then press '−3000 ENTER' 4 times to read off the first five terms. Thus: 48 000, 45 000, 42 000, 39 000, 36 000

f Using your CAS, enter 25 000, then press '×0.9 – 550 ENTER' 4 times to read off the first five terms. Thus: 25 000, 21 950, 19 205, 16 734.50, 14 511.05

6 a $A_0 = 2$
$A_1 = A_0 + 2 = 2 + 2 = 4$
$A_2 = A_1 + 2 = 4 + 2 = 6$

b $B_0 = 11$
$B_1 = B_0 - 3 = 11 - 3 = 8$
$B_2 = B_1 - 3 = 8 - 3 = 5$
$B_3 = B_2 - 3 = 5 - 3 = 2$
$B_4 = B_3 - 3 = 2 - 3 = -1$

c $C_0 = 1$
$C_1 = 3 \times C_0 = 3 \times 1 = 3$
$C_2 = 3 \times C_1 = 3 \times 3 = 9$
$C_3 = 3 \times C_2 = 3 \times 9 = 27$

d $D_0 = 3$
$D_1 = 2 \times D_0 + 1 = 2 \times 3 + 1 = 7$
$D_2 = 2 \times D_1 + 1 = 2 \times 7 + 1 = 15$
$D_3 = 2 \times D_2 + 1 = 2 \times 15 + 1 = 31$
$D_4 = 2 \times D_3 + 1 = 2 \times 31 + 1 = 63$
$D_5 = 2 \times D_4 + 1 = 2 \times 63 + 1 = 127$

7 a Starting value of 4: $V_0 = 4$
Adding 2: $V_{n+1} = V_n + 2$

b Starting value of 24: $V_0 = 24$
Subtracting 4: $V_{n+1} = V_n - 4$

c Starting value of 2: $V_0 = 2$
Multiplying by 3: $V_{n+1} = 3V_n$

8 **a** As the sequence starts with the value 5, $V_0 = 5$. The terms increase by 5 so $V_{n+1} = V_n + 5$.

b As the sequence starts with the value 13, $V_0 = 13$. The terms decrease by 4 so $V_{n+1} = V_n - 4$.

c As the sequence starts with the value 1, $V_0 = 1$. To generate the next term, the current term is multiplied by 4 so $V_{n+1} = 4V_n$.

d As the sequence starts with the value 64, $V_0 = 64$. To generate the next term, the current term is multiplied by 0.5 (or halved) so $V_{n+1} = 0.5V_n$.

9 Type in '150' on your CAS and press ENTER. Then type '×0.6 − 5' and press ENTER until you obtain a negative answer.
Count the total number of positive terms, including the initial term of 150.
150, 85, 46, 22.6, 8.56, 0.136, −4.9184.
Thus there are 6 positive terms.

10 Type in '30' on your CAS and press ENTER. Then type '×1.2 + 2' and press ENTER until you obtain a negative answer.
As the terms are increasing in value, there are no negative terms.

11 Applying the rule recursively:
$A_0 = 3$
$A_1 = 4 \times A_0 + 1 = 4 \times 3 + 1 = 13$
$A_2 = 4 \times A_1 + 1 = 4 \times 13 + 1 = 53$
$A_3 = 4 \times A_2 + 1 = 4 \times 53 + 1 = 213$
$A_4 = 4 \times A_3 + 1 = 4 \times 213 + 1 = 853$
Thus, $A_4 = 853$ so the answer is **E**.

12 Type in '15' on your CAS and press ENTER. Then type '+4' and press ENTER until you obtain 51. Once you do, count the number of times that you added 4.
Since 4 was added 9 times, then $A_9 = 51$ so the answer is **D**.

13 The first term in the sequence is 3 so $B_0 = 3$.
The difference between the first two terms is $7 - 3 = 4$ but the difference between the second and third term is $15 - 7 = 8$ so we don't have a constant difference. This means that the answer isn't **A** or **B**.
The third option also doesn't work since $B_1 = 3 \times B_0 - 1 = 3 \times 3 - 1 = 8$ which isn't true.
The fourth option also doesn't work since $B_1 = 4 \times B_0 - 5 = 4 \times 3 - 5 = 7$ and then $B_2 = 4 \times B_1 - 5 = 4 \times 7 - 5 = 23$ which is not true.
The fifth option does work since:
$B_0 = 3$
$B_1 = 2 \times B_0 + 1 = 2 \times 3 + 1 = 7$
$B_2 = 2 \times B_1 + 1 = 2 \times 7 + 1 = 15$
$B_3 = 2 \times B_2 + 1 = 2 \times 15 + 1 = 31$
$B_4 = 2 \times B_3 + 1 = 2 \times 31 + 1 = 63$

Solutions to Exercise 7B

1 a Applying the rule:

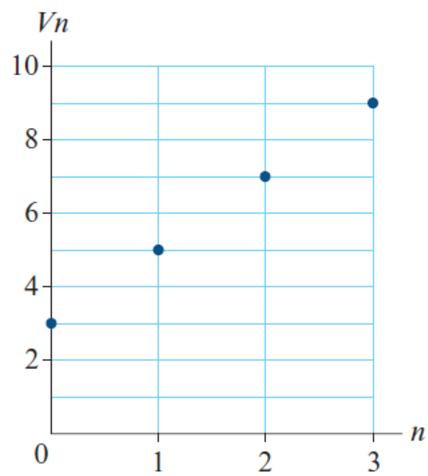
$V_0 = 3$

$V_1 = V_0 + 2 = 3 + 2 = 5$

$V_2 = V_1 + 2 = 5 + 2 = 7$

$V_3 = V_2 + 2 = 7 + 2 = 9$

So the first four terms are 3, 5, 7, 9



b Applying the rule:

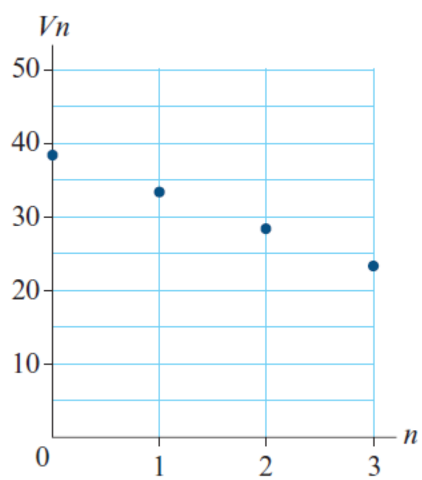
$V_0 = 38$

$V_1 = V_0 - 5 = 38 - 5 = 33$

$V_2 = V_1 - 5 = 333 - 5 = 28$

$V_3 = V_2 - 5 = 28 - 5 = 23$

So the first four terms are 38, 33, 28, 23



2 a Since Ashwin invests \$8000 into the account, this is the starting value. $V_0 = 8000$.

b Substitute $r = 4$ and $V_0 = 8000$ into the rule:

$$D = \frac{r}{100}V_0 = \frac{4}{100} \times 8000 = 320$$

c Substitude the values into the recurrence relation.

$$V_0 = 8000, \quad V_{n+1} = V_n + 320$$

3 a Since Huang invests \$41 000 in an account, this gives the starting value. Hence, $H_0 = 41\ 000$.

b Substitute $r = 6.2$ and $H_0 = 41\ 000$ into the rule:

$$D = \frac{r}{100}H_0 = \frac{6.2}{100} \times 41\ 000 = 2542$$

Hence, Huang's investment earns \$2542 each year.

c Substitude the values into the recurrence relation.

$$H_0 = 41\ 000, \quad H_{n+1} = H_n + 2542$$

4 a Apply the rule three times:

$V_0 = 2000$

$V_1 = V_0 + 76 = 2000 + 76 = 2076$

$V_2 = V_1 + 76 = 2076 + 76 = 2152$

$V_3 = V_2 + 76 = 2152 + 76 = 2228$

Thus, the value of the investment after 3 years is \$2228.

b Enter 2000 onto your CAS. Press '+76 ENTER' until the value first exceeds 3000. Count the number of times that you press ENTER.

2000, 2076, 2152, 2228, 2304, 2380, 2456, 2532, 2608, 2684, 2760, 2836, 2912, 2988, 3064
Since 76 is added fourteen times it will take 14 years for the value to first exceed $3000.

5 a Apply the rule three times $V_0 = 7000$
$V_1 = V_0 + 518 = 7000 + 518 = 7518$
$V_2 = V_1 + 518 = 7518 + 518 = 8036$
$V_3 = V_2 + 518 = 8036 + 518 = 8554$
Thus, the value of the investment after 3 years is $8554.

b Enter 7000 onto your CAS. Press '+518 ENTER' until the value first exceeds 10 000. Count the number of times that you press ENTER.
7000, 7518, 8036, 8554, 9072, 9590, 10 108
Since 76 is added fourteen times it will take 6 years for the value to first exceed $10 000.

6 a **i** The principal of the investment is given by V_0 which is $15 000.

ii The amount of interest earned each year is the amount that is added to the investment each year which is $525.

iii $\frac{525}{15\,000} \times 100\% = 3.5\%$

b Enter 15 000 onto your CAS. Press '+525 ENTER' until the value first exceeds 30 000. Count the number of times that you press ENTER.
Alternatively, you could find how many times you need to add 525 to increase the value by 15 000 (30 000 – 15 000 = 15 000) by dividing 15 000 by 525 which gives 28.57. This means that it will take 29 years for the value to increase by 15 000 and first exceed $30 000. (Note: 28 years is not quite enough!)

7 a Substitute $r = 15$ and $C_0 = 82\,000$ (the original purchase price of the cherry picker) into the rule:
$$D = \frac{r}{100}C_0 = \frac{15}{100} \times 82\,000 = 12\,300$$
Hence, the cherry picker depreciates by $12 300 each year.

b Using the value of $C_0 = 82\,000$ and $D = 12\,3000$ and substituting into the recurrence relation gives:
$$C_0 = 82\,000, \quad C_{n+1} = C_n + 12\,300$$

8 a Substitute $r = 8$ and $W_0 = 2800$ (the original purchase price of the cherry picker) into the rule:
$$D = \frac{r}{100}W_0 = \frac{8}{100} \times 2800 = 224$$
Hence, Wendy's dental chair will depreciate by $224 each year.

b Since the initial value of the chair is $2800, $W_0 = 2800$.

c Using the value of $W_0 = 82\,000$ and $D = 224$ and substituting into the recurrence relation gives:
$$W_0 = 2800, \quad W_{n+1} = W_n + 224$$

9 a Apply the rule three times $V_0 = 2500$
$V_1 = V_0 - 400 = 2500 - 400 = 2100$
$V_2 = V_1 - 400 = 2100 - 400 = 1700$

$V_3 = V_2 - 400 = 1700 - 400 = 1300$
Thus, the value of the investment after 1, 2 and 3 years respectively is \$2100, \$1700 and \$1300.

b Enter 2500 onto your CAS. Press '-400 ENTER' until the value first goes below 1000. Count the number of times that you press ENTER.
2500, 2100, 1700, 1300, 1000
Thus the answer is 4 since 400 is subtracted 4 times to reach \$1000.

10 a Apply the rule three times:
$V_0 = 23\,000$
$V_1 = V_0 - 805 = 23\,000 - 805 = 22\,195$
$V_2 = V_1 - 805 = 22\,195 - 805 = 21\,390$
$V_3 = V_2 - 805 = 21\,390 - 805 = 20\,585$
Thus, the value of the investment after 1, 2 and 3 years respectively is \$22 195, \$21 390 and \$20 585.

b Enter 23 000 onto your CAS. Press '-805 ENTER' until the value first goes below 10 000. Count the number of times that you press ENTER.
23 000, 22 195, 21 390, 20 585,
19 780, 18 975, 18 170, 17 365,
16 560, 15 755, 14 950, 14 145,
13 340, 12 535, 11 730, 10 925,
10 120, 9315
Thus the answer is 17 since 805 is subtracted 17 times to first get below \$10 000.

11 a i The television was purchased for the starting value given by $V_0 = 1500$ so it initially cost 1500.

ii The value of the television declined by \$102 each year so annual depreciation is \$102.

iii To find the annual percentage depreciation we calculate

$$\frac{102}{1500} \times 100 = 6.8\%$$

b Enter 1500 onto your CAS. Press '-102 ENTER' 8 times to find the value after 8 years. This gives \$684.

c Enter 1500 onto your CAS. Press '-102 ENTER' until the value first goes below 100. Count the number of times that 102 was subtracted. Since 102 was subtracted 14 times (to give 72), it takes 14 years for the television to first be below 100.

12 a The starting price of the minibus is given as \$32 600 so $M_0 = 32\,000$.

b With a starting value of \$32 000 (so so $M_0 = 32\,000$) and depreciation of \$10 per trip (so $D = 100$, the recurrence relation can be written as

$$M_0 = 32\,000, \quad M_{n+1} = M_n - 10$$

13 **a** Apply the rule to find the first four terms:
$V_0 = 450$
$V_1 = V_0 - 0.05 = 450 - 0.05 = 449.95$
$V_2 = V_1 - 0.05 = 449.95 - 0.05 = 449.90$
$V_3 = V_2 - 0.05 = 449.90 - 0.05 = 449.85$
$V_4 = V_3 - 0.05 = 449.85 - 0.05 = 449.80$
Thus, the first five terms are 450, 449.95, 449.90, 449.85, 449.80.

b Enter 450 onto your CAS. Press '-0.05 ENTER' twenty times to find that the value of the printer after printing 20 pages is $449.

14 **a** $V_0 = 48\ 000$
$V_1 = V_0 - 200 = 48\ 000 - 200 = 47\ 800$
$V_2 = V_1 - 200 = 47\ 800 - 200 = 47\ 600$
$V_3 = V_2 - 200 = 47\ 600 - 200 = 47\ 400$
Thus, the value after 1000, 2000 and 3000 respectively is $47 800, $47 600 and $47 400.

b To find the value of the van after 15 000 km, we need to subtract $200 from $48 000 15 times. Calculating: $48\ 000 - 15 \times 200 = 45\ 000$. Thus, the van is worth $45 000 after 15 000 km.

c There are several ways to solve this. First, you can use your CAS to enter in the original value (48 000) then press '−200 ENTER' until the value is first less than $43 000. A second method is to use your CAS to solve $43\ 000 = 45\ 000 - 200 \times n$ and solve for n which gives $n = 25$. A third method is to note that $48\ 000 - 43\ 000 = 5\ 000$ so the van has declined by $5000. Given it loses $200 per 1000 km, it must lose $200 on $5000 \div 200 = 25$ times.
All of these methods reveal that 200 is deducted 25 times so the van must have travelled $10 \times 1000 = 25\ 000$ km.

15 **a** The blender depreciates by $0.02 per juice and in total, depreciates by $144 per year. Thus, the number of juices produced in a year is:

$$\frac{144}{0.02} = 7200$$

b First we calculate how long it takes for the blender to reach a value of 0. This is done using the CAS to find that it takes 10 years. Second, since 7200 juices are made each year for ten years, then in total, $7200 \times 10 = 72\ 000$ juices are made with the blender before the blender reaches zero.

c First calculate how much the blender depreciates by after 36 000 juices:

$$36\ 000 \times 0.02 = \$720$$

$1440-$720=$720

d Since the blender depreciates by $144 per year, we calculate $\frac{144}{1440} \times 100 = 10\%$

16 The coffee machine depreciated by 720 − 586 = $134 in five years. Since 670 coffees are made each year (on average), then 3350 coffees were made over the five years. This means that the coffee machine depreciated by $134 ÷ 3350 = 0.04 cents per coffee. Thus, the answer is **C**.

17 Since the Tandoori oven depreciates by $405 each year and the initial value of the oven was $4500, then the depreciation rate is:

$$\frac{405}{4500} \times 100 = 9\%$$

Thus, the answer is **C**.

18 The initial value of the bike is $5500 so $V_0 = 5500$. Calculating 10% of the purchase price gives:

$$10\% \times 5500 = \$550$$

this means that 550 is subtracted so $V_{n+1} = V_n - 550$. Thus, the answer is **B**.

Solutions to Exercise 7C

1 **a** The starting term is 4 and then 2 is added in each iteration. This means that after n iterations, 2n is added to the starting value. Thus, $A_n = 4 + 2n$.
Substituting in $n = 20$ gives:
$A_{20} = 4 + 2 \times 20 = 44$.

b The starting term is 10 and then 3 is subtracted in each iteration. This means that after n iterations, 3n is subtracted from the starting value.
Thus, $A_n = 10 - 3n$.
Substituting in $n = 20$ gives:
$A_{20} = 10 - 3 \times 20 = -50$.

c The starting term is 5 and then 8 is added in each iteration. This means that after n iterations, 8n is added to the starting value. Thus, $A_n = 5 + 8n$.
Substituting in $n = 20$ gives:
$A_{20} = 5 + 8 \times 20 = 165$.

d The starting term is 300 and then 18 is subtracted in each iteration. This means that after n iterations, 18n is subtracted from the starting value.
Thus, $A_n = 300 - 18n$.
Substituting in $n = 20$ gives:
$A_{20} = 300 - 18 \times 20 = -60$.

2 **a** The starting value is $5000 so $V_0 = 5000$.

b An annual interest rate of 5.4% means that $270 is charged each year in simple interest.

c Given the starting value, $V_0 = 5000$. Since $270 is added each year, then the value of the investment after n years has increased by $270n$. Thus, $V_n = 5000 + 270n$.

d Here, $n = 9$ so $V_9 = 5000 + 270 \times 9 = 7430$ so the value after 9 years is $7430.

3 **a** Since Anthony borrows $12 000, then this is the starting value so $V_0 = 12\,000$.

b An annual interest rate of 7.2% means that $12\,000 \times 7.2\% = \$864$ interest is charged each year.

c With a charge of $864 each year, then after n years, $864n$ is added to the balance of the loan. Thus the value of the loan after n years is $V_n = 12\,000 + 864n$.

d Here, $n = 9$ so $V_9 =$
$12\,000 + 864 \times 9 = 19\,776$ so the value after 9 years is $19 776.

4 **a** The principal of a loan is the starting value. By reading from the rule, the starting value is $8000.

b The value of the loan increases by 512 each year so the interest charged each year is $512.

c **i** The value of the loan after 12 years is given when $n = 12$ so $V_{12} = 8000 + 512 \times 12 = 14\,144$.

ii Double the value of the loan is $16 000. To find when this first happens, we solve $16\,000 = 8000 + 512 \times n$ for n on the CAS. This gives $n = 15.625$ so it will take 16 years for the value of the loan to double.

5 a The principal of a investment is the starting value. By reading from the rule, the starting value is $2000.

b The value of the investment increases by 70 each year so the interest earned each year is $70.

c i The value of the investment after 6 years is given when $n = 6$ so $V_6 = 2000 + 70 \times 6 = 2420$.

ii Double the value of the investment is $4000. To find when this first happens, we solve $4000 = 2000 + 70 \times n$ for n on the CAS. This gives $n = 28.57...$ so it will take 29 years for the value of the loan to double.

6 a The starting value is the purchase price of the computer $5600.

b The computer depreciates by 22.5% per year. This means that the computer loses $5600 \times 22.5\% = \$1260$ each year.

c Since the computer's value declines by $1260 each year then the value after n years declines by $1260n$ so it is $V_n = 5600 - 1260n$.

d The value of the computer after 3 years is found when $n = 3$ which is given by $V_3 = 5600 - 1260 \times 3 = 1820$. Thus, the value is $1820.

7 a The starting value is the initial cost of the machine $7000.

b The machine depreciates by 17.5% per year. This means that the machine loses $7000 \times 17.5\% = \$1225$ each year.

c Since the machine's value declines by $1225 each year then the value after n years declines by $1225n$ so it is $V_n = 7000 - 1225n$.

d On your CAS, solve $0 = 7000 - 1225n$ for n gives $n = 5.714...$ means that it will take 5 full years (plus there will be a bit left over).

8 a Reading from the rule, the purchase price of the sewing machine is $1700.

b The value goes down by 212.5 so the value of the sewing machine must be decreasing by $212.50 each year.

c Using the rule with $n = 4$, $V_4 = 1700 - 212.50 \times 4 = 850$ so the value after 4 years is $850.

d Using the rule with $n = 7$, $V_7 = 1700 - 212.50 \times 7 = 850$ so the value after 7 years is $212.50.

e On your CAS, solve $0 = 1700 - 212.5n$ for n gives $n = 8$ means that it will take 8 years for the sewing machine to reach a value of 0.

9 a From the rule, we can read off the starting value as 65 000. This means that the harvester was purchased for $65 000.

b From the rule, the value declines by 3250 each iteration. This means that the harvester is depreciated by $3250 each year.

c The annual percentage depreciation for the harvester is calculated as:

$$\frac{3250}{65\,000} \times 100 = 5\%$$

d To find the value of the harvester after 7 years, substitute $n = 7$ into the rule: $V_7 = 65\,000 - 3250 \times 7 = 42\,250$.

e To find when the harvester will reach a value of $29 250, solve $29\,250 = 65\,000 - 3250 \times n$ for n to give $n = 10.923....$ This means that it will take 11 years for the value to first decline to $29250.

10 a From the rule, we can read off the starting value as 29 000. This means that the taxi was purchased for $29 000.

b From the rule, the value declines by 0.25 each iteration. This means that the taxi declines by $0.25 (25 cents) per kilometer.

c To find the value of the taxi after 20 000 kilometers of travel, substitute $n = 20\,000$ into the rule: $V_{20\,000} = 29\,000 - 0.25 \times 20\,000 = 24\,000$. Thus, the taxi is valued at $24 000.

d To find when the taxi will reach a value of $5000, solve $5000 = 29\,000 - 0.25n$ for n to give $n = 96\,000$. This means that the taxi will be worth $5000 after it has travelled 96 000 kilometers.

11 a Over the first year, the value of the car declines by $35\,400 - 25\,700 = 9700$ Thus the total depreciation in one year is $9700.

b Since the car travels 25 000 kilometers in the first year and declines in value by $9700, the rate of depreciation per kilometer is $9700 \div 25\,000 = \$0.388$.

c Since the car declines by $0.388 per kilometer and it starts with a value of $35 400, the rule for the value after n km is $V_n = 35\,400 - 0.388n$.

d Using CAS, solve $6688 = 35\,400 - 0.388n$ for n to give $n = 74\,000$. This means that the car has travelled 74 000 kilometers before it reaches a value of $6688.

12 a i The printing machine declines by $110\,000 - 2500 = 107\,500$ after printing 4 million

pages so it declines by
$107\ 500 \div 4\ 000\ 000 = 0.026875$ per page.

ii The machine declines by $0.026875 \times 1\ 500\ 000 =$ \$40 312.50 after printing 1.5 million pages. This means that the value is $110\ 000 - 40\ 312.50 =$ \$69 687.50 after printing 1.5 million pages.

iii The machine declines by $0.026875 \times 750\ 000 =$ \$20 156.25 after 750 000 pages. Thus, if the machine prints 750 000 pages each year then the annual depreciation is \$20 156.25

b If the machine prints 750 000 pages each year then it depreciates by \$20 156.25 each year so after 5 years, the value would be $110\ 000 - 20\ 156.25 \times 5 = 9218.75$

c First, set up a rule using the starting value of \$110 000 and the depreciation per page of \$0.026875. Thus, the rule is $V_n = 110\ 000 - 0.026875 \times n$. Now use the CAS to solve $70\ 053 = 110\ 000 - 0.026875 \times n$ for n to get $n = 1\ 486\ 400$ so the machine will have printed 1 486 400 pages when the machine is valued at \$70 053.

13 If the bike is depreciating by 10% each year, then the bike is valued at 90% of the previous year. Thus
$V_0 = 3800$
$V_1 = 0.9 \times V_0 = 0.9 \times 3800 = 3420$
$V_2 = 0.9 \times V_1 = 0.9 \times 3420 = 3078$
$V_3 = 0.9 \times V_2 = 0.9 \times 3078 = 2770.20$
$V_4 = 0.9 \times V_3 = 0.9 \times 2770.20 = 2493.18$
Thus, the answer is **D**.

14 Here, the starting value is \$4280 so $V_0 = 4280$. The value is increasing by 25 for each iteration so $V_{n+1} = V_n + 25$. Hence, the answer is **C**.

Solutions to Exercise 7D

1 **a** Use the recursion relation:

$V_0 = 2$

$V_1 = 2 \times V_0 = 2 \times 2 = 4$

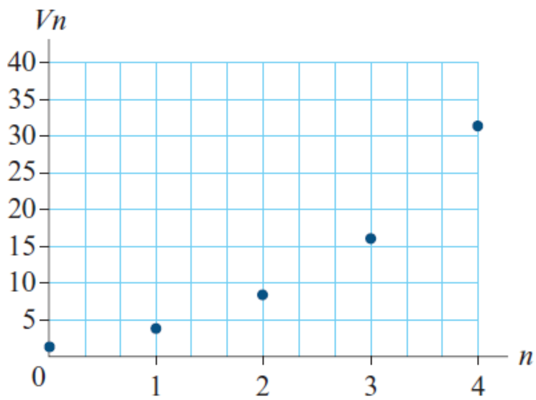
$V_2 = 2 \times V_1 = 2 \times 4 = 8$

$V_3 = 2 \times V_2 = 2 \times 6 = 16$

$V_4 = 2 \times V_3 = 2 \times 8 = 32$

Thus, the first five terms are 2, 4, 8, 16 and 32.

These can be graphed as points on the axis n and V_n.



b Use the recursion relation:

$V_0 = 3$

$V_1 = 3 \times V_0 = 3 \times 3 = 9$

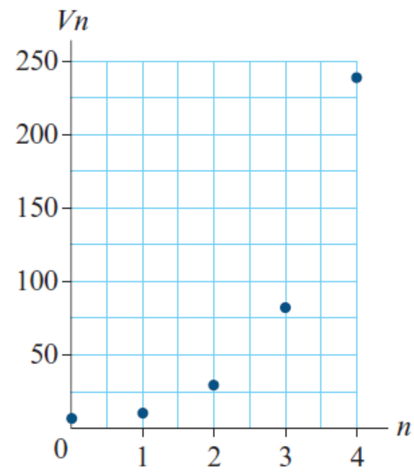
$V_2 = 3 \times V_1 = 3 \times 9 = 27$

$V_3 = 3 \times V_2 = 3 \times 27 = 81$

$V_4 = 3 \times V_3 = 3 \times 81 = 243$

Thus, the first five terms are 3, 9, 27, 81 and 243.

These can be graphed as points on the axis n and V_n.



c Use the recursion relation:

$V_0 = 100$

$V_1 = 0.1 \times V_0 = 0.1 \times 100 = 10$

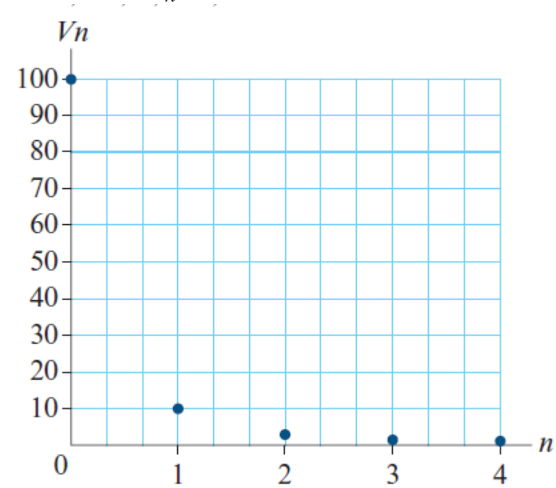
$V_2 = 0.1 \times V_1 = 0.1 \times 10 = 1$

$V_3 = 0.1 \times V_2 = 0.1 \times 1 = 0.1$

$V_4 = 0.1 \times V_3 = 0.1 \times 0.1 = 0.01$

Thus, the first five terms are 100, 10, 1, 0.1 and 0.01.

These can be graphed as points on the axis n and V_n.



2 a Using the recurrence relation:
$V_0 = 6000$
$V_1 = 1.042 \times V_0 = 1.042 \times 6000 = 6252$
$V_2 = 1.042 \times V_1 = 1.042 \times 6252 = 6514.58$ (round currency to 2 decimal places)
$V_3 = 1.042 \times V_2 = 1.042 \times 6514.58 = 6788.20$

b Enter 6000 on the CAS. Press '×1.042 ENTER' until you first reach a value greater than 8000: 6000, 6252, 6514.58, 6788.20, 7073.30, 7370.38, 7679.94, 8002.49. Since you multiply by 1.042 seven times, it will take seven years for the investment to reach $8000.

3 a Using the recurrence relation:
$V_0 = 20\,000$
$V_1 = 1.063 \times V_0 = 1.063 \times 20\,000 = 21\,260$
$V_2 = 1.063 \times V_1 = 1.063 \times 21\,260 = 22\,599.38$
$V_3 = 1.063 \times V_2 = 1.063 \times 22\,599.38 = 24\,023.14$

b Enter 20 000 on the CAS. Press '×1.063 ENTER' until you first reach a value greater than 30 000. Since you multiply by 1.063 seven times, it will take seven years for the investment to reach $30 000.

4 a Since Sue invests $5000, the starting value is $V_0 = 5000$.

b Substituting $r = 6.8$ gives:

$$R = 1 + \frac{6.8}{100} = 1.068$$

c Substituting into the general form from parts **a** and **b** gives $V_0 = 5000$ and $V_{n+1} = 1.068V_n$.

d Using the recurrence relation:
$V_0 = 5000$
$V_1 = 1.068 \times V_0 = 1.068 \times 5000 = 5340$
$V_2 = 1.068 \times V_1 = 1.068 \times 5340 = 5603.12$
$V_3 = 1.068 \times V_2 = 1.068 \times 5603.12 = 6090.93$
$V_4 = 1.068 \times V_3 = 1.068 \times 6090.93 = 6505.12$
$V_5 = 1.068 \times V_4 = 1.068 \times 6505.12 = 6947.46$
Thus, the value of the investment after 5 years is $6947.46.

e The amount of interest earned from the investment is $6947.46 - 5000 = \$1947.46$

5 a The starting value of the loan is $18 000 which is the principal.

b Using $r = 9.4$, we can find the value of R:

$$R = 1 + \frac{9.4}{100} = 1.094$$

c Using the starting value of 18 000 and $R = 1.094$, the recurrence relation is $V_0 = 18\,000$ (the initial value is the principal) and $V_{n+1} = 1.094V_n$ (the next value is found by multiplying the current value by 1.094).

d Using the recurrence relation:
$V_0 = 18\ 000$
$V_1 = 1.094 \times V_0 = 1.094 \times 18\ 000 = 19\ 692$
$V_2 = 1.094 \times V_1 = 1.094 \times 19\ 692 = 21\ 543.05$
$V_3 = 1.094 \times V_2 = 1.094 \times 21\ 543.05 = 23\ 568.09$
$V_4 = 1.094 \times V_3 = 1.094 \times 23\ 568.09 = 25\ 783.50$
This means that the loan has a value of \$25 783.50 after four years.

e To find when the loan will first be valued at more than \$25 000 we can look at **d** to see how many times the iteration needs to be applied. We can see that V_3 is less than \$25 000 while V_4 is above \$25 000 so it will take 4 iterations or 4 years to have a value greater than \$25 000.

6 Since the motorcylce was purchased for \$9000, $V_0 = 9000$. The annual depreciation rate is 3.5% so

$$R = 1 - \frac{3.5}{100} = 0.965$$

Substituting R into $V_{n+1} = R \times V_n$ gives $V_{n+1} = 0.965V_n$.

7 Since the minibus was initially valued at \$28 600, $M_0 = 28\ 600$. The annual depreciation rate is 7.4% so

$$R = 1 - \frac{7.4}{100} = 0.926$$

Substituting R into $M_{n+1} = R \times M_n$ gives $M_{n+1} = 0.926M_n$.

8 a Since the furniture was purchased for \$18 000, $V_0 = 18\ 000$. Depreciation of 4.5% means that the furniture next period will be worth $1 - 4.5\% = 95.5\%$ so $V_{n+1} = 0.955V_n$.

b Using the recurrence relation and substituting the values of n:
$V_0 = 18\ 000$
$V_1 = 0.955 \times V_0 = 0.955 \times 18\ 000 = 17\ 190$
$V_2 = 0.955 \times V_1 = 0.955 \times 17\ 190 = 16\ 416.45$
$V_3 = 0.955 \times V_2 = 0.955 \times 16\ 416.45 = 15\ 677.71$
$V_4 = 0.955 \times V_3 = 0.955 \times 15\ 677.71 = 14\ 972.21$
$V_5 = 0.955 \times V_4 = 0.955 \times 14\ 972.21 = 14\ 298.46$

c From **d**, the value of the furniture after 3 years is V_3 which is \$15 677.71.

d The total depreciation after five years is 18 000 − 14 298.46 = \$3701.54

9 a Since the wedding dress was purchased \$4000, $W_0 = 4000$. The depreciation rate of 4.1% means that the value of the dress next year is only worth 1 − 4.1% = 95.9% of the current value. Hence $W_{n+1} = 0.959W_n$.

b Using the recurrence relation:
$W_0 = 4000$
$W_1 = 0.959 \times W_0 = 0.959 \times 4000 = 3836$
$W_2 = 0.959 \times W_1 = 0.959 \times 3836 = 3678.72$
$W_3 = 0.959 \times W_2 = 0.959 \times 3678.72 = 3527.90$

c Continuing with the recurrence relation, $W_4 = 3383.25$ and $W_5 = 3244.54$ This means that the depreciation is $4000 - 3244.54 = 755.46$.

10 a Since the computer server was purchased for \$13 420, then $S_0 = 13\,420$. The depreciation rate of 11.2% means that the value of the server next year is only worth $1 - 11.2\% = 88.8\%$ of the current value. Hence $S_{n+1} = 0.888S_n$.

b Using the recurrence relation and substituting the values of n:
$S_0 = 13\,420$
$S_1 = 0.888 \times S_0 = 0.888 \times 13\,420 = 11\,916.96$
$S_2 = 0.888 \times S_1 = 0.888 \times 11\,916.96 = 10\,582.26$
$S_3 = 0.888 \times S_2 = 0.888 \times 10\,582.26 = 9397.05$
$S_4 = 0.888 \times S_3 = 0.888 \times 9397.05 = 8344.58$
$S_5 = 0.888 \times S_4 = 0.888 \times 8344.58 = 7409.99$

c From **c**, the value of the server after 5 years is S_5 which is \$7409.99.

d The depreciation in the third year is $S_2 - S_3 = 10\,582.26 - 9397.05 =$ \$1185.21

11 Since the next value is calculating by multiplying the current value by 1.02, then $R = 1.02$ so

$$1.02 = 1 + \frac{r}{100}$$

means that $r = 2$. Thus, the interest rate is 2%. The answer is **C**.

12 Applying the recurrence relation:
$R_0 = 6000$
$R_1 = 1.0384 \times 6000 = 6230.40$
$R_2 = 1.0384 \times 6230.40 = 6469.65$
$R_3 = 1.0384 \times 6469.65 = 6718.08$
Thus, the value after three years is \$6718.08. Since the starting value was \$6000, he investment has earned \$6718.08 − \$6000 = \$718.08 in three years. The answer is **E**.

13 Enter the initial value of 28 000 onto the calculator. Since the investment is compounding at a rate of 6.2% per year, type '×1.062 ENTER' until the value first exceeds \$56 000 (that is, double 28 000). Since you press ENTER 12 times, it will take 12 years. Thus, the answer is **E**.

14 Applying the recurrence relation:
$J_0 = 18\,000$
$J_1 = 0.9 \times 18\,000 = 16\,200$
$J_2 = 0.9 \times 16\,200 = 14580$
$J_3 = 0.9 \times 14\,580 = 13\,122$
This means that in the third year, the jetski depreciates by $14\,508 - 13\,122 = 1458$. Thus, the answer is **C**.

Solutions to Exercise 7E

1 a Since the multiplier is 2, we multiply the initial value (of 6) by 2^n so $V_n = 2^n \times 6$.
Substituting in $n = 4$ gives
$V_n = 2^4 \times 6 = 96$.

b Since the multiplier is 3 we multiply the initial value (of 10) by 3^n so $V_n = 3^n \times 10$.
Substituting in $n = 4$ gives
$V_n = 3^4 \times 10 = 810$.

c Since the multiplier is 0.5, we multiply the initial value (of 1) by 0.5^n so $V_n = 0.5^n \times 1$.
Substituting in $n = 4$ gives
$V_n = 0.5^4 \times 1 = 0.625$.

d Since the multiplier is 0.25, we multiply the initial value (of 80) by 0.25^n so $V_n = 0.25^n \times 80$.
Substituting in $n = 4$ gives
$V_n = 0.25^4 \times 80 = 0.3125$.

2 a i The amount of money invested is given by V_0 as \$3000.

ii To find the next value, we multiply the current value by 1.1. This is the same as 110% which means that the value increases by 10%.

b Applying the rule:
V_0
$V_1 = 1.1 \times V_0$
$V_2 = 1.1 \times V_1 = 1.1 \times (1.1 \times V_0) = 1.1^2 \times V_0$
$V_3 = 1.1 \times V_2 = 1.1 \times (1.1^2 \times V_0) = 1.1^3 \times V_0$
Thus, $V_n = 1.1^n \times V_0$ or
$V_n = 1.1^n \times 3000$.

c Using the rule when $n = 5$, $V_5 = 1.1^5 \times V_0 = 1.1^5 \times 3000 = \$4831.53$

3 a i The amount of money borrowed is given by V_0 as \$2000.

ii To find the next value, we multiply the current value by 1.06. This is the same as 106% which means that the value increases by 6%.

b Applying the rule:
$V_1 = 1.06 \times V_0$
$V_2 = 1.06 \times V_1 = 1.06 \times (1.06 \times V_0) = 1.06^2 \times V_0$
$V_3 = 1.06 \times V_2 = 1.06 \times (1.06^2 \times V_0) = 1.06^3 \times V_0$
Thus, $V_n = 1.06^n \times V_0$ or
$V_n = 1.06^n \times 2000$.

c Using the rule when $n = 4$, $V_4 = 1.06^4 \times V_0 = 1.06^4 \times 2000 = \$2524.95$

d First find the value of the loan after 6 years by substituting $n = 6$ into the rule: $V_6 = 1.06^6 \times 2000 = \$2837.04$
Since the principal of the loan was only \$2000, then the total interest paid over the six years is $2837.04 - 2000 = \$837.04$.

4 a Since Pacey invests \$8000 then $V_0 = 8000$. The account earns 12.5% compounding each year so the value of the next year is found by multiplying the current term by:

$$R = 1 + \frac{12.5}{100} = 1.125$$

Thus, $V_{n+1} = 1.125V_n$. The rule is then $V_n = 1.125 \times V_0$ or $V_n = 1.125^n \times 8000$.

b Subsitute $n = 3$ $V_3 = 1.125 \times 8000 =$ \$11 390.63

c The interest earned is found by comparing V_3 to the original investment: 11 390.63 − 8000 = \$3390.63.

d The interest earned in the third year is the difference between V_3 and V_2. $V_2 = 10\,125$ so 11 390.63 − 10 125 = 1265.63,

5 a i The purchase price is given by V_0 as \$1200.

ii To find the next value, we multiply the current value by 0.88. This is the same as 88% which means that the value decreases by 12%.

b Applying the rule:
$V_1 = 0.88 \times V_0$
$V_2 = 0.88 \times V_1 = 0.88 \times (0.88 \times V_0) = 0.88^2 \times V_0$
$V_3 = 0.88 \times V_2 = 0.88 \times (0.88^2 \times V_0) = 0.88^3 \times V_0$
Thus, $V_n = 0.88^n \times V_0$ or
$V_n = 0.88^n \times 1200$.

c Using the rule when $n = 7$, $V_7 = 0.88^7 \times V_0 = 0.88^7 \times 1200 =$ \$490.41

6 a Since the car was purchased for \$38 500 then $V_0 = 38\,500$. The car depreciates by 9.5% compounding each year so the value of the next year is found by multiplying the current term by:

$$R = 1 - \frac{9.5}{100} = 0.905$$

Thus, $V_{n+1} = 0.905V_n$. The rule is then $V_n = 0.905^n \times V_0$ or $V_n = 0.905^n \times 38\,500$.

b Subsitute $n = 5$ $V_5 = 0.905 \times 38500 =$ \$23 372.42

c The total depreciation is found by comparing V_5 to the original investment: 23 372.42 − 38 500 = \$15 127.58.

7 Since the investment was \$3500 then $V_0 = 3500$. An interest rate of 6.75% means that $R = 1.0675$ so the rule is $V_n = 1.0675^n \times 3500$.
To find how long it takes for the investment to reach \$5179.35, use your CAS to solve $5179.35 = 1.0675^n \times 3500$ for n gives $n = 5.9999999$ so it will take 6 years.

8 Since the investment was \$200 then $V_0 = 200$. An interest rate of 4.75% means that $R = 1.0475$ so the rule is $V_n = 1.0475^n \times 200$.
To find how long it takes for the investment to reach \$20 000, use your CAS to solve $20\,000 = 1.0475^n \times 200$ for n gives $n = 99.2357..$ so it will take 100 years.

9 Since the investment was \$1000 then $V_0 = 1000$. With an interest rate of $r\%$ the rule is

$$V_n = (1 + \frac{r}{100})^n \times 1000$$

After 12 years, the value is \$1601.03 so when $n = 12$, $V_{12} = 1601.03$. Substituting in gives:

$$1601.03 = (1 + \frac{r}{100})^{12} \times 1000$$

which can be solved on the CAS for r which gives $r = 3.9999...$ so the interest rate is 4%.
Note that r must be positive so reject the other solution that your CAS gives you.

10 Since the value of the car was \$8000 then $V_0 = 8000$. With a depreciation rate of $r\%$ the rule is

$$V_n = (1 - \frac{r}{100})^n \times 8000$$

After 3 years, the value is \$6645 so when $n = 3$, $V_3 = 6645$. Substituting in gives:

$$6645 = (1 - \frac{r}{100})^3 \times 8000$$

which can be solved on the CAS for r which gives $r = 5.998...$ so the interest rate is 6%.

11 Since the interest rate is 6.8% the rule is $V_n = 1.068^n \times V_0$.
After 4 years, the value is \$12 000 so when $n = 4$, $V_4 = 12\,000$. Substituting in gives $12\,000 = 1.068^4 \times V_0$ which can be solved on the CAS for V_0 which gives $V_0 = 9223.51$ so the initial investment is \$9223.51.

12 Since the depreciation rate is 8.2% the rule is $V_n = 0.918^n \times V_0$.
After 10 years, the value is \$13 770 so when $n = 10$, $V_4 = 13\,770$. Substituting in gives $13\,770 = 0.918^{10} \times V_0$ which can be solved on the CAS for V_0 which gives $V_0 = 32\,397.17$ so the initial investment is \$32 397.17

13 Substitute $V_0 = 68\,000$, $n = 5$ and $V_5 = 37\,971.60$ into the rule

$$V_n = (1 - \frac{r}{100})^n \times V_0$$

to give

$$37\,971.60 = \left(1 - \frac{r}{100}\right)^5 \times 68\,000$$

and solve on the CAS for r to give $r = 11$ so the depreciation rate is 11%.
Thus, the answer is **C**.

14 Substitute $V_0 = 15\,000$, $r = 5.8$ and $V_n = 30\,000$ (double the initial value) into

$$V_n = (1 + \frac{r}{100})^n \times V_0$$

to give $30\,000 = 1.058^n \times 15\,000$ which is solved on the CAS for n to give $n = 12.29$ This means that it will take at least 13 years so the answer is **E**.

Solutions to Exercise 7F

1 a Since there are 12 months in a year, 4.8% annually is equivalent to

$$\frac{4.8}{12} = 0.4\%$$

per month.

b Since there are 4 quarters in a year, 8.3% annually is equivalent to

$$\frac{8.3}{4} = 2.08\%$$

per month.

c Since there are 26 fortnights in a year, 10.4% annually is equivalent to

$$\frac{10.4}{26} = 0.4\%$$

per fortnight.

d Since there are 52 weeks in a year, 7.4% annually is equivalent to

$$\frac{7.4}{52} = 0.14\%$$

per week (correct to two decimal places).

a Since there are 365 days in a year, 12.7% annually is equivalent to

$$\frac{12.7}{365} = 0.03\%$$

per day (correct to two decimal places).

2 a Since there are 12 months in a year, 0.54% monthly is equivalent to $0.54 \times 12 = 6.48\%$ annually.

b Since there are 4 quarters in a year, 1.45% quarterly is equivalent to $5.8 \times 4 = 5.8\%$ annually.

c Since there are 26 fortnights in a year, 0.57% fortnightly is equivalent to $0.57 \times 26 = 14.82\%$ annually.

d Since there are 52 weeks in a year, 0.19% weekly is equivalent to $0.19 \times 52 = 9.88\%$ annually.

e Since there are 365 days in a year, 0.022% daily is equivalent to $0.022 \times 365 = 8.03\%$ annually.

3 a Since the initial loan was for \$8000, $V_0 = 8000$. With interest compounding yearly, the compounding period is yearly so $R = 1.048$ and $V_{n+1} = 1.048 \times V_n$.

b Since the initial loan was for \$8000, $V_0 = 8000$. With interest compounding quarterly, there are four compounding periods each year so

$$R = 1 + \frac{4.8}{100 \times 4} = 1.012$$

and $V_{n+1} = 1.012 \times V_n$.

c Since the initial loan was for \$8000, $V_0 = 8000$. With interest compounding monthly, there are twelve compounding periods each year so

$$R = 1 + \frac{4.8}{100 \times 12} = 1.004$$

and $V_{n+1} = 1.004 \times V_n$.

4 a Since the principal is \$20 000, then $V_0 = 20\,000$. With interest compounding monthly, there are twelve compounding periods each year so

$$R = 1 + \frac{6}{100 \times 12} = 1.005$$

and $V_{n+1} = 1.005 \times V_n$.

b Using the recurrence relation and the formula $V_n = R^n \times V_0$, then $V_n = 1.005^n \times 20\,000$.

c There are 60 months in 5 years so $n = 60$. Substituting into the rule, $V_{60} = 1.005^{60} \times 20\,000 = \$26977.00$

5 a Since the principal is \$8000, then $V_0 = 8000$. With interest compounding quarterly, there are four compounding periods each year so

$$R = 1 + \frac{4.8}{100 \times 4} = 1.012$$

and $V_{n+1} = 1.012 \times V_n$.

b Using the recurrence relation and the formula $V_n = R^n \times V_0$, then $V_n = 1.012^n \times 8000$.

c There are 12 quarters in 3 years so $n = 12$. Substituting into the rule, $V_{12} = 1.012^{12} \times 8000 = \$9231.16$

6 a Since the principal is \$7600, then $V_0 = 7600$. With interest compounding monthly, there are twelve compounding periods each year so

$$R = 1 + \frac{6}{100 \times 12} = 1.005$$

and $V_{n+1} = 1.005 \times V_n$.

b Using the recurrence relation and the formula $V_{n+1} = R^n \times V_0$, then $V_{n+1} = 1.005^n \times 7600$.

c There are 5 months, $n = 5$. Substituting into the rule, $V_5 = 1.005^5 \times 7600 = \$7791.91$

d To find when Wayne's investment first reaches \$15 200 (double \$7600), use your CAS to solve $15\,200 = 1.005^n \times 7600$ for n. This gives $n = 138.9757...$ so it will take 139 months for Wayne's investment to first double in value.

7 a Since the principal is \$3500, then $V_0 = 3500$. With interest compounding quarterly, there are four compounding periods each year so

$$R = 1 + \frac{8}{100 \times 4} = 1.02$$

and $V_{n+1} = 1.02 \times V_n$.

b Using the recurrence relation and the formula $V_n = R^n \times V_0$, then $V_n = 1.02^n \times 3500$.
There are 4 quarters in one year so $n = 4$. Substituting into the rule, $V_4 = 1.02^4 \times 3500 = \$3788.51$

8 a Use the formula for effective interest rates where $r = 4.6$ and $n = 4$:

$$r_{eff} = \left(\left(1 + \frac{4.6}{100 \times 4}\right)^4 - 1\right) \times 100\%.$$

That is, $r = 4.67996...$ so the effective interest rate is 4.68% (correct to two decimal places).

b Use the formula for effective interest rates where $r = 4.6$ and $n = 12$:

$$r_{eff} = \left(\left(1 + \frac{4.6}{100 \times 12}\right)^{12} - 1\right) \times 100\%$$

which is 4.69823... so the effective interest rate is 4.70% (correct to two decimal places).

c Since Brenda is investing, she wants the highest possible effective interest rate so she should choose the second option of compounding monthly.

9 a Use the formula for effective interest rates where $r = 7.94$ and $n = 26$:

$$r_{eff} = \left(\left(1 + \frac{7.94}{100 \times 26}\right)^{26} - 1\right) \times 100\%$$

which is equal to 8.2506... Thus, the effective interest rate is 8.25% (correct to two decimal places).

b Use the formula for effective interest rates where $r = 7.94$ and $n = 12$:

$$r_{eff} = \left(\left(1 + \frac{7.94}{100 \times 12}\right)^{12} - 1\right) \times 100\%$$

which is equal to 8.2354... Thus, the effective interest rate is 8.24% (correct to two decimal places).

c Since Stella is borrowing money, she wants the lowest possible effective interest rate so she should choose the second option of compounding monthly.

10 a Use the formula for effective interest rates where $r = 8.3$ and $n = 12$:

$$r_{eff} = \left(\left(1 + \frac{8.3}{100 \times 12}\right)^{12} - 1\right) \times 100\%$$

That is, 8.623... so the effective interest rate is 8.62% (correct to two decimal places).

Repeat for $r = 7.8$ and $n = 52$:

$$r_{eff} = \left(\left(1 + \frac{7.8}{100 \times 52}\right)^{52} - 1\right) \times 100\%$$

That is, 8.105... so the effective interest rate is 8.11% (correct to two decimal places).

b From **Option A:** $n = 12$ since there are 12 months in a year.

$$V_{12} = \left(1 + \frac{8.3}{100 \times 12}\right)^{1} 2 \times 35\,000$$

That is, 38 018.0990... so the value of the loan is \$38 018.10 from option A meaning that Luke owes 38 018.10 – 35 000 = \$3018.10 in interest.

From **Option B:** $n = 52$ since there are 52 months in a year.

$$V_{12} = \left(1 + \frac{7.8}{100 \times 52}\right)^{5} 2 \times 35\,000$$

That is, 37 837.0817... so the value of the loan is \$37 837.08 from option B meaning that Luke owes \$2837.08 in interest.

c Luke should choose Option B as this is a loan and so he will wish to pay the least amount possible.

11 a Use the formula for effective interest rates where $r = 5.3$ and $n = 12$:

$$r_{eff} = \left(\left(1 + \frac{5.3}{100 \times 12}\right)^{12} - 1\right) \times 100\%$$

This gives: 5.4306..... so the effective interest rate is 5.43% (correct to two decimal places).

Repeat for $r = 5.5$ and $n = 4$:

$$r_{eff} = \left(\left(1 + \frac{5.5}{100 \times 4}\right)^4 - 1\right) \times 100\%$$

This gives 5.61448... so the effective interest rate is 5.61% (correct to two decimal places).

b From **Option A:** $n = 12$ since there are 12 months in a year.

$$V_{12} = \left(1 + \frac{5.3}{100 \times 12}\right)^1 2 \times 140\,000$$

This gives 147 602.9243... so the investment is worth $147 602.92 from option A meaning interest of $7603 (to the nearest dollar).
From **Option B:** $n = 4$ since there are 4 quarters in a year.

$$V_4 = \left(1 + \frac{5.5}{100 \times 4}\right)^4 \times 140\,000$$

That is, 147 860.273285... so the investment is worth $147 860.27 from option B meaning interest of $7860.

c Sharon should choose Option B as this is an investment and so she will wish for the value to be as high as possible.

12 a Using your CAS calculator follow Example 32 where $r = 6.2$ and $n = 12$ since the investment is compounding monthly to give 6.38%.

b Using your CAS calculator follow Example 32 where $r = 8.4$ and $n = 365$ since the investment is compounding monthly to give 8.76%.

c Using your CAS calculator follow Example 32 where $r = 4.8$ and $n = 52$ since the investment is compounding monthly to give 4.91%.

d Using your CAS calculator follow Example 32 where $r = 12.5$ and $n = 4$ since the investment is compounding monthly to give 13.10%.

13 Use the function on your CAS with the interest rate of 7% and the compounding period of 12 to get 6.78%

14 Since Chung invests $3300, $V_0 = 3300$. To find the interest rate per compounding period, divide by 4 since there are 4 quarters in a year:

$$R = 1 + \frac{4.8}{100 \times 4} = 1.004$$

This means that the next value is obtained by multiplying the current value by 1.004 so $V_{n+1} = 1.004 \times V_n$.
Hence the answer is **E**.

15 Use the formula for effective interest rates where $r = 6.8$ and $n = 4$:

$$r_{eff} = \left(\left(1 + \frac{6.8}{100 \times 4}\right)^4 - 1\right) \times 100\%$$

This gives: 6.9753735521.... so the effective interest rate is 6.98% (correct to two decimal places).
Hence the answer is **C**.

16 Using your CAS solve:

$$5214.09 = \left(1 + \frac{r}{100 \times 12}\right)^{12} \times 5000$$

for r so $r = 4.1999999...$

To then find the effective interest rate, either use the formula or your CAS given the value of r and that $n = 12$ so the effective interest rate is 4.28%
Thus the answer is **E**.

17 Using your CAS solve

$$26\,253 = \left(1 + \frac{r}{100 \times 12}\right)^{12} \times 25\,000$$

for r so $r = 4.900423....$

To then find the effective interest rate, either use the formula or your CAS given the value of r and that $n = 12$.
This gives an effective interest rate of 5.0119999...
The difference between the interest rate per annum and the effective interest rate is
5.011999... − 4.900423... = 0.1157685...
Thus the answer is 0.112 given in **A**

Solutions to Review: Multiple-choice questions

1 $V_0 = 5$
$V_1 = V_0 - 3 = 5 - 3 = 2$
$V_2 = V_1 - 3 = 2 - 3 = -1$
$V_3 = V_2 - 3 = -1 - 3 = -4$
$V_4 = V_3 - 3 = -4 - 3 = -7$
Thus, the sequence generated is $5, 2, -1, -4, -7$. **C**

2 Enter 2 onto your CAS and then type '$\times 2 + 8$ four times. This gives $V_4 = 152$. **E**

3 Enter 5 onto your CAS and then type '$\times 3 - 6$ three times. This gives $V_3 = 57$. **D**

4 The starting value is 2 since Brian starts with 2 trees. Thus, $T_0 = 2$. Since Brian plants 3 more trees each month, 3 is added to the total number of trees. $T_{n+1} = T_n + 3$. **C**

5 Since the initial investment is \$1000, the graph starts at 1000 on the y-axis. The graph is earning \$5 interest each month so the graph should be a straight line with a gradient of 5 to show an increase of \$5 for each month. **A**

6 The total amount of depreciation is $18\,990 - 15\,990 = 3000$. As it travelled 20 000 kilometers, the depreciation per kilometer is $3000 \div 20\,000 = \$0.15$ **A**

7 Since Arthur's initial investment is \$2000, $V_0 = 2000$. Simple interest of 5.1% means that Arthur's investment will increase by $5.1\% \times 2000 = \$102$ is added each year. Thus, the next value is 102 more than the current value so $V_{n+1} = V_n + 102$. **D**

8 Converting 4.6% into quarterly, monthly, fortnightly and daily rates:
Quarterly: $4.6\% \div 4 = 1.15\%$
Monthly: $4.6\% \div 12 = 0.8333...\%$
Fortnightly: $4.6\% \div 26 = 0.176923...\%$
Weekly: $4.6\% \div 52 = 0.08846...\%$
Now compare these to the options given. **A**

9 After n iterations, 16 has been subtracted n times from the initial value of 40. **B**

10 Calculate 1098 as a percentage of the initial value:
$$\frac{1098}{18\,000} \times 100 = 6.1\%$$
Since n represents the year since the car was purchased, the car depreciates by 6.1% per annum. **A**

11 Depreciation each year is $8\% \times 2800 = \$224$ Thus, after four years the car has depreciated by $224 \times 4 = \$896$. **D**

12 An interest rate of 4.57% means that $R = 1.0457$ so the rule is $V_n = 1.0457^n \times 6000$. Then solve $8000 = 1.0457^n \times 6000$ for n so $n = 6.4377...$ so it takes 7 years to reach \$8000. **C**

13 Since the machine is depreciating by 8% every year, the value of the machine after one year is $1 - 8\% = 0.92$.

Thus, $P_{n+1} = 0.92 \times P_n$. **C**

14 Since the initial investment was \$50 000, the graph begins at a height above zero. This excludes option A. The line representing the investment is increasing at an increasing rate . **B**

15 From the recurrence relation, $V_0 = 4500$ and $R = 0.86$. Substituting into the form for the rule $V_n = R^n \times V_0$ gives $V_n = 0.86^n \times 4500$. **A**

16 Since the investment earned \$4000 in interest after 10 years, $V_{10} = 8000 + 4000 = 12\,000$ solving $12\,000 = R^{10} \times 8000$ for R gives $R = 1.0413797...$
To find the interest, solve

$$1.0413797 = 1 + \frac{r}{100}$$

for r to get $r = 4.13797$ which is closest to 4.14. **B**

17 12.6% annually compounding monthly has an R value of

$$R = 1 + \frac{12.6}{100 \times 12} = 1.0105$$

C

18 Using the formula where $r = 5.4$ and $n = 4$ gives

$$r_{eff} = \left(\left(1 + \frac{5.4}{100 \times 4}\right)^4 - 1\right) \times 100\%.$$

which is equal to 5.51033... **D**

19 Setting up the rule:
$V_n = 0.915^n \times 74\,500$ so
$V_5 = \$47\,781.716$.
Hence the car has depreciated by
$74\,500 - 47\,781.72 = \$26\,718.28$ **B**

20 First find R:

$$R = 1 + \frac{8.75}{100 \times 12} = 1.00729...$$

Now using the rule
$V_n = 1.00729...^n \times 6500$ and solving $13\,056 = 1.00729^n \times 6500$ for n gives $n = 96.01985...$ which is the equivalent of 8 years. **C**

Solutions to Review: Extended-response questions

1 **a** Since Jack borrows \$20 000, $V_0 = 20\,000$. Simple interest of 9.4% calculated annually means that the loan increases by 9.4% × 20 000 = \$1880 each year. Hence the next value is 1880 more than the current value so $V_{n+1} = V_n + 1880$.

b $V_0 = 20\,000$
$V_1 = V_0 + 1880 = 20\,000 + 1880 = 21\,880$
$V_2 = V_1 + 1880 = 21\,880 + 1880 = 23\,760$
$V_3 = V_2 + 1880 = 23\,760 + 1880 = 25\,640$
$V_4 = V_3 + 1880 = 25\,640 + 1880 = 27\,520$
$V_5 = V_4 + 1880 = 27\,520 + 1880 = 29\,400$
Thus Jack will need to pay back \$29 400 after five years.

c The initial value remains unchanged so $W_0 = 20\,000$. If the next value increases by 9.4%, we multiply the current value by 1.094. Thus, $W_{n+1} = 1.094 \times W_n$.

d As the rule is applied n times, this means that we are multiplying by 1.094 on n occassions to find V_n. Thus, $V_n = 1.094^n \times 20\,000$.

e The value of the loan after five years is given when $n = 5$ so $V_5 = 1.094^5 \times 20\,000 =$ \$31 341.27

2 Set up a rule where $V_0 = 300$ and

$$R = 1 + \frac{18}{100 \times 12} = 1.015$$

Thus the rule for the value after n months is $V_n = 1.015^n \times 300$. To find the value after 6 months, let $n = 6$ so $V_6 = 1.015^6 \times 300 =$ \$328.03

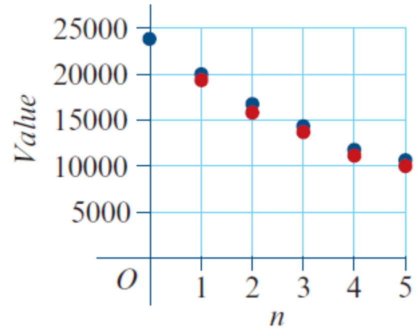
3 **a** **i** Kelly's car depreciates by 12% × 22 500 = 2700 each year and has a starting value of \$22 500 so $V_0 = 22\,500$ and $V_{n+1} = V_n - 2700$.

ii Since 5 years have passed since Kelly bought the car, we want to find V_5. This can be done by applying the rule 5 times: $V_5 = 9000$ so the car is currently worth \$9000.

b **i** Reducing balance of 16% means that 84% of the value is left next year. Thus, the recurrence relation is $V_0 = 22\,500$, $V_{n+1} = 0.84 \times V_n$.

ii Since 5 years have passed since Kelly bought the car, we want to find V_5. This can be done by applying the rule 5 times: $V_5 = 9409.77$ so the car is currently worth \$9409.77.

c Calculate the value of V_0, V_1, V_2, ... V_5 and plot on the same axis as shown.



4 a Since the vacuum cleaner value decreases by $10 for every 50 offices, it decreases by $10 \div 50 = 0.2$ for each office it cleans. That is, it depreciates by $0.20 per clean.

b The initial value of the cleaner is $650 so $V_0 = 650$. The value declines by 0.2 for each office so $V_{n+1} = V_n - 0.2$.

c The number of offices cleaned by the vacuum cleaner is $10 \times 5 \times 40 = 2000$. We can convert the recurrence relation into the rule $V_n = 650 - 0.2n$. Substituting $n = 2000$ into the rule gives $V_{2000} = 650 - 0.2 \times 2000 = 250$ so the vacuum cleaner is worth $250 after 1 year.

5 a Simple interest gives $6.3\% \times 5000 = \$315$ per year so $315 \times 5 = \$1575$ over five years. The total value in the Company A account would be $5000 + 1575 = \$6575$.

b 6.1% compound interest applied annually on a $5000 investment has a rule $V_n = 1.061^n \times 5000$. The value after five years is given when $n = 5$ so $V_5 = \$6722.75$.

c We want to find a simple interest rate such that the value after five years is $6722.75. This means that the investment must earn $6722.75 - 5000 = 1722.75$ over five years, or $1722.75 \div 5 = \$344.55$. To receive this amount in interest each year, calculate $344.55 \div 6722.75 \times 100\% = 6.891\%$. Thus, Company A would need to offer simple interest with an annual rate of 6.9%.

6 a The initial amount borrowed is $30 000 so $V_0 = 30\ 000$. The interest rate is 9%, compounding monthly so

$$R = 1 + \frac{9}{100 \times 12} = 1.0075$$

Thus, $V_{n+1} = 1.0075 \times V_n$

b Applying the rule five times to find V_5 gives $31 142.00.

c 1 year is equivalent to 12 months so we apply the recurrence relation 12 times.

Alternatively, we can rewrite the recurrence relation as a rule for V_n. This is $V_n = 1.0075^n \times 30\,000$. Substituting in $n = 12$ gives \$32 814.21

d Repaying the loan after 18 months is the same as substituting $n = 18$ into the rule. This gives $V_{18} = \$34\,318.81$

7 First find the value of R given $r = 11.65$ and that the investment compounds twice a year.

$$R = 1 + \frac{11.65}{100 \times 2} = 1.05825$$

Thus, the rule can be written as: $V_n = 1.05825^n \times V_0$. After 21 years, the investment has compounded 42 times (twice per year) and is valued at \$2529.14. We use the CAS to solve: $2529.14 = 1.05825^{42} \times V_0$ for V_0 and get $V_0 = 234.566...$ Thus, the man must have invested \$234.57 when she was born.

8 Setting up the rule given $V_0 = 18\,000$, that interest compounds quarterly and that it is invested for 2 years (8 quarters) to be worth $V_8 = 19\,300$, then we solve

$$19\,300 = \left(1 + \frac{r}{100 \times 4}\right)^8 \times 18\,000$$

for r gives $r = 3.5\%$. That is, the interest rate must be 3.5% per annum.

Solutions to Exercise 8A

1 **a** $A_0 = 2, A_{n+1} = 2A_n + 1$

$A_0 = 2$

$A_1 = 2 \times 2 + 1 = 5$

$A_2 = 2 \times 5 + 1 = 11$

$A_3 = 2 \times 11 + 1 = 23$

$A_4 = 2 \times 23 + 1 = 47$

Or

2	2
$2 \cdot 2 + 1$	5
$5 \cdot 2 + 1$	11
$11 \cdot 2 + 1$	23
$23 \cdot 2 + 1$	47

Gives 2, 5, 11, 23, 47

b $B_0 = 50, B_{n+1} = 2B_n - 10$

$B_0 = 50$

$B_1 = 2 \times 50 - 10 = 90$

$B_2 = 2 \times 90 - 10 = 170$

$B_3 = 2 \times 170 - 10 = 330$

$B_4 = 2 \times 330 - 10 = 650$

Or

50	50
$50 \cdot 2 - 10$	90
$90 \cdot 2 - 10$	170
$170 \cdot 2 - 10$	330
$330 \cdot 2 - 10$	650

Gives 50, 90, 170, 330, 650

c $C_0 = 128, C_{n+1} = 0.5C_n + 32$

$C_0 = 128$

$C_1 = 0.5 \times 128 - 10 = 96$

$C_2 = 0.5 \times 96 - 10 = 80$

$C_3 = 0.5 \times 80 - 10 = 72$

$C_4 = 0.5 \times 72 - 10 = 68$

Or

128	128
$128 \cdot 0.5 + 32$	96
$96 \cdot 0.5 + 32$	80
$80 \cdot 0.5 + 32$	72
$72 \cdot 0.5 + 32$	68

Gives 128, 96, 80, 72, 68

2 **a** The initial value of the investment is \$500. This gives $V_0 = 500$.

b Molly adds \$100 per year. This gives $D = 100$.

c Since the interest rate is 3% per annum, compounding annually, then $r = 3$ and $p = 1$. The growth multiplier is

$$R = 1 + \frac{3}{100 \times 1}$$
$$= 1.03$$

d Substituting the initial value for V_0, the additional payment for D and $R = 1.03$ into the recurrence relation gives:

$$V_0 = 500, \qquad V_{n+1} = 1.03V_n + 100$$

3 **a** The initial value of the investment is \$300 000. This gives $V_0 = 300\,000$.

b Jane adds \$50 000 per year. This gives $D = 50\,000$.

c Since the interest rate is 5.2% per annum, compounding annually, then $r = 5.2$ and $p = 1$. The growth multiplier is

$$R = 1 + \frac{5.2}{100 \times 1}$$
$$= 1.052$$

d Substituting the initial value for V_0, the additional payment for D and $R = 1.052$ into the recurrence relation gives:

$$V_0 = 300\,000, \quad V_{n+1} = 1.052V_n + 50\,000$$

4 **a** Since the interest rate is 3.6% per annum, compounding monthly, then $r = 3.6$ and $p = 12$. The growth multiplier is

$$R = 1 + \frac{3.6}{100 \times 12}$$
$$= 1.003$$

b Substituting $V_0 = 3500$, the additional payment for $D = 150$ and $R = 1.003$ into the recurrence relation gives:

$$V_0 = 3500, \quad V_{n+1} = 1.003V_n + 150$$

c $V_0 = 3500$
$V_1 = 1.003 \times 3500 + 150 = 3660.50$
$V_2 = 1.003 \times 3660.50 + 150 = 3821.48$
Thus, the investment is worth \$3821.48 after two months.

5 **a** Since the interest rate is 3.2% per annum, compounding quarterly, then $r = 3.2$ and $p = 4$. The growth multiplier is

$$R = 1 + \frac{3.2}{100 \times 4}$$
$$= 1.008$$

Substituting $V_0 = 1700$, $D = 100$ and $R = 1.008$ into the recurrence relation gives:

$$V_0 = 1700, \quad V_{n+1} = 1.008V_n + 100$$

b $V_0 = 1700$
$V_1 = 1.008 \times 1700 + 100 = 1813.60$
$V_2 = 1.008 \times 1813.60 + 100 = 1928.11$
$V_3 = 1.008 \times 1928.11 + 100 = 2043.53$
$V_4 = 1.008 \times 2043.53 + 100 = 2159.88$
$V_5 = 1.008 \times 2159.88 + 100 = 2277.16$
$V_6 = 1.008 \times 2277.16 + 100 = 2395.38$
Thus, the investment is worth \$2395.38 after six quarters.

6 Since Sarah's initial investment is \$1500, then $V_0 = 1500$.
She adds \$4 each day so $D = 4$.
Since the interest rate is 7.3% per annum, compounding daily, then $r = 7.3$ and $p = 365$. The growth multiplier is

$$R = 1 + \frac{7.3}{100 \times 365}$$
$$= 1.0002$$

Substituting $V_0 = 1500$, $D = 4$ and $R = 1.0002$ into the recurrence relation gives:

$$V_0 = 1500, \quad V_{n+1} = 1.002V_n + 4$$

7 Since Rachel's initial investment is \$24 000, then $V_0 = 24\,000$.
She adds \$500 each month so $D = 500$.
Since the interest rate is 6% per annum, compounding monthly, then $r = 6$ and $p = 12$. The growth multiplier is

$$R = 1 + \frac{6}{100 \times 12}$$
$$= 1.005$$

Substituting $V_0 = 24\,000$, $D = 500$ and $R = 1.005$ into the recurrence relation gives:

$$V_0 = 24\,000, \quad V_{n+1} = 1.005V_n + 500$$

$V_0 = 24\,000$
$V_1 = 1.005 \times 24\,000 + 500 = 24\,620$
$V_2 = 1.005 \times 24\,620 + 500 = 25\,243.10$
$V_3 = 1.005 \times 25\,243.10 + 500 = 25\,869.32$
$V_4 = 1.005 \times 25\,869.32 + 500 = 26\,498.66$
$V_5 = 1.005 \times 26\,498.66 + 500 = 27\,131.16$
$V_6 = 1.005 \times 27\,131.16 + 500 = 27\,766.81$
Thus, the investment is worth \$27 766.81 after six months.

8 **a** The principal is \$2000.

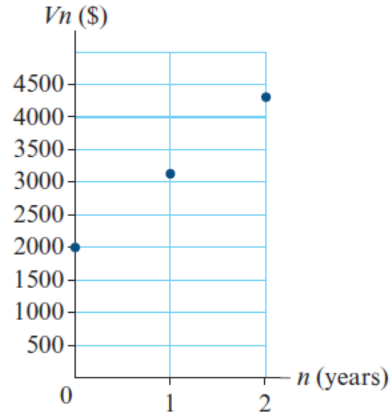
b \$1000 is added to the investment each year.

c $V_0 = 2000$
$V_1 = 1.08 \times 2000 + 1000 = 3160$
$V_2 = 1.08 \times 3160 + 1000 = 4412.80$
Thus, the investment is worth \$4412.80 after two months.

d

n	0	1	2
V_n	2000	3160	4412.80

Plotting the values from the table gives the following graph:



9 **a** The principal is \$20 000.

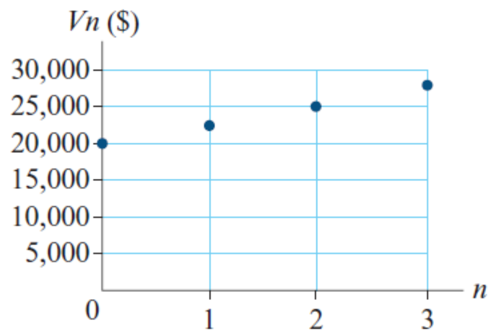
b \$2000 is added to the investment each quarter.

c $V_0 = 20\,000$
$V_1 = 1.025 \times 20\,000 + 2000 = 22\,500$
$V_2 = 1.025 \times 22\,500 + 2000 = 25\,062.50$
$V_3 = 1.025 \times 25\,062.50 + 2000 = 27\,689.06$
Thus, the investment is worth \$27 689.06 after three quarters.

d

n	0	1	2	3
V_n	20 000	22 500	25 062.50	27 689.06

Plotting the values from the table gives the following graph:



10 Since $R = 1.08$ and $p = 1$ (compounding annually), we solve

$$1.08 = 1 + \frac{r}{100 \times 1}$$

for r so $r = 8$.
Thus, the annual interest rate is 8%.

11 Since $R = 1.025$ and $p = 4$ (compounding quarterley), we solve

$$1.025 = 1 + \frac{r}{100 \times 4}$$

for r so $r = 10$.
Thus, the annual interest rate is 10%.

12 Since \$1500 is added in the recurrence relation, the answer is **C**.

13 $V_0 = 36\,000$
$V_1 = 1.008 \times 36\,000 + 200 = 36\,488$
$V_2 = 1.008 \times 36\,488 + 200 = 36\,979.90$
$V_3 = 1.008 \times 36\,979.90 + 200 =$
$37\,475.74$
The increase in the third quarter is given as:

$$\begin{aligned} V_3 - V_2 &= 37\,475.74 - 36\,979.90 \\ &= 495.84 \end{aligned}$$

Thus, the answer is **B**.

14 $V_0 = 10\,000, V_{n+1} = V_n + 1500$ is increasing.
$V_0 = 10\,000, V_{n+1} = V_n - 1500$ is decreasing so the asset is depreciating.
$V_0 = 10\,000, V_{n+1} = 1.15V_n - 1500$ is neither increasing or decreasing.
$V_0 = 10\,000, V_{n+1} = 1.125V_n - 1500$ is decreasing so the asset is depreciating.
$V_0 = 10\,000, V_{n+1} = 1.25V_n - 1500$ is increasing.
Thus, the second and fourth recurrence relations give a decreasing sequence of terms and so the asset they represent is depreciating.
Thus, there are two assets that are depreciating and so the answer is **C**.

Solutions to Exercise 8B

1 Since Brooke borrows \$5000, the principal is \$5000 so $V_0 = 5000$.
Additional payments of \$1400 are made so $D = 1400$.
An interest rate of 5.4% per annum, compounding annually means that $r = 5.4$ and $p = 1$. Thus, the growth multiplier is

$$R = 1 + \frac{5.4}{100 \times 1}$$
$$= 1.054$$

Substituting V_0, D and R gives:

$$V_0 = 5000, \quad V_{n+1} = 1.054V_n - 1400$$

2 **a** Since Jackson borrows \$2000, $V_0 = 2000$. He makes monthly payments of \$339 so $D = 339$.

b Since the interest rate is 6% per annum and the interest compounds monthly, $r = 6$ and $p = 12$. Thus, the growth multiplier is

$$R = 1 + \frac{6}{100 \times 12}$$
$$= 1.005$$

c Substituting V_0, D and R gives:

$$V_0 = 2000, \quad V_{n+1} = 1.005V_n - 339$$

3 **a** Since Benjmain borrows \$10 000, $V_0 = 10\,000$. He makes monthly payments of \$2600 so $D = 2600$.
Since the interest rate is 12% per annum and the interest compounds quarterly, $r = 12$ and $p = 4$. Thus, the growth multiplier is

$$R = 1 + \frac{12}{100 \times 4}$$
$$= 1.03$$

Substituting V_0, D and R gives:

$$V_0 = 10\,000, \quad V_{n+1} = 1.03V_n - 2600$$

b $V_0 = 10\,000$
$V_1 = 1.03 \times 10\,000 - 2600 = 7700$
$V_2 = 1.03 \times 7700 - 2600 = 5331$
Thus, the balance of the loan after two payments have been made is \$5331.

4 Since \$3500 is borrowed, $V_0 = 3500$.
Regular payments of \$280 are made so $D = 280$.
Since the annual interest rate is 4.8% per annum and the interest compounds monthly, $r = 4.8$ and $p = 12$. Thus, the growth multiplier is

$$R = 1 + \frac{4.8}{100 \times 12}$$
$$= 1.004$$

Substituting V_0, D and R gives:

$$V_0 = 3500, \quad V_{n+1} = 1.004V_n - 280$$

5 Since \$150 000 is borrowed, $V_0 = 150\,000$.
Regular payments of \$650 are made so $D = 650$.
Since the annual interest rate is 3.64% per annum and the interest compounds fortnightly, $r = 3.64$ and $p = 26$. Thus,

the growth multiplier is

$$R = 1 + \frac{3.64}{100 \times 26}$$
$$= 1.0014$$

Substituting V_0, D and R gives:

$$V_0 = 150\,000, \quad V_{n+1} = 1.0014V_n - 650$$

6 **a** Since \$235 000 is borrowed, $V_0 = 235\,000$.
Regular daily payments of \$150 are made so $D = 150$.
Since the annual interest rate is 3.65% per annum and the interest compounds daily, $r = 3.65$ and $p = 365$. Thus, the growth multiplier is

$$R = 1 + \frac{3.65}{100 \times 365}$$
$$= 1.0001$$

Substituting V_0, D and R gives:

$$V_0 = 235\,000, \quad V_{n+1} = 1.0001V_n - 150$$

b $V_0 = 235\,000$
$V_1 = 1.0001 \times 235\,000 - 150 = 234\,873.50$
$V_2 = 1.0001 \times 234\,873.50 - 150 = 234\,746.98735$
$V_3 = 1.0001 \times 234\,746.98735 - 150 = 234\,620.462049$
Thus, the balance of the loan after three days is \$234 620.46.

7 **a** The initial balance can be read from the recurrence relation as \$2500.

b The regular payment can be read from the recurrence relation as \$626.

c Solving

$$1.08 = 1 + \frac{r}{100 \times p}$$

for r when $p = 1$ gives $r = 8$. Thus, the annual interest rate is 8%.

d $V_0 = 2500$
$V_1 = 1.08 \times 2500 - 626 = 2074$
$V_2 = 1.08 \times 2074 - 626 = 1613.92$
$V_3 = 1.08 \times 1613.92 - 626 = 1117.0336$
Thus the balance of the loan after three years is \$1117.03.

8 **a** The initial balance can be read from the recurrence relation as \$5000.

b The regular payment can be read from the recurrence relation as \$865.

c Solving

$$1.01 = 1 + \frac{r}{100 \times p}$$

for r when $p = 12$ gives $r = 12$. Thus, the annual interest rate is 12%.

d $V_0 = 5000$
$V_1 = 1.01 \times 5000 - 865 = 4185$
$V_2 = 1.01 \times 2074 - 865 = 3361.85$
Thus the balance of the loan after three years is \$3361.85.

9 **a** Since Mark invests \$20 000 then $V_0 = 20\,000$.
He receives a payment of \$3375 each year so $D = 3375$.

b Since the annual interest rate is 7.2% and interest compounds yearly,

$r = 7.2$ and $p = 1$.

$$R = 1 + \frac{7.2}{100 \times 1} = 1.072$$

c Substituting V_0, D and R gives:

$$V_0 = 20\,000, \quad V_{n+1} = 1.072V_n - 3375$$

10 **a** Sandra invests \$750 000 in the annuity and receives regular payments of \$4100 so $V_0 = 750\,000$ and $D = 4100$.

b Since the annual interest rate is 5.4% and interest compounds monthly, $r = 5.4$ and $p = 12$.

$$R = 1 + \frac{5.4}{100 \times 12} = 1.0045$$

c Substituting V_0, D and R gives:

$$V_0 = 750\,000, \quad V_{n+1} = 1.0045V_n - 4100$$

11 **a** Helen invests \$40 000 in the annuity and receives regular payments of \$10 380 so $V_0 = 40\,000$ and $D = 10\,380$.
Since the annual interest rate is 6% and interest compounds quarterly, $r = 6$ and $p = 4$.

$$R = 1 + \frac{6}{100 \times 4} = 1.015$$

Substituting V_0, D and R gives:

$$V_0 = 40\,000, \quad V_{n+1} = 1.015V_n - 10\,380$$

b $V_0 = 40\,000$
$V_1 = 1.015 \times 40\,000 - 380 = 30\,220$
$V_2 = 1.015 \times 30\,220 - 380 =$ 20 293.30
$V_3 = 1.015 \times 20\,293.30 - 380 =$ 10217.6995
Thus, the value of the annuity after 3 quarters is \$10 217.70.

12 **a** The initial balance can be read from the recurrence relation as \$5000.

b The regular payment received can be read from the recurrence relation as \$1030.

c Solving

$$1.01 = 1 + \frac{r}{100 \times p}$$

for r when $p = 12$ gives $r = 12$.
Thus, the annual interest rate is 12%.

d Using a CAS:

5000	5000
5000 · 1.01 – 1030	4020
4020 · 1.01 – 1030	3030.20
3030.20 · 1.01 – 1030	2030.502

Thus the balance of the annuity after three payments is \$2030.50.

e The amount paid out in the first three months is $3 \times 1030 =$ \$3090.

13 **a** Using a CAS:

6000	6000
6000 · 1.005 – 1500	4530
4530 · 1.005 – 1500	3052.65

Thus, the value of the annuity after two payments is \$3052.65

b Since payments of \$1500 are made quarterly, the total payments made in one year is:

$$\$1500 \times 4 = \$6000$$

14 a The initial balance can be read from the recurrence relation as \$1 000 000.

b Jeff receives a regular payment of \$4000 which can be read from the recurrence relation.

c Solving

$$1.0024 = 1 + \frac{r}{100 \times p}$$

for r when $p = 12$ gives $r = 2.88$.
Thus, the annual interest rate is 2.88%.

b $V_0 = 1\,000\,000$
$V_1 = 1.0024 \times 1\,000\,000 - 4000 = 998\,400$
$V_2 = 1.0024 \times 998\,400 - 4000 = 996\,796.16$
Thus, the value of the annuity after 2 months is \$996 796.16.

15 a Esme receives a regular monthly payment of \$18 400 which can be read from the recurrence relation.

b Solving

$$1.0055 = 1 + \frac{r}{100 \times p}$$

for r when $p = 12$ gives $r = 6.6$.
Thus, the annual interest rate is 6.6%.

c Using a CAS type in 100 000 and press ENTER. Then type ×1.055 − 18 400 and press ENTER until the value first falls below 18 400. Count the number of times you pressed ENTER.
Thus, the balance of the annuity first falls below the regular payment amount of \$18 400 after 5 payments. It reaches a value of \$9762.84.

16 The initial loan is for \$640 000 so $V_0 = 640\,000$.
Matthew pays \$3946.05 each month, so $D = 3946.05$ which **reduces** the value of the loan.
With an annual interest rate of 4.2% compounding monthly, $r = 4.2$ and $p = 12$. Thus,

$$R = 1 + \frac{4.2}{100 \times 12}$$
$$= 1.0035$$

Substituting V_0, D and R gives:

$$V_0 = 640\,000, \quad V_{n+1} = 1.0035V_n - 10\,3946.05$$

Thus, the answer is **D**.

17 Using the recurrence relation,
$V_0 = 3800$
$V_1 = 1.002 \times 3800 - 480 = 3327.60$
$V_2 = 1.002 \times 3327.60 - 480 = 2854.2552$
$V_3 = 1.002 \times 2854.2552 - 480 = 2379.9637$
Thus, the value of the annuity after 3 month is \$2379.96, which is closest to \$2380.
Thus, the answer is **E**.

18 Since the annual interest rate is 4.2% which compounds monthly, $r = 4.2$ and $p = 12$, then

$$R = 1 + \frac{4.2}{100 \times 12}$$
$$= 1.0035$$

With an initial investment of \$9200 ($V_0 = 9200$) and a regular monthly payment of \$620 ($D = 620$), the recurrence relation can be written as:

$$V_0 = 9200, \quad V_{n+1} = 1.0035V_n - 620$$

where V_n is the value of the annuity after n months.
Calculating $V_1 = 1.0035 \times 9200 - 620 =$ 8612.20.
Thus, in the second month, interest is applied to \$8612.20 to give:

$$8612.20 \times \frac{4.2}{100 \times 12} = 30.1422$$

Thus, the amount of interest earned is closest to \$30.
Thus, the answer is **A**.

Solutions to Exercise 8C

1 a The principal of the loan is stated as $14 000.

b The interest charged in the first year is

$$\text{Interest} = 11\% \text{ of principal}$$
$$= \frac{11}{100} \times 14\,000$$
$$= \$1540$$

Thus, $1540 interest is charged in the first year.

c The principal reduction is the difference between the payment made and the interest changed.

$$\text{Reduction} = \text{Payment} - \text{Interest}$$
$$= 1800 - 1540$$
$$= \$260$$

Thus, the principal reduction is $260 for the first payment.

d The new balance is the difference between the previous balance and the principal reduction.

$$\text{Balance} = \text{Old Balance} - \text{Reduction}$$
$$= 14\,000 - 260$$
$$= \$13\,740$$

Thus, the new balance is $13 740.

e We can now complete the table using these values.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	14 000.00
1	1800.00	1540.00	260.00	13 740

f For the second payment:

$$\text{Interest} = 11\% \text{ of } \$13\,740$$
$$= \frac{11}{100} \times 13\,740$$
$$= \$1511.40$$

$$\text{Reduction} = \text{Payment} - \text{Interest}$$
$$= 1800 - 1511.40$$
$$= \$288.60$$

$$\text{Balance} = \text{Old Balance} - \text{Reduction}$$
$$= 13\,740 - 288.60$$
$$= \$13\,451.40$$

For the third payment:

$$\text{Interest} = 11\% \text{ of } \$13\,451.40$$
$$= \frac{11}{100} \times 13\,451.40$$
$$= \$1479.65$$

$$\text{Reduction} = \text{Payment} - \text{Interest}$$
$$= 1800 - 1479.65$$
$$= \$320.35$$

$$\text{Balance} = \text{Old Balance} - \text{Reduction}$$
$$= 13\,451.40 - 320.35$$
$$= \$13\,131.05$$

Payment number	Payment	Interest	Principal redu
0	0.00	0.00	0.00
1	1800.00	1540.00	260.00
2	1800.00	1511.40	288.60
3	1800.00	1479.65	320.35

2 a The principal of the loan is stated as $20 000.

b Substituting $r = 6$ and $p = 12$ gives:

$$\frac{6}{100 \times 12} = 0.005$$

Thus, the interest rate for each month is 0.5%.

c The interest charged in the first year is

$$\begin{aligned}\text{Interest} &= 0.5\% \text{ of principal} \\ &= \frac{0.5}{100} \times 12\,000 \\ &= \$60\end{aligned}$$

Thus, \$60 interest is charged in the first month.

d The principal reduction is the difference between the payment made and the interest changed.

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest} \\ &= 300 - 60 \\ &= \$240\end{aligned}$$

Thus, the principal reduction is \$240 for the first payment.

e The new balance is the difference between the previous balance and the principal reduction.

$$\begin{aligned}\text{Balance} &= \text{Old Balance} - \text{Reduction} \\ &= 12\,000 - 240 \\ &= \$11\,760\end{aligned}$$

Thus, the new balance is \$11 760.

f We can now complete the table using these values.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	12 000.00
1	300.00	60.00	240.00	11 760.00

g For the second payment:

$$\begin{aligned}\text{Interest} &= 0.5\% \text{ of } \$11\,760 \\ &= \frac{0.5}{100} \times 11\,760 \\ &= \$58.80\end{aligned}$$

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest} \\ &= 300 - 58.80 \\ &= \$241.20\end{aligned}$$

$$\begin{aligned}\text{Balance} &= \text{Old Balance} - \text{Reduction} \\ &= 11\,760 - 241.20 \\ &= \$11\,518.80\end{aligned}$$

For the third payment:

$$\begin{aligned}\text{Interest} &= 0.5\% \text{ of } \$11\,518.80 \\ &= \frac{0.5}{100} \times 11\,518.80 \\ &= \$57.59\end{aligned}$$

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest} \\ &= 300 - 57.59 \\ &= \$242.41\end{aligned}$$

$$\begin{aligned}\text{Balance} &= \text{Old Balance} - \text{Reduction} \\ &= 11\,518.80 - 242.41 \\ &= \$11\,276.39\end{aligned}$$

Payment number	Payment	Interest	Principal redu
0	0.00	0.00	0.00
1	300.00	60.00	240.00
2	300.00	58.80	241.20
3	300.00	57.59	242.41

3 **a** The principal of the loan is stated as \$36 000.

b Substituting $r = 8$ and $p = 4$ gives:

$$\frac{8}{100 \times 4} = 0.02$$

Thus, the interest rate for each quarter is 2%.

c For the first payment:

Interest = 2% of principal

$= \frac{2}{100} \times 36\,000$

= $720

Reduction = Payment − Interest

= 1000 – 720

= $280

Balance = Old Balance − Reduction

= 36 000 – 280

= $35 720

For the second payment:

Interest = 2% of $35 720

$= \frac{2}{100} \times 35\,720$

= $714.40

Reduction = Payment − Interest

= 1000 – 714.40

= $285.60

Balance = Old Balance − Reduction

= 35 720 – 285.60

= $35 434.40

For the third payment:

Interest = 2% of $35 434.40

$= \frac{2}{100} \times 35\,434.40$

= $708.69

Reduction = Payment − Interest

= 1000 – 708.69

= $291.31

Balance = Old Balance − Reduction

= 35 434.40 – 291.31

= $35 143.09

Payment number	Payment	Interest	Principal redu
0	0.00	0.00	0.00
1	1000.00	720.00	280.00
1	1000.00	714.40	285.60
1	1000.00	708.69	291.31

4 **a** Given a principal of $2000 and an interest payment of $20, the monthly interest rate is

$$\frac{20}{2000} \times 100\% = 1\%$$

Thus, the interest rate for each month is 1%.

b *A* corresponds to the interest charged on the balance of $1675.

Interest = 1% of principal

$= \frac{1}{100} \times 1675$

= $16.75

Thus, $A = 16.75$.

B corresponds to the principal reduction from the fourth paymenbt of $345 when $10.15 interest is charged.

Reduction = Payment − Interest

= 345 – 10.15

= $334.85

Thus, $B = 334.85$.

C corresponds to the balance after 5 payments have been made. Since the previous balance was $680.37 and hte principal reduction was $338.20, the new balance is the difference between the two.

$$\begin{aligned}\text{Balance} &= \text{Old Balance} - \text{Reduction}\\ &= 680.37 - 338.20\\ &= \$342.17\end{aligned}$$

Thus, $C = 342.17$.

5 a The principal of the loan was \$4000 and the interest paid in the first quarter was \$100.

b The quarterly interest rate is

$$\frac{100}{4000} \times 100\% = 2.5\%$$

Thus, the interest rate for each quarter is 2.5%.

c A corresponds to the interest charged on the balance of \$2592.44

$$\begin{aligned}\text{Interest} &= 2.5\% \text{ of principal}\\ &= \frac{2.5}{100} \times 2592.44\\ &= \$64.81\end{aligned}$$

Thus, $A = 64.81$.
B corresponds to the principal reduction from the fifth payment of \$557.65 when \$52.49 interest is charged.

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest}\\ &= 557.65 - 52.49\\ &= \$505.16\end{aligned}$$

Thus, $B = 505.16$.
C corresponds to the balance after the 6th payment.

$$\begin{aligned}\text{Balance} &= \text{Old Balance} - \text{Reduction}\\ &= 1594.44 - 517.79\\ &= \$1076.65\end{aligned}$$

Thus, $C = 1076.65$.
D corresponds to the interest charged on the balance of \$1076.65 (found as C).

$$\begin{aligned}\text{Interest} &= 2.5\% \text{ of principal}\\ &= \frac{2.5}{100} \times 1076.65\\ &= \$26.92\end{aligned}$$

Thus, $D = 26.92$.
E corresponds to the principal reduction from the seventh payment of \$557.65 when \$26.92 interest is charged.

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest}\\ &= 557.65 - 26.92\\ &= \$530.73\end{aligned}$$

Thus, $E = 530.73$.
Alternatively, the principal reduction can be calculated by finding the difference between the new and old balance.

$$\begin{aligned}\text{Reduction} &= \text{Old balance} - \text{New balance}\\ &= 1076.65 - 545.92\\ &= \$527.73\end{aligned}$$

6 A corresponds to the principal reduction from the second payment of \$1800 when \$1378.74 interest is charged.

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest}\\ &= 1800 - 1378.74\\ &= \$421.26\end{aligned}$$

Thus, $A = 421.26$.
B corresponds to the balance after the third payment. To find, first find the interest so that you can find the

reduction and then balance:

$$\text{Interest} = \frac{3.6}{100 \times 12} \times 459\,158.74$$
$$= \$1377.48$$

$$\text{Reduction} = \text{Payment} - \text{Interest}$$
$$= 1800 - 1377.48$$
$$= \$422.52$$

$$\text{Balance} = \text{Old Balance} - \text{Reduction}$$
$$= 459\,158.74 - 422.52$$
$$= \$458\,736.22$$

Thus, $B = 458\,736.22$.

7 a i Reading from the table, the interest from payment 1 is \$15.

ii The monthly interest rate is calculated using $r = 3$ and $p = 12$:

$$\frac{3}{100 \times 12} = 0.025$$

Thus, the interest rate for each month is 0.25%.

b A corresponds to the interest charged on the balance of \$5012.77.

$$\text{Interest} = 0.25\% \text{ of principal}$$
$$= \frac{0.25}{100} \times 5012.77$$
$$= \$12.53$$

Thus, $A = 12.53$.
B corresponds to the principal reduction corresponding to the third payment.

$$\text{Reduction} = \text{Payment} - \text{Interest}$$
$$= 508 - 12.53$$
$$= \$495.47$$

Thus, $B = 495.47$.
C corresponds to the balance after the 3rd payment.

$$\text{Balance} = \text{Old Balance} - \text{Reduction}$$
$$= 5012.77 - 495.47$$
$$= \$4517.30$$

Thus, $C = 4517.30$.

8 a Given a principal of \$5000 and an interest payment of \$50, the monthly interest rate is

$$\frac{50}{5000} \times 100\% = 1\%$$

Thus, the interest rate for each month is 1%.

b A corresponds to the interest charged on the balance of \$5301.50.

$$\text{Interest} = 1\% \text{ of principal}$$
$$= \frac{1}{100} \times 5301.50$$
$$= \$53.02$$

Thus, $A = 53.02$.
B corresponds to the principal increase.

$$\text{Increase} = \text{Payment} + \text{Interest}$$
$$= 100 - 53.02$$
$$= \$153.02$$

Thus, $B = 153.02$.
C corresponds to the balance following the third payment.

$$\text{Balance} = \text{Old Balance} + \text{Increase}$$
$$= 5301.50 - 153.02$$
$$= \$5454.52$$

Thus, $C = 5454.52$.

9 The principal reduction can be found by considering the payment made and the interest earned.

$$\begin{aligned}\text{Reduction} &= \text{Payment} + \text{Interest}\\ &= 51\,801.82 - 16\,740.31\\ &= \$35\,061.51\end{aligned}$$

Thus, the principal has reduced by \$35 061.51.
Thus, the answer is **D**.

10 Since the interest paid is \$1800 on a balance of \$200 000, we can calculate the interest rate as:

$$\frac{1800}{200\,000} \times 100\% = 0.9\%$$

Thus, the interest rate for each period is 0.9%.
Thus, since the annual rate is 3.6%, $3.6 \div 0.9 = 4$, so there are four payments made each year so the payments must be quarterly.
Thus, the answer is **D**.

11 First, calculate the interest rate per payment period:

$$\frac{50.50}{10\,100} \times 100\% = 0.5\%$$

Using this interest rate, the interest applied to the balance of \$10 301.50 is:

$$\frac{0.5}{100} \times 10\,301.50 = \$51.51$$

By comparing the new and old balance, the principal increase can be calculated.

$$\begin{aligned}\text{Increase} &= 10\,533.01 - 10\,301.50\\ &= \$251.51\end{aligned}$$

Subtracting the interest from the increase gives the payment that was made:

$$\begin{aligned}\text{Payment} &= \text{Increase} - \text{Interest}\\ &= 251.51 - 51.51\\ &= 200\end{aligned}$$

Thus, the answer is **D**.

Solutions to Exercise 8D

1 First find the interest on the balance of \$532.85.

$$\frac{4}{100} \times 532.85 = \$21.31$$

The final payment must cover the balance (\$532.85) plus the interest (\$21.31).

$$\begin{aligned} \text{Final Payment} &= \text{Balance} + \text{Interest} \\ &= 532.85 + 21.31 \\ &= 554.16 \end{aligned}$$

Thus, the final payment is \$554.16

2 First find the interest on the balance of \$1257.57.

$$\frac{4.8}{100 \times 12} \times 1257.57 = \$5.03$$

The final payment must cover the balance (\$1257.57) plus the interest (\$5.03).

$$\begin{aligned} \text{Final Payment} &= \text{Balance} + \text{Interest} \\ &= 1257.57 + 5.03 \\ &= 1262.60 \end{aligned}$$

Thus, the final payment is \$1262.60

3 First find the interest on the balance of \$659.60.

$$\frac{5}{100} \times 659.60 = \$32.98$$

The final payment must cover the balance (\$659.60) plus the interest (\$32.98).

$$\begin{aligned} \text{Final Payment} &= \text{Balance} + \text{Interest} \\ &= 659.60 + 32.98 \\ &= 692.58 \end{aligned}$$

Thus, the final payment is \$692.58

4 For the first payment:

$$\begin{aligned} \text{Interest} &= \frac{5.4}{100 \times 12} \times 4500 \\ &= \$20.25 \end{aligned}$$

$$\begin{aligned} \text{Reduction} &= \text{Payment} - \text{Interest} \\ &= 760 - 20.25 \\ &= \$739.75 \end{aligned}$$

$$\begin{aligned} \text{Balance} &= \text{Old Balance} - \text{Reduction} \\ &= 4500 - 739.75 \\ &= \$3760.25 \end{aligned}$$

For the second payment:

$$\begin{aligned} \text{Interest} &= \frac{5.4}{100 \times 12} \times 3760.25 \\ &= \$16.92 \end{aligned}$$

$$\begin{aligned} \text{Reduction} &= \text{Payment} - \text{Interest} \\ &= 760 - 16.92 \\ &= \$743.08 \end{aligned}$$

$$\begin{aligned} \text{Balance} &= \text{Old Balance} - \text{Reduction} \\ &= 3760.25 - 743.08 \\ &= \$3017.17 \end{aligned}$$

For the third payment:

$$\begin{aligned} \text{Interest} &= \frac{5.4}{100 \times 12} \times 3017.17 \\ &= \$13.58 \end{aligned}$$

$$\begin{aligned} \text{Reduction} &= \text{Payment} - \text{Interest} \\ &= 760 - 13.58 \\ &= \$746.42 \end{aligned}$$

$$\begin{aligned} \text{Balance} &= \text{Old Balance} - \text{Reduction} \\ &= 3017.17 - 746.42 \\ &= \$2270.75 \end{aligned}$$

For the fourth payment:

$$\text{Interest} = \frac{5.4}{100 \times 12} \times 2270.75$$
$$= \$10.22$$

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest}\\ &= 760 - 10.22\\ &= \$749.78\end{aligned}$$

$$\begin{aligned}\text{Balance} &= \text{Old Balance} - \text{Reduction}\\ &= 2270.75 - 749.78\\ &= \$1520.97\end{aligned}$$

For the fifth payment:

$$\text{Interest} = \frac{5.4}{100 \times 12} \times 1520.97$$
$$= \$6.84$$

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest}\\ &= 760 - 6.84\\ &= \$753.16\end{aligned}$$

$$\begin{aligned}\text{Balance} &= \text{Old Balance} - \text{Reduction}\\ &= 1520.97 - 753.16\\ &= \$767.81\end{aligned}$$

For the sixth payment:

$$\text{Interest} = \frac{5.4}{100 \times 12} \times 767.81$$
$$= \$3.46$$

$$\text{Reduction} = \text{Old Balance} = \$767.81$$

to fully exhaust the annuity.

$$\begin{aligned}\text{Payment} &= \text{Reduction} - \text{Interest}\\ &= 767.81 + 3.46\\ &= \$771.27\end{aligned}$$

5 a A corresponds to the interest earned on \$15 044.14.

$$\frac{4.8}{100 \times 12} \times 15\,044.14 = \$180.53$$

Thus, $A = 180.53$.

b B corresponds to the principal increase from payment 4.

$$\begin{aligned}\text{Increase} &= \text{Payment} + \text{Interest}\\ &= 1200 + 180.53\\ &= \$1380.53\end{aligned}$$

c The total interest earned on the investment is found by adding up the interest earned.

$$\begin{aligned}&= 132 + 147.98 + 164.16 + 180.53\\ &= \$624.67\end{aligned}$$

Alternatively, we can calculate how much the balance has increased by and the payments that have been made.

$$\begin{aligned}\text{Change in balance} &= 16\,424.67 - 11\,000\\ &= \$5424.67\end{aligned}$$

$$\begin{aligned}\text{Payments made} &= 4 \times 1200\\ &= \$4800\end{aligned}$$

$$\begin{aligned}\text{Difference} &= 5424.67 - 4800\\ &= \$624.67\end{aligned}$$

6 a For the final balance to be 0, the principal reduction from payment 12 must be \$343.97.
Thus, $C = 343.97$.
The interest charged is calculated on the balance.

$$\frac{6}{100 \times 12} \times 343.97 = \$1.72$$

Thus, $B = 1.72$.

To find the total final payment:

$$\begin{aligned}\text{Payment} &= \text{Reduction} - \text{Interest}\\ &= 343.97 + 1.72\\ &= \$345.69\end{aligned}$$

Thus, $A = 345.69$.

b Total payment is:

$$\begin{aligned}&= 11 \times 344.14 + 345.69\\ &= \$4131.23\end{aligned}$$

c Total interest is

$$\begin{aligned}&= \text{Total payment} - \text{Principal}\\ &= 4131.23 - 4000\\ &= \$131.23\end{aligned}$$

7 a B corresponds to the interest on \$3868.37:

$$\frac{3.6}{100 \times 4} \times 3868.37 = \$34.82$$

Thus, $B = 34.82$.
C corresponds to the principal reduction which must be \$3868.37 to make the balance 0 after the payment.
Thus, $C = 3868.37$.
A corresponds to the final payment which is the sum of the reduction and the interest earned.

$$\begin{aligned}\text{Payment} &= \text{Reduction} + \text{Interest}\\ &= 3868.37 + 34.82\\ &= \$3903.19\end{aligned}$$

Thus, $A = 3903.19$.

b The total payment received from the annuity is the sum of the first seven regular payments of \$3903.50 and the final payment of \$3903.19.

$$\begin{aligned}\text{Total Payment} &= 7 \times 3903.50 + 3903.19\\ &= \$31\,227.69\end{aligned}$$

c The total interest earned by the annutiy is the difference btween the amount paid out and the principal (initial investment).

$$\begin{aligned}\text{Total interest} &= 31\,227.69 - 30\,000\\ &= \$1227.69\end{aligned}$$

8 For the first payment:

$$\begin{aligned}\text{Interest} &= \frac{6.6}{100 \times 12} \times 12\,000\\ &= \$66\end{aligned}$$

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest}\\ &= 3040 - 66\\ &= \$2974\end{aligned}$$

$$\begin{aligned}\text{Balance} &= \text{Old Balance} - \text{Reduction}\\ &= 12\,000 - 2974\\ &= \$9026\end{aligned}$$

For the second payment:

$$\begin{aligned}\text{Interest} &= \frac{6.6}{100 \times 12} \times 9026\\ &= \$49.64\end{aligned}$$

$$\begin{aligned}\text{Reduction} &= \text{Payment} - \text{Interest}\\ &= 3040 - 49.64\\ &= \$2990.36\end{aligned}$$

$$\begin{aligned}\text{Balance} &= \text{Old Balance} - \text{Reduction}\\ &= 12\,000 - 2990.36\\ &= \$6035.64\end{aligned}$$

For the third payment:

$$\text{Interest} = \frac{6.6}{100 \times 12} \times 6035.64$$
$$= \$33.20$$

$$\text{Reduction} = \text{Payment} - \text{Interest}$$
$$= 3040 - 33.20$$
$$= \$3006.80$$

$$\text{Balance} = \text{Old Balance} - \text{Reduction}$$
$$= 6035.64 - 3006.80$$
$$= \$3028.84$$

For the fourth payment:

$$\text{Interest} = \frac{6.6}{100 \times 12} \times 3028.84$$
$$= \$16.66$$

$$\text{Balance} = 0$$

$$\text{Reduction} = \$3028.84$$

$$\text{Final Payment} = \text{Reduction} + \text{Interest}$$
$$= 3028.84 - 16.66$$
$$= \$3045.50$$

Thus, to find the total payments, add up all four payments:

$$\text{Total Payments} = 3 \times 3040 + 3045.50$$
$$= \$12\,165.50$$

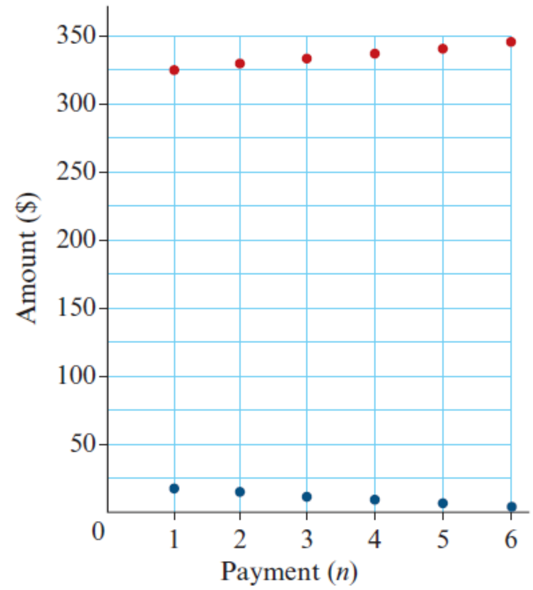
The total interest earned is the sum of all interest earned:

$$\text{Total Interest} = 66 + 49.64 + 33.20 + 16.66$$
$$= \$165.50$$

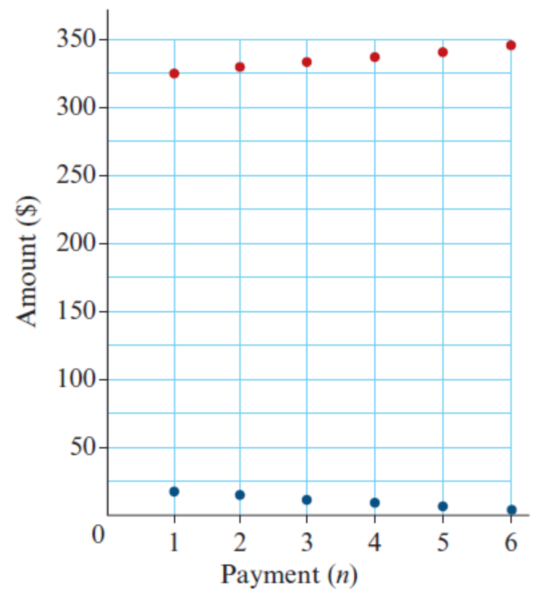
Alternatively, you can calculate the difference between the payments made and the initial investment:

$$\text{Total Interest} = 12\,165.50 - 12\,000$$
$$= \$165.50$$

9 Plot points on graph as shown.



10 Plot points on graph as shown.



11 The annuity earns interest on the balance of \$1534.48 of

$$\frac{6}{100 \times 12} \times 1534.48 = \$22.25$$

Thus, the final payment is the balance

after the third payment plus the interest.

$$\text{Final Payment} = \text{Balance} + \text{Interest}$$
$$= 1534.48 - 22.25$$
$$= \$1556.73$$

Thus, the answer is **E**.

12 For the first payment:

$$\text{Interest} = \frac{6}{100 \times 12} \times 15\,000$$
$$= \$75$$

$$\text{Reduction} = \text{Payment} - \text{Interest}$$
$$= 3796.99 - 75$$
$$= \$3721.99$$

$$\text{Balance} = \text{Old Balance} - \text{Reduction}$$
$$= 15\,000 - 3721.99$$
$$= \$11\,278.01$$

For the second payment:

$$\text{Interest} = \frac{6}{100 \times 12} \times 11\,278.01$$
$$= \$56.39$$

$$\text{Reduction} = \text{Payment} - \text{Interest}$$
$$= 3796.99 - 56.39$$
$$= \$3740.60$$

$$\text{Balance} = \text{Old Balance} - \text{Reduction}$$
$$= 11\,278.01 - 3740.60$$
$$= \$7537.41$$

For the third payment:

$$\text{Interest} = \frac{6}{100 \times 12} \times 7537.411$$
$$= \$37.69$$

$$\text{Reduction} = \text{Payment} - \text{Interest}$$
$$= 3796.99 - 37.69$$
$$= \$3759.30$$

$$\text{Balance} = \text{Old Balance} - \text{Reduction}$$
$$= 7537.41 - 3759.30$$
$$= \$3778.11$$

For the fourth payment:

$$\text{Interest} = \frac{6}{100 \times 12} \times 3778.11$$
$$= \$18.89$$

Thus, the total interest is found by summing up the four amounts of interest:

$$\text{Total Interest} = 75 + 56.39 + 37.69 + 18.89$$
$$= \$187.97$$

The closest answer to this is \$188.
Thus, the answer is **C**.

Solutions to Exercise 8E

1 a **PV** is negative as Wanda is giving money to the bank.

b **PMT** is negative as Wanda is making regular annual investments.

c Using financial solver:
N: 10 (10 years of investment)
I: 7.1 (Interest rate is 7.1% per year)
PV: −20000 (Amount initially invested)
PMT: −6000 (Amount invested each period)
FV: BLANK (This is what we're trying to find)
PpY: 1 (Interest is compounding once per year)
CpY: 1 (Interest is compounding once per year)
Hence, FV is 123003.547.. so the value of the investment after 10 years is $123 003.55

d Using financial solver:
N: 30 (30 years of investment)
I: 7.1 (Interest rate is 7.1% per year)
PV: −20000 (Amount initially invested)
PMT: −6000 (Amount invested each period)
FV: BLANK (This is what we're trying to find)
PpY: 1 (Interest is compounding once per year)
CpY: 1 (Interest is compounding once per year)
Hence, FV is 733636.829.. so the value of the investment after 30 years is $733 636.83

2 a Using financial solver:
N: 5 (5 months of investment)
I: 4.9 (Interest rate is 4.9% per year)
PV: −20000 (Amount initially invested)
PMT: −380 (Amount invested each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is 22327.26... so the value of the investment after 5 months is $22 327.26

b Using financial solver:
N: 36 (36 months in 3 years of investment)
I: 4.9 (Interest rate is 4.9% per year)
PV: −20000 (Amount initially invested)
PMT: −380 (Amount invested each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is 37864.4996555... so the value of the investment after 3 years is $37 864.50

3 a **PV** is positive as Barry is receiving money from the bank.

b **FV** is negative as Barry is paying money to the bank to repay the loan.

c Using financial solver:
N: 6 (6 months of borrowing)
I: 4.5 (Interest rate is 4.5% per year)
PV: 8000 (Amount initially borrowed)
PMT: −350 (Amount repaid each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is −6061.9097... so the amount that Barry owes after 6 months is $6061.91

4 **a** Using financial solver:
N: 3 (3 months of borrowing)
I: 7.8 (Interest rate is 7.8% per year)
PV: 25 000 (Amount initially borrowed)
PMT: −1200 (Amount repaid each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is −21867.2249... so the amount that Suzanne owes after 3 months is $21 867.22

b Using financial solver:
N: 12 (12 months in one year of borrowing)
I: 7.8 (Interest rate is 7.8% per year)
PV: 25 000 (Amount initially borrowed)
PMT: −1200 (Amount repaid each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is −12095.126422... so the amount that Suzanne owes after 3 months is $12 095.13

5 **a** Using financial solver:
N: 24 (24 quarters in 6 years of borrowing)
I: 8.3 (Interest rate is 8.3% per year)
PV: 240 000 (Amount initially borrowed)
PMT: −5442.90 (Amount repaid each period)
FV: BLANK (This is what we're trying to find)
PpY: 4 (Interest is compounding four times per year)
CpY: 4 (Interest is compounding four times per year)
Hence, FV is −225 788.13... so the amount that Rachel owes after 6 years is $225 788.13

b Using financial solver, find the final value if Rachel makes 120 regular payments:
N: 12 (120 regular payments)
I: 7.8 (Interest rate is 7.8% per year)
PV: 25 000 (Amount initially borrowed)

PMT: −5442.90 (Amount repaid each period)
FV: BLANK (This is what we're trying to find)
PpY: 4 (Interest is compounding twelve times per year)
CpY: 4 (Interest is compounding twelve times per year)
Hence, FV is −9.99163... so the amount that Rachel would still owe if she made 120 regular payments is \$9.99. This means that her final payment (payment 120) must be \$9.99 more than the regular payments. Thus, the final payment is \$5442.90 + \$9.99 = \$5452.89

6 a Using financial solver:
N: 52 (52 weeks in one year of borrowing)
I: 4.6 (Interest rate is 4.6% per year)
PV: 50 000 (Amount initially borrowed)
PMT: −343.27 (Amount repaid each period)
FV: BLANK (This is what we're trying to find)
PpY: 52 (Interest is compounding four times per year)
CpY: 52 (Interest is compounding four times per year)
Hence, FV is −34 093.95880... so the amount that David owes after 1 year is \$34 093.96

b Using financial solver, find the final value if David makes regular payments for 3 years:
N: 156 (120 regular payments)
I: 4.6 (Interest rate is 4.6% per year)
PV: 50 000 (Amount initially borrowed)
PMT: −343.27 (Amount repaid each period)
FV: BLANK (This is what we're trying to find)
PpY: 52 (Interest is compounding four times per year)
CpY: 52 (Interest is compounding four times per year)
Hence, FV is −1.373922... so the amount that David would still owe if he made regular payments for 3 years is \$1.37. This means that his final payment must be \$1.37 more than the regular payments. Thus, the final payment is \$343.27 + \$1.37 = \$344.64

7 a **PV** is negative as Kazou is handing over money for the annuity.

b Using financial solver:
N: 5 (5 years of annuity)
I: 6.1 (Interest rate is 6.1% per year)
PV: −50000 (Amount initially invested)
PMT: 6825.61 (Amount received each period)
FV: BLANK (This is what we're trying to find)
PpY: 1 (Interest is compounding once per year)
CpY: 1 (Interest is compounding once per year)
Hence, FV is 28674.000216... so the amount in the annuity after 5 years is \$28 674

c Using financial solver:
N: 10 (10 years of annuity)
I: 6.1 (Interest rate is 6.1% per year)
PV: −50000 (Amount initially invested)
PMT: 6825.61 (Amount received each period)
FV: BLANK (This is what we're trying to find)
PpY: 1 (Interest is compounding once per year)
CpY: 1 (Interest is compounding once per year)
Hence, FV is 0.129686... so the amount that Kazou has in the annuity after receiving regular payments for 10 years is \$0.13.
This means that his final payment must be \$0.13 more than the regular payments. Thus, the final payment is \$6825.61 + \$0.13 = \$6825.74

c Using financial solver:
N: 12 (12 months of annuity)
I: 7.2 (Interest rate is 7.2% per year)
PV: −20000 (Amount initially invested)
PMT: 1732.37 (Amount received each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding once per year)
CpY: 12 (Interest is compounding once per year)
Hence, FV is 0.1174448... so the amount that Eliza has in the annuity after receiving regular payments for 10 years is \$0.12.
This means that his final payment must be \$0.12 more than the regular payments. Thus, the final payment is \$1732.37 + \$0.12 = \$1732.49

8 a Using financial solver:
N: 3 (3 months of annuity)
I: 7.2 (Interest rate is 7.2% per year)
PV: −20000 (Amount initially invested)
PMT: 1732.37 (Amount received each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding once per year)
CpY: 12 (Interest is compounding once per year)
Hence, FV is 15133.80929468... so the amount in the annuity after 3 months is \$15 133.81

9 a Using financial solver:
N: 26 (26 weeks of annuity)
I: 6.8 (Interest rate is 6.8% per year)
PV: −15000 (Amount initially invested)
PMT: 298.57 (Amount received each period)
FV: BLANK (This is what we're trying to find)
PpY: 52 (Interest is compounding 52 times per year)
CpY: 52 (Interest is compounding 52 times per year)
Hence, FV is 7627.374633... so the amount in the annuity after 26 weeks is \$7627.37

c Using financial solver:
N: 52 (52 weeks of annuity)
I: 6.8 (Interest rate is 6.8% per year)
PV: −15000 (Amount initially invested)
PMT: 298.57 (Amount received each period)
FV: BLANK (This is what we're trying to find)
PpY: 52 (Interest is compounding 52 times per year)
CpY: 52 (Interest is compounding 52 times per year)
Hence, FV is −0.06066389....
Since the amount is negative, Ezra's final payment will be **LESS** than his regular payment by $0.16. Thus, the final payment is $298.57 − $0.06 = $298.51

10 Using financial solver:
N: 72 (72 months in 6 years)
I: 2.8 (Interest rate is 2.8% per year)
PV: −3000 (Amount initially invested)
PMT: −200 (Amount invested each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding 12 times per year)
CpY: 12 (Interest is compounding 12 times per year)
Hence, FV is 19208.555792373 so the amount in the investment after six years is $19 208.56.
Thus the answer is **E**.

11 Using financial solver:
N: 240 (240 months in 20 years)
I: 4.24 (Interest rate is 4.24% per year)
PV: 450000 (Amount initially borrowed)
PMT: −2784 (Amount invested each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding 12 times per year)
CpY: 12 (Interest is compounding 12 times per year)
Hence, FV is −58.5730967.... Since the amount is negative, Ezra's final payment will be **MORE** than his regular payment by $58.57. Thus, the final payment is $2784 + $2842.57 = $
Thus the answer is **E**.

12 Using financial solver:
N: 24 (24 months in two years)
I: 7.3 (Interest rate is 7.3% per year)
PV: −75000 (Amount initially invested)
PMT: 2326 (Amount received each period)
FV: BLANK (This is what we're trying to find)
PpY: 12 (Interest is compounding 52 times per year)
CpY: 12 (Interest is compounding 52 times per year)
Hence, FV is 26842.057877... Thus, the value of the annuity is $26 842.06.
Thus the answer is **B**.

Solutions to Exercise 8F

1 Using financial solver:
N: 10 (Investment is for 10 years)
I: BLANK (This is what we're trying to find)
PV: −15000 (Amount initially invested)
PMT: −4000 (Amount invested each period)
FV: 100000 (Final value)
PpY: 1 (Interest is compounding once per year)
CpY: 1 (Interest is compounding once per year)
Hence, I is 8.3929044... Thus, the annual interest rate is 8.39%.

2 **a** Using financial solver:
N: 60 (Investment is for 5 years)
I: BLANK (This is what we're trying to find)
PV: −30000 (Amount initially invested)
PMT: −400 (Amount invested each period)
FV: 60000 (Final value)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, I is 2.7019216... Thus, the annual interest rate is 2.7%.

b **i** Using financial solver:
N: 12 (Investment is for 1 year)
I: 3.2 (Interest rate is 3.2% per year)
PV: −30000 (Amount initially invested)
PMT: BLANK (This is what we're trying to find)
FV: 40000 (Final value)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, PMT is 741.181633...
Thus, the amount that should be added each month is \$741.19 (as \$741.18 is insuffficent to reach \$40 000).

ii Using financial solver:
N: BLANK (This is what we're trying to find)
I: 3.2 (Interest rate is 3.2% per year)
PV: −30000 (Amount initially invested)
PMT: −1000 (Amount invested each period)
FV: 100000 (Final value)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, N is 59.86507... Thus, the number of months required for the investment to reach \$100 000 is 60 months (as 59 months is insufficient to reach \$100 000).

3 **a** Using financial solver:
N: 12 (Investment is for 1 year)
I: 4.7 (Interest rate is 4.7% per year)
PV: −7500 (Amount initially invested)
PMT: BLANK (This is what we're

trying to find)
FV: 13991.15 (Final value after 1 year)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, PMT is 500.003386... Thus, the amount that should be added each month is \$500.00.

b **i** Since Kelven invests \$500 each month then he invests a total of \$500 × 12 = \$6000 in 1 year.

ii The investment has increased by 13 991.15 − 7500 = 6491.15 over the year.

iii Given that Kelven added \$6000 to the investment, the amount of interest earned in the year was 6491.15 − 6000 = \$491.15.

c Using financial solver:
N: BLANK (This is what we're trying to find)
I: 4.7 (Interest rate is 4.7% per year)
PV: −7500 (Amount initially invested)
PMT: −500 (Kelven adds \$500 each month)
FV: 20000 (Final value required)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, N is 22.628... Thus, it will take at 23 months for the value of the investment to reach at least \$20 000.

4 Using financial solver:
N: BLANK (This is what we're trying to find)
I: 9.5 (Interest rate is 9.5% per year)
PV: 20000 (Amount initially borrowed)
PMT: −450 (Dan makes payments of \$450 each month)
FV: 0 (Final value required)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, N is 54.9916... Thus, it will take at 55 months for the value of the loan to reach to be fully paid (consisting of 54 payments of \$450 and then a final smaller payment).

5 **a** Using financial solver:
N: 144 (144 months in 12 years)
I: 10.25 (Interest rate is 10.25% per year)
PV: 240000 (Amount initially borrowed)
PMT: −450 (Payments of \$450 each month)
FV: BLANK (This is what we're finding)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is 197793.8516... Thus, the value remaining on the loan after 12 years is \$197 793.85

b **i** Using financial solver:
N: 360 (Find the final value assuming 360 payments)
I: 10.25 (Interest rate is 10.25%

per year)
PV: 240000 (Amount initially borrowed)
PMT: −2150.64 (Payments of \$2150.64 each month)
FV: BLANK (This is what we're finding)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is −7.41803.. so if 360 payments of \$2150.64 are made, then \$7.42 is still owing so the final payment must be 2150.64 + 7.42 = \$2158.06

ii The total amount that is repaid is $359 \times 2150.64 + 2158.06 =$ \$774 237.82.

iii The total amount of interest that is repaid is 774 237.82 − 240 000 = \$534 237.82

6 a Using financial solver:
N: 30 (30 monthly repayments)
I: 6.8 (Annual interest rate)
PV: 17000 (Amount initially borrowed)
PMT: BLANK (This is what we're finding)
FV: 0 (Aiming to repay the loan)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, PMT is −617.7975955.. the regular payment for the first 29 payments is \$617.80.

b Using financial solver:
N: 30 (30 monthly repayments)
I: 6.8 (Annual interest rate)
PV: 17000 (Amount initially borrowed)
PMT: −617.80 (Regular monthly payment)
FV: BLANK (This is what we're looking for)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is 0.078387.. so the final payment should be \$0.08 less than the regular payment. Hence the final payment is 617.80 − 0.08 = \$617.72

c The total repayments would be $29 \times 617.80 + 617.72 =$ \$18 533.92.

d Total interest is the difference between what is borrowed and what is repaid: 18 533.92 − 17 000 = 1533.92.

7 a i Using financial solver:
N: 40 (40 quarterly repayments)
I: 8.6 (Annual interest rate)
PV: 140000 (Amount initially borrowed)
PMT: BLANK (This is what we're looking for)
FV: 0 (We want the loan fully repaid)
PpY: 4 (Interest is compounding four times per year)
CpY: 4 (Interest is compounding four times per year)
Hence, FV is −5253.390222.. so

the regular repayment amount is $5253.39 per quarter.
Given this regular payment, we can find what the final value is if Cale makes this payment for 10 years.
Using financial solver:
N: 40 (40 quarterly repayments)
I: 8.6 (Annual interest rate)
PV: 140000 (Amount initially borrowed)
PMT: −5253.39 (Regular repayment)
FV: BLANK (This is what we are looking for)
PpY: 4 (Interest is compounding four times per year)
CpY: 4 (Interest is compounding four times per year)
Hence, FV is 0.0138749.... so the regular payment should be $0.01 more than the regular payment. Hence the final payment is 5253.39 + 0.01 = $5253.40

ii The total amount that Cale pays is $39 \times 5253.39 + 5253.40 =$ 210 135.61.

b **i** Using financial solver:
N: 60 (60 quarterly repayments in 15 years)
I: 8.6 (Annual interest rate)
PV: 140000 (Amount initially borrowed)
PMT: BLANK (This is what we're looking for)
FV: 0 (We want the loan fully repaid)
PpY: 4 (Interest is compounding four times per year)
CpY: 4 (Interest is compounding four times per year)
Hence, FV is −4175.10528999... so the regular repayment amount is $4175.11 per quarter.

ii Given this regular payment, we can find what the final value is if Cale makes this payment for 15 years.
Using financial solver:
N: 60 (60 quarterly repayments)
I: 8.6 (Annual interest rate)
PV: 140000 (Amount initially borrowed)
PMT: −4175.11 (Regular repayment)
FV: BLANK (This is what we are looking for)
PpY: 4 (Interest is compounding four times per year)
CpY: 4 (Interest is compounding four times per year)
Hence, FV is 0.5659578481.... so the regular payment should be $0.57 less than the regular payment. Hence the final payment is 4175.11 − 0.57 = $4174.54

iii The total amount that Cale pays is $59 \times 4175.11 + 4174.54 =$ $250 506.03.

8 First find the regular payment by calculating PMT, assuming 650 equal payments. Using financial solver:
N: 650 (650 fortnightly payments)
I: 5.2 (Annual interest rate)
PV: 250000 (Amount initially borrowed)
PMT: BLANK (This is what we're

looking for)
FV: 0 (We want the loan fully repaid)
PpY: 26 (Interest is compounding twenty-six times per year)
CpY: 26 (Interest is compounding twenty-six times per year)
Hence, PMT is −687.6499306... so the regular repayment amount is \$687.65 per fortnight.
If Lorenzo makes this regular payment, we calculate his final payment using financial solver:
N: 650 (650 fortnightly payments)
I: 5.2 (Annual interest rate)
PV: 250000 (Amount initially borrowed)
PMT: 687.65 (Regular payment)
FV: BLANK (This is what we're looking for)
PpY: 26 (Interest is compounding twenty-six times per year)
CpY: 26 (Interest is compounding twenty-six times per year)
Hence, FV is 0.092405.... so the regular payment should be \$0.09 less than the regular payment. Hence the final payment is 687.65 − 0.09 = \$687.56 Combining, we find that the total payment is:
$649 \times 687.65 + 687.56 = \$446\,972.41$

9 First find the regular payment by calculating PMT, assuming 24 equal payments. Using financial solver:
N: 24 (24 monthly payments)
I: 4.9 (Annual interest rate)
PV: 50000 (Amount initially borrowed)
PMT: BLANK (This is what we're looking for)
FV: 0 (We want the loan fully repaid)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, PMT is −2191.33095... so the regular repayment amount is \$2191.33 per month.
If Joan makes this regular payment, we calculate her final payment using financial solver:
N: 24 (24 monthly payments)
I: 4.9 (Annual interest rate)
PV: 50000 (Amount initially borrowed)
PMT: −2191.33 (Regular repayment)
FV: BLANK (We want the loan fully repaid)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is −0.0239288.... so the regular payment should be \$0.02 more than the regular payment. Hence the final payment is 2191.33 + 0.02 = \$2191.35 Combining, we find that the total payment is:
$23 \times 2191.33 + 2191.35 = \$52\,591.94$
Since the loan was for \$50 000, the total interest is the additional amount repaid:
$52\,591.94 - 50\,000 = \$2591.94$

10 **a** Using financial solver:
N: 48 (48 months in 4 years)
I: BLANK (This is what we're trying to find)
PV: −100000 (Amount Olek initially invests)
PMT: 2500 (Olek receives \$2500 each month)
FV: 0 (We want the loan fully repaid)
PpY: 12 (Interest is compounding twelve times per year)

CpY: 12 (Interest is compounding twelve times per year)
Hence, I is 9.241766... so the interest rate correct to two decimal places is 9.24%.

b Using financial solver:
N: 48 (48 months in 4 years)
I: 6 (Assume interest rate is 6%)
PV: −100000 (Amount Olek initially invests)
PMT: BLANK (This is what we're trying to find)
FV: 0 (We want the loan fully repaid)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, PMT is 2348.50290... so the amount that Olek receives each month is $2348.50.

c Using financial solver:
N: BLANK (This is what we're trying to find)
I: 6 (Assume interest rate is 6%)
PV: −100000 (Amount Olek initially invests)
PMT: 2000 (Amount Olek receives each month)
FV: 0 (We want the loan fully repaid)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, N is 57.68012495... so the number of months Olek receives his full payment is 57 months. (Note: In the 58th month Olek will receive a smaller amount to fully exhaust the annuity.)

11 Using financial solver:
N: BLANK (This is what we're trying to find)
I: 4.3 (Assume interest rate is 4.3%)
PV: −300000 (Amount Sophia initially invests)
PMT: 5000 (Amount Sophia receives each quarter)
FV: 0 (We want the loan fully repaid)
PpY: 4 (Interest is compounding four times per year)
CpY: 4 (Interest is compounding four times per year)
Hence, N is 96.855267... so the number of quarters that Sophia receives at least $5000 is 96 quarters. (Note: In the 97th quarter Sophia will receive a smaller amount to fully exhaust the annuity.)

12 **a** Using financial solver:
N: 12 (12 months in one year)
I: 4.7 (Assume interest rate is 4.7%)
PV: −500000 (Amount Kai initially invests)
PMT: BLANK (This is what we're trying to find)
FV: 474965.28 (Value of the annuity after 12 months)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, PMT is 3999.999814... so the amount that Kai receives each month is $4000.

b Over 12 months, Kai receives $12 \times 4000 = \$48\,000$.
Since his annuity declines by $500\,000 - 474965.28 = \$25\,034.72$,

the amount of interest earned is
48 000 − 25 034.72 = $22 965.28.

13 Using financial solver:
N: BLANK (This is what we're trying to find)
I: 3.1 (Assume interest rate is 3.1%)
PV: −3000 (Amount Simone initially invests)
PMT: −250 (An additional $250 is invested each month)
FV: 30000 (Final value of the investment)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, N is 92.82857... so the number of months that it will take for the investment to reach $30 000 is 93.
Thus, the answer is **E**.

14 Using financial solver:
N: 240 (240 months in 20 years)
I: BLANK (This is what we're trying to find)
PV: 480000 (Amount Lachlan borrows)
PMT: −3075.72 (Monthly repayments)
FV: 0 (We want the loan fully repaid)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, I is 4.650001849... so the interest rate correct to two decimal places is 4.65%. Thus, the answer is **C**.

15 Using financial solver:
N: 12 (12 months in one year)
I: 6.3 (Assume interest rate is 6.3%)
PV: −85000 (Amount Audrey initially invests)
PMT: BLANK (This is what we're trying to find)
FV: 71983.41 (Value of the annuity after 12 months)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, PMT is 1499.9999986... so the amount that Audrey receives each month as a regular payment is $1500.
Thus, over one year she receives 1500 × 12 = $18 000. Since the change in the value of the annutiy is 85 000 − 71 983.41 = $13 016.59
In one year, the annuity has paid out $18 000 and declined by $13 016.59 so the total amount of interest earned is the difference of these: $4983.41.
Thus, the answer is **B**

Solutions to Exercise 8G

1 Find the value of Danielle's investment after five years.

Financial Solver

N: 5
I: 7.6
PV: −8000
PMT: −1000
FV: BLANK (solve for this)
PpY: 1
CpY: 1

Solution: FV = 17358.54109 . . .

This value is now used to calculate the next seven years (to reach the total of twelve years because 12 − 5 = 7); we reinvest with all the decimal places

Financial Solver

N: 7
I: 7.6
PV: −17358.54109 . . . (this is now negative because we are reinvesting it)
PMT: −2000
FV: BLANK (solve for this)
PpY: 1
CpY: 1

Solution: FV = 46615.20846 . . .
Rounded to nearest cent gives $46615.21

2 Find the value after ten years ($10 \times 12 = 120$ months)

Financial Solver

N: 120
I: 4.8
PV: −20000
PMT: −200
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 63016.94852 . . . , which is now copy-and-pasted back in (with a negative sign) for the next ten years, but now the payment is $500.

Financial Solver

N: 120
I: 4.8
PV: −63016.94852 . . .
PMT: −500
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 178558.59703 . . .
Rounded to nearest cent gives $178558.60

3 **a** Find the value after 10 years:

Financial Solver

N: 120 (10×12)
I: 6
PV: 0
PMT: −500
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 81939.67340. . . Rounded to nearest cent gives $81939.67

b We use the *rounded* value because the money is withdrawn and placed in a different account (must withdraw to the nearest cent; cannot with draw 0.34032… cents.)

Financial Solver

N: 120
I: 6
PV: −81939.67
PMT: 500
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 67141.09458…
Rounded to nearest cent gives $67141.09

4 Find the value after four years.

Financial Solver

N: 48 (4×12)
I: 10.5
PV: 35000
PMT: −349.43
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = −32437.90262…
So after four years, he still owes $32437.90…, which becomes the new PV.
We now see what would happen if Julien makes 192 payments of $418.66.

Financial Solver

N: 192
I: 13.75
PV: 32437.90262…
PMT: −418.66
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 2.28687…, rounded to nearest cent gives $2.29 so we have overpaid the loan by this amount. Therefore the final payment should be reduced by $2.29, giving 418.66 − 2.29 = $416.37 as the last payment.

5 Find the value at the end of 7 years.

Financial Solver

N: 84 (7×12)
I: 7.5
PV: 500000
PMT: −3694.96
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = −437286.23457…
Use this value for the next seven years:

Financial Solver

N: 84
I: 8.5
PV: 437286.23457…
PMT: −3959.44
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = −338807.90034… So, correct to the nearest cent, they still owe $338807.90

6 **a** Because none of the loan is actually being paid off, the FV is just the negative of the PV. (This part could also be solved using interest-only loans methods.)

Financial Solver

N: 60
I: 8.5
PV: 750000
PMT: (solve for this)
FV: −750000
PpY: 12
CpY: 12

Solution: PMT = −5312.50 So the regular monthly payment is \$5312.50

b Find the regular payment by setting FV to zero.

Financial Solver

N: 300
I: 9.4
PV: 750000
PMT: (solve for this)
FV: 0
PpY: 12
CpY: 12

Solution: PMT = −6500.66573 . . . So the regular payment is \$6500.67 (it is not possible to have a regular payment with more than two decimal places)

c Use the payment found above to get the value after 300 payments

Financial Solver

N: 300
I: 9.4
PV: 750000
PMT: −6500.67
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 5.11781 . . . Rounded to nearest cent gives \$5.12, so the loan has been overpaid by this amount.
Therefore the final payment should be *reduced* to 6500.67 − 5.12 = \$6495.55

d For the first five years ($12 \times 5 = 60$ months), Zian was paying \$5312.50 per month. For the next 299 months, he was paying \$6500.67 per month. In the last month he paid \$6495.55. So the total paid was
$(60 \times 5312.50) + (299 \times 6500.67) + (1 \times 6495.55) = \$2268945.88$

e Zian paid a total of \$2268945.88 in repayments but received \$750000 initially. The interest is the difference in these values, which is 2268945.88 − 750000 = \$1518945.88

7 **a** First, find the value after two years of receiving the regular payment.

Financial Solver

N: 24
I: 6.4
PV: −80000
PMT: 692.50
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 73212.95107 . . .
Now we change to solve for N to see how long it lasts.

Financial Solver

N: (solve for this)
I: 6.2
PV: −73212.95107 . . .
PMT: 692.50
FV: 0
PpY: 12
CpY: 12

Solution: N = 153.33149 . . . Therefore, she can afford 153 more months with this regular payment (round down).

b If you imagine she has 154 more payments in total (153 full payments and 1 smaller payment) we find the future value assuming she withdrew $692.50 each month.

Financial Solver

N: 154
I: 6.2
PV: −73212.95107 . . .
PMT: 692.5
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = −462.54170 . . .
If she did this, the annuity is overdrawn $462.54 (i.e., she would owe $462.54), so the final payment to leave a FV of zero would be 692.50 − 462.54 = $229.96

8 a Calculating the growth multiplier first, we have $R = 1 + \frac{r}{100 \times p} =$ $1 + \frac{5.4}{100 \times 12} = 1.0045$
The initial value, V_0, is 125000 (dollars), and the value of the annuity is decreasing by $850 each month as Ethan withdraws.
This gives us the recurrence relation

$$V_0 = 125000, \quad V_{n+1} = 1.0045V_n - 850$$

b First, find the value of the annuity after two years at the interest rate of 5.4% p.a.

Financial Solver

N: 24
I: 5.4
PV: −125000
PMT: 850
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 117730.85805 . . . So, to the nearest cent he has $117730.86 still in the annuity. (Because the final answer is required to the nearest dollar, it is okay to round intermediate values to the nearest cent.)
We now want Ethan to continue to receive $850 for 18 years at the lower interest rate, so we need to find out how much money he would need in his account to achieve this (i.e., find the Present Value at the start of the 18 years.)

Financial Solver

N: 216
I: 4.1
PV: BLANK (solve for this)
PMT: 850
FV: 0
PpY: 12
CpY: 12

Solution: PV = −129696.52451 . . .
So we need him to have invested \$129696.52 to achieve this.
Since his annuity naturally reached \$117730.86 after two years but he needs it to be at \$129696.52 to last 18 years, he should add 129696.52 − 117730.86 = 11965.66 dollars, which is \$11966, to the nearest dollar.

9 **a** Calculating the growth multiplier first, we have $R = 1 + \frac{r}{100 \times p} =$
$1 + \frac{6}{100 \times 12} = 1.005$
The initial value, S_0, is 150000 (dollars), and there are no additional withdrawals or deposits so we don't add or subtract.
This gives us the recurrence relation
$S_0 = 150000, \quad S_{n+1} = 1.005 S_n$

b *Financial Solver*

N: 120
I: 6
PV: −150000
PMT: 0
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 272909.51010 . . .
Rounded to nearest cent gives \$272909.51
An alternative in this case is to use the explicit rule $S_n = 150000 \times 1.005^n$, is to find $S_{120} = 150000 \times 1.005^{120} =$ \$272909.51

c Since Sameep withdraws the money we can use the rounded value from (b) rather than keep all the decimal places, then solve for the interest rate.

Financial Solver

N: 120
I: BLANK (solve for this)
PV: −272909.51
PMT: 2600
FV: 0
PpY: 12
CpY: 12

Solution: I = 2.719133 . . . , which is 2.72% correct to two decimal places.

10 **a** Find the future value after 18 years with annual compounding.

Financial Solver

N: 18
I: 4
PV: −2000
PMT: −1000
FV: BLANK (solve for this)
PpY: 1
CpY: 1

Solution: FV = 29697.04591 . . .
Rounded to nearest cent gives \$29697.05

b For the next three years (as Marcus goes from 18 to 21), the interest rate changes to 5% and the PMT changes to zero (since the grandparents are no longer contributing.)

Financial Solver

N: 3
I: 5
PV: −29697.04591 . . .
PMT: 0
FV: BLANK (solve for this)
PpY: 1
CpY: 1

Solution: FV = 34378.04277 . . .
Rounded to nearest dollar gives $34378

11 **a** Find the value of the account after one year (12 months)

Financial Solver

N: 12
I: 4.9
PV: −1000
PMT: −200
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 3504.75599 . . .
Rounded to nearest cent gives $3504.76

b Jessica has deposited a total of $200 \times 12 = \$2400$ over the first year.
The account has increased by $3504.76 - 1000 = \$2504.76$
Therefore the interest earned is $2504.76 - 2400 = \$104.76$

c Find the value of the account after three years at the lower interest rate:

Financial Solver

N: 36
I: 4.9
PV: −1000
PMT: −200
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 8897.14073 . . .
Now change the interest rate (to 6%) and the regular deposit (to $350) for the next two years.

Financial Solver

N: 24
I: 6
PV: −8897.14073 . . .
PMT: −350
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 18929.68349 . . .
Rounded to nearest cent gives $18929.68

d Going back to when she first started paying $350 per month, the account had a value of 8897.14073 . . . , so we put that in and solve for N.

Financial Solver

N: (solve for this)
I: 6
PV: −8897.14073 . . .
PMT: −350
FV: 35000
PpY: 12
CpY: 12

Solution: N = 57.30586 . . .

It is not possible to make 57.30586 ... payments, so we need to consider whether to round up or down. If we round down and Jessica only puts in 57 payments, this would not be enough (it would be $34839.95), so we actually need to round up and have 58 payments if we want to exceed $35000 (it would actually be $35364.14 after the 58th payment)

12 First, find the value after two years:

Financial Solver

N: 24
I: 4.31
PV: 500000
PMT: −4200
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = −439852.77656 ...
Next, change the regular payment to $5361.49 and the number of months to $8 \times 12 = 96$

Financial Solver

N: 96
I: BLANK (solve for this)
PV: 439852.77656 ...
PMT: −5361.49
FV: 0
PpY: 12
CpY: 12

Solution: I = 4.00001 ..., which is closest to 4.00%, therefore B is the answer.

13 We need to work from the very final value back to get the initial value.
For the last 20 years of the investment, we know the future value so we work out how much was in the account at the start of that 20-year period.

Financial Solver

N: 240
I: 2.7
PV: BLANK (solve for this)
PMT: 0
FV: 876485.10
PpY: 12
CpY: 12

Solution: PV = −511080.08246 ...
So the amount twenty years ago (10 years after the investment started) was $511080.08, to the nearest cent.
Now we can use this with the changed interest rate of 3.1% to find the value invested.

Financial Solver

N: 120
I: 3.1
PV: BLANK (solve for this)
PMT: 0
FV: 511080.08246 ...
PpY: 12
CpY: 12

Solution: PV = −374999.99906 ...
So the amount invested originally was closest to $375000, therefore B is the answer.
An alternative method is to use an explicit rule (since there are no additional deposits) with x as the unknown initial value, and solve

$$x \times \left(1 + \frac{3.1}{1200}\right)^{10\times12} \times \left(1 + \frac{2.7}{1200}\right)^{20\times12} =$$

876485.10
which has the solution
$x = 374999.99906\ldots$ (same as above).

14 First, find the value two years from now.

Financial Solver

N: 24
I: 3.6
PV: −227727.96
PMT: −2500
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = 306818.95857 . . .

Next, find the payment that would allow Calvin to reach his goal of $800000 after 10 more years.

Financial Solver

N: 120
I: 3.3
PV: −306818.95857 . . .
PMT: (solve for this)
FV: 800000
PpY: 12
CpY: 12

Solution: PMT = −2630.79277 . . .
So, Calvin's new monthly payment is closest to $2630, therefore E is the answer.

Solutions to Exercise 8H

1 Interest paid in one month is:

$$\frac{7.2}{100 \times 12} \times 100\,000 = \$600$$

2 Interest paid in one month is:

$$\frac{8.4}{100 \times 12} \times 50\,000 = \$350$$

3 Interest paid in one fortnight is:

$$\frac{5.46}{100 \times 26} \times 220\,000 = \$462$$

4 Interest paid in one quarter is:

$$\frac{4.95}{100 \times 4} \times 180\,000 = \$2227.50$$

Thus, over a 5 year period (which has 20 quarters), she pays $2227.50 \times 20 = \$44\,550$

5 a Interest paid in one month is:

$$\frac{9.25}{100 \times 12} \times 30\,000 = \$231.25$$

Thus, over one year, Jackson pays $231.25 \times 12 = \$2775$

b Jackson needs to cover the initial cost of the painting and the interest so he must sell the painting for at least $32 775.

6 a Interest paid in one month is:

$$\frac{5.11}{100 \times 12} \times 600\,000 = \$2555$$

b Over 10 years, Ric will have made 120 monthly payments of $2555 which is a total of $2555 \times 120 = \$306\,600$

c Since Ric paid $600 000 initially and then $306 600 in interest, he has paid a total of $906 600. To make a profit of $100 000, he must sell the property for at least $1 06 600.

7 a Interest paid in one month is:

$$\frac{6.24}{100 \times 12} \times 35\,000 = \$182$$

b Since Mindy is only paying the interest, the balance of the loan remains unchanged. Thus, it is $35 000.

c Using financial solver:
N: 180 (180 monthly repayments)
I: 6.24 (Annual interest rate)
PV: 35000 (Amount initially borrowed)
PMT: −300 (Regular repayment)
FV: BLANK (We want the loan fully repaid)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Since FV is 27.52311... then the final payment will be $27.52 less than the regular payment.
Thus, the final payment will be $300 - 27.52 = \$272.48$.

d Over the 20 years, Mindy pays interest only for the first five years then pays 179 regular payments of $300 then a final payment of $272.48. Thus, she pays:
$182 \times 12 \times 5 + 300 \times 179 + 272.48 =$

$64 892.48.

8 Solve for the principal, V_0 on your CAS:

$$\frac{5.3}{100} \times V_0 = \$2120$$

gives $V_0 = 40\ 000$ so the principal is $40 000.

9 Solve for the principal, V_0 on your CAS:

$$\frac{6.6}{100 \times 12} \times V_0 = \$88$$

gives $V_0 = 16\ 000$ so the principal is $16 000.

10 Note that $2352 over two years means receiving 2352 ÷ 24 = $98 per month. Solve for the principal, V_0 on your CAS:

$$\frac{4.2}{100 \times 12} \times V_0 = \$98$$

gives $V_0 = 28\ 000$ so the principal is $28 000.

11 Solve for the annual interest rate, r on your CAS:

$$\frac{r}{100} \times 4000 = \$116$$

gives $r = 2.9$ so the annual interest rate is 2.9%.

12 Solve for the annual interest rate, r on your CAS:

$$\frac{r}{100 \times 12} \times 12\ 000 = \$36$$

gives $r = 3.6$ so the annual interest rate is 3.6%.

13 Note that $3360 over two years means receiving 3360 ÷ 24 = $140 per month. Solve for the annual interest rate, r on your CAS:

$$\frac{r}{100 \times 12} \times 35\ 000 = \$140$$

gives $r = 4.8$ so the annual interest rate is 4.8%.

14 **a** Interest paid in one month is:

$$\frac{4.92}{100 \times 12} \times 320\ 000 = \$1312$$

b Over five years, Svetlana paid: 1312 × 5 × 12 = $78 720.

c Since the loan is interest-only, Svelana paid $86 400 in the second half of the loan.

d There are 60 months in 5 years. Since Svetlana pays a total of $86 400 then she pays 86 400 ÷ 60 = $1440 per month.

e Solve for the annual interest rate, r on your CAS:

$$\frac{r}{100 \times 12} \times 320\ 000 = \$1440$$

gives $r = 5.4$ so the annual interest rate is 5.4%.

15 Since Matthew has an interest-only loan, he will not pay any of the principal. As such, the balance of his loan remains at $310 000. This means the answer is **C**.

16 First find the monthly repayment on a 20 year loan using a financial solver:

N: 240 (240 months in twenty years)
I: 4.82 (Initial interest rate of 4.82%)
PV: 780000 (Amount Eve initially borrows)
PMT: BLANK (This is what we're trying to find)
FV: 0 (Eve wants to repay the loan)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, PMT is 5070.4116218722... so Eve would need to pay \$5070.41 each month for 20 years to repay the loan fully.
The interest rate then changes after two years meaning that there are 18 years left (216 months). Using financial solver:
N: 216 (12 months in one year)
I: BLANK (This is what we're looking for)
PV: 780000 (Initial loan)
PMT: −5070.41 (This is the regular payment)
FV: 0 (Eve wants to repay the loan)
PpY: 12 (Interest is paid twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, r is 3.9971907685... so the interest rate for the remaining 18 years is 4%. Thus the answer is **B**.

17 Since the loan is an interest only loan, $V_1 = V_0 = 56\,000$. Substitute this into the recurrence relation and solve for D:

$$56\,000 = 1.0034 \times 56\,000 - D$$

Thus, $D = 190.40$ so Jason must repay \$190.40 each month. Over five years, he must repay: $190.40 \times 5 \times 12 = \$11\,424$. Thus the answer is **D**.

Solutions to Exercise 8I

1 **a** Interest paid in one quarter is:

$$\frac{6.1}{100 \times 4} \times 642\,000 = \$9790.50$$

b As the perpetuity only pays out interest, the balance of the perpetuity remains unchanged and so after five quarters is \$642 000.

c As the perpetuity only pays out interest, the balance of the perpetuity remains unchanged and so after 10 quarters is \$642 000.

2 **a** Interest paid in one month is:

$$\frac{5.76}{100 \times 12} \times 1\,000\,000 = \$4800$$

b In the first year, Craig receives 12 monthly payments: $4800 \times 12 = \$57\,600$.

3 **a** Interest paid in one month is:

$$\frac{3.6}{100 \times 12} \times 720\,000 = \$2160$$

b Since the perpetuity makes a payment that is calculated based on the interest, an increase in the interest rate results in an increase in the payment made.

4 Solve for the principal, V_0 on your CAS:

$$\frac{2.5}{100} \times V_0 = \$2500$$

gives $V_0 = 100\,000$ so Geoff should invest \$100 000.

5 Solve for the principal, V_0 on your CAS:

$$\frac{4.8}{100 \times 4} \times V_0 = \$600$$

gives $V_0 = 50\,000$ so Barbara should invest \$50 000.

6 **a** Solve for the principal, V_0 on your CAS:

$$\frac{5.2}{100 \times 12} \times V_0 = \$2340$$

gives $V_0 = 540\,000$ so Omar invested \$540 000 in the perpetuity.

b Hence, Omar invested
$920\,000 - 540\,000 = \$380\,000$
in the annuity.

c Using a financial solver:
N: 36 (36 months in three years)
I: 4.8 (Initial interest rate of 4.82%)
PV: −380000 (Amount Omar initially invests)
PMT: −340 (Amount that Omar adds each month)
FV: BLANK (Eve wants to repay the loan)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence, FV is 451866.881734...
so Omar has \$451 866.88 in his investment after three years.

7 Solve for the annual interest rate, r on your CAS:

$$\frac{r}{100} \times 80000 = \$2400$$

gives $r = 3$ so the annual interest rate is 3%.

8 Solve for the annual interest rate, r on your CAS:

$$\frac{r}{100} \times 12000 = \$750$$

gives $r = 6.25$ so the annual interest rate is 6.25%.

9 Solve for the annual interest rate, r on your CAS:

$$\frac{r}{100 \times 12} \times 694400 = \$3645.6$$

gives $r = 6.3$ so the annual interest rate is 6.3%.

10 **a** Marco receives a regular payment of \$1487.50 each month so in the course of a year he receives $1487.5 \times 12 = \$17\,850$.

b Since the perpetuity only pays the interest earned, the balance remains the same. Thus, after one year, the balance of the perpetuity is still \$350 000

c Since the initial balance in the perpetuity is \$350 000, $M_0 = 350\,000$. The perpetuity earns interest at a rate of 5.1% per annum, compounding monthly. Thus

$$R = 1 + \frac{5.1}{100 \times 12} = 1.00425$$

In addition, the perpetuity pays out \$1487.50 each month so $D = 1487.5$. Thus

$M_{n+1} = 1.00425 \times M_n - 1487.5$

11 **a** Interest paid in one month from Option A is:

$$\frac{3.6}{100 \times 12} \times 200\,000 = \$600$$

b Interest paid in one year from Option B is:

$$\frac{3.8}{100} \times 200\,000 = \$7600$$

c Option A pays \$600 per month or $600 \times 12 = \$7200$ per year. Since Option B pays \$7600 per year, Option B pays more.

d Since the initial value of the investment is \$200 000, $Z_0 = 200\,000$. Option B has an annual interest rate of 3.8%, compounding annually so $R = 1.038$. Since it pays \$7600 each year then $D = 7600$. Combining this gives

$Z_{n+1} = 1.038Z_n - 7600$.

12 To find which recurrence relation is a perpetuity, we need to find when the interest earned is equal to the payment made.
We can disregard a and d as a payment is made **to** the perpetuity rather than from it.
For b, $1.005 \times 100\,000 = 100\,500$ so \$500 must be paid out. This is the case so b is correct.
For c, $R < 1$ so the value of the investment is declining so this can be disregarded.
For e, $1.103 \times 200\,000 = 220\,000$ so \$20 000 must be paid out. This is not the case so this answer can be rejected.
Thus the answer is **B**

13 The amount that Aaliyah receives each month is:

$$\frac{5.2}{100 \times 12} \times 120\,000 = \$520$$

In two years there are 24 months so the amount Aaliyah receives in two years is $520 \times 24 = \$12\,480$. Hence the answer is **D**.

Solutions to Review: Multiple-choice questions

1 Since the initial investment is for \$18 000, $V_0 = 18\,000$.
An interest rate of 6.8% compounding yearly means that $R = 1.068$ while a regular addition of \$2500 each year means that $D = 2500$. As such, $V_{n+1} = 1.068V_n + 2500$. **C**

2 In this case, $R = 1.007$. Solving for r:

$$1.007 = 1 + \frac{r}{100 \times 12}$$

gives $r = 8.4$. **D**

3 Using the recurrence relation:
$V_1 = 1.045 \times 20\,000 + 500 = 21\,400$
$V_2 = 1.045 \times 21\,400 + 500 = 22\,863$
Thus the increase in the second month is $22\,863 - 21\,400 = \$1463$.
D

4 Using financial solver:
N: 5 (5 monthly repayments)
I: 6.4 (Annual interest rate)
PV: 28000 (Amount initially borrowed)
PMT: −1200 (Regular repayment)
FV: BLANK (What we want to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Since FV is 22690.331457... so the amount owing after 5 months is \$22 690.33. **B**

5 Using financial solver:
N: 25 (25 monthly repayments)
I: 6.4 (Annual interest rate)
PV: 28000 (Amount initially borrowed)
PMT: −1200 (Regular repayment)
FV: BLANK (What we want to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Since FV is 18.532544... meaning that the final payment is \$18.53 less than the regular payment. Thus, the final payment is $1200 - 18.53 = \$1181.47$ **D**

6 Using financial solver:
N: 12 (12 quarterly repayments)
I: 10 (Annual interest rate)
PV: 6000 (Amount initially borrowed)
PMT: BLANK (What we want to find)
FV: 0 (We want the loan fully repaid)
PpY: 4 (Interest is compounding four times per year)
CpY: 4 (Interest is compounding four times per year)
Since PMT is −584.92276193... the regular quarterly payment is \$584.92. **C**

7 Using financial solver:
N: 24 (24 monthly repayments)
I: 12 (Annual interest rate)
PV: 12000 (Amount initially borrowed)
PMT: −266.90 (Regular repayment)
FV: BLANK (What we want to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding

twelve times per year)
Since FV is −8037.598013... so Paula still owes $8037.60 at the end of the second year which is closest to $8040 from the options listed. **D**

8 Using financial solver:
N: 3 (3 monthly repayments)
I: 7.2 (Annual interest rate)
PV: −12000 (Amount initially borrowed)
PMT: 239 (Regular payment)
FV: BLANK (What we want to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Since FV is 11495.987988... so the value of Ayush's annuity after 3 months is $11 495.99. **B**

9 Using your CAS, type in the initial value of 50 000 and press enter. Then press '×1.0035 − 925.30 6 times gives V_6 = 45 458.62. **C**

10 Read the balance off the table in the first row (Payment number 0) to find that the principal is $40 000. **C**

11 The periodic payment can be read off the table by looking at the payment column. **D**

12 The principal reduction for the third payment can be read off the table by looking at the row associated with payment number 3 and the column associated with the principal reduction. **E**

13 To find the interest rate, use the interest charged ($160) on the principal ($40 000).

$$\frac{160}{40\,000} \times 100\% \times 12 = 4.8\%$$

Thus, the annual interest rate is 4.8%
D

14 Using financial solver:
N: 4 (4 quarterly repayments)
I: 3.6 (Annual interest rate)
PV: 1500 (Amount initially borrowed)
PMT: −383.45 (Regular repayment)
FV: BLANK (What we want to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Since FV is −0.1025665... so there is still $0.10 owing on the loan. Thus, Amir must pay $0.10 more in her final payment. Thus the final payment is 383.45 + 0.10 = $383.55
E

15 Using financial solver:
N: 12 (12 monthly repayments)
I: 7 (Annual interest rate)
PV: −35 300 (Amount initially invested)
PMT: 220 (Regular payment)
FV: BLANK (What we want to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Since FV is 35125.47109... so there is still $35125.47 in the account after 1 year. **C**

16 Using financial solver:
N: BLANK (What we want to find)
I: 3.9 (Annual interest rate)
PV: −5000 (Amount initially invested)
PMT: −1200 (Regular amount added)
FV: 20 000 (Final balance)
PpY: 1 (Interest is compounding once per year)
CpY: 1 (Interest is compounding once per year)
Since N is 9.153534... so it will take at least 10 years for the account to reach a value of $20 000. **D**

17 Using financial solver:
N: 120 (120 monthly repayments)
I: 2.8 (Annual interest rate)
PV: −65 000 (Amount initially invested)
PMT: 0 (Regular repayment)
FV: BLANK (What we want to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence FV is 85 975.39. This, along with an additional $20 000 is invested for a further 10 years:
N: 120 (120 monthly repayments)
I: 3.2 (Annual interest rate)
PV: −105 975.39 (Amount initially invested)
PMT: 0 (Regular payment)
FV: BLANK (What we want to find)
PpY: 12 (Interest is compounding twelve times per year)
CpY: 12 (Interest is compounding twelve times per year)
Hence FV is $145 879.51 **E**

18 The monthly amount is:

$$\frac{5.9}{100 \times 12} \times 175\ 000 = \$860.42$$

The closest answer is $860 **C**

19 Solving for V_0:

$$\frac{3.4}{100} \times V_0 = 400$$

gives $V_0 = 11764.7058824$. Hence the closes amount is $11 765. **E**

20 The monthly amount is:

$$\frac{4.8}{100 \times 12} \times 74\ 000 = \$296$$

Hence over two years,
$296 \times 24 = \$7104$ **D**

Solutions to Review: Extended-response questions

1 **a** Since Josie initially borrows \$250 000, $V_0 = 250\ 000$.

An interest rate of 4.8% that compounds monthly means that

$$R = 1 + \frac{4.8}{100 \times 12} = 1.004$$

Monthly payments of \$1800 mean that $D = 1800$. Thus, $V_{n+1} = 1.004 \times V_n - 1800$

b Using finance solver ($n = 12$, $r = 4.8$, PV=250 000, PMT=−1800, $PPY = CCY = 12$) or your CAS and applying the rule 12 times gives \$240 185.96.l

c Using finance solver:

N: BLANK (What we want to find)

I: 4.8 (Annual interest rate)

PV: 250 000 (Amount initially borrowed)

PMT: −1800 (Regular repayment)

FV: −200000 (Final payment)

PpY: 12 (Interest is compounding twelve times per year)

CpY: 12 (Number of times interest compounds each year)

Hence N is 55.897385... so it will take at least 56 months.

d **i** The interest on one month would be:

$$\frac{4.8}{100 \times 12} \times 250\ 000 = \$1000$$

ii For a year the interest would be $1000 \times 12 = \$12\ 000$

iii As the loan is interest-free, the principal remains unchanged so she still owes \$250 000 on the loan.

2 **a** The interest on one month would be:

$$\frac{6.25}{100 \times 12} \times 150\ 000 = \$781.25$$

b Using finance solver:

N: 12 (12 regular payments)

I: 6.25 (Annual interest rate)

PV: −150 000 (Amount initially invested)

PMT: 1000 (Regular payment received)

FV: BLANK (What we want to find)

PpY: 12 (Number of payments per year)

CpY: 12 (Number of times interest compounds each year)

Hence FV is 147298.48378... so after twelve months, the value of the annuity is 147 298.48.

c Using finance solver:
N: BLANK (What we want to find)
I: 6.25 (Annual interest rate)
PV: −150 000 (Amount initially borrowed)
PMT: 2000 (Regular payment received)
FV: 100000 (Final value)
PpY: 12 (Number of payments per year)
CpY: 12 (Number of times interest compounds each year)
Hence N is 37.27814... so it will take at least 38 months for the annuity to first reduce below $100 000.

d **i** Using finance solver:
N: BLANK (What we want to find)
I: 6.25 (Annual interest rate)
PV: −150 000 (Amount initially invested)
PMT: 4000 (Regular payment received)
FV: 0 (Final value)
PpY: 12 (Number of payments per year)
CpY: 12 (Number of times interest compounds each year)
Hence N is 41.8304... so there will be 41 regular payments of $4000 followed by a smaller final payment.

ii Using finance solver:
N: 42 (Number of payments)
I: 6.25 (Annual interest rate)
PV: −150 000 (Amount initially invested)
PMT: 4000 (Regular payment received)
FV: BLANK (What we want to find)
PpY: 12 (Number of payments per year)
CpY: 12 (Number of times interest compounds each year)
Hence FV is −676.9316... so the final payment is $676.93 less than the regular payment. Thus the final payment is 4000 – 676.93 = $3323.07

3 **a** Using finance solver:
N: 20 (Number of payments)
I: 11 (Annual interest rate)
PV: 10 000 (Amount initially borrowed)
PMT: −656.72 (Regular repayment)
FV: BLANK (What we want to find)
PpY: 4 (Number of payments per year)
CpY: 4 (Number of times interest compounds each year)

Hence FV is 0.07057... so the final payment is \$0.07 less than the regular payment.
Thus, the regular payment is 656.72 – 0.07 = \$656.65

b The total repayments for the loan consist of 19 regular payments plus the final payment: 19 × 656.72 + 656.65 = \$13 134

c The total interest paid is the difference between the total repayments and the amount of the loan: 13 134.33 − 10 000 = \$3134

4 a Using finance solver:
N: BLANK (What we want to find)
I: 10.8 (Annual interest rate)
PV: 24800 (Amount initially borrowed)
PMT: −750 (Regular repayment)
FV: 0 (Final value)
PpY: 12 (Number of payments per year)
CpY: 12 (Number of times interest compounds)
Hence N is 39.4266... so there will be a total of 40 payments.

b Using finance solver:
N: 40 (Number of payments)
I: 10.8 (Annual interest rate)
PV: 24800 (Amount initially borrowed)
PMT: −750 (Regular repayment)
FV: BLANK (What we want to find)
PpY: 12 (Number of payments per year)
CpY: 12 (Number of times interest compounds each year)
Hence FV is 429.21977... so the final payment is \$429.22 less than the regular repayment: 750 – 429.22 = \$320.78.

c Since the total interest paid is the difference between the total repayments and the value of the loan, we first find the total repayments.
Total repayments: 39 × 750 + 320.78 = \$29 570.78.
Interest paid: 29 570.78 − 24 800 = \$4770.78

5 a The amount of interest owing on \$100 000 is:

$$\frac{9.6}{100 \times 4} \times 100\ 000 = \$2400$$

The amount paid in a regular payment is found using a financial solver:
Using finance solver:
N: 100 (Number of payments)
I: 9.6 (Annual interest rate)

PV: 100000 (Amount initially borrowed)
PMT: BLANK (What we want to find)
FV: 0 (Final value)
PpY: 4 (Number of payments per year)
CpY: 4 (Number of times interest compounds each year)
Hence PMT is −2647.03847 so the regular quarterly payment is \$2647.04.
Thus the amount paid off the principal is 2647.04 − 2400 = \$247.04

b Using finance solver:
N: 40 (Number of payments)
I: 9.6 (Annual interest rate)
PV: 100000 (Amount initially borrowed)
PMT: −2647.04 (Regular payment)
FV: BLANK (What we want to find)
PpY: 4 (Number of payments per year)
CpY: 4 (Number of times interest compounds each year)
Hence FV is −83 713.37 so Elsa should pay a lump sum of \$83.713.37 at the end of 10 years, in addition to the regular payment.

6 a First calculate R using $r = 4.5$ given monthly compounding

$$R = 1 + \frac{4.5}{100 \times 12} = 1.00375$$

So after 10 years: $1.00375^{10 \times 12} \times 750\,000 = \$1\,175\,244.58$

b From **a**, Helen withdraws \$1 175 244.58 from the investment account and places this in an annuity. Using finance solver:
N: BLANK (What we want)
I: 3.5 (Annual interest rate)
PV: −1 175 244.58 (Amount initially invested)
PMT: −6000 (Regular amount received)
FV: 0 (Final value)
PpY: 12 (Number of payments per year)
CpY: 12 (Number of times interest compounds each year)
Hence N is 290.8221336... so there will be 290 payments of \$6000 (followed by a final payment)

c Placing \$1 100 000 in a perpetuity at 3.6% per year, compounding monthly, means that the monthly payment is

$$\frac{3.6}{100 \times 12} \times 1\,100\,000 = \$3300$$

Solutions to Review: Multiple-choice questions

1 Using the rule:
$A_0 = 5$
$A_1 = 2 \times 5 - 1 = 9$
$A_2 = 2 \times 9 - 1 = 17$
$A_3 = 2 \times 17 - 1 = 33$
Hence, $A_3 = 33$. **E**

2 Note that 270 is added to the previous term and the previous term is not multiplied. This tells us that the sequence is arithmetic and so represents simple interest.
$T_0 = 6000$ which tells us that the initial investment or loan is for \$6000.
Calculating
$$\frac{270}{6000} \times 100\% = 4.5\%$$
so the annual interest rate is 4.5% **C**

3 7.2% compounding monthly means
$$R = 1 + \frac{7.2}{100 \times 12} = 1.006$$
Thus, $R + 1.06$ **C**

4 Applying 10% depreciation each year using the rule:
$V_0 = 9000$
$V_1 = 0.9 \times 9000 = 8100$
$V_2 = 0.9 \times 8100 = 7290$
$V_3 = 0.9 \times 7290 = 6561$
$V_4 = 0.9 \times 6561 = 5904.9$
Thus, the last line is incorrect. **E**

5 The rule takes the form:
$V_{n+1} = R^n \times V_0$. We know that $V_0 = 14\,000$, that $n = 4$ (since interest compounds quarterly) and that $V_4 = 14\,686.98$. Then solving $14\,686.98 = R^4 \times 14\,000$ gives $R = 1.012$.
Note that
$$1.012 = 1 + \frac{r}{100 \times 4}$$
so $r = 4.8192$ is the annual percentage interest rate.
Using your CAS to find the effective interest rate we get 4.9%. **C**

6 After two years, value of the caravan is:
Year 0: \$75 000
Year 1: $0.88 \times 75\,000 = \$66\,000$
Year 2: $0.88 \times 66\,000 = \$58\,080$
Using $n = 3$, $V_3 = 41\,638.56$
and $V_0 = 58\,080$, we can solve $41\,638.56 = R^3 \times 58\,080$ for R to get $R = 0.894999...$ Thus, the depreciation is closest to
$(1 - 0.894999) \times 100\% = 10.5\%$ **D**

7 Using financial solver:
N: BLANK (What we want to find)
I: 3.6 (Annual interest rate)
PV: −375 000 (Amount invested)
PMT: −400 (Additional amounts)
FV: 650 000 (Final balance)
PPY: 12 (Payments per year)
CPY: 12 (Compounding periods per year)
Thus, N is 144.3564... so it takes 145 months. **E**

8 Using financial solver:

N: 60 (Number of repayments)
I: 7.9 (Annual interest rate)
PV: 54 000 (Amount borrowed)
PMT: −1092 (Repayment amounts)
FV: BLANK (Final balance)
PPY: 12 (Payments per year)
CPY: 12 (Compounding periods per year)
Thus, FV is −25.1196755... so Carlos needs to pay an additional $25.12 in his final payment. Thus, the final payment is $1092 + 25.12 = \$1117.12$. **D**

9 Note that solving

$$1.002 = 1 + \frac{r}{100 \times 12}$$

for r gives $r = 2.4\%$ as an annual interest rate, compounding monthly. Further, since

$$\frac{350}{175\,000} \times 100 = 0.2$$

we can see that the value is unchanged. Thus, the recurrence relation represents a perpetuity with an interest rate of 2.4% compounding monthly. **A**

10 Calculate the interest over the balance for each period as follows:

$$\frac{1742.64}{402\,148.19} \times 100 = 0.433327$$
$$\frac{1709.03}{394\,390.83} \times 100 = 0.433327$$
$$\frac{1675.27}{386\,599.86} \times 100 = 0.433327$$
$$\frac{1609.79}{378\,775.13} \times 100 = 0.4249988$$

Since this rate is different, the interest rate has changed so the first payment with the new interest rate is payment 17. **D**

11 First, note that $D_0 = 225\,000$ since she invests $225 000.
To find R, note that $r = 5.2$ and that interest compounds quarterly

$$R = 1 + \frac{5.2}{100 \times 4} = 1.013$$

This, combined with the fact that 3800 is deducted each quarter means that $V_{n+1} = 1.103V_n - 3800$. **D**

12 Solve

$$\frac{4.3}{100 \times 4} \times V_0 = 675.10$$

for V_0 gives $62 800. **D**

13 Using financial solver: N: BLANK (Number of repayments)
I: 3.1 (Annual interest rate)
PV: −317 922.75 (Amount invested)
PMT: −420 (Regular investments)
FV: 720 000 (Final balance)
PPY: 26 (Payments per year)
CPY: 26 (Compounding periods per year)
Thus, N is 394.407098... so it will take 395 fortnights for the account to reach at least $720 000. **E**

14 Solve

$$\frac{1080}{180\,000} \times 100 = 0.6$$

Then each month the interest rate is 0.6% so the annual interest rate is $0.6 \times 12 = 7.2\%$ **E**

15 For the first year using financial solver: N: 12 (Number of repayments)
I: 10.4 (Annual interest rate)
PV: 32 000 (Amount borrowed)
PMT: −380 (Regular repayments)
FV: BLANK (Final balance)
PPY: 12 (Payments per year)

CPY: 12 (Compounding periods per year)
Thus, FV is −30707.5446... at the end of the first year.
Now use a financial solver for the second year: N: 12 (Number of repayments)
I: 3.1 (Annual interest rate)
PV: 30 707.54 (Amount borrowed)
PMT: −510 (Regular repayments)
FV: BLANK (Final balance)
PPY: 12 (Payments per year)
CPY: 12 (Compounding periods per year)
Thus, FV is −27637.52222... at the end of the second year.
Over the two year period, Maya has paid $380 \times 12 + 510 \times 12 = \$10\,680$ and the value of the loan has gone down $4362.48. Thus, Maya has paid $10\,680 - 4362.48 = \$6317.52$ in interest. **D**

Solutions to Review: Extended-response questions

1 **a** The initial investment is given by $A_0 = 8500$ (the value after 0 quarters); she invested \$8500 initially.

b Her account at the end of the second quarter had a value of \$8722.44 (reading A_2 from the question), from an initial value of \$8500. No additional payments were made so the interest is the difference, i.e., $8722.44 - 8500 = \$222.44$

c The next value each time is calculated by multiplying the current value by 1.013, and the initial value is 8500 so the recurrence relation is

$$A_0 = 8500, \quad A_{n+1} = 1.013A_n$$

d Using the formula $R = 1 + \dfrac{r}{100 \times p}$ we can use $p = 4$ (since it is quarterly) and solve the equation $1.013 = 1 + \dfrac{r}{100 \times 4}$ for r (using CAS or algebra).
The solution of $r = 5.2$ means the annual interest rate is 5.2%

e This could be solved using algebra or financial solver (also by trial-and-error but that is not shown here.) We could solve the inequality $8500 \times 1.013^n \geq 10000$ (using CAS or algebra) which gives $n \geq 12.58254\ldots$, so 13 quarters are required to ensure the value is at least \$10000.

Financial Solver

N: (solve for this)
I: 5.2
PV: −8500
PMT: 0
FV: 10000
PpY: 4
CpY: 4

Solution: N = 12.58254 . . . , which needs to be rounded up to ensure we have at least \$10000. Therefore it is at the end of quarter number 13 that she has enough.

2 **a** To show this using recursion, we need to write out each step from V_0 using the recurrence relation, stating the calculation used and the result each time.
$V_0 = 25\,000$
$V_1 = 25\,000 - 936 = 24\,064$
$V_2 = 24\,064 - 964 = 23\,128$
$V_3 = 23\,128 - 964 = 22\,192$ (dollars) as required

b We know the value depreciates by \$0.50 per hour of use (given in the question), and that it also depreciates by \$936 each year (seen from the recurrence relation). Therefore each year it is being used for $936 \div 0.5 = 1872$ hours each year.
We are told that it is being used for the same amount of time each of the 52 weeks, so each week it is being used $1872 \div 52 = 36$ hours.

3 **a** This is given by the balance at the very start (top row of the table), which is \$260 000.

b Calculate the interest of any payment divided by the balance at the end of the previous payment (so you need to look at the interest in one row and the balance in the row above it.)
So the first calculation is $1170 \div 260\,000 = 0.0045$ (or $1164.59 \div 259\,020 = 0.0045$, or $1161.16 \div 258\,035.59 = 0.0045$)
To convert to an annual interest rate, multiply by 12 (because they are monthly) and then multiply by 100 to get a percentage.
So the second calculation would be $0.0045 \times 12 \times 100 = 5.4$, therefore the interest rate is 5.4% p.a.
You could also combine into one calculation like $\frac{1170}{260\,000} \times 12 \times 100 = 5.4$

c A is the interest, which can be calculated using 0.0045×257046.75 (the number 0.0045 comes from $1170 \div 260\,000$, or you could use the provided interest rate of 5.4% per annum to calculate $\frac{r}{100 \times p} = \frac{5.4}{100 \times 12} = 0.0045$). Therefore
$A = 0.0045 \times 257046.75 = 1156.71$
B is the principal reduction, which is $2150 - A = 2150 - 1156.71 = 993.29$
C is the new balance, which is $257\,046.75 - 993.29 = 256\,053.46$

4 **a** Use the financial solver to see how many payments of \$1350 it takes to get the value of the loan down to zero dollars (fully repaid).

Financial Solver

N: (solve for this)
I: 3.6
PV: 240000
PMT: −1350
FV: 0
PpY: 26
CpY: 26

Solution: N = 204.21739 . . . , so that means the first 204 payments will be the full amount of \$1350.

b 7 years of fortnightly payments is $7 \times 26 = 182$.

Financial Solver

N: 182
I: 3.6
PV: 240000
PMT: −1350
FV: BLANK (solve for this)
PpY: 26
CpY: 26

Solution: FV = −29516.72532 . . .
So after seven years, Millie still owes $29516.73 (the remaining balance of the loan), which means this is the amount she will pay.

c The amount of interest depends on how much is owed. When Millie owes the most money that is when she will pay the most interest. This occurs in the first year because in later years, the balance is lower (hence the name *reducing balance* loan.)

5 a The value was decreased by $50 after 20 shifts, so the value of the scooter at this point was 4450 − 50 = $4400

b In 20 shifts, the scooter was depreciated by $50, so a calculation to get the depreciation per shift is $50 \div 20 = 2.50$

c The initial value is $4450 and it is depreciating by $2.50 each shift, so a rule would be $H_n = 4450 - 2.5 \times n$. (This could also be modelled with a recurrence relation but that is not what is being asked for here.)

d $H_{200} = 4450 - 2.5 \times 200 = 3950$, so the value after 200 evening shifts is is $3950.

e Solving $4450 - 2.5n < 3000$ for n using CAS or by hand gives $n > 580$.
The first value of n greater than 580 is 581, so it takes 581 shifts to first fall *below* $3000. (Note that $n = 580$ is incorrect because then $4450 - 2.5 \times 580 = 3000$ so the scooter's value is exactly $3000, which is not below $3000).

6 a *Financial Solver*

N: 4
I: 3.54
PV: −65000
PMT: (solve for this)
FV: 43592.5
PpY: 12
CpY: 12

Solution: PMT = 5520.00108 . . . So the monthly payment is $5520.00

b We need to find how long the annuity will last from the balance of $43 592.50 (given in the question as the balance after four months) until it has run out.

Financial Solver

N: (solve for this)
I: 3.54
PV: −43592.5
PMT: 2000
FV: 0
PpY: 12
CpY: 12

Solution: N = 22.56176 . . .
So the artist will receive 22 months of the $2000 payment followed by a final payment that is smaller. (The last payment is $1124.26, but the question does not ask for this so it is inadvisable to include that in your answer.)

Solutions to Exercise 10A

1 **a** 2 rows and 3 columns.$\therefore$ order$= 2\times 3$

b 1 row and 3 columns.$\therefore$ order$= 1\times 3$

c 3 rows and 2 columns.$\therefore$ order$= 3\times 2$

d 3 rows and 1 column.$\therefore$ order$= 3\times 1$

e 3 rows and 3 columns.$\therefore$ order$= 3\times 3$

2 **a** 2 rows and 3 columns.$\therefore$ order$= 2\times 3$

b 4 rows and 1 column.$\therefore$ order$= 4\times 1$

c 1 row and 3 columns.$\therefore$ order$= 1\times 3$

3 **a** A 2×6 matrix has two rows and 6 columns, so each row has 6 elements. Thus a 2×6 matrix has $2\times 6 = 12$ elements.

b A 3×5 matrix has three rows and 5 columns, so each row has 5 elements. Thus a 3×5 matrix has $3\times 5 = 15$ elements.

c A 7×4 matrix has t7 rows and 4 columns, so each row has 4 elements. Thus a 7×4 matrix has $7\times 4 = 28$ elements.

4 A $m\times n$ matrix has mn elements. From this it follows matrices of the orders 12×1, 12×1, 6×2, 2×6, 3×4, 4×3 all have 12 elements.

5 To find the transpose of a matrix, interchange its rows and columns. Thus,

a $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

b $\begin{bmatrix} 3 \\ 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \end{bmatrix}$

c $\begin{bmatrix} 9 & 1 & 0 & 7 \\ 8 & 9 & 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 9 & 8 \\ 1 & 9 \\ 0 & 1 \\ 7 & 5 \end{bmatrix}$

6 **a** square matrix

b column matrix

c row matrix

7 **a** $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8 **a** Square matrices have the same number of rows as columns. The square matrices are C and E (they have dimensions 2×2 and 3×3 respectively).

b Matrix B has 3 rows.

c The row matrix is A (it contains a single row of elements only).

d The column matrix is B (it contains a single column of elements only).

e Matrix D has 4 rows and 2 columns.

f The order of matrix E is 3×3. It has 3 rows and 3 columns.

g The order of matrix A is 1×5. It has 1 row and 5 columns.

h The order of matrix B is 3×1. It has 3 rows and 1 column.

i The order of matrix D is 4×2. It has 4 rows and 2 columns.

j There are 9 elements in matrix E.

k There are 5 elements in matrix A.

l $a_{14} = 0$: a_{14} is the element in row 1 and column 4

m $b_{31} = 1$: b_{31} is the element in row 3 and column 1

n $c_{11} = 0$.

o $d_{41} = 4$.

p $e_{22} = -1$.

q $d_{32} = 3$.

r $b_{11} = 3$.

s $c_{12} = 1$.

9 Set up a blank 3×2 matrix and call it B.

$$B = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

The rule $b_{ij} = i \times j$ tells us to find the value each element, multiply its row number by its column number.

$$B = \begin{bmatrix} 1 \times 1 = 1 & 1 \times 2 = 2 \\ 2 \times 1 = 2 & 2 \times 2 = 4 \\ 3 \times 1 = 3 & 3 \times 2 = 6 \end{bmatrix}$$

10 Set up a blank 4×1 column matrix and call it C. $C = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{bmatrix}$

The rule $c_{ij} = i+2j$ tells us to find the value each element, add twice its column number to its row number.

$$C = \begin{bmatrix} 1 + 2 \times 1 = 3 \\ 2 + 2 \times 1 = 4 \\ 3 + 2 \times 1 = 5 \\ 4 + 2 \times 1 = 6 \end{bmatrix}$$

11 Set up a blank 3×2 matrix and call it D.

$$D = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

The rule $d_{ij} = i \times j$ tells us to find the value each element, subtract three times its column number from its row number.

$$D = \begin{bmatrix} 1 - 3 \times 1 = -2 & 1 - 3 \times 2 = -5 \\ 2 - 3 \times 1 = -1 & 2 - 3 \times 2 = -4 \\ 3 - 3 \times 1 = 0 & 3 - 3 \times 2 = -3 \end{bmatrix}$$

12 Set up a blank 1×3 matrix and call it D.

$E = \begin{bmatrix} \dots & \dots & \dots \end{bmatrix}$

The rule $e_{ij} = i + j^2$ tells us to find the value each element, the square of its column number to its row number

$E =$

$\begin{bmatrix} 1 + 1^2 = 2 & 1 + 2^2 = 5 & 1 + 3^2 = 10 \end{bmatrix}$

13 Set up a blank 2×2 matrix and call it F.

$F = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$

The rule $f_{ij} = i \times j$ tells us to find the value each element, add the row and column number and square the result

$D = \begin{bmatrix} (1+1)^2 = 4 & (1+2)^2 = 9 \\ (2+1)^2 = 9 & (2+2)^2 = 16 \end{bmatrix}$

14 a Follow the instructions on page 371 (TI) or page 371 (CAS) to enter the matrix and determine the transpose.

$B^T = \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}$

b As for **a** above. $C^T = \begin{bmatrix} 4 & -2 \\ -4 & 6 \end{bmatrix}$

c As for **a** above.

$E^T = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

d As for **a** above.

$F^T = \begin{bmatrix} 2 & 3 \end{bmatrix}$

Exam 1 type questions

15 C

The second and third matrices are diagonal matrices.

16 D

$m_{ij} = 3i + 2j$

$m_{11} = 5 \quad m_{12} = 7 \quad m_{13} = 9$

$m_{21} = 8 \quad m_{22} = 10 \quad m_{23} = 12$

$m_{31} = 11 \quad m_{32} = 13 \quad m_{33} = 15$

17 B

The column entries in a given row go up by 3 and the row entries in a given column go up by 2. Now check:

First row entries

$2 + 3 = 5 \quad 2 + 6 = 8 \quad 2 + 9 = 11$

Second row entries

$4 + 3 = 7 \quad 4 + 6 = 10 \quad 4 + 9 = 13$

Third row entries

$6 + 3 = 9 \quad 6 + 6 = 12 \quad 6 + 9 = 15$

18 C

19 B Only B works

Solutions to Exercise 10B

1 a See Example 7for an illustrative example .
Construct the matrix by enclosing the numbers in the table in square brackets to form the square matrix.
$\begin{bmatrix} 4 & 2 & 1 \\ 6 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$ Order: 3×3: the matrix has 3 rows and 3 columns.

b Construct the matrix by enclosing the numbers in row B of the table in square brackets to form the row matrix. $[6 \quad 2 \quad 3]$ Order: 1×3: the matrix has 1 row and 3 columns.

c Construct the matrix by enclosing the numbers in the 'Computers' column in square brackets to form the column matrix.
$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$
Order: 3×1: the matrix has 3 rows and 1 column.
The sum of the elements will represent the total number of computers owned by members of the three households.

2 a See Example 7 for an illustrative example.
Construct the matrix by enclosing the numbers in the table in square brackets to form the rectangular matrix.
$\begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix}$ Order: 2×3: 2 rows and 3 columns

b Row matrix: $[24 \quad 32 \quad 11]$
Order = 1×3.

c Column matrix: $\begin{bmatrix} 24 \\ 32 \end{bmatrix}$ Order: 2×1.
The sum of the elements will represent the total number of small cars sold by both dealers.

3 a 4×4

b $\begin{bmatrix} 430 & 380 & 950 & 900 \end{bmatrix}$
The sum is the tootal exports ofexporter B.

c $\begin{bmatrix} 370 \\ 950 \\ 150 \\ 470 \end{bmatrix}$ Order is 4×1

4 $\begin{bmatrix} 200 & 110 \\ 180 & 117 \\ 135 & 98 \\ 110 & 89 \\ 56 & 53 \\ 28 & 33 \end{bmatrix}$

5 See Example 8 for an illustrative example.
The 2×8 matrix representing the paired digits listed one under the other is as follows:
$\begin{bmatrix} 3 & 5 & 8 & 7 & 0 & 2 & 3 & 6 \\ 4 & 2 & 2 & 9 & 0 & 0 & 0 & 9 \end{bmatrix}$

6 $\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

7 See Example 9 for an illustrative example.
The matrices for the given graphs are given below:

a $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

8 See Example 9 for an illustrative example.
With Town 1 as the first row and column, Town 2 as the second and Town 3 as the third, the following matrix can be created: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

9 See Example 10 for an illustrative example.

a $f_{34} = 1$, so girls 3 and 4 are friends.

b $f_{25} = 0$, so girls 2 and 5 are **not** friends.

c The sum of row 3 elements will tell us the total number of friends for girl 3: three friends.

d The girl with the least friends is girl 1, with only 1 friend. The girl with the most friends is girl 3, with 3 friends.

10 **a** **i** Polar bears eat cod

ii Polar bears are not eaten by seals, cod or other polar bears

b $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Exam 1 type questions

11 **C**
Row 2, Column 2 entry 154.

12 **B**
By inspection the largest number is 154.
It occurs in row 2, column 2.
Goatmilk and Bunchof

Solutions to Exercise 10C

1 $a + 2 = 11 \Rightarrow a = 9$
$b - 1 = 6 \Rightarrow b = 7$

2 **a** The only matrices to be equal are C and F They are the same order and contain identical elements in identical positions.

b A and B have the same order as each other (1×2); C and F are of the same order (2×3); D and E are of the same order (2×2).

c Matrices that can be added and subtracted must be of the same order as each other.
Thus, the 3 pairs of matrices – A and B, C and F, and D and E – can all be added to and subtracted from each other.

d **i** $A + B = \begin{bmatrix} 3 + 1 = 4 & 3 + 1 = 4 \end{bmatrix}$

ii $D + E = \begin{bmatrix} 0 + 1 = 1 & 1 + 0 = 1 \\ -1 + 2 = 1 & 2 + -1 = 1 \end{bmatrix}$

iii $C - F = \begin{bmatrix} 0 - 0 = 0 & 1 - 1 = 0 & 4 - 4 = 0 \\ 3 - 3 = 0 & 2 - 2 = 0 & 1 - 1 = 0 \end{bmatrix}$

iv $A - B = \begin{bmatrix} 1 - 3 = -2 & 3 - 1 = 2 \end{bmatrix}$

v $E - D = \begin{bmatrix} 1 - 0 = 1 & 0 - 1 = -1 \\ 2 - -1 = 3 & -1 - 2 = -3 \end{bmatrix}$

vi $3B = [3 \times 3 = 9 \quad 3 \times 1 = 3]$

vii $4F =$
$$\begin{bmatrix} 4 \times 0 = 0 & 4 \times 1 = 4 & 4 \times 4 = 16 \\ 4 \times 3 = 12 & 4 \times 2 = 8 & 4 \times 1 = 4 \end{bmatrix}$$

viii $3C + F = 4F(\text{ since } C = F)$
$= \begin{bmatrix} 0 & 4 & 16 \\ 12 & 8 & 4 \end{bmatrix}$

ix $4A - 2B = [4 \times 1 - 2 \times 3 = -2 \quad 4 \times 3 - 2 \times 1 = 10]$

x $E + F$ is not possible. The matrices are not of the same order, so this addition cannot be performed and thus is undefined.

3 $2\begin{bmatrix} 8 & 0 \\ -4 & 2 \end{bmatrix} - 8\begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$
$= \begin{bmatrix} 16 & 0 \\ -8 & 4 \end{bmatrix} - \begin{bmatrix} 16 & 0 \\ -8 & 4 \end{bmatrix}$
$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4 These example are designed to be done quickly by hand.

a $\begin{bmatrix} 1 + 4 = 5 & 2 + 3 = 5 \\ 4 + 1 = 5 & 3 + 2 = 5 \end{bmatrix}$

b $\begin{bmatrix} 1 - 4 = -3 & 2 - 3 = -1 \\ 4 - 1 = 3 & 3 - 2 = 1 \end{bmatrix}$

c $\begin{bmatrix} 1 + 8 = 9 & 2 + 6 = 8 \\ 4 + 2 = 6 & 3 + 4 = 7 \end{bmatrix}$

d $\begin{bmatrix} 1 - 1 = 0 & 1 - 1 = 0 \end{bmatrix}$

e $\begin{bmatrix} 0 + 1 = 1 \\ 1 + 0 = 1 \end{bmatrix}$

f $\begin{bmatrix} 0+2=2 \\ 3+0=3 \end{bmatrix}$

g $\begin{bmatrix} 0-2=-2 \\ 3-0=3 \end{bmatrix}$

h Matrices are not of the same order so this addition cannot be performed and thus is undefined.

5 Following the instructions given on page 494 (TI) or pages 494 – 495 (CASIO), enter the matrices involved into your calculator, and perform the required computations to obtain the answers.

6 $2\begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 12 & 6 \end{bmatrix}$

$8 + x = 10 \Rightarrow x = 2$

$4 + y = 8 \Rightarrow y = 4$

$6 + z = 12 \Rightarrow z = 6$

$10 + w = 6 \Rightarrow w = -4$

7 a $A = \begin{bmatrix} 2.4 \\ 3.5 \\ 1.6 \end{bmatrix} B = \begin{bmatrix} 2.8 \\ 3.4 \\ 1.8 \end{bmatrix} C = \begin{bmatrix} 2.5 \\ 2.6 \\ 1.7 \end{bmatrix}$

$D = \begin{bmatrix} 3.4 \\ 4.1 \\ 2.1 \end{bmatrix}$

b The sum of the 4 matrices represents the total sale of CDs in a year for each of the three stores.

$\begin{bmatrix} 2.4+2.8+1.5+3.4=11.1 \\ 3.5+3.4+2.6+4.1=13.6 \\ 1.6+1.8+1.7+2.1=7.2 \end{bmatrix}$

8 a $A = \begin{bmatrix} 16 & 104 & 86 \\ 75 & 34 & 94 \end{bmatrix}$

$B = \begin{bmatrix} 24 & 124 & 100 \\ 70 & 41 & 96 \end{bmatrix}$

b $C = A + B =$ $C = A + B =$

$\begin{bmatrix} 16+24=40 & 104+124=228 & 86+100=186 \\ 75+70=145 & 34+41=75 & 94+96=190 \end{bmatrix}$

Matrix C represents the total numbers of males and females enrolled in all the programs for the two years of data given.

c $D = B - A =$

$\begin{bmatrix} 24-16=8 & 124-104=20 & 100-86=14 \\ 70-75=-5 & 41-34=7 & 96-94=2 \end{bmatrix}$

Matrix D represents the increase in the number of males and females in all the programs from 2005 to 2006. The negative element in the matrix represents a decrease in numbers in that category from 2005 to 2006.

d E = 2B

$= \begin{bmatrix} 2\times 24=48 & 2\times 124=248 & 2\times 100=200 \\ 2\times 70=140 & 2\times 41=82 & 2\times 96=192 \end{bmatrix}$

Exam 1 type questions

9 C

$\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -4+4-6+4 & 2+2-2+2 \\ -3+(-3)-3+3 & 1+(-1)-1+1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 4 \\ -6 & 0 \end{bmatrix}$

10 C $2M - 2N = 2\begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 4 \\ -6 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 8 \\ 6 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$

11 E

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 7 \\ 5 & 8 & 11 \\ 7 & 11 & 15 \end{bmatrix}$$

Solutions to Exercise 10D

1 a i AB-yes: $(1 \times 2)\ (2 \times 1)$

ii BA-yes: $(2 \times 1)\ (1 \times 2)$

iii AC-no: $(1 \times 1)\ (1 \times 3)$

iv CE-yes: $(1 \times 3)\ (3 \times 1)$

v EC-yes: $(3 \times 1)\ (1 \times 3)$

vi EA-yes $(3 \times 1)\ (1 \times 2)$

vii DB-yes: $(2 \times 2)\ (2 \times 1)$

viii CD-no: $(1 \times 3)\ (2 \times 2)$

b i $AB = [1 \times 3 + 3 \times 1 = 6]$

ii $CE = [1 \times 2 + 0 \times 1 + -1 \times 0 = 2]$

iii $DB = \begin{bmatrix} 0 \times 3 + 1 \times 1 = 1 \\ -1 \times 3 + 2 \times 1 = -1 \end{bmatrix}$

iv $AD =$
$\begin{bmatrix} 1 \times 0 + (-1) \times 3 = -3 & 1 \times 1 + 3 \times 2 = 7 \end{bmatrix}$

c Following the instructions given on pages 504–505 (TI and CASIO), enter the matrices involved into your calculator, and perform the required computations to obtain the answers.

2 a $[0 \times 1 + 2 \times 0 = 1] = [0]$

b $[1 \times 1 + 0 \times 2 = 1] = [1]$

c $[2 \times 1 + 0 \times 3 + 1 \times 1 = 1] = [3]$

d $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$
$\begin{bmatrix} 1 \times 1 + 2 \times 0 = 1 \\ 3 \times 1 + 4 \times 0 = 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

e $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$
$\begin{bmatrix} 1 \times 4 + 1 \times 1 = 5 & 1 \times 3 + 1 \times 2 = 5 \\ 0 \times 4 + 1 \times 1 = 1 & 0 \times 3 + 1 \times 2 = 2 \end{bmatrix} =$
$\begin{bmatrix} 5 & 5 \\ 1 & 2 \end{bmatrix}$

f $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} =$
$\begin{bmatrix} 1 \times 2 + 0 \times 0 + 1 \times 1 = 3 \\ 0 \times 2 + 1 \times 0 + 0 \times 1 = 0 \\ 1 \times 2 + 1 \times 0 + 0 \times 1 = 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$

3 Following the instructions given on pages 504 – 505 (TI and CASIO), enter the matrices involved into your calculator, and perform the required computations to obtain the answers.

4 a To sum the rows of a 3×2 matrix, post multiply by the 2×1 column matrix $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b To sum the columns of a 3×2 matrix, pre multiply by the 1×3 row matrix $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

5 $\begin{bmatrix} 7 & 1 & 2 \\ 1 & 2 & 2 \\ 8 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 13 \end{bmatrix}$

6 To sum the columns of a 3×3 matrix, pre multiply by the 1×3 row matrix $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 2 \\ 1 & 7 & 3 \\ 8 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 9 + 1\times 1 + 1\times 8 & 1\times 0 + 1\times 7 + 1\times 3 & 1\times 2 + 1\times 3 + 1\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 10 & 9 \end{bmatrix}$$

7 **a** To sum the rows of a 5×5 matrix, *post* multiply by the 1×5 summing matrix (a column matrix of 1s).

$$\begin{bmatrix} 2 & 4 & 1 & 7 & 8 \\ 1 & 9 & 0 & 0 & 2 \\ 3 & 4 & 3 & 3 & 5 \\ 2 & 1 & 1 & 1 & 7 \\ 5 & 3 & 6 & 7 & 9 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2+4+1+7+8 \\ 1+9+0+0+2 \\ 3+4+3+3+5 \\ 2+1+1+1+7 \\ 5+3+6+7+9 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \\ 18 \\ 12 \\ 30 \end{bmatrix}.$$

b To sum the rows of a 3×5 matrix, *pre* multiply by the 1×3 summing matrix (a row matrix of 1s).

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 5 & 1 & 2 & 1 \\ 0 & 3 & 4 & 5 & 1 \\ 4 & 2 & 1 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+4 & 5+3+2 & 1+4+1 & 2+5+7 & 1+!+9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 10 & 6 & 14 & 11 \end{bmatrix}$$

8 $RP =$

$$\begin{bmatrix} 4\times 2 + 1\times 1 + 0\times 0 = 9 \\ 3\times 2 + 1\times 1 + 1\times 0 = 7 \\ 3\times 2 + 0\times 1 + 2\times 0 = 6 \\ 1\times 2 + 2\times 1 + 2\times 0 = 4 \\ 1\times 2 + 1\times 1 + 3\times 0 = 3 \\ 0\times 2 + 1\times 1 + 4\times 0 = 1 \end{bmatrix}$$

9 $TE =$

$$\begin{bmatrix} 10\times 25 + 20\times 40 + 30\times 65 = 3000 \\ 15\times 25 + 20\times 40 + 25\times 65 = 2800 \\ 20\times 25 + 20\times 40 + 20\times 65 = 2600 \\ 30\times 25 + 20\times 40 + 10\times 65 = 2200 \end{bmatrix}$$

10 **a** Q has 2 rows and 3 columns, so the order of Q is 2×3.
$M = QP$

$M = QP$

b **i** $$= \begin{bmatrix} 25 & 34 & 19 \\ 30 & 45 & 25 \end{bmatrix}\begin{bmatrix} 2.50 \\ 1.80 \\ 3.20 \end{bmatrix}$$

$$= \begin{bmatrix} 184.5 & 236 \end{bmatrix}$$

ii The total revenue from selling A, B and C.

c PQ is not defined as the no of columns of P(1) is **not equal** to the number of rows of Q(2).

11 $$\begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix} = \begin{bmatrix} 110\,000 \\ 116\,000 \\ 154\,000 \\ 58\,000 \end{bmatrix}$$

XY is the total sales amount of each of the dealers

12 **a** $$\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times 1 + 12\times 2 \\ 2.50\times 1 + 3.00\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$$

1×5 min plus 2×12 min means 29 min for one milkshake and two banana splits.
The total cost is \$8.50.

b

$$\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times 1 + 12\times 2 & 5\times 2 + 12\times 1 & 5\times 0 + 12\times 1 \\ 2.5\times 1 + 3\times 2 & 2.5\times 2 + 3\times 1 & 2.5\times 0 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix}$$

The matrix shows that John spent 29 min and \$8.50, one friend spent 22 min and \$8.00 (2 milkshakes and 1 banana split) while the other friend spent 12 min and \$3.00 (no milkshakes and 1 banana split).

13 a The 2×3 matrix is: $\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}$

The rows correspond to the semesters and the columns to the forms of assessment.

b The percentages of the three components can be represented in the 3×1 matrix: $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$

c Multiplying the two matrices gives the semester scores.

$$\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 79.2 \\ 80.4 \end{bmatrix}$$

Notice that multiplication of a 2×3 matrix by a 3×1 matrix results in a 2×1 matrix.

d For Chemistry the result is given by the following multiplication.

$$\begin{bmatrix} 86 & 82 & 84 \\ 81 & 80 & 70 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 83.8 \\ 75.2 \end{bmatrix}$$

e The aggregate of the four marks is 318.6. This is below 320.

f Three marks will be required to obtain an aggregate of marks above 320.

14, 15,16, & 17] Enter the relevant matrix or matrices into your calculator, use worked Example 23 as a model for performing these computations.

Exam 1 type questions

18 A

$$\begin{bmatrix} 6 & 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} =$$

$$\begin{bmatrix} 6\times 3 + 2\times 6 + 0\times 9 \end{bmatrix} = \begin{bmatrix} 30 \end{bmatrix}$$

19 C

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 26 \\ 14 & 16 & 40 \\ 5 & 7 & 14 \end{bmatrix}$$

20 D

$7 \times 6 + 9 \times 8$

Solutions to Exercise 10E

1 a i i 2×2 identity matrix: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ii ii 3×3 identity matrix: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii iii 4×4 identity matrix: $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b $AI = \begin{bmatrix} 1 \times 1 + 2 \times 0 = 1 & 1 \times 0 + 2 \times 1 = 2 \\ 0 \times 1 + 3 \times 0 = 0 & 0 \times 0 + 3 \times 1 = 3 \end{bmatrix} = A$

$IA = \begin{bmatrix} 1 \times 1 + 0 \times 0 = 1 & 1 \times 2 + 0 \times 3 = 2 \\ 0 \times 1 + 1 \times 0 = 0 & 0 \times 2 + 1 \times 3 = 3 \end{bmatrix} = A/$

Thus, we can see that $AI = IA = A$.

c

$$CI = \begin{bmatrix} 1 \times 1 + 2 \times 0 + 0 \times 0 = 1 & 1 \times 0 + 2 \times 1 + 0 \times 0 = 2 & 1 \times 0 + 2 \times 0 + 0 \times 1 = 0 \\ 3 \times 1 + 1 \times 0 + 0 \times 0 = 3 & 3 \times 0 + 1 \times 1 + 0 \times 0 = 1 & 3 \times 0 + 1 \times 0 + 0 \times 1 = 0 \\ 0 \times 1 + 1 \times 0 + 2 \times 0 = 0 & 0 \times 0 + 1 \times 1 + 2 \times 0 = 1 & 0 \times 0 + 1 \times 0 + 2 \times 1 = 2 \end{bmatrix} = C$$

$$IC = \begin{bmatrix} 1 \times 1 + 0 \times 3 + 0 \times 0 = 1 & 1 \times 2 + 0 \times 1 + 0 \times 1 = 2 & 1 \times 0 + 0 \times 0 + 0 \times 2 = 0 \\ 0 \times 1 + 1 \times 3 + 0 \times 0 = 3 & 0 \times 2 + 1 \times 1 + 0 \times 1 = 1 & 0 \times 0 + 1 \times 0 + 0 \times 2 = 0 \\ 0 \times 1 + 0 \times 3 + 1 \times 0 = 0 & 0 \times 2 + 0 \times 1 + 1 \times 1 = 1 & 0 \times 0 + 0 \times 0 + 1 \times 2 = 2 \end{bmatrix} = C$$

Thus, we can see that $CI = IC = C$.

2 Multiplying all the pairs of matrices results in the 2×2 identity matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

eg $= \begin{bmatrix} 1 \times 2 + 1 \times -1 = 1 & 1 \times 2 + 2 \times -1 = 0 \\ 1 \times 2 + 2 \times -1 = 0 & 1 \times -1 + 2 \times -1 = 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

3 a $\det(A) = 1 \times 3 - 0 \times 2 = 3$

b $\det(B) = 0 \times 4 - 1 \times 3 = -3$

c $\det(C) = 1 \times 4 - 2 \times 2 = 0$

d $\det(D) = -1 \times 4 - 2 \times 2 = -8$

4 Follow the instructions on page 516 (TI) or 517 (CASIO) to generate the following inverse matrices using your calculator.

a $A^{-1} = \begin{bmatrix} \frac{10}{11} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

b $B^{-1} = \begin{bmatrix} \frac{20}{9} & \frac{1}{18} \\ -\frac{50}{9} & \frac{1}{9} \end{bmatrix}$

c D^{-1} does not exist, since $\det(D) = 0$.

d $E^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

5 **a** $B + X = C$

$$X = C - B$$

$$= \begin{bmatrix} -6 & -1 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -3 \\ 1 & 3 \end{bmatrix}$$

b $BX = C$

$$X = B^{-1}C$$

$$= \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -6 & -1 \\ 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -28 & -15 \\ 39 & 22 \end{bmatrix}$$

c $XB = C$

$$X = CB^{-1}$$

$$= \begin{bmatrix} -6 & -1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 9 \\ -9 & 8 \end{bmatrix}$$

d $BX = D$

$$X = B^{-1}D$$

$$= \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

e $AX = E$

$$X = A^{-1}E$$

$$= \begin{bmatrix} -1 & -2 & \frac{5}{2} \\ 0 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{2} \\ \frac{5}{2} \\ -\frac{3}{2} \end{bmatrix}$$

f $BX + \begin{bmatrix} 7 \\ 6 \end{bmatrix} = D$

$$BX = D - \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$X = B^{-1} \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ 10 \end{bmatrix}$$

6 $A\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}5 & 6\\12 & 14\end{bmatrix}$

$$A = \begin{bmatrix}5 & 6\\12 & 14\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}^{-1}$$

$$= \begin{bmatrix}-1 & 2\\-3 & 5\end{bmatrix}$$

7 a $AX = C$

$A^{-1}AX = A^{-1}C$

$IX = A^{-1}C$

b $ABX = C$

$A^{-1}ABX = A^{-1}C$

$IBX = A^{-1}C$

$BX = A^{-1}C$

$B^{-1}BX = B^{-1}A^{-1}C$

$X = B^{-1}A^{-1}C$

c $AXB = C$

$A^{-1}AXB = A^{-1}C$

$IXB = A^{-1}C$

$XB = A^{-1}C$

$XBB^{-1} = A^{-1}CB^{-1}$

$X = A^{-1}CB^{-1}$

d $A(X + B) = C$

$A^{-1}A(X + B) = A^{-1}C$

$I(X + B) = A^{-1}C$

$(X + B) = A^{-1}C$

$X = A^{-1}C - B$

e $AX + B = C$

$$AX = C - B$$

$$A^{-1}AX == A^{-1}(C - B)$$

$$X = A^{-1}(C - B)$$

f $XA + B = A$

$$XA = A - B$$

$$XAA^{-1} = (A - B)A^{-1}$$

$$X = I - BA^{-1}$$

8 $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 15000 \\ 20000 \\ 10000 \end{bmatrix}$$

$$= \begin{bmatrix} -5000 \\ 15000 \\ 0 \end{bmatrix}$$

9

Spray	P	Q	R
Barrels	$\frac{8}{13}$	$\frac{46}{39}$	$\frac{12}{13}$

10 **a** $A^{-1} = \begin{bmatrix} 0.1 & 0.25 & -0.4 \\ 0.3 & -0.75 & 0.8 \\ -0.2 & 0.5 & -0.2 \end{bmatrix}$

b Let a, b and c be the rate of assembly of products P, Q and R respectively.

Let $\mathbf{X} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Then $\mathbf{AX} = \mathbf{K}$

Therefore,

$\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

$$\mathbf{X} = \frac{1}{20}\begin{bmatrix} 2 & 5 & -8 \\ 6 & -15 & 16 \\ -4 & 10 & -4 \end{bmatrix} \begin{bmatrix} 95 \\ 80 \\ 40 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 13.5 \\ 0.5 \\ 13 \end{bmatrix}$$

11 Suppose Brad, Flynn and Lina employ x, y and z workers respectively. The there contractors need to supply the warehouse with 310 dresses, 175 slacks and 175 shirts, so x, y and z must satisfy the matrix equation

$$\begin{bmatrix} 3 & 6 & 10 \\ 3 & 4 & 5 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 310 \\ 175 \\ 175 \end{bmatrix}$$

which is in the for $\mathbf{AX} = \mathbf{B}$, there A is the 3×3 matrix, $\mathbf{X}$ is the column matrix of the variables and $\mathbf{B}$ is the column matrix of the numbers required.

The solution is given by: $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

Use a calculator to find A^{-1}, then multiply by $\mathbf{B}$ to find $\mathbf{X}$.

$$A^{-1} = \frac{1}{20}\begin{bmatrix} -10 & 30 & -10 \\ -5 & 5 & 5 \\ 10 & -18 & 2 \end{bmatrix}$$

$$\mathbf{X} = \frac{1}{20}\begin{bmatrix} -10 & 30 & -10 \\ -5 & 5 & 5 \\ 10 & -18 & 2 \end{bmatrix}\begin{bmatrix} 310 \\ 175 \\ 175 \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ 10 \\ 15 \end{bmatrix}$$

So Brad need 20 workers, Flynn need 10 workers and Lina need 15 workers.

Exam 1 type questions

12 A

$2 \times 1 - 2 \times (-1) = 4$

13 E

Use calculator.

14 D

$$3\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix} + X = \begin{bmatrix} 14 & 12 \\ 18 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 6 \\ 18 & 21 \end{bmatrix} + X = \begin{bmatrix} 14 & 12 \\ 18 & 22 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}$$

Solutions to Exercise 10F

1 B only: a permutation matrix is a square binary matrix with only one '1' per row.

2 a $PX =$

$$\begin{bmatrix} S & H & U & T \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} H & U & T & S \end{bmatrix}.$$

b $$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

I: the identity matrix

so

$P^4X = I^4X = X$

3 $$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} C \\ D \\ A \\ B \end{bmatrix}$$

4 $$PA = \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

$$A = P^{-1}\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

$$= \begin{bmatrix} X \\ W \\ Z \\ Y \end{bmatrix}$$

5 a Follow the instructions in Example 24 (page 400) to construct the following 3×3 communication matrix.

$$C = \begin{array}{c} \begin{matrix} M & F & L \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} M \\ F \\ L \end{matrix}$$

b Follow the instructions in Example 24 (page 400) to construct the following 3×3 communication matrix.

Use your calculator to square the communication matrix to obtain

$$C^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

c To find the total number of ways Mei can communicate with Freya, calculate

$$T = C + C^2$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{matrix} M & F & L \end{matrix}$$

$$= \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{matrix} M \\ F \\ L \end{matrix}$$

From the matrix, we can see that there are 2 ways that Mei can send a message to Freya (Row M, Row F).

6 a In the communication matrix, a zero indicates there is no *direct* communicate link. For example, from the diagram we see that there is no way for a person in tower 1 to directly communicate with a person in tower 4 and this is indicated in the matrix by placing a '0' in row T1, column T4.

b The '1' in row T1, column T2, and row T3, column T2, indicate that a person in towers 1 and 3 can both communicate directly with a person in tower 2.

c From the diagram, it can be seen that the missing element in

- column T2 is zero, because there is no direct communication link between tower T2 and tower T4.

- column T3 is a '1' because there is a direct communication link between tower T2 and tower T3.

d The C^2 matrix gives the number of ways a person can communicate with a person in another tower via a third tower. For example, the '1' in row T1, column T3 indicates that there is one two-step communication link between tower T1 and tower T3. From this diagram, we see that this two-step communication link is T1→T2→T3.

e 6 (=1 + 2 + 2 +1): A redundant communication link is one in which a person eventually end up receiving the message they sent. The communication links in the leading diagonal of the matrix are all redundant communication links. For example the 1 in row T1, column 1, represents the redundant communication link: T1→T2→T1.

f The matrix $T = C + C^2$ shows the total number of one and two-step links between pairs of tower.

g Tower 1 and tower 4 (T1→T2→T3→T4): for a person to communicate with a person in tower 4, they need to first communicate with a person in tower 2 who will need to pass the message onto person in tower 3. This person can then pass the message onto the person in tower 4 tower link and vice versa.

7 Follow the instructions given in Example 24 (page 400) to arrive at the matrix but note that there are both uni-directional and bi-directional communication links. Because of this, it is very important to be very clear in defining whether the source of the communication is representing by the rows or the columns. In the solution below, the rows represent the source of the communication. For example, the ‘1’ in row A, column E, indicates that a message can be sent *from* A *to* E.

$$\begin{array}{cc} & \textit{To} \\ & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \textit{From} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Exam 1 type questions

8 D

There is no direct communication from Adam to David (not A)
There is no direct communication from Bertie to David (not B)
There is no direct communication from Catherine to David (not C)
D works
There is no direct communication from Catherine to Bertie (not E)

9 A

Solutions to Exercise 10G

1 a Follow the instructions given in Example 24 (page 400) to arrive at the matrix. Use the row labels to indicate the winners and the column labels to represent the losers as in the answer shown below.

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \textit{winners} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

To find the total one-step dominances for each team, sum the rows: $A(2)$, $B(1)$, $C(0)$, $D(3)$

Ranking the players according to one-step dominances: D, A, B, C

b To find the total two-step dominances for each team, square the one-step dominance matrix and sum the rows.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

To rank the teams by one and two-step dominances, add the one-step and two-step dominance matrices and sum the rows.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{array}{c} 3 \\ 1 \\ 0 \\ 6 \end{array}$$

Ranking the teams using both one and two step dominances we have: $D(6)$, $A(3)$, B(1), $C(0)$ (the same ranking as before).

2 a To rank the players according to one-step dominances, find the total one-step dominances for each of the players by summing the rows and rank the players from highest to lowest according to their total one-step dominances.

$$\text{winners}\ \begin{matrix} & \text{losers} \\ & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix} = D$$

To find the total one-step dominances for each player, sum the rows: $A(4)$, $B(1)$, $C(0)$, $D(2)$, $E(2)$

Ranking the players according to one-step dominances: *A; D* & *E* equal; *B; C*

b To find the total two-step dominances for each player, square the one-step dominance matrix and sum the rows.

$$D^2 = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix} \begin{matrix} 5 \\ 0 \\ 0 \\ 1 \\ 3 \end{matrix}$$

c To rank the players by one and two-step dominances, add the one-step and two-step dominance matrices and sum the rows. $T = D + D^2 = \begin{bmatrix} 0 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 \end{bmatrix} \begin{matrix} 9 \\ 1 \\ 0 \\ 3 \\ 5 \end{matrix}$

Ranking the players using both one and two step dominances we have: $A(9)$, $E(5)$, D(3), B(1), $C(0)$.

3 a

$$\mathbf{D} = \begin{matrix} & \begin{matrix} A & B & C & D & E \end{matrix} & \text{Score} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} & \begin{matrix} 2 \\ 3 \\ 1 \\ 1 \\ 3 \end{matrix} \end{matrix}$$

- Using the one-step dominance matrix, we see that Bea and Eve are equal first.

b

$$\mathbf{D}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}$$

c

$$\mathbf{D} + \mathbf{D}^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Score} \\ 4 \\ 9 \\ 2 \\ 4 \\ 7 \end{array}$$

The matrix $\mathbf{D} + \mathbf{D}^2$ gives the following ranking:

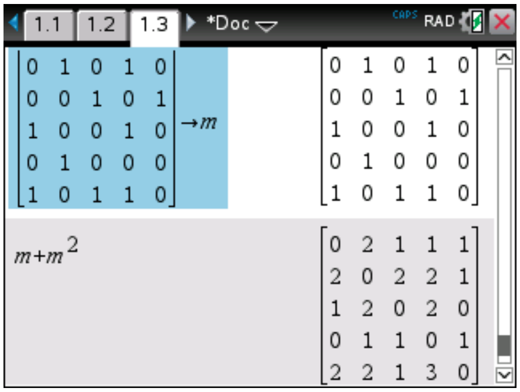
Rank	Player	Score
First	Bea	9
Second	Eve	7
Equal third	Ann and Deb	4
Fifth	Cat	2

4 a 10

b
- Ash defeats Carl and Dot
- Ben defeats Ash, Carl and Dot
- Carl defeats Dot
- Dot defeats Elle
- Elle defeats Ash, Ben and Carl

c Add the 1's across a row to determine dominance score. Scores: Ash 2, Ben 3, Carl 1, Dot 1, Elle 3
Ben = Elle, Ash, Carl = Dot

5 a $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

b 

Scores: $A - 5, B - 7, C - 5, D - 3, E - 8$ $E,\ B,\ A = C,\ D$

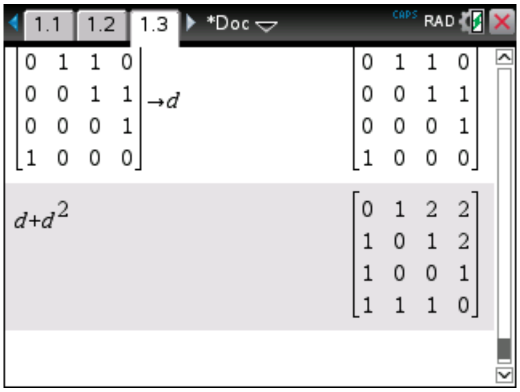
6 **a** $M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$M^2 = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$M + M^2 = \begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

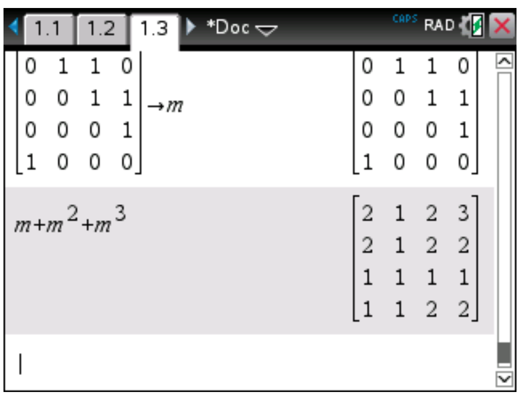
b Person A (5)
Person B (1)
Person C(0)
Person D (3)
Person A is the most inflential.

7 **a** $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

b 

Scores: $A-5, B=4, C-2, D-3$

$A,\ B,\ D,\ C$

8 

Scores: $A-8, B=7, C-4, D-6$

$A,\ B,\ D,\ C$

$A,\ B,\ D,\ C$

9 $E,\ B,\ A=C,\ D$

Exam 1 type questions

10 C

If B lost to A then A defeated B

If C defeated A then A lost to C

If B defeated D then D lost to B

11 A

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$M + M^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

Totals

Alpha: 3

Beta: 8

Gamma: 2

Delta: 6

Epsilon: 7

12 **A** Trial and error

Chapter Review: Multiple-choice questions

1 W is the row matrix as it contains a single row. $\Rightarrow$ **C**

2 Square matrices have the same number of rows as columns. U and Y are both 2×2 square matrices. $\Rightarrow$ **D**

3 The order of matrix X is 2 rows by 3 columns = 2×3.

$\Rightarrow$ **B**

4 U and Y can be added as they are both 2×2 matrices. $\Rightarrow$ **D**

5 XY is undefined as X is a 2×3 matrix and Y is a 2×2 matrix. $\Rightarrow$ **D**

6 $-2Y =$

$$\begin{bmatrix} -2 \times 0 = 0 & -2 \times 1 = -2 \\ -2 \times -1 = 2 & -2 \times 2 = -4 \end{bmatrix} \Rightarrow \textbf{A}$$

7 X is a 2×3 matrix and Z is a 3×1 matrix so XZ is 2×1 matrix. $\Rightarrow$ **B**

8 To obtain U^T from U, interchange the rows and columns in U.
If

$$U = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

then

$$U^T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ **C**

9 $a^{23} = 3 \Rightarrow$ **D**

10 $$\begin{bmatrix} 4 - -1 = 5 & 0 - 0 = 0 \\ -2 - 1 = -3 & 2 - 1 = 1 \end{bmatrix} \Rightarrow \textbf{E}$$

11 $[1 \times 3 + 2 \times 2 + 3 \times 1 = 10] \Rightarrow$ **A**

12 V cannot be raised to a power as it is not a square matrix. $\Rightarrow$**B**

13 $\det(U) = 2 \times 1 - 1 \times 0 = 2 \Rightarrow$**D**

14 $\det(Y) = 1 \times 4 - 2 \times 2 = 0$.
Thus, the inverse of Y is undefined.
$\Rightarrow$**E**

15 $\det(U) = 2$

$$U^{-1} = \begin{bmatrix} \frac{1}{2} = 0.5 & \frac{0}{2} = 0 \\ \frac{-1}{2} = -0.5 & \frac{2}{2} = 1 \end{bmatrix} \Rightarrow \textbf{A}$$

16 UW: U & W are both 2×2 matrices
$\Rightarrow$**D**

17 $$\begin{bmatrix} 1 \times 2 + 2 \times 1 = 4 \\ 3 \times 2 + 4 \times 1 = 10 \end{bmatrix} \Rightarrow \textbf{D}$$

18 X is 3×2, Y is 2×3 and Z is 2×2
XY is defined as columns of X match rows of Y.
YX is defined as columns of Y match rows of X.
XZ is defined as columns of X match rows of Z, this result in a 3×2 matrix, therefore $XZ - 2X$ is defined.
YX results in a matrix of order 2×2, this has the same order as matrix Z, therefore $YX + 2Z$ is defined.
$XY - YX$: XY is a 3×3 matrix, YX is a 2×2 matrix, so $XY - YX$ is not defined.
$\Rightarrow$ **E**

19 The mean of 3, 5, 2, 4 is given by $\frac{3 + 5 + 2 + 4}{4}$ which gives a mean of $\frac{14}{4}$.
The matrix that displays the mean

must have an order of 1×1. The only option which gives a 1×1 matrix is D. $\Rightarrow$ **D**

20 **C**

21 **A**

22 **C**

23 $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

$\Rightarrow$ **C**

24 **C** $M^T = \begin{bmatrix} 3 & 4 & 7 & 9 \\ 0 & 1 & 2 & 6 \end{bmatrix}$

25 **A**

NM is a 5×6 matrix so $NM - 2N$ is defined. A is correct.

MN is not defined. So answer is not B.

M^2N is not defined. So answer is not C.

N^T is a matrix of order 6×5 so N^TN is not defined. So answer is not D.

M^T has order 6×6, so M^TN is not defined. So answer is not E.

26 Row 1:C is to go from the top to third so '1' is in the third position.

Row 2: B is to go from the second to second so '1' is in the second position.

Row 3: A is to go from the third tofirst so '1' is in the first position.

Row 4: D is to go from the fourth tofourth position first so '1' is in the fourth position.

C

27 **C** By trial and error.

28 **C**

29 Use the matrix to eliminate the incorrect responses.

A: *A* and *D* from the matrix we see that a car can travel from *A* to *D* (there is a '1' in column *A* and row *D*) but not from *D* to *A* (there is a '0' in column *D* and row *A*), so this cannot be the solution.

B: *B* and *C* from the matrix we see that a car can travel from *B* to *C* (there is a '1' in column *B* and row *C*) but not from *C* to *B* (there is a '0' in column *C* and row *B*) , so this cannot be the solution.

C: *C* and *D* from the matrix we see that a car cannot travel from *C* to *D* (there is a '0' in column *C* and row *D*) so this cannot be the solution.

D: *D* and *E* from the matrix we see that a car can travel from *D* to *E* (there is a '1' in column *D* and row *E*) but not from *D* to *E* (there is a '0' in column *D* and row *E*), so this cannot be the solution.

E: *C* and *E* from the matrix we see that a car can travel from *C* to *E* (there is a '1' in column *C* and row *E*). It can also travel from *E* to *C* (there is a '1' in column *E* and row *C*), so this is the solution.

$\Rightarrow$ **E**

Chapter Review: Extended-response questions

1 a ■ Construct a square matrix with as many rows as there are towns and number both the rows and columns.

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

■ Count the number of roads connecting each pair of two towns and insert this number into the appropriate cell of the matrix to obtain the matrix below.

For example, for town 1, there are no roads connecting town 1 to itself so insert a 0 in row 1 column 1, and so on to obtain.

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

b Follow the procedure for **1a** but start with a matrix with 4 rows and 4 columns because there are 4 towns.

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

c Follow the procedure for **1a**, but start with a matrix with 2 rows and 2 columns because there are two towns. As there are no roads connecting the two towns, it will be a matrix of zeros.

2 a $C = \begin{bmatrix} 30.45 \\ 2.54 \end{bmatrix}$

C is a 2×1 matrix; 2 rows and 1 column.

b $J = [\,5\ 4\,]$

J is a 1×2 matrix; 1 row and 2 columns.

c JC is defined, as the number of columns in J equals the number of rows in C.

d $JC = [\,5\ 4\,]\begin{bmatrix} 30.45 \\ 2.54 \end{bmatrix} = [162.41]$

Jodie's height is 162.41 cm.

e $HC = \begin{bmatrix} 5 & 8 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 30.45 \\ 2.54 \end{bmatrix} = \begin{bmatrix} 172.57 \\ 185.24 \end{bmatrix}$

The heights of the two people are 172.57 cm and 185.24 cm.

3 a Bookshop 1 carries 456 non-fiction paperbacks.

b $A = \begin{bmatrix} 334 & 876 \\ 213 & 456 \end{bmatrix}$ Order of A is 2×2.

c $B = \begin{bmatrix} 354 & 987 \\ 314 & 586 \end{bmatrix}$

d $C = A + B$

$$= \begin{bmatrix} 334 + 354 & 876 + 987 \\ 213 + 314 & 456 + 586 \end{bmatrix} = \begin{bmatrix} 688 & 1863 \\ 527 & 1042 \end{bmatrix}$$

C represents the total number of each type of book which is in stock at either bookshop. (Assume no common titles.)

e i $E = \begin{bmatrix} 45.00 \\ 18.50 \end{bmatrix}$ The order of E is 2×1, two rows, 1 coulmn.

ii $AE = \begin{bmatrix} 334 & 876 \\ 213 & 456 \end{bmatrix} \begin{bmatrix} 45.00 \\ 18.50 \end{bmatrix} = \begin{bmatrix} 31236 \\ 18021 \end{bmatrix}$

iii The product represents the total value of fiction and non-fiction books in Bookshop 1. The fiction books are worth \$31236. The non-fiction books are worth \$18021.

f $2A = 2\begin{bmatrix} 334 & 876 \\ 213 & 456 \end{bmatrix} = \begin{bmatrix} 668 & 1752 \\ 426 & 912 \end{bmatrix}$

4 a P has order 1×5.

b i $R = NP$

$$= \begin{bmatrix} 600 \\ 320 \end{bmatrix} \begin{bmatrix} 0.15 & 0.225 & 0.275 & 0.25 & 0.10 \end{bmatrix}$$

$$= \begin{bmatrix} 90 & 135 & 165 & 150 & 60 \\ 48 & 72 & 88 & 80 & 32 \end{bmatrix}$$

ii The first row of NP gives the number of students in each grade category in Mathematics starting at A and going through to E.
The second row of NP gives the number of students in each grade category in Physics starting at A and going through to E. R_{13} is the number of Mathematics students wo got a C.

c i $F = \begin{bmatrix} 220 & 197 \end{bmatrix}$
F_{11} is the Mathematics fee and F_{12} the Physics fee.

ii $T = \begin{bmatrix} 220 & 197 \end{bmatrix} \times \begin{bmatrix} 600 \\ 320 \end{bmatrix} = \begin{bmatrix} 195\,040 \end{bmatrix}$

The total fees paid are \$195 040.

5 a $N = \begin{bmatrix} 8 & 6 & 1 \end{bmatrix}$

b $P = N \times G = \begin{bmatrix} 8 & 6 & 1 \end{bmatrix} \times \begin{bmatrix} 25 & 50 & 75 \end{bmatrix} = \begin{bmatrix} 575 \end{bmatrix}$

c The total score.

6 a Total number of tonnes 20 + 20 + 40 = 80

b Total number of tonnes 20 + 40 + 40 = 100

c Total revenue = 46 000 + 34 000 + 106 000 = 186 000

d i 3×1

ii The price per tonne of each of the minerals

iii Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the matrix representing the selling price per tonne of each the minerals p, q, r. That is, the price per tonne of p is \$$x$, the price per tonne of q is \$$y$ and the price per tonne of r is \$$z$. Then we have the matrix equation:

$$\begin{bmatrix} 20 & 20 & 40 \\ 0 & 40 & 20 \\ 60 & 40 & 60 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 46\,000 \\ 34\,000 \\ 106\,000 \end{bmatrix}$$

We will use a calculator to find the inverse matrix of $\begin{bmatrix} 20 & 20 & 40 \\ 0 & 40 & 20 \\ 60 & 40 & 60 \end{bmatrix}$.

$$\begin{bmatrix} 20 & 20 & 40 \\ 0 & 40 & 20 \\ 60 & 40 & 60 \end{bmatrix}^{-1} = \frac{1}{100} \begin{bmatrix} -4 & -1 & 3 \\ -3 & 3 & 1 \\ 6 & -1 & -2 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{100} \begin{bmatrix} -4 & -1 & 3 \\ -3 & 3 & 1 \\ 6 & -1 & -2 \end{bmatrix} \begin{bmatrix} 46\,000 \\ 34\,000 \\ 106\,000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1000 \\ 700 \\ 300 \end{bmatrix}$$

Solutions to Exercise 11A

1 **a** $$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.4 & 0.55 \\ 0.6 & 0.45 \end{bmatrix} \end{array}$$

b $$T = \begin{array}{c} \\ X \\ Y \end{array}\begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{bmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{bmatrix} \end{array}$$

c $$T = \begin{array}{c} \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 0.6 & 0.15 & 0.22 \\ 0.1 & 0.7 & 0.23 \\ 0.3 & 0.15 & 0.55 \end{bmatrix} \end{array}$$

d $$T = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.45 & 0.35 & 0.15 \\ 0.25 & 0.45 & 0.2 \\ 0.3 & 0.20 & 0.65 \end{bmatrix} \end{array}$$

2 $$T = \begin{array}{c} \\ O \\ B \end{array}\begin{array}{c} \begin{array}{cc} O & B \end{array} \\ \begin{bmatrix} 0.96 & 0.98 \\ 0.04 & 0.02 \end{bmatrix} \end{array}$$

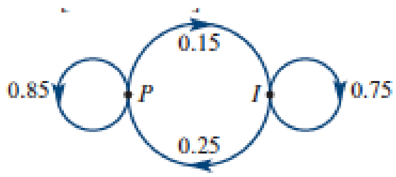
3 $$T = \begin{array}{c} \\ F \\ P \end{array}\begin{array}{c} \begin{array}{cc} F & P \end{array} \\ \begin{bmatrix} 0.8 & 0.14 \\ 0.2 & 0.86 \end{bmatrix} \end{array}$$

4 **B**

Solutions to 11B

1 a a

$$T = \begin{bmatrix} P & I \\ 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix} \begin{matrix} \\ P \\ I \end{matrix} \quad \text{next time}$$



b 85% of 80 = 0.85 × 80 = 68

c 25% of 60 = 0.25 × 60 = 15

d 0.15 × 120 + 0.75 × 40 = 48

2 Reading from table:

a **i** 10%

ii 80%

iii 10%

b **i** 80% of 850 = 0.80 × 850 = 680

ii 10% of 850 = 0.10 × 850 = 85

c **i** 00% of 1150 = 1 × 1150 = 1150

ii 0% of 1150 = 0 × 1150 = 0

iii 0% of 1150 = 0 × 1150 = 0

d All (100%) of the sea birds that nest at site A this year will nest at site A next year.

3 a **i** 91% of 84 000 = 76 440

ii 9% of 84 000 = 7560

b **i** 22% of 25 000 = 5500

ii 22% of 5500 = 1210

iii 22% of 1210 = 266

4 a **i** 30 × 0.6 = 18 fleas stay at A.

ii 30 × 0.2 = 6 fleas go to B.

iii 30 × 0.2 = 6 fleas go to C.

b,c $\begin{bmatrix} 0.60 & 0.10 & 0.70 \\ 0.20 & 0.80 & 0.10 \\ 0.20 & 0.10 & 0.20 \end{bmatrix} \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix} = \begin{bmatrix} 84 \\ 66 \\ 30 \end{bmatrix}$

i 84 at A

ii 66 at B

iii 30 at C

The product $T \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix}$ gives the column vector with the number of fleas at each location.

d **i** 0.7 × 30 = 21

ii 0.1 × 30 = 3

iii 0.2 × 30 = 6

e $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} T \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 84 \\ 66 \\ 30 \end{bmatrix} = \begin{bmatrix} 180 \end{bmatrix}$

This gives the total number of fleas.

Exam 1 style questions

5 **B** $\begin{bmatrix} 0.60 & 0.20 & 0.40 \\ 0.10 & 0.70 & 0.10 \\ 0.30 & 0.10 & 0.50 \end{bmatrix}$

In the first week let x be the number of students that do sport, y be the number that do outdoor activities and z be the number that do First Aid.

The number of students who not change = $0.6x + 0.7y + 0.5z =$ $0.5(x + y + z) + 0.1x + 0.2y \geq 60$

6 **E**

From the matrix $C \rightarrow E \rightarrow D \rightarrow B \rightarrow A$

Solutions to Exercise 11C

1 a $S_1 = TS_0 = \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 200 \\ 400 \end{bmatrix} = \begin{bmatrix} 380 \\ 220 \end{bmatrix}$

b $S_2 = TS_1 = \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 380 \\ 220 \end{bmatrix} = \begin{bmatrix} 398 \\ 202 \end{bmatrix}$

c $S_3 = TS_2$

$= \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 398 \\ 202 \end{bmatrix}$

$= \begin{bmatrix} 399.8 \\ 200.2 \end{bmatrix}$

2 a $S_5 = T^5 S_0$

$= \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}^5 \begin{bmatrix} 200 \\ 400 \end{bmatrix}$

$= \begin{bmatrix} 399.998 \\ 200.002 \end{bmatrix}$

b $S_7 = T^7 S_0$

$= \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}^7 \begin{bmatrix} 200 \\ 400 \end{bmatrix}$

$= \begin{bmatrix} 400 \\ 200 \end{bmatrix}$

c $S_{12} = T^{12} S_0$

$= \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}^{12} \begin{bmatrix} 200 \\ 400 \end{bmatrix}$

$= \begin{bmatrix} 400 \\ 200 \end{bmatrix}$

3 $T = \begin{bmatrix} 0.65 & 0.4 \\ 0.35 & 0.6 \end{bmatrix}$

$TS_4 = S_5$

$\therefore S_4 = T^{-1} S_5$

$S_4 = \begin{bmatrix} 2.4 & -1.6 \\ -1.4 & 2.6 \end{bmatrix}\begin{bmatrix} 5461 \\ 4779 \end{bmatrix}$

$= \begin{bmatrix} 5460 \\ 4780 \end{bmatrix}$

$TS_3 = S_4$

$\therefore S_3 = T^{-1} S_4$

$S_3 = \begin{bmatrix} 2.4 & -1.6 \\ -1.4 & 2.6 \end{bmatrix}\begin{bmatrix} 5460 \\ 4780 \end{bmatrix}$

$= \begin{bmatrix} 5456 \\ 4784 \end{bmatrix}$

4 a i $S_1 = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 130 \\ 170 \end{bmatrix}$

ii $S_2 = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \times \begin{bmatrix} 130 \\ 170 \end{bmatrix} = \begin{bmatrix} 151 \\ 149 \end{bmatrix}$

iii $S_3 = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \times \begin{bmatrix} 151 \\ 149 \end{bmatrix} = \begin{bmatrix} 165.7 \\ 134.3 \end{bmatrix}$

b $T^5 = \begin{bmatrix} 0.72269 & 0.55462 \\ 0.27731 & 0.44538 \end{bmatrix}$

c i $S_2 = \begin{bmatrix} 0.83 & 0.34 \\ 0.17 & 0.66 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 151 \\ 149 \end{bmatrix}$

ii $S_3 = \begin{bmatrix} 0.781 & 0.438 \\ 0.219 & 0.562 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 165.7 \\ 134.3 \end{bmatrix}$

iii $S_7 = \begin{bmatrix} 0.6941 & 0.6118 \\ 0.3059 & 0.3882 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 191.76 \\ 108.24 \end{bmatrix}$

d $S_{10} = T^{10}S_0 = \begin{bmatrix} 197.175 \\ 102.825 \end{bmatrix}$

$S_{15} = T^{15}S_0 = \begin{bmatrix} 199.525 \\ 100.475 \end{bmatrix}$

$S_{21} = T^{21}S_0 = \begin{bmatrix} 199.92 \\ 100.06 \end{bmatrix}$

$S_{22} = T^{22}S_0 = \begin{bmatrix} 199.96 \\ 100.04 \end{bmatrix}$

As n increases $S_n \rightarrow \begin{bmatrix} 200 \\ 100 \end{bmatrix}$

5 a i $S_1 = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 180 \\ 130 \\ 290 \end{bmatrix}$

ii $S_2 = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 180 \\ 130 \\ 290 \end{bmatrix} = \begin{bmatrix} 207 \\ 136 \\ 257 \end{bmatrix}$

iii $S_3 = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 207 \\ 136 \\ 257 \end{bmatrix} = \begin{bmatrix} 225 \\ 132.1 \\ 242.9 \end{bmatrix}$

b i $S_2 = \begin{bmatrix} 0.58 & 0.37 & 0.25 \\ 0.19 & 0.24 & 0.23 \\ 0.23 & 0.39 & 0.52 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 207 \\ 136 \\ 257 \end{bmatrix}$

ii $S_3 = \begin{bmatrix} 0.505 & 0.394 & 0.319 \\ 0.204 & 0.215 & 0.229 \\ 0.291 & 0.391 & 0.452 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 225 \\ 132.1 \\ 242.9 \end{bmatrix}$

iii $S_7 = \begin{bmatrix} 0.4210 & 0.4099 & 0.4027 \\ 0.2145 & 0.2159 & 0.2169 \\ 0.3645 & 0.3742 & 0.3805 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 244.9 \\ 129.7 \\ 225.4 \end{bmatrix}$

c $S_{10} = T^{10}S_0 = \begin{bmatrix} 246.67 \\ 129.46 \\ 223.86 \end{bmatrix}$

$S_{15} = T^{15}S_0 = \begin{bmatrix} 247.04 \\ 129.42 \\ 223.55 \end{bmatrix}$

$S_{17} = T^{17}S_0 = \begin{bmatrix} 247.05 \\ 129.41 \\ 223.54 \end{bmatrix}$

$S_{18} = T^{18}S_0 = \begin{bmatrix} 247.06 \\ 129.41 \\ 223.53 \end{bmatrix}$

As n increases $S_n \rightarrow \begin{bmatrix} 247.1 \\ 129.4 \\ 223.5 \end{bmatrix}$

6 **a** Let J be the first row and column and P be the second row and column.

$$T = \begin{bmatrix} 0.80 & 0.25 \\ 0.20 & 0.75 \end{bmatrix}$$

b Let the first row be J and the second row be P.

$$S_0 = \begin{bmatrix} 400 \\ 400 \end{bmatrix}$$

c $S_1 = T \times S_0$

$$S_1 = \begin{bmatrix} 0.80 & 0.25 \\ 0.20 & 0.75 \end{bmatrix} \times \begin{bmatrix} 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 420 \\ 380 \end{bmatrix}$$

Thus 420 people are expected to go to Jill's next week and 380 to Pete's.

d $S_5 = T^5 \times S_0$

$$\begin{bmatrix} 0.5779 & 0.5276 \\ 0.4221 & 0.4724 \end{bmatrix} \times \begin{bmatrix} 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 442.2 \\ 357.8 \end{bmatrix}$$

Thus 442 people are expected to go to Jill's after 5 weeks and 358 to Pete's.

e As n increases we find that $S_n \rightarrow \begin{bmatrix} 444.4 \\ 355.6 \end{bmatrix}$

Thus, in the long term, 444 people are expected to go to Jill's each week and 356 to Pete's.

7 **a** Let H be the first row and column and U be the second row and column.

$$T = \begin{bmatrix} 0.90 & 0.60 \\ 0.10 & 0.40 \end{bmatrix}$$

b Let the first row be J and the second row be P.

$$S_0 = \begin{bmatrix} 1500 \\ 500 \end{bmatrix}$$

c $S_1 = T \times S_0$

$S_1 = TS_0$

$$= \begin{bmatrix} 0.90 & 0.60 \\ 0.10 & 0.40 \end{bmatrix} \begin{bmatrix} 1500 \\ 500 \end{bmatrix}$$

$$= \begin{bmatrix} 1650 \\ 350 \end{bmatrix}$$

The next day, are expected 1650 to be happy and 350 sad.

d $S_4 = T^4 S_0$

$$= \begin{bmatrix} 0.90 & 0.60 \\ 0.10 & 0.40 \end{bmatrix}^4 \begin{bmatrix} 1500 \\ 500 \end{bmatrix}$$

$$= \begin{bmatrix} 1712.55 \\ 287.45 \end{bmatrix}$$

After 4 days, 1713 happy and 287 sad.

e As n increases we find that $S_n \rightarrow \begin{bmatrix} 1714.3 \\ 285.7 \end{bmatrix}$

In the long term, 1714 are expected to be happy and 286 sad.

8 **a**

$$S_0 = \begin{bmatrix} 1200 \\ 600 \\ 200 \end{bmatrix} \begin{matrix} \textit{happy} \\ \textit{neither} \\ \textit{sad} \end{matrix}$$

b

$S_1 = TS_0$

$$= \begin{bmatrix} 0.80 & 0.40 & 0.35 \\ 0.15 & 0.30 & 0.40 \\ 0.05 & 0.30 & 0.25 \end{bmatrix} \begin{bmatrix} 1200 \\ 600 \\ 200 \end{bmatrix}$$

$$= \begin{bmatrix} 1270 \\ 440 \\ 290 \end{bmatrix}$$

1270 people are expected to be happy

c

$S_5 = T^5 S_0$

$= \begin{bmatrix} 0.80 & 0.40 & 0.35 \\ 0.15 & 0.30 & 0.40 \\ 0.05 & 0.30 & 0.25 \end{bmatrix}^5 \begin{bmatrix} 1200 \\ 600 \\ 200 \end{bmatrix}$

$= \begin{bmatrix} 1310.33 \\ 429.82 \\ 259.85 \end{bmatrix}$

1310 people are expected to be happy

d

As n increases we find that $S_n \to \begin{bmatrix} 1311.7 \\ 429.1 \\ 259.1 \end{bmatrix}$

In the long term, 1312 are expected to be happy, 429 neither happy nor sad and 260 sad.

Exam 1 style questions

9 A

$\begin{bmatrix} 0.42 & 0.56 \\ 0.58 & 0.44 \end{bmatrix} \begin{bmatrix} 25 \\ 75 \end{bmatrix} = \begin{bmatrix} 52.5 \\ 47.5 \end{bmatrix}$

10 E $\begin{bmatrix} 0.32 & 0.16 \\ 0.68 & 0.84 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$0.68 \times 72 + 0.84y = y$

$0.16y = 0.68 \times 72$

$y = 306$

11 A

$T = \begin{bmatrix} x & 0.4 & y \\ 0.6 & z & 0.4 \\ 0.1 & 0.2 & w \end{bmatrix}$

From T you obtain $x = 0.3$ and $z = 0.4$ because columns must add to one.

Also $y + w = 0.6$

$\begin{bmatrix} 0.3 & 0.4 & y \\ 0.6 & 0.4 & 0.4 \\ 0.1 & 0.2 & w \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 14 \\ 26 \\ 20 \end{bmatrix}$

Therefore ,

$\begin{bmatrix} 30y + 11 \\ 26 \\ 30w + 5 \end{bmatrix} = \begin{bmatrix} 14 \\ 26 \\ 20 \end{bmatrix}$

Hence $y = 0.1$

and $w = 0.5$

Solutions to Exercise 11D

1 a i $S_1 = TS_0 = \begin{bmatrix} 80 \\ 120 \end{bmatrix}$

ii $S_2 = TS_1 = \begin{bmatrix} 72 \\ 128 \end{bmatrix}$

$S_3 = TS_2 = \begin{bmatrix} 68.8 \\ 131.2 \end{bmatrix}$

b i

$$S_1 = TS_0 + R = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 90 \\ 125 \end{bmatrix}$$

ii

$$S_2 = TS_1 + R = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} 90 \\ 125 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 79 \\ 136 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 89 \\ 141 \end{bmatrix}$$

c i

$$S_1 = TS_0 - B = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

ii

$$S_2 = TS_1 - B = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 100 \\ 100 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 80 \\ 120 \end{bmatrix} - \begin{bmatrix} -20 \\ 20 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

2 a i

$$S_1 = TS_0 = \begin{bmatrix} 11500 \\ 8500 \\ 10000 \end{bmatrix}$$

ii

$$S_2 = TS_1 = \begin{bmatrix} 12800 \\ 7300 \\ 9850 \end{bmatrix}$$

7300 birds at site B

b $A : 30\,000$ $B : 0$ $C : 0$ Explanation included in answer.

c **i**

$$S_2 = TS_1 + N = \begin{bmatrix} 9500 \\ 9500 \\ 11000 \end{bmatrix}$$

ii

$$S_3 = TS_2 + N = \begin{bmatrix} 9000 \\ 9150 \\ 11850 \end{bmatrix}$$

iii $S_4 = TS_3 + N = \begin{bmatrix} 8507.5 \\ 8912.5 \\ 12580 \end{bmatrix}$

Exam 1 style questions

3 **C**

$S_1 = TS_0 - C$

$\therefore C = TS_0 - S_1$

$$C = TS_0 - S_1$$

$$= \begin{bmatrix} 0.6 & 0.75 & 0.1 \\ 0.2 & 0.2 & 0.1 \\ 0.2 & 0.05 & 0.8 \end{bmatrix} \begin{bmatrix} 2000 \\ 1000 \\ 1000 \end{bmatrix} - \begin{bmatrix} 1975 \\ 650 \\ 1125 \end{bmatrix}$$

$$= \begin{bmatrix} 75 \\ 50 \\ 125 \end{bmatrix}$$

$S_2 = TS_1 - C$

$$= \begin{bmatrix} 0.6 & 0.75 & 0.1 \\ 0.2 & 0.2 & 0.1 \\ 0.2 & 0.05 & 0.8 \end{bmatrix} \begin{bmatrix} 1975 \\ 650 \\ 1125 \end{bmatrix} - \begin{bmatrix} 75 \\ 50 \\ 125 \end{bmatrix}$$

$$= \begin{bmatrix} 1710 \\ 587.5 \\ 1202.5 \end{bmatrix}$$

4 B

$X_3 = TX_2 - D$

$\therefore TX_2 = X_3 + D$

$X_2 = T^{-1}X_3 + D$

$$= \begin{bmatrix} 0.6 & 0.75 & 0.1 \\ 0.2 & 0.2 & 0.1 \\ 0.2 & 0.05 & 0.8 \end{bmatrix}^{-1} \left(\begin{bmatrix} 9830 \\ 11130 \\ 7830 \end{bmatrix} + \begin{bmatrix} 70 \\ 70 \\ 70 \end{bmatrix} \right)$$

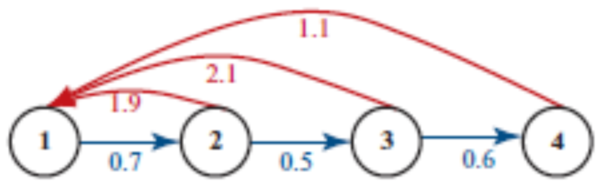
$$= \begin{bmatrix} 4000 \\ 10000 \\ 15000 \end{bmatrix}$$

Solutions to Exercise 11E

1 a i 1.9

ii 0.6

b



c i

$$S_1 = \begin{bmatrix} 0 & 1.9 & 2.1 & 1.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 510 \\ 70 \\ 50 \\ 60 \end{bmatrix}$$

ii

$$S_3 = \begin{bmatrix} 0 & 1.9 & 2.1 & 1.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix}^3 \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 784.8 \\ 212.8 \\ 178.5 \\ 21 \end{bmatrix}$$

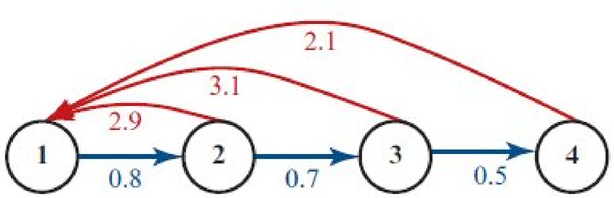
iii

$$S_{20} = \begin{bmatrix} 0 & 1.9 & 2.1 & 1.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix}^{20} \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 208\,276 \\ 103\,876 \\ 36\,984 \\ 15\,815.8 \end{bmatrix}$$

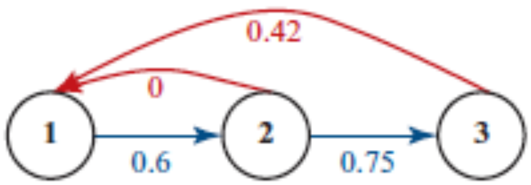
d $S_8 = LS_7$

$$S_8 = \begin{bmatrix} 0 & 1.9 & 2.1 & 1.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 2613 \\ 1200 \\ 485 \\ 168 \end{bmatrix} = \begin{bmatrix} 3483.3 \\ 1829.1 \\ 600 \\ 291 \end{bmatrix}$$

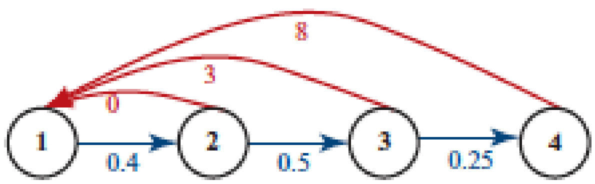
2 a



b



c



3 a $\begin{bmatrix} 0 & 1.3 & 2.4 \\ 0.7 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$

b $\begin{bmatrix} 0 & 2.3 & 3 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$

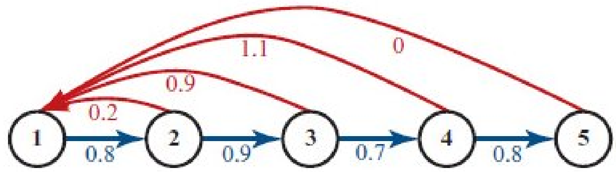
c $\begin{bmatrix} 0 & 1.4 & 2.6 & 0.6 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{bmatrix}$

4 a

$$S_0 = \begin{bmatrix} 15 \\ 20 \\ 30 \\ 15 \\ 10 \end{bmatrix}$$

b $\begin{bmatrix} 0 & 0.2 & 0.9 & 1.1 & 0 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \end{bmatrix}$

c



d **i**

$$S_1 = \begin{bmatrix} 0 & 0.2 & 0.9 & 1.1 & 0 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \end{bmatrix} \begin{bmatrix} 15 \\ 20 \\ 30 \\ 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 47.5 \\ 12 \\ 18 \\ 21 \\ 12 \end{bmatrix}$$

ii

$$S_5 = \begin{bmatrix} 0 & 0.2 & 0.9 & 1.1 & 0 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \end{bmatrix}^5 \begin{bmatrix} 15 \\ 20 \\ 30 \\ 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 58.34 \\ 36.61 \\ 22.45 \\ 21.02 \\ 19.15 \end{bmatrix}$$

e 37, 4 – 8 female kangaroos

f $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 20 \\ 30 \\ 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 90 \end{bmatrix}$ The initial population is 90.

g **i** $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 47.5 \\ 12 \\ 18 \\ 21 \\ 12 \end{bmatrix} = \begin{bmatrix} 110.5 \end{bmatrix}$

111 after one year.

ii $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 75.95 \\ 59.34 \\ 51.33 \\ 28.36 \\ 20.42 \end{bmatrix} = \begin{bmatrix} 157.58 \end{bmatrix}$

158 after five years.

iii $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}75.95\\59.34\\51.33\\28.36\\20.42\end{bmatrix}=\begin{bmatrix}235.398\end{bmatrix}$

235 after ten years.

h **i** $1.1 \times 90 = 99$

ii $1.1^5 \times 90 = 144.95$

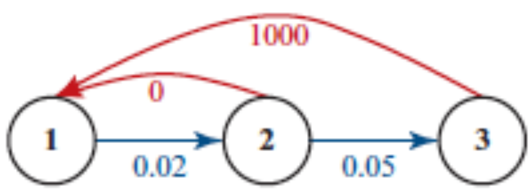
iii $1.1^{10} \times 90 = 233.437$

It appears to be increasing at about this rate. The rate is acually approximately 1.09924. This gives a much better approximation.

5 **a** $\begin{bmatrix}0\\0\\50\end{bmatrix}$

b $\begin{bmatrix}0 & 0 & 1000\\0.02 & 0 & 0\\0 & 0.05 & 0\end{bmatrix}$

c



d **i** $S_1 = LS_0 = \begin{bmatrix}50\,000\\0\\0\end{bmatrix}$

ii $S_3 = L^3S_0 = \begin{bmatrix}0\\0\\50\end{bmatrix}$

iii $S_4 = L^4S_0 = \begin{bmatrix}50\,000\\0\\0\end{bmatrix}$

e **i** $S_1 = LS_0 = \begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix} \begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 50\,000 \\ 1 \\ 5 \end{bmatrix}$

ii $S_3 = L^3 S_0 = \begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix}^3 \begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix}$

iii $S_4 = L^4 S_0 = \begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix}^4 \begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 50\,000 \\ 1 \\ 5 \end{bmatrix}$

Note that L^3 is the 3×3 identity matrix.

6 a **i** $S_5 = L^5 S_0 = \begin{bmatrix} 0 & 2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}^5 \begin{bmatrix} 204 \\ 96 \\ 23 \end{bmatrix} = \begin{bmatrix} 269 \\ 127.438 \\ 30.0625 \end{bmatrix}$

ii $S_{10} = L^{10} S_0 = \begin{bmatrix} 0 & 2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}^{10} \begin{bmatrix} 204 \\ 96 \\ 23 \end{bmatrix} = \begin{bmatrix} 356.17 \\ 168.238 \\ 39.8203 \end{bmatrix}$

iii $S_{20} = L^{20} S_0 = \begin{bmatrix} 0 & 2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}^{20} \begin{bmatrix} 204 \\ 96 \\ 23 \end{bmatrix} = \begin{bmatrix} 622.475 \\ 294.244 \\ 69.5867 \end{bmatrix}$

b After 5 years: 426.5

After 10 years: 564.229

After 20 years: 986.306

$\left(\frac{564}{426.5}\right)^{\frac{1}{5}} \approx 1.057$

$\left(\frac{986.306}{564.229}\right)^{\frac{1}{10}} \approx 1.057$

$S_{21} = L^{21} S_0 = \begin{bmatrix} 0 & 2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}^{21} \begin{bmatrix} 204 \\ 96 \\ 23 \end{bmatrix} = \begin{bmatrix} 658.075 \\ 311.238 \\ 73.5611 \end{bmatrix}$

$\frac{658.075}{622.475} = \frac{311.238}{294.244} = \frac{73.5611}{69.5867} \approx 1.057$

7 a i $\begin{bmatrix} 0 & 2.5 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 1400 \\ 240 \\ 100 \end{bmatrix}$

ii $\begin{bmatrix} 0 & 2.5 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}^2 \begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 700 \\ 840 \\ 60 \end{bmatrix}$

iii $\begin{bmatrix} 0 & 2.5 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}^3 \begin{bmatrix} 2160 \\ 420 \\ 210 \end{bmatrix} = \begin{bmatrix} 700 \\ 840 \\ 60 \end{bmatrix}$

b i $\begin{bmatrix} 0 & 2.5 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\begin{bmatrix} 767 \\ 362 \\ 71 \end{bmatrix} = \begin{bmatrix} 976 \\ 460 \\ 91 \end{bmatrix}$

ii $\begin{bmatrix} 0 & 2.5 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}^2 \begin{bmatrix} 767 \\ 362 \\ 71 \end{bmatrix} = \begin{bmatrix} 1241 \\ 586 \\ 115 \end{bmatrix}$

iii $\begin{bmatrix} 0 & 2.5 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}^3 \begin{bmatrix} 767 \\ 362 \\ 71 \end{bmatrix} = \begin{bmatrix} 1579 \\ 745 \\ 146 \end{bmatrix}$

c i $1.27 \times \begin{bmatrix} 767 \\ 362 \\ 71 \end{bmatrix} = \begin{bmatrix} 974 \\ 460 \\ 90 \end{bmatrix}$

ii $1.27^2 \times \begin{bmatrix} 767 \\ 362 \\ 71 \end{bmatrix} = \begin{bmatrix} 1237 \\ 584 \\ 115 \end{bmatrix}$

iii $1.27^3 \times \begin{bmatrix} 767 \\ 362 \\ 71 \end{bmatrix} = \begin{bmatrix} 1571 \\ 742 \\ 145 \end{bmatrix}$

The population is increasing at about 27% every three years.
A better approximation is 27.1977444 %

8 a i $LS_0 = \begin{bmatrix} 0 \\ 300 \\ 0 \end{bmatrix}$

ii $L^2 S_0 = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$

iii $L^3 S_0 = \begin{bmatrix} 1200 \\ 0 \\ 0 \end{bmatrix}$

b Population cycles through three states

c **i** Population decreases by 50% every three time periods

ii Population increases by 25% every three time periods

9 **a** **i** $\begin{bmatrix} 2800 \\ 200 \\ 200 \\ 40 \end{bmatrix}$

ii $\begin{bmatrix} 1080 \\ 1400 \\ 100 \\ 20 \end{bmatrix}$

iii $\begin{bmatrix} 4440 \\ 540 \\ 700 \\ 10 \end{bmatrix}$

10 **a** **i** When $B_3 = 10$, $L^3 \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} = L^6 \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} = L^9 \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} \cdots = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$

Every 3 years, the population returns to 1000 newborns

ii When $B_3 = 15$, $\mathbf{L}^3 \begin{bmatrix} 1500 \\ 0 \\ 0 \end{bmatrix}, L^6 \begin{bmatrix} 2250 \\ 0 \\ 0 \end{bmatrix}, L^9 \begin{bmatrix} 3375 \\ 0 \\ 0 \end{bmatrix}$

Every 3 years, the population increases by 50% and returns to only newborns

iii Every 3 years, the population decreases by 40% and returns to only newborns

b Every three yesrs the number of newborns double - zero in other agegroups.

Exam 1 style questions

11 D

$$\begin{bmatrix} 0.9 & 2.5 & 0.4 \\ 0.3 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}^7 \begin{bmatrix} 130 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 1962.53 \\ 407.387 \\ 126.953 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1962.53 \\ 407.387 \\ 126.953 \end{bmatrix} = \begin{bmatrix} 2496.87 \end{bmatrix}$$

Population is closest to 2500.

12 D

$$\begin{bmatrix} 0 & 2 & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix} \begin{bmatrix} 16 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2b + 8 \\ 16c \\ 4d \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 2 \end{bmatrix}$$

$2b + 8 = 16 \Rightarrow b = 4$

$16c = 4 \Rightarrow c = 0.25$

$4d = 2 \Rightarrow d = 0.5$

13 A

$$\begin{bmatrix} 0 & 2 & 1.5 \\ 0.44 & 0 & 0 \\ 0 & 0.55 & 0 \end{bmatrix} \begin{bmatrix} 1000 \\ 400 \\ 200 \end{bmatrix} = \begin{bmatrix} 1100 \\ 440 \\ 220 \end{bmatrix}$$

$$\frac{1100}{1000} = \frac{440}{400} = \frac{220}{200} = 1.1$$

Chapter Review: Multiple-choice questions

1 From the transition diagram:

$$T = \begin{bmatrix} 0.75 & 0.05 \\ 0.25 & 0.95 \end{bmatrix} \quad \Rightarrow \mathbf{B}$$

2 From the transition diagram:

$$T = \begin{bmatrix} 0.75 & 0.05 & 0.30 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \Rightarrow \mathbf{A}$$

3 $S_1 = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} =$

$\begin{bmatrix} 160 \\ 140 \end{bmatrix}$

4 $T^2 = \begin{bmatrix} 0.56 & 0.55 \\ 0.44 & 0.45 \end{bmatrix} \quad \Rightarrow \mathbf{B}$

5 $S_3 = \begin{bmatrix} 0.556 & 0.555 \\ 0.444 & 0.445 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \end{bmatrix} =$

$\begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$

6 $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$: Obtain by evaluating $T_n S_0$ for increasing large values of n until there is little or no change in the state matrix (the steady or equilibrium state).

$\Rightarrow$**C**

7 $L_1 = TS_0 + B$

$$= \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \end{bmatrix} + \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 170 \\ 160 \end{bmatrix} \Rightarrow \mathbf{C}$$

8 $P_1 = TS_0 - 2B$

$$= \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \end{bmatrix} - 2\begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 140 \\ 100 \end{bmatrix} \Rightarrow \mathbf{A}$$

9 $S_2 = GS_1 = \begin{bmatrix} -10 \\ 25 \end{bmatrix} \Rightarrow \mathbf{B}$

10

$S_5 = TS_4$

$T^{-1}S_5 = T^{-1}TS_4$

$T^{-1}S_5 = S_4$

$$S_4 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}^{-1}\begin{bmatrix} 22 \\ 18 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} \Rightarrow \mathbf{B}$$

11 Eventually all of the birds will settle at location B because the ‘1’ in row B column B, tells us that once a bird settles at location B it will never leave.

$\Rightarrow$ B

12 From the transition matrix, the number of people who plan to change who they vote for after 1 week is:

$0.25 \times 5692 + 0.24 \times 3450 = 2251$

$\Rightarrow$ C

13 $S_{10} = TS_0$

$$= \begin{bmatrix} 0.75 & 0.24 \\ 0.25 & 0.76 \end{bmatrix}\begin{bmatrix} 5692 \\ 3450 \end{bmatrix}$$

$$= \begin{bmatrix} 4479.15... \\ 4662.84... \end{bmatrix}$$

$$\approx \begin{bmatrix} 4479 \\ 4663 \end{bmatrix} \begin{matrix} Rob \\ Anna \end{matrix}$$

$\therefore$ Anna wins by: $4663 - 4479 = 182$ votes $\Rightarrow$ E

Chapter Review: Extended-response questions

1 **a** **i** $A_2 = \begin{bmatrix} 1.2 & -0.4 \\ 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} 500 \\ 240 \end{bmatrix} = \begin{bmatrix} 504 \\ 244 \end{bmatrix}$

ii $504 + 244 = 748$

iii $A_8 = \begin{bmatrix} 1.2 & -0.4 \\ 0.2 & 0.6 \end{bmatrix}^8 \begin{bmatrix} 500 \\ 240 \end{bmatrix} = \begin{bmatrix} 517 \\ 257 \end{bmatrix}$

b The attendance stabilises. $A_{80} = \begin{bmatrix} 1.2 & -0.4 \\ 0.2 & 0.6 \end{bmatrix}^{80} \begin{bmatrix} 500 \\ 240 \end{bmatrix} = \begin{bmatrix} 520 \\ 260 \end{bmatrix}$

c The attendance stabilises to a very small number. $A_{80} = \begin{bmatrix} 1.2 & -0.4 \\ 0.2 & 0.6 \end{bmatrix}^{80} \begin{bmatrix} 500 \\ 490 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$

2
$$\begin{array}{c|ccc} & Y & M & O \\ \hline Y & 0.1 & 0.2 & 0.4 \\ M & 0.9 & 0 & 0 \\ O & 0 & 0.8 & 0.6 \end{array}$$

3 **a**
$$\begin{bmatrix} 0 & 0.1 & 0.9 & 0.2 & 0 & 0 & 0 & 0 \\ 0.98 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.95 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.95 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 \end{bmatrix}$$

b
$$\mathbf{P}_2 = \begin{bmatrix} 104.5 \\ 107.8 \\ 0 \\ 90.25 \\ 85.5 \\ 31.5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P}_3 = \begin{bmatrix} 28.83 \\ 102.41 \\ 102.41 \\ 0 \\ 81.225 \\ 59.85 \\ 15.75 \\ 0 \end{bmatrix}$$

c This Leslie matrix has a dominant eigenvalue 1.035 (See chapter 18)Rate of growth may be estimated numerically.

4 a $\begin{bmatrix} 0.2 & 0.5 & 0.6 & 0.4 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \end{bmatrix}$

b $\begin{bmatrix} 0.2 & 0.5 & 0.6 & 0.4 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \end{bmatrix}^{20} \begin{bmatrix} 800 \\ 800 \\ 800 \\ 800 \end{bmatrix} \approx \begin{bmatrix} 980 \\ 692 \\ 488 \\ 344 \end{bmatrix}$

c $\begin{bmatrix} 0.2 & 0.5 & 0.6 & 0.4 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \end{bmatrix}^{21} \begin{bmatrix} 800 \\ 800 \\ 800 \\ 800 \end{bmatrix} \approx \begin{bmatrix} 972 \\ 686 \\ 484 \\ 342 \end{bmatrix}$,

Rate of growth = 0.992.

d $\begin{bmatrix} 0.3 & 0.6 & 0.7 & 0.5 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \end{bmatrix}^{21} \begin{bmatrix} 972 \\ 686 \\ 484 \\ 343 \end{bmatrix} \approx \begin{bmatrix} 7467 \\ 4762 \\ 3038 \\ 1938 \end{bmatrix}$

e $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.8 & 0.5 \\ 0.85 & 0 & 0 & 0 \\ 0 & 0.85 & 0 & 0 \\ 0 & 0 & 0.85 & 0 \end{bmatrix}^{8} \begin{bmatrix} 7467 \\ 4762 \\ 3038 \\ 1938 \end{bmatrix} = \begin{bmatrix} 119\,557 \end{bmatrix}$

Day 50

Solutions to Multiple-choice questions

1 $\begin{bmatrix}1 & 4\\2 & 3\\6 & 2\end{bmatrix}\begin{bmatrix}2 & 3 & 4\\1 & 0 & 6\end{bmatrix}=\begin{bmatrix}6 & 3 & 28\\7 & 6 & 36\\14 & 18 & 36\end{bmatrix}$

$q_{33} = 6\times 4 + 2\times 6$

$\Rightarrow$**E**

2 $\begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\end{bmatrix}^n\begin{bmatrix}D\\O\\G\\S\end{bmatrix}=\begin{bmatrix}D\\O\\G\\S\end{bmatrix}$

$\Rightarrow$**D**

3 Using $2i + 4j$

$\begin{bmatrix}2\times 1+4\times 1 & 2\times 1+4\times 2 & 2\times 1+4\times 3\\2\times 2+4\times 1 & 2\times 2+4\times 2 & 2\times 3+4\times 3\\2\times 3+4\times 1 & 2\times 3+4\times 2 & 2\times 3+4\times 3\end{bmatrix}$

$=\begin{bmatrix}6 & 10 & 14\\8 & 12 & 16\\10 & 14 & 18\end{bmatrix}$

$\Rightarrow$**A**

4 multiply out the matrices (by hand, is quicker) to obtain the matrix [S T O P]

$\Rightarrow$**A**

5 multiply out the matrices (by hand, is quicker) to obtain the matrix:

$[\,1\times 1+0\times 0+1\times 0+0\times 1+0\times 1]$

$=[1]$

$\Rightarrow$**B**

6 $C = B\times A$

B is a 3×4 matrix and A a 4×3 matrix.

Therefore C is a 4×4 matrix. $\Rightarrow$**C**

7 14 in position 1 goes to position 4

12 in position 3 goes to position 3

3 in position 2 goes to position 2

2 in position 4 goes to position 1

Therefore $P=\begin{bmatrix}0 & 0 & 0 & 1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\1 & 0 & 0 & 0\end{bmatrix}$

$\Rightarrow$**D**

8 Both matrices have inverses, so they must be square.

Thus $p = q$ and $q = r$.

Both products XY^{--1} and $X^{--1}Y$ are defined so the matrices X and Y must be of the same order.

Thus $p = q = r$.

$\Rightarrow$**C**

9 From the diagram, the matrix can be seen to be: $\begin{bmatrix}0 & 0 & 1 & 1\\0 & 0 & 1 & 1\\1 & 1 & 0 & 1\\1 & 1 & 1 & 0\end{bmatrix}$

$\Rightarrow$**C**

10 The order of A is 3×5

The order of B is 3×4

$\Rightarrow$order of B^T is 4×3

The order of C is 4×5

$\Rightarrow$order of C^T is 5×4

Order of $C^T\times B^T$ is 5×3.

The order of A is 3×5

Order of $(C^T\times B^T)\times A$ is 5×5.

$\Rightarrow$**D**

11 Only B is false since
$(A - B)(A + B) = A(A + B) - B(A + B)$
$= A^2 + AB - BA - B^2$.
In general $AB \neq BA$

$\Rightarrow$**B**

12 B^T is a 5×8 matrix. Therefore AB^T is defined.
AB is not defined and BA is a n 8×5 matrix. All other choices are not possible.t

$\Rightarrow$**A**

13 $S_1 = TS_0 = \begin{bmatrix} 90 \\ 110 \end{bmatrix}$

$\Rightarrow$**A**

14

$$S_1 = T^5 S_0 = \begin{bmatrix} 93.1 \\ 106.9 \end{bmatrix}$$

$\Rightarrow$**B**

15

$S_{\text{steady state}} = T^n S_0$ for n large
Using successively large values of n,
$S_{\text{steady state}} \approx \begin{bmatrix} 94.1 \\ 105.9 \end{bmatrix}$

$\Rightarrow$C

16 **C**

$$P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Position 1 goes to position 3
Position 2 goes to position 4
Position 3 goes to position 1
Position 4 goes to position 2

17 10% of people in Gym C go to Gym A from month to month. Therefore 30 go from C to A. $\Rightarrow$C

18 The long term numbers are described by the column matrix:

$$\begin{bmatrix} 215.787 \\ 350.864 \\ 299.86 \\ 333.489 \end{bmatrix}$$

$\Rightarrow$E

19 $S_{12} = \begin{bmatrix} 215.794 \\ 350.88 \\ 299.858 \\ 333.468 \end{bmatrix}$

Therefore total at B and C isclosest to 650

$\Rightarrow$D

20 $T^2 \begin{bmatrix} 0.35 \\ 0.56 \end{bmatrix} = \begin{bmatrix} 0.665875 \\ 0.334125 \end{bmatrix}$ $\Rightarrow$C

21 $M + M^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 2 & 0 \end{bmatrix}$

Therefore ranking B, E, D, A, C

$\Rightarrow$A

22 $\begin{bmatrix} 0.65 & 0.75 \\ 0.35 & 0.3 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 0.65x + 0.7z \\ 0.35x + 0.3z \end{bmatrix}$

$0.7z = 14 \Rightarrow z = 20$
$0.65x + 14 = x \Rightarrow 0.35x = 14 \Rightarrow x = 40$
Therefore fleet size is 60. $\Rightarrow$D

23 $S_{n+1} = T \times S_n - C$
Therefore,

$S_1 = TS_0 - C$ and $C = TS_0 - S_1$

$$C = \begin{bmatrix} 22 \\ 19 \\ 3 \end{bmatrix}$$

$$S_2 = TS_1 - C = \begin{bmatrix} 35 \\ 25 \\ 12 \end{bmatrix}$$

$\Rightarrow$B

24 $\begin{bmatrix} 0.65 & 0.70 & 0.5 \\ 0.2 & 0.1 & 0.25 \\ 0.15 & 0.2 & 0.25 \end{bmatrix} \begin{bmatrix} 70 \\ 0 \\ 30 \end{bmatrix} = \begin{bmatrix} 60.5 \\ 21.5 \\ 18 \end{bmatrix}$

The number who don't change their mode of transport

$= 0.65 \times 60.5 + 0.1 \times 21.5 + 0.25 \times 18$

$= 45.975$

$\Rightarrow$D

25 From the diagram A

Chapter Review: 12B Extended-response questions

1 **a** $C = \begin{bmatrix} 1.316 \\ 1.818 \\ 0.167 \end{bmatrix} \begin{matrix} \text{US dollar rate} \\ \text{Euro rate} \\ \text{HK dollar rate} \end{matrix}$

Order of C is 3×1 (3 rows, 1 column).

b $H = [\,102 \quad 262 \quad 516\,]$

Order of H is 1×3.

(1 rows, 3 columns)

c The matrix product HC is defined as H has order 1×3 and C has order 3×1.

The order of HC is 1×1.

d **i** $HC = [102 \quad 262 \quad 516] \begin{bmatrix} 1.316 \\ 1.818 \\ 0.167 \end{bmatrix}$

$= [102 \times 1.316 + 262 \times 1.818 + 516 \times 0.167]$

$= 696.72$

ii This is the total amount converted to Australian dollars.

Solution integrated with answer.

e $MC = \begin{bmatrix} 125 & 216 & 54 \\ 0 & 34 & 453 \\ 0 & 356 & 0 \end{bmatrix} \begin{bmatrix} 1.316 \\ 1.818 \\ 0.167 \end{bmatrix}$

$= \begin{bmatrix} 566.21 \\ 137.46 \\ 647.21 \end{bmatrix}$

Person 1 receives \$566.21, person 2 receives \$137.46 and person 3 receives \$647.21.

2 **a**

$$\begin{matrix} & & \textit{From} & \\ & & \textit{Blue} & \textit{Green} \\ \textit{To} & \begin{matrix} \textit{Blue} \\ \textit{Green} \end{matrix} & \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix.} & \begin{bmatrix} 0.28 \\ 0.72 \end{bmatrix.} \end{matrix}$$

b $S_0 = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix} \begin{matrix} \textit{Blue} \\ \textit{Green} \end{matrix}$

c $\begin{bmatrix} 0.67 & 0.28 \\ 0.33 & 0.72 \end{bmatrix} \begin{bmatrix} 4000 \\ 6000 \end{bmatrix} = \begin{bmatrix} 4360 \\ 5640 \end{bmatrix}$

Tomorrow, we expect 4360 fish in Lake Blue and 5640 fish in Lake Green.

d $\begin{bmatrix} 0.67 & 0.28 \\ 0.33 & 0.72 \end{bmatrix}^{30} \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$

$= \begin{bmatrix} 4555.156 \\ 5444.844 \end{bmatrix} \approx \begin{bmatrix} 4555 \\ 5445 \end{bmatrix} \begin{matrix} \textit{Blue} \\ \textit{Green} \end{matrix}$ (to the nearest whole number)

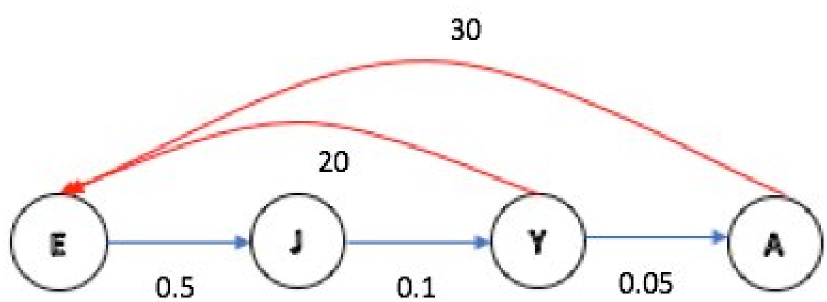
In three days, we expect 4555 fish in Lake Blue and 5445 fish in Lake Green.

e Estimate the steady state solution by trying successively large values of n until there is little change between successive state matrices: $n = 30$ has been used here.

$\begin{bmatrix} 0.67 & 0.28 \\ 0.33 & 0.72 \end{bmatrix}^{30} \begin{bmatrix} 4000 \\ 6000 \end{bmatrix} \approx \begin{bmatrix} 4590 \\ 5410 \end{bmatrix} \begin{matrix} \textit{blue} \\ \textit{green} \end{matrix}$ (to the nearest whole number)

In the long term, we expect 4590 fish in Lake Blue and 5410 fish in Lake Green.

3 a



b 20 + 30 = 50 eggs produced each week

c $\begin{bmatrix} 0 & 0 & 20 & 30 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{bmatrix} \begin{bmatrix} 897 \\ 438 \\ 43 \\ 2 \end{bmatrix} = \begin{bmatrix} 920 \\ 448.5 \\ 43.8 \\ 2.15 \end{bmatrix}$

There are 920 insect eggs after one week.

d $\begin{bmatrix} 0 & 0 & 20 & 30 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{bmatrix}^{5} \begin{bmatrix} 897 \\ 438 \\ 43 \\ 2 \end{bmatrix} = \begin{bmatrix} 1009.5 \\ 493.638 \\ 48.135 \\ 2.35125 \end{bmatrix}$

First time is five week

e 5%

f **i** $S_8 = \begin{bmatrix} 0 & 0 & 20 & 30 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{bmatrix}^8 \begin{bmatrix} 897 \\ 438 \\ 43 \\ 2 \end{bmatrix} = \begin{bmatrix} 1083.55 \\ 529.739 \\ 51.6619 \\ 2.52375 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1083.55 \\ 529.739 \\ 51.6619 \\ 2.52375 \end{bmatrix} = [1667.47]$

ii $S_9 = \begin{bmatrix} 0 & 0 & 20 & 30 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{bmatrix}^9 \begin{bmatrix} 897 \\ 438 \\ 43 \\ 2 \end{bmatrix} = \begin{bmatrix} 1108.95 \\ 541.773 \\ 52.9739 \\ 2.58309 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1108.95 \\ 541.773 \\ 52.9739 \\ 2.58309 \end{bmatrix} = [1706.28]$

g $\frac{1108.95}{1083.55} \approx \frac{541.773}{529.739} \approx \frac{52.9739}{51.6619} \approx \frac{2.68309}{2.52375} \approx 1.023$

4 **a** **i** $S_2 = TS_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.20 & 0.7 \end{bmatrix} \begin{bmatrix} 110 \\ 40 \end{bmatrix} = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$

ii $S_3 = TS_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.20 & 0.7 \end{bmatrix} \begin{bmatrix} 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 95 \\ 55 \end{bmatrix}$

b $S_n = T^n S_1$

c $S_4 = T^4 S_1 = \begin{bmatrix} 91.25 \\ 58.75 \end{bmatrix}$ and $S_5 = T^5 S_1 = \begin{bmatrix} 90.625 \\ 59.375 \end{bmatrix}$

It takes 5 weeks.

d $S_{50} = T^{50} S_0 = \begin{bmatrix} 90 \\ 60 \end{bmatrix}$

5 **a** **i** 3×1

ii $k = 1.2$

b **i** A and B and A and C

ii $D \to B \to A \to C$

iii $C \to A \to B$ and $C \to D \to B$

c **i** $S_1 = TS_0 = \begin{bmatrix} 0.7 & 0.8 & 0.5 \\ 0.05 & 0.1 & 0.2 \\ 0.25 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 200 \\ 150 \\ 250 \end{bmatrix} = \begin{bmatrix} 385 \\ 75 \\ 140 \end{bmatrix}$

ii $0.7 \times 200 + 0.1 \times 150 + 0.3 \times 250 = 230$

iii August shoppers: $S_2 = TS_1 = \begin{bmatrix} 399.5 \\ 54.75 \\ 145.75 \end{bmatrix}$

September shoppers: $S_3 = TS_2 = \begin{bmatrix} 396.325 \\ 54.6 \\ 149.075 \end{bmatrix}$

Shoppers that go Cleanup in August and then again in September = 0.7×399.5

Percentage of shoppers = $\dfrac{0.7 \times 399.5}{396.325} \times \dfrac{100}{1} = 70.5608\%$

d $S_1 = TS_0 + B \Rightarrow B = S_1 - TS_0 = \begin{bmatrix} 20 \\ 15 \\ 25 \end{bmatrix}$

6 **a** 4×2

b **i** Adding all entries = 435

ii Percentage of seats occupied = $\dfrac{315}{435} \times \dfrac{100}{1} = 72.418\%$

c $L = \begin{bmatrix} 60 \\ 120 \\ 50 \\ 85 \end{bmatrix}$

$Total = Q \times L = \begin{bmatrix} 5115 \end{bmatrix}$

7 **a** $S_1 = TS_0 = \begin{bmatrix} 0.6 & 0.2 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.6 \\ 0.1 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.3 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 30 \\ 30 \\ 30 \\ 30 \end{bmatrix} = \begin{bmatrix} 39 \\ 36 \\ 21 \\ 24 \end{bmatrix}$

$$S_2 = TS_1 = \begin{bmatrix} 0.6 & 0.2 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.6 \\ 0.1 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.3 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 39 \\ 36 \\ 21 \\ 24 \end{bmatrix} = \begin{bmatrix} 41.4 \\ 35.1 \\ 20.1 \\ 23.4 \end{bmatrix}$$

b $S_{10} = T^{10} S_0 = \begin{bmatrix} 43 \\ 35 \\ 20 \\ 23 \end{bmatrix}$

c $0.1 \times 30 = 3$

d $0.6 \times 30 + 0.3 \times 30 + 0.2 \times 30 + 0.1 \times 30 = 36$

e 39 people did pottery in February.
10% of them moved to weaving = 3.9
Rounding to the nearst whole number: 4 people moved from pottery to weaving from February to March.

f $S_1 = \begin{bmatrix} 39 \\ 36 \\ 21 \\ 24 \end{bmatrix}$.

Therefore number of people who stayed in the same activity from February to March $= 0.6 \times 39 + 0.3 \times 36 + 0.2 \times 21 + 0.1 \times 24 = 40.8(41)$

8 a 550

b The number of sandwiches sold in week 2

c Hamburgers: \$8250 ÷ 550 = \$15
Fish and Chips: \$9100 ÷ 650 = \$14
Sandwichs: \$2220 ÷ 185 = \$12

d $L = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$.

Solutions to Exercise 13A

1 **a** **i** There are three edges connected to town A, so $\deg(A) = 3$.

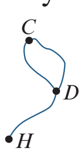
ii There are two edges connected to town B, so $\deg(B) = 2$.

iii There is one edge connected to town H, so $\deg(H) = 1$.

b sum of degrees
$= \deg(A) + \deg(B)$
$\quad + \deg(C) + \deg(D) + \deg(H)$
$= 3 + 2 + 4 + 4 + 1$
$= 14$
Alternatively: The total number of edges for this graph is 7.
The sum of the degrees of a graph
$= 2 \times$ the total number of edges
$= 2 \times 7 = 14$

c If the edge connecting towns D and H was removed, town H would not be connected to the other towns, directly or indirectly. Every other edge, if removed, would not result in one or more vertices being isolated from the other vertices. Therefore, for this graph one bridge exists between towns D and H.

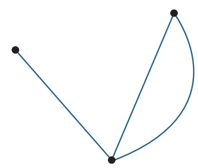
d A possible subgraph that contains only towns H, D and C is



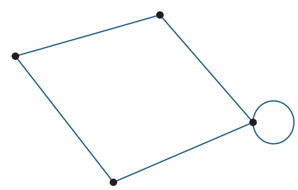
Note: Some subgraphs are possible where vertices are isolated. These subgraphs have not been shown.

2 *Note: only one alternative has been shown for the answers to the following questions. Others are possible.*

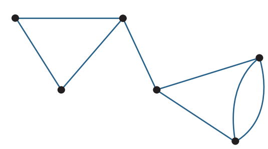
a



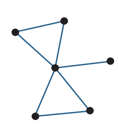
b



c



d



3 **a** The graphs in **ii**, **iii** and **iv** have all connections between vertices the same, but the graph in **i** does not. For example, they have two edges between A and C but the graph in i does not.

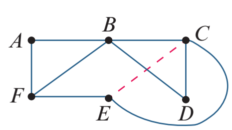
Graph **i** is not isomorphic to the others.

b The graphs in **i**, **iii** and **iv** have all connections between vertices the same, but the graph in **ii** does not. For example, they do not have an edge between A and C but the graph in **ii** does.
Graph **ii** is not isomorphic to the others.

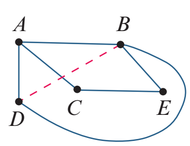
c The graphs in **i**, **iii** and **iv** have all connections between vertices the same, but the graph in **ii** does not. For example, they do not have an edge between E and C but the graph in **ii** does.
Graph **ii** is not isomorphic to the others.

4 *Note: In the graphs for this question, dotted edges show the edges that are repositioned in order to demonstrate the planar nature of the graphs. There are other solutions possible.*

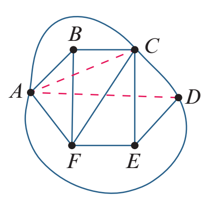
a



b



c



d This graph is non-planar and cannot be redrawn.

5 a i There are eight vertices, so $v = 8$.
There are six faces, so $f = 6$.
There are twelve edges, $e = 12$.

ii $v + f = e + 2$
$8 + 6 = 12 + 2$
$14 = 14$
Euler's formula is verified.

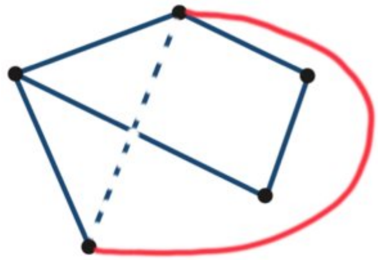
b i There are six vertices, so $v = 6$.
There are eight faces, so $f = 8$.
There are twelve edges, $e = 12$.

ii $v + f = e + 2$
$6 + 8 = 12 + 2$
$14 = 14$
Euler's formula is verified.

c i There are seven vertices, so $v = 7$.
There are seven faces, so $f = 7$.
There are twelve edges, $e = 12$.

ii $v + f = e + 2$
$7 + 7 = 12 + 2$
$14 = 14$
Euler's formula is verified.

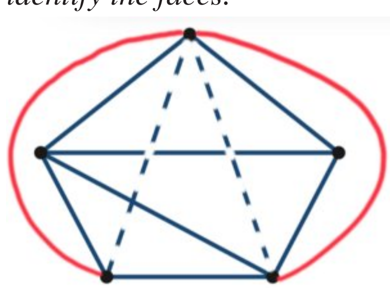
d **i** There are five vertices, so $v = 5$.
There are three faces, so $f = 3$.
Note: the graph must be redrawn without any edges crossing to identify the faces.



There are six edges, $e = 6$.

ii $v + f = e + 2$
$5 + 3 = 6 + 2$
$8 = 8$
Euler's formula is verified.

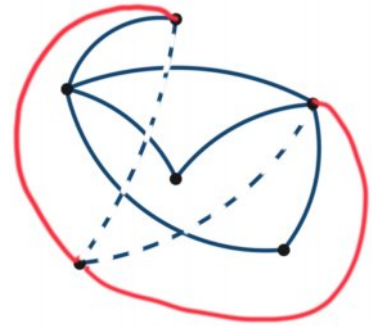
e **i** There are five vertices, so $v = 5$.
There are six faces, so $f = 6$.
Note: the graph must be redrawn without any edges crossing to identify the faces.



There are nine edges, $e = 9$.

ii $v + f = e + 2$
$5 + 6 = 9 + 2$
$11 = 11$
Euler's formula is verified.

f **i** There are six vertices, so $v = 6$.
There are four faces, so $f = 4$.
Note: the graph must be redrawn without any edges crossing to identify the faces.



There are eight edges, $e = 8$.

ii $v + f = e + 2$
$6 + 4 = 8 + 2$
$10 = 10$
Euler's formula is verified.

6 **a** $v + f = e + 2$
$8 + f = 10 + 2$
$8 + f = 12$
$f = 12 - 8$
$f = 4$

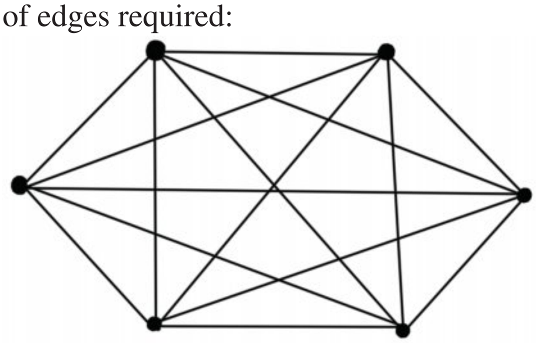
b $v + f = e + 2$
$v + 4 = 14 + 2$
$v + 4 = 16$
$v = 16 - 4$
$v = 12$

c $v + f = e + 2$
$10 + 11 = e + 2$
$21 = e + 2$
$21 - 2 = e$
$e = 19$

7 Connected planar graph, therefore Euler's formula will be verified:
$v + f = e + 2$
$v = 8$ and $e = 13$
$8 + f = 13 + 2$
$f = 15 - 8$
$f = 7$

8 The total number of edges for this graph is 7.
The sum of the degrees of a graph
= 2 × the total number of edges
$= 2 \times 7 = 14$

9 A complete graph is where all vertices are connected directly to all other vertices in the graph.
Method 1
Draw this graph and count the number of edges required:



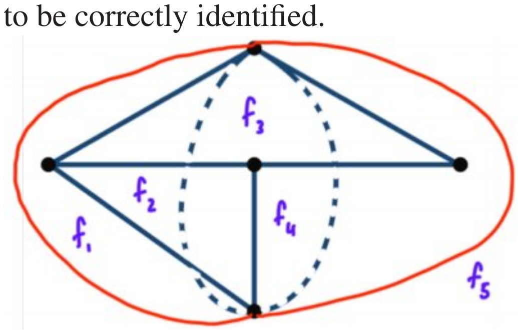
Method 2
A complete graph with n vertices will have $\frac{n(n-1)}{2}$ edges.
This graph has six vertices, so $n = 6$ and the graph will have $\frac{6(6-1)}{2} = \frac{6(5)}{2} = \frac{30}{2} = 15$ edges.

Exam 1 style questions

10 Identify the degree of each vertex (count how many edges are connected to each vertex). There is one vertex with degree 2, two vertices with degree 3 and three vertices with degree 4.
C

11 Planar graph, therefore Euler's formula applies: $v + f = e + 2$
For this graph $f = 4$.
$v + 4 = e + 2$
Test each option to find the correct number of vertices and edges that hold true for the formula above.
Option E: $v = 5$ and $e = 7$
$5 + 4 = 7 + 2$
$9 = 9$
Euler's formula is verified.
E

12 The graph must be redrawn without any edges crossing for the number of faces to be correctly identified.



There are five faces.
C

13 Consider each option:

- The graph is planar. *TRUE*; the graph is connected and it can be drawn so no edges are crossing.

- The graph contains a bridge. *FALSE*; if any one edge is removed, the graph remains connected.

- It is a simple graph. *FALSE*; the graph has multiple edges (two or more edges that connect the same vertices).

- The sum of degrees of the vertices is 16. *TRUE*; the total number of edges is 8 and the sum of the degrees of a graph $= 2 \times$ the total number of edges $= 2 \times 8 = 16$

- It is a complete graph. *FALSE*; every vertex is not connected directly to every other vertex in the graph.

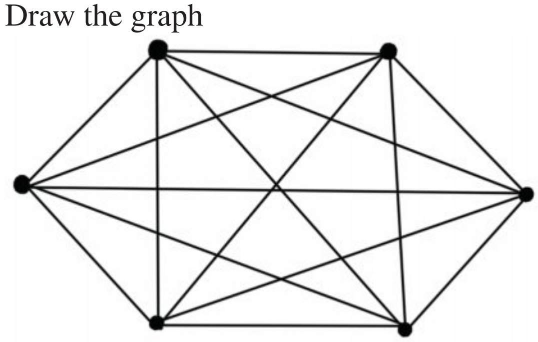
There are two true statements.

B

14 A complete graph must have every vertex connected directly to every other vertex in the graph with an edge.

Method 1

Draw the graph



Method 2

A complete graph with n vertices will have $\frac{n(n-1)}{2}$ edges.

This graph has six vertices, so $n = 6$ and the graph will have $\frac{6(6-1)}{2} = \frac{6(5)}{2} = \frac{30}{2} = 15$ edges. There are already 3 edges, therefore 12 more edges must be added to make this a complete graph.

E

15 This graph has 6 vertices. The minimum number of edges required for this graph to be connected is 5. The graph has 15 edges (this is found by either counting the number of edges in the graph or considering a complete graph with n vertices will have $\frac{n(n-1)}{2}$ edges, where this graph has six vertices, so $n = 6$ and the graph will have $\frac{6(6-1)}{2} = \frac{6(5)}{2} = \frac{30}{2} = 15$ edges.).

There are 15 edges, so 10 must be removed in order for the graph with 6 vertices to be connected with the minimum number of edges (5).

C

Solutions to Exercise 13B

1 *For each of the following, a square matrix is constructed where the number of rows and columns correspond to each of the vertices in the graph. Each row and column are labelled with the letter it represents.*

a
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

c
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

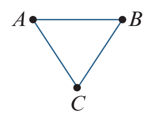
d
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

e
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

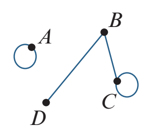
f
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

2 *For each of the following, a graph is drawn using a vertex for each row/column of the matrix.*

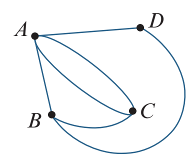
a



b



c



3 The zero in row C, column A show that vertex C is not connected to vertex A. There is a zero in row C, column B and row C, column C as well. This means that C is not connected to any other vertex, so it is isolated.

4 If every vertex has a loop, there will be a '1' in every position along the main diagonal (top left, to bottom right), that is in position (A,A), (B,B), . . .

5 The graph

- has no loops, so the diagonal will be all zeros
- has no duplicate edges, so there will only be '0' or '1'
- is complete, so every vertex is connected to every other vertex. Every position in the matrix will be a '1', except for the diagonal.

The adjacency matrix for the graph is

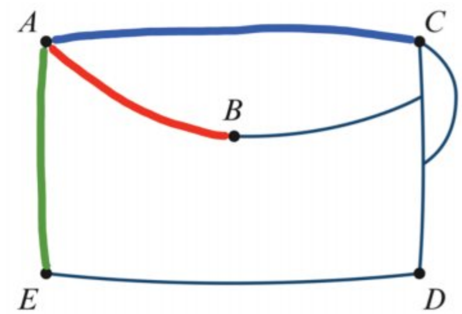
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

Exam 1 style questions

6 Construct a matrix to represent the graph. It will have 5 rows and 5 columns for the 5 vertices:

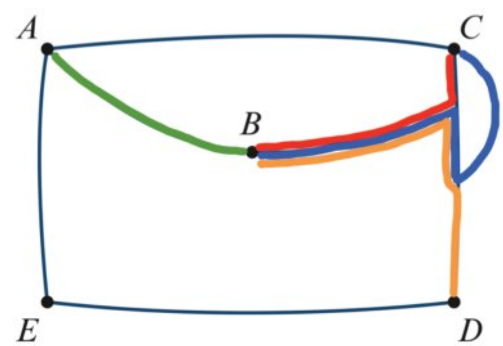
Vertex *A* has:

- one connection to *B*
- one connection to *C*
- no connection to *D*
- one connection to *E*



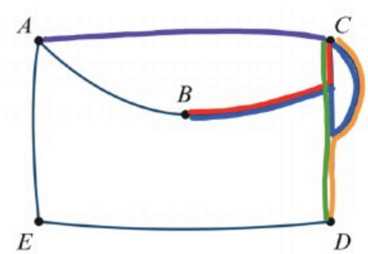
Vertex *B* has:

- one connection to *A*
- two different connections to *C*
- one connection to *D*
- no connection to *E*



Vertex *C* has:

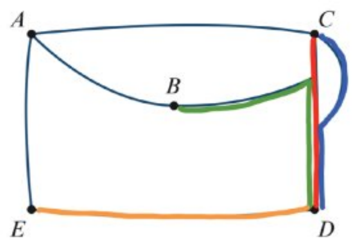
- one connection to *A*
- two different connections to *B*
- a *loop*
- two different connections to *D*
- no connection to *E*



Vertex *D*

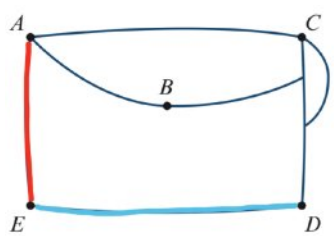
has:

- no connection to A
- one connection to B
- two different connections to C
- one connection to E



Vertex E has:

- one connection to A
- no connection to B
- no connection to C
- one connection to D



The adjacency matrix that represents this graph is:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

B

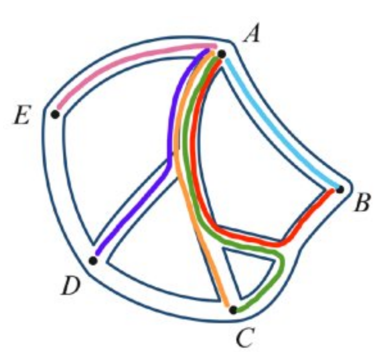
7 Look for key features in the matrix to eliminate incorrect options.

- Row A column B has a 3, thereofre there should be 3 edges between vertices A and B; eliminate options **B** and **C**
- Row B column B has a zero, therefore there should be no loop at vertex B; eliminate option **D**
- Row D column D has a zero, therefore there should be no loop at vertex D; eliminate option **E**

Final remaining graph is correct.
A

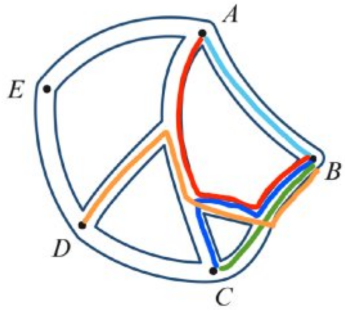
8 Construct a matrix to represent the graph. It will have 5 rows and 5 columns for the 5 vertices:
Vertex A has:

- two different connections to B
- two different connections to C
- one connection to D
- one connection to E



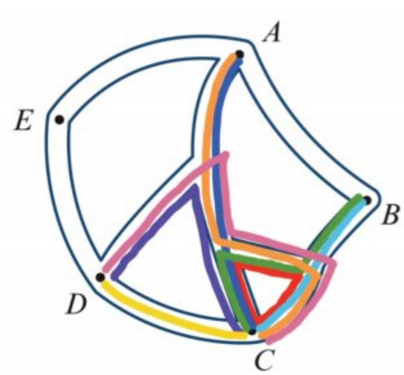
Vertex B has:

- two different connections to A
- two different connections to C
- one connection to D
- no connection to E



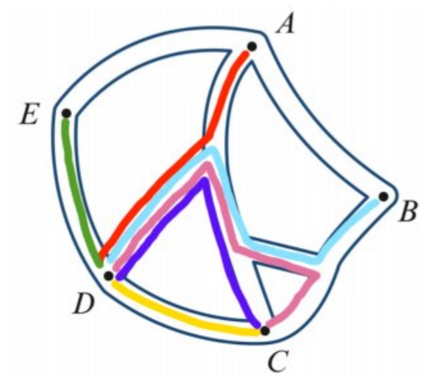
Vertex C has:

- two different connections to A
- two different connections to B
- a *loop*
- three different connections to D
- no connection to E



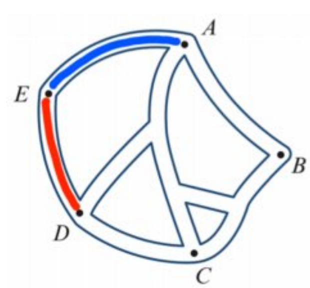
Vertex D has:

- one connection to A
- one connection to B
- three different connections to C
- one connection to E



Vertex E has:

- one connection to A
- no connection to B
- no connections to C
- one connection to D



From the information listed above, the matrix that represents the map is:

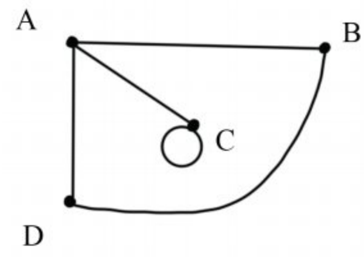
	A	B	C	D	E
A	0	2	2	1	1
B	2	0	2	1	0
C	2	2	1	3	0
D	1	1	3	0	1
E	1	0	0	1	0

The number '1' appears 9 times.

C

9 *Use the matrix constructed for the solution to Question 8 above*. The numbers ’2’ and ’3’ appear 8 times.
C

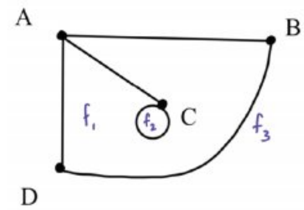
10 Draw the graph represented for the given adjacency matrix.



- the graph above is connected, as there are no isolated vertices; **A** is *TRUE*
- there is a loop at vertex *C*; **B** is *TRUE*
- there are no multiple edges (duplicate edges) between any of the vertices; **C** is *FALSE*
- the graph is connected and can be drawn with no edges crossing; **D** is *TRUE*
- the edge connecting vertices *A* and *C* is a bridge, because if it were removed, the vertex *C* would be isolated from the other vertices; **E** is *TRUE*

Only one option was not true.
C

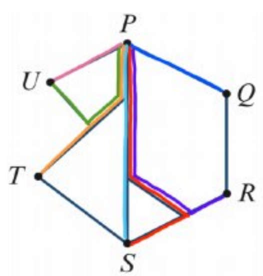
11 *Use the graph drawn for the solution to Question 10 above*:



B

12 Construct a matrix to represent the graph. It will have 6 rows and 6 columns for the 6 vertices:
Vertex *P* has:

- one connection to *Q*
- one connection to *R*
- two different connections to *S*
- one connection to *T*
- two different connections to *U*

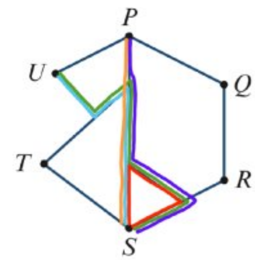


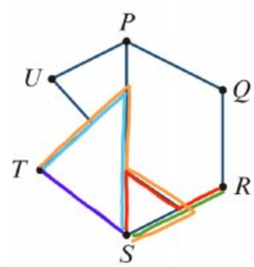
From this information, eliminate the incorrect options **A** and **B**

Rows *Q* and *R* are identical for the remaining options **C,D** and **E**.

Vertex S has:

- two different connections to P
- no connection to Q
- two different connections to R
- a loop
- three different connections to T
- two different connections to U





From this information, eliminate the remaining incorrect options **C,D**

Only one matrix correctly represents this map.

E

Solutions to Exercise 13C

Note: There are multiple possible answers to the questions in this exercise.

1 **a** This walk starts and ends at different vertices so it is not a cycle, nor circuit. The walk does not repeat edges, nor vertices, so it is a **path**.

b This walk starts and ends at different vertices so is not a cycle nor circuit. The walk has a repeated vertex, but not a repeated edge so the walk is a **trail**.

c This walk starts and ends at different vertices so it is not a cycle, nor circuit. The walk does not repeat edges, nor vertices, so it is a **path**.

d This walk starts and ends at the same vertex, so it could be a circuit or a cycle. however, there is a repeated vertex and a repeated edge, so it will be neither. It cannot be a trail or a path because of the repeated edge and vertex, so this walk is only a **walk**.

e This walk starts and ends at different vertices, so it is not a cycle, nor circuit. The walk has a repeated vertex but not a repeated edge, so this walk is a **trail**.

f This walk starts and ends at different vertices, so it is not a cycle, nor circuit. The walk has not repeated edge and no repeated vertex, so the walk is a **path**.

2 **a** This walk starts and ends at different vertices, so it is not a cycle, nor circuit. The walk has a repeated edge and vertex, so it is a **walk** only.

b This walk starts and ends at the same vertex, so it could be a cycle or a circuit. There is no repeated edge, nor vertex, so the walk is a **cycle**.

c This walk starts and ends at different vertices, so it is not a cycle, nor circuit, There are no repeated edges, nor vertices, so the walk is a **path**.

d This walk starts and ends at different vertices, so it is not a cycle, nor circuit. There are repeated edges and vertices, so the walk is a **walk** only.

e This walk starts and ends at different vertices, so it is not a cycle, nor circuit. There are no repeated edges, nor vertices, so the walk is a **path**.

f This walk starts and ends at different vertices, so it is not a cycle, nor circuit. There are repeated edges and vertices, so the walk is a **walk** only.

g This walk starts and ends at the same vertex, so it could be a cycle or a circuit. There is no repeated edge, however there is a repeated vertex, so the walk is a **circuit**.

h This walk starts and ends at the same vertex, so it could be a cycle or a circuit. There is no repeated edge, nor vertex, so the walk is a **cycle**.

3 **a** **i** This graph has two odd-degree vertices (A and E) and so it will have an eulerian trail.

ii One possible eulerian trail for this graph is $A–B–E–D–B–C–D–A–E$. There are other trails possible.

b **i** This graph has all vertices of odd degree. Neither an eulerian trail, nor eulerian circuit, are possible.

c **i** This graph has two odd-degree vertices (A and F) and so it will have an eulerian trail.

ii One possible eulerian trail for this graph is $A–C–E–C–B–D–E–F$. There are other trails possible

d **i** This graph has all vertices of even degree. Both an eulerian trail and an eulerian circuit are possible.

ii One possible eulerian circuit for this graph is $A–B–C–D–E–C–A$. There are other trails and circuits possible

e **i** This graph has all vertices of even degree. Both an eulerian trail and an eulerian circuit are possible.

ii One possible eulerian circuit for this graph is $F–E–A–B–E–D–C–B–D–F$. There are other trails and circuits possible.

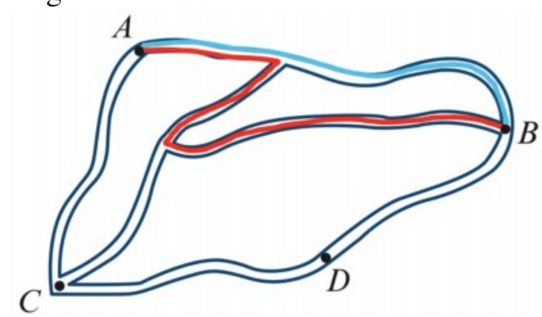
4 **a** A hamiltonian cycle for this graph is $A–B–C–F–I–H–E–G–D–A$.

b A hamiltonian cycle for this graph is $A–B–C–D–E–F–A$.

c A hamiltonian cycle for this graph is $A–B–D–C–E–A$.

5 $F–A–B–C–D–E–H–G$.

6 **a** A vehicle can travel between town A and town B in two ways, without visiting any other town. See the diagram below:

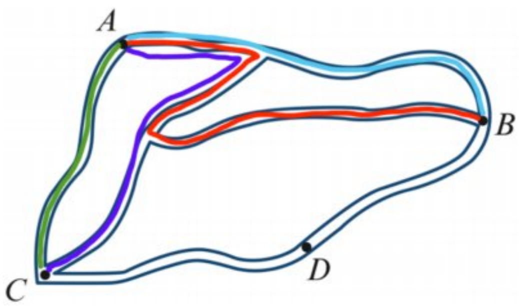


b By inspection, there are 7 different trails from town A to town D. Where there are different routes between two towns, the route is shown by a subscript.

$A–C_1–D$
$A–C_2–D$
$A–B_1–D$
$A–B_2–D$
$A–B_1–C–D$
$A–C_1– B_1–D$
$A–C_1– B_2–D$

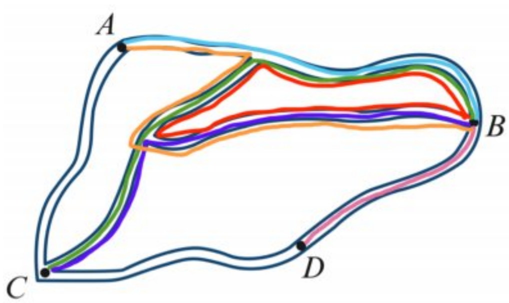
c Start with 4 vertices to represent the 4 towns.
Vertex *A* has:

- two different connections to *B*
- two different connections to *C*
- no connections to *D*



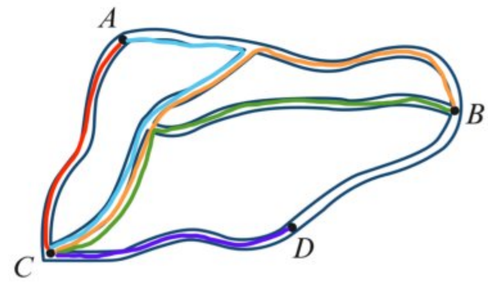
Vertex *B* has:

- two different connections to *A*
- two different connections to *C*
- one connection to *D*
- one loop



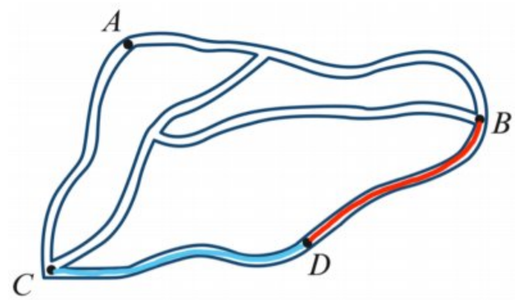
Vertex *C* has:

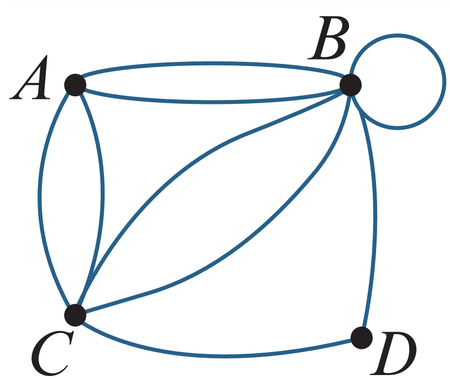
- two different connections to *A*
- two different connections to *B*
- one connection to *D*



Vertex *D* has:

- no connections to *A*
- one connection to *B*
- one connection to *C*





d An eulerian circuit is not possible through this network because there are some odd-degree vertices (*B* and *C*).

7 a $v = 9$, $e = 12$, $f = 5$; $v + f = e + 2$
$9 + 5 = 12 + 2$
$14 = 14$
Euler's formula is verified.

b **i** The walk described will start and end at the same vertex, so it could be either a circuit or a cycle. The walk will not repeat any vertex, therefore it is a cycle and because the organisers will visit *every* vertex the walk is classified as a **Hamiltonian cycle**.

ii Lake Bolac - Streatham - Nerrin Nerrin - Woorndoo - Mortlake - Hexham - Chatworth - Glen-thompson - Wickliffe - Lake Bolac. The reverse of this is the second possible option.

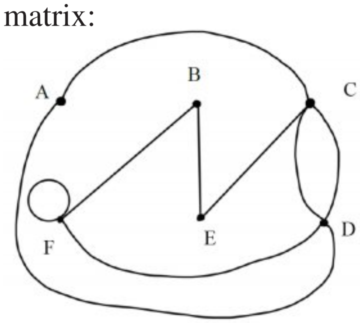
c **i** The walk described will start and end at the same vertex, so it could be either a circuit or a cycle. The walk will pass through every edge which describes an **Eulerian circuit**.

ii All vertices must have an *even* degree for an Eulerian circuit to be possible. The vertices representing the towns of Wickliffe and Lake Bolac both have an *odd* degree, therefore an Eulerian circuit is not possible.

d **i** The proposed race planned in part **c** was an *Eulerian circuit*, however it was not possible because the vertices representing the towns of Wickliffe and Lake Bolac both have an *odd* degree; for an Eulerian circuit *all* vertices must have an *even* degree. For an Eulerian circuit to be possible, the road connecting **Wickliffe and Lake Bolac** could be travelled along twice. Travelling along a road twice can be interpreted as *adding* an extra edge to the graph, thus all vertices would have an even degree and the proposed Eulerian circuit from part **c** would be possible.

ii Lake Bolac - Streatham - Nerrin Nerrin - Woorndoo - Mortlake - Hexham - Chatsworth - Woorn-doo - Lake Bolac - Wickliffe - Chatsworth - Glenthompson - Wickliffe - Lake Bolac

8 Draw the graph represented by the matrix:



a Yes, as the graph above shows, all vertices are either directly or indirectly connected to all other vertices.

b Yes, the graph can be drawn with no edges crossing.

c No, there are no edges which, if removed, would result in a disconnected graph.

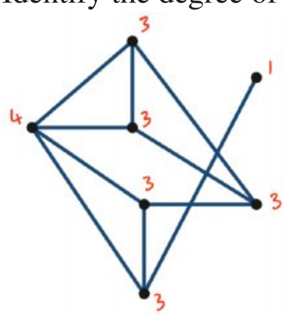
d Yes, as all vertices have an *even* degree.

e Yes, as all vertices have an *even* degree.

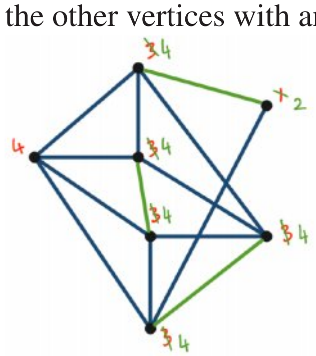
f Yes, a walk is possible where all vertices are visited without repeating any vertices.

g Yes, a walk is possible, where all vertices are visited without repeating any vertices, starting and ending at the same vertex.

9 For an Eulerian circuit to exist, all vertices must have an *even* degree. Identify the degree of each vertex:



Of the 7 vertices, 6 of them have an odd degree. When an edge is added between two vertices, the degree of each of those vertices increases by 1; when an edge is added to a vertex with an odd degree, it will change to an even degree as the number of edges connected to it has increased by 1. Consider connecting 3 vertices with an odd degree with one of the other vertices with an odd degree:



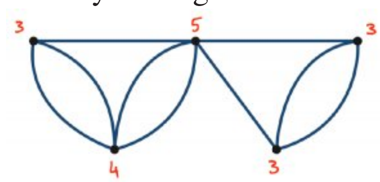
Adding 3 edges to this graph results in all vertices having an even degree; this could not be achieved with a smaller number of edges.

Exam 1 style questions

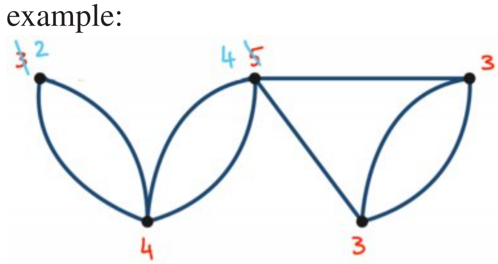
10 For an Eulerian circuit to exist, all vertices must have an *even* degree. Options A, B, D, E are graphs that have at least one vertex with an odd degree. *Note: A loop contributes 2 to the degree of a vertex.*
C

11 The walk $DEFACBD$ is not a Hamiltonian cycle for this graph as there is no edge connecting vertex A to vertex C.
D

12 Identify the degree of each vertex:



An Eulerian trail is possible if there are *zero* or *two* vertices with an odd degree. The original graph has 5 vertices and 4 of them have an odd degree. When removing an edge from the graph, the degree of two vertices will change. For example:



The vertex with an even degree should not have any of its edges removed, as it is only connected to vertices with an odd degree, therefore the removal of any of the edges connected to the vertex with an even degree in this graph will not result in a change in number of vertices with an odd degree.
There are five edges that connect two vertices with an odd degree; if any of these five edges were removed, the total number of vertices with an odd degree would decrease to 2 out of the 5 vertices, which would make an Eulerian trail possible for the new graph.
E

13 The walk $ABCDEF$ is the only Hamiltonian path for the graph. All other options either visit a vertex more than once (options **C,D,E**) or do not visit every vertex (option **A**).
B

14 For an Eulerian circuit to be possible, all vertices must have an *even* degree. Vertices C and E have an odd degree. If an edge was added between these vertices, the degree of these vertices would increase by 1 and then all vertices in the graph would have an *even* degree.
A

Solutions to Exercise 13D

1 **a** The edge showing a weight of 12 is between town D and town E.

b C to D via B, means C to B (8 minutes) followed by B to D (9 minutes) for a total of 9 + 8 = 17 minutes.

c D to E direct is 12 minutes
D to E via B is 9 + 11 = 20 minutes.
By driving direct, the motorist will save 20 – 12 = 8 minutes.

d The options for travelling from A to E and visiting all towns exactly once are:
$A–C–D–B–E$ for a total time of 45 minutes.
$A–B–C–D–E$ for a total time of 36 minutes.
$A–C–B–D–E$ for a total time of 44 minutes.
The shortest time is 36 minutes.

2 By inspection, the shortest path from A to E will be $A–C–D–E$ for a length of 11.

3 **a** The path $A–B–E–H–I$ has length:
5 + 9 + 12 + 8
= 34 kilometres

b The circuit $F–E–D–H–E–A–C–F$ has length:
6 + 6 + 10 + 12 + 8 + 4 + 10
= 56 kilometres

c The shortest cycle starting and ending at E is via B and A, $E–B–A–E$ for a distance of:
9 + 5 + 8
= 22 kilometres

d Shortest path from A to I (by inspection) is either $A–C–F–G–I$ for a distance of 4 + 10 + 4 + 8
= 26 kilometres
or $A–E–F–G–I$ for a distance of
8 + 6 + 4 + 8
= 26 kilometres

4 **a** The shortest path from S to F is $S–B–D–F$ with length 12.

b The shortest path from S to F is $S–A–C–D–F$ with length 10.

c The shortest path from S to F is $S–B–D–F$ with length 15.

d The shortest path from S to F is $S–A–E–G–F$ with length 19.

5 The length of the shortest path between town A and B is 19 kilometres.

Exam 1 style questions

6 The shortest path from C to E is
$C – D – A – E$
B

7 The shortest path from B to G is
$B – A – D – C – G$
E

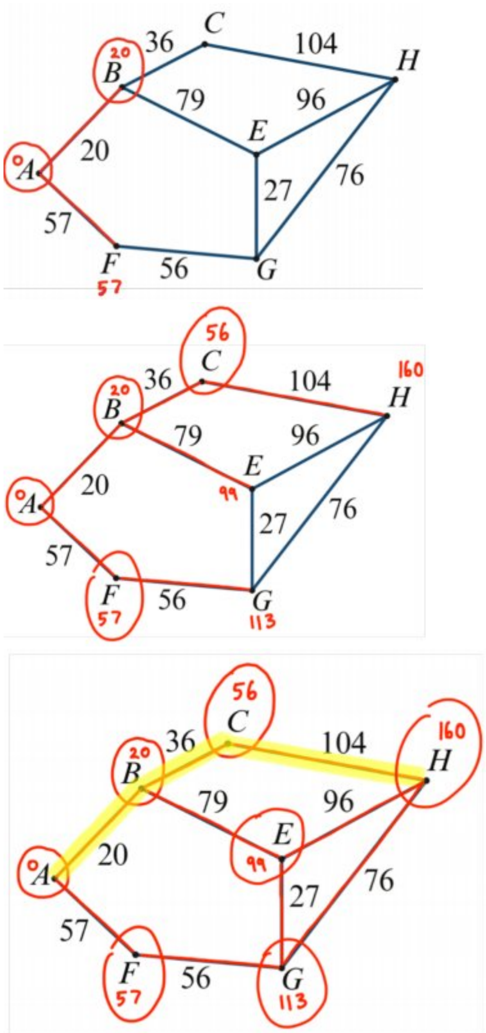
8 The shortest path from $Home$ to $School$

is $Home - E - C - B - D - School$ which equates to a length of $30 + 14 + 10 + 26 = 80$ minutes.
A

8 The shortest path from *Home* to *School* is either $Home - E - C - B - D - School$ or $Home - E - C - B - School$.
E

Solutions to Exercise 13E

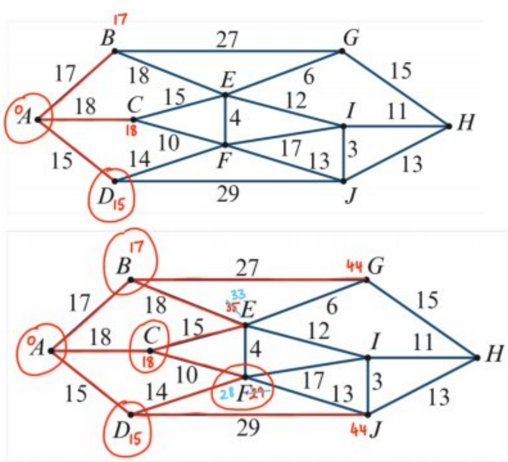
1 a

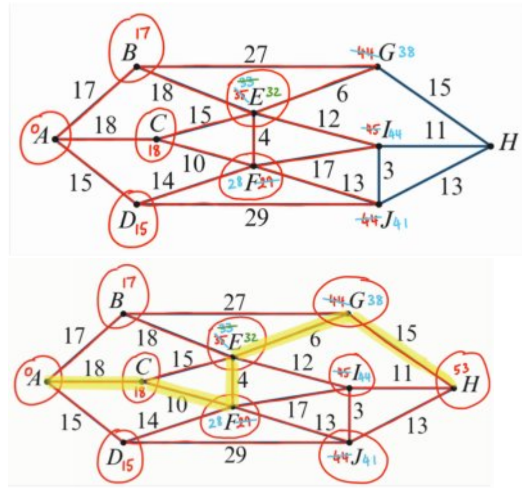


Shortest path= $A - B - C - H$

Length of shortest path = 160

b

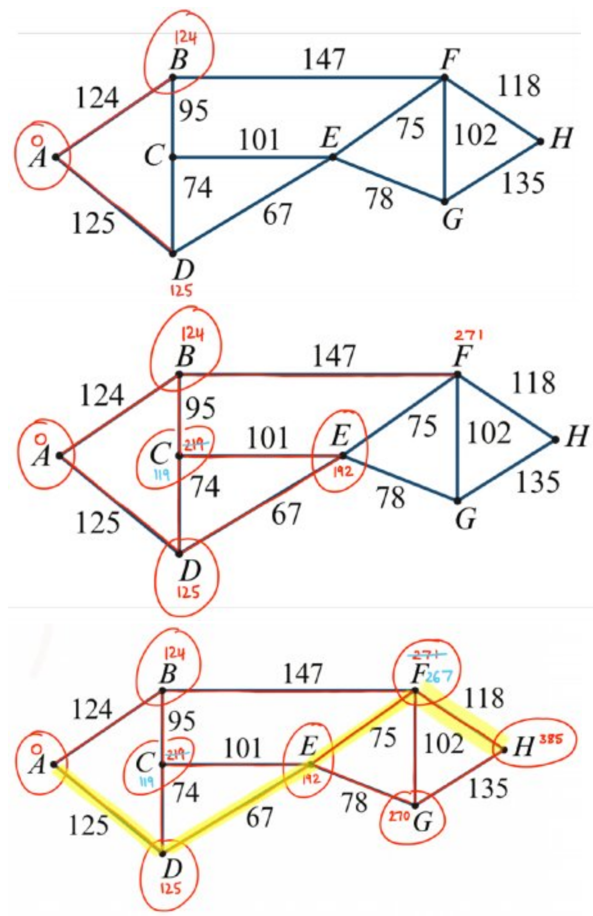




Shortest path= $A - C - F - E - G - H$

Length of shortest path = 53

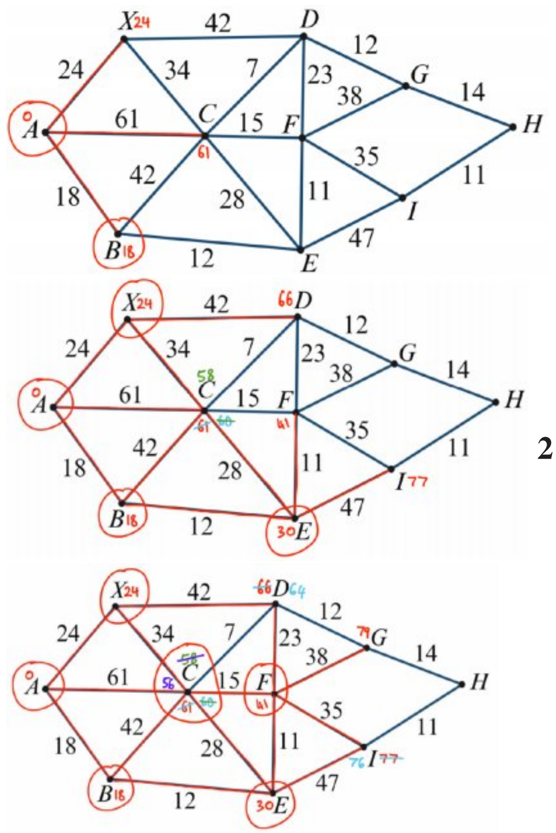
c

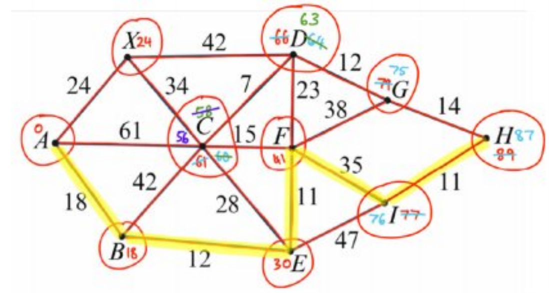


Shortest path= $A - D - E - F - H$

Length of shortest path = 385

d

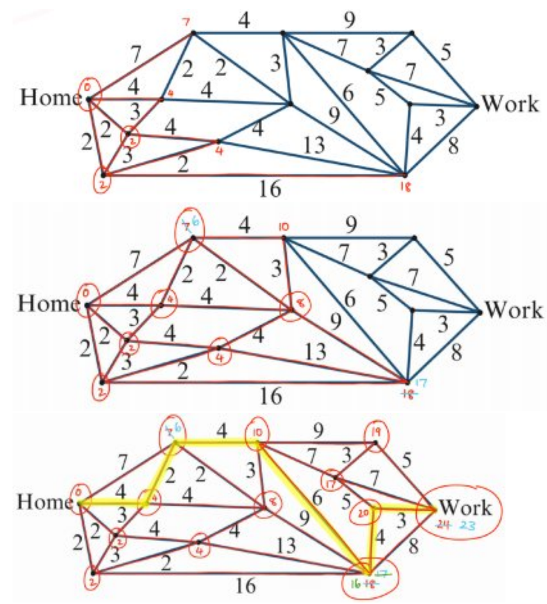




Shortest path= $A - B - E - F - I - H$

Length of shortest path = 87

2

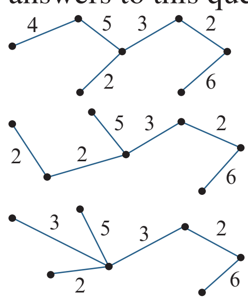


Shortest path = 23 minutes

Solutions to Exercise 13F

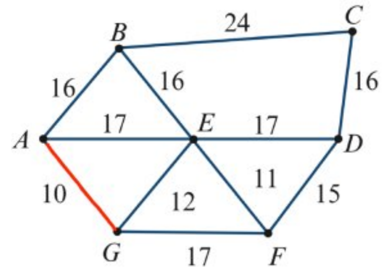
1 a There are 7 vertices, so the spanning tree will have 7 – 1 = 6 edges.
The network has 12 edges, so 12 – 6 = 6 edges must be removed.

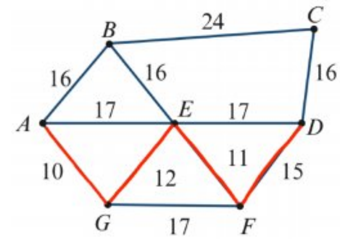
b Note: other trees are possible as answers to this question.

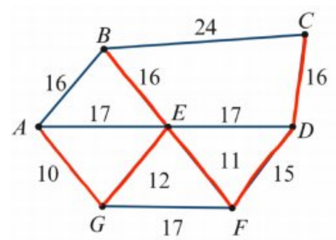


c The first graph has weight:
4 + 5 + 2 + 3 + 2 + 6 = 22
The second graph has weight:
2 + 2 + 5 + 3 + 2 + 6 = 20
The third graph has weight:
3 + 2 + 5 + 3 + 2 + 6 = 21

b

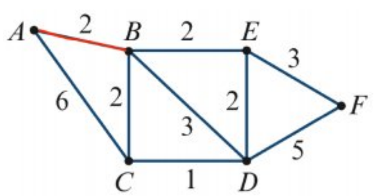


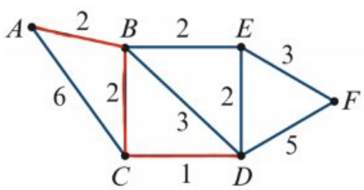


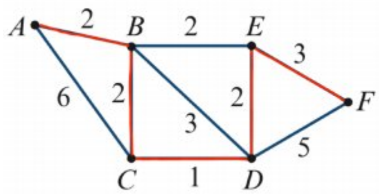


Total weight
= 16 + 10 + 12 + 11 + 15 + 16
= 80

2 a

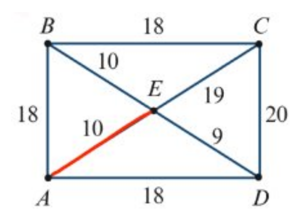


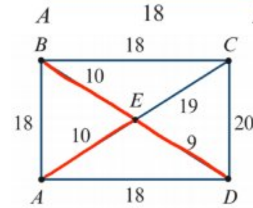


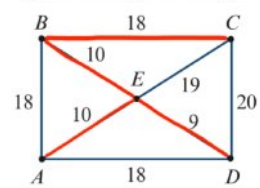


Total weight = 2 + 2 + 2 + 1 + 3
= 10

c

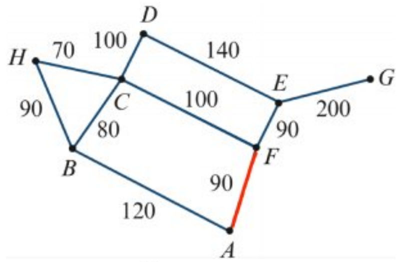


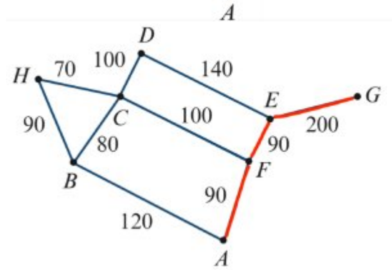


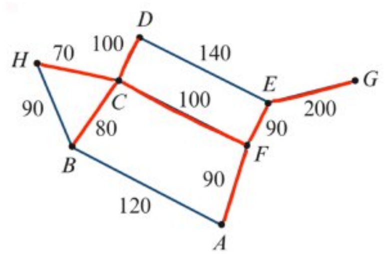


Total weight = 18 + 10 + 10 + 9
= 47

d

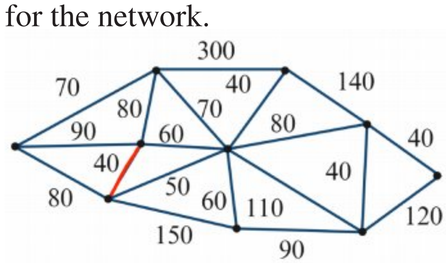


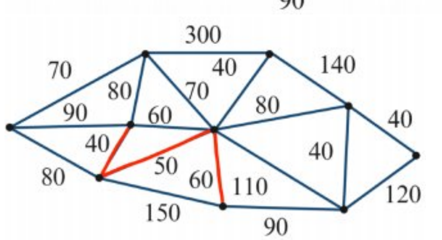


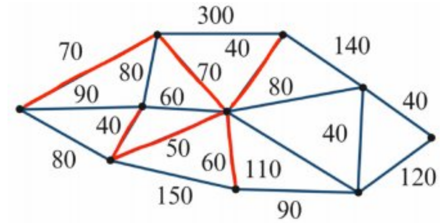


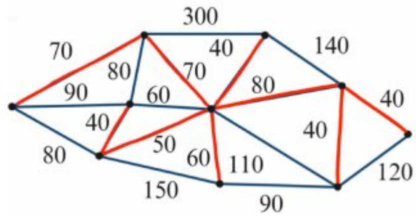
Total weight
= 70 + 80 + 100 + 100 + 90 + 90 + 200 = 730

3 The shortest length of pipe required to connect all water storages will be the weight of the minimum spanning tree for the network.



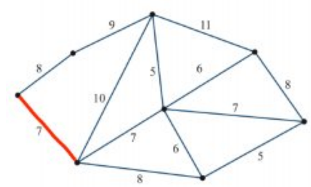


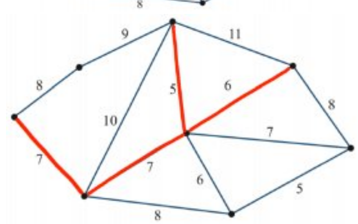


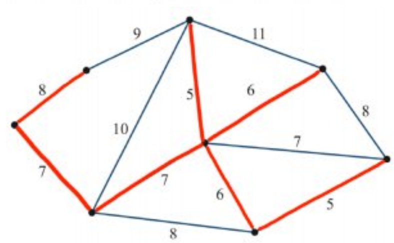


The weight of the minimum spanning tree, shown in red above, is
weight
= 70 + 70 + 40 + 50 + 40 + 60 + 80 + 40 + 40 = 490

4 Using Prim's algorithm, determine a minimum spanning tree:

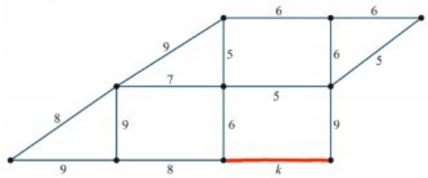


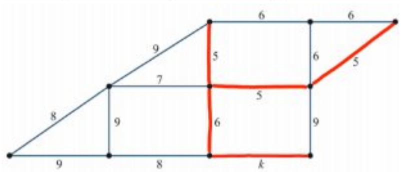


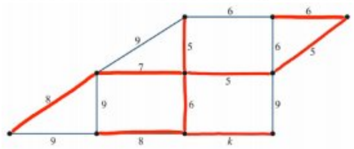


Length of minimum spanning tree= 8 + 7 + 7 + 5 + 6 + 6 + 5 = 44
A

5 Using Prim's algorithm, determine a minimum spanning tree. Start at the edge with weight *k*; although you do not know the value of *k* it is stated that it is included in the minimum spanning tree:







Total length of the minimum spanning tree = 58

$6 + 5 + 5 + 5 + 6 + 8 + 7 + 8 + k = 58$

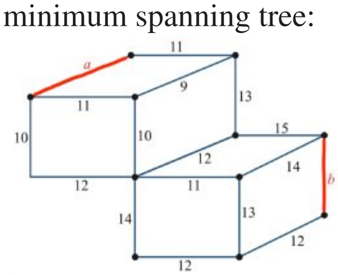
$50 + k = 58$

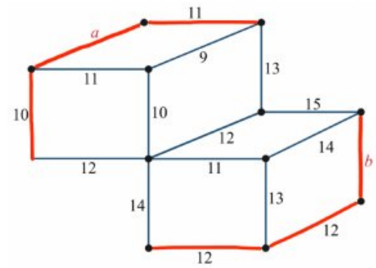
$k = 58 - 50$

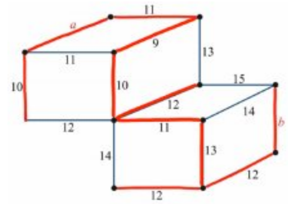
$k = 8$

C

6 Using Prim's algorithm, determine a minimum spanning tree. Start at the edges with weight *a* and *b*; although you do not know the values of *a* or *b*, it is stated that they are included in the minimum spanning tree:







Total length of the minimum spanning tree = 124

$a + b + 10 + 11 + 9 + 10 + 12 + 11 + 13 + 12 + 12 = 124$

$a + b + 100 = 124$

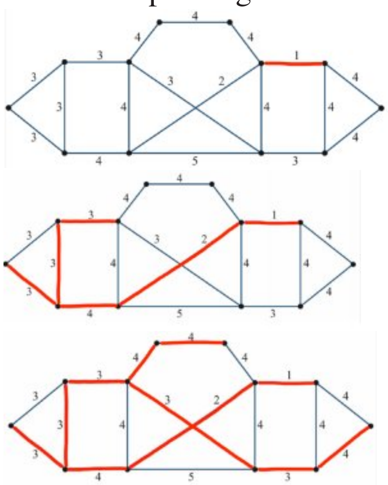
$a + b = 124 - 100$

$a + b = 24$

The sum of *a* and *b* must equal 24; eliminate the incorrect options **A,D,E**. Of the two remaining options, consider position of *a* and *b* and whether they would be included in the minimum spanning tree when applying Prim's algorithm. If $a = 12$, the edge with weight *a* would not be included in the minimum spanning tree, as the vertices that edge connects to could be included in the minimum spanning tree using edges with a smaller weight, thus option **B** is incorrect. From the given list of options, $a = 10$ and $b = 14$.

C

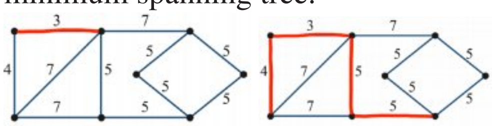
7 Using Prim's algorithm, determine a minimum spanning tree.



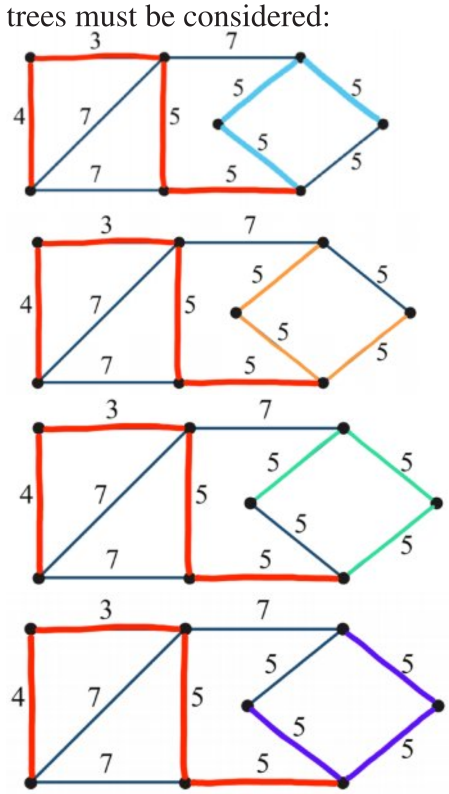
Count the number of edges with weight 4 that are not included in the minimum spanning tree found.

E

8 Using Prim's algorithm, determine a minimum spanning tree.



At this stage, there are multiple minimum spanning trees possible. For this question **all** possible minimum spanning trees must be considered:



There are four different minimum spanning trees possible.

D

Solutions to Review: Multiple-choice questions

1 Seven vertices can be connected with six edges, one less than the number of vertices. **C**

2 **A** This graph has a cycle so is not a tree.

B This graph has a cycle so is not a tree.

C This graph is a spanning tree.

D This graph has a cycle so is not a tree.

E This graph is a tree but does not include the vertex 2 so it is not a spanning tree. **C**

3 P has degree 2
Q has degree 5
R has degree 3
S has degree 4
T has degree 4
U has degree 2 **A**

4 $v = 15$ and $f = 12$
$v - e + f = 2$
$15 - e + 12 = 2$
$-e = 2 - 15 - 12$
$-e = -25$
$e = 25$ **D**

5 An eulerian circuit will exist if all of the vertices have an even degree.

A has two odd-degree vertices.

B has all even-degree vertices.

C has all even-degree vertices.

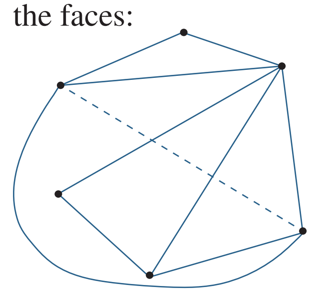
D has all even-degree vertices.

E has all even-degree vertices. **A**

6 $v = 8$ and $e = 13$
$v - e + f = 2$
$8 - 13 + f = 2$
$F = 2 - 8 + 13$
$f = 7$ **C**

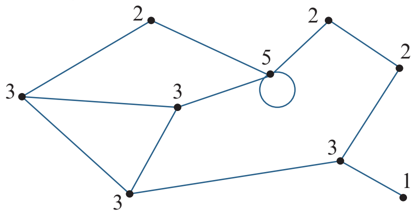
7 Hamiltonian cycle starts and ends at the same vertex, so it cannot be option E. Hamiltonian cycles pass through every vertex only once, so it cannot be option A (visits E multiple times) or option C (visits A multiple times), Hamiltonian cycles pass through every vertex in the graph, so it cannot be option D which does not visit vertex F. **B**

8 The graph is planar and must be redrawn without edges crossing before counting the faces:



There are five regions defined by the graph in planar form. **B**

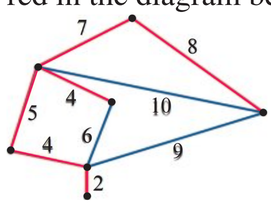
9 The graph is drawn below, with the degrees of each vertex written beside them.



The sum of the degrees is
3 + 2 + 3 + 3 + 5 + 2 + 2 + 3 + 1
= 24 **E**

10 An eulerian trail exists if there are exactly
two odd-degree vertices in a graph. The graph currently has four odd-degree vertices, that is *A*, *E*, *C*, *D*.
Joining two of these by an edge would make their degree even. **B**

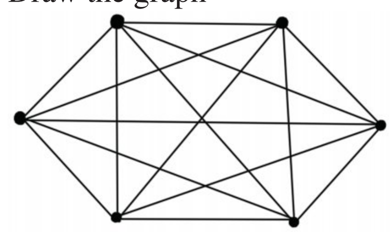
11 The minimum spanning tree is shown in red in the diagram below:



The length of the minimum spanning tree
= 2 + 4 + 5 + 4 + 7 + 8 = 30 **A**

12 Eulerian circuit will be possible if all of the vertices have an even degree, so it could be option **A** or **B**.
By inspection, a hamiltonian cycle is possible only in option **A**. **A**

13 A graph with six vertices, connected with the minimum number of edges, is a tree with 5 edges.
A complete graph has every vertex connected to every other vertex; the number of edges for a complete graph with six vertices can found by either:
Method 1
Draw the graph



Method 2
A complete graph with *n* vertices will have $\frac{n(n-1)}{2}$ edges. This graph has six vertices, so $n = 6$ and the graph will have $\frac{6(6-1)}{2} = \frac{6(5)}{2} = \frac{30}{2} = 15$ edges.
Starting with 5 edges, 10 extra edges must be added to the graph to make this a complete graph. **C**

14 In the given graph,

- *A* is directly connected to *B* in two ways (could be only option B)
- *A* is directly connected to *C* in one way
- *A* is directly connected to *D* in one way
- *B* is directly connected to *C* in one way
- *B* is directly connected to *D* in one way
- *C* is directly connected to *D* in two ways **B**

15 An eulerian circuit exists if all the vertices have an even degree.
Option **A** has two odd-degree vertices.
Option **B** has all even-degree vertices.
Option **C** has two odd-degree vertices.
Option **D** has two odd-degree vertices.
Option **E** has two odd-degree vertices. **B**

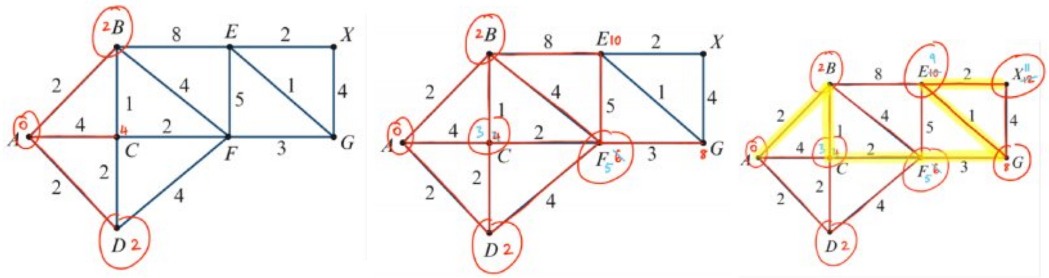
16 Consider some different paths from S to T:

- $S-A-F-T = 22 + 15 + 8 = 45$
- $S-A-D-G-T = 22+3+2+10 = 37$
- $S-B-D-G-T = 18+9+2+10 = 39$
- $S-B-E-G-T = 18+3+x+10 = 31+x$
- $S-B-H-T = 18+3+7+11 = 39$
- $S-C-E-G-T = 17+6+x+10 = 33+x$
- $S-C-E-H-T = 17+6+7+11 = 41$
- $S-C-H-T = 17+15+11 = 43$

The path with the shortest distance is $S-B-E-G-T$ with a distance of $31 + x$. Given the shortest distance is 36 metres, $x = 5$. **B**

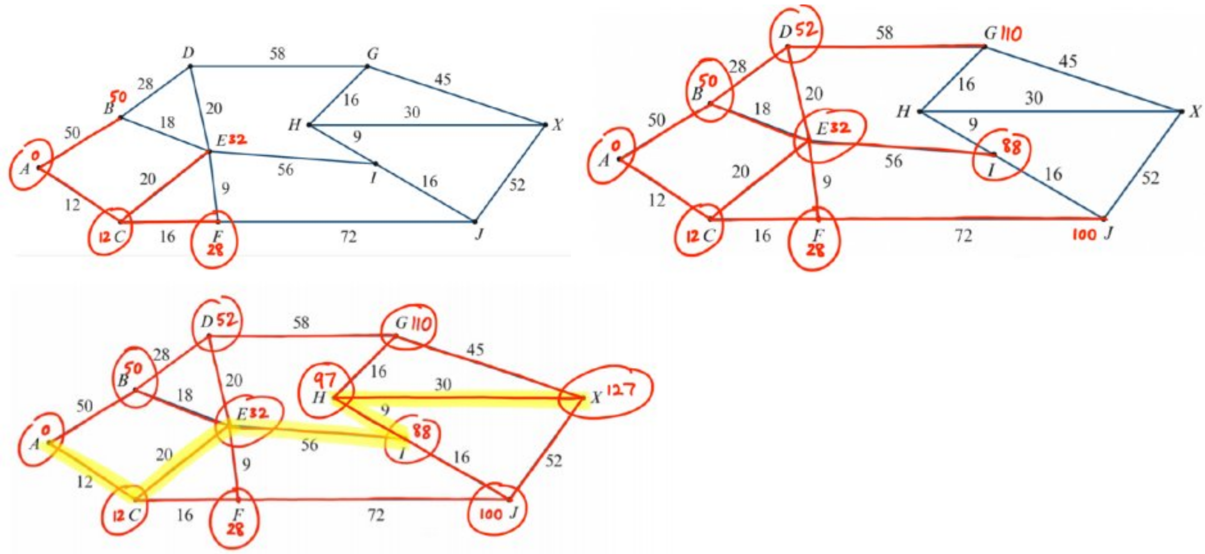
Chapter Review: Extended-response questions

1 a



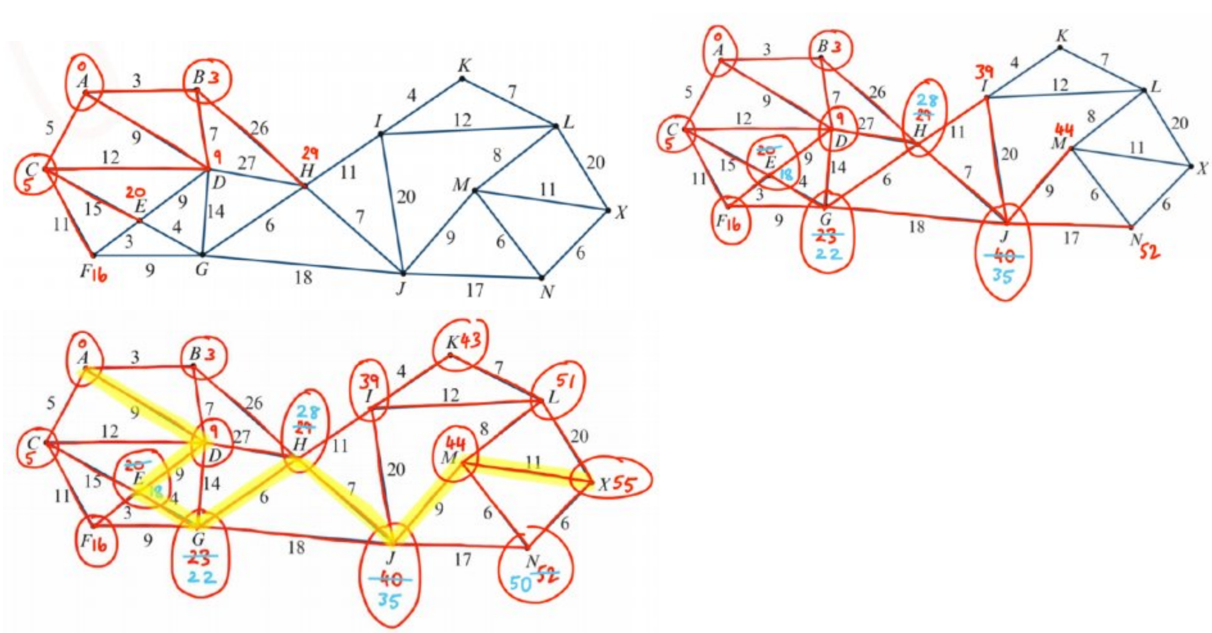
The shortest path from A to X is: $A - B - C - F - G - E - X$.
The length of the shortest path = 2 + 1 + 2 + 3 + 1 + 2 = 11.

b



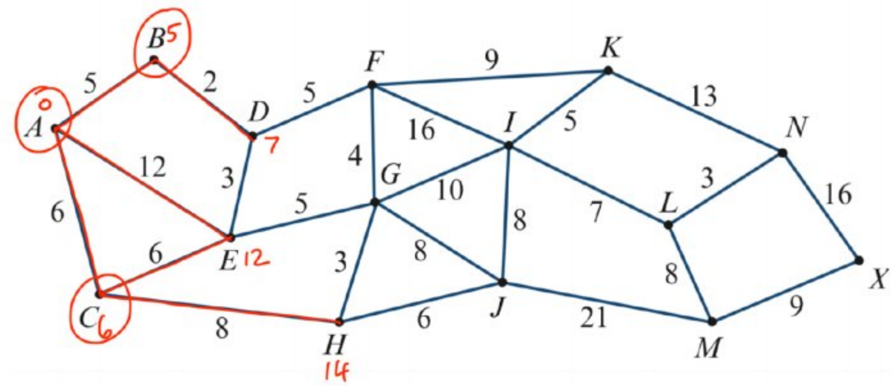
The shortest path from A to X is: $A - C - E - I - H - X$.
The length of the shortest path = 12 + 20 + 56 + 9 + 30 = 127.

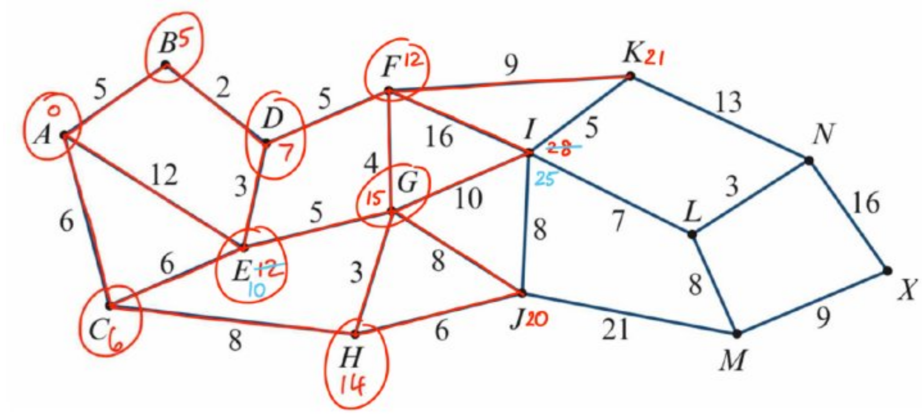
c

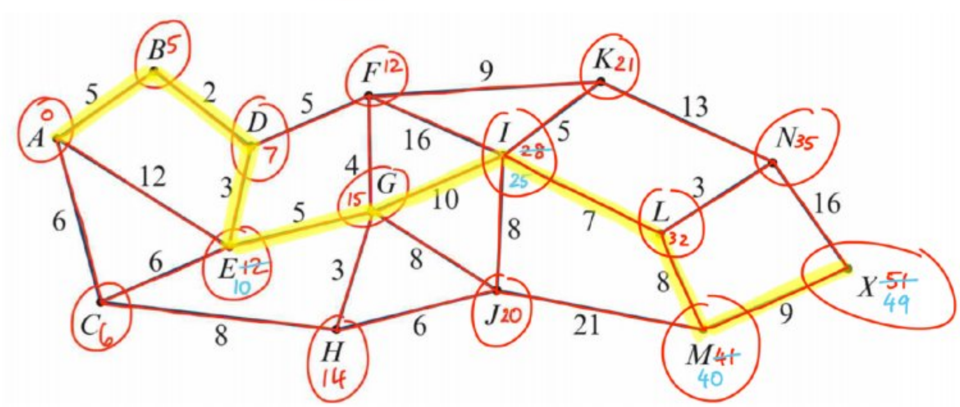


The shortest path from A to X is: $A - D - E - G - H - J - M - X$.
The length of the shortest path = 9 + 9 + 4 + 6 + 7 + 9 + 11 = 55.

d

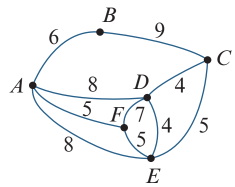




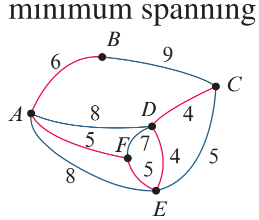


The shortest path from A to X is: $A - B - D - E - G - I - L - M - X$.
The length of the shortest path = 5 + 2 + 3 + 5 + 10 + 7 + 8 + 9 = 49.

2 a i The only edge missing from the graph is the direct connection between vertex E and vertex C. There is only one direct connection between these vertices, of length 5, so this must be added to the graph.



ii The cable should be laid along the minimum spanning tree for the graph. The minimum spanning tree is shown in red below:



The weight of the minimum spanning tree
= 4 + 5 + 4 + 5 + 6
= 24
The minimum length of cable required is 24 kilometres.

iii There is:

- one connection between *D* and *C*
- no loop at *D*
- one connection between *D* and *E*
- one connection between *D* and *F*
- one connection between *E* and *C*
- no loop at *E*
- one connection between *E* and *F*
- no connection between *F* and *C*
- no loop at *F*

The matrix is

$$\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b **i** The route
$A - B - A - F - E - D - C - E - F - A$
has distance
6 + 6 + 5 + (3+2) + 4 + 4 + 5 + (2+3) + 5
= 45 kilometres

ii This route is not a hamiltonian cycle because some of the vertices are visited more than once, namely *A*, *F* and *E*.

iii There are many answers, but one possible route is:
$A - B - C - D - F - E - A$

iv The distance travelled will vary depending on the answer for part iii. The route in part **iii** above has distance
6 + 9 + 4 + 7 + 5 + 8
= 39 kilometres

c Starting at *A* and returning to *A* by
travelling each track once is an example of an eulerian circuit. This can only occur if all vertices are of even degree.
At the moment, vertex *C* and *F* both have odd degrees, so joining them by a new path will make them both have an even degree, making an eulerian circuit possible.

3 **a** The vertex representing Melville has 4 edges connected directly to it, so the degree of this vertex is 4.

b There is a total of 9 edges in this graph. The sum of the degrees of a graph
= 2 × the total number of edges
= 2 × 9 = 18

c $v = 6, e = 9, f = 5$
$v + f = e + 2$
$6 + 5 = 9 + 2$
$11 = 11$
Euler's formula is verified for this graph.

d No, the salesperson has visited a vertex more than once (Melville), therefore this is not a Hamiltonian cycle.

e Hamiltonian path, because all vertices were visited without repeating any vertices and the starting vertex is different to the ending vertex.

f Bartow - Stratmoore - Melville - Kenton - Osburn - Croghon - Bartow.
The shortest distance is 71 kilometres.

g Melville - Croghon - Bartow - Stratmoore - Osburn - Kenton - Melville.
The shortest distance is 58 kilometres.

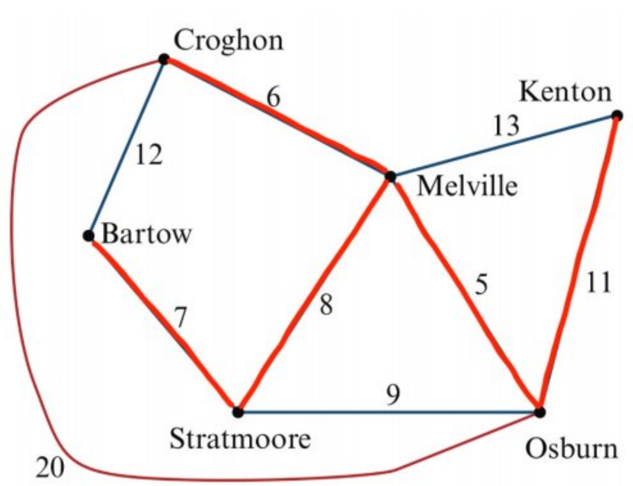
h An Eulerian circuit is possible is all vertices have an *even* degree. The vertices that represent Croghon and Stratmoore both have an *odd* degree.

i This walk described is an Eulerian trail. The inspector could start their route at either Croghon or Stratmoore, because these are the only two vertices with an *odd* degree. One option is: Stratmoore - Osburn - Kenton - Melville - Osburn - Croghon - Bartow - Stratmoore - Melville - Croghon (the reverse is also acceptable)

j The total distance travelled by the inspector is
$9 + 11 + 13 + 5 + 20 + 12 + 7 + 8 + 6 = 91$ kilometres.
91 km at 60 km/hr
$91 \div 60 = 1.51666...$ hours = 1 hour and $(0.51666... \times 60)$ minutes = 1 hour and 31 minutes.
1 hour and 31 minutes *BEFORE* 5:00pm.
Therefore 3:29 pm.

k Minimum spanning tree.

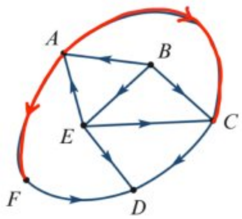
l



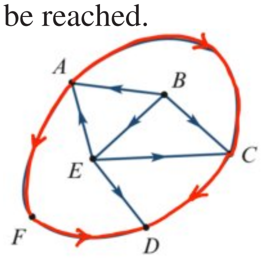
Total length of the minimum spanning tree = $11 + 5 + 8 + 7 + 6 = 37$ kilometres.

Solutions to Exercise 14A

1 **a** Starting at vertex A there are two edges, directed towards vertices C and F.

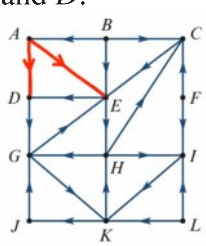


From vertices C and F, vertex D can be reached.

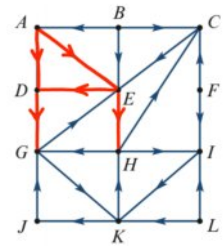


No other vertices can be reached from vertex A directly nor indirectly. There are 3 vertices that can be reached from vertex A (C, F, D).

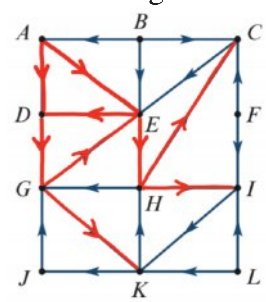
b Starting at vertex A, there are two edges, directed towards vertices E and D.



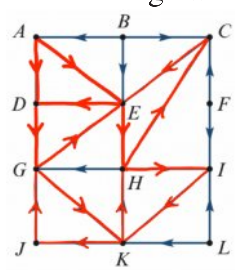
Vertex H is connected by a directed edge with E, likewise vertex G is connected by a directed edge with D.



Vertices K, I and C are connected by directed edges with vertices G and H.

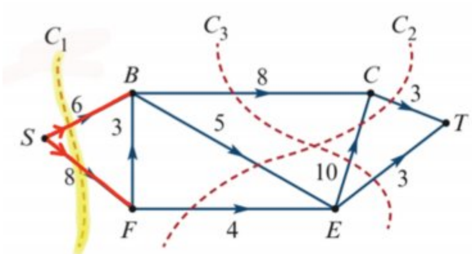


Finally, vertex J is connected by a directed edge with vertex K.

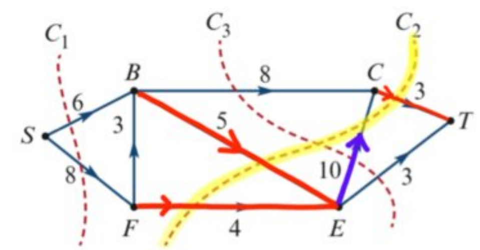


No other vertices can be reached from vertex A directly nor indirectly. There are 8 vertices that can be reached from vertex A (D, E, G, H, I, C, K, J).

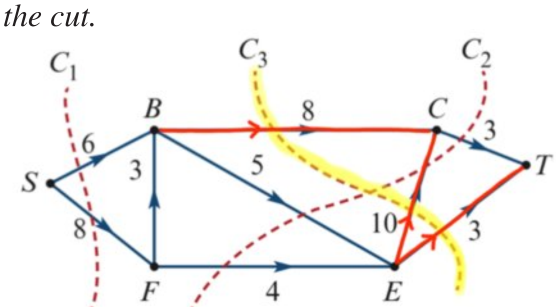
2



The capacity of $C1$
= 6 + 8
= 14

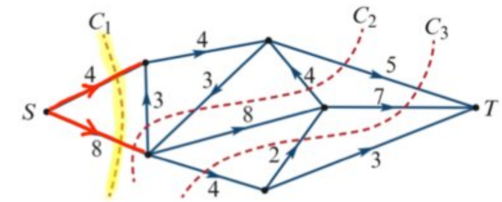


The capacity of $C2$
= 3 + 5 + 4
= 12
Note: the edge from E to C is not counted as the flow along this edge is from the sink side to the source side of the cut.

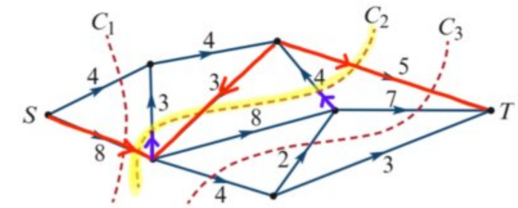


The capacity of $C3$
= 8 + 10 + 3
= 21

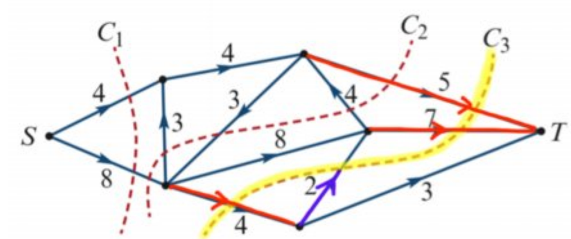
3



The capacity of $C1$
= 4 + 8
= 12

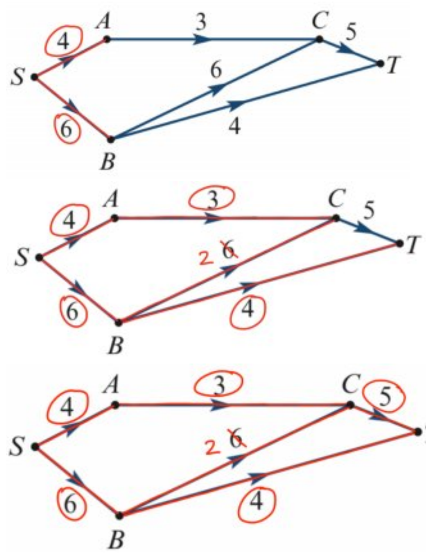


The capacity of $C2$
= 5 + 3 + 8
= 16
Note: there are two edges where the flow is from the sink side to the source side of the cut. These edges have capacity 4 and 3, and both are not counted in the calculation.



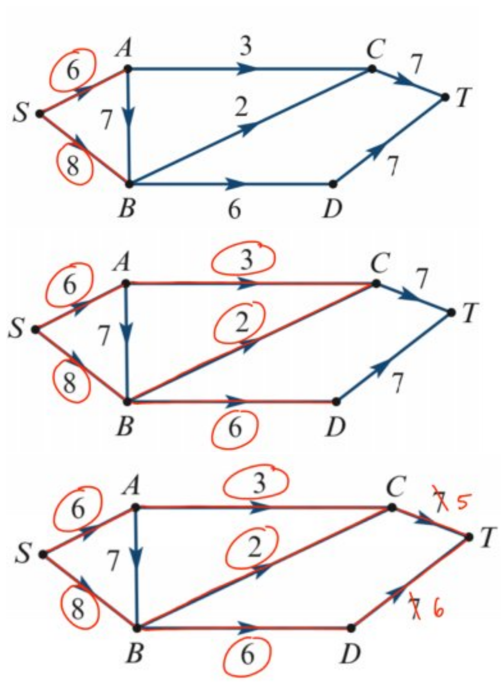
The capacity of $C3$
= 5 +7 + 4
= 16
Note: there is one edge where the flow is from the sink side to the source side of the cut. This edge has capacity 2 and this is not counted in the calculation.

4 a



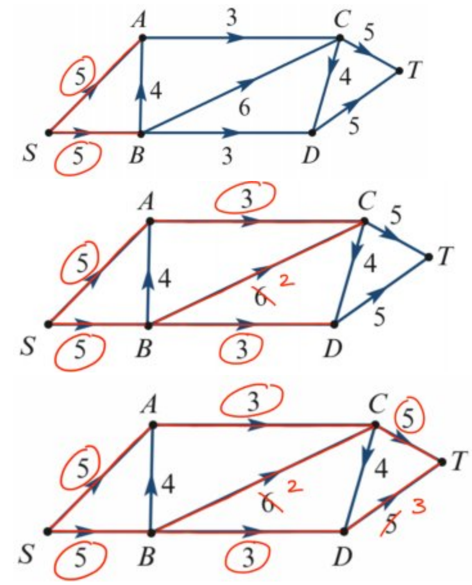
Maximum flow = 5 + 4 = 9

b



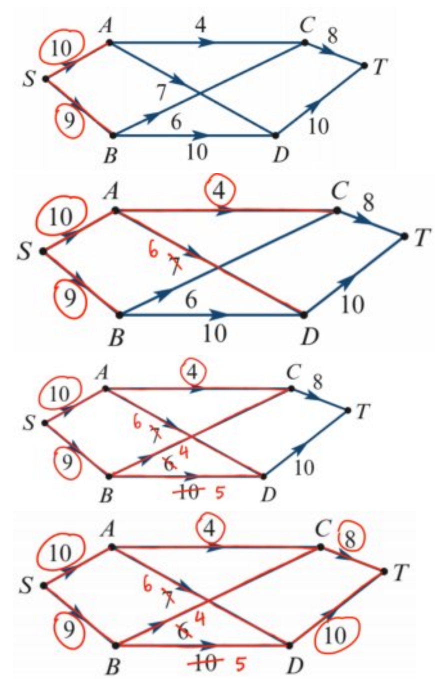
Maximum flow = 5 + 6 = 11

c



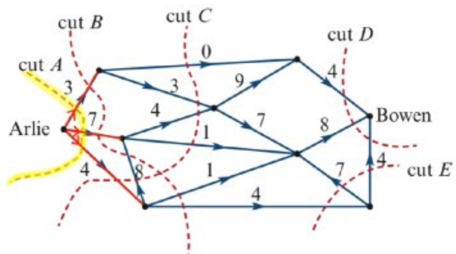
Maximum flow = 3 + 5 = 8

d

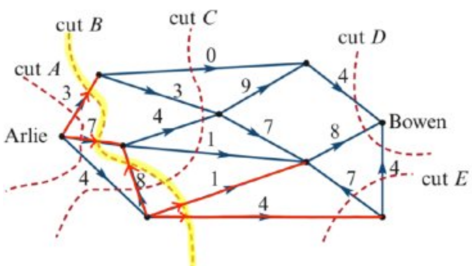


Maximum flow = 8 + 10 = 18

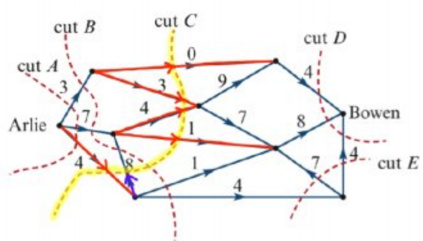
5 a



The capacity of cut A
$= 3 + 7 + 4$
$= 14$

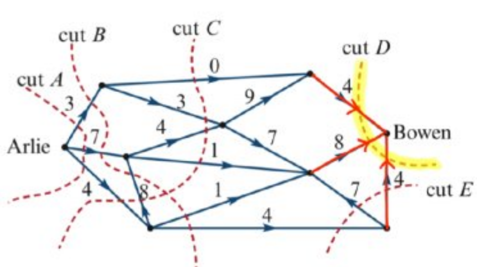


The capacity of cut B
$= 3 + 7 + 8 + 1 + 4$
$= 23$



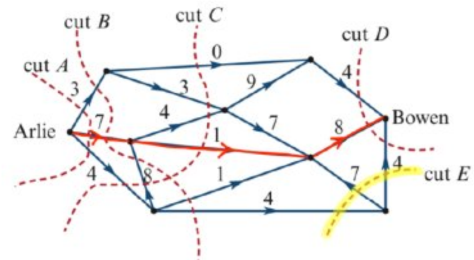
The capacity of cut C
$= 0 + 3 + 4 + 1 + 4$
$= 12$

Note: the edge with capacity 8 is not counted as the flow along this edge is from the sink side to the source side of the cut.

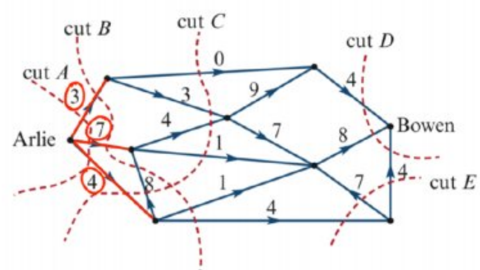


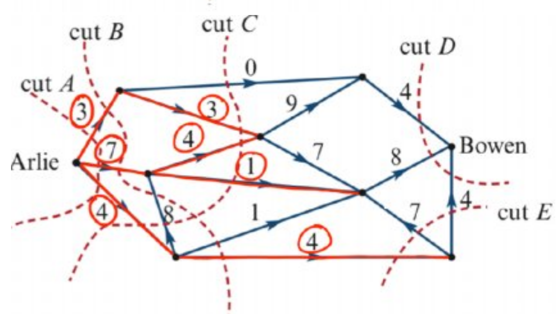
The capacity of cut D
$= 4 + 8 + 4$
16

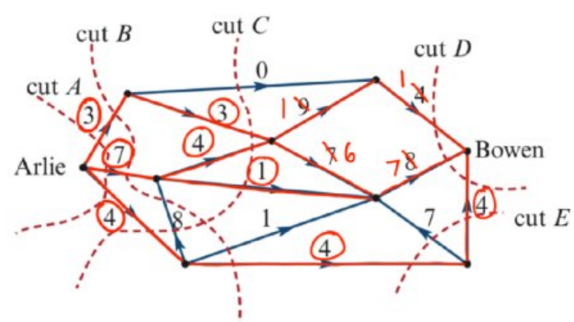
b Cut E is not valid because it does not isolate the flow from Arlie (source) to Bowen (sink).



c

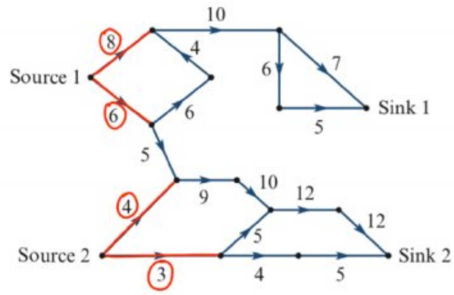


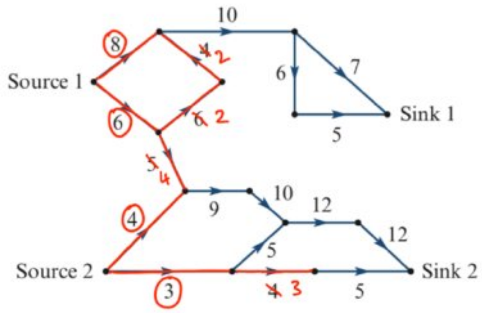


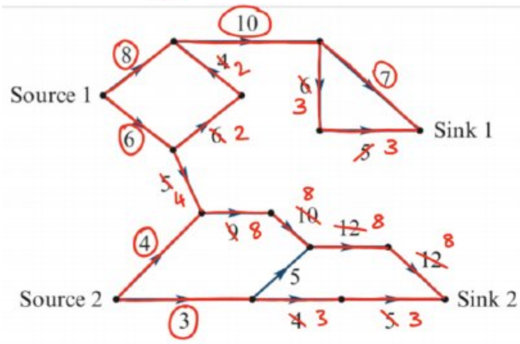


Maximum flow $= 7 + 4 + 1 = 12$.
The maximum number of seats from Arlie to Bowen is 12.

6 a

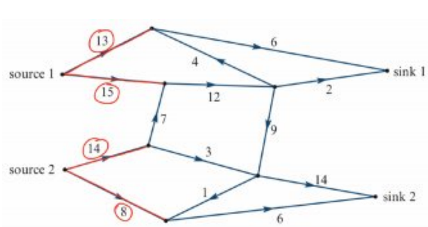


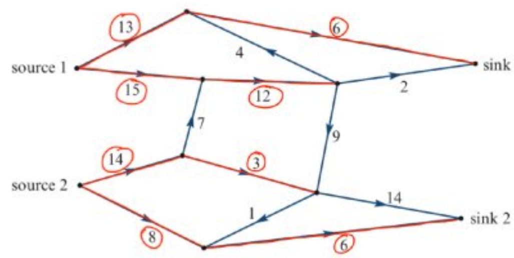


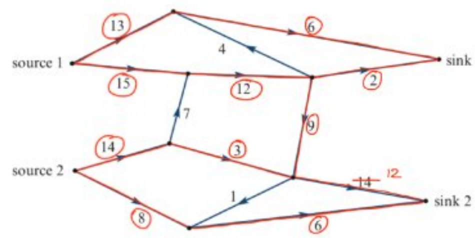


The maximum flow to sink 1
= 7 + 3
= 10
The maximum flow to sink 2
= 8 + 3
=11

b





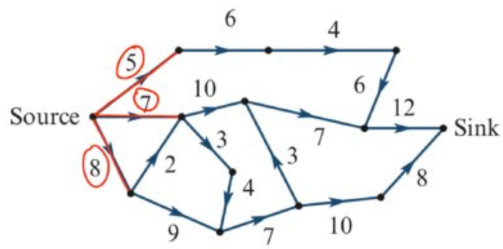


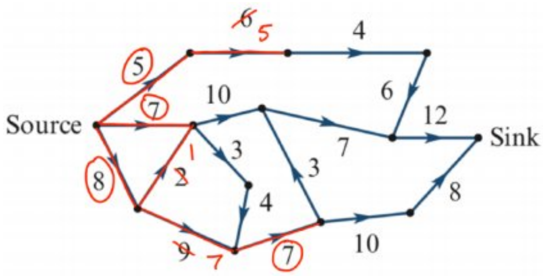
The maximum flow to sink 1
= 6 + 2
= 8
The maximum flow to sink 2
= 12 + 6
=18

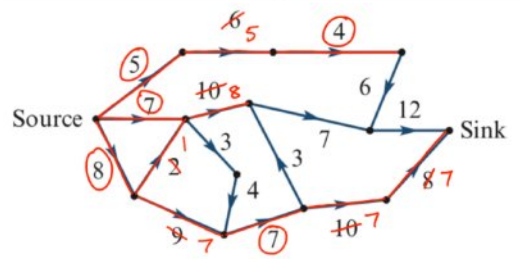
7 a There are 9 different paths from the *source* to the *sink*:

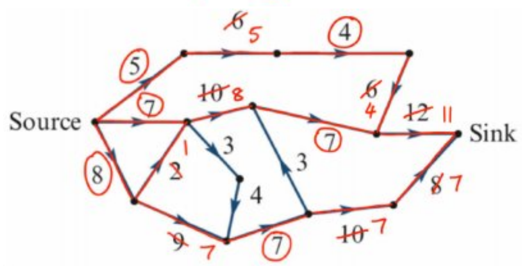
- Source - 5 - 6 - 4 - 6 - 12 - Sink
- Source - 7 - 10 - 7 - 12 - Sink
- Source - 7 - 3 - 4 - 7 - 3 - 7 - 12 - Sink
- Source - 7 - 3 - 4 - 7 - 10 - 8 - Sink
- Source - 8 - 2 - 10 - 7 - 12 - Sink
- Source - 8 - 2 - 3 - 4 - 7 - 3 - 7 - 12 - Sink
- Source - 8 - 2 - 3 - 4 - 7 - 10 - 8 - Sink
- Source - 8 - 9 - 7 - 3 - 7 - 12 - Sink
- Source - 8 - 9 - 7 - 10 - 8 - Sink

b



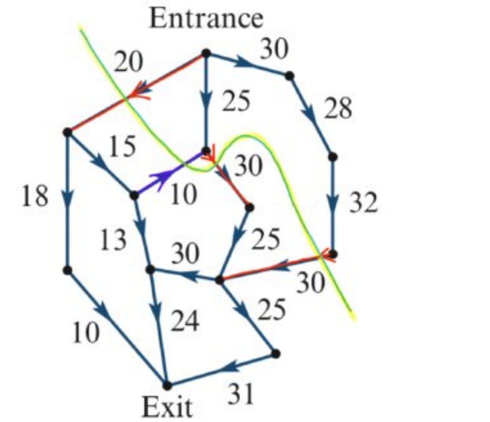






Maximum flow = 11 + 7 = 18.

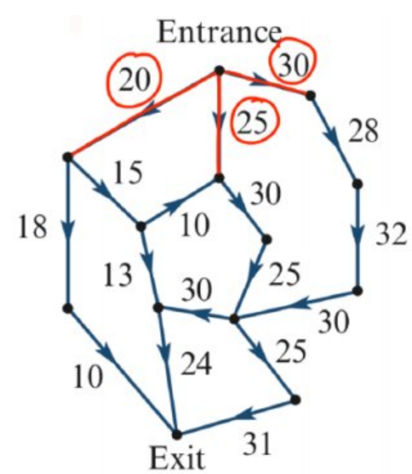
8 a

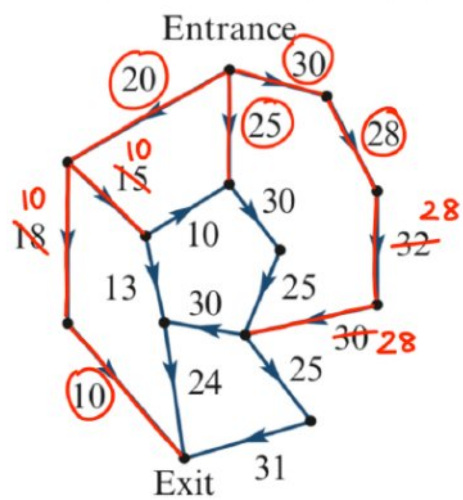


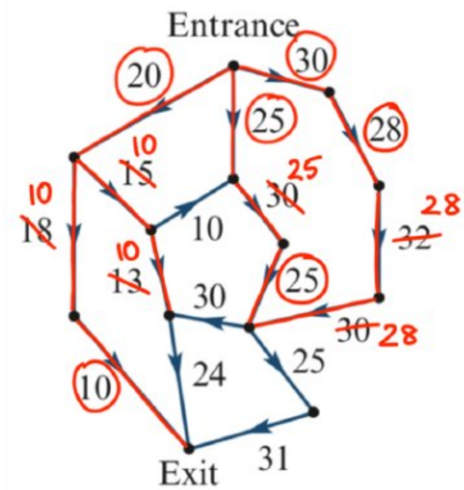
The capacity of the cut shown above
= 20 + 30 + 30 = 80

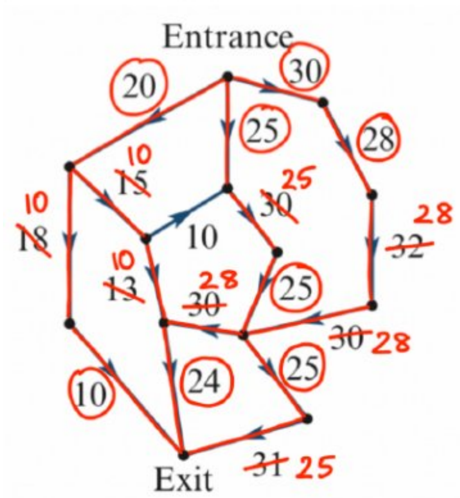
Note: the edge with capacity 10 is not counted as the flow along this edge is from the sink side to the source side of the cut.

b



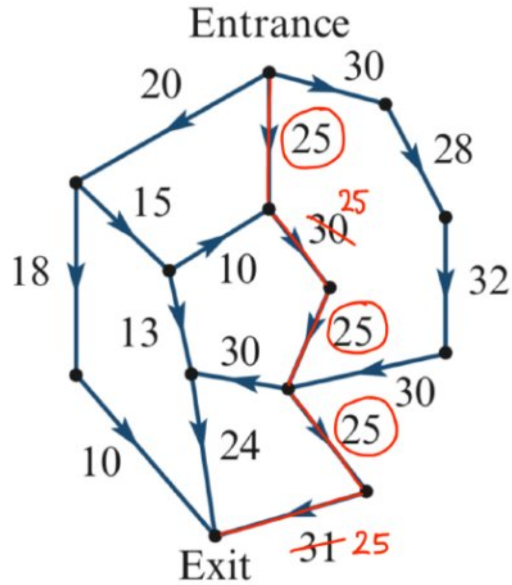


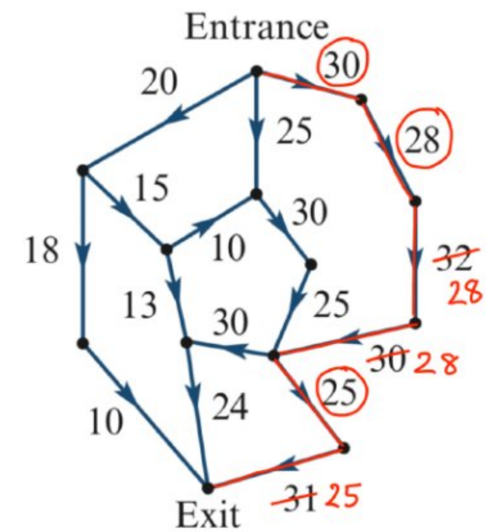




Maximum flow = 10 + 24 + 25 = 59.

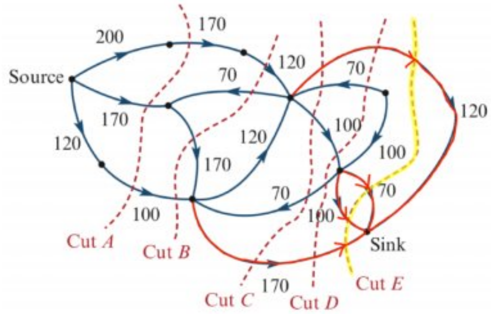
c Must consider the largest group of students that can follow one path from the *Entrance* down to the *Exit*. There are two potential paths that allow a maximum of 25 students to walk through the museum as one group every 30 minutes:





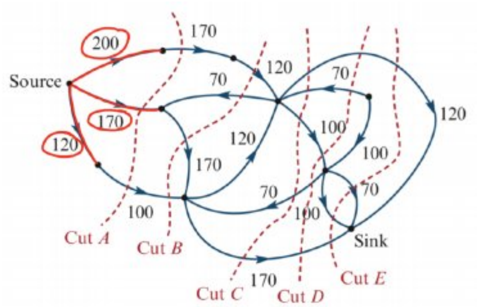
The largest group of students that can walk together through the museum is 25.

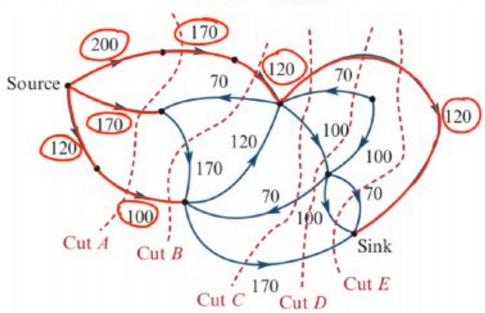
9

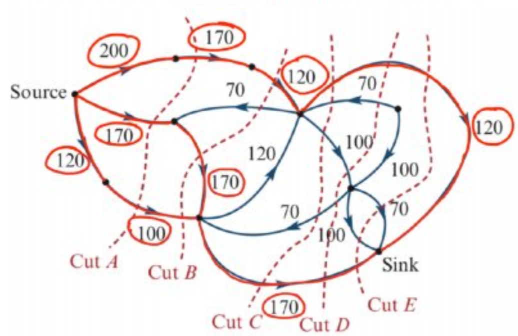


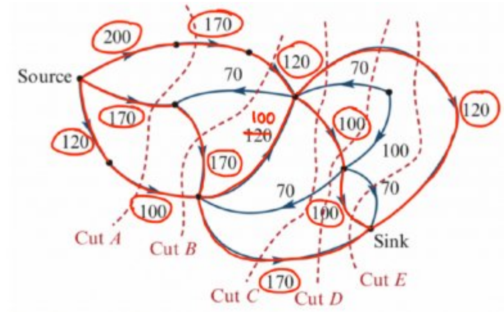
The capacity of *Cut E*
= 120 + 70 + 100 + 170 = 460
E

10









Maximum flow
= 120 + 100 + 170 = 390.

Capacity of *Cut A*
= 170 + 170 + 100 = 440
Capacity of *Cut B*
= 120 + 170 + 100 = 390
Capacity of *Cut C*
= 120 + 100 + 170 = 390
Capacity of *Cut D*
= 120 + 100 + 170 = 390
Capacity of *Cut E*
= 120 + 70 + 100 + 170 = 460 There-

fore 3 cuts (B, C, D) have a capacity equal to the maximum flow of this network.
D

11

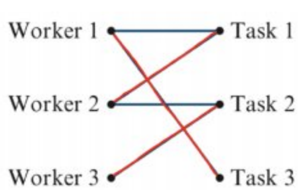
	if $x = 4$	if $x = 6$	if $x = 8$
Cut A = 44	44	44	**44**
Cut $B = 37 + x$	**41**	**43**	45
Cut $C = 41 + x$	45	47	49
Cut D = 46	46	46	46
Cut E = 51	51	51	51

If x = 4, the maximum flow is given by *Cut B* with the minimum capacity of 41.
If x = 6, the maximum flow is given by *Cut B* with the minimum capacity of 43.
If x = 8, the maximum flow is given by *Cut A* with the minimum capacity of 44.
B

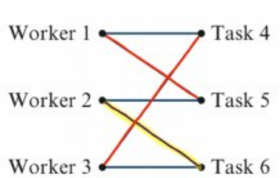
Solutions to Exercise 14B

1 **a** First allocate the Tasks that can only be completed by a certain Worker; consider vertices with only one edge connected to it. Then allocate Tasks to corresponding Workers with the remaining possible options.

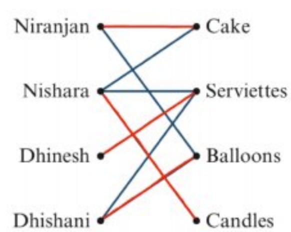
- Worker 1 must be allocated to Task 3, as there is only one edge connected to the vertex representing Task 3
- Worker 3 must be allocated Task 2, as there is only one edge connected to the vertex representing Worker 3
- As a result, Worker 2 must be allocated to Worker Task 1



b
- Worker 2 must be allocated Task 6
- Worker 3 cannot be allocated to Task 6, so they must be allocated to Task 4
- As a result, Worker 1 must be allocated to Task 5



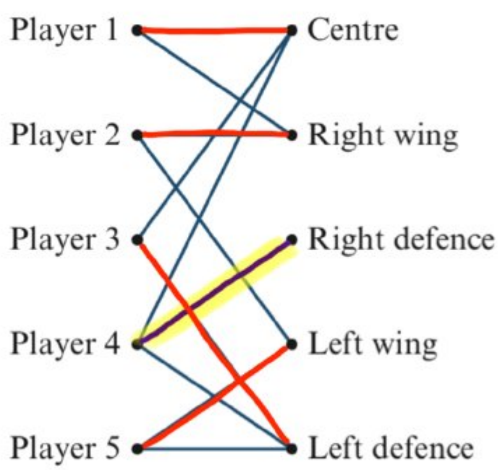
2
- Dhinesh must be allocated to the Serviettes, as there is one edge connected to the vertex representing Dhinesh
- Nishara must be allocated to the Candles, as there is one edge connected to the vertex representing Candles
- Dhishani cannot be allocated to the Serviettes, so she must be allocated Balloons
- As a result, Niranjan must be allocated to Cake



3 There is one position that must be allocated first; Player 4 must be allocated Right defence, as there is only one edge connected to the vertex representing Right defence. Player 4 could have been allocated to Centre; the choice of which player to allocate to Centre can determine the overall allocation of the team.

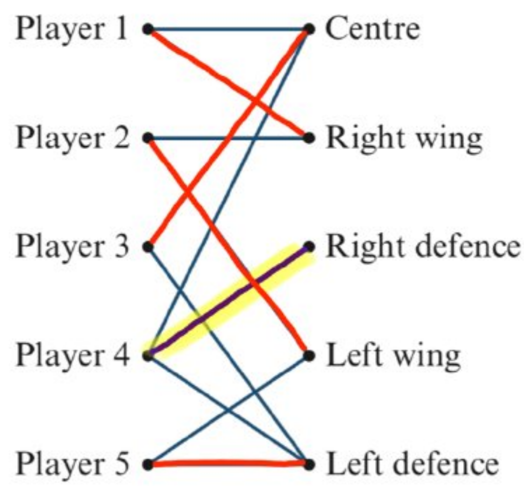
If Player 1 is allocated to Centre:

- Player 3 must be allocated Left defence, as Player 3 could not be allocated to Centre
- Player 5 must be allocated Left wing, as they cannot be allocated to their other possible option of Left defence
- Finally, Player 2 must be allocated Right wing, as there are no other positions available for allocation



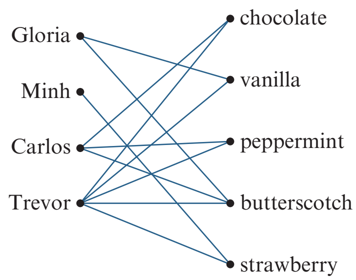
If Player 3 is allocated to Centre:

- Player 1 must be allocated Right wing, as Player 1 could not be allocated to Centre
- Player 2 must be allocated Left wing, as they cannot be allocated to their other possible option of Right wing
- Finally, Player 5 must be allocated Left defence, as there are no other positions available for allocation



4 a A bipartite graph can be used to display this information because there are two distinct groups of objects that are, in some way, connected to each other.
One group is the people and the other group is the flavours of ice-cream.
Each of the people are connected to one or more ice-cream flavours.

b List the people on the left and the flavours on the right with a vertex for each.
Gloria likes vanilla and butterscotch, so join Gloria's vertex to each of the vertices for these flavours.
Minh only likes strawberry, so join Minh's vertex to the vertex for strawberry.
Similarly, join Carlos' vertex to the vertex for chocolate, peppermint and butterscotch. Join Trevor's vertex to the vertex for every flavour.
The completed bipartite graph is below:



c Trevor is connected to all five ice-cream flavours. The degree of this vertex is 5.

5 a

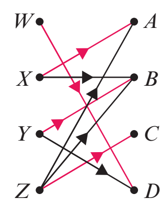
	A	*B*	*C*	*D*	
W	110	95	140	80	−80
X	105	82	145	80	−80
Y	125	78	140	75	−75
Z	115	90	135	85	−85

	A	*B*	*C*	*D*
W	30	15	60	0
X	25	2	65	0
Y	50	3	65	0
Z	30	5	50	0

− 25 − 2 −50

	A	*B*	*C*	*D*
W	5	13	10	0
X	0	0	15	0
Y	25	1	15	0
Z	5	3	0	0

	A	*B*	*C*	*D*
W	4	12	9	0
X	0	0	15	1
Y	24	0	14	0
Z	0	0	0	1



W must be allocated to *D*, so *Y* cannot.

Y must then be allocated to *B*, so *X* cannot.

X must be allocated to *A*

Z must be allocated to *C*

Allocation: *W* – *D*, *Y* – *B*, *X* – *A*, *Z* – *C*

b

	A	*B*	*C*	*D*	
W	2	4	3	5	−2
X	3	5	3	4	−3
Y	2	3	4	2	−2
Z	2	4	2	3	−2

	A	*B*	*C*	*D*
W	0	2	1	3
X	0	2	0	1
Y	0	1	2	0
Z	0	2	0	1

	A	*B*	*C*	*D*
W	0	1	1	3
X	0	1	0	1
Y	0	0	2	0
Z	0	1	0	1

	A	*B*	*C*	*D*
W	0	0	1	2
W	0	0	0	0
W	1	0	3	0
W	0	0	0	0

Multiple allocations are possible, but all will have the same minimum cost. One possible allocation is: *W* to *A* (2), *X* to *B* (5), *Y* to *D* (2) and *Z* to *C* (2)

Minimum cost = 2 + 5 + 2 + 2 = 11

6

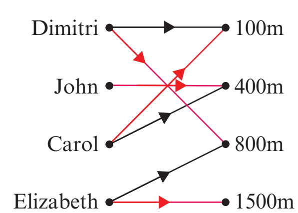
Student	***100 m***	***400 m***	***800 m***	***1500 m***	
Dimitri	11	62	144	379	−11
John	13	60	146	359	−13
Carol	12	61	149	369	−12
Elizabeth	13	63	142	349	−13

Student	***100 m***	***400 m***	***800 m***	***1500 m***
Dimitri	0	51	133	368
John	0	47	133	346
Carol	0	49	137	357
Elizabeth	0	50	129	336
		−47	−129	−336

Student	***100 m***	***400 m***	***800 m***	***1500 m***
Dimitri	0	4	4	32
John	0	0	4	10
Carol	0	2	8	21
Elizabeth	0	3	0	0

Student	***100 m***	***400 m***	***800 m***	***1500 m***
Dimitri	0	2	2	30
John	2	0	4	10
Carol	0	0	6	19
Elizabeth	2	3	0	0

Student	***100 m***	***400 m***	***800 m***	***1500 m***
Dimitri	0	2	0	28
John	2	0	2	8
Carol	0	0	4	17
Elizabeth	4	5	0	0



John must be allocated to 400 m, so Carol cannot.

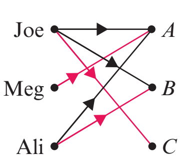
If Carol cannot be allocated to 400 m, she must be allocated to 100 m, so Dimitri cannot.
If Dimitri cannot be allocated to 100m, he must be allocated to 800 m, leaving 1500 m for Elizabeth
The “best” student allocation is:
Dimitri – 800 m, John – 400 m,
Carol – 100 m, Elizabeth – 1500 m

7

	Job			
Student	***A***	***B***	***C***	
Joe	20	20	36	−20
Meg	16	20	44	−16
Ali	26	26	44	−26

	Job		
Student	***A***	***B***	***C***
Joe	0	0	16
Meg	0	4	28
Ali	0	0	18
			−16

	Job		
Student	***A***	***B***	***C***
Joe	0	0	0
Meg	0	4	12
Ali	0	0	2



Meg must be allocated to job *A*, so Joe and Ali cannot.
If Ali cannot be allocated to job *A*, he must be allocated to job *B*, so Joe cannot.
If Joe cannot be allocated to job *A*, nor job *B*, he must be allocated to job *C*.

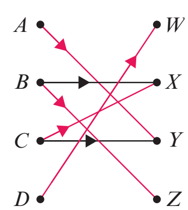
The allocation that minimises the time taken to complete the jobs is:
Meg – *A*, Ali – *B*, Joe – *C*

8

	Machine				
Operator	***W***	***X***	***Y***	***Z***	
A	38	35	26	54	−26
B	32	29	32	26	−26
C	44	26	23	35	−23
D	20	26	32	29	−20

	Machine			
Operator	***W***	***X***	***Y***	***Z***
A	12	9	0	28
B	6	3	6	0
C	21	3	0	12
D	0	6	12	9
		-3		

	Machine			
Operator	***W***	***X***	***Y***	***Z***
A	9	6	0	28
B	3	0	6	0
C	18	0	0	12
D	0	6	15	12



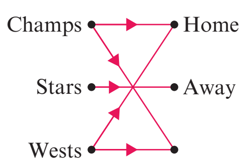
A must be allocated to *Y*, so *C* cannot.
If *C* cannot be allocated to *Y*, *C* must be allocated to *X* and so *B* cannot.
If *B* cannot be allocated to *X*, *B* must be allocated to *Z*.
D must be allocated to *W*.
The allocation of machinists to machines that minimises the total cost is:
A – *Y*, *B* – *Z*, *C* – *X*, *D* – *W*

9

Team	*Home*	*Away*	*Neutral*	
Champs	10	9	8	−8
Stars	7	4	5	−4
West	8	7	6	−6

Team	*Home*	*Away*	*Neutral*
Champs	2	1	0
Stars	3	0	2
West	2	1	0
	-2		

Team	*Home*	*Away*	*Neutral*
Champs	0	1	0
Stars	1	0	2
West	0	1	2



Stars must play at the away ground. Both Champs and Wests can play at either Home or Neutral, so there are two possible allocations:

Champs – Home (10), Stars – Away (4) and Wests – Neutral (6) for a total of $20 000.

or

Champs – Neutral (8), Stars – Away (4) and Wests – Home (8) for a total of $20 000.

10

Service	**Motorist**				
Vehicle	**Jess**	**Mark**	**Raj**	**Karla**	
A	18	15	15	16	−15
B	7	17	11	13	−7
C	25	19	18	21	−18
D	9	22	19	23	−9

Service	**Motorist**			
Vehicle	**Jess**	**Mark**	**Raj**	**Karla**
A	3	0	0	1
B	0	10	4	6
C	7	1	0	3
D	0	13	10	14

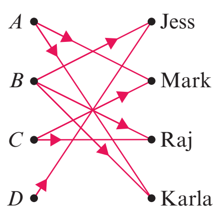
Service	**Motorist**			
Vehicle	**Jess**	**Mark**	**Raj**	**Karla**
A	3	0	0	1
B	0	10	4	6
C	7	1	0	3
D	0	13	10	14

Service	**Motorist**			
Vehicle	**Jess**	**Mark**	**Raj**	**Karla**
A	3	0	0	0
B	0	10	4	5
C	7	1	0	2
D	0	13	10	13

Service	**Motorist**			
Vehicle	**Jess**	**Mark**	**Raj**	**Karla**
A	7	0	0	0
B	0	6	0	1
C	11	1	0	2
D	0	9	6	9

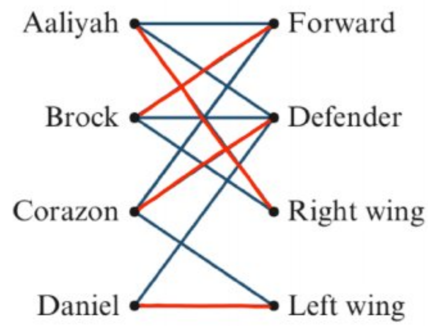
Service Vehicle	Motorist			
	Jess	**Mark**	**Raj**	**Karla**
A	7	0	0	0
B	0	6	0	1
C	11	1	0	2
D	0	9	6	9

Service Vehicle	Motorist			
	Jess	**Mark**	**Raj**	**Karla**
A	8	0	1	0
B	0	5	0	0
C	11	0	0	1
D	0	8	6	8

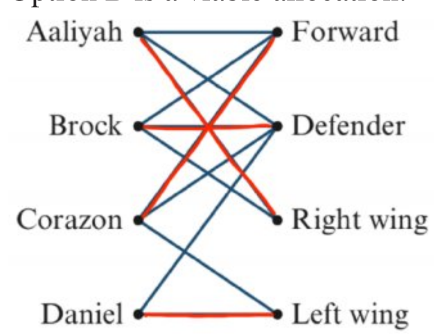


A must go to Mark or Karla. *B* can go to Jess, Raj or Karla. *C* must go to Mark or Raj. *D* must go to Jess.
The allocations of vehicle to motorist that minimise the total distance travelled are:
A – Karla (16), *B* – Raj (11), *C*– Mark (19) and *D* – Jess (9) OR
A – Mark (15), *B* – Karla (13), *C*– Raj (18) and *D* – Jess (9), both for a total of 55 km.

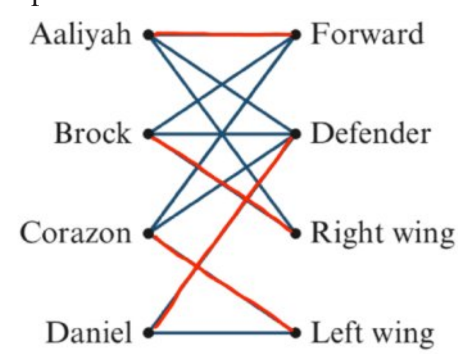
11 Option A is a viable allocation:



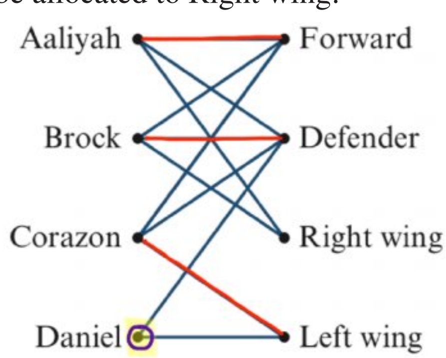
Option B is a viable allocation:



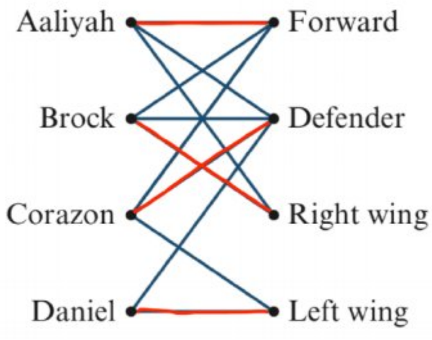
Option C is a viable allocation:



Option D is **not** a viable allocation, as Daniel cannot be allocated to Right wing:



Option E is a viable allocation:



D

12 Perform the Hungarian Algorithm to obtain the optimal allocation:

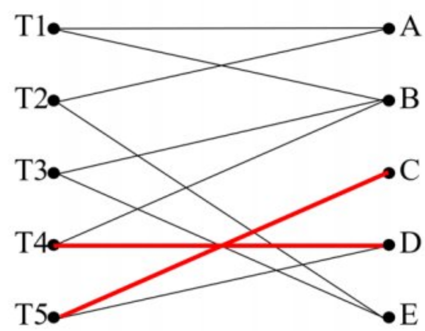
	A	B	C	D	E	
T1	1	2	2	5	4	−1
T2	4	9	7	11	6	−4
T3	5	3	3	9	4	−3
T4	8	5	6	6	7	−5
T5	5	8	4	6	9	−4

	A	B	C	D	E
T1	0	1	1	4	3
T2	0	5	3	7	2
T3	2	0	0	6	1
T4	3	0	1	1	2
T5	1	4	0	2	5
				−1	−1

	A	B	C	D	E
T1	0	1	1	3	2
T2	0	5	3	6	1
T3	2	0	0	5	0
T4	3	0	1	0	1
T5	1	4	0	1	4

	A	B	C	D	E
T1	0	0	1	2	1
T2	0	4	3	5	0
T3	3	0	1	5	0
T4	4	0	2	0	1
T5	1	3	0	0	3

	A	B	C	D	E
T1	0	0	1	2	1
T2	0	4	3	5	0
T3	3	0	1	5	0
T4	4	0	2	0	1
T5	1	3	0	0	3



Multiple allocations are possible (which will all result in the same overall minimum completion time for the tasks), however in every scenario Task 5 must be completed by Carmen and Task 4 must be completed by Dexter.
One allocation is: Task 1 = Anita = 1 hour, Task 2 = Electra = 6 hours, Task 3 = Brad = 3 hours, Task 4 = Dexter = 6 hours, Task 5 = Carmen = 4 hours. If this allocation is followed, then Anita will be the first person to finish. A second allocation is: Task 1 = Brad = 2 hours, Task 2 = Anita = 4 hours, Task 3 = Electra = 4 hours, Task 4 = Dexter = 6 hours, Task 5 = Carmen = 4 hours. If this allocation is followed, then Brad will be the first person to finish. In any other allocation, Anita or Brad will always finish their task before the ther people at the bank.
A

13 In the previous question, when the Hungarian algorithm was performed, the minimum total time for all tasks was found to be 20 hours.
Perform the Hungarian Algorithm with the new time for Task 5 to be completed by Electra, to obtain the optimal allocation:

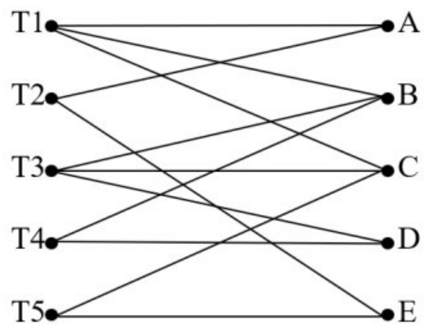
	A	B	C	D	E	
T1	1	2	2	5	4	−1
T2	4	9	7	11	6	−4
T3	5	3	3	9	4	−3
T4	8	5	6	6	7	−5
T5	5	8	4	6	4	−4

	A	B	C	D	E
T1	0	1	1	4	3
T2	0	5	3	7	2
T3	2	0	0	6	1
T4	3	0	1	1	2
T5	1	4	0	2	0
				−1	

	A	B	C	D	E
T1	0	1	1	3	2
T2	0	5	3	6	1
T3	2	0	0	5	0
T4	3	0	1	0	1
T5	1	4	0	1	0

	A	B	C	D	E
T1	0	0	0	2	1
T2	0	4	2	5	0
T3	3	0	0	5	0
T4	4	0	1	0	1
T5	2	4	0	1	0

	A	B	C	D	E
T1	0	0	0	2	1
T2	0	4	2	5	0
T3	3	0	0	5	0
T4	4	0	1	0	1
T5	2	4	0	1	0



Multiple allocations are possible, however every allocation will result in the same minimum total time for all tasks to be completed. One possible allocation is: Task 1 = Carmen = 2 hours, Task 2 = Anita = 4 hours, Task 3 = Brad = 3 hours, Task 4 = Dexter = 6 hours, Task 5 = Electra = 4 hours. This allocation takes a total of 19 hours, which is 1 hour less than the original allocation where Electra took 9 hours to complete Task 5.
E

14 If $p = 10$:

	Job 1	Job 2	Job 3	Job 4
Xena	5	3	7	10
Wilson	1	2	5	6
Yasmine	1	7	1	5
Zachary	4	7	6	10

The allocation by the manager will not produce the minimum total completion time. When $p = 10$, following the manager's allocation, Wilson = Job 1 = 1 hour and Zachary = Job 4 = 10 hours, giving a total of 11 hours. If this allocation was swapped whereby Wilson = Job 4 = 6 hours and Zachary = Job 1 = 4 hours, this would give a total of 10 hours, therefore when $p = 10$ the manager's inital allocation of jobs would not result in the minimum total completion time.

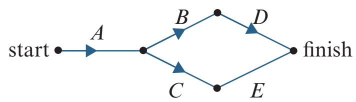
If $p = 9$:

	Job 1	Job 2	Job 3	Job 4
Xena	5	3	7	9
Wilson	1	2	5	6
Yasmine	1	7	1	5
Zachary	4	7	6	9

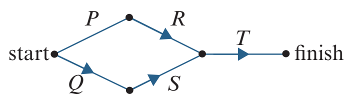
The allocation by the manager will produce the minimum total completion time. When $p = 9$, following the manager's allocation, Zachary's completion time of 9 hours for Job 4 cannot be swapped with any of the previously allocated jobs to result in a shorter completion time. For example, when $p = 9$, Zachary and Wilson take a total of 10 hours to complete their allocated jobs and if they swapped jobs, they would still take a total of 10 hours to complete each job (Wilson = Job 4 = 6 hours, Zachary = Job 1 = 4 hours). This will be true for all values of p less than or equal to 9, therefore the manager's initial allocation will only achieve the minimum completion time if the value of p is not greater than 9 hours.
D

Solutions to Exercise 14C

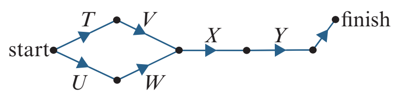
1 a



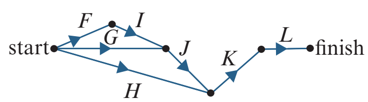
b



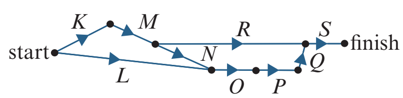
c



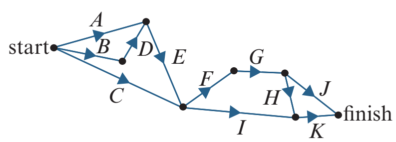
d



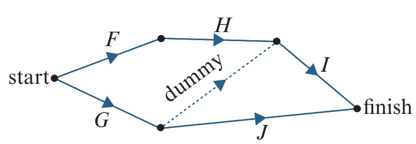
e



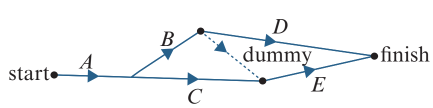
f



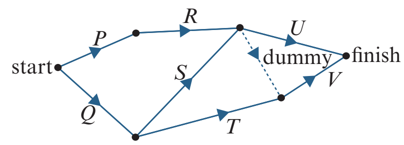
2 a



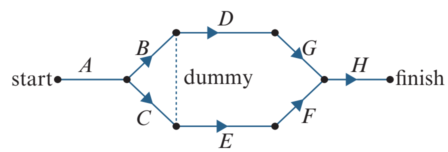
b



c



d



3 a

Activity	*Immediate predecessors*
A	-
B	-
C	*A*
D	*A*
E	*B,C*
f	*D*
G	*E*

b

Activity	*Immediate predecessors*
P	-
Q	*P*
R	*p*
S	*Q*
T	*Q*
U	*S,V*
V	*R*
W	*R*
X	*T,U*

c

Activity	*Immediate predecessors*
J	-
K	-
L	*J*
M	*N*
N	*K*
O	*K*
P	*N*
Q	*L,M*
R	*P*
S	*Q,R*
T	*Q*

d

Activity	*Immediate predecessors*
A	-
B	-
C	*A*
D	*A*
E	*D,B*
f	*C,E*
G	*D,B*
H	*B*

e

Activity	*Immediate predecessors*
P	-
Q	*P*
R	*p*
S	*Q*
T	*Q*
U	*R*
V	*S*
W	*S,T*
X	*U*
Y	*W*
Z	*V,X,Y*

f

Activity	*Immediate predecessors*
A	-
B	*A*
C	*A*
D	*A*
E	*B*
f	*C,D*
G	*D*
H	*E,F,G*
I	*G*
J	*I*
K	*H*

4 a "Remove broken component" is activity *C*.
Look at activity *C* in the activity network.
Activity *C* follows immediately from activity *A*.
Activity *A* is an immediate predecessor of activity *C*.
"Remove panel" is an immediate predecessor of "Remove broken component".

b "Install new component" is activity *F*.
Look at activity *F* in the activity network.
Activity *F* follows immediately from activity *B* and the dummy that follows activity *D*.
Activities *B* and *D* are immediate predecessors of activity *F*.
"Order component" and "Pound out dent" are immediate predecessors of "Install new component".

5 **a**

Activity	*Immediate predecessors*
A	–
B	–
C	–
D	*A*
E	*B, F*
F	*C*
G	*B, F*
H	*D, E*
I	*H*
J	*I, K*
K	*G*
L	*G*
M	*H*
N	*J, L*
O	*N*

b $A - D - H - M$

$A - D - H - I - J - N - O$

c $B - E - H - M$

$B - E - H - I - J - N - O$

$B - G - K - J - N - O$

$B - G - L - N - O$

d $C - F - E - H - M$

$C - F - E - H - I - J - N - O$

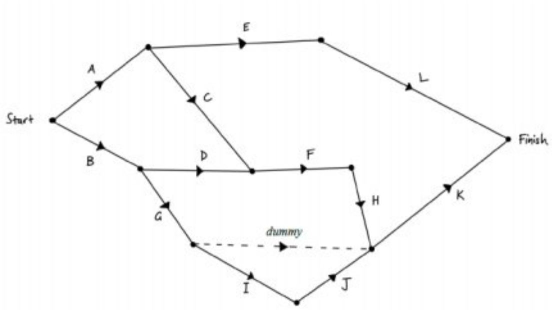
$C - F - G - K - J - N - O$

$C - F - G - L - N - O$

6 **a** D, F, H

b A, B, C, D, E, F, G, H

7 **a** Draw an acitvity network for this project:



The dummy activity must be drawn from the end of activity G because activities I and K share activity G as an immediate predecessor.

b The dummy activity must be drawn to the start of activity K. As seen in the diagram above, activity K shares some but not all immediate predecessors with activity I.

c It is necessary to include a dummy activity because there are activities that share some, but not all, of their immediate predecessors. Activity I has activity G as an immediate predecessor, whereas activity K has activities H, J and G as immediate predecessors.

8 There are four paths from *start* to finish that start with activity B are:

$B - D - E - G - M$

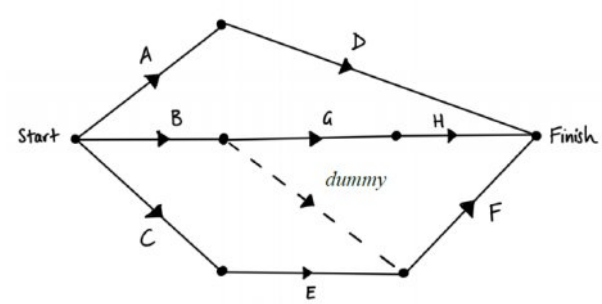
$B - D - E - H - L$

$B - D - F - I - J - L$

$B - D - F - I - K$

D

9 Draw an acitvity network for this project:



The dummy activity will drawn from the end of activity B to the start of activity F

A

10 Using the activity network drawn for Question 9 above, there are 4 paths from *start* to *finish*:

$A - D$

$B - G - H$

$B - F$

$C - E - F$.

A

Solutions to Exercise 14D

1 **a** Use forward scanning,
$p = 8 + 4$
$p = 12$

b Use forward scanning,
$w = 4 + 6$
$w = 10$

c Using forward scanning,
$m + 4 = 12$
$m = 12 - 4$
$m = 8$
Using backward scanning,
$n = 12 - 4$
$n = 8$

d Using forward scanning,
$c = 6 + 5$
$c = 11$
Using backward scanning,
$a = 15 - 5$
$a = 10$
Using backward scanning,
$b - 3 = 15$
$b = 15 + 3$
$b = 18$

e Using forward scanning,
$f = 3 + 6$
$f = 9$
Using forward scanning, g is the largest of:
$g = 5 + 7$ or $g = f + 0$
$g = 12$ or $g = 9$
So, $g = 12$

f Using forward scanning,
$4 + q = 12$
$q = 12 - 4$
$q = 8$
Using backward scanning,
$9 - p = 4$
$p = 9 - 4$
$p = 5$
Using backward scanning,
$12 - r = 9$
$12 - 9 = r$
$r = 3$
Using forward scanning,
$n + r = 12$
$n + 3 = 12$
$n = 12 - 3$
$n = 9$

2 **a** Using forward scanning,
6 + duration of $A = 9$
duration of $A = 9 - 6$
duration of $A = 3$

b The critical path follows activities that have no float time. The two numbers in the boxes at the start of these activities will be the same.
The critical path is: $A - C$

c Float time = LST – EST
= 11 – 6
= 5

d LST for D is the second number in the boxes at the start of activity D.
LST for activity $D = 13$

e Using backward scanning,
15 – duration of $D = 13$
Duration of $D = 15 - 13$
Duration of $D = 2$

3 a Using forward scanning,
0 + duration of B = 12
Duration of B = 12

b LST for E is the second number in the boxes at the start of activity E.
LST for E = 10

c EST for E is the first number in the boxes at the start of activity E.
EST for E = 9

d Float time for E = LST – EST
= 10 – 9
= 1

e Using forward scanning,
0 + duration A = 3
duration A = 3

f Using forward scanning,
0 + duration D = 9
duration D = 9

4 a The critical path follows activities to boxes where the EST and LST are the same.
The critical path is: $D – E – F$

b Non-critical activities are A, B, C
Float A = LST (B) – duration of A
= 4 – 3
= 1
Float B = LST (B) – EST (B)
= 4 – 3
= 1
Float C = LST (F) – duration of C
= 22 – 7
= 15

5 a The critical path follows activities to boxes that have EST and LST the same.
The critical path is: $B – E – F – H – J$

b Non-critical activities are A, C, D, G, I
Float A = LST(D) – duration of A
= 11 – 10
= 1
Float C = LST(J) – duration of C
= 17 – 3
= 14
Float D = LST (D) – EST (D)
= 11 – 10
= 1
Float G = LST(I) – EST(G) – duration G
= 15 – 13 – 1
= 1
Float I = LST(J) – EST(I) – duration I
= 17 – 14 – 2
= 1

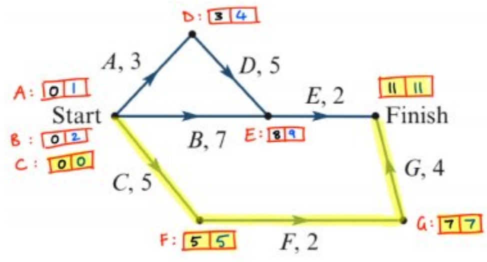
6 a Use information in the activity network.

Activity	*Duration(weeks)*	*Immediate predecessors*
A	3	-
B	6	-
C	6	*A*,*B*
D	5	*B*
E	7	*C*,*D*
f	1	*D*
G	3	*E*
H	3	*F*
I	2	*B*

b The critical path follows activities to boxes where the EST and LST are the same.

The critical path is: $B - C - E - G$

7



The answers for each of the following questions can be determined from inspection of the diagram above, where critical path analysis using forward and backward scanning have been used.

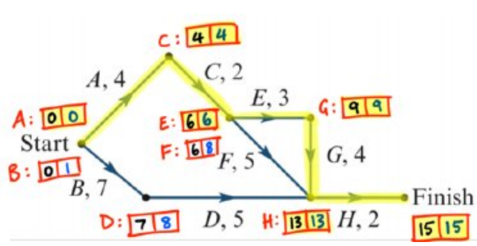
a EST of E = 8 days

b Minimum completion time = 11 days

c The critical path is the path where all activities have a float time of zero; EST = LST. The critical path is *C-F-G*

d Float time = LST − EST. Only Activity B has a float time of 2 days (2 − 0 = 2)

8



The answers for each of the following questions can be determined from inspection of the diagram above, where critical path analysis using forward and backward scanning have been used.

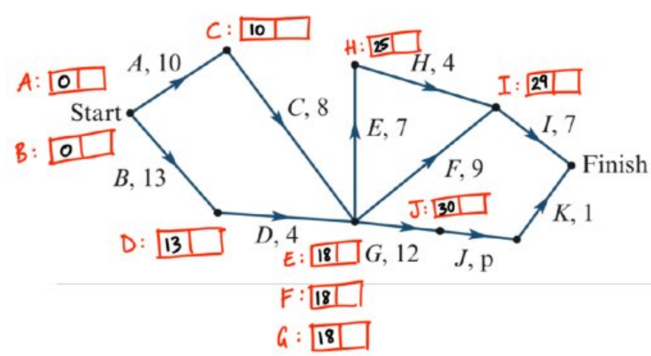
a The immediate predecessors of Activity H refers to the three edges directed towards the vertex where the edge representing Activity H begins; the three acticities are D, F, G.

b EST of H = 13

c Minimum completion time = 15

d The duration of time an activity can be delayed by, without affecting the minimum completion time of the project, is also referred to as the *Float time*. Float time for an activity = LST − EST; Activity F has the largest float time (8 − 6 = 2) which means is can be delayed longer than any other activity without affecting the minimum completion time of the project.

9



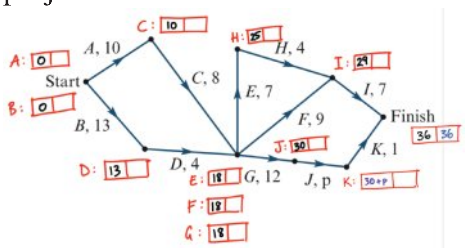
The answers for each of the following questions can be determined from inspection of the diagram above, where critical path analysis using forward scanning have been used. In this project, the duration of activity J is initially unknown, therefore the minimum completion time and backward scanning will only be considered for parts *b* and *c*.

a **i** EST for H = 25

ii EST for I = 29

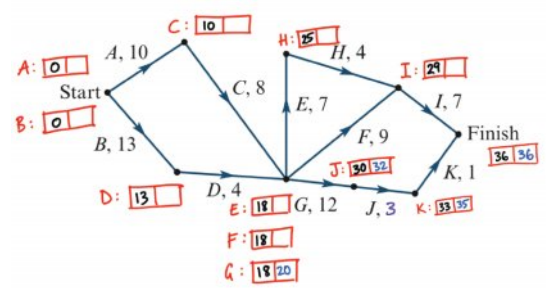
iii EST for J = 30

b Complete forward scanning for the diagram above, including p as an unknown value, knowing the minimum completion time for the project is 36:



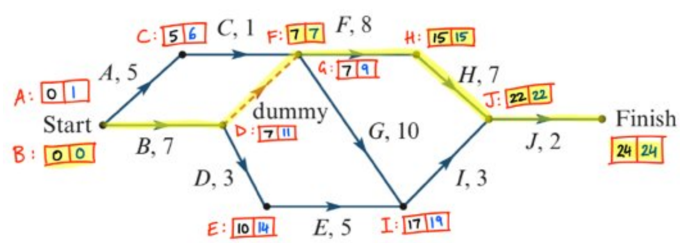
At the *Finish* vertex, there are two directed edges connected; activities *I* and *K*. Activity *I* has an EST of 29 and a duration of 7; it can be determined that the LST for activity *I* will be 29, resulting in a float time of zero and thus will be included in the critical path for this project. Activity *K* has a, EST of 30 + p and a duration of 1. The LST of activity *K* is 35, therefore to create more than one critical path, the value of p (the duration of activity *J*) must be 5. If it was greater than 5, the overall minimum completion time for the project would be greater than 36 and the path using activity *I* as the final activity would no longer be a critical path. $p = 5$.

c Complete forward and backward scanning for the diagram above, including $p = 3$ for the duration of activity *J*:



Float time of activity *G*
= LST − EST
= 20 − 18
= 2

10



The answers for each of the following questions can be determined from inspection of the diagram above, where critical path analysis using forward and backward scanning have been used.

a The immediate predecessors of activity *G* are the two edges directed towards the vertex where the edge representing activity *G* begins; be careful of the dummy activity, as this is not an activity. It is required if two activities share some, but not all, of their immediate predecessors. The dummy activity allows activity *B* to be an immediate predecessor for activity *G*. The immediate predecessors of activity *G* are *C* and *B*. *Note: we* *do not indicate the dummy activity is an immediate predecessor.*

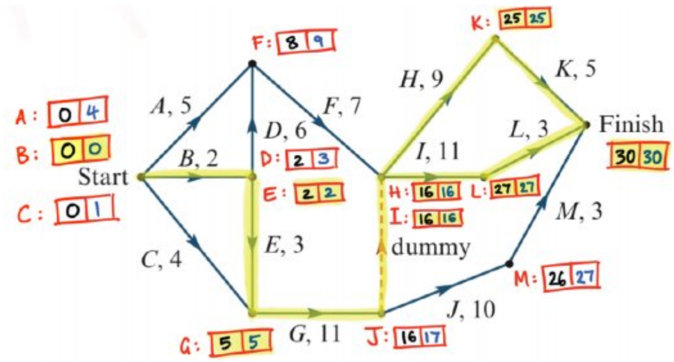
b *Note: this question is not exclusively asking about immediate predecessors.* To determine all activites that must be completed before activity I can begin, start by considering the immediate predecessors, then backtrack and identify the immediate predecessors for those activities. Activity I has G and E as immediate predecessors. Activity G has C and B as immediate predecessors. Activity C has A as an immediate predecessor. Activity E has D as an immediate predecessor. Therefore the activities that must be completed before activity I can begin are: activities A, B, C, D, E, G.

c The critical path is $B - F - H - J$

d Float time of activity E
$= \text{LST} - \text{EST}$
$= 14 - 10$
$= 4$

e Activities that have a float time of 2 or more can have their duration increase by 2 and not affect the minimum completion time of the project. There are four activities with a float time of 2 or more: D, E, G, I

11



The answers for each of the following questions can be determined from inspection of the diagram above, where critical path analysis using forward and backward scanning have been used.

a

Activity	*Immediate predecessors*
A	–
B	–
C	–
D	*B*
E	*B*
F	*A, D*
G	*C, E*
H	*F, G*
I	*F, G*
J	*G*
K	*H*
L	*I*
M	*J*

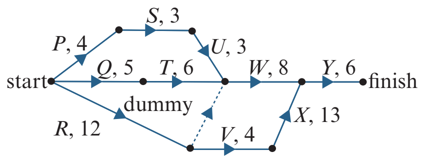
b 3 activities have an EST of 16 (activities H, I, J)

c LST of activity $F = 9$

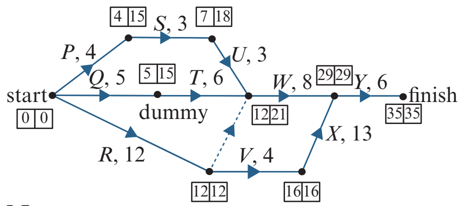
d The critical path includes all activities from $Start$ to $Finish$ that have a float time of zero. The two paths are: $B - E - G - H - K$ and $B - G - I - L$. Note, the dummy is never listed as part of the critical path.

e Activities that have a float time of 1 or more can be delayed by 1 and not affect the minimum completion time of the project. There are six activities with a float time of 1 or more: A, C, D, F, J, M

12 a



b



Note:

LST (P) = LST (S) – duration P

= 15 – 4

= 11

LST (Q) = LST (T) – duration Q

= 15 – 5

= 10

A table of EST and LST for each activity is shown below:

Activity	***EST***	***LST***
P	0	11
Q	0	10
R	0	0
S	4	15
T	5	15
U	7	18
V	12	12
W	12	21
X	16	16
Y	29	29

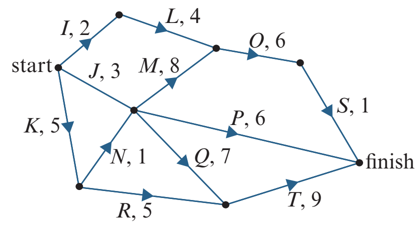
c The critical path follows activities to boxes where the EST and LST are the same.

The critical path is: $R – V – X – Y$

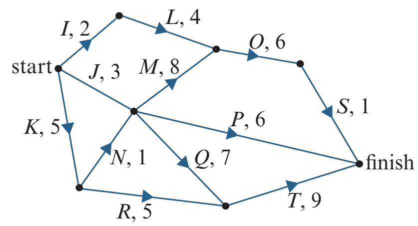
d The minimum time to complete the project is the value in the boxes at the finish.

Minimum project completion time is 35 weeks.

13 a



b



Note:

LST (I) = LST (L) – duration I

= 11 – 2

= 9

LST (J) = LST (M, P or Q) – duration J

= 6 – 3

= 3

LST (M) = LST (O) – duration M

= 15 – 8

= 7

LST (P) = LST (*finish*) – duration P

= 22 – 6

= 16

LST (Q) = LST (T) – duration Q

= 13 – 7

= 6

LST (R) = LST (T) – duration R

= 13 – 5

= 8

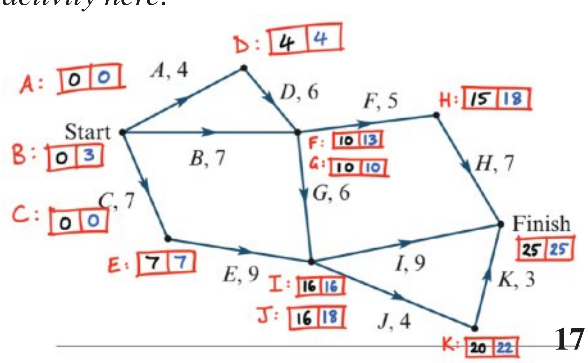
A table of EST and LST for each activity is shown below:

Activity	*EST*	*LST*
I	0	9
J	0	3
K	0	0
L	2	11
M	6	7
N	5	5
O	14	15
P	6	16
Q	6	6
R	5	8
S	20	21
T	13	13

c The critical path follows activities to boxes where the EST and LST are the same.
The critical path is: $K - N - Q - T$

d The minimum time to complete the project is the value in the boxes at the finish.
Minimum project completion time is 22 weeks.

14 *A crticial path anaylysis, using forward and backward scanning, was used on the activity network given for Questions 14, 15 and 16. Answers will be in reference to the EST and LST found for each activity here:*



EST for activity J = 16 days

E

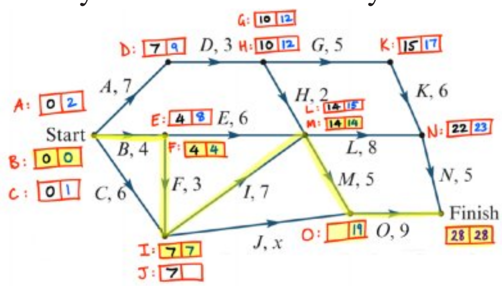
15
- Activities A, B, C each have zero immediate predecessors
- Activities D and E have one immediate predecessor each
- Activities F and G share two immediate predecessors (B, D)
- Activity H has one immediate predecessor
- Activities I and J share two immediate predecessors (G, E)
- Activity K has one immediate predecessor

Therefore 4 activities (F, G, I, J) are the only activities with two immediate predecessors each.
D

16 An activity can be delayed and not affect the minimum completion time of the project if it has a float time greater than zero (in other words, not along a critical path). There are two critical paths: $A - D - G - I$ and $C - E - I$.
Five activities have a float time greater than zero and are not included in either of the critical paths: B, F, H, J, K.
Five activites can be delayed without affecting the minimum completion time of the project.
C

17 Perform a critical analysis, using forward scanning and backward scanning.

Although the duration of J is unknown, the minimum completion time for the project is given (28 weeks), therefore all but two values of the crticial analysis can intially be determined: EST of activity O and LST of activity J:

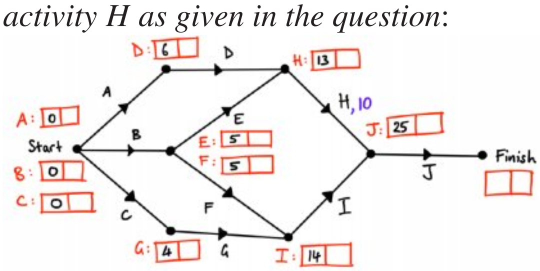


Activity J has an EST of 7 weeks and

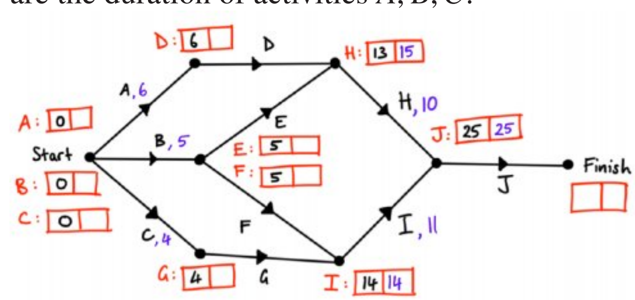
activity O has an LST of 19 week. If the duration of activity J was 12 weeks or less, the minimum completion time for the project would remain 28 weeks as the EST of activity O would remain 19 because of the duration of activity M and all activities leading to activity M are fixed. If the duration of activity J is 12 or less, the original critical path will still be the critical path for the project, as it will have the longest duration of all possible paths from $Start$ to $Finish$.

A

18 Using the precedence table, draw an activity network for the project and include the EST for each activity *Note: do not forget to include the duration of activity H as given in the question*:



Knowing activity H has a duration of 10 days, the LST of activity H can be determined (15 days) and given that activity H has a float time of 2 days, activity I must be along the critical path and have a float time of 0 days. Activities D, E, F and G are each connected to an acitivty with zero immediate predecessors (A, B, C), therefore the EST of D, E, F, G are the duration of activities A, B, C.



If Activity H has an EST of 13 days, activity D, with its EST of 6 days, can have a maximum duration of 7 days and a minimum of 1 day (a duration of zero days should never be considered). If the duration of activity D was 7 days, then its LST would be 8 days ($15 - 7 = 8$) and it could be delayed by 2 days (Delay or Float time = LST − EST = $8 - 6 = 2$). If the duration of activity D was 1 day, then its LST would be 14 days ($15 - 1 = 14$) and it could be delayed by 8 days (Delay or Float time = LST − EST = $14 - 6 = 8$). Therefore the maximum length of the delay for activity D is 8 days.

D

Solutions to Exercise 14E

1 a

Path	**Duration (hours)**
$A-D$	17
$B-E-F$	20
$B-E-G-I$	21 ©
$C-H-I$	16

b The critical path and minimum completion time refer to the path with the longest duration of time. From the table above, the critical path is $B-E-G-I$ with a duration of 21 hours.

c

Path	**Duration (hours)**	**Max reduction: E by 3**
$A-D$	17	17
$B-E-F$	20	17
$B-E-G-I$	21 ©	18 ©
$C-H-I$	16	16

The new minimum completion time for this project is 18 hours.

2 a

Path	**Duration (days)**
$A-B-D-G$	19
$A-B-E$	20
$A-B-F-H$	21 ©
$A-C-E$	19
$A-C-F-H$	20

The critical path is the path with the longest duration of time. From the table above, the critical path is $A-B-F-H$

b The minimum completion time for the project is the duration of the crtical path. 21 days.

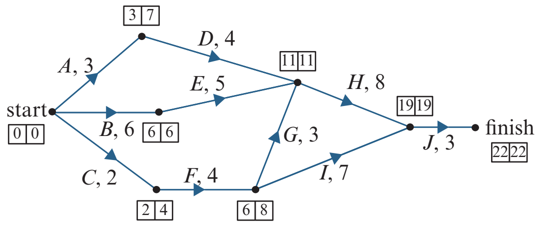
c

Path	**Duration (days)**	**Max reduction: B by 3**
$A-B-D-G$	19	16
$A-B-E$	20	17
$A-B-F-H$	21 ©	18
$A-C-E$	19	19
$A-C-F-H$	20	20 ©

The new minimum completion time for the project is 20 days.

d The new minimum completion time for the project is 20 days. Four of the five possible paths from $Start$ to $Finish$ originally had a duration of 20 days or less. There is no value in reducing the completion time for those four paths as this will incur an unnecessary cost. Paths that originally have a duration of 20 days or more must have activity B crashed by the minimum amount to achieve the mminimum completion time. The path $A-B-F-H$ originally had a duration of 21 days, therefore activity B would only need to be crashed by 1 day to achieve the greatest reduction in time taken to complete the project. As previously stated, reducing the completion time to less than 20 would incur an unnnecessary cost, as the minimum completion time of anoth path where crashing cannot take place prevents the overall completion time for the project to be achieved in a shorter amount of time. As the crashing will cost \$100 per day, the minimum cost to achieve the greatest reduction in time is \$100.

3 a



The critical path follows activities to boxes where the EST and LST are the same.

The critical path is: $B - E - H - J$

b Total minimum completion time for A and D is 3 + 4 = 7 hours.

Total minimum completion time for activities C, F and G is 2 + 4 + 3 = 9 hours

Reduction in completion time for activity E must not cause the total completion time for activities B and E to drop below 9 hours.

Completion time for activity E cannot be lower than 3.

The maximum reduction in completion time for activity E is 2 hours.

c Total minimum completion time for activities C, F and I is 2 + 4 + 7 = 13 hours.

Activity H can be reduced in duration to ensure duration of activity H + 11 is not lower than 13

Minimum duration of activity H is 2.

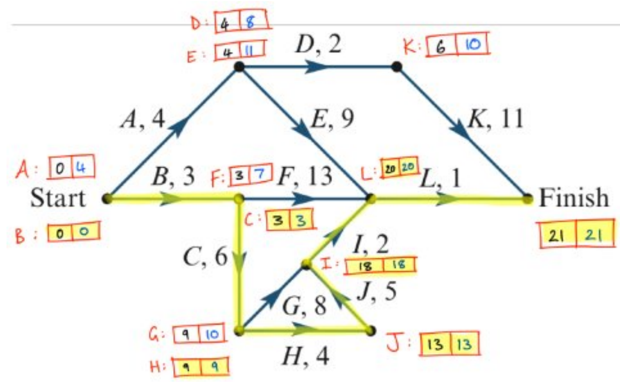
Maximum reduction in completion time for activity H is 6 hours.

d

Path	Duration (hours)	Max reduction: every activity by 2
$A - D - H - J$	18	10
$B - E - H - J$	22 ©	14 ©
$C - F - G - H - J$	20	10
$C - F - I - J$	16	8

New minimum completion time for the project is 14 hours.

4 a Perform a critical analysis of the activity network using forward and backward scanning:



b

Path	Duration (hours)
$A - D - K$	17
$A - E - L$	14
$B - F - L$	17
$B - C - G - I - L$	20
$B - C - H - J - I - L$	21 ©

Initially the minimum completion time for the project is 21 hours. Consider every activity and crash by its maximum number of hours (reduce to 1 hour) and examine the impact on the overall minimum completion time of the project:

- Activities A, D, E, F, G, K should not be considered as they do not lie along the critical path. If any of these activites were crashed, the minimum completion time would remain as 21 hours

- Activity *L* cannot be considered as it has a duration of 1 hour and cannot be reduced to a time less than 1 hour.

- Activity B can only be crashed by a maximum of 2 hours. This will result in a minimum completion time of 19 hours

- Activity C can only be crashed by a maximum of 5 hours. This will result in a minimum completion time of 17 hours, as an initial non-critical path will now become the critical path (both $A - D - K$ and $B - F - L$)

- Activity H can only be crashed by a maximum of 3 hours. This will result in a minimum completion time of 20 hours as there will be a new critical path $B - C - G - I - L$

- Activity I can only be crashed by a maximum of 1 hour. This will result in a minimum completion time of 20 hours

- Activity J can only be crashed by a maximum of 4 hours. This will result in a minimum completion time of 20 hours as there will be a new critical path $B - C - G - I - L$

The minimum time the project can be completed in, if only one activity can be crashed, is 17 hours. This is achieved by crashing activity C by a maximum of 5 hours.

c Activity G can be crashed by a maximum of 7 hours and activity J can be crashed by a maximum of 4 hours:

Path	**Duration (hours)**	**Max reduction: G by 7 and J by 4**
$A - D - K$	17	17 ©
$A - E - L$	14	14
$B - F - L$	17	17 ©
$B - C - G - I - L$	20	13
$B - C - H - J - I - L$	21 ©	17 ©

The new minimum completion time, after crashing, is 17 hours. There is at least one path that cannot be reduced to a shorter amount of time.

- The original critical path $B - C - H - J - I - L$ can be reduced from 21 hours to 17 hours if activity J is crashed by 4 hours.

- The path $B - C - G - I - L$ can be reduced from 20 hours to 17 hours if activity G is crashed by 3 hours. *Note: although this path can be reduced to 13 hours, this would incur unnecessary cost as the new overall minimum completion time for the project is 17 hours. At least one path cannot be reduced further than 17 hours, therefore we should not consider any reductions that results in a completion time less than 17 hours.*

- The three remaining paths do not have activities G or J along them, so they cannot be reduced by crashing. They also originally had completion times of 17 hours or less, therefore there is no needs to investigate further.

The minimum cost of reducing the completion time of this project as much as possible is

$= G \times 3 + J \times 4$
$= 200 \times 3 + 150 \times 4$
$= \$1200$

5 a

Path	Duration (days)	Max reduction: D, E, H by 2
$A-D-F-G$	23	21
$A-D-H$	22	20
$B-F-G$	22	22 ©
$B-H$	21	19
$C-E-F-G$	24 ©	22 ©
$C-E-H$	23	19

The new minimum completion time for this project is 22 days.

b
- The original critical path $C-E-F-G$ can be reduced from 24 to 22 days if activity E is crashed by 2 days
- The crashing of activity E will also reduce the time of the path $C-E-H$
- the path $A-D-F-G$ must be crashed by 1 day to reduce its completion time from 23 to 22 days. This can be achieved if activity D is crashed by 1 day.
- All other paths originally had a completion time of 22 days or less, so no further investigation is required

The minimum cost of reducing the completion time of this project as much as possible is
$= E \times 2 + D \times 1$
$= 350 \times 2 + 170 \times 1$
$= \$870$

6 a The immediate predecessors of activity G are C, D, H. The dummy activity is used because activities F and G share some, but not all immediate predecessors; activity F only has C as an immediate predecessor.

b

Path	Duration (days)
$A-C-F-J$	21
$A-C-G-I$ - J	23
$A-C-G-K$	22
$A-D-G-I-J$	22
$A-D-G-K$	21
$B-E-H-G-I-J$	24 ©
$B-E-H-G-K$	23

There is one critical path. If any of the activities along the critical path were crashed, more than one critical path would be created. The activities along the crtical path are: B, E, H, G, I, J

c i

Path	Duration (days)	Max reduction: B, E, G, H, I by 1
$A-C-F-J$	21	21 ©
$A-C-G-I$ - J	23	21 ©
$A-C-G-K$	22	21 ©
$A-D-G-I-J$	22	20
$A-D-G-K$	21	20
$B-E-H-G-I-J$	24 ©	20
$B-E-H-G-K$	23	20

The minimum number of hours in which the project could be completed, with crashing, is 21 hours.

ii All paths should have a duration of 21 hours or less. When investigating which activities to crash, start with the paths that have a duration closest to the minimum completion time, then work your way up to the paths that require the greatest reduction in completion time. Don't forget, when you choose to crash an activity, it will reduce the time of all paths that activity is used in.

- Do not consider the two paths that originally had a duration of 21 hours: $A-C-F-J$ and $A-D-G-K$.

- The path $A-C-G-K$ originally had a duration of 22 hours. To reduce this path to 21 hours, one activity must be crashed. The only activity that can be crashed along this path is activity G. Crash G by 1 hour; this will also reduce the duration of all paths with activity G by 1 hour.

- The path $A-C-G-I-J$ originally had a duration of 23 hours. As previously stated, activity G will be crashed by 1 hour and there is only one other option for crashing along this path; activity I must be crashed by 1 hour to reduce the duration of this path to 21 hours. All paths with activity I will also be reduced by 1 hour.

- The path $B-E-H-G-K$ originally had a duration of 23 hours. Activities E, G, H can be crashed by 1 hour each. Previously, it has been decided that acitivity G will be reduced by 1 hour. To reduce the duration of this path, either activity E or I can be crashed by 1 hour. The cost of crashing either of these activities is equal, so consider the final path to determine which activity to crash

- The path $B-E-H-G-I-J$ originally has a duration of 24 hours. Activities E, G, H, I can be crashed to reduce the duration to 21 hours. Previously, activities G and I were chosen to be crashed by 1 hour each. Similar to the previous path investigated, crashing either activity E or I by 1 hour would reduce the duration of this path to 21 hours. As the cost of crashing either of these activities is equal, the minimum cost of crashing can be determined by choosing either one of these activities; crash activitiy E by 1 hour.

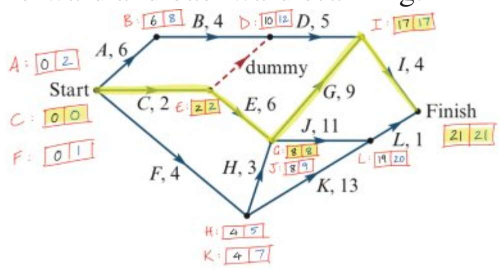
The minimum cost of completing this project in the minimum number of hours is
$= G \times 1 + I \times 1 + E \times 1$
$= 150 \times 1 + 150 \times 1 + 150 \times 1$
$= \$450$

7 a

Path	**Duration (weeks)**
$A-B-D-I$	19
$C-D-I$	11
$C-E-G-I$	21 ©
$C-E-J-L$	20
$F-H-G-I$	20
$F-H-J-L$	19
$F-K-L$	18

The project can be completed in 21 weeks.

b Perform a critical analysis using forward and backward scanning:



The EST for *D* is 10 weeks. This was a sum of the duration of activities *A* and *B*, as that was the path from the *Start* to activity *D* with the longest duration.

c The LST for *H* is 5 weeks. When backward scanning, if there are two LST options at a vertex, you must use the smallest value, before subtracting the duration of activity *H*.

d Activity *K* has a float time of 3 weeks and is the only activity with a float time greater than two weeks. Activity *K* has an EST of 4 weeks and a LST of 7 weeks.

e

Path	Duration (weeks)	Max reduction: D, E, G, H, J by 2
$A-B-D-I$	19	17
$C-D-I$	11	9
$C-E-G-I$	21 ©	17
$C-E-J-L$	20	16
$F-H-G-I$	20	16
$F-H-J-L$	19	15
$F-K-L$	18	18 ©

The minimum number of weeks in which the project could be completed, with crashing, is 18 weeks.

All paths should have a duration of 18 weeks or less. When investigating which activities to crash, start with the paths that have a duration closest to the minimum completion time, then work your way up to the paths that require the greatest reduction in completion time. Don't forget, when you choose to crash an activity, it will reduce the time of all paths that activity is used in.

- Do not consider the two paths that originally had a duration of 18 weeks or less: $C-D-I$ and $F-K-L$.
- The path $A-B-D-I$ originally had a duration of 19 weeks. Acitvity *D* is the only option for crashing to reduce the completion time of this path; crash activity *D* by 1 week.
- The path $F-H-J-L$ originaly had a duration of 19 weeks. Activities *H* or *J* can be crashed by 1 week to reduce the completion time of this path. Keep this in mind when investigating the other paths.
- The path $C-E-J-L$ originally had a duration of 20 weeks. Activities *E* and *J* can be crashed by 2 or 1 week each to reduce the completion time of this path.
- The path $F-H-G-I$ originally had a duration of 20 weeks. Activities *G* and *H* can be crashed by 2 or 1 week each to reduce the

completion time of this path.

- The path $C - E - G - I$ originally had a duration of 21 weeks. Activities E and G can be crashed by 2 or 1 week each to reduce the completion time of this path.

The information above, with all possible crashing scenarios, is summarised in the following table:

Path	Original duration	Crashing scenarios to achieve 18 weeks		
$A - B - D - I$	19	D by 1 = 2000		
$F - H - J - L$	19	H by 1 = 1500	J by 1 = 3000	
$C - E - J - L$	20	E by 2 = 2000	E by 1 J by 1 = 4000	J by 2 = 6000
$F - H - G - I$	20	G by 2 = 1000	G by 1 H by 1 = 2000	H by 2 = 3000
$C - E - G - I$	21	E by 2 G by 1 = 2500	E by 1 G by 2 = 2000	

Consider the cheapest options for each path and how there is overlap between crashing scenarios (e.g. crashing activities H, E and G will impact multiple paths). The cheapest scenario would be to crash D by 1, H by 1, E by 2 and G by 1. The minimum cost for the greatest reduction in time is:

$= D \times 1 + H \times 1 + E \times 2 + G \times 1$

$= 2000 \times 1 + 1500 \times 1 + 1000 \times 2 + 500 \times 1$

$= \$6000$

8 **a**

Path	Duration (days)
$A - C - H - I - N$	27
$A - C - H - J - M - O$	27
$A - C - G - M - O$	21
$A - C - F - K - M - O$	28
$A - C - F - L - O$	26
$B - D - H - I - N$	28
$B - D - H - J - M - O$	28
$B - D - G - M - O$	22
$B - D - F - K - M - O$	29 ©
$B - D - F - L - O$	27
$B - E - K - M - O$	28
$B - E - L - O$	26

The project can be completed in 29 weeks.

b The critical path is $B - D - F - K - M - O$. There are six activities along the critical path.

c From the table above, there are four paths with a duration of 28 days. They are: $A - C - F - K - M - O$, $B - D - H - I - N$, $B - D - H - J - M - O$, $B - E - K - M - O$.

d

Path	Duration (days)	Max reduction: H, J, K, L, M by 2
$A - C - H - I - N$	27	25
$A - C - H - J - M - O$	27	21
$A - C - G - M - O$	21	19
$A - C - F - K - M - O$	28	24
$A - C - F - L - O$	26	24
$B - D - H - I - N$	28	26 ©
$B - D - H - J - M - O$	28	22
$B - D - G - M - O$	22	20
$B - D - F - K - M - O$	29 ©	25
$B - D - F - L - O$	27	25
$B - E - K - M - O$	28	24
$B - E - L - O$	26	24

The new minimum completion time for the project is 26 days.
Four of the paths originally had a completion time of 26 days or less, so they do not need to be considered when crashing: $A - C - G - M - O$, $A - C - F - L - O$, $B - D - G - M - O$ and $B - E - L - O$.

Path	Duration (days)	Crashing scenarios to achieve 26 days					
$A-C-H-I-N$	27	H by 1					
$A-C-H-J-M-O$	27	H by 1	J by 1	M by 1			
$A-C-F-K-M-O$	28	K by 2	K by 1 M by 1	M by 2			
$B-D-H-I-N$	28	H by 2					
$B-D-H-J-M-O$	28	H by 2	H by 1 J by 1	H by 1 M by 1	J by 2	J by 1 M by 1	M by 2
$B-D-F-K-M-O$	29 ©	K by 2 M by 1	K by 1 M by 2				
$B-D-F-L-O$	27	L by 1					
$B-E-K-M-O$	28	K by 2	K by 1 M by 1	M by 2			

From the table above, activity H must be crashed by 2 and activity L must be crashed by 1, otherwise there will be paths that cannot be reduced to 26 days. Activity J will not be crashed as it only appears in one possible path to be crashed, whereas activities K and M are in common with multiple paths. Considering there is no difference in cost of crashing activities K and M there are mutiple options for crashing: K by 2 and M by 1 or K by 1 and M by 2 (as dictated by the original critical path $B-D-F-K-M-O$).

There are two possible crashing scenarios for this project, that would achieve the minimum completion time of 26 days:
H by 2, J by 0, K by 2, L by 1, M by 1
or
H by 2, J by 0, K by 1, L by 1, M by 2.

9

Path	Duration (hours)
$A-E-I$	17
$A-C-G$	19 ©
$B-D-F-G$	18
$B-D-H-J$	17

The critical path is the path with the longest duration of time; $A-C-G$.
B

10 Trial and error each option.

Path	Duration (hours)	Max reduction: C, G by 1
$A-E-I$	17	17 ©
$A-C-G$	19 ©	17 ©
$B-D-F-G$	18	17 ©
$B-D-H-J$	17	17 ©

B

11 Trial and error each option.

Path	Duration (weeks)	Max reduction: I by 3 and J by 1
$A-C-H-L$	21	21
$A-C-G-I-M$	29	26 ©
$B-D-F-I-M$	28	25
$B-E-J-I-M$	30 ©	26 ©
$B-E-K-M$	26	26 ©

D

Chapter Review: Solutions to Multiple-choice questions

1 Identify the shortest path using inspection, or Dijkstra's algorithm

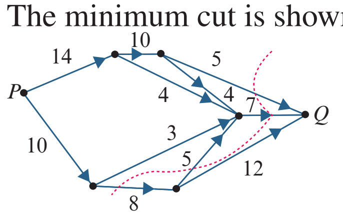
	B	*C*	*D*	*E*	*F*	*G*	*H*	*Z*
A	10	12	×	×	×	×	×	×
B	10	12	17	19	15	×	×	×
C	10	12	17	19	15	22	×	×
F	10	12	17	19	15	18	22	26
D	10	12	17	19	15	18	22	26
G	10	12	17	19	15	18	22	26
E	10	12	17	19	15	18	22	26
H	10	12	17	19	15	18	22	26

D

2 Capacity of cut = 3 + 4 + 3
= 10

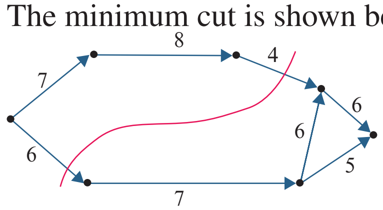
Note: one of the edges with capacity of 4 is not counted as it flows from the sink side of the cut to the source side of the cut. **D**

3 The minimum cut is shown below:



maximum flow = capacity of cut
= 5 + 7 + 8
= 20 **A**

4 The minimum cut is shown below:



maximum flow = capacity of cut
= 6 + 4
= 10 **A**

5 **A:** Travis – basketball and volleyball
Miriam – basketball, athletics, tennis
Swimming is missing so **A** is false.

B: Miriam played 3 sports
Fulvia played 2 sports
total: 5 sports

Andrew played 2 sports
Travis played 2 sports
total: 4 sports

Miriam and Fulvia played more sports than Andrew and Travis so **B** is false

C: Kieran played 3 sports
Miriam played 3 sports so **C** is true.

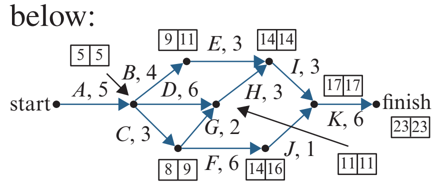
D: Kieran and Travis played basketball, volleyball, swimming, athletics and tennis (5 different sports).

Miriam and Fulvia played swimming, athletics, basketball and tennis (4 different sports).

Travis and Kieran played more sports than Miriam and Fulvia so **D** is false.

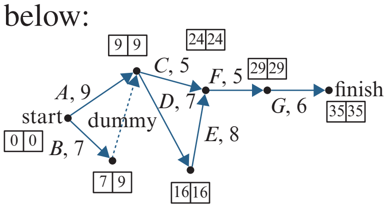
E: Andrew played the same number of sports as Travis and Fulvia so he did not play fewer sports than all others. **E** is false. **C**

6 The critical path analysis is shown below:



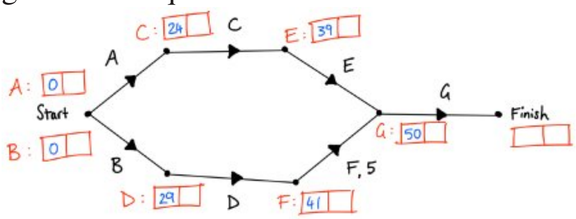
The critical path is: $A - D - H - I - K$ **B**

7 The critical path analysis is shown below:

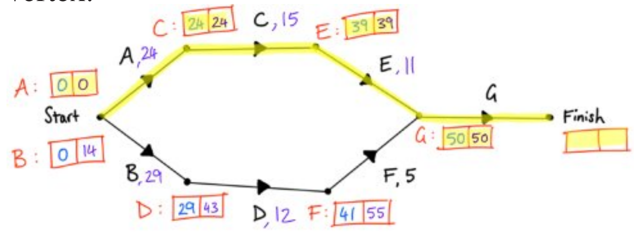


The earliest start time for activity *F* is 24. **E**

8 The information in the table has been used to construct the following incomplete activity network, in addition to the duration of activity *F* given in the question.

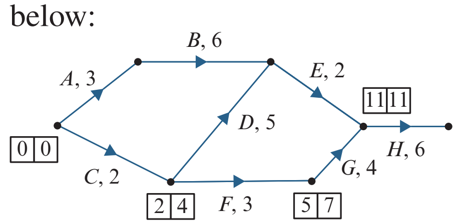


Given activity *F* has a duration of 5 hours and and an EST of 41, it cannot be on the critical path, as this sum does not equate to the EST of activity *G*. Likewise, the LST of activity *G*, minus the duration of activity *F*, does not equal the EST of activity *F*, thus there is a float time and if an activity is along the critical path, it must have a float time of zero. If activity *F* is not along the critical path, then activity *E* must be along the critical path with a duration of 11 hours to ensure its LST = 39 hours (LST of G − duration of E = 50 − 11 = 39 = EST of E). Although the completion time for this project is unknown, backward scanning can be used to find the LST of each activity as activity *G* is the only activity which leads to the *Finish* vertex.



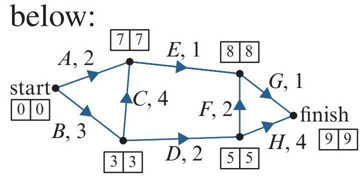
The EST of activity *F* can increase to 45 hours without affecting the completion time of the project, because activity *F* has a duration of 5 hours and activity *G* has an EST of 50 hours. Activity *D* has a duration of 12 hours. If increased by 4 hours to 16 hours, this would increase the EST of *F* from 41 to 45 hours (EST of *D* + new duration of *D* = 29 + 16 = 45). **D**

9 The critical path analysis is shown below:



The completion time of the project is 17 weeks. **D**

10 The critical path analysis is shown below:



The EST for activity *G* is 8. **E**

Chapter Review: Extended-response questions

1 a Perform the Hungarian algorithm:

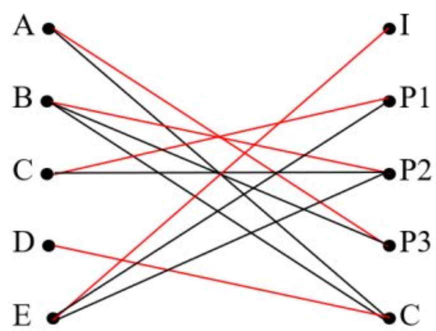
	I	P1	P2	P3	C	
A	16	14	19	9	9	−9
B	17	18	10	9	9	−9
C	9	8	6	15	8	−6
D	11	12	11	16	6	−6
E	10	10	8	15	8	−8

	I	P1	P2	P3	C
A	7	5	10	0	0
B	8	9	1	0	0
C	3	2	0	9	2
D	5	6	5	10	0
E	2	2	0	7	0
	−2	−2			

	I	P1	P2	P3	C
A	5	3	10	0	0
B	6	7	1	0	0
C	1	0	0	9	2
D	3	4	5	10	0
E	0	0	0	7	0

	I	P1	P2	P3	C
A	4	2	9	0	0
B	5	6	0	0	0
C	1	0	0	10	3
D	2	3	4	10	0
E	0	0	0	8	1

	I	P1	P2	P3	C
A	4	2	9	0	0
B	5	6	0	0	0
C	1	0	0	10	3
D	2	3	4	10	0
E	0	0	0	8	1



- Alvin should write **Paragraph 3**
- Billy should write **Paragraph 2**
- Chloe should write **Paragraph 1**
- Danielle should write the **Conclusion**
- Elena should write the **Introduction**

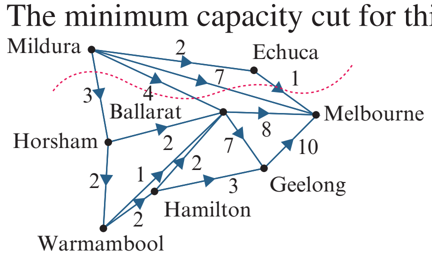
b A,P3 + B,P2 + C,P1 + D,Con + E,Into = 9 + 10 + 8 + 6 + 10 = 43 minutes

2 a All edges that this cut crosses flow from the source side of the cut to the sink side of the cut, so all of them are used in the capacity calculation.

Cut capacity = 1 + 7 + 8 + 10

= 26

b The minimum capacity cut for this network is shown below:



The maximum flow of passengers from Mildura to Melbourne is the capacity of the minimum capacity cut.

Maximum flow = 1 + 7 + 4 + 3

= 15

3 Use the Hungarian algorithm to determine the allocation for minimum time.

Swimmer	*Backstroke*	*Backstroke*	*Butterfly*	*Freestyle*	
Rob	76	78	70	62	−62
Joel	74	80	66	62	−62
Henk	72	76	68	58	−58
Sav	78	80	66	60	−60

Swimmer	*Backstroke*	*Backstroke*	*Butterfly*	*Freestyle*
Rob	14	16	8	0
Joel	12	18	4	0
Henk	14	18	10	0
Sav	18	20	6	0
	−12	−16	−4	

Swimmer	*Backstroke*	*Backstroke*	*Butterfly*	*Freestyle*
Rob	2	0	4	0
Joel	0	2	0	0
Henk	2	2	6	0
Sav	6	4	2	0

Swimmer	*Backstroke*	*Backstroke*	*Butterfly*	*Freestyle*
Rob	2	0	4	2
Joel	0	2	0	2
Henk	0	0	4	0
Sav	4	2	0	0

Rob must be allocated to Breaststroke, so Henk cannot.
There are two different allocations possible now:

1. If Joel is allocated to Backstroke, Henk cannot, so Henk must be allocated to Freestyle and Sav to Butterfly.

2. If Joel is allocated to Butterfly, Sav cannot, so Sav must be allocated to Freestyle and Henk to Backstroke.

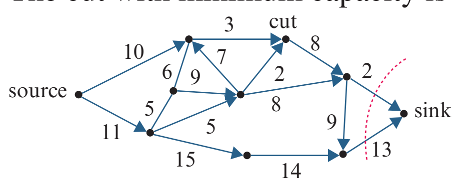
Allocation 1:
Rob – Breaststroke, Joel – Backstroke, Henk – Freestyle, Sav – Butterfly
Allocation 2:
Rob – Breaststroke, Joel – Butterfly, Henk – Backstroke, Sav – Freestyle
Total time for both allocations is 276.

4 a The edge with capacity 9 flows from the sink side of the cut to the source side of the cut, so will not be counted in the capacity calculation.

Cut capacity $= 3 + 2 + 8 + 13$

$= 26$

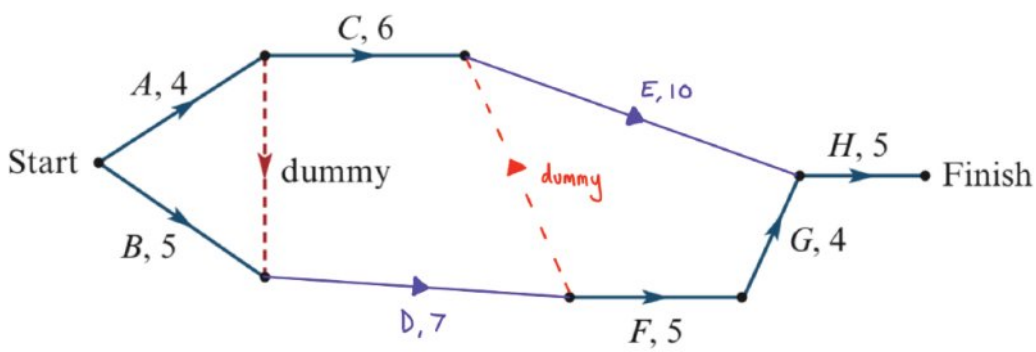
b The maximum flow through this network is equal to the minimum cut capacity. The cut with minimum capacity is shown below:



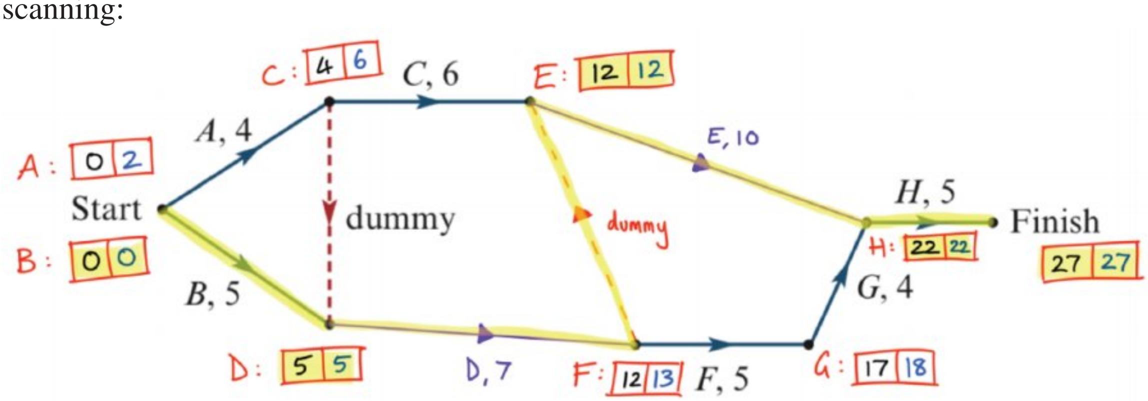
Cut capacity $= 2 + 13$

$= 15$

5 a Consider the immediate predecessors from the given table. Activity D must begin at the end of activities B and A (using the dummy activity) and go towards activity F. A dummy activity is needed because activities E and F share some, but not all, immediate predecessors. Activity E must start at the end of activity C and finish at the beginning of activity H.



b Perform a critical analysis of the activity network, using forward and backward scanning:



The EST of $E = 12$ hours

c Float time of G = LST − EST = 18 − 17 = 1 hour

d Using the diagram above, four activities have a non-zero float time (activities A, C, F, G).

e $B - D - E - H$

f 27 hours

g **i**

Path	Duration (hours)	Max reduction: E by 2
$A-C-E-H$	25	23
$A-D-E-H$	26	24
$A-D-F-G-H$	25	25
$B-D-E-H$	27 ©	25
$B-D-F-G-H$	26	26 ©

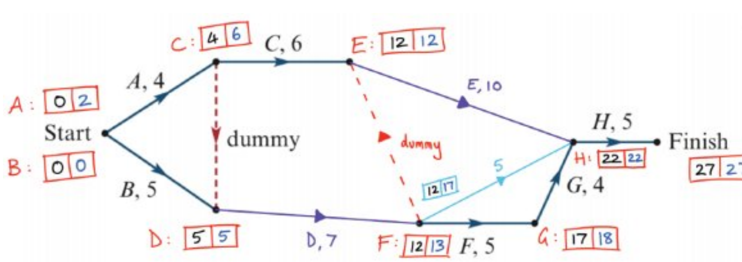
After crashing activity E, the critical path is $B - D - F - G - H$

ii

Path	Duration (hours)	Max reduction: E by 2 and A,B,D by 1
$A-C-E-H$	25	22
$A-D-E-H$	26	22
$A-D-F-G-H$	25	23
$B-D-E-H$	27 ©	23
$B-D-F-G-H$	26	24 ©

The minimum number of hours in which the project can now be completed in is 24 hours.

h



The new activity has an EST of 12 hours, therefore the new activity must start at the beginning of either activity E or activity F. The new activity has a duration of 5 hours and if it has an LST of 17 hours, it must finish at an acitivty with an LST of 22 hours (22 − 5 = 17); this can only be activity H. By convention, only straight edges are used in activity networks, as VCAA exams avoid networks where multiple activities have the same immediate predecessors and are directed towards the same activity. Therefore the new activity cannot be drawn as an identical line to activity E. The new activity must be drawn as a straight directed edge from the end of activity **D** to the start of activity **H**.

Solutions to Review: Multiple-choice questions

1 It is possible to travel from F and back again without passing through another vertex, so there is a loop at F. Answer is either **A**, **B** or **C**.
There are two ways to travel from G directly to F, so the answer is either **A** or **C**.
There are two ways to travel from G directly to H so the answer is **A**.
Answer: **A**

2 Apply the Hungarian algorithm.

Name	*A*	*B*	*C*	*D*	*E*	
Francis	12	15	99	10	14	-10
David	10	9	10	7	12	-7
Hermam	99	10	11	6	12	-6
Indire	8	8	12	9	99	-8
Natalie	8	99	9	8	11	-8

Name	*A*	*B*	*C*	*D*	*E*
Francis	2	5	89	0	4
David	3	2	3	0	5
Hermam	93	4	5	0	5
Indire	0	0	4	1	91
Natalie	0	91	1	0	3
			-1		-3

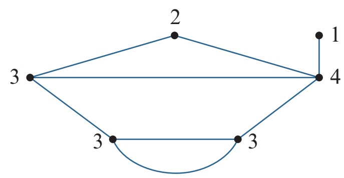
Name	*A*	*B*	*C*	*D*	*E*
Francis	2	5	88	0	1
David	3	2	2	0	2
Hermam	93	4	4	0	3
Indire	0	0	3	1	88
Natalie	0	91	0	0	0

Name	*A*	*B*	*C*	*D*	*E*
Francis	1	4	87	0	0
David	2	1	1	0	1
Hermam	92	3	3	0	2
Indire	0	0	3	2	88
Natalie	0	91	0	1	0

Name	*A*	*B*	*C*	*D*	*E*
Francis	0	3	86	0	0
David	2	0	0	0	1
Hermam	91	2	2	0	2
Indire	0	0	3	3	89
Natalie	0	91	0	2	1

Francis is the only person who can be allocated task E.
Answer: **E**

3 The degree of each vertex is shown in the diagram below:



Sum of degrees

= 3 + 2 + 4 + 1 + 3 + 3

= 16

Answer: **E**

4 An eulerian trail will be possible if there are exactly two vertices of odd-degree with all others even-degree.

S, *Z*, *U* and *W* are all odd-degree vertices, so joining two of these will cause those joined to be even-degree, leaving only two odd-degree vertices.

Answer: **B**

5 Let n be the number of vertices.

$v = n, f = n$ since the number of vertices and faces is equal.

$e = 20$

$v - e + f = 2$

$n - 20 + n = 2$

$2n = 2 + 20$

$2n = 22$

$n = 11$

The number of vertices and faces is 11.

Answer: **C**

6 Connecting water pipes with the shortest length of pipe involves connecting each point for water in a minimal spanning tree.

Answer: **B**

7 Analyse each option separately:

A: This statement is true.

B: Activities or tasks in a project can happen simultaneously with tasks on the critical path. The critical path activities do not have to be completed before any other activities can start.

C: Tasks not on the critical path have slack time, which means decreasing their completion time will have no effect on the length of the project.

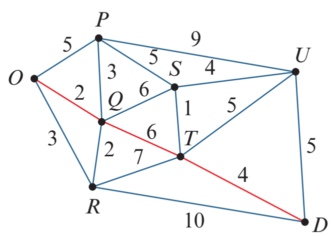
D: The critical path can be a single activity.

E: Some projects can have multiple critical paths. Answer: **A**

8 The shortest path from O to D can be determined using Dijsktra's algorithm, or by inspection.
Using Dijkstra's algorithm:

	P	Q	R	S	T	U	D
O	5	$\boxed{2}$	3	×	×	×	×
Q	5	$\boxed{2}$	$\boxed{3}$	8	8	×	×
R	$\boxed{5}$	$\boxed{2}$	$\boxed{3}$	8	8	×	13
P	$\boxed{5}$	$\boxed{2}$	$\boxed{3}$	$\boxed{8}$	8	14	13
S	$\boxed{5}$	$\boxed{2}$	$\boxed{3}$	$\boxed{8}$	$\boxed{8}$	12	13
T	$\boxed{5}$	$\boxed{2}$	$\boxed{3}$	$\boxed{8}$	$\boxed{8}$	12	$\boxed{12}$

The shortest path is shown in red in the diagram below:



The length of the shortest path
$= 2 + 6 + 4$
$= 12$
Answer: **B**

9 Ann has visited two resorts. Matt has visited one. Tom has visited two. Maria has visited three.

A : Ann and Maria have visited 5 in total.
Matt and Tom have visited 3 in total.
Ann and Maria have visited more than Matt and Tom, not fewer.

B: Matt and Tom have visited 3, not 4.

C: Maria has visited more, not fewer, than anyone else.

D: Neither Ann nor Maria have visited Mt Hutt.

E: Ann and Tom have visited 3 of the 5 resorts, Matt and Maria have visited 4 of them. This option is true.
Answer: **E**

10 In this pipeline, each town needs to be connected to any other town. This can be represented by a minimum spanning tree.
Answer: **C**

11 Consider each option separately.

A: French has two translators, Greek has one. True statement.

B: Sally can translate Spanish, Turkish and French. Kate can translate Italian and Greek. Five languages in total. True statement.

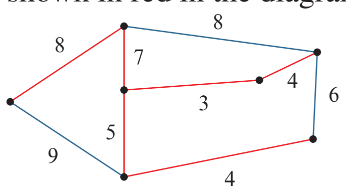
C: John can translate Spanish, Italian and Turkish. Greg can translate French and Turkish. Four different languages. True statement.

D: Kate and John can translate 4 different languages. Sally and Greg can translate 3 different languages. True statement.

E: Sally and John can translate 4 different languages. Kate and Greg can translate 5 different languages. False statement.
Answer: **E**

12 O can donate to any type so vertex O must be connected to O, A, B and AB.
The answer must be either option **D** or **E**.Each type can donate to its own type, so O must be connected to O, A to A, B to B and AB to AB. Option **E** has AB to B – however there is no such information in the question to support this. Thus **D** relays all the information given.
Answer: **D**

13 The minimum spanning tree is shown in red in the diagram below.



The weight of the minimum spanning tree
$= 8 + 7 + 5 + 3 + 4 + 4$
$= 31$
Answer: **D**

14 Euler's formula, where $e = 12$ and $f = 4$
$v + f = e + 2$
$v + 4 = 12 + 2$
$v + 4 = 14$
$v = 14 - 4$
$v = 10$
Answer: **B**

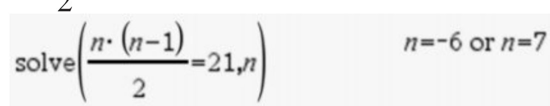
15 A complete graph with n vertices will have $\frac{n(n-1)}{2}$ edges.
This graph has twenty vertices, so $n = 20$ and the graph will have

$$\frac{20(20-1)}{2} = \frac{20(19)}{2} = \frac{380}{2}$$

$= 190$ edges.
Answer: **E**

16 A complete graph with n vertices will have $\frac{n(n-1)}{2}$ edges.
Use a CAS calculator to solve the following equation:

$$\frac{n(n-1)}{2} = 21$$



As n represents the number of vertices, reject the negative solution.
$n = 7$
Answer: **A**

17 A tree with n vertices has $n - 1$ edges.
When $n = 4$, the number of edges for the tree is $4 - 1 = 3$ edges
Answer: **C**

18 A tree with n vertices has $n - 1$ edges.
When the number of edges of a tree is 13, solve the following equation:
$n - 1 = 13$
$n = 13 + 1$
$n = 14$
The tree will have 14 vertices.
Answer: **C**

19 An eulerian circuit exists if all vertices have an even degree. In the graph, vertices *A* and *D* have an odd degree, so joining them with an edge will make them both have an even degree. Answer: **C**

20 Consider all options separately.

A : *A* and *B* can occur simultaneously. False statement.

B: A does not lead into *F* so *A* does not need to be completed before *F* can start. False statement.

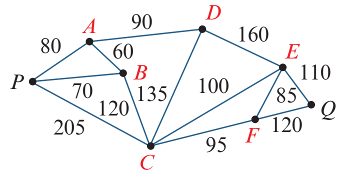
C: *E* and *F* can be completed independently. False statement.

D: *E* and *F* must both be completed before *H* can begin, but they do not necessarily have to finish at the same time. False statement.

E: *E* follows after *A* in the network so this is a true statement. Answer: **E**

21 There is one isolated vertex in the graph, so there will be one row and one column consisting of all zeros. There is one loop, so there will be one "1" along the diagonal. Answer: **B**

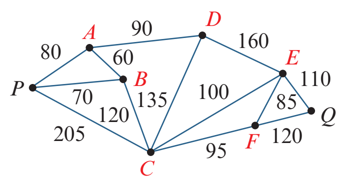
22 The shortest path, or path of minimum cost in this example, from *P* to *Q* can be determined using Dijsktra's algorithm, or by inspection. To use Dijkstra's algorithm, the vertices must be labelled with names, as shown below.



Using Dijkstra's algorithm:

	A	*B*	*C*	*D*	*E*	*F*	*Q*
P	80	70	205	×	×	×	×
B	80	70	190	×	×	×	×
Q	80	70	190	170	×	×	×
D	80	70	190	170	330	×	×
C	80	70	190	170	290	285	×
F	80	70	190	170	290	285	405
E	80	70	190	170	290	285	400

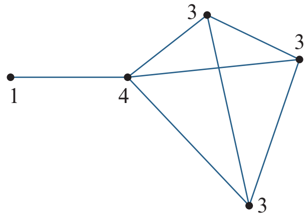
The shortest path is shown in red in the diagram below.



Minimum cost = 70 + 120 + 100 + 110
= 400
Answer: **A**

23 The capacity of the cut
= 8 + 2 + 3
= 13
Note: one of the edges with weight 3 passes from the sink side of the cut to the source side of the cut, so is not counted in the cut capacity calculation.
Answer: **D**

24 The degree of each of the vertices are shown on the graph below.



The sum of the degrees

= 1 + 4 + 3 + 3 + 3

= 14

Answer: **C**

25 $v = 9$, $e = 20$

$v - e + f = 2$

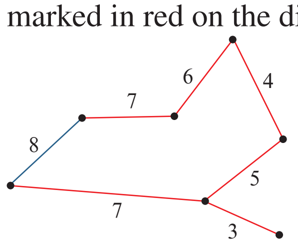
$9 - 20 + f = 2$

$f = 2 - 9 + 20$

$f = 13$

Answer: **B**

26 The minimum spanning tree is marked in red on the diagram below.



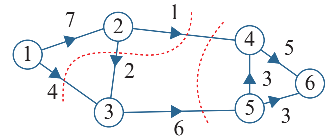
Weight of minimum spanning tree

= 7 + 6 + 4 + 5 + 3 + 7

= 32

Answer: **C**

27 There are two minimum cuts (same capacity) in this diagram. Both are shown below.



Maximum flow

= 4 + 2 + 1

= 7

or

Maximum flow

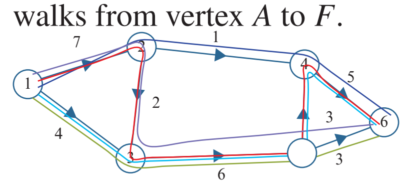
= 1 + 6

= 7

Answer: **C**

27 The diagram below shows all the possible

walks from vertex *A* to *F*.

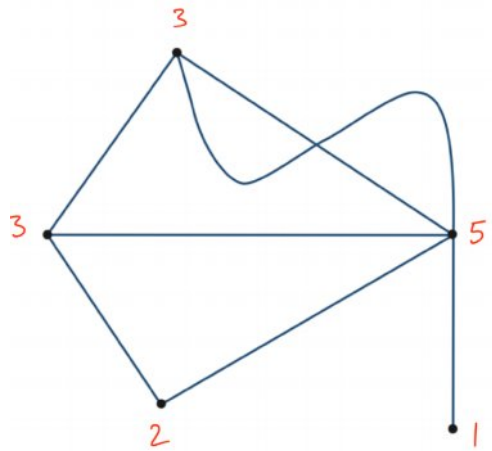


There are five different walks.

Answer: **E**

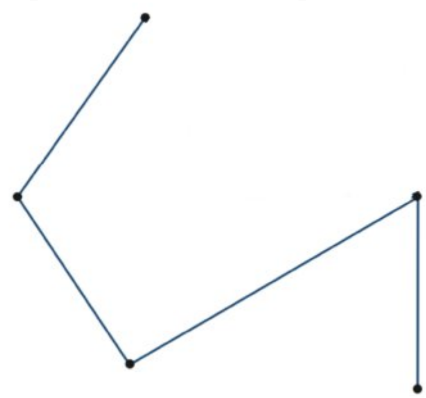
Solutions to Review: Extended-response questions

1 a Determine the degree of each vertex.



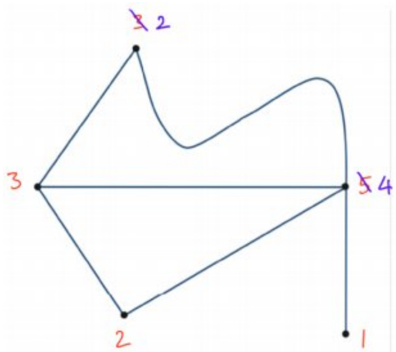
The sum of degrees= 3 + 3 + 2 + 1 + 5 = 14
Alternatively, Sum of degrees = 2 × Total number of edges = 2 × 7 = 14

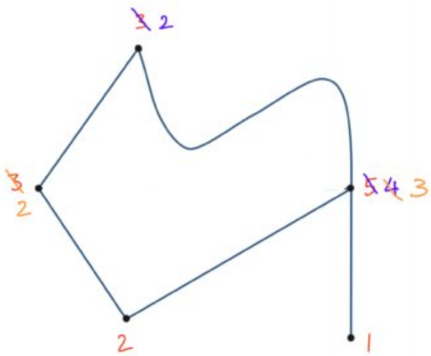
b For the graph to be connected, every vertex connected to every other vertex, either directly or indirectly via other vertices. This can be achieved using the minimum number of edges if the graph was redrawn as a spanning tree; the spanning tree for a graph with 5 vertices requires 4 edges. Here is one example:

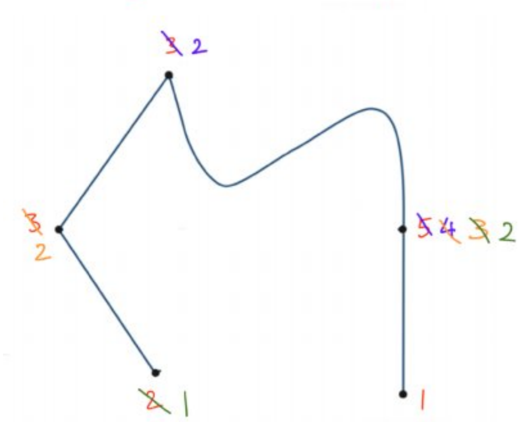


Originally the graph had 7 edges, this tree has 4 edges, therefore the maximum number of edges than can be removed so that the graph remains connected is 3.

c For an Eulerian trail to exist, there must be exactly zero or two odd vertices. From **part a** above, we noted the degree of each vertex; four vertices have an odd degree and one vertex has an even degree, thus there is no Eulerian trail for the original graph. If edges were removed, there would be an Eulerian trail, because for every edge removed, the degree of the two vertices the edge was connected to would result in a decrease of the degree of those vertices by 1 each. Here is one example of how this can occur:

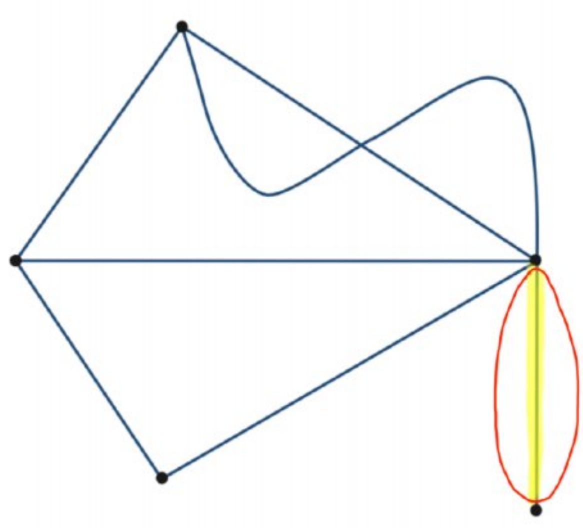




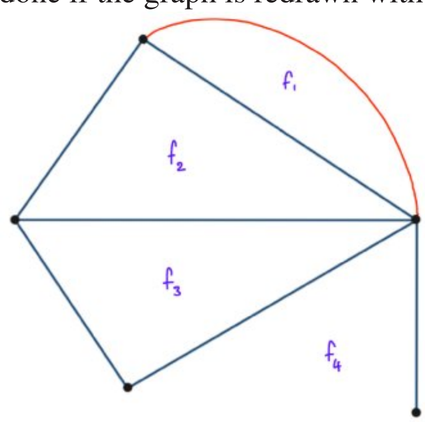


The maximum number of edges that can be removed for an Eulerian trail to exist is 3.

d A bridge is an edge in a connected graph that, if removed, will cause the graph to be disconnected. There is only one edge that can be classified in this way:



e To verify Euler's formula, the number of faces must be determined; this can only be done if the graph is redrawn with no edges intersecting, except at the vertices.



Note: you must count the space outside of the graph as a face. Euler's formula, where $v = 5$, $e = 7$ and $f = 4$

$v + f = e + 2$

$5 + 4 = 7 + 2$

$9 = 9$

Therefore, Euler's formula is verified,

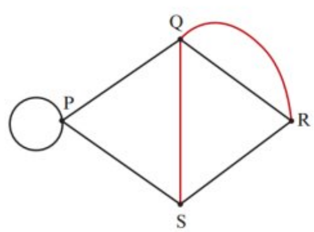
2 Using the matrix, verify each row with the given graph. Starting with row P, the three given elements in the matrix are correctly represented in the graph and the missing value will have a non-zero value if vertex P in the graph connects with itself; the graph shows there is a loop at vertex P therefore this missing value is **1**.

Next, row Q, only two of the three given elements are correctly represented in the graph; there is a '1' in row Q column S however there is no edge that represents this in the graph. **An edge must be added to the graph, connecting vertices Q and S**. The missing value in row Q is in column R; there is one edge connecting the vertices Q and R in the graph. Therfore the missing value is row Q column R can be **1**.

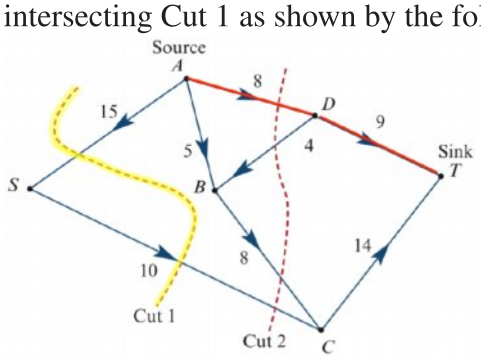
Only three of the four elements in row R are correctly represented in the graph; the 2 in row R column Q means there should be two edges connecting vertices R and Q. Therefore **one edge must be added to the graph, connecting R and Q** and this will also mean the unknown value is row Q column R must be **2**.

All four elements in row S are correctly represented in the graph. After the analysis of row Q, an edge was added between vertices Q and S.

	P	Q	R	S
P	1	1	0	1
Q	1	0	2	1
R	0	2	0	1
S	1	1	1	0

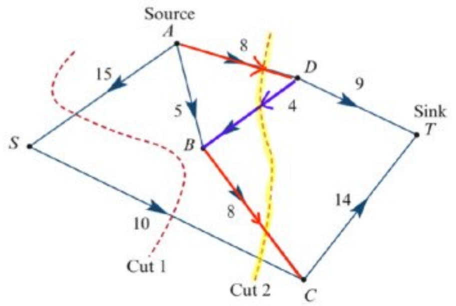


3 **a** Cut 1 is not a valid cut because it does not isolate the *Source* from the *Sink*; there is at least one path where flow is able to travel from the *Source* to the *Sink* without intersecting Cut 1 as shown by the following example:



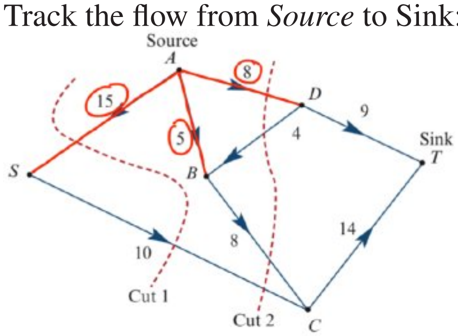
Note: there are multiple paths that verify this answer, however only one is required to support this answer.

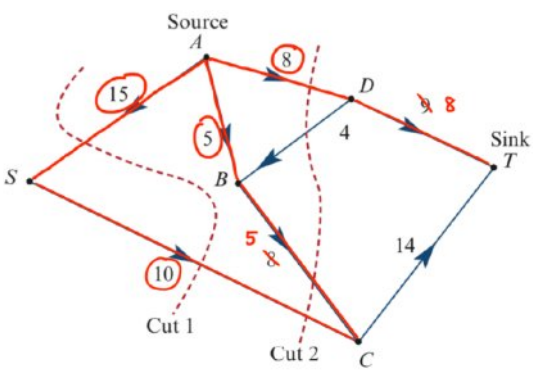
b Consider the edges Cut 2 intersects with and note if the flow is coming from the *Source* side of the cut or the *Sink* side of the cut:

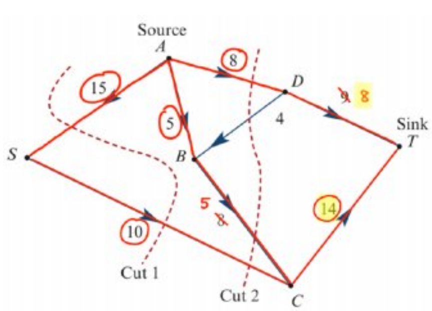


The edge connecting vertices *D* and *B* carries flow from the *Sink* side of the cut and should be included in the calculation of the capacity of the cut.
Capacity of Cut 2 = 8 + 8 = 16.

c Track the flow from *Source* to Sink:







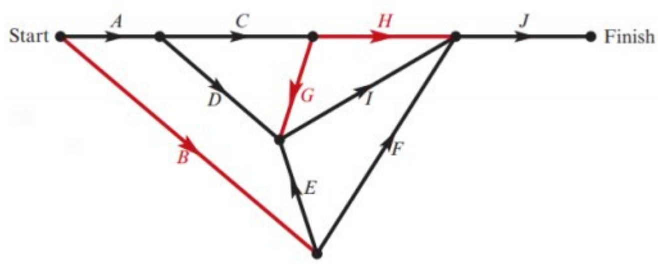
The maximum flow of the network = 14 + 8 = 22

4 Three activites are missing from the activity network: B, G, H.
Activity B has no immediate predecessors, therefore it must start at the vertex labelled *Start*. Activity B is the immediate predecessor for activities E and F therefore it must finish at the beginning of activities E and F.

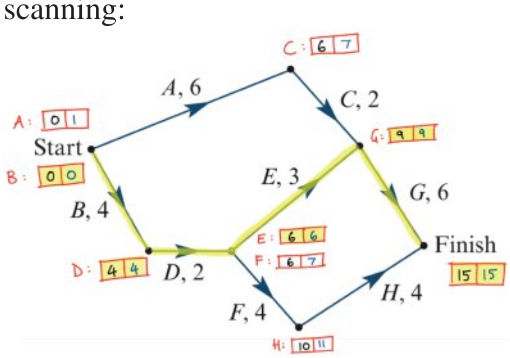
Activities G and H both have activity C as an immediate predecessor, therefore they must both start at the end of activity C.

Activity G is one of the immediate predecessors of activity I, therefore it must finish at the beginning of activity I.

Activity H is one of the immediate predecessors of activity J, therefore it must finish at the beginning of activity J.



5 **a** Perform a critical analysis of the activity network, using forward and backward scanning:



The EST of $G = 9$

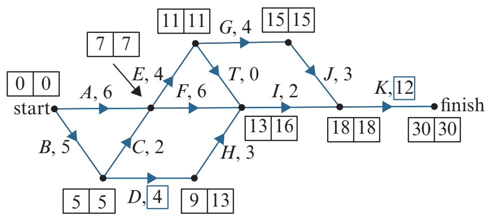
b From the diagram above, the LST of $F = 7$

c The float time of C = LST of C − EST of C = 7 − 6 = 1

d The critical path is the path from *Start* to *Finish* where all activities have a float time of zero = $B - D - E - G$

e The minimum completion time for the project = the duration of the longest path from *Start* to *Finish* = duration of the critical path = 15

6 The least time to complete the project is 30 hrs.
Activity K has an EST of 18 hrs, so
duration K = 30 – 18
= 12
Activity D has an EST of 5.
Activity H has an EST of 9.
duration D = EST H – EST D
= 9 – 5
= 4
The critical path analysis can now be completed.



LST A = LST E (or F) – duration A
= 7 – 6
= 1
LST F = LST I – duration F
= 16 – 6
= 10

a The completed table is

Activity	Completion time (hours)	Earliest starting time (hours)	Latest starting time (hours)
A	6	0	**1**
B	5	0	0
C	2	5	5
D	**4**	5	9
E	4	7	7
F	6	7	**10**
G	4	11	11
H	3	9	13
I	2	13	16
J	3	15	15
K	**12**	18	18

b The critical path is $B – C – E – G – J – K$

7 **a** **i** By inspection, or using Dijkstra's
algorithm below the shortest route from *P* to *U* has length 2.1 km.

	Q	***R***	***S***	***T***	***U***
P	0.8	0.7	×	×	×
R	0.8	0.7	×	1.5	×
Q	0.8	0.7	1.7	1.5	×
T	0.8	0.7	1.7	1.5	2.1
S	0.8	0.7	1.7	1.5	2.1

ii The path required is a hamiltonian path. There are multiple answers:
P – *Q* – *R* – *T* – *S* – *U*,
P – *R* – *Q* – *S* – *T* – *U*,
P – *R* – *Q* – *T* – *S* – *U*,
P – *R* – *T* – *Q* – *S* – *U*.

b **i** There are two eulerian trails through
this network:
R – *Q* – *P* – *R* – ***T* – *Q* – *S*** – *T* – *U* – *S*
or
R – *Q* – *P* – *R* – ***T* – *S* – *Q*** – *T* – *U* – *S*

ii By travelling an eulerian path, the
technician would travel along each of the roads only once. This would minimise the total distance travelled and would avoid having to travel down a street that has already been checked.

8 **a** A planar graph can be drawn so that none of the edges in that graph intersect, except at the vertices. The edges in a planar graph can be drawn so that they do not cross over.

b $v = 7$ (one for each town)
$e = 11$ (one for each road)
$f = 6$ (one for each subregion)
$v - e + f = 7 - 11 + 6$
$= 2$
Euler's rule has been verified.

c The inspector starts in town *B*, which has degree three (odd) in the graph.
The inspector must travel every road to inspect it. To do this in the least total distance, she would need to travel each road only once, that is, follow and eulerian trail. This is possible because there are exactly two odd-degree vertices in the graph, *B* (where she starts) and *C* (where she must end). The inspector will end in town *C*.

d The distance travelled will be the sum of all the edge weights
distance = 18 + 19 + 32 + 29 + 20 + 25 +
28 + 21 + 56 + 16 + 33
= 297 km

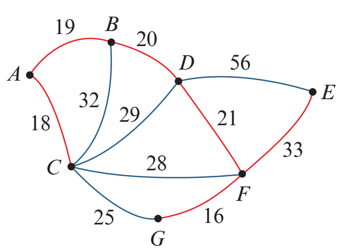
e The route travelled is not unique, because there are two different eulerian trails through this graph.
One is: $B - A - C - B - D - E - F - D -$
$C - F - G - C$
Another is: $B - C - A - B - D - E - F -$
$D - C - F - G - C$

f The shortest path from E to A can be found using inspection, or Dijkstra's algorithm as shown below.

	A	*B*	*C*	*D*	*F*	*G*
E	×	×	×	56	33	49
F	×	×	61	54	33	49
G	×	×	61	54	33	49
D	×	74	61	54	33	49
C	79	74	61	54	33	49
B	79	74	61	54	33	49

The shortest path from E to A has length 79 km (along the route $E - F - C - A$).

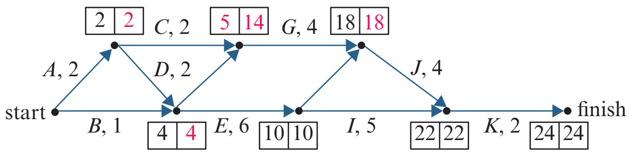
g The minimal length of cable required will form a minimal spanning tree. This is shown in red in the diagram below.



The minimal spanning tree has weight
= 18 + 19 + 20 + 21 + 33 + 16
= 127
The minimal length of cable required is 127 km.

h A hamiltonian cycle would be
$C - A - B - D - E - F - G - C$
(or the reverse).
The total distance travelled on this cycle
= 18 + 19 + 20 + 56 + 33 + 16 + 25
= 187 km

9 The completed critical path analysis for the network is shown in red in the diagram below.



a The EST for G is 5 hours.

b The shortest time required to complete the project is 24 hours.

c Float I = LST I – EST I

= LST K – duration I – EST I

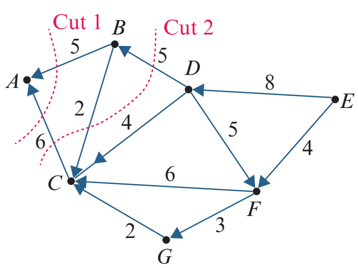
= 22 – 5 – 10

= 7 hours

10 The maximum flow from E to A is the minimum cut capacity.

Note: Take care with this question because the source is on the right of the diagram and the sink is on the left.

There are two minimum cuts for this network and both are shown in the diagram below.



Cut 1 capacity = 6 + 5 = 11

Cut 2 capacity = 5 + 6 = 11

Note: the edge from B to C is not counted in the calculation of the cut capacity for Cut 2 because the flow is from the sink side of the cut (left) to the source side of the cut (right).

The maximum flow from E to A

= 11 megalitres per day.

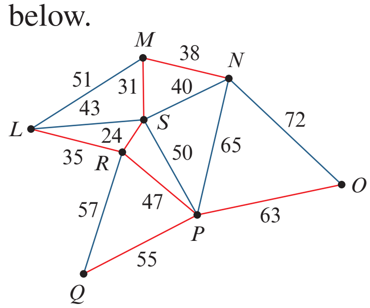
11 **a** The shortest path from S to O is best found using inspection in this question, or Dijkstra's algorithm as shown below.

	L	M	N	O	P	Q	R
S	43	31	40	×	50	×	24
R	43	31	40	×	50	81	24
M	43	31	40	×	50	81	24
N	43	31	40	112	50	81	24
L	43	31	40	112	50	81	24
P	43	31	40	112	50	81	24
Q	43	31	40	112	50	81	24

The shortest path from S to O has length 112 km (along the route $S - N - O$).

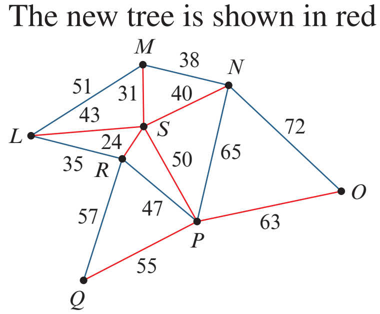
b **i** The network within a graph that links all the vertices to give the shortest overall length is called the minimum spanning tree.

ii The minimum spanning tree for this network is shown in red in the diagram below.



iii The minimum length of pipe required
= total weight of minimum spanning tree
= 35 + 24 + 47 + 55 + 63 + 31 + 38
= 293 kilometres

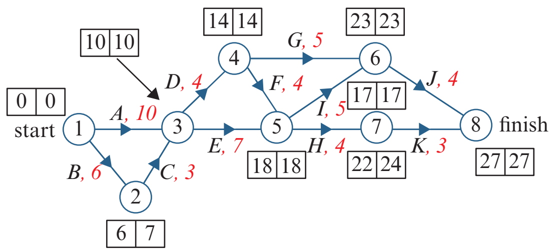
c In the new tree, S will need to be connected directly to R, L, M, N and P.
O and Q will be connected with the shortest possible pipe length.
The new tree is shown in red in the diagram below.



The minimum length of pipe required
= 43 + 24 + 31 + 40 + 50 + 55 + 63
= 306 km

12 **a** From the activity network activities A and C lead directly to the start of activity E and so are immediate predecessors. Although it is not an *immediate* predecessor, activity B must be finished before E can start (because C cannot begin until it is). Activities A, B and C must be finished before activity E can start.

b The activity network with durations and completed critical path analysis is shown in the diagram below.



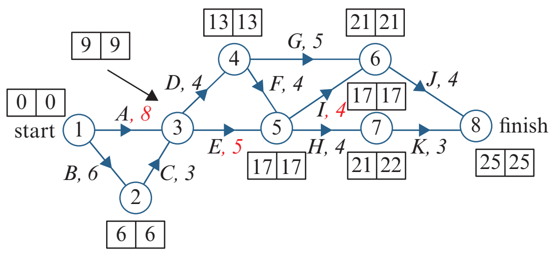
LST B = LST C – duration B
= 7 – 6
= 1
EST E = 10 (from diagram)
LST I = LST J – duration I
= 23 – 5
= 18
The completed table is below.

task	EST	*LST*
A	0	0
B	0	**1**
C	6	7
D	10	10
E	**10**	11
F	14	14
G	14	18
H	18	20
I	18	**18**
J	23	24
K	22	24

c **i** Using the critical path analysis above,
or the tasks for which EST and LST are the same in the table above, the critical path can be determined as
$A – D – F – I – J$

ii The length of the critical path is the sum of the individual durations
= 10 + 4 + 4 + 5 + 4
= 27 months
or it can just be read from the critical path analysis (last box).

d The new durations and critical path analysis after reduction in duration of given activities is shown below.



i The new critical path, from the diagram above, is
$B - C - D - F - I - J$

ii The time taken to complete the project with the new durations is 25 months (from the diagram above).

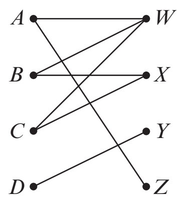
13 **a** The completion of the Hungarian algorithm is shown below.

Camp site	*W*	*X*	*Y*	*Z*	
A	30	70	60	20	-20
B	40	30	50	80	-30
C	50	40	60	50	-40
D	60	70	30	70	-30

Camp site	*W*	*X*	*Y*	*Z*
A	10	50	40	0
B	10	0	20	50
C	10	0	20	10
D	30	40	0	40
	-10			

Camp site	*W*	*X*	*Y*	*Z*
A	0	50	40	0
B	0	0	20	50
C	0	0	20	10
D	20	40	0	40

Draw a bipartite graph to see possible allocations.



D must be allocated to *Y*.
A must be allocated to *Z* so cannot be allocated to *W*.
B and *C* can be allocated to the same two residents so there are two different allocations possible.
Allocate:
D – *Y*, *A* – *Z*, *B* – *W*, *C* – *X*
or
D – *Y*, *A* – *Z*, *B* – *X*, *C* – *W*

b The cost will be the same regardless of which of the two allocations are used to calculate it.
Using the first allocation

Camp site	*W*	*X*	*Y*	*Z*
A	30	70	60	20
B	40	30	50	80
C	50	40	60	50
D	60	70	30	70

Cost = \$20 + \$40 + \$40 + \$30
= \$130

14 a Crash all activities by the maximum reduction possible to identify the new minimum completion time and critical path:

Path	**Duration (days)**	**Max reduction: E, J by 2 and F, H by 3**
A – *B* – *D* – *G* – *I*	14	14
A – *B* – *E* – *G* – *I*	15	13
A – *C* – *E* – *G* – *I*	17	15 ©
A – *C* – *F* – *H* – *I*	19 ©	13
A – *C* – *F* – *J*	14	9

The new minimum completion time for this project is 15 weeks.

b Consider all possible options for crashing to achieve the new minimum completion time of 15 weeks:

Path	**Original duration**	**Crashing scenarios to achieve 15 weeks**		
$A-B-D-G-I$	14			
$A-B-E-G-I$	15			
$A-C-E-G-I$	17	E by 2 = 2000		
$A-C-F-H-I$	19 ©	F by 3 H by 1 = 6500	F by 2 H by 2 = 7000	F by 1 H by 3 = 7500
$A-C-F-J$	14			

The minimum cost to achieve the maximum reduction in time will require activity E to be crashed by 2 weeks, activity F to be crashed by 3 weeks and activity H to be crashed by 1 week

$= E \times 2 + F \times 3 + H \times 1 = 1000 \times 2 + 1500 \times 3 + 2000 \times 1 = \$8500$

c 3 activities will be crashed and have their duration reduced to achieve the new minimum completion time: activities E, F, H

Chapter Review: 16B Exam 2 questions

Data analysis, probability and statistics

1 a Using your calculator, mean = 54.042, stand dev = 2.717

b $z = \dfrac{x - \bar{x}}{s} = \dfrac{51 - 54.042}{2.717} = -1.1$

c i $z = -2 \Rightarrow x = 54.042 - 2 \times 2.717 = 48.6$ kg

ii Using the 68-95-99.7% rule, we expect 2.5% to be less than $z = -2$.

2 a EV: *number of distractions*, RV: *time*

b *time*

c i Using the summary statistics in the table, IQR= 28.2 − 22.0 = 6.2 seconds

ii The value of 38.0 seconds is more than the upper fence, which is:
Upper fence= $28.2 + 1.5 \times 6.2 = 37.5$

d From the boxplots:
Slow: There is one person who took more than 36 seconds (the outlier).
A few: There is no-one who took more than 36 seconds.
Many: Q_3=36, so 25% of this group took more than 36 seconds. There are 36 people in this group (from the table), so there are $36 \times \dfrac{25}{100} = 9$ people who took more than 36 seconds.
Total = 1 + 9 = 10 people.

e Since the medians are different for each group we need only to quote and compare the medians to provide evidence that there is an association ⇒
From this information it can be concluded that the time taken to complete the task is associated with the number of distractions. The median time taken by the group who completed the task with no distractions was 25.0 seconds, faster than the group with a few distractions which has a median time of 26.2 seconds, which was in turn faster than the group with many distractions which took a median time of 29.2 seconds to complete the task.

3 a $r = \dfrac{bs_x}{s_y} = 0.0218 \times \dfrac{22.367}{0.52941} = 0.92103$
$r^2 = (0.92103)^2 = 84.8\%$

b 84.8% of the variation in *fuel consumption* can be explained by the variation in *speed*.

c *fuel consumption* = 6.827 × 0.0218 × 100 = 9.0 litres/km

d slope= 0.0218. On average, for each additional 1 km/hr increase in the *speed* of the car, the *fuel consumption* increases by 0.0218 litres/km.

e predicted value = 6.827 × 0.0218 × 72 = 8.40, actual value = 8.30.
Thus residual = −0.10.

4 a There is a strong, non-linear relationship between *efficiency* and *enthusiasm*.

b Compare the scatterplot to the Circle of Transformations on page 239 ⇒ $\log y, \frac{1}{y}, x^2$

c lUse your calculator to apply a log transformation to the variable *efficiency*, and to find the equation of the least squares regression line:
log(*efficiency*)= 0.0205 + 0.0860 × *enthusiasm*

d log(*efficiency*)= 0.0205 + 0.0860 × 9.3 = 0.8203
⇒ *efficiency*= $10^{0.8203}$ = 6.611

5 a Data have been given for two years. We need to determine the seasonal indices for each year separately, and then average over the two years to give the best estimate of the seasonal indices. We start be working our the quarterly averages for each year:

2020: quarterly average = $\dfrac{52 + 59 + 68 + 27}{4} = 51.5$

2021: quarterly average = $\dfrac{57 + 65 + 75 + 29}{4} = 56.5$

Seasonal indices:

Year	Q1	Q2	Q3	Q4
2020	$\frac{52}{51.5} = 1.010$	$\frac{59}{51.5} = 1.146$	$\frac{68}{51.5} = 1.320$	$\frac{27}{51.5} = 0.524$
2021	$\frac{57}{56.5} = 1.009$	$\frac{65}{56.5} = 1.150$	$\frac{75}{56.5} = 1.327$	$\frac{29}{56.5} = 0.513$

Average and round to two decimal places to find the final SI's:

	Q1	Q2	Q3	Q4
SI	1.01	1.15	1.32	0.52

b To deseasonalise the data, divide by the seasonal indices:

Year	Q1	Q2	Q3	Q4
2022	$\frac{63}{1.01} = 62$	$\frac{69}{1.15} = 60$	$\frac{80}{1.32} = 61$	$\frac{33}{0.52} = 63$

6 a centred 4 mean smoothed value $= \dfrac{\frac{126.3+116.4+128.7+143.4}{4} + \frac{116.4+128.7+143.4+141.1}{4}}{2}$

$= \dfrac{128.7 + 132.4}{2}$

$= 130.6$ cents/litre

b The 5 median smoothed value is the median for the years 2017, 2018, 2019, 2020, 202, this can be read from the time series plot for Victoria (the red line), or from the table of values.
Looking at the plot, the median is the point at 2019 = 141.1 cents/litre

c i Victoria, slope= 2.08. On average, the price of fuel in Victoria is increasing by 2.08 cents/litre each year.

ii NT, slope= 2.20. On average, the price of fuel in the NT is increasing by 2.20 cents/litre each year.

d i *petrol price*$= -4071.53 + 2.08699 \times 2026 = \$156.7$ cents per litre

ii *petrol price*$= -4290.07 + 2.20128 \times 2026 = \$169.7$ cents per litre

e The difference is predicted to increase over time. The cost of petrol in the NT is already higher than the cost of petrol in Victoria, and the cost is increasing at a higher rate in the NT (on average 2.20 cents/litre each year) than it is increasing in Victoria (on average 2.08 cents/litre each year).

Finance

7 a First find $R = 1 - \dfrac{7.4}{100} = 0.926$. A rule for the value after n years is $V_n = 9500 \times 0.926^n$, so the value after five years is $9500 \times 0.926^5 = \$6468.13$

b Solving $9500 \times R^5 = 6890$ gives $R = 0.93778$, and then solving $0.93778 = 1 - \dfrac{r}{100}$ gives $r = 6.2\%$ correct to one decimal place.
This could also be solved directly by solving $9500 \times \left(1 - \dfrac{r}{100}\right)^5 = 6890$

c i We can step through $J_0 = 9500$, $J_1 = 9500 - 855 = 8645$, etc., or just using a direct rule, $J_5 = 9500 - 855 \times 5 = \$5225$ is the value after five years.

ii It is depreciating by \$855 per year from an initial value of \$9500, so we need to find 855 as a percentage of 9500. This is $\dfrac{855}{9500} \times 100 = 9\%$ flat rate.

8 a A direct rule can be used to find the balance after 5 months. $M_n = 2600 \times 1.003^n$, so $M_5 = 2600 \times 1.003^5 = 2639.23$ is the value after 5 months. Since no additional payments are made, the interest is $2639.23 - 2600 = \$39.23$

b We can set this up using financial solver with a future value of $5500.

Financial Solver

N: 18
I: BLANK (solve for this)
PV: −2600
PMT: −140
FV: 5500
PpY: 12
CpY: 12

Solution: I = 6.41949 . . . , so the annual interest rate is 6.42% correct to two decimal places.

c Calculating the growth multiplier first, we have $R = 1 + \frac{r}{100 \times p} = 1 + \frac{3}{100 \times 12} = 1.0025$

Combined with the initial value of $2600 and the fact we are adding $140 to the account each month, this gives us the recurrence relation

$$V_0 = 2600, \quad V_{n+1} = 1.0025V_n + 140$$

9 a Use $D = \frac{r}{100 \times p} \times V_0$ to get the regular payment per compounding period.

In this case $V_0 = 12\,000$ (initial amount borrowed), $r = 7.9$ and $p = 12$ (monthly) so the value is $D = \frac{7.9}{100 \times 12} \times 12\,000 = 79$, thus the regular payment is $79.

An alternative method is to use the financial solver with one period where $12 000 is the initial value borrowed and then $12 000 is the amount owed after one month (because the balance stays constant for an interest-only loan).

Financial Solver

N: 1
I: 7.9
PV: 12000
PMT: (solve for this)
FV: −12000
PpY: 12
CpY: 12

Solution: PMT = −79, so the regular payment is $79.

b Setting this up and solving for the regular payment on financial solver:

Financial Solver

N: 48
I: 7.9
PV: 12000
PMT: (solve for this)
FV: 2946.24
PpY: 12
CpY: 12

Solution: PMT = −344.78414 . . . , so the regular payment is $344.78

c First, try to find the regular payment to repay the loan in 36 months ($3 \times 12 = 36$)

Financial Solver

N: 36
I: 7.9
PV: 12000
PMT: (solve for this)
FV: 0
PpY: 12
CpY: 12

Solution: PMT = −375.48304 . . . Rounding to the nearest cent, this means the regular payment should be $375.48 for the first 35 months.
The final payment is found by seeing what happens if we pay $375.48 for all 36 months.

Financial Solver

N: 36
I: 7.9
PV: 12000
PMT: −375.48
FV: BLANK (solve for this)
PpY: 12
CpY: 12

Solution: FV = −0.12335 . . . , so that means Allesandro still owes 12 cents.
Therefore the final payment would be 375.48 + 0.12 = $375.60
(Note: because we had to round down the regular payment from 375.48304 . . . to 375.48, this also implies that the final payment will need to be higher than the regular payment in order to compensate.)

Matrices

10 **a** W is a 1×3 matrix

b $Q = P \times W = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix} \begin{bmatrix} 1500 & 2500 & 3200 \end{bmatrix} = \begin{bmatrix} 150 & 250 & 320 \\ 100 & 500 & 640 \\ 1050 & 1750 & 2240 \end{bmatrix}$

The colums correspomd to the three supermarkets and the rows correspond to the 'type of shopping'. For example, 150 people shopped at HSL but only from the meat counter.

$q_{32} = 1750$ is the number of people who shopped at Radcliffs in several sections.

c $TA = \begin{bmatrix} 150 & 250 & 600 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 60 \end{bmatrix} = \$64\ 000$

d **i** $S_1 = TS_0 = \begin{bmatrix} 0.13 & 0.8 & 0.2 \\ 0.7 & 0.1 & 0.2 \\ 0.17 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} 1500 \\ 2500 \\ 3200 \end{bmatrix} = \begin{bmatrix} 2835 \\ 1940 \\ 2425 \end{bmatrix}$

$S_1 = T^{20} S_0 = \begin{bmatrix} 0.13 & 0.8 & 0.2 \\ 0.7 & 0.1 & 0.2 \\ 0.17 & 0.1 & 0.6 \end{bmatrix}^{20} \begin{bmatrix} 1500 \\ 2500 \\ 3200 \end{bmatrix} = \begin{bmatrix} 2791.27 \\ 2577.94 \\ 1830.79 \end{bmatrix}$

ii $S_{100} = \begin{bmatrix} 2791.33 \\ 2577.88 \\ 1830.79 \end{bmatrix}$ The matrix stabilises at these larger values of n.

11 **a** $L = \begin{bmatrix} 0 & 0.9 & 0.7 \\ 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$

b $S_0 = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$

a $S_1 = \begin{bmatrix} 90 \\ 0 \\ 80 \end{bmatrix}, S_2 = \begin{bmatrix} 56 \\ 81 \\ 0 \end{bmatrix}, S_3 = \begin{bmatrix} 73 \\ 50 \\ 65 \end{bmatrix}, S_4 = \begin{bmatrix} 91 \\ 66 \\ 40 \end{bmatrix}$;

number of cases is increasing and spreading through the three stages

b $S_{40} = \begin{bmatrix} 5309 \\ 4258 \\ 3036 \end{bmatrix}, S_{41} = \begin{bmatrix} 5957 \\ 4778 \\ 3406 \end{bmatrix},$ 1.122

c Not sufficient to eradicate disease; growth rate after 40 weeks is approximately 1.027

d Sufficient to eradicate disease; growth rate after 40 weeks is approximately 0.94

12 **a** $S_1 = \begin{bmatrix} 29.6 \\ 14.4 \\ 14.4 \\ 21.6 \\ 6.4 \\ 0.0 \end{bmatrix}, S_{40} = \begin{bmatrix} 96.11 \\ 56.97 \\ 48.88 \\ 42.69 \\ 33.15 \\ 19.30 \end{bmatrix}, S_{41} = \begin{bmatrix} 99.04 \\ 57.67 \\ 50.37 \\ 43.99 \\ 34.15 \\ 19.89 \end{bmatrix}$

c $S_0 = \begin{bmatrix} 715 \\ 416 \\ 363 \\ 317 \\ 247 \\ 144 \end{bmatrix}, S_1 = \begin{bmatrix} 735.9 \\ 429.0 \\ 374.4 \\ 326.7 \\ 253.6 \\ 148.2 \end{bmatrix}$

13 **a** Order of A is 3×1

b $C \times A = \begin{bmatrix} 34 & 42 & 60 \end{bmatrix} \begin{bmatrix} 200 \\ 150 \\ 80 \end{bmatrix} = \begin{bmatrix} 17\,500 \end{bmatrix}$

Total income if all seats are sold.

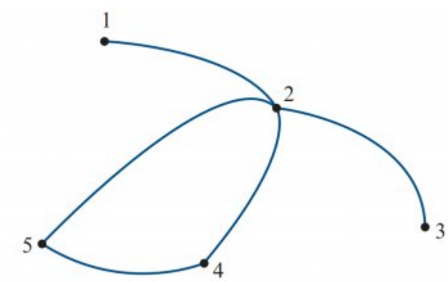
c $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 200 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 80 \end{bmatrix} \begin{bmatrix} 34 \\ 42 \\ 60 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6800 \\ 6300 \\ 4800 \end{bmatrix} = \begin{bmatrix} 6800 + 4800 \end{bmatrix}$ Income from stalls and dress circle.

Networks and decision maths

14 a i Each region of the garden, labelled 1 to 5, represents a vertex in the graph. If a region borders another region, an edge must be used to connect the vertices that represent the bordering regions. There should be:

- One edge between vertices 1 and 2
- One edge between vertices 2 and 3
- One edge between vertices 2 and 4
- One edge between vertices 2 and 5
- One edge between vertices 4 and 5



ii Sum of degrees = 2× total number of edges = $2 \times 5 = 10$

b i An Eulerian circuit is only possible if all vertices have an *even* degree. Vertices D and E have an odd degree (each vertex has a degree of 3).

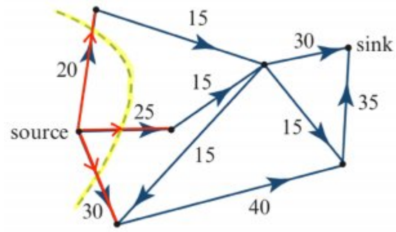
ii A Hamiltonian path must travel to all vertices, without visiting any vertex more than once. The shortest Hamiltonian path can begin at either vertex E or F.

iii An Eulerian trail must travel every edge of the graph and is possible if there are exactly zero or two *odd* vertices; vertices D and E are the only two vertices with an odd degree in this graph. Some possible Eulerian trails are:

- $E - A - B - C - D - F - E - D$
- $E - A - B - C - D - E - F - D$
- $E - F - D - E - A - B - C - D$
- $E - F - D - C - B - A - E - D$

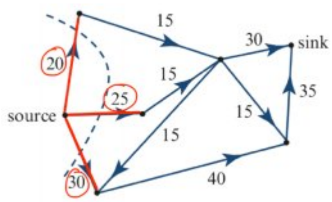
The reverse of any of the trails listed above are also possible.

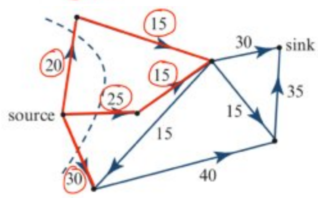
c **i** Only edges with flow from the *Source* side of the cut towards to *Sink* side of the cut contribute to the capacity of the cut:

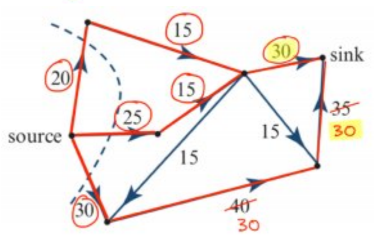


Capacity of Cut = 20 + 25 + 30 = 75

ii







Maximum flow = 30 + 30 = 60

15 **a** **i** Vertex *E* is connected to two vertices: *C* and *F*. Swimmer E had competed again 2 of these swimmers before joining the club.

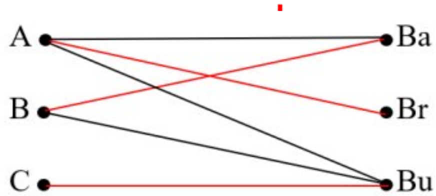
ii Vertex *C* is connected to vertices *A* and *B*. Swimmer C had competed with swimmers A and B before joining the club.

b Perform the Hungarian algorithm:

	Ba	**Br**	**Bu**	
A	72	74	66	−66
B	68	72	62	−62
C	70	76	62	−62

	Ba	**Br**	**Bu**
A	6	8	0
B	6	10	0
C	8	14	0
	−6	−8	

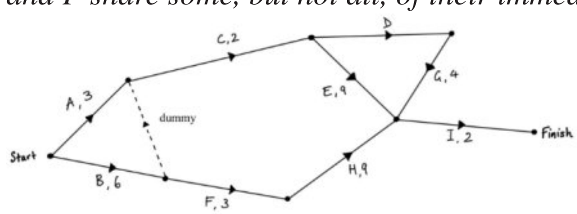
	Ba	**Br**	**Bu**
A	0	0	0
B	0	2	0
C	2	6	0



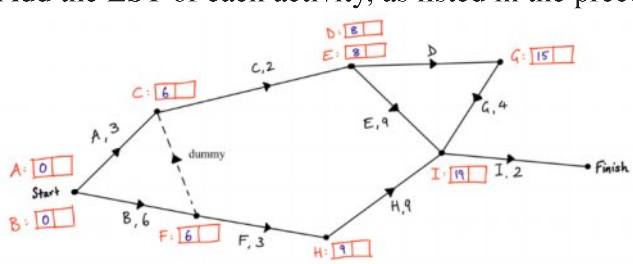
The optimal allocation should be:

- Swimmer A = Breaststroke
- Swimmer B = Backstroke
- Swimmer C = Butterfly

c **i** Draw the activity network, using the precedence table with the immediate predecessors listed. *Note: a dummy activity will be needed because activites C and F share some, but not all, of their immediate predecessors)*



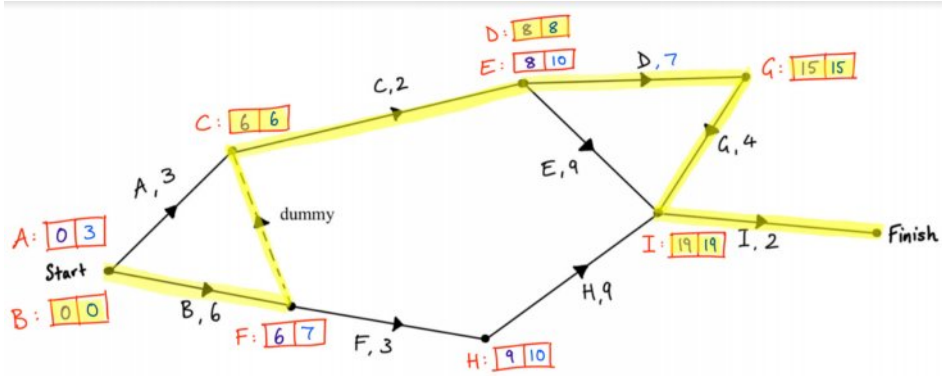
Add the EST of each activity, as listed in the precedence table.



From the diagram, it can be seen that the EST of activity *G* is determined by the sum of the EST and duration of activity *D*. Activity *D* has an EST of 8 months, therefore the duration of activity *D* must be 7 months to result in an EST of 15 months for activity *G*.

ii Completed in the previous **part i**.

iii Complete a critical analysis using forward and backward scanning on the activity network drawn earlier:



The four activities that have a float time (i.e. EST does not equal LST and they are not along the critical path) are: A, E, F, H.

iv List every path from *Start* to *Finish* and determine the original duration of each path. Only activities along the critical path should be in consideration of crashing as crashing an activity on a non-critical path that is not common to the critical path will not reduce the minimum completion time of the project. The duration of any activity may be crashed down to 1 month as this is the minimum duration for all activities in this network (cannot reduce further than 1 month).

	Duration (months)					
Path	**Original**	**If B crashed by 5**	**If C crashed by 1**	**If D crashed by 6**	**If G crashed by 3**	**If I crashed by 1**
$A-C-D-G-I$	18	18 ©	17	12	15	17
$A-C-E-I$	16	16	15	16	16	15
$B-C-D-G-I$	21 ©	16	20 ©	15	18	20 ©
$B-C-E-I$	19	14	19	19	19	18
$B-F-H-I$	20	15	20 ©	20	20 ©	19

The greatest reduction in completion time occurs when activity B is crashed by 5 months. The minimum time that the project can be completed when only one activity is crashed is 18 months.

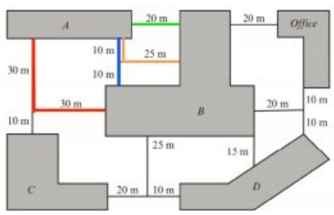
16 a A Hamiltonian cycle must travel to all vertices, start and end at the same vertex and not travel to any vertex more than once (except the starting vertex). Two possibilities are: $A - B - D - E - C - A$ and the reverse.

b An Eulerian trail must travel along every edge without travelling along any edge more than once. It can start and finish at different vertices. An Eulerian trail is possible if zero or only two vertices have an *odd* degree; vertices B and D have are the only two vertices with an *odd* degree. The Eulerian trail must begin at one of the odd vertices and end at the other. Some possible options are: $B - A - C - E - C - B - D - E - D$, $B - C - A - D - E - C - E - D$,

$B - D - E - C - B - A - C - E - D$ and the reverse of each.

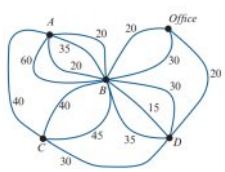
c An Eulerian trail travels along all edges, therefore sum the weight of all edges to determine the total distance travelled.
$2.4 + 3 + 3 + 3.2 + 4.2 + 1.6 + 1.8 + 2.1 = 21.3$ km
If walking speed is 3 km/hr, the time taken to travel along all edges in the graph is
$= \dfrac{21.3}{3} = 7.1$ hours = 7 hours and 6 minutes.
If the theme park closes at 5 p.m. then the latest the family can visit all rides is 7 hours and 6 minutes earlier than 5 p.m., which is 9:54 a.m.

17 a i



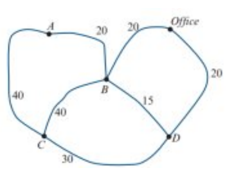
There are 4 different ways a student can walk directly from building A to building B.

ii



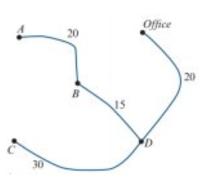
iii The buildings immediately adjacent to building C are represented as the vertices connected to vertex C via one edge in the graph drawn in the previous part:
A, B, D

b i



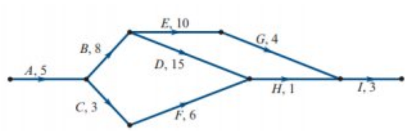
ii Total distance = 40 + 40 + 30 + 15 + 20 + 20 + 20 = 185
Total cost = 240×185 = $44 400

c i

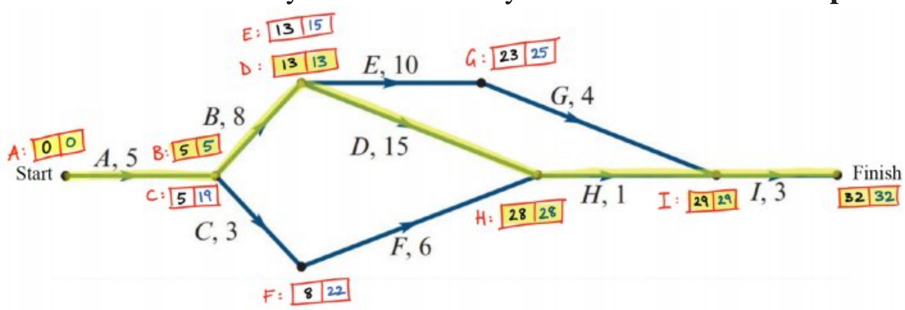


ii Total distance = 30 + 15 + 20 + 20 = 85
Total cost = 240 × 85 = \$20 400

18 a



b Perform a critical analysis of the activity network constructed in **part a**.



The shortest time Anthony can expect to create his robot is 32 weeks.

c **i**

Activity	Duration	EST	LST	Float
A	5	0	0	0
B	8	5	5	0
C	3	5	19	14
D	15	13	13	0
E	10	13	15	2
F	6	8	22	14
G	4	23	25	2
H	1	28	28	0
I	3	29	29	0

ii The critical path encompasses all activities with a float time of zero weeks:
$A - B - D - H - I$.

d **i** Activity A = Research robot design and control. Reducing the duration of activity A from 5 weeks to 3 weeks = Crash activity A by 2 weeks.

Path	**Duration (weeks)**	**Decrease A by 2**
$A - B - E - G - I$	30	28
$A - B - D - H - I$	32 ©	30 ©
$A - C - F - H - I$	18	16

The minimum completion time would decrease from 32 weeks to 30 weeks.

ii Activity F = Construct and program the remote control. Increasing the duration of activity F from 6 weeks to 10 weeks = Increase the duration of activity F by 4 weeks.

Path	Duration (weeks)	Increase F by 4
A – B – E – G – I	30	30
A – B – D – H – I	32 ©	32 ©
A – C – F – H – I	18	22

The minimum completion time would remain the same at 32 weeks.

iii Activity *D* = Construct and assemble the robot. Increasing the duration of activity *D* from 15 weeks to 20 weeks = Increase the duration of activity *D* by 5 weeks.

Path	Duration (weeks)	Increase D by 5
A – B – E – G – I	30	30
A – B – D – H – I	32 ©	37 ©
A – C – F – H – I	18	18

The minimum completion time would increase from 32 weeks to 37 weeks.