MATHSQUEST 12

FURTHER MATHEMATICS

VCE UNITS 3 AND 4

MATHSQUEST 12
FURTHER MATHEMATICS

FIFTH EDITION | VCE UNITS 3 AND 4

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Introduction

At Jacaranda, we are deeply committed to the ideal that learning brings life-changing benefits to all students. By continuing to provide resources for Mathematics of exceptional and proven quality, we ensure that all VCE students have the best opportunity to excel and to realise their full potential.

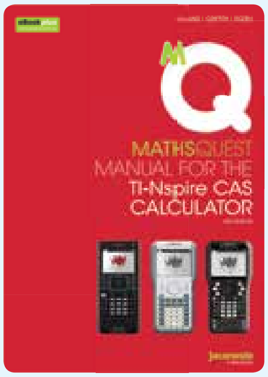
Maths Quest 12 Further Mathematics VCE Units 3 and 4 Fifth edition comprehensively covers the requirements of the revised Study Design 2016–2018.

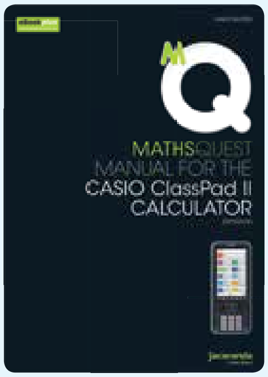
Features of the new *Maths Quest* series

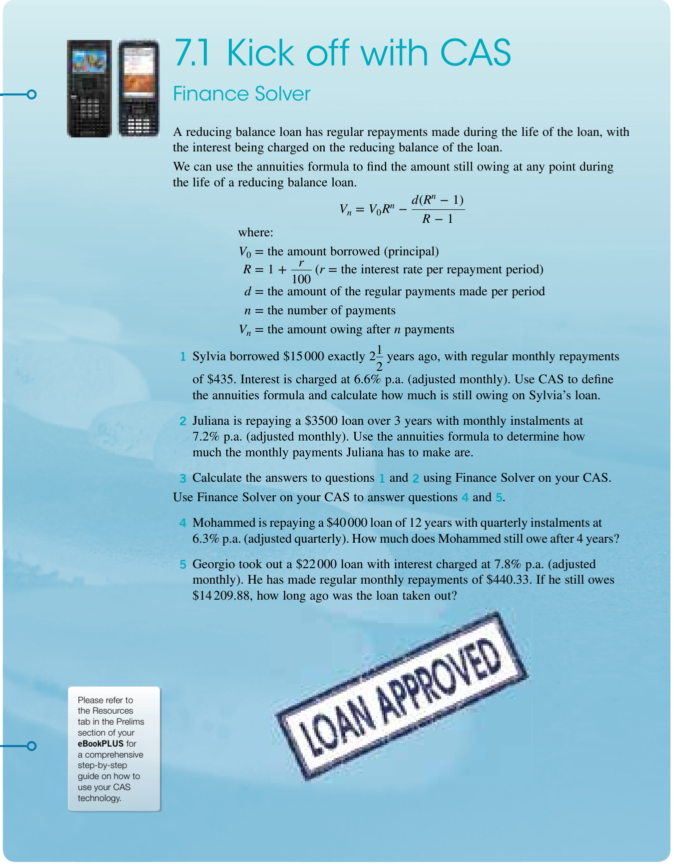
CAS technology

Each topic opens with an engaging **Kick off with CAS** activity designed to stimulate students' interest and curiosity and to highlight the important applications of CAS technology in developing deep understanding of the mathematical concepts presented.

For up-to-date, step-by-step instructions on how to use CAS technology, we have provided the *Manual for the TI-Nspire CAS calculator* and the *Manual for the Casio ClassPad II* in the Prelims section of the eBook.







7.1 Kick off with CAS

Finance Solver

A reducing balance loan has regular repayments made during the life of the loan, with the interest being charged on the reducing balance of the loan.

We can use the annuities formula to find the amount still owing at any point during the life of a reducing balance loan.

$$V_n = V_0R^n - \frac{d(R^n - 1)}{R - 1}$$

where:

V_0 = the amount borrowed (principal)
$R = 1 + \frac{r}{100}$ (r = the interest rate per repayment period)
d = the amount of the regular payments made per period
n = the number of payments
V_n = the amount owing after n payments

1 Sylvia borrowed \$15 000 exactly $2\frac{1}{2}$ years ago, with regular monthly repayments of \$435. Interest is charged at 6.6% p.a. (adjusted monthly). Use CAS to define the annuities formula and calculate how much is still owing on Sylvia's loan.

2 Juliana is repaying a \$3500 loan over 3 years with monthly instalments at 7.2% p.a. (adjusted monthly). Use the annuities formula to determine how much the monthly payments Juliana has to make are.

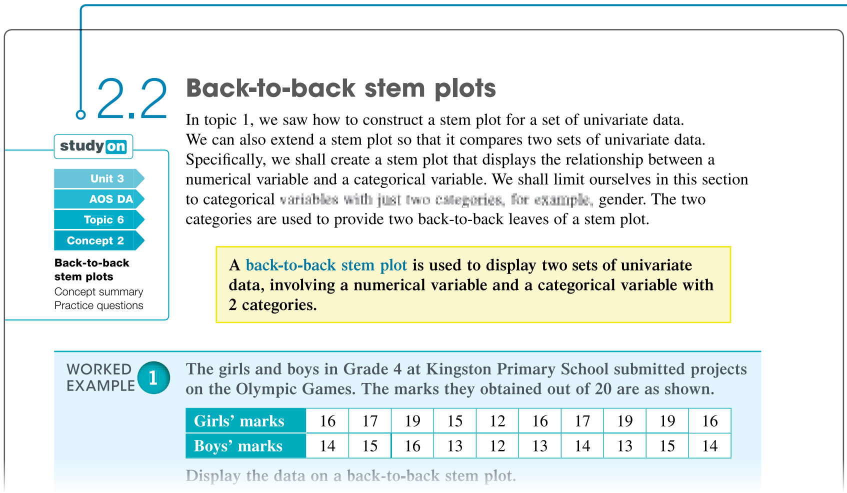
3 Calculate the answers to questions 1 and 2 using Finance Solver on your CAS.

Use Finance Solver on your CAS to answer questions 4 and 5.

4 Mohammed is repaying a \$40 000 loan of 12 years with quarterly instalments at 6.3% p.a. (adjusted quarterly). How much does Mohammed still owe after 4 years?

5 Georgio took out a \$22 000 loan with interest charged at 7.8% p.a. (adjusted monthly). He has made regular monthly repayments of \$440.33. If he still owes \$14 209.88, how long ago was the loan taken out?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.



2.2 **Back-to-back stem plots**

In topic 1, we saw how to construct a stem plot for a set of univariate data. We can also extend a stem plot so that it compares two sets of univariate data. Specifically, we shall create a stem plot that displays the relationship between a numerical variable and a categorical variable. We shall limit ourselves in this section to categorical variables with just two categories, for example, gender. The two categories are used to provide two back-to-back leaves of a stem plot.

Back-to-back stem plots
Concept summary
Practice questions

A back-to-back stem plot is used to display two sets of univariate data, involving a numerical variable and a categorical variable with 2 categories.

WORKED EXAMPLE 1

The girls and boys in Grade 4 at Kingston Primary School submitted projects on the Olympic Games. The marks they obtained out of 20 are as shown.

Girls' marks	16	17	19	15	12	16	17	19	19	16
Boys' marks	14	15	16	13	12	13	14	13	15	14

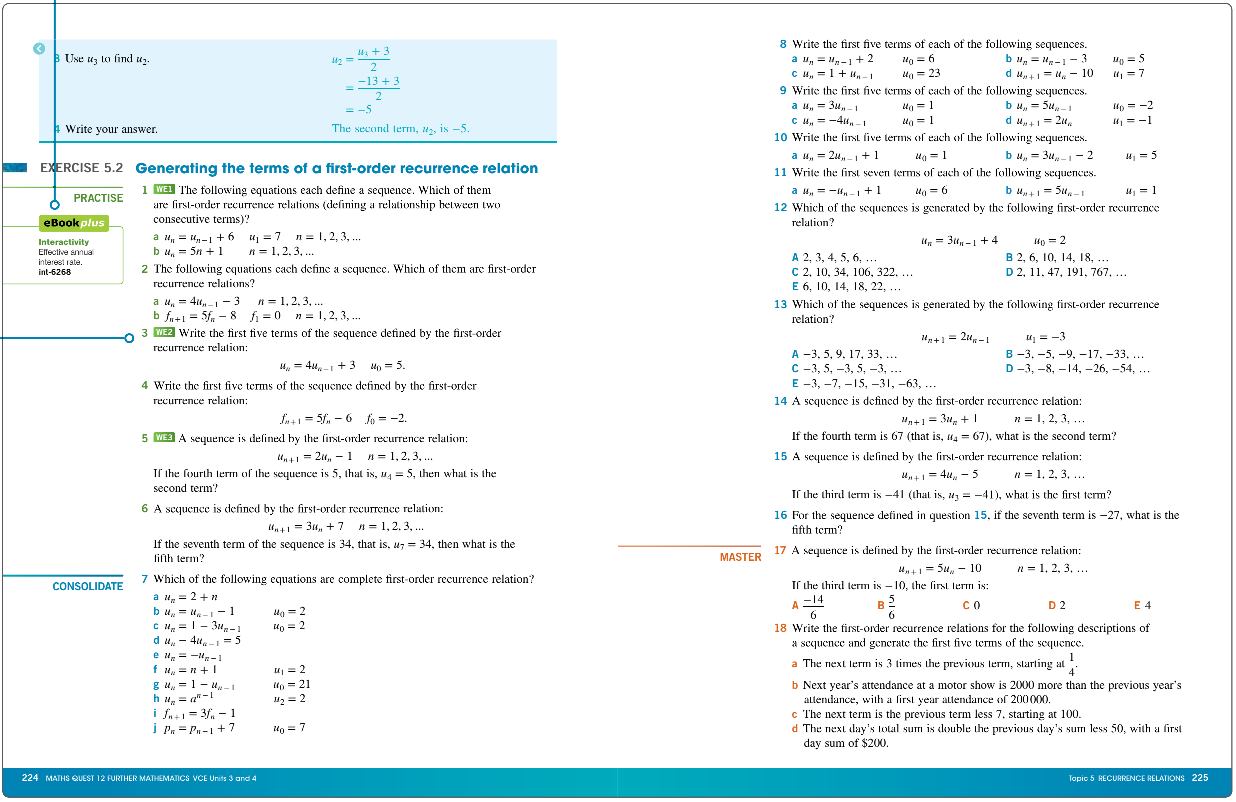
Display the data on a back-to-back stem plot.

studyON links

Link to **studyON**, an interactive and highly visual study, revision and exam practice tool for instant feedback and on-demand progress reports.

Interactivities

Many **NEW** interactivities in the resources tab of the eBookPLUS bring difficult concepts to life to engage and excite and to consolidate understanding.



3 Use u_3 to find u_2.

$$u_2 = \frac{u_3 + 3}{2} = \frac{-13 + 3}{2} = -5$$

4 Write your answer.

The second term, u_2, is −5.

EXERCISE 5.2 Generating the terms of a first-order recurrence relation

PRACTISE

eBook*plus*

Interactivity
Effective annual interest rate.
int-6268

1 **WE1** The following equations each define a sequence. Which of them are first-order recurrence relations (defining a relationship between two consecutive terms)?

a $u_n = u_{n-1} + 6 \quad u_1 = 7 \quad n = 1, 2, 3, ...$

b $u_n = 5n + 1 \quad n = 1, 2, 3, ...$

2 The following equations each define a sequence. Which of them are first-order recurrence relations?

a $u_n = 4u_{n-1} - 3 \quad n = 1, 2, 3, ...$

b $f_{n+1} = 5f_n - 8 \quad f_1 = 0 \quad n = 1, 2, 3, ...$

3 **WE2** Write the first five terms of the sequence defined by the first-order recurrence relation:

$$u_n = 4u_{n-1} + 3 \quad u_0 = 5.$$

4 Write the first five terms of the sequence defined by the first-order recurrence relation:

$$f_{n+1} = 5f_n - 6 \quad f_0 = -2.$$

5 **WE3** A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 2u_n - 1 \quad n = 1, 2, 3, ...$$

If the fourth term of the sequence is 5, that is, $u_4 = 5$, then what is the second term?

6 A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 3u_n + 7 \quad n = 1, 2, 3, ...$$

If the seventh term of the sequence is 34, that is, $u_7 = 34$, then what is the fifth term?

CONSOLIDATE

7 Which of the following equations are complete first-order recurrence relation?

a $u_n = 2 + n$

b $u_n = u_{n-1} - 1 \quad u_0 = 2$

c $u_n = 1 - 3u_{n-1} \quad u_0 = 2$

d $u_n - 4u_{n-1} = 5$

e $u_n = -u_{n-1}$

f $u_n = n + 1 \quad u_1 = 2$

g $u_n = 1 - u_{n-1} \quad u_0 = 21$

h $u_n = a^{n-1} \quad u_2 = 2$

i $f_{n+1} = 3f_n - 1$

j $p_n = p_{n-1} + 7 \quad u_0 = 7$

8 Write the first five terms of each of the following sequences.

a $u_n = u_{n-1} + 2 \quad u_0 = 6$

b $u_n = u_{n-1} - 3 \quad u_0 = 5$

c $u_n = 1 + u_{n-1} \quad u_0 = 23$

d $u_{n+1} = u_n - 10 \quad u_1 = 7$

9 Write the first five terms of each of the following sequences.

a $u_n = 3u_{n-1} \quad u_0 = 1$

b $u_n = 5u_{n-1} \quad u_0 = -2$

c $u_n = -4u_{n-1} \quad u_0 = 1$

d $u_{n+1} = 2u_n \quad u_1 = -1$

10 Write the first five terms of each of the following sequences.

a $u_n = 2u_{n-1} + 1 \quad u_0 = 1$

b $u_n = 3u_{n-1} - 2 \quad u_1 = 5$

11 Write the first seven terms of each of the following sequences.

a $u_n = -u_{n-1} + 1 \quad u_0 = 6$

b $u_{n+1} = 5u_{n-1} \quad u_1 = 1$

12 Which of the sequences is generated by the following first-order recurrence relation?

$$u_n = 3u_{n-1} + 4 \quad u_0 = 2$$

A 2, 3, 4, 5, 6, …

B 2, 6, 10, 14, 18, …

C 2, 10, 34, 106, 322, …

D 2, 11, 47, 191, 767, …

E 6, 10, 14, 18, 22, …

13 Which of the sequences is generated by the following first-order recurrence relation?

$$u_{n+1} = 2u_{n-1} \quad u_1 = -3$$

A −3, 5, 9, 17, 33, …

B −3, −5, −9, −17, −33, …

C −3, 5, −3, 5, −3, …

D −3, −8, −14, −26, −54, …

E −3, −7, −15, −31, −63, …

14 A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 3u_n + 1 \quad n = 1, 2, 3, \ldots$$

If the fourth term is 67 (that is, $u_4 = 67$), what is the second term?

15 A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 4u_n - 5 \quad n = 1, 2, 3, \ldots$$

If the third term is −41 (that is, $u_3 = -41$), what is the first term?

16 For the sequence defined in question 15, if the seventh term is −27, what is the fifth term?

MASTER

17 A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 5u_n - 10 \quad n = 1, 2, 3, \ldots$$

If the third term is −10, the first term is:

A $\frac{-14}{6}$ B $\frac{5}{6}$ C 0 D 2 E 4

18 Write the first-order recurrence relations for the following descriptions of a sequence and generate the first five terms of the sequence.

a The next term is 3 times the previous term, starting at $\frac{1}{4}$.

b Next year's attendance at a motor show is 2000 more than the previous year's attendance, with a first year attendance of 200 000.

c The next term is the previous term less 7, starting at 100.

d The next day's total sum is double the previous day's sum less 50, with a first day sum of $200.

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Topic 5 RECURRENCE RELATIONS 225

Graded questions

A wide variety of questions at **Practise**, **Consolidate** and **Master** levels allow students to build, apply and extend their knowledge independently and progressively.

Review

Each topic concludes with a customisable **Review**, available in the resources tab of the **eBookPLUS**, giving students the opportunity to revise key concepts covered throughout the topic. A variety of typical question types is available including short-answer, multiple-choice and extended response.

eBook*plus* **ONLINE ONLY** 1.12 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

Summary

A comprehensive and fully customisable topic summary is available in the resources tab of the e**BookPLUS**, enabling students to add study notes and key information relevant to their personal study needs.

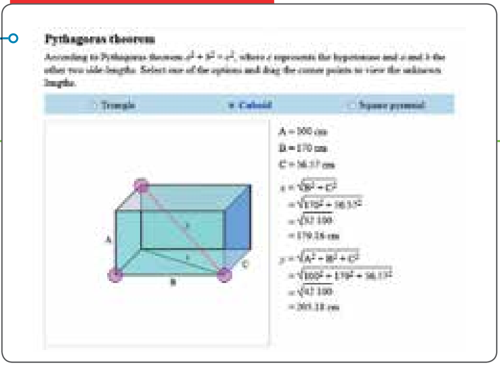
eBook*plus*

The **eBookPLUS** is available for students and teachers and contains:

- the full text online in HTML format, including PDFs of all topics
- the *Manual for the TI-Nspire CAS calculator* for step-by-step instructions
- the *Manual for the Casio ClassPad II calculator* for step-by-step instructions
- **interactivities** to bring concepts to life
- topic reviews in a customisable format
- topic summaries in a customisable format
- links to **studyON**.

Interactivities bring concepts to life

Interactivity



Pythagoras theorem

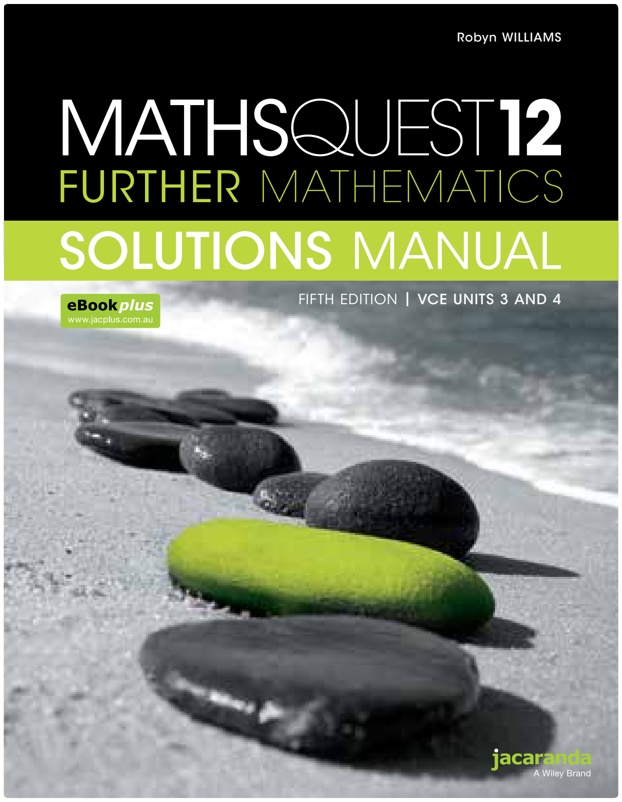
According to Pythagoras theorem $a^2 + b^2 = c^2$, where c represents the hypotenuse and a and b the other two side lengths. Select one of the options and drag the corner points to view the unknown lengths.

○ Triangle ● Cuboid ○ Square pyramid

eGuide*plus*

The **eGuidePLUS** is available for teachers and contains:

- the full **eBookPLUS**
- a Work Program to assist with planning and preparation
- School-assessed Coursework — Application task and Modelling and Problem-solving tasks, including fully worked solutions
- two tests per topic with fully worked solutions.



Maths Quest 12 Further Mathematics Solutions Manual VCE Units 3 and 4 Fifth edition

Available to students and teachers to purchase separately, the Solutions Manual provides fully worked solutions to every question in the corresponding student text. The Solutions Manual is designed to encourage student independence and to model best practice. Teachers will benefit by saving preparation and correction time.

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studyON features:

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- **Sit past VCAA exams** (Units 3 & 4) or **topic tests** (Units 1 & 2) in exam-like situations.
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- **Contact** John Wiley & Sons Australia, Ltd.
  Email: support@jacplus.com.au
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Minimum requirements

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com**.

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1

Univariate data

1.1 Kick off with CAS

Exploring histograms with CAS

Histograms can be used to display numerical data sets. They show the shape and distribution of a data set, and can be used to gather information about the data set, such as the range of the data set and the value of the median (middle) data point.

1 The following data set details the age of all of the guests at a wedding.

Age	Frequency
0–	4
10–	6
20–	18
30–	27
40–	14
50–	12
60–	6
70–	3
80–	2

Use CAS to draw a histogram displaying this data set.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

2 Comment on the shape of the histogram. What does this tell you about the distribution of this data set?

3 a Use CAS to display the mean of the data set on your histogram.
  b What is the value of the mean?

4 a Use CAS to display the median of the data set on your histogram.
  b What is the value of the median?

1.2 Types of data

Univariate data are data that contain one variable. That is, the information deals with only one quantity that changes. Therefore, the number of cars sold by a car salesman during one week is an example of *univariate* data. Sets of data that contain two variables are called **bivariate data** and those that contain more than two variables are called *multivariate* data.

Data can be classified as either numerical or categorical. The methods we use to display data depend on the type of information we are dealing with.

studyon

Unit 3
AOS DA
Topic 1
Concept 2

Defining univariate data
Concept summary
Practice questions

studyon

Unit 3
AOS DA
Topic 1
Concept 1

Classification of data
Concept summary
Practice questions

Types of data

Numerical data is data that has been assigned a numeric value. Numerical data can be:

- **discrete** — data that can be counted but that can have only a particular value, for example the number of pieces of fruit in a bowl
- **continuous** — data that is not restricted to any particular value, for example the temperature outside, which is measured on a continuous scale.

Categorical data is data split into two or more categories. Categorical data can be:

- **nominal** — data that can be arranged into categories but not ordered, for example arranging shoes by colour or athletes by gender
- **ordinal** — data that can be arranged into categories that have an order, for example levels of education from high school to post-graduate degrees.

Discrete and continuous data

Data are said to be *discrete* when a variable can take only certain fixed values. For example, if we counted the number of children per household in a particular suburb, the data obtained would always be whole numbers starting from zero. A value in between, such as 2.5, would clearly not be possible. If objects can be counted, then the data are discrete.

Continuous data are obtained when a variable takes any value between two values. If the heights of students in a school were obtained, then the data could consist of any values between the smallest and largest heights. The values recorded would be restricted only by the precision of the measuring instrument. If variables can be measured, then the data are continuous.

WORKED EXAMPLE 1

Which of the following is *not* numerical data?

A Maths test results

B Ages

C AFL football teams

D Heights of students in a class

E Lengths of bacterium

THINK	**WRITE**
1 To be numerical data, it is to be measurable or countable.	
2 Look for the data that does not fit the criteria.	A: Measurable B: Countable C: Names, so not measurable or countable D: Measurable E: Measurable
3 Answer the question.	The data that is not numerical is AFL football teams. The correct option is C.

WORKED EXAMPLE 2

Which of the following is *not* discrete data?

A **Number of students older than 17.5 years old**

B **Number of girls in a class**

C **Number of questions correct in a multiple choice test**

D **Number of students above 180 cm in a class**

E **Height of the tallest student in a class**

THINK	**WRITE**
1 To be discrete data, it is to be a whole number (countable).	
2 Look for the data that does not fit the criteria.	A: Whole number (countable) B: Whole number (countable) C: Whole number (countable) D: Whole number (countable) E: May not be a whole number (measurable)
3 Answer the question.	The data that is not discrete is the height of the tallest student in a class. The correct option is E.

EXERCISE 1.2 Types of data

PRACTISE

1 **WE1** Which of the following is *not* numerical data?

A Number of students in a class
B The number of supporters at an AFL match
C The amount of rainfall in a day
D Finishing positions in the Melbourne Cup
E The number of coconuts on a palm tree

2 Which of the following is *not* categorical data?

A Preferred political party
B Gender
C Hair colour
D Salaries
E Religion

3 **WE2** Which of the following is *not* discrete data?

A Number of players in a netball team
B Number of goals scored in a football match
C The average temperature in March
D The number of Melbourne Storm members
E The number of twins in Year 12

4 Which of the following is *not* continuous data?

A The weight of a person
B The number of shots missed in a basketball game
C The height of a sunflower in a garden
D The length of a cricket pitch
E The time taken to run 100 m

CONSOLIDATE

5 Write whether each of the following represents numerical or categorical data.

a The heights, in centimetres, of a group of children
b The diameters, in millimetres, of a collection of ball bearings
c The numbers of visitors to an exhibit each day
d The modes of transport that students in Year 12 take to school
e The 10 most-watched television programs in a week
f The occupations of a group of 30-year-olds

6 Which of the following represent categorical data?

a The numbers of subjects offered to VCE students at various schools
b Life expectancies
c Species of fish
d Blood groups
e Years of birth
f Countries of birth
g Tax brackets

7 For each set of numerical data identified in question **5** above, state whether the data are discrete or continuous.

8 An example of a numerical variable is:

A attitude to 4-yearly elections (for or against)
B year level of students
C the total attendance at Carlton football matches
D position in a queue at the pie stall
E television channel numbers shown on a dial

9 The weight of each truck-load of woodchips delivered to the wharf during a one-month period was recorded. This is an example of:

A categorical and discrete data
B discrete data
C continuous and numerical data
D continuous and categorical data
E numerical and discrete data

10 When reading the menu at the local Chinese restaurant, you notice that the dishes are divided into sections. The sections are labelled chicken, beef, duck, vegetarian and seafood. What type of data is this?

11 NASA collects data on the distance to other stars in the universe. The distance is measured in light years. What type of data is being collected?

A Discrete
B Continuous
C Nominal
D Ordinal
E Bivariate

12 The number of blue, red, yellow and purple flowers in an award winning display is counted. What type of data is being collected?

A Nominal **B** Ordinal **C** Discrete
D Continuous **E** Bivariate

13 Students in a performing arts class watch a piece of modern dance and are then asked to rate the quality of the dance as poor, average, above average or excellent. What type of data is being collected?

14 Given the set of data: 12, 6, 21, 15, 8, 2, describe what type of numerical data this data set is.

MASTER

15 If a tennis tournament seeds the players to organise the draw, what type of categorical data is this?

16 The height of the players in the basketball team is measured. The data collected is best described as:

A categorical and nominal **B** categorical and numerical
C numerical and discrete **D** numerical and continuous
E none of the above

1.3 Stem plots

studyon
Unit 3
AOS DA
Topic 2
Concept 4
Stem plots
Concept summary
Practice questions

eBookplus
Interactivity
Stem plots
int-6242

eBookplus
Interactivity
Create stem plots
int-6495

A *stem-and-leaf plot*, or **stem plot** for short, is a way of ordering and displaying a set of data, with the advantage that all of the raw data is kept. Since all the individual values of the data are being listed, it is only suitable for smaller data sets (up to about 50 observations).

The following stem plot shows the ages of people attending an advanced computer class.

Stem	Leaf
1	6
2	2 2 3
3	0 2 4 6
4	2 3 6 7
5	3 7
6	1

Key: $2|2 = 22$ years old

The ages of the members of the class are 16, 22, 22, 23, 30, 32, 34, 36, 42, 43, 46, 47, 53, 57 and 61.

A stem plot is constructed by splitting the numerals of a record into two parts — the stem, which in this case is the first digit, and the leaf, which is always the last digit.

With your stem-and-leaf plot it is important to include a key so it is clear what the data values represent.

In cases where there are numerous leaves attached to one stem (meaning that the data is heavily concentrated in one area), the stem can be subdivided. Stems are commonly subdivided into halves or fifths. By splitting the stems, we get a clearer picture about the data variation.

WORKED EXAMPLE 3

The number of cars sold in a week at a large car dealership over a 20-week period is given as shown.

16	12	8	7	26	32	15	51	29	45
19	11	6	15	32	18	43	31	23	23

Construct a stem plot to display the number of cars sold in a week at the dealership.

THINK

1 In this example the observations are one- or two-digit numbers and so the stems will be the digits referring to the 'tens', and the leaves will be the digits referring to the units.

Work out the lowest and highest numbers in the data in order to determine what the stems will be.

WRITE

Lowest number = 6
Highest number = 51
Use stems from 0 to 5.

2 Before we construct an ordered stem plot, construct an unordered stem plot by listing the leaf digits in the order they appear in the data.

Stem	Leaf
0	8 7 6
1	6 2 5 9 1 5 8
2	6 9 3 3
3	2 2 1
4	5 3
5	1

3 Now rearrange the leaf digits in numerical order to create an ordered stem plot.

Include a key so that the data can be understood by anyone viewing the stem plot.

Stem	Leaf
0	6 7 8
1	1 2 5 5 6 8 9
2	3 3 6 9
3	1 2 2
4	3 5
5	1

Key: 2|3 = 23 cars

WORKED EXAMPLE 4

The masses (in kilograms) of the members of an Under-17 football squad are given as shown.

70.3	65.1	72.9	66.9	68.6	69.6	70.8
72.4	74.1	75.3	75.6	69.7	66.2	71.2
68.3	69.7	71.3	68.3	70.5	72.4	71.8

Display the data in a stem plot.

<table>
<tr><th>THINK</th><th>WRITE</th></tr>
<tr><td>1 In this case the observations contain 3 digits. The last digit always becomes the leaf and so in this case the digit referring to the tenths becomes the leaf and the two preceding digits become the stem.
Work out the lowest and highest numbers in the data in order to determine what the stems will be.</td><td>Lowest number = 65.1
Highest number = 75.6
Use stems from 65 to 75.</td></tr>
<tr><td>2 Construct an unordered stem plot. Note that the decimal points are omitted since we are aiming to present a quick visual summary of data.</td><td><table>
<tr><th>Stem</th><th>Leaf</th></tr>
<tr><td>65</td><td>1</td></tr>
<tr><td>66</td><td>9 2</td></tr>
<tr><td>67</td><td></td></tr>
<tr><td>68</td><td>6 3 3</td></tr>
<tr><td>69</td><td>6 7 7</td></tr>
<tr><td>70</td><td>3 8 5</td></tr>
<tr><td>71</td><td>2 3 8</td></tr>
<tr><td>72</td><td>9 4 4</td></tr>
<tr><td>73</td><td></td></tr>
<tr><td>74</td><td>1</td></tr>
<tr><td>75</td><td>3 6</td></tr>
</table></td></tr>
<tr><td>3 Construct an ordered stem plot. Provide a key.</td><td><table>
<tr><th>Stem</th><th>Leaf</th></tr>
<tr><td>65</td><td>1</td></tr>
<tr><td>66</td><td>2 9</td></tr>
<tr><td>67</td><td></td></tr>
<tr><td>68</td><td>3 3 6</td></tr>
<tr><td>69</td><td>6 7 7</td></tr>
<tr><td>70</td><td>3 5 8</td></tr>
<tr><td>71</td><td>2 3 8</td></tr>
<tr><td>72</td><td>4 4 9</td></tr>
<tr><td>73</td><td></td></tr>
<tr><td>74</td><td>1</td></tr>
<tr><td>75</td><td>3 6</td></tr>
</table>Key: 74|1 = 74.1 kg</td></tr>
</table>

WORKED EXAMPLE 5

A set of golf scores for a group of professional golfers trialling a new 18-hole golf course is shown on the following stem plot.

Stem	Leaf
6	1 6 6 7 8 9 9 9
7	0 1 1 2 2 3 7

Key: 6|1 = 61

Produce another stem plot for these data by splitting the stems into:

a halves

b fifths.

THINK

a By splitting the stem 6 into halves, any leaf digits in the range 0–4 appear next to the 6, and any leaf digits in the range 5–9 appear next to the 6*. Likewise for the stem 7.

WRITE

a

Stem	Leaf
6	1
6*	6 6 7 8 9 9 9
7	0 1 1 2 2 3
7*	7

Key: 6|1 = 61

b Alternatively, to split the stems into fifths, each stem would appear 5 times. Any 0s or 1s are recorded next to the first 6. Any 2s or 3s are recorded next to the second 6. Any 4s or 5s are recorded next to the third 6. Any 6s or 7s are recorded next to the fourth 6 and, finally, any 8s or 9s are recorded next to the fifth 6.

This process would be repeated for those observations with a stem of 7.

b

Stem	Leaf
6	1
6	
6	
6	6 6 7
6	8 9 9 9
7	0 1 1
7	2 2 3
7	
7	7
7	

Key: 6|1 = 61

EXERCISE 1.3 Stem plots

PRACTISE

1 **WE3** The number of iPads sold in a month from a department store over 16 weeks is shown.

28	31	18	48	38	25	21	16
33	42	35	39	49	30	29	28

Construct a stem plot to display the number of iPads sold over the 16 weeks.

2 The money (correct to the nearest dollar) earned each week by a busker over an 18-week period is shown below. Construct a stem plot for the busker's weekly earnings. What can you say about the busker's earnings?

5	19	11	27	23	35	18	42	29
31	52	43	37	41	39	45	32	36

3 **WE4** The test scores (as percentages) of a student in a Year 12 Further Maths class are shown.

88.0	86.8	92.1	89.8	92.6	90.4	98.3	94.3	87.7
94.9	98.9	92.0	90.2	97.0	90.9	98.5	92.2	90.8

Display the data in a stem plot.

4 The heights of members of a squad of basketballers are given below in metres. Construct a stem plot for these data.

1.96	1.85	2.03	2.21	2.17	1.89	1.99	1.87
1.95	2.03	2.09	2.05	2.01	1.96	1.97	1.91

5 **WE5** LeBron James's scores for his last 16 games are shown in the following stem plot.

Stem	Leaf
2	0 2 2 5 6 8 8
3	3 3 3 7 7 8 9 9 9

Key: 2|0 = 20

Produce another stem plot for these data by splitting the stems into:

a halves **b** fifths.

6 The data below give the head circumference (correct to the nearest cm) of 16 four-year-old girls.

48	49	47	52	51	50	49	48
50	50	53	52	43	47	49	50

Construct a stem plot for head circumference, using:

a the stems 4 and 5 **b** the stems 4 and 5 split into halves

c the stems 4 and 5 split into fifths.

CONSOLIDATE

7 For each of the following, write down all the pieces of data shown on the stem plot. The key used for each stem plot is 3|2 = 32.

a

Stem	Leaf
0	1 2
0*	5 8
1	2 3 3
1*	6 6 7
2	1 3 4
2*	5 5 6 7
3	0 2

b

Stem	Leaf
1	0 1
2	3 3
3	0 5 9
4	1 2 7
5	5
6	2

c

Stem	Leaf
10	1 2
11	5 8
12	2 3 3
13	6 6 7
14	1 3 4
15	5 5 6 7

d

Stem	Leaf
5	0 1
5	3 3
5	4 5 5
5	6 6 7
5	9

e

Stem	Leaf
0	1 4
0*	5 8
1	0 2
1*	6 9 9
2	1 1
2*	5 9

8 The ages of those attending an embroidery class are given below. Construct a stem plot for these data and draw a conclusion from it.

39	68	51	57	63	51	37	42
63	49	52	61	58	59	49	53

9 The observations shown on the stem plot at right are:

A 4 10 27 28 29 31 34 36 41
B 14 10 27 28 29 29 31 34 36 41 41
C 4 22 27 28 29 29 30 31 34 36 41 41
D 14 22 27 28 29 30 30 31 34 36 41 41
E 4 2 27 28 29 29 30 31 34 36 41

Stem	Leaf
0	4
1	
2	2 7 8 9 9
3	0 1 4 6
4	1 1

Key: 2|5 = 25

10 The ages of the mothers of a class of children attending an inner-city kindergarten are given below. Construct a stem plot for these data. Based on your display, comment on the statement 'Parents of kindergarten children are young' (less than 30 years old).

32	30	19	28	25	29	32
28	29	34	32	35	39	30
37	33	29	35	38	33	

11 The number of hit outs made by each of the principal ruckmen in each of the AFL teams for Round 11 is recorded below. Construct a stem plot to display these data. Which teams had the three highest scoring ruckmen?

Team	Number of hit outs	Team	Number of hit outs
Collingwood	19	Adelaide	32
Bulldogs	41	St Kilda	34
Kangaroos	29	Essendon	31
Port Adelaide	24	Carlton	26
Geelong	21	West Coast	29
Sydney	31	Fremantle	22
Melbourne	40	Hawthorn	33
Brisbane	25	Richmond	28

12 The 2015 weekly median rental price for a 2-bedroom unit in a number of Melbourne suburbs is given in the following table. Construct a stem plot for these data and comment on it.

Suburb	Weekly rental ($)	Suburb	Weekly rental ($)
Alphington	400	Moonee Ponds	373
Box Hill	365	Newport	380
Brunswick	410	North Melbourne	421
Burwood	390	Northcote	430
Clayton	350	Preston	351
Essendon	350	St Kilda	450
Hampton	430	Surrey Hills	380
Ivanhoe	395	Williamstown	330
Kensington	406	Windsor	423
Malvern	415	Yarraville	390

13 A random sample of 20 screws is taken and the length of each is recorded to the nearest millimetre below.

23	15	18	17	17	19	22
19	20	16	20	21	19	23
17	19	21	23	20	21	

Construct a stem plot for screw length using:

a the stems 1 and 2

b the stems 1 and 2 split into halves

c the stems 1 and 2 split into fifths.

Use your plots to help you comment on the screw lengths.

14 The first twenty scores that came into the clubhouse in a local golf tournament are shown. Construct a stem plot for these data.

102	98	83	92	85	99	104	112	88	91
78	87	90	94	83	93	72	100	92	88

15 Golf handicaps are designed to even up golfers on their abilities. Their handicap is subtracted from their score to create a net score. Construct a stem plot on the golfer's net scores below and comment on how well the golfers are handicapped.

76, 76, 73, 74, 69, 72, 73, 86, 73, 72,
75, 74, 77, 73, 75, 75, 71, 71, 68, 67

16 The following data represents percentages for a recent Further Maths test.

63	71	70	89	88	69	76	83	93	80	73
77	91	75	81	84	87	78	97	89	98	60

Construct a stem plot for the test percentages, using:

a the stems 6, 7, 8 and 9

b the stems 6, 7, 8 and 9 split into halves.

MASTER

17 The following data was collected from a company that compared the battery life (measured in minutes) of two different Ultrabook computers. To complete the test they ran a series of programs on the two computers and measured how long it took for the batteries to go from 100% to 0%.

Computer 1	358	376	392	345	381	405	363	380	352	391	410	366
Computer 2	348	355	361	342	355	362	353	358	340	346	357	352

a Draw a back-to-back stem plot (using the same stem) of the battery life of the two Ultrabook computers.

b Use the stem plot to compare and comment on the battery life of the two Ultrabook computers.

18 The heights of 20 Year 8 and Year 10 students (to the nearest centimetre), chosen at random, are measured. The data collected is shown in the table below.

Year 8	151	162	148	153	165	157	172	168	155	164	175	161	155	160	149	155	163	171	166	150
Year 10	167	164	172	158	169	159	174	177	165	156	154	160	178	176	182	152	167	185	173	178

a Draw a back-to-back stem plot of the data.

b Comment on what the stem plot tells you about the heights of Year 8 and Year 10 students.

1.4 Dot plots, frequency tables and histograms, and bar charts

study on

Unit 3
AOS DA
Topic 2
Concept 3

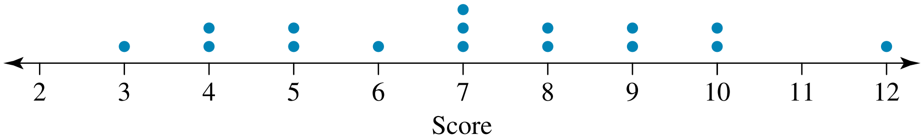
Dot plots
Concept summary
Practice questions

Dot plots, frequency **histograms** and **bar charts** display data in graphical form.

Dot plots

In picture graphs, a single picture represents each data value. Similarly, in dot plots, a single dot represents each data value. Dot plots are used to display discrete data where values are not spread out very much. They are also used to display categorical data.

When representing discrete data, dot plots have a scaled horizontal axis and each data value is indicated by a dot above this scale. The end result is a set of vertical 'lines' of evenly-spaced dots.



WORKED EXAMPLE 6

The number of hours per week spent on art by 18 students is given as shown.

4	0	3	1	3	4	2	2	3
4	1	3	2	5	3	2	1	0

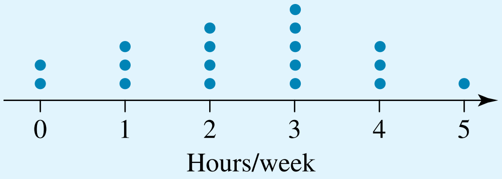
Display the data as a dot plot.

THINK

1 Determine the lowest and highest scores and then draw a suitable scale.

2 Represent each score by a dot on the scale.

DRAW



study on

Unit 3
AOS DA
Topic 2
Concept 5

Frequency histograms
Concept summary
Practice questions

Frequency tables and histograms

A histogram is a useful way of displaying large data sets (say, over 50 observations). The vertical axis on the histogram displays the frequency and the horizontal axis displays class intervals of the variable (for example, height or income).

The vertical bars in a histogram are adjacent with no gaps between them, as we generally consider the numerical data scale along the horizontal axis as continuous. Note, however, that histograms can also represent discrete data. It is common practice to leave a small gap before the first bar of a histogram.

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Interactivity
Create a histogram
int-6494

When data are given in raw form — that is, just as a list of figures in no particular order — it is helpful to first construct a **frequency table** before constructing a histogram.

WORKED EXAMPLE 7

The data below show the distribution of masses (in kilograms) of 60 students in Year 7 at Northwood Secondary College. Construct a frequency histogram to display the data more clearly.

45.7	45.8	45.9	48.2	48.3	48.4	34.2	52.4	52.3	51.8	45.7	56.8	56.3	60.2	44.2
53.8	43.5	57.2	38.7	48.5	49.6	56.9	43.8	58.3	52.4	54.3	48.6	53.7	58.7	57.6
45.7	39.8	42.5	42.9	59.2	53.2	48.2	36.2	47.2	46.7	58.7	53.1	52.1	54.3	51.3
51.9	54.6	58.7	58.7	39.7	43.1	56.2	43.0	56.3	62.3	46.3	52.4	61.2	48.2	58.3

THINK

1 First construct a frequency table. The lowest data value is 34.2 and the highest is 62.3. Divide the data into class intervals. If we started the first class interval at, say, 30 kg and ended the last class interval at 65 kg, we would have a range of 35. If each interval was 5 kg, we would then have 7 intervals which is a reasonable number of class intervals.

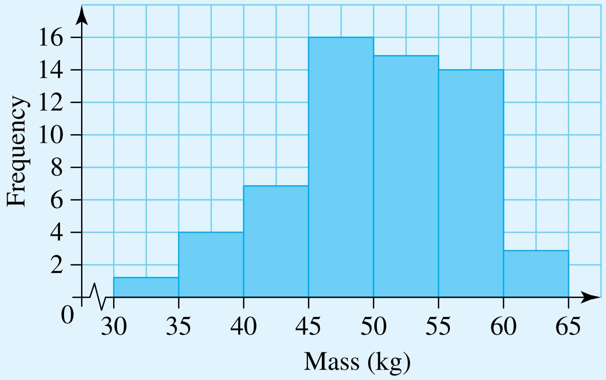
While there are no set rules about how many intervals there should be, somewhere between about 5 and 15 class intervals is usual. Complete a tally column using one mark for each value in the appropriate interval. Add up the tally marks and write them in the frequency column.

2 Check that the frequency column totals 60. The data are in a much clearer form now.

3 A histogram can be constructed.

WRITE/DRAW

Class interval	Tally	Frequency
30–	|	1
35–	||||	4
40–	𝍸 ||	7
45–	𝍸 𝍸 𝍸 |	16
50–	𝍸 𝍸 𝍸	15
55–	𝍸 𝍸 ||||	14
60–	|||	3
	Total	60



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Interactivity
Dot plots, frequency tables and histograms, and bar charts
int-6243

When constructing a histogram to represent continuous data, as in Worked example 7, the bars will sit between two values on the horizontal axis which represent the class intervals. When dealing with discrete data the bars should appear above the middle of the value they're representing.

WORKED EXAMPLE 8

The marks out of 20 received by 30 students for a book-review assignment are given in the frequency table.

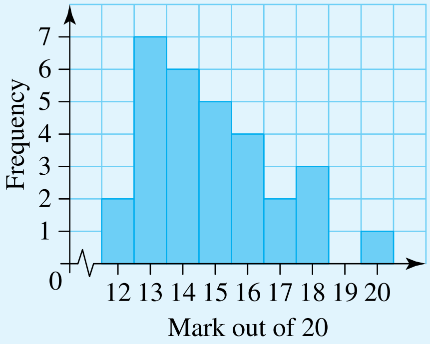
Mark	12	13	14	15	16	17	18	19	20
Frequency	2	7	6	5	4	2	3	0	1

Display these data on a histogram.

THINK

In this case we are dealing with integer values (discrete data). Since the horizontal axis should show a class interval, we extend the base of each of the columns on the histogram halfway either side of each score.

DRAW



studyon

Unit 3
AOS DA
Topic 2
Concept 1

Bar charts
Concept summary
Practice questions

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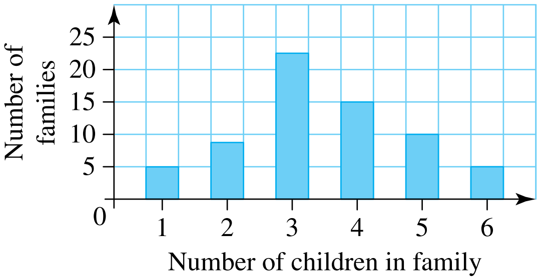
Interactivity
Create a bar chart
int-6493

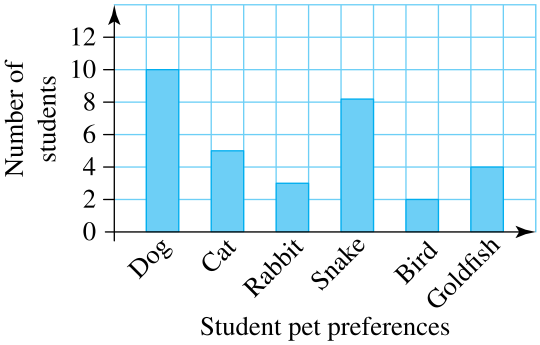
Bar charts

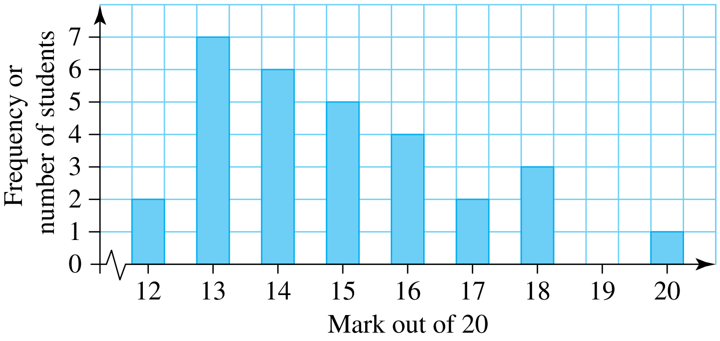
A *bar chart* consists of bars of equal width *separated* by small, equal spaces and may be arranged either *horizontally* or *vertically*. Bar charts are often used to display categorical data.

In bar charts, the frequency is graphed against a variable as shown in both figures.

The variable may or may not be numerical. However, if it is, the variable should represent discrete data because the scale is broken by the gaps between the bars.







The bar chart shown represents the data presented in Worked example 8. It could also have been drawn with horizontal bars (rows).

study on

Unit 3
AOS DA
Topic 2
Concept 2

Segmented graphs
Concept summary
Practice questions

Segmented bar charts

A **segmented bar chart** is a single bar which is used to represent all the data being studied. It is divided into segments, each segment representing a particular group of the data. Generally, the information is presented as percentages and so the total bar length represents 100% of the data.

WORKED EXAMPLE 9

The table shown represents fatal road accidents in Australia. Construct a segmented bar chart to represent this data.

	Accidents involving fatalities								
Year	NSW	Vic.	Qld	SA	WA	Tas.	NT	ACT	Aust.
2008	376	278	293	87	189	38	67	12	**1340**

Source: Australian Bureau of Statistics 2010, *Year Book Australia 2009–10*, cat. no. 1301.0. ABS. Canberra, table 24.20, p. 638.

THINK

1 To draw a segmented bar chart the data needs to be converted to percentages.

WRITE/DRAW

State	Number of accidents	Percentage
NSW	376	376 ÷ 1340 × 100% = 28.1%
Vic.	278	278 ÷ 1340 × 100% = 20.7%
Qld	293	293 ÷ 1340 × 100% = 21.9%
SA	87	87 ÷ 1340 × 100% = 6.5%
WA	189	189 ÷ 1340 × 100% = 14.1%
Tas.	38	38 ÷ 1340 × 100% = 2.8%
NT	67	67 ÷ 1340 × 100% = 5.0%
ACT	14	12 ÷ 1340 × 100% = 0.9%

2 To draw the segmented bar chart to scale decide on its overall length, let's say 100 mm.

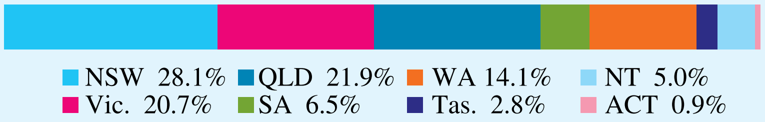
Measure a line 100 mm in length.

3 Therefore NSW = 28.1%, represented by 28.1 mm. Vic = 20.7%, represented by 20.7 mm and so on.

Measure off each segment and check it adds to the set 100 mm.

4 Draw the answer and colour code it to represent each of the states and territories.

The segmented bar chart is drawn to scale. An appropriate scale would be constructed by drawing the total bar 100 mm long, so that 1 mm represents 1%. That is, accidents in NSW w ould be represented by a segment of 28.1 mm, those in Victoria by a segment of 20.7 mm and so on. Each segment is then labelled directly, or a key may be used.



Using a log (base 10) scale

Sometimes a data set will contain data points that vary so much in size that plotting them using a traditional scale becomes very difficult. For example, if we are studying the population of different cities in Australia we might end up with the following data points:

studyon

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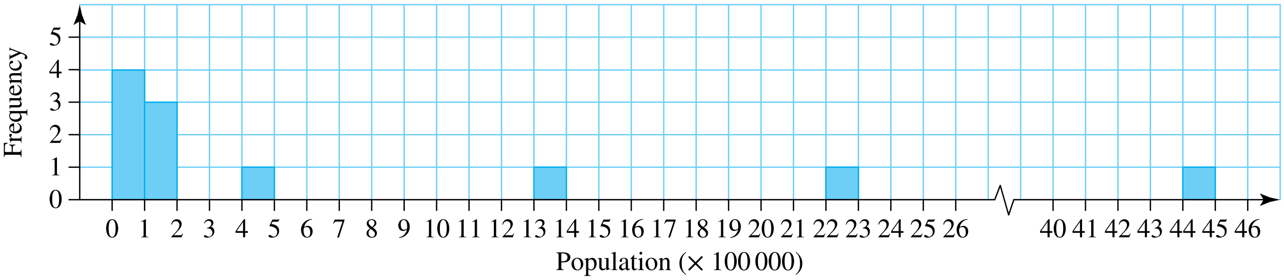
City	Population
Adelaide	1 304 631
Ballarat	98 543
Brisbane	2 274 460
Cairns	146 778
Darwin	140 400
Geelong	184 182
Launceston	86 393
Melbourne	4 440 328
Newcastle	430 755
Shepparton	49 079
Wagga Wagga	55 364

A histogram splitting the data into class intervals of 100 000 would then appear as follows:



A way to overcome this is to write the numbers in **logarithmic (log) form**. The log of a number is the power of 10 which creates this number.

$$\log(10) = \log(10^1) = 1$$
$$\log(100) = \log(10^2) = 2$$
$$\log(1000) = \log(10^3) = 3$$
$$\vdots$$
$$\log(10^n) = n$$

Not all logarithmic values are integers, so use the log key on CAS to determine exact logarithmic values.

For example, from our previous example showing the population of different Australian cities:

$\log(4\,440\,328) = 6.65$ (correct to 2 decimal places)

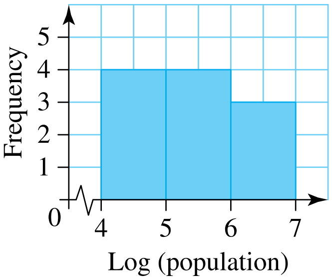
$\log(184\,182) = 5.27$ (correct to 2 decimal places)

$\log(49\,079) = 4.69$ (correct to 2 decimal places)

and so on…

We can then group our data using class intervals based on log values (from 4 to 7) to come up with the following frequency table and histogram.

Population	Log (population)	Frequency
10 000–	4–5	4
100 000–	5–6	4
1 000 000–	6–7	3



WORKED EXAMPLE 10

The following table shows the average weights of 10 different adult mammals.

Mammal	Weight (kg)
African elephant	4800
Black rhinoceros	1100
Blue whale	136 000
Giraffe	800
Gorilla	140
Humpback whale	30 000
Lynx	23
Orang-utan	64
Polar bear	475
Tasmanian devil	7

Display the data in a histogram using a log base 10 scale.

THINK

1 Using CAS, calculate the logarithmic values of all of the weights, e.g.:

log(4800) = 3.68 (correct to 2 decimal places)

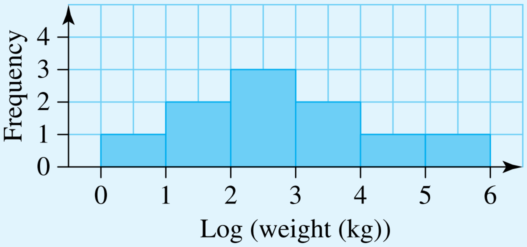
2 Group the logarithmic weights into class intervals and create a frequency table for the groupings.

WRITE/DRAW

Weight	Log (weight (kg))
4800	3.68
1100	3.04
136 000	5.13
800	2.90
140	2.15
30 000	4.48
23	1.36
64	1.81
475	2.68
7	0.85

Log (weight (kg))	Frequency
0–1	1
1–2	2
2–3	3
3–4	2
4–5	1
5–6	1

3 Construct a histogram of the data set.



Interpreting log (base 10) values

If we are given values in logarithmic form, by raising 10 to the power of the logarithmic number we can determine the conventional number.

For example, the number 3467 in log (base 10) form is 3.54, and $10^{3.54} = 3467$.

We can use this fact to compare values in log (base 10) form, as shown in the following worked example.

WORKED EXAMPLE 11

The Richter Scale measures the magnitude of earthquakes using a log (base 10) scale.

How many times stronger is an earthquake of magnitude 7.4 than one of magnitude 5.2? Give your answer correct to the nearest whole number.

THINK	WRITE
1 Calculate the difference between the magnitude of the two earthquakes.	$7.4 - 5.2 = 2.2$
2 Raise 10 to the power of the difference in magnitudes.	$10^{2.2} = 158.49$ (correct to 2 decimal places)
3 Express the answer in words.	The earthquake of magnitude 7.4 is 158 times stronger than the earthquake of magnitude 5.2.

EXERCISE 1.4 Dot plots, frequency tables and histograms, and bar charts

PRACTISE

1 **WE6** The number of questions completed for maths homework each night by 16 students is shown below.

5	6	5	9	10	10	6	8
10	9	8	5	7	8	7	9

Display the data as a dot plot.

2 The data below represent the number of hours each week that 40 teenagers spent on household chores. Display these data on a bar chart and a dot plot.

2 5 2 0 8 7 8 5 1 0 2 1 8 0 4 2 2 9 8 5
7 5 4 2 1 2 9 8 1 2 8 5 8 10 0 3 4 5 2 8

3 **WE7** The data shows the distribution of heights (in cm) of 40 students in Year 12. Construct a frequency histogram to display the data more clearly.

167	172	184	180	178	166	154	150	164	161
187	159	182	177	172	163	179	181	170	176
177	162	172	184	188	179	189	192	164	160
166	169	163	185	178	183	190	170	168	159

4 Construct a frequency table for each of the following sets of data.

a 4.3 4.5 4.7 4.9 5.1 5.3 5.5 5.6 5.2 3.6 2.5 4.3 2.5 3.7 4.5 6.3 1.3

b 11 13 15 15 16 18 20 21 22 21 18 19 20 16 18 20 16 10 23 24 25 27 28 30 35 28 27 26 29 30 31 24 28 29 20 30 32 33 29 30 31 33 34

c 0.4 0.5 0.7 0.8 0.8 0.9 1.0 1.1 1.2 1.0 1.3 0.4 0.3 0.9 0.6

Using the frequency tables above, construct a histogram by hand for each set of data.

5 **WE8** The number of fish caught by 30 anglers in a fishing competition are given in the frequency table below.

Fish	0	1	2	3	4	5	7
Frequency	4	7	4	6	5	3	1

Display these data on a histogram.

6 The number of fatal car accidents in Victoria each week is given in the frequency table for a year.

Fatalities	0	1	2	4	6	7	9	10	13
Frequency	12	3	6	10	7	6	5	2	1

Display these data on a histogram.

7 **WE9** The following table shows how many goals each of the 18 AFL team's leading goal kickers scored in the 2014 regular season. Construct a segmented bar chart to represent this data.

Club	**Goals**
Adelaide Crows	51
Brisbane Lions	33
Carlton	29
Collingwood	39
Essendon	27
Fremantle	49
Geelong Cats	62
Gold Coast Suns	46
GWS Giants	29
Hawthorn	62
Melbourne	20
North Melbourne	41
Port Adelaide	62
Richmond	58
St Kilda	49
Sydney Swans	67
West Coast Eagles	61
Western Bulldogs	37

8 Information about adult participation in sport and physical activities in 2005–06 is shown in the following table. Draw a segmented bar graph to compare the participation of all persons from various age groups. Comment on the statement, 'Only young people participate in sport and physical activities'.

Participation in sport and physical activities[(a)] — 2005–06

	Males		Females		Persons	
Age group (years)	**Number (×1000)**	**Participation rate (%)**	**Number (×1000)**	**Participation rate (%)**	**Number (×1000)**	**Participation rate (%)**
18–24	735.2	73.3	671.3	71.8	1406.4	72.6
25–34	1054.5	76.3	1033.9	74.0	2088.3	75.1
35–44	975.4	66.7	1035.9	69.1	2011.2	68.0
45–54	871.8	63.5	923.4	65.7	1795.2	64.6
55–64	670.1	60.4	716.3	64.6	1386.5	62.5
65 and over	591.0	50.8	652.9	48.2	1243.9	49.4
Total	**4898**	**64.6**	**5033.7**	**64.4**	**9931.5**	**64.5**

(a) Relates to persons aged 18 years and over who participated in sport or physical activity as a player during the 12 months prior to interview.

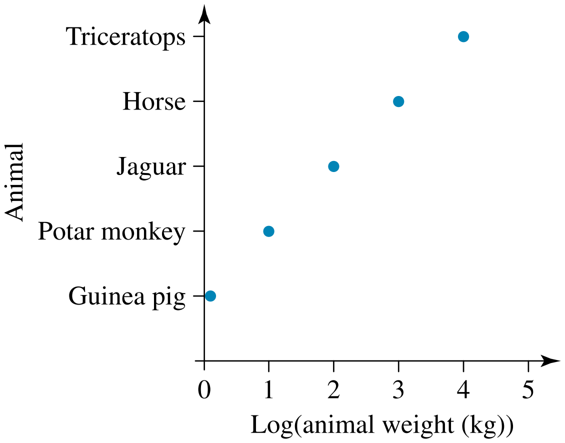
Source: Participation in Sport and Physical Activities, Australia, 2005–06 (4177.0). Viewed 10 October 2008 <http://abs.gov.au/Ausstats>

9 **WE10** The following table shows the average weights of 10 different adult mammals.

Mammal	**Weight (kg)**
Black wallaroo	18
Capybara	55
Cougar	63
Fin whale	70 000
Lion	175
Ocelot	9
Pygmy rabbit	0.4
Red deer	200
Quokka	4
Water buffalo	725

Display the data in a histogram using a log base 10 scale, using class intervals of width 1.

10 The following graph shows the weights of animals.



If a gorilla has a weight of 207 kilograms then its weight is between that of:

A Potar monkey and jaguar.

B horse and triceratops.

C guinea pig and Potar monkey.

D jaguar and horse.

E none of the above.

11 **WE11** The Richter scale measures the magnitude of earthquakes using a log (base 10) scale.

How many times stronger is an earthquake of magnitude 8.1 than one of magnitude 6.9? Give your answer correct to the nearest whole number.

12 The pH scale measures acidity using a log (base 10) scale. For each decrease in pH of 1, the acidity of a substance increases by a factor of 10.

If a liquid's pH value decreases by 0.7, by how much has the acidity of the liquid increased?

CONSOLIDATE

13 Using CAS, construct a histogram for each of the sets of data given in question **4**. Compare this histogram with the one drawn for question **4**.

14 The following table shows a variety of top speeds.

F1 racing car	370 000 m/h
V8 supercar	300 000 m/h
Cheetah	64 000 m/h
Space shuttle	28 000 000 m/h
Usain Bolt	34 000 m/h

The correct value, to 2 decimal places, for a cheetah's top speed using a log (base 10) scale would be:

A 5.57 **B** 5.47 **C** 7.45 **D** 4.81 **E** 11.07

15 The number of hours spent on homework for a group of 20 Year 12 students each week is shown.

15	15	18	21	20	21	24	24	20	18
18	21	22	20	24	20	24	21	18	15

Display the data as a dot plot.

16 The number of dogs at a RSPCA kennel each week is as shown. Construct a frequency table for these data.

7, 6, 2, 12, 7, 9, 12, 10, 5, 7, 9, 4, 5, 9, 3, 2, 10, 8, 9, 7, 9, 10, 9, 4, 3, 8, 9, 3, 7, 9

17 Using the frequency table in question **16**, construct a histogram by hand.

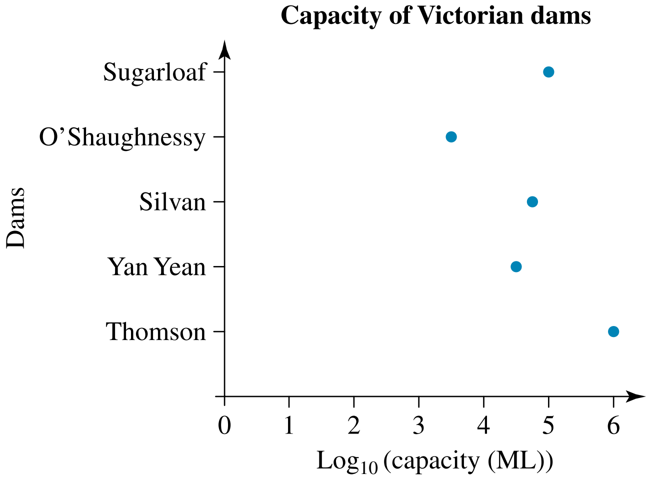
18 Using CAS construct a histogram from the data in question **16**, and compare it to the histogram in question **17**.

19 A group of students was surveyed, asking how many children were in their family. The data is shown in the table.

Number of children	1	2	3	4	5	6	9
Number of families	12	18	24	10	8	3	1

Construct a bar chart that displays the data.

20 The following graph represents the capacity of five Victorian dams.



a The capacity of Thomson Dam is closest to:
A 100 000 ML **B** 1 000 000 ML **C** 500 000 ML
D 5000 ML **E** 40 000 ML

b The capacity of Sugarloaf Dam is closest to:
A 100 000 ML **B** 1 000 000 ML **C** 500 000 ML
D 5000 ML **E** 40 000 ML

c The capacity of Silvan Dam is between which range?
A 1 and 10 ML **B** 10 and 100 ML
C 1000 and 10 000 ML **D** 10 000 and 100 000 ML
E 100 000 and 1 000 000 ML

The table shows the flow rate of world famous waterfalls. Use this table to complete questions **21** and **22**.

Victoria Falls	1 088 m^3/s
Niagara Falls	2 407 m^3/s
Celilo Falls	5 415 m^3/s
Khane Phapheng Falls	11 610 m^3/s
Boyoma Falls	17 000 m^3/s

21 The correct value, correct to 2 decimal places, to be plotted for Niagara Falls' flow rate using a log (base 10) scale would be:

A 3.38 **B** 3.04 **C** 4.06 **D** 4.23 **E** 5.10

22 The correct value, correct to 2 decimal places, to be plotted for Victoria Falls' flow rate using a log (base 10) scale would be:

A 3.38 **B** 3.04 **C** 4.06 **D** 4.23 **E** 5.10

MASTER

23 Using the information provided in the table below:

a calculate the proportion of residents who travelled in 2005 to each of the countries listed

b draw a segmented bar graph showing the major destinations of Australians travelling abroad in 2005.

Short-term resident departures by major destinations

	2004 (× 1000)	2005 (× 1000)	2006 (× 1000)	2007 (× 1000)	2008 (× 1000)
New Zealand	815.8	835.4	864.7	902.1	921.1
United States of America	376.1	426.3	440.3	479.1	492.3
United Kingdom	375.1	404.2	412.8	428.5	420.3
Indonesia	335.1	319.7	194.9	282.6	380.7
China (excluding Special Administrative Regions (SARs))	182.0	235.1	251.0	284.3	277.3
Thailand	188.2	202.7	288.0	374.4	404.1
Fiji	175.4	196.9	202.4	200.3	236.2
Singapore	159.0	188.5	210.9	221.5	217.8
Hong Kong (SAR of China)	152.6	185.7	196.3	206.5	213.1
Malaysia	144.4	159.8	168.0	181.3	191.0

Source: Australian Bureau of Statistics 2010, *Year Book Australia 2009–10*, cat. no. 1301.0, ABS, Canberra, table 23.12, p. 621.

24 The number of people who attended the Melbourne Grand Prix are shown.

Year	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
Attendance (000s)	360	359	301	301	303	287	305	298	313	323

a Construct a bar chart by hand to display the data.

b Use CAS to construct a bar chart and compare the two charts.

1.5 Describing the shape of stem plots and histograms

Symmetric distributions

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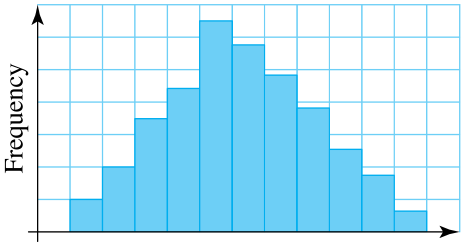
The data shown in the histogram below can be described as **symmetric**.

There is a single peak and the data trail off on both sides of this peak in roughly the same fashion.



Similarly, in the stem plot at right, the distribution of the data could be described as symmetric.

The single peak for these data occur at the stem 3. On either side of the peak, the number of observations reduces in approximately matching fashion.

Stem	Leaf
0	7
1	2 3
2	2 4 5 7 9
3	0 2 3 6 8 8
4	4 7 8 9 9
5	2 7 8
6	1 3

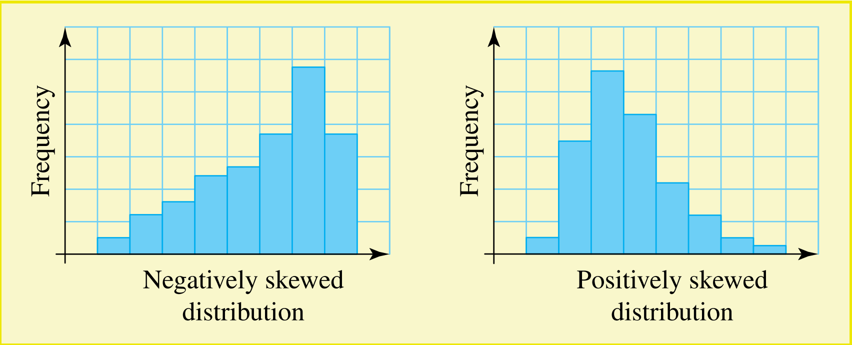
Key: 0|7 = 7

Skewed distributions

Each of the histograms shown on next page are examples of skewed distributions.

The figure on the left shows data which are **negatively skewed**. The data in this case peak to the right and trail off to the left.

The figure on the right shows **positively skewed** data. The data in this case peak to the left and trail off to the right.



Negatively skewed distribution

Positively skewed distribution

WORKED EXAMPLE 12

The ages of a group of people who were taking out their first home loan is shown below.

Stem	Leaf
1	9 9
2	1 2 4 6 7 8 8 9
3	0 1 1 2 3 4 7
4	1 3 5 6
5	2 3
6	7

Key: 1|9 = 19 years old

Describe the shape of the distribution of these data.

THINK	WRITE
Check whether the distribution is symmetric or skewed. The peak of the data occurs at the stem 2. The data trail off as the stems increase in value. This seems reasonable since most people would take out a home loan early in life to give themselves time to pay it off.	The data are positively skewed.

EXERCISE 1.5 Describing the shape of stem plots and histograms

PRACTISE

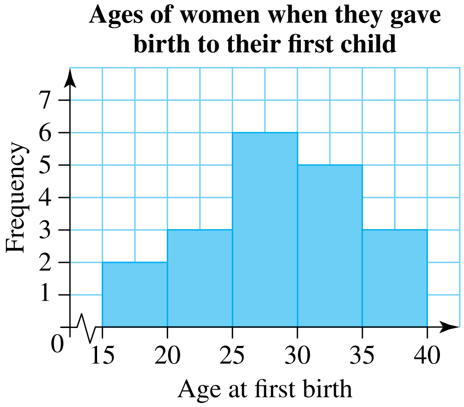
1 **WE12** The ages of a group of people when they bought their first car are shown.

Stem	Leaf
1	7 7 8 8 8 8 9 9
2	0 0 1 2 3 6 7 8 9
3	1 4 7 9
4	4 8
5	3

Key: 1|7 = 17 years old

Describe the shape of the distribution of these data.

2 The ages of women when they gave birth to their first child is shown.
Describe the shape of the distribution of the data.



CONSOLIDATE

3 For each of the following stem plots, describe the shape of the distribution of the data.

a

Stem	Leaf
0	1 3
1	2 4 7
2	3 4 4 7 8
3	2 5 7 9 9 9 9
4	1 3 6 7
5	0 4
6	4 7
7	1

Key: 1|2 = 12

b

Stem	Leaf
1	3
2	6
3	3 8
4	2 6 8 8 9
5	4 7 7 7 8 9 9
6	0 2 2 4 5

Key: 2|6 = 2.6

c

Stem	Leaf
2	3 5 5 6 7 8 9 9
3	0 2 2 3 4 6 6 7 8 8
4	2 2 4 5 6 6 6 7 9
5	0 3 3 5 6
6	2 4
7	5 9
8	2
9	7
10	

Key: 10|4 = 104

d

Stem	Leaf
1	
1*	5
2	1 4
2*	5 7 8 8 9
3	1 2 2 3 3 3 4 4
3*	5 5 5 6
4	3 4
4*	

Key: 2|4 = 24

e

Stem	Leaf
3	
3	8 9
4	0 0 1 1 1
4	2 3 3 3 3 3
4	4 5 5 5
4	6 7
4	8

Key: 4|3 = 0.43

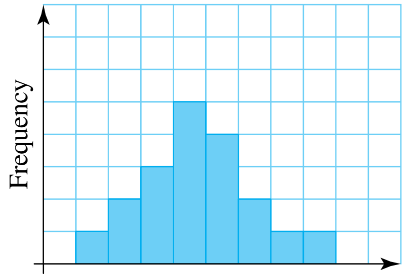
f

Stem	Leaf
60	2 5 8
61	1 3 3 6 7 8 9
62	0 1 2 4 6 7 8 8 9
63	2 2 4 5 7 8
64	3 6 7
65	4 5 8
66	3 5
67	4

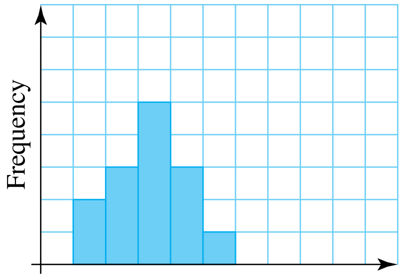
Key: 62|3 = 623

4 For each of the following histograms, describe the shape of the distribution of the data and comment on the existence of any outliers.

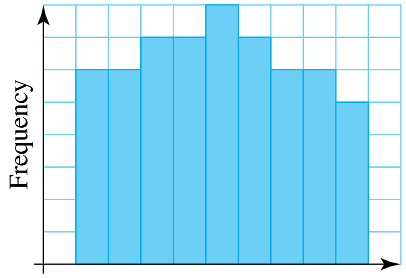
a



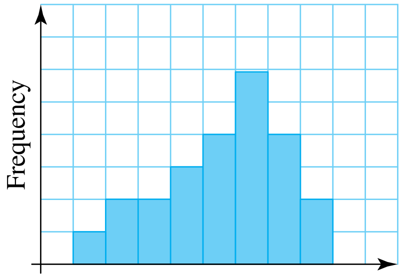
b



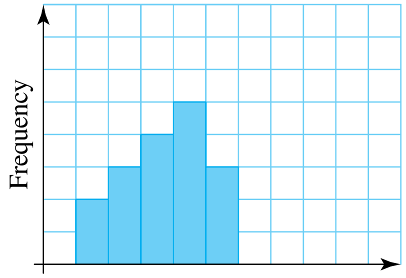
c



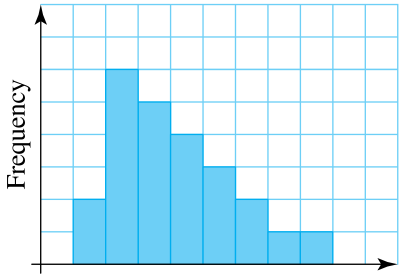
d



e



f



5 The distribution of the data shown in this stem plot could be described as:

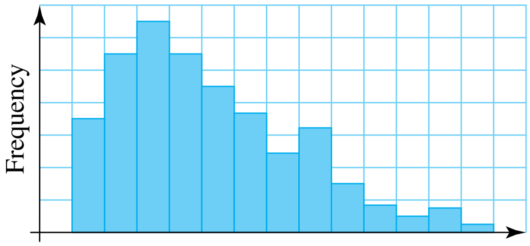
A negatively skewed
B negatively skewed and symmetric
C positively skewed
D positively skewed and symmetric
E symmetric

Stem	Leaf
0	1
0	2
0	4 4 5
0	6 6 6 7
0	8 8 8 8 9 9
1	0 0 0 1 1 1 1
1	2 2 2 3 3 3
1	4 4 5 5
1	6 7 7
1	8 9

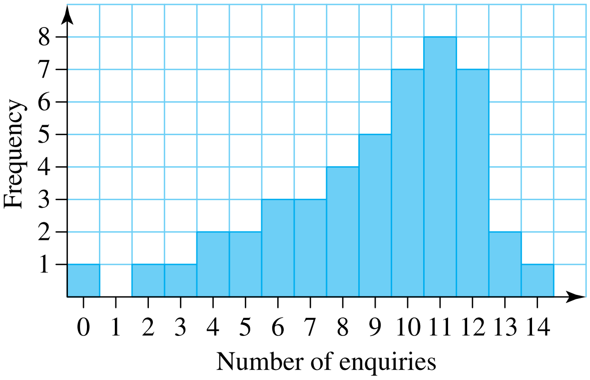
Key: 1|8 = 18

6 The distribution of the data shown in the histogram could be described as:

A negatively skewed
B negatively skewed and symmetric
C positively skewed
D positively skewed and symmetric
E symmetric



7 The average number of product enquiries per day received by a group of small businesses who advertised in the Yellow Pages telephone directory is given at right. Describe the shape of the distribution of these data.



8 The number of nights per month spent interstate by a group of flight attendants is shown in the stem plot. Describe the shape of distribution of these data and explain what this tells us about the number of nights per month spent interstate by this group of flight attendants.

Stem	Leaf
0	0 0 1 1
0	2 2 3 3 3 3 3 3 3 3
0	4 4 5 5 5 5 5
0	6 6 6 6 7
0	8 8 8 9
1	0 0 1
1	4 4
1	5 5
1	7

Key: 1|4 = 14 nights

9 The mass (correct to the nearest kilogram) of each dog at a dog obedience school is shown in the stem plot.

a Describe the shape of the distribution of these data.
b What does this information tell us about this group of dogs?

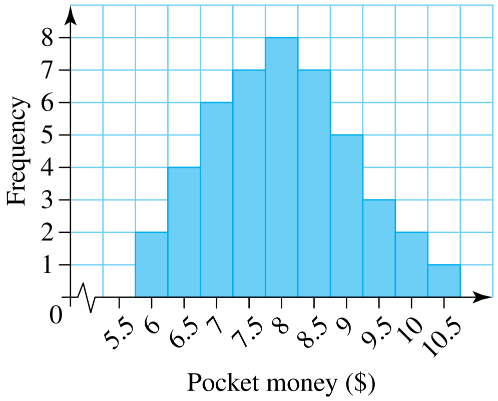
Stem	Leaf
0	4
0*	5 7 9
1	1 2 4 4
1*	5 6 6 7 8 9
2	1 2 2 3
2*	6 7

Key: 0|4 = 4 kg

10 The amount of pocket money (correct to the nearest 50 cents) received each week by students in a Grade 6 class is illustrated in the histogram.

a Describe the shape of the distribution of these data.

b What conclusions can you reach about the amount of pocket money received weekly by this group of students?



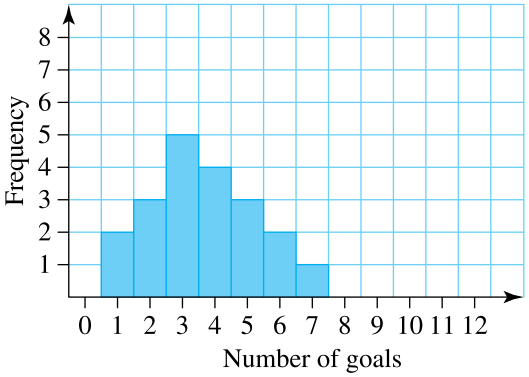
11 Statistics were collected over 3 AFL games on the number of goals kicked by forwards over 3 weeks. This is displayed in the histogram.

a Describe the shape of the histogram.

b Use the histogram to determine:

i the number of players who kicked 3 or more goals over the 3 weeks

ii the percentage of players who kicked between 2 and 6 goals inclusive over the 3 weeks.



12 The number of hours a group of students exercise each week is shown in the stem plot.

a Describe the shape of the distribution of these data.

b What does this sample data tell us about this group of students?

Stem	Leaf
0	0 0 0 0 1 1 1
0	2 2 2 3
0	4 4
0	6
0	8 8 9
1	0 0 1
1	2 2 2 3

Key: 0|1 = 1

MASTER

13 The stem plot shows the age of players in two bowling teams.

a Describe the shape of the distribution of Club A and Club B.

b What comments can you make about the make-up of Club A compared to Club B?

c How many players are over the age of 70 from:

i Club A **ii** Club B?

Club A	Stem	Club B
1	4	
	5	5 7 8 9
	6	0 2 3 5 7
6 5 4 3	7	0 1 2
8 6 5 4 3 2 1	8	8
	9	0

Key: 5|5 = 55

14 The following table shows the number of cars sold at a dealership over eight months.

Month	April	May	June	July	August	September	October	November
Cars sold	9	14	27	21	12	14	10	18

a Display the data on a bar chart.

b Describe the shape of the distribution of these data.

c What does this sample data tell us about car sales over these months?

d Explain why the most cars were sold in the month of June.

1.6 The median, the interquartile range, the range and the mode

After displaying data using a histogram or stem plot, we can make even more sense of the data by calculating what are called *summary statistics*. Summary statistics are used because they give us an idea about:

1. where the centre of the distribution is
2. how the distribution is spread out.

We will look first at four summary statistics — the **median**, the **interquartile range**, the **range** and the **mode** — which require that the data be in ordered form before they can be calculated.

The median

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The *median* is the midpoint of an ordered set of data. Half the data are less than or equal to the median.

Consider the set of data: 2 5 6 8 11 12 15. These data are in ordered form (that is, from lowest to highest). There are 7 observations. The median in this case is the middle or fourth score; that is, 8.

Consider the set of data: 1 3 5 6 7 8 8 9 10 12. These data are in ordered form also; however, in this case there is an even number of scores. The median of this set lies halfway between the 5th score (7) and the 6th score (8). So the median is 7.5.

$\left(\text{Alternatively, median} = \frac{7+8}{2} = 7.5.\right)$

When there are n records in a set of ordered data, the median can be located at the $\left(\frac{n+1}{2}\right)$ th position.

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Interactivity
The median, the interquartile range, the range and the mode **int-6244**

Checking this against our previous example, we have $n = 10$; that is, there were 10 observations in the set. The median was located at the $\left(\frac{10+1}{2}\right) = 5.5$th position; that is, halfway between the 5th and the 6th terms.

A stem plot provides a quick way of locating a median since the data in a stem plot are already ordered.

WORKED EXAMPLE 13

Consider the stem plot below which contains 22 observations. What is the median?

Stem	Leaf
2	3 3
2*	5 7 9
3	1 3 3 4 4
3*	5 8 9 9
4	0 2 2
4*	6 8 8 8 9

Key: 3|4 = 34

THINK	WRITE
1 Find the median position, where $n = 22$.	$$\begin{aligned}\text{Median} &= \left(\frac{n+1}{2}\right)\text{th position}\\ &= \left(\frac{22+1}{2}\right)\text{th position}\\ &= 11.5\text{th position}\end{aligned}$$
2 Find the 11th and 12th terms.	11th term = 35 12th term = 38
3 The median is halfway between the 11th and 12th terms.	Median = 36.5

The interquartile range

We have seen that the median divides a set of data in half. Similarly, **quartiles** divide a set of data in quarters. The symbols used to refer to these quartiles are Q_1, Q_2 and Q_3. The middle quartile, Q_2, is the median.

The interquartile range IQR = $Q_3 - Q_1$.

The interquartile range gives us the range of the middle 50% of values in a set of data.

There are four steps to locating Q_1 and Q_3.

Step 1. Write down the data in ordered form from lowest to highest.

Step 2. Locate the median; that is, locate Q_2.

Step 3. Now consider just the lower half of the set of data. Find the middle score. This score is Q_1.

Step 4. Now consider just the upper half of the set of data. Find the middle score. This score is Q_3.

The four cases given below illustrate this method.

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Case 1

Consider data containing the 6 observations: 3 6 10 12 15 21.

The data are already ordered. The median is 11.

Consider the lower half of the set, which is 3 6 10. The middle score is 6, so $Q_1 = 6$.

Consider the upper half of the set, which is 12 15 21. The middle score is 15, so $Q_3 = 15$.

Case 2

Consider a set of data containing the 7 observations: 4 9 11 13 17 23 30.

The data are already ordered. The median is 13.

Consider the lower half of the set, which is 4 9 11. The middle score is 9, so $Q_1 = 9$.

Consider the upper half of the set, which is 17 23 30. The middle score is 23, so $Q_3 = 23$.

Case 3

Consider a set of data containing the 8 observations: 1 3 9 10 15 17 21 26.

The data are already ordered. The median is 12.5.

Consider the lower half of the set, which is 1 3 9 10. The middle score is 6, so $Q_1 = 6$.

Consider the upper half of the set, which is 15 17 21 26. The middle score is 19, so $Q_3 = 19$.

Case 4

Consider a set of data containing the 9 observations: 2 7 13 14 17 19 21 25 29.

The data are already ordered. The median is 17.

Consider the lower half of the set, which is 2 7 13 14. The middle score is 10, so $Q_1 = 10$.

Consider the upper half of the set, which is 19 21 25 29. The middle score is 23, so $Q_3 = 23$.

WORKED EXAMPLE 14

The ages of the patients who attended the casualty department of an inner-suburban hospital on one particular afternoon are shown below.

14	3	27	42	19	17	73
60	62	21	23	2	5	58
33	19	81	59	25	17	69

Find the interquartile range of these data.

THINK	WRITE
1 Order the data.	2 3 5 14 17 17 19 19 21 23 25 27 33 42 58 59 60 62 69 73 81
2 Find the median.	The median is 25 since ten scores lie below it and ten lie above it.
3 Find the middle score of the lower half of the data.	For the scores 2 3 5 14 17 17 19 19 21 23, the middle score is 17. So, $Q_1 = 17$.
4 Find the middle score of the upper half of the data.	For the scores 27 33 42 58 59 60 62 69 73 81, the middle score is 59.5. So, $Q_3 = 59.5$.
5 Calculate the interquartile range.	$\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &= 59.5 - 17 \\ &= 42.5\end{aligned}$

CAS can be a fast way of locating quartiles and hence finding the value of the interquartile range.

WORKED EXAMPLE 15

Parents are often shocked at the amount of money their children spend. The data below give the amount spent (correct to the nearest whole dollar) by each child in a group that was taken on an excursion to the Royal Melbourne Show.

15	12	17	23	21	19	16
11	17	18	23	24	25	21
20	37	17	25	22	21	19

Calculate the interquartile range for these data.

THINK	WRITE
1 Enter the data into CAS to generate one-variable statistics. Copy down the values of the first and third quartiles.	$Q_1 = 17$ and $Q_3 = 23$
2 Calculate the interquartile range.	So, $\text{IQR} = Q_3 - Q_1$ $= 23 - 17$ $= 6$

The range

The *range* of a set of data is the difference between the highest and lowest values in that set.

It is usually not too difficult to locate the highest and lowest values in a set of data. Only when there is a very large number of observations might the job be made more difficult. In Worked example 15, the minimum and maximum values were 11 and 37, respectively. The range, therefore, can be calculated as follows.

$$\begin{aligned}\text{Range} &= \max_x - \min_x \\ &= 37 - 11 \\ &= 26\end{aligned}$$

While the range gives us some idea about the spread of the data, it is not very informative since it gives us no idea of how the data are distributed between the highest and lowest values.

Now let us look at another measure of the centre of a set of data: the mode.

The mode

The mode is the score that occurs most often; that is, it is the score with the highest frequency. If there is more than one score with the highest frequency, then all scores with that frequency are the modes.

The mode is a weak measure of the centre of data because it may be a value that is close to the extremes of the data. If we consider the set of data in Worked example 13, the mode is 48 since it occurs three times and hence is the score with the highest frequency. In Worked example 14 there are two modes, 17 and 19, because they equally occur most frequently.

EXERCISE 1.6 The median, the interquartile range, the range and the mode

PRACTISE

1 **WE13** The stem plot shows 30 observations. What is the median value?

Stem	Leaf
2	1 1 3 4 4 4
2*	5 5 7 8 9
3	0 0 1 3 3 3 3
3*	6 6 7 9
4	0 1 1
4*	6 7 9 9 9

Key: 2|1 = 21

2 The following data represents the number of goals scored by a netball team over the course of a 16 game season. What was the team's median number of goals for the season?

28	36	24	46	37	21	49	32
33	41	47	29	45	52	37	24

3 **WE14** From the following data find the interquartile range.

33	21	39	45	31	28	15	13	16
21	49	26	29	30	21	37	27	19
12	15	24	33	37	10	23	28	39

4 The ages of a sample of people surveyed at a concert are shown.

21	25	24	18	19	16	19	27	32	24
15	20	31	24	29	33	27	18	19	21

Find the interquartile range of these data.

5 **WE15** The data shows the amount of money spent (to the nearest dollar) at the school canteen by a group of students in a week.

3	5	7	12	15	10	8	9	21	5
7	9	13	15	7	3	4	2	11	8

Calculate the interquartile range for the data set.

6 The amount of money, in millions, changing hands through a large stocks company, in one-minute intervals, was recorded as follows.

45.8	48.9	46.4	45.7	43.8	49.1	42.7	43.1	45.3	48.6
41.9	40.0	45.9	44.7	43.9	45.1	47.1	49.7	42.9	45.1

Calculate the interquartile range for these data.

CONSOLIDATE

7 Write the median, the range and the mode of the sets of data shown in the following stem plots. The key for each stem plot is 3|4 = 34.

a

Stem	Leaf
0	7
1	2 3
2	2 4 5 7 9
3	0 2 3 6 8 8
4	4 7 8 9 9
5	2 7 8
6	1 3

b

Stem	Leaf
0	0 0 1 1
0	2 2 3 3
0	4 4 5 5 5 5 5 5 5 5
0	6 6 6 6 7
0	8 8 8 9
1	0 0 1
1	3 3
1	5 5
1	7
1	

c

Stem	Leaf
0	1
0	2
0	4 4 5
0	6 6 6 7
0	8 8 8 8 9 9
1	0 0 0 1 1 1 1
1	2 2 2 3 3 3
1	4 4 5 5
1	6 7 7
1	8 9

d

Stem	Leaf
3	1
3	
3	
3	6
3	8 9
4	0 0 1 1 1
4	2 2 3 3 3 3
4	4 5 5 5
4	6 7
4	9

e

Stem	Leaf
60	2 5 8
61	1 3 3 6 7 8 9
62	0 1 2 4 6 7 8 8 9
63	2 2 4 5 7 8
64	3 6 7
65	4 5 8
66	3 5
67	4

8 For each of the following sets of data, write the median and the range.

a 2 4 6 7 9

b 12 15 17 19 21

c 3 4 5 6 7 8 9

d 3 5 7 8 12 13 15 16

e 12 13 15 16 18 19 21 23 24 26

f 3 8 4 2 1 6 5

g 16 21 14 28 23 15 11 19 25

h 7 4 3 4 9 5 10 4 2 11

9 **a** The number of cars that used the drive-in at a McBurger restaurant during each hour, from 7.00 am until 10.00 pm on a particular day, is shown below.

14 18 8 9 12 24 25 15 18 25 24 21 25 24 14

Find the interquartile range of this set of data.

b On the same day, the number of cars stopping during each hour that the nearby Kenny’s Fried Chicken restaurant was open is shown below.

7 9 13 16 19 12 11 18 20 19 21 20 18 10 14

Find the interquartile range of these data.

c What do these values suggest about the two restaurants?

10 Write down a set of data for which $n = 5$, the median is 6 and the range is 7. Is this the only set of data with these parameters?

11 Give an example of a data set where:

a the lower quartile equals the lowest score
b the IQR is zero.

12 The quartiles for a set of data are calculated and found to be $Q_1 = 13$, $Q_2 = 18$ and $Q_3 = 25$. Which of the following statements is true?

A The interquartile range of the data is 5.
B The interquartile range of the data is 7.
C The interquartile range of the data is 12.
D The median is 12.
E The median is 19.

13 For each of the following sets of data find the median, the interquartile range, the range and the mode.

a

16	12	8	7	26	32	15	51	29	45
19	11	6	15	32	18	43	31	23	23

b

22	25	27	36	31	32	39	29	20	30
23	25	21	19	29	28	31	27	22	29

c

1.2	2.3	4.1	2.4	1.5	3.7	6.1	2.4	3.6	1.2
6.1	3.7	5.4	3.7	5.2	3.8	6.3	7.1	4.9	

14 For each set of data shown in the stem plots, find the median, the interquartile range, the range and the mode. Compare these values for both data sets.

a

Stem	Leaf
2	3 5 5 6 7 8 9 9
3	0 2 2 3 4 6 6 7 8 8
4	2 2 4 5 6 6 6 7 9
5	0 3 3 5 6
6	2 4
7	5 9
8	2
9	7
10	
11	4

Key: 4|2 = 42

b

Stem	Leaf
1	4
1*	
2	1 4
2*	5 7 8 8 9
3	1 2 2 2 4 4 4 4
3*	5 5 5 6
4	3 4
4*	

Key: 2|1 = 21

15 For the data in the stem plots shown, find the range, median, mode and interquartile range.

a

Stem	Leaf
0	1
1	1 4 7
2	3 4 6 7
3	4 6 7 8 9
4	2 3 6 6
5	2 3 5
6	7
7	3

Key: 0|1 = 1

b

Stem	Leaf
40	3 5 7
41	1 1 1 3 4 6 7 8 9
42	0 2 3 6 7 8 9
43	2 3 3 6 8
44	1 2
45	0

Key: 40|3 = 403

16 From the following set of data, find the median and mode.

4, 7, 9, 12, 15, 2, 3, 7, 9, 4, 7, 9, 2, 8, 13, 5, 3, 7, 5, 9, 7, 10

MASTER

17 The following data represents distances (in metres) thrown during a javelin-throwing competition. Use CAS to calculate the interquartile range and median.

40.3	42.8	41.0	50.3	52.2	46.1	44.5
41.6	44.3	47.4	45.1	48.8	46.1	44.5
45.3	42.9	41.1	49.0	47.5	40.8	51.1

18 The data below shows the distribution of golf scores for one day of an amateur tournament.

a Use CAS to calculate the median, interquartile range, mode and range.

b Comment on what you suggest the average handicap of the players listed should be if par for the course is 72.

111	93	103	85	81	90
101	95	84	93	101	85
87	85	93	100	86	91
93	95	93	99	95	93
92	96	93	97	93	93
97	96	92	100	95	104

1.7 Boxplots

The five number summary statistics that we looked at in the previous section ($\min_x$, Q_1, Q_2, Q_3, $\max_x$) can be illustrated very neatly in a special diagram known as a **boxplot** (or *box-and-whisker* diagram). The diagram is made up of a box with straight lines (whiskers) extending from opposite sides of the box.

A boxplot displays the minimum and maximum values of the data together with the quartiles and is drawn with a labelled scale. The length of the box is given by the interquartile range. A boxplot gives us a very clear visual display of how the data are spread out.

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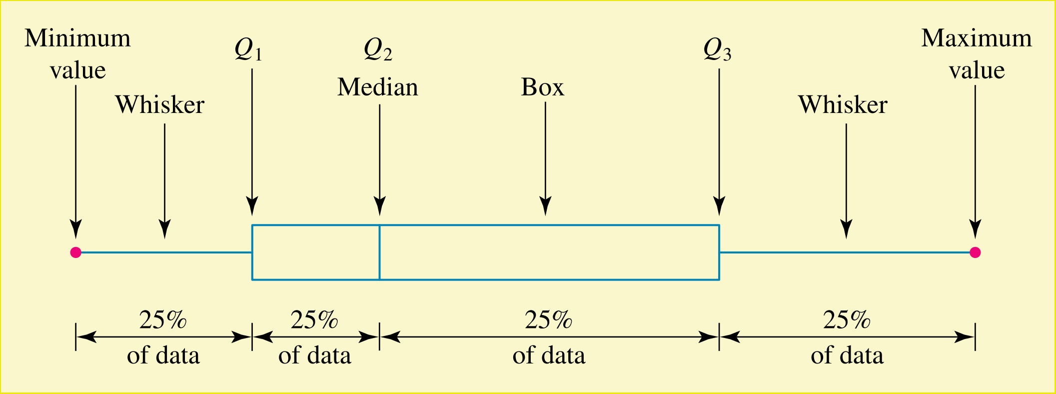
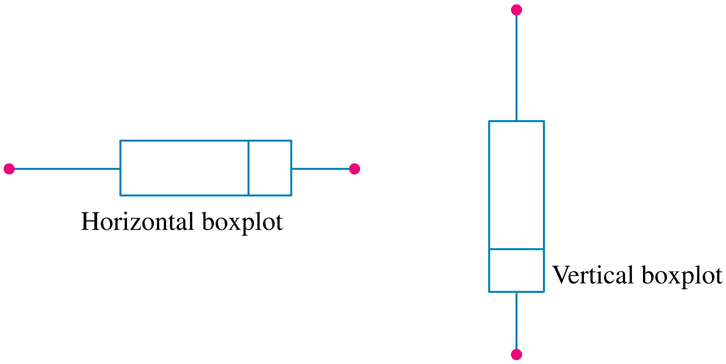
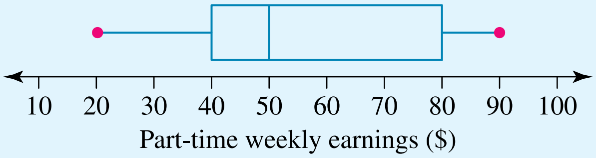


Boxplots can be drawn horizontally or vertically.



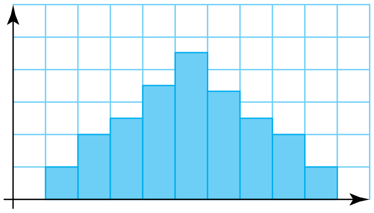
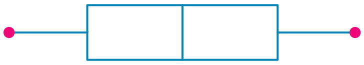
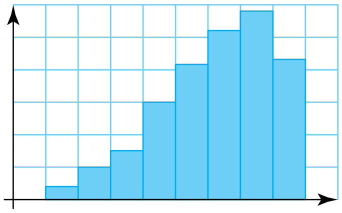
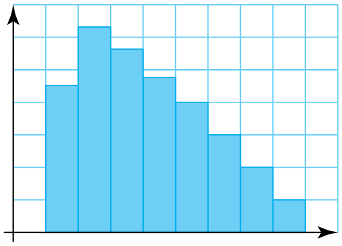
WORKED EXAMPLE 16

The boxplot at right shows the distribution of the part-time weekly earnings of a group of Year 12 students. Write down the range, the median and the interquartile range for these data.



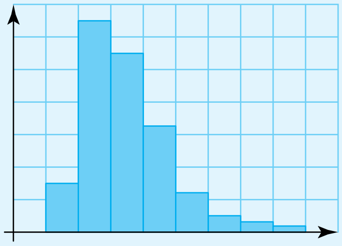
THINK	WRITE
1 Range = Maximum value − Minimum value. The minimum value is 20 and the maximum value is 90.	Range = 90 − 20 = 70
2 The median is located at the bar inside the box.	Median = 50
3 The ends of the box are at 40 and 80. IQR = $Q_3 - Q_1$	$Q_1 = 40$ and $Q_3 = 80$ IQR = 80 − 40 = 40

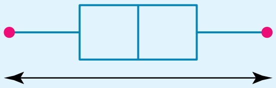
Earlier, we noted three general types of shape for histograms and stem plots: symmetric, negatively skewed and positively skewed. It is useful to compare the corresponding boxplots of distributions with such shapes.

 Symmetric histogram	 Symmetric boxplot The median in the centre of the boxplot.
 Negatively skewed histogram	 Negatively skewed boxplot The median to the right of the centre of the boxplot.
 Positively skewed histogram	 Positively skewed boxplot The median to the left of the centre of the boxplot.

WORKED EXAMPLE 17

Explain whether or not the histogram and the boxplot shown below could represent the same data.





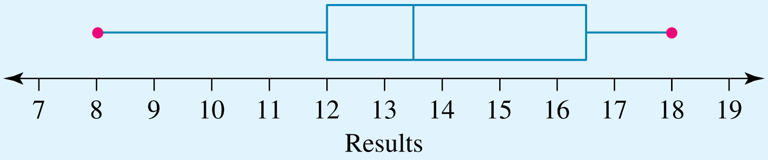
THINK	WRITE
The histogram shows a distribution which is positively skewed. The boxplot shows a distribution which is approximately symmetric.	The histogram and the boxplot could not represent the same data since the histogram shows a distribution that is positively skewed and the boxplot shows a distribution that is approximately symmetric.

WORKED EXAMPLE 18

The results (out of 20) of oral tests in a Year 12 Indonesian class are:

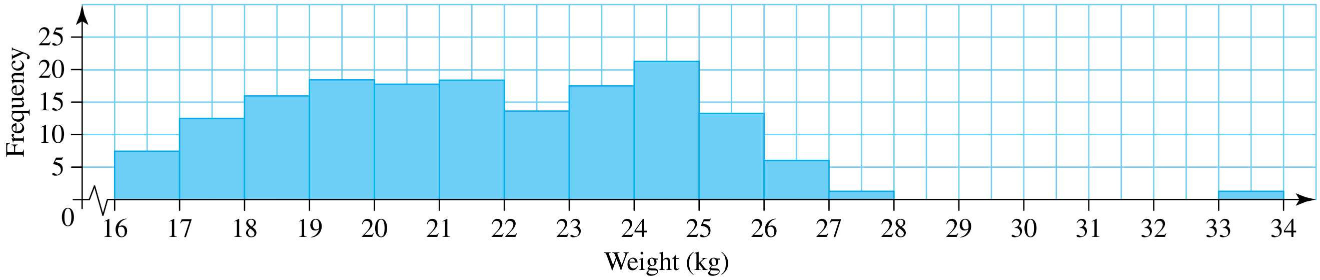
15 12 17 8 13 18 14 16 17 13 11 12

Display these data using a boxplot and discuss the shape obtained.

THINK	WRITE/DRAW
1 Find the lowest and highest scores, Q_1, the median (Q_2) and Q_3 by first ordering the data.	8 11 12 12 13 13 14 15 16 17 17 18 The median score is 13.5. The lower half of the scores are 8 11 12 12 13 13. So, $Q_1 = 12$ The upper half of the scores are 14 15 16 17 17 18. So, $Q_3 = 16.5$ The lowest score is 8. The highest score is 18.
2 Using these five number summary statistics, draw the boxplot.	
3 Consider the spread of each quarter of the data.	The scores are grouped around 12 and 13, as well as around 17 and 18 with 25% of the data in each section. The scores are more spread elsewhere.

Outliers

When one observation lies well away from other observations in a set, we call it an **outlier**. Sometimes an outlier occurs because data have been incorrectly obtained or misread. For example, here we see a histogram showing the weights of a group of 5-year-old boys.



The outlier, 33, may have occurred because a weight was incorrectly recorded as 33 rather than 23 or perhaps there was a boy in this group who, for some medical reason, weighed a lot more than his counterparts. When an outlier occurs, the reasons for its occurrence should be checked.

The lower and upper fences

We can identify outliers by calculating the values of the **lower and upper fences** in a data set. Values which lie either below the lower fence or above the upper fence are outliers.

The lower fence = $Q_1 - 1.5 \times \text{IQR}$
The upper fence = $Q_3 + 1.5 \times \text{IQR}$

An outlier is not included in the boxplot, but should instead be plotted as a point beyond the end of the whisker.

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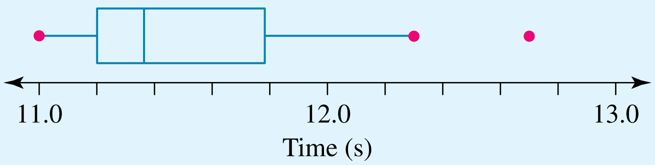
WORKED EXAMPLE 19

The times (in seconds) achieved by the 12 fastest runners in the 100-m sprint at a school athletics meeting are listed below.

11.2	12.3	11.5	11.0	11.6	11.4
11.9	11.2	12.7	11.3	11.2	11.3

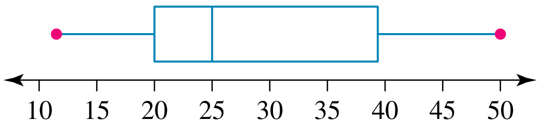
Draw a boxplot to represent the data, describe the shape of the distribution and comment on the existence of any outliers.

THINK	WRITE/DRAW
1 Determine the five number summary statistics by first ordering the data and obtaining the interquartile range.	11.0 11.2 11.2 11.2 11.3 11.3 11.4 11.5 11.6 11.9 12.3 12.7 Lowest score = 11.0 Highest score = 12.7 Median = Q_2 = 11.35 Q_1 = 11.2 Q_3 = 11.75 IQR = 11.75 − 11.2 = 0.55
2 Identify any outliers by calculating the values of the lower and upper fences.	$Q_1 - 1.5 \times \text{IQR} = 11.2 - 1.5 \times 0.55$ $= 10.375$ The lowest score lies above the lower fence of 10.375, so there is no outlier below. $Q_3 + 1.5 \times \text{IQR} = 11.75 + 1.5 \times 0.55$ $= 12.575$ The score 12.7 lies above the upper fence of 12.575, so it is an outlier and 12.3 becomes the end of the upper whisker.
3 Draw the boxplot with the outlier.	
4 Describe the shape of the distribution. Data peak to the left and trail off to the right with one outlier.	The data are positively skewed with 12.7 seconds being an outlier. This may be due to incorrect timing or recording but more likely the top eleven runners were significantly faster than the other competitors in the event.

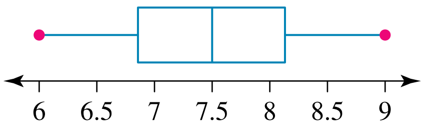
EXERCISE 1.7 Boxplots

PRACTISE

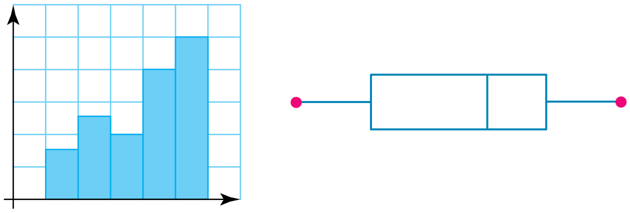
1 **WE16** Write down the range, median and interquartile range for the data in the boxplot shown.



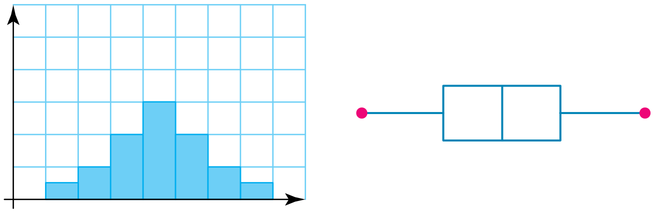
2 Find the median, range and interquartile range of the data displayed in the boxplot shown.



3 **WE17** Explain whether or not the histogram and the boxplot shown could represent the same data.



4 Do the histogram and boxplot represent the same data? Explain.



5 **WE18** The results for a Physics test (out of 50) are shown.

32	38	42	40	37	26	46	36	50	41	48	50	40	38	32	35	28	30

Display the data using a boxplot and discuss the shape obtained.

6 The numbers of hours Year 12 students spend at their part-time job per week are shown.

4	8	6	5	12	8	16	4	7	10	8	20	12	7	6	4	8

Display the data using a boxplot and discuss what the boxplot shows in relation to part-time work of Year 12 students.

7 **WE19** The heights jumped (in metres) at a school high jump competition are listed:

1.35	1.30	1.40	1.38	1.45	1.48
1.30	1.36	1.45	1.75	1.46	1.40

a Draw a boxplot to represent the data.

b Describe the shape of the distribution and comment on the existence of any outliers by finding the lower and upper fences.

8 The amount of fuel (in litres) used at a petrol pump for 16 cars is listed:

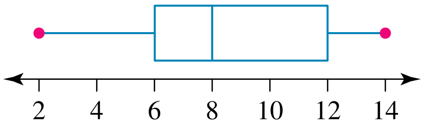
48.5	55.1	61.2	58.5	46.9	49.2	57.3	49.9
51.6	30.3	45.9	50.2	52.6	47.0	55.5	60.3

Draw a boxplot to represent the data and label any outliers.

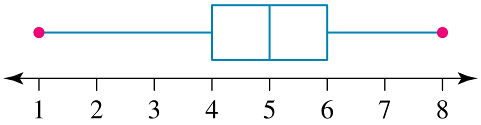
CONSOLIDATE

9 For the boxplots shown, write down the range, the interquartile range and the median of the distributions which each one represents.

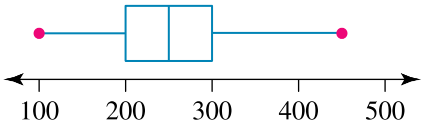
a



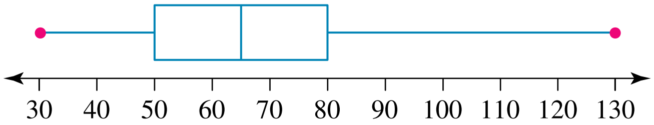
b



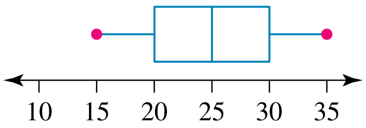
c



d

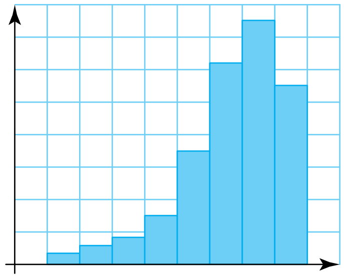


e

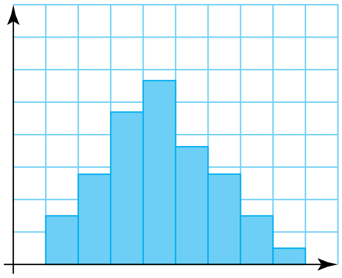


10 Match each histogram below with the boxplot which could show the same distribution.

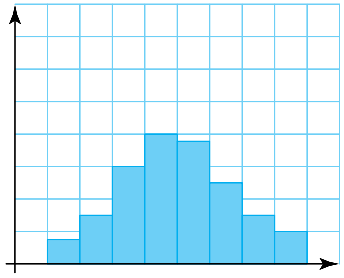
a



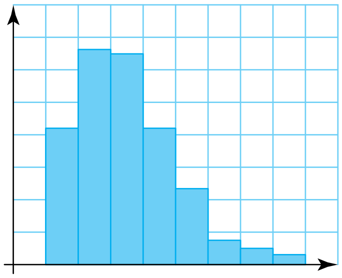
b



c



d

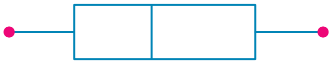


i

ii

iii

iv



11 For each of the following sets of data, construct a boxplot.

a 3 5 6 8 8 9 12 14 17 18

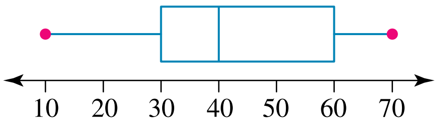
b 3 4 4 5 5 6 7 7 7 8 8 9 9 10 10 12

c 4.3 4.5 4.7 4.9 5.1 5.3 5.5 5.6

d 11 13 15 15 16 18 20 21 22 21 18 19 20 16 18 20

e 0.4 0.5 0.7 0.8 0.8 0.9 1.0 1.1 1.2 1.0 1.3

12 For the distribution shown in the boxplot below, it is true to say that:



A the median is 30
B the median is 45
C the interquartile range is 10
D the interquartile range is 30
E the interquartile range is 60

13 The number of clients seen each day over a 15-day period by a tax consultant is:

3 5 2 7 5 6 4 3 4 5 6 6 4 3 4

Represent these data on a boxplot.

14 The maximum daily temperatures (in °C) for the month of October in Melbourne are:

18 26 28 23 16 19 21 27 31 23 24 26 21 18 26 27
23 21 24 20 19 25 27 32 29 21 16 19 23 25 27

Represent these data on a boxplot.

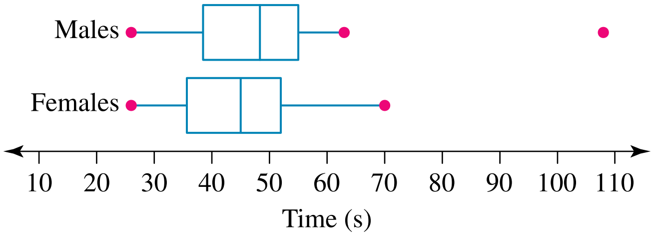
15 The number of rides that 16 children had at the annual show are listed below.

8 5 9 4 9 0 8 7 9 2 8 7 9 6 7 8

a Draw a boxplot to represent the data, describe the shape of the distribution and comment on the existence of any outliers.
b Use CAS to draw a boxplot for these data.

16 A concentration test was carried out on 40 students in Year 12 across Australia. The test involved the use of a computer mouse and the ability to recognise multiple images. The less time required to complete the activity, the better the student's ability to concentrate.

The data are shown by the parallel boxplots.



a Identify two similar properties of the concentration spans for boys and girls.
b Find the interquartile range for boys and girls.
c Comment on the existence of an outlier in the boys' data by finding the lower and upper fences.

17 The weights of 15 boxes (in kilograms) being moved from one house to another are as follows:

5, 7, 10, 15, 13, 14, 17, 20, 9, 4, 11, 12, 18, 21, 15.

Draw a boxplot to display the data.

18 You work in the marketing department of a perfume company. You completed a survey of people who purchased your perfume, asking them how many times a week they used it. Analyse the data by drawing a boxplot and comment on the existence of any outliers by finding the lower and upper fences.

7	2	5	4	7	5	7	2
5	4	3	5	7	7	9	8
5	6	5	3	15	8	7	5

MASTER

19 For the data set shown:

11	11	14	16	19	22	24	25
25	27	28	28	36	38	38	39

a construct a boxplot by hand

b comment on the presence of outliers by finding the lower and upper fences

c construct a boxplot using CAS and compare the two boxplots.

20 From the stem plot shown construct a boxplot using CAS and comment on any outliers if they exist in the data.

Stem	Leaf
2	3
3	
4	
5	
6	1 5 8
7	2 5 5 7
8	1 2 3 3 6 7 7
9	1 1 3 7 7
10	6 7 9
11	1 3

Key: 2|3 = 23

1.8 The mean of a sample

study on

Unit 3

AOS DA

Topic 3

Concept 1

Measures of centre—mean
Concept summary
Practice questions

When dealing with sets of data, we are always working with either the **population** or a **sample** from the population. The means of a data set representing the population and a sample are calculated in the same way; however, they are represented by different symbols. The **mean** of a population is represented by the Greek letter μ (mu) and the mean of a sample is represented by $\bar{x}$ (x-bar), a lower case x with a bar on top. In this section we will only use $\bar{x}$ to represent the mean.

The *mean* of a set of data is what is referred to in everyday language as the *average*.

For the set of data {4, 7, 9, 12, 18}:

$$\begin{aligned}\bar{x} &= \frac{4 + 7 + 9 + 12 + 18}{5}\\ &= 10.\end{aligned}$$

The formal definition of the mean is:

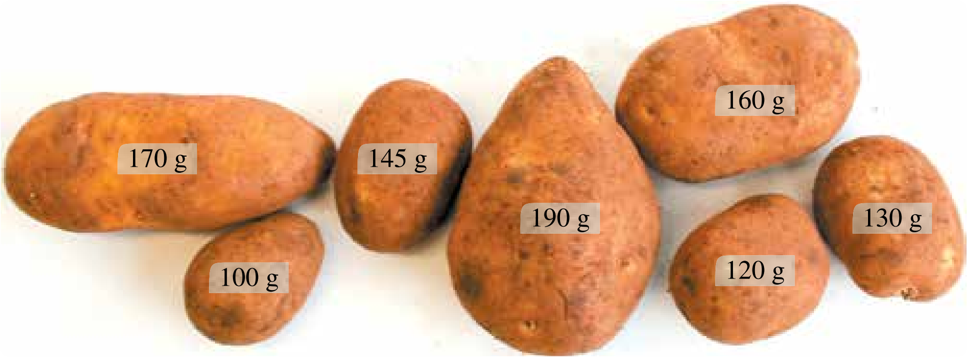
$$\bar{x} = \frac{\sum x}{n}$$

where $\sum x$ represents the sum of all of the observations in the data set and n represents the number of observations in the data set.

Note that the symbol, $\sum$ is the Greek letter, sigma, which represents 'the sum of'.

The mean is also referred to as a *summary statistic* and is a measure of the centre of a distribution. The mean is the point about which the distribution 'balances'.

Consider the masses of 7 potatoes, given in grams, in the photograph.



The mean is 145 g. The observations 130 and 160 'balance' each other since they are each 15 g from the mean. Similarly, the observations 120 and 170 'balance' each other since they are each 25 g from the mean, as do the observations 100 and 190. Note that the median is also 145 g. That is, for this set of data the mean and the median give the same value for the centre. This is because the distribution is symmetric.

Now consider two cases in which the distribution of data is not symmetric.

Case 1

Consider the masses of a different set of 7 potatoes, given in grams below.

100 105 110 115 120 160 200

The median of this distribution is 115 g and the mean is 130 g. There are 5 observations that are less than the mean and only 2 that are more. In other words, the mean does not give us a good indication of the *centre* of the distribution. However, there is still a 'balance' between observations below the mean and those above, in terms of the spread of all the observations from the mean. Therefore, the mean is still useful to give a measure of the central tendency of the distribution but in cases where the distribution is skewed, the median gives a better indication of the centre. For a positively skewed distribution, as in the previous case, the mean will be greater than the median. For a negatively skewed distribution the mean will be less than the median.

Case 2

Consider the data below, showing the weekly income (to the nearest \$10) of 10 families living in a suburban street.

\$600 \$1340 \$1360 \$1380 \$1400 \$1420 \$1420 \$1440 \$1460 \$1500

In this case, $\bar{x} = \dfrac{13\,320}{10} = \$1332$, and the median is \$1410.

One of the values in this set, \$600, is clearly an outlier. As a result, the value of the mean is below the weekly income of the other 9 households. In such a case the mean is not very useful in establishing the centre; however, the 'balance' still remains for this negatively skewed distribution.

The mean is calculated by using the values of the observations and because of this it becomes a less reliable measure of the centre of the distribution when the distribution is skewed or contains an outlier. Because the median is based on the order of the observations rather than their value, it is a better measure of the centre of such distributions.

WORKED EXAMPLE 20

Calculate the mean of the set of data shown.

10, 12, 15, 16, 18, 19, 22, 25, 27, 29

THINK	WRITE
1 Write the formula for calculating the mean, where $\sum x$ is the sum of all scores; n is the number of scores in the set.	$\bar{x} = \frac{\sum x}{n}$ $= \frac{10 + 12 + 15 + 16 + 18 + 19 + 22 + 25 + 27 + 29}{10}$ $\bar{x} = 19.3$
2 Substitute the values into the formula and evaluate.	The mean, $\bar{x}$, is 19.3.

When calculating the mean of a data set, sometimes the answer you calculate will contain a long stream of digits after the decimal point.

For example, if we are calculating the mean of the data set

44, 38, 55, 61, 48, 32, 49

Then the mean would be:

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n}\\ &= \frac{327}{7}\\ &= 46.714\,285\,71\ldots\end{aligned}$$

In this case it makes sense to round the answer to either a given number of decimal places, or a given number of **significant figures**.

Rounding to a given number of significant figures

When rounding to a given number of signified figures, we are rounding to the digits in a number which are regarded as 'significant'.

To determine which digits are significant, we can observe the following rules:

- All digits greater than zero are significant
- Leading zeros can be ignored (they are placeholders and are not significant)
- Zeros included between other digits are significant
- Zeros included after decimal digits are significant
- Trailing zeros for integers are not significant (unless specified otherwise)

The following examples show how these rules work:

0.003 561 — leading digits are ignored, so this has 4 significant figures

70.036 — zeros between other digits are significant, so this has 5 significant figures

5.320 — zeros included after decimal digits are significant, so this has 4 significant figures

450 000 — trailing zeros are not significant, so this has 2 significant figures

78 000.0 — the zero after the decimal point is considered significant, so the zeros between other numbers are also significant; this has 6 significant figures

As when rounding to a given number of decimal places, when rounding to a given number of significant figures consider the digit after the specified number of figures. If it is 5 or above, round the final digit up; if it is 4 or below, keep the final digit as is.

5067.37 — rounded to 2 significant figures is 5100

3199.01 — rounded to 4 significant figures is 3199

0.004 931 — rounded to 3 significant figures is 0.004 93

1 020 004 — rounded to 2 significant figures is 1 000 000

Calculating the mean of grouped data

When data are presented in a frequency table with class intervals and we don't know what the raw data are, we employ another method to find the mean of these grouped data. This other method is shown in the worked example that follows and uses the midpoints of the class intervals to represent the raw data.

Recall that the Greek letter sigma, $\sum$, represents 'the sum of'. So, $\sum f$ means the sum of the frequencies and is the total of all the numbers in the frequency column.

To find the mean for grouped data,

$$\bar{x} = \frac{\sum(f \times m)}{\sum f}$$

where f represents the frequency of the data and m represents the midpoint of the class interval of the grouped data.

WORKED EXAMPLE 21

The ages of a group of 30 people attending a superannuation seminar are recorded in the frequency table.

Age (class intervals)	Frequency (f)
20–29	1
30–39	6
40–49	13
50–59	6
60–69	3
70–79	1

Calculate the mean age of those attending the seminar. Give your answer correct to 3 significant figures.

THINK

WRITE

1 Since we don't have individual raw ages, but rather a class interval, we need to decide on one particular age to represent each interval. We use the midpoint, m, of the class interval. Add an extra column to the table to display these.

The midpoint of the first interval is $\frac{20 + 29}{2} = 24.5$, the midpoint of the second interval is 34.5 and so on.

Age (class intervals)	Frequency (f)	Midpoint class interval (m)	$f \times m$
20–29	1	24.5	24.5
30–39	6	34.5	207
40–49	13	44.5	578.5
50–59	6	54.5	327
60–69	3	64.5	193.5
70–79	1	74.5	74.5
	$\sum f = 30$		$\sum(f \times m) = 1405$

2 Multiply each of the midpoints by the frequency and display these values in another column headed $f \times m$. For the first interval we have $24.5 \times 1 = 24.5$. For the second interval we have $34.5 \times 6 = 207$ and so on.

3 Sum the product of the midpoints and the frequencies in the $f \times m$ column.

$24.5 + 207 + 578.5 + 327 + 193.5 + 74.5 = 1405$

4 Divide this sum by the total number of people attending the seminar (given by the sum of the frequency column).

$$\text{So, } \bar{x} = \frac{1405}{30} = 46.833\ldots \approx 46.8 \text{ (correct to 3 significant figures).}$$

EXERCISE 1.8 The mean of a sample

PRACTISE

1 **WE20** Calculate the mean of the data set shown.

9, 12, 14, 16, 18, 19, 20, 25, 29, 33, 35, 36, 39

2 Calculate the mean of the data set shown.

5.5, 6.3, 7.7, 8.3, 9.7, 6.7, 12.9, 10.5, 9.9, 5.1

3 **WE21** The number of hamburgers sold at a take-away food shop is recorded in the frequency table shown.

Hamburgers sold (class intervals)	Frequency (f)
0–4	6
5–9	8
10–14	2
15–19	3
20–24	5

Calculate the mean number of hamburgers sold each day. Give your answer correct to 4 significant figures.

4 The ages of 100 supporters who attended the grand final parade are recorded in the frequency table shown.

Age (class intervals)	Frequency (f)
0–9	10
10–19	21
20–29	30
30–39	18
40–49	17
50–59	4

Calculate the mean age of those attending the parade. Give your answer correct to 2 decimal places.

CONSOLIDATE

5 Find the mean of each of the following sets of data.

a 5 6 8 8 9 (correct to 2 significant figures)
b 3 4 4 5 5 6 7 7 7 8 8 9 9 10 10 12 (correct to 4 significant figures)
c 4.3 4.5 4.7 4.9 5.1 5.3 5.5 5.6 (correct to 5 significant figures)
d 11 13 15 15 16 18 20 21 22 (correct to 1 decimal place)
e 0.4 0.5 0.7 0.8 0.8 0.9 1.0 1.1 1.2 1.0 1.3 (correct to 4 decimal places)

6 Calculate the mean of each of the following and explain whether or not it gives us a good indication of the centre of the data.

a 0.7 0.8 0.85 0.9 0.92 2.3
b 14 16 16 17 17 17 19 20
c 23 24 28 29 33 34 37 39
d 2 15 17 18 18 19 20

7 The number of people attending sculpture classes at the local TAFE college for each week during the first semester is given.

15 12 15 11 14 8 14 15 11 10
7 11 12 14 15 14 15 9 10 11

What is the mean number of people attending each week? (Express your answer correct to 2 significant figures.)

8 The ages of a group of junior pilots joining an international airline are indicated in the stem plot below.

Stem	Leaf
2	1
2	2
2	4 5
2	6 6 7
2	8 8 8 9
3	0 1 1
3	2 3
3	4 4
3	6
3	8

Key: 2|1 = 21 years

The mean age of this group of pilots is:

A 20 **B** 28 **C** 29 **D** 29.15 **E** 29.5

9 The number of people present each week at a 15-week horticultural course is given by the stem plot at right.

Stem	Leaf
0	4
0*	7
1	2 4
1*	5 5 6 7 8
2	1 2 4
2*	7 7 7

Key: 2|4 = 24 people

The mean number of people attending each week was closest to:

A 17.7 **B** 18 **C** 19.5
D 20 **E** 21.2

10 For each of the following, write down whether the mean or the median would provide a better indication of the centre of the distribution.

a A positively skewed distribution
b A symmetric distribution
c A distribution with an outlier
d A negatively skewed distribution

11 Find the mean of each set of data given.

a

Class interval	Frequency (f)
0–	1
10–	3
20–	6
30–	17
40–	12
50–	5

b

Class interval	Frequency (f)
0–	2
5–	5
10–	7
15–	13
20–	8
25–	6

c

Class interval	Frequency (f)
0–	2
50–	7
100–	8
150–	14
200–	12
250–	5

d

Class interval	Frequency (f)
1–	14
7–	19
13–	23
19–	22
25–	20
31–	14

12 The ages of people attending a beginner's course in karate are indicated in the following frequency table.

a What is the mean age of those attending the course? (Express your answer correct to 1 decimal place.)

b Calculate the median. What does this value, compared to the mean, suggest about the shape of the distribution?

Age	Frequency (f)
10–14	5
15–19	5
20–24	7
25–29	4
30–34	3
35–39	2
40–44	2
45–49	1

13 The number of papers sold each morning from a newsagent is recorded in the frequency table shown.

Age (class intervals)	Frequency (f)
50–99	2
100–149	18
150–199	25
200–249	33
250–299	21
300–349	9
350–399	2

Calculate the mean number of papers sold over this period.

14 The number of fish eaten by seals at Sea Haven on a daily basis is shown. Calculate the mean number of fish eaten per day.

Number of fish	Frequency (f)
0–9	2
10–19	4
20–29	5
30–39	18
40–49	19
50–59	24

MASTER

15 A shipping container is filled with cargo and each piece of cargo is weighed prior to being packed on the container. Using class intervals of 10 kg, calculate the mean weight (in kg) of the pieces of cargo. Give your answer correct to four significant figures.

78.3	67.3	82.8	44.3	90.5	57.3	55.9	42.3	48.8	63.4	69.7
70.4	77.1	79.4	47.6	52.9	45.4	60.1	73.4	88.6	41.9	63.7

16 The number of cups of coffee drunk by 176 Year 12 students in the two weeks leading to their exams is shown.

Number of cups	Frequency (f)
0–9	3
10–19	5
20–29	7
30–39	24
40–49	29
50–59	41
60–69	32
70–79	30
80–89	3
90–99	2

a What is the mean number of cups of coffee drunk in the two-week period?
b Calculate the median.
c What does the median value when compared to the mean value suggest about the shape of the distribution?

1.9 Standard deviation of a sample

study on

Unit 3

AOS DA

Topic 3

Concept 4

Measures of spread—standard deviation
Concept summary
Practice questions

eBook plus

Interactivity
The mean and the standard deviation
int-6246

The **standard deviation** gives us a measure of how data are spread around the mean. For the set of data {8, 10, 11, 12, 12, 13}, the mean, $\bar{x} = 11$.

The amount that each observation 'deviates' (that is, differs) from the mean is calculated and shown in the table at right.

Particular observation (x)	Deviation from the mean ($x - \bar{x}$)
8	$8 - 11 = -3$
10	$10 - 11 = -1$
11	$11 - 11 = 0$
12	$12 - 11 = 1$
12	$12 - 11 = 1$
13	$13 - 11 = 2$

The deviations from the mean are either positive or negative depending on whether the particular observation is lower or higher in value than the mean. If we were to add all the deviations from the mean we would obtain zero.

If we square the deviations from the mean we will overcome the problem of positive and negative deviations cancelling each other out. With this in mind, a quantity known as sample **variance** (s^2) is defined:

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}.$$

Technically, this formula for variance is used when the data set is a sub-set of a larger population; that is, a sample of the population.

Variance gives the average of the squared deviations and is also a measure of spread. A far more useful measure of spread, however, is the standard deviation, which is the square root of variance (s). One reason for it being more useful is that it takes the same unit as the observations (for example, cm or number of people). Variance would square the units, for example, cm^2 or number of people squared, which is not very practical.

Other advantages of the standard deviation will be dealt with later in the topic.

In summary,

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$

where s represents sample standard deviation

$\sum$ represents 'the sum of'

x represents an observation

$\bar{x}$ represents the mean

n represents the number of observations.

Note: This is the formula for the standard deviation of a sample. The standard deviation for a population is given by σ (sigma) and is calculated using a slightly different formula, which is outside the scope of this course.

While some of the theory or formulas associated with standard deviation may look complex, the calculation of this measure of spread is straightforward using CAS. Manual computation of standard deviation is therefore rarely necessary, however application of the formula is required knowledge.

WORKED EXAMPLE 22

The price (in cents) per litre of petrol at a service station was recorded each Friday over a 15-week period. The data are given below.

152.4	**160.2**	**159.6**	**168.6**	**161.4**	**156.6**	**164.8**	**162.6**
161.0	**156.4**	**159.0**	**160.2**	**162.6**	**168.4**	**166.8**	

Calculate the standard deviation for this set of data, correct to 2 decimal places.

THINK	WRITE
1 Enter the data into CAS to determine the sample statistics.	$s = 4.515\ 92$
2 Round the value correct to 2 decimal places.	$= 4.52$ cents/L

WORKED EXAMPLE 23

The number of students attending SRC meetings during the term is given in the stem plot shown. Calculate the standard deviation for this set of data, correct to 4 significant figures, by using the formula $s = \sqrt{\dfrac{\sum(x-\bar{x})^2}{n-1}}$.

Stem	Leaf
0	4
0*	8 8
1	1 3 4
1*	5 8
2	3
2*	5

Key: 1|4 = 14 students

THINK	WRITE
1 Calculate the value of the mean ($\bar{x}$).	$\bar{x} = \dfrac{\sum x}{n}$ $= \dfrac{4+8+8+11+13+14+15+18+23+25}{10}$ $= 13.9$
2 Set up a table to calculate the values of $(x - \bar{x})^2$.	

x	$x - \bar{x}$	$(x - \bar{x})^2$
4	–9.9	98.01
8	–5.9	34.81
8	–5.9	34.81
11	–2.9	8.41
13	–0.9	0.81
14	0.1	0.01
15	1.1	1.21
18	4.1	16.81
23	9.1	82.81
25	11.1	123.21
	$\sum(x - \bar{x})^2 = 400.9$	

3 Enter the values into the formula to calculate the standard deviation (s).

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$

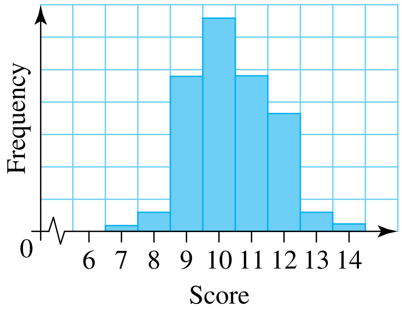
$$= \sqrt{\frac{400.9}{9}}$$

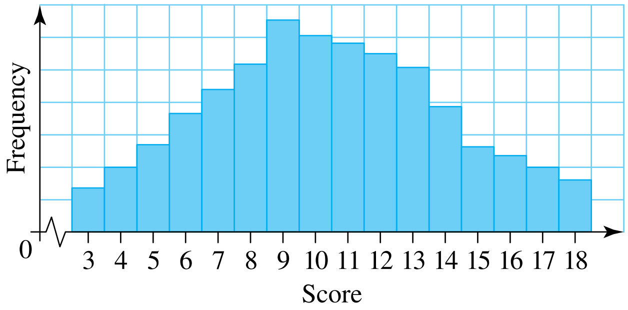
$$= 6.6741\ldots$$

4 Round the value correct to 4 significant figures.

$= 6.674$ (correct to 4 significant figures)

The standard deviation is a measure of the spread of data from the mean. Consider the two sets of data shown.





Each set of data has a mean of 10. The top set of data has a standard deviation of 1 and the bottom set of data has a standard deviation of 3.

As we can see, the larger the standard deviation, the more spread are the data from the mean.

EXERCISE 1.9 Standard deviation of a sample

PRACTISE

1 **WE22** The Australian dollar is often compared to the US dollar. The value of the Australian dollar compared to the US dollar each week over a six-month period is shown.

97.2	96.8	98.0	98.3	97.5	95.9	96.1	95.6	95.0	94.8	94.6	94.9
93.9	93.3	92.9	90.6	90.4	89.6	88.1	88.3	89.5	88.8	88.5	88.1

Calculate the standard deviation for this set of data, correct to 2 decimal places.

2 The test results for a maths class are shown.

67	98	75	81	70	64	55	52	78	90	76
92	78	80	83	59	67	45	78	48	82	62

Calculate the standard deviation for the set of data, correct to 2 decimal places.

3 **WE23** Calculate the standard deviation of the data set shown, correct to 4 significant places, by using the formula

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}.$$

Stem	Leaf
1	1 1
1*	6 9
2	4
2*	5 5 8
3	6 9

Key: 2|4 = 24

4 The number of students attending the service learning meetings in preparation for the year's fundraising activities is shown by the stem plot. Calculate the standard deviation for this set of data, correct to 3 decimal places, by using the formula

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}.$$

Stem	Leaf
1	2 3 3
1*	6 7 8
2	1 4
2*	7
3	
3*	5

Key: 1|3 = 13

CONSOLIDATE

5 For each of the following sets of data, calculate the standard deviation correct to 2 decimal places.

a 3 4 4.7 5.1 6 6.2

b 7 9 10 10 11 13 13 14

c 12.9 17.2 17.9 20.2 26.4 28.9

d 41 43 44 45 45 46 47 49

e 0.30 0.32 0.37 0.39 0.41 0.43 0.45

6 First-quarter profit increases for 8 leading companies are given below as percentages.

2.3 0.8 1.6 2.1 1.7 1.3 1.4 1.9

Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

7 The heights in metres of a group of army recruits are given below.

1.8 1.95 1.87 1.77 1.75 1.79 1.81 1.83 1.76 1.80 1.92 1.87 1.85 1.83

Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

8 Times (correct to the nearest tenth of a second) for the heats in the 100 m sprint at the school sports carnival are given at right.

Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

Stem	Leaf
11	0
11	2 3
11	4 4 5
11	6 6
11	8 8 9
12	0 1
12	2 2 3
12	4 4
12	6
12	9

Key: 11|0 = 11.0 s

9 The number of outgoing phone calls from an office each day over a 4-week period is shown in the stem plot below.

Stem	Leaf
0	8 9
1	3 4 7 9
2	0 1 3 7 7
3	3 4
4	1 5 6 7 8
5	3 8

Key: 2|1 = 21 calls

Calculate the standard deviation for this set of data and express your answer correct to 4 significant figures.

10 A new legal aid service has been operational for only 5 weeks. The number of people who have made use of the service each day during this period is set out at right.

The standard deviation (to 2 decimal places) of these data is:

A 6.00
B 6.34
C 6.47
D 15.44
E 16.00

Stem	Leaf
0	2 4
0*	7 7 9
1	0 1 4 4 4 4
1*	5 6 6 7 8 8 9
2	1 2 2 3 3 3
2*	7

Key: 1|0 = 10 people

11 The speed of 20 cars (in km/h) is monitored along a stretch of road that is a designated 80 km/h zone. Calculate the standard deviation of the data, correct to 2 decimal places.

80, 82, 77, 75, 80, 80, 81, 78, 79, 78, 80, 80, 85, 70, 79, 81, 81, 80, 80, 80

12 Thirty pens are randomly selected off the conveyor belt at the factory and are tested to see how long they will last, in hours. Calculate the standard deviation of the data shown, correct to 3 decimal places.

20, 32, 38, 22, 25, 34, 47, 31, 26, 29, 30, 36, 28, 40, 31,

26, 37, 38, 32, 36, 35, 25, 29, 30, 40, 35, 38, 39, 37, 30

13 Calculate the standard deviation of the data shown, correct to 2 decimal places, representing the temperature of the soil around 25 germinating seedlings:

28.9	27.4	23.6	25.6	21.1	22.9	29.6	25.7	27.4
23.6	22.4	24.6	21.8	26.4	24.9	25.0	23.5	26.1
23.7	25.3	29.3	23.5	22.0	27.9	23.6		

14 Aptitude tests are often used by companies to help decide who to employ. An employer gave 30 potential employees an aptitude test with a total of 90 marks. The scores achieved are shown.

67, 67, 68, 68, 68, 69, 69, 72, 72, 73, 73, 74, 74, 75, 75,

77, 78, 78, 78, 79, 79, 79, 81, 81, 81, 82, 83, 83, 83, 86

Calculate the mean and standard deviation of the data, correct to 1 decimal place.

MASTER

15 The number of players attending basketball try-out sessions is shown by the stem plot. Calculate the standard deviation for this set of data, correct to 4 significant figures.

Stem	Leaf
2	1
2	
2	
2	6
2	8 9
3	0 0 1 1 1
3	2 3 3 3 3 3
3	4 4 5 5
3	6 7
3	8

Key: 2|1 = 21

16 The scores obtained out of 20 at a dancing competition are shown. Calculate the standard deviation of the scores, correct to 3 decimal places.

18.5	16.5	18.0	12.5	13.0	18.0	15.5	17.5	18.5	19.0
17.0	12.5	16.5	13.5	19.0	20.0	17.5	19.5	16.0	15.5

1.10 Populations and simple random samples

Populations

studyon

Unit 3
AOS DA
Topic 5
Concept 1

Population
Concept summary
Practice questions

A group of Year 12 students decide to base their statistical investigation for a maths project on what their contemporaries — that is, other Year 12 students — spend per year on Christmas and birthday presents for their family members. One of their early decisions is to decide what the population is going to be for their investigation. That is, are they looking at Year 12 students in Australia or in Victoria or in metropolitan Melbourne or in their suburb or just in their school? In practice, it is difficult to look at a large population unless, of course, you have a lot of resources available to you! The students decide that their population will be the Year 12 students at their school. This means that any conclusions they draw as a result of their investigation can be generalised to Year 12 students at their school but not beyond that.

Samples

studyon

Unit 3
AOS DA
Topic 5
Concept 2

Sample and simple random sample
Concept summary
Practice questions

Given that there are 95 students in Year 12 at the school, it would be too time-consuming to interview all of them. A smaller group known as a *sample* is therefore taken from the population. The way in which this smaller group is chosen is of paramount importance. For the investigation to have credibility, the sample should be a random selection from the population and every member of that population should have an equal chance of being chosen in the sample. Also, the selection of one person from the population should not affect whether or not another person is chosen; that is, the selections should be independent. A **simple random sample** provides such a sample.

The students conducting the investigation decide to choose a sample of 12 fellow students. While it would be simplest to choose 12 of their mates as the sample, this would introduce **bias** since they would not be representative of the population as a whole.

The students obtain a list of names of the 95 students in Year 12. They then write next to the name of each student a number from 1 to 95. Using a calculator, the students generate 12 random numbers between 1 and 95. Alternatively, the students could have used a table of random numbers. Any point on the table can be taken as the starting point. The students decide which direction to move through the table; for example, across the table to the right or to the left or down. Once a direction is chosen, they must stay with that movement and write down the 2-digit numbers as they go along.

The numbers chosen by the students are then matched to the numbers on the name list and the students in their sample can be identified.

These 12 students are then asked what they spent in the last year on family presents.

The students conducting the investigation can then record the data.

Random numbers can also be generated with the aid of CAS.

WORKED EXAMPLE 24

Generate 5 random numbers (integers) between 1 and 50.

THINK	WRITE
1 Find the appropriate menu in CAS to generate random integers.	
2 Generate 5 random numbers between 1 and 50.	
3 An example of a set of numbers is displayed.	{48, 46, 8, 26, 21}.

The mean of a data set which represents a population is μ.

The mean of a data set which represents a sample is $\bar{x}$.

The standard deviation of a data set which represents a population is σ.

The standard deviation of a data set which represents a sample is s.

Displaying the data

The raw data for a sample of 12 students from our population of 95 are given below in dollars.

25 30 35 38 34 22 30 40 35 25 32 40

Since there are not many responses, a stem plot is an appropriate way of displaying the data.

To summarise and comment further on the sample, it is useful to use some of the summary statistics covered earlier in this topic. The most efficient way to calculate these is to use CAS. Using the steps outlined in the previous sections, we obtain a list of summary statistics for these data.

Stem	Leaf
2	2
2*	5 5
3*	0 0 2 4
3*	5 5 8
4	0 0

Key: 2|2 = 22 dollars

$$\begin{aligned} \bar{x} &= 32.2 \\ s &= 6 \\ Q_1 &= 27.5 \\ \text{median} &= 33 \\ Q_3 &= 36.5 \end{aligned}$$

To measure the centre of the distribution, the median and the mean are used. Since there are no outliers and the distribution is approximately symmetric, the mean is quite a good measure of the centre of the distribution. Also, the mean and the median are quite close in value.

To measure the spread of the distribution, the standard deviation and the interquartile range are used. Since $s = 6$, and since the distribution is approximately bell-shaped, we would expect that approximately 95% of the data lie between $32.2 + 12 = 44.2$ and $32.2 - 12 = 20.2$ (as shown in section 1.11). It is perhaps a little surprising to think that 95% of students spend between \$20.20 and \$44.20 on family presents. One might have expected there to be greater variation on what students spend. The data, in that sense, are quite bunched.

The interquartile range is equal to $36.5 - 27.5 = 9$. This means that 50% of those in the sample spent within \$9 of each other on family presents. Again, one might have expected a greater variation in what students spent. It would be interesting to know whether students confer about what they spend and therefore whether they tended to allocate about the same amount of money to spend.

At another school, the same investigation was undertaken and the results are shown in the following stem plot.

The summary statistics for these data are as follows:

$\bar{x} = 47.5$, $s = 16.3$, $Q_1 = 35$, median $= 50$, $Q_3 = 60$.

Stem	Leaf
2	0
2*	5 5
3	
3*	5
4	
4*	5 5
5	0 0
5*	5 5
6	0 0
6*	5
7	
7*	5

Key: 2|2 = 22 dollars

The distribution is approximately symmetric, albeit very spread out. The mean and the median are therefore reasonably close and give us an indication of the centre of the distribution. The mean value for this set of data is higher than for the data obtained at the other school. This indicates that students at this school in this year level, in general, spend more than their counterparts at the other school. Reasons for this might be that this school is in a higher socio-economic area and students receive greater allowances, or perhaps at this school there is a higher proportion of students from cultures where spending more money on family presents is usual.

The range of money spent on family presents at this school and at this particular year level is \$55. This is certainly much higher than at the other school. The interquartile range at this school is \$25. That is, the middle 50% of students spend within \$25 of each other which is greater than the students at the other school.

EXERCISE 1.10 Populations and simple random samples

PRACTISE

1 **WE24** Generate 5 random numbers (integers) between 1 and 100.

2 Generate 10 random numbers (integers) between 1 and 250.

CONSOLIDATE

3 Students are selecting a sample of students at their school to complete an investigation. Which of the following are examples of choosing this sample randomly?

A Choosing students queuing at the tuckshop
B Assigning numbers to a list of student names and using a random number table to select random numbers
C Calling for volunteers
D Choosing the girls in an all-girls science class
E Choosing students in a bus on the way home

4 Generate 10 random numbers (integers) between 1 and 100.

5 Generate 20 random numbers (integers) between 1 and 500.

6 Which is larger: A population or a sample? Explain why.

7 When selecting students for a simple random sample of a year level, the students selected should be:

A of similar age
B a group of mates
C independent
D female
E the tallest students

8 The students selected for a simple random sample of a year level should be selected by:

A a group of mates
B a group who all dance
C a selection of males
D the students with the best test results
E using random numbers

9 Would the mean be a good measure of the centre of the distribution shown at right? Explain.

Stem	Leaf
0	7
1	0 2
2	3 6 7 8 8
3	2 4 5 7 9 9
4	2 3 4 7 8
5	1 3 7
6	2 8

Key: 1|2 = 12

10 The mean is a good measure of the centre of a distribution if the data is:

A skewed left
B symmetric
C skewed right
D has outliers
E bimodal

11 The interquartile range is 12, since $Q_1 = 24$ and $Q_3 = 36$. The percentage of data that fit between 24 and 36 is:

A 12%
B 30%
C 50%
D 68%
E 95%

12 Conduct an investigation into how much money students in your year level earn per week (this might be an allowance or a wage). Write a report on your findings, ensuring you include:

a an explanation of the population for your investigation
b the manner in which your sample was selected
c the number in your sample
d your results as raw data
e your results in a stem plot or histogram
f the summary statistics for your data.

Comment on your results based on the summary statistics.

MASTER

13 Repeat question 12, but this time investigate the following for students in your year level:

a the number of hours spent on homework each week
b the number of hours spent working in part-time jobs.

14 Conduct a similar investigation to that which you completed in questions 12 and 13; however, this time sample students in another year group. Compare these data with those obtained for your year level.

1.11 The 68–95–99.7% rule and *z*-scores

The 68–95–99.7% rule

studyon

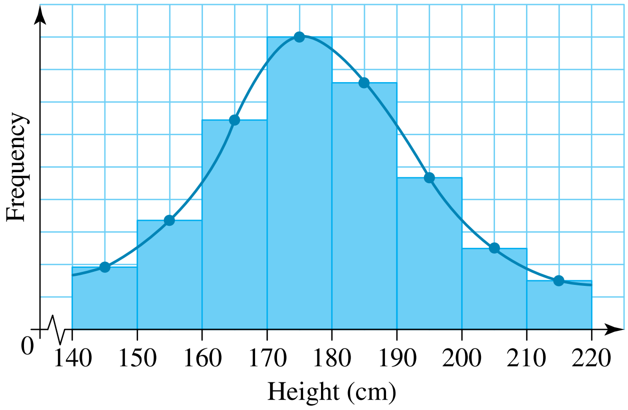
Unit 3
AOS DA
Topic 4
Concept 1

Normal distributions—the 68–95–99.7% rule
Concept summary
Practice questions

eBookplus

Interactivity
68–95–99.7% rule in a normal distribution
int-6247

The heights of a large number of students (a population) at a graduation ceremony were recorded and are shown in the histogram at right.



This set of data is approximately symmetric and has what is termed a *bell* shape. Many sets of data fall into this category and are often referred to as **normal distributions**. Examples are birth weights and people's heights. Data which are normally distributed have their symmetrical, bell-shaped distribution centred on the mean value, μ (the mean of the population).

A feature of this type of distribution is that we can predict what percentage of the data lie 1, 2 or 3 standard deviations (σ, the standard deviation of the population) either side of the mean using what is termed the 68–95–99.7% rule.

The 68–95–99.7% rule for a bell-shaped curve states that approximately:
1. 68% of data lie within 1 standard deviation either side of the mean
2. 95% of data lie within 2 standard deviations either side of the mean
3. 99.7% of data lie within 3 standard deviations either side of the mean.

Figure 1

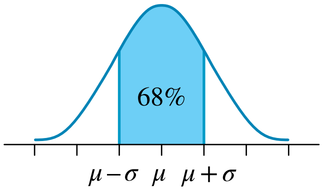


Figure 2

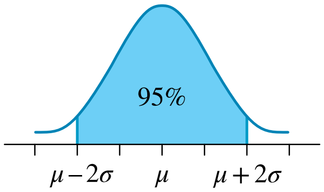
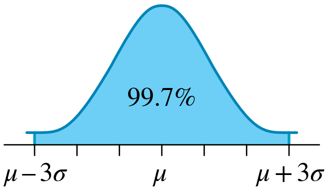


Figure 3



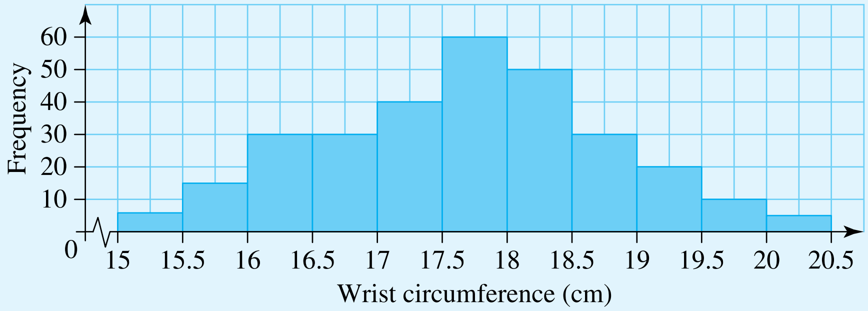
In Figure 1, 68% of the data shown lie between the value which is 1 standard deviation below the mean, that is $\mu - \sigma$, and the value which is 1 standard deviation above the mean, that is, $\mu + \sigma$.

In Figure 2, 95% of the data shown lie between the value which is 2 standard deviations below the mean, that is, $\mu - 2\sigma$, and the value which is 2 standard deviations above the mean, that is $\mu + 2\sigma$.

In Figure 3, 99.7% of the data shown lie between the value which is 3 standard deviations below the mean, that is, $\mu - 3\sigma$, and the value which is 3 standard deviations above the mean, that is, $\mu + 3\sigma$.

WORKED EXAMPLE 25

The wrist circumferences of a large group of people were recorded and the results are shown in the histogram below. The mean of the set of data is 17.7 and the standard deviation is 0.9. Write down the wrist circumferences between which we would expect approximately:



a **68% of the group to lie**

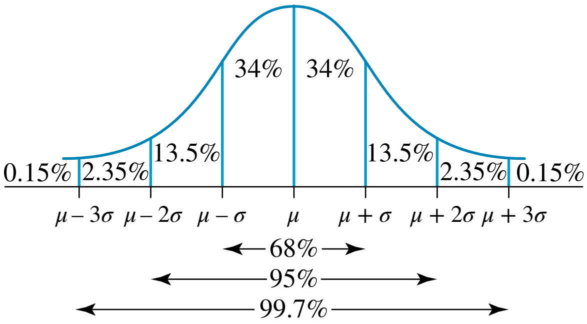
b **95% of the group to lie**

c **99.7% of the group to lie.**

THINK	WRITE
a The distribution can be described as approximately bell-shaped and therefore the 68–95–99.7% rule can be applied. Approximately 68% of the people have a wrist circumference between $\mu - \sigma$ and $\mu + \sigma$ (or one standard deviation either side of the mean).	**a** $\mu - \sigma = 17.7 - 0.9 = 16.8$ $\mu + \sigma = 17.7 + 0.9 = 18.6$ So approximately 68% of the people have a wrist size between 16.8 and 18.6 cm.
b Similarly, approximately 95% of the people have a wrist size between $\mu - 2\sigma$ and $\mu + 2\sigma$.	**b** $\mu - 2\sigma = 17.7 - 1.8 = 15.9$ $\mu + 2\sigma = 17.7 + 1.8 = 19.5$ Approximately 95% of people have a wrist size between 15.9 cm and 19.5 cm.
c Similarly, approximately 99.7% of the people have a wrist size between $\mu - 3\sigma$ and $\mu + 3\sigma$.	**c** $\mu - 3\sigma = 17.7 - 2.7 = 15.0$ $\mu + 3\sigma = 17.7 + 2.7 = 20.4$ Approximately 99.7% of people have a wrist size between 15.0 cm and 20.4 cm.

Using the 68–95–99.7% rule, we can work out the various percentages of the distribution which lie between the mean and 1 standard deviation from the mean and between the mean and 2 standard deviations from the mean and so on. The diagram at right summarises this.

Note that 50% of the data lie below the mean and 50% lie above the mean due to the symmetry of the distribution about the mean.



WORKED EXAMPLE 26

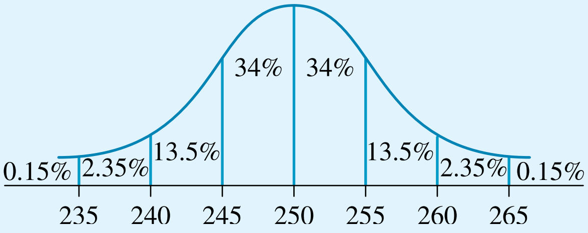
The distribution of the masses of a large number of packets of 'Fibre-fill' breakfast cereal is known to be bell-shaped with a mean of 250 g and a standard deviation of 5 g. Find the percentage of Fibre-fill packets with a mass which is:

a **less than 260 g**

b **less than 245 g**

c **more than 240 g**

d **between 240 g and 255 g.**

THINK	WRITE/DRAW
1 Draw the bell-shaped curve. Label the axis. $\mu = 250$, $\mu + \sigma = 255$, $\mu + 2\sigma = 260$ etc.	
a 260 g is 2 standard deviations above the mean. Using the summary diagram, we can find the percentage of data which is less than 260 g.	**a** Mass of 260 g is 2 standard deviations above the mean. Percentage of distribution less than 260 g is 13.5% + 34% + 34% + 13.5% + 2.35% + 0.15% = 97.5% or 13.5% + 34% + 50% = 97.5%
b 245 g is 1 standard deviation below the mean.	**b** Mass of 245 g is 1 standard deviation below the mean. Percentage of distribution less than 245 g is 13.5% + 2.35% + 0.15% = 16% or 50% − 34% = 16%
c 240 g is 2 standard deviations below the mean.	**c** Mass of 240 g is 2 standard deviations below the mean. Percentage of distribution more than 240 g is 13.5% + 34% + 34% + 13.5% + 2.35% + 0.15% = 97.5% or 13.5% + 34% + 50% = 97.5%
d Now, 240 g is 2 standard deviations below the mean while 255 g is 1 standard deviation above the mean.	**d** Mass of 240 g is 2 standard deviations below the mean. Mass of 255 g is 1 standard deviation above the mean. Percentage of distribution between 240 g and 255 g is 13.5% + 34% + 34% = 81.5%

WORKED EXAMPLE 27

The number of matches in a box is not always the same. When a sample of boxes was studied it was found that the number of matches in a box approximated a normal (bell-shaped) distribution with a mean number of matches of 50 and a standard deviation of 2. In a sample of 200 boxes, how many would be expected to have more than 48 matches?

THINK	WRITE
1 Find the percentage of boxes with more than 48 matches. Since 48 = 50 − 2, the score of 48 is 1 standard deviation below the mean.	48 matches is 1 standard deviation below the mean. Percentage of boxes with more than 48 matches = 34% + 50% = 84%
2 Find 84% of the total sample.	Number of boxes = 84% of 200 = 168 boxes

Standard *z*-scores

studyon

Unit 3
AOS DA
Topic 4
Concept 2

Standard *z*-scores
Concept summary
Practice questions

To find a comparison between scores in a particular distribution or in different distributions, we use the ***z*-score**. The z-score (also called the *standardised score*) indicates the position of a certain score in relation to the mean.

A z-score of 0 indicates that the score obtained is equal to the mean, a negative z-score indicates that the score is below the mean and a positive z-score indicates a score above the mean.

The z-score measures the distance from the mean in terms of the standard deviation. A score that is exactly one standard deviation above the mean has a z-score of 1. A score that is exactly one standard deviation below the mean has a z-score of −1.

To calculate a z-score we use the formula:

$$z = \frac{x - \mu}{\sigma}$$

where x = the score, μ = the mean of the population and σ = the standard deviation of the population.

WORKED EXAMPLE 28

In an IQ test, the mean IQ is 100 and the standard deviation is 15. Dale's test results give an IQ of 130. Calculate this as a z-score.

THINK	WRITE
1 Write the formula.	$z = \frac{x - \mu}{\sigma}$
2 Substitute for x, μ and σ.	$= \frac{130 - 100}{15}$
3 Calculate the z-score.	$= 2$

Dale's z-score is 2, meaning that his IQ is exactly two standard deviations above the mean.

Not all z-scores will be whole numbers; in fact most will not be. A whole number indicates only that the score is an exact number of standard deviations above or below the mean.

Using Worked example 28, an IQ of 88 would be represented by a z-score of −0.8, as shown below.

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ &= \frac{88 - 100}{15} \\ &= -0.8 \end{aligned}$$

The negative value indicates that the IQ of 88 is below the mean but by less than one standard deviation.

Comparing data

studyon
Unit 3
AOS DA
Topic 4
Concept 3
Comparing data values
Concept summary
Practice questions

An important use of z-scores is to compare scores from different data sets. Suppose that in your maths exam your result was 74 and in English your result was 63. In which subject did you achieve the better result?

At first glance, it may appear that the maths result is better, but this does not take into account the difficulty of the test. A mark of 63 on a difficult English test may in fact be a better result than 74 if it was an easy maths test.

The only way that we can fairly compare the results is by comparing each result with its mean and standard deviation. This is done by converting each result to a z-score.

If, for maths, $\mu = 60$ and $\sigma = 12$, then

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ &= \frac{74 - 60}{12} \\ &= 1.17 \end{aligned}$$

And if, for English, $\mu = 50$ and $\sigma = 8$, then

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ &= \frac{63 - 50}{8} \\ &= 1.625 \end{aligned}$$

The English result is better because the higher z-score shows that the 63 is higher in comparison to the mean of each subject.

WORKED EXAMPLE 29

Janine scored 82 in her physics exam and 78 in her chemistry exam. In physics, $\mu = 62$ and $\sigma = 10$, while in chemistry, $\mu = 66$ and $\sigma = 5$.

a Write both results as a standardised score.

b Which is the better result? Explain your answer.

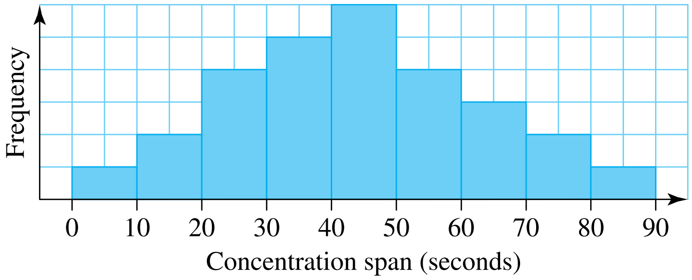
THINK	WRITE
a 1 Write the formula for each subject.	**a** Physics: $z = \frac{x - \mu}{\sigma}$ Chemistry: $z = \frac{x - \mu}{\sigma}$
2 Substitute for x, μ and σ.	$= \frac{82 - 62}{10}$ $= \frac{78 - 66}{5}$
3 Calculate each z-score.	$= 2$ $= 2.4$
b Explain that the subject with the highest z-score is the better result.	**b** The chemistry result is better because of the higher z-score.

In each example the circumstances must be analysed carefully to see whether a higher or lower *z*-score is better. For example, if we were comparing times for runners over different distances, the lower *z*-score would be the better one.

EXERCISE 1.11 The 68–95–99.7% rule and *z*-scores

PRACTISE

1 **WE25** The concentration ability of a large group of adults is tested during a short task which they are asked to complete. The length of the concentration span of those involved during the task is shown.



The mean, μ, is 49 seconds and the standard deviation, σ, is 14 seconds.

Write down the values between which we would expect approximately:

a 68% of the group's concentration spans to fall

b 95% of the group's concentration spans to fall

c 99.7% of the group's concentration spans to fall.

2 The monthly rainfall in Mathmania Island was found to follow a bell-shaped curve with a mean of 45 mm and a standard deviation of 1.7 mm. Write down the rainfall range which we would expect approximately:

a 68% of the group to lie

b 95% of the group to lie

c 99.7% of the group to lie.

3 **WE26** The distribution of masses of potato chips is known to follow a bell-shaped curve with a mean of 200 g and a standard deviation of 7 g. Find the percentage of the potato chips with a mass which is:

a more than 214 g

b more than 200 g

c less than 193 g

d between 193 g and 214 g.

4 The distribution of heights of a group of Melbourne-based employees who work for a large international company is bell-shaped. The data have a mean of 160 cm and a standard deviation of 10 cm.
Find the percentage of this group of employees who are:

a less than 170 cm tall

b less than 140 cm tall

c greater than 150 cm tall

d between 130 cm and 180 cm in height.

5 **WE27** The number of marbles in a bag is not always the same. From a sample of boxes it was found that the number of marbles in a bag approximated a normal distribution with a mean number of 20 and a standard deviation of 1. In a sample of 500 bags, how many would be expected to have more than 19 marbles?

6 The volume of fruit juice in a certain type of container is not always the same. When a sample of these containers was studied it was found that the volume of juice they contained approximated a normal distribution with a mean of 250 mL and a standard deviation of 5 mL. In a sample of 400 containers, how many would be expected to have a volume of:

a more than 245 mL
b less than 240 mL
c between 240 and 260 mL?

7 **WE28** In a Physics test on electric power, the mean result for the class was 76% and a standard deviation of 9%. Drew's result was 97%. Calculate his mark as a z-score.

8 In a maths exam, the mean score is 60 and the standard deviation is 12. Chifune's mark is 96. Calculate her mark as a z-score.

9 **WE29** Bella's Specialist Maths mark was 83 and her English mark was 88. In Specialist Maths the mean was 67 with a standard deviation of 9, while in English the mean was 58 with a standard deviation of 14.

a Convert the marks in each subject to a z-score.
b Which subject is the better result for Bella? Explain.

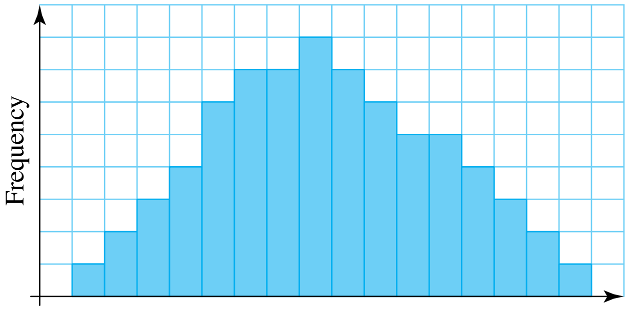
10 Ken's English mark was 75 and his maths mark was 72. In English, the mean was 65 with a standard deviation of 8, while in maths the mean mark was 56 with a standard deviation of 12.

a Convert the mark in each subject to a z-score.
b In which subject did Ken perform better? Explain your answer.

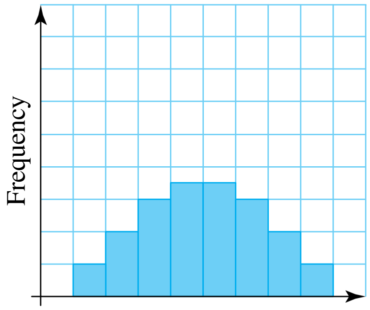
CONSOLIDATE

11 In each of the following, decide whether or not the distribution is approximately bell-shaped.

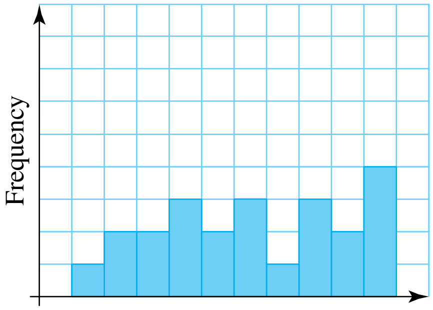
a



b



c



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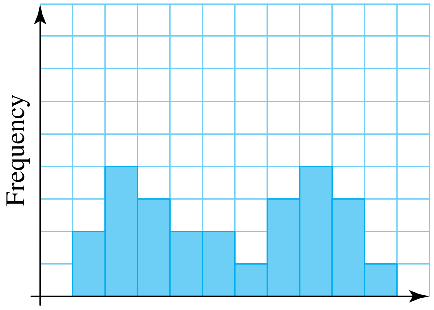
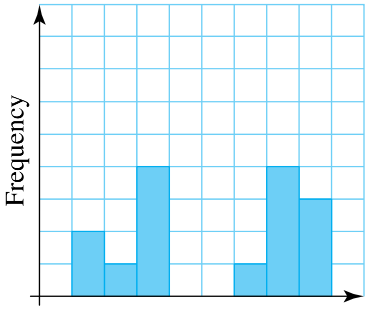


e



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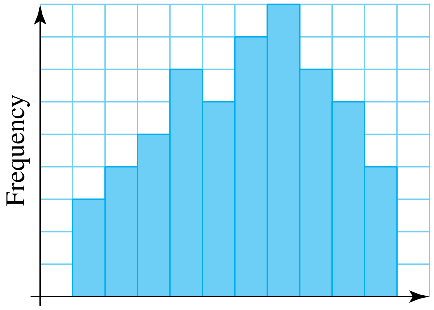
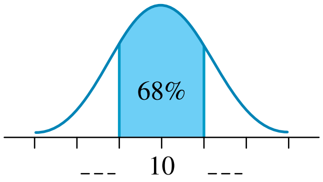
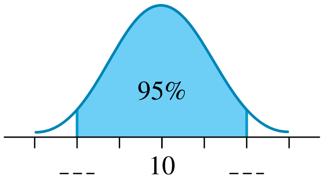


12 Copy and complete the entries on the horizontal scale of the following distributions, given that $\mu = 10$ and $\sigma = 2$.

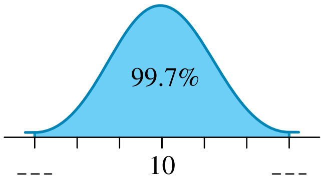
a



b

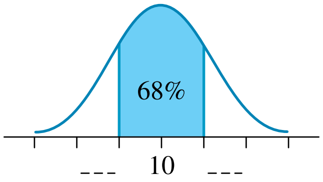


c

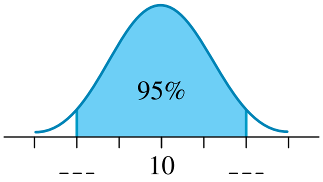


13 Copy and complete the entries on the horizontal scale of the following distributions, given that $\mu = 5$ and $\sigma = 1.3$.

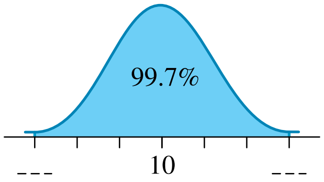
a



b



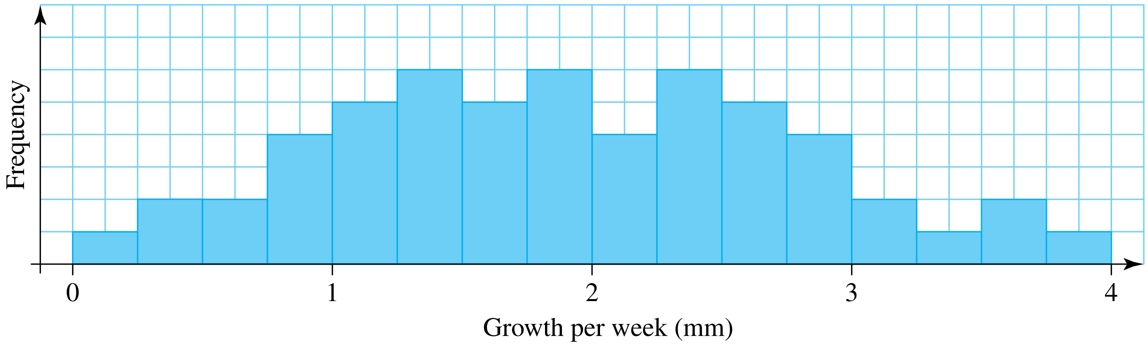
c



14 A research scientist measured the rate of hair growth in a group of hamsters. The findings are shown in the histogram.

The mean growth per week was 1.9 mm and the standard deviation was 0.6 mm. Write down the hair growth rates between which approximately:

a 68% of the values fall
b 95% of the values fall
c 99.7% of the values fall.



15 The heights of the seedlings sold in a nursery have a bell-shaped distribution. The mean height is 7 cm and the standard deviation is 2.

Write down the values between which approximately:

a 68% of seedling heights will lie
b 95% of seedling heights will lie
c 99.7% of seedling heights will lie.

16 A distribution of scores is bell-shaped and the mean score is 26. It is known that 95% of scores lie between 21 and 31.

It is true to say that:

A 68% of the scores lie between 23 and 28
B 97.5% of the scores lie between 23.5 and 28.5

C the standard deviation is 2.5
D 99.7% of the scores lie between 16 and 36
E the standard deviation is 5

17 The number of days taken off in a year by employees of a large company has a distribution which is approximately bell-shaped. The mean and standard deviation of this data are shown below.

Mean = 9 days Standard deviation = 2 days

Find the percentage of employees of this company who, in a year, take off:

a more than 15 days
b fewer than 5 days
c more than 7 days
d between 3 and 11 days
e between 7 and 13 days.

18 A particular bolt is manufactured such that the length is not always the same. The distribution of the lengths of the bolts is approximately bell-shaped with a mean length of 2.5 cm and a standard deviation of 1 mm.

a In a sample of 2000 bolts, how many would be expected to have a length:
  i between 2.4 cm and 2.6 cm
  ii less than 2.7 cm
  iii between 2.6 cm and 2.8 cm?

b The manufacturer rejects bolts which have a length of less than 2.3 cm or a length of greater than 2.7 cm. In a sample of 2000 bolts, how many would the manufacturer expect to reject?

19 In a major exam, every subject has a mean score of 60 and a standard deviation of 12.5. Clarissa obtains the following marks on her exams. Express each as a z-score.

a English 54
b Maths 78
c Biology 61
d Geography 32
e Art 95

20 In a normal distribution the mean is 58. A score of 70 corresponds to a standardised score of 1.5. The standard deviation of the distribution is:

A 6 **B** 8 **C** 10 **D** 12 **E** 9

MASTER

21 In the first maths test of the year, the mean mark was 60 and the standard deviation was 12. In the second test, the mean was 55 and the standard deviation was 15. Barbara scored 54 in the first test and 50 in the second test. In which test did Barbara do better? Explain your answer.

22 The table below shows the average number of eggs laid per week by a random sample of chickens with 3 different types of living conditions.

Number of eggs per week		
Cage chickens	**Barn chickens**	**Free range chickens**
5.0	4.8	4.2
4.9	4.6	3.8
5.5	4.3	4.1

(continued)

Number of eggs per week		
Cage chickens	**Barn chickens**	**Free range chickens**
5.4	4.7	4.0
5.1	4.2	4.1
5.8	3.9	4.4
5.6	4.9	4.3
5.2	4.1	4.2
4.7	4.0	4.3
4.9	4.4	3.9
5.0	4.5	3.9
5.1	4.6	4.0
5.4	4.1	4.1
5.5	4.2	4.1

a Copy and complete the following table by calculating the mean and standard deviation of barn chickens and free range chickens correct to 1 decimal place.

Living conditions	Cage	Barn	Free range
Mean	5.2		
Standard deviation	0.3		

b A particular free range chicken lays an average of 4.3 eggs per week. Calculate the z-score relative to this sample correct to 3 significant figures.

The number of eggs laid by free range chickens is normally distributed. A free range chicken has a z-score of 1.

c Approximately what percentage of chickens lay fewer eggs than this chicken?

d Referring to the table showing the number of eggs per week, prepare five-number summaries for each set of data.
  i State the median of each set of data.
  ii What could be concluded about the egg-producing capabilities of chickens in different living conditions?

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1.12 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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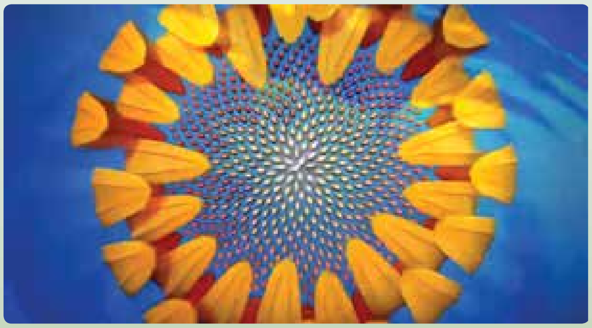
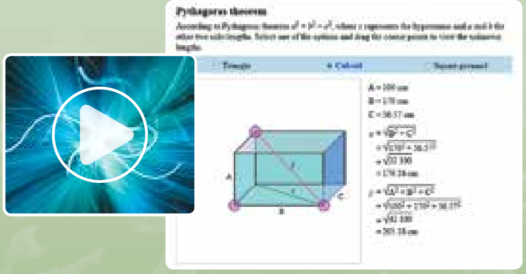
A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

1 Answers

EXERCISE 1.2

1 D

2 D

3 C

4 B

5 Numerical: **a**, **b**, **c**

6 Categorical: **c**, **d**, **e**, **f**, **g**

7 Discrete: **c**
Continuous: **a**, **b**

8 C

9 C

10 Categorical

11 B

12 A

13 Categorical and ordinal

14 Discrete

15 Ordinal

16 D

EXERCISE 1.3

1

Stem	Leaf
1	6 8
2	1 5 8 8 9
3	0 1 3 5 8 9
4	2 8 9

Key: 1|6 = 16

2

Stem	Leaf
0	5
1	1 8 9
2	3 7 9
3	1 2 5 6 7 9
4	1 2 3 5
5	2

Key: 0|5 = $5

The busker's earnings are inconsistent.

3

Stem	Leaf
86	8
87	7
88	0
89	8
90	2 4 8 9
91	
92	0 1 2 6
93	
94	3 9
95	
96	
97	0
98	3 5 9

Key: 86|8 = 86.8%

4

Stem	Leaf
18	5 7 9
19	1 5 6 6 7 9
20	1 3 3 5 9
21	7
22	1

Key: 18|5 = 1.85 cm

5 a

Stem	Leaf
2	0 2 2
2*	5 6 8 8
3	3 3 3
3*	7 7 8 9 9 9

Key: 2|0 = 20 points

b

Stem	Leaf
2	0
2	2 2
2	5
2	6
2	8 8
3	
3	3 3 3
3	
3	7 7
3	8 9 9 9

Key: 2|0 = 20 cm

6 a

Stem	Leaf
4	3 7 7 8 8 9 9 9
5	0 0 0 0 1 2 2 3

Key: 4|3 = 43 cm

b

Stem	Leaf
4	3
4*	7 7 8 8 9 9 9
5	0 0 0 0 1 2 2 3
5*	

Key: 4|3 = 43 cm

c

Stem	Leaf
4	
4	3
4	
4	7 7
4	8 8 9 9 9
5	0 0 0 0 1
5	2 2 3
5	
5	
5	

Key: 4|3 = 43 cm

7 a 1 2 5 8 12 13 13 16
16 17 21 23 24 25 25 26
27 30 32

b 10 11 23 23 30 35 39 41
42 47 55 62

c 101 102 115 118 122 123
123 136 136 137 141 143
144 155 155 156 157

d 50 51 53 53 54 55 55 56
56 57 59

e 1 4 5 8 10 12 16 19 19
21 21 25 29

8

Stem	Leaf
3	7 9
4	2 9 9
5	1 1 2 3 7 8 9
6	1 3 3 8

Key: 3|7 = 37 years

It seems to be an activity for older people.

9 C

10

Stem	Leaf
1*	9
2	
2*	5 8 8 9 9 9
3	0 0 2 2 2 3 3 4
3*	5 5 7 8 9

Key: 2|5 = 25 years

More than half of the parents are 30 or older with a considerable spread of ages, so this statement is not very accurate.

11

Stem	Leaf
1*	9
2	1 2 4
2*	5 6 8 9 9
3	1 1 2 3 4
3*	
4	0 1

Key: 2|1 = 21 hit outs

Bulldogs, Melbourne, St Kilda

12

Stem	Leaf
33	0
34	
35	0 0 1
36	5
37	3
38	0 0
39	0 0 5
40	0 6
41	0 5
42	1 3
43	0 0
44	
45	0

Key: 33|0 = $330

The stem plot shows a fairly even spread of rental prices with no obvious outliers.

13 a

Stem	Leaf
1	5 6 7 7 7 8 9 9 9 9
2	0 0 0 1 1 1 2 3 3 3

Key: 1|5 = 15 mm

b

Stem	Leaf
1	
1*	5 6 7 7 7 8 9 9 9 9
2	0 0 0 1 1 1 2 3 3 3
2*	

Key: 1|5 = 15 mm

c

Stem	Leaf
1	
1	
1	5
1	6 7 7 7
1	8 9 9 9 9
2	0 0 0 1 1 1
2	2 3 3 3
2	
2	
2	

Key: 1|5 = 15 mm

Values are bunched together; they vary little.

14

Stem	Leaf
7	2 8
8	3 3 5 7 8 8
9	0 1 2 2 3 4 8 9
10	0 2 4
11	2

Key: 7|2 = 72 shots

15

Stem	Leaf
6	
6*	8 9
7	1 1 2 2 3 3 3 3 4 4
7*	5 5 5 6 6 7
8	
8*	6

Key: 7|1 = 71 net score

The handicapper has done a good job as most of the net scores are around the same scores; that is, in the 70s.

16 a

Stem	Leaf
6	0 3 9
7	0 1 3 5 6 7 8
8	0 1 3 4 7 8 9 9
9	1 3 7 8

Key: 6|0 = 60%

b

Stem	Leaf
6	0 3
6*	9
7	0 1 3
7*	5 6 7 8
8	0 1 3 4
8*	7 8 9 9
9	1 3
9*	7 8

Key: 6|0 = 60%

17 a

Computer 1	Stem	Computer 2
5	34	0 2 6 8
8 2	35	2 3 5 5 7 8
6 3	36	1 2
6	37	
1 0	38	
2 1	39	
5	40	
0	41	

Key: 34|0 = 340 minutes

b Computer 1 lasts longer but is not as consistent. Computer 2 is more consistent but doesn't last as long.

18 a

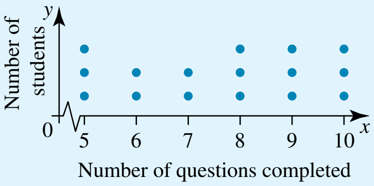
Year 8	Stem	Year 10
9 8	14	
7 5 5 5 3 1 0	15	2 4 6 8 9
8 6 5 4 3 2 1 0	16	0 4 5 7 7 9
5 2 1	17	2 3 4 6 7 8 8
	18	2 5

Key: 14|8 = 148 cm

b As you would expect the Year 10 students are generally taller than the Year 8 students; however, there is a large overlap in the heights.

EXERCISE 1.4

1



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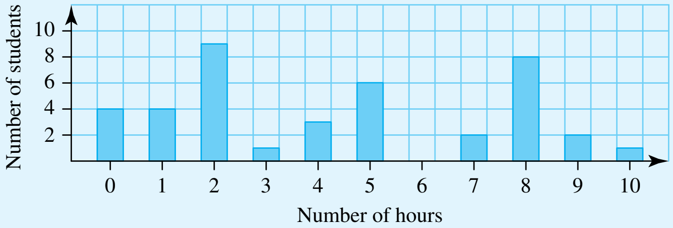

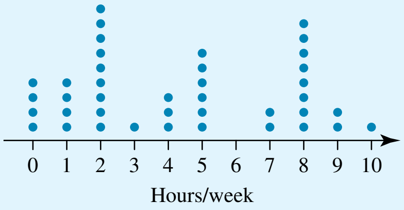

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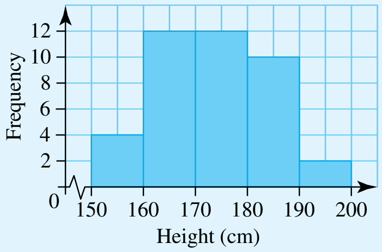
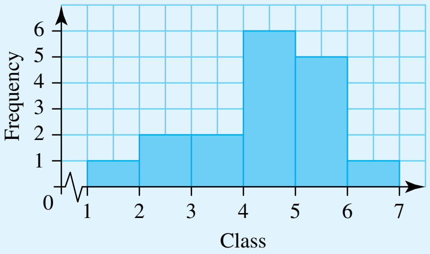


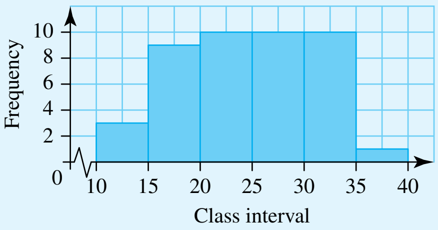
4 a

Class	Frequency
1–	1
2–	2
3–	2
4–	6
5–	5
6–	1



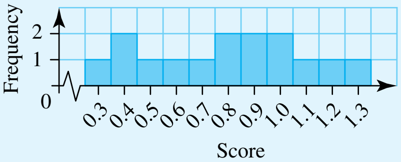
b

Class interval	Frequency
10–	3
15–	9
20–	10
25–	10
30–	10
35–	1



c

Score	Frequency
0.3	1
0.4	2
0.5	1
0.6	1
0.7	1
0.8	2
0.9	2
1.0	2
1.1	1
1.2	1
1.3	1



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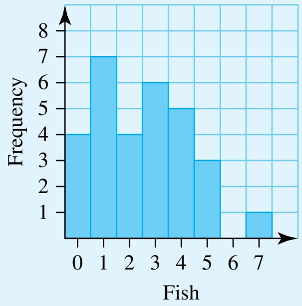

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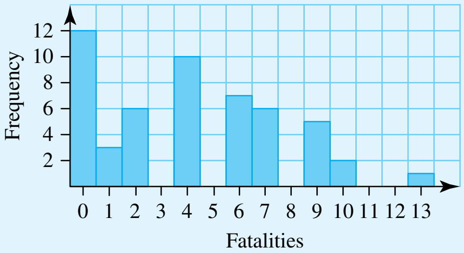

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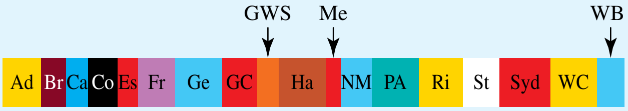
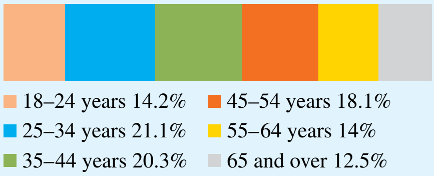
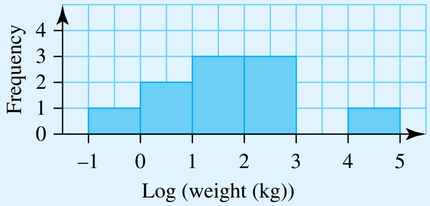


8 Participation in activities



The statement seems untrue as there are similar participation rates for all ages. However, the data don't indicate types of activities.

9



10 D

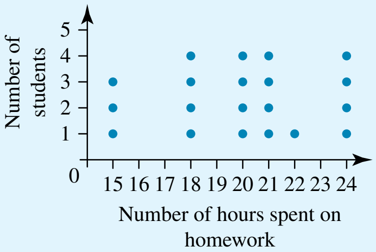
11 16

12 5 times

13 Check your histograms against those shown in the answer to question **4**.

14 D

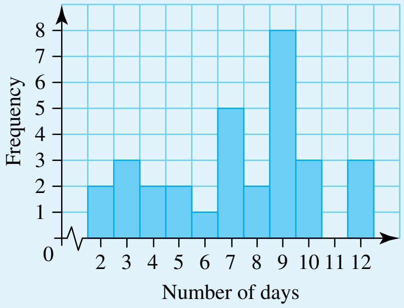
15



16

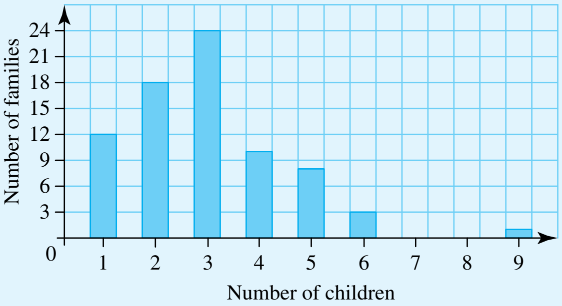
Number of days	Tally	Frequency
2	||	2
3	|||	3
4	||	2
5	||	2
6	|	1
7	𝍸	5
8	||	2
9	𝍸 |||	8
10	|||	3
11		0
12	||	2
		30

17



18 Check your histogram against that shown in the answer to question **17**.

19



20 a B **b** A **c** D

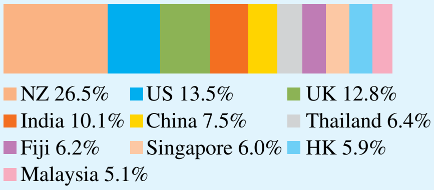
21 A

22 B

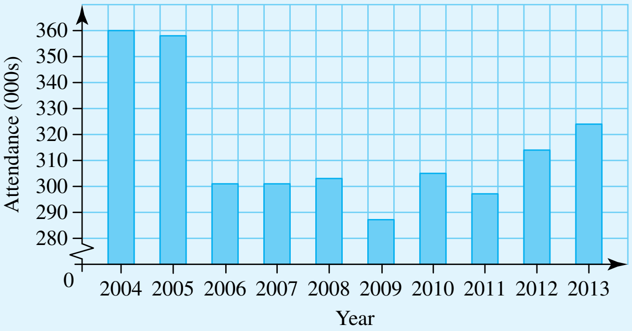
23 a

NZ	26.5%
US	13.5%
UK	12.8%
India	10.1%
China	7.5%
Thailand	6.4%
Fiji	6.2%
Singapore	6.0%
HK	5.9%
Malaysia	5.1%

b



24 a



b Check your bar chart against that shown in the answer to part **a**.

EXERCISE 1.5

1 Positively skewed

2 Negatively skewed

3 a Symmetric **b** Negatively skewed

c Positively skewed **d** Symmetric

e Symmetric **f** Positively skewed

4 a Symmetric, no outliers

b Symmetric, no outliers

c Symmetric, no outliers

d Negatively skewed, no outliers

e Negatively skewed, no outliers

f Positively skewed, no outliers

5 E **6** C

7 Negatively skewed

8 Positively skewed. This tells us that most of the flight attendants in this group spend a similar number of nights (between 2 and 5) interstate per month. A few stay away more than this and a very few stay away a lot more.

9 a Symmetric

b This tells us that there are few low-weight dogs and few heavy dogs but most dogs have a weight in the range of 10 to 19 kg.

10 a Symmetric

b Most students receive about $8 (give or take $2).

11 a Positively skewed

b i 15 **ii** 85%

12 a Positively skewed

b Since most of the data is linked to the lower stems, this suggests that some students do little exercise, but those students who exercise, do quite a bit each week. This could represent the students in teams or in training squads.

13 a Club A: negatively skewed

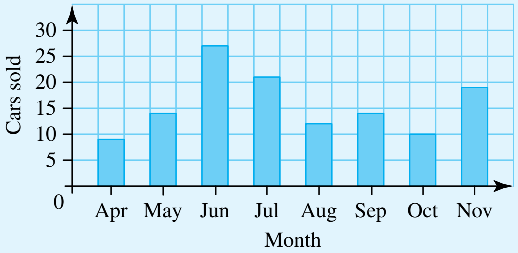
Club B: positively skewed

b Since Club A has more members of its bowling team at the higher stems as compared to Club B; you could say Club A has the older team as compared to Club B.

c i Club A: 11 members over 70 years of age

ii Club B: 4 members over 70 years of age.

14 a



b Positively skewed

c June, July and November represent the months with the highest number of sales.

d This is when the end of financial year sales occur.

EXERCISE 1.6

1 Median = 33 **2** Median = 36.5 goals

3 IQR = 14 **4** IQR = 8

5 IQR = 6.5 **6** IQR = 3.3

7

	Median	Range	Mode
a	37	56	38, 49
b	5	17	5
c	11	18	8, 11
d	42.5	18	43
e	628	72	613, 628, 632

8

	Median	Range
a	6	7
b	17	9
c	6	6
d	10	13
e	18.5	14
f	4	7
g	19	17
h	4.5	9
i	23	21

9 a 10

b 8

c The IQRs (middle 50%) are similar for the two restaurants, but they don't give any indication about the number of cars in each data set.

10 An example is 2 3 6 8 9. There are many others.

11 a The lowest score occurs several times. An example is 2 2 2 3 5 6.

b There are several data points that have the median value. An example is 3 5 5 5 5 5 7.

12 C

13

	Median	Interquartile range	Range	Mode
a	21	18	45	15, 23, 32
b	27.5	8	20	29
c	3.7	3	5.9	3.7

14

	Median	Interquartile range	Range	Mode
a	42	21	91	46
b	32	7	30	34

The data in set **a** have a greater spread than in set **b**, although the medians are similar. The spread of the middle 50% (IQR) of data for set **a** is bigger than for set **b** but the difference is not as great as the spread for all the data (range).

15 a Range = 72, Median = 37.5, Mode = 46, IQR = 22

b Range = 47

Median = 422

Mode = 411

IQR = 20

16 Median = 7, Mode = 7

17 $Q_1 = 42.2$, $Q_3 = 48.15$, IQR = 5.95, Median = 45.1

18 a Median = 93, $Q_1 = 91.5$, $Q_3 = 97$, IQR = 5.5, Range = 30, Mode = 93

b The average handicap of the golfer's should be around 21.

EXERCISE 1.7

1 Range = 39

Median = 25

IQR = 19

2 Range = 3

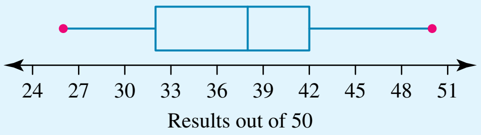
Median = 7.5

IQR = 1.4

3 They could represent the same data.

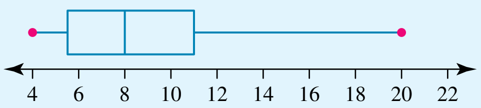
4 They could represent the same data.

5



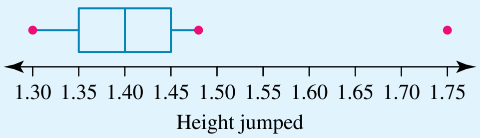
Negatively skewed; 50% of results are between 32 and 42.

6



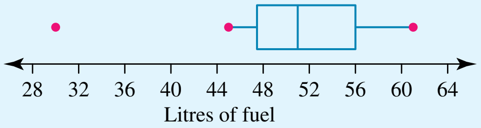
Fairly symmetrical.

7 a



b The data is symmetrical and 1.75 is an outlier.

8



30.3 is an outlier.

9

	Range	Interquartile range	Median
a	12	6	8
b	7	2	5
c	350	100	250
d	100	30	65
e	20	10	25

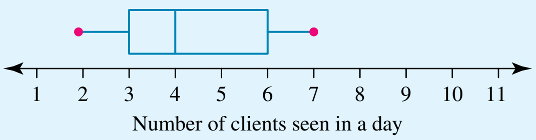
10 a iii **b** iv **c** i **d** ii

11 The boxplots should show the following:

	Minimum value	Q_1	Median	Q_3	Maximum value
a	3	6	8.5	14	18
b	3	5	7	9	12
c	4.3	4.6	5	5.4	5.6
d	11	15.5	18	20	22
e	0.4	0.7	0.9	1.1	1.3

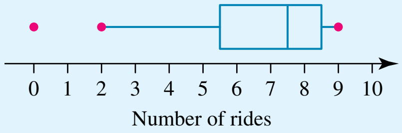
12 D

13



14 See boxplot at foot of the page*

15 a



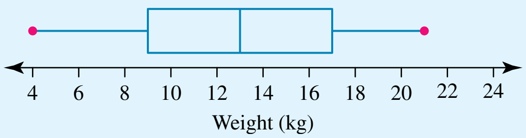
The data are negatively skewed with an outlier on the lower end. The reason for the outlier may be that the person wasn't at the show for long or possibly didn't like the rides.

16 a Two similar properties: both sets of data have the same minimum value and similar IQR value.

b Boys IQR = 16
Girls IQR = 16.5

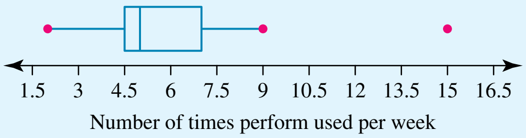
c The reason for an outlier in the boys' data may be that the student did not understand how to do the test, or he stopped during the test rather than working continuously.

17 Median = 13, $Q_1 = 9$, $Q_3 = 17$, $\text{Min}_x = 4$, $\text{Max}_x = 21$

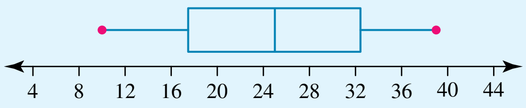


18 Median = 5
$Q_1 = 4.5$
$Q_3 = 7$
$\text{Min}_x = 2, \text{Max}_x = 15$
IQR = 2.5

15 is an outlier



19 a Median = 25
$Q_1 = 17.5$
$Q_3 = 32$
$\text{Min}_x = 11, \text{Max}_x = 39$
IQR = 14.5

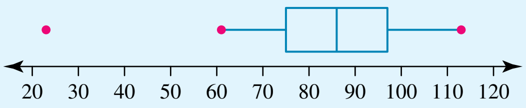


b No outliers

c Check your boxplot against that shown in the answer to part **a**.

20 Median = 86
$Q_1 = 75$
$Q_3 = 97$
$\text{Min}_x = 23, \text{Max}_x = 113$
IQR = 22

23 is an outlier



EXERCISE 1.8

1 23.46

2 8.26

3 10.54

4 26.80

5 a 7.2 **b** 7.125 **c** 4.9875
d 16.7 **e** 0.8818

6 a $1.078\dot{3}$ No, because of the outlier.
b 17 Yes
c 30.875 Yes
d 15.57 No, because of the outlier.

7 12

8 D

9 A

10 a Median **b** Mean **c** Median **d** Median

11 a 36.09 **b** 16.63 **c** 168.25 **d** 18.55

12 a 24.4
b Median = 22

The distribution is positively skewed — confirmed by the table and the boxplot.

13 214.5 papers

14 Approximately 41 fish

15 63.14 kg

16 a Approximately 53 cups
b The median is 54.5, approximately 55 cups.
c The data is negatively skewed.

EXERCISE 1.9

1 3.54 cents

2 14.27%

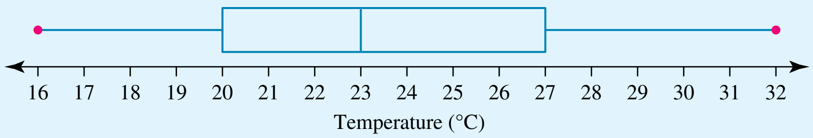
3 9.489

4 7.306

5 a 1.21
b 2.36

---

*14



c 6.01

d 2.45

e 0.06

6 0.48%

7 0.06 m

8 0.51 seconds

9 15.49

10 C

11 2.96 km/h

12 6.067 pens

13 2.39 °C

14 $\bar{x} = 75.7, s = 5.6$

15 3.786 players

16 2.331

EXERCISE 1.10

1 Answers will vary.

2 Answers will vary.

3 B

4 Answers will vary.

5 Answers will vary.

6 Population is larger, since a sample is taken from the population.

7 C

8 E

9 Yes, because the distribution is reasonably symmetric with no outliers

10 B

11 C

12 Answers will vary.

13 Answers will vary.

14 Answers will vary.

EXERCISE 1.11

1 a 68% of group's concentration span falls between 35 secs and 63 secs

b 95% of group's concentration span falls between 21 secs and 77 secs

c 99.7% of group's concentration span falls between 7 secs and 91 secs

2 a 68% of the group to lie between 43.3 mm and 46.7 mm

b 95% of the group to lie between 41.6 mm and 48.4 mm

c 99.7% of the group to lie between 39.9 mm and 50.1 mm

3 a 2.50% b 50% c 16% d 81.5%

4 a 84% b 2.5% c 84% d 97.35%

5 420 bags

6 a 336 containers b 10 containers

c 380 containers.

7 2.33

8 3

9 a Specialist: $\mu = 67, \sigma = 9$

English: $\mu = 58, \sigma = 14$

$z_s = 1.78, z_e = 2.14$

b English has the higher result as it has the higher z-score.

10 a English 1.25, Maths 1.33

b Maths mark is better as it has a higher z-score.

11 a Yes b Yes c No

d No e No f Yes

12 a 8 and 12 b 6 and 14 c 4 and 16

13 a 3.7 and 6.3 b 2.4 and 7.6 c 1.1 and 8.9

14 a 1.3 mm and 2.5 mm

b 0.7 mm and 3.1 mm

c 0.1 mm and 3.7 mm

15 a 5 and 9 b 3 and 11 c 1 and 13

16 C

17 a 0.15% b 2.5% c 84%

d 83.85% e 81.5%

18 a i 1360 ii 1950 iii 317

b 100

19 a −0.48 b 1.44 c 0.08

d −2.24 e 2.8

20 B

21 Second test, Barbara's z-score was −0.33 compared to −0.5 in the first test.

22 a Barn: $\mu = 4.4 \quad \sigma = 0.3$

FR: $\mu = 4.1 \quad \sigma = 0.2$

b 1.18

c 84%

d

	Cage	Barn	Free range
Min_x	4.7	3.9	3.8
Q_1	5	4.1	4
med	5.15	4.35	4.1
Q_3	5.5	4.6	4.2
Max_x	5.8	4.9	4.4

i Cage: 5.15

Barn: 4.35

Free: 4.1

ii It could be concluded that the more space a chicken has, the fewer eggs it lays because the median is greatest for cage eggs.

2

Comparing data sets

2.1 Kick off with CAS

Exploring parallel boxplots with CAS

Parallel boxplots can be used to compare and contrast key information about two different numerical data sets.

1 Use CAS to draw parallel boxplots of the following two data sets, which detail the time it takes for two groups of individuals to complete an obstacle course (rounded to the nearest minute).

Group A: 18, 22, 24, 17, 22, 27, 15, 20, 25, 19, 26, 19, 23, 26, 18, 20, 27, 24, 16
Group B: 21, 22, 19, 21, 17, 21, 18, 24, 21, 20, 18, 24, 35, 22, 19, 17, 23, 20, 19

2 Use your parallel boxplots from question 1 to answer the following questions.

- a Which group has the larger range of data?
- b Which group has the larger interquartile range of data?
- c Which group has the higher median value?

3 One of the data sets has an outlier.

- a How is this marked on the parallel boxplot?
- b State the value of the outlier.

4 a Use CAS to draw a parallel boxplot of the following two data sets.

Set A: 41, 46, 38, 44, 49, 39, 50, 47, 47, 42, 53, 44, 46, 35, 39
Set B: 35, 31, 39, 41, 37, 43, 29, 40, 36, 38, 42, 33, 34, 30, 37

- b Which data set has the largest range?
- c Which data set has an interquartile range of 8?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

2.2 Back-to-back stem plots

studyon

Unit 3
AOS DA
Topic 6
Concept 2

Back-to-back stem plots
Concept summary
Practice questions

In topic 1, we saw how to construct a stem plot for a set of univariate data. We can also extend a stem plot so that it compares two sets of univariate data. Specifically, we shall create a stem plot that displays the relationship between a numerical variable and a categorical variable. We shall limit ourselves in this section to categorical variables with just two categories, for example, gender. The two categories are used to provide two back-to-back leaves of a stem plot.

A back-to-back stem plot is used to display two sets of univariate data, involving a numerical variable and a categorical variable with 2 categories.

WORKED EXAMPLE 1

The girls and boys in Grade 4 at Kingston Primary School submitted projects on the Olympic Games. The marks they obtained out of 20 are as shown.

Girls' marks	16	17	19	15	12	16	17	19	19	16
Boys' marks	14	15	16	13	12	13	14	13	15	14

Display the data on a back-to-back stem plot.

THINK

1 Identify the highest and lowest scores in order to decide on the stems.

WRITE

Highest score = 19
Lowest score = 12
Use a stem of 1, divided into fifths.

THINK

2 Create an unordered stem plot first. Put the boys' scores on the left, and the girls' scores on the right.

WRITE

Leaf Boys	Stem	Leaf Girls
	1	
3 2 3 3	1	2
4 5 4 5 4	1	5
6	1	6 7 6 7 6
	1	9 9 9

Key: 1|2 = 12

THINK

3 Now order the stem plot. The scores on the left should increase in value from right to left, while the scores on the right should increase in value from left to right.

WRITE

Leaf Boys	Stem	Leaf Girls
3 3 3 2	1	2
5 5 4 4 4	1	5
6	1	6 6 6 7 7
	1	9 9 9

Key: 1|2 = 12

eBook*plus*

Interactivity
Back-to-back stem plots
int-6252

The back-to-back stem plot allows us to make some visual comparisons of the two distributions. In Worked example 1, the centre of the distribution for the girls is higher than the centre of the distribution for the boys. The spread of each of the distributions seems to be about the same. For the boys, the scores are grouped around the 12–15 mark; for the girls, they are grouped around the 16–19 mark. On the whole, we can conclude that the girls obtained better scores than the boys did.

To get a more precise picture of the centre and spread of each of the distributions, we can use the summary statistics discussed in topic 1. Specifically, we are interested in:

1. the mean and the median (to measure the centre of the distributions), and
2. the interquartile range and the standard deviation (to measure the spread of the distributions).

We saw in topic 1 that the calculation of these summary statistics is very straightforward using CAS.

WORKED EXAMPLE 2

The number of 'how to vote' cards handed out by various Australian Labor Party and Liberal Party volunteers during the course of a polling day is as shown.

Labor	180	233	246	252	263	270	229	238	226	211
	193	202	210	222	257	247	234	226	214	204
Liberal	204	215	226	253	263	272	285	245	267	275
	287	273	266	233	244	250	261	272	280	279

Display the data using a back-to-back stem plot and use this, together with summary statistics, to compare the distributions of the number of cards handed out by the Labor and Liberal volunteers.

THINK

1 Construct the stem plot.

WRITE

Leaf Labor	Stem	Leaf Liberal
0	18	
3	19	
4 2	20	4
4 1 0	21	5
9 6 6 2	22	6
8 4 3	23	3
7 6	24	4 5
7 2	25	0 3
3	26	1 3 6 7
0	27	2 2 3 5 9
	28	0 5 7

Key: 18|0 = 180

2 Use CAS to obtain summary statistics for each party. Record the mean, median, IQR and standard deviation in the table. (IQR = $Q_3 - Q_1$)

	Labor	Liberal
Mean	227.9	257.5
Median	227.5	264.5
IQR	36	29.5
Standard deviation	23.9	23.4

3 Comment on the relationship.

From the stem plot we see that the Labor distribution is symmetric and therefore the mean and the median are very close, whereas the Liberal distribution is negatively skewed.

Since the distribution is skewed, the median is a better indicator of the centre of the distribution than the mean.

Comparing the medians therefore, we have the median number of cards handed out for Labor at 228 and for Liberal at 265, which is a big difference.

The standard deviations were similar, as were the interquartile ranges. There was not a lot of difference in the spread of the data.

In essence, the Liberal party volunteers handed out more 'how to vote' cards than the Labor party volunteers did.

EXERCISE 2.2 Back-to-back stem plots

PRACTISE

1 **WE1** Boys and girls submitted an assignment on the history of the ANZACs. The results out of 40 are shown.

Girls' results	30	35	31	32	38	33	35	30
Boys' results	34	33	37	39	31	32	39	36

Display the data on a back-to-back stem plot.

2 The marks obtained out of 50 by students in Physics and Chemistry are shown. Display the data on a back-to-back stem plot.

Physics	32	45	48	32	24	30	41	29	44	45	36	34	28	49
Chemistry	46	31	38	28	45	49	34	45	47	33	30	21	32	28

3 **WE2** The number of promotional pamphlets handed out for company A and company B by a number of their reps is shown.

Company A	144	156	132	138	148	160	141	134	132	142	132	134	168	149
Company B	146	131	138	155	145	153	134	153	138	133	130	162	148	160

Display the data using a back-to-back stem plot and use this, together with summary statistics, to compare the number of pamphlets handed out by each company.

4 A comparison of student achievements (out of 100) in History and English was recorded and the results shown.

History	75	78	42	92	59	67	78	82	84	64	77	98
English	78	80	57	96	58	71	74	87	79	62	75	100

a Draw a back-to-back stem plot.
b Use summary statistics and the stem plot to comment on the two subjects.

CONSOLIDATE

5 The marks out of 50 obtained for the end-of-term test by the students in German and French classes are given as shown. Display the data on a back-to-back stem plot.

German	20	38	45	21	30	39	41	22	27	33	30	21	25	32	37	42	26	31	25	37
French	23	25	36	46	44	39	38	24	25	42	38	34	28	31	44	30	35	48	43	34

6 The birth masses of 10 boys and 10 girls (in kilograms, correct to the nearest 100 grams) are recorded in the table. Display the data on a back-to-back stem plot.

Boys	3.4	5.0	4.2	3.7	4.9	3.4	3.8	4.8	3.6	4.3
Girls	3.0	2.7	3.7	3.3	4.0	3.1	2.6	3.2	3.6	3.1

7 The number of delivery trucks making deliveries to a supermarket each day over a 2-week period was recorded for two neighbouring supermarkets — supermarket A and supermarket B. The data are shown in the table.

A	11	15	20	25	12	16	21	27	16	17	17	22	23	24
B	10	15	20	25	30	35	16	31	32	21	23	26	28	29

a Display the data on a back-to-back stem plot.
b Use the stem plot, together with some summary statistics, to compare the distributions of the number of trucks delivering to supermarkets A and B.

8 The marks out of 20 obtained by males and females for a science test in a Year 10 class are given.

Females	12	13	14	14	15	15	16	17
Males	10	12	13	14	14	15	17	19

a Display the data on a back-to-back stem plot.
b Use the stem plot, together with some summary statistics, to compare the distributions of the marks of the males and the females.

9 The end-of-year English marks for 10 students in an English class were compared over 2 years. The marks for 2011 and for the same students in 2012 are as shown.

2011	30	31	35	37	39	41	41	42	43	46
2012	22	26	27	28	30	31	31	33	34	36

a Display the data on a back-to-back stem plot.
b Use the stem plot, together with some summary statistics, to compare the distributions of the marks obtained by the students in 2011 and 2012.

10 The age and gender of a group of people attending a fitness class are recorded.

Female	23	24	25	26	27	28	30	31
Male	22	25	30	31	36	37	42	46

a Display the data on a back-to-back stem plot.

b Use the stem plot, together with some summary statistics, to compare the distributions of the ages of the female members to the male members of the fitness class.

11 The scores on a board game for a group of kindergarten children and for a group of children in a preparatory school are given as shown.

Kindergarten	3	13	14	25	28	32	36	41	47	50
Prep. school	5	12	17	25	27	32	35	44	46	52

a Display the data on a back-to-back stem plot.

b Use the stem plot, together with some summary statistics, to compare the distributions of the scores of the kindergarten children compared to the preparatory school children.

12 A pair of variables that could be displayed on a back-to-back stem plot is:

A the height of a student and the number of people in the student's household
B the time put into completing an assignment and a pass or fail score on the assignment
C the weight of a businessman and his age
D the religion of an adult and the person's head circumference
E the income of an employee and the time the employee has worked for the company

13 A back-to-back stem plot is a useful way of displaying the relationship between:

A the proximity to markets in kilometres and the cost of fresh foods on average per kilogram
B height and head circumference
C age and attitude to gambling (for or against)
D weight and age
E the money spent during a day of shopping and the number of shops visited on that day

14 The score out of 100 a group of males and females received when going for their licence are shown. Construct a back-to-back stem plot of the data.

Male	86	92	100	90	94	82	72	90	88	94	76	80
Female	94	96	72	80	84	92	83	88	90	70	81	83

MASTER

15 A back-to-back stem plot is used to display two sets of data, involving which two variables?

A increasing variables
B discrete and numerical variables
C continuous and categorical variables
D numerical and categorical variables
E numerical and continuous variables

16 The study scores (out of 50) of students who studied both Mathematical Methods and Further Mathematics are shown.

Methods	28	34	41	36	33	39	44	40	39	42	36	31	29	44
Further	30	37	38	41	35	43	44	46	43	48	37	31	28	48

a Display the data in a back-to-back stem plot.

b Use the stem plot, together with some summary statistics, to compare the distributions of the scores for Mathematical Methods compared to Further Mathematics.

2.3 Parallel boxplots and dot plots

We saw in the previous section that we could display relationships between a numerical variable and a categorical variable with just two categories, using a back-to-back stem plot.

When we want to display a relationship between a numerical variable and a categorical variable with two or *more* categories, **parallel boxplots** or **parallel dot plots** can be used.

Parallel boxplots are obtained by constructing individual boxplots for each distribution and positioning them on a common scale.

Parallel dot plots are obtained by constructing individual dot plots for each distribution and positioning them on a common scale.

Construction of individual boxplots was discussed in detail in topic 1. In this section we concentrate on comparing distributions represented by a number of boxplots (that is, on the interpretation of parallel boxplots).

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WORKED EXAMPLE 3

The four Year 7 classes at Western Secondary College complete the same end-of-year maths test. The marks, expressed as percentages for the four classes, are given as shown.

7A	40	43	45	47	50	52	53	54	57	60	69	63	63	68	70	75	80	85	89	90
7B	60	62	63	64	70	73	74	76	77	77	78	82	85	87	89	90	92	95	97	97
7C	50	51	53	55	57	60	63	65	67	69	70	72	73	74	76	80	82	82	85	89
7D	40	42	43	45	50	53	55	59	60	61	69	73	74	75	80	81	82	83	84	90

Display the data using parallel boxplots. Use this to describe any similarities or differences in the distributions of the marks between the four classes.

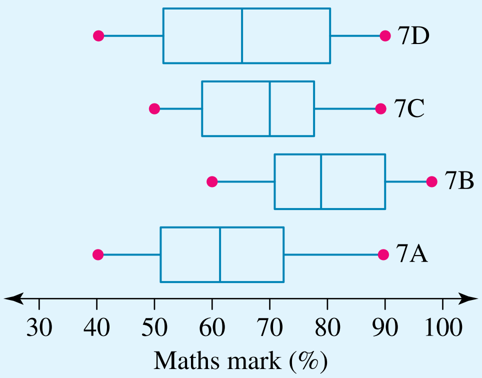
THINK

1 Use CAS to determine the five number summary for each data set.

WRITE/DRAW

	7A	7B	7C	7D
Min.	40	60	50	40
Q_1	51	71.5	58.5	51.5
Median = Q_2	61.5	77.5	69.5	65
Q_3	72.5	89.5	78	80.5
Max.	90	97	89	90

2 Draw the boxplots, labelling each class. All four boxplots share a common scale.



3 Describe the similarities and differences between the four distributions.

Class 7B had the highest median mark and the range of the distribution was only 37. The lowest mark in 7B was 60.

We notice that the median of 7A's marks is 61.5. So, 50% of students in 7A received less than 61.5. This means that about half of 7A had scores that were less than the lowest score in 7B.

The range of marks in 7A was the same as that of 7D with the highest scores in each equal (90), and the lowest scores in each equal (40). However, the median mark in 7D (65) was slightly higher than the median mark in 7A (61.5) so, despite a similar range, more students in 7D received a higher mark than in 7A.

While 7D had a top score that was higher than that of 7C, the median score in 7C (69.5) was higher than that of 7D and almost 25% of scores in 7D were less than the lowest score in 7C. In summary, 7B did best, followed by 7C, then 7D and finally 7A.

EXERCISE 2.3 Parallel boxplots and dot plots

PRACTISE

1 **WE3** The times run for a 100 m race in grade 6 are shown for both boys and girls. The times are expressed in seconds.

Boys	15.5	16.1	14.5	16.9	18.1	14.3	13.8	15.9	16.4	17.3	18.8	17.9	16.1
Girls	16.7	18.4	19.4	20.1	16.3	14.8	17.3	20.3	19.6	18.4	16.5	17.2	16.0

Display the data using parallel boxplots and use this to describe any similarities or differences between the boys' and girls' performances.

2 A teacher taught two Year 10 maths classes and wanted to see how they compared on the end of year examination. The marks are expressed as percentages.

10A	67	73	45	59	67	89	42	56	68	75	87	94	80	98
10D	76	82	62	58	40	55	69	71	89	95	100	84	70	66

Display the data using parallel boxplots and parallel dot plots. Use this to describe any similarities or differences between the two classes.

CONSOLIDATE

3 The heights (in cm) of students in 9A, 10A and 11A were recorded and are shown in the table.

9A	120	126	131	138	140	143	146	147	150
10A	140	143	146	147	149	151	153	156	162
11A	151	153	154	158	160	163	164	166	167

9A	156	157	158	158	160	162	164	165	170
10A	164	165	167	168	170	173	175	176	180
11A	169	169	172	175	180	187	189	193	199

a Construct parallel boxplots to show the data.

b Use the boxplots to compare the distributions of height for the 3 classes.

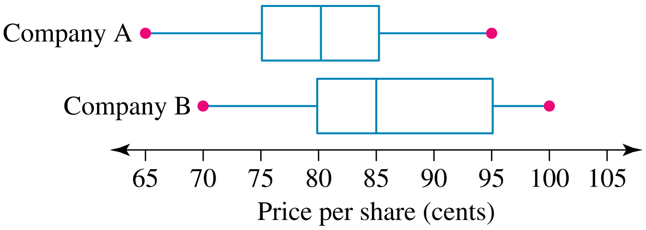
4 The amounts of money contributed annually to superannuation schemes by people in 3 different age groups are as shown.

20–29	2000	3100	5000	5500	6200	6500	6700	7000	9200	10 000
30–39	4000	5200	6000	6300	6800	7000	8000	9000	10 300	12 000
40–49	10 000	11 200	12 000	13 300	13 500	13 700	13 900	14 000	14 300	15 000

a Construct parallel boxplots to show the data.

b Use the boxplots to comment on the distributions.

5 The daily share price of two companies was recorded over a period of one month. The results are presented as parallel boxplots.



State whether each of the following statements is true or false.

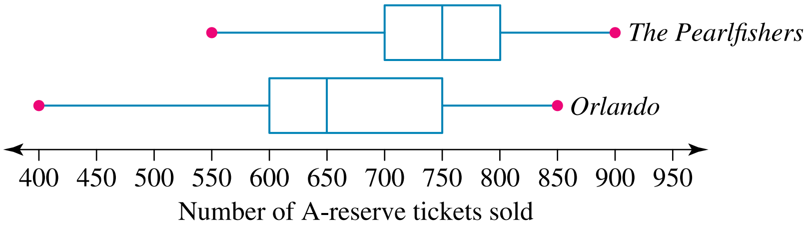
a The distribution of share prices for Company A is symmetrical.

b On 25% of all occasions, share prices for Company B equalled or exceeded the highest price recorded for Company A.

c The spread of the share prices was the same for both companies.

d 75% of share prices for Company B were at least as high as the median share price for Company A.

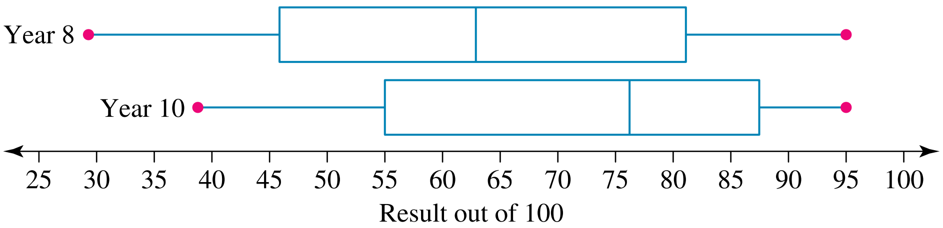
6 Last year, the spring season at the Sydney Opera House included two major productions: *The Pearlfishers* and *Orlando*. The number of A-reserve tickets sold for each performance of the two operas is shown as parallel boxplots.



a Which of the two productions proved to be more popular with the public, assuming A-reserve ticket sales reflect total ticket sales? Explain your answer.

b Which production had a larger variability in the number of patrons purchasing A-reserve tickets? Support your answer with the necessary calculations.

7 The results for a maths test given to classes in two different year levels, one in Year 8 and the other in Year 10, are given by the parallel boxplots.



The percentage of Year 10 students who obtained a mark greater than 87 was:

A 2% **B** 5% **C** 20% **D** 25% **E** 75%

8 From the parallel boxplots in question **7**, it can be concluded that:

A the Year 8 results were similar to the Year 10 results
B the Year 8 results were lower than the Year 10 results and less variable
C the Year 8 results were lower than the Year 10 results and more variable
D the Year 8 results were higher than the Year 10 results and less variable
E the Year 8 results were higher than the Year 10 results and more variable

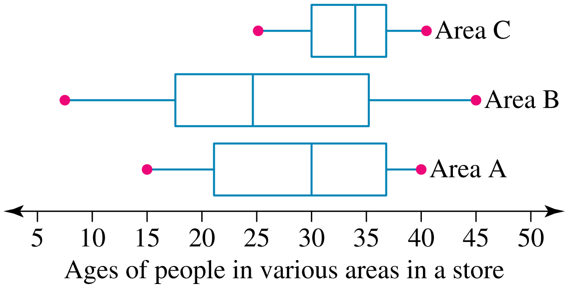
9 The scores of 10 competitors on two consecutive days of a diving competition are recorded below:

Day 1	5.4	4.1	5.4	5.6	4.9
	5.6	5.4	6.0	5.8	6.0
Day 2	4.9	5.1	5.3	5.8	5.7
	5.2	5.8	5.4	5.5	6.0

Construct parallel dot plots to show the data and comment on the divers' results over the two days.

The following figure relates to questions **10** to **12**.

The ages of customers in different areas of a department store are as shown.



10 Which area has the largest range from Q_2 (the median) to Q_3: Area A, Area B or Area C?

11 Which area has the largest range: Area A, Area B or Area C?

12 Which area has the highest median age: Area A, Area B or Area C?

MASTER

13 The numbers of jars of vitamin A, B, C and multi-vitamins sold per week by a local chemist are shown in the table.

Vitamin A	5	6	7	7	8	8	9	11	13	14
Vitamin B	10	10	11	12	14	15	15	15	17	19
Vitamin C	8	8	9	9	9	10	11	12	12	13
Multi-vitamins	12	13	13	15	16	16	17	19	19	20

Construct parallel boxplots to display the data and use it to compare the distributions of sales for the 4 types of vitamin.

14 Eleven golfers in a golf tournament play 18 holes each day. The scores for each of the golfers on the four days are given below. Display this data using parallel boxplots.

Thursday	Friday	Saturday	Sunday
70	77	81	70
71	78	83	81
75	81	84	81
79	82	84	88
80	83	86	88
81	83	86	89
83	85	87	90
83	85	87	90
84	85	87	91
85	88	88	93
90	89	89	94

2.4 Two-way (contingency) frequency tables and segmented bar charts

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Unit 3

AOS DA

Topic 6

Concept 4

Two-way frequency tables and segmented bar charts

Concept summary

Practice questions

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Two-way tables and segmented bar graphs

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When we are examining the relationship between two categorical variables, **two-way (or contingency) tables** are an excellent tool. Consider the following example.

Once the two-way table is formed, **marginal distributions** and **conditional distributions** can both be found. Marginal distributions are the sums (totals) of the row or the column and are found in the margins of the table. The conditional distribution is the sub-population (sample) and this is found in the middle of the table.

If we were to look at mobile phone preference as shown in the table the marginal distributions are the totals, as shown by the green highlighted numbers.

	Apple	Samsung	Nokia	Total
Men	13	9	3	25
Women	17	7	1	25
Total	30	16	4	50

The conditional distribution is the sub-population, so if we are looking at people who prefer Samsung, the conditional distribution is shown by the purple highlighted numbers.

	Apple	Samsung	Nokia	Total
Men	13	9	3	25
Women	17	7	1	25
Total	30	16	4	50

WORKED EXAMPLE 4

At a local shopping centre, 34 females and 23 males were asked which of the two major political parties they preferred. Eighteen females and 12 males preferred Labor. Display these data in a two-way (contingency) table, and calculate the party preference for males and females.

THINK

WRITE

1 Draw a table. Record the respondents' sex in the columns and party preference in the rows of the table.

Party preference	Female	Male	Total
Labor			
Liberal			
Total			

2 We know that 34 females and 23 males were asked. Put this information into the table and fill in the total.

We also know that 18 females and 12 males preferred Labor. Put this information in the table and find the total of people who preferred Labor.

Party preference	Female	Male	Total
Labor	18	12	30
Liberal			
Total	34	23	57

3 Fill in the remaining cells. For example, to find the number of females who preferred the Liberals, subtract the number of females preferring Labor from the total number of females asked: 34 − 18 = 16.

Party preference	Female	Male	Total
Labor	18	12	30
Liberal	16	11	27
Total	34	23	57

4 Marginal distributions for party preference for males and females refers to percentage (probability) of each party. For Labor there are 30 out of a total of 57.

Labor: $\frac{30}{57} = 0.53$

5 For Liberal there are 27 out of a total of 57.

Liberal: $\frac{27}{57} = 0.47$

In Worked example 4, we have a very clear breakdown of data. We know how many females preferred Labor, how many females preferred the Liberals, how many males preferred Labor and how many males preferred the Liberals.

If we wish to compare the number of females who prefer Labor with the number of males who prefer Labor, we must be careful. While 12 males preferred Labor compared to 18 females, there were fewer males than females being asked. That is, only 23 males were asked for their opinion, compared to 34 females.

To overcome this problem, we can express the figures in the table as percentages.

WORKED EXAMPLE 5

Fifty-seven people in a local shopping centre were asked whether they preferred the Australian Labor Party or the Liberal Party. The results are as shown.

Convert the numbers in this table to percentages.

Party preference	Female	Male	Total
Labor	18	12	30
Liberal	16	11	27
Total	34	23	57

THINK

Draw the table, omitting the 'total' column.

Fill in the table by expressing the number in each cell as a percentage of its column's total. For example, to obtain the percentage of males who prefer Labor, divide the number of males who prefer Labor by the total number of males and multiply by 100%.

$\frac{12}{23} \times 100\% = 52.2\%$ (correct to 1 decimal place)

WRITE

Party preference	Female	Male
Labor	52.9	52.2
Liberal	47.1	47.8
Total	100.0	100.0

We could have also calculated percentages from the table rows, rather than columns. To do that we would, for example, have divided the number of females who preferred Labor (18) by the total number of people who preferred Labor (30) and so on. The table shows this:

Party preference	Female	Male	Total
Labor	60.0	40.0	100
Liberal	59.3	40.7	100

By doing this we have obtained the percentage of people who were female and preferred Labor (60%), and the percentage of people who were male and preferred Labor (40%), and so on. This highlights facts different from those shown in the previous table. In other words, different results can be obtained by calculating percentages from a table in different ways.

In the above example, the respondent's gender is referred to as the **explanatory variable**, and the party preference as the **response variable**.

As a general rule, when the explanatory variable is placed in the columns of the table, the percentages should be calculated in columns.

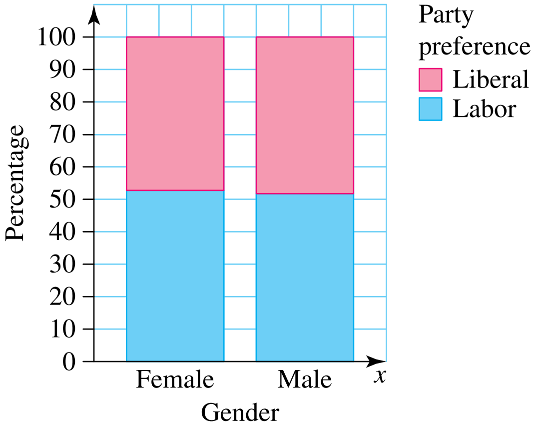
Comparing percentages in each row of a two-way table allows us to establish whether a relationship exists between the two categorical variables that are being examined. As we can see from the table in Worked example 5, the percentage of females who preferred Labor is about the same as that of males. Likewise, the percentage of females and males preferring the Liberal Party are almost equal. This indicates that for the group of people participating in the survey, party preference is not related to gender.

Segmented bar charts

When comparing two categorical variables, it can be useful to represent the results from a two-way table (in percentage form) graphically. We can do this using **segmented bar charts**.

A segmented bar chart consists of two or more columns, each of which matches one column in the two-way table. Each column is subdivided into segments, corresponding to each cell in that column.

For example, the data from Worked example 5 can be displayed using the segmented bar chart shown.



The segmented bar chart is a powerful visual aid for comparing and examining the relationship between two categorical variables.

WORKED EXAMPLE 6

Sixty-seven primary and 47 secondary school students were asked about their attitude to the number of school holidays which should be given. They were asked whether there should be fewer, the same number, or more school holidays. Five primary students and 2 secondary students wanted fewer holidays, 29 primary and 9 secondary students thought they had enough holidays (that is, they chose the same number) and the rest thought they needed to be given more holidays.

Present these data in percentage form in a two-way table and a segmented bar chart. Compare the opinions of the primary and the secondary students.

THINK

1 Put the data in a table. First, fill in the given information, then find the missing information by subtracting the appropriate numbers from the totals.

WRITE/DRAW

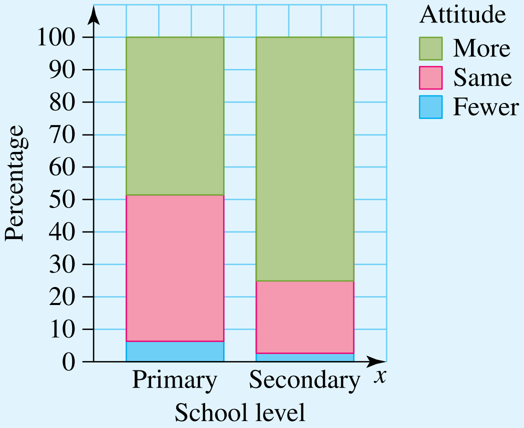
Attitude	Primary	Secondary	Total
Fewer	5	2	7
Same	29	9	38
More	33	36	69
Total	67	47	114

2 Calculate the percentages. Since the explanatory variable (the level of the student: primary or secondary) has been placed in the columns of the table, we calculate the percentages in columns. For example, to obtain the percentage of primary students who wanted fewer holidays, divide the number of such students by the total number of primary students and multiply by 100%.

That is, $\frac{5}{67} \times 100\% = 7.5\%$.

Attitude	Primary	Secondary
Fewer	7.5	4.3
Same	43.3	19.1
More	49.2	76.6
Total	100.0	100.0

3 Rule out the set of axes. (The vertical axis shows percentages from 0 to 100, while the horizontal axis represents the categories from the columns of the table.) Draw two columns to represent each category — primary and secondary. Columns must be the same width and height (up to 100%). Divide each column into segments so that the height of each segment is equal to the percentage in the corresponding cell of the table. Add a legend to the graph.



4 Comment on the results.

Secondary students were much keener on having more holidays than were primary students.

EXERCISE 2.4 Two-way (contingency) frequency tables and segmented bar charts

PRACTISE

1 **WE4** A group of 60 people, 38 females and 22 males, were asked whether they prefer an Apple or Samsung phone. Twenty-three females and 15 males said they preferred an Apple phone. Display this data in a two-way (contingency) table and calculate the marginal distribution for phone preference for males and females.

2 A group of 387 females and 263 males were asked their preference from Coke and Pepsi. Two hundred and twenty-one females preferred Coke, whereas 108 males preferred Pepsi. Display this data in a two-way (contingency) table and calculate the conditional distribution of drink preference among females.

3 **WE5** A group of 60 people were asked their preferences on phones. The results are shown.

Convert the numbers in this table to percentages.

Phone	Females	Males	Total
Apple	23	15	38
Samsung	15	7	22
Total	38	22	60

4 A group of 650 people were asked their preferences on soft drink. The results are shown.

Convert the numbers in this table to percentages.

Drink	Females	Males	Total
Pepsi	221	155	376
Coke	166	108	274
Total	387	263	650

5 **WE6** Sixty-one females and 57 males were asked which they prefer off the menu: entrée, main or dessert. Seven males and 18 females preferred entrée, while 31 males and 16 females said they preferred the main course, with the remainder having dessert as their preferred preference.

Present these data in percentage form in a two-way table and a segmented bar chart. Compare the opinions of the males and females on their preferences.

6 Ninety-three people less than 40 years of age and 102 people aged 40 and over were asked where their priority financially is, given the three options 'mortgage', 'superannuation' or 'investing'. Eighteen people in the 40 and over category and 42 people in the less than 40 years category identified mortgage as their priority, whereas 21 people under 40 years of age and 33 people aged 40 and over said investment was most important. The rest suggested superannuation was their most important priority.

Present these data in percentage form in a two-way table and segmented bar chart. Compare the opinions of the under 40s to the people aged 40 and over on their priority to their finances.

CONSOLIDATE

7 In a survey, 139 women and 102 men were asked whether they approved or disapproved of a proposed freeway. Thirty-seven women and 79 men approved of the freeway. Display these data in a two-way table (not as percentages), and calculate the approval or disapproval of the proposed freeway for men and women.

8 Students at a secondary school were asked whether the length of lessons should be 45 minutes or 1 hour. Ninety-three senior students (Years 10–12) were asked and it was found 60 preferred 1-hour lessons, whereas of the 86 junior students (Years 7–9), 36 preferred 1-hour lessons. Display these data in a two-way table (not as percentages), and calculate the conditional distribution on length of lessons and senior students.

9 For each of the following two-way frequency tables, complete the missing entries.

a

Attitude	Female	Male	Total
For	25	i	47
Against	ii	iii	iv
Total	51	v	92

b

Attitude	Female	Male	Total
For	**i**	**ii**	21
Against	**iii**	21	**iv**
Total	**v**	30	63

c

Party preference	Female	Male
Labor	**i**	42%
Liberal	53%	**ii**
Total	**iii**	**iv**

10 Sixty single men and women were asked whether they prefer to rent by themselves, or to share accommodation with friends. The results are shown below.

Preference	Men	Women	Total
Rent by themselves	12	23	35
Share with friends	9	16	25
Total	21	39	60

Convert the numbers in this table to percentages.

The information in the following two-way frequency table relates to questions **11** and **12**.

The data show the reactions of administrative staff and technical staff to an upgrade of the computer systems at a large corporation.

Attitude	Administrative staff	Technical staff	Total
For	53	98	151
Against	37	31	68
Total	90	129	219

11 From the previous table, we can conclude that:

A 53% of administrative staff were for the upgrade
B 37% of administrative staff were for the upgrade
C 37% of administrative staff were against the upgrade
D 59% of administrative staff were for the upgrade
E 54% of administrative staff were against the upgrade

12 From the previous table, we can conclude that:

A 98% of technical staff were for the upgrade
B 65% of technical staff were for the upgrade
C 76% of technical staff were for the upgrade
D 31% of technical staff were against the upgrade
E 14% of technical staff were against the upgrade

13 Delegates at the respective Liberal Party and Australian Labor Party conferences were surveyed on whether or not they believed that marijuana should be legalised. Sixty-two Liberal delegates were surveyed and 40 of them were against legalisation. Seventy-one Labor delegates were surveyed and 43 were against legalisation.

Present the data in percentage form in a two-way frequency table. Comment on any differences between the reactions of the Liberal and Labor delegates.

14 Use the results in question **13** to draw a segmented bar chart.

The information in the following table relates to questions **15**–**18**.

The amount of waste recycled by 100 townships across Australia was rated as low, medium or high and the size of the town as small, mid-sized or large.

The results of the ratings are:

Amount of waste recycled	**Type of town**		
	Small	**Mid-sized**	**Large**
Low	6	7	4
Medium	8	31	5
High	5	16	18

15 The percentage of mid-sized towns rated as having a high level of waste recycling is closest to:

A 41% **B** 25% **C** 30% **D** 17% **E** 50%

16 The variables, *Amount of waste recycled* and *Type of town*, as used in this rating are:

A both categorical variables
B both numerical variables
C numerical and categorical respectively
D categorical and numerical respectively
E neither categorical nor numerical variables

MASTER

17 Calculate the conditional distribution for amount of waste and large towns.

18 Calculate the percentage of small towns rated as having a high level of waste recycling.

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2.5 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

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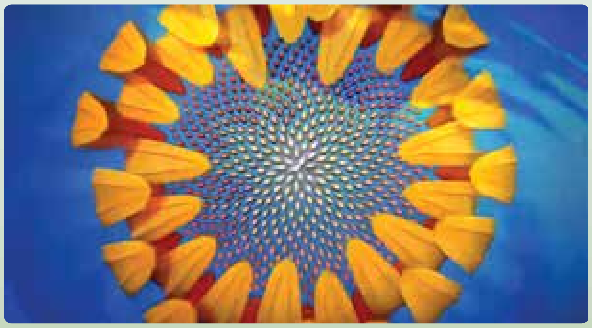
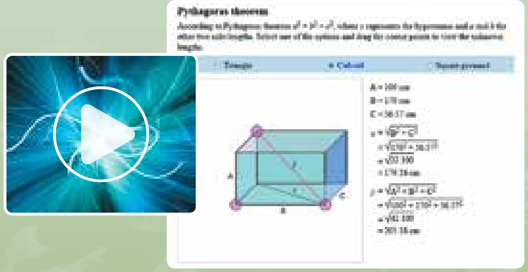
A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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2 Answers

EXERCISE 2.2

1 Key 3|1 = 31

Leaf Boys	Stem	Leaf Girls
1	3	0 0 1
3 2	3	2 3
4	3	5 5
7 6	3	
9 9	3	8

2 Key 2|4 = 24

Physics Leaf	Stem	Chemistry Leaf
4	2	1
9 8	2*	8 8
4 2 2 0	3	0 1 2 3 4
6	3*	8
4 1	4	
9 8 5 5	4*	5 5 6 7 9

3 Key 13|0 = 130

Company A Leaf	Stem	Company B Leaf
4 4 2 2 2	13	0 1 3 4
8	13*	8 8
4 2 1	14	
9 8	14*	5 6 8
	15	3 3
6	15*	5
0	16	0 2
8	16*	

	Company A	Company B
Mean	143.57	144.71
Median	141.5	145.5
IQR	149 − 134 = 15	153 − 134 = 19
Standard deviation	11.42	10.87

They are both positively skewed. The median is a better indicator of the centre of the distribution than the mean. This shows Company B handing out more pamphlets, taking into account that the IQR and the standard deviations are quite similar.

4 a Key 5|7 = 57

History Leaf	Stem	English Leaf
2	4	
9	5	7 8
7 4	6	2
8 8 7 5	7	1 4 5 8 9
4 2	8	0 7
8 2	9	6
	10	0

b

	History	English
Mean	74.67	76.42
Median	77.5	76.5
IQR	83 − 65.5 = 17.5	83.5 − 66.5 = 17
Standard deviation	15.07	13.60

History has a slightly higher median; however, English has a slightly higher mean. Their standard deviations are similar, so overall the results are quite similar.

5 Key: 2|3 = 23

Leaf German	Stem	Leaf French
2 1 1 0	2	3 4
7 6 5 5	2*	5 5 8
3 2 1 0 0	3	0 1 4 4
9 8 7 7	3*	5 6 8 8 9
2 1	4	2 3 4 4
5	4*	6 8

6 Key: 2*|7 = 2.7 (kg)

Leaf Boys	Stem	Leaf Girls
	2*	6 7
4 4	3	0 1 1 2 3
8 7 6	3*	6 7
3 2	4	0
9 8	4*	
0	5	

7 a Key: 2*|5 = 25 trucks

Leaf A	Stem	Leaf B
2 1	1	0
7 7 6 6 5	1*	5 6
4 3 2 1 0	2	0 1 3
7 5	2*	5 6 8 9
	3	0 1 2
	3*	5

b For supermarket A the mean is 19, the median is 18.5, the standard deviation is 4.9 and the interquartile range is 7. The distribution is symmetric.

For supermarket B the mean is 24.4, the median is 25.5, the standard deviation is 7.2 and the interquartile range is 10. The distribution is symmetric.

The centre and spread of the distribution of supermarket B is higher than that of supermarket A.

There is greater variation in the number of trucks arriving at supermarket B.

8 a Key: 1|2 = 12 marks

Leaf Females	Stem	Leaf Males
	1	0
3 2	1	2 3
5 5 4 4	1	4 4 5
7 6	1	7
	1	9

b For the marks of the females, the mean is 14.5, the median is 14.5, the standard deviation is 1.6 and the interquartile range is 2. The distribution is symmetric.

For the marks of the males, the mean is 14.25, the median is 14, the standard deviation is 2.8 and the interquartile range is 3.5. The distribution is symmetric.

The centre of each distribution is about the same. The spread of marks for the boys is greater, however. This means that there is a wider variation in the abilities of the boys compared to the abilities of the girls.

9 a Key: 2*|6 = 26 marks

Leaf 2011	Stem	Leaf 2012
	2	2
	2*	6 7 8
1 0	3	0 1 1 3 4
9 7 5	3*	6
3 2 1 1	4	
6	4*	

b The distribution of marks for 2011 and for 2012 are each symmetric.

For the 2011 marks, the mean is 38.5, the median is 40, the standard deviation is 5.2 and the interquartile range is 7. The distribution is symmetric.

For the 2012 marks, the mean is 29.8, the median is 30.5, the standard deviation is 4.2 and the interquartile range is 6.

The spread of each of the distributions is much the same, but the centre of each distribution is quite different with the centre of the 2012 distribution lower. The work may have become a lot harder!

10 a Key: 3*|6 = 36 years old

Leaf Female	Stem	Leaf Male
4 3	2	2
8 7 6 5	2*	5
1 0	3	0 1
	3*	6 7
	4	2
	4*	6

b For the distribution of the females, the mean is 26.75, the median is 26.5, the standard deviation is 2.8 and the interquartile range is 4.5.

For the distribution of the males, the mean is 33.6, the median is 33.5, the standard deviation is 8.2 and the interquartile range is 12.

The centre of the distributions is very different: it is much higher for the males. The spread of the ages of the females who attend the fitness class is very small but very large for males.

11 a Key: 5|0 = 50 points

Leaf Kindergarten	Stem	Leaf Prep.
3	0	5
4 3	1	2 7
8 5	2	5 7
6 2	3	2 5
7 1	4	4 6
0	5	2

b For the distribution of scores of the kindergarten children, the mean is 28.9, the median is 30, the standard deviation is 15.4 and the interquartile range is 27.

For the distribution of scores for the prep. children, the mean is 29.5, the median is 29.5, the standard deviation is 15.3 and the interquartile range is 27.

The distributions are very similar. There is not a lot of difference between the way the kindergarten children and the prep. children scored.

12 B

13 C

14 Key: 7|2 = 72

Male Leaf	Stem	Female Leaf
2	7	0 2
6	7*	
2 0	8	0 1 3 3 4
8 6	8*	8
4 4 2 0 0	9*	0 2 4
	9*	6
0	10	

15 D

16 a Key: 3|1 = 31

Mathematical Methods Leaf	Stem	Further Mathematics Leaf
9 8	2*	8
4 3 1	3	0 1
9 9 6 6	3*	5 7 7 8
4 4 2 1 0	4	1 3 3 4
	4*	6 8 8

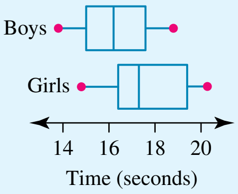
	Mathematical Methods	Further Mathematics
Mean	36.86	39.21
Median	37.5	39.5
IQR	41 − 33 = 8	44 − 35 = 9
Standard deviation	5.29	6.58

b Mathematical Methods has a slightly lower IQR and standard deviation. It was found that Further Mathematics had a greater mean (39.21) as compared to Mathematical Methods (39.5), as well as a greater median; 39.5 as compared to 36.86. This suggests that students do better in Further Mathematics as compared to Mathematical Methods by an average of two study scores.

EXERCISE 2.3

Note: When comparing and contrasting data sets, answers will naturally vary. It is good practice to discuss your conclusions in a group to consider different viewpoints.

1

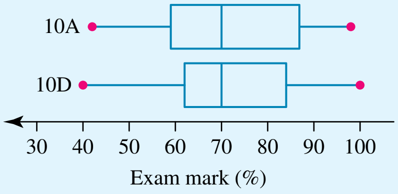


Boys: 13.8, 15, 16.1, 17.6, 18.8

Girls: 14.8, 16.4, 17.3, 19.5, 20.3

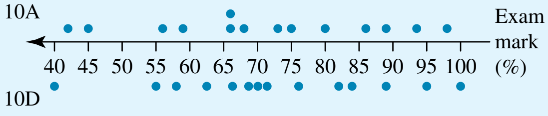
From the boxplots we can see that the boys have a significantly lower median. The boys' median is lower than Q_1 of the girls' time; that is; the lowest 25% of times for the girls is greater than the lowest 50% of times for the boys.

2



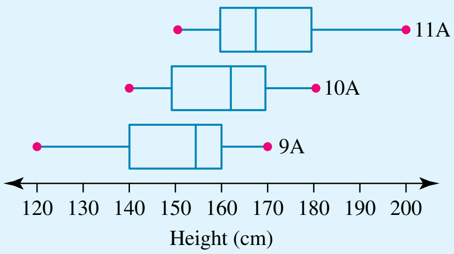
10A: 42, 59, 70.5, 87, 98

10D: 40, 62, 70.5, 84, 100



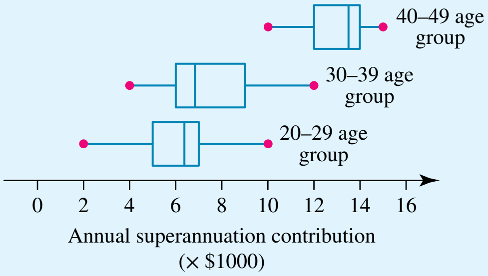
From the boxplots you can see the medians are the same but 10D has a higher mean. 10D also has the highest score of 100%, but 10D also has the lowest score. Since Q_1 and Q_3 are closer together for 10D their results are more consistent around the median. The parallel dot plot confirms this but doesn't give you any further information.

3 a



b Clearly, the median height increases from Year 9 to Year 11. There is greater variation in 9A's distribution than in 10A's. There is a wide range of heights in the lower 25% of the distribution of 9A's distribution. There is a greater variation in 11A's distribution than in 10A's, with a wide range of heights in the top 25% of the 11A distribution.

4 a



b Clearly, there is a great jump in contributions to superannuation for people in their 40s. The spread of contributions for that age group is smaller than for people in their 20s or 30s, suggesting that a high proportion of people in their 40s are conscious of superannuation. For people in their 20s and 30s, the range is greater, indicating a range of interest in contributing to super.

5 a True **b** True

c False **d** True

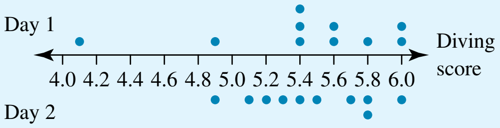
6 a *The Pearlfishers*, which had a significantly higher medium number of A-reserve tickets sold, as well as a higher minimum and maximum number of A-reserve tickets sold.

b *Orlando*, which had both a larger range and IQR of A-reserve tickets sold.

7 D

8 C

9



The dives on day 1 were more consistent than the dives on day 2 with most of the dives between 5.4 and 6.0

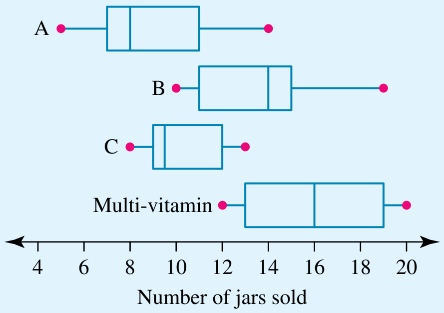
(inclusive), despite two lower dives. Day 2 was more spread with dives from 4.9 to 6.0 (inclusive). It must be noted that there were no very low scoring dives on the second day.

10 B

11 B

12 C

13

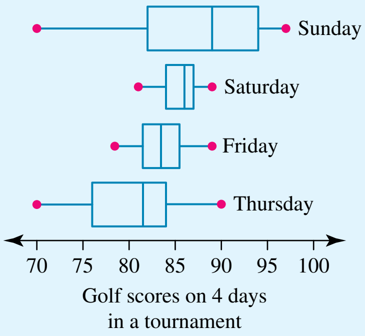


Overall, the biggest sales were of multi-vitamins, followed by vitamin B, then C and finally vitamin A.

14 For all four days, the median is the 6th score.

For all four days, Q_1 is the 3rd score. For all four days, Q_3 is the 9th score.

Day	Min.	Max.	Range	Median
Thursday	70	90	20	81
Friday	77	89	12	83
Saturday	81	89	8	86
Sunday	70	94	24	89



EXERCISE 2.4

1 Note that black data is given in the question; red data are the answers.

Phone	Female	Male	Total
Apple	23	15	38
Samsung	15	7	22
Total	38	22	60

Marginal distribution: Apple = 0.63 Samsung = 0.37

2 Note that black data is given in the question; red data are the answers.

Drink	Female	Male	Total
Coke	221	155	376
Pepsi	166	108	274
Total	387	263	650

Conditional distribution: Females who prefer Coke = 0.57

Females who prefer Pepsi = 0.43

3

Phone	Female	Male
Apple	60.5%	68.2%
Samsung	39.5%	31.8%
Total	100%	100%

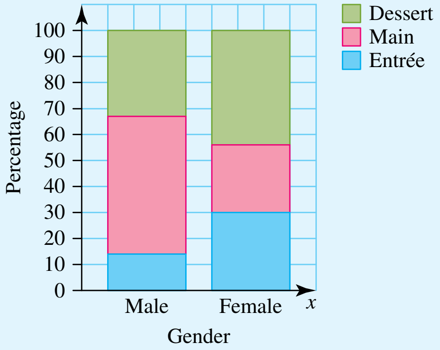
4

Drink	Female	Male
Coke	57.1%	58.9%
Pepsi	42.9%	41.1%
Total	100%	100%

5

Choice	Male	Female	Total
Entrée	8	18	26
Main	31	16	47
Dessert	19	27	46
Total	58	61	118

Choice	Male	Female
Entrée	14	30
Main	53	26
Dessert	33	44
Total	100	100

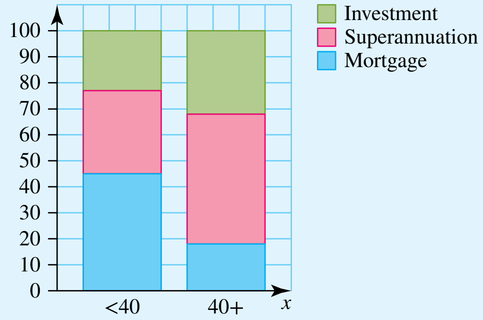


Males enjoy main meal the most compared to females who prefer their dessert the most.

6

Choice	<40	40+	Total
Mortgage	42	18	60
Superannuation	30	51	81
Investment	21	33	54
Total	93	102	195

Choice	<40	40+
Mortgage	45	18
Superannuation	32	50
Investment	23	32
Total	100	100



The under 40s have a focus on their mortgage, whereas the 40 and overs prioritise their superannuation.

7

Attitude	Female	Male	Total
For	37	79	116
Against	102	23	125
Total	139	102	241

Marginal distribution: For = 0.48, Against = 0.52

8

Lesson length	Junior	Senior	Total
45 minutes	50	33	83
1 hour	36	60	96
Total	86	93	179

Conditional distribution:

Senior students prefer 45 mins = 0.35,
Senior students prefer an hour = 0.65

9 a i 22 ii 26 iii 19
iv 45 v 41
b i 12 ii 9 iii 21
iv 42 v 33
c i 47% ii 58%
iii 100% iv 100%

10

Preference	Men	Women
Rent by themselves	57%	59%
Share with friends	43%	41%
Total	100%	100%

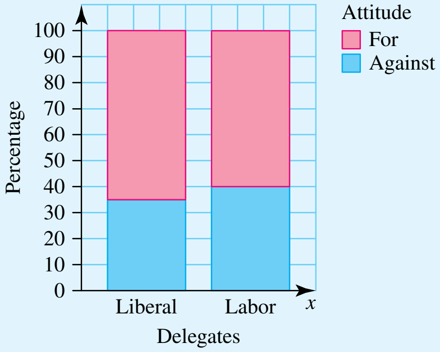
11 D

12 C

13

Attitude	Liberal	Labor
For	35.5	39.4
Against	64.5	60.6
Total	100.0	100.0

14



There is not a lot of difference in the reactions.

15 C

16 A

17 Conditional distribution:

Large town and no waste = 0.15

Large town and medium waste = 0.19

Large town and high waste = 0.67

Note: rounding causes the total to be greater than 100%.

18 26.32%

3

Introduction to regression

3.1 Kick off with CAS

Lines of best fit with CAS

Least-squares regression allows us to fit a line of best fit to a scatterplot. We can then use this line of best fit to make predictions about the data.

1 Using CAS, plot a scatterplot of the following data set, which indicates the temperature (x) and the number of visitors at a popular beach (y).

x	21	26	33	24	35	16	22	30	39	34	22	19
y	95	154	212	141	173	40	104	193	177	238	131	75

2 If there appears to be a linear relationship between x and y, use CAS to add a least-squares regression line of best fit to the data set.

3 What does the line of best fit tell you about the relationship between the x- and y-values?

4 Use the line of best fit to predict y-values given the following x-values:

a 37 b 24 c 17.

Are there any limitations on the data points?

5 The line of best fit can be extended beyond the limits of the original data set. Would you feel comfortable making predictions outside of the scope of the original data set?

6 Use CAS to plot scatterplots of the following sets of data, and if there appears to be a linear relationship, plot a least-squares regression line of best fit:

a

x	62	74	59	77	91	104	79	85	55	74	90	83
y	108	83	127	90	62	55	86	70	141	92	59	77

b

x	2.2	1.7	0.4	2.6	–0.3	1.5	3.1	1.1	0.8	2.9	0.7	–0.8
y	45	39	22	50	9	33	66	34	21	56	27	6

c

x	40	66	38	55	47	61	34	49	53	69	43	58
y	89	112	93	90	75	106	101	77	86	120	81	99

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

3.2 Response (dependent) and explanatory (independent) variables

studyon

Unit 3
AOS DA
Topic 6
Concept 1

Defining two variable data
Concept summary
Practice questions

A set of data involving two variables where one affects the other is called bivariate data. If the values of one variable 'respond' to the values of another variable, then the former variable is referred to as the response (dependent) variable. So an explanatory (independent) variable is a factor that influences the response (dependent) variable.

When a relationship between two sets of variables is being examined, it is important to know which one of the two variables responds to the other. Most often we can make a judgement about this, although sometimes it may not be possible.

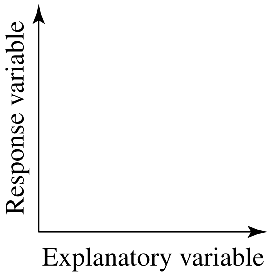
Consider the case where a study compared the heights of company employees against their annual salaries. Common sense would suggest that the height of a company employee would not respond to the person's annual salary nor would the annual salary of a company employee respond to the person's height. In this case, it is not appropriate to designate one variable as explanatory and one as response.

In the case where the ages of company employees are compared with their annual salaries, you might reasonably expect that the annual salary of an employee would depend on the person's age. In this case, the age of the employee is the explanatory variable and the salary of the employee is the response variable.

It is useful to identify the explanatory and response variables where possible, since it is the usual practice when displaying data on a graph to place the explanatory variable on the horizontal axis and the response variable on the vertical axis.

The explanatory variable is a factor that influences the response variable.

When displaying data on a graph place the explanatory variable on the horizontal axis and the response variable on the vertical axis.



WORKED EXAMPLE 1

For each of the following pairs of variables, identify the explanatory (independent) variable and the response (dependent) variable. If it is not possible to identify this, then write 'not appropriate'.

a **The number of visitors at a local swimming pool and the daily temperature**

b **The blood group of a person and his or her favourite TV channel**

THINK	WRITE
a It is reasonable to expect that the number of visitors at the swimming pool on any day will respond to the temperature on that day (and not the other way around).	**a** Daily temperature is the explanatory variable; number of visitors at a local swimming pool is the response variable.
b Common sense suggests that the blood type of a person does not respond to the person's TV channel preferences. Similarly, the choice of a TV channel does not respond to a person's blood type.	**b** Not appropriate

EXERCISE 3.2 Response (dependent) and explanatory (independent) variables

PRACTISE

1 **WE1** For each of the following pairs of variables, identify the explanatory (independent) and the response (dependent) variable. If it is not possible to identify this, then write ‘not appropriate’.

- a The number of air conditioners sold and the daily temperature
- b The age of a person and their favourite colour

2 For each of the following pairs of variables, identify the explanatory (independent) and the response (dependent) variable. If it is not possible to identify the variables, then write ‘not appropriate’.

- a The size of a crowd and the teams that are playing
- b The net score of a round of golf and the golfer’s handicap

CONSOLIDATE

3 For each of the following pairs of variables, identify the explanatory variable and the response variable. If it is not possible to identify this, then write ‘not appropriate’.

- a The age of an AFL footballer and his annual salary
- b The growth of a plant and the amount of fertiliser it receives
- c The number of books read in a week and the eye colour of the readers
- d The voting intentions of a woman and her weekly consumption of red meat
- e The number of members in a household and the size of the house

4 For each of the following pairs of variables, identify the explanatory variable and the response variable. If it is not possible to identify this, then write ‘not appropriate’.

- a The month of the year and the electricity bill for that month
- b The mark obtained for a maths test and the number of hours spent preparing for the test
- c The mark obtained for a maths test and the mark obtained for an English test
- d The cost of grapes (in dollars per kilogram) and the season of the year

5 In a scientific experiment, the explanatory variable was the amount of sleep (in hours) a new mother got per night during the first month following the birth of her baby. The response variable would most likely have been:

- A the number of times (per night) the baby woke up for a feed
- B the blood pressure of the baby
- C the mother’s reaction time (in seconds) to a certain stimulus
- D the level of alertness of the baby
- E the amount of time (in hours) spent by the mother on reading

6 A paediatrician investigated the relationship between the amount of time children aged two to five spend outdoors and the annual number of visits to his clinic. Which one of the following statements is not true?

- A When graphed, the amount of time spent outdoors should be shown on the horizontal axis.
- B The annual number of visits to the paediatric clinic is the response variable.
- C It is impossible to identify the explanatory variable in this case.
- D The amount of time spent outdoors is the explanatory variable.
- E The annual number of visits to the paediatric clinic should be shown on the vertical axis.

7 Alex works as a personal trainer at the local gym. He wishes to analyse the relationship between the number of weekly training sessions and the weekly weight loss of his clients. Which one of the following statements is correct?

- A When graphed, the number of weekly training sessions should be shown on the vertical axis, as it is the response variable.
- B When graphed, the weekly weight loss should be shown on the vertical axis, as it is the explanatory variable.
- C When graphed, the weekly weight loss should be shown on the horizontal axis, as it is the explanatory variable.
- D When graphed, the number of weekly training sessions should be shown on the horizontal axis, as it is the explanatory variable.
- E It is impossible to identify the response variable in this case.

Answer questions 8 to 12 as true or false.

8 When graphing data, the explanatory variable should be placed on the x-axis.

9 The response variable is the same as the dependent variable.

10 If variable A changes due to a change in variable B, then variable A is the response variable.

11 When graphing data the response variable should be placed on the x-axis.

12 The independent variable is the same as the response variable.

MASTER

13 If two variables investigated are the number of minutes on a basketball court and the number of points scored:

- a which is the explanatory variable
- b which is the response variable?

14 Callum decorated his house with Christmas lights for everyone to enjoy. He investigated two variables, the number of Christmas lights he has and the size of his electricity bill.

- a Which is the response variable?
- b If Callum was to graph the data, what should be on the x-axis?
- c Which is the explanatory variable?
- d On the graph, what variable should go on the y-axis?

3.3 Scatterplots

Fitting straight lines to bivariate data

The process of ‘fitting’ straight lines to bivariate data enables us to analyse relationships between the data and possibly make predictions based on the given data set.

We will consider the most common technique for fitting a straight line and determining its equation, namely least squares.

The **linear relationship** expressed as an equation is often referred to as the *linear regression equation* or line. Recall that when we display bivariate data as a **scatterplot**, the explanatory variable is placed on the horizontal axis and the response variable is placed on the vertical axis.

Scatterplots

studyon

Unit 3
AOS DA
Topic 6
Concept 5

Scatterplots
Concept summary
Practice questions

eBookplus

Interactivity
Scatterplots
int-6250

eBookplus

Interactivity
Create scatterplots
int-6497

We often want to know if there is a relationship between two numerical variables. A scatterplot, which gives a visual display of the relationship between two variables, provides a good starting point.

Consider the data obtained from last year's 12B class at Northbank Secondary College. Each student in this class of 29 students was asked to give an estimate of the average number of hours they studied per week during Year 12. They were also asked for the ATAR score they obtained.

Average hours of study	ATAR score
18	59
16	67
22	74
27	90
15	62
28	89
18	71
19	60
22	84
30	98
14	54
17	72
14	63
19	72
20	58
10	47
28	85
25	75
18	63
19	61
17	59
16	76
14	59
29	89
30	93
30	96
23	82
26	35
22	78

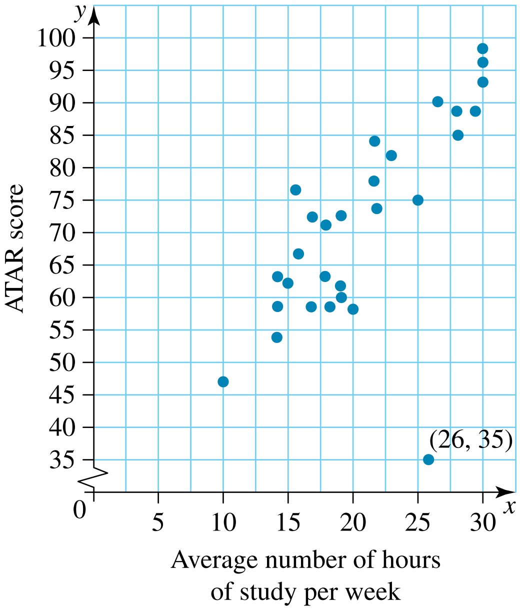
The figure below shows the data plotted on a scatterplot.

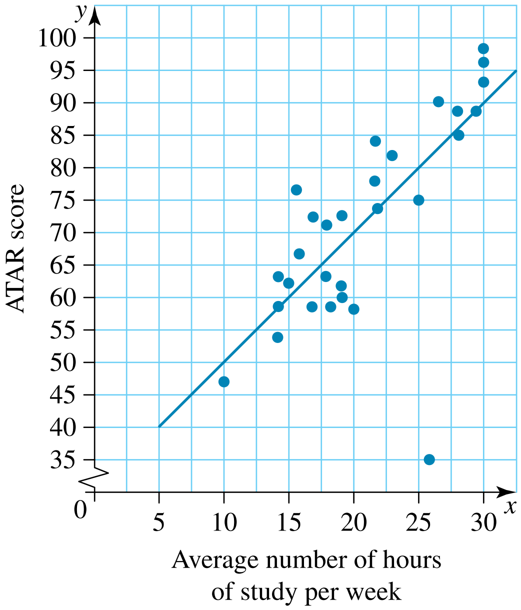
It is reasonable to think that the number of hours of study put in each week by students would affect their ATAR scores and so the number of hours of study per week is the explanatory (independent) variable and appears on the horizontal axis. The ATAR score is the response (dependent) variable and appears on the vertical axis.

There are 29 points on the scatterplot. Each point represents the number of hours of study and the ATAR score of one student.

In analysing the scatterplot we look for a pattern in the way the points lie. Certain patterns tell us that certain relationships exist between the two variables. This is referred to as **correlation**. We look at what type of correlation exists and how strong it is.

In the diagram we see some sort of pattern: the points are spread in a rough corridor from bottom left to top right. We refer to data following such a direction as having a *positive relationship*. This tells us that as the average number of hours studied per week increases, the ATAR score increases.





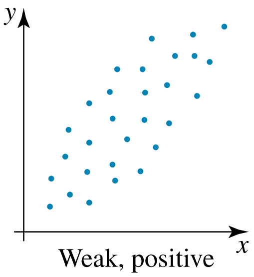
The point (26, 35) is an outlier. It stands out because it is well away from the other points and clearly is not part of the 'corridor' referred to previously. This outlier may have occurred because a student exaggerated the number of hours he or she worked in a week or perhaps there was a recording error. This needs to be checked.

We could describe the rest of the data as having a *linear* form as the straight line in the diagram indicates.

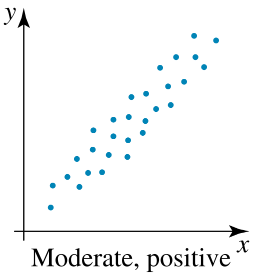
When describing the relationship between two variables displayed on a scatterplot, we need to comment on:

(a) the direction — whether it is positive or negative
(b) the form — whether it is linear or non-linear
(c) the strength — whether it is strong, moderate or weak
(d) possible outliers.

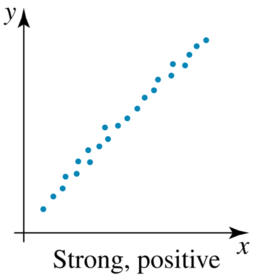
Here is a gallery of scatterplots showing the various patterns we look for.



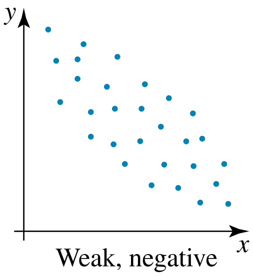
Weak, positive linear relationship



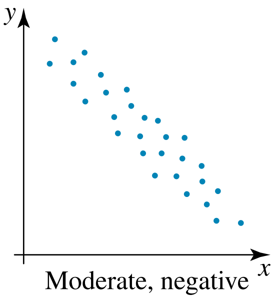
Moderate, positive linear relationship



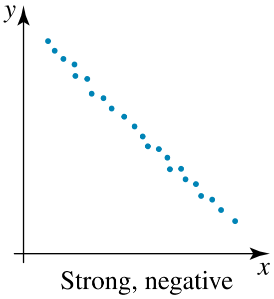
Strong, positive linear relationship



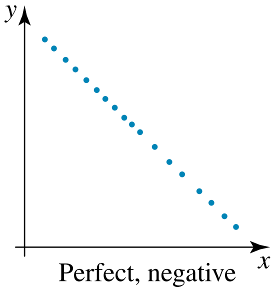
Weak, negative linear relationship



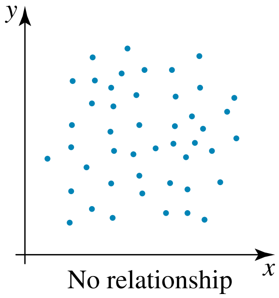
Moderate, negative linear relationship



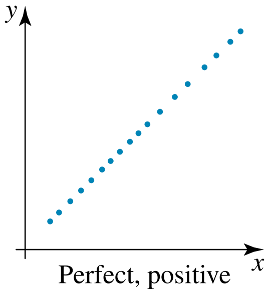
Strong, negative linear relationship



Perfect, negative linear relationship



No relationship

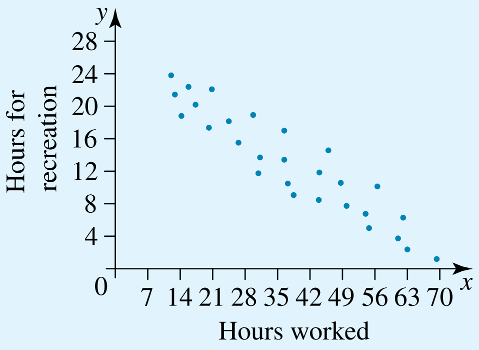


Perfect, positive linear relationship

WORKED EXAMPLE 2

The scatterplot shows the number of hours people spend at work each week and the number of hours people get to spend on recreational activities during the week.

Decide whether or not a relationship exists between the variables and, if it does, comment on whether it is positive or negative; weak, moderate or strong; and whether or not it has a linear form.



THINK

1. The points on the scatterplot are spread in a certain pattern, namely in a rough corridor from the top left to the bottom right corner. This tells us that as the work hours increase, the recreation hours decrease.
2. The corridor is straight (that is, it would be reasonable to fit a straight line into it).
3. The points are neither too tight nor too dispersed.
4. The pattern resembles the central diagram in the gallery of scatterplots shown previously.

WRITE

There is a moderate, negative linear relationship between the two variables.

WORKED EXAMPLE 3

Data showing the average weekly number of hours studied by each student in 12B at Northbank Secondary College and the corresponding height of each student (correct to the nearest tenth of a metre) are given in the table.

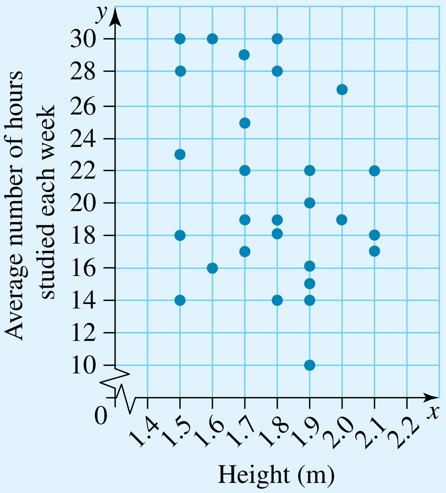
Average hours of study	18	16	22	27	15	28	18	20	10	28	25	18	19	17	19	22	30	14	17	14	19	16	14	29	30	30	23	22
Height (m)	1.5	1.9	1.7	2.0	1.9	1.8	2.1	1.9	1.9	1.5	1.7	1.8	1.8	2.1	2.0	1.9	1.6	1.5	1.7	1.8	1.7	1.6	1.9	1.7	1.8	1.5	1.5	2.1

Construct a scatterplot for the data and use it to comment on the direction, form and strength of any relationship between the number of hours studied and the height of the students.

THINK

1. CAS can be used to assist you in drawing a scatterplot.

WRITE/DRAW

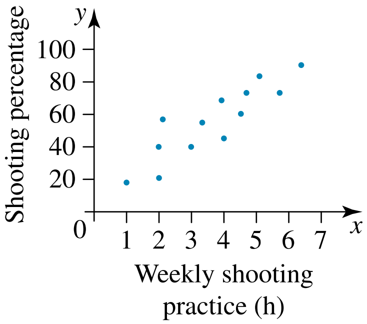


2 Comment on the direction of any relationship.	There is no relationship; the points appear to be randomly placed.
3 Comment on the form of the relationship.	There is no form, no linear trend, no quadratic trend, just a random placement of points.
4 Comment on the strength of any relationship.	Since there is no relationship, strength is not relevant.
5 Draw a conclusion.	Clearly, from the graph, the number of hours spent studying for VCE has no relation to how tall you might be.

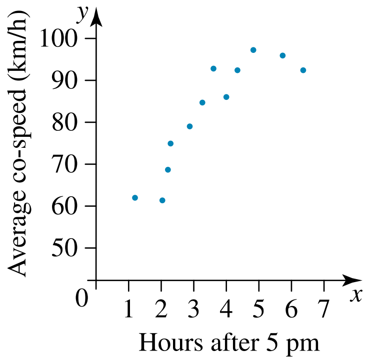
EXERCISE 3.3 Scatterplots

PRACTISE

1 **WE2** The scatterplot shown represents the number of hours of basketball practice each week and a player's shooting percentage. Decide whether or not a relationship exists between the variables and, if it does, comment on whether it is positive or negative; weak, moderate or strong; and whether or not it is linear form.



2 The scatterplot shown shows the hours after 5 pm and the average speed of cars on a freeway. Explain the direction, form and strength of the relationship of the two variables.



3 **WE3** Data on the height of a person and the length of their hair is shown. Construct a scatterplot for the data and use it to comment on the direction, form and strength of any relationship between the height of a person and the length of their hair.

Height (cm)	158	164	184	173	194	160	198	186	166
Hair length (cm)	18	12	5	10	7	3	10	6	14

4 The following table shows data on hours spent watching television per week and your age. Use the data to construct a scatterplot and use it to comment on the direction, form and strength of any relationship between the two variables.

Age (yr)	12	25	61	42	18	21	33	15	29
TV per week (h)	23	30	26	18	12	30	20	19	26

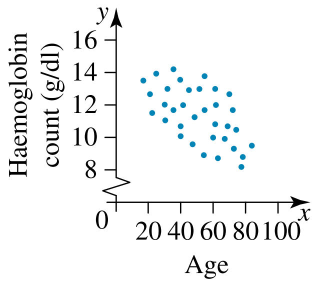
CONSOLIDATE

5 For each of the following pairs of variables, write down whether or not you would reasonably expect a relationship to exist between the pair and, if so, comment on whether it would be a positive or negative association.

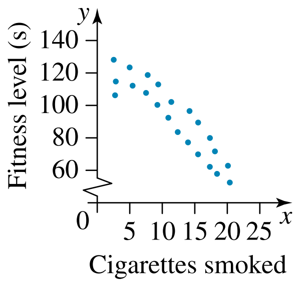
a Time spent in a supermarket and total money spent
b Income and value of car driven
c Number of children living in a house and time spent cleaning the house
d Age and number of hours of competitive sport played per week
e Amount spent on petrol each week and distance travelled by car each week
f Number of hours spent in front of a computer each week and time spent playing the piano each week
g Amount spent on weekly groceries and time spent gardening each week

6 For each of the scatterplots, describe whether or not a relationship exists between the variables and, if it does, comment on whether it is positive or negative, whether it is weak, moderate or strong and whether or not it has a linear form.

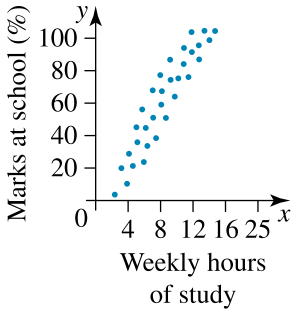
a



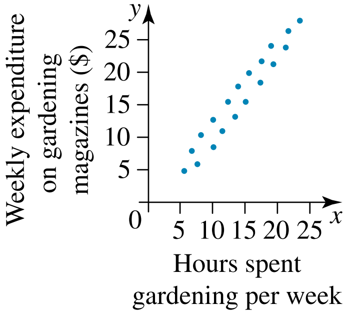
b



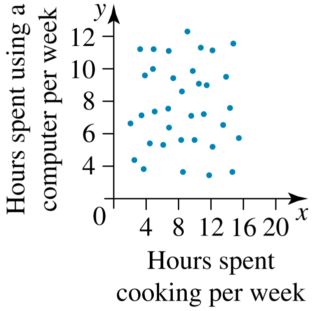
c



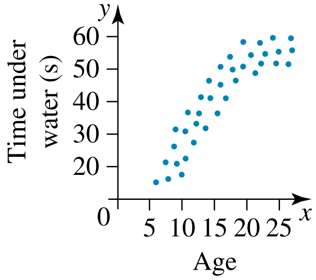
d



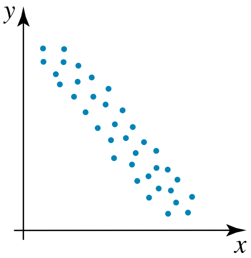
e



f



7 From the scatterplot shown, it would be reasonable to observe that:



A as the value of x increases, the value of y increases
B as the value of x increases, the value of y decreases
C as the value of x increases, the value of y remains the same
D as the value of x remains the same, the value of y increases
E there is no relationship between x and y

8 The population of a municipality (to the nearest ten thousand) together with the number of primary schools in that particular municipality is given below for 11 municipalities.

Population (× 1000)	110	130	130	140	150	160	170	170	180	180	190
Number of primary schools	4	4	6	5	6	8	6	7	8	9	8

Construct a scatterplot for the data and use it to comment on the direction, form and strength of any relationship between the population and the number of primary schools.

9 The table contains data for the time taken to do a paving job and the cost of the job.

Construct a scatterplot for the data. Comment on whether a relationship exists between the time taken and the cost. If there is a relationship describe it.

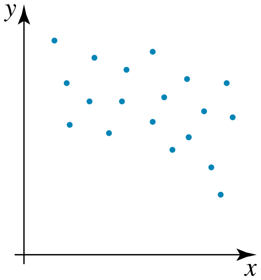
Time taken (hours)	Cost of job ($)
5	1000
7	1000
5	1500
8	1200
10	2000
13	2500
15	2800
20	3200
18	2800
25	4000
23	3000

10 The table shows the time of booking (how many days in advance) of the tickets for a musical performance and the corresponding row number in A-reserve seating.

Time of booking	Row number	Time of booking	Row number	Time of booking	Row number
5	15	14	12	25	3
6	15	14	10	28	2
7	15	17	11	29	2
7	14	20	10	29	1
8	14	21	8	30	1
11	13	22	5	31	1
13	13	24	4		

Construct a scatterplot for the data. Comment on whether a relationship exists between the time of booking and the number of the row and, if there is a relationship, describe it.

11 The correlation of this scatterplot is:



A weak, positive, linear
B no correlation
C strong, positive linear
D weak, negative, linear
E strong, negative, linear

12 Draw a scatterplot to display the following data:

a by hand **b** using CAS.

Number of dry-cleaning items	1	2	3	4	5	6	7
Cost ($)	12	16	19	20	22	24	25

13 Draw a scatterplot to display the following data:

a by hand **b** using CAS.

Maximum daily temperature (°C)	26	28	19	17	32	36	33	23	24	18
Number of drinks sold	135	156	98	87	184	133	175	122	130	101

14 Describe the correlation between:

a the number of dry-cleaning items and the cost in question **12**
b the maximum daily temperature and the number of drinks sold in question **13**.

MASTER

15 Draw a scatterplot and describe the correlation for the following data.

	NSW	VIC	QLD	SA	WA	TAS	NT	ACT
Population	7 500 600	5 821 000	4 708 000	1 682 000	2 565 000	514 000	243 000	385 000
Area of land (km^2)	800 628	227 010	1 723 936	978 810	2 526 786	64 519	1 335 742	2 358

16 The table at right contains data giving the time taken to engineer a finished product from the raw recording (of a song, say) and the length of the finished product.

a Construct a scatterplot for these data.
b Comment on whether a relationship exists between the time spent engineering and the length of the finished recording.

Time spent engineering in studio (hours)	Finished length of recording (minutes)
1	3
2	4
3	10
4	12
5	20
6	16
7	18
8	25
9	30
10	28
11	35
12	36
13	39
14	42
15	45

3.4 Pearson's product–moment correlation coefficient

studyon

Unit 3
AOS DA
Topic 6
Concept 6

Pearson's product–moment correlation coefficient (*r*)
Concept summary
Practice questions

eBookplus

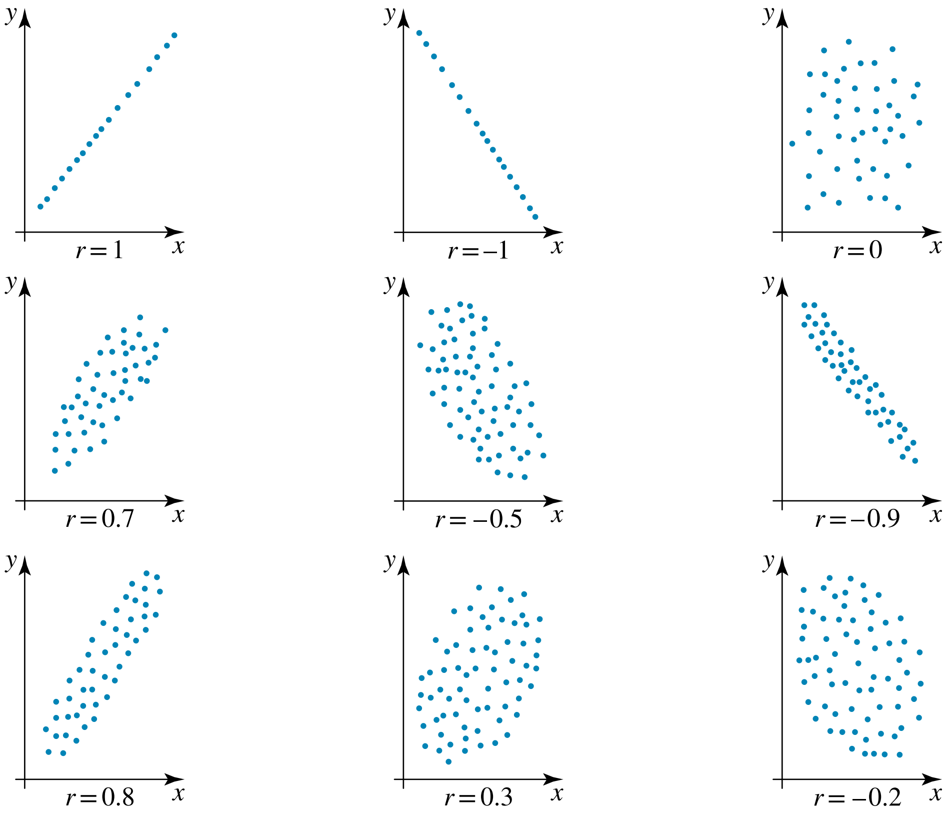
Interactivity
Pearson's product–moment correlation coefficient and the coefficient of determination
int-6251

In the previous section, we estimated the strength of association by looking at a scatterplot and forming a judgement about whether the correlation between the variables was positive or negative and whether the correlation was weak, moderate or strong.

A more precise tool for measuring correlation between two variables is **Pearson's product–moment correlation coefficient**. This coefficient is used to measure the strength of *linear relationships* between variables.

The symbol for Pearson's product–moment correlation coefficient is r. The value of r ranges from −1 to 1; that is, $-1 \leq r \leq 1$.

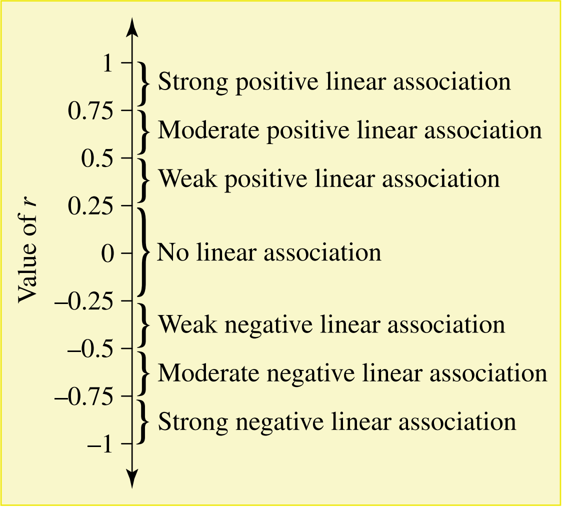
Following is a gallery of scatterplots with the corresponding value of r for each.



The two extreme values of r (1 and −1) are shown in the first two diagrams respectively.

From these diagrams we can see that a value of $r = 1$ or -1 means that there is perfect linear association between the variables.

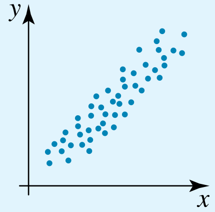
The value of the Pearson's product–moment correlation coefficient indicates the strength of the linear relationship between two variables. The diagram at right gives a rough guide to the strength of the correlation based on the value of r.



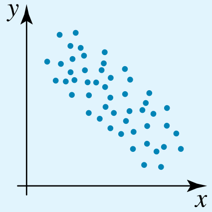
WORKED EXAMPLE 4

For each of the following:

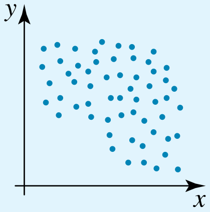
a



b



c



i Estimate the value of Pearson's product–moment correlation coefficient (r) from the scatterplot.

ii Use this to comment on the strength and direction of the relationship between the two variables.

THINK	WRITE
a i Compare these scatterplots with those in the gallery of scatterplots shown previously and estimate the value of r.	**a i** $r \approx 0.9$
ii Comment on the strength and direction of the relationship.	**ii** The relationship can be described as a strong, positive, linear relationship.
b Repeat parts **i** and **ii** as in **a**.	**b i** $r \approx -0.7$
	ii The relationship can be described as a moderate, negative, linear relationship.
c Repeat parts **i** and **ii** as in **a**.	**c i** $r \approx -0.1$
	ii There is almost no linear relationship.

Note that the symbol ≈ means 'approximately equal to'. We use it instead of the = sign to emphasise that the value (in this case r) is only an estimate.

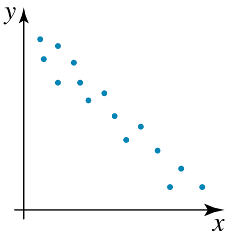
In completing Worked example 4 above, we notice that estimating the value of r from a scatterplot is rather like making an informed guess. In the next section, we will see how to obtain the actual value of r.

EXERCISE 3.4 Pearson's product–moment correlation coefficient

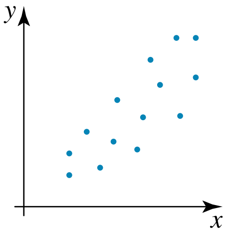
PRACTISE

1 **WE4** For each of the following:

a



b



i estimate the Pearson's product–moment correlation coefficient (r) from the scatterplot.

ii use this to comment on the strength and direction of the relationship between the two variables.

2 What type of linear relationship does each of the following values of r suggest?

a 0.85 **b** −0.3

CONSOLIDATE

3 What type of linear relationship does each of the following values of r suggest?

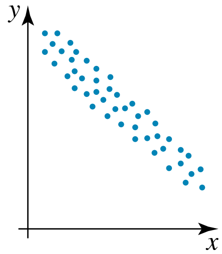
a 0.21 **b** 0.65 **c** -1 **d** -0.78

4 What type of linear relationship does each of the following values of r suggest?

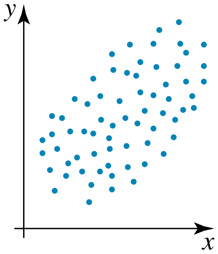
a 1 **b** 0.9 **c** -0.34 **d** -0.1

5 For each of the following:

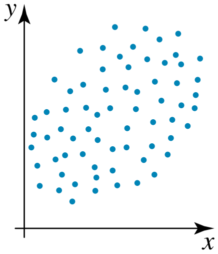
a



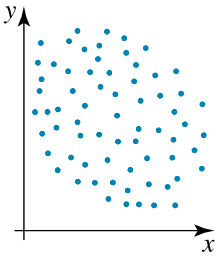
b



c



d

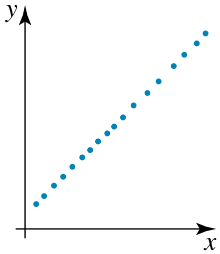


i estimate the value of Pearson's product–moment correlation coefficient (r), from the scatterplot.

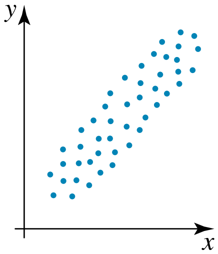
ii use this to comment on the strength and direction of the relationship between the two variables.

6 For each of the following:

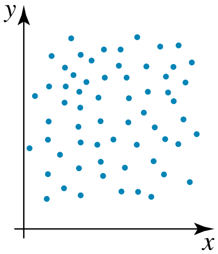
a



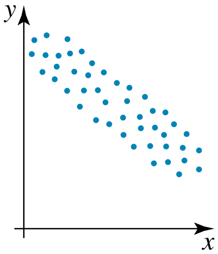
b



c



d

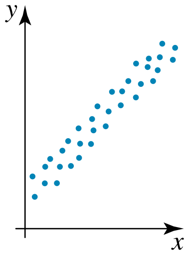


i estimate the value of Pearson's product–moment correlation coefficient (r), from the scatterplot.

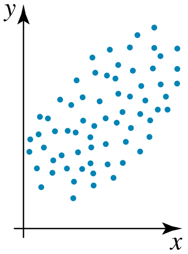
ii use this to comment on the strength and direction of the relationship between the two variables.

7 A set of data relating the variables x and y is found to have an r value of 0.62. The scatterplot that could represent the data is:

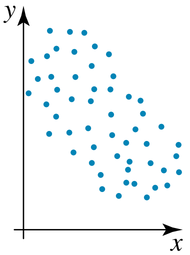
A



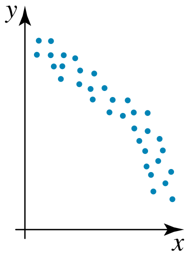
B



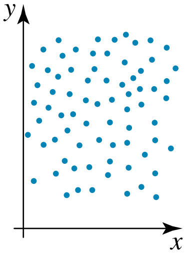
C



D



E



8 A set of data relating the variables x and y is found to have an r value of -0.45. A true statement about the relationship between x and y is:

A There is a strong linear relationship between x and y and when the x-values increase, the y-values tend to increase also.

B There is a moderate linear relationship between x and y and when the x-values increase, the y-values tend to increase also.

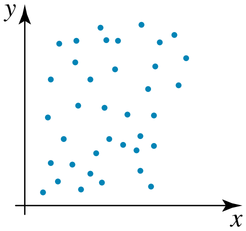
C There is a moderate linear relationship between x and y and when the x-values increase, the y-values tend to decrease.

D There is a weak linear relationship between x and y and when the x-values increase, the y-values tend to increase also.

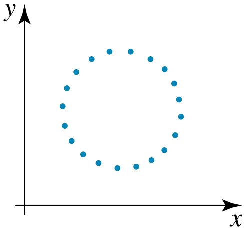
E There is a weak linear relationship between x and y and when the x-values increase, the y-values tend to decrease.

9 From the scatterplots shown estimate the value of r and comment on the strength and direction of the relationship between the two variables.

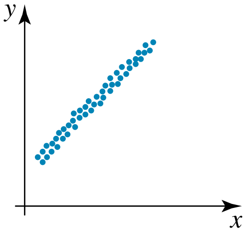
a



b



c



10 A weak, negative, linear association between two variables would have an r value closest to:

A −0.55 **B** 0.55 **C** −0.65 **D** −0.45 **E** 0.45

11 Which of the following is *not* a Pearson product–moment correlation coefficient?

A 1.0 **B** 0.99 **C** −1.1 **D** −0.01 **E** 0

12 Draw a scatterplot that has a Pearson product–moment correlation coefficient of approximately −0.7.

MASTER

13 If two variables have an r value of 1, then they are said to have:

A a strong positive linear relationship
B a strong negative linear relationship
C a perfect positive relationship
D a perfect negative linear relationship
E a perfect positive linear relationship

14 Which is the correct ascending order of positive values of r?

A Strong, Moderate, Weak, No linear association
B Weak, Strong, Moderate, No linear association
C No linear association, Weak, Moderate, Strong
D No linear association, Moderate, Weak, Strong
E Strong, Weak, Moderate, No linear association

3.5 Calculating *r* and the coefficient of determination

Pearson's product–moment correlation coefficient (*r*)

The formula for calculating Pearson's correlation coefficient r is as follows:

$$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

where n is the number of pairs of data in the set
s_x is the standard deviation of the x-values
s_y is the standard deviation of the y-values
$\bar{x}$ is the mean of the x-values
$\bar{y}$ is the mean of the y-values.

The calculation of r is often done using CAS.

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Unit 3
AOS DA
Topic 6
Concept 7

Use and interpretation of *r*
Concept summary
Practice questions

There are two important limitations on the use of *r*. First, since *r* measures the strength of a linear relationship, it would be inappropriate to calculate *r* for data which are not linear — for example, data which a scatterplot shows to be in a quadratic form.

Second, outliers can bias the value of *r*. Consequently, if a set of linear data contains an outlier, then *r* is not a reliable measure of the strength of that linear relationship.

The calculation of *r* is applicable to sets of bivariate data which are known to be linear in form and which do not have outliers.

With those two provisos, it is good practice to draw a scatterplot for a set of data to check for a linear form and an absence of outliers before *r* is calculated. Having a scatterplot in front of you is also useful because it enables you to estimate what the value of *r* might be — as you did in the previous exercise, and thus you can check that your workings are correct.

WORKED EXAMPLE 5

The heights (in centimetres) of 21 football players were recorded against the number of marks they took in a game of football. The data are shown in the following table.

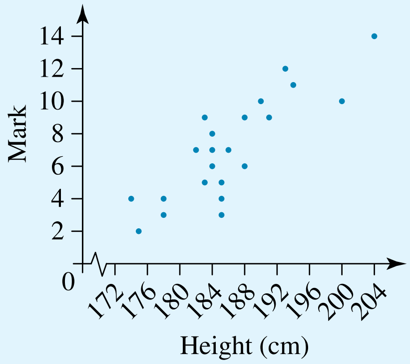
a **Construct a scatterplot for the data.**

b **Comment on the correlation between the heights of players and the number of marks that they take, and estimate the value of *r*.**

c **Calculate *r* and use it to comment on the relationship between the heights of players and the number of marks they take in a game.**

Height (cm)	Number of marks taken
184	6
194	11
185	3
175	2
186	7
183	5
174	4
200	10
188	9
184	7
188	6

Height (cm)	Number of marks taken
182	7
185	5
183	9
191	9
177	3
184	8
178	4
190	10
193	12
204	14

THINK	WRITE/DRAW
a Height is the explanatory variable, so plot it on the x-axis; the number of marks is the response variable, so show it on the y-axis.	**a** 
b Comment on the correlation between the variables and estimate the value of r.	**b** The data show what appears to be a linear form of moderate strength. We might expect $r \approx 0.8$.
c 1 Because there is a linear form and there are no outliers, the calculation of r is appropriate.	**c**
2 Use CAS to find the value of r. Round correct to 2 decimal places.	$r = 0.859\,311...$ ≈ 0.86
3 The value of $r = 0.86$ indicates a strong positive linear relationship.	$r = 0.86$. This indicates there is a strong positive linear association between the height of a player and the number of marks he takes in a game. That is, the taller the player, the more marks we might expect him to take.

Correlation and causation

In Worked example 5 we saw that $r = 0.86$. While we are entitled to say that there is a strong association between the height of a footballer and the number of marks he takes, we cannot assert that the height of a footballer causes him to take a lot of marks. Being tall might assist in taking marks, but there will be many other factors which come into play; for example, skill level, accuracy of passes from teammates, abilities of the opposing team, and so on.

So, while establishing a high degree of correlation between two variables may be interesting and can often flag the need for further, more detailed investigation, it in no way gives us any basis to comment on whether or not one variable *causes* particular values in another variable.

As we have looked at earlier in this topic, *correlation* is a statistic measure that defines the size and direction of the relationship between two variables. **Causation** states that one event is the result of the occurrence of the other event (or variable). This is also referred to as **cause and effect**, where one event is the cause and this makes another event happen, this being the effect.

An example of a cause and effect relationship could be an alarm going off (cause — happens first) and a person waking up (effect — happens later). It is also important to realise that a high correlation does not imply causation. For example, a person smoking could have a high correlation with alcoholism but it is not necessarily the cause of it, thus they are different.

One way to test for causality is experimentally, where a control study is the most effective. This involves splitting the sample or population data and making one a control group (e.g. one group gets a placebo and the other get some form of medication). Another way is via an observational study which also compares against a control variable, but the researcher has no control over the experiment (e.g. smokers and non-smokers who develop lung cancer). They have no control over whether they develop lung cancer or not.

Non-causal explanations

studyon

Unit 3
AOS DA
Topic 6
Concept 8

Non-causal explanations
Concept summary
Practice questions

Although we may observe a strong correlation between two variables, this does not necessarily mean that an association exists. In some cases the correlation between two variables can be explained by a common response variable which provides the association. For example, a study may show that there is a strong correlation between house sizes and the life expectancy of home owners. While a bigger house will not directly lead to a longer life expectancy, a common response variable, the income of the house owner, provides a direct link to both variables and is more likely to be the underlying cause for the observed correlation.

In other cases there may be hidden, confounding reasons for an observed correlation between two variables. For example, a lack of exercise may provide a strong correlation to heart failure, but other hidden variables such as nutrition and lifestyle might have a stronger influence.

Finally, an association between two variables may be purely down to coincidence. The larger a data set is, the less chance there is that coincidence will have an impact.

When looking at correlation and causation be sure to consider all of the possible explanations before jumping to conclusions. In professional research, many similar tests are often carried out to try to identify the exact cause for a shown correlation between two variables.

The coefficient of determination (r^2)

studyon

Unit 3
AOS DA
Topic 7
Concept 4

The coefficient of determination (r^2)
Concept summary
Practice questions

The **coefficient of determination** is given by r^2. It is very easy to calculate, we merely square Pearson's product–moment correlation coefficient (r). The value of the coefficient of determination ranges from 0 to 1; that is, $0 \leq r^2 \leq 1$.

The coefficient of determination is useful when we have two variables which have a linear relationship. It tells us the proportion of variation in one variable which can be explained by the variation in the other variable.

The coefficient of determination provides a measure of how well the linear rule linking the two variables (x and y) predicts the value of y when we are given the value of x.

WORKED EXAMPLE 6

A set of data giving the number of police traffic patrols on duty and the number of fatalities for the region was recorded and a correlation coefficient of $r = -0.8$ was found.

a **Calculate the coefficient of determination and interpret its value.**

b **If it was a causal relationship state the most likely variable to be the cause and effect.**

THINK	WRITE
a 1 Calculate the coefficient of determination by squaring the given value of r.	**a** Coefficient of determination $= r^2$ $= (-0.8)^2$ $= 0.64$
2 Interpret your result.	We can conclude from this that 64% of the variation in the number of fatalities can be explained by the variation in the number of police traffic patrols on duty. This means that the number of police traffic patrols on duty is a major factor in predicting the number of fatalities.
b 1 The two variables are: **i** the number of police on traffic patrols, and **ii** the number of fatalities.	**b**
2 Cause happens first and it has an effect later on.	FIRST: Number of police cars on patrol (cause) LATER: Number of fatalities (effect)

Note: In Worked example 6, 64% of the variation in the number of fatalities was due to the variation in the number of police cars on duty and 36% was due to other factors; for example, days of the week or hour of the day.

EXERCISE 3.5 Calculating *r* and the coefficient of determination

PRACTISE

1 **WE5** The heights (cm) of basketball players were recorded against the number of points scored in a game. The data are shown in the following table.

Height (cm)	Points scored	Height (cm)	Points scored
194	6	201	13
203	4	196	10
208	18	205	20
198	22	215	14
195	2	203	3

a Construct a scatterplot of the data.

b Comment on the correlation between the heights of basketballers and the number of points scored, and estimate the value of r.

c Calculate the r value and use it to comment on the relationship between heights of players and the number of points scored in a game.

2 The following table shows the gestation time and the birth mass of 10 babies.

Gestation time (weeks)	31	32	33	34	35	36	37	38	39	40
Birth mass (kg)	1.08	1.47	1.82	2.06	2.23	2.54	2.75	3.11	3.08	3.37

a Construct a scatterplot of the data.

b Comment on the correlation between the 'gestation time' and 'birth mass', and estimate the value of r.

c Calculate the r value and use it to comment on the relationship between gestation time and birth mass.

3 **WE6** Data on the number of booze buses in use and the number of drivers registering a blood alcohol reading over 0.05 was recorded and a correlation coefficient of $r = 0.77$ was found.

a Calculate the coefficient of determination and interpret its value.

b If there was a causal relationship, state the most likely variable to be the cause and the variable to be the effect.

4 An experiment was conducted that looked at the number of books read by a student and their spelling skills. If this was a cause and effect relationship, what variable most likely represents the cause and what variable represents the effect?

CONSOLIDATE

5 The yearly salary (× \$1000) and the number of votes polled in the Brownlow medal count are given below for 10 footballers.

Yearly salary (× \$1000)	360	400	320	500	380	420	340	300	280	360
Number of votes	24	15	33	10	16	23	14	21	31	28

a Construct a scatterplot for the data.

b Comment on the correlation of salary and the number of votes and make an estimate of r.

c Calculate r and use it to comment on the relationship between yearly salary and number of votes.

6 A set of data, obtained from 40 smokers, gives the number of cigarettes smoked per day and the number of visits per year to the doctor. The Pearson's correlation coefficient for these data was found to be 0.87. Calculate the coefficient of determination for the data and interpret its value.

7 Data giving the annual advertising budgets (× \$1000) and the yearly profit increases (%) of 8 companies are shown below.

Annual advertising budget (× \$1000)	11	14	15	17	20	25	25	27
Yearly profit increase (%)	2.2	2.2	3.2	4.6	5.7	6.9	7.9	9.3

a Construct a scatterplot for these data.

b Comment on the correlation of the advertising budget and profit increase and make an estimate of r.

c Calculate r.

d Calculate the coefficient of determination.

e Write the proportion of the variation in the yearly profit increase that can be explained by the variation in the advertising budget.

8 Data showing the number of tourists visiting a small country in a month and the corresponding average monthly exchange rate for the country's currency against the American dollar are as given.

Number of tourists (× 1000)	2	3	4	5	7	8	8	10
Exchange rate	1.2	1.1	0.9	0.9	0.8	0.8	0.7	0.6

a Construct a scatterplot for the data.
b Comment on the correlation between the number of tourists and the exchange rate and give an estimate of *r*.
c Calculate *r*.
d Calculate the coefficient of determination.
e Write the proportion of the variation in the number of tourists that can be explained by the exchange rate.

9 Data showing the number of people in 9 households against weekly grocery costs are given below.

Number of people in household	2	5	6	3	4	5	2	6	3
Weekly grocery costs ($)	60	180	210	120	150	160	65	200	90

a Construct a scatterplot for the data.
b Comment on the correlation of the number of people in a household and the weekly grocery costs and give an estimate of *r*.
c If this is a causal relationship state the most likely variable to be the cause and which to be the effect.
d Calculate *r*.
e Calculate the coefficient of determination.
f Write the proportion of the variation in the weekly grocery costs that can be explained by the variation in the number of people in a household.

10 Data showing the number of people on 8 fundraising committees and the annual funds raised are given in the table.

Number of people on committee	3	6	4	8	5	7	3	6
Annual funds raised ($)	4500	8500	6100	12 500	7200	10 000	4700	8800

a Construct a scatterplot for these data.
b Comment on the correlation between the number of people on a committee and the funds raised and make an estimate of *r*.
c Calculate *r*.
d Based on the value of *r* obtained in part **c**, would it be appropriate to conclude that the increase in the number of people on the fundraising committee causes the increase in the amount of funds raised?
e Calculate the coefficient of determination.
f Write the proportion of the variation in the funds raised that can be explained by the variation in the number of people on a committee.

The following information applies to questions **11** and **12**. A set of data was obtained from a large group of women with children under 5 years of age. They were asked the number of hours they worked per week and the amount of money they spent on child care. The results were recorded and the value of Pearson's correlation coefficient was found to be 0.92.

11 Which of the following is not true?

A The relationship between the number of working hours and the amount of money spent on child care is linear.
B There is a positive correlation between the number of working hours and the amount of money spent on child care.
C The correlation between the number of working hours and the amount of money spent on child care can be classified as strong.
D As the number of working hours increases, the amount spent on child care increases as well.
E The increase in the number of hours worked causes the increase in the amount of money spent on child care.

12 Which of the following is not true?

A The coefficient of determination is about 0.85.
B The number of working hours is the major factor in predicting the amount of money spent on child care.
C About 85% of the variation in the number of hours worked can be explained by the variation in the amount of money spent on child care.
D Apart from number of hours worked, there could be other factors affecting the amount of money spent on child care.
E About $\frac{17}{20}$ of the variation in the amount of money spent on child care can be explained by the variation in the number of hours worked.

13 To experimentally test if a relationship is a cause and effect relationship the data is usually:

A randomly selected
B split to produce a control study
C split into genders
D split into age categories
E kept as small as possible

14 A study into the unemployment rate in different Melbourne studies found a negative correlation between the unemployment rate in a suburb and the average salary of adult workers in the same suburb.

Using your knowledge of correlation and causation, explain whether this is an example of cause and effect. If not, what non-causal explanations might explain the correlation?

MASTER

15 The main problem when using an observational study to determine causality is:

A collecting the data
B splitting the data
C controlling the control group
D getting a high enough coefficient of determination
E none of the above

16 An investigation is undertaken with people following the Certain Slim diet to explore the link between weeks of dieting and total weight loss. The data are shown in the table.

Total weight loss (kg)	Number of weeks on the diet
1.5	1
4.5	5
9	8
3	3
6	6
8	9
3.5	4
3	2
6.5	7
8.5	10
4	4
6.5	6
10	9
2.5	2
6	5

a Display the data on a scatterplot.
b Describe the association between the two variables in terms of direction, form and strength.
c Is it appropriate to use Pearson's correlation coefficient to explain the link between the number of weeks on the Certain Slim diet and total weight loss?
d Estimate the value of Pearson's correlation coefficient from the scatterplot.
e Calculate the value of this coefficient.
f Is the total weight loss affected by the number of weeks staying on the diet?
g Calculate the value of the coefficient of determination.
h What does the coefficient of determination say about the relationship between total weight loss and the number of weeks on the Certain Slim diet?

3.6 Fitting a straight line — least-squares regression

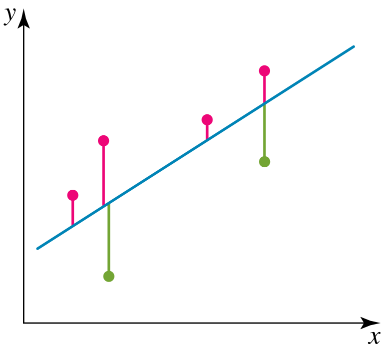
studyon

Unit 3
AOS DA
Topic 7
Concept 1

Least-squares regression
Concept summary
Practice questions

A method for finding the equation of a straight line which is fitted to data is known as the method of **least-squares regression**. It is used when data show a linear relationship and have no obvious outliers.

To understand the underlying theory behind least-squares, consider the regression line shown.



We wish to minimise the total of the vertical lines, or 'errors' in some way. For example, balancing the errors above and below the line. This is reasonable, but for sophisticated mathematical reasons it is preferable to minimise the sum of the *squares* of each of these errors. This is the essential mathematics of least-squares regression.

The calculation of the equation of a least-squares regression line is simple using CAS.

WORKED EXAMPLE 7

A study shows the more calls a teenager makes on their mobile phone, the less time they spend on each call. Find the equation of the linear regression line for the number of calls made plotted against call time in minutes using the least-squares method on CAS. Express coefficients correct to 2 decimal places, and calculate the coefficient of determination to assess the strength of the association.

Number of minutes (x)	1	3	4	7	10	12	14	15
Number of calls (y)	11	9	10	6	8	4	3	1

THINK	WRITE
1 Enter the data into CAS to find the equation of least squares regression line.	$y = 11.7327 - 0.634\,271x$
2 Write the equation with coefficients expressed to 2 decimal places.	$y = 11.73 - 0.63x$
3 Write the equation in terms of the variable names. Replace x with number of minutes and y with number of calls.	Number of calls = 11.73 − 0.63 × no. of minutes
4 Read the r^2 value from your calculator.	$r^2 = 0.87$, so we can conclude that 87% of the variation in y can be explained by the variation in x. Therefore the strength of the linear association between y and x is strong.

Calculating the least-squares regression line by hand

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Interactivity
Fitting a straight line using least-squares regression
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The least-squares regression equation minimises the average deviation of the points in the data set from the line of best fit. This can be shown using the following summary data and formulas to arithmetically determine the least-squares regression equation.

Summary data needed:

$\bar{x}$ the mean of the explanatory variable (x-variable)

$\bar{y}$ the mean of the response variable (y-variable)

s_x the standard deviation of the explanatory variable

s_y the standard deviation of the response variable

r Pearson's product–moment correlation coefficient.

Formula to use:

The general form of the least-squares regression line is

$$y = a + bx$$

where the slope of the regression line is $b = r\dfrac{s_y}{s_x}$

the y-intercept of the regression line is $a = \bar{y} - b\bar{x}$.

Alternatively, if the general form is given as $y = mx + c$, then $m = r\dfrac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$.

WORKED EXAMPLE 8

A study to find a relationship between the height of husbands and the height of their wives revealed the following details.

Mean height of the husbands: 180 cm

Mean height of the wives: 169 cm

Standard deviation of the height of the husbands: 5.3 cm

Standard deviation of the height of the wives: 4.8 cm

Correlation coefficient, $r = 0.85$

The form of the least-squares regression line is to be:

$$\text{Height of wife} = a + b \times \text{height of husband}$$

- **a** **Which variable is the response variable?**
- **b** **Calculate the value of b (correct to 2 significant figures).**
- **c** **Calculate the value of a (correct to 4 significant figures).**
- **d** **Use the equation of the regression line to predict the height of a wife whose husband is 195 cm tall (correct to the nearest cm).**

THINK	WRITE
a Recall that the response variable is the subject of the equation in $y = a + bx$ form; that is, y.	**a** The response variable is the height of the wife.
b **1** The value of b is the gradient of the regression line. Write the formula and state the required values.	**b** $b = r\frac{s_y}{s_x}$ $r = 0.85$, $s_y = 4.8$ and $s_x = 5.3$
2 Substitute the values into the formula and evaluate b.	$= 0.85 \times \frac{4.8}{5.3}$ $= 0.7698$ ≈ 0.77
c **1** The value of a is the y-intercept of the regression line. Write the formula and state the required values.	**c** $a = \bar{y} - b\bar{x}$ $\bar{y} = 169$, $\bar{x} = 180$ and $b = 0.7698$ (from part **b**)
2 Substitute the values into the formula and evaluate a.	$= 169 - 0.7698 \times 180$ $= 30.436$ ≈ 30.44
d **1** State the equation of the regression line, using the values calculated from parts **b** and **c**. In this equation, y represents the height of the wife and x represents the height of the husband.	**d** $y = 30.44 + 0.77x$ or height of wife = 30.44 + 0.77 × height of husband
2 The height of the husband is 195 cm, so substitute $x = 195$ into the equation and evaluate.	$= 30.44 + 0.77 \times 195$ $= 180.59$
3 Write a statement, rounding your answer correct to the nearest cm.	Using the equation of the regression line found, the wife's height would be 181 cm.

EXERCISE 3.6 Fitting a straight line — least-squares regression

PRACTISE

1 **WE7** A study shows that as the temperature increases the sales of air conditioners increase. Find the equation of the linear regression line for the number of air conditioners sold per week plotted against the temperature in °C using the least-squares method on CAS. Also find the coefficient of determination. Express the values correct to 2 decimal places and comment on the association between temperature and air conditioner sales.

Temperature °C (*x*)	21	23	25	28	30	32	35	38
Air conditioner sales (*y*)	3	7	8	14	17	23	25	37

2 Consider the following data set: x represents the month, y represents the number of dialysis patients treated.

x	1	2	3	4	5	6	7	8
y	5	9	7	14	14	19	21	23

Using CAS find the equation of the linear regression line and the coefficient of determination, with values correct to 2 decimal places.

3 **WE8** A study was conducted to find the relationship between the height of Year 12 boys and the height of Year 12 girls. The following details were found.

Mean height of the boys: 182 cm

Mean height of the girls: 166 cm

Standard deviation of the height of boys: 6.1 cm

Standard deviation of the height of girls: 5.2 cm

Correlation coefficient, $r = 0.82$

The form of the least squares regression line is to be:

$$\text{Height of Year 12 Boy} = a + b \times \text{height of Year 12 girl}$$

a Which is the explanatory variable?
b Calculate the value of b (correct to 2 significant figures).
c Calculate the value of a (correct to 2 decimal places).

4 Given the summary details $\bar{x} = 4.4$, $s_x = 1.2$, $\bar{y} = 10.5$, $s_y = 1.4$ and $r = -0.67$, find the value of b and a for the equation of the regression line $y = a + bx$.

CONSOLIDATE

5 Find the equation of the linear regression line for the following data set using the least-squares method, and comment on the strength of the association.

x	4	6	7	9	10	12	15	17
y	10	8	13	15	14	18	19	23

6 Find the equation of the linear regression line for the following data set using the least-squares method.

x	1	2	3	4	5	6	7	8	9
y	35	28	22	16	19	14	9	7	2

7 Find the equation of the linear regression line for the following data set using the least-squares method.

x	−4	−2	−1	0	1	2	4	5	5	7
y	6	7	3	10	16	9	12	16	11	21

8 The following summary details were calculated from a study to find a relationship between mathematics exam marks and English exam marks from the results of 120 Year 12 students.

Mean mathematics exam mark = 64%

Mean English exam mark = 74%

Standard deviation of mathematics exam mark = 14.5%

Standard deviation of English exam mark = 9.8%

Correlation coefficient, $r = 0.64$

The form of the least-squares regression line is to be:

$$\text{Mathematics exam mark} = a + b \times \text{English exam mark}$$

- **a** Which variable is the response variable (y-variable)?
- **b** Calculate the value of b for the least-squares regression line (correct to 2 decimal places).
- **c** Calculate the value of a for the least-squares regression line (correct to 2 decimal places).
- **d** Use the regression line to predict the expected mathematics exam mark if a student scores 85% in an English exam (correct to the nearest percentage).

9 Find the least-squares regression equations, given the following summary data.

- **a** $\bar{x} = 5.6$ $s_x = 1.2$ $\bar{y} = 110.4$ $s_y = 5.7$ $r = 0.7$
- **b** $\bar{x} = 110.4$ $s_x = 5.7$ $\bar{y} = 5.6$ $s_y = 1.2$ $r = -0.7$
- **c** $\bar{x} = 25$ $s_x = 4.2$ $\bar{y} = 10200$ $s_y = 250$ $r = 0.88$
- **d** $\bar{x} = 10$ $s_x = 1$ $\bar{y} = 20$ $s_y = 2$ $r = -0.5$

10 Repeat questions **5**, **6** and **7**, collecting the values for $\bar{x}$, s_x, $\bar{y}$, s_y and r from CAS. Use these data to find the least-squares regression equation. Compare your answers to the ones obtained earlier from questions **5**, **6** and **7**. What do you notice?

11 A mathematician is interested in the behaviour patterns of her kitten, and collects the following data on two variables. Help her manipulate the data.

x	1	2	3	4	5	6	7	8	9	10
y	20	18	16	14	12	10	8	6	4	2

- **a** Fit a least-squares regression line.
- **b** Comment on any interesting features of this line.
- **c** Now fit the 'opposite regression line', namely:

x	20	18	16	14	12	10	8	6	4	2
y	1	2	3	4	5	6	7	8	9	10

12 The best estimate of the least-squares regression line for the scatterplot is:

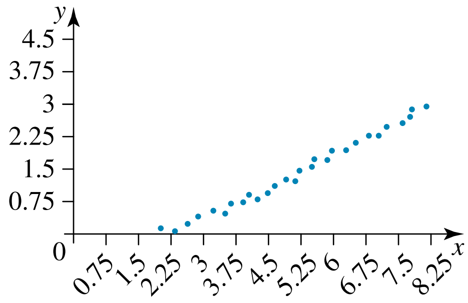
A $y = 2x$

B $y = \frac{1}{2}x$

C $y = 2 + \frac{1}{2}x$

D $y = -2 + \frac{1}{2}x$

E $y = -1 + \frac{1}{2}x$



13 The life span of adult males in a certain country over the last 220 years has been recorded.

Year	1780	1800	1820	1840	1860	1880	1900	1920	1940	1960	1980	2000
Life span (years)	51.2	52.4	51.7	53.2	53.1	54.7	59.9	62.7	63.2	66.8	72.7	79.2

a Fit a least-squares regression line to these data.
b Plot the data and the regression line on a scatterplot.
c Do the data really look linear? Discuss.

14 The price of a long distance telephone call changes as the duration of the call increases. The cost of a sample of calls from Melbourne to Slovenia are summarised in the table.

Cost of call ($)	1.25	1.85	2.25	2.50	3.25	3.70	4.30	4.90	5.80
Duration of call (seconds)	30	110	250	260	300	350	420	500	600

Cost of call ($)	7.50	8.00	9.25	10.00	12.00	13.00	14.00	16.00	18.00
Duration of call (seconds)	840	1000	1140	1200	1500	1860	2400	3600	7200

a What is the explanatory variable likely to be?
b Fit a least-squares regression line to the data.
c View the data on a scatterplot and comment on the reliability of the regression line in predicting the cost of telephone calls. (That is, consider whether the regression line you found proves that costs of calls and duration of calls are related.)
d Calculate the coefficient of determination and comment on the linear relationship between duration and cost of call.

MASTER

15 In a study to find a relationship between the height of plants and the hours of daylight they were exposed to, the following summary details were obtained.

Mean height of plants = 40 cm

Mean hours of daylight = 8 hours

Standard deviation of plant height = 5 cm

Standard deviation of daylight hours = 3 hours

Pearson's correlation coefficient = 0.9

The most appropriate regression equation is:

A height of plant (cm) = −13.6 + 0.54 × hours of daylight
B height of plant (cm) = −8.5 + 0.34 × hours of daylight
C height of plant (cm) = 2.1 + 0.18 × hours of daylight
D height of plant (cm) = 28.0 + 1.50 × hours of daylight
E height of plant (cm) = 35.68 + 0.54 × hours of daylight

16 Consider the following data set.

x	1	2	3	4	5	6
y	12	16	17	21	25	29

a Perform a least-squares regression on the first two points only.
b Now add the 3rd point and repeat.
c Repeat for the 4th, 5th and 6th points.
d Comment on your results.

3.7 Interpretation, interpolation and extrapolation

Interpreting slope and intercept (b and a)

study on

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Interpretation of slope and intercepts
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Once you have a linear regression line, the **slope** and **intercept** can give important information about the data set.

The slope (b) indicates the change in the response variable as the explanatory variable increases by 1 unit.

The y-intercept indicates the value of the response variable when the explanatory variable = 0.

WORKED EXAMPLE 9

In the study of the growth of a species of bacterium, it is assumed that the growth is linear. However, it is very expensive to measure the number of bacteria in a sample. Given the data listed, find:

a the equation, describing the relationship between the two variables
b the rate at which bacteria are growing
c the number of bacteria at the start of the experiment.

Day of experiment	1	4	5	9	11
Number of bacteria	500	1000	1100	2100	2500

THINK	WRITE
a 1 Find the equation of the least-squares regression line using CAS.	**a**
2 Replace x and y with the variables in question.	Number of bacteria = 202.5 + 206.25 × day of experiment

b The rate at which bacteria are growing is given by the gradient of the least-squares regression.

b b is 206.25, hence on average, the number of bacteria increases by approximately 206 per day.

c The number of bacteria at the start of the experiment is given by the y-intercept of the least-squares regression line.

c The y-intercept is 202.5, hence the initial number of bacteria present was approximately 203.

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Interpolation and extrapolation
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Interpolation and extrapolation

As we have already observed, any linear regression method produces a linear equation in the form:

$$y = a + bx$$

where b is the gradient and a is the y-intercept.

This equation can be used to 'predict' the y-value for a given value of x. Of course, these are only approximations, since the regression line itself is only an estimate of the 'true' relationship between the bivariate data. However, they can still be used, in some cases, to provide additional information about the data set (that is, make predictions).

There are two types of prediction: **interpolation** and **extrapolation**.

Interpolation

Interpolation is the use of the regression line to predict values within the range of data in a set, that is, the values that are *in between the values* already in the data set. If the data are highly linear (r near $+1$ or -1) then we can be confident that our interpolated value is quite accurate. If the data are not highly linear (r near 0) then our confidence is duly reduced. For example, medical information collected from a patient every third day would establish data for day 3, 6, 9, . . . and so on. After performing regression analysis, it is likely that an interpolation for day 4 would be accurate, given a high r value.

Extrapolation

Extrapolation is the use of the regression line to predict values outside the range of data in a set, that is, values that are *smaller than the smallest value* already in the data set or *larger than the largest value*.

Two problems may arise in attempting to extrapolate from a data set. Firstly, it may not be reasonable to extrapolate too far away from the given data values. For example, suppose there is a weather data set for 5 days. Even if it is highly linear (r near $+1$ or -1) a regression line used to predict the same data 15 days in the future is highly risky. Weather has a habit of randomly fluctuating and patterns rarely stay stable for very long.

Secondly, the data may be highly linear in a narrow band of the given data set. For example, there may be data on stopping distances for a train at speeds of between 30 and 60 km/h. Even if they are highly linear in this range, it is unlikely that things are similar at very low speeds (0–15 km/h) or high speeds (over 100 km/h).

Generally, one should feel *more confident about the accuracy of a prediction derived from interpolation* than one derived from extrapolation. Of course, it still depends upon the correlation coefficient (r). The closer to linearity the data are, the more confident our predictions in all cases.

Interpolation is the use of the regression line to predict values within the range of data in a set.

Extrapolation is the use of the regression line to predict values outside the range of data in a set.

WORKED EXAMPLE 10

Using interpolation and the following data set, predict the height of an 8-year-old girl.

Age (years)	1	3	5	7	9	11
Height (cm)	60	76	115	126	141	148

THINK	WRITE
1 Find the equation of the least-squares regression line using your calculator. (Age is the explanatory variable and height is the response one.)	$y = 55.63 + 9.23x$
2 Replace x and y with the variables in question.	Height = 55.63 + 9.23 × age
3 Substitute 8 for age into the equation and evaluate.	When age = 8, Height = 55.63 + 9.23 × 8 = 129.5 (cm)
4 Write the answer.	At age 8, the predicted height is 129.5 cm.

WORKED EXAMPLE 11

Use extrapolation and the data from Worked example 10 to predict the height of the girl when she turns 15. Discuss the reliability of this prediction.

THINK	WRITE
1 Use the regression equation to calculate the girl's height at age 15.	Height = 55.63 + 9.23 × age = 55.63 + 9.23 × 15 = 194.08 cm
2 Analyse the result.	Since we have extrapolated the result (that is, since the greatest age in our data set is 11 and we are predicting outside the data set) we cannot claim that the prediction is reliable.

EXERCISE 3.7 Interpretation, interpolation and extrapolation

PRACTISE

1 **WE9** A study on the growth in height of a monkey in its first six months is assumed to be linear. Given the data shown, find:

a the equation, describing the relationship between the two variables
b the rate at which the monkey is growing
c the height of the monkey at birth.

Month from birth	1	2	3	4	5	6
Height (cm)	15	19	23	27	30	32

2 The outside temperature is assumed to increase linearly with time after 6 am. Given the data shown, find:

a the equation, describing the relationship between the two variables
b the rate at which the temperature is increasing
c the temperature at 6 am.

Hours after 6 am	0.5	1.5	3	3.5	5
Temperature (°C)	15	18	22	23	28

3 WE10 Using interpolation and the following data set, predict the height of a 10-year-old boy.

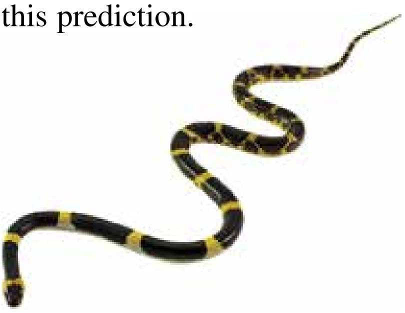
Age (years)	1	3	4	8	11	12
Height (cm)	65	82	92	140	157	165

4 Using interpolation and the following data set, predict the length of Matt's pet snake when it is 15 months old.

Age (months)	1	3	5	8	12	18
Length (cm)	48	60	71	93	117	159

5 WE11 Use extrapolation and the data from question 3 to predict the height of the boy when he turns 16. Discuss the reliability of this prediction.

6 Use extrapolation and the data from question 4 to predict the length of Matt's pet snake when it is 2 years old. Discuss the reliability of the prediction.



CONSOLIDATE

7 A drug company wishes to test the effectiveness of a drug to increase red blood cell counts in people who have a low count. The following data are collected.

Day of experiment	4	5	6	7	8	9
Red blood cell count	210	240	230	260	260	290

Find:

a the equation, describing the relationship between the variables in the form $y = a + bx$
b the rate at which the red blood cell count was changing
c the red blood cell count at the beginning of the experiment (that is, on day 0).

8 A wildlife exhibition is held over 6 weekends and features still and live displays. The number of live animals that are being exhibited varies each weekend. The number of animals participating, together with the number of visitors to the exhibition each weekend, is as shown.

Number of animals	6	4	8	5	7	6
Number of visitors	311	220	413	280	379	334

Find:

a the rate of increase of visitors as the number of live animals is increased by 1
b the predicted number of visitors if there are no live animals.

9 An electrical goods warehouse produces the following data showing the selling price of electrical goods to retailers and the volume of those sales.

Selling price ($)	60	80	100	120	140	160	200	220	240	260
Sales volume (× 1000)	400	300	275	250	210	190	150	100	50	0

Perform a least-squares regression analysis and discuss the meaning of the gradient and y-intercept.

10 A study of the dining-out habits of various income groups in a particular suburb produces the results shown in the table.

Weekly income ($)	100	200	300	400	500	600	700	800	900	1000
Number of restaurant visits per year	5.8	2.6	1.4	1.2	6	4.8	11.6	4.4	12.2	9

Use the data to predict:

a the number of visits per year by a person on a weekly income of $680
b the number of visits per year by a person on a weekly income of $2000.

11 Fit a least-squares regression line to the following data.

x	0	1	2	4	5	6	8	10
y	2	3	7	12	17	21	27	35

Find:

a the regression equation
b y when $x = 3$
c y when $x = 12$
d x when $y = 7$
e x when $y = 25$.
f Which of **b** to **e** above are extrapolations?

12 The following table represents the costs for shipping a consignment of shoes from Melbourne factories. The cost is given in terms of distance from Melbourne. There are two factories that can be used. The data are summarised in the table.

Distance from Melbourne (km)	10	20	30	40	50	60	70	80
Factory 1 cost ($)	70	70	90	100	110	120	150	180
Factory 2 cost ($)	70	75	80	100	100	115	125	135

a Find the least-squares regression equation for each factory.
b Which factory is likely to have the lowest cost to ship to a shop in Melbourne (i.e. distance from Melbourne = 0 km)?

c Which factory is likely to have the lowest cost to ship to Mytown, 115 kilometres from Melbourne?

d Which factory has the most ‘linear’ shipping rates?

13 A factory produces calculators. The least-squares regression line for cost of production (C) as a function of numbers of calculators (n) produced is given by:

$$C = 600 + 7.76n$$

Furthermore, this function is deemed accurate when producing between 100 and 1000 calculators.

a Find the cost to produce 200 calculators.

b How many calculators can be produced for \$2000?

c Find the cost to produce 10 000 calculators.

d What are the ‘fixed’ costs for this production?

e Which of **a** to **c** above is an interpolation?

14 A study of the relationship between IQ and results in a mathematics exam produced the following results. Unfortunately, some of the data were lost. Copy and complete the table by using the least-squares equation with the data that were supplied.

Note: Only use (x, y) pairs if both are in the table.

IQ	80		92	102	105		107	111	115	121
Test result (%)	56	60	68	65		74	71	73		92

15 The least-squares regression line for a starting salary (s) as a function of number of years of schooling (n) is given by the rule: $s = 37\,000 + 1800n$.

a Find the salary for a person who completed 10 years of schooling.

b Find the salary for a person who completed 12 years of schooling.

c Find the salary for a person who completed 15 years of schooling.

d Mary earned \$60 800. What was her likely schooling experience?

e Discuss the reasonableness of predicting salary on the basis of years of schooling.

16 Fit a least-squares regression to the following data.

q	0	1	3	7	10	15
r	12	18	27	49	64	93

Find:

a the regression equation.

b r when $q = 4$

c r when $q = 18$

d q when $r = 100$.

e Which of **b** to **d** is extrapolation?

MASTER

17 A plumbing company’s charges follow the least-squares regression line:

$$C = 180 + 80n$$

where C is the total cost and n is the number of hours of work. This function is accurate for a single 8 hour day.

a Find the total cost if the plumber worked for 3 hours.

b If the total charge was \$1250, how long did the plumber work, correct to 2 decimal places?

c Find the total cost if the plumber worked 9 hours and 30 minutes.

d What is the call-out fee (the cost to come out before they start any work)?

e Which of **a** to **c** is an extrapolation?

18 A comparison was investigated between AFL memberships sold and the amount of money spent on advertising by the club.

Advertising (in millions $)	2.3	1.8	1.2	0.8	1.6	0.6	1.0
Members	81 363	67 947	58 846	55 597	62 295	54 946	57 295

a Find the least-squares regression equation. Round the coefficient to the nearest whole number.

b Using the least-squares equation if $2 million was spent on advertising, how many members would you expect to have? Is this extrapolation or interpolation?

c If you wanted 70 000 members, how much would you expect to have to pay on advertising?

d Calculate the coefficient of determination and use it to explain the association between membership numbers and the amount of money spent on advertising.

3.8 Residual analysis

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Residual analysis
Concept summary
Practice questions

There are situations where the mere fitting of a regression line to some data is not enough to convince us that the data set is *truly* linear. Even if the correlation is close to +1 or −1 it still may not be convincing enough.

The next stage is to analyse the **residuals**, or deviations, of each data point from the straight line. A residual is the vertical difference between each data point and the regression line.

Calculating residuals

A sociologist gathers data on the heights of brothers and sisters in families from different cultural backgrounds. He enters his records in the table shown.

x	2	3	5	8	9	9
y	3	7	12	10	12	16

He then plots each point, and fits a regression line as shown in Figure 1, which follows. He then decides to calculate the residuals.

The *residuals* are simply the vertical distances from the line to each point. These lines are shown as green and pink bars in Figure 2.

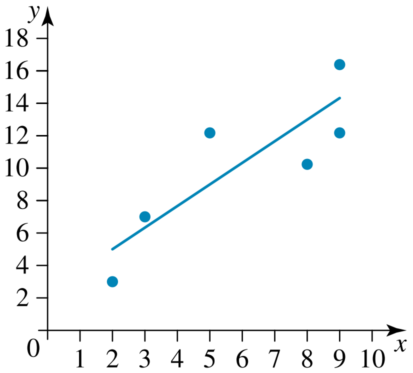

Figure 1

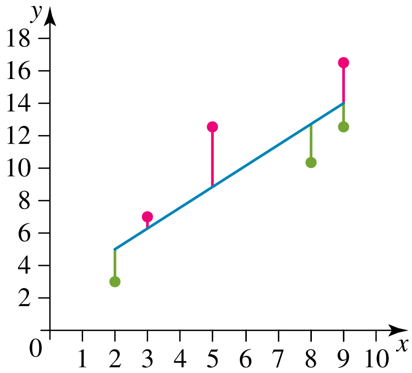

Figure 2

Finally, he calculates the residuals for each data point. This is done in two steps.

Step 1. He calculates the *predicted* value of y by using the regression equation.

Step 2. He calculates the *difference* between this predicted value and the original value.

A residual = actual y-value − predicted y-value

WORKED EXAMPLE 12

Consider the data set shown. Find the equation of the least-squares regression line and calculate the residuals.

x	1	2	3	4	5	6	7	8	9	10
y	5	6	8	15	24	47	77	112	187	309

THINK

1 Find the equation of a least-squares regression line using a calculator.

2 Use the equation of the least-squares regression line to calculate the predicted y-values (these are labelled as y_{pred}) for every x-value in the table. That is, substitute each x-value into the equation and evaluate record results in the table.

3 Calculate residuals for each point by subtracting predicted y-values from the actual y-value. (That is, residual = observed y-value − predicted y-value). Record results in the table.

WRITE

$y = -78.7 + 28.7x$

x-values	1	2	3	4	5
y-values	5.0	6.0	8.0	15.0	24.0
Predicted y-values	−50.05	−21.38	7.3	35.98	64.66
Residuals ($y - y_{\text{pred}}$)	55.05	27.38	0.7	−20.98	−40.66

x-values	6	7	8	9	10
y-values	47.0	77.0	112.0	187.0	309.0
Predicted y-values	93.34	122.02	150.7	179.38	208.06
Residuals ($y - y_{\text{pred}}$)	−46.34	−45.02	−38.7	7.62	100.94

Notes

1. The residuals may be determined by ($y - y_{\text{pred}}$); that is, the actual values minus the predicted values.
2. The *sum* of all the residuals *always* adds to 0 (or very close to 0 after rounding), when least-squares regression is used. This can act as a check for our calculations.

Introduction to residual analysis

As we observed in Worked example 12, there is not really a good fit between the data and the least-squares regression line; however, there seems to be a pattern in the residuals. How can we observe this pattern in more detail?

The answer is to plot the *residuals* themselves against the *original x-values*. If there is a pattern, it should become clearer after they are plotted.

Types of residual plots

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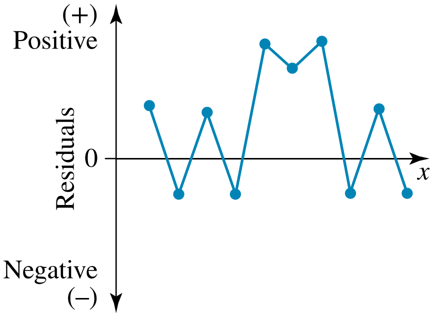
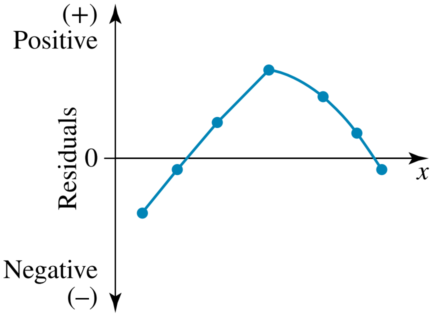
There are three basic types of **residual plots**. Each type indicates whether or not a linear relationship exists between the two variables under investigation.

Note: The points are joined together to see the patterns more clearly.

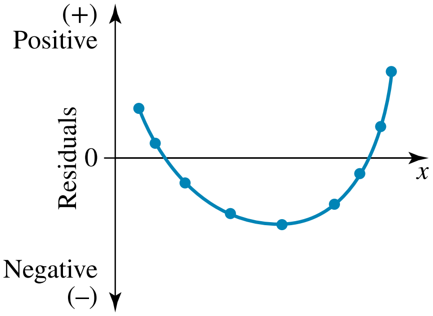
The points of the residuals are randomly scattered above and below the x-axis. The original data probably have a *linear* relationship.



The points of the residuals show a curved pattern (∩), with a series of negative, then positive and back to negative residuals along the x-axis. The original data probably have a **non-linear relationship**. Transformation of the data may be required.



The points of the residuals show a curved pattern (∪), with a series of positive, then negative and back to positive residuals along the x-axis. The original data probably have a *non-linear* relationship. Transformation of the data may be required.



The transformation of data suggested in the last two residual plots will be studied in more detail in the next section.

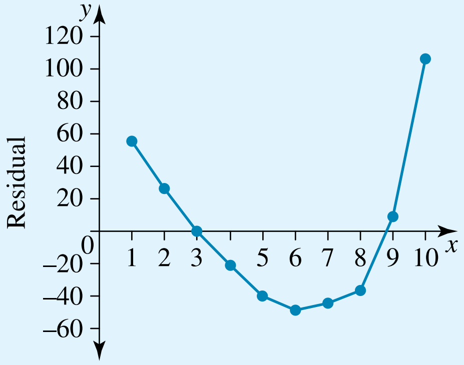
WORKED EXAMPLE 13 Using the same data as in Worked example 12, plot the residuals and discuss the features of the residual plot.

THINK / **WRITE/DRAW**

1 Generate a table of values of residuals against x.

x-values	1	2	3	4	5
Residuals ($y - y_{\text{pred}}$)	55.05	27.38	0.7	−20.98	−40.66
x-values	6	7	8	9	10
Residuals ($y - y_{\text{pred}}$)	−46.34	−45.02	−38.7	7.62	100.94

2 Plot the residuals against x. To see the pattern clearer, join the consecutive points with straight line segments.



3 If the relationship was linear the residuals would be scattered randomly above and below the line. However, in this instance there is a pattern which looks somewhat like a parabola. This should indicate that the data were not really linear, but were more likely to be quadratic. Comment on the residual plot and its relevance.

The residual plot indicates a distinct pattern suggesting that a non-linear model could be more appropriate.

EXERCISE 3.8 Residual analysis

PRACTISE

1 **WE12** Consider the data set shown. Find the equation of the least-squares regression line and calculate the residuals.

x	1	2	3	4	5	6	7	8	9
y	12	20	35	40	50	67	83	88	93

2 From the data shown, find the equation of the least-squares regression line and calculate the residuals.

x	5	7	10	12	15	18	25	30	40
y	45	61	89	122	161	177	243	333	366

3 **WE13** Using the same data from question 1, plot the residuals and discuss the features of the residual plot. Is your result consistent with the coefficient of determination?

4 Using the same data from question **2**, plot the residuals and discuss the features of the residual plot. Is your result consistent with the coefficient of determination?

CONSOLIDATE

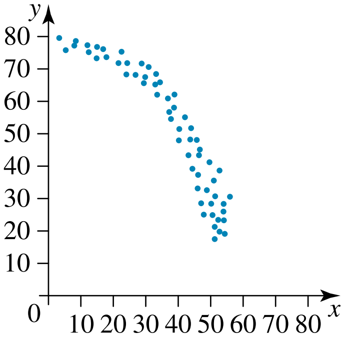
5 Find the residuals for the following data.

x	1	2	3	4	5	6
y	1	9.7	12.7	13.7	14.4	14.5

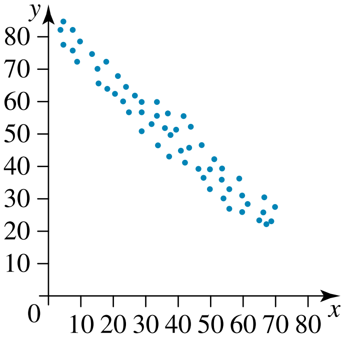
6 For the results of question **5**, plot the residuals and discuss whether the relationship between x and y is linear.

7 Which of the following scatterplots shows linear relationship between the variables?

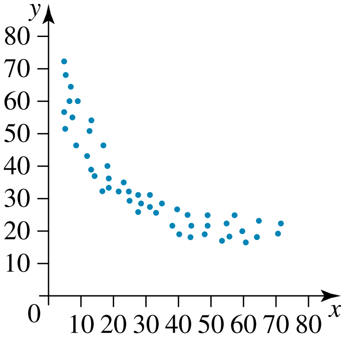
i



ii



iii



A All of them
B None of them
C **i** and **iii** only
D **ii** only
E **ii** and **iii** only

8 Consider the following table from a survey conducted at a new computer manufacturing factory. It shows the percentage of defective computers produced on 8 different days after the opening of the factory.

Day	2	4	5	7	8	9	10	11
Defective rate (%)	15	10	12	4	9	7	3	4

a The results of least-squares regression were: $b = -1.19$, $a = 16.34$, $r = -0.87$. Given $y = a + bx$, use the above information to calculate the predicted defective rates (y_{pred}).
b Find the residuals ($y - y_{\text{pred}}$).
c Plot the residuals and comment on the likely linearity of the data.
d Estimate the defective rate after the first day of the factory's operation.
e Estimate when the defective rate will be at zero. Comment on this result.

9 The following data represent the number of tourists booked into a hotel in central Queensland during the first week of a drought. (Assume Monday = 1.)

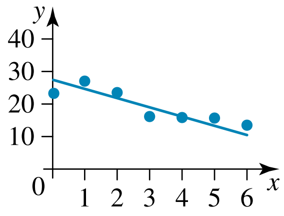
Day	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Bookings in hotel	158	124	74	56	31	35	22

The results of least-squares regression were:
$b = -22.5$, $a = 161.3$, $r = -0.94$, where $y = a + bx$.

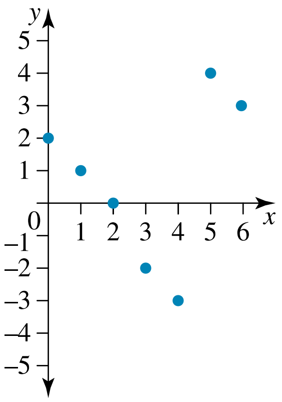
a Find the predicted hotel bookings (y_{pred}) for each day of the week.
b Find the residuals ($y - y_{\text{pred}}$).
c Plot the residuals and comment on the likely linearity of the data.
d Would this regression line be a typical one for this hotel?

10 A least-squares regression is fitted to the points shown in the scatterplot.

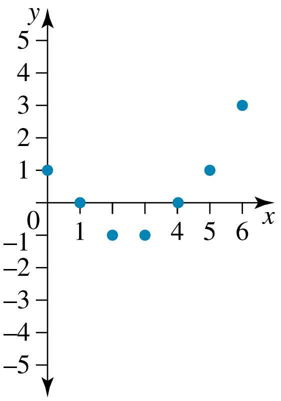
Which of the following looks most similar to the residual plot for the data?



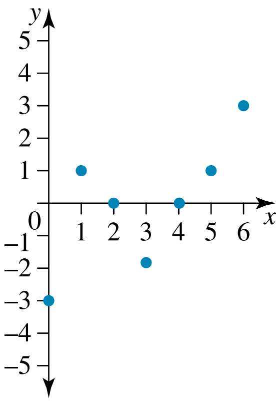
A



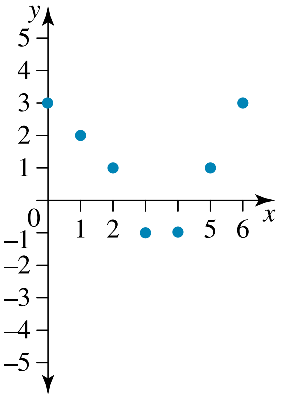
B



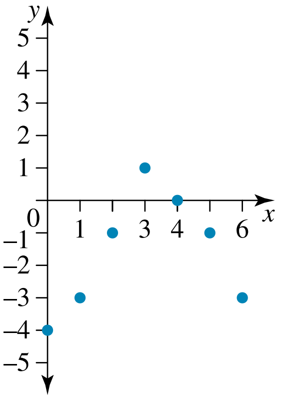
C



D



E



11 From each table of residuals, decide whether or not the relationship between the variables is likely to be linear.

a

x	y	Residuals
1	2	−1.34
2	4	−0.3
3	7	−0.1
4	11	0.2
5	21	0.97
6	20	2.3
7	19	1.2
8	15	−0.15
9	12	−0.9
10	6	−2.8

b

x	y	Residuals
23	56	0.12
21	50	−0.56
19	43	1.30
16	41	0.20
14	37	−1.45
11	31	2.16
9	28	−0.22
6	22	−3.56
4	19	2.19
3	17	−1.05

c

x	y	Residuals
1.2	23	0.045
1.6	25	0.003
1.8	24	−0.023
2.0	26	−0.089
2.2	28	−0.15
2.6	29	−0.98
2.7	34	−0.34
2.9	42	−0.01
3.0	56	0.45
3.1	64	1.23

12 Consider the following data set.

x	0	1	2	3	4	5	6	7	8	9	10
y	1	4	15	33	60	94	134	180	240	300	390

a Plot the data and fit a least-squares regression line.

b Find the correlation coefficient and interpret its value.

c Calculate the coefficient of determination and explain its meaning.

d Find the residuals.

e Construct the residual plot and use it to comment on the appropriateness of the assumption that the relationship between the variables is linear.

13 Find the residuals for the following data set.

m	12	37	35	41	55	69	77	90
P	2.5	21.6	52.3	89.1	100.7	110.3	112.4	113.7

14 For the data in question **13**, plot the residuals and comment whether the relationship between x and y is linear.

MASTER

15 Calculate the residuals of the following data.

k	1.6	2.5	5.9	7.7	8.1	9.7	10.3	15.4
D	22.5	37.8	41.5	66.9	82.5	88.7	91.6	120.4

16 For the data in question **15**:

a plot the residuals and comment whether the relationship between x and y is linear.

b calculate the coefficient of determination and explain is meaning.

3.9 Transforming to linearity

studyon

Unit 3
AOS DA
Topic 7
Concept 7

Transformation of data
Concept summary
Practice questions

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Interactivity
Transforming to linearity
int-6253

Although linear regression might produce a 'good' fit (high r value) to a set of data, the data set may still be non-linear. To remove (as much as is possible) such *non-linearity*, the data can be transformed.

Either the x-values, y-values, or both may be transformed in some way so that the transformed data are more linear. This enables more accurate predictions (extrapolations and interpolations) from the regression equation. In Further Mathematics, six transformations are studied:

Logarithmic transformations:	y **versus** $\log_{10}(x)$	$\log_{10}(y)$ **versus** x
Quadratic transformations:	y **versus** x^2	y^2 **versus** x
Reciprocal transformations:	y **versus** $\frac{1}{x}$	$\frac{1}{y}$ **versus** x

Choosing the correct transformations

To decide on an appropriate transformation, examine the points on a scatterplot with high values of x and/or y (that is, away from the origin) and decide for each axis whether it needs to be stretched or compressed to make the points line up. The best way to see which of the transformations to use is to look at a number of 'data patterns'.

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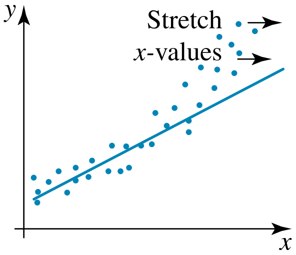
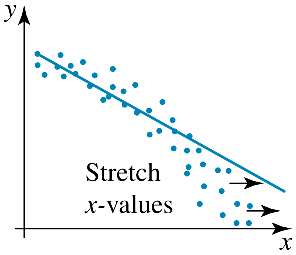
Interactivity
Linearising data
int-6491

Quadratic transformations

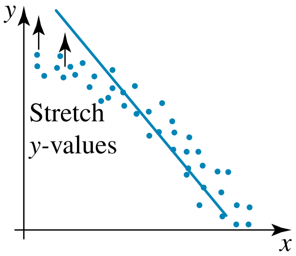
1. Use y versus x^2 transformation.



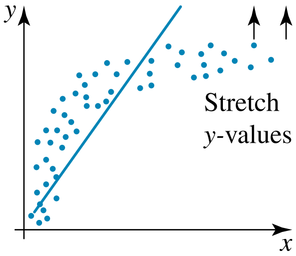
2. Use y versus x^2 transformation.



3. Use y^2 versus x transformation.

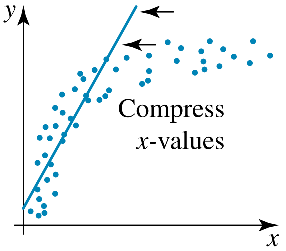


4. Use y^2 versus x transformation.

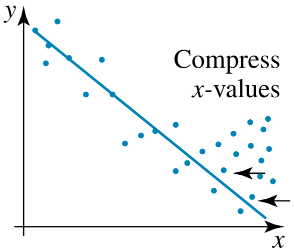


Logarithmic and reciprocal transformations

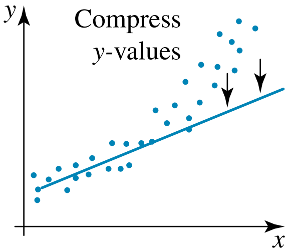
1. Use y versus $\log_{10}(x)$ or y versus $\frac{1}{x}$ transformation.



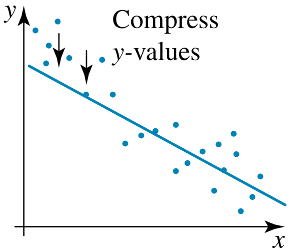
2. Use y versus $\log_{10}(x)$ or y versus $\frac{1}{x}$ transformation.



3. Use $\log_{10}(y)$ versus x or $\frac{1}{y}$ versus x transformation.



4. Use $\log_{10}(y)$ versus x or $\frac{1}{y}$ versus x transformation.



Testing transformations

As there are at least two possible transformations for any given non-linear scatterplot, the decision as to which is the best comes from the coefficient of correlation. The least-squares regression equation that has a Pearson correlation coefficient closest to 1 or −1 should be considered as the most appropriate. However, there may be very little difference so common sense needs to be applied. It is sometimes more useful to use a linear function rather than one of the six non-linear functions.

WORKED EXAMPLE 14

a **Plot the following data on a scatterplot, consider the shape of the graph and apply a quadratic transformation.**

b **Calculate the equation of the least-squares regression line for the transformed data.**

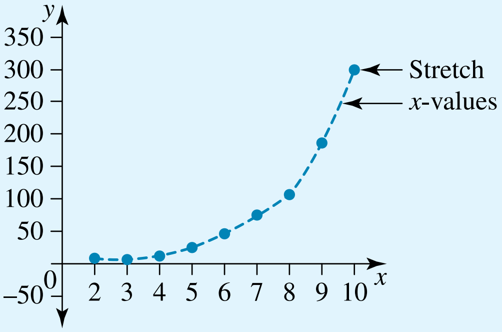
x	1	2	3	4	5	6	7	8	9	10
y	5	6	8	15	24	47	77	112	187	309

THINK

WRITE/DRAW

1 Plot the data to check that a quadratic transformation is suitable.

Looking at the shape of the graph, the best option is to stretch the x-axis. This requires an x^2 transformation.



Stretch x-values

2 Square the x-values to give a transformed data set by using CAS.

x^2	1	4	9	16	25	36	49	61	81	100
y	5	6	8	15	24	47	77	112	187	309

3 Find the equation of the least-squares regression line for the transformed data.

Using CAS:

y-intercept (a) = −28.0

gradient (b) = 2.78

correlation (r) = 0.95.

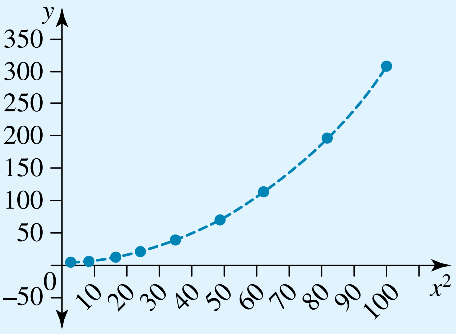
$y = a + bx$

$y = -28.0 + 2.78x_T$ where $x_T = x^2$; that is,

$y = -28.0 + 2.78x^2$

4 Plot the new transformed data.

Note: These data are still not truly linear, but are ‘less’ parabolic. Perhaps another transformation would improve things even further. This could involve transforming the y-values, such as $\log_{10}(y)$, and applying another linear regression.



WORKED EXAMPLE 15

a **Tranform the data by applying a logarithmic transformation to the y-variable.**

b **Calculate the equation of the least squares regression line for the transformed data.**

c **Comment on the value of r.**

Time after operation (h)	x	1	2	3	4	5	6	7	8
Heart rate (beats/min)	y	100	80	65	55	50	51	48	46

THINK	WRITE
a Transform the y data by calculating the log of y-values or, in this problem, the log of heart rate.	**a** (table below)
b **1** Use a calculator to find the equation of least-squares regression line for x and log y.	**b** $\log_{10}(y) = 1.98 - 0.05x$
2 Rewrite the equation in terms of the variables in question.	$\log_{10}$ (heart rate) = 1.98 − 0.05 × time (i.e. time = number of hours after the operation.)
c State the value of r and comment on the result.	**c** $r = -0.93$ There is a slight improvement of the correlation coefficient that resulted from applying logarithmic transformation.

Time	x	1	2	3	4	5	6	7	8
log (heart rate)	log y	2	1.903	1.813	1.740	1.694	1.708	1.681	1.663

Further investigation

Often all appropriate transformations need to be performed to choose the best one. Extend Worked example 15 by compressing the y data using the reciprocals of the y data or even compress the x data. Go back to the steps for transforming the data. Did you get a better r value and thus a more reliable line of best fit? (*Hint:* The best transformation gives $r = -0.98$.)

Using the transformed line for predictions

Once the appropriate model has been established and the equation of least-squares regression line has been found, the equation can be used for predictions.

WORKED EXAMPLE 16

a **Using CAS, apply a reciprocal transformation to the following data.**

b **Use the transformed regression equation to predict the number of students wearing a jumper when the temperature is 12 °C.**

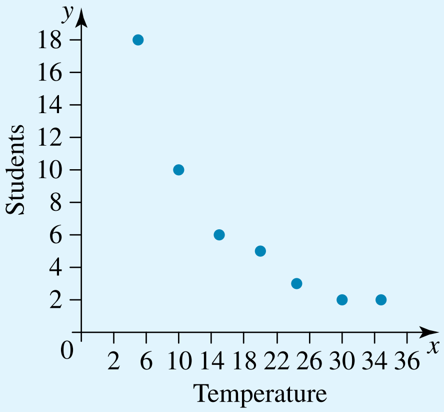
Temperature (°C)	x	5	10	15	20	25	30	35
Number of students in a class wearing jumpers	y	18	10	6	5	3	2	2

THINK | **WRITE/DRAW**

a **1** Construct the scatterplot. Temperature is the explanatory variable, while the number of students wearing jumpers is the response one.

Therefore, put *temperature* on the horizontal axis and *students* on the vertical axis.

a



2 The x-values should be compressed, so it may be appropriate to transform the x-data by calculating the reciprocal of temperature. Reciprocate each x-value $\left(\text{that is, find } \frac{1}{x}\right)$.

$\frac{1}{\text{Temperature}}$	$\frac{1}{x}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{20}$	$\frac{1}{25}$	$\frac{1}{30}$	$\frac{1}{35}$
Number of students wearing jumpers	y	18	10	6	5	3	2	2

3 Use CAS to find the equation of the least-squares regression line for $\frac{1}{x}$ and y.

$y = -0.4354 + 94.583x_T$, where $x_T = \frac{1}{x}$ or

$$y = -0.4354 + \frac{94.583}{x}$$

4 Replace x and y with the variables in question.

The number of students in class wearing jumpers

$$= -0.4354 + \frac{94.583}{\text{Temperature}}$$

b **1** Substitute 12 for x into the equation of the regression line and evaluate.

b Number of students wearing jumpers

$$= -0.4354 + \frac{94.583}{\text{Temperature}}$$

$$= -0.4354 + \frac{94.583}{12}$$

$$= 7.447$$

2 Write your answer to the nearest whole number.

7 students are predicted to wear jumpers when the temperature is 12 °C.

EXERCISE 3.9 Transforming to linearity

PRACTISE

1 WE14 **a** Plot the data on a scatterplot, consider the shape of the graph and apply a quadratic transformation.

b Calculate the equation of the least-squares regression line for the transformed data.

x	1	2	3	4	5	6	7	8	9
y	12	19	29	47	63	85	114	144	178

2 **a** Plot the following data on a scatterplot, consider the shape of the graph and apply a quadratic transformation.

b Calculate the equation of the least-squares regression line for the transformed data.

x	3	5	9	12	16	21	24	33
y	5	12	38	75	132	209	291	578

3 **WE15** Apply a logarithmic transformation to the following data, which represents a speed of a car as a function of time, by transforming the y-variable.

Time (s)	1	2	3	4	5	6	7	8
Speed (ms^{-1})	90	71	55	45	39	35	32	30

4 Apply a logarithmic transformation to the following data by transforming the y-variable.

x	5	10	15	20	25	30	35	40
y	1000	500	225	147	99	70	59	56

5 **WE16** **a** Using CAS, apply a reciprocal transformation to the x-variable of the following data.

Time after 6 pm (h)	1	2	3	4	5	6	7	8
Temperature (°C)	32	22	16	11	9	8	7	7

b Use the transformed regression equation to predict the temperature at 10.30 pm.

6 **a** Using CAS, apply a reciprocal transformation to the x-variable of the following data.

x	2	5	7	9	10	13	15	18
y	120	50	33	15	9	5	2	1

b Use the transformed regression equation to predict y when $x = 12$.

c Use the transformed regression equation to predict x when $y = 20$.

CONSOLIDATE

7 Apply a quadratic (x^2) transformation to the following data set. The regression line has been determined as $y = 186 - 27.7x$ with $r = -0.91$.

x	2	3	4	5	7	9
y	96	95	92	90	14	−100

8 The *average* heights of 50 girls of various ages were measured as follows.

Age group (years)	9	10	11	12	13	14	15	16	17	18
Average height (cm)	128	144	148	154	158	161	165	164	166	167

The original linear regression yielded:

$$\text{Height} = 104.7 + 3.76 \times \text{age, with } r = 0.92.$$

a Plot the original data and regression line.

b Apply a $\log_{10}(x)$ transformation.

c Perform regression analysis on the transformed data and comment on your results.

9 **a** Use the transformed data from question **8** to predict the heights of girls of the following ages:

- **i** 7 years old
- **ii** 10.5 years old
- **iii** 20 years old.

b Which of the predictions in part **a** were obtained by interpolating?

10 Comment on the suitability of transforming the data of question **8** in order to improve predictions of heights for girls under 8 years old or over 18.

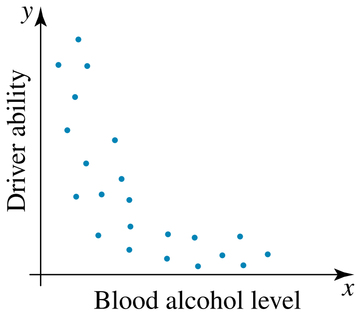
11 **a** Apply a reciprocal transformation to the following data obtained by a physics student studying light intensity.

Distance from light source (metres)	1	2	3	4	5	10
Intensity (candlepower)	90	60	28	22	20	12

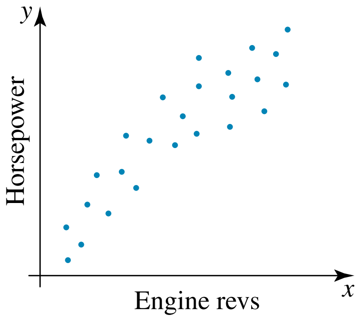
b Use the transformed regression equation to predict the intensity at a distance of 20 metres.

12 For each of the following scatterplots suggest an appropriate transformation(s).

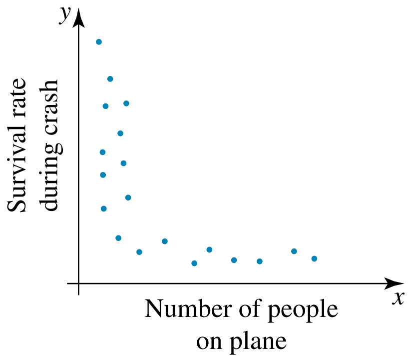
a



b



c



13 Use the equation $y = -12.5 + 0.2x^2$, found after transformation, to predict values of y for the given x-value (correct to 2 decimal places):

a $x = 2.5$ **b** $x = -2.5$.

14 Use the equation $y = -25 + 1.12 \log_{10}(x)$, found after transformation, to predict values of y for the given x-value (correct to 2 decimal places):

a $x = 2.5$ **b** $x = -2.5$ **c** $x = 0$.

15 Use the equation $\log_{10}(y) = 0.03 + 0.2x$, found after transformation, to predict values of y for the given x-value (correct to 2 decimal places):

a $x = 2.5$ **b** $x = -2.5$.

16 Use the equation $\frac{1}{y} = 12.5 + 0.2x$, found after transformation, to predict values of y for the given x-value (correct to 2 decimal places):

a $x = 2.5$ **b** $x = -2.5$.

MASTER

17 The seeds in the sunflower are arranged in spirals for a compact head. Counting the number of seeds in the successive circles starting from the centre and moving outwards, the following number of seeds were counted.

Circle	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Number of seeds	3	5	8	13	21	34	55	89	144	233

a Plot the data and fit a least-squares regression line.

b Find the correlation coefficient and interpret its value.

c Using the equation of the regression line, predict the number of seeds in the 11th circle.

d Find the residuals.

e Construct the residual plot. Is the relation between the number of the circle and the number of seeds linear?

f What type of transformation could be applied to:

  i the x-values? Explain why.

  ii the y-values? Explain why.

18 Apply a $\log_{10}(y)$ transformation to the data used in question **17**.

a Fit a least-squares regression line to the transformed data and plot it with the data.

b Find the correlation coefficient. Is there an improvement? Why?

c Find the equation of the least-squares regression line for the transformed data.

d Calculate the coefficient of determination and interpret its value.

e Using the equation of the regression line for the transformed data, predict the number of seeds for the 11th circle.

f How does this compare with the prediction from question **17**?

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3.10 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

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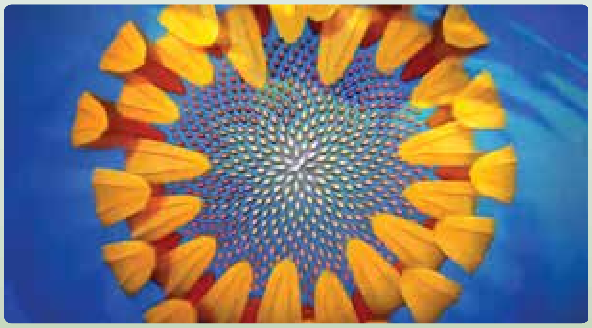
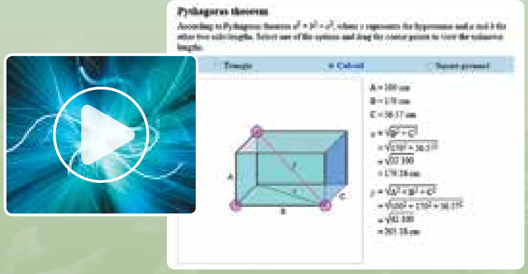
A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

3 Answers

EXERCISE 3.2

1 a Daily temperature = explanatory variable, air conditioners sold = response variable

b Not appropriate

2 a Size of the crowd = response variable, teams playing = explanatory variable

b Net score of a round of golf = response variable, golfer's handicap = explanatory variable

3 a Explanatory — age, response — salary

b Explanatory — amount of fertiliser, response — growth

c Not appropriate

d Not appropriate

e Explanatory — number in household, response — size of house

4 a Explanatory — month of the year, response — size of electricity bill

b Explanatory — number of hours, response — mark on the test

c Not appropriate

d Explanatory — season, response — cost

5 C

6 C

7 D

8 True

9 True

10 True

11 False

12 False

13 a Minutes on the court: explanatory variable

b Points scored: response variable

14 a Response variable: electricity bill

b Number of Christmas lights should be on the x-axis.

c Explanatory variable: number of Christmas lights

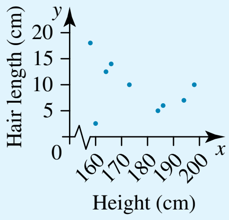
d Electricity bill should be on the y-axis.

EXERCISE 3.3

1 Moderate positive linear relationship

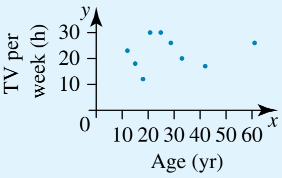
2 Moderate positive linear relationship

3



No relationship

4



No relationship

5 a Yes — positive association

b Yes — positive association

c Yes — positive association

d Yes — negative association

e Yes — positive association

f Yes — negative association

g No — no association

6 a Weak, negative association of linear form

b Moderate, negative association of linear form

c Moderate, positive association of linear form

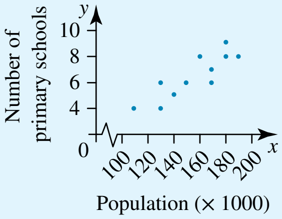
d Strong, positive association of linear form

e No association

f Non-linear association

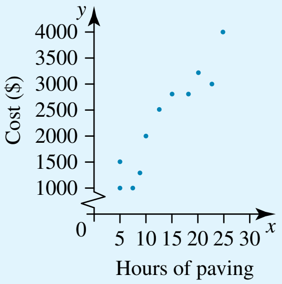
7 B

8



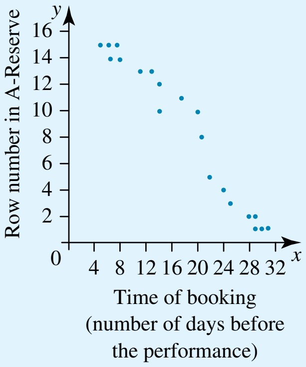
Moderate positive association of linear form, no outliers

9



Strong positive association of linear form, no outliers

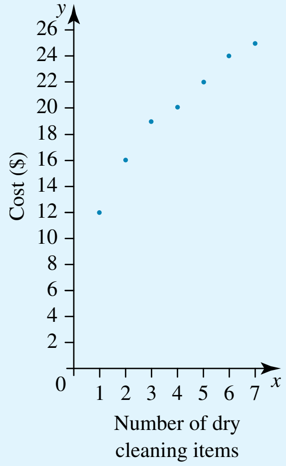
10



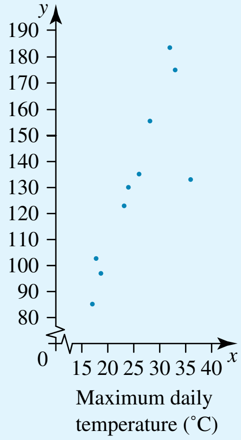
Strong negative association of linear form, no outliers

11 D

12 a, b



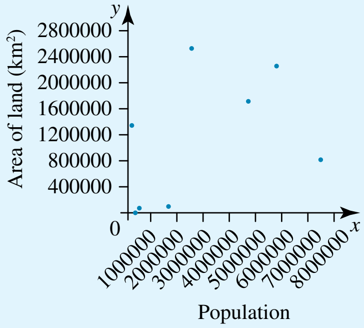
13 a, b



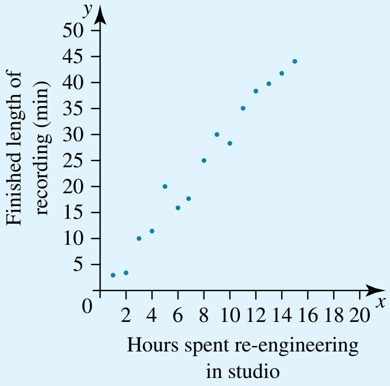
14 a There is a strong positive correlation between the dry cleaning items and the cost.

b There is a strong positive correlation between the maximum daily temperature and the number of drinks sold.

15 There appears to be no correlation between population and area of the various states and territories.



16 a



b There is a relationship. It is strong, positive and linear.

EXERCISE 3.4

1 a i $r = -0.9$

ii Moderate, negative, linear

b i $r = 0.7$

ii Strong, positive, linear

2 a Strong, positive, linear

b Weak, negative, linear

3 a No association

b Moderate positive

c Strong negative

d Strong negative

4 a Strong positive

b Strong positive

c Weak negative

d No association

5 a i $r \approx -0.8$

ii Strong, negative, linear association

b i $r \approx 0.6$

ii Moderate, positive, linear association

c i $r \approx 0.2$

ii No linear association

d i $r \approx -0.2$

ii No linear association

6 a i $r = 1$

ii Perfect, positive, linear association

b i $r \approx 0.8$

ii Strong, positive, linear association

c i $r \approx 0$

ii No linear association

d i $r \approx -0.7$

ii Moderate, negative, linear association

7 B

8 E

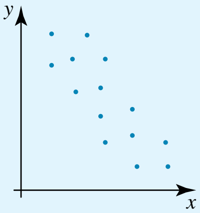
9 a $r = 0.1$: no linear relationship

b $r = 0.2$: no linear relationship

c $r = 0.95$: strong, positive linear relationship

10 D

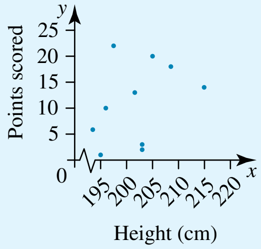
11 C

12 

13 E

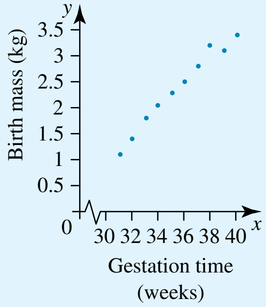
14 C

EXERCISE 3.5

1 a 

b No linear relationship, $r \approx 0.3$

c $r \approx 0.36$, weak, positive linear relationship

2 a 

b Strong, positive, linear relationship. $r \approx 0.95$

c $r \approx 0.99$, very strong linear relationship

3 a Coefficient of determination = 0.59

We can then conclude that 59% of the variation in the number of people found to have a blood alcohol reading over 0.05 can be explained by the variation in the number of booze buses in use. Thus the number of booze buses in use is a factor in predicting the number of drivers with a reading over 0.05.

b First: Number of booze buses in use (CAUSE)

Later: Number of drivers with a blood alcohol reading over 0.05 (EFFECT)

4 First: Number of books read (CAUSE)

Later: Spelling ability (EFFECT)

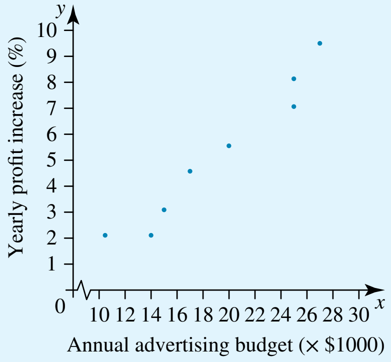
Note: This is the most likely case, but there could be an argument that spelling ability affects the number of books read.

5 a 

b There is moderate, negative linear association. r is approximately −0.6.

c $r = -0.66$. There is a moderate negative linear association between the yearly salary and the number of votes. That is, the larger the yearly salary of the player, the fewer the number of votes we might expect to see.

6 Coefficient of determination is 0.7569. The portion of variation in the number of visits to the doctor that can be explained by the variation in the number of cigarettes smoked is about 76%.

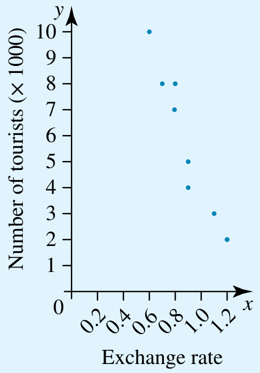
7 a 

b There is strong, positive linear association. r is approximately 0.8.

c $r = 0.98$

d Coefficient of determination is 0.96.

e The proportion of the variation in the yearly profit increase that can be explained by the variation in the advertising budget is 96%.

8 a 

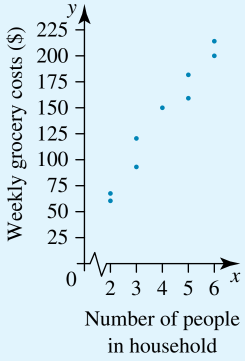
b There is strong, negative association of a linear form and r is approximately −0.9.

c $r = -0.96$

d Coefficient of determination is 0.92.

e The proportion of the variation in the number of tourists that can be explained by the variation in the exchange rate is 92%.

9 a



b There is strong, positive association of a linear form and r is approximately 0.9.

c First: Number of people in household (CAUSE)

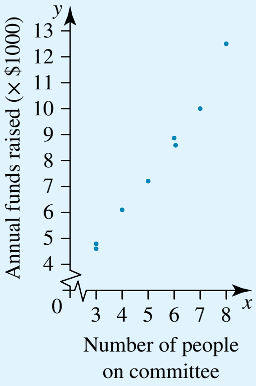
Later: Weekly grocery costs (EFFECT)

d $r = 0.98$

e Coefficient of determination is 0.96.

f The proportion of the variation in the weekly grocery costs that can be explained by the variation in the number of people in the household is 96%.

10 a



b There is almost perfect positive association of a linear form and r is nearly 1.

c $r = 0.99$

d No. High degree of correlation does not mean we can comment on whether one variable causes particular values in another.

e Coefficient of determination is 0.98.

f The proportion of the variation in the funds raised that can be explained by the variation in the number of people on the committee is 98%.

11 E

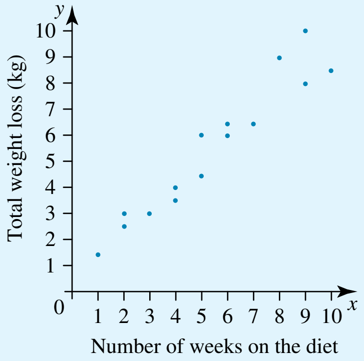
12 C

13 B

14 This is not an example of cause and effect. A common response variable, the education level of adults in a suburb, would provide a direct link to both variables.

15 C

16 a



b The scatterplot shows strong, positive association of linear form.

c It is appropriate since the scatterplot indicates association showing linear form and there are no outliers.

d $r \approx 0.9$

e $r = 0.96$

f We cannot say whether total weight loss is affected by the number of weeks people stayed on the Certain Slim diet. We can only note the degree of correlation.

g $r^2 = 0.92$

h The coefficient of determination tells us that 92% of the variation in total weight loss can be explained by the variation in the number of weeks on the Certain Slim diet.

EXERCISE 3.6

1 $y = -37.57 + 1.87x$ or air conditioner sales = –37.57 + 1.87 × (temperature), $r^2 = 0.97$, strong linear association

2 $y = 2.11 + 2.64x$ or number of dialysis patients = 2.11 + 2.64 × (month), $r^2 = 0.95$, strong linear association

3 a Independent variable = ‘height of Year 12 girl’.

b $b = 0.96$

c $a = 22.64$

4 $b = -0.78167$, $a = 13.94$

$y = 13.94 - 0.78x$

5 $y = 4.57 + 1.04x$

6 $y = 35.47 - 3.72x$

7 $y = 9.06 + 1.20x$

8 a The mathematics exam mark

b 0.95 **c** −6.07 **d** 75%

9 a $y = 91.78 + 3.33x$

b $y = 21.87 - 0.15x$

c $y = 8890.48 + 52.38x$

d $y = 30 - x$

10 The least-squares regression equations are exactly the same as obtained in questions **5**, **6** and **7**.

11 a $y = 22 - 2x$

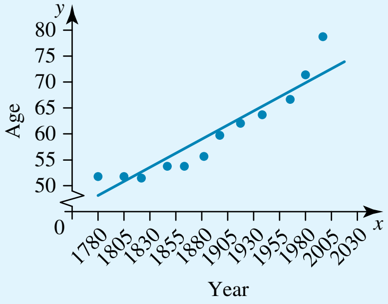
b A 'perfect' fit

c $y = 11 - 0.5x$

12 E

13 a $y = -164.7 + 0.119x$

b

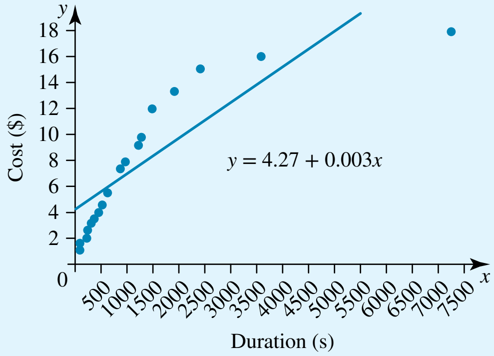


c The data definitely are not linear; there are big increases from 1880–1920, 1940–2000.

14 a Duration of call

b Cost of call (\$) = \$4.27 + \$0.002 57 × duration of call (sec)

c



The line does not fit closely for all data points. The equation is not reliable due to outliers. If you eliminate the last two calls then there is a direct relationship.

d $r^2 = 0.73$. Therefore the linear association is only moderate.

15 D

16 a $y = 8 + 4x$, perfect fit, but meaningless

b $y = 10 + 2.5x$, good fit, but almost meaningless

c $y = 9.5 + 2.8x$, $y = 8.9 + 3.1x$, $y = 8.4 + 3.3x$, good fit.

d The answers appear to be converging towards a 'correct' line.

EXERCISE 3.7

1 a $y = 12.13 + 3.49x$ or monkey height (cm) = 12.13 + 3.49 × (month from birth)

b 3.49 cm/month

c 12.13 cm

2 a $y = 13.56 + 2.83x$ or temperature (°C) = 13.56 + 2.83 × (time after 6 a.m.)

b 2.83°C/hr

c 13.56°C

3 $y = 56.03 + 9.35x$ or height (cm) = 56.03 + 9.35 × (age)

Height (cm) = 149.53 cm

4 $y = 40.10 + 6.54x$ or length (cm) = 40.10 + 6.54 × (months)

Length (cm) = 138.20 cm

5 Height (cm) = 205.63 cm

We cannot claim that the prediction is reliable, as it uses extrapolation.

6 Length (cm) = 197.06 cm

We cannot claim that the prediction is reliable, as it uses extrapolation.

7 a $y = 157.3 + 14x$

b 14 cells per day

c 157

8 a 48.5, or 49 people per extra animal

b 31.8, or 32 visitors

9 $y = 464 - 1.72x$, $r = -0.98$. Gradient shows a drop of 1720 sales for every \$1 increase in the price of the item. Clearly, the y-intercept is nonsensical in this case since an item is not going to be sold for \$0! This is a case where extrapolation of the line makes no sense.

10 a 7

b 18

11 a $y = 0.286 + 3.381x$

b 10.4

c 40.9

d 1.99

e 7.31

f c

12 a Factory 1: $y = 43.21 + 1.51x$
Factory 2: $y = 56.61 + 0.96x$

b Factory 1 is cheaper at \$43.21 (compared to Factory 2 at \$56.61).

c Factory 2 is cheaper at \$167.47 (compared to Factory 1 at \$216.86).

d Factory 2 is marginally more linear (Factory 1: $r = 0.97$; Factory 2: $r = 0.99$).

13 a \$2152

b 180

c \$78 200

d \$600

e **a**, **b** only

14

IQ	Test result (%)
80	56
87	60
92	68
102	65
105	73
106	74
107	71
111	73
115	80
121	92

15 a \$55 000 **b** \$58 600

c \$64 000 **d** About 13 years

e Various answers

16 a $r = 11.73 + 5.35q$ **b** $r = 33.13$

c $r = 108.03$ **d** $q = 16.50$

e **c** and **d**

17 a $C = \$420$ **b** $n = 13.38$ hours

c $C = \$940$ **d** Call out fee = \$180

e **b** and **c**

18 a $y = 42956 + 14.795 \times$ (millions spent on advertising)
or
Members = 42 956 + 14 795
× (millions spent on advertising)

b 72 546 members

Interpolation

c \$1.83 million

d $r^2 = 0.90$, strong linear relationship

EXERCISE 3.8

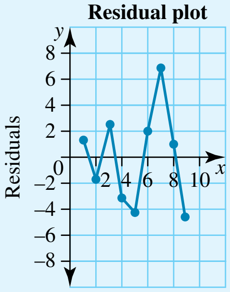
1 Using CAS: $y = -0.03 + 10.85x$

See table at foot of the page.*

2 Using CAS: $y = 0.69 + 9.82x$

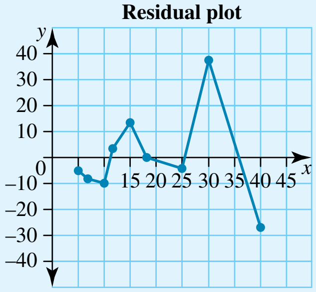
See table at foot of the page.*

3



$r^2 = 0.98$, consistent with a linear relationship

4



$r^2 = 0.98$, consistent with a linear relationship

5

x	y	y_{pred}	Residuals
1	1	5.1	−4.1
2	9.7	7.46	2.24
3	12.7	9.82	2.88
4	13.7	12.18	1.52
5	14.4	14.54	−0.14
6	14.5	16.9	−2.4

---

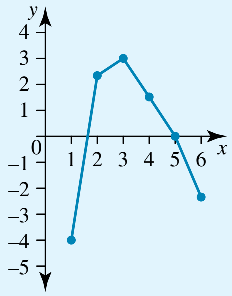
*1

x	1	2	3	4	5	6	7	8	9
y	12	20	35	40	50	67	83	88	93
predicted y	10.82	21.67	32.52	43.37	54.22	65.07	75.92	86.77	97.62
Residual ($y - y_{\text{predicted}}$)	1.18	−1.67	2.48	−3.37	−4.22	1.93	7.08	1.23	−4.62

*2

x	5	7	10	12	15	18	25	30	40
y	45	61	89	122	161	177	243	333	366
predicted y	49.79	69.43	98.89	118.53	147.99	177.45	246.19	295.29	393.49
Residual ($y - y_{\text{predicted}}$)	−4.79	−8.43	−9.39	3.47	13.01	−0.45	−3.19	37.71	−27.49

6 By examining the original scatterplot, and residual plot, data are clearly not linear.

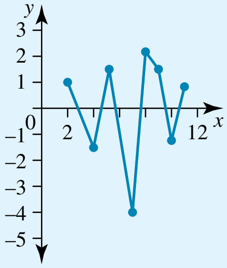


7 D

8 a, b

Day	Defective rate (%)	y_{pred}	Residuals
2	15	13.96	1.04
4	10	11.58	−1.58
5	12	10.39	1.61
7	4	8.01	−4.01
8	9	6.82	2.18
9	7	5.63	1.37
10	3	4.44	−1.44
11	4	3.25	0.75

c



No apparent pattern in the residuals — likely to be linear

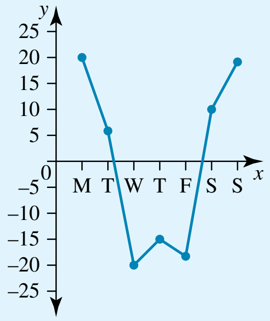
d 15.15%

e 13.7 days. Unlikely that extrapolation that far from data points is accurate. Unlikely that there would be 0% defectives.

9 a, b

Day	Bookings in hotel	y_{pred}	Residuals
1	158	138.8	19.2
2	124	116.3	7.7
3	74	93.8	−19.8
4	56	71.3	−15.3
5	31	48.8	−17.8
6	35	26.3	8.7
7	22	3.8	18.2

c



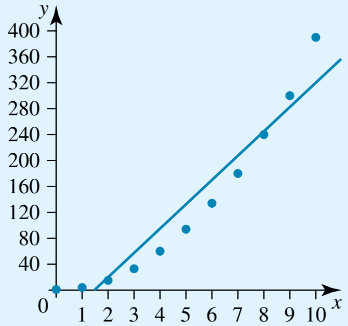
Slight pattern in residuals — may not be linear

d Decline in occupancy likely due to drought — an atypical event.

10 C

11 a Non-linear **b** Linear **c** Non-linear

12 a $y = -57.73 + 37.93x$



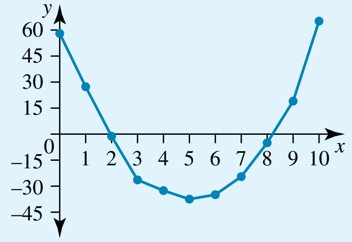
b $r = 0.958$. This means that there is a strong positive relationship between variables x and y.

c 0.9177, therefore 91.8% of the variation in y can be explained by the variation in x.

d

x	Residuals
0	58.7
1	23.8
2	−3.1
3	−23.1
4	−34.0
5	−37.9
6	−35.8
7	−27.8
8	−5.7
9	16.4
10	68.5

e

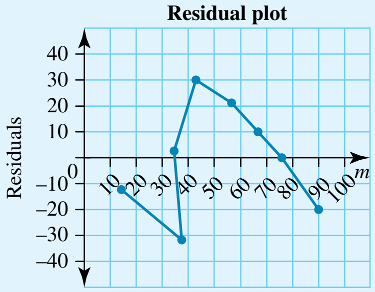


There is a clear pattern; the relationship between the variables is non-linear.

13 $P = -3.94 + 1.52m$

See table at foot of the page.*

14

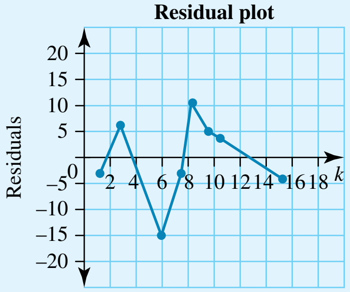


Since the data shows a curved pattern, the original data probably have a non-linear relationship.

15 Using CAS: $D = 13.72 + 7.22k$

See table at foot of the page.*

16 a

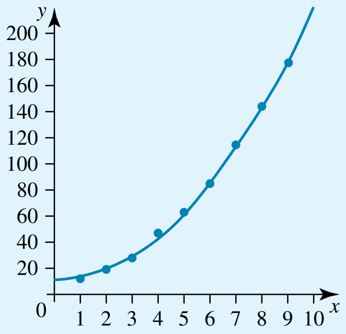


Since the data randomly jumps from above to below the k-axis; the data probably has a linear relationship.

b $r^2 = 0.94$; 94% of variation in D can be explained by variation in k.

EXERCISE 3.9

1 a



Apply an x^2 transformation to stretch the x-axis.

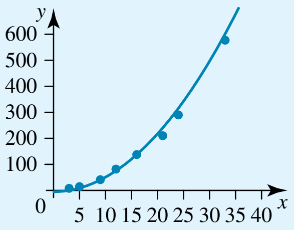
b

x^2	1	4	9	16	25	36	49	64	81
y	12	19	29	47	63	85	114	144	178

$y = 11.14 + 2.07x^2$ with $r = 0.99976$

This transformation has improved the correlation coefficient from 0.97 to 0.99976; thus the transformed equation is a better fit of the data.

2 a



Apply an x^2 transformation to stretch the x-axis.

b

x^2	9	25	81	144	256	441	576	1089
y	5	12	38	75	132	209	291	578

$y = -4.86 + 0.53x^2$ with $r = 0.9990$

This transformation has improved the correlation coefficient from 0.96 to 0.9990, thus the transformed equation is a better fit of the data.

3

Time (sec)	1	2	3	4	5	6	7	8
Log_{10} Speed (ms^{-1})	1.95	1.85	1.74	1.65	1.59	1.54	1.51	1.48

$\log_{10}(\text{speed}) = 1.97 - 0.07 \times \text{time}$ with $r = -0.97$

Therefore there is an improvement of the correlation coefficient that resulted from applying a logarithmic transformation.

4

x	5	10	15	20	25	30	35	40
$\log_{10} y$	3	2.70	2.35	2.17	2.00	1.85	1.77	1.75

$\log_{10} y = 3.01 - 0.04x$ with $r = -0.96$

Therefore there is a significant improvement of the correlation coefficient that resulted from applying a logarithmic transformation.

---

*13

m	12	37	35	41	55	69	77	90
P	2.5	21.6	52.3	89.1	100.7	110.3	112.4	113.7
$P_{\text{predicted}}$	14.30	52.30	49.26	58.38	79.66	100.94	113.10	132.86
Residual ($P - P_{\text{predicted}}$)	−11.8	−30.7	3.04	30.72	21.04	9.36	−0.7	−19.16

*15

k	1.6	2.5	5.9	7.7	8.1	9.7	10.3	15.4
D	22.5	37.8	41.5	66.9	82.5	88.7	91.6	120.4
$D_{\text{predicted}}$	25.27	31.77	56.32	69.31	72.20	83.75	88.09	124.91
Residual ($D - D_{\text{predicted}}$)	−2.77	6.03	−14.82	−2.41	10.30	4.95	3.51	−4.51

5 a

$\frac{1}{\text{Time after 6 pm (hr)}}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
Temperature (°C)	32	22	16	11	9	8	7	7

$y = 3.85 + 29.87x_T$, where $x_T = \dfrac{1}{x}$

or

$$\text{Temperature} = 3.85 + \frac{29.87}{\text{Time after 6 pm}}$$

b Temperature = 10.49 °C

6 a

$\frac{1}{x}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{13}$	$\frac{1}{15}$	$\frac{1}{18}$
y	120	50	33	15	9	5	2	1

$y = -13.51 + 273.78x_T$, where $x_T = \dfrac{1}{x}$

or

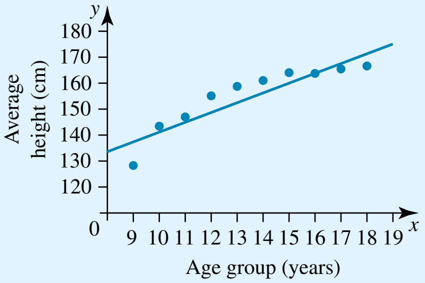
$$y = -13.51 + \frac{273.78}{x}$$

b $y = 9.31$

c $y = 8.17$

7 $y = 128.15 - 2.62x_T$ where $x_T = x^2$, $r = -0.97$, which shows some improvement.

8 a



b

log (age group)	Average height (cm)
0.954	128
1	144
1.041	148
1.079	154
1.114	158
1.146	161
1.176	165
1.204	164
1.230	166
1.255	167

c $y = 24.21 + 117.2x_T$ where $x_T = \log_{10}(x)$, $r = 0.95$, most non-linearity removed.

9 a i 123.3 cm **ii** 143.9 cm **iii** 176.7 cm

b a ii

10 Normal growth is linear only within given range; eventually the girl stops growing. Thus logarithmic transformation is a big improvement over the original regression.

11 a $y = 2.572 + 90.867x_T$ where

$x_T = \dfrac{1}{x} (r = 0.9788)$

b The intensity is 7.1 candlepower.

12 a Compress the y- or x-values using logs or reciprocals.

b Stretch the y-values using y^2 or compress the x-values using logs or reciprocals.

c Compress the y- or x-values using logs or reciprocals.

13 a −11.25 **b** −11.25

14 a −24.55

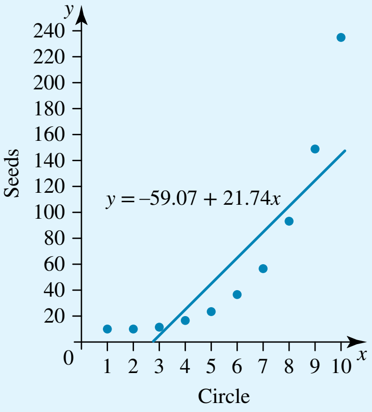
b Cannot take the log of a negative number.

c Cannot take the log of zero.

15 a 3.39 **b** 0.34

16 a −0.08 **b** −0.08

17 a



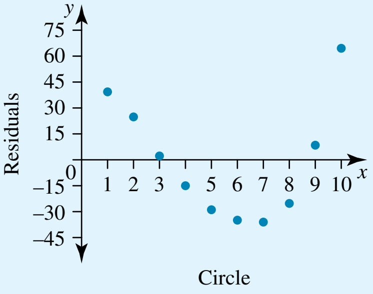
b $r = 0.87$, which means it is a strong and positive relationship.

c 180

d

Circle	Seeds	Residual
1	3	40.33
2	5	20.59
3	8	1.85
4	13	−14.89
5	21	−28.63
6	34	−37.37
7	55	−38.11
8	89	−25.84
9	144	7.41
10	233	74.67

e

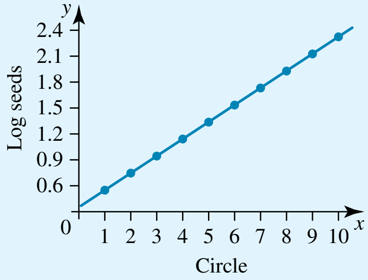


No, the relationship is not linear.

f **i** We can stretch the x-values towards linearity by using an x^2 transformation.

ii We can compress the y-values towards linearity by using either a $\log_{10}(y)$ or a $\frac{1}{y}$ transformation.

18 **a**



b $r = 0.9999$, this is an almost perfect relation.

c $\log_{10}(y) = 0.2721 + 0.2097x$

$\log_{10}$ (number of seeds) = 0.2721 + 0.2097 × circle number

d 0.9999, 99.99% (100.0%) of variation in number of seeds is due to number of circles. This is a perfect relation, often found in nature (see the Golden Ratio).

e 378

f This is a much better prediction as it follows a steep upward trend.

4

Time series

4.1 Kick off with CAS

Time series graphs with CAS

A time series graph displays data where the *x*-variable is time. The value of the intervals in a time series can vary, from hours to days, weeks, months or years.

1 Use CAS to plot a time series graph for the following data set, which displays the average maximum quarterly temperature in Melbourne over a 36-month period, starting from January 2011.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Temp. (°C)	24	17	15	21	25	17	16	22	25	18	16	20

2 Is there a pattern to your time series graph? If so, try to describe this pattern.

3 Use CAS to plot a comparative time series on a separate set of axes for the following data set, which displays the average maximum quarterly temperature in Auckland over the same 36-month period.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Temp. (°C)	23	18	15	19	22	17	16	20	23	17	15	19

4 Use CAS to plot a third time series on a separate set of axes for the following data set, which displays the average maximum quarterly temperature in New York over the same 36-month period.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Temp. (°C)	7	22	27	12	6	21	28	13	7	23	26	11

5 Are the patterns similar in all three time series? If not, what are the differences?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

4.2 Time series and trend lines

studyon

Unit 3
AOS DA
Topic 8
Concept 1

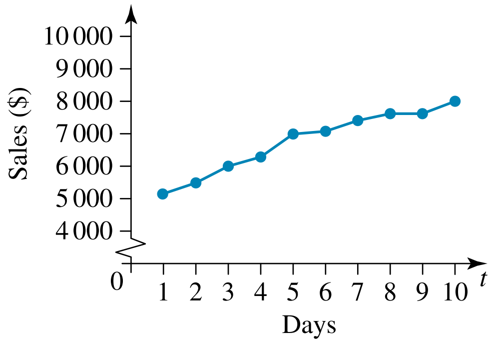
Nature of time series data
Concept summary
Practice questions

In previous topics we looked at bivariate, or (x, y), data where both x and y could vary independently. In this topic we shall consider cases where the x-variable is time and, generally, where time goes up in even increments such as hours, days, weeks or years. In these cases we have what is called a **time series**. The main purpose of a time series is to see how some quantity varies with time. For example, a company may wish to record its daily sales figures over a 10-day period.

Time	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9	Day 10
Sales ($)	5200	5600	6100	6200	7000	7100	7500	7700	7700	8000

We could also make a graph of this time series as shown at right.

As can be seen from this graph, there seems to be a trend upwards — clearly, this company is increasing its revenues.



Types of time series

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Concept 2

Long-term direction trends
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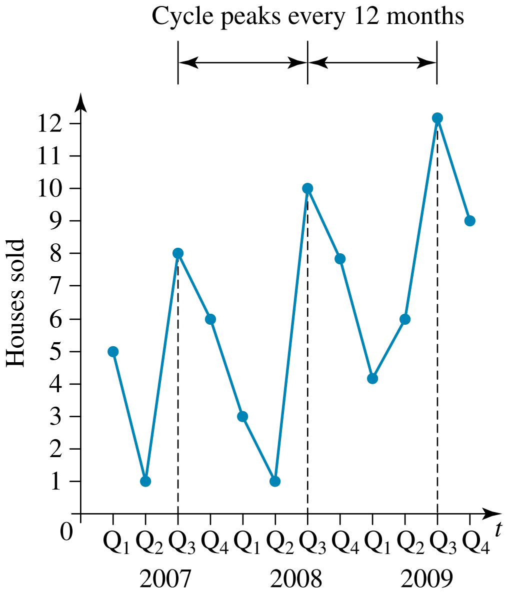
Practice questions

Seasonal fluctuations

Certain data seem to fluctuate during the year, as the seasons change. Consequently, this is termed a **seasonal time series**. The most obvious example of this would be total rainfall during summer, autumn, winter and spring in a year.

The name *seasonal* is not specific to the seasons of a year. It could also be related to other constant periods of highs and lows. For example, sales figures at a fast-food store could be consistently higher on Saturdays and Sundays and drop off during the weekdays. Here the seasons are days of the week and repeat once every week.



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Seasonal trends
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Practice questions

A key feature of seasonal fluctuations is that the seasons occur at the same time each cycle.

Here are some common seasonal periods.

	Seasons	Cycle	Example
Seasons	Winter, spring, summer, autumn	Four seasons in a *year*	Rainfall
Months	Jan., Feb., Mar., …, Nov., Dec.	12 months in a *year*	Grocery store monthly sales figures
Quarters	1st quarter (Q_1), 2nd quarter (Q_2), 3rd quarter (Q_3), 4th quarter (Q_4)	Four quarters in a *year*	Quarterly expenditure figures of a company

(*continued*)

	Seasons	Cycle	Example
Days	Monday to Friday	Five days in a *week*	Daily sales for a store open from Monday to Friday only
Days	Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday	Seven days in a *week*	Number of hamburgers sold at a takeaway store daily

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Cyclic trends
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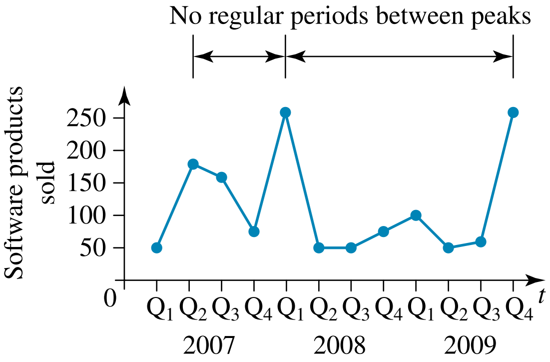
Cyclic fluctuations



Like seasonal time series, **cyclic time series** show fluctuations upwards and downwards, but not according to season. Businesses often have cycles where at times profits increase, then decline, then increase again. A good example of this would be the sales of a new major software product. At first, sales are slow; then they pick up as the product becomes popular. When enough people have bought the product, sales may fall off until a new version of the product comes on the market, causing sales to increase again. This cycle can be repeated many times, which is why there are many versions of some software products.

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Irregular fluctuations
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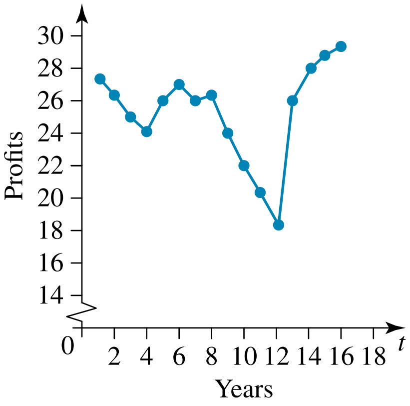
Irregular fluctuations



Fluctuations may seem to occur at random. This can be caused by external events such as floods, wars, new technologies or inventions, or anything else that results from random causes. There is no obvious way to predict the direction of the time series or even when it changes direction.

In the figure shown, there are a couple of minor fluctuations at $t = 4$ and $t = 8$, and a major one at $t = 12$. The major fluctuation could have been caused by a change in government which positively affected profits.

Plotting time series

To make better judgements about the type of time series, data in tabular form need to be plotted on a time series plot. This is similar to a scatterplot with some notable differences.

1. The explanatory variable is always *time*. This may be in days, days of the week, time of the day, weeks, months, quarters, years and so on. Thus, the x-axis variable is *time*.
2. As the periods are often labels and not numerical, the x-axis may be scaled using these period labels.
3. The points are connected. As they occur in chronological order, joining the points assists in identifying the type of time series and if a trend exists.

So that we can enter the data into CAS, time periods that are labels (and not numerical) need to be converted to numerals. For this, an **association table** is needed.

An association table summarises how the time periods are to be converted to numerical values. The first point is converted to 1, the second to 2 and so on, until the series is fully converted. Here are two examples.

Example 1

Week 1 Mon.	Week 1 Tues.	Week 1 Wed.	Week 1 Thurs.	Week 1 Fri.	Week 1 Sat.	Week 1 Sun.	Week 2 Mon.	Week 2 Tues.
1	2	3	4	5	6	7	8	9

Example 2

Jan. 2009	Feb. 2009	Mar. 2009	Apr. 2009	May 2009	June 2009	July 2009	Aug. 2009
1	2	3	4	5	6	7	8

Fitting trend lines

After we have plotted a time series graph, if there is a noticeable trend (upward, downward or flat) we can add a trend line, in the form of a straight line, to the data.

The trend lines we plot in this topic will be made using the least squares regression method we covered in topic 3. While these trend lines can be calculated manually, it is expected that you will use CAS to quicken this process.

WORKED EXAMPLE 1

The following table displays the school fees collected over a 10-week period. Plot the data and decide on the type of time series pattern. If there is a trend, fit a straight line.

Week beginning	8 Jan.	15 Jan.	22 Jan.	29 Jan.	5 Feb.	12 Feb.	19 Feb.	26 Feb.	5 Mar.	12 Mar.
$ × 1000	1.5	2.5	14.0	4.5	13.0	4.5	8.5	0.5	5.0	1.0

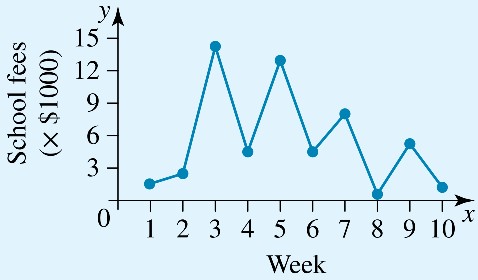
THINK

1 Set up an association table. One method is to add another row and enter the numerical time code for each of the ten weeks starting at 1, 2, …, through to 10.

WRITE/DRAW

Week beginning	8 Jan.	15 Jan.	22 Jan.	29 Jan.	5 Feb.	12 Feb.	19 Feb.	26 Feb.	5 Mar.	12 Mar.
$ × 1000	1.5	2.5	14.0	4.5	13.0	4.5	8.5	0.5	5.0	1.0
Time code	1	2	3	4	5	6	7	8	9	10

2 Construct a plot of the data. Place weeks on the horizontal axis and school fees on the vertical axis.



3 Identify the pattern as either seasonal, cyclical or irregular. If there is a trend, is it upwards or downwards?

The school fees can be classified as cyclical or irregular with a downward trend. This is evident by the reducing totals in school fees collected.

4 Use CAS to fit a least-squares regression line.

$y = 7.2 - 0.31x$, where y represents school fees in thousands of dollars and $x = 1$ corresponds to week beginning on 8th of January.

EXERCISE 4.2 Time series and trend lines

PRACTISE

1 **WE1** The following table shows the sales for the first 8 years of a new business. Plot the data and decide on the type of time-series pattern. If there is a trend, fit a straight line.

Year	1	2	3	4	5	6	7	8
Sales ($)	1326	1438	1376	1398	1412	1445	1477	1464

2 Data was recorded about the number of families who moved from Melbourne to Ballarat over the last 10 years. Plot the data and decide on the type of time series pattern. If there is a secular trend, fit a straight line.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Number moved	97	118	125	106	144	155	162	140	158	170

CONSOLIDATE

For questions 3 to 7, identify whether the time series are likely to be seasonal, cyclic or irregular and if they are also displaying a trend:

3 the amount of rainfall, per month, in Western Victoria

4 the number of soldiers in the United States army, measured annually

5 the number of people living in Australia, measured annually

6 the share price of BHP Billiton, measured monthly

7 the number of seats held by the Liberal Party in Federal Parliament.

8 The following table shows the temperature in Victoria over a 10-day period.

Day	1	2	3	4	5	6	7	8	9	10
Temperature (°C)	38	35	34	30	28	27	23	20	19	18

Fit a trend line to the data.

9 A wildlife park ranger is travelling on safari towards the centre of a wildlife park. Each day (t), he records the number of sightings (y) of zebra that he notes. He draws up the table shown.

t	1	2	3	4	5	6	7	8	9	10	11	12
y	6	9	13	8	9	14	15	17	14	11	15	19

Fit a trend line to the data.

10 The monthly share prices of a recently privatised telephone company were recorded as follows.

Date	Jan. 09	Feb. 09	Mar. 09	Apr. 09	May 09	June 09	July 09	Aug. 09
Price ($)	2.50	2.70	3.00	3.20	3.60	3.70	3.90	4.20

Graph the data (let 1 = Jan., 2 = Feb., … and so on) and fit a trend line to the data. Comment on the feasibility of predicting share prices for the following year.

11 Plot the following monthly sales data for umbrellas. Discuss the type of time series reflected by the data and the limitations of a trend line.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Sales	5	10	15	40	70	95	100	90	60	35	20	10

12 Consider the data in the table shown, which represent the price of oranges over a 19-week period.

Week	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Price (cents)	40	45	53	46	40	45	62	58	67	60	72	60	64	78	74	66	78	81	80

a Fit a straight trend line to the data.

b Predict the price in week 25.

MASTER

13 The following table represents the quarterly sales figures (in $000s) of a popular software product. Plot the data and fit a trend line. Discuss the type of time series best reflected by these data.

Quarter	Q_1-07	Q_2-07	Q_3-07	Q_4-07	Q_1-08	Q_2-08	Q_3-08	Q_4-08	Q_1-09	Q_2-09	Q_3-09	Q_4-09
Sales	120	135	150	145	140	120	100	110	120	140	190	220

14 The number of employees at the Comnatpac Bank was recorded over a 10-month period. Plot and fit a trend line to the data. What would you say about the trend?

Month	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Employees	6100	5700	5400	5200	4800	4400	4200	4000	3700	3300

4.3 Trend lines and forecasting

Association tables and forecasting

An association table is often required to convert period labels to a numerical value, so that a straight-line equation can be calculated. It is best to set up an extra row if data are in tabular form, or to change the labels shown on the axis of a time-series plot to numerical values. Here are three examples.

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Time series trend lines and forecasting
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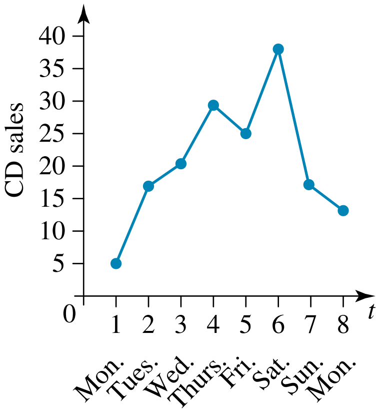
Example 1

Year	2006	2007	2008	2009
Time code	1	2	3	4

Example 2

1st quarter 2008	1
2nd quarter 2008	2
3rd quarter 2008	3
4th quarter 2008	4
1st quarter 2009	5
2nd quarter 2009	6

Example 3



For **forecasting**, use the association table to devise a time code for any period in the future. This time code will then be used in the straight-line equation.

From the three examples we can calculate that for:

Example 1: 2013 would have a time code of 8
Example 2: 1st quarter 2010 would have a time code of 9
Example 3: Monday week 4 would have a time code of 22.

WORKED EXAMPLE 2

A new tanning salon has opened in a shopping centre, with customer numbers for its first days shown in the following table. Fit a straight line to the data set using the least-squares regression method.

	Week 1							Week 2		
Period	**Mon.**	**Tues.**	**Wed.**	**Thurs.**	**Fri.**	**Sat.**	**Sun.**	**Mon.**	**Tues.**	**Wed.**
Number of customers	9	9	11	13	16	18	19	20	23	27

Use the equation of the straight line to predict the number of customers for:

a **Monday week 4**

b **Thursday week 2.**

THINK

1 Complete an association table, where Monday week 1 is 1, Tuesday week 1 is 2, Wednesday week 2 is 10.

WRITE

	Week 1							Week 2		
Period	**Mon.**	**Tues.**	**Wed.**	**Thurs.**	**Fri.**	**Sat.**	**Sun.**	**Mon.**	**Tues.**	**Wed.**
Number of customers	9	9	11	13	16	18	19	20	23	27
Time code	1	2	3	4	5	6	7	8	9	10

2 Use a calculator to find the equation of least-squares regression line.	$y = 5.67 + 1.97x$ Number of customers = 5.67 + 1.97 × time code where time code 1 corresponds to Monday of week 1.
3 a For Monday week 4, the time code is 22. Substitute $t = 22$ into the equation and evaluate. Round to the nearest integer.	**a** Number of customers = 5.67 + 1.97 × time code = 5.67 + 1.97 × 22 = 49.01 Number of customers = 49
b For Thursday week 2, the time code is 11. Substitute $t = 11$ into the equation and evaluate. Round to the nearest whole number.	**b** Number of customers = 5.67 + 1.97 × time code = 5.67 + 1.97 × 11 = 27.34 Number of customers = 27

Note: Remember that forecasting is an extrapolation and if going too far into the future, the prediction is not reliable, as the trend may change.

Once an equation has been determined for a time series, it can be used to analyse the situation.

For the period given in Worked example 2, the equation is:

$$\text{Number of customers} = 5.67 + 1.97 \times \text{time code.}$$

The y-intercept (5.67) has no real meaning, as it represents the time code of zero, which is the day before the opening of the salon. The gradient or rate of change is of more importance. It indicates that the number of customers is changing; in this instance, growing by approximately 2 customers per day (gradient of 1.97).

WORKED EXAMPLE 3

The forecast equation for calculating share prices, y, in a sugar company was obtained from data of the share prices over the past 5 years. The equation is $y = 1.56 + 0.42t$, where $t = 1$ represents the year 2001, $t = 2$ represents the year 2002 and so on.

a Rewrite the equation putting it in the context of the question.

b Interpret the values of the gradient and y-intercept.

c Predict the share price in 2013.

THINK	WRITE
a The x-variable represents the time codes and the y-variable represents the share price in dollars.	**a** Share price = \$1.56 + \$0.42 × time code Time code $t = 1$ is 2001, $t = 2$ is 2002 and so on.

b The y-intercept of 1.56 represents the starting value; that is, when $t = 0$. The gradient of 0.42 represents the rate of change in share price with respect to time. That is, it will grow as it has a positive gradient.	**b** The y-intercept of \$1.56 represents the approximate value of the shares in 2000. The gradient of \$0.42 means that on average the share price will grow by \$0.42 (42 cents) each year.
c If $t = 1$ is 2001, then for 2013, the time code will be $t = 13$. Substitute into the equation given.	**c** $\begin{aligned}\text{Share price} &= \$1.56 + \$0.42 \times \text{time code}\\ &= \$1.56 + \$0.42 \times 13\\ &= \$1.56 + \$5.46\\ &= \$7.02\end{aligned}$

Before we fit a straight line to a data set using least-squares regression it is useful to draw a scatterplot. This is beneficial since it can:

1. Demonstrate how close the points are to a straight line, or if a curve is a better fit for the data.
2. Demonstrate if there is an outlier in the data set that could affect the least-squares regression.

If there is an outlier in the data and we are using the equation to make a prediction, then this can bias our prediction. If the outlier was removed from the data then this is more likely to give a better prediction since the least-squares regression will fit the data more closely.

WORKED EXAMPLE 4

The following table shows the sales for the first 8 years of the new business from question 1, Exercise 4.2.

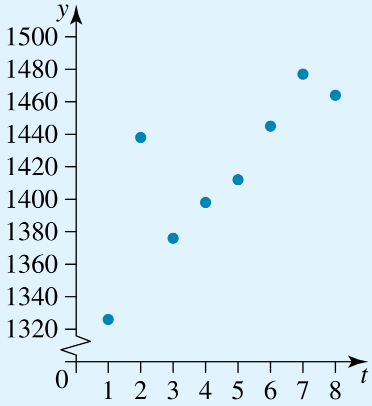
a Plot the data and decide if there are any outliers.

b Re-plot the data after removing any outliers. Explain a possible reason for the outlier.

c If there is a trend, fit a straight line to the data using a least-squares regression method.

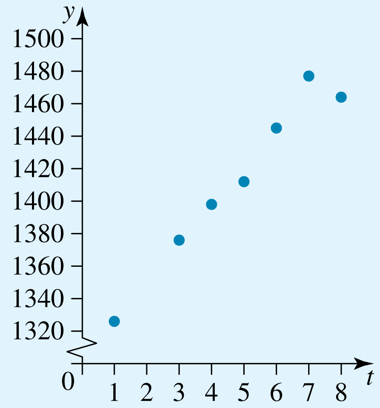
d Compare the gradient and y-intercept to that of question 1, Exercise 4.2.

Year	1	2	3	4	5	6	7	8
Sales (\$)	1326	1438	1376	1398	1412	1445	1477	1464

THINK	WRITE/DRAW
a Plot all the points on a scatterplot.	**a** 

b Looking at the scatterplot it appears that the second point is an outlier from a linear regression. Thus we remove this point and plot another scatterplot.

b



Since we don't know the year in the second year of business or the type of business it is difficult to pinpoint a reason; however, they may have released a new line or developed something new to get this jump in sales. There could be many reasons for this outlier.

c This looks like an upward trend, so use CAS to complete a least-squares regression on the reduced data set.

c The linear trend line is $y = 1309.35 + 21.55x$, where y is the sales in \$ and $x = 1$ corresponds to 1 year after the new business opening.

d Compare the gradient and y-intercept to that of question **1**, Exercise 4.2, where the gradient was 16.45 and the y-intercept was 1342.96.

d By removing the outlier the gradient has increased and the y-intercept has decreased. This would now be an improved trend line to make predictions from.

EXERCISE 4.3 Trend lines and forecasting

PRACTISE

1 **WE2** The number of people who watched the Channel 9 news over a fortnight is shown in the following table. Fit a straight line to the data set using a least-squares regression method.

Day	M	T	W	Th	F	S	Su
Viewers (1 000 000s)	1.20	1.18	1.16	1.18	0.9	0.75	1.0

Day	M	T	W	Th	F	S	Su
Viewers (1 000 000s)	1.21	1.23	1.19	1.16	0.95	0.68	0.98

Use the equation of the straight line to predict the number of viewers for:

a Wednesday week 3

b Monday week 4.

2 Data was recorded on the number of road fatalities in Australia in 2009 and 2010.

Month and year	01/09	02/09	03/09	04/09	05/09	06/09
Driver fatalities	115	117	132	153	130	131

Month and year	07/09	08/09	09/09	10/09	11/09	12/09
Driver fatalities	109	117	113	143	107	124

Month and year	01/10	02/10	03/10	04/10	05/10	06/10
Driver fatalities	126	98	104	115	134	111

Month and year	07/10	08/10	09/10	10/10	11/10	12/10
Driver fatalities	107	94	104	121	119	120

Source: Australian Government, Department of Infrastructure and Regional Development

Fit a straight line to the data set using the least-squares regression method and use the straight line to predict the number of fatalities in:

a June 2015

b April 2018.

3 WE3 The forecast equation for calculating the share prices, y, of a bank company was obtained from the data of share prices over the past 7 years. The equation is $y = 22.74 + 0.28t$, where $t = 1$ represents the year 2007.

a Rewrite the equation putting it in the context of the question.

b Interpret the values of the gradient and y-intercept.

c Predict the share price in 2017.

4 The forecast equation for calculating the share prices, y, of a mining company was obtained from the data of share prices over the past 4 years. The equation is $y = 18.57 - 0.1t$, where $t = 1$ represents the year 2010.

a Rewrite the equation putting it in the context of the question.

b Interpret the values of the gradient and y-intercept.

c Predict the share price in 2019.

5 WE4 The following table shows the All Ordinaries values from the Australian stock market over 8 years. The All Ordinaries consists of the 500 largest eligible companies.

Year	1995	1996	1997	1998	1999	2000	2001	2002
All Ords	1925	2140	2435	2500	2820	3200	2950	3050

a Plot the data and decide if there are any outliers. Complete a least-squares regression including all data points.

b Re-plot the data after removing any outliers.

c If there is a trend, fit a straight line to the data using a least-squares regression method.

d Compare the gradient and y-intercept of the two regressions.

6 The following table shows the stock price for Apple in 2009–10.

Month 2010	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.
Stock price ($)	192.06	204.62	235.00	261.09	256.88	251.53	257.25	243.10	283.75

a Plot the data and decide if there are any outliers. Complete a least-squares regression including all data points.

b Re-plot the data after removing any outliers.

c If there is a trend, fit a straight line to the data using a least-squares regression method.

d Compare the gradient and y-intercept of the two regressions.

CONSOLIDATE

7 The following table represents the number of cars remaining to be completed on an assembly line. Fit a straight line to the following data using the least-squares regression method.

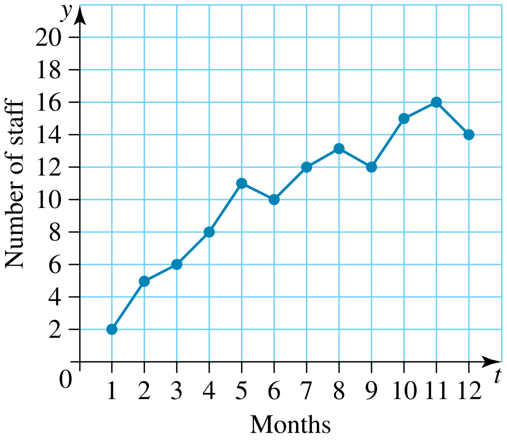
Time (hours)	1	2	3	4	5	6	7	8	9
Cars remaining	32	26	27	23	16	17	13	10	9

a Predict the number of cars remaining to be completed after 11 hours.

b At what rate is the numbers of cars on the assembly line being reduced by?

8 From the equation of the trend line, it should be possible to predict when there are no cars left on the assembly line. This is done by finding the value of t which makes $y = 0$. Using the equation from question **7**, find the time when there will be no cars left on the assembly line.

9 When the MicroHard Company first started, it employed only one person. The company has consistently grown, so that after 12 months there are 14 people working there. The time-series data are shown by the graph.



a Fit a straight line to the data using the least-squares regression method.

b Predict the number of employees after a *further* 12 months.

10 The table below shows the share price of MicroHard during a volatile period in the stock market. Using CAS:

a fit a least-squares regression line.

b What type of time series is this?

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price ($)	2.75	3.30	3.15	2.25	2.10	1.80	1.50	2.70	4.10	4.20	3.55	1.65	2.60	2.95	3.25	3.70

11 The following time series shows the number of internet websites on a webring over a 9-month period. Plot the data and fit a line using the least-squares regression method. Comment on this line as a predictor of further growth.

Time (months)	1	2	3	4	5	6	7	8	9
Sites (millions)	2.00	2.20	2.50	3.10	3.60	4.70	6.10	7.20	8.50

12 The forecast equation for calculating prices, y, of shares in a steel company was obtained from data of the share prices of the past 6 years. The equation is.

$$y = 2.56 + 0.72t$$

where $t = 1$ represents the year 2010, $t = 2$ represents the year 2011 and so on.

a Rewrite the equation putting it in the context of the question.

b Interpret the values of the gradient and the y-intercept.

c Predict the share price in 2020.

13 The Teeny-Tiny-Tot Company has started to make prams. Its sales figures for the first 8 months are given in the table shown.

Date	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.
Sales	65	95	130	115	145	170	190	220

a Using the sequence Jan. = 1, Feb. = 2, …, calculate the equation of the trend line using the least-squares regression method.
b Plot the data points and the trend line on the same set of axes.
c Use the trend line equation to predict the company's sales for December.
d Comment on the suitability of the trend line as a predictor of future trends, supporting your arguments with mathematical statements.

14 The sales figures of Harold Courtenay's latest novel (in thousands of units) are given in the table shown. The book was released a week before the first figures were collected.

Time (weeks)	1	2	3	4	5	6	7	8	9
Sales (× 1000)	1	3	5	17	21	25	28	27	26

a Calculate the equation of the trend line for these data using the least-squares regression method.
b Plot the data points and the trend line on the same set of axes.
c Use the trend line equation to predict the sales for weeks 10, 12 and 14.
d Comment on the suitability of the trend line as a predictor of future trends, supporting your arguments with mathematical statements.

15 The average quarterly price of coffee (per 100 kg) has been recorded for 3 years.

Quarter	Q_1-07	Q_2-07	Q_3-07	Q_4-07	Q_1-08	Q_2-08	Q_3-08	Q_4-08	Q_1-09	Q_2-09	Q_3-09	Q_4-09
Price ($)	358	323	316	336	369	333	328	351	389	387	393	402

a Calculate the equation of the trend line for these data using the least-squares regression method.
b Plot the data points and the trend line on the same set of axes.
c Use the trend line equation to predict the price for the next quarter.
d Comment on the suitability of the trend line as a predictor of future trends, supporting your arguments with mathematical statements.

16 A mathematics teacher gives her students a test each month for 10 months, and the class average is recorded. The tests are carefully designed to be of similar difficulty.

Test	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.
Mark (%)	57	63	62	67	65	68	70	72	74	77

a Calculate the equation of the trend line for these data using the least-squares regression method.
b Plot the data points and the trend line on the same set of axes.

c Use the trend line equation to predict the results for the last exam in December.

d Comment on the suitability of the trend line as a predictor of future trends, supporting your arguments with mathematical statements.

MASTER

17 The average cost of a hotel room in Sydney in 2000 is shown in the table.

Month 2000	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.
Hotel price ($)	250	240	235	237	239	230	228	237	332

a Plot the data and decide if there are any outliers. Complete a least-squares regression including all data points.

b Re-plot the data after removing any outliers.

c If there is a trend, fit a straight line to the data using a least-squares regression method.

d Compare the gradient and y-intercept of the two regressions and give a possible explanation for the outlier.

18 The percentage of homes that have a television is shown in the table.

Year	1970	1971	1972	1973	1974	1975	1976	1977	1978
Percentage	42	45	48	54	58	72	77	83	90

a Plot the data and decide if there are any outliers.

b Fit a straight line to the data using a least-squares regression method.

c Explain any reason for the change in the data from its original trend.

4.4 Smoothing time series

When the data fluctuates a lot, it is often hard to see the underlying trend. In order to reveal the trend, we may need to try and remove some of these fluctuations before attempting to fit the trend line. This process is referred to as smoothing.

There are two basic techniques for smoothing *random* or *cyclical* variation: **median smoothing** and **moving-mean smoothing**.

Median smoothing is preferred where there are small data sets, as it can be done graphically on a time-series plot. Also, for data sets with many outliers due to the volatile random or cyclical trend, median smoothing is preferred.

We have seen earlier that the median is not affected by outliers, while the mean is.

Moving-mean smoothing is an option that is preferred for data sets with few random fluctuations.

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Unit 3

AOS DA

Topic 9

Concept 3

Moving-mean and moving-median smoothing

Concept summary

Practice questions

Moving-mean smoothing

This technique relies on the principle that the means of data can be used to represent the original data. When applied to time series, a number of data points are averaged, then we move on to another group of data points in a systematic fashion and average them, and so on. It is generally quite simple. Consider the following example:

Notice how the third column in the table is computed from the first two columns.

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Interactivity
Smoothing: moving mean method
int-6255

1. Take the first three y-values (i.e. first, second and third) and find their average $\left(\frac{12 + 10 + 15}{3} = 12.3\right)$; place the result against $t = 2$.
2. Move down one line and again take three y-values (i.e. second, third and fourth). Find their average $\left(\frac{10 + 15 + 13}{3} = 12.7\right)$ and place the result against $t = 3$.
3. Continue moving down the table until you reach the last three points.

Time (t)	Data (y)	Moving mean
1	12	
2	10	$\frac{12 + 10 + 15}{3} = 12.3$
3	15	$\frac{10 + 15 + 13}{3} = 12.7$
4	13	$\frac{15 + 13 + 16}{3} = 14.7$
5	16	$\frac{13 + 16 + 13}{3} = 14.0$
6	13	$\frac{16 + 13 + 18}{3} = 15.7$
7	18	$\frac{13 + 18 + 21}{3} = 17.3$
8	21	$\frac{18 + 21 + 19}{3} = 19.3$
9	19	

As we use three points to average, moving down the table from top to bottom, the process is called a *3-point moving-mean smoothing.*

The number of points averaged at a time may vary: we could have a 4-point smoothing, a 5-point smoothing or even an 11-point smoothing. Although it is preferable to choose an odd number, such as 3 or 5, it is possible to choose even numbers as well, with a slight change in the method. Later in the topic, we will discuss how to choose the number of points for smoothing.

Moving-mean smoothing with odd numbers of points

As previously shown, the method for smoothing with an odd number (3, 5, ...) can be done in a vertical tabular form. The time values must be equally spaced, but they don't have to be in the sequence 1, 2, 3.

Note: There are fewer smoothed points than original ones. For a 3-point smooth, 1 point at either end is 'lost'; while for a 5-point smooth, 2 points at either end are 'lost'.

WORKED EXAMPLE 5

The temperature of a sick patient is measured every 2 hours and the results are recorded.

a **Use a 3-point moving-mean technique to smooth the data.**

b **Plot both original and smoothed data on the same set of axes.**

c **Predict the temperature for 18 hours using the last smoothed value.**

Time (hours)	2	4	6	8	10	12	14	16
Temp. (°C)	36.5	37.2	36.9	37.1	37.3	37.2	37.5	37.8

THINK | **WRITE/DRAW**

a **1** Put the data in a table.

2 Calculate a 3-point moving-mean for each data point.

Note: The 'lost' values are at $t = 2$ and $t = 16$. Therefore, the first point plotted is (4, 36.87).

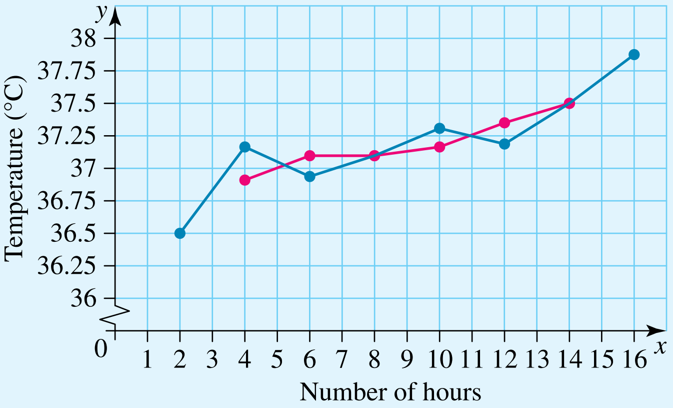
a

Time (h)	Temp. (°C)	Smoothed temp. (°C)
2	36.5	
4	37.2	$\frac{1}{3}(36.5 + 37.2 + 36.9) = 36.87$
6	36.9	$\frac{1}{3}(37.2 + 36.9 + 37.1) = 37.07$
8	37.1	$\frac{1}{3}(36.9 + 37.1 + 37.3) = 37.1$
10	37.3	$\frac{1}{3}(37.1 + 37.3 + 37.2) = 37.2$
12	37.2	$\frac{1}{3}(37.3 + 37.2 + 37.5) = 37.33$
14	37.5	$\frac{1}{3}(37.2 + 37.5 + 37.8) = 37.5$
16	37.8	

b **1** Plot the data. The smoothed line is the pink one.

Note: The smoothed data start at the 2nd time point and finish at the 7th point.

b



2 Comment on the result.

The smoothed line has removed much of the fluctuation of the original time series and, in fact, clearly exposes the upwards trend in temperature.

c Last smoothed data point is 37.5.

c The temperature at 18 hours is predicted to be 37.5°C.

Prediction using moving means

Because the moving mean does not generate a single linear equation, there are limited possibilities for using the resultant smoothed data for prediction. However, there are two things that can be done.

1. Predict the next value — use the last smoothed value to predict the next time point. In Worked example 5, our prediction for $t = 18$ would be temperature = 37.5. This is not necessarily an accurate prediction but it is the best we can do without a linear trend equation.
2. Fit a single straight line to the smoothed data — using the least-squares regression technique, one could find a single equation for the smoothed data points. This is often the preferred technique.

How many points should be in the moving mean?

When smoothing data, it is important to decide on the number of points to be used. Several factors that should be taken into account are discussed below. Here are some basic hints. (n = the number of points in the time series and p = the number of points taken at a time to be averaged.)

1. For small data sets, the value of p should be *considerably* smaller than n. For example, if $n = 7$, p should be no more than about 4.
2. The most common practice if there is a cyclic variation that we want to remove, is to let $p \approx$ length of cycle. If the data shows seasonal variation, p should be proportional to the number of seasons. For example, if there are quarterly data with seasonal variation, use $p = 4$.

   Other preferred choices for number of points used in the smoothing procedures include:

Monthly sales figures	use 12 point (centred)
Daily sales for a store open each day of the week	use 7 point
Daily sales for a store open from Monday to Friday only	use 5 point
Quarterly electricity consumption figures	use 4 point (centred)

3. It is *always* preferable to use an odd value of p, regardless of whether n is even or odd.
4. The larger the value of p, the smoother the trend line of the resulting data becomes. More of the fluctuations will be removed. However, you can go too far as the bigger p is, the more data points are lost: when $p = 3$, we lose 2 points, if $p = 5$ we lose 4 points, if $p = 7$ we lose 6 points, etc. Therefore if the set is small, we may lose nearly all data by selecting a large value of p!

EXERCISE 4.4 Smoothing time series

PRACTISE

1 **WE5** The temperature of a pool was measured every hour over a summer's day and the results recorded.

a Use a 3-point moving-mean technique to smooth the data.
b Plot both the original and smoothed data on the same set of axes.
c Predict the temperature for 9 hours using the last smoothed value.

Time (h)	1	2	3	4	5	6	7	8
Temp (°C)	20.3	21.8	20.9	22.0	23.4	24.9	24.3	25.3

2 The membership numbers of the Collingwood Football Club over the past eight years are shown in the table.

Year (t)	2007	2008	2009	2010	2011	2012	2013	2014
Members (m)	38 587	42 498	45 972	57 408	71 271	72 688	78 427	72 170

a Use a 3-point moving-mean technique to smooth the data.
b Plot both the original and smoothed data on the same set of axes.
c Predict the number of members in 2016 using the last smoothed value.

CONSOLIDATE

3 The following table represents sales of a textbook.

Year (*t*)	2002	2003	2004	2005	2006	2007	2008	2009
Sales (*y*)	2250	2600	2400	2750	2900	2450	3100	3400

a Use a 3-point moving mean technique to smooth the data.
b Plot both the original and smoothed data.
c Predict the sales for 2010 using the last smoothed value.

4 The sales of a certain car seem to have been declining in recent months. The management wishes to find out if this is the case.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Sales	120	70	100	110	90	80	70	90	80	100	60	60

a Using a 3-point moving mean, smooth the data and comment on the result. Use Jan. = 1, Feb. = 2, ...
b Using the least-squares regression method, find the equation of the trend line for the smoothed data.
c Use the equation to predict the number of sales for March next year. Comment on the prediction.

5 Perform a 5-point moving mean smoothing on the data from question 4 and discuss the result.

6 Consider the quarterly rainfall data shown. Rainfall has been measured over a 3-year period. Perform a 3-point moving mean and comment on whether there is an underlying trend.

Time (*t*)	Spring 2006	Summer 2006	Autumn 2007	Winter 2007	Spring 2007	Summer 2007	Autumn 2008	Winter 2008	Spring 2008	Summer 2008	Autumn 2009	Winter 2009
Rainfall (mm)	100	50	65	120	90	50	60	110	85	40	50	100

7 The attendance at Bendigo Football Club games was recorded over 10 years. Management wishes to see if there is a trend.

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Attendance (× 1000)	75	72	69	74	66	72	61	64	69	65

a Perform a 3-point moving mean smoothing on the data and comment on the result.
b Using least-squares regression on the smoothed data, find the equation of the trend line.
c Use the equation from b to predict the attendance in 2011. Comment on the prediction.

8 Use technology to complete a 3-point moving mean smoothing on the following data which represent sales figures for a 21-week period.

Week	Sales	Smoothed data
1	34	
2	27	
3	31	
4	37	
5	41	
6	29	
7	32	
8	37	
9	47	
10	38	
11	41	

Week	Sales	Smoothed data
12	44	
13	47	
14	49	
15	41	
16	52	
17	48	
18	44	
19	49	
20	56	
21	54	

9 Coffee price data are shown in the table. Perform a 3-point moving average to smooth the data. Plot the smoothed and original data and comment on your result.

Quarter	Q_1-07	Q_2-07	Q_3-07	Q_4-07	Q_1-08	Q_2-08
Price ($)	358	323	316	336	369	333

Quarter	Q_3-08	Q_4-08	Q_1-09	Q_2-09	Q_3-09	Q_4-09
Price ($)	328	351	389	387	393	402

10 The sales of a new car can vary due to the effect of advertising and promotion. The sales figures for Nassin Motor Company's new sedan are shown in the table. Use 5-point moving means to smooth the data. Plot the data, and use the last smoothed value to predict sales for the next month.

Month	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Sales	141	270	234	357	267	387	288	303	367	465	398

11 A large building site requires varying numbers of workers. The weekly employment figures over the last 7 weeks have been recorded. By performing a 3-point moving mean smoothing, predict the number of people required for the next week.

Week	Employees
1	67
2	78
3	54
4	82
5	69
6	88
7	94

12 If you were to smooth quarterly gas bill data over a number of years, you would be best to use:

A 3-point **B** 4-point (centred) **C** 5-point
D 6-point (centred) **E** 7-point

MASTER

13 Use a spreadsheet or CAS to perform a 5-point moving mean smoothing on the data in question **8** and discuss your result.

14 Use a spreadsheet or CAS to perform a 5-point moving mean smoothing on the data in question **9** and discuss your result.

4.5 Smoothing with an even number of points

As mentioned in the previous section, it is usually preferable to use an odd number of points. However, there are situations when an even number of points should be used — that is, a 4-point, 6-point or even 12-point moving mean. When we used an odd number of points, the result was automatically centred; that is, the y-data had the same t-values as the original (except at the first and last 'lost' points). This does not occur with an even-point smoothing, as shown in the following example of a 4-point moving mean.

Time	y-value	4-point mean (smoothed value)	
		Calculation	Result
2006	6		
2007	10		
		(6 + 10 + 14 + 12) ÷ 4	10.5
2008	14		
		(10 + 14 + 12 + 11) ÷ 4	11.75
2009	12		
		(14 + 12 + 11 + 15) ÷ 4	13
2010	11		
		(12 + 11 + 15 + 16) ÷ 4	13.5
2011	15		
2012	16		

Observe that the first mean (10.5) is not aligned with any particular year — it is aligned with 2007.5! Also note that there are now three 'lost' values (the seven original records reduced to four). In other words, the moving mean is not centred properly. To align the data correctly, an additional step needs to be performed; this is called *centring*.

Use the following procedure to centre the data:

Step 1. Find the mean of the first two smoothed points and align it with the 3rd time point.

Step 2. Find the mean of the next two smoothed points and align it with the 4th time point.

Step 3. Repeat, leaving two blank entries at both top and bottom of the table.

This is demonstrated in the following table, using the data from the previous table.

Time	y-value	4-point mean (smoothed value)		4-point mean after centring	
		Calculation	Result	Calculation	Result
2006	6				
2007	10				
		(6 + 10 + 14 + 12) ÷ 4	10.5		
2008	14			(10.5 + 11.75) ÷ 2	11.125
		(10 + 14 + 12 + 11) ÷ 4	11.75		
2009	12			(11.75 + 13) ÷ 2	12.375
		(14 + 12 + 11 + 15) ÷ 4	13		
2010	11			(13 + 13.5) ÷ 2	13.25
		(12 + 11 + 15 + 16) ÷ 4	13.5		
2011	15				
2012	16				

The first mean (11.125) is now aligned with 2008, the second (12.375) aligned with 2009 and so on. This process not only introduces an extra step, but an *extra averaging* (or smoothing) as well. It is usually preferable to stick with an odd-point smoothing to reduce these difficulties.

WORKED EXAMPLE 6

The quarterly sales figures for a dress shop (in thousands of dollars) were recorded over a 2-year period. Perform a centred 4-point moving mean smoothing and plot the result. Comment on any trends that have been revealed.

Time	Summer	Autumn	Winter	Spring	Summer	Autumn	Winter	Spring
Sales (× $1000)	27	22	19	25	31	25	22	29

THINK

1 Arrange the data in a table.

Note: Code the time column.

2 Calculate a 4-point moving mean in column 3.

WRITE/DRAW

Time	Sales	4-point moving mean	4-point centred moving mean
1	27		
2	22		
		(27 + 22 + 19 + 25) ÷ 4 = 23.25	
3	19		(23.25 + 24.25) ÷ 2 = 23.75
		(22 + 19 + 25 + 31) ÷ 4 = 24.25	
4	25		(24.25 + 25.00) ÷ 2 = 24.63
		(19 + 25 + 31 + 25) ÷ 4 = 25.00	

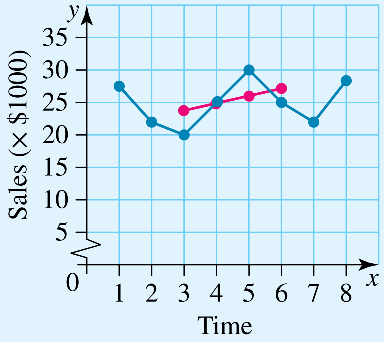
(continued)

3 Average the pairs of averages to find the 4-point centred data. This is done in column 4.

Time	Sales	4-point moving mean	4-point centred moving mean
5	31		(25.00 + 25.75) ÷ 2 = 25.38
		(25 + 31 + 25 + 22) ÷ 4 = 25.75	
6	25		(25.75 + 26.75) ÷ 2 = 26.25
		(31 + 25 + 22 + 29) ÷ 4 = 26.75	
7	22		
8	29		

4 Plot the data. The smoothed line is the red one.

Note: The smoothed data start at the 3rd time point and finish at the 6th point.



5 Interpret the results. Observe the steadily increasing trend (even with only four smoothed points) that was not obvious from the original data.

EXERCISE 4.5 Smoothing with an even number of points

PRACTISE

1 **WE6** The quarterly sales figures for a shoe shop (in thousands of dollars) were recorded over a 2-year period.

a Perform a centred 4-point moving mean smoothing and plot the results.

b Comment on any trend that has been revealed.

Time	Summer	Autumn	Winter	Spring	Summer	Autumn	Winter	Spring
Sales (× $1000)	59	48	43	50	63	52	47	61

2 The quarterly electricity cost figures for a shop were recorded over a 2-year period.

a Perform a centred 4-point moving mean smoothing and plot the results.

b Comment on any trend that has been revealed.

Time	Q_1-14	Q_2-14	Q_3-14	Q_4-14	Q_1-15	Q_2-15	Q_3-15	Q_4-15
Cost ($)	554	503	467	587	636	533	493	684

CONSOLIDATE

3 a Perform a 4-point centred moving mean to smooth the following data and plot the result.

b Comment on any trends that you find.

t	1	2	3	4	5	6	7	8	9	10
y	75	54	62	60	70	45	54	59	62	64

4 The price of oranges fluctuates from season to season. Data have been recorded for 3 years. Perform a 4-point centred moving mean, plot the data and comment on any trends.

t	Autumn 2007	Winter 2007	Spring 2007	Summer 2007	Autumn 2008	Winter 2008
Price	45	67	51	44	52	76

t	Spring 2008	Summer 2008	Autumn 2009	Winter 2009	Spring 2009	Summer 2009
Price	63	48	58	80	66	52

5 a Use technology to complete the following table. The time series represents the temperature of a hospital patient over 15 days.

<table>
<tr><th>Day</th><th>Temperature</th><th>4-point moving average</th><th>4-point centred moving average</th></tr>
<tr><td rowspan="2">1</td><td rowspan="2">36.6</td><td rowspan="3"></td><td rowspan="2"></td></tr>
<tr></tr>
<tr><td rowspan="2">2</td><td rowspan="2">36.4</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">36.75</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">36.8</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">36.825</td></tr>
<tr><td rowspan="2">4</td><td rowspan="2">37.2</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">36.85</td></tr>
<tr><td rowspan="2">5</td><td rowspan="2">36.9</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">36.95</td></tr>
<tr><td rowspan="2">6</td><td rowspan="2">36.5</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">37</td></tr>
<tr><td rowspan="2">7</td><td rowspan="2">37.2</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">37.05</td></tr>
<tr><td rowspan="2">8</td><td rowspan="2">37.4</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">37.275</td></tr>
<tr><td rowspan="2">9</td><td rowspan="2">37.1</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">37.375</td></tr>
<tr><td rowspan="2">10</td><td rowspan="2">37.4</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">37.25</td></tr>
<tr><td rowspan="2">11</td><td rowspan="2">37.6</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">37.275</td></tr>
<tr><td rowspan="2">12</td><td rowspan="2">36.9</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">37.325</td></tr>
<tr><td rowspan="2">13</td><td rowspan="2">37.2</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">37.15</td></tr>
<tr><td rowspan="2">14</td><td rowspan="2">37.6</td><td rowspan="2"></td></tr>
<tr><td rowspan="2"></td></tr>
<tr><td rowspan="2">15</td><td rowspan="2">36.9</td><td rowspan="2"></td></tr>
<tr><td></td></tr>
</table>

b Using the smoothed data, find the equation of the least-squares regression line.

c Use the trend line to predict the temperature of the patient on day 16.

6 The sales of summer clothing vary according to the season. The following table gives seasonal sales data (in thousands of dollars) for 3 years at a Darryl Jones department store.

Season	Q_3-06	Q_4-06	Q_1-07	Q_2-07	Q_3-07	Q_4-07
Sales	78	92	90	73	62	85

Season	Q_1-08	Q_2-08	Q_3-08	Q_4-08	Q_1-09	Q_2-09
Sales	83	70	61	78	74	59

a Calculate a 4-point centred moving average.

b Plot the original and smoothed data on the same set of axes.

c Determine if there is an underlying upwards or downwards trend.

7 Calculate a 6-point centred moving mean on the data from question **5**.

8 An athlete wishes to measure her performance in running a 1 km race. She records her times over the last 10 days.

Day	1	2	3	4	5	6	7	8	9	10
Time (s)	188	179	183	180	173	171	182	168	171	166

a Perform a 4-point centred moving mean smoothing.
b Plot the original and smoothed data on the same set of axes.
c Determine if there is a significant improvement in her times.

9 The following table shows the share price index of Industrial Companies during an unstable fortnight's trading. By calculating a 4-point centred moving mean, determine if there seems to be an upward or downward trend.

Day	1	2	3	4	5	6	7	8	9	10
Index	678	762	692	714	689	687	772	685	688	712

10 If data was given to you at the start of each season of the year, it would be best to smooth the data by using:

A 3 point
B 4 point centred
C 5 point
D 6 point centred
E 7 point

11 If you start with 12 points of data and carry out a 4-point centred moving average, the number of points you end up with is:

A 8
B 7
C 6
D 5
E 4

12 You have data points for each day in the month of April and complete a 6-point centred moving average. The number of points you end up with is:

A 30
B 28
C 26
D 24
E 22

MASTER

13 Calculate a 6-point centred moving mean on the data from question **8**.

14 **a** Use the smoothed data in question **9** to find the least-squares regression line.
b Use the trend line to predict the price index of the Industrial Company after 15 days.

4.6 Median smoothing from a graph

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Interactivity
Smoothing: median method
int-6256

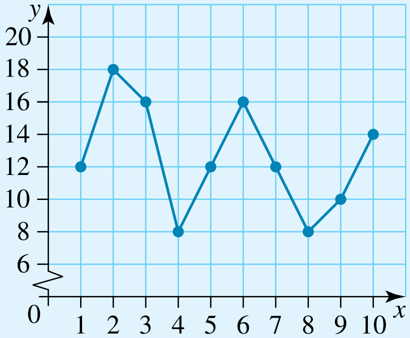
An alternative to moving-average smoothing is to replace the averaging of a group of points with the median of each group. It is a faster technique requiring no calculations and in the scope of this course, will only be addressed graphically.

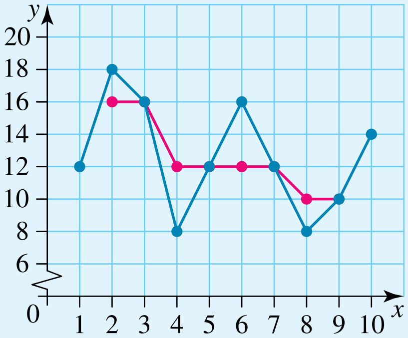
Generally, the effect of median smoothing is to remove some random fluctuations. It performs poorly on cyclical or seasonal fluctuations — unless the size of the range being used (3, 5, 7, ... points) is chosen carefully.

Provided the graph has clearly marked data points, it is possible to find a median smooth directly from it.

WORKED EXAMPLE 7

Perform a 3-point median smoothing on the graph of a time series below.

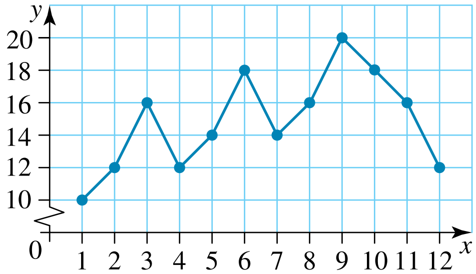


THINK	WRITE/DRAW
1 Read the data values and compute the median.	The 1st group of 3 points is: 12, 18, 16 — median = 16. The 2nd group of 3 points is: 18, 16, 8 — median = 16. The 3rd group of 3 points is: 16, 8, 12 — median = 12. The 4th group of 3 points is: 8, 12, 16 — median = 12. The 5th group of 3 points is: 12, 16, 12 — median = 12. The 6th group of 3 points is: 16, 12, 8 — median = 12. The 7th group of 3 points is: 12, 8, 10 — median = 10. The 8th group of 3 points is: 8, 10, 14 — median = 10.
2 Plot the medians on the graph. *Note:* Median smoothing has indicated a downward trend that is probably not in the real time series. This indicates that moving-average smoothing would be the preferred option.	

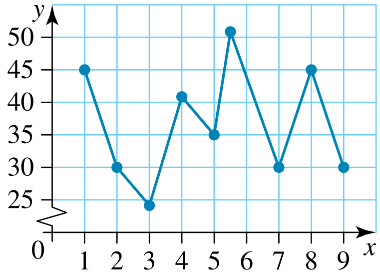
EXERCISE 4.6 Median smoothing from a graph

PRACTISE

1 **WE7** Perform a 3-point median smoothing on the graph of the time series shown.



2 Perform a 3-point median smoothing on the graph of the time series shown.

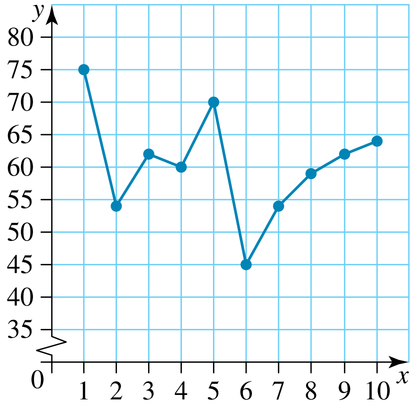


CONSOLIDATE

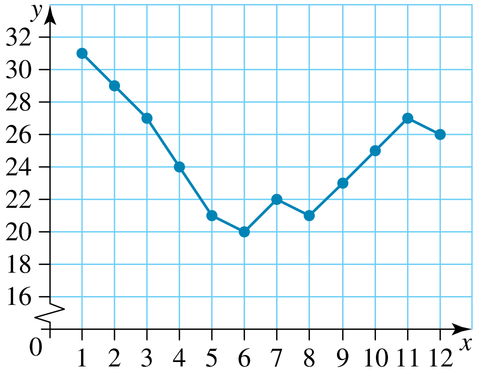
3 **a** If you have a 12-point time series and perform a 3-point median smoothing on the data, how many data points would you be left with?

b If you have an 11-point time series and perform a 5-point median smoothing on the data, how many data points would you be left with?

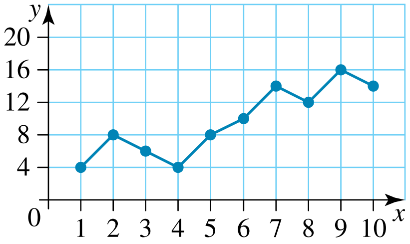
4 Perform a 3-point median smoothing on the graph below and plot the result. Comment on any trends that you find. These are the same data as in question **3**, Exercise 4.5, so compare the graphs of the median smooth with the moving-mean smooth.



5 The maximum daily temperatures for a year were recorded as a monthly average. Perform a 3-point median smoothing on the following graph. Comment on your result.

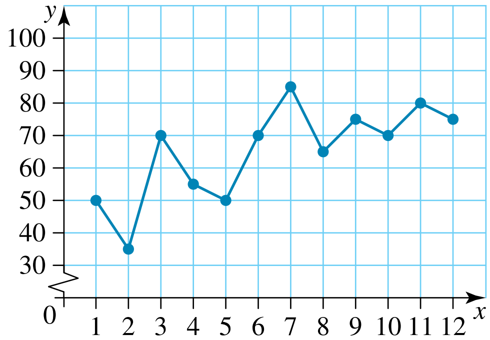


6 Perform a 3-point median smoothing on the graphical time series shown below.



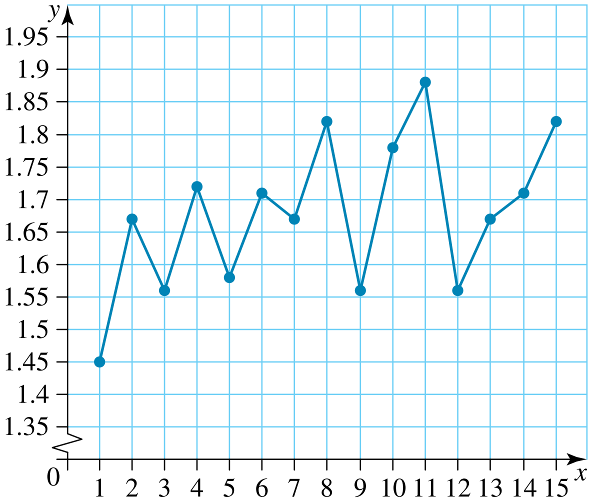
7 Comment on the effectiveness of the 3-point median smoothing in question **6**.

8 Perform a 3-point median smoothing on the graphical time series shown below.

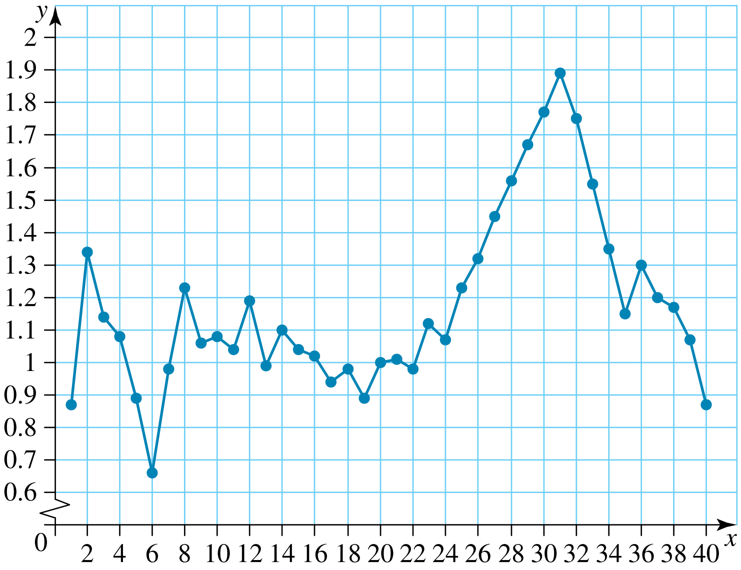


9 Comment on the effectiveness of the 3-point median smoothing in question **8**.

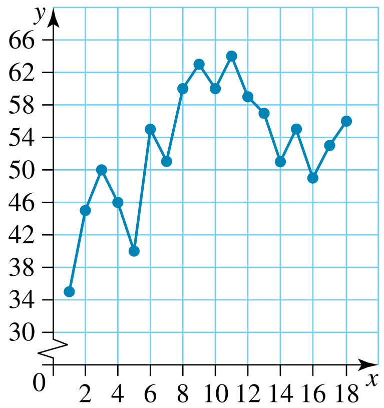
10 Perform a 3-point median smoothing on the graph shown, which represents the share price of the HAL computer company over the last 15 days.



11 Perform a 5-point median smoothing on the graph, which represents the share price of the Pear-Shaped Computer Company over an 8-week trading period.

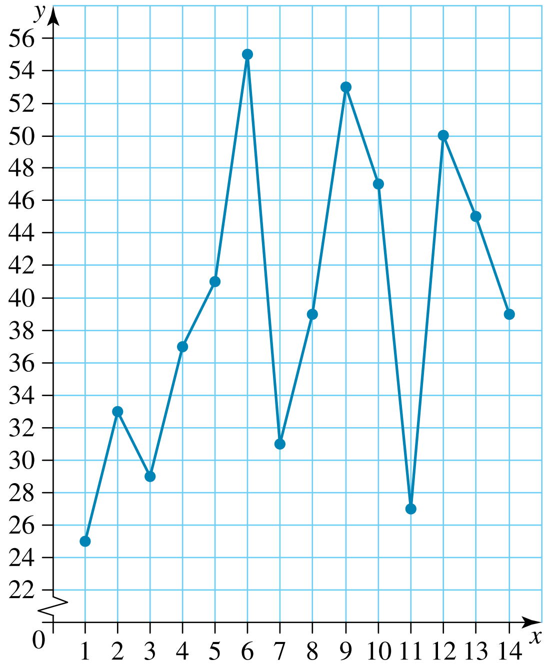


12 Perform a 5-point median smoothing on the following time series.



MASTER

13 Perform a 3-point median smoothing on the following data.



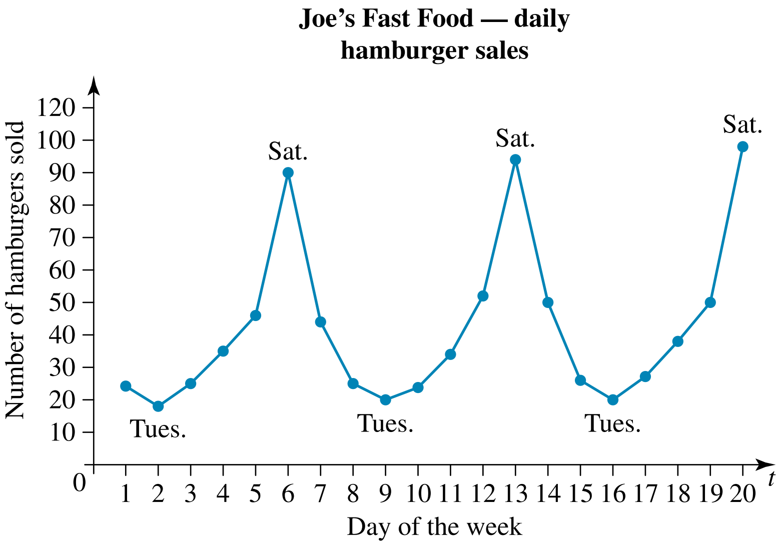
14 Perform a 5-point median smoothing on the data from question **13**. Comment on the differences of the two smoothing techniques.

4.7 Seasonal adjustment

A seasonal time series is similar to a cyclical time series where there are defined peaks and troughs in the time-series data, except for one notable difference.

Seasonal time series have a fixed and regular period of time between one peak and the next peak in the data values. Conversely, there is a fixed and regular period of time between one trough and the next trough.

As we have seen in the sections on fitting a straight line to a time series, it is difficult to find an effective linear equation for such data. As well, the sections on smoothing indicated that seasonal data may not lend themselves to the techniques of moving-mean or median smoothing. We may just have to accept that the data vary from season to season and treat each record individually.



For example, the unemployment rate in Australia is often quoted as '6.8% — seasonally adjusted'. The Government has accepted that each season has its own time series, more or less independent of the other seasons. How can we remove the effect of the season on our time series? The technique of seasonally adjusting, or **deseasonalising**, will modify the original time series, hopefully removing the seasonal variation, and exposing any other fluctuations (cyclic or irregular) which may be 'hidden' by seasonal variation.

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Interactivity
Seasonal adjustment
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Deseasonalising time series

The process of deseasonalising time series data involves calculating **seasonal indices**. A seasonal index compares a particular season to the average season.

That is, the seasonal index measures by what factor a particular season is above or below the average of all seasons for the cycle. For example:

Seasonal index = 1.3 means that season is 1.3 times the average season (that is, the figures for this season are 30% above the seasonal average). It is a peak or high season.

Seasonal index = 0.7 means that season is 0.7 times the average season (that is the figures for this season are 30% below the seasonal average). It is a trough or low season.

Seasonal index = 1.0 means that season is the same as the average season. It is neither a peak nor a trough.

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To deseasonalise the data, we divide each value by the corresponding seasonal index. That is,

$$\textbf{Deseasonalised figure or value} = \frac{\textbf{actual original figure or value}}{\textbf{seasonal index}}.$$

The method of deseasonalising time series is best demonstrated with an example. Observe carefully the various steps, which must be performed in the order shown.

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Unit 3
AOS DA
Topic 9
Concept 2
Deseasonalising data
Concept summary
Practice questions

WORKED EXAMPLE 8

Unemployment figures have been collected over a 5-year period and presented in this table. It is difficult to see any trends.

Season	2005	2006	2007	2008	2009
Summer	6.2	6.5	6.4	6.7	6.9
Autumn	8.1	7.9	8.3	8.5	8.1
Winter	8.0	8.2	7.9	8.2	8.3
Spring	7.2	7.7	7.5	7.7	7.6

a Calculate the seasonal indices.

b Deseasonalise the data using the seasonal indices.

c Plot the original and deseasonalised data.

d Comment on your results, supporting your statements with mathematical evidence.

THINK

a 1 Find the yearly averages over the four seasons for each year and put them in a table.

WRITE/DRAW

a 2005: $(6.2 + 8.1 + 8.0 + 7.2) \div 4 = 7.375$
2006: $(6.5 + 7.9 + 8.2 + 7.7) \div 4 = 7.575$
2007: $(6.4 + 8.3 + 7.9 + 7.5) \div 4 = 7.525$
2008: $(6.7 + 8.5 + 8.2 + 7.7) \div 4 = 7.775$
2009: $(6.9 + 8.1 + 8.3 + 7.6) \div 4 = 7.725$

Year	2005	2006	2007	2008	2009
Average	7.375	7.575	7.525	7.775	7.725

2 Divide each term in the original time series by its yearly average. That is, divide each value for 2005 by the yearly average for 2005 (i.e. by 7.375); next divide each value for 2006 by the yearly average for 2006 (i.e. by 7.575) etc.

Summer 2005: $6.2 \div 7.375 = 0.8407$
Autumn 2005: $8.1 \div 7.375 = 1.0983$
Winter 2005: $8.0 \div 7.375 = 1.0847$
Spring 2005: $7.2 \div 7.375 = 0.9763$
Summer 2006: $6.5 \div 7.575 = 0.8581$
⋮
Spring 2009: $7.6 \div 7.725 = 0.9838$

Season	2005	2006	2007	2008	2009
Summer	0.8407	0.8581	0.8505	0.8617	0.8932
Autumn	1.0983	1.0429	1.1030	1.0932	1.0485
Winter	1.0847	1.0825	1.0498	1.0547	1.0744
Spring	0.9763	1.0165	0.9967	0.9904	0.9838

3 Determine the seasonal averages from this second table. That is, find the average of all five values for summer; next find the average of all values for autumn and so on. These are called *seasonal indices*.

Summer:
$(0.8407 + 0.8581 + 0.8505 + 0.8617 + 0.8932) \div 5 = 0.8608$

Autumn:
$(1.0983 + 1.0429 + 1.1030 + 1.0932 + 1.0485) \div 5 = 1.0772$

Winter:
$(1.0847 + 1.0825 + 1.0498 + 1.0547 + 1.0744) \div 5 = 1.0692$

Spring:
$(0.9763 + 1.0165 + 0.9967 + 0.9904 + 0.9838) \div 5 = 0.9927$

Season	Summer	Autumn	Winter	Spring
Seasonal index	0.8608	1.0772	1.0692	0.9927

b Divide each term in the original series by its seasonal index. That is, divide all summer figures by summer seasonal index (0.8608), all autumn figures by the autumn seasonal index (1.0772) and so on. This gives the seasonally adjusted or deseasonalised time series.

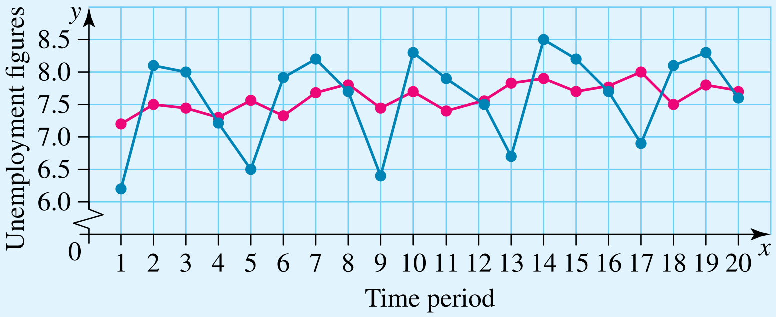
Note: Your answers may vary a little, depending upon how and when you rounded your calculations.

b Summer 05: 6.2 ÷ 0.8608 = 7.2023
Autumn 05: 8.1 ÷ 1.0772 = 7.5195
⋮
Spring 09: 7.6 ÷ 0.9927 = 7.6557

	2005	2006	2007	2008	2009
Summer	7.202	7.551	7.435	7.783	8.015
Autumn	7.520	7.334	7.705	7.891	7.520
Winter	7.482	7.669	7.388	7.669	7.763
Spring	7.253	7.756	7.555	7.756	7.656

c Graph the original and the seasonally adjusted (deseasonalised) time series.

c



d Note that most, but not all, of the seasonal variation has been removed. However, by using least-squares, we could more confidently fit a straight line to the deseasonalised data.

d There appears to be a slight upward trend in unemployment figures.

Forecasting with seasonal time series

In the previous section we smoothed out the seasonal variation and are now able to see any trends more clearly. If there is an upward or downward secular trend, then a straight line equation can be calculated and used for making predictions into the future. Using the least-squares regression method on the *deseasonalised* data is always preferred.

Once the equation of the regression line for deseasonalised data has been obtained, it can be used for **forecasting**.

However, the prediction obtained using such an equation will also be deseasonalised or smoothed out to the average season. But as we have the relevant seasonal indices, we should be able to use it to remove the smoothing; that is, to **re-seasonalise** the predicted value.

The formula for re-seasonalising is:

Re-seasonalised figure or value = deseasonalised figure or value × seasonal index.

WORKED EXAMPLE 9

Use the deseasonalised data from Worked example 8 to find the equation of the straight line for the deseasonalised data using the least-squares regression method. Predict the unemployment figure for summer in 2010. The deseasonalised data are reproduced in the table. (The seasonal index for summer is 0.8608.)

	2005	2006	2007	2008	2009
Summer	7.202	7.551	7.435	7.783	8.015
Autumn	7.520	7.334	7.705	7.891	7.520
Winter	7.482	7.669	7.388	7.669	7.763
Spring	7.253	7.756	7.555	7.756	7.656

THINK	WRITE
1 Use CAS to find the equation of the least-squares regression line for the deseasonalised data.	Deseasonalised unemployment (%) = 7.357 + 0.0227 × time code, where time code 1 represents summer 2005.
2 Using the association table, the summer of 2010 will be represented by $t = 21$. Substitute into the equation.	Summer of 2010: Deseasonalised unemployment (%) $= 7.357 + 0.0227 \times 21$ $= 7.834\%$
3 The predicted value is very high for summer. Re-seasonalise by using the seasonal index for summer, which was 0.8608. That is, this was a season or period of low unemployment.	Seasonalised value = deseasonalised value × seasonal index $= 7.834 \times 0.8608$ $= 6.74\%$

Note: Forecasting comes with limitations and cannot be used for an infinite time period. It should be used only for a small time period after your most recent recorded data.

WORKED EXAMPLE 10

Quarterly sales figures for a pool chemical supplier between 2007 and 2012 were used to determine the following seasonal indices.

Season	1st quarter	2nd quarter	3rd quarter	4th quarter
Seasonal index	1.8	1.2	0.2	0.8

Using the seasonal indices provided in the table, calculate the following.

a **Find the deseasonalised figure if the actual sales figure for the second quarter in 2011 was $4680.**

b **Find the deseasonalised figure if the actual sales figure for the third quarter in 2011 was $800.**

c **Find the predicted value if the deseasonalised predicted value for the first quarter in 2013 is expected to be $4000.**

THINK	WRITE
Use the formula for deseasonalising.	
a Use the 2nd quarter seasonal index.	**a** Deseasonalised figure $= \dfrac{\text{actual figure}}{\text{seasonal index}}$ $= \dfrac{4680}{1.2}$ $= \$3900$
b Use the 3rd quarter seasonal index to obtain the deseasonalised figure.	**b** Deseasonalised figure $= \dfrac{\text{actual figure}}{\text{seasonal index}}$ $= \dfrac{800}{0.2}$ $= \$4000$
c Use the re-seasonalising formula and select the 1st quarter seasonal index.	**c** Re-seasonalised figure = deseasonalised figure × seasonal index $= 4000 \times 1.8$ $= \$7200$ The forecast sales figure for the first quarter in 2013 is \$7200.

Seasonal indices

Finally, it should be noted that the sum of all the seasonal indices gives a specific result, which can be used to answer certain types of queries.

The sum of the seasonal indices is equal to the number of seasons.

This can be summarised as follows.

Type of data	Number of seasons	Cycle	Sum of all the seasonal indices
Monthly figures	12	A year	12
Quarterly figures	4	A year	4
Fortnightly figures	26	A year	26
Daily figures for data from Monday to Friday only	5	A week	5
Daily figures for data from Monday to Sunday	7	A week	7

WORKED EXAMPLE 11

A fast food store that is open seven days a week has the following seasonal indices.

Season	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Index	0.5	0.2	0.5	0.6		2.2	1.1

The index for Friday has not been recorded. Calculate the missing index.

THINK	WRITE
1 The sum of the seasonal indices is equal to the number of seasons.	There are 7 seasons (Monday to Sunday), therefore the sum of indices is 7.
2 The missing index is the sum of all the other seasons subtracted from the total.	Friday index $= 7 - \text{(sum of all the other indices)}$ $= 7 - (0.5 + 0.2 + 0.5 + 0.6 + 2.2 + 1.1)$ $= 7 - 5.1$ $= 1.9$

EXERCISE 4.7 Seasonal adjustment

Note: Your answers may vary slightly, depending upon rounding. Try to round correct to 4 decimal places for all intermediate calculations.

PRACTISE

1 WE8 The prices of 1 litre of milk has been collected over a 5-year period and is presented in the table. It is difficult to see any trends, other than seasonal ones.

Season	2010	2011	2012	2013	2014
Summer	1.55	1.60	1.50	1.60	1.45
Autumn	1.50	1.55	1.50	1.45	1.40
Winter	1.65	1.75	1.70	1.70	1.60
Spring	1.70	1.80	1.75	1.80	1.65

a Calculate the seasonal indices.
b Deseasonalise the data using the seasonal indices.
c Plot the original and deseasonalised data.
d Comment on your results, supporting your statements with mathematical evidence.

2 The average prices of 1 litre of petrol have been collected over a 5-year period and are presented in the table. It is difficult to see any trends, other than seasonal ones.

Season	2010	2011	2012	2013	2014
Summer	1.60	1.60	1.57	1.63	1.58
Autumn	1.48	1.50	1.46	1.31	1.27
Winter	1.40	1.35	1.39	1.28	1.19
Spring	1.65	1.60	1.63	1.69	1.59

a Calculate the seasonal indices.
b Deseasonalise the data using the seasonal indices.
c Plot the original and deseasonalised data.
d Comment on your results, supporting your statements with mathematical evidence.

3 WE9 Use the deseasonalised data from question **1** to find the equation of the straight line for the deseasonalised data using the least-squares regression method. Predict the price of 1 litre of milk for summer 2015. The deseasonalised data are reproduced at right. (The seasonal index for summer is 0.96).

	2010	2011	2012	2013	2014
Summer	1.61	1.67	1.56	1.67	1.51
Autumn	1.63	1.68	1.63	1.58	1.52
Winter	1.57	1.67	1.62	1.62	1.52
Spring	1.57	1.67	1.62	1.67	1.53

4 Use the deseasonalised data from question **2** to find the equation of the straight line for the deseasonalised data using the least-squares regression method. Predict the price of 1 litre of petrol for summer 2015. The deseasonalised data are reproduced at right. (The seasonal index for summer is 1.0731).

	2010	2011	2012	2013	2014
Summer	1.49	1.49	1.46	1.52	1.47
Autumn	1.57	1.59	1.55	1.39	1.35
Winter	1.58	1.52	1.57	1.44	1.34
Spring	1.50	1.46	1.49	1.54	1.45

5 **WE10** Quarterly sales figures of a pet supplier between 2010 and 2015 were used to determine the following seasonal indices.

Season	1st quarter	2nd quarter	3rd quarter	4th quarter
Seasonal index	1.4	1.2	0.6	0.8

Using the seasonal indices provided in the table, calculate the following.

a Find the deseasonalised figure if the actual sales figure for the second quarter in 2014 was $4345.

b Find the deseasonalised figure if the actual sales figure for the third quarter in 2014 was $950.

c Find the predicted value if the deseasonalised predicted value for the first quarter in 2016 is expected to be $5890.

6 Quarterly sales figures of a surf shop between 2010 and 2015 were used to determine the following seasonal indices.

Season	1st quarter	2nd quarter	3rd quarter	4th quarter
Seasonal index	1.7	1.3	0.4	0.6

Using the seasonal indices provided in the table, calculate the following.

a Find the deseasonalised figure if the actual sales figure for the second quarter in 2014 was $8945.

b Find the deseasonalised figure if the actual sales figure for the third quarter in 2014 was $3250.

c Find the predicted value if the deseasonalised predicted value for the first quarter in 2016 is expected to be $7950.

7 **WE11** A pizza shop that is open seven days a week has the following seasonal indices. The index for Friday has not been recorded. Calculate the missing index.

Season	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Index	0.4	0.3	0.6	0.8		1.8	1.3

8 A cinema open seven days a week has the following seasonal indices. Calculate the missing index.

Season	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Index	0.4	1.1	0.5	0.6	1.3		1.6

CONSOLIDATE

9 The price of sugar ($/kg) has been recorded over 3 years on a seasonal basis.

Season	2007	2008	2009
Summer	1.03	0.98	0.95
Autumn	1.26	1.25	1.21
Winter	1.36	1.34	1.29
Spring	1.14	1.07	1.04

a Compute the seasonal indices.
b Deseasonalise the data using the seasonal indices.
c Plot the original and deseasonalised data.
d Comment on your results, supporting your statements with mathematical evidence.

10 Data on the total seasonal rainfall (in mm) have been accumulated over a 6-year period.

Season	2004	2005	2006	2007	2008	2009
Summer	103	97	95	117	118	120
Autumn	93	84	82	100	99	98
Winter	143	124	121	156	155	151
Spring	123	109	107	125	122	124

a Compute the seasonal indices.
b Deseasonalise the original time series.
c Plot the original and deseasonalised time series.
d Comment on your result, supporting your statements with mathematical evidence.

11 It is known that young people (18–25) have problems in finding work; these problems are different from those facing older people. The youth unemployment statistics are recorded separately from the overall data. Using the youth unemployment figures for five years shown:

Season	2005	2006	2007	2008	2009
Summer	7.6	7.7	7.8	7.7	7.9
Autumn	10.9	11.3	11.9	12.6	13.1
Winter	11.7	12.4	12.8	13.5	13.9
Spring	9.9	10.5	10.8	11.4	11.9

a compute the seasonal indices
b deseasonalise the time series
c plot the original and deseasonalised time series
d comment on your result, supporting your statements with mathematical evidence.

12 The unemployment rate in a successful European economy is given in the table as a percentage.

Quarter	1	2	3	4
2007	5.8	4.9	3.5	6.7
2008	6.1	5.1	3.2	6.5
2009	5.7	4.5	4.1	7.1

a Compute the seasonal indices.

b Deseasonalise the time series.

c Plot the original and deseasonalised time series.

d Find the equation of the line-of-best-fit for the deseasonalised data using the least-squares method.

e Use the equation of the line from part **d** to predict the unemployment rate for:

i quarter 1 in 2010 **ii** quarter 3 in 2014.

13 It is possible to seasonally adjust time series for other than the usual 4 seasons. Consider an expensive restaurant that wishes to study its customer patterns on a daily basis. In this case a 'season' is a single day and there are 7 seasons in a weekly cycle. Data are total revenue each day shown in the table which follows. Modify the spreadsheet solution to allow for these 7 seasons and deseasonalise the following data over a 5-week period. Comment on your result, supporting your statements with mathematical evidence.

Season	Week 1	Week 2	Week 3	Week 4	Week 5
Monday	1036	1089	1064	1134	1042
Tuesday	1103	1046	1085	1207	1156
Wednesday	1450	1324	1487	1378	1408
Thursday	1645	1734	1790	1804	1789
Friday	2078	2204	2215	2184	2167
Saturday	2467	2478	2504	2526	2589
Sunday	1895	1786	1824	1784	1755

14 A line-of-best-fit for deseasonalised data was given as:

$$\text{Deseasonalised monthly sales} = 10\,000 + 1500 \times \text{time code}$$

where June 2012 represents $t = 1$.

The predicted actual expected sales figure for June 2013, if the June seasonal index is 0.8, would be:

A \$23 600 **B** \$29 500 **C** \$19 500

D \$36 875 **E** \$35 000

15 The following table gives the deseasonalised figures and corresponding seasonal indices for umbrella sales.

Season	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Number of umbrellas (deseasonalised)	24	24	25	26	25	27	27	28	30	31	33	34
Index	1.15	0.90	0.20	0.20	0.35	0.45	3.0	2.10	2.15	0.95	0.40	0.15

a Find the equation of the straight line for the deseasonalised data using the least-squares regression method.

b Predict the umbrella sales for January the following year.

16 Quarterly sales figures for an ice-cream parlour between 2010 and 2012 were used to determine the following seasonal indices.

Season	1st quarter	2nd quarter	3rd quarter	4th quarter
Seasonal index	1.50	1.00	0.25	1.25

Using the seasonal indices provided in the table, calculate the following.

a Find the deseasonalised figure, if the actual sales figure for the second quarter in 2011 was $3000.

b Find the deseasonalised figure, if the actual sales figure for the third quarter in 2011 was $800.

c Find the predicted value, if the deseasonalised predicted value for the first quarter in 2013 is expected to be $3200.

17 A newsagency store that is open seven days a week has the following seasonal indices.

Season	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Index	0.5	0.2		0.6	1.5	2.2	1.1

Find the value of the missing index.

18 Complete the following table of seasonal indices.

Season	Summer	Autumn	Winter	Spring
Index	1.23	0.89		1.45

MASTER

Questions **19** and **20** relate to the following table, which contains the seasonal indices for the monthly sales of spring water in a particular supermarket.

Season	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Index	1.05		1.0	1.0	0.95	0.85	0.8	0.9	0.95	1.05	1.10	1.15

19 The seasonal index missing from the table is:

A 1.0
B 1.05
C 1.10
D 1.15
E 1.20

20 If the actual sales figure for June 2012 was $102 000, then the deseasonalised figure would be:

A $96 900
B $86 700
C $107 368.42
D $120 000
E $102 000

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The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

To access eBookPLUS activities, log on to

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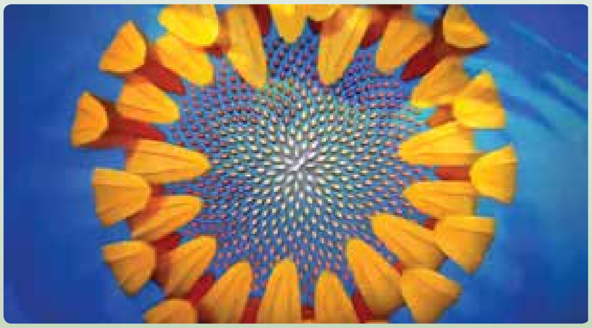
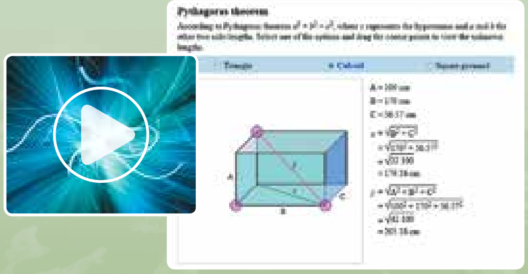
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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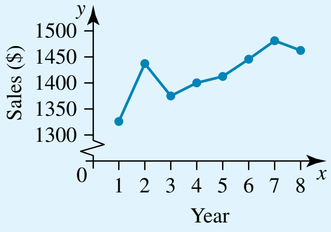
4 Answers

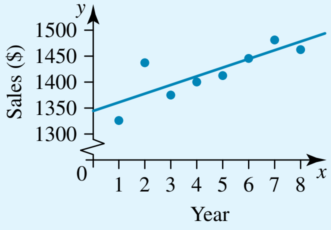
EXERCISE 4.2

Note: Your answers may vary slightly due to using the ‘by eye’ method.

1

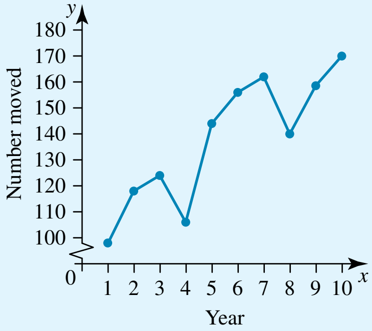


Upward trend with a possible outlier

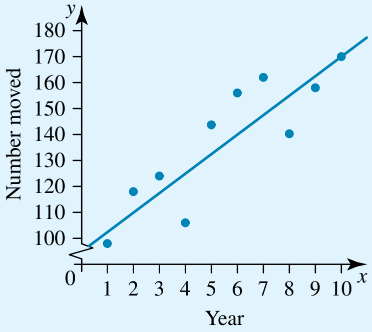


$y = 1342.96 + 16.45x$

2



Irregular with an upward trend



$y = 97.8 + 7.22x$

3 Seasonal

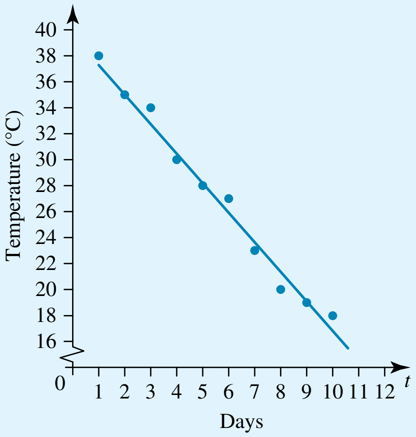
4 Irregular

5 Upwards trend

6 Irregular or cyclical with slight upward trend

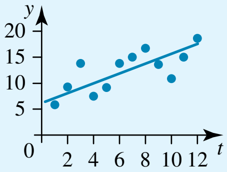
7 Cyclical

8 Definite downward trend



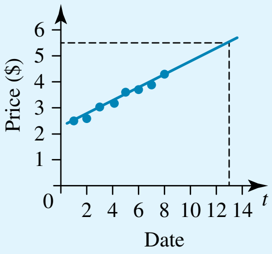
$y = 40 - 2.33x$

9 Although there are some random variations, the time series could also be cyclical, with an upward trend.



$y = 7.09 + 0.83x$

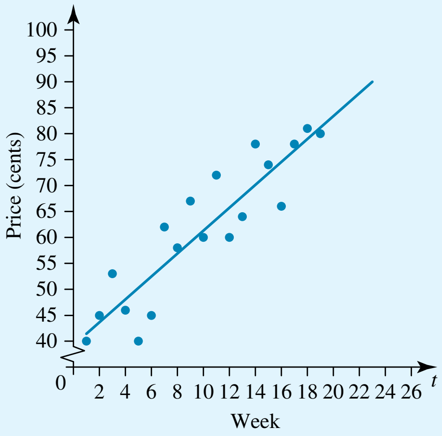
10 Predicting a share price in the following year is an extrapolated value (outside the plotted values) and can only be treated as an approximate value at best.



$y = 2.26 + 0.24x$

11 The time series is cyclical, which means that it is difficult to fit an appropriate trend line to the data. There is no noticeable trend to the data.

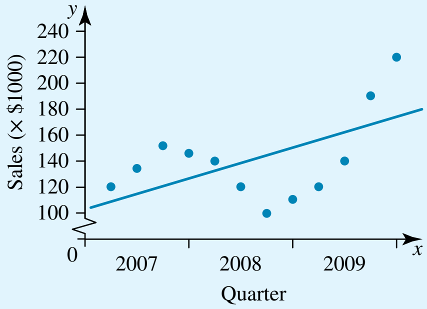
12 a



$y = 39.42 + 2.21x$

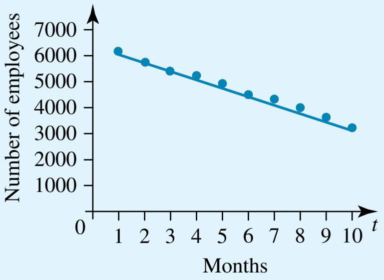
b Prediction for $t = 25$ is about 95 cents.

13 Difficult to fit an accurate trend line, due to likely cyclical nature of software sales business.



$y = 111.52 + 4.51x$

14 At current rate (about 300/month), bank will have no employees in another year! Although not likely, there is a clear downward trend.



$y = 6333.33 - 300.61x$

EXERCISE 4.3

1 $y = 1.18 - 0.02x$ or

Millions of viewers $= 1.18 - 0.02 \times$ time code

a Number of viewers = 840 000

b Number of viewers = 740 000

2 $y = 128.1 - 0.77x$ or Driver fatalities $= 128.1 - 0.77 \times$ time code

a Driver fatalities = 77

b Driver fatalities = 51

3 a Share price (\$) = \$22.74 + \$0.28 × time code

Time code of $t = 1$ represents 2007

b The y-intercept of \$22.74 represents the approximate value of the shares in 2006. The gradient of \$0.28 means that on average the share price will grow by \$0.28 each year.

c Share price (\$) = \$25.82

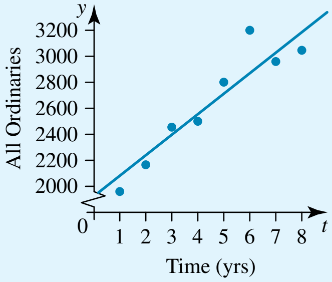
4 a Share price (\$) = \$18.57 − \$0.1 × time code

Time code of $t = 1$ represents 2010

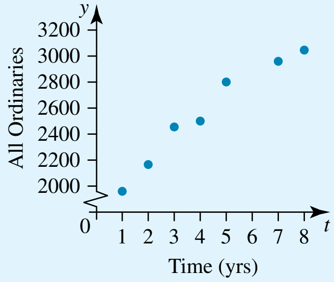
b The y-intercept of \$18.57 represents the approximate value of the shares in 2009. The gradient of −\$0.1 means that on average the share price will decline by \$0.1 (10 cents) each year.

c Share price (\$) = \$17.57

5 a $y = 1848.57 + 173.1x$, where y = All Ordinaries value and x = time, where $x = 1$ is 1995.



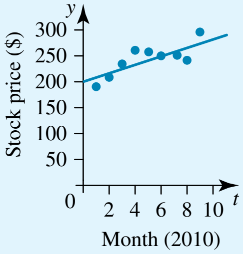
b Taking out the 2000 data value:



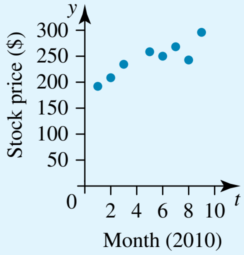
c $y = 1862.17 + 159.49x$

d The gradient decreased and the y-intercept is higher when the outlier is taken out.

6 a $y = 199.71 + 8.62x$, where y = stock price (\$) and x = month, where $x = 1$ is January 2010.



b The April value is a possible outlier.



c $y = 193.72 + 9.13x$

d The gradient increased and the y-intercept was lower when the April value was taken out.

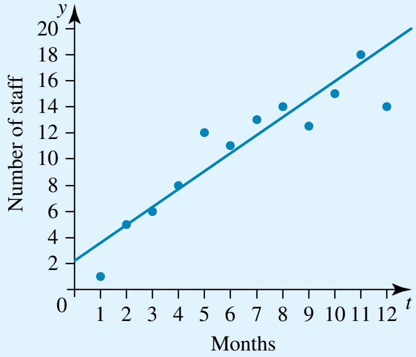
7 $y = 33.72 - 2.9t$

a 2

b 2.9 per hour

8 11.63 hours

9 a $y = 3.02 + 1.13x$, where y = numbers of staff and x = months, where $x = 1$ is the first month of business.

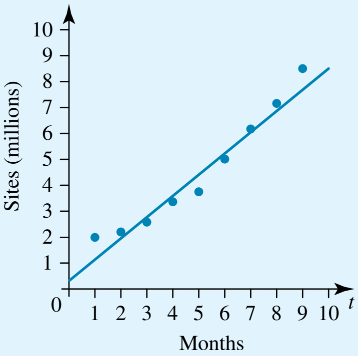


b $y = 30.14$, approximately 30 staff

10 a Least-squares: $y = 2.48 + 0.044t$

b Probably a random (or cyclical) time series.

11 $y = 0.283 + 0.83x$; given the exponential nature of data, it is a very poor predictor.



12 a Share price = \$2.56 + \$0.72 × time code

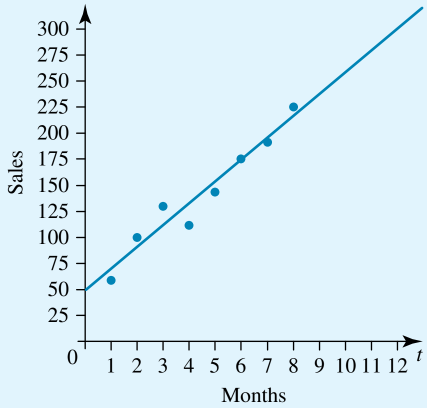
Time code $t = 1$ represents 2010, $t = 2$ represents 2011 and so on.

b The y-intercept of \$2.56 represents the approximate value of the shares in 2009. The gradient of +\$0.72 means that the share value will grow by \$0.72 (72 cents) each year.

c \$10.48

13 a $y = 49.64 + 20.36t$

b

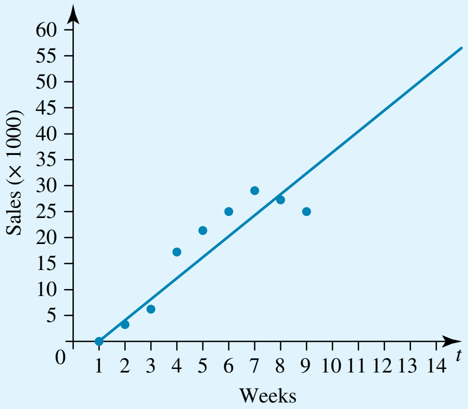


c y(Dec.) = 293.96 (294)

d The trend line fits the data well, so it is a good predictor of future trends.

14 a $y = -1.83 + 3.77t$

b

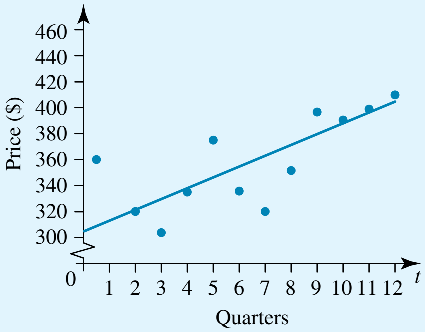


c $y(10) = 35.87$, $y(12) = 43.41$, $y(14) = 50.95$

d A poor predictor, given nature of data

15 a $y = 315.8 + 6.35t$

b

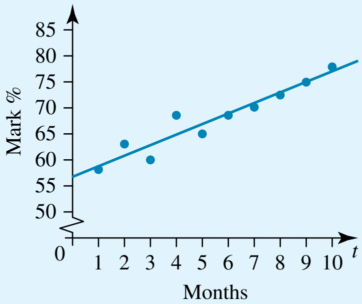


c \$398

d The first 2 years displayed seasonal fluctuations, the last year an upward trend, so overall trend line would be a poor predictor.

16 a $y = 56.9 + 1.93t$

b

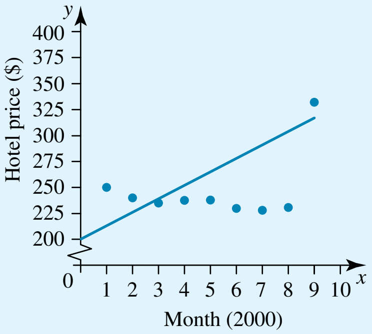


c 78%

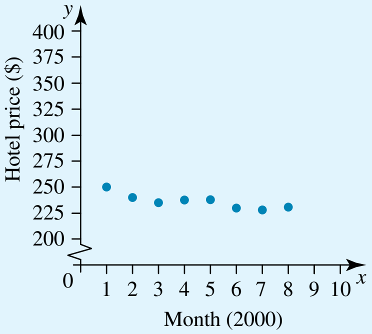
d Given fairly even increase in averages, trend line is an excellent predictor.

17 a $y = 222.72 + 4.97x$, where y = hotel price (\$) and x = months, where $x = 1$ is January 2000.

Month (2000)	1	2	3	4	5	6	7	8	9
Hotel price ($)	250	240	235	237	239	230	228	237	332



b Outlier = September value

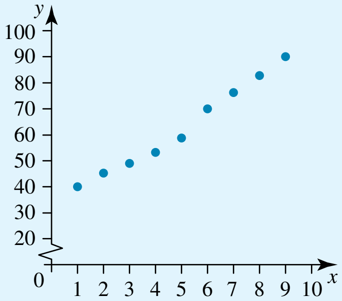


c $y = 245.79 - 1.95x$

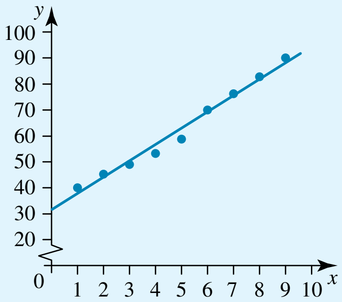
d Gradient (positive to negative), y-intercept (increased)

The most likely reason for the steep jump in hotel price in the month of September 2000 is that this is when the Olympics were held in Sydney.

18 a No obvious outliers



b $y = 31.39 + 6.37x$, where y = percentage of homes with TV and x = year, where $x = 1$ is 1970.



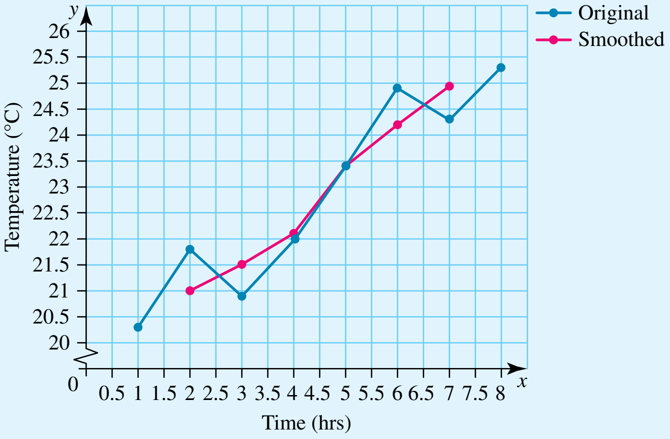
c The data does jump significantly from 1974 going to 1975. The possible reason is that this is when colour TV was introduced into Australia.

EXERCISE 4.4

1 a

Time (h)	Temp (°C)	Smoothed temperature (°C)
1	20.3	
2	21.8	$\frac{1}{3}(20.3 + 21.8 + 20.9) = 21$
3	20.9	$\frac{1}{3}(21.8 + 20.9 + 22.0) = 21.57$
4	22.0	$\frac{1}{3}(20.9 + 22.0 + 23.4) = 22.1$
5	23.4	$\frac{1}{3}(22.0 + 23.4 + 24.9) = 23.43$
6	24.9	$\frac{1}{3}(23.4 + 24.9 + 24.3) = 24.2$
7	24.3	$\frac{1}{3}(24.9 + 24.3 + 25.3) = 24.83$
8	25.3	

b

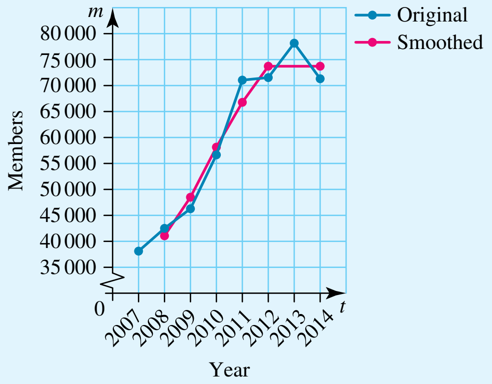


c 24.83 °C

2 a

Year (t)	Members (m)	Smoothed members (m)
2007	38 587	
2008	42 498	$\frac{1}{3}(38587 + 42498 + 45972) = 42352$
2009	45 972	$\frac{1}{3}(42498 + 45972 + 57408) = 48626$
2010	57 408	$\frac{1}{3}(45972 + 57408 + 71271) = 58217$
2011	71 271	$\frac{1}{3}(57408 + 71271 + 72688) = 67122$
2012	72 688	$\frac{1}{3}(71271 + 72688 + 78427) = 74129$
2013	78 427	$\frac{1}{3}(72688 + 78427 + 72170) = 74428$
2014	72 170	

b

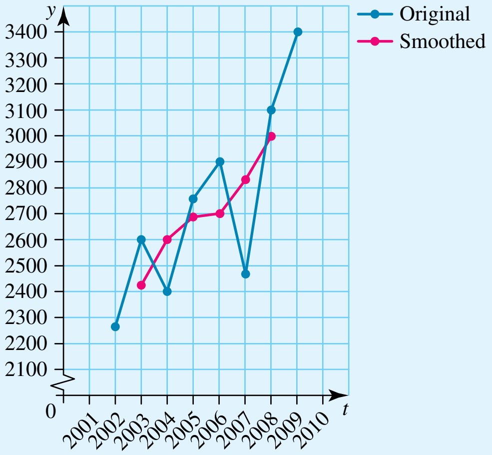


c 74 428 members

3 a Smoothed data:

2417	2583	2683	2700	2817	2983

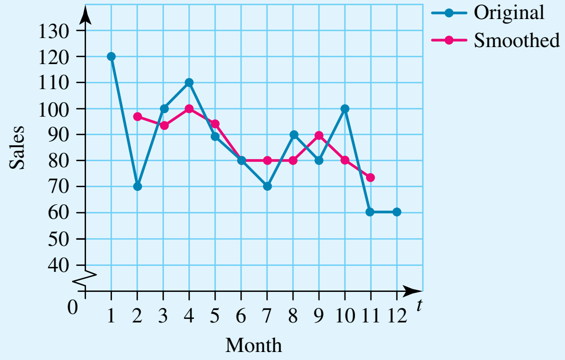
b



c 2983

4 a Smoothed data: possible downward trend, but still fluctuations.

96.7	93.3	100	93.3	80	80	80	90	80	73.3

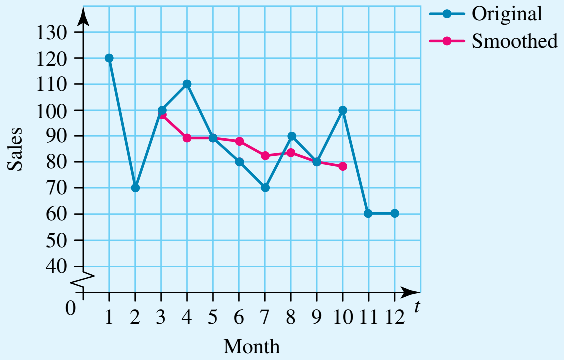


b Price = 102.17 − 2.39t

c 66.32, the data are cyclical and the prediction is based on smoothed data that have removed this trend.

5

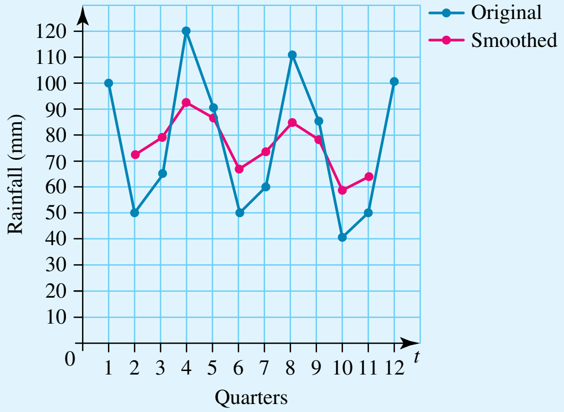
98	90	90	88	82	84	80	78



Smoothed data: definite downward trend is now apparent.

6

71.7	78.3	91.7	86.7	66.7	73.3	85.0	78.3	58.3	63.3



Smoothed data did not remove seasonal fluctuation; from the figure, there may be a slight trend downward.

7 a

72	71.7	69.7	70.7	66.3	65.7	64.7	66

Smoothed data: shows a clear downward trend.

b Attendance = 74.47 − 1.11x

c 61 150, this is a reasonable prediction as long as the trend continues to decline as given by the negative gradient.

8

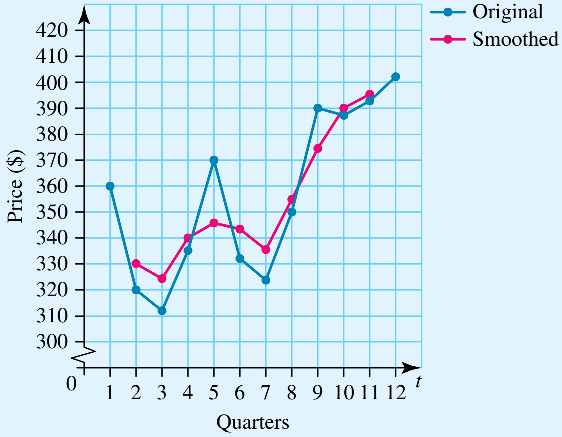
Week	Sales	Smoothed data
1	34	
2	27	30.67
3	31	31.67
4	37	36.33
5	41	35.67
6	29	34
7	32	32.67
8	37	38.67
9	47	40.67
10	38	42
11	41	41
12	44	44

(continued)

Week	Sales	Smoothed data
13	47	46.67
14	49	45.67
15	41	47.33
16	52	47
17	48	48
18	44	47
19	49	49.67
20	56	53
21	54	

9

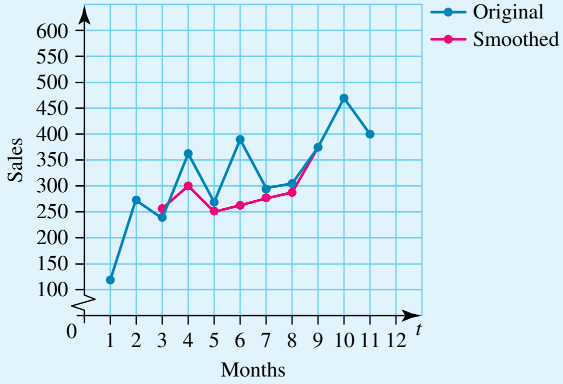
332.3	325	340.3	346	343.3	337.3	356	375.7	389.7	394



Smoothed data: Some but not all seasonal fluctuation removed.

10

253.8	303	306.6	320.4	322.4	362	364.2



Smoothed data: most random variation smoothed, slight upward trend possible. Prediction for 12th month is 364.

11

66.33	71.33	68.33	79.67	83.67

Smoothed data; prediction for week 7 is 84.

12 B

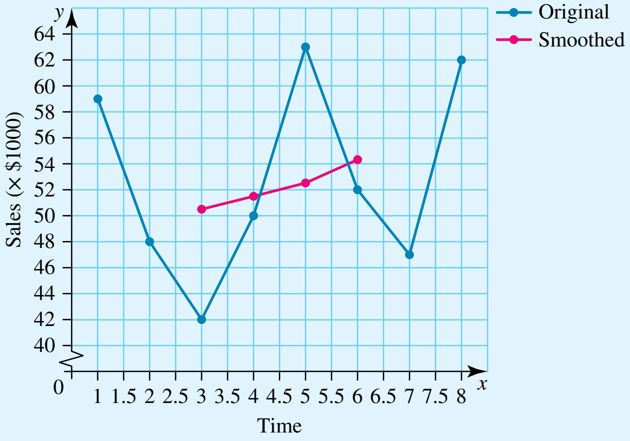
13 The 5-point moving mean smoothing has made it more evident that there is an upward trend in the data.

14 It has further smoothed out the seasonal fluctuations.

EXERCISE 4.5

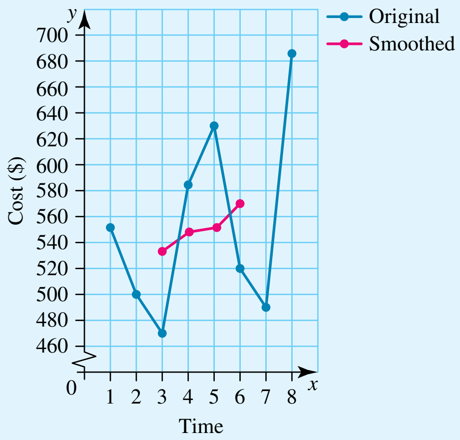
1 a

Time	Sale	4-point moving mean	4-point centred moving mean
1	59		
2	48		
		$\frac{(59 + 48 + 43 + 50)}{4} = 50$	
3	43		$\frac{(50 + 51)}{2} = 50.5$
		$\frac{(48 + 43 + 50 + 63)}{4} = 51$	
4	50		$\frac{(51 + 52)}{2} = 51.5$
		$\frac{(43 + 50 + 63 + 52)}{4} = 52$	
5	63		$\frac{(52 + 53)}{2} = 52.5$
		$\frac{(50 + 63 + 52 + 47)}{4} = 53$	
6	52		$\frac{(53 + 55.75)}{2} = 54.38$
		$\frac{(63 + 52 + 47 + 61)}{4} = 55.75$	
7	47		
8	61		



b The data shows a steady increasing trend.

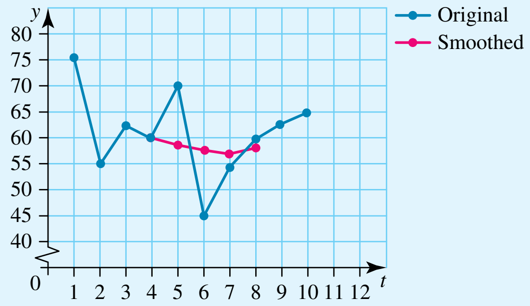
2 a See table at foot of the page*



b This data shows a steady increasing trend. This is not obvious with the original data.

3 a

t	3	4	5	6	7	8
y	62.125	60.375	58.25	57.125	56	57.375



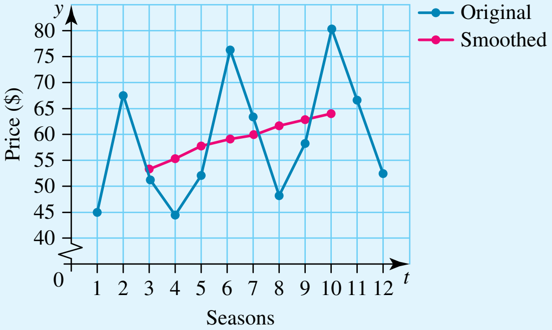
b Smoothed data indicated a general downward trend, possibly with cyclic fluctuations in original data.

***2 a**

Time	Sale	4-point moving mean	4-point centred moving mean
1	554		
2	503		
		$\frac{(554 + 503 + 467 + 587)}{4} = 527.75$	
3	467		$\frac{(527.75 + 548.25)}{2} = 538$
		$\frac{(503 + 467 + 587 + 636)}{4} = 548.25$	
4	587		$\frac{(548.25 + 555.75)}{2} = 552$
		$\frac{(467 + 587 + 636 + 533)}{4} = 555.75$	
5	636		$\frac{(555.75 + 562.25)}{2} = 559$
		$\frac{(587 + 636 + 533 + 493)}{4} = 562.25$	
6	533		$\frac{(562.25 + 586.50)}{2} = 574.38$
		$\frac{(636 + 533 + 493 + 684)}{4} = 586.5$	
7	493		
8	684		

4

t	Price ($)
Autumn ’07	
Winter ’07	
Spring ’07	52.625
Summer ’07	54.63
Autumn ’08	57.25
Winter ’08	59.25
Spring ’08	60.50
Summer ’08	61.75
Autumn ’09	62.63
Winter ’09	63.50
Spring ’09	
Summer ’09	



Smoothed data indicate a strong upward trend of almost 12 cents over 3 years.

5 a

Day	Temperature	4-point moving mean	4-point centred moving mean
1	36.6		
2	36.4	36.75	
3	36.8	36.825	36.7875
4	37.2	36.85	36.8375
5	36.9	36.95	36.9
6	36.5	37	36.975
7	37.2	37.05	37.025
8	37.4	37.275	37.1625
9	37.1	37.375	37.325
10	37.4	37.25	37.3125
11	37.6	37.275	37.2625
12	36.9	37.325	37.3
13	37.2	37.15	37.2375
14	37.6		
15	36.9		

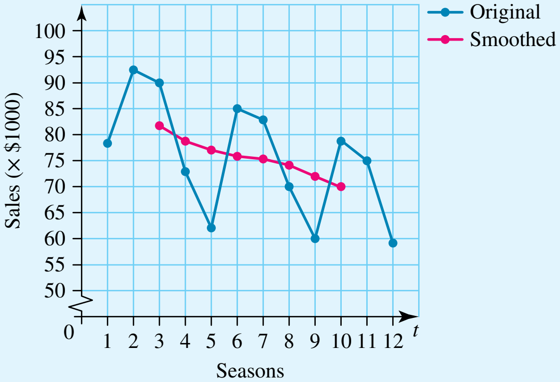
b Temperature = $36.654 + 0.056d$

c 37.6 °C

6 a

Season	Smoothed
Q_3-06	
Q_4-06	
Q_1-07	81.25
Q_2-07	78.375
Q_3-07	76.625
Q_4-07	75.375
Q_1-08	74.88
Q_2-08	73.875
Q_3-08	71.875
Q_4-08	69.375
Q_1-09	
Q_2-09	

b



c Smoothed data indicate a clear downward trend.

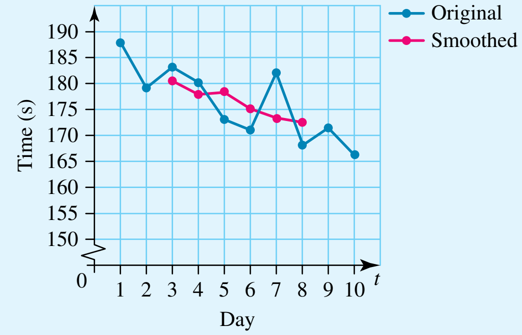
7

Day	Smoothed
1	
2	
3	
4	36.78
5	36.92
6	37.03
7	37.07
8	37.14
9	37.24
10	37.27
11	37.29
12	37.29
13	
14	
15	

8 a

Day	Smoothed
1	
2	
3	180.625
4	177.75
5	176.625
6	175
7	173.25
8	172.375
9	
10	

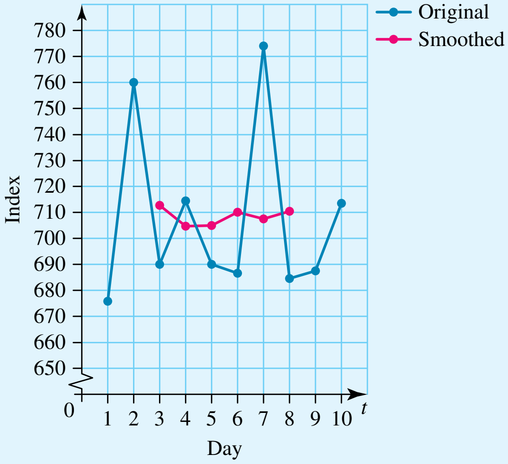
b



c Yes, there is a significant improvement in times.

9

Day	Index
1	
2	
3	712.875
4	704.875
5	705.5
6	711.875
7	708.125
8	711.125
9	
10	



Smoothing indicates a flat trend (neither upward or downward).

10 B

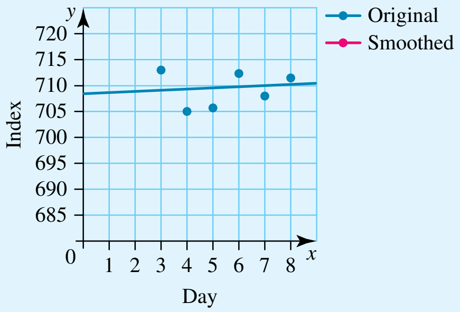
11 A

12 D

13

Day	Time (s)	6-point moving average	6-point centred moving average
1	188		
2	179		
3	183	179	
4	180	178	178.5
5	173	176.17	177.08
6	171	174.17	175.17
7	182	171.83	173
8	168		
9	171		
10	166		

14 a

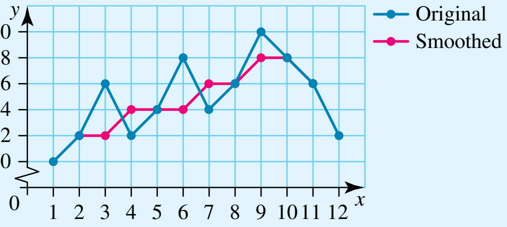


$y = 707.9 + 0.21x$, where y = price index and x = day number.

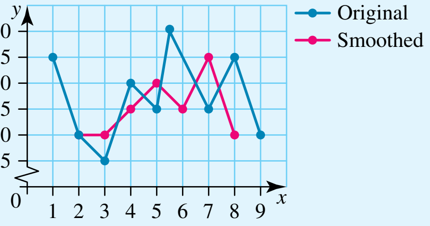
b $y = 711.05$ therefore price index is approximately 711 after 15 days.

EXERCISE 4.6

1

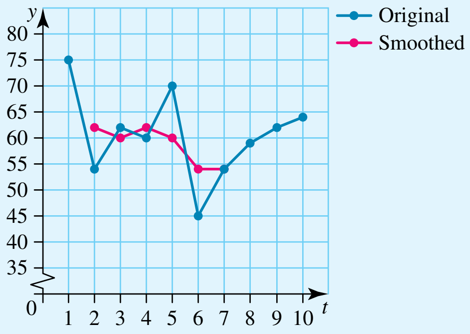


2



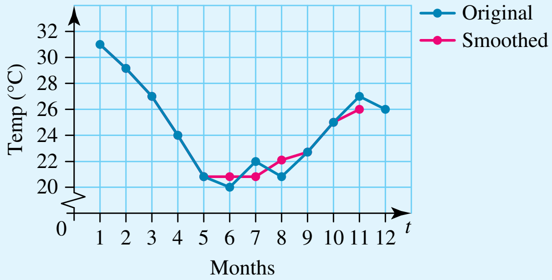
3 a 10 **b** 7

4



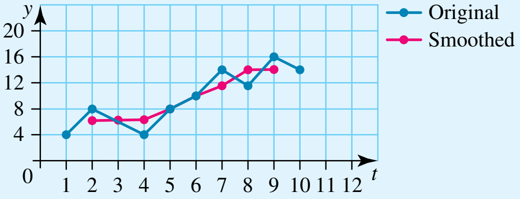
Not nearly as effective as moving mean smooth in demonstrating trend.

5



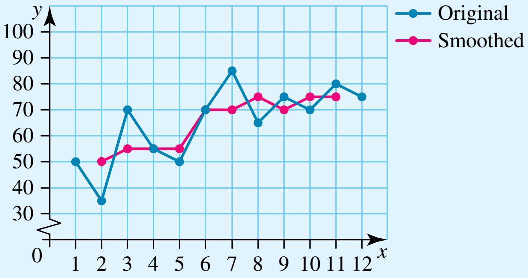
Smoothing had virtually no effect on data — only minor variations smoothed.

6



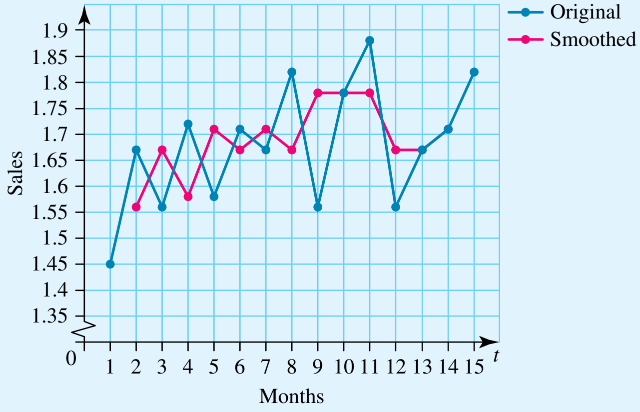
7 Effective at smoothing out small random variations.

8

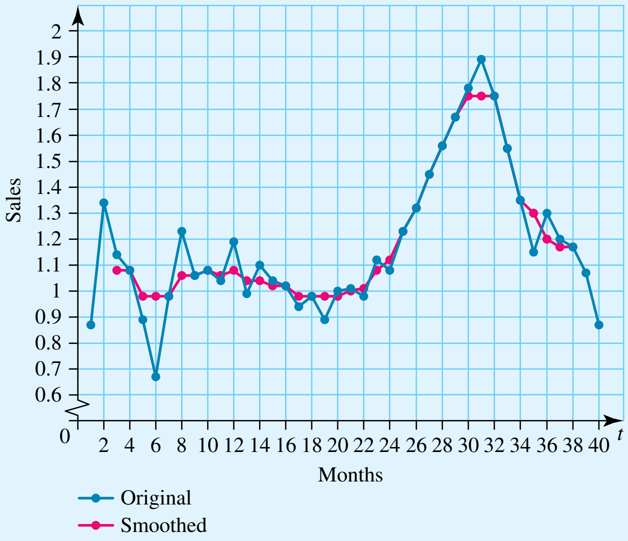


9 Smoothed out much of the variation, indicates a slight upward trend.

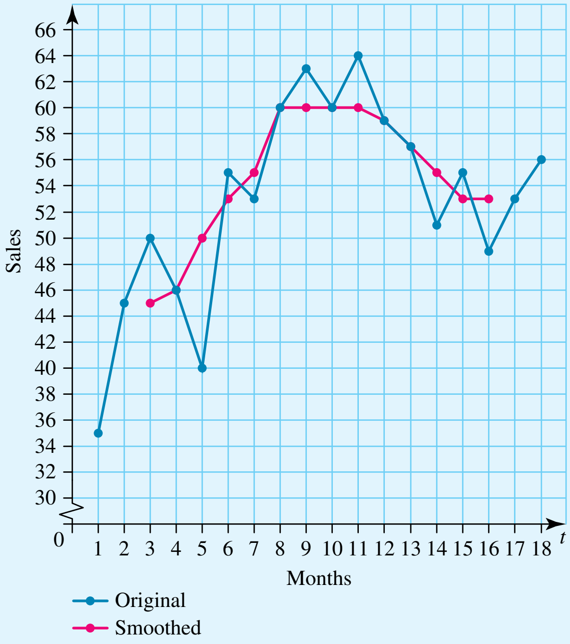
10



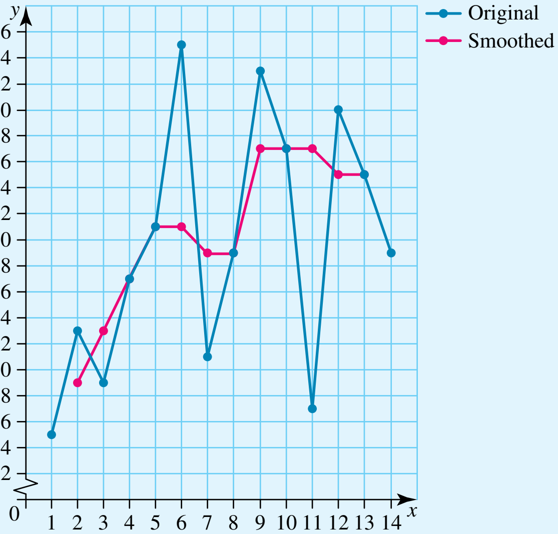
11



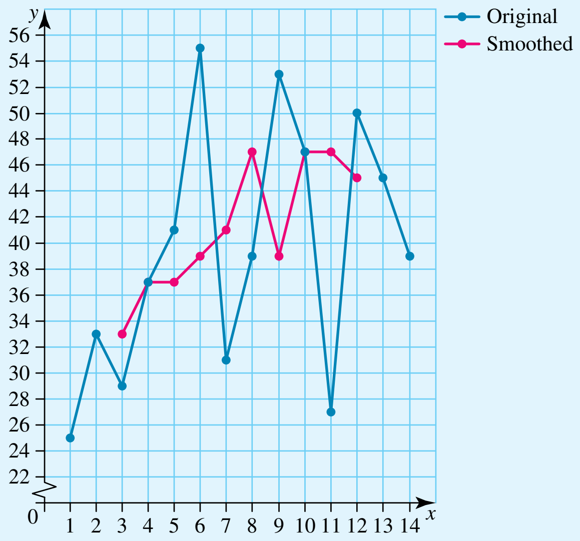
12



13



14



The 3-point median smoothing in question **13** reduces the fluctuations more than the 5-point median smoothing in question **14**.

EXERCISE 4.7

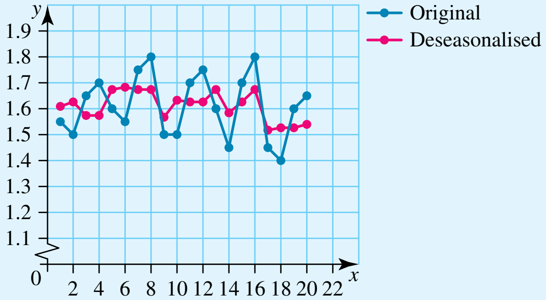
1 a

Season	Summer	Autumn	Winter	Spring
Seasonal index	0.96	0.92	1.05	1.08

b

Season	**2010**	**2011**	**2012**	**2013**	**2014**
Summer	1.61	1.67	1.56	1.67	1.51
Autumn	1.63	1.68	1.63	1.58	1.52
Winter	1.57	1.67	1.62	1.62	1.52
Spring	1.57	1.67	1.62	1.67	1.53

c



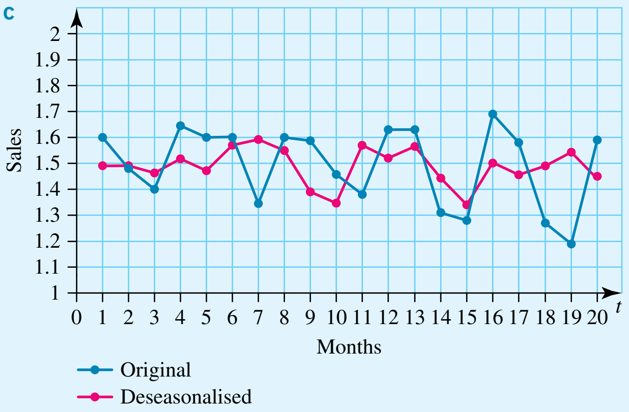
d There seems to be a very slight downward trend in milk prices.

2 a Seasonal indices: 1.0731, 0.9423, 0.8874, 1.0971

b

Season	**2010**	**2011**	**2012**	**2013**	**2014**
Summer	1.4910	1.4910	1.4630	1.5189	1.4723
Autumn	1.5706	1.5918	1.5493	1.3902	1.3477
Winter	1.5777	1.5213	1.5664	1.4424	1.3410
Spring	1.5039	1.4583	1.4857	1.5404	1.4492

c



d There appears to be a slight downward trend in the price of 1 litre of petrol.

3 Deseasonalised 1 L of milk cost = 1.6527 − 0.00445 × Time code

where time code 1 represents summer 2010

Predicted summer 2015 price = $1.50

4 Deseasonalised 1 L of petrol cost = 1.5659 − 0.00738 × Time code

where time code 1 represents summer 2010

Predicted summer 2015 price = $1.51

5 a $3620.83 **b** $1583.33 **c** $8246

6 a $6880.77 **b** $8125 **c** $13 515

7 1.8

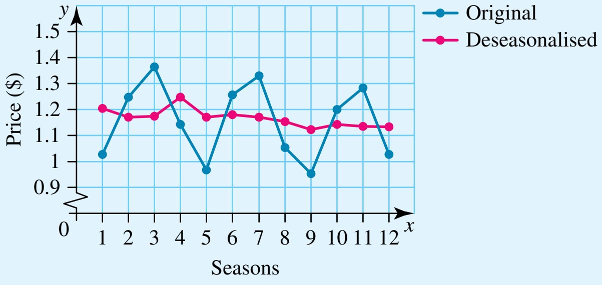
8 1.5

9 a Seasonal indices: 0.8504, 1.0692, 1.1467, 0.9336

b

Season	**2007**	**2008**	**2009**
Summer	1.211	1.152	1.117
Autumn	1.178	1.169	1.132
Winter	1.186	1.169	1.125
Spring	1.221	1.146	1.114

c



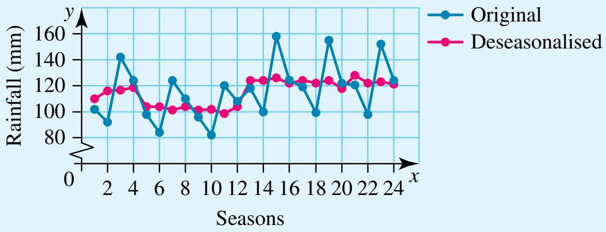
d Slight trend downwards

10 a Seasonal indices 0.9394, 0.8044, 1.2274, 1.0288

b

Season	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**
Summer	109.643	103.257	101.128	124.548	125.612	127.741
Autumn	115.614	104.426	101.939	124.316	123.023	121.830
Winter	116.506	101.027	98.582	127.098	126.283	123.024
Spring	119.557	105.949	104.005	121.501	118.585	120.529

c



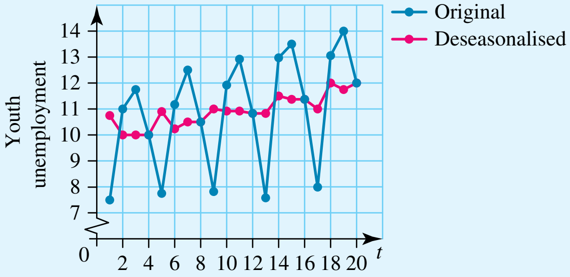
d Probable drought in 2005–06

11 a Seasonal indices: 0.7141, 1.100, 1.1832, 1.0027

b

Season	2005	2006	2007	2008	2009
Summer	10.6431	10.7832	10.9232	10.7832	11.0633
Autumn	9.9090	10.2726	10.8181	11.4544	11.9090
Winter	9.8884	10.4800	10.8181	11.4097	11.7478
Spring	9.8732	10.4716	10.7708	11.3692	11.8678

c



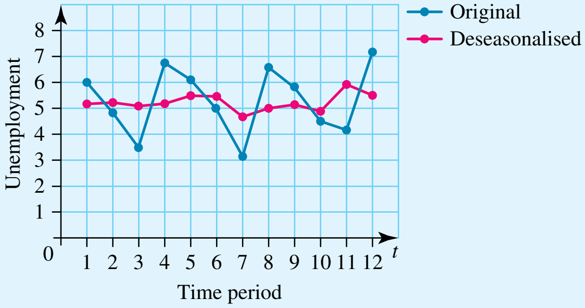
d Youth unemployment increases in all seasons except in summer.

12 a Seasonal indices: 1.1143, 0.9183, 0.6829, 1.2845

b

Quarter	1	2	3	4
2007	5.205	5.336	5.125	5.216
2008	5.474	5.554	4.686	5.060
2009	5.115	4.900	6.004	5.528

c

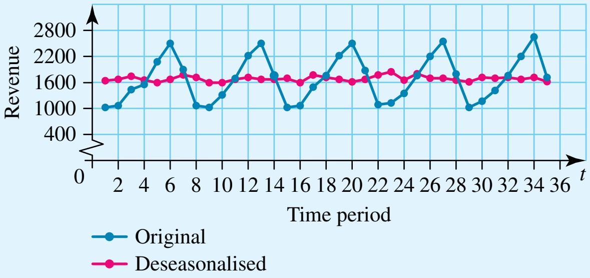


d Deseasonalised unemployment rate
$= 5.1448 + 0.0188 \times t$

e i 6.0 **ii** 3.9

13 Seasonal indices: 0.6341, 0.6613, 0.8329, 1.0354, 1.2822, 1.4850, 1.0692. Restaurant should probably close Mon.–Tues., steady sales over 5-week period — see following table.

Season	Week 1	Week 2	Week 3	Week 4	Week 5
Monday	1633.86	1717.45	1678.02	1788.42	1643.33
Tuesday	1667.88	1581.69	1640.66	1825.14	1748.02
Wednesday	1740.94	1589.66	1785.36	1654.49	1690.51
Thursday	1588.81	1674.77	1728.86	1742.38	1727.89
Friday	1620.72	1718.99	1727.57	1703.39	1690.13
Saturday	1661.32	1668.72	1686.23	1701.05	1743.47
Sunday	1772.29	1670.35	1705.89	1668.48	1641.36



14 A

15 a Deseasonalised umbrella sales $= 21.8788 + 0.9161 \times t$

b 39

16 a \$3000

b \$3200

c \$4800

17 0.9

18 0.43

19 E

20 D

5

Recurrence relations

5.1 Kick off with CAS

Generating terms of sequences with CAS

First-order recurrence relations relate a term in a sequence to the previous term in the same sequence. You can use recurrence relations to generate all of the terms in a sequence, given a starting (or initial) value.

1 Use CAS to generate a table of the first 10 terms of a sequence if the starting value is 7 and each term is formed by adding 3 to the previous term.

Term	Value
1	7
2	
3	
4	
5	
6	
7	
8	
9	
10	

2 Use CAS to generate the first 10 terms of a sequence if the starting value is 14 and each term is formed by subtracting 5 from the previous term.

Term	Value
1	14
2	
3	
4	
5	
6	
7	
8	
9	
10	

3 Use CAS to generate the first 10 terms of a sequence where the starting value is 1.5 and each term is formed by multiplying the previous term by 2.

4 Use CAS to generate the first 10 terms of a sequence where the starting value is 4374 and each term is formed by dividing the previous term by 3.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

5.2 Generating the terms of a first-order recurrence relation

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Unit 3
AOS R&FM
Topic 1
Concept 1

Generating a sequence
Concept summary
Practice questions

eBookplus

Interactivity
Initial values and first-order recurrence relations
int-6262

A **first-order recurrence relation** relates a term in a **sequence** to the previous term in the same sequence, which means that we only need an **initial value** to be able to generate all remaining terms of a sequence.

In a recurrence relation the *n*th term is represented by u_n, with the next term after u_n being represented by u_{n+1}, and the term directly before u_n being represented by u_{n-1}.

The initial value of the sequence is represented by either u_0 or u_1.

We can define the sequence 1, 5, 9, 13, 17, … with the following equation:

$$u_{n+1} = u_n + 4 \qquad u_1 = 1.$$

This expression is read as ‘the next term is the previous term plus 4, starting at 1’.

Or, transposing the above equation, we get:

$$u_{n+1} - u_n = 4 \qquad u_1 = 1.$$

A first-order recurrence relation defines a relationship between two successive terms of a sequence, for example, between:

u_n, the previous term $\qquad$ u_{n+1}, the next term.

Another notation that can be used is:

u_{n-1}, the previous term $\qquad$ u_n, the next term.

The first term can be represented by either u_0 or u_1.

Throughout this topic we will use either notation format as short hand for *next term*, *previous term* and *first term*.

WORKED EXAMPLE 1

The following equations each define a sequence. Which of them are first-order recurrence relations (defining a relationship between two consecutive terms)?

a $u_n = u_{n-1} + 2 \qquad u_1 = 3 \qquad n = 1, 2, 3, \ldots$

b $u_n = 4 + 2n \qquad n = 1, 2, 3, \ldots$

c $f_{n+1} = 3f_n \qquad n = 1, 2, 3, \ldots$

THINK	WRITE
a The equation contains the consecutive terms u_n and u_{n-1} to describe a pattern with a known term.	a This is a first-order recurrence relation. It has the pattern $u_n = u_{n-1} + 2$ and a starting or first term of $u_1 = 3$.
b The equation contains only the u_n term. There is no u_{n+1} or u_{n-1} term.	b This is not a first-order recurrence relation because it does not describe the relationship between two consecutive terms.
c The equation contains the consecutive terms f_n and f_{n+1} to describe a pattern but has no known first or starting term.	c This is an incomplete first-order recurrence relation. It has no first or starting term, so a sequence cannot be commenced.

Given a fully defined first-order recurrence relation (pattern and a known term) we can generate the other terms of the sequence.

Starting term

Earlier, it was stated that a starting term was required to fully define a sequence. As can be seen below, the same pattern with a different starting point gives a different set of numbers.

$$u_{n+1} = u_n + 2 \qquad u_1 = 3 \qquad \text{gives } 3, 5, 7, 9, 11, \ldots$$
$$u_{n+1} = u_n + 2 \qquad u_1 = 2 \qquad \text{gives } 2, 4, 6, 8, 10, \ldots$$

WORKED EXAMPLE 2

Write the first five terms of the sequence defined by the first-order recurrence relation:

$$u_n = 3u_{n-1} + 5 \qquad u_0 = 2.$$

THINK	WRITE
1 Since we know the u_0 or starting term, we can generate the next term, u_1, using the pattern: The next term is 3 × the previous term + 5.	$u_n = 3u_{n-1} + 5 \qquad u_0 = 2$ $u_1 = 3u_0 + 5$ $= 3 \times 2 + 5$ $= 11$
2 Now we can continue generating the next term, u_2, and so on.	$u_2 = 3u_1 + 5$ $= 3 \times 11 + 5$ $= 38$ $u_3 = 3u_2 + 5$ $= 3 \times 38 + 5$ $= 119$ $u_4 = 3u_3 + 5$ $= 3 \times 119 + 5$ $= 362$
3 Write your answer.	The sequence is 2, 11, 38, 119, 362.

WORKED EXAMPLE 3

A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 2u_n - 3 \qquad n = 1, 2, 3, \ldots$$

If the fourth term of the sequence is −29, that is, $u_4 = -29$, then what is the second term?

THINK	WRITE
1 Transpose the equation to make the previous term, u_n, the subject.	$u_n = \dfrac{u_{n+1} + 3}{2}$
2 Use u_4 to find u_3 by substituting into the transposed equation.	$u_3 = \dfrac{u_4 + 3}{2}$ $= \dfrac{-29 + 3}{2}$ $= -13$

3 Use u_3 to find u_2.

$$u_2 = \frac{u_3 + 3}{2} = \frac{-13 + 3}{2} = -5$$

4 Write your answer.

The second term, u_2, is -5.

EXERCISE 5.2 Generating the terms of a first-order recurrence relation

PRACTISE

1 **WE1** The following equations each define a sequence. Which of them are first-order recurrence relations (defining a relationship between two consecutive terms)?

a $u_n = u_{n-1} + 6 \quad u_1 = 7 \quad n = 1, 2, 3, \ldots$
b $u_n = 5n + 1 \quad n = 1, 2, 3, \ldots$

2 The following equations each define a sequence. Which of them are first-order recurrence relations?

a $u_n = 4u_{n-1} - 3 \quad n = 1, 2, 3, \ldots$
b $f_{n+1} = 5f_n - 8 \quad f_1 = 0 \quad n = 1, 2, 3, \ldots$

3 **WE2** Write the first five terms of the sequence defined by the first-order recurrence relation:

$$u_n = 4u_{n-1} + 3 \quad u_0 = 5.$$

4 Write the first five terms of the sequence defined by the first-order recurrence relation:

$$f_{n+1} = 5f_n - 6 \quad f_0 = -2.$$

5 **WE3** A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 2u_n - 1 \quad n = 1, 2, 3, \ldots$$

If the fourth term of the sequence is 5, that is, $u_4 = 5$, then what is the second term?

6 A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 3u_n + 7 \quad n = 1, 2, 3, \ldots$$

If the seventh term of the sequence is 34, that is, $u_7 = 34$, then what is the fifth term?

CONSOLIDATE

7 Which of the following equations are complete first-order recurrence relation?

a $u_n = 2 + n$
b $u_n = u_{n-1} - 1 \quad u_0 = 2$
c $u_n = 1 - 3u_{n-1} \quad u_0 = 2$
d $u_n - 4u_{n-1} = 5$
e $u_n = -u_{n-1}$
f $u_n = n + 1 \quad u_1 = 2$
g $u_n = 1 - u_{n-1} \quad u_0 = 21$
h $u_n = a^{n-1} \quad u_2 = 2$
i $f_{n+1} = 3f_n - 1$
j $p_n = p_{n-1} + 7 \quad u_0 = 7$

8 Write the first five terms of each of the following sequences.

a $u_n = u_{n-1} + 2 \qquad u_0 = 6$ **b** $u_n = u_{n-1} - 3 \qquad u_0 = 5$

c $u_n = 1 + u_{n-1} \qquad u_0 = 23$ **d** $u_{n+1} = u_n - 10 \qquad u_1 = 7$

9 Write the first five terms of each of the following sequences.

a $u_n = 3u_{n-1} \qquad u_0 = 1$ **b** $u_n = 5u_{n-1} \qquad u_0 = -2$

c $u_n = -4u_{n-1} \qquad u_0 = 1$ **d** $u_{n+1} = 2u_n \qquad u_1 = -1$

10 Write the first five terms of each of the following sequences.

a $u_n = 2u_{n-1} + 1 \qquad u_0 = 1$ **b** $u_n = 3u_{n-1} - 2 \qquad u_1 = 5$

11 Write the first seven terms of each of the following sequences.

a $u_n = -u_{n-1} + 1 \qquad u_0 = 6$ **b** $u_{n+1} = 5u_{n-1} \qquad u_1 = 1$

12 Which of the sequences is generated by the following first-order recurrence relation?

$$u_n = 3u_{n-1} + 4 \qquad u_0 = 2$$

A 2, 3, 4, 5, 6, … **B** 2, 6, 10, 14, 18, …

C 2, 10, 34, 106, 322, … **D** 2, 11, 47, 191, 767, …

E 6, 10, 14, 18, 22, …

13 Which of the sequences is generated by the following first-order recurrence relation?

$$u_{n+1} = 2u_n - 1 \qquad u_1 = -3$$

A −3, 5, 9, 17, 33, … **B** −3, −5, −9, −17, −33, …

C −3, 5, −3, 5, −3, … **D** −3, −8, −14, −26, −54, …

E −3, −7, −15, −31, −63, …

14 A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 3u_n + 1 \qquad n = 1, 2, 3, \ldots$$

If the fourth term is 67 (that is, $u_4 = 67$), what is the second term?

15 A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 4u_n - 5 \qquad n = 1, 2, 3, \ldots$$

If the third term is −41 (that is, $u_3 = -41$), what is the first term?

16 For the sequence defined in question **15**, if the seventh term is −27, what is the fifth term?

MASTER

17 A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 5u_n - 10 \qquad n = 1, 2, 3, \ldots$$

If the third term is −10, the first term is:

A $\frac{-14}{6}$ **B** $\frac{5}{6}$ **C** 0 **D** 2 **E** 4

18 Write the first-order recurrence relations for the following descriptions of a sequence and generate the first five terms of the sequence.

a The next term is 3 times the previous term, starting at $\frac{1}{4}$.

b Next year's attendance at a motor show is 2000 more than the previous year's attendance, with a first year attendance of 200 000.

c The next term is the previous term less 7, starting at 100.

d The next day's total sum is double the previous day's sum less 50, with a first day sum of $200.

5.3 First-order linear recurrence relations

First-order linear recurrence relations with a common difference

Consider the sequence 3, 7, 11, 15, 19, …

The **common difference**, d, is the value between consecutive terms in the sequence:

$$d = u_2 - u_1 = u_3 - u_2 = u_4 - u_3 = \ldots$$
$$d = 7 - 3 = 11 - 7 = 15 - 11 = +4$$

The common difference is +4.

This sequence may be defined by the first-order linear recurrence relation:

$$u_{n+1} - u_n = 4 \qquad u_1 = 3.$$

Rewriting this, we obtain:

$$u_{n+1} = u_n + 4 \qquad u_1 = 3.$$

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Unit 3
AOS R&FM
Topic 1
Concept 2

First-order linear recurrence relations
Concept summary
Practice questions

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Interactivity
First-order recurrence relations with a common difference
int-6264

A sequence with a common difference of d may be defined by a first-order linear recurrence relation of the form:

$$u_{n+1} = u_n + d \qquad (\text{or } u_{n+1} - u_n = d)$$

where d is the common difference and for

$d > 0$ it is an increasing sequence

$d < 0$ it is a decreasing sequence.

WORKED EXAMPLE 4

Express each of the following sequences as first-order recurrence relations.

a 7, 12, 17, 22, 27, …

b 9, 3, −3, −9, −15, …

THINK	WRITE
a 1 Write the sequence.	**a** 7, 12, 17, 22, 27, …
2 Check for a common difference.	$d = u_4 - u_3 = 22 - 17 = 5$ $\quad d = u_3 - u_2 = 17 - 12 = 5$ $\quad d = u_2 - u_1 = 12 - 7 = 5$
3 There is a common difference of 5 and the first term is 7.	The first-order recurrence relation is given by: $u_{n+1} = u_n + d$ $u_{n+1} = u_n + 5 \qquad u_1 = 7$
b 1 Write the sequence.	**b** 9, 3, −3, −9, −15, …
2 Check for a common difference.	$d = u_4 - u_3 = -9 - -3 = -6$ $\quad d = u_3 - u_2 = -3 - 3 = -6$ $\quad d = u_2 - u_1 = 3 - 9 = -6$
3 There is a common difference of −6 and the first term is 9.	The first-order recurrence relation is given by: $u_{n+1} = u_n - 6 \qquad u_1 = 9$

WORKED EXAMPLE 5

Express the following sequence as a first-order recurrence relation.

$$u_n = -3n - 2 \qquad n = 1, 2, 3, 4, 5, \ldots$$

THINK	WRITE
1 Generate the sequence using the given rule.	$n = 1, 2, 3, 4, 5, \ldots$ $\quad u_n = -3n - 2$ $n = 1$ $\quad u_1 = -3 \times 1 - 2 = -3 - 2 = -5$ $n = 2$ $\quad u_2 = -3 \times 2 - 2 = -6 - 2 = -8$ $n = 3$ $\quad u_3 = -3 \times 3 - 2 = -9 - 2 = -11$ $n = 4$ $\quad u_4 = -3 \times 4 - 2 = -12 - 2 = -14$ The sequence is $-5, -8, -11, -14, \ldots$
2 There is a common difference of -3 and the first term is -5. Write the first-order recurrence relation.	The first order difference equation is: $u_{n+1} = u_n - 3 \qquad u_1 = -5$

First-order linear recurrence relations with a common ratio

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Interactivity
First-order recurrence relations with a common ratio
int-6263

Not all sequences have a common difference (i.e. a sequence increasing or decreasing by adding or subtracting the same number to find your next term). You might also find that a sequence increases or decreases by multiplying the terms by a **common ratio**.

Consider the sequence 1, 3, 9, 27, 81, … The common ratio is found by:

$$R = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \ldots$$

$$R = \frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \ldots$$

$$= 3$$

The common ratio is 3.

This sequence may be defined by the first-order linear recurrence relation:

$$u_{n+1} = 3u_n \qquad u_1 = 1$$

A sequence with a common ratio of R may be defined by a first-order linear recurrence relation of the form:

$$u_{n+1} = Ru_n$$

where R is the common ratio

$R > 1$ is an increasing sequence

$0 < R < 1$ is a decreasing sequence

$R < 0$ is a sequence alternating between positive and negative values.

WORKED EXAMPLE 6

Express each of the following sequences as first-order recurrence relations.

a **1, 5, 25, 125, 625, …** **b** **3, −6, 12, −24, 48 …**

THINK	WRITE
a **1** There is a common ratio of 5 and the first term is 1.	**a** $R = \frac{5}{1} = \frac{25}{5} = \frac{125}{25} = \ldots$ $= 5$ $u_1 = 1$
2 Write the first-order recurrence relation.	The first-order recurrence relation is given by: $u_{n+1} = 5u_n \qquad u_1 = 1$
b **1** There is a common ratio of −2 and the first term is 3.	**b** $R = \frac{-6}{3} = \frac{12}{-6} = \frac{-24}{12} = \ldots$ $= -2$ $u_1 = 3$
2 Write the first-order recurrence relation.	The first-order recurrence relation is given by: $u_{n+1} = -2u_n \qquad u_1 = 3$

WORKED EXAMPLE 7

Express each of the following sequences as first-order recurrence relations.

a $u_n = 2(7)^{n-1} \qquad n = 1, 2, 3, 4, \ldots$

b $u_n = -3(2)^{n-1} \qquad n = 1, 2, 3, 4, \ldots$

THINK	WRITE
a **1** Generate the sequence using the given rule.	**a** $n = 1, 2, 3, 4, \ldots \qquad u_n = 2(7)^{n-1}$ $n = 1 \qquad u_1 = 2(7)^{1-1}$ $= 2 \times 7^0$ $= 2 \times 1$ $= 2$ $n = 2 \qquad u_1 = 2(7)^{2-1}$ $= 2 \times 7^1$ $= 2 \times 7$ $= 14$ $n = 3 \qquad u_1 = 2(7)^{3-1}$ $= 2 \times 7^2$ $= 2 \times 49$ $= 98$ $n = 4 \qquad u_1 = 2(7)^{4-1}$ $= 2 \times 7^3$ $= 2 \times 343$ $= 686$
2 There is a common ratio of 7 and the first term is 2.	$R = 7, u_1 = 2$

3 Write the first-order recurrence relation.

The first-order recurrence relation is given by:

$u_{n+1} = 7u_n \quad u_1 = 2$

b **1** Generate the sequence using the given rule.

b $n = 1, 2, 3, 4, \ldots \quad u_n = -3(2)^{n-1}$

$n = 1 \quad u_1 = -3(2)^{1-1}$

$= -3 \times 2^0$

$= -3 \times 1$

$= -3$

$n = 2 \quad u_1 = -3(2)^{2-1}$

$= -3 \times 2^1$

$= -3 \times 2$

$= -6$

$n = 3 \quad u_1 = -3(2)^{3-1}$

$= -3 \times 2^2$

$= -3 \times 4$

$= -12$

$n = 4 \quad u_1 = -3(2)^{4-1}$

$= -3 \times 2^3$

$= -3 \times 8$

$= -24$

2 There is a common ratio of 2 and the first term is −3.

$R = 2,\ u_1 = -3$

3 Write the first-order recurrence relation.

The first-order recurrence relation is given by:

$u_{n+1} = 2u_n \quad u_1 = -3$

EXERCISE 5.3 First-order linear recurrence relations

PRACTISE

1 **WE4** Express each of the following sequences as first-order recurrence relations.

a 1, 4, 7, 10, 13, …

b 12, 8, 4, 0, −4, …

2 Express each of the following sequences as first-order recurrence relations.

a −24, −19, −14, −9, −4, …

b 55, 66, 77, 88, 99, …

3 **WE5** Express the following sequence as a first-order recurrence relation.

$$u_n = -5n - 3 \qquad n = 1, 2, 3, 4, 5, \ldots$$

4 Express the sequence as a first-order recurrence relation.

$$u_n = \frac{1}{2}n + 4 \qquad n = 1, 2, 3, 4, 5, \ldots$$

5 **WE6** Express each of the following sequences as first-order recurrence relations.

a 2, 10, 20, 40, 80, …

b 4, −8, 16, −32, 64, …

6 Express each of the following sequences as first-order recurrence relations.

a 3, 15, 75, 375, 1875, …

b 200, −100, 50, −25, 12.5, …

7 **WE7** Express each of the following sequences as first-order recurrence relations.

a $u_n = 2(5)^{n-1} \quad n = 1, 2, 3, 4, \ldots$

b $u_n = -3(4)^{n-1} \quad n = 1, 2, 3, 4, \ldots$

8 Express each of the following sequences as first-order recurrence relations.

a $u_n = 3(2)^{n-1} \quad n = 1, 2, 3, 4, \ldots$ **b** $u_n = -2(3)^{n-1} \quad n = 1, 2, 3, 4, \ldots$

CONSOLIDATE

9 Express each of the following sequences as first-order recurrence relations.

a 1, 3, 5, 7, 9, … **b** 3, 10, 17, 24, 31, … **c** 12, 5, −2, −9, −16, …
d 1, 0.5, 0, −0.5, −1, … **e** 2, 6, 10, 14, 18, … **f** −2, 2, 6, 10, 14, …
g 6, 1, −4, −9, −14, … **h** 4, 10.5, 17, 23.5, 30, …

10 The sequence −6, −3, 0, 3, 6, … can be defined by the first-order recurrence relation:

A $u_{n+1} = u_n - 3 \quad u_0 = -6$ **B** $u_{n+1} = u_n + 3 \quad u_1 = -6$
C $u_{n+1} = 3u_n \quad u_1 = -3$ **D** $u_{n+1} = 3u_n - 1 \quad u_0 = -3$
E $u_{n+1} = 3u_n \quad u_1 = 3$

11 Express each of the following sequences as first-order recurrence relations.

a $u_n = -n - 3 \quad n = 1, 2, 3, \ldots$ **b** $u_n = 2n + 1 \quad n = 1, 2, 3, \ldots$
c $u_n = 3n - 4 \quad n = 1, 2, 3, \ldots$ **d** $u_n = -2n + 6 \quad n = 1, 2, 3, \ldots$

12 The sequence defined by $u_n = -2n + 3$, $n = 1, 2, 3, \ldots$, can be defined by the first-order recurrence relation:

A $u_{n+1} = u_n - 2 \quad u_1 = 1$ **B** $u_{n+1} = -2u_n \quad u_1 = 3$
C $u_{n+1} = -2u_n \quad u_n = 1$ **D** $u_{n+1} = u_n + 3 \quad u_1 = 1$
E $u_{n+1} = -2u_n + 3 \quad u_1 = 1$

13 Express each of the following sequences as a first-order recurrence relation.

a 12, 10, 8, 6, 4, … **b** 1, 2, 3, 4, 5, 6, …

14 Express each of the following sequences as first-order recurrence relations.

a 5, 10, 20, 40, 80, … **b** 1, 6, 36, 216, 1296, …
c −3, 3, −3, 3, −3, … **d** −3, −12, −48, −192, −768, …
e 2, 6, 18, 54, 162, … **f** 5, −5, 5, −5, 5, …
g −2, −8, −32, −128, −512, … **h** 5, −15, 45, −135, 405, …

15 The sequence −2, 6, −18, 54, −162, … can be defined by the first-order recurrence relation:

A $u_{n+1} = -2u_n \quad u_1 = -2$ **B** $u_{n+1} = -2u_n \quad u_1 = 3$
C $u_{n+1} = -3u_n \quad u_1 = 3$ **D** $u_{n+1} = -3u_n \quad u_1 = -2$
E $u_{n+1} = 3u_n \quad u_1 = 2$

16 Express each of the following sequences as first-order recurrence relations.

a $u_n = 2(3)^{n-1} \quad n = 1, 2, 3, \ldots$ **b** $u_n = -3(4)^{n-1} \quad n = 1, 2, 3, \ldots$
c $u_n = 0.5(-1)^{n-1} \quad n = 1, 2, 3, \ldots$ **d** $u_n = 3(5)^{n-1} \quad n = 1, 2, 3, \ldots$
e $u_n = -5(2)^{n-1} \quad n = 1, 2, 3, \ldots$ **f** $u_n = 0.1(-3)^{n-1} \quad n = 1, 2, 3, \ldots$

17 The sequence $u_n = -4(1)^{n-1}$ $n = 1, 2, 3, \ldots$ can be defined by the first-order recurrence relation:

A $u_{n+1} = u_n \quad u_1 = -4$ **B** $u_{n+1} = u_n - 1 \quad u_1 = -4$
C $u_{n+1} = 4u_n \quad u_1 = 1$ **D** $u_{n+1} = -4u_n + 3 \quad u_1 = -4$
E $u_{n+1} = -4u_n \quad u_1 = -1$

18 Express each of the following sequences as a first-order recurrence relation.

a −4, 4, −4, 4, −4, … **b** 2, −14, 98, −686, 4802, …
c 2, 6, 18, 54, 162, … **d** −1, 1, −1, 1, −1, …
e 5, 10, 20, 40, 80, …

MASTER

19 Express each of the following sequences as recurrence relations.

a 4, 10.5, 17, 23.5, 30, … **b** 15, 10, 5, 0, −5, −10, …
c 1, 5, 9, 13, 17, 21, …

20 Express each of the sequences as recurrence relations.

a $u_n = 4(2)^{n-1}$, $n = 1, 2, 3, \ldots$ **b** $u_n = -3(4)^{n-1}$, $n = 1, 2, 3, \ldots$
c $u_n = 2(-6)^{n-1}$, $n = 1, 2, 3, \ldots$ **d** $u_n = -0.1(8)^{n-1}$, $n = 1, 2, 3, \ldots$
e $u_n = 3.5(-10)^{n-1}$, $n = 1, 2, 3, \ldots$

5.4 Graphs of first-order recurrence relations

Certain quantities in nature and business may change in a uniform way (forming a pattern).

This change may be an increase, as in the case of:

$$u_{n+1} = u_n + 2 \qquad u_1 = 3,$$

or it may be a decrease, as in the case of:

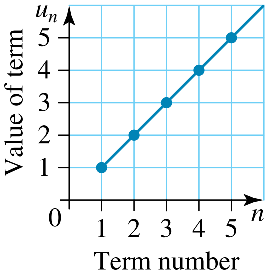
$$u_{n+1} = u_n - 2 \qquad u_1 = 3.$$

These patterns can be modelled by graphs that, in turn, can be used to recognise patterns in the real world.

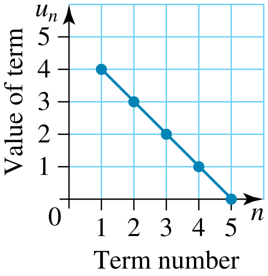
A graph of the equation could be drawn to represent a situation, and by using the graph the situation can be analysed to find, for example, the next term in the pattern.

First-order recurrence relations: $u_{n+1} = u_n + b$ (arithmetic patterns)

The sequences of a first-order recurrence relation $u_{n+1} = u_n + b$ are distinguished by a constant increase or decrease.



An increasing pattern or a positive common difference gives an upward sequence of points.



A decreasing pattern or a negative common difference gives a downward sequence of points.

WORKED EXAMPLE 8

On a graph, show the first five terms of the sequence described by the first-order recurrence relation:

$$u_{n+1} = u_n - 3 \qquad u_1 = -5.$$

THINK

1 Generate the values of each of the five terms of the sequence.

WRITE/DRAW

$u_{n+1} = u_n - 3$ $u_1 = -5$

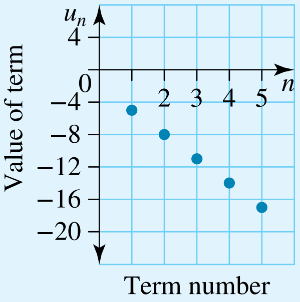
$u_2 = u_1 - 3$
$= -5 - 3$
$= -8$

$u_3 = u_2 - 3$
$= -8 - 3$
$= -11$

$u_4 = u_3 - 3$
$= -11 - 3$
$= -14$

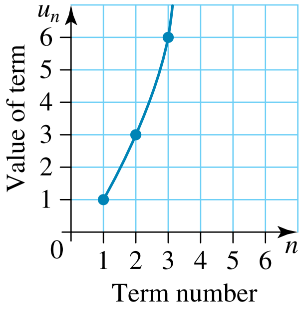
$u_5 = u_4 - 3$
$= -14 - 3$
$= -17$

2 Graph these first five terms. The value of the term is plotted on the y-axis, and the term number is plotted on the x-axis.

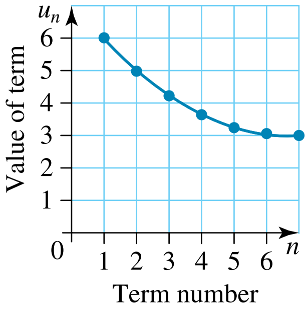


First-order recurrence relations: $u_{n+1} = Ru_n$

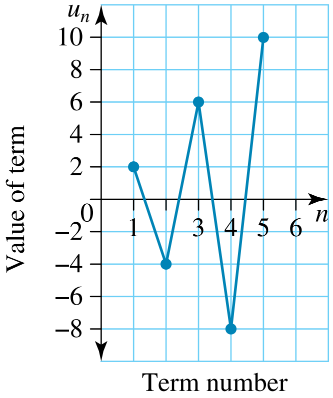
The sequences of a first-order recurrence relation $u_{n+1} = Ru_n$ are distinguished by a curved line or a saw form.



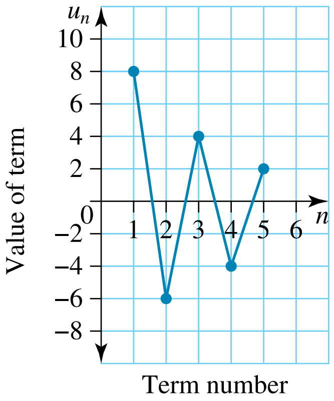
An increasing pattern or a positive common ratio greater than 1 ($R > 1$) gives an upward curved line.



A decreasing pattern or a positive fractional common ratio ($0 < R < 1$) gives a downward curved line.



An increasing saw pattern occurs when the common ratio is a negative value less than −1 ($R < -1$).



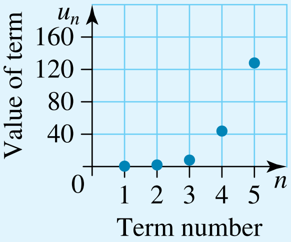
A decreasing saw pattern occurs when the common ratio is a negative fraction ($-1 < R < 0$).

Note: Values of n in recurrence relations are integer values, so they should not usually be joined by lines. They have been joined here only for the purpose of showing the patterns in these relations.

WORKED EXAMPLE 9

On a graph, show the first six terms of the sequence described by the first-order recurrence relation:

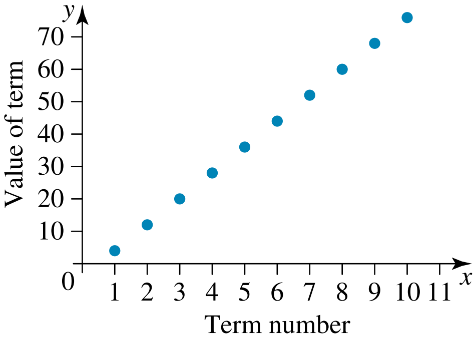
$$u_{n+1} = 4u_n \qquad u_1 = 0.5.$$

THINK	WRITE/DRAW
1 Generate the six terms of the sequence.	$u_{n+1} = 4u_n \qquad u_1 = 0.5$ $u_2 = 4u_1 = 4 \times 0.5 = 2$ $u_3 = 4u_2 = 4 \times 2 = 8$ $u_4 = 4u_3 = 4 \times 8 = 32$ $u_5 = 4u_4 = 4 \times 32 = 128$ $u_6 = 4u_5 = 4 \times 128 = 512$
2 Graph these terms. *Note:* The sixth term is not included in this graph to more clearly illustrate the relationship between the terms.	

Interpretation of the graph of first-order recurrence relations

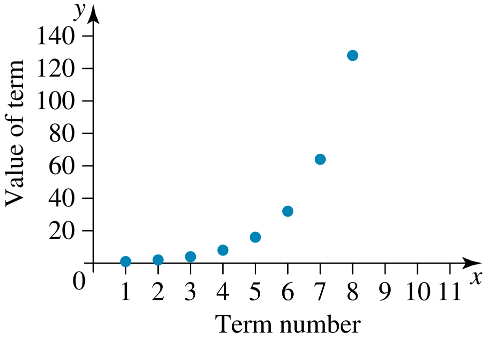
Straight or linear

A straight line or linear pattern is given by first-order recurrence relations of the form $u_{n+1} = u_n + d$



Non-linear (exponential)

A non-linear pattern is generated by first-order recurrence relations of the form $u_{n+1} = Ru_n$



Starting term

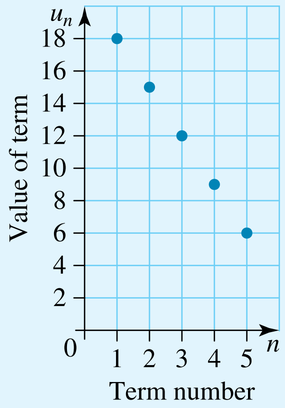
Earlier, the need for a starting term to be given to fully define a sequence was stated. As can be seen below, the same pattern but a different starting point gives a different set of numbers.

$u_{n+1} = u_n + 2 \qquad u_1 = 3 \qquad$ gives 3, 5, 7, 9, 11, …

$u_{n+1} = u_n + 2 \qquad u_1 = 2 \qquad$ gives 2, 4, 6, 8, 10, …

WORKED EXAMPLE 10

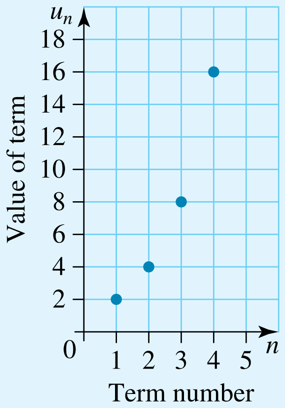
The first five terms of a sequence are plotted on the graph. Write the first-order recurrence relation that defines this sequence.



THINK	WRITE
1 Read from the graph the first five terms of the sequence.	The sequence from the graph is: 18, 15, 12, 9, 6, …
2 Notice that the graph is linear and there is a common difference of −3 between each term.	$u_{n+1} = u_n + d$ Common difference, $d = -3$ $u_{n+1} = u_n - 3 \qquad (\text{or } u_{n+1} - u_n = -3)$
3 Write your answer including the value of one of the terms (usually the first), as well as the rule defining the first order difference equation.	$u_{n+1} = u_n - 3 \qquad u_1 = 18$

WORKED EXAMPLE 11

The first four terms of a sequence are plotted on the graph. Write the first-order recurrence relation that defines this sequence.



THINK	WRITE
1 Read the terms of the sequence from the graph.	The sequence is 2, 4, 8, 16, …
2 The graph is non-linear and there is a common ratio of 2, that is, for the next term, multiply the previous term by 2.	$u_{n+1} = R \times u_n$ Common ratio, $R = 2$ $u_{n+1} = 2u_n$
3 Define the first term.	$u_1 = 2$
4 Write your answer.	$u_{n+1} = 2u_n \quad u_1 = 2$

WORKED EXAMPLE 12

The first five terms of a sequence are plotted on the graph shown. Which of the following first-order recurrence relations could describe the sequence?

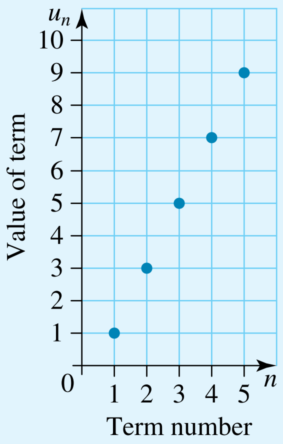
A $u_{n+1} = u_n + 1$ with $u_1 = 1$

B $u_{n+1} = u_n + 2$ with $u_1 = 1$

C $u_{n+1} = 2u_n$ with $u_1 = 1$

D $u_{n+1} = u_n + 1$ with $u_1 = 2$

E $u_{n+1} = u_n + 2$ with $u_1 = 2$



THINK	WRITE
1 Eliminate the options systematically. Examine the first term given by the graph to decide if it is $u_1 = 1$ or $u_1 = 2$.	The coordinates of the first point on the graph are (1, 1). The first term is $u_1 = 1$. Eliminate options **D** and **E**.
2 Observe any pattern between each successive point on the graph.	There is a common difference of 2 or $u_{n+1} = u_n + 2$.
3 Option **B** gives both the correct pattern and first term.	The answer is **B**.

EXERCISE 5.4 Graphs of first-order recurrence relations

PRACTISE

1 **WE8** On a graph, show the first five terms of a sequence described by the first-order recurrence relation:

$$u_{n+1} = u_n - 1 \qquad u_1 = -2.$$

2 On a graph, show the first five terms of a sequence described by the first-order recurrence relation:

$$u_{n+1} = u_n + 3 \qquad u_1 = 1.$$

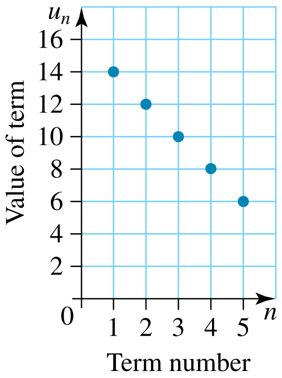
3 **WE9** On a graph, show the first six terms of a sequence described by the first-order recurrence relation:

$$u_{n+1} = 3u_n \qquad u_1 = 2.$$

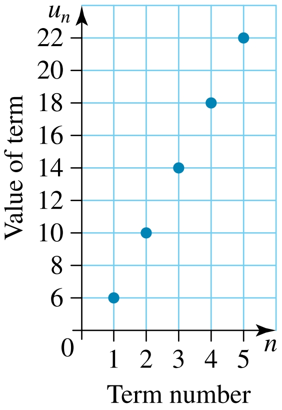
4 On a graph, show the first six terms of a sequence described by the first-order recurrence relation:

$$u_{n+1} = \frac{1}{2}u_n \qquad u_1 = 36.$$

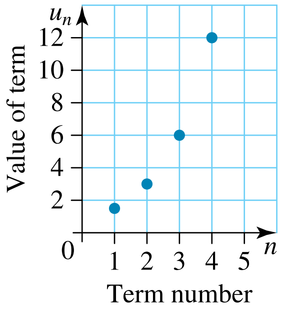
5 **WE10** The first five terms of a sequence are plotted on the graph shown. Write the first-order recurrence relation that defines this sequence.



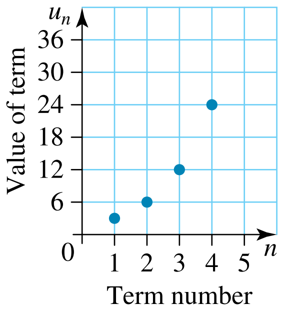
6 The first five terms of a sequence are plotted on the graph shown. Write the first-order recurrence relation that defines this sequence.



7 **WE11** The first four terms of a sequence are plotted on the graph shown. Write the first-order recurrence relation that defines this sequence.

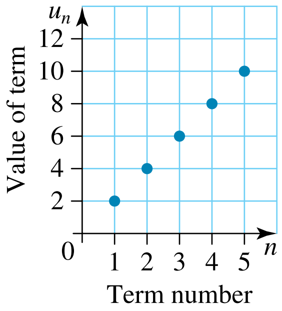


8 The first four terms of a sequence are plotted on the graph shown. Write the first-order recurrence relation that defines this sequence.

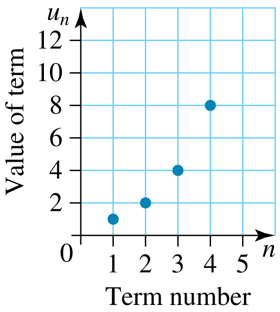


9 **WE12** The first five terms of a sequence are plotted on the graph shown. Which of the following first-order recurrence relations could describe the sequence?

A $u_{n+1} = u_n + 1 \quad u_1 = 2$
B $u_{n+1} = u_n + 2 \quad u_1 = 2$
C $u_{n+1} = u_n + 3 \quad u_1 = 1$
D $u_{n+1} = u_n + 1 \quad u_1 = 1$
E $u_{n+1} = u_n + 2 \quad u_1 = 1$



10 The first four terms of a sequence are plotted on the graph shown. Which of the following first-order recurrence relations could describe the sequence?



A $u_{n+1} = 2u_n \quad u_1 = 2$ **B** $u_{n+1} = 2u_n \quad u_1 = 1$
C $u_{n+1} = 2u_n \quad u_1 = 0$ **D** $u_{n+1} = 3u_n \quad u_1 = 1$
E $u_{n+1} = 3u_n \quad u_1 = 2$

CONSOLIDATE

11 For each of the following, plot the first five terms of the sequence defined by the first-order recurrence relation.

a $u_{n+1} = u_n + 3 \quad u_1 = 1$ **b** $u_{n+1} = u_n + 7 \quad u_1 = 5$
c $u_{n+1} = u_n - 3 \quad u_1 = 17$

12 For each of the following, plot the first five terms of the sequence defined by the first-order recurrence relation.

a $u_{n+1} = -2u_n \quad u_1 = -0.5$ **b** $u_n = 0.5u_{n-1} \quad u_1 = 16$
c $u_{n+1} = 2.5u_n \quad u_1 = 2$

13 For each of the following, plot the first four terms of the sequence defined by the first-order recurrence relation.

a $u_{n+1} = 3u_n - 4 \quad u_1 = -3$ **b** $u_n = 2u_{n-1} + 0.5 \quad u_1 = 2$
c $u_n = 2 + 5u_{n-1} \quad u_1 = 0.2$

14 For each of the following, plot the first four terms of the sequence defined by the first-order recurrence relation.

a $u_{n+1} = 100 - 3u_n \quad u_1 = 20$ **b** $u_{n+1} = u_n - 50 \quad u_1 = 100$
c $u_n = 0.1u_{n-1} \quad u_1 = 10$

15 On graphs, show the first 5 terms of the sequences defined by the following recurrence relations.

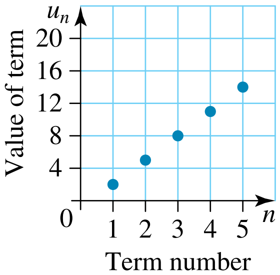
a $u_{n+1} = u_n + 3, \quad u_1 = 1, \quad n = 1, 2, 3, \ldots$
b $u_{n+1} = u_n - 3, \quad u_1 = 17, \quad n = 1, 2, 3, \ldots$
c $u_{n+1} = u_n - 15, \quad u_1 = 75, \quad n = 1, 2, 3, \ldots$
d $u_{n+1} = u_n + 10, \quad u_1 = 80, \quad n = 1, 2, 3, \ldots$

16 On graphs, show the first 4 terms of the sequences defined by the following recurrence relations.

a $u_{n+1} = 6u_n, \quad u_1 = 1, \quad n = 1, 2, 3, \ldots$
b $u_{n+1} = 3u_n, \quad u_1 = -1, \quad n = 1, 2, 3, \ldots$
c $u_{n+1} = 1.5u_n, \quad u_0 = 1, \quad n = 0, 1, 2, 3, \ldots$
d $u_{n+1} = 0.5u_n, \quad u_1 = 10, \quad n = 1, 2, 3, \ldots$

17 For each of the following graphs, write a first-order recurrence relation that defines the sequence plotted on the graph.

a



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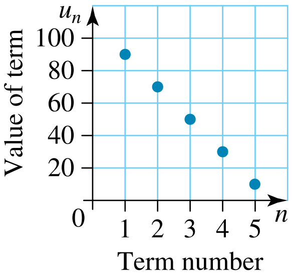
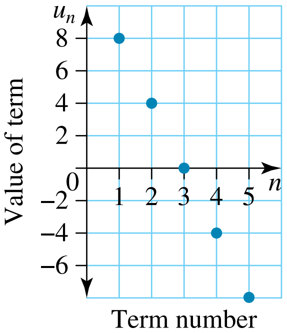
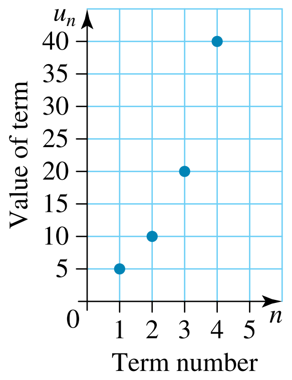


c

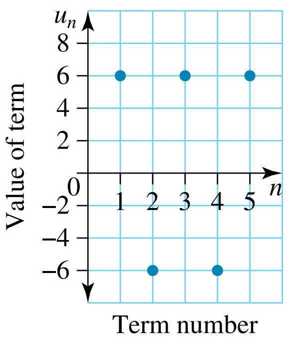


18 For each of the following graphs, write a first-order recurrence relation that defines the sequence plotted on the graph.

a



b



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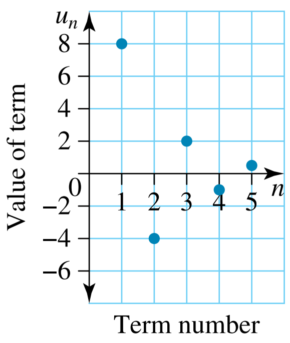


19 The first five terms of a sequence are plotted on the graph.

Which of the following first-order recurrence relations could describe the sequence?

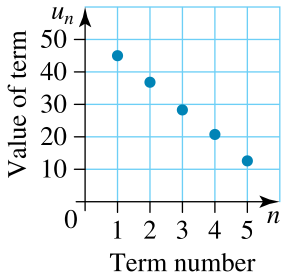
A $u_{n+1} = u_n - 8 \qquad u_1 = 8$

B $u_{n+1} = u_n + 8 \qquad u_1 = 8$

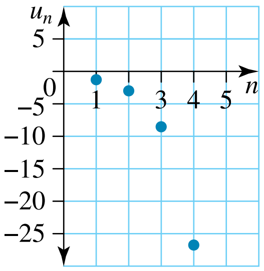
C $u_{n+1} = u_n - 8 \qquad u_1 = 45$

D $u_{n+1} = u_n + 8 \qquad u_1 = 45$

E $u_{n+1} = 8u_n \qquad u_1 = 45$



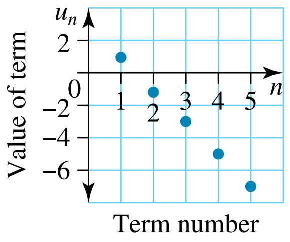
20 Write the first-order recurrence relation that defines the sequence plotted on the graph shown.



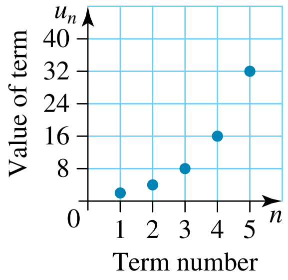
MASTER

21 Graphs of the first five terms of first-order recurrence relations are shown below together with the first-order recurrence relations. Match the graph with the first-order recurrence relation by writing the letter corresponding to the graph together with the number corresponding to the first-order recurrence relation.

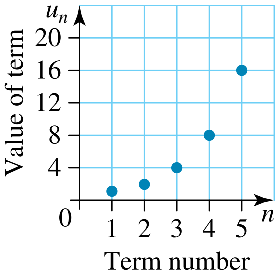
a



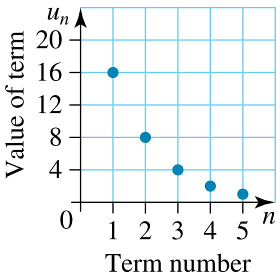
b



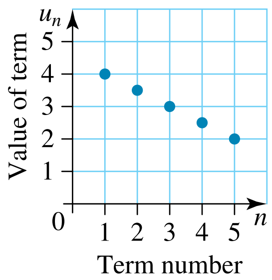
c



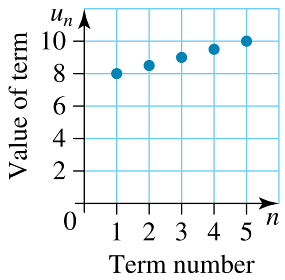
d



e



f



i $u_{n+1} = u_n + \frac{1}{2} \quad u_1 = 8$

ii $u_{n+1} = u_n - 2 \quad u_1 = 1$

iii $u_{n+1} = 2u_n \quad u_1 = 1$

iv $u_{n+1} = \frac{1}{2}u_n \quad u_1 = 16$

v $u_{n+1} = 2u_n \quad u_1 = 2$

vi $u_{n+1} = u_n - \frac{1}{2} \quad u_1 = 4$

22 The first few terms of four sequences are plotted on the graphs below. Match the following first-order recurrence relations with the graphs.

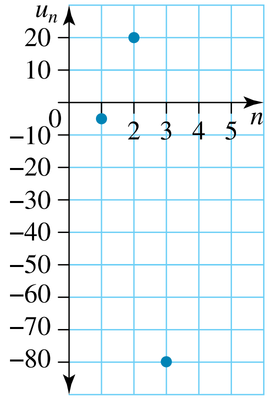
i $u_{n+1} = u_n + 2, \quad u_1 = 1, \quad n = 1, 2, 3, \ldots$

ii $u_{n+1} = 3u_n, \quad u_1 = 1, \quad n = 1, 2, 3, \ldots$

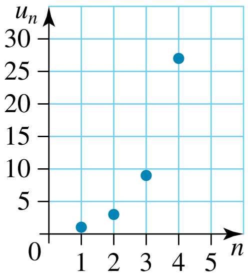
iii $u_{n+1} = -4u_n, \quad u_1 = -5, \quad n = 1, 2, 3, \ldots$

iv $u_{n+1} = u_n - 4, \quad u_1 = 1, \quad n = 1, 2, 3, \ldots$

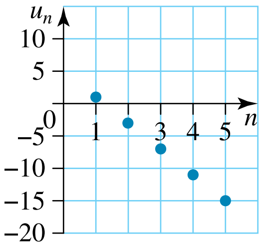
a



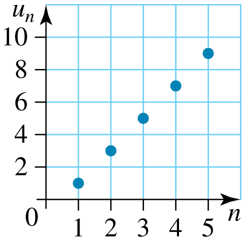
b



c



d



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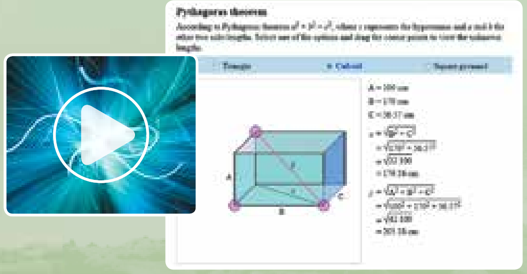

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5 Answers

EXERCISE 5.2

1 a This is a first-order recurrence relation.

b This is not a first-order recurrence relation.

2 a This is not a first-order recurrence relation.

b This is a first-order recurrence relation.

3 $u_n = 4u_{n-1} + 3, u_0 = 5$

The sequence is 5, 23, 95, 383, 1535.

4 $f_{n+1} = 5f_n - 6, f_0 = -2$

The sequence is −2, −16, −86, −436, −2186.

5 The second term is 2.

6 The fifth term is $\frac{2}{3}$.

7 b, c, g, j

8 a 6, 8, 10, 12, 14

b 5, 2, −1, −4, −7

c 23, 24, 25, 26, 27

d 7, −3, −13, −23, −33

9 a 1, 3, 9, 27, 81

b −2, −10, −50, −250, −1250

c 1, −4, 16, −64, 256

d −1, −2, −4, −8, −16

10 a 1, 3, 7, 15, 31

b 5, 13, 37, 109, 325

11 a The first seven terms are 6, −5, 6, −5, 6, −5, 6.

b The first seven terms are 1, 5, 25, 125, 625, 3125, 15 625.

12 C

13 E

14 7

15 −1

16 $-\frac{1}{8}$

17 D

18 a $u_{n+1} = 3u_n, u_0 = \frac{1}{4}; \frac{1}{4}, \frac{3}{4}, 2\frac{1}{4}, 6\frac{3}{4}, 20\frac{1}{4}$

b $u_{n+1} = u_n + 2000, u_0 = 200\,000$; 200 000, 202 000, 204 000, 206 000, 208 000

c $u_{n+1} = u_n - 7, u_0 = 100$; 100, 93, 86, 79, 72

d $u_{n+1} = 2u_n - 50, u_0 = \$200$; \$200, \$350, \$650, \$1250, \$2450

EXERCISE 5.3

1 a $u_{n+1} = u_n + 3, \quad u_1 = 1$

b $u_{n+1} = u_n - 4, \quad u_1 = 12$

2 a $u_{n+1} = u_n + 5, \quad u_1 = -24$

b $u_{n+1} = u_n + 11, \quad u_1 = 55$

3 The sequence is −8, −13, −18, −23, −28, ...

The first-order recurrence relation is $u_{n+1} = u_n - 5, u_1 = -8$.

4 The sequence is $4\frac{1}{2}, 5, 5\frac{1}{2}, 6, 6\frac{1}{2}, \ldots$

The first-order recurrence relation is $u_{n+1} = u_n + \frac{1}{2}, u_1 = 4\frac{1}{2}$.

5 a $u_{n+1} = 5u_n, \quad u_1 = 1$ **b** $u_{n+1} = -2u_n, \quad u_1 = 4$

6 a $u_{n+1} = 5u_n, \quad u_1 = 3$

b $u_{n+1} = -\frac{1}{2}u_n, \quad u_1 = 200$

7 a $u_{n+1} = 5u_n \quad u_1 = 2$ **b** $u_{n+1} = 4u_n \quad u_1 = -3$

8 a $u_{n+1} = 2u_n \quad u_1 = 3$ **b** $u_{n+1} = 3u_n \quad u_1 = -2$

9 a $u_{n+1} = u_n + 2 \quad u_1 = 1$

b $u_{n+1} = u_n + 7 \quad u_1 = 3$

c $u_{n+1} = u_n - 7 \quad u_1 = 12$

d $u_{n+1} = u_n - 0.5 \quad u_1 = 1$

e $u_{n+1} = u_n + 4 \quad u_1 = 2$

f $u_{n+1} = u_n + 4 \quad u_1 = -2$

g $u_{n+1} = u_n - 5 \quad u_1 = 6$

h $u_{n+1} = u_n + 6.5 \quad u_1 = 4$

10 B

11 a $u_{n+1} = u_n - 1 \quad u_1 = -4$

b $u_{n+1} = u_n + 2 \quad u_1 = 3$

c $u_{n+1} = u_n + 3 \quad u_1 = -1$

d $u_{n+1} = u_n - 2 \quad u_1 = 4$

12 A

13 a $u_{n+1} = u_n - 2, \quad u_1 = 12, \quad n = 1, 2, 3, \ldots$

b $u_{n+1} = u_n + 1, \quad u_1 = 1, \quad n = 1, 2, 3, \ldots$

14 a $u_{n+1} = 2u_n \quad u_1 = 5$

b $u_{n+1} = 6u_n \quad u_1 = 1$

c $u_{n+1} = -u_n \quad u_1 = -3$

d $u_{n+1} = 4u_n \quad u_1 = -3$

e $u_{n+1} = 3u_n \quad u_1 = 2$

f $u_{n+1} = -u_n \quad u_1 = 5$

g $u_{n+1} = 4u_n \quad u_1 = -2$

h $u_{n+1} = -3u_n \quad u_1 = 5$

15 D

16 a $u_{n+1} = 3u_n \quad u_1 = 2$

b $u_{n+1} = 4u_n \quad u_1 = -3$

c $u_{n+1} = -u_n \quad u_1 = 0.5$

d $u_{n+1} = 5u_n \quad u_1 = 3$

e $u_{n+1} = 2u_n \quad u_1 = -5$

f $u_{n+1} = -3u_n \quad u_1 = 0.1$

17 A

18 a $u_{n+1} = -u_n, \quad u_1 = -4, \quad n = 1, 2, 3, \ldots$

b $u_{n+1} = -7u_n, \quad u_1 = 2, \quad n = 1, 2, 3, \ldots$

c $u_{n+1} = 3u_n, \quad u_1 = 2, \quad n = 1, 2, 3, \ldots$

d $u_{n+1} = -u_n, \quad u_1 = -1, \quad n = 1, 2, 3, \ldots$

e $u_{n+1} = 2u_n, \quad u_1 = 5, \quad n = 1, 2, 3, \ldots$

19 a $u_{n+1} = u_n + 6.5, \quad u_1 = 4, \quad n = 1, 2, 3, \ldots$

b $u_{n+1} = u_n - 5, \quad u_1 = 15, \quad n = 1, 2, 3, \ldots$

c $u_{n+1} = u_n + 4, \quad u_1 = 1, \quad n = 1, 2, 3, \ldots$

20 a $u_{n+1} = 2u_n, \quad u_1 = 4, \quad n = 1, 2, 3, \ldots$

b $u_{n+1} = 4u_n, \quad u_1 = -3, \quad n = 1, 2, 3, \ldots$

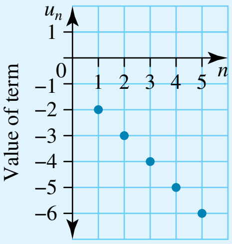
c $u_{n+1} = -6u_n, \quad u_1 = 2, \quad n = 1, 2, 3, \ldots$

d $u_{n+1} = 8u_n, \quad u_1 = -0.1, \quad n = 1, 2, 3, \ldots$

e $u_{n+1} = -10u_n, \quad u_1 = 3.5, \quad n = 1, 2, 3, \ldots$

EXERCISE 5.4

1



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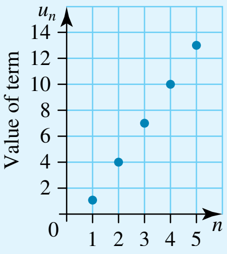

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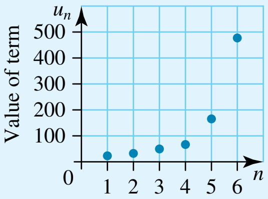

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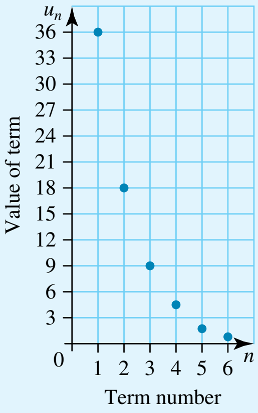


5 $u_{n+1} = u_n - 2 \qquad u_1 = 14$

6 $u_{n+1} = u_n + 4 \qquad u_1 = 6$

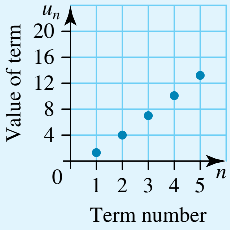
7 $u_{n+1} = 2u_n \qquad u_1 = 1.5$

8 $u_{n+1} = 2u_n \qquad u_1 = 3$

9 B

10 B

11 a



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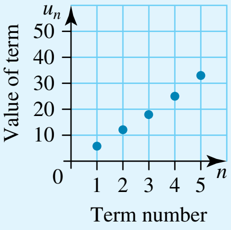

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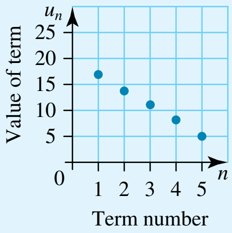
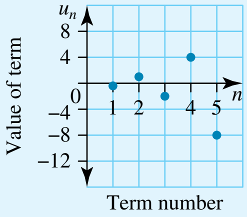


12 a



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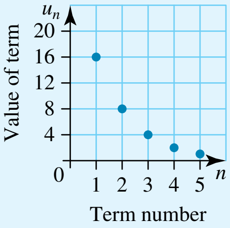

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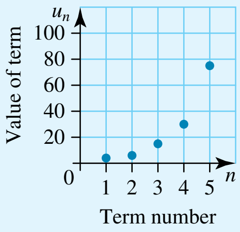
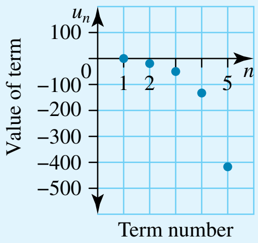


13 a



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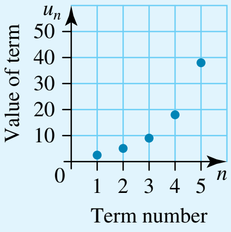

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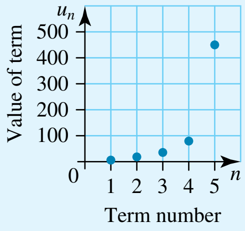
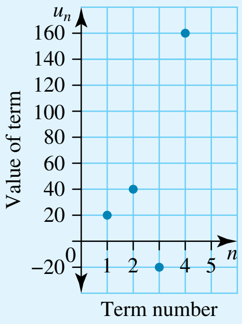


14 a



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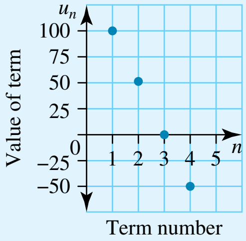

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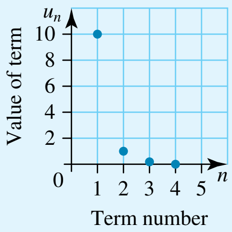

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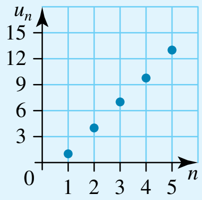

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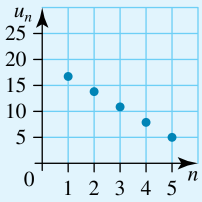

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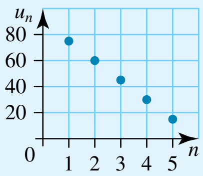

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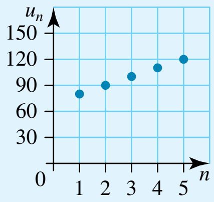

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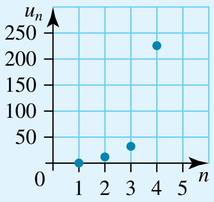

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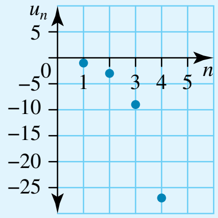

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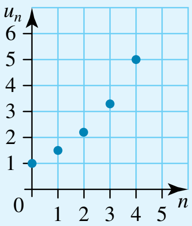

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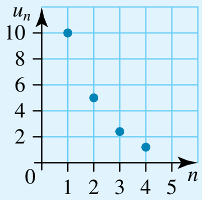


17 a $u_{n+1} = u_n + 3 \qquad u_1 = 2$

b $u_{n+1} = u_n - 20 \qquad u_1 = 90$

c $u_{n+1} = u_n - 4 \qquad u_1 = 8$

18 a $u_{n+1} = 2u_n \qquad u_1 = 5$

b $u_{n+1} = -u_n \qquad u_1 = 6$

c $u_{n+1} = -0.5u_n \qquad u_1 = 8$

19 C

20 $u_{n+1} = 3u_n \qquad u_1 = -1$

21 a **ii**

b **v**

c **iii**

d **iv**

e **vi**

f **i**

22 **i** matches to **d**.

ii matches to **b**.

iii matches to **a**.

iv matches to **c**.

6

Interest and depreciation

6.1 Kick off with CAS

Calculating interest with CAS

A recursion relation links one term in a series to the previous or next term in the same series.

We can use recursion equations to model financial situations, such as the amount of interest accrued in a bank account for a number of consecutive years.

1 \$3000 was placed into a bank account earning simple interest at a rate of 5% per annum.

  a Using CAS, define the recurrence relation $V_{n+1} = V_n + 150$, $V_0 = 3000$ that represents this investment.

  b Use your recurrence relation defined in part a to complete the following table which represents the amount of money in the bank account after n years.

  c Use CAS to plot a graph of the amount of money in the bank after the first 5 years (using the figures in the table).

Years	Amount in bank account (\$)
0	3000
1	
2	
3	
4	
5	

2 \$3000 was placed into a bank account earning compound interest at a rate of 5% per annum.

  a Using CAS, define the recurrence relation $V_{n+1} = 1.05V_n$, $V_0 = 3000$ that represents this investment.

  b Use your recurrence relation defined in part a to complete the following table which represents the amount of money in the bank account after n years.

  c Use CAS to plot a graph of the amount of money in the bank after the first 5 years (using the figures in the table).

Years	Amount in bank account (\$)
0	3000
1	
2	
3	
4	
5	

3 Comment on the shape of the two graphs drawn in questions 1 and 2.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

6.2 Simple interest

studyon

Unit 3

AOS R&FM

Topic 2

Concept 1

Simple interest

Concept summary

Practice questions

People often wish to buy goods and services that they cannot afford to pay for at the time, but which they are confident they can pay for over a period of time. The options open to these people include paying by credit card (usually at a very high interest rate), lay-by (where the goods are paid off over a period of time with no interest charged but no access to or use of the goods until the last payment is made), hire-purchase, or a loan from the bank.

The last two options usually attract what is called **simple interest**. This is the amount of money charged by the financial institution for the use of its money. It is calculated as a percentage of the money borrowed multiplied by the number of periods (usually years) over which the money is borrowed.

As an example, Monica wishes to purchase a television for \$550, but does not currently have the cash to pay for it. She makes an agreement to borrow the money from a bank at 12% p.a. (per year) simple interest and pay it back over a period of 5 years. The amount of interest Monica will be charged on top of the \$550 is:

$$\$550 \times 12\% \times 5 \text{ years which is } \$330.$$

Therefore, Monica is really paying \$550 + \$330 = \$880 for the television.

Besides taking out a loan, you can also make an investment. One type of investment is depositing money into an account with a bank or financial institution for a period of time. The financial institution uses your money and in return adds interest to the account at the end of the period.

Simple interest can be represented by a first-order linear recurrence relation, where V_n represents the value of the investment after n time periods, V_0 is the initial (or starting) amount and d is the amount of interest earned per period:

$$V_{n+1} = V_n + d, d = \frac{V_0 \times r}{100},$$

where V_n represents the value of the investment after n time periods, d is the amount of interest earned per period, V_0 is the initial (or starting) amount and r is the interest rate.

You can also calculate the total amount of a simple interest loan or investment by using:

Total amount of loan or investment = initial amount or principal + interest

$$V_n = V_0 + I$$

Simple interest is the percentage of the amount borrowed or invested multiplied by the number of time periods (usually years). The amount is added to the principal either as payment for the use of the money borrowed or as return on money invested.

$$I = \frac{V_0 r n}{100}$$

I = simple interest charged or earned (\$)

V_0 = principal (money invested or loaned) (\$)

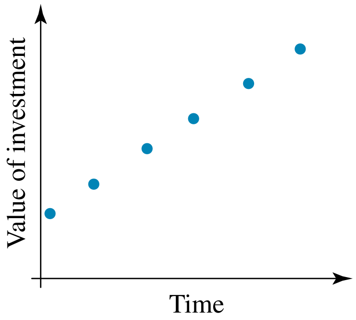
r = rate of interest per period (% per period)

n = the number of periods (years, months, days) over which the agreement operates

In the case of simple interest, the total value of investment increases by the same amount per period. Therefore, if the values of the investment at the end of each time period are plotted, a straight line graph is formed.

Hint: The interest rate, r, and time period, n, must be stated and calculated in the same time terms; for example:

1. 4% per annum for 18 months must be calculated over $1\frac{1}{2}$ years, as the interest rate period is stated in years (per annum);
2. 1% per month for 2 years must be calculated over 24 months, as the interest rate period is stated in months.



WORKED EXAMPLE 1

\$325 is invested in a simple interest account for 5 years at 3% p.a. (per year).

a Set up a recurrence relation to find the value of the investment after n years.

b Use the recurrence relation from part **a** to find the value of the investment at the end of each of the first 5 years.

THINK

WRITE

a 1 Write the formula to calculate the amount of interest earned per period (d).

a $d = \frac{V_0 \times r}{100}$

2 List the values of V_0 and r.

$V_0 = 325,\ r = 3$

3 Substitute into the formula and evaluate.

$$d = \frac{325 \times 3}{100}$$
$$= 9.75$$

4 Use the values of d and V_0 to set up your recurrence relation.

$V_{n+1} = V_n + 9.75,\ V_0 = 325$

b 1 Set up a table to find the value of the investment for up to $n = 5$.

2 Use the recurrence relation from part **a** to complete the table.

b

$n + 1$	V_n (\$)	V_{n+1} (\$)
1	325	325 + 9.75 = 334.75
2	334.75	334.75 + 9.75 = 344.50
3	344.50	344.50 + 9.75 = 354.25
4	354.25	354.25 + 9.75 = 364
5	364	364 + 9.75 = 373.75

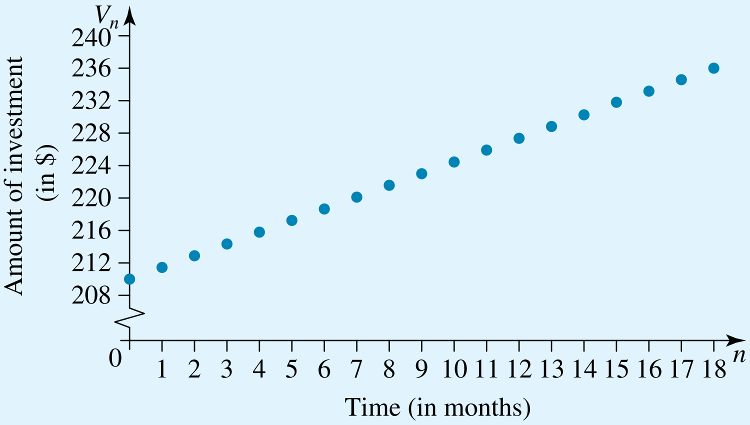
3 Write your answer.	The value of the investment at the end of each of the first 5 years is: $334.75, $344.50, $354.25, $364 and $373.75.

WORKED EXAMPLE 2

Jan invested $210 with a building society in a fixed deposit account that paid 8% p.a. simple interest for 18 months.

a **How much did she receive after the 18 months?**

b **Represent the account balance for each of the 18 months graphically.**

THINK	**WRITE/DRAW**
a **1** Write the simple interest formula.	**a** $I = \frac{V_0 r n}{100}$
2 List the values of V_0, r and n. Check that r and n are in the same time terms. Need to convert 18 months into years.	$V_0 = \$210$ $r = 8\%$ per year $n = 18$ months $= 1\frac{1}{2}$ years
3 Substitute the values of the pronumerals into the formula and evaluate.	$I = \frac{210 \times 8 \times 1.5}{100}$ $= 25.2$
4 Write the answer.	The interest charged is $25.20.
5 Add the interest to the principal (total amount received).	$V_n = V_0 + I$ $= 210 + 25.20$ $= 235.20$
6 Write your answer.	Total amount received at the end of 18 months is $235.20.
b **1** As we are dealing with simple interest, the value of the investment increases each month by the same amount. To find the monthly increase, divide the total interest earned by the number of months.	**b** Increase per month $= \frac{25.20}{18}$ $= \$1.40$
2 Draw a set of axes. Put time (in months) on the horizontal axis and the amount of investment (in $) on the vertical axis. Plot the points: the initial value of investment is $210 and it grows by $1.40 each month. (The last point has coordinates (18, 235.20).)	

Finding V_0, r and n

In many cases we may wish to find the principal, interest rate or period of a loan. In these situations it is necessary to rearrange or transpose the simple interest formula after (or before) substitution, as the following example illustrates.

WORKED EXAMPLE 3 **A bank offers 9% p.a. simple interest on an investment. At the end of 4 years the total interest earned was $215. How much was invested?**

THINK	WRITE
1 Write the simple interest formula.	$I = \frac{V_0 rn}{100}$
2 List the values of I, r and n. Check that r and n are in the same time terms.	$I = \$215$ $r = 9\%$ per year $n = 4$ years
3 Substitute into the formula.	$I = \frac{V_0 \times r \times n}{100}$ $215 = \frac{V_0 \times 9 \times 4}{100}$
4 Make V_0 the subject by multiplying both sides by 100 and dividing both sides by (9×4).	$V_0 = \frac{215 \times 100}{9 \times 4}$ $= 597.22$
5 Write your answer in the correct units.	The amount invested was $597.22.

Transposed simple interest formula

eBook*plus*

Interactivity
Simple interest
int-6461

It may be easier to use the transposed formula when finding V_0, r or n.

To find the principal: $\boldsymbol{V_0 = \frac{100 \times I}{r \times n}}$

To find the interest rate: $\boldsymbol{r = \frac{100 \times I}{V_0 \times n}}$

To find the period of the loan or investment: $\boldsymbol{n = \frac{100 \times I}{V_0 \times r}}$

WORKED EXAMPLE 4 **When $720 is invested for 36 months it earns $205.20 simple interest. Find the yearly interest rate.**

THINK	WRITE
1 Write the simple interest formula, where rate is the subject.	$r = \frac{100 \times I}{V_0 \times n}$
2 List the values of V_0, I and n. n must be expressed in years so that r can be evaluated in % per year.	$V_0 = \$720$ $I = \$205.20$ $n = 36$ months $= 3$ years
3 Substitute into the formula and evaluate.	$r = \frac{100 \times 205.20}{720 \times 3}$ $= 9.5$
4 Write your answer.	The interest rate offered is 9.5% per annum.

WORKED EXAMPLE 5 An amount of \$255 was invested at 8.5% p.a. How long will it take, to the nearest year, to earn \$86.70 in interest?

THINK	WRITE
1 Write the simple interest formula, where time is the subject.	$n = \dfrac{100 \times I}{V_0 \times r}$
2 Substitute the values of V_0, I and r. The rate, r is expressed per annum so time, n, will be evaluated in the same time terms, namely years.	$V_0 = \$255$ $I = \$86.70$ $r = 8.5\%$ p.a.
3 Substitute into the formula and evaluate.	$n = \dfrac{100 \times 86.70}{255 \times 8.5}$ $= 4$
4 Write your answer.	It will take 4 years.

EXERCISE 6.2 Simple interest

PRACTISE

1 **WE1** \$1020 is invested in a simple interest account for 5 years at 8.5% p.a.
  a Set up a recurrence relation to find the value of the investment after n years.
  b Use the recurrence relation from part **a** to find the value of the investment at the end of each of the first 5 years.

2 \$713 is invested in a simple interest account for 5 years at 6.75% p.a.
  a Set up a recurrence relation to find the value of the investment after n years.
  b Use the recurrence relation from part **a** to find the value of the investment at the end of each of the first 5 years.

3 a **WE2** Find the amount to which the investment has grown if \$1020 is invested at $12\frac{1}{2}\%$ p.a. for 2 years.
  b Represent the account balance for each of the first 12 months graphically.

4 a Find the amount to which the investment has grown if \$713 is invested at $6\frac{3}{4}\%$ p.a. for 7 years.
  b Represent the account balance for each of the first 12 months graphically.

5 **WE3** Find the principal invested if simple interest is 7% p.a., earning a total of \$1232 interest over 4 years.

6 Find the principal invested if simple interest is 8% p.a., earning a total of \$651 interest over 18 months.

7 **WE4** Find the interest rate offered, expressed in % p.a., for an investment of \$5000 earning a total of \$1250 interest for 4 years.

8 Find the interest rate offered, expressed in % p.a., for a loan of \$150 with a \$20 interest charge for 2 months.

9 **WE5** Find the period of time (to the nearest month) for which the principal was invested or borrowed.

Loan of \$6000 at simple interest of 7% p.a. with an interest charge of \$630.

10 Find the period of time (to the nearest month) for which the principal was invested or borrowed.

Loan of \$100 at simple interest of 24% p.a. with an interest charge of \$6.

CONSOLIDATE

11 Find the value of the following investments at the end of each year of the investment by using a recurrence relation.

- **a** \$680 for 4 years at 5% p.a.
- **b** \$210 for 3 years at 9% p.a.

12 Find the interest charged or earned on the following loans and investments:

- **a** \$690 loaned at 12% p.a. simple interest for 15 months
- **b** \$7500 invested for 3 years at 1% per month simple interest
- **c** \$25 000 borrowed for 13 weeks at 0.1% per week simple interest
- **d** \$250 invested at $1\frac{3}{4}$% per month for $2\frac{1}{2}$ years.

13 Find the amount to which each investment has grown after the investment periods shown in the following situations:

- **a** \$300 invested at 10% p.a. simple interest for 24 months
- **b** \$750 invested for 3 years at 1% per month simple interest
- **c** \$20 000 invested for 3 years and 6 months at 11% p.a. simple interest.

14 Silvio invested the \$1500 he won in Lotto with an insurance company bond that pays $12\frac{1}{4}$% p.a. simple interest provided he keeps the bond for 5 years.

- **a** What is Silvio's total return from the bond at the end of the 5 years?
- **b** Represent the balance at the end of each year graphically.

15 Jill and John decide to borrow money to improve their boat, but cannot agree which loan is the better value. They would like to borrow \$2550. Jill goes to the Big-4 Bank and finds that they will lend her the money at $11\frac{1}{3}$% p.a. simple interest for 3 years. John finds that the Friendly Building Society will lend the \$2550 to them at 1% per month simple interest for the 3 years.

- **a** Which institution offers the better rate over the 3 years?
- **b** Explain why.

16 The value of a simple interest investment at the end of year 2 is \$3377. At the end of year 3 the investment is worth \$3530.50.

Use a recurrence relation to work out how much was invested.

17 Find the interest rate offered. Express rates in % per annum.

Loan of \$10 000, with a \$2000 total interest charge, for 2 years.

18 Find the period of time (to the nearest month) for which the principal was invested or borrowed.

Investment of \$1000 at simple interest of 5% p.a. earning \$50.

19 **a** Lennie earned \$576 in interest when she invested in a fund paying 9.5% simple interest for 4 years. How much did Lennie invest originally?

b Lennie's sister Lisa also earned \$576 interest at 9% simple interest, but she had to invest it for only 3 years. What was Lisa's initial investment?

20 James needed to earn \$225 interest. He invested \$2500 in an account earning simple interest at a rate of 4.5% p.a. paid monthly. How many months will it take James to achieve his aim?

MASTER

21 Carol has \$3000 to invest. Her aim is to earn \$450 in interest at a rate of 5% p.a. Over what term would she invest?

22 A loan of \$1000 is taken over 5 years. The simple interest is calculated monthly. The total amount repaid for this loan is \$1800. The simple interest rate per year on this loan is closest to:

A 8.9%
B 16%
C 36%
D 5%
E 11.1%

6.3 Compound interest tables

As you have seen in *simple interest* calculations, the principal does not change throughout the life of the investment. Interest is added at the end.

For investments, if interest is added to the initial amount (principal) at the end of an interest-bearing period, then both the interest and the principal earn further interest during the next period, which in turn is added to the balance. This process continues for the life of the investment. The interest is said to be compounded.

The result is that the balance of the account increases at regular intervals and so too does the interest earned.

Let the starting amount be V_n. Then the amount at the start of the next compounding period is V_{n+1}.

Consider \$1000 invested for 4 years at an interest rate of 12% p.a. with interest compounded annually (added on each year). What will be the final balance of this account?

Time period $(n + 1)$	V_n (\$)	Interest (\$)	V_{n+1} (\$)
1	1000	12% of 1000 = 120	1000 + 120 = 1120
2	1120	12% of 1120 = 134.40	1120 + 134.40 = 1254.40
3	1254.40	12% of 1254.40 = 150.53	1254.40 + 150.53 = 1404.93
4	1404.93	12% of 1404.93 = 168.59	1404.93 + 168.59 = 1573.52
5	1573.52	12% of 1573.52 = 188.82	1573.52 + 188.82 = 1762.34

So the balance after 5 years is \$1762.34.

During the total period of an investment, interest may be compounded many times, so a formula has been derived to make calculations easier.

In the above example the principal is increased by 12% each year. That is, the end of year balance = 112% or 1.12 of the start of the year balance.

Now let us look at how this growth or compounding factor of 1.12 is applied in the example.

Time period	Balance ($)
1	$1120 = 1000 \times 1.12 \quad = 1000(1.12)^1$
2	$1254.40 = 1120 \times 1.12 \quad = 1000 \times 1.12 \times 1.12$ $= 1000(1.12)^2$
3	$1404.93 = 1254.40 \times 1.12 = 1000 \times 1.12 \times 1.12 \times 1.12$ $= 1000(1.12)^3$
4	$1573.52 = 1404.93 \times 1.12 = 1000 \times 1.12 \times 1.12 \times 1.12 \times 1.12$ $= 1000(1.12)^4$
5	$1762.34 = 1573.52 \times 1.12 = 1000 \times 1.12 \times 1.12 \times 1.12 \times 1.12 \times 1.12$ $= 1000(1.12)^5$

If this investment continued for n years the final balance would be:

$$1000(1.12)^n = 1000(1 + 0.12)^n = 1000\left(1 + \frac{12}{100}\right)^n.$$

WORKED EXAMPLE 6

Laura invested $2500 for 5 years at an interest rate of 8% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

Time period ($n + 1$)	V_n ($)	Interest ($)	V_{n+1} ($)
1	2500	A% of 2500 = 200	2700
2	B	8% of C = 216	D
3	2916	8% of 2916 = 233.28	3149.28
4	3149.28	8% of 3149.28 = 251.94	E
5	F	8% of 3401.22 = 272.10	3673.32

THINK	WRITE
A: The percentage interest per annum earned on the investment.	A = 8
B: The principal at the start of the second year is the balance at the end of the first.	B = 2700
C: 8% interest is earned on the principal at the start of the second year.	C = B = 2700
D: The balance is the sum of the principal at the start of the year and the interest earned.	D = 2700 + 216 = 2916
E: The final balance is the sum of the principal at the start of the 4th year and the interest earned.	E = 3149.28 + 251.94 = 3401.22
F: The principle at the start of the fifth year is the balance at the end of the fourth year.	F = 3401.22

EXERCISE 6.3 Compound interest tables

PRACTISE

1 **WE6** Fraser invested \$7500 for 4 years at an interest rate of 6% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

Time period (n + 1)	V_n (\$)	Interest (\$)	V_{n+1} (\$)
1	7500	A% of 7500 = 450	7950
2	B	6% of C = 477	D
3	8427	6% of 8427 = 505.62	8932.62
4	8932.62	6% of 8932.62 = 535.96	E
5	F	6% of 9468.58 = 568.11	10

2 Bob invested \$3250 for 4 years at an interest rate of 7.5% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

Time period (n + 1)	V_n (\$)	Interest (\$)	V_{n+1} (\$)
1	3250	A% of 3250 = 243.75	3493.75
2	B	7.5% of C = 262.03	D
3	3755.78	7.5% of 3755.78 = 281.68	4037.46
4	4037.46	7.5% of 4037.46 = 302.81	E
5	F	7.5% of 4340.27 = 325.52	4665.79

CONSOLIDATE

The following information relates to questions 3 to 6. An investment of \$3000 was made over 3 years at an interest rate of 5% p.a. with interest compounding annually.

3 The interest earned in the first year is:

A \$50 **B** \$100 **C** \$150
D \$200 **E** \$300

4 The balance at the end of the first year is:

A \$3000 **B** \$3050 **C** \$3100
D \$3150 **E** \$3200

5 The principal at the start of the second year is:

A \$3300 **B** \$3200 **C** \$3150
D \$3250 **E** \$3100

6 The interest earned during the second year is:

A \$100 **B** \$125 **C** \$150
D \$157.50 **E** \$172.50

7 If \$4500 is invested for 10 years at 12% p.a. with interest compounding annually and the interest earned in the third year was \$677.38, then the interest earned in the fourth year is closest to:

A \$600 **B** \$625 **C** \$650
D \$677 **E** \$759

8 Declan invested \$5750 for 4 years at an interest rate of 8% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

Time period ($n + 1$)	V_n (\$)	Interest (\$)	V_{n+1} (\$)
1	5750	A% of 5750 = 460	6210
2	B	8% of C = 496.80	D
3	6706.80	8% of 6706.80 = 536.54	7243.34
4	7243.34	8% of 7243.34 = 579.47	E
5	F	8% of 7822.81 = 625.82	8448.63

9 Alex invested \$12 000 for 4 years at an interest rate of 7.5% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

Time period ($n + 1$)	V_n (\$)	Interest (\$)	V_{n+1} (\$)
1	12 000	7.5% of 12 000 = 900	12 900
2	12 900	7.5% of 12 900 = A	B
3	C	7.5% of 13 867.50 = 1040.06	14 907.56
4	14 907.56	7.5% of 14 907.56 = 1118.07	D
5	E	7.5% of 16 025.63 = 1201.92	F

10 Sarina invested \$5500 for 3 years at an interest rate of 6.5% p.a. with interest compounded annually. Complete the table shown.

Time period ($n + 1$)	V_n (\$)	Interest (\$)	V_{n+1} (\$)
1	5500	6.5% of 5500 =	
2			
3			
4			
5			

11 Fredrick invested \$15 000 for 5 years at an interest rate of 4.25% p.a. with interest compounded annually. Complete the table shown to find the value of his investment after 5 years.

Time period	Balance (\$)
1	$15\,000 \times 1.0425 = 15\,000(1.0425)^1$ $= 15\,637.5$
2	$15\,637.5 \times 1.0425 = 15\,000 \times 1.0425 \times 1.0425$ $= 15\,000(1.0425)^2$ $= 16\,302.09$
3	
4	
5	

12 Joel invested \$25 000 for 5 years at an interest rate of 6.5% p.a. with interest compounded annually. Complete the table shown to find the value of his investment after 5 years.

Time period	Balance (\$)
1	$25\,000 \times 1.065 = 25\,000(1.065)^1$ $= 26\,625$
2	$26\,625 \times 1.065 = 25\,000 \times 1.065 \times 1.065$ $= 25\,000(1.065)^2$ $= 28\,355.63$
3	
4	
5	

MASTER

13 An investment of \$10 000 was made for 5 years at an interest rate of 5% p.a. with interest compounded quarterly. Complete the table shown to find the value of the investment after 5 years.

Time period	Balance (\$)
1	$10\,000\left(1+\dfrac{0.05}{4}\right)^{1\times4} = 10\,509.45$
2	$10\,000\left(1+\dfrac{0.05}{4}\right)^{2\times4} = 11\,044.86$
3	
4	
5	

14 An investment of \$8500 was made for 5 years at an interest rate of 7.5% p.a. with interest compounded monthly. Complete the table shown to find the value of the investment after 5 years.

Time period	Balance (\$)
1	$8500\left(1+\dfrac{0.075}{12}\right)^{1\times12} = 9159.88$
2	
3	
4	
5	

6.4 Compound interest formula

From the previous exercise we saw that we could write the value of the investment in terms of its previous value. This can be expressed as the recurrence relation:

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$$V_{n+1} = V_n R$$

where V_{n+1} is the amount of the investment 1 time period after V_n, R is the growth or compounding factor $\left(= 1 + \frac{r}{100}\right)$ and r is interest rate per period.

This pattern can be expanded further to write the value of the investment in terms of the initial investment. This is known as the compound interest formula.

$V_n = V_0R^n$ where
V_n = **final or total amount (\$)**
V_0 = **principal (\$)**
R = **growth or compounding factor** $\left(= 1 + \frac{r}{100}\right)$
r = **interest rate per period**
n = **number of interest-bearing periods**

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Note that the compound interest formula gives the *total amount* in an account, not just the interest earned as in the simple interest formula.

To find the total interest compounded, I:

$I = V_n - V_0$ where
V_n = **final or total amount (\$)**
V_0 = **principal (\$)**

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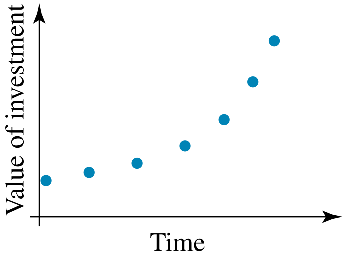
Interactivity
Simple and compound interest
int-6265

If compound interest is used, the value of the investment at the end of each period grows by an increasing amount. Therefore, when plotted, the values of the investment at the end of each period form an exponential curve.

Now let us consider how the formula is used.



WORKED EXAMPLE 7

$5000 is invested for 4 years at 6.5% p.a., interest compounded annually.

a Generate the compound interest formula for this investment.

b Find the amount in the balance after 4 years and the interest earned over this period.

THINK	WRITE
a 1 Write the compound interest formula.	a $V_n = V_0\left(1 + \frac{r}{100}\right)^n$
2 List the values of n, r and P.	$n = 4$ $r = 6.5$ $V_0 = 5000$

3 Substitute into the formula.	$V_n = 5000\left(1 + \frac{6.5}{100}\right)^n$
4 Simplify.	$V_n = 5000(1.065)^n$
b **1** Substitute $n = 4$ into the formula.	**b** $V_4 = 5000(1.065)^4$
2 Evaluate correct to 2 decimal places.	$= \$6432.33$
3 Subtract the principal from the balance.	$I = V_4 - V_0$ $= 6432.33 - 5000$ $= \$1432.33$
4 Write your answer.	The amount of interest earned is \$1432.33 and the balance is \$6432.33.

Non-annual compounding

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Interactivity
Non-annual compounding
int-6462

In Worked example 7, interest was compounded annually. However, in many cases the interest is compounded more often than once a year, for example, quarterly (every 3 months), weekly, or daily. In these situations n and r still have their usual meanings and we calculate them as follows.

Number of interest periods, n = number of years × number of interest periods per year

$$\textbf{Interest rate per period, } r = \frac{\textbf{nominal interest rate per annum}}{\textbf{number of interest periods per year}}$$

Note: Nominal interest rate per annum is simply the annual interest rate advertised by a financial institution.

WORKED EXAMPLE 8

If \$3200 is invested for 5 years at 6% p.a., interest compounded quarterly:

a **find the number of interest-bearing periods, n**

b **find the interest rate per period, r**

c **find the balance of the account after 5 years**

d **graphically represent the balance at the end of each quarter for 5 years. Describe the shape of the graph.**

THINK	WRITE/DRAW
a Calculate n.	**a** $n = 5 \text{ (years)} \times 4 \text{ (quarters)}$ $= 20$
b Convert % p.a. to % per quarter to match the time over which the interest is calculated. Divide r% p.a. by the number of compounding periods per year, namely 4. Write as a decimal.	**b** $r\% = \frac{6\% \text{ p.a.}}{4}$ $= 1.5\% \text{ per quarter}$ $r = 1.5$

c **1** Write the compound interest formula.

c $V_n = V_0\left(1 + \frac{r}{100}\right)^n$

2 List the values of V_0, r and n.

$V_0 = \$3200,\ r = 1.5,\ n = 20$

3 Substitute into the formula.

$V_{20} = \$3200\left(1 + \frac{1.5}{100}\right)^{20}$

4 Simplify.

$= 3200(1.015)^{20}$

5 Evaluate correct to 2 decimal places.

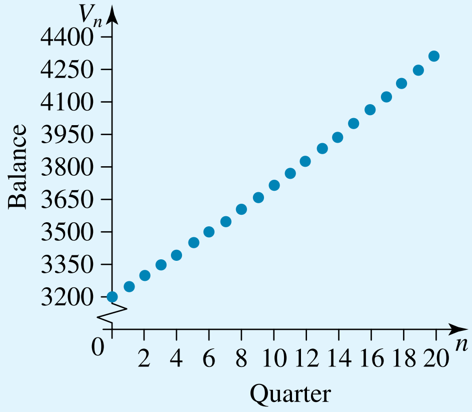
$= \$4309.94$

6 Write your answer.

Balance of account after 5 years is \$4309.94.

d **1** Using CAS, find the balance at the end of each quarter and plot these values on the set of axes. (The first point is (0, 3200), which represents the principal.)

d



2 Comment on the shape of the graph.

The graph is exponential as the interest is added at the end of each quarter and the following interest is calculated on the *new* balance.

The situation often arises where we require a certain amount of money by a future date. It may be to pay for a holiday or to finance the purchase of a car. It is then necessary to know what principal should be invested now in order that it will increase in value to the desired final balance within the time available.

WORKED EXAMPLE 9

Find the principal that will grow to \$4000 in 6 years, if interest is added quarterly at 6.5% p.a.

THINK

WRITE

1 Calculate n (there are 4 quarters in a year).

$n = 6 \times 4$
$= 24$

2 Calculate r.

$r = \frac{6.5}{4}$
$= 1.625$

3 List the value of V_{24}.

$V_{24} = 4000$

4 Write the compound interest formula, substitute and simplify.

$V_{24} = V_0\left(1 + \frac{r}{100}\right)^n$

$4000 = V_0\left(1 + \frac{1.625}{100}\right)^{24}$

$4000 = V_0(1.016\,25)^{24}$

5 Transpose to isolate V_0. $\quad V_0 = \dfrac{4000}{(1.016\,25)^{24}}$

6 Evaluate correct to 2 decimal places. $\quad = 2716.73$

7 Write a summary statement. $\quad$ \$2716.73 would need to be invested.

EXERCISE 6.4 Compound interest formula

PRACTISE

1 **WE7** Find the amount in the account and interest earned after \$2500 is invested for 5 years at 7.5% p.a., interest compounded annually.

2 Find the amount in the account and interest earned after \$6750 is invested for 7 years at 5.25% p.a., interest compounded annually.

3 **WE8** If \$4200 is invested for 3 years at 7% p.a., interest compounded quarterly:

- a find the number of interest-bearing periods, n
- b find the interest rate per period, r
- c find the balance of the account after 3 years
- d graphically represent the balance at the end of each quarter for 3 years. Describe the shape of the graph.

4 If \$7500 is invested for 2 years at 5.5% p.a., interest compounded monthly:

- a find the number of interest-bearing periods, n
- b find the interest rate per period, r
- c find the balance of the account after 2 years
- d graphically represent the balance at the end of each month for 2 years. Describe the shape of the graph.

5 **WE9** Find the principal that will grow to \$5000 in 5 years, if interest is added quarterly at 7.5% p.a.

6 Find the principal that will grow to \$6300 in 7 years, if interest is added monthly at 5.5% p.a.

CONSOLIDATE

For the following questions, use the compound interest formula to calculate the answer, then check your answer using Finance Solver on your CAS.

7 Use the compound interest formula to find the amount, V_n, when:

a $V_0 = \$500, n = 2, r = 8$

b $V_0 = \$1000, n = 4, r = 13$

c $V_0 = \$3600, n = 3, r = 7.5$

d $V_0 = \$2915, n = 5, r = 5.25$.

8 Using a recurrence relation, find: i the balance, and ii the interest earned (interest compounded annually) after:

- a \$2000 is invested for 1 year at 7.5% p.a.
- b \$2000 is invested for 2 years at 7.5% p.a.
- c \$2000 is invested for 6 years at 7.5% p.a.

9 Find the number of interest-bearing periods, n, if interest is compounded:

- a annually for 5 years
- b quarterly for 5 years
- c semi-annually for 4 years
- d monthly for 6 years
- e 6-monthly for $4\frac{1}{2}$ years
- f quarterly for 3 years and 9 months.

10 Find the interest rate per period, r, if the annual rate is:

- **a** 6% and interest is compounded quarterly
- **b** 4% and interest is compounded half-yearly
- **c** 18% and interest is compounded monthly
- **d** 7% and interest is compounded quarterly.

11 \$1500 is invested for 2 years into an account paying 8% p.a. Find the balance if:

- **a** interest is compounded yearly
- **b** interest is compounded quarterly
- **c** interest is compounded monthly
- **d** interest is compounded weekly.
- **e** Compare your answers to parts **a**–**d**

12 Use the recurrence relation $V_{n+1} = 1.045V_n$ to answer the following questions.

- **a** If the balance in an account after 1 year is \$2612.50, what will the balance be after 3 years?
- **b** If the balance in an account after 2 years is \$4368.10, what will the balance be after 5 years?
- **c** If the balance in an account after 2 years is \$6552.15, what was the initial investment?

13 Find the amount that accrues in an account which pays compound interest at a nominal rate of:

- **a** 7% p.a. if \$2600 is invested for 3 years (compounded monthly)
- **b** 8% p.a. if \$3500 is invested for 4 years (compounded monthly)
- **c** 11% p.a. if \$960 is invested for $5\frac{1}{2}$ years (compounded fortnightly)
- **d** 7.3% p.a. if \$2370 is invested for 5 years (compounded weekly)
- **e** 15.25% p.a. if \$4605 is invested for 2 years (compounded daily).

14 The greatest return is likely to be made if interest is compounded:

A annually **B** semi-annually **C** quarterly
D monthly **E** fortnightly

15 If \$12 000 is invested for $4\frac{1}{2}$ years at 6.75% p.a., compounded fortnightly, the amount of interest that would accrue would be closest to:

A \$3600 **B** \$4200 **C** \$5000
D \$12 100 **E** \$16 300

16 Use the compound interest formula to find the principal, V_0, when:

- **a** $V_n = \$5000,\ r = 9,\ n = 4$
- **b** $V_n = \$2600,\ r = 8.2,\ n = 3$
- **c** $V_n = \$3550,\ r = 1.5,\ n = 12$
- **d** $V_n = \$6661.15,\ r = 0.8,\ n = 36$

MASTER

17 Find the principal that will grow to:

- **a** \$3000 in 4 years, if interest is compounded 6-monthly at 9.5% p.a.
- **b** \$2000 in 3 years, if interest is compounded quarterly at 9% p.a.
- **c** \$5600 in $5\frac{1}{4}$ years, if interest is compounded quarterly at 8.7% p.a.
- **d** \$10 000 in $4\frac{1}{4}$ years, if interest is compounded monthly at 15% p.a.

18 Find the interest accrued in each case in question **17**.

6.5 Finding rate or time for compound interest

Sometimes we know how much we can afford to invest as well as the amount we want to have at a future date. Using the compound interest formula we can calculate the interest rate that is needed to increase the value of our investment to the amount we desire. This allows us to 'shop around' various financial institutions for an account which provides the interest rate we want.

We must first find the interest rate per period, r, and convert this to the corresponding nominal rate per annum.

WORKED EXAMPLE 10

Find the interest rate per annum (correct to 2 decimal places) that would enable an investment of \$3000 to grow to \$4000 over 2 years if interest is compounded quarterly.

THINK	WRITE
1 List the values of V_n, V_0 and n. For this example, n needs to represent quarters of a year and therefore r will be evaluated in % per quarter.	$V_n = \$4000$ $V_0 = \$3000$ $n = 2 \times 4$ $= 8$
2 Write the compound interest formula and substitute the known values.	$V_n = V_0R^n$ $4000 = 3000R^8$
3 Divide V_n by V_0.	$\frac{4000}{3000} = R^8$
4 Obtain R to the power of 1, that is, raise both sides to the power of $\frac{1}{8}$.	$\left(\frac{4}{3}\right)^{\frac{1}{8}} = (R^8)^{\frac{1}{8}} = R$
5 Replace R with $1 + \frac{r}{100}$.	$\left(\frac{4}{3}\right)^{\frac{1}{8}} = 1 + \frac{r}{100}$
6 Isolate r and evaluate.	$\frac{r}{100} = \left(\frac{4}{3}\right)^{\frac{1}{8}} - 1$ $= 0.0366146$ $r = 3.66146$ $r\% = 3.66146\%$ per quarter
7 Multiply r by the number of interest periods per year to get the annual rate (correct to 2 decimal places).	Annual rate = $r\%$ per quarter × 4 = 3.66146% per quarter × 4 = 14.65% per annum
8 Write your answer.	Interest rate of 14.65% p.a. is required, correct to 2 decimal places.

Note: Worked example 10 requires a number of operations to find the solution. This is one of the reasons why most financial institutions use finance software for efficient and error-free calculations. Your CAS has a finance function called **Finance Solver**. This can be used for compound interest calculations as shown in the worked examples in this section. Finance Solver will also be used extensively in the remaining sections of this topic.

Finding time in compound interest

We have seen how the compound interest formula, $V_n = V_0R^n$, where $R = 1 + \frac{r}{100}$ can be manipulated to solve situations where V_n, V_0 and r were unknown.

To find n, the number of interest-bearing periods, that is, to find the time period of an investment, is more difficult. Possible methods to solve for n include:

1. trial and error
2. logarithms
3. using Finance Solver on CAS.

The value obtained for n may be a whole number, but it is more likely to be a decimal. That is, the time required will lie somewhere between two consecutive integers. The smaller of the two integers represents insufficient time for the investment to amount to the balance desired; the larger integer represents more than enough time.

In practice, if this is the case, an investor may choose to:

a. withdraw funds as soon as the final balance is reached, in which case a fee may be imposed for early withdrawal
b. withdraw funds at the first integral value of n after the final balance is reached; that is, when the investment matures.

In this section, we will use Finance Solver to calculate the time period of an investment.

WORKED EXAMPLE 11 **How long it will take \$2000 to amount to \$3500 at 8% p.a. with interest compounded annually?**

THINK	WRITE
1 State the values of V_n, V_0 and r.	$V_n = \$3500$, $V_0 = \$2000$ and $r = 8\%$ p.a.
2 Use the Finance Solver on CAS to enter the following values: n (N:) = unknown r (I(%):) = 8 V_0 (PV:) = −2000 V_n (FV:) = 3500 PpY: = 1 CpY: = 1	
3 Solve for n.	$n = 7.27$ years
4 Interest is compounded annually, so n represents years. Raise n to the next whole year.	As the interest is compounded annually, $n = 8$ years.
5 Write your answer.	It will take 8 years for \$2000 to amount to \$3500.

WORKED EXAMPLE 12 **Calculate the number of interest-bearing periods, n, required and hence the time it will take \$3600 to amount to \$5100 at a rate of 7% p.a., with interest compounded quarterly.**

THINK	WRITE
1 State the values of V_n, V_0 and r.	$V_n = \$5100$, $V_0 = \$3600$ and $r = 7\%$ p.a.
2 Enter the following values into the Finance Solver on your CAS: n (N:) = unknown r (I(%):) = 7 V_0 (PV:) = −3600 V_n (FV:) = 5100 PpY: = 4 CpY: = 4	
3 Solve for n.	$n = 20.08$ quarters
4 Write your answer using more meaningful units.	As the interest is compounded annually, $n = 21$ quarters.
5 Answer the question fully.	It will take $5\frac{1}{4}$ years for \$3600 to amount to \$5100.

EXERCISE 6.5 Finding rate or time for compound interest

PRACTISE

1 **WE10** Find the interest rate per annum (correct to 2 decimal places) that would enable an investment of \$4000 to grow to \$5000 over 2 years if interest is compounded quarterly.

2 Find the interest rate per annum (correct to 2 decimal places) that would enable an investment of \$7500 to grow to \$10 000 over 3 years if interest is compounded monthly.

3 **WE11** How long will it take \$3000 to amount to \$4500 at 7% p.a. with interest compounded annually?

4 How long will it take \$7300 to amount to \$10 000 at 7.5% p.a. with interest compounded annually?

5 **WE12** Calculate the number of interest-bearing periods, n, required and hence the time it will take \$4700 to amount to \$6100 at a rate of 9% p.a., with interest compounded quarterly.

6 Calculate the number of interest-bearing periods, n, required and hence the time it will take \$3800 to amount to \$6300 at a rate of 15% p.a., with interest compounded quarterly.

CONSOLIDATE

7 Lillian wishes to have \$24 000 in a bank account after 6 years so that she can buy a new car. The account pays interest at 15.5% p.a. compounded quarterly. The amount (correct to the nearest dollar) that Lillian should deposit in the account now, if she is to reach her target, is:

A \$3720 **B** \$9637 **C** \$10 109
D \$12 117 **E** \$22 320

8 Find the interest rates per annum (correct to 2 decimal places) that would enable investments of:

a \$2000 to grow to \$3000 over 3 years if interest is compounded 6-monthly
b \$12 000 to grow to \$15 000 over 4 years (interest compounded quarterly).

9 Find the interest rates per annum (correct to 2 decimal places) that would enable investments of:

a \$25 000 to grow to \$40 000 over $2\frac{1}{2}$ years (compounded monthly)

b \$43 000 to grow to \$60 000 over $4\frac{1}{2}$ years (compounded fortnightly)

c \$1400 to grow to \$1950 over 2 years (compounded weekly).

10 What is the minimum interest rate per annum (compounded quarterly) needed for \$2300 to grow to at least \$3200 in 4 years?

A 6% p.a. **B** 7% p.a. **C** 9% p.a. **D** 8% p.a. **E** 10% p.a.

11 Use Finance Solver on CAS to find out how long it will take (with interest compounded annually) for:

a \$2000 to amount to \$3173.75 at 8% p.a.
b \$9250 to amount to \$16 565.34 at 6% p.a.

12 Use Finance Solver on CAS to find out how long it will take (with interest compounded annually) for:

a \$850 to amount to \$1000 at 7% p.a.
b \$12 000 to amount to \$20 500 at 13.25% p.a.

13 Calculate the number of interest-bearing periods, n, required, and hence the time in more meaningful terms when:

a $V_n = \$2100$, $V_0 = \$1200$, $r = 3\%$ per half-year
b $V_n = \$13\,500$, $V_0 = \$8300$, $r = 2.5\%$ per quarter
c $V_n = \$16\,900$, $V_0 = \$9600$, $r = 1\%$ per month.

14 Calculate the number of interest-bearing periods, n, required, and hence the time in more meaningful terms when:

a $V_n = \$24\,000$, $V_0 = \$16\,750$, $r = 0.25\%$ per fortnight
b \$7800 is to amount to \$10 000 at a rate of 8% p.a. (compounded quarterly)
c \$800 is to amount to \$1900 at a rate of 11% p.a. (compounded quarterly).

15 Wanda has invested \$1600 in an account at a rate of 10.4% p.a., interest compounded quarterly. How long will it take to reach \$2200?

16 What will be the least number of interest periods, n, required for \$6470 to grow to at least \$9000 in an account with interest paid at 6.5% p.a. and compounded half-yearly?

A 10 **B** 11 **C** 12 **D** 20 **E** 22

MASTER

17 About how long would it take for:

a \$1400 to accrue \$300 interest at 8% p.a., interest compounded monthly?
b \$8000 to accrue \$4400 interest at 9.6% p.a., interest compounded fortnightly?

18 Jennifer and Dawn each want to save \$15 000 for a car. Jennifer has \$11 000 to invest in an account with her bank which pays 8% p.a., interest compounded quarterly. Dawn's credit union has offered her 11% p.a., interest compounded quarterly.

a How long will it take Jennifer to reach her target?
b How much will Dawn need to invest in order to reach her target at the same time as Jennifer? Assume their accounts were opened at the same time.

6.6 Flat rate depreciation

Many items such as antiques, jewellery or real estate increase in value (appreciate or increase in capital gain) with time. On the other hand, items such as computers, vehicles or machinery decrease in value (depreciate) with time as a result of wear and tear, advances in technology or a lack of demand for those specific items.

The estimated loss in value of assets is called **depreciation**. Each financial year a business will set aside money equal to the depreciation of an item in order to cover the cost of the eventual replacement of that item. The estimated value of an item at any point in time is called its **future value** (or book value).

When the value becomes zero, the item is said to be *written off*. At the end of an item's useful or effective life (as a contributor to a company's income) its future value is then called its **scrap value**.

Future value = cost price − total depreciation to that time
When book value = \$0, then the item is said to be written off.
Scrap value is the book value of an item at the end of its useful life.

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Interactivity
Depreciation: flat rate, reducing balance, unit cost
int-6266

There are 3 methods by which depreciation can be calculated. They are:

1. flat rate depreciation
2. reducing balance depreciation
3. unit cost depreciation.

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Unit 3
AOS R&FM
Topic 1
Concept 3

Using recurrence relations to find depreciation values
Concept summary
Practice questions

Flat rate (straight line) depreciation

If an item depreciates by the **flat rate method**, then its value decreases by a fixed amount each unit time interval, generally each year. This depreciation value may be expressed in dollars or as a percentage of the original cost price.

This method of depreciation may also be referred to as *prime cost depreciation*. Since the depreciation is the same for each unit time interval, the flat rate method is an example of straight line (linear) decay. The relationship can be represented by the recurrence relation:

$$V_{n+1} = V_n - d$$

where V_n is the value of the asset after n depreciating periods and d is the depreciation each time period.

We can also look at the future value of an asset after n periods of depreciation, which can be calculated by:

$$V_n = V_0 - nd$$

We can use this relationship to analyse flat rate depreciation or we can use a depreciation schedule (table) which can then be used to draw a graph of future value against time. The schedule displays the future value after each unit time interval, that is:

Time, n	Depreciation, d	Future value, V_n

WORKED EXAMPLE 13

Fast Word Printing Company bought a new printing press for $15 000 and chose to depreciate it by the flat rate method. The depreciation was 15% of the prime cost price each year and its useful life was 5 years.

a **Find the annual depreciation.**

b **Set up a recurrence relation to represent the depreciation.**

c **Draw a depreciation schedule for the useful life of the press and use it to draw a graph of book value against time.**

d **Generate the relationship between the book value and time and use it to find the scrap value.**

THINK

a **1** State the cost price.

2 Find the depreciation rate as 15% of the prime cost price.

3 Write your answer.

b Write the recurrence relation for flat rate depreciation and substitute in the value for *d*.

c **1** Draw a depreciation schedule for 0–5 years, using depreciation of 2250 each year and a starting value of $15 000.

2 Draw a graph of the tabled values for future value against time.

WRITE/DRAW

a $V_0 = 15\,000$

$$d = V_0 \times \frac{r}{100}$$

$$= 15\,000 \times \frac{15}{100}$$

$$= 2250$$

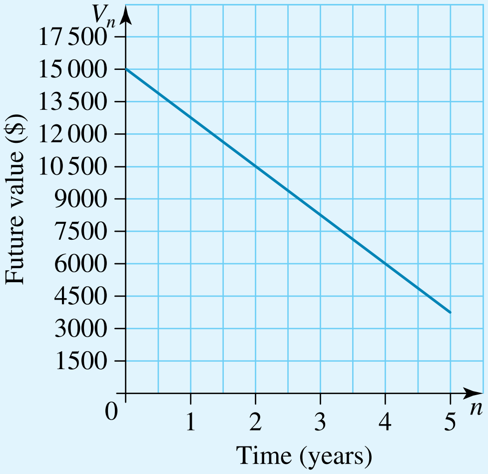
Annual depreciation is $2250.

b $V_{n+1} = V_n - d$

$V_{n+1} = V_n - 2250$

c

Time, *n* (years)	Depreciation, *d* ($)	Future value, V_n ($)
0		15 000
1	2250	12 750
2	2250	10 500
3	2250	8 250
4	2250	6 000
5	2250	3 750



d **1** Set up the equation: $V_n = V_0 - nd$. State d and V_0.	**d** $d = 2250$ $V_0 = 15\,000$ $V_n = 15\,000 - 2250n$
2 The press is scrapped after 5 years so substitute $n = 5$ into the equation.	$V_5 = 15\,000 - 2250(5)$ $= 15\,000 - 11\,250$ $= 3750$
3 Write your answer.	The scrap value is $3750.

The depreciation schedule gives the scrap value, as can be seen in Worked example 13. So too does a graph of book value against time, since it is only drawn for the item's useful life and its end point is the scrap value.

Businesses need to keep records of depreciation for tax purposes on a year-to-year basis. What if an individual wants to investigate the rate at which an item has depreciated over many years? An example is the rate at which a private car has depreciated. If a straight line depreciation model is chosen, then the following example demonstrates its application.

WORKED EXAMPLE 14

Jarrod bought his car 5 years ago for $15 000. Its current market value is $7500. Assuming straight line depreciation, find:

a the car's annual depreciation rate

b the relationship between the future value and time, and use it to find when the car will have a value of $3000.

THINK	WRITE
a **1** Find the total depreciation over the 5 years and thus the rate of depreciation.	**a** Total depreciation = cost price − current value = $15 000 − $7500 = $7500 $\text{Rate of depreciation} = \frac{\text{total depreciation}}{\text{number of years}}$ $= \frac{\$7500}{5 \text{ years}}$ = $1500 per year
2 Write your answer.	The annual depreciation rate is $1500.
b **1** Set up the future value equation.	**b** $V_n = V_0 - nd$ $V_n = 15\,000 - 1500n$
2 Use the equation and substitute $V_n = \$3000$ and transpose the equation to find n.	When $V_n = 3000$, $3000 = 15\,000 - 1500n$ $-1500n = 3000 - 15\,000$ $-1500n = -12\,000$ $n = \frac{-12\,000}{-1500}$ $= 8$

3 Write your answer.	The depreciation equation for the car is $V_n = 15\,000 - 1500n$. The future value will reach \$3000 when the car is 8 years old.

EXERCISE 6.6 Flat rate depreciation

PRACTISE

1 **WE13** A mining company bought a vehicle for \$25 000 and chose to depreciate it by the flat rate method. The depreciation was 15% of the cost price each year and its useful life was 5 years.

- a Find the annual depreciation.
- b Set up a recurrence relation to represent the depreciation.
- c Draw a depreciation schedule for the item's useful life and draw a graph of future value against time.
- d Find the relationship between future value and time. Use it to find the scrap value.

2 Write the future value functions for the following items.

- a A \$30 000 car that is depreciated by 20% p.a. flat rate
- b A \$2000 display unit depreciated by 10% of its prime cost
- c A \$6000 piano depreciated by \$500 p.a.

3 **WE14** For the situations described below, and using a straight line depreciation model, find:

- i the annual rate of depreciation
- ii the relationship between the future value and time and use it to find at what age the item will be written off, that is, have a value of \$0.
- a A car purchased for \$50 000 with a current value of \$25 000; it is now 5 years old.
- b A stereo unit bought for \$850 seven years ago; it now has a current value of \$150.
- c A refrigerator with a current value of \$285 bought 10 years ago for \$1235

4 A second-hand car is currently on sale for \$22 500. Its value is expected to *depreciate* by \$3200 per year.

- a Write an equation that will predict the car's price (V_n) in the future (n).
- b Work out the expected value of the car after 5 years.

CONSOLIDATE

5 All Clean carpet cleaners bought a cleaner for \$10 000 and chose to depreciate it by the flat rate method. The depreciation was 15% of the cost price each year and its useful life was 5 years.

- a Find the annual depreciation.
- b Draw a depreciation schedule for the item's useful life and draw a graph of future value against time.

For the situations outlined in questions 6 and 7:

- a draw a depreciation schedule for the item's useful life and draw a graph of future value against time
- b find the relationship between future value and time. Use it to find the scrap value.

6 A farming company chose to depreciate a tractor by the prime cost method and the annual depreciation was \$4000. The tractor was purchased for \$45 000 and its useful life was 10 years.

7 A winery chose to depreciate a corking machine, that cost \$13 500 when new, by the prime cost method. The annual depreciation was \$2000 and its useful life was 6 years.

For the situations outlined in questions **8** and **9**:

a find the annual depreciation

b set up a recurrence relation to represent the depreciation

c draw a depreciation schedule for the item's useful life and draw a graph of future value against time

d generate the relationship between future value and time. Use it to find how long it will take for the item to reach its scrap value.

8 Machinery is bought for \$7750 and depreciated by the flat rate method. The depreciation is 20% of the cost price each year and its scrap value is \$1550.

9 An excavation company buys a digger for \$92 000 and depreciates it by the flat rate method. The depreciation is 15% of the cost price per year and its scrap value is \$9200.

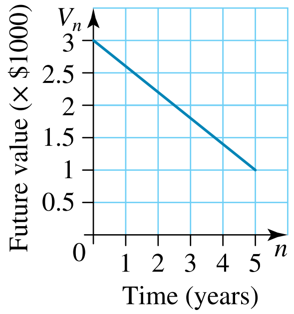
10 Each of the following graphs represents the flat rate depreciation of four particular items. In each case determine:

i the cost price of the item

ii the annual depreciation

iii the time taken for the item to reach its scrap value or to be written off.

a



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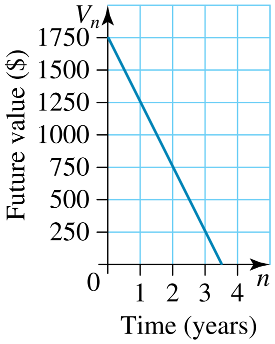

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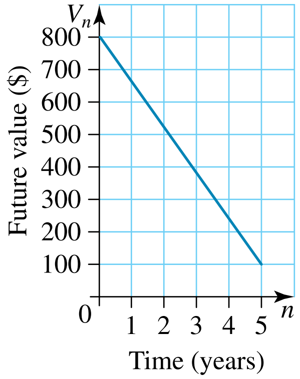

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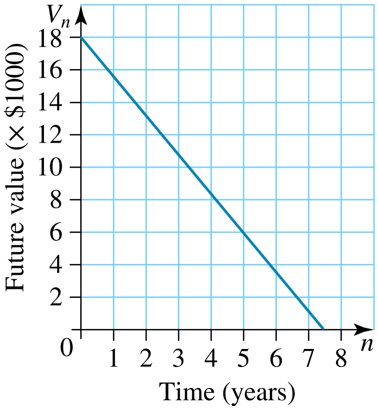


11 A 1-tonne truck, bought for \$31 000, was depreciated using the flat rate method. If the scrap value of \$5000 was reached after 5 years, the annual depreciation would be:

A \$31

B \$1000

C \$5200

D \$6200

E \$26 000

12 The depreciation of a piece of machinery is given by the equation, $V_n = 6000 - 450n$. The machinery will have a future value of \$2400 after:

A 7 years
B 8 years
C 9 years
D 10 years
E 24 years

13 The depreciation of a computer is given by the equation, $V_n = 3450 - 280n$. After how many years will the computer have a future value of \$1770?

14 Listed below are the depreciation equations for 5 different items. Which item would be written off in the least amount of time?

A $V_n = 7000 - 650n$
B $V_n = 7000 - 750n$
C $V_n = 6000 - 650n$
D $V_n = 6000 - 750n$
E $V_n = 6000 - 850n$

MASTER

15 A business buys two different photocopiers at the same time. One costs \$2200 and is to be depreciated by \$225 per annum. It also has a scrap value of \$400. The other costs \$3600 and is to be depreciated by \$310 per annum. This one has a scrap value of \$500.

a Which machine would need to be replaced first?
b How much later would the other machine need to be replaced?

16 A car valued at \$20 000 was bought 5 years ago for \$45 000. The straight line depreciation model is represented by:

A $V_n = 45\,000 - 20\,000n$
B $V_n = 45\,000 - 5000n$
C $V_n = 45\,000 - 4000n$
D $V_n = 20\,000n$
E $V_n = 45\,000 - 25\,000n$

6.7 Reducing balance depreciation

If an item depreciates by the **reducing balance depreciation** method then its value decreases by a fixed rate each unit time interval, generally each year. This rate is a percentage of the previous value of the item.

Reducing balance depreciation is also known as *diminishing value depreciation*.

Reducing balance depreciation can be expressed by the recurrence relation:

$$V_{n+1} = RV_n$$

where V_n is the value of the asset after n depreciating periods and $R = 1 - \frac{r}{100}$, where r is the depreciation rate.

WORKED EXAMPLE 15

Suppose the new \$15 000 printing press considered in Worked example 13 was depreciated by the reducing balance method at a rate of 20% p.a. of the previous value.

a **Generate a depreciation schedule using a recurrence relation for the first 5 years of work for the press.**

b **What is the future value after 5 years?**

c **Draw a graph of future value against time.**

THINK | **WRITE/DRAW**

a 1 Calculate the value of R.

a
$$\begin{aligned} R &= 1 - \frac{r}{100} \\ &= 1 - \frac{20}{100} \\ &= 0.8 \end{aligned}$$

2 Write the recurrence relation for reducing balance depreciation and substitute in the known information.

$$V_{n+1} = RV_n$$
$$V_{n+1} = 0.8V_n,\ V_0 = 15\,000$$

3 Use the recurrence relation to calculate the future value for the first 5 years (up to $n = 5$).

$$\begin{aligned} V_1 &= 0.8V_0 \\ &= 0.8 \times 15\,000 \\ &= 12\,000 \\ V_2 &= 0.8V_1 \\ &= 0.8 \times 12\,000 \\ &= 9600 \\ V_3 &= 0.8V_2 \\ &= 0.8 \times 9600 \\ &= 7680 \\ V_4 &= 0.8V_3 \\ &= 0.8 \times 7680 \\ &= 6144 \\ V_5 &= 0.8V_4 \\ &= 0.8 \times 6144 \\ &= 4915.2 \end{aligned}$$

4 Draw the depreciation schedule.

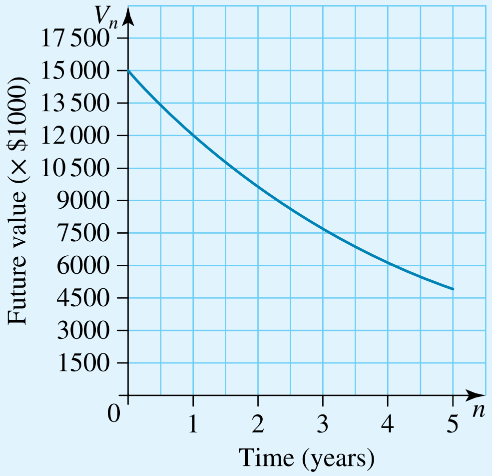
Time, n (years)	Future value, V_n (\$)
0	15 000
1	12 000
2	9600
3	7680
4	6144
5	4915.20

b State the future value after 5 years from the depreciation schedule.

b The future value of the press after 5 years will be \$4915.20.

c Draw a graph of the future value against time.

c



It is clear from the graph and the schedule that the reducing balance depreciation results in greater depreciation during the early stages of the asset's life (the future value drops more quickly at the start since the annual depreciation falls from \$3000 in year 1 to \$1228.80 in year 4).

Currently, the Australian Taxation Office allows depreciation of an asset as a tax deduction. This means that the annual depreciation reduces the amount of tax paid by a business in that year. The higher the depreciation, the greater the tax benefit. Therefore, depreciating an asset by the reducing balance method allows a greater tax benefit for a business in the beginning of an asset's life rather than towards the end. In contrast, flat rate depreciation remains constant throughout the asset's life. People have a choice as to whether they depreciate an item by the flat rate or reducing balance methods, but once a method is applied to an article it cannot be changed for the life of that article. The percentage depreciation rates, which are set by the Australian Taxation Office, vary from one item to another but for each item the rate applied for the reducing balance method is greater than that for the flat rate method.

Let us compare depreciation for both methods.

WORKED EXAMPLE 16

A transport business has bought a new bus for \$60 000. The business has the choice of depreciating the bus by a flat rate of 20% of the cost price each year or by 30% of the previous value each year.

a **Generate depreciation schedules using both methods for a life of 5 years.**

b **Draw graphs of the future value against time for both methods on the same set of axes.**

c **After how many years does the reducing balance future value become greater than the flat rate future value?**

THINK

a **1** Calculate the flat rate depreciation per year.

2 Generate a flat rate depreciation schedule for 0–5 years.

WRITE/DRAW

a $d = 20\%$ of \$60 000

$= \$12\,000$ per year

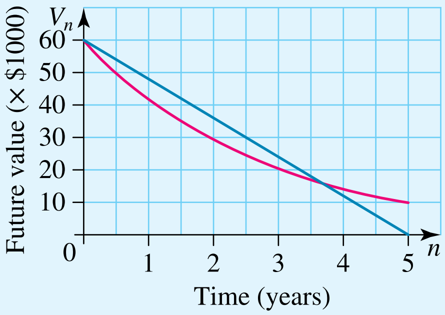
Time, n (years)	Depreciation, d (\$)	Future value, V_n (\$)
0	—	60 000
1	12 000	48 000
2	12 000	36 000
3	12 000	24 000
4	12 000	12 000
5	12 000	0

3 Generate a reducing balance depreciation schedule. Annual depreciation is 30% of the previous value, so to calculate the future value multiply the previous value by 0.7. Continue to calculate the future value for a period of 5 years.

Time, n (years)	Future value, V_n (\$)
0	60 000
1	$60\,000 \times 0.7 = 42\,000$
2	$42\,000 \times 0.7 = 29\,400$
3	$29\,400 \times 0.7 = 20\,580$
4	$20\,580 \times 0.7 = 14\,406$
5	$14\,406 \times 0.7 = 10\,084.20$

b Draw graphs using values for V_n and n from the schedules. In this instance the blue line is the flat rate value and the pink curve is the reducing balance value.

b



c Look at the graph to see when the reducing balance curve lies above the flat rate line. State the first whole year after this point of intersection.

c The future value for the reducing balance method is greater than that of the flat rate method after 4 years.

We can write a general formula for reducing balance depreciation which is similar to the compound interest formula, except that the rate is subtracted rather than added to 1.

The reducing balance depreciation formula is:

$$V_n = V_0 R^n$$

V_n = book value after time, n

R = rate of depreciation $\left(= 1 - \frac{r}{100}\right)$

V_0 = cost price

n = time since purchase

That is, given the cost price and depreciation rate we can find the future value (including scrap value) of an article at any time after purchase.

Let us now see how we can use this formula.

WORKED EXAMPLE 17

The printing press from Worked example 13 was depreciated by the reducing balance method at 20% p.a. What will be the future value and total depreciation of the press after 5 years if it cost \$15 000 new?

THINK	WRITE
1 State V_0, r and n.	$V_0 = 15\,000,\ r = 20,\ n = 5$
2 Calculate the value of R.	$R = 1 - \frac{20}{100}$ $= 0.8$

3 Substitute into the depreciation formula and simplify.	$V_n = V_0R^n$ $V_5 = 15\,000(0.8)^5$
4 Evaluate.	$= 4915.2$
5 Total depreciation is: cost price − future value.	Total depreciation $= V_0 - V_5$ $= 15\,000 - 4915.2$ $= 10\,084.8$
6 Write a summary statement.	The future value of the press after 5 years will be \$4915.20 and its total depreciation will be \$10 084.80.

Effective life

The situation may arise where the scrap value is known and we want to know how long it will be before an item reaches this value; that is, its useful or **effective life**. So, in the reducing balance formula $V_n = V_0R^n$, n is needed.

WORKED EXAMPLE 18

A photocopier purchased for \$8000 depreciates by 25% p.a. by the reducing balance method. If the photocopier has a scrap value of \$1200, how long will it be before this value is reached?

THINK	WRITE
1 State the values of V_n, V_0 and r.	$V_n = \$1200,\ V_0 = \$8000$ and $r = 25\%$
2 Calculate the value of R.	$R = 1 - \frac{25}{100}$ $= 0.75$
3 Substitute the values of the pronumerals into the formula and simplify.	$V_n = V_0R^n$ $1200 = 8000 \times (0.75)^n$ $0.15 = (0.75)^n$
4 Use CAS to find the value of n.	$n = 6.59$ years
5 Interest is compounded annually, so n represents years. Raise n to the next whole year.	As the depreciation is calculated once a year, $n = 7$ years.
6 Write your answer.	It will take 7 years for the photocopier to reach its scrap value.

EXERCISE 6.7 Reducing balance depreciation

PRACTISE

1 **WE15** A laptop was bought new for \$1500 and it depreciates by the reducing balance method at a rate of 17% p.a. of the previous book value.

- a Generate a depreciation schedule using a recurrence relation for the first 5 years.
- b What is the future value after 5 years?
- c Draw a graph of future value against time.

2 A road bike was bought new for \$3300 and it depreciates by the reducing balance method at a rate of 13% p.a. of the previous value.

- **a** Generate a depreciation schedule using a recurrence relation for the first 5 years.
- **b** What is the future value after 5 years?
- **c** Draw a graph of future value against time.

3 **WE16** A taxi company bought a new car for \$29 990. The company has the choice of depreciating the car by a flat rate of 20% of the cost price each year or by 27% of the previous value each year.

- **a** Generate depreciation schedules using both methods for a life of 5 years.
- **b** Draw graphs of the future value against time for both methods on the same axis.
- **c** After how many whole years does the reducing balance future value become greater than the flat rate future value?

4 A sailing rental company bought a new catamaran for \$23 000. The company has the choice of depreciating the catamaran by a flat rate of 20% of the cost price each year or by 28% of the previous value each year.

- **a** Generate depreciation schedules using both methods for a life of 5 years.
- **b** Draw graphs of the future value against time for both methods on the same axis.
- **c** After how many whole years does the reducing balance future value become greater than the flat rate future value?

5 **WE17** Using the reducing balance formula, find V_n (correct to 2 decimal places) given $V_0 = 45\,000$, $r = 15$, $n = 6$.

6 Using the reducing balance formula, find V_n (correct to 2 decimal places) given $V_0 = 2675$, $r = 22.5$, $n = 5$.

7 **WE18** Use the Finance Solver to find n (correct to 2 decimal places), given $V_n = \$900$, $V_0 = \$4500$, $r = 25$.

8 Use the Finance Solver to find n (correct to 2 decimal places), given $V_n = \$1500$, $V_0 = \$7600$, $r = 15$.

CONSOLIDATE

9 A farming company chose to depreciate its new \$60 000 bulldozer by the reducing balance method at a rate of 20% p.a. of the previous value.

- **a** Write the recurrence relation that represents this depreciation.
- **b** Draw a depreciation schedule for the first 5 years of the bulldozer's life.
- **c** What is its future value after 5 years?
- **d** Draw a graph of future value against time.

10 A retail store chose to depreciate its new \$4000 computer by the reducing balance method at a rate of 40% p.a. of the previous value.

- **a** Write the recurrence relation that represents this depreciation.
- **b** Draw a depreciation schedule for the first 5 years of the computer's life.
- **c** What is its future value after 5 years?
- **d** Draw a graph of future value against time.

11 A café buys a cash register for \$550. The owner has the choice of depreciating the register by the flat rate method (at 20% of the cost price each year) or the reducing balance method (at 30% of the previous value each year).

a Draw depreciation schedules for both methods for a life of 5 years.
b Draw graphs of future value against time for both methods on the same set of axes.
c After how many whole years does the reducing balance future value become greater than the flat rate future value?

12 Speedy Cabs taxi service has bought a new taxi for \$30 000. The company has the choice of depreciating the taxi by the flat rate method (at $33\frac{1}{3}\%$ of the cost price each year) or the diminishing value method (at 50% of the previous value each year).

a Draw depreciation schedules for both methods for 3 years.
b Draw graphs of future value against time for both methods on the same set of axes.
c After how many whole years does the reducing balance future value become greater than the flat rate future value?

13 Using the reducing balance formula, find V_n (correct to 2 decimal places) given:

a $V_0 = 20\,000$, $r = 20$, $n = 4$ **b** $V_0 = 30\,000$, $r = 25$, $n = 4$.

Check your answers using CAS.

14 A refrigerator costing \$1200 new is depreciated by the reducing balance method at 20% a year. After 4 years its future value will be:

A \$240 **B** \$491.52 **C** \$960 **D** \$1105 **E** \$2488.32

15 The items below are depreciated by the reducing balance method at 25% p.a. What will be the future value and total depreciation of:

a a TV after 8 years, if it cost \$1150 new
b a photocopier after 4 years, if it cost \$3740 new
c carpets after 6 years, if they cost \$7320 new?

16 The items below are depreciated at 30% p.a. by the reducing balance method. What will be the future value and total depreciation of:

a a lawn mower after 5 years, if it cost \$685 new
b a truck after 4 years, if it cost \$32 500 new
c a washing machine after 3 years, if it cost \$1075 new?

17 After 7 years, a new \$3000 photocopier, which devalues by 25% of its value each year, will have depreciated by:

A \$400.45 **B** \$750 **C** \$2250 **D** \$2599.55 **E** \$2750

18 New office furniture valued at \$17 500 is subjected to reducing balance depreciation of 20% p.a. and will reach its scrap value in 15 years. The scrap value will be:

A less than \$300
B between \$300 and \$400
C between \$400 and \$500
D between \$500 and \$600
E between \$600 and \$700

MASTER

19 A new chainsaw bought for \$1250 has a useful life of only 3 years. If it depreciates annually by a 60% reducing balance rate, its scrap value will be:

A \$0 **B** \$60 **C** \$80 **D** \$250 **E** \$270

20 An item is depreciated by using reducing balance depreciation. Use Finance Solver to find n (correct to 2 decimal places), given:

a $V_n = \$3000,\ V_0 = \$40\,000,\ r = 20$ **b** $V_n = \$500,\ V_0 = \$3000,\ r = 30$

6.8 Unit cost depreciation

The flat rate and reducing balance depreciations of an item are based on the age of the item. With the **unit cost method**, the depreciation is based on the possible maximum output (units) of the item. For instance, the useful life of a truck could be expressed in terms of the distance travelled rather than a fixed number of years — for example, 120 000 kilometres rather than 6 years. The actual depreciation of the truck for the financial year would be a measure of the number of kilometres travelled. (The value of the truck decreases by a certain amount for each kilometre travelled.)

The future value over time using unit cost depreciation can be expressed by the recurrence relation:

$$V_{n+1} = V_n - d$$

where V_n is the value of the asset after n outputs and d is the depreciation per output.

WORKED EXAMPLE 19

A motorbike purchased for \$12 000 depreciates at a rate of \$14 per 100 km driven.

a **Set up a recurrence relation to represent the depreciation.**

b **Use the recurrence relation to generate a depreciation schedule for the future value of the bike after it has been driven for 100 km, 200 km, 300 km, 400 km and 500 km.**

THINK	WRITE
a 1 Write the recurrence relation for unit cost depreciation as well as the known information.	**a** $V_{n+1} = V_n - d$ $V_0 = 12\,000,\ d = 14$
2 Substitute the values into the recurrence relation.	$V_{n+1} = V_n - 14,\ V_0 = 12\,000$
b 1 Use the recurrence relation to calculate the future value for the first 5 outputs (up to 500 km).	**b** $V_1 = V_0 - 14$ $= 12\,000 - 14$ $= 11\,986$ $V_2 = V_1 - 14$ $= 11\,986 - 14$ $= 11\,972$ $V_3 = V_2 - 14$ $= 11\,972 - 14$ $= 11\,958$

$V_4 = V_3 - 14$
$= 11958 - 14$
$= 11944$
$V_5 = V_4 - 14$
$= 11944 - 14$
$= 11930$

2 Draw the depreciation schedule.

Distance driven (km)	Outputs (n)	Future value, V_n (\$)
100	1	11 986
200	2	11 972
300	3	11 958
400	4	11 944
500	5	11 930

WORKED EXAMPLE **20**

A taxi is bought for \$31 000 and it depreciates by 28.4 cents per kilometre driven. In one year the car is driven 15 614 km. Find:

a the annual depreciation for this particular year

b its useful life if its scrap value is \$12 000.

THINK	WRITE
a 1 Depreciation amount = distance travelled × rate	**a** depreciation = 15 614 × \$0.284 = \$4434.38
2 Write a summary statement.	Annual depreciation for the year is \$4434.38.
b 1 Total depreciation = cost price − scrap value Distance travelled $= \dfrac{\text{total depreciation}}{\text{rate of depreciation}}$ where rate of depreciation = 28.4 cents/km = \$0.284 per km	**b** Total depreciation = 31 000 − 12 000 = \$19 000 Distance travelled $= \dfrac{19\,000}{0.284}$ = 66 901 km
2 State your answer.	The taxi has a useful life of 66 901 km.

WORKED EXAMPLE **21**

A photocopier purchased for \$10 800 depreciates at a rate of 20 cents for every 100 copies made. In its first year of use 500 000 copies were made and in its second year, 550 000. Find:

a the depreciation each year

b the future value at the end of the second year.

THINK	WRITE
a To find the depreciation, identify the rate and number of copies made. Express the rate of 20 cents per 100 copies in a simpler form of dollars per 100 copies, that is, \$0.20 per 100 copies or $\frac{0.20}{100\text{ copies}}$.	**a** Depreciation = copies made × rate $\text{depreciation}_{\text{1st year}} = 500\,000 \times \frac{0.20}{100\text{ copies}}$ $= \$1000$ Depreciation in the first year is \$1000. $\text{depreciation}_{\text{2nd year}} = 550\,000 \times \frac{0.20}{100\text{ copies}}$ $= \$1100$ Depreciation in the second year is \$1100.
b Future value = cost price − total depreciation	**b** Total depreciation after 2 years = 1000 + 1100 = \$2100 Book value = 10 800 − 2100 = \$8700

The future value after *n* outputs using unit cost depreciation can be expressed as:

$$V_n = V_0 - nd$$

where V_n is the value of the asset after n outputs and d is the depreciation per output.

WORKED EXAMPLE 22

The initial cost of a vehicle was \$27 850 and its scrap value is \$5050. If the vehicle needs to be replaced after travelling 80 000 km (useful life):

a **find the depreciation rate (depreciation (\$) per km)**

b **find the amount of depreciation in a year when 16 497 km were travelled**

c **set up an equation to determine the value of the car after travelling *n* km**

d **find the future value after it has been used for a total of 60 000 km**

e **set up a schedule table listing future value for every 20 000 km.**

THINK	WRITE
a **1** To find the depreciation rate, first find the total depreciation. Total amount of depreciation = cost price − scrap value	**a** Total amount of depreciation = 27 850 − 5050 = \$22 800
2 Find the rate of depreciation. It is common to express rates in cents per use if less than a dollar.	$\text{Depreciation rate} = \frac{\text{total depreciation}}{\text{total distance travelled}}$ $= \frac{22\,800}{80\,000}$ $= \$0.285 \text{ per km}$ $= 28.5 \text{ cents per km}$

b Find the amount of depreciation using the rate calculated.

Amount of depreciation is always expressed in dollars.

b Amount of depreciation

$$\begin{aligned} &= \text{amount of use} \times \text{rate of depreciation} \\ &= 16\,497 \times 28.5 \\ &= 470\,165 \text{ cents} \\ &= \$4701.65 \end{aligned}$$

c 1 Write the equation for unit cost depreciation after n outputs as well as the known information.

c $V_n = V_0 - nd$

$V_0 = 27\,850,\ d = 0.285$

2 Substitute the values into the equation.

$V_n = 27\,850 - 0.285d$

d 1 Use the equation from part **c** to find the future value when $n = 60\,000$.

d
$$\begin{aligned} V_{60\,000} &= 27\,850 - 0.285 \times 60\,000 \\ &= 27\,850 - 17\,100 \\ &= 10\,750 \end{aligned}$$

2 Write your answer.

The future value after the car has been used for 60 000 km is $10 750.

e Calculate the future value for every 20 000 km of use and summarise in a table.

e

Use, n (km)	Future value, V_n (\$)
0	\$27 850
20 000	$V_1 = 27\,850 - (20\,000 \times 0.285)$ $= \$22\,150$
40 000	$V_2 = 27\,850 - (20\,000 \times 0.285 \times 2)$ $= \$16\,450$
60 000	$V_3 = 27\,850 - (20\,000 \times 0.285 \times 3)$ $= \$10\,750$
80 000	$V_4 = 27\,850 - (20\,000 \times 0.285 \times 4)$ $= \$5\,050$

EXERCISE 6.8 Unit cost depreciation

PRACTISE

1 WE19 A washing machine purchased for $800 depreciates at a rate of 35 cents per wash.

- **a** Set up a recurrence relation to represent the depreciation.
- **b** Use the recurrence relation to generate a depreciation schedule for the future value of the washing machine after each of the first 5 washes.

2 An air conditioning unit purchased for $1400 depreciates at a rate of 65 cents per hour of use.

- **a** Set up a recurrence relation to represent the depreciation.
- **b** Use the recurrence relation to generate a depreciation schedule for the future value of the air conditioning unit at the end of each of the first 5 hours of use.

3 WE20 Below are the depreciation details for a vehicle. Find:

- **a** the annual depreciation in the first year
- **b** the useful life (km).

Purchase price (\$)	Scrap value (\$)	Rate of depreciation (cents/km)	Distance travelled in first year (km)
29 600	12 000	28.5	14 000

4 A taxi was purchased for \$42 000 and depreciates by 25 cents per km driven. During its first year the taxi travelled 64 000 km; during its second year it travelled 56 000 km. Find:

a the depreciation in each of the first 2 years

b how far the car had travelled if its total depreciation was \$20 000.

5 **WE21** A photocopier is bought for \$8600 and it depreciates at a rate of 22 cents for every 100 copies made. In its first year of use, 400 000 copies are made and in its second year, 480 000 copies are made. Find:

a the depreciation for each year

b the future value at the end of the second year.

6 A printing machine was purchased for \$38 000 and depreciated at a rate of \$1.50 per million pages printed. In its first year 385 million pages were printed and 496 million pages were printed in its second year. Find:

a the depreciation for each year

b the future value at the end of the second year.

7 **WE22** A delivery service purchases a van for \$30 000 and it is expected that the van will be written off after travelling 200 000 km. It is estimated that the van will travel 1600 km each week.

a Find the depreciation rate (charge per km).

b Find how long it will take for the van to be written off.

c Set up an equation to determine the value of the van after travelling n kilometre.

d Find the distance travelled for the van to depreciate by \$13 800.

e Find its future value after it has travelled 160 000 kilometres.

f Set up a schedule table for the value of the van for every 20 000 kilometres.

8 A car bought for \$28 395 depreciates at a rate of 23.6 cents for every km travelled. Copy and complete the table below.

Time (years)	Distance travelled, n (km)	Depreciation (\$)	Future value at end of year, V_n (\$)
1	13 290		
2	15 650		
3	14 175		
4	9 674		
5	16 588		

CONSOLIDATE

9 Below are depreciation details for 2 vehicles. In each case find:

i the annual depreciation in the first year

ii the useful life (km).

	Purchase price (\$)	Scrap value (\$)	Rate of depreciation (cents/km)	Distance travelled in first year (km)
a	25 000	10 000	26	12 600
b	21 400	8 000	21.6	13 700

In each situation in questions **10** to **12**, find:

a the annual depreciation in the first year

b the item's useful life.

10 A company buys a \$32 000 car which depreciates at a rate of 23 cents per km driven. It covers 15 340 km in the first year and has a scrap value of \$9500.

11 A new taxi is worth \$29 500 and it depreciates at 27.2 cents per km travelled. In its first year of use it travelled 28 461 km. Its scrap value was \$8200.

12 A photocopier purchased for \$7200 depreciates at a rate of \$1.50 per 1000 copies made. In its first year of use, 620 000 copies were made and in its second year, 540 000 were made. Find:

a the depreciation for each year
b the future value at the end of the second year.

13 A photocopier bought for \$11 300 depreciates at a rate of 2.5 cents for every 10 copies made.

a Set up a recurrence relation to represent the depreciation.
b Copy and complete the table below.

Time (years)	Outputs, n (10 copies)	Annual depreciation (\$)	Future value at end of year, V_n (\$)
1	35 000		
2	42 500		
3	37 620		
4	29 104		
5	38 562		

14 A corking machine bought for \$14 750 depreciates at a rate of \$2.50 for every 100 bottles corked.

a Set up a recurrence relation to represent the depreciation.
b Copy and complete the table below.

Time (years)	Outputs, n (100 bottles corked)	Depreciation (\$)	Future value at end of year, V_n (\$)
1	400		
2	425		
3	467		
4	382.5		
5	430.6		

15 A vehicle is bought for \$25 900 and it depreciates at a rate of 21.6 cents per km driven. After its first year of use, in which it travels 13 690 km, the future value of the vehicle is closest to:

A \$1000 **B** \$3000 **C** \$20 000 **D** \$23 000 **E** \$25 000

In each situation in questions **16** and **17**, find:

a the depreciation for each year
b the future value at the end of the second year.

16 A company van is purchased for \$32 600 and it depreciates at a rate of 24.8 cents per km driven. In its first year of use the van travels 15 620 km and it travels 16 045 km in its second year.

17 A taxi is bought for \$35 099 and it depreciates at a rate of 29.2 cents per km driven. It travels 21 216 km in its first year of use and 19 950 km in its second year.

18 A car is bought for \$35 000 and a scrap value of \$10 000 is set for it. The following three options for depreciating the car are available:

- **i** flat rate of 10% of the purchase price each year
- **ii** 20% p.a. of the reducing balance
- **iii** 25 cents per km driven (the car travels an average of 10 000 km per year).

a Which method will enable the car to reach its scrap value soonest?

b If the car is used in a business the annual depreciation can be claimed as a tax deduction. What would the tax deduction be in the first year of use for each of the depreciation methods?

c How would your answers to part **b** vary for the 5th year of use?

MASTER

19 A machine which was bought for \$8500 was depreciated at the rate of 2 cents per unit produced. By the time the book value had decreased to \$2000, the number of units produced would be:

A 75 000
B 100 000
C 125 000
D 300 000
E 325 000

20 An \$8500 machine depreciates by 2 cents per unit. By the time the machine had depreciated by \$5000, it would have produced:

A 275 000 units
B 250 000 units
C 225 000 units
D 175 000 units
E 150 000 units

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6.9 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

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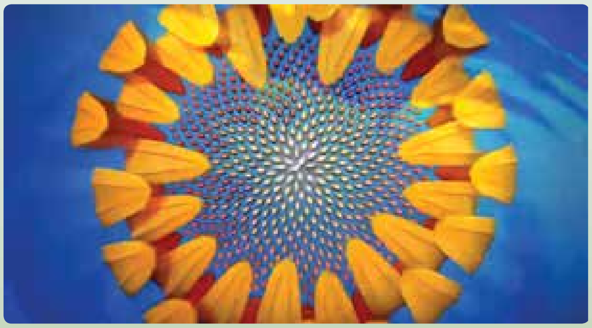
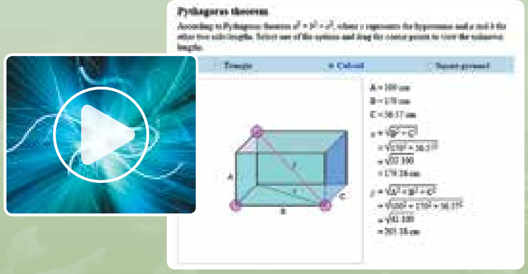
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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6 Answers

EXERCISE 6.2

1 a $V_{n+1} = V_n + 86.7,\ V_0 = 1020$

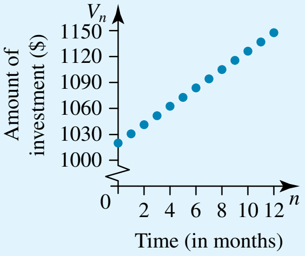
b $1106.70, $1193.40, $1280.10, $1366.80 and $1453.50

2 a $V_{n+1} = V_n + 48.13,\ V_0 = 713$

b $761.13, $809.26, $857.39, $905.52 and $953.65

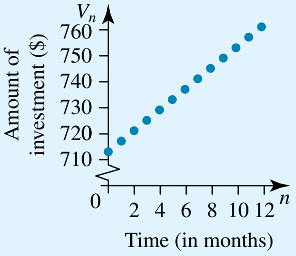
3 a $1275

b



4 a $1049.89

b



5 $4400 **6** $5425

7 6.25% **8** 80%

9 18 months **10** 3 months

11 a $714, $748, $782 and $816

b $228.90, $247.80 and $266.70

12 a $103.50 **b** $2700 **c** $325 **d** $131.25

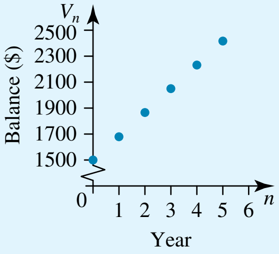
13 a $360

b $1020

c $27 700

14 a $2418.75

b



15 a The Big-4 Bank offers the better rate.

b The Big-4 Bank charges 11$\frac{1}{3}$% p.a. for a loan while The Friendly Building Society charges 12% (= 12 × 1% per month).

16 $3070

17 10%

18 1 year

19 a $1515.79 **b** $2133.33

20 24 months

21 3 years

22 B

EXERCISE 6.3

1 A = 6
B = 7950
C = 7950
D = 8427
E = 9468.58
F = 9468.58

2 A = 7.5
B = 3493.75
C = 3493.75
D = 3755.78
E = 4340.27
F = 4340.27

3 C

4 D

5 C

6 D

7 E

8 A = 8%
B = 6210
C = 6210
D = 6706.80
E = 7882.81
F = 7882.81

9 A = 967.50
B = 13 867.50
C = 13 867.50
D = 16 025.63
E = 16 025.63
F = 17 227.55

10

Time period	Principal, *P* ($)	Interest ($)	Balance ($)
1	5500	357.50	5857.50
2	5857.50	380.74	6238.24
3	6238.24	405.49	6643.72
4	6643.72	431.84	7075.56
5	7075.56	459.91	7535.47

11

Time period	Balance ($)
1	15 637.50
2	16 302.09
3	16 994.93
4	17 717.22
5	18 470.20

12

Time period	Balance ($)
1	26 625
2	28 355.63
3	30 198.74
4	32 161.66
5	34 252.17

13

Time period	Balance ($)
1	10 509.45
2	11 044.86
3	11 607.55
4	12 198.90
5	12 820.37

14

Time period	Balance ($)
1	9 159.88
2	9 870.98
3	10 637.29
4	11 463.09
5	12 353.00

EXERCISE 6.4

1 $3589.07

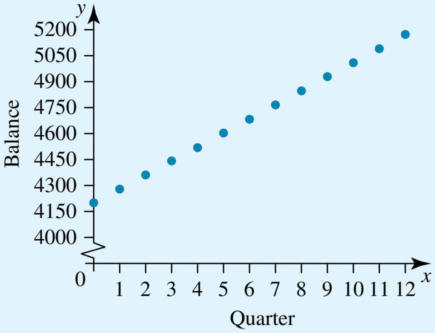
2 $9657.36

3 a 12

b 1.75%

c $5172.05

d



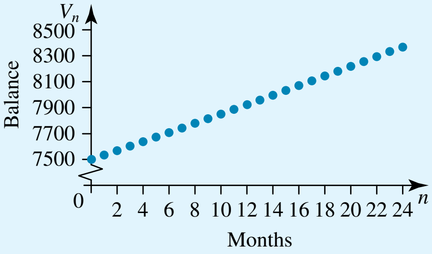
Despite it being difficult to see on the graph, the graph is exponential.

4 a 24

b 0.4583%

c $8369.92

d



Despite it being difficult to see on the graph, the graph is exponential.

5 $3448.40

6 $4290.73

7 a $583.20 **b** $1630.47

c $4472.27 **d** $3764.86

8 a i $2150 **ii** $150

b i $2311.25 **ii** $311.25

c i $3086.60 **ii** $1086.60

9 a 5 **b** 20 **c** 8

d 72 **e** 9 **f** 15

10 a 1.5% **b** 2%

c 1.5% **d** 1.75%

11 a $1749.60 **b** $1757.49

c $1759.33 **d** $1760.24

e The balance increases as the compounding periods become more frequent.

12 a $2852.92

b $4984.72

c $6000

13 a $605.60 **b** $1314.84 **c** $795.77

d $1043.10 **e** $1641.82

14 E

15 B

16 a $3542.13 **b** $2052.54

c $2969.18 **d** $5000

17 a $2069.61 **b** $1531.33

c $3564.10 **d** $5307.05

18 a $930.39 **b** $468.67

c $2035.90 **d** $4692.95

EXERCISE 6.5

1 11.31%

2 9.63%

3 Approx. 6 years

4 Approx. 5 years

5 Approx. 3 years

6 Approx. 3.5 years

7 B

8 a 13.98% **b** 5.62%

9 a 18.95% **b** 7.41% **c** 16.59%

10 8.34%, that is, C

11 a 6 years **b** 10 years

12 a 3 years **b** 5 years

13 a 19, $9\frac{1}{2}$ years **b** 20, 5 years **c** 57, $4\frac{3}{4}$ years

14 a 145, 5 years 15 fortnights

b 13, $3\frac{1}{4}$ years

c 32, 8 years

15 13, $3\frac{1}{4}$ years

16 B

17 a $n = 30$, $2\frac{1}{2}$ years

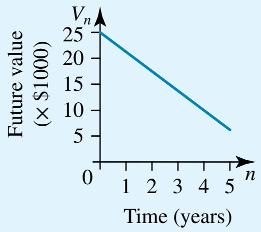
b $n = 119$, 4 years 15 fortnights

18 a 4 years **b** 9718.11

EXERCISE 6.6

1 a \$3750 **b** $V_{n+1} = V_n - 3750$

c



d $V_n = 25\,000 - 3750n$, \$6250

2 a $V_n = 30\,000 - 6000n$

b $V_n = 2000 - 200n$

c $V_n = 6000 - 500n$

3 a i \$5000 per year **ii** 10 years old

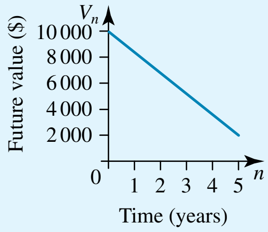
b i \$100 per year **ii** 8.5 years old

c i \$95 per year **ii** 13 years old

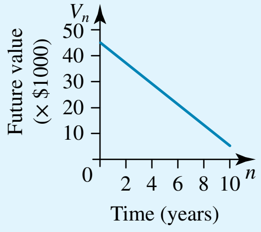
4 a $V_n = 22\,500 - 3200n$ **b** \$6500

5 a \$1500

b

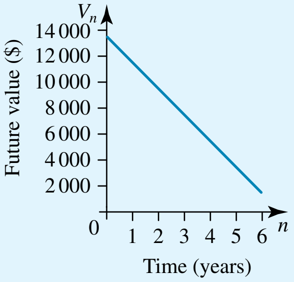


6 a



b $V_n = 45\,000 - 4000n$, scrap value is \$5000

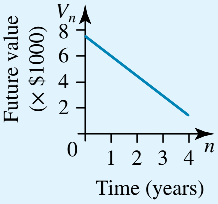
7 a



b $V_n = 13\,500 - 2000n$, scrap value is \$1500

8 a \$1550 **b** $V_{n+1} = V_n - 1550$

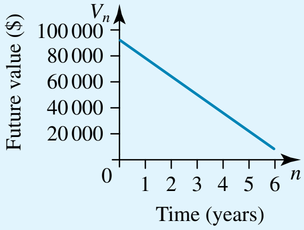
c



d $V_n = 7750 - 1550n$, 4 years

9 a \$13 800 **b** $V_{n+1} = V_n - 13\,800$

c



d $V_n = 92\,000 - 13\,800n$, 6 years

10 a i \$3000 **ii** \$400 **iii** 5 years

b i \$1750 **ii** \$500 **iii** $3\frac{1}{2}$ years

c i \$800 **ii** \$140 **iii** 5 years

d i \$18 000 **ii** \$2400 **iii** $7\frac{1}{2}$ years

11 C **12** B

13 6 **14** E

15 a The cheaper machine **b** 2 years

16 B

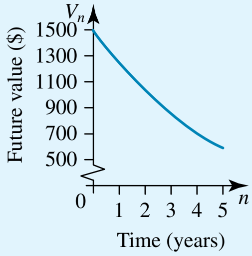
EXERCISE 6.7

1 a

Time, n (years)	Future value, V_n (\$)
0	1500
1	1245.00
2	1033.35
3	857.68
4	711.87
5	590.86

b \$590.86

c

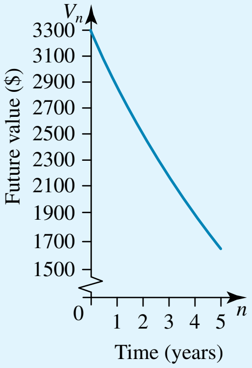


2 a

Time, n (years)	Future value, V_n (\$)
0	3300.00
1	2871.00
2	2497.77
3	2173.06
4	1890.56
5	1644.79

b \$1644.79

c

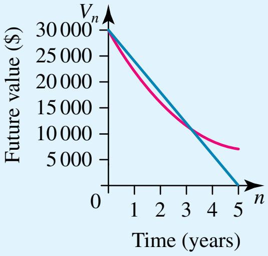


3 **a**

Time, n (years)	Depreciation (\$)	Future value, V_n (\$)
0	—	29 990
1	5998	23 992
2	5998	17 994
3	5998	11 996
4	5998	5 998
5	5998	0

Time, n (years)	Future value, V_n (\$)
0	29 990.00
1	21 892.70
2	15 981.67
3	11 666.62
4	8 516.63
5	6217.14

b



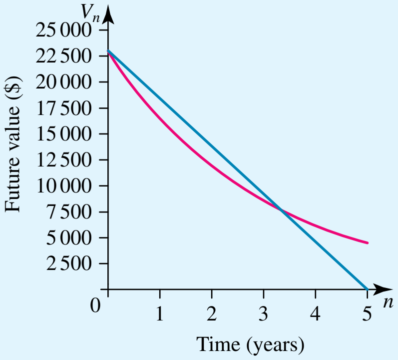
c The future value for the reducing balance method is greater than the flat rate method after 4 years.

4 **a**

Time, n (years)	Depreciation (\$)	Future value, V_n (\$)
0	-	23 000
1	4600	18 400
2	4600	13 800
3	4600	9 200
4	4600	4 600
5	4600	0

Time, n (years)	Future value, V_n (\$)
0	23 000.00
1	16 560.00
2	11 923.20
3	8 584.70
4	6 180.98
5	4 450.31

b



c The future value for the reducing balance method is greater than the flat rate method after 4 years.

5 \$16 971.73

6 \$747.88

7 5.59

8 9.98

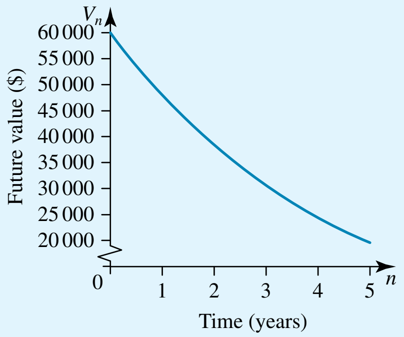
9 **a** $V_{n+1} = 0.8V_n,\ V_0 = 60\,000$

b

Time, n (years)	Future value, V_n (\$)
0	60 000.00
1	48 000.00
2	38 400.00
3	30 720.00
4	24 576.00
5	19 660.80

c \$19 660.80

d



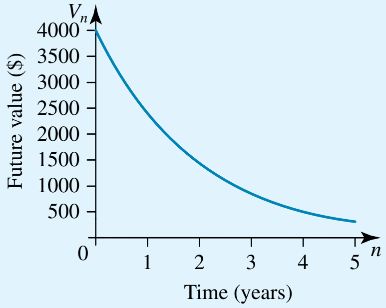
10 a $V_{n+1} = 0.6V_n, V_0 = 4000$

b

Time, n (years)	Depreciation ($)	Future value, V_n ($)
0	—	4000.00
1	1600	2400.00
2	960	1440.00
3	576	864.00
4	345.60	518.40
5	207.36	311.04

c $311.04

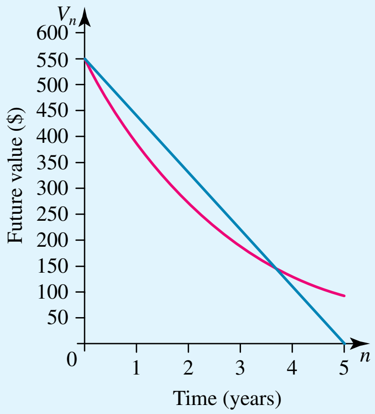
d



11 a

Flat rate			Reducing balance		
Time (years)	Dep. ($)	Future value ($)	Time (years)	Dep. ($)	Future value ($)
0	—	550	0	—	550.00
1	110	440	1	165	385.00
2	110	330	2	115.50	269.50
3	110	220	3	80.85	188.65
4	110	110	4	56.60	132.05
5	110	0	5	39.62	92.43

b

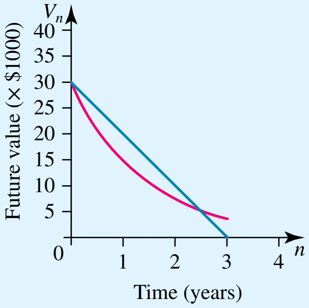


c 4 years

12 a

Flat rate			Reducing balance		
Time (years)	Dep. ($)	Future value ($)	Time (years)	Dep. ($)	Future value ($)
0	—	30 000	0	—	30 000
1	10 000	20 000	1	15 000	15 000
2	10 000	10 000	2	7 500	7 500
3	10 000	0	3	3 750	3 750

b



c 3 years

13 a $8192 **b** $9492.19

14 B

15 a $115.13, $1034.87 **b** $1183.36, $2556.64

c $1302.80, $6017.20

16 a $115.13, $569.87 **b** $7803.25, $24 696.75

c $368.73, $706.27

17 D

18 E

19 C

20 a 11.61 **b** 5.02

EXERCISE 6.8

1 a $V_{n+1} = V_n - 0.35n, V_0 = 800$

b

Number of washes (n)	Future value, V_n ($)
1	799.65
2	799.30
3	798.95
4	798.60
5	798.25

2 a $V_{n+1} = V_n - 0.65n, V_0 = 1400$

b

Hours of use (n)	Future value, V_n ($)
1	1399.35
2	1398.70
3	1398.05
4	1397.40
5	1396.75

3 a \$3990

b 61 754 km

4 a \$16 000, \$14 000

b 80 000 km

5 a \$880, \$1056

b \$6664

6 a \$577.50, \$744

b \$36 678.50

7 a 15 cents per km

b 2 years 21 weeks

c $V_n = 30000 - 0.15n$

d 92 000 km

e \$6000

f

Distance, *n* (km)	Value (\$)
0	30 000
20 000	27 000
40 000	24 000
60 000	21 000
80 000	18 000
100 000	15 000
120 000	12 000
140 000	9 000
160 000	6 000
180 000	3 000
200 000	0

8

Depreciation (\$)	Future value at end of year, V_n (\$)
3136.44	25 258.56
3693.40	21 565.16
3345.30	18 219.86
2283.06	15 936.80
3914.77	12 022.03

9 a **i** \$3276

ii 57 692 km

b **i** \$2959.20

ii 62 037 km

10 a \$3528.20

b 97 826 km

11 a \$7741.39

b 78 309 km

12 a \$930, \$810

b \$5460

13 a $V_{n+1} = V_n - 0.0025n,\ V_0 = 11300$

b

Annual depreciation (\$)	Future value at end of year, V_n (\$)
875	10 425
1062.50	9 362.50
940.50	8 422
727.60	7 694.40
964.05	6 730.35

14 a $V_{n+1} = V_n - 0.025n,\ V_0 = 14750$

b

Depreciation (\$)	Future value at end of year, V_n (\$)
1000	13 750
1062.50	12 687.50
1167.50	11 520
956.25	10 563.75
1076.50	9 487.25

15 D

16 a \$3873.76, \$3979.16

b \$24 747.08

17 a \$6195.07, \$5825.40

b \$23 078.53

18 a Reducing balance depreciation (6 years compared to 7+ years and 10 years)

b **i** \$3500

ii \$7000

iii \$2500

c **i** \$3500 (same)

ii \$2867.20 (\$4132.80 less)

iii \$2500 (same)

19 E

20 B

7

Loans, investments and asset values

7.1 Kick off with CAS

Finance Solver

A reducing balance loan has regular repayments made during the life of the loan, with the interest being charged on the reducing balance of the loan.

We can use the annuities formula to find the amount still owing at any point during the life of a reducing balance loan.

$$V_n = V_0R^n - \frac{d(R^n - 1)}{R - 1}$$

where:

V_0 = the amount borrowed (principal)

$R = 1 + \frac{r}{100}$ (r = the interest rate per repayment period)

d = the amount of the regular payments made per period

n = the number of payments

V_n = the amount owing after n payments

1 Sylvia borrowed \$15 000 exactly $2\frac{1}{2}$ years ago, with regular monthly repayments of \$435. Interest is charged at 6.6% p.a. (adjusted monthly). Use CAS to define the annuities formula and calculate how much is still owing on Sylvia's loan.

2 Juliana is repaying a \$3500 loan over 3 years with monthly instalments at 7.2% p.a. (adjusted monthly). Use the annuities formula to determine how much the monthly payments Juliana has to make are.

3 Calculate the answers to questions 1 and 2 using Finance Solver on your CAS.

Use Finance Solver on your CAS to answer questions 4 and 5.

4 Mohammed is repaying a \$40 000 loan of 12 years with quarterly instalments at 6.3% p.a. (adjusted quarterly). How much does Mohammed still owe after 4 years?

5 Georgio took out a \$22 000 loan with interest charged at 7.8% p.a. (adjusted monthly). He has made regular monthly repayments of \$440.33. If he still owes \$14 209.88, how long ago was the loan taken out?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

7.2 Reducing balance loans I

Annuities

studyon

Unit 3
AOS R&FM
Topic 2
Concept 7

Annuities amortisation
Concept summary
Practice questions

An annuity is an investment or a loan that has regular and constant payments over a period of time. Let's first look at this as a recurrence relation that calculates the value of an annuity after each time period.

$$V_{n+1} = V_n R - d$$

where:

V_{n+1} = **Amount left after** $n + 1$ **payments**
V_n = **Amount at time** n
$R = \left(1 + \frac{r}{100}\right)$, **where** r **is the interest rate per period**
d = **Payment amount**

When we invest money with a financial institution the institution pays us interest because it is using our money to lend to others. Conversely, when we borrow money from an institution we are using the institution's money and so it charges us interest.

studyon

Unit 3
AOS R&FM
Topic 2
Concept 6

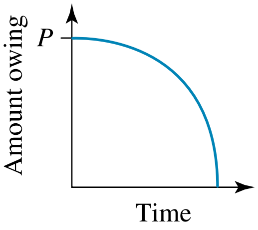
Reducing balance loans
Concept summary
Practice questions

In **reducing balance loans**, interest is usually charged every month by the financial institution and repayments are made by the borrower on a regular basis. These repayments nearly always amount to more than the interest for the same period of time and so the amount still owing is reduced. Since the amount still owing is continually decreasing and interest is calculated on the current balance but debited monthly, the amount of interest charged also decreases throughout the life of the loan.

This means that less of the amount borrowed is paid off in the early stages of the loan compared to the end. If we graphed the amount owing against time for a loan it would look like the graph at right. That is, the rate at which the loan is paid off increases as the loan progresses.

The terms below are often used when talking about reducing balance loans:

Principal, V_0 = amount borrowed (\$)
Balance, V_n = amount still owing (\$)
Term = life of the loan (years)
To discharge a loan = to pay off a loan (that is, V_n = \$0).



It is possible to have an 'interest only' loan account whereby the repayments equal the interest added and so the balance doesn't reduce. This option is available to a borrower who wants to make the smallest repayment possible.

Although the focus of this section is reducing balance loans, note that the theory behind reducing balance loans can also be applied to other situations such as superannuation payouts, for people during retirement, and bursaries. In each of these situations a *lump sum* is realised at the start of a period of time and *regular payments* are made during that time. Regular payments are called **annuities**. So these situations are often called *annuities in arrears* because the annuity follows the realisation of the lump sum.

WORKED EXAMPLE 1

A loan of $100 000 is taken out over 15 years at a rate of 7.5% p.a. (interest debited monthly) and is to be paid back monthly with $927 instalments. Complete the table below for the first five payments.

$n+1$	V_n	d	V_{n+1}
1	$100 000		
2			
3			
4			
5			

THINK	WRITE
1 State the initial values for V_0, r and d.	$V_0 = \$100\,000$ $r = \frac{7.5}{12}$ $= 0.625$ $d = \$927$
2 Evaluate V_1.	$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$ $V_1 = 100\,000\left(1 + \frac{0.625}{100}\right) - 927$ $= \$99\,698$
3 Evaluate V_2.	$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$ $V_2 = 99\,698\left(1 + \frac{0.625}{100}\right) - 927$ $= \$99\,394.11$
4 Evaluate V_3.	$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$ $V_3 = 99\,394.11\left(1 + \frac{0.625}{100}\right) - 927$ $= \$99\,088.32$
5 Evaluate V_4.	$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$ $V_4 = 99\,088.32\left(1 + \frac{0.625}{100}\right) - 927$ $= \$98\,780.62$
6 Evaluate V_5.	$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$ $V_4 = 98\,780.62\left(1 + \frac{0.625}{100}\right) - 927$ $= \$98\,471.00$

7 Complete the table.

$n+1$	V_n	d	V_{n+1}
1	$V_0 = \$100\,000$	\$927	$V_1 = \$99\,698$
2	$V_1 = \$99\,698$	\$927	$V_2 = \$99\,394.11$
3	$V_2 = \$99\,394.11$	\$927	$V_3 = \$99\,088.32$
4	$V_3 = \$99\,088.32$	\$927	$V_4 = \$98\,780.62$
5	$V_4 = \$98\,780.62$	\$927	$V_5 = \$98\,471.00$

The annuities formula

The annuities formula can be used to find the amount still owing at any point in time during the term of a reducing balance loan. When a consumer borrows money from a financial institution, that person contracts to make regular payments or annuities in order to repay the sum borrowed over time.

The amount owing in a loan account for n repayments is given by the annuities formula:

$$V_n = V_0 R^n - \frac{d(R^n - 1)}{R - 1}$$

where:

V_0 = **the amount borrowed (principal)**

R = **the compounding or growth factor for the amount borrowed**

$= 1 + \frac{r}{100}$ **(r = the interest rate per repayment period)**

d = **the amount of the regular payments made per period**

n = **the number of payments**

V_n = **the amount owing after n payments**

Note: Showing how the annuities formula is developed is not required in the Further Mathematics course.

WORKED EXAMPLE 2

A loan of \$50 000 is taken out over 20 years at a rate of 6% p.a. (interest debited monthly) and is to be repaid with monthly instalments of \$358.22. Find the amount still owing after 10 years.

THINK	WRITE
1 State the loan amount, V_0, and the regular repayment, d.	$V_0 = 50\,000$ $d = 358.22$
2 Find the number of payments, n, the interest rate per month, r, and the growth factor, R.	$n = 10 \times 12$ $= 120$ $r = \frac{6}{12}$ $= 0.5$ $R = 1 + \frac{r}{100}$ $= 1.005$

3 Substitute into the annuities formula.

$$V_{120} = V_0R^n - \frac{d(R^n - 1)}{R - 1}$$
$$= 50\,000(1.005)^{120} - \frac{358.22(1.005^{120} - 1)}{1.005 - 1}$$

4 Evaluate V_{120}.

$V_{120} = \$32\,264.98$

5 Write a statement.

The amount still owing after 10 years will be \$32 264.98.

Note: If R is a recurring decimal, place the value in memory and bracket R if needed when evaluating V_n.

Note that, even though 10 years is the halfway point of the term of the loan, more than half of the original \$50 000 is still owing.

Finance Solver or your CAS can be used in calculations involving the annuities formula in the same way it was used in compound interest calculations.

WORKED EXAMPLE 3

Rob wants to borrow \$2800 for a new sound system at 7.5% p.a., interest adjusted monthly.

a What would be Rob's monthly repayment if the loan is fully repaid in $1\frac{1}{2}$ years?

b What would be the total interest charged?

THINK	WRITE
a 1 State the value of V_0, n, r and R.	**a** $V_0 = 2800$ $n = 18$ $r = \frac{7.5}{12}$ $= 0.625$ $R = 1 + \frac{0.625}{100}$ $= 1.006\,25$
2 Substitute into the annuities formula to find the regular monthly repayment, d.	$d = \frac{V_0R^n(R - 1)}{R^n - 1}$ $= \frac{2800(1.006\,25)^{18}(1.006\,25 - 1)}{1.006\,25^{18} - 1}$
3 Evaluate d.	$d = \$164.95$
4 Write a statement.	The monthly regular payment is \$164.95 over 18 months.
b 1 Total interest = total repayments − amount borrowed	**b** Total interest = 164.95 × 18 − 2800 = 2969.10 − 2800 = \$169.10
2 Write a statement.	The total interest on the \$2800 loan over 18 months is \$169.10.

WORKED EXAMPLE 4

Josh borrows $12 000 for some home office equipment. He agrees to repay the loan over 4 years with monthly instalments at 7.8% p.a. (adjusted monthly). Find:

a **the instalment value**

b **the principal repaid and interest paid during the:**

i **10th repayment** **ii** **40th repayment.**

THINK	WRITE
a 1 State the value of V_0, n, r and R.	**a** $V_0 = 12\,000$ $n = 4 \times 12$ $= 48$ $r = \frac{7.8}{12}$ $= 0.65$ $R = 1 + \frac{0.65}{100}$ $= 1.0065$
2 Substitute into the annuities formula to find the monthly repayment, d.	$d = \frac{V_0R^n(R-1)}{R^n-1}$ $= \frac{12\,000(1.0065)^{48}(1.0065-1)}{1.0065^{48}-1}$
3 Evaluate d.	$d = \$291.83$
4 Write a statement.	The monthly repayment over a 4-year period is \$291.83.
b i 1 Find the amount owing after 9 months.	**b i**
(a) State V_0, n, R.	$V_0 = 12\,000$, $n = 9$, $R = 1.0065$
(b) Substitute into the annuities formula.	$V_n = V_0R^n - \frac{d(R^n-1)}{R-1}$ $V_9 = 12\,000(1.0065)^9 - \frac{291.83(1.0065^9-1)}{1.0065-1}$
2 Evaluate V_9.	$V_9 = \$10\,024.73$
3 Find the amount owing after 10 months. Substitute (change $n = 9$ to $n = 10$) and evaluate.	$V_{10} = 12\,000(1.0065)^{10} - \frac{291.83(1.0065^{10}-1)}{1.0065-1}$ $= \$9798.06$
4 Principal repaid $= V_9 - V_{10}$	Principal repaid $= 10\,024.73 - 9798.06$ $= \$226.67$
5 Interest paid = repayment − principal repaid	Total interest $= \$291.83 - 226.67$ $= \$65.16$
6 Write a statement.	In the 10th repayment, \$226.67 principal is repaid and \$65.16 interest is paid.

ii **1** Repeat steps **1**–**6** for V_{39} and V_{40}.

ii $V_{39} = 12\,000(1.0065)^{39} - \dfrac{291.83(1.0065^{39} - 1)}{1.0065 - 1}$

$= \$2543.10$

$V_{40} = 12\,000(1.0065)^{40} - \dfrac{291.83(1.0065^{40} - 1)}{1.0065 - 1}$

$V_{40} = \$2267.80$

Principal repaid $= V_{39} - V_{40}$
$= 2543.10 - 2267.80$
$= \$275.30$
Interest $= 291.83 - 275.30$
$= \$16.53$

2 Write a statement.

In the 40th repayment, \$275.30 principal is repaid and \$16.53 interest is paid.

EXERCISE 7.2 Reducing balance loans I

PRACTISE

1 **WE1** A loan of \$25 000 is taken out over 10 years at a rate of 5.5% p.a. (interest debited monthly) and is to be paid back monthly with \$271.32 instalments. Complete the table below for the first five payments.

$n+1$	V_n	d	V_{n+1}
1	\$25 000		
2			
3			
4			
5			

2 Paul wanted to buy a new road bike so he took out a \$10 000 loan over 6 years at a rate of 8% p.a. (interest debited quarterly) and is to be paid back quarterly with instalments of \$528.71. Complete the table below for the first five payments and state how much he owes at the end of this time.

$n+1$	V_n	d	V_{n+1}
1	10 000		
2			
3			
4			
5			

3 **WE2** Use the annuities formula,

$V_n = V_0R^n - \dfrac{d(R^n - 1)}{R - 1}$, to find V_n, given the following.

$V_0 = \$300$, $r = 1\%$, $d = \$9$ p.a., $n = 36$

4 Use the annuities formula,

$$V_n = V_0R^n - \frac{d(R^n - 1)}{R - 1},$$ to find V_n, given the following.

$V_0 = \$1450$, $R = 1.001$, $n = 20$, $d = \$75$

5 **WE3** Macca wants to borrow \$3500 for a new motorbike at 6.8% p.a., interest adjusted monthly.

a What would be Macca's monthly repayment if the loan is fully repaid in 2 years?

b What would be the total interest charged?

6 Willow wants to borrow \$2400 for a new computer at 6.3% p.a., interest adjusted monthly.

a What would be Willow's monthly repayment if the loan is fully repaid in 2 years?

b What would be the total interest charged?

7 **WE4** Grace has borrowed \$18 000 to buy a car. She agrees to repay the reducing balance loan over 5 years with monthly instalments at 8.1% p.a. (adjusted monthly). Find:

a the instalment value

b the principal repaid and the interest paid during:

i the 10th repayment ii the 50th repayment.

8 Sarah borrows \$8000 for an overseas holiday. She agrees to repay the loan over 4 years with monthly instalments at 6.8% p.a. (adjusted monthly). Find:

a the instalment value.

b the principal repaid and interest paid during the:

i 10th repayment ii 40th repayment.

CONSOLIDATE

9 Tamara took out a loan for \$23 000 to pay for her wedding over 7 years at a rate of 6.5% p.a. (interest debited monthly), which is to be paid back monthly with instalments of \$341.54.

a Write down a recurrence relation to model this loan.

b Use your recurrence relation to find how much Tamara owes after her third payment was made.

10 Noah took out a personal loan of \$2000 to buy a new air conditioning unit, with the loan having an interest rate of 12% p.a. He will repay the loan with 8 equal monthly payments of \$261.

a Write down a recurrence relation to model this loan.

b Use the recurrence relation to determine how much is still owed after 4 months (4 payments).

c Is the loan paid out exactly after 8 months? If not, how much will Noah need to add to the last payment to fully repay the loan?

11 A loan of \$65 000 is taken out over 20 years at a rate of 12% p.a. (interest debited monthly) and is to be repaid with monthly instalments of \$715.71. Find the amount still owing after:

a 5 years b 10 years c 15 years.

12 A loan of \$52 000 is taken out over 15 years at a rate of 13% p.a. (interest debited fortnightly) and is to be repaid with fortnightly instalments of \$303.37. Find the amount still owing after:

a 4 years
b 8 years
c 12 years.

13 Link borrows \$48 000, taken out over 10 years and to be repaid in monthly instalments. (*Note:* As the interest rate increases, the monthly repayment increases if the loan period is to remain the same.) Find the amount still owing after 5 years if interest is debited monthly at a rate of:

a 6% p.a. and the repayment is \$532.90
b 9% p.a. and the repayment is \$608.04
c 12% p.a. and the repayment is \$688.66.

14 Peter wants to borrow \$8000 for a second-hand car and his bank offers him a personal loan for that amount at an interest rate of 13% p.a., interest debited fortnightly, with fortnightly repayments of \$124.11 over 3 years. After 2 years he wants to calculate how much he still owes by using the annuities formula.

a Which of the following equations should he use?

A $V_{78} = 8000(1.005)^{78} - \dfrac{124.11(1.005^{78} - 1)}{1.005 - 1}$

B $V_{52} = 8000(1.05)^{52} - \dfrac{124.11(1.05^{52} - 1)}{1.05 - 1}$

C $V_{52} = 8000(1.005)^{52} - \dfrac{124.11(1.005^{52} - 1)}{1.005 - 1}$

D $V_{78} = 8000(1.05)^{78} - \dfrac{124.11(1.05^{78} - 1)}{1.05 - 1}$

E $V_{52} = 8000(0.005)^{52} - \dfrac{124.11(1.05^{52} - 1)}{1.005 - 1}$

b The actual amount that Peter still owes after 2 years is closest to:

A \$2500 **B** \$3000 **C** \$3500
D \$4000 **E** \$4500

15 Ben took out a loan for \$20 000 to buy a new car. The contract required that he repay the loan over 5 years with monthly instalments of \$421.02. After $2\frac{1}{2}$ years Ben used the annuities formula to obtain the expression below to calculate the amount he still owed.

$$V_{30} = 20000(1.008)^{30} - \frac{421.02(1.008^{30} - 1)}{0.008}$$

The interest rate per annum charged by the bank for this reducing balance loan is:

A 1.008% **B** 0.008% **C** 0.096%
D 9.6% **E** 12.096%

16 Sergio's reducing balance loan of $12 000 has interest charged at 9% p.a., interest adjusted monthly. Find:

i the monthly repayment **ii** the total interest charged

if the loan is fully repaid in:

a 2 years **b** 3 years

c 4 years **d** $4\frac{1}{2}$ years.

17 A loan of $94 000 is to be repaid over 20 years. Find:

i the repayment value **ii** the total interest charged

if the loan is repaid:

a weekly at 13% p.a., interest adjusted weekly
b fortnightly at 13% p.a., interest adjusted fortnightly
c monthly at 13% p.a., interest adjusted monthly
d quarterly at 13% p.a., interest adjusted quarterly.

18 Which of the following would decrease the total amount of interest paid during the life of a loan? (There may be more than one answer.)

A A fall in the interest rate
B A decrease in the frequency of repayments
C A greater amount borrowed
D A decrease in the term of the loan
E A rise in the interest rate

MASTER

19 Gail has agreed to repay a $74 000 reducing balance loan with fortnightly instalments over 20 years at 9.75% p.a. (adjusted fortnightly). Find:

a the instalment value
b the principal repaid and the interest paid during:
  i the 1st repayment **ii** the 500th repayment.

20 Terry is repaying a $52 000 loan over 15 years with quarterly instalments at 6.25% p.a. (adjusted quarterly). Currently, $5\frac{1}{2}$ years have passed since the loan was drawn down (money borrowed). How much does Terry still owe?

7.3 Reducing balance loans II

Number of repayments

The situation often arises in reducing balance loans when a potential borrower knows how much needs to be borrowed as well as the amount that can be repaid each month. The person then wants to know how long the loan needs to be to accommodate these conditions, that is, to determine the number of repayments, n, required. As with compound interest, n is calculated using Finance Solver on your CAS.

WORKED EXAMPLE 5

A reducing balance loan of $60 000 is to be repaid with monthly instalments of $483.36 at an interest rate of 7.5% p.a. (debited monthly). Find:

a **the number of monthly repayments (and, hence, the term of the loan in more meaningful units) needed to repay the loan in full**

b **the total interest charged.**

THINK	WRITE
a 1 Using Finance Solver on your CAS enter the appropriate values: n (N:) = unknown r (I(%):) = 7.5 V_0 (PV:) = 60 000 d (Pmt:) = −483.36 V_n (FV:) = 0 PpY: = 12 (monthly) CpY: = 12 (monthly)	**a** $n = 239.995\,307\,038\,33$
2 Solve for n.	$n = 240$ months
3 Interpret the results.	$\text{Time} = \frac{240}{12}$ years $= 20$ years
4 Write a statement.	Term of loan needed is 20 years.
b 1 Total interest = Total repayments − Principal repaid	**b** $\text{Interest} = 483.36 \times 240 - 60\,000$ $= \$56\,006.40$
2 Write a statement.	Total interest charged on the loan is $56 006.40.

Sometimes we may want to find the time for only part of the loan term. The procedure that is followed is the same as in Worked example 5; however, V_n is zero only if we are calculating the time to repay the loan in full. Otherwise we should consider the amount still owing at that time.

WORKED EXAMPLE 6

Some time ago, Petra borrowed $14 000 to buy a car. Interest on this reducing balance loan has been charged at 9.2% p.a. (adjusted monthly) and she has been paying $446.50 each month to service the loan. Currently she still owes $9753.92. How long ago did Petra borrow the money?

THINK	WRITE
1 Identify V_n, V_0, d and r and enter the following values into Finance Solver on your CAS: n (N:) = unknown r (I(%):) = 9.2 V_0 (PV:) = 14 000 d (Pmt:) = −446.50 V_n (FV:) = −9753.92 PpY: = 12 (monthly payments) CpY: = 12 (monthly compounds)	$n = 11.999\,995\,037\,662$
2 Solve for n.	$n = 12$ months
3 Interpret the results.	Time = 1 year
4 Write a statement.	Petra has had the loan for the past 12 months.

In the situations investigated so far, we have considered calculating only the time from the start of the loan to a later date (including repayment in full). In fact, it doesn't matter what period of the loan is considered; we can still use Finance Solver as we have already done. In using CAS to do this, we can use V_n and V_0 such that they have the following meanings:

V_n = amount owing at the end of the time period
V_0 = amount owing at the start of the time period.

WORKED EXAMPLE 7

A loan of $11 000 is being repaid by monthly instalments of $362.74 with interest being charged at 11.5% p.a. (debited monthly). Currently, the amount owing is $7744.05. How much longer will it take to:

a reduce the amount outstanding to $2105.11

b repay the loan in full?

THINK	WRITE
a 1 Let V_0 be the amount still owing at the start of the time period. State V_n, V_0, d and r.	**a** $V_n = 2105.11$, $V_0 = 7744.05$, $Q = 362.74$, $r = 11.5\%$ p.a.
2 Using the Finance Solver on your CAS, enter the following values: n (N:) = unknown r (I(%):) = 11.5 V_0 (PV:) = 7744.05 d (Pmt:) = −362.74 V_n (FV:) = −2105.11 PpY: = 12 CpY: = 12	$n = 17.999\,988\,603\,29$
3 Solve for n.	$n = 18$ months
4 Interpret the results and write a statement.	It will take another $1\frac{1}{2}$ years to reduce the amount owing to $2105.11.
b 1 Repeat part **a**, entering the appropriate values into Finance Solver. Enter FV: = 0 to represent the loan is fully repaid.	**b** $n = 23.999\,534\,856\,457$ $= 24$ months
2 Write a statement.	It will take another 2 years to repay the loan in full.

Effects of changing the repayment

Since most loans are taken over a long period of time it is probable that a borrower's financial situation will change during this time. For instance, a borrower may receive a pay rise and so their take home pay is greater per week or fortnight. The person may then choose to increase the value of the repayments made to service the loan.

It may also be that a person's financial situation deteriorates, in which case he/she may request from their institution that the repayment value be decreased.

In this section we will look at the effect that changing the repayment value has on the term of the loan and the total interest paid.

WORKED EXAMPLE 8

A reducing balance loan of \$16 000 has a term of 5 years. It is to be repaid by monthly instalments at a rate of 8.4% p.a. (debited monthly).

a **Find the repayment value.**

b **What will be the term of the loan if the repayment is increased to \$393.62?**

c **Calculate the total interest paid for repayments of \$393.62.**

d **By how much does the interest figure in c differ from that paid for the original offer?**

THINK	WRITE
a **1** (a) Write the equation for d.	**a** $d = \dfrac{V_0R^n(R-1)}{R^n-1}$
(b) Give the values of V_0, n, r and R.	$V_0 = 16\,000$, $n = 5 \times 12 = 60$ $r = \dfrac{8.4}{12} = 0.7$ $R = 1.007$
2 Substitute into the annuities formula to evaluate d.	$d = \dfrac{16000(1.007)^{60}(1.007-1)}{1.007^{60}-1}$ $= \$327.49$
3 Write a statement.	\$16 000 to be paid off in 5 years at 8.4% p.a. will need monthly repayments of \$327.49.
b **1** Using Finance Solver on your CAS, enter the following values: n (N:) = unknown r (I(%):) = 8.4 V_0 (PV:) = 16 000 d (Pmt:) = −393.62 V_n (FV:) = 0 PpY: = 12 CpY: = 12	**b** $n = 47.999\,695\,088\,867$
2 Solve for n.	$n = 48$ months
3 Interpret the results.	Time $= \dfrac{48}{12}$ years $= 4$ years
4 Write a statement.	The new term of the loan would be 4 years.
c Interest paid = total repayments − principal repaid	**c** When term = 4 years, $d = 393.62$. Interest $= 48 \times 393.62 - 16\,000$ $= \$2893.76$

d 1 (a) Review the known quantities.	**d** When term = 5 years, $d = 327.49$. Interest $= 60 \times 327.49 - 16\,000$ $= \$3649.40$
(b) Find the interest difference.	Interest difference $= 3649.40 - 2893.76$ $= \$755.64$
2 Write a statement.	If the repayment is increased from \$327.49 to \$393.62 per month then \$755.64 is saved in interest payments.

If a borrower does increase the value of each repayment and if all other variables remain the same, then the term of the loan is reduced. Conversely, a decrease in the repayment value increases the term of the loan. There are two stages to the loan, each with a different repayment.

WORKED EXAMPLE 9

Brad borrowed \$22 000 to start a business and agreed to repay the loan over 10 years with quarterly instalments of \$783.22 and interest debited at 7.4% p.a. However, after 6 years of the loan Brad decided to increase the repayment value to \$879.59. Find:

a **the actual term of the loan**

b **the total interest paid**

c **the interest savings achieved by increasing the repayment value.**

THINK	**WRITE**
a 1 To find A after 6 years:	**a**
(a) First identify V_0, d, n, r and R.	$V_0 = 22\,000$, $d = 783.22$ $n = 6 \times 4$ $= 24$ $r = \frac{7.4}{4}$ $= 1.85$ $R = 1.0185$
(b) Find V_{24} (the balance owing after 6 years).	$V_{24} = 22\,000(1.0185)^{24} - \frac{783.22(1.0185^{24} - 1)}{1.0185 - 1}$ $= \$10\,761.83$
2 Now find the n value to reduce \$10 761.83 to zero; that is, the remaining part of the loan. Identify V_0 and d.	$V_0 = 10\,761.83$, $d = 879.59$
3 Using Finance Solver, enter the appropriate values. n (N:) = unknown r (I(%):) = 7.4 V_0 (PV:) = 10 761.9 d (Pmt:) = −879.59 V_n (FV:) = 0 PpY: = 4 CpY: = 4	$n = 14.000\,122\,313\,731$

4 Solve for n.	$n = 14$ quarters
5 Interpret the results.	$\text{Time} = 3\frac{1}{2}$ years
6 Find the total term of the loan.	$\text{Total term} = 6 + 3\frac{1}{2}$ $= 9\frac{1}{2}$ years
b For the two-repayment scenario: Interest paid = Total repayments − Principal repaid In this case, in two stages.	**b** For the two-repayment scenario: $\text{Interest} = 783.22 \times 24 + 879.59 \times 14 - 22\,000$ $= \$9111.54$
c **1** For the same repayment scenario: $d = 783.22$ for 10 years.	**c** For the same repayment scenario: $\text{Interest} = 783.22 \times 40 - 22\,000$ $= \$9328.80$
2 Find the difference between the two scenarios.	$\text{Interest difference} = 9328.80 - 9111.54$ $= \$217.26$
3 Write a statement.	Brad will save $217.26 interest by increasing his repayment value.

EXERCISE 7.3 Reducing balance loans II

PRACTISE

1 **WE5** A reducing balance loan of $50 000 is to be repaid with monthly instalments of $424.52 at an interest rate of 7% p.a. (debited monthly). Find:

a the number of monthly repayments (and, hence, the term of the loan in more meaningful units) needed to repay the loan in full

b the total interest charged.

2 A loan of $2400 is taken out with a reducing balance interest rate of 4.5% per annum with interest debited monthly. The borrower wishes to pay instalments of $154.82 per month. What, correct to the nearest month, would be the term of such a loan?

3 **WE6** Simon borrowed $8000. Interest on this reducing balance loan has been charged at 8.7% p.a. (adjusted monthly) and he has been paying $368.45 each month to service the loan. Currently he still owes $5489.56. How long ago did Simon borrow the money?

4 Simon borrowed $11 000. Interest on this reducing balance loan has been charged at 6.5% p.a. (adjusted monthly) and he has been paying $409.50 each month to service the loan. Currently he still owes $5565.48. How long ago did Simon borrow the money?

5 **WE7** A loan of $15 000 is being repaid by monthly instalments of $423.82 with interest being charged at 11.5% p.a. (debited monthly). Currently, the amount owing is $8357.65. How much longer will it take to:

a reduce the amount outstanding to $2450.15

b repay the loan in full?

6 A loan of $9000 is being repaid by monthly instalments of $273.56 with interest being charged at 8.8% p.a. (debited monthly). Currently, the amount owing is $6900.86. How much longer will it take to:

a reduce the amount outstanding to $1670.48

b repay the loan in full?

7 **WE8** A reducing balance loan of $18 000 has a term of 5 years. It is to be repaid by monthly instalments at a rate of 7.8% p.a. (debited monthly).

- a Find the repayment value.
- b What will be the term of the loan if the repayment is increased to $390.50?
- c Calculate the total interest paid for repayments of $390.50.
- d By how much does the interest figure in **c** differ from that paid for the original offer?

8 A reducing balance loan of $25 000 has a term of 5 years. It is to be repaid by fortnightly instalments at a rate of 6.5% p.a. (debited fortnightly).

- a Find the repayment value.
- b What will be the term of the loan if the repayment is increased to $245?
- c Calculate the total interest paid for repayments of $245.
- d By how much does the interest figure in **c** differ from that paid for the original offer?

9 **WE9** James borrowed $21 000 for some home renovations and agreed to pay the loan over 7.5 years with quarterly instalments of $899.41 and interest debited at 6.8% p.a. However, after 6 years of the loan James decided to increase the repayment value to $1070.41. Find:

- a the actual term of the loan
- b the total interest paid
- c the interest savings achieved by increasing the repayment value.

10 Gabriel borrowed $17 000 for some new furniture and agreed to pay the loan over 8 years with quarterly instalments of $670.29 and interest debited at 5.9% p.a. However, after 6 years of the loan Gabriel decided to increase the repayment value to $1724.02. Find:

- a the actual term of the loan
- b the total interest paid
- c the interest savings achieved by increasing the repayment value.

CONSOLIDATE

11 Jim has a reducing balance loan of $3500 that he is using for a holiday and has agreed to repay it by monthly instalments of $206.35 at a rate of 7.6% p.a. (interest debited monthly). Find:

- a the number of repayments needed to repay in full and this time in years
- b the total interest charged.

12 Aimee has borrowed $5500 for some new outdoor furniture. She is to repay the reducing balance loan by quarterly instalments of $861.29 with interest debited quarterly at 9.4% p.a. Find:

- a how long it will take Aimee to repay the loan in full
- b the total interest charged.

13 Melpomeni's loan of $22 000 was taken out some time ago. Interest has been charged at 7.8% p.a. (adjusted monthly) and monthly repayments of $443.98 have serviced the loan. If the amount still owing is $14 209.88:

- a how long ago was the loan taken out
- b what was the term of the loan?

14 Some time ago, Elizabeth took out a loan of $25 000. Interest has been charged at 10.5% p.a. (adjusted monthly) and monthly repayments of $537.35 have serviced the loan. If the amount still owing is $11 586.64:

a how long ago was the loan taken out

b what was the term of the loan?

15 A reducing balance loan of $80 000 is taken out at 7.9% p.a. (adjusted monthly) to finance the purchase of a boat. It is to be repaid with monthly instalments of $639.84. The loan will be paid in full in:

A 10 years **B** 15 years **C** 20 years **D** 22 years **E** 25 years

16 Stuart has decided to borrow $85 000 to set up a business that makes garden ornaments. He will repay this amount plus interest, charged at 6.6% p.a. (debited monthly), over 20 years with monthly instalments of $638.75. If he wanted to use the annuities formula to find out how long it would be before the amount he still owed fell below $50 000, the equation that he should use is:

A $50\,000 = 85\,000(1.0055)^n - \dfrac{638.75(1.0055^{240} - 1)}{1.0055 - 1}$

B $50\,000 = 85\,000(1.0055)^n - \dfrac{638.75(1.0055^{n} - 1)}{1.0055 - 1}$

C $85\,000 = 50\,000(1.0055)^n - \dfrac{638.75(1.0055^{n} - 1)}{1.0055 - 1}$

D $0 = 85\,000(1.0055)^n - \dfrac{638.75(1.0055^{n} - 1)}{1.0055 - 1}$

E $0 = 85\,000(1.0055)^n - \dfrac{638.75(1.0055^{n} - 1)}{1.0055 - 1}$

17 Gila's reducing balance loan of $9000 is to be repaid by monthly instalments of $230.43 with interest charged at 10.5% p.a. (debited monthly).

a Currently, the amount owing is $8069.78. How much longer will it take to:

  i reduce the amount owing to $3822.20 **ii** repay the loan in full?

b Some time later the amount owing has fallen to $3226.06. How much longer will it take to:

  i reduce the amount owing to $1341.23 **ii** repay the loan in full?

18 Megan wanted to borrow $50 000 and was offered a reducing balance loan over 20 years at 6.9% p.a. (adjusted monthly) with monthly instalments.

a What will be the monthly repayment value?

b What would be the term of the loan if instead the repayment was:

  i increased to $577.97 **ii** increased to $486.33

  iii decreased to $361.85 **iv** decreased to $352.90?

c In each case in **b** above, calculate the total interest paid.

d For each case above, calculate the interest difference from the original offer.

19 Jack borrowed $20 000 and agreed to repay the loan over 10 years with quarterly instalments of $750.48 with interest debited quarterly at 8.6% p.a. However, after 5 years he decided to increase the repayment value. Find:

i the actual term of the loan

ii the total interest paid

iii the interest saving achieved by increasing the repayment if the quarterly repayment was increased to:

a $901.48 b $1154.34.

20 Robin borrowed $25 000 and agreed to repay this reducing balance loan over 10 years with quarterly instalments of $975.06, interest being charged at 9.5% p.a. After 4 years Robin increased her repayment value to $1167.17. The term of her loan will be closest to:

A 6 years B 7 years C 8 years D 9 years E 10 years

MASTER

21 A loan of $25 000 is repaid in 2 stages over $8\frac{3}{4}$ years with quarterly instalments. For the first 4 years the repayment was $975.06 and was increased to $1167.17 for the remaining time. The total amount of interest charged would be closest to:

A $9000 B $10 000 C $11 000 D $12 000 E $13 000

22 Anne is repaying a $26 000 loan over 8 years with monthly instalments of $383.61 at 9.2% p.a., debited monthly on the outstanding balance. She has made 2 years' worth of repayments but would like to repay the loan in full in the next 5 years. Find:

a the amount that she still owes

b the monthly repayment value needed to repay in full.

7.4 Reducing balance loans III

When paying off a loan it is often wise to follow its progress through the life of the loan. The **amortisation** of the loan can be tracked on a step-by-step basis by following the payments made, the interest and reduction in the principal.

WORKED EXAMPLE 10

Sharyn takes out a loan of $5500 to pay for solar heating for her pool. The loan is to be paid in full over 3 years with quarterly payments at 6% p.a.

a Calculate the quarterly payment required.

b Complete an amortisation table for the loan with the following headings.

Payment no.	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	5500			

THINK	WRITE
a Using Finance Solver, enter the appropriate values. n (N:) = 12 r (I(%):) = 6 V_0 (PV:) = −5500 d (Pmt:) = unknown V_n (FV:) = 0 PpY: = 4 CpY: = 4	a $d = 504.239\ldots$ So the quarterly payments are $504.24.

b 1 Place $504.24 in the first payment column. Calculate the interest with the initial principal of $5500. Loan outstanding is the addition of principal and interest less the payment made. Complete the first line as shown.

b $I = 5500\left(\frac{0.06}{4}\right)$
$= 82.50$

Payment	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	5500	82.50	504.24	5078.26

2 Complete the following:
Principal outstanding
= previous loan

Interest due: calculated as previous with the new principal

Payment: stays the same
Loan outstanding
= principal + interest
− payment

Payment	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	5500	82.50	504.24	5078.26
2	5078.26	76.17	504.24	4650.19

3 Complete the table following the previous steps.

Payment	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	5500	82.50	504.24	5078.26
2	5078.26	76.17	504.24	4650.19
3	4650.19	69.75	504.24	4215.70
4	4215.70	63.24	504.24	3774.70
5	3774.70	56.62	504.24	3327.08
6	3327.08	49.91	504.24	2872.75
7	2872.75	43.09	504.24	2411.60
8	2411.60	36.17	504.24	1943.53
9	1943.53	29.15	504.24	1468.44
10	1468.44	22.03	504.24	986.23
11	986.23	14.79	504.24	496.78
12	496.78	7.45	504.24	−0.01

Frequency of repayments

In this section we investigate the effect on the actual term of the loan, and on the total amount of interest charged, of making more frequent repayments. While the value of the repayment will change, the actual outlay will not. For example, a $3000 quarterly (each 3 months) repayment will be compared to a $1000 monthly repayment. That is, the same amount is repaid during the same period of time in each case. So the only variable will be how often repayments are made. In all cases in this section interest will be charged just before a repayment is made, although this may not be the case in practice.

WORKED EXAMPLE 11

Tessa wants to buy a dress shop. She borrows \$15 000 at 8.5% p.a. (debited prior to each repayment) of the reducing balance. She can afford quarterly repayments of \$928.45 and this will pay the loan in full in exactly 5 years.

One-third of the quarterly repayment gives the equivalent monthly repayment of \$309.48. The equivalent fortnightly repayment is \$142.84.

Find:

i **the term of the loan and**

ii **the amount still owing prior to the last payment if Tessa made repayments:**

  a **monthly**
  b **fortnightly.**

THINK	WRITE
a i 1 Identify the given values. Enter the appropriate values using Finance Solver. Remember that PpY: = 12 and CpY: = 12 for monthly repayments.	**a i** For monthly repayments: $V_0 = 15\,000$, $d = 309.48$, $I = 8.5\%$ p.a., $n = ?$
2 Solve for n.	$n = 59.582\,518\,723\,273$
3 The value obtained for n is 59.58 which means that a 60th repayment is required. That is, $n = 60$.	$n = 60$ months Term of loan = 5 years
ii 1 To find the amount still owing prior to the last payment, find V_n when $n = 59$. Enter the appropriate values using Finance Solver: $n = 59$ $I = 8.5$ $PV = 15\,000$ $Pmt = -309.48$ PpY: = 12 CpY: = 12	**ii** $V_{59} = -179.273\,603\,537\,66$
2 Solve for V_{59}.	
3 State the amount still owing.	The amount still owing prior to the last payment is \$179.27.
b i 1 Enter the appropriate values using Finance Solver on your CAS. Remember that PpY: = 26 and CpY: = 26 for fortnightly repayments.	**b i** For fortnightly repayments: $V_0 = 15\,000$, $d = 142.84$, $I = 8.5\%$ p.a., $n = ?$

2 Solve for n. $n = 128.847\,104\,593\,38$

3 The value obtained for n is 128.85 which means that a 129th repayment is required.
That is, $n = 129$.

$n = 129$ fortnights
Term of loan = 4 years, 25 fortnights

ii 1 To find the amount still owing prior to the last payment, find V_n (or FV) when $n = 128$. Enter the appropriate values using Finance Solver:

$n = 128$
$I = 8.5$
$PV = 15\,000$
$Pmt = -142.84$
PpY: = 26
CpY: = 26

ii $V_{128} = -120.636\,212\,821\,95$

2 Solve for V_{128}.

3 State the amount still owing.

The amount still owing prior to the last payment is $120.64.

It can be seen from Worked example 11 that while the same outlay is maintained there may be a slight decrease in the term of a loan when repayments are made more often. Let us now find what the saving is for such a loan. In this situation we should consider the final (partial) payment separately because the amount of interest that it attracts is less than a complete repayment, d.

The calculation of the total interest paid is now calculated as usual:

Total interest = total repayments − principal repaid.

WORKED EXAMPLE 12

In Worked example 11, Tessa's $15 000 loan at 8.5% p.a. gave the following three scenarios:

1. quarterly repayments of $928.45 for 5 years

2. monthly repayments of $309.48 for 59 months with $179.27 still outstanding

3. fortnightly repayments of $142.84 for 128 fortnights with $120.64 still owing.

Compare the total interest paid by Tessa if she repaid her loan:

a quarterly **b monthly** **c fortnightly.**

THINK	WRITE
a For quarterly repayments Total interest = total repayments − principal repaid	a For quarterly repayments: total interest $= 928.45 \times 20 - 15\,000$ $= \$3569$

b **1** For monthly repayments Find r and V_{59}. (Refer to Worked example 11.)	**b** For monthly repayments: $r = \frac{8.5}{12}\% = 0.7083\%$ $V_{59} = \$179.27$
2 Calculate the interest on V_{59} to find the final repayment.	$\text{Interest on } V_{59} = 0.7083\% \text{ of } \$179.27$ $= 0.007\,083 \times 179.27$ $= \$1.27$ $\text{Final repayment} = 179.27 + 1.27$ $= \$180.54$
3 Calculate the total interest paid.	$\text{Total interest} = 309.48 \times 59 + 180.54 - 15\,000$ $= \$3439.86$
c **1** Fortnightly repayments Find r and V_{128}. (Refer to Worked example 11.)	**c** For fortnightly repayments: $r = \frac{8.5}{26}\% = 0.3269\%$ $V_{128} = 120.64$
2 Calculate the interest on V_{128} to find the final repayment.	$\text{Interest on } V_{128} = 0.3269\% \text{ of } \$120.64$ $= 0.003\,269 \times 120.64$ $= \$0.39$ $\text{Final repayment} = 120.64 + 0.39$ $= \$121.03$
3 Calculate the total interest paid.	$\text{Total interest} = 142.84 \times 128 + 121.03 - 15\,000$ $= \$3404.55$
4 Calculate the interest saving with monthly repayments over quarterly repayments.	$\text{Monthly interest saving} = 3569 - 3439.86$ $= \$129.14$
5 Calculate the interest saving with fortnightly repayments over quarterly repayments.	$\text{Fortnightly interest saving} = 3569 - 3404.55$ $= \$164.45$
6 Write a comparison statement.	Tessa saves \$164.45 if she repays fortnightly rather than quarterly and \$129.14 if she repays monthly rather than quarterly.

The slight time savings calculated in Worked example 11 when repayments were made more often have now been transformed to money savings. The saving increases as the frequency of repayment increases. This is because the amount outstanding is reduced more often and so the amount of interest added is slightly less. A saving of \$164 over 5 years, out of more than \$18 000 repaid, might not seem much but the saving increases as the term of the loan increases and as the amount borrowed increases.

Changing the rate

Of all the variables associated with reducing balance loans, the one that is most likely to change during the term of a loan is the interest rate. These rates rarely stay the same for the life of a loan; for most loans the rate will change several times.

The Reserve Bank of Australia is the main monetary authority of the Federal Government and, as such, is the overall guiding influence on monetary factors in the Australian economy. Consequently, it indirectly controls the lending interest rates of financial institutions here.

There is usually some variation in rates between institutions; for example, a lower rate may be designed to attract more customers. Within each institution there are rate variations as well for different types of reducing balance loans. Banks advertise their loan rates to attract customers.

In this section we investigate the effect that changing the interest rate has on the term of the loan and on the total interest paid. It should be remembered that as the interest rate increases so too will the term (if d remains constant) of the loan since more interest needs to be paid.

First, let us simply compare loan situations by varying only the rate.

WORKED EXAMPLE 13

A reducing balance loan of $18 000 has been taken out over 5 years at 8% p.a. (adjusted monthly) with monthly repayments of $364.98.

a What is the total interest paid?

b If, instead, the rate was 9% p.a. (adjusted monthly) and the repayments remained the same, what would be:

i the term of the loan

ii the total amount of interest paid?

THINK	WRITE
a For 8% p.a.: Total interest = total repayments − principal repaid	**a** For 8% p.a.: $\text{Total interest} = 364.98 \times 60 - 18\,000$ $= \$3898.80$
b i Using the Finance Solver, enter the appropriate values. Remember that PpY: and CpY: will both equal 12 for monthly repayments. N: = 61.81 means 61 full repayments plus a final lesser payment.	**b i** For 9% p.a.: $V_0 = 18\,000$, $d = 364.98$, $I = 9$, $V_n = 0$, $n = ?$ $n = 61.810\,665\,384\,123$ $n = 62$ months Term = 5 years, 2 months
ii 1 Find V_{61} to calculate the amount still owing. Enter the following values: $n = 61$ $I = 9$ $PV = 18\,000$ $Pmt = -364.98$ PpY: = 12 CpY: = 12 Solve for FV and interpret the result.	**ii** $FV = -293.886\,672\,877$ The amount still owing after 61 repayments is $293.88.

2 Calculate the interest on V_{61} to find the final repayment. $r = \frac{9}{12} = 0.75\%$.	Interest on final repayment $= 0.75\%$ of \$293.88 $= \$2.20$ Final repayment $= 293.88 + 2.20$ $= \$296.08$
3 Total interest = total repayments − principal repaid.	Total interest paid $= 364.98 \times 61 + 296.08 - 18\,000$ $= \$4559.86$

In Worked example 13 the rate was increased by only 1% p.a. on \$18 000 for only 5 years, yet the amount of interest paid has increased from \$3898.80 to \$4559.86, a difference of \$661.06. This difference takes on even more significant proportions over a longer period of time and with a larger principal.

Let us now consider varying the rate during the term of the loan.

WORKED EXAMPLE 14

Natsuko and Hymie took out a loan for home renovations. The loan of \$42 000 was due to run for 10 years and attract interest at 7% p.a., debited quarterly on the outstanding balance. Repayments of \$1468.83 were made each quarter. After 4 years the rate changed to 8% p.a. (debited quarterly). The repayment value didn't change.

a **Find the amount outstanding when the rate changed.**

b **Find the actual term of the loan.**

c **Compare the total interest paid to what it would have been if the rate had remained at 7% p.a. for the 10 years.**

THINK	WRITE
a 1 Rate changes after 4 years; that is, $n = 16$. State V_0 (PV:), d (Pmt:), interest (I(%):) and n (N:).	a $V_0 = 42\,000$, $d = 1468.83$, $I = 7\%$, $n = 16$, $V_n = ?$
2 Find V_{16} (FV:) using a CAS calculator.	FV $= -285\,84.356\,811\,602$ The amount outstanding when the rate changed is \$28 584.36.
b 1 Find n to repay \$28 584.36 in full at the new rate. (Use Finance Solver to enter the following values: $I = 8$, $PV = 28\,584.36$, $Pmt = -1468.83$, FV = 0, PpY: = 4 and CpY: = 4)	b New interest rate of 8%: $n = -24.896\,005\,939\,422$ $= 25$ quarters Time $= 6\frac{1}{4}$ years

2 Find the total term of the loan, that is, time at 7% plus time at 8%.	Term = 4 years + $6\frac{1}{4}$ years = $10\frac{1}{4}$ years
c 1 Find V_{24} to calculate the amount still owing. (That is, on your calculator change the value of *n* to 24 and solve for FV.)	**c** FV = −1291.669 456 564 649 The amount still owing after the 24th repayment is \$1291.61.
2 Calculate the interest on the outstanding amount to find the final repayment. $r = \frac{8}{4} = 2\%$	Interest on final payment = 2% of \$1291.61 = \$25.83 Final repayment = 1291.61 + 25.83 = \$1317.44
3 Find the total interest for the rate change scenario. The number of repayments is 40 at \$1468.83 plus 1 at \$1317.44.	For the rate change scenario, total interest = 1468.83 × 40 + 1317.44 − 42 000 = \$18 070.64
4 Calculate the total interest if the rate remained at 7%.	For the rate at 7% only, total interest = 1468.83 × 40 − 42 000 = \$16 753.20
5 Find the interest difference between the two scenarios.	Interest difference = 18 070.64 − 16 753.20 = \$1317.44
6 Write a comparison statement.	An extra \$1317.44 interest will be paid due to the interest rate change from 7% p.a. to 8% p.a.

In the situations studied so far the repayment value, *d*, remained the same, even though the rate varied. In practice, this is what happens if the rate decreases and so the term of the loan decreases. However, when the rate increases, financial institutions will generally increase the repayment value to maintain the original term of the loan. This was discussed in the section ‘Effects of changing the repayment’. If this is not done the term of the loan can increase quite dramatically. In fact this may occur to such an extent that the repayments are insufficient to cover the interest added, so that the amount outstanding *increases*.

Consider a \$44 000 loan over 15 years at 10% p.a. (monthly).

Monthly repayments = \$472.83

After 5 years the amount owing = \$35 779.02

Suppose the interest rate rises dramatically to 16% p.a.

After a further 10 years under these conditions, the amount owing = \$37 014.72

That is, the amount owing has increased.

This situation is not beneficial to either the lender or the borrower.

Interest only loans

Interest only loans are loans where the borrower makes only the minimum repayment equal to the interest charged on the loan. As the initial principal and amount owing is the same for the period of this loan, we could use either the simple interest formula or CAS to solve problems of this type. When using Finance Solver, the present value (PV:) and future value (FV:) are entered as the same amount. Note that the future value is negative to indicate money owed to the bank.

This type of loan is used by two kinds of borrowers: investors in shares and/or property or families that are experiencing financial difficulties and seek short-term relief from high repayment schedules.

WORKED EXAMPLE 15

Jade wishes to borrow \$40 000 to invest in shares. She uses an interest only loan to minimise her repayments and hopes to realise a capital gain when she sells the shares at a higher value. The term of the loan is 6.9% p.a. compounded monthly with monthly repayments equal to the interest charged.

a **Calculate the monthly interest-only repayment.**

b **If, in 3 years, she sells the shares for \$50 000, calculate the profit she would make on this investment strategy.**

THINK	WRITE
a 1 Identify V_0, r and n, where n is equal to one payment period.	**a** $V_0 = 40\,000$, $r = 6.9\%$ p.a. $n = 1 \text{ month} = \frac{1}{12}\text{th year}$
2 Evaluate I using the simple interest formula.	$I = \frac{V_0 rn}{100}$ $= \frac{40\,000 \times 6.9 \times \frac{1}{12}}{100}$ $= \$230$
3 Write your answer.	The monthly repayment to pay the interest only for the loan is \$230.
b 1 Find whether the capital gain on the shares exceeds the amount paid in interest.	**b** Capital gain = selling price − purchase price = \$50 000 − \$40 000 = \$10 000 Total interest charged = repayment × number of payments = \$230 × 36 = \$8280 Profit = Capital gain − Loan cost = \$10 000 − \$8280 = \$1720
2 Write a statement.	Jade will make a profit of \$1720.

EXERCISE 7.4 Reducing balance loans III

PRACTISE

1 WE10 Nghia borrows \$8000 to upgrade his car. The loan is to be paid in full over 3 years with quarterly payments at 8% p.a.

a Calculate the quarterly payment required.

b Complete an amortisation table for the loan with the following headings.

Payment no.	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	8000			

2 Chau borrows $7500 to pay her university fees. The loan is to be paid in full over 2 years with bi-monthly payments at 9.5% p.a.

a Calculate the bi-monthly payment required.
b Complete an amortisation table for the loan with the following headings.
c Calculate the total interest paid on the loan.

Payment no.	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	7500			

3 WE11 Bella wants to buy a shoe shop. She borrows $18 000 at 8.3% p.a. (debited prior to each repayment) of the reducing balance. She can afford quarterly repayments of $1108.80 and this will pay the loan in full in exactly 5 years.

One third of the quarterly repayment gives the equivalent monthly repayment of $369.60.

The equivalent fortnightly repayment is $170.58.

Find:

i the term of the loan and
ii the amount still owing prior to the last payment if Bella made repayments:
  a monthly   b fortnightly.

4 Simba wants to invest money in shares. He borrows $25 000 at 7.2% p.a. (debited prior to each repayment) of the reducing balance. He can afford quarterly repayments of $1292.02 and this will pay the loan in full in exactly 6 years.

One third of the quarterly repayment gives the equivalent monthly repayment of $430.67.

The equivalent fortnightly repayment is $198.77.

Find:

i the term of the loan and
ii the amount still owing prior to the last payment if Simba made repayments:
  a monthly   b fortnightly.

5 WE12 In question 3, Bella's $18 000 loan at 8.3% p.a. gave the following scenarios:

1. quarterly repayments of $1108.8 for 5 years.
2. monthly repayments of $369.60 for 59 months with $217.37 outstanding.
3. fortnightly repayments of $170.58 for 128 months with $149.63 still owing.

Compare the total interest paid by Bella if she repaid her loan:

a quarterly   b monthly   c fortnightly.

6 In question 4, Simba's $25 000 loan at 7.2% p.a. gave the following scenarios:

1. quarterly repayments of $1292.02 for 6 years.
2. monthly repayments of $430.67 for 71 months with $246.17 outstanding.
3. fortnightly repayments of $198.77 for 154 fortnights with $164.36 still owing.

Compare the total interest paid by Simba if she repaid his loan:

a quarterly   b monthly   c fortnightly.

7 **WE13** A reducing balance loan of \$20 000 has been taken out over 5 years at 10% p.a. (adjusted monthly) with monthly repayments of \$424.94.

a What is the total interest paid?

b If, instead, the rate was 11% p.a. (adjusted monthly) and the repayments remained the same, what would be:

i the term of the loan

ii the total amount of interest paid?

8 A reducing balance loan of \$16 500 has been taken out over 4 years at 8% p.a. (adjusted monthly) with monthly repayments of \$402.81.

a What is the total interest paid?

b If, instead, the rate was 9% p.a. (adjusted monthly) and the repayments remained the same, what would be:

i the term of the loan

ii the total amount of interest paid?

9 **WE14** Mi Lin took out a loan for a new kitchen. The loan of \$32 000 was due to run for 10 years and attract interest at 8% p.a., debited quarterly on the outstanding balance. Repayments of \$1169.78 were made each quarter. After 4 years the rate changed to 9% p.a. (debited quarterly). The repayment value didn't change.

a Find the amount outstanding when the rate changed.

b Find the actual term of the loan.

c Compare the total interest paid to what it would have been if the rate had remained at 8% p.a. for 10 years.

10 Roger took out a loan to get his large pine trees removed after they were affected after a massive storm. The loan of \$28 000 was due to run for 8 years and attract interest at 6.5% p.a., debited monthly on the outstanding balance. Repayments of \$374.81 were made each month. After 3 years the rate changed to 7.5% p.a. (debited monthly). The repayment value didn't change.

a Find the amount outstanding when the rate changed.

b Find the actual term of the loan.

c Compare the total interest paid to what it would have been if the rate had remained at 6.5% p.a. for 8 years.

11 **WE15** Jodie wishes to borrow \$30 000 to invest in shares. She uses an interest only loan minimising her repayments and hopes to realise a capital gain when she sells the shares at a higher value. The term of the loan is 5.8% p.a. compounded monthly with monthly repayments equal to the interest charged.

a Calculate the monthly interest only repayments.

b If, in 3 years, she sells the shares for \$35 000, calculate the profit she would make on this investment strategy.

12 Max wishes to borrow \$50 000 to invest in shares. He uses an interest only loan minimising his repayments and hopes to realise a capital gain when he sells the shares. The term of the loan is 7.5% p.a. compounded monthly with monthly repayments equal to the interest charged.

a Calculate the monthly interest only repayments.

b If, in 3 years, he sells the shares for \$63 000, calculate the profit he would make on this investment.

CONSOLIDATE

13 Harper borrows \$2000 to purchase new cricket gear. The loan is to be paid in full over 1 year with quarterly payments at an interest rate of 7.5% p.a.

a Calculate the quarterly payment required.
b Complete an amortisation table for the loan using a first-order recurrence relation for each payment.
c What is the principal outstanding after the third payment?
d Calculate the total amount paid on the loan.
e Calculate the total interest paid on the loan.

14 Grace takes a loan out for \$3750 to purchase new curtains for her house. The loan is to be paid in full over 1 year with bi-monthly payments at a rate of 6% p.a.

a Calculate the bi-monthly payment required.
b Complete an amortisation table for the loan using a first-order recurrence relation.
c If the payments were monthly instead of bi-monthly what payments are required?
d Complete an amortisation table for the loan with monthly payments.
e Calculate the interest paid on the loan for both monthly and bi-monthly payments and comment on the answers.

15 Phul has a reducing balance loan of \$40 000. The loan has interest charged at 8% p.a. (debited before each repayment) and can be repaid by quarterly instalments of \$1462.23 over exactly 10 years. The equivalent monthly repayment is \$487.41 and the equivalent fortnightly one is \$224.96. Find the term of the loan and the amount still owing before the final repayment if repayments are made:

a monthly **b** fortnightly.

16 A loan of \$25 000 attracts interest at 8.25% p.a. on the outstanding balance and the following four scenarios are available:

1. half-yearly repayments of \$3101.48 for 5 years
2. quarterly repayments of \$1550.74 for $4\frac{3}{4}$ years with \$1217.93 still owing
3. monthly repayments of \$516.91 for 58 months with \$512.33 still owing
4. fortnightly repayments of \$238.58 for 127 fortnights with \$140.27 still owing.

Compare the total interest paid if the loan is repaid:

a half-yearly **b** quarterly **c** monthly **d** fortnightly.

The following information refers to questions **17** and **18**. Betty has borrowed \$65 000 to finance her plant and flower nursery. Betty chooses to repay the loan, which attracts interest at 9.3% p.a. on the outstanding balance, by fortnightly repayments of \$309.66 rather than the equivalent monthly repayment of \$670.92.

17 The term of the loan will be:

A 14 years 11 months
B 14 years 25 fortnights
C 15 years
D 15 years 1 fortnight
E 15 years 1 month

18 The amount Betty will save is closest to:

A \$240 **B** \$260 **C** \$270
D \$300 **E** \$320

19 A reducing balance loan of \$25 000 has been taken out over 5 years at 8% p.a. (adjusted monthly) with monthly repayments of \$506.91.

a What is the total interest paid?

b If, instead, the rate was 9% p.a. (adjusted monthly) with the same repayments maintained, what would be:

i the term of the loan now **ii** the total interest paid?

c If, instead, the rate was 10% p.a. (adjusted monthly) with the same repayments maintained, what would be:

i the term of the loan now **ii** the total interest paid?

20 Andrew's reducing balance loan of \$52 000 over 15 years attracts interest at 11.75% p.a. (adjusted fortnightly). Repayments of \$283.92 per fortnight are made.

During the loan the interest rate is increased to 12.5% p.a. (adjusted fortnightly), but the fortnightly repayment remains unchanged. Find:

a the amount outstanding when the rate changes

b the actual term of the loan

c the total interest paid compared to what it would have been if the rate had remained at 11.75% p.a. for the 15 years if the rate changed after:

i 2 years **ii** $12\frac{1}{2}$ years.

The following information relates to questions **21** and **22**. Clint's \$28 000 loan for his house extensions has interest debited every month at 12% p.a. of the outstanding balance. The loan was due to run for 10 years and he was to make repayments of \$401.72 per month to service the loan. After he had made 50 repayments his credit union reduced the interest rate to 10.75% p.a. (adjusted monthly) for the remainder of the loan.

21 If Clint maintained the monthly repayment, the term of the loan would be:

A $9\frac{3}{4}$ years **B** $10\frac{1}{4}$ years **C** $10\frac{3}{4}$ years **D** 11 years **E** 12 years

22 The total interest paid by Clint would lie between:

A \$18 700 and \$18 800 **B** \$18 800 and \$18 900
C \$18 900 and \$19 000 **D** \$46 800 and \$46 900
E \$46 900 and \$47 000

MASTER

23 The Risky brothers want to invest in \$140 000 worth of shares. They use other people's money and take out an interest only loan from the bank. The loan is at 10.8% p.a. compounded quarterly with quarterly repayments.

a Calculate the quarterly repayment amount.

b If in 1 year they sell the shares for \$152 000, calculate the amount of profit or loss they made on this investment strategy.

24 The Bigs have had a new addition to the family and John, the father, takes 12 months of leave from work to stay at home. To financially cope, they ask their bank manager for an interest only loan for this period on the outstanding amount on their home loan, which is currently \$210 000. If the terms of the interest only loan are 6.79% p.a. compounded fortnightly, the fortnightly repayments will be closest to:

A \$548.42 **B** \$1188.25 **C** \$14 637
D \$4879 **E** \$21 032.18

7.5 Reducing balance and flat rate loan comparisons

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Interactivity
Reducing balance and flat rate loan comparison
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As we have seen in previous sections, with reducing balance loans, interest is calculated on the current balance and debited to the loan account at regular intervals just before repayments are made. Since the balance continually reduces, the amount of interest charged also reduces.

In contrast, **flat rate loans** charge a fixed amount of interest as a percentage of the original amount borrowed. This is calculated at the start of a loan and added to the amount borrowed. Since it is a flat rate based on a fixed amount, the simple interest formula is used to calculate the interest:

$$I = \frac{V_0 rn}{100}.$$

Let us compare the two types of loan under similar circumstances.

WORKED EXAMPLE 16

A loan of $12 000 is taken out over 5 years at 12% p.a. Find:

a the monthly repayment

b the total amount of interest paid

if the money is borrowed on:

i a flat rate loan **ii a reducing balance loan.**

THINK	WRITE
a i 1 For a flat rate loan: State V_0, r and n. Find the interest by using $I = \frac{V_0 rn}{100}$.	**a i** For a flat rate loan: $V_0 = 12\,000,\ r = 12,\ n = 5$ $I = \frac{12\,000 \times 12 \times 5}{100}$ $= \$7200$
2 Find the amount to repay, V_n. Total repaid = $V_0 + I$.	$V_n = V_0 + I$ $= 12\,000 + 7200$ $= \$19\,200$
3 Find the monthly repayment. (First find n.)	$n = 12 \times 5$ $= 60$ Repayment $= 19\,200 \div 60$ $= \$320$/month
b i State the total amount of interest paid.	**b i** The total amount of interest paid is $7200.
a ii For a reducing balance loan: (a) Find V_0, n, r and R.	**a ii** For a reducing balance loan: $V_0 = 12\,000,\quad n = 5 \times 12$ $= 60$ $r = \frac{12}{12}$ $= 1$ $R = 1.01$
(b) Find the monthly repayment, d, using $d = \frac{V_0 R^n (R - 1)}{R^n - 1}$.	$d = \frac{12\,000(1.01)^{60}(1.01 - 1)}{1.01^{60} - 1}$ $= \$266.93$/month The monthly repayment is $266.93.

b ii Find the total interest paid (total repayments − principal repaid).	**b ii** Total interest paid $= 266.93 \times 60 - 12\,000$ $= \$4015.80$

In Worked example 16, the difference between the two loan types is significant. For the reducing balance loan, each month \$53.07 less is repaid and overall \$3184.20 less interest is paid.

The percentage saving over this short loan is:

$$\begin{aligned}\text{percentage interest saving} &= \frac{3184.20}{7200} \times 100\% \\ &= 44.23\%.\end{aligned}$$

Choosing a reducing balance loan rather than a flat rate loan results in a smaller repayment value or a shorter term and in both cases an interest saving. Now let us consider what flat rate of interest is equivalent to the rate for a reducing balance loan.

WORKED EXAMPLE 17

A reducing balance loan of \$25 000 is repaid over 8 years with monthly instalments and interest charged at 9% p.a. (debited monthly).

Find:

a **the repayment value**

b **the total amount of interest paid**

c **the equivalent flat rate of interest for a loan in which all other variables are the same.**

THINK	WRITE
a For a reducing balance loan: First find V_0, n, r and R.	**a** For a reducing balance loan: $V_0 = 25\,000,\ n = 8 \times 12$ $= 96.$ $r = \frac{9}{12} = 0.75$ $R = 1.0075$
Find d.	$d = \frac{V_0R^n(R-1)}{R^n - 1}$ $= \frac{25\,000(1.0075)^{96}(1.0075 - 1)}{(1.0075^{96} - 1)}$ $= \$366.26$
b Total interest = total repayments − principal repaid	**b** Total interest $= 366.26 \times 96 - 25\,000$ $= \$10\,160.96$
c For a flat rate loan: (i) State I, V_0 and n.	**c** For a flat rate loan: I = interest from reducing balance loan $= \$10\,160.96$ $V_0 = 25\,000,\ n = 8$
(ii) Find r using $I = \frac{V_0rn}{100}$.	$10\,160.96 = \frac{25\,000 \times r \times 8}{100}$ $= 2000 \times r$ $r = 5.08\%$ The equivalent flat rate of interest is 5.08%.

Worked example 17 illustrates that an interest rate of 9% p.a. on the outstanding balance is equivalent to a flat rate of only 5.08% p.a., which again is a major difference between the two loan types.

Finally, we consider the effect on the amount that can be borrowed at a given rate for both types of loan.

WORKED EXAMPLE 18

A loan of \$76 000 is repaid over 20 years by quarterly instalments of \$2205.98 and interest is charged quarterly at 10% p.a. of the outstanding balance.

Find:

a **the total amount of interest paid**

b **the amount which can be borrowed on a flat rate loan in which all other variables are the same as above**

c **the difference in the amount borrowed between the two types of loan.**

THINK	WRITE
a **1** For a reducing balance loan: Find the total interest.	**a** For a reducing balance loan: $n = 20 \times 4$ $= 80$ $\text{Interest} = 2205.98 \times 80 - 76\,000$ $= \$100\,478.40$
2 Write a statement.	The total amount of interest paid is \$100 478.40.
b **1** For a flat rate loan: (a) State I, r and n. I is the same as for the reducing balance loan. (b) Use the simple interest formula to find V_0.	**b** For a flat rate loan: $I = 100\,478.40,\ r = 10,\ n = 20$ $I = \dfrac{V_0 rn}{100}$ $100\,478.40 = \dfrac{V_0 \times 10 \times 20}{100}$ $= 2 \times V_0$ $V_0 = \$50\,239.20$
2 Write a statement.	For a flat rate loan, \$50 239.20 can be borrowed.
c **1** Find the difference between the principals for the two loan types.	**c** The difference in the amount borrowed $= 76\,000 - 50\,239.20$ $= \$25\,760.80$
2 Write a statement.	Under the same conditions a \$76 000 reducing balance loan is equivalent to a \$50 239.20 flat rate loan.

The greater financial benefit of the reducing balance loan over the flat rate loan is again evident, this time in terms of the amount that can be borrowed in the first place.

EXERCISE 7.5 Reducing balance and flat rate loan comparisons

PRACTISE

1 **WE16** A loan of \$10 000 is taken out over 5 years at 10% p.a. Find:

- **a** the monthly repayment
- **b** the total amount of interest paid if the money is borrowed on:
  - **i** a flat rate loan
  - **ii** a reducing balance loan.

2 A loan of \$17 550 is taken out over 7 years at 8.3% p.a. Find:

a the monthly repayment
b the total amount of interest paid

if the money is borrowed on:

i a flat rate loan
ii a reducing balance loan.

3 **WE17** A reducing balance loan of \$24 000 is repaid over 10 years with monthly instalments and interest charged at 6.6% p.a. (debited monthly). Calculate:

a the repayment value
b the total amount of interest paid
c the equivalent flat rate of interest p.a. for a loan in which all other variables are the same.

4 A reducing balance loan of \$22 000 is repaid over 7 years with monthly instalments and interest charged at 8% p.a. (debited monthly).

Find:

a the repayment value
b the total amount of interest paid
c the equivalent flat rate of interest for a loan in which all other variables are the same.

5 **WE18** Alice takes out a reducing balance loan for \$27 000 and will repay it over 5 years by quarterly instalments of \$1899.75 at an interest rate of 14% p.a. (debited quarterly). Calculate:

a the total amount of interest to be paid
b the amount which Alice could have borrowed if, instead, she had chosen a flat rate loan with the same details except the principal
c the difference in the amount borrowed for the two different loan scenarios.

6 Sarah's reducing balance loan of \$9000 is repaid over 2 years with fortnightly instalments and interest charged at 8.4% p.a. (debited fortnightly). If a flat rate loan was repaid over the same time, at the same rate and with the same repayment value, the amount borrowed would be closest to:

A \$4500 **B** \$5000 **C** \$5500 **D** \$8500 **E** \$9000

CONSOLIDATE

7 Calculate the monthly repayment for money borrowed on:

a a flat rate loan
b a reducing balance loan

if the loan was for:

i \$15 000 over 5 years at 9% p.a. **ii** \$30 000 over 10 years at 8% p.a.

8 For the loan situations outlined in question **7** calculate the total amount of interest paid for both types of loan.

9 Calculate the quarterly repayment for money borrowed on:

a a flat rate loan
b a reducing balance loan

if the loan was for:

i \$8000 over 3 years at 6% p.a. **ii** \$28 000 over 8 years at 10% p.a.

10 For the loan situations outlined in question **9** calculate the total amount of interest paid for both types of loan.

11 Minnie's loan of \$6000 is taken out over 2 years at 11% p.a. Calculate:

a the monthly repayment
b the total amount of interest paid
if the money was borrowed on:
i a flat rate loan   **ii** a reducing balance loan.

12 The flat rate loans outlined below are repaid by monthly instalments. For each loan, calculate:

i the total amount of interest paid
ii the monthly repayment
iii the term of a reducing balance loan which has the same principal, monthly repayment and interest rate
iv the interest saving achieved by using the reducing balance option in part **iii**.

a \$14 000 borrowed over 5 years at 9% p.a.
b \$21 000 borrowed over 5 years at 8% p.a.
c \$90 000 borrowed over 20 years at 12% p.a.

13 Mike borrows \$24 000 over 6 years at a flat rate of 10% p.a. and agrees to repay the loan with monthly repayments.

a Calculate the total interest charged.
b Calculate Mike's monthly repayment.
c Calculate the term of the loan if Mike had borrowed the money on a reducing balance loan which had the same monthly repayment and interest rate.
d Calculate the interest saving Mike would have achieved if he had used the reducing balance option.

14 If \$11 000 is repaid over 5 years with monthly instalments, the loan that would require the greatest repayment value would be:

A a flat rate loan at 9% p.a.
B a flat rate loan at 10% p.a.
C a reducing balance loan at 11% p.a.
D a reducing balance loan at 12% p.a.
E a reducing balance loan at 13% p.a.

15 Aaron repaid a loan of \$14 000 over 3 years with quarterly instalments and interest charged at 7.6% p.a. of the reducing balance, debited quarterly. Calculate:

a the repayment value
b the total amount of interest paid
c the rate of interest for the equivalent flat rate loan in which all other variables were the same.

16 A reducing balance loan of \$19 000 is repaid over 4 years with monthly instalments and interest charged at 7.5% p.a. (debited monthly). The flat rate of interest that would allow \$19 000 to be borrowed over the same time and with the same repayments would be closest to:

A 7.5% p.a.   **B** 7% p.a.   **C** 4% p.a.   **D** 4.5% p.a.   **E** 5% p.a.

MASTER

17 Rachel repaid a loan of \$46 000 over 10 years with fortnightly instalments and interest charged at 9.3% p.a. of the reducing balance, debited fortnightly. Calculate:

a the repayment value
b the total amount of interest paid
c the rate of interest for the equivalent flat rate loan in which all other variables were the same.

18 A loan is repaid over 20 years by monthly instalments with interest charged monthly at 9% p.a. of the outstanding balance. Calculate:

i the total amount of interest paid

ii the amount which can be borrowed on a flat rate loan in which all other variables are the same as above

if the amount borrowed on the reducing balance loan was:

a $65 000 and the repayment is $584.82

b $84 000 and the repayment is $755.77

c $54 000 and the repayment is $485.85.

7.6 Effective annual interest rate

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Unit 3
AOS R&FM
Topic 2
Concept 4

Nominal and effective interest rates
Concept summary
Practice questions

Previously we have looked at paying off a loan at a set interest rate, however we have found the amount of interest paid would vary with different compounding terms (daily, weekly, monthly, etc.). The **effective annual interest rate** is used to compare the annual interest between loans with these different compounding terms.

To calculate the effective annual interest rate, use the formula:

$$r = \left(1 + \frac{i}{n}\right)^n - 1$$

where

r **= the effective annual interest rate**

i **= the nominal rate, as a decimal**

n **= the number of compounding periods per year**

For a loan of $100 at 10% p.a. compounding quarterly over 2 years.

The effective annual interest rate:

$$\begin{aligned} r &= \left(1 + \frac{i}{n}\right)^n - 1 \\ &= \left(1 + \frac{0.10}{4}\right)^4 - 1 \\ &= 0.1038 \\ &= 10.38\% \end{aligned}$$

This means that the effective annual interest rate is actually 10.38% and not 10%. The comparison between the two can be shown in the following table.

Period	Amount owing ($)	Annual effective rate calculation ($)
1	$100\left(1 + \frac{0.10}{4}\right) = 102.50$	
2	$102.50\left(1 + \frac{0.10}{4}\right) = 105.06$	
3	$105.06\left(1 + \frac{0.10}{4}\right) = 107.69$	
4 (Year 1)	$107.69\left(1 + \frac{0.10}{4}\right) = 110.38$	$100\left(1 + \frac{10.38}{100}\right)^1 = 110.38$
5	$110.38\left(1 + \frac{0.10}{4}\right) = 113.14$	

Period	Amount owing ($)	Annual effective rate calculation ($)
6	$113.14\left(1+\frac{0.10}{4}\right)=115.97$	
7	$115.97\left(1+\frac{0.10}{4}\right)=118.87$	
8 (Year 2)	$118.87\left(1+\frac{0.10}{4}\right)=121.84$	$100\left(1+\frac{10.38}{100}\right)^2=121.84$

WORKED EXAMPLE 19

Jason decides to borrow money for a holiday. If a personal loan is taken over 4 years with equal quarterly repayments compounding at 12% p.a., calculate the effective annual rate of interest (correct to 2 decimal places).

THINK	WRITE
1 Write the values for i and n.	$n = 4$ (since quarterly) $i = 0.12$ (12% as a decimal)
2 Write the formula for effective annual rate of interest.	$r=\left(1+\frac{i}{n}\right)^n-1$
3 Substitute $n = 4$ and $i = 0.12$.	$r=\left(1+\frac{0.12}{4}\right)^4-1$ $= 0.1255$ $= 12.55\%$
4 Write your answer.	The effective annual interest rate is 12.55% p.a. for a loan of 12%, correct to 2 decimal places.

EXERCISE 7.6 Effective annual interest rate

PRACTISE

1 **WE19** Brad wants to take his family to Africa. He decides to take out a loan over 4 years with equal quarterly repayments compounding at 11% p.a. Calculate the effective annual rate of interest (correct to 1 decimal place).

2 Caitlin wants to landscape her garden. She decides to take out a loan over 3 years with equal quarterly repayments compounding at 14% p.a. Calculate the effective annual rate of interest (correct to 1 decimal place).

CONSOLIDATE

3 William is to purchase a new video recorder. If William pays $125 monthly instalments over 3 years at an interest rate of 11.5% p.a. compound interest, what effective annual interest rate is he paying? Give your answer correct to 2 decimal places.

4

Item	Cash price($)	Deposit($)	Monthly instalment ($)	Compound interest rate	Term of loan
a Television	$875	$150		8% p.a.	2 years
b New car	$23 990	$2000		10% p.a.	5 years
c Clothing	$550	$100		7.5% p.a.	1 year
d Refrigerator	$1020	$50		$6\frac{3}{4}\%$ p.a.	2 years

For each of the items in the above table, calculate:

- **i** the monthly instalment on each item
- **ii** the total amount paid over the period given for each item
- **iii** the total amount of interest charged on each item
- **iv** the effective annual interest rate.

5 A camera valued at $1200 is purchased via a loan. A deposit of $200 is required and equal monthly instalments of $64 are paid over the 18-month agreed period. Calculate:

- **a** the compound interest rate per annum
- **b** the effective annual interest rate.

The following information relates to questions **6** and **7**.

The bank approves a personal loan of $5000. An interest rate of 12.5% p.a. is charged, with repayments to be made over a 9-month period in equal weekly instalments.

6 Calculate the weekly instalment.

7 Calculate the effective annual interest rate.

8 Calculate the effective annual interest rate on a loan of $1000 if the monthly repayments are $45 and the loan is to be repaid over 2 years. (*Hint:* First calculate the compound interest rate.)

9 For a compound interest rate of 4.85% p.a. charged on a loan with monthly repayments over 2 years, the effective rate of interest is:

A 4.96% **B** 4.85% **C** 9.3% **D** 5.12% **E** 9.6%

10 A reducing balance loan was used to purchase a home theatre system valued at $2500. If the loan is paid off with quarterly repayments over 3 years at 9.6% p.a., then the effective annual interest rate is closest to:

A 10.03% **B** 8.75% **C** 5.2% **D** 9.6% **E** 9.95%

11 If the effective annual interest rate is 8.5% p.a. on a loan with monthly repayments over 4 years, then the compound interest rate is closest to:

A 9.23% **B** 10% **C** 4.3%
D 8.19% **E** 7.95%

12 Carefully read the advertisement for the purchase of the phone at right and calculate:

- **a** the compound interest rate
- **b** the effective annual interest rate
- **c** the total cost under the loan plan
- **d** the increase in cost over a cash sale.

MASTER

13 Sarah had two options to borrow $10 000 that is to be paid back in 2 years. The two options are:

1. Compounding interest of 9% p.a. compounding monthly.
2. An effective annual interest rate of 9.3%.

With calculations, show which option she should take.

14 Calculate the effective annual interest rate that would require the same amount to be paid back as a $25 000 loan at 8.5% p.a. compounding monthly for 5 years.

7.7 Perpetuities

A **perpetuity** is an annuity where a permanently invested sum of money provides regular payments that continue forever.

Many *scholarships* or *grants* offered to students at universities are provided by funds known as perpetuities.

The funds last for an indefinite period of time as long as the amount paid out is no more than the interest earned on the initial lump sum deposited. The type of investment that is used to earn the interest is usually a bond, which offers a fixed interest amount, paid on a regular basis, over a long period of time. Wealthy people who wish to encourage and support a worthwhile cause usually set up these perpetuities.

The balance of the amount invested does not change and is the same for an indefinite period.

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The perpetuity formula is:

$$d = \frac{V_0 r}{100}$$

where

d = the amount of the regular payment per period ($)

V_0 = the principal ($)

r = the interest rate earned per period (%).

Notes

1. The period of the regular payment must be the same as the period of the given interest rate.
2. Finance Solver can be used in calculations involving perpetuities. As the principal does not change, the present value (PV: or negative cash flow) and the future value (FV: or positive cash flow) are entered as the same amount, but with opposite signs.

WORKED EXAMPLE 20

Robert wishes to use part of his wealth to set up a scholarship fund to help young students from his town further their education at university. Robert invests $200 000 in a bond that offers a long-term guaranteed interest rate of 4% p.a. If the interest is calculated once a year, then the annual amount provided as scholarship will be:

A $188 000
B $288 000
C $666.67
D $8000
E $4000

THINK	WRITE
1 Write the perpetuity formula.	$d = \frac{V_0 r}{100}$
2 List the values of V_0 and r.	V_0 = \$200 000 and r = 4% p.a.
3 Substitute the values into the formula and calculate the amount provided.	$d = \frac{\$200000 \times 4}{100}$ $= 8000$
4 Select the appropriate answer.	The annual amount provided for the scholarship is $8000. Therefore **D** is the correct answer.

Finding V_0 and r

As was the case with earlier sections in this topic, there are calculations where we need to find the principal (V_0) or interest rate (r) needed to provide a certain regular payment (d). For example, how much needs to be invested at 3% p.a. interest to provide a \$10 000 annual grant, or what interest rate is needed so that \$100 000 will provide a \$4000 yearly scholarship indefinitely? Other calculations involve finding what extra amount could be granted annually as a scholarship if the interest is compounded monthly in each year rather than once a year and the scholarship paid in two equal six-monthly instalments.

The perpetuity formula can be transposed to:

$$V_0 = \frac{100 \times d}{r} \text{ and } r = \frac{100 \times d}{V_0}$$

If the frequency of the payments each year is not the same as the compounding period of the given interest rate, then Finance Solver is to be used with different values for PpY and CpY.

Notes:

1. The principal must be known to use Finance Solver.
2. Finance Solver gives the interest rate per annum.

WORKED EXAMPLE 21

A Rotary Club has \$100 000 to set up a perpetuity as a grant for the local junior sporting clubs. The club invests in bonds that return 5.2% p.a. compounded annually.

a **Find the amount of the annual grant.**
b **What interest rate (compounded annually) would be required if the perpetuity is to provide \$6000 each year?**

The Rotary Club wants to investigate other possible arrangements for the structure of the grant.

c **How much extra would the annual grant amount to if the original interest rate was compounded monthly?**

d **What interest rate (compounded monthly) would be required to provide 4 equal payments of \$1500 every 3 months? Give your answer correct to 2 decimal places.**

THINK	WRITE
a 1 Write the perpetuity formula and list the values of V_0 and r.	**a** $d = \frac{V_0 r}{100}$ $V_0 = \$100\,000$ and $r = 5.2\%$ p.a.
2 Substitute the values into the formula and find the value of the annual grant.	$d = \frac{\$100\,000 \times 5.2}{100}$ $= 5200$
3 Write a statement.	The amount of the annual grant is \$5200.

b **1** Write the perpetuity formula and list the values of V_0 and d.

b $r = \dfrac{100 \times d}{V_0}$

$V_0 = \$100\,000$ and

$d = \$6000$

2 Substitute the values into the formula and find the interest rate.

$$r = \frac{100 \times 6000}{100\,000}$$
$$= 6$$

3 Write a statement.

For a \$100 000 perpetuity to provide \$6000 a year, the bond needs to offer an interest rate of 6% p.a.

c **1** As the frequency of the payment is not the same as the compounding period, the perpetuity formula cannot be used. Use Finance Solver and enter the values as follows.

n (N:) = 1

r (I(%):) = 5.2

V_0 (PV:) = −100 000

d (Pmt:) = unknown

V_n (FV:) = 100 000

PpY: = 1 (one payment per year)

CpY: = 12 (there are 12 compound periods per year)

Solve for Pmt.

c Pmt = 5325.741 057 054

If the interest was compounded monthly, the annual grant would amount to \$5325.74.

2 Compare the sizes of the 2 grants and write a statement.

The extra amount is \$5325.74 − \$5200 = \$125.74. If the interest is compounded monthly, the annual grant would increase by \$125.74.

d **1** As the frequency of the payment is not the same as the compounding period, the perpetuity formula cannot be used. Use Finance Solver and enter the values as follows.

n (N:) = 1

r (I(%):) = unknown

V_0 (PV:) = −100 000

d (Pmt:) = 1500

V_n (FV:) = 100 000

PpY: = 4 (four payments per year)

CpY: = 12 (there are 12 compound periods per year)

Solve for I to find the required interest rate.

d I = 5.970 247 527 183

$r = 5.97$

2 Write a statement.	An interest rate of 5.97% p.a. compounded annually is needed to provide four equal payments of $1500, correct to 2 decimal places.

WORKED EXAMPLE 22 A benefactor of a college has been approached to provide a Year 7 scholarship of $1000 per term. He is able to get a financial institution to offer a long-term interest rate of 8% per annum. What is the principal that needs to be invested?

THINK	WRITE
1 Write the perpetuity formula and list the values of d and r. Both d and r need to be expressed in the same period of time.	$V_0 = \dfrac{100 \times d}{r}$ $d = \$1000$ per term (4 terms per year) $R = 8\%$ p.a. $= \dfrac{8}{4}$ $= 2\%$ per term
2 Substitute the values into the formula and find the value of the annual grant.	$V_0 = \dfrac{100 \times 1000}{2}$ $= 50\,000$
3 Write a statement.	The principal that needs to be invested to provide a scholarship of $1000 per term at an annual interest rate of 8% is $50 000.

Note that Finance Solver cannot be used in the previous worked example as the principal is not known. (Both PV: and FV: would be unknowns.)

EXERCISE 7.7 Perpetuities

1 **WE20** Chris wants to invest $150 000 in a bond that offers a long-term guaranteed interest rate of 5% p.a. If the interest is calculated once a year, then the annual interest earned is:

A $142 500 **B** $7500 **C** $130 000
D $6500 **E** $8000

2 Freda chose to invest $95 000 in a bond that offers a long-term guaranteed interest rate of 3.8% p.a. If the interest is calculated once a year, then the annual interest earned is:

A $25 000 **B** $4845 **C** $91 390
D $3610 **E** $5430

3 **WE21** A charity has $75 000 to set up a perpetuity as a grant to help homeless people.

The charity invests in bonds that return 4.8% p.a. compounded annually.

a Find the amount of the annual grant.
b What interest rate (compounded annually) would be required if the perpetuity is to provide $4800 each year?

The charity wants to investigate other possible arrangements for the structure of the grant.

c How much extra would the annual grant amount to if the original interest rate was compounded monthly?

d What interest rate (compounded monthly) would be required to provide 4 equal payments of $1200 every 3 months? Give your answer correct to 2 decimal places.

4 A family wants to use $125 000 to set up a perpetuity as a scholarship fund at their old school.

The charity invests in bonds that return 5.7% p.a. compounded annually.

a Find the amount of the annual grant.

b What interest rate (compounded annually) would be required if the perpetuity is to provide $8000 each year?

The charity wants to investigate other possible arrangements for the structure of the grant.

c How much extra would the annual grant amount to if the original interest rate was compounded monthly?

d What interest rate (compounded monthly) would be required to provide 4 equal payments of $2000 every 3 months? Give your answer correct to 2 decimal places.

5 **WE22** A benefactor of a school had been approached to provide a Year 12 scholarship of $5000 per term. She is able to get a financial institution to offer a long-term interest rate of 6.25% per annum. What is the principal that needs to be invested?

6 A benefactor of a college had been approached to provide a Year 11 scholarship of $2000 per term. He is able to get a financial institution to offer a long-term interest rate of 7.5% per annum. What is the principal that needs to be invested?

CONSOLIDATE

7 The owner of a search engine company uses part of his wealth to set up research grants to help young Australian scientists with their endeavours. He invests $350 000 in a bond that offers a long-term guarantee of 5% p.a. If the interest is calculated once a year, then the annual amount provided as a research grant will be:

A $70 000 **B** $17 500 **C** $1750 **D** $7000 **E** $3500

8 Use the perpetuity formula to calculate the annual payment as specified in each of the following situations.

a $400 000 invested at 4% p.a., paid once a year

b $300 000 invested at 1% per quarter, paid 4 times each year

9 Use the perpetuity formula to calculate the annual payment as specified in each of the following situations.

a $100 000 invested at 12% p.a., calculated monthly, paid out monthly

b $2 million invested at 6% p.a., compounded quarterly, paid out every 3 months

10 Check your answers to questions **8** and **9** using Finance Solver on your CAS.

11 An AFL club has $80 000 to set up a perpetuity as a grant for the local senior sporting clubs. The club invests in bonds that return 4% p.a. compounded annually.

a Find the amount of the annual grant.

b What interest rate (compounded annually) would be required if the perpetuity is to provide $5000 each year?

The club wishes to investigate other possible arrangements for the structure of the grant.

c How much extra would the annual grant amount to if the original interest rate was compounded quarterly?

d What interest rate (compounded monthly) would be required to provide 12 equal monthly payments of $400?

12 Prachi invested $15 000 in a fund that will earn on average 10.5% p.a. over a 1 year period with interest calculated monthly. On top of the initial investment Prachi contributes $250 at the start of each month after the initial investment. Complete the table to find the value of his investment at the end of the sixth month.

Time period	Principal	Interest earned	Balance
1	15 000		
2			
3			
4			
5			
6			

13 Use the perpetuity formula to calculate the interest rate (p.a.) required for each of the following perpetuities.

a $400 000 provides $5000 per annum with interest compounded annually.

b Half a million dollars provides $1000 each month with interest compounded monthly.

c $800 000 provides $30 000 every six months with interest compounded biannually.

d $100 000 provides $200 per fortnight with interest calculated fortnightly.

14 Check your answers to question **13** using Finance Solver on your CAS.

15 Use Finance Solver on your CAS to calculate the interest rate required for each of the following perpetuities. Give your answers correct to 2 decimal places.

a $400 000 provides $5000 per annum with interest compounded monthly.

b Half a million dollars provides $1000 each month with interest compounded annually.

c $800 000 provides $30 000 every six months with interest compounded quarterly.

d $100 000 provides $200 per fortnight with interest compounded monthly.

16 A benefactor of a college has been approached to provide a Year 9 scholarship of $200 per month. He is able to get a financial institution to offer a long-term interest rate of 3.6% per annum, compounded monthly. What is the principal that needs to be invested?

MASTER

17 The total amount given by the perpetuity in question **16** over a 50-year period is:

A $10 000
B $2400
C $57 000
D $66 700
E $120 000

18 Use the perpetuity formula to calculate the initial sum to be invested in a perpetuity specified as follows:

- **a** A \$1200 per annum grant from a fund offering 6% p.a. compounded annually
- **b** A \$10 000 per annum grant from a fund offering 4.5% p.a. compounded annually
- **c** A \$300 per month scholarship from a fund offering 0.5% per month
- **d** A \$120 per month grant from a fund offering 3% p.a. compounded monthly.

7.8 Annuity investments

There are many different ways to invest your money. One of the ways is a managed fund, where you invest an initial principal and rely on the fund managers to invest wisely and hope on a positive percentage return. There are risks involved and losses may occur, along with varying percentage returns from year to year or month to month. Another option with a managed fund is to continue to contribute to the fund, increasing the principal and thus increasing the interest earned. This is another example of a first-order recurrence relation.

WORKED EXAMPLE 23

Johnathan invested \$5000 in a managed fund that will earn an average of 8% p.a. over a 2 year period with interest calculated monthly. If Johnathan contributes \$100 at the start of the second, third, fourth and fifth months, complete the table to find the value of his investment at the end of the fifth month.

Time period	Principal (\$)	Interest earned (\$)	Balance (\$)
1	5000		
2			
3			
4			
5			

THINK

WRITE

1 Calculate the interest earned in the first month, allowing for the monthly interest.

$$I = 5000\left(\frac{0.08}{12}\right)$$
$$= 33.33$$

2 Calculate the balance by adding the initial principal and the interest earned.

$$\text{Balance} = 5000 + 33.33$$
$$= 5033.33$$

3 The new principal at the start of the second period is the sum of the balance at the end of the first period and the \$100 added.

$$\text{Principal} = 5033.33 + 100$$
$$= 5133.33$$

4 Repeat the above steps to complete the table.

Time period	Principal ($)	Interest earned ($)	Balance ($)
1	5000	33.33	5033.33
2	5133.33	34.22	5167.55
3	5267.55	35.12	5302.67
4	5402.67	36.02	5438.69
5	5538.69	36.92	5575.61

5 Write the answer to the value of the investment after 5 months.

After five months the investment is worth $5575.61.

A savings plan, like a Christmas Club account, is an investment where an initial sum as well as regular deposits are made. The interest earned is calculated regularly on the balance of the investment, which increases with each regular deposit (annuity). This is similar to reducing balance loans with the main difference being that the principal amount is *growing*.

An **annuity investment** is an investment that has regular deposits made over a period of time. Let's first look at this as a recurrence relation that calculates the value of the annuity after each time period.

$$V_{n+1} = V_n R + d$$

$$= V_n\left(1 + \frac{r}{100}\right) + d$$

Where: V_{n+1} **= Amount after** $n + 1$ **payments**

V_n **= Amount at time** n

r **= Interest rate per period**

d **= Deposit amount**

WORKED EXAMPLE 24

An initial deposit of $1000 was made on an investment taken out over 5 years at a rate of 5.04% p.a. (interest calculated monthly), and an additional deposit of $100 is made each month. Complete the table below for the first five deposits and calculate how much interest had been earned over this time.

$n+1$	V_n	d	V_{n+1}
1	$1000		
2			
3			
4			
5			

<table>
<tr><th>THINK</th><th>WRITE</th></tr>
<tr><td>1 State the initial values for V_0, r and d.</td><td>$V_0 = \$1000$
$r = \frac{5.04}{12} = 0.42$
$d = \$100$</td></tr>
<tr><td>2 Evaluate V_1.</td><td>$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) + d$
$V_1 = 1000\left(1 + \frac{0.42}{100}\right) + 100$
$= \$1104.20$</td></tr>
<tr><td>3 Evaluate V_2.</td><td>$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) + d$
$V_2 = 1104.20\left(1 + \frac{0.42}{100}\right) + 100$
$= \$1208.84$</td></tr>
<tr><td>4 Evaluate V_3.</td><td>$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) + d$
$V_3 = 1208.84\left(1 + \frac{0.42}{100}\right) + 100$
$= \$1313.92$</td></tr>
<tr><td>5 Evaluate V_4.</td><td>$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) + d$
$V_4 = 1313.92\left(1 + \frac{0.42}{100}\right) + 100$
$= \$1419.44$</td></tr>
<tr><td>6 Evaluate V_5.</td><td>$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) + d$
$V_5 = 1419.44\left(1 + \frac{0.42}{100}\right) + 100$
$= \$1525.40$</td></tr>
<tr><td>7 Complete the table.</td><td>
<table>
<tr><th>$n+1$</th><th>V_n</th><th>d</th><th>V_{n+1}</th></tr>
<tr><td>1</td><td>$V_0 = \$1000$</td><td>$100</td><td>$V_1 = \$1104.20$</td></tr>
<tr><td>2</td><td>$V_1 = \$1104.20$</td><td>$100</td><td>$V_2 = \$1208.84$</td></tr>
<tr><td>3</td><td>$V_2 = \$1208.84$</td><td>$100</td><td>$V_3 = \$1313.92$</td></tr>
<tr><td>4</td><td>$V_3 = \$1313.92$</td><td>$100</td><td>$V_4 = \$1419.44$</td></tr>
<tr><td>5</td><td>$V_4 = \$1419.44$</td><td>$100</td><td>$V_5 = \$1525.40$</td></tr>
</table>
</td></tr>
<tr><td>8 Calculate the interest by subtracting the initial investment ($1000) and the amount deposited (5 × $100) from the amount at the end of the 5 deposits.</td><td>$I = \$1525.40 - \$1000 - \$500$
$= \$25.40$</td></tr>
<tr><td>9 State the answer.</td><td>The interest earned over the first five months was $25.40.</td></tr>
</table>

Superannuation

Most, if not all, Australians will have to provide for themselves in their retirement rather than relying on the government's age pensions. To provide for their future, all working Australians have a **superannuation** fund into which money is contributed by their employers, and optionally topped up by the employee, each pay period. The sum accumulates over many years, until retirement age, when the money can be withdrawn. The funds can then be placed into an annuity or perpetuity that pays for the retiree's living expenses and lifestyle.

This is where superannuation calculations become tricky. You need to work out the amount of money you must save to give you that retirement income. This depends on all sorts of variables, such as the initial deposit, regular instalments, investment returns, inflation and tax rates.

The accumulated money, deposited by the workers, is invested in shares and properties, over many years by financial institutions (also known as superannuation fund managers). The performance of these superannuation fund managers varies from year to year. For the scope of this exercise, we will assume a constant rate of return (interest rates remain the same). Also, the effects of inflation and taxation will not be considered.

The money that accumulates in these annuity investments can be calculated using the annuities formula in a similar way to that used in reducing balance loans. The difference is that the amount (V_n) *grows with the addition of a regular payment* (d). The formula is:

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$$V_n = V_0R^n + \frac{d(R^n - 1)}{R - 1}$$

where:

V_0 **= the initial amount invested**

R **= the compounding or growth factor**

$= 1 + \frac{r}{100}$ **(r = the interest rate per payment period)**

d **= the amount of the regular deposits made per period**

n **= the number of deposits**

V_n **= the balance after n deposits.**

Finance Solver can also be used in a similar way to reducing balance loans, with one difference — the cash flows are reversed (opposite signs).

WORKED EXAMPLE 25

Helen currently has \$2000 in a savings account that is averaging an interest rate of 8% p.a. compounded annually. She wants to calculate the amount that she will receive in 5 years when she plans to go on an overseas trip.

a **If she deposits \$6000 each year, find (correct to the nearest \$1000) the amount available for her overseas trip.**

b **If she places her \$2000 and increases her deposits to \$7000 each year into a different savings account that can offer 9% p.a. compounded annually, find (correct to the nearest \$1000) the amount available for her overseas trip.**

c **Calculate the extra amount saved by investing \$7000 each year at 9% p.a. compared with \$6000 each year at 8% p.a.**

THINK	WRITE
a **1** State the value of V_0, d, n, r and R.	**a** $V_0 = \$2000,$ $d = \$6000,$ $n = 5,$ $r = 8$ and $R = 1.08$
2 Write the annuities formula and substitute in the values.	$V_n = V_0R^n + \dfrac{d(R^n - 1)}{R - 1}$ $V_5 = 2000 \times 1.08^5 + \dfrac{6000(1.08^5 - 1)}{1.08 - 1}$
3 Evaluate V_5.	$= \$38\,138.26$
4 Write a statement, rounding the answer correctly.	The final balance of the investment after 5 years is \$38 000, correct to the nearest \$1000.
b **1** State the value of V_0, d, n, r and R.	**b** $V_0 = \$2000,$ $d = \$7000,$ $n = 5,$ $r = 9$ and $R = 1.09$
2 Write the annuities formula and substitute in the values.	$V_n = PR^n + \dfrac{d(R^n - 1)}{R - 1}$ $V_5 = 2000 \times 1.09^5 + \dfrac{7000(1.09^5 - 1)}{1.09 - 1}$
3 Evaluate V_5.	$= \$44\,970.22$
4 Write a statement, rounding the answer correctly.	The final balance of the investment after 5 years, if Helen deposits \$7000 each year into an account offering 9% p.a., would be \$45 000, correct to the nearest \$1000.
c The extra amount saved is the difference between the amounts found in parts **a** and **b**.	**c** The extra amount is \$45 000 − \$38 000 = \$7000.

Planning for retirement

When do you want to retire, and how many years will you want to spend in retirement? How much money will you need? An Australian male is now expected to live to 80 years of age and an Australian female to 84 years of age. So, most people will need to plan on 20 to 25 years in retirement. A common target is

60 to 65 per cent of your pre-retirement income (bearing in mind that you won't have to pay for expenses such as commuting, work clothes and, hopefully, mortgage repayments by then, and you may be entitled to a part pension and/or tax concessions). So, if you earn \$60 000 a year now, a starting point might be to think of a retirement income of about \$36 000 a year. Remember, these figures are in today's dollars.

Planning for retirement is an issue that you'll need to revise regularly, maybe with a financial planner. The annuities formula and Finance Solver can be used to calculate how much money is needed under different financial situations.

WORKED EXAMPLE 26

Andrew is aged 45 and is planning to retire at 65 years of age. He estimates that he needs \$480 000 to provide for his retirement. His current superannuation fund has a balance of \$60 000 and is delivering 7% p.a. compounded monthly.

a **Find the monthly contributions needed to meet the retirement lump sum target.**

b **If in the final ten years before retirement, Andrew doubles his monthly contribution calculated from a, find the new lump sum amount available for retirement.**

c **How much extra could Andrew expect if the interest rate from part b is increased to 9% p.a. (for the final 10 years) compounded monthly? Round the answer correct to the nearest \$1000.**

THINK	WRITE
a **1** Identify the initial amount (V_0) and the final amount (V_n).	**a** $V_0 = \$60\,000,\ V_n = \$480\,000$
2 Find the number of payments, n, the interest rate per month, r, and the growth factor, R.	$n = 20 \times 12 = 240$ $r = 7\%$ per annum $= \frac{7}{12}\%$ per annum $= 0.58\dot{3}$ $R = 1 + \frac{r}{100}$ $= 1.005\,8\dot{3}$
3 Write the annuities formula and substitute in the values.	$V_n = V_0R^n + \frac{d(R^n - 1)}{R - 1}$ $480\,000 = 60\,000 \times 1.005\,8\dot{3}^{240} + \frac{d(1.005\,8\dot{3}^{240} - 1)}{(1.005\,8\dot{3} - 1)}$ $480\,000 = 242\,324.33 + \frac{d \times 3.038\,738\,849}{0.005\,8\dot{3}}$ $= 242\,324.33 + d \times 520.93$
4 Evaluate d.	$d = \$456.26$
5 Write a statement.	The monthly contribution to achieve a retirement lump sum of \$480 000 is \$456.26.

b **1** We need to find the balance after the first ten years with $d = \$456.26$ and $n = 120$. Enter the values into the formula and solve for V_{120}.

b $$V_n = V_0R^n + \frac{d(R^n - 1)}{R - 1}$$

$$V_{120} = 60\,000 \times 1.0058\dot{3}^{120} + \frac{456.26 \times (1.0058\dot{3}^{120} - 1)}{1.0058\dot{3} - 1}$$

$$= \$199\,551.36$$

2 State the values used for the final ten years and substitute them into the formula.

$V_0 = \$199\,551.36$, $n = 120$, $R = 1.0058\dot{3}$ and

$d = \$912.52 \quad (2 \times 456.26)$

$$V_n = V_0R^n + \frac{d(R^n - 1)}{R - 1}$$

$$V_{120} = 199\,551.36 \times 1.0058\dot{3}^{120} + \frac{912.52 \times (1.0058\dot{3}^{120} - 1)}{1.0058\dot{3} - 1}$$

3 State the new value of V_{120}.

$$= \$558\,974.01$$

4 Write a statement.

The new lump sum will be \$558 974.00.

c **1** Calculate the new growth factor.

c $r = 9\%$ per annum

$= \frac{9}{12}\%$ per month

$= 0.75$

$R = 1 + \frac{r}{100}$

$= 1.0075$

2 Substitute the values into the formula for V_n and evaluate.

$$V_n = V_0R^n + \frac{d(R^n - 1)}{R - 1}$$

$$V_{120} = 199\,551.36 \times 1.0075^{120} + \frac{912.52 \times (1.0075^{120}) - 1}{1.0075 - 1}$$

$$= \$665\,757.29$$

3 Find how much extra is expected by finding the difference between the two amounts.

The difference expected is \$665 757.29 − \$558 974.01 = \$106 783.28.

4 Write a statement, rounding the answer appropriately.

If the interest rate is increased to 9% for the final 10 years, Andrew could expect an extra \$107 000, correct to the nearest \$1000.

Once a lump sum has been realised, the funds are transferred or rolled over to a suitable annuity. This annuity will then provide a regular income to live on. There are two options:

1. Perpetuities. As seen in the previous exercise, these annuities provide a regular payment forever. This has two benefits. Firstly, it will provide for the retiree no matter how long they live and secondly, the perpetuity could be willed to relatives who in turn will collect the same annuity indefinitely.

2. Annuity — reducing balance. This is the same as reducing balance loans except the fund manager borrows the money and pays the retiree a regular income for a specified number of years. The main disadvantage is if the retiree outlives the term of the reducing balance annuity; that is, the money runs out.

WORKED EXAMPLE 27

Jarrod is aged 50 and is planning to retire at 55. His annual salary is \$70 000 and his employer contributions are 9% of his gross monthly income. Jarrod also contributes a further \$500 a month as a salary sacrifice (that is, he pays \$500 from his salary into the superannuation fund). The superfund has been returning an interest rate of 7.2% p.a. compounded monthly and his current balance in the superfund is \$255 000.

a **Calculate Jarrod's total monthly contribution to the superannuation fund.**

b **Calculate the lump sum that he can receive for his planned retirement at age 55.**

Jarrod has two options for setting up an annuity to provide a regular income after he retires at 55.

1. **A perpetuity that offers monthly payments at 8% p.a. compounded monthly.**
2. **A reducing balance annuity, also paid monthly at 8% p.a., compounded monthly.**

c **Calculate the monthly annuity using option 1. Express the annual salary from this option as a percentage of his current salary.**

d **Calculate the monthly annuity using option 2 if the fund needs to last for 25 years. Express the annual salary from this option as a percentage of his current salary.**

THINK	WRITE
a 1 Calculate the total contributions made by Jarrod's employer.	**a** The employer contribution is 9% of the gross monthly income. 9% of $\frac{70000}{12}$ $= 0.09 \times 5833.33$ $= \$525$
2 Calculate the total monthly contribution made to the superannuation fund.	The total monthly contributions are the employer contributions and Jarrod's contribution. That is, \$525 + \$500 = \$1025.
b 1 State the value of V_0, d, n and R.	**b** $V_0 = \$255\,000$, $d = \$1025$, $n = 60$, $R = 1.006$
2 Write the annuities formula and substitute in the values of the pronumerals. Evaluate.	$V_n = V_0 R^n + \frac{d(R^n - 1)}{R - 1}$ $V_{60} = 255\,000 \times 1.006^{60} + \frac{1025(1.006^{60} - 1)}{1.006^{60} - 1}$ $= \$438\,869.90$

3	Write a statement.	The lump sum available for retirement is \$438 869.90.
c 1	Write the perpetuity formula and list the values of V_0 and r.	**c** $d = \frac{V_0 r}{100}$ $\quad V_0 = \$438\,869.90$ and $r = 8\%$ p.a.
2	Substitute the values into the formula and find the value of the annual payment.	$d = \frac{438\,869.90 \times 8}{100}$ $= \$35\,109.59$ per year
3	Calculate the monthly payment.	The monthly payment is $\frac{35\,109.59}{12} = \$2925.80$.
4	Express the yearly payment as a percentage of his current annual salary.	$\frac{35\,109.59}{70\,000} \times 100\%$ $= 50.16\%$
5	Write a statement.	The perpetuity will provide \$2925.80 a month which is equivalent to 50.16% of his current salary (in today's value of the money).
d 1	State the values of V_0, A, n and R.	**d** $V_0 = \$438\,869.90$, $A = 0$, $n = 300(25 \times 12)$, $R = 1.00\dot{6}$
2	Write the annuities formula and substitute in the values.	$V_n = V_0 R^n - \frac{d(R^n - 1)}{R - 1}$ $0 = 438\,869.90 \times 1.00\dot{6}^{300} - \frac{d(1.00\dot{6}^{300} - 1)}{1.00\dot{6} - 1}$
3	Calculate the monthly payment.	$d = \$3387.27$ The monthly payment is \$3387.27.
4	Express the yearly payment as a percentage of his current annual salary.	The yearly payment will be \$3387.27 × 12 = \$40 647.24. As a percentage of his current annual salary, $\frac{40\,647.24}{70\,000} \times 100\% = 58.07\%$
5	Write a statement.	The reducing balance annuity will provide \$3387.27 a month which is equivalent to 58.07% of his current salary (in today's value of the money).

The treatment of superannuation in this exercise presents the basics only, but includes most of the mathematical techniques for analysing a retirement plan and understanding annuity investments. There are other issues or factors such as taxation laws, the effects of inflation and other sources that can contribute to a retirement plan. These are not required for the Further Mathematics course.

EXERCISE 7.8 Annuity investments

PRACTISE

1 **WE23** Clifford invested \$7000 in a managed fund that will earn on average 10% p.a. over a 3 year period with interest calculated monthly. If Clifford contributes \$150 at the start of each month after the initial investment, complete the table to find the value of his investment at the end of the fifth month.

Time period	Principal (\$)	Interest earned (\$)	Balance (\$)
1	7000		
2			
3			
4			
5			

2 Pauline invested \$10 000 in a fund that will earn on average 12% p.a. over a 1 year period with interest calculated monthly. If Pauline contributes \$200 at the start of each month after the initial investment, complete the table to find the value of her investment at the end of the sixth month.

Time period	Principal (\$)	Interest earned (\$)	Balance (\$)
1	10 000		
2			
3			
4			
5			
6			

3 **WE24** An initial deposit of \$5000 was made on an investment taken out over 10 years at a rate of 7.5% p.a. (interest calculated monthly) and an additional deposit of \$100 is made each month. Complete the table below for the first five deposits and calculate how much interest had been earned over this time.

$n + 1$	V_n	d	V_{n+1}
1	\$5000		
2			
3			
4			
5			

4 An initial deposit of \$10 000 was made on an investment taken out over 7 years at a rate of 8% p.a. (interest calculated monthly) and an additional deposit of \$150 is made each month. Complete the table below for the first five deposits and calculate how much interest had been earned during the fifth month.

$n+1$	V_n	d	V_{n+1}
1	\$10 000		
2			
3			
4			
5			

5 **WE25** Sharyn has \$2500 in a savings account that is averaging an interest rate of 6.5% p.a. compounded annually. She wants to calculate the amount she will receive in 5 years when she plans to buy a car.

a If she deposits \$6500 each year, find (correct to the nearest \$1000) the amount available for her car.

b If she places her \$2500 and deposits \$8000 each year into a different savings account that can offer 7% p.a. compounded annually, find (correct to the nearest \$1000) the amount available for her car.

c Calculate the extra amount saved by investing \$8000 each year at 7% p.a. compared with \$6500 each year at 6.5% p.a.

6 Rhonda has \$4000 in a savings account that is averaging an interest rate of 5.5% p.a. compounded annually. She wants to calculate the amount she will receive in 5 years when she plans to go on a holiday.

a If she deposits \$6000 each year, find (correct to the nearest \$1000) the amount available for her car.

b If she places her \$4000 and deposits \$8000 each year into a different savings account that can offer 6.5% p.a. compounded annually, find (correct to the nearest \$1000) the amount available for her car.

c Calculate the extra amount saved by investing \$8000 each year at 7% p.a. compared with \$6000 each year at 5.5% p.a.

7 **WE26** Peter is 48 and is planning to retire at 65 years of age. He estimates that he needs \$550 000 to retire at 65. His current superannuation fund has a balance of \$55 000 and is delivering 8% p.a. compounded monthly.

a Find the monthly contributions needed to meet the retirement lump sum target.

b If in the final ten years before retirement Peter doubles his monthly contribution calculated from a, find the new lump sum amount available for retirement.

c How much extra could Peter expect if the interest rate from part b is increased to 10% p.a. (for the final 10 years) compounded monthly? Round the answer correct to the nearest \$1000.

8 Patricia is 54 and is planning to retire at 65 years of age. She estimates that she needs \$600 000 to retire at 65. Her current superannuation fund has a balance of \$115 000 and is delivering 9.5% p.a. compounded monthly.

a Find the monthly contributions needed to meet the retirement lump sum target.

b If in the final five years before retirement Patricia doubles her monthly contribution calculated from a, find the new lump sum amount available for retirement.

c How much extra could Patricia expect if the interest rate from part b is increased to 11% p.a. (for the final 5 years) compounded monthly? Round the answer correct to the nearest \$1000.

9 **WE27** George is 50 and has an ambitious plan to retire at age 55. His annual salary is $85 000 and his employer contributions are 9% of his gross monthly income. George also contributes a further $400 a month as a salary sacrifice (that is, he pays $400 from his salary into the superannuation fund). The superfund has been returning an interest rate of 8.7% p.a. compounded monthly and his current balance in the superfund is $230 000.

a Calculate the total monthly contributions to George's superannuation fund.
b Calculate the lump sum that he can receive for his planned retirement at age 55.

George has two options for setting up an annuity to provide a regular income after he retires at 55.

1. A perpetuity that offers monthly payments at 9% p.a., compounded monthly.
2. A reducing balance annuity, also paid monthly at 9% p.a., compounded monthly.

c Calculate the monthly annuity using option 1. Express the annual salary from this option as a percentage of his current salary.
d Calculate the monthly annuity using option 2 if the fund needs to last 32 years. Express the annual salary from this option as a percentage of his current salary.

10 Katrina is 48 and hopes to retire at age 53. Her annual salary is $78 000 and her employer contributions are 12% of her gross monthly income. Katrina also contributes a further $500 a month as a salary sacrifice (that is, she pays $500 from her salary into the superannuation fund). The super fund has been returning an interest rate of 9.8% p.a. compounded monthly and her current balance in the super fund is $180 000.

a Calculate Katrina's total monthly contribution to the superannuation fund.
b Calculate the lump sum that she can receive for her planned retirement at age 53.

Katrina has two options for setting up an annuity to provide a regular income after she retires at 53.

1. A perpetuity that offers monthly payments at 10.5% p.a., compounded monthly.
2. A reducing balance annuity, also paid monthly at 10.5% p.a., compounded monthly.

c Calculate the monthly annuity using option 1. Express the annual salary from this option as a percentage of her current salary.
d Calculate the monthly annuity using option 2 if the fund needs to last 26 years. Express the annual salary from this option as a percentage of her current salary.

CONSOLIDATE

11 An initial deposit of $20 000 was made on an investment taken out over 3 years at a rate of 6% p.a. (interest calculated monthly), and an additional deposit of $200 is made each month. Complete the table below for the first five deposits and calculate the amount of interest that was earned:

a in the second month
b in the fifth month
c over the first five months.

$n+1$	V_n	d	V_{n+1}
1	\$20 000		
2			
3			
4			
5			

12 Barbara currently has \$60 000 in an investment account that is averaging an interest rate of 6% p.a., compounded annually. She wants to calculate the amount that she will receive after 20 years.

- **a** If she deposits \$9000 each year, find (correct to the nearest \$1000) the amount available for her after 20 years.
- **b** If she places her \$60 000 and increases her deposits to \$10 000 each year into a different savings account that can offer 8% p.a. compounded annually, find (correct to the nearest \$1000) the amount available for her after 20 years.
- **c** Calculate the extra amount saved by investing \$10 000 each year at 8% p.a. compared with \$9000 each year at 6% p.a.

13 Justin is aged 38 and is planning to retire at 60 years of age. He estimates that he needs \$680 000 to provide for his retirement. His current superannuation fund has a balance of \$40 000 and is delivering 5% p.a. compounded monthly.

- **a** Find the monthly contributions needed to meet the retirement lump sum target.
- **b** If in the final ten years before retirement, Justin doubles his monthly contribution calculated from **a**, find the new lump sum amount target for his retirement.
- **c** How much extra could Justin expect if the interest rate from part **b** is increased to 8% p.a. compounded monthly (for the final 10 years)? Round the answer correct to the nearest \$1000.

14 Find the contributions required to meet the following superannuation goals.

- **a** A final payout of \$800 000 if the current balance is \$300 000 with 10 years to go, in a superannuation fund delivering 6% p.a. compounded annually. Contributions are to be paid once a year.
- **b** A final payout of \$375 000 if the current sum is \$60 000 with 20 years to go. The contribution is paid monthly into a fund averaging 8.4% p.a. compounded monthly.
- **c** A final payout of \$1 million if the current sum is \$160 000 with 20 years to go. The contribution is paid monthly into a fund averaging 8.4% p.a. compounded monthly.

15 For each of the superannuation plans in question **14**, find:

- **i** the total contributions made during the period stated
- **ii** the interest earned during the period stated.

16 A superannuation plan for the past 30 years has had the following four stages. Calculate the total amount in the superannuation fund at the end of each stage.

a Stage 1: An initial amount of \$0 earning interest at 4% p.a. compounded annually with annual contributions of \$3000 for ten years.

b Stage 2: The final balance from Stage 1 now earning interest at 6% p.a. compounded monthly with monthly contributions of \$600 for twelve years.

c Stage 3: The final balance from Stage 2 now earning interest at 12% p.a. compounded monthly with monthly contributions of \$1000 for two years.

d Stage 4: The final balance from Stage 3 now earning interest at 9% p.a. compounded monthly with monthly contributions of \$1000 for the final six years.

17 Mr Rookie is aged 25 and is planning to retire at 55 years of age. He estimates that he needs \$880 000 to provide for his retirement. His current superannuation fund has a balance of \$600 and is delivering 7% p.a. compounded monthly.

a Find the monthly contributions needed to meet the retirement lump sum target.

b If in the final ten years before retirement, Mr Rookie adds a further \$300 to his monthly contribution calculated from **a**, calculate the new lump sum for retirement.

18 Jomar is aged 48 and is planning to retire at 55. His annual salary is \$50 000 and his employer contributions are 9% of his gross monthly income. Jomar also contributes a further \$440 a month as a salary sacrifice. The superannuation fund has been returning an interest rate of 9.6% p.a. compounded monthly and his current balance in the fund is \$110 000.

a Calculate the total monthly contribution to the superannuation fund.

b Calculate the lump sum that he can get when he retires at age 55.

Jomar has two options for setting up an annuity to provide a regular income after he retires at 55.

1. A perpetuity that offers monthly payments at 10% p.a. compounded monthly
2. A reducing balance annuity also paid monthly at 9% p.a. compounded monthly

c Calculate the monthly annuity using option 1. Express the annual salary from this option as a percentage of his current salary.

d Calculate the monthly annuity using option 2 if the fund needs to last for 20 years. Express the annual salary from this option as a percentage of his current salary.

e Choose the better option and explain why.

19 Find the annual salary in retirement for the following investments.

a An initial amount of \$30 000 earning 7.65% p.a. with annual contributions of \$10 000 for the next 27 years. The retirement income comes from a perpetuity fund offering 6.4% p.a.

b An initial amount of \$300 000 earning 9.4% p.a. with annual contributions of \$12 000 for the next 5 years. The retirement income is paid monthly from a reducing balance annuity that has a term of 15 years offering 10% p.a. compounded monthly.

c An initial amount of \$0 earning 7.2% p.a. compounded monthly with monthly contributions of \$1200 for the next 15 years. The retirement income is paid monthly from a reducing balance annuity that has a term of 20 years offering 8% p.a. compounded monthly.

20 Claire is aged 48 and is planning to retire at 65. Her annual salary is \$60 000 and her employer contributions are 10% of her gross monthly income. The superannuation fund has been returning an interest rate of 9.6% p.a. compounded monthly. Claire's current balance is \$92 200 which she wants to grow to \$800 000. The extra amount that Claire will have to contribute each month to ensure this final payout is achieved is closest to:

A \$0 **B** \$1990 **C** \$650 **D** \$150 **E** \$240

MASTER

21 Lee has \$80 000 in an investment with monthly contributions of \$850 earning an interest rate of 8% p.a., interest debited quarterly. Lee is aged 42 and wishes to retire at 60 years of age.

Which of the following equations should he use to find the final value of the investment?

A $V_{72} = 80000 \times 1.08^{72} + \dfrac{850(1.08^{72} - 1)}{(1.08 - 1)}$

B $V_{72} = 80000 \times 1.02^{72} + \dfrac{2550(1.02^{72} - 1)}{(1.02 - 1)}$

C $V_{72} = 80000 \times 1.02^{72} + \dfrac{850(1.02^{72} - 1)}{(1.02 - 1)}$

D $V_{18} = 80000 \times 1.08^{18} + \dfrac{2550(1.08^{18} - 1)}{(1.08 - 1)}$

E $V_{72} = 80000 \times 1.02^{72} + \dfrac{3440(1.02^{72} - 1)}{(1.02 - 1)}$

22 Use CAS to calculate the unknown value in each of the following annuity investments.

a A superannuation fund's performance is often measured by the interest rate returned on superannuation investments. What is the required interest rate compounded monthly, if a current balance of \$100 000 is to mature to \$800 000, if there is 20 years to go with monthly contributions of \$408 per month?

b A fund that returned 5.2% p.a. compounded monthly had grown from \$25 000 to \$250 000 where the monthly contributions to the fund were \$600. How long did it take for this fund to achieve this?

c An investment account has a current balance of \$156 000, which has had \$6100 annual contributions earning 6% p.a. compounded annually. What was the initial amount in the account 8 years ago?

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7.9 Review

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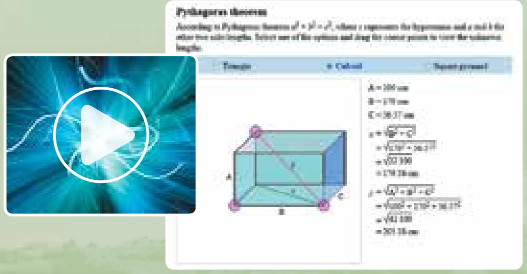

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7 Answers

EXERCISE 7.2

1

$n+1$	V_n	d	V_{n+1}
1	\$25 000	\$271.32	\$24 843.26
2	\$24 843.26	\$271.32	\$24 685.80
3	\$24 685.80	\$271.32	\$24 527.62
4	\$24 527.62	\$271.32	\$24 368.72
5	\$24 368.72	\$271.32	\$24 209.09

2

$n+1$	V_n	d	V_{n+1}
1	\$10 000	\$528.71	\$9671.29
2	\$9671.29	\$528.71	\$9336.01
3	\$9336.01	\$528.71	\$8994.02
4	\$8994.02	\$528.71	\$8645.19
5	\$8645.19	\$528.71	\$8289.38

Paul owes \$8289.38 after 5 payments.

3 \$41.54

4 −\$35.06

5 a \$156.39 **b** \$253.36

6 a \$106.68 **b** \$160.35

7 a \$365.84

b i \$259.59, \$106.25 **ii** \$339.75, \$26.09

8 a \$190.83

b i \$153.09, \$37.74 **ii** \$181.36, \$9.47

9 a $V_{n+1} = V_n\left(1 + \frac{13}{2400}\right) - 341.54,\ V_0 = 23\,000$

b \$22 345.60

10 a $V_{n+1} = V_n\left(1 + \frac{1}{100}\right) - 261,\ V_0 = 2000$

b \$1021.45

c \$3.16

11 a \$59 633.49 **b** \$49 884.16 **c** \$32 172.59

12 a \$46 102.98 **b** \$36 196.88 **c** \$19 556.12

13 a \$27 564.36 **b** \$29 291.80 **c** \$30 958.81

14 a C **b** B

15 D

16 a i \$548.22 **ii** \$1157.28

b i \$381.60 **ii** \$1737.60

c i \$298.62 **ii** \$2333.76

d i \$271.07 **ii** \$2637.78

17 a i \$253.92 **ii** \$170 076.80

b i \$507.97 **ii** \$170 144.40

c i \$1101.28 **ii** \$170 307.20

d i \$3311.33 **ii** \$170 906.40

18 A, D

19 a \$323.73

b i \$46.23, \$277.50 **ii** \$299.28, \$24.45

20 \$38 231.10

EXERCISE 7.3

1 a 16 years and 8 months **b** \$34 904

2 16 months

3 $n \approx 8$ months

4 $n \approx 15$ months

5 a $n \approx 16$ months **b** $n \approx 22$ months

6 a $n \approx 22$ months **b** $n \approx 28$ months

7 a \$363.25 **b** $n \approx 55$ months

c \$3477.50 **d** \$317.50

8 a \$225.49 **b** $n \approx 118$ fortnights

c \$3910 **d** \$403.70

9 a $n \approx 29$ quarters $\left(7\frac{1}{4}\text{ years}\right)$

b \$5937.89

c \$44.41

10 a $6\frac{3}{4}$ years **b** \$4259.02 **c** \$190.26

11 a $n = 18$, $1\frac{1}{2}$ years **b** \$214.30

12 a $n = 7$, $1\frac{3}{4}$ years **b** \$529.03

13 a $n = 24$, 2 years **b** 5 years

14 a $n = 36$, 3 years **b** 5 years

15 D

16 B

17 a i 2 years **ii** $3\frac{1}{2}$ years

b i $\frac{3}{4}$ year **ii** $1\frac{1}{4}$ years

18 a \$384.65

b i 10 years **ii** 13 years

iii 23 years **iv** $24\frac{1}{2}$ years

	i	ii	iii	iv
c	\$19 356.40	\$25 867.48	\$49 870.60	\$53 752.60
d	\$22 959.60 less	\$16 448.52 less	\$7554.60 more	\$11 436.00 more

19

	i	ii	iii
a	9 years	\$9433.28	\$585.92
b	8 years	\$8861.68	\$1157.52

20 D

21 E

22 a \$21 164.60 **b** \$441.40

EXERCISE 7.4

1 a $756.48

b

Payment no.	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	8000	160	756.48	7403.52
2	7403.52	148.07	756.48	6795.11
3	6795.11	135.90	756.48	6174.53
4	6174.53	123.49	756.48	5541.54
5	5541.54	110.83	756.48	4895.89
6	4895.89	97.92	756.48	4237.33
7	4237.33	84.75	756.48	3565.60
8	3565.60	71.31	756.48	2880.43
9	2880.43	57.61	756.48	2181.56
10	2181.56	43.63	756.48	1468.71
11	1468.71	29.37	756.48	741.60
12	741.60	14.83	756.48	−0.04

2 a $691.17

b

Payment no.	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	7500	118.75	691.17	6927.58
2	6927.58	109.69	691.17	6346.10
3	6346.10	100.48	691.17	5755.41
4	5755.41	91.13	691.17	5155.36
5	5155.36	81.63	691.17	4545.82
6	4545.82	71.98	691.17	3926.63
7	3926.63	62.17	691.17	3297.63
8	3297.63	52.21	691.17	2658.67
9	2658.67	42.10	691.17	2009.60
10	2009.60	31.82	691.17	1350.24
11	1350.24	21.38	691.17	680.45
12	680.45	10.77	691.17	0.06

c $794.10

3 a i Term of loan = 5 years

ii $217.37

b i Term of loan = 4 years 25 fortnights

ii $149.63

4 a i Term of loan = 6 years

ii $246.17

b i Term of loan = 5 years 25 fortnights

ii $164.36

5 a $4176 **b** $4025.27 **c** $3984.35

6 a $6008.48 **b** $5825.22 **c** $5775.40

7 a $5496.40

b i Term = 5 years and 2 months

ii $6292.40

8 a $2834.88

b i Term = 4 years and 2 months

ii $8120.60

9 a $22 125.28 **b** $10\frac{1}{4}$ years **c** $1070.37

10 a $19 156.45

b Term = 8 years + 2 months

c $660.18 more

11 a $145 **b** Jodie made a loss of $220.

12 a $312.50 **b** Profit = $1750

13 a $523.66

b

Payment no.	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	2000	37.50	523.66	1513.84
2	1513.84	28.38	523.66	1018.56
3	1018.56	19.10	523.66	514.00
4	514.00	9.64	523.66	−0.02

c $514.00

d $2094.62

e $94.62

14 a $647.06

b

Payment no.	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	3750	37.50	647.06	3140.44
2	3140.44	31.40	647.06	2524.78
3	2524.78	25.25	647.06	1902.97
4	1902.97	19.03	647.06	1274.94
5	1274.94	12.75	647.06	640.63
6	640.63	6.41	647.06	−0.02

c $322.75

d

Payment no.	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	3750	18.75	322.75	3446.00
2	3446.00	17.23	322.75	3140.48
3	3140.48	15.70	322.75	2833.43
4	2833.43	14.17	322.75	2524.85
5	2524.85	12.62	322.75	2214.71
6	2214.71	11.07	322.75	1903.05
7	1903.05	9.52	322.75	1589.81
8	1589.81	7.95	322.75	1275.01
9	1275.01	6.38	322.75	958.64
10	958.64	4.79	322.75	640.68
11	640.68	3.20	322.75	321.13
12	321.13	1.61	322.75	−0.01

e Monthly: $122.99, Bi-monthly: $132.34

The interest paid on the monthly payments is almost $10 less than the interest paid on the bi-monthly payments.

15 a 10 years, A_{119} = $102.61

b 9 years 24 fortnights, A_{257} = $185.59

16

	Frequency	Total interest ($)	Saving ($)
a	Half-yearly	6014.80	–
b	Quarterly	5707.11	$307.69 saving on half-yearly
c	Monthly	5496.63	$518.17 saving on half-yearly
d	Fornightly	5440.38	$574.42 saving on half-yearly

17 C

18 B

19 a $5414.60

b i 5 years 2 months ii $6333.26

c i 5 years 4 months ii $7331.64

20 a i $49 139.65 ii $15 959.17

b i 16 years 9 fortnights

ii 15 years 1 fortnight

c $58 728.80

i $9663.89 more ii $194.87 more

21 A

22 C

23 a $1260 b $6960

24 A

EXERCISE 7.5

1 a i $250 ii $212.27

b i $5000 ii $2736.20

2 a i $330.32

ii $275.99

b i $10 196.55

ii $5633.16

3 a $273.74 b $8848.80 c 3.69% p.a.

4 a $343.34 b $6840.56 c 4.44%.

5 a $10 995 b $15 707.14 c $11 292.86

6 A

7 a i $362.50 ii $450

b i $311.38 ii $363.98

8

	a	b
i	$6 750	$3 682.80
ii	$24 000	$13 677.60

9

	a	b
i	$786.67	$733.44
ii	$1575	$1281.51

10

	a	b
i	$1 440	$801.28
ii	$22 400	$13 008.32

11

	i	ii
a	$305	$279.65
b	$1320	$711.60

12

	i	ii	iii	iv
a	$6 300	$338.33	4 yrs 2 mths	$3 475.29
b	$8 400	$490	4 yrs 3 mths	$4 586.57
c	$216 000	$1275	$10\frac{1}{4}$ yrs	$149 189.70

13 a $14 400 b $533.33

c $4\frac{3}{4}$ years d $8194.04

14 B

15 a $1315.72 b $1788.64 c 4.26% p.a.

16 C

17 a $272.06 b $24 735.60 c 5.38% p.a.

18

	i	ii
a	$75 356.80	$41 864.89
b	$97 384.80	$54 102.67
c	$62 604	$34 780

EXERCISE 7.6

1 11.5%

2 14.8%

3 12.13%

4 **a** **i** $32.79 **ii** $936.96
iii $61.96 **iv** 8.30%
b **i** $467.22 **ii** $30 033.20
iii $6043.20 **iv** 10.47%
c **i** $39.04 **ii** $568.48
iii $18.48 **iv** 7.76%
d **i** $43.32 **ii** $1089.68
iii $69.68 **iv** 6.96%

5 **a** 18.41% **b** 20.05%

6 $134.46

7 13.30%

8 7.76%

9 A

10 A

11 D

12 **a** 22.33% **b** 24.96%
c $1029.60 **d** $200.60

13 The effective annual interest rate of 9.3% is the better option (the compound interest rate has an effective annual interest rate of 9.38%).

14 8.84%

EXERCISE 7.7

1 B

2 D

3 **a** $3600 **b** 6.4 % p.a.
c $3680.27 **d** 6.37 %

4 **a** $7125 **b** 6.4 % p.a.
c $7314.12 **d** 6.37 %

5 $320 000

6 $106 666.67

7 B

8 **a** $16 000 per year
b $3000 per quarter

9 **a** $1000 per month
b $30 000 per quarter

10 (8) **a** N: = 1
I%: = 4
PV: = −400 000
Pmt: = 16 000
FV: = 400 000
PpY: = 1
CpY: = 1
PmtAt: = END

b N: = 1
I%: = 4
PV: = −300 000
Pmt: = 3000
FV: = 300 000
PpY: = 4
CpY: = 4
PmtAt: = END

10 (9) **a** N: = 1
I%: = 12
PV: = −100 000
Pmt: = 1000
FV: = 100 000
PpY: = 12
CpY: = 12
PmtAt: = END

b N: = 1
I%: = 6
PV: = −2 000 000
Pmt: = 30 000
FV: = 2 000 000
PpY: = 4
CpY: = 4
PmtAt: = END

11 **a** $3200 per year **b** 6.25%
c $48.32 **d** 6%

12 $17 088.13

13 **a** 1.25% **b** 2.40%
c 7.50% **d** 5.20%

14 **a** N: = 1
I%: = 1.25
PV: = −400 000
Pmt: = 5000
FV: = 400 000
PpY: = 1
CpY: = 1
PmtAt: = END

b N: = 1
I%: = 2.4
PV: = −500 000
Pmt: = 1000
FV: = 500 000
PpY: = 12
CpY: = 12
PmtAt: = END

c N: = 1
I%: = 7.5
PV: = −800 000
Pmt: = 30 000
FV: = 800 000
PpY: = 2
CpY: = 2
PmtAt: = END

d N: = 1
I%: = 5.2
PV: = −100 000
Pmt: = 200
FV: = 100 000
PpY: = 26
CpY: = 26
PmtAt: = END

15 **a** N: = 1
I%: = 1.242895218
PV: = −400 000
Pmt: = 5000
FV: = 400 000
PpY: = 1
CpY: = 12
PmtAt: = END
r= 1.24% p.a.

b N: = 1
I%: = 2.426576795
PV: = −500 000
Pmt: = 1000
FV: = 500 000
PpY: = 12
CpY: = 1
PmtAt: = END
r= 2.43% p.a.

c N: = 1
I%: = 7.430975749
PV: = −800 000
Pmt: = 30 000
FV: = 800 000
PpY: = 2
CpY: = 4
PmtAt: = END
r= 7.43% p.a.

d N: = 1
I%: = 5.20606734
PV: = −100 000
Pmt: = 200
FV: = 100 000
PpY: = 26
CpY: = 12
PmtAt: = END
r= 5.21% p.a.

16 $66 666.67

17 E

18 **a** $20 000 **b** $222 222.22
c $60 000 **d** $48 000

EXERCISE 7.8

1 $7909.18

2 $11 645.60

3

$n+1$	V_n	d	V_{n+1}
1	$5000	$100	$5131.25
2	$5131.25	$100	$5263.32
3	$5263.22	$100	$5396.22
4	$5396.22	$100	$5529.95
5	$5529.95	$100	$5664.51

$164.51

4

$n+1$	V_n	d	V_{n+1}
1	$10 000	$150	$10 216.67
2	$10 216.67	$150	$10 434.78
3	$10 434.78	$150	$10 654.35
4	$10 654.35	$150	$10 875.38
5	$10 875.38	$150	$11 097.88

$72.50

5 **a** $40 000
b $50 000
c $10 000

6 **a** $39 000
b $51 000
c $12 000

7 **a** $779.71
b $692 646.51
c $124 000

8 **a** $d = \$1185.65$
b $690 610.64
c $47 000

9 **a** $1037.50 **b** $432 423.72
c 45.79% **d** 48.54%

10 **a** $1280 **b** $391 830.07
c 52.75% **d** 56.47%

11

$n+1$	V_n	d	V_{n+1}
1	$20 000	$200	$20 300
2	$20 300	$200	$20 601.50
3	$20 601.50	$200	$20 904.51
4	$20 904.51	$200	$21 209.03
5	$21 209.03	$200	$21 515.08

a $101.50 **b** $106.05 **c** $515.08

12 **a** $523 000 **b** $737 000 **c** $214 000

13 **a** $1168.46 **b** $861 442.14 **c** $238 000

14 **a** $19 933.98 per year **b** $88.74 per month
c $236.64 per month

15 **a** **i** $199 339.80 **ii** $300 660.20
b **i** $21 297.60 **ii** $293 702.40
c **i** $56 793.60 **ii** $783 206.40

16 **a** $36 018.32 **b** $199 954.70
c $280 862.88 **d** $575 999.51

17 **a** $717.34 **b** $931 925.99

18 **a** $815
b $311 900.16
c $2599.17 per month, 62.4%
d $2806.25 per month, 67.4%
e Answers will vary.

19 **a** $66 904.69 **b** $69 958.08
c $38 848.08

20 D

21 B

22 **a** 8.5%
b Approximately 16 years and 8 months
c $59 996.59

8

Matrices

8.1 Kick off with CAS

Dominance matrices

We can use dominance matrices to find which vertex in a matrix is the most dominant. This can have many purposes, for example determining the order of teams in a round robin tournament.

1 Five soccer teams took part in a round robin tournament with each team playing each other team once. The results were as follows:

- B defeated C and E
- A defeated B, C, and D
- C defeated E
- D defeated B, C, and E
- E defeated A.

Given the above results, try to rank the 5 teams in order.

2 These results can be represented in a 5 × 5 matrix. Complete the following matrix by placing a 1 in the elements where the team in the rows defeated the opposing teams in the columns. Place a 0 in all other elements.

$$\begin{array}{cc} & \begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \end{array} \\ D = \begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \end{array} & \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} \end{array}$$

3 a Using CAS, define the completed matrix from question 2.

b Use CAS to determine the matrix D^2.

c Use CAS to determine the matrix $D + D^2$.

4 Sum each row of the matrix found in question 3c.

5 Use your answer to question 4 to rank the teams in order (the team with the highest sum is the most dominant team, and so on).

6 Compare your answer to question 5 with your answer to question 1.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

8.2 Matrix representation

A **matrix** (plural *matrices*) is a rectangular array of numbers arranged in rows and columns. The numbers in a matrix are called the **elements** of the matrix.

The matrix shown here has 3 *rows* and 2 *columns*. We say that it is a 3 × 2 **rectangular matrix** and its **order** is 3 by 2.

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 0 & 4 \end{bmatrix}$$

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Unit 4
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Topic 1
Concept 1

Defining a matrix
Concept summary
Practice questions

Capital letters will be used to represent matrices.

In general, a matrix with m rows and n columns is known as a $m \times n$ matrix.

The elements in a matrix are referred to by the row and then by the column position.

The element in the second row and the first column of matrix A above is –1. This is represented as $a_{21} = -1$ or the 2,1 element.

In general, the elements of matrix A are referred to as a_{ij}, where i refers to the row position and j refers to the column position.

$$\text{That is, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ \cdot & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix} \text{ for the matrix of order } m \times n.$$

The 2,1 element is a_{21} and the 3,2 element is a_{32}.

Types of matrices

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Unit 4
AOS M1
Topic 1
Concept 2

Matrix types
Concept summary
Practice questions

A matrix with one row is called a **row matrix** or *row vector*.

$$B = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \text{ is a } 1 \times 3 \text{ row matrix.}$$

A matrix with one column is called a **column matrix** or *column vector*.

$$C = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ is a } 3 \times 1 \text{ column matrix.}$$

A matrix with an equal number of rows and columns is called a **square matrix**.

$$D = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 2 & -1 \\ 3 & -2 & 0 \end{bmatrix} \text{ is a } 3 \times 3 \text{ square matrix.}$$

$$E = \begin{bmatrix} 4 & 2 \\ -5 & 7 \end{bmatrix} \text{ is a } 2 \times 2 \text{ square matrix.}$$

A square matrix that has only non-zero elements on the main diagonal is called a **diagonal matrix**.

$$F = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ is a } 3 \times 3 \text{ diagonal matrix.}$$

A **zero matrix** consists only of elements that are 0.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is a } 2 \times 2 \text{ zero matrix.}$$

Two matrices are *equal* if they are of the same order and all corresponding elements are equal.

$$\overset{A}{\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & -1 & 3 & 5 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & -1 & 3 & 5 \end{bmatrix}}$$

These two 2×4 matrices are equal as the corresponding elements are equal; that is, $A = B$ as $a_{11} = b_{11} = 1$, $a_{12} = b_{12} = 0$, ...

A **symmetrical matrix** needs to be a square matrix.

Matrix A is symmetric if $A = A^T$, where A^T is the transpose matrix.

The elements with respect to the main diagonal are symmetric in a symmetric matrix, so $a_{ij} = a_{ji}$ for all indices i and j.

The following matrix is a 3×3 symmetric matrix, where the black numbers represent the main diagonal.

$$\begin{bmatrix} 1 & 3 & -7 \\ 3 & 5 & 9 \\ -7 & 9 & 2 \end{bmatrix}$$

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Unit 4
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Concept 3

More matrix types
Concept summary
Practice questions

A **triangular matrix** is a type of square matrix.

A triangular matrix can be labelled a **lower triangular matrix** if all elements above the main diagonal are zero. However, if all elements below the main diagonal are zero, then it is called an **upper triangular matrix**.

The following are both 3×3 triangular matrices.

$$\begin{bmatrix} 3 & 0 & 0 \\ 8 & 5 & 0 \\ 7 & -2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & -3 & -1 \\ 0 & 5 & 8 \\ 0 & 0 & 2 \end{bmatrix}$$

Lower triangular matrix Upper triangular matrix.

A matrix that consists only of elements that are either 0 or 1 is known as a **binary matrix**. They are also referred to as (0, 1), Boolean or logical matrices.

WORKED EXAMPLE 1

For each of the following matrices, give the order and the appropriate name (if it can be categorised). Where possible, write the 2,1 element and the position of the number 3 in x_{ij} form.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 7 & 3 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 9 & 7 & -5 \\ 7 & -1 & 4 \\ -5 & 4 & 3 \end{bmatrix}, E = \begin{bmatrix} -2 & 12 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & 6 \end{bmatrix}$$

THINK	WRITE
1 A has 3 rows and 3 columns, with the numbered elements in the main diagonal. A zero is in the 2,1 position and a 3 is in position 1,1.	A is a 3×3 diagonal matrix. The element in the 2,1 position is a 0 and the number 3 is represented by a_{11}.
2 B has 1 row and 3 columns. There is no element in the 2,1 position and 3 is in position 1,2.	B is a 1×3 row matrix. There is no element in the 2,1 position and the number 3 is represented by b_{12}.
3 C has 2 rows and 2 columns. All elements are 0 or 1. A 0 is in the 2,1 position and there is no number 3 in the matrix.	C is a 2×2 binary matrix. The element in the 2,1 position is a 0 and there is no number 3 in the matrix.
4 D has 3 rows and 3 columns. The elements are symmetric about the main diagonal. A 7 is in the 2,1 position and a 3 is in position 3,3.	D is a 3×3 symmetrical matrix. The element in the 2,1 position is a 7 and the number 3 is represented by d_{33}.
5 E has 3 rows and 3 columns. The elements have zeros below the main diagonal. A zero is in the 2,1 position and a 3 is in position 1,3.	E is a 3×3 upper triangular matrix. The element in the 2,1 position is a 0 and the number 3 is represented by e_{13}.

Using matrices to store data

Matrices are useful for recording and storing bivariate data (data that depend on two categories). They can also keep track of the coefficients of systems of numbers in a simple two-dimensional format. For storing data, whether the data are organised in a row or column need not be specified; they are shown in the same manner as the original data format.

Distances between 3 local townships (kilometres)			
	Town A	**Town B**	**Town C**
Town A	0	23	17
Town B	23	0	43
Town C	17	43	0

$$\begin{bmatrix} 0 & 23 & 17 \\ 23 & 0 & 43 \\ 17 & 43 & 0 \end{bmatrix}$$

Labelling convention is not important as the matrix is simply being used to store data.

However, when performing mathematical processes using the data stored in matrices, the defining of the rows and columns must follow formal conventions. Two common examples where the columns and rows must follow a well-established convention are two-way frequency tables and simultaneous equations.

Two-way frequency tables follow the convention of organising explanatory variable headings in the columns and response variable headings in the rows.

Attitude	Primary	Secondary
Fewer	5	2
Same	29	9
More	33	36
Total	67	47

$$= \begin{bmatrix} 5 & 2 \\ 29 & 9 \\ 33 & 36 \\ 67 & 47 \end{bmatrix}$$

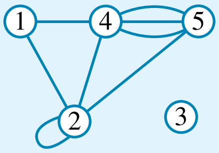
Simultaneous equations follow the convention of organising the variables into columns and the coefficients into rows.

$$\begin{array}{l} 3x + 2y = 5 \\ 2x - 6y = 2 \end{array} = \begin{bmatrix} 3 & 2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Matrices are usually presented without headings or labels on the rows and columns.

WORKED EXAMPLE 2

a **Generate a matrix to show the number of major country roads between five nearby townships in the network shown.**



b **Generate a matrix to represent the following contingency table for party preferences of females and males.**

Party preference	Female	Male
Labor	18	12
Liberal	16	11
Total	34	23

c **Generate a 2 × 2 matrix to represent the information provided in the following scenario. Omit the totals from your matrix.**

‘In a survey, 139 women and 102 men were asked whether they approved or disapproved of a proposed freeway. Thirty-seven women and 79 men approved of the freeway.’

THINK

a **1** Set up a blank 5 × 5 matrix. Thus, there are 25 entries inside the matrix. Label the rows and columns for accuracy.

WRITE

a

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

2 Consider vertex 1. It is connected to vertices 2 and 4 once each, so put a 1 in the corresponding columns of row 1 (shown in red) and in the corresponding rows of column 1 (shown in blue).

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} & 1 & & 1 & \\ 1 & & & & \\ & & & & \\ 1 & & & & \\ & & & & \end{bmatrix} \end{array}$$

3 Consider vertex 2. It is connected to itself once and vertices 1, 4 and 5 once each. Put a 1 in the corresponding columns of row 2 (shown in red) and in the corresponding rows of column 2 (shown in blue).

Note: Some elements, such as m_{12} and m_{21}, will have been previously filled.

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} & 1 & & 1 & \\ 1 & 1 & & 1 & 1 \\ & & & & \\ 1 & 1 & & & \\ & 1 & & & \end{bmatrix} \end{array}$$

4 Vertex 3 is not connected to any other vertex, so fill in column 3 and row 3 with zeros (shown in red).

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ 1 & & 1 & 0 & 1 & \\ 2 & 1 & 1 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & & \\ 5 & & 1 & 0 & & \end{array}$$

5 Vertex 4 is connected to vertices 1 and 2 once. This was recorded in previous steps. Vertex 4 is also connected to vertex 5 three times, so put a 3 in elements m_{45} and m_{54} (shown in red).

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ 1 & & 1 & 0 & 1 & \\ 2 & 1 & 1 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & & 3 \\ 5 & & 1 & 0 & 3 & \end{array}$$

6 Vertex 5 has all its connections already entered, as they were considered in previous steps, so complete the matrix by placing a 0 in all unoccupied places, indicating no connections between the vertices.

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 0 & 3 \\ 5 & 0 & 1 & 0 & 3 & 0 \end{array}$$

7 Remove the labels to complete the matrix. Check your result by comparing the entries in the matrix with the original network representation. This is best done on a vertex-by-vertex basis.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 & 0 \end{bmatrix}$$

b Set up a 3×2 matrix and record the data as shown with labels removed.

b $\begin{bmatrix} 18 & 12 \\ 16 & 11 \\ 34 & 23 \end{bmatrix}$

c **1** Set up a 2×2 matrix to record the data. For accuracy, you may label the rows and columns but remember to remove them in your final answer. Record the independent variable, the respondent's gender, in the columns and the dependent variable, party preference, in the rows.

c

$$\begin{array}{r|cc|} & \text{m} & \text{f} \\ \text{approve} & & \\ \text{disapprove} & & \end{array}$$

2 We know that, of the 139 females, 37 approved of the freeway and the remainder (139 – 37 = 102) disapproved of the freeway. Enter this information into m_{12} and m_{22}.

$$\begin{array}{r|cc|} & \text{m} & \text{f} \\ \text{approve} & & 37 \\ \text{disapprove} & & 102 \end{array}$$

3 We also know that, of the 102 males, 79 males approved of the freeway and the remainder (102 – 79 = 23) disapproved of the freeway. Put this information into m_{11} and m_{21}. Remove any labels on the rows or columns to complete the matrix.

$$\begin{bmatrix} 79 & 37 \\ 23 & 102 \end{bmatrix}$$

Matrices are not just useful for storing data. They can be added, subtracted, multiplied (but not divided) and generally manipulated to extract greater information from the data. For example, we can use matrices to quickly calculate percentages in the two-way frequency table used in Worked example 2. This will be examined more closely later in the topic.

EXERCISE 8.2 Matrix representation

PRACTISE

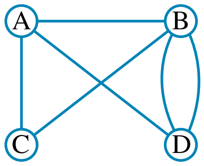
1 **WE1** For each of the following matrices, give the order and the appropriate name (if it can be categorised). Where possible, write the 2,1 element and the position of the number 3 in x_{ij} form.

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 3 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 1 & -5 \\ 1 & -12 & 6 \\ -5 & 6 & 13 \end{bmatrix}, E = \begin{bmatrix} -2 & 0 & 0 \\ 5 & 4 & 0 \\ -10 & 3 & 6 \end{bmatrix}$$

2 For each of the following matrices, give the order and the appropriate name (if it can be categorised). Where possible, write the 1,2 element and the position of the number 3 in x_{ij} form.

$$A = \begin{bmatrix} 3 & -5 & 2 \\ 0 & 5 & -12 \\ 0 & 0 & 15 \end{bmatrix}, B = \begin{bmatrix} -1 & -3 & 1 \end{bmatrix}, C = \begin{bmatrix} 7 & 10 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 12 & 7 & -17 \\ 7 & -9 & 3 \\ -17 & 3 & 12 \end{bmatrix}$$

3 **WE2** Represent the following network as a matrix.



4 Represent the following contingency table for sport preference as a matrix.

Sport preference	Men	Women
AFL	26	22
Cricket	15	9
Total	41	31

CONSOLIDATE

5 For each of the following:

i state the order and type of the matrix

ii where possible, write the 2,1 element

iii state the position of the number 3 in x_{ij} form.

a $\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 2 & 0 \\ 0 & -3 \end{bmatrix}$

b $\begin{bmatrix} 7 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

c $\begin{bmatrix} & 2 & \\ 1 & 4 & 2 \\ & 3 & \end{bmatrix}$

d $\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$

e $\begin{bmatrix} 3 & 1 & 2 \end{bmatrix}$

f $\begin{bmatrix} -1 & -4 & -12 \\ -4 & 15 & 3 \\ -12 & 3 & 4 \end{bmatrix}$

6 For each of the following matrices, give the order and the appropriate name (if it can be categorised). Where possible, write the 2,2 element and the position of the number 7 in x_{ij} form.

$$A = \begin{bmatrix} -13 & 0 & 0 \\ -11 & 4 & 0 \\ 10 & 5 & 7 \end{bmatrix},\ B = \begin{bmatrix} 0 & 7 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 6 & 0 & 5 \\ 0 & 0 & -9 \\ 5 & -9 & 7 \end{bmatrix}$$

7 State the a_{21} element in each of the matrices given.

a $\begin{bmatrix} 4 & -1 & 30 \\ 1 & -5 & 1 \\ 5 & 6 & 3 \end{bmatrix}$

b $\begin{bmatrix} -9 & 2 & 5 \\ -5 & 4 & 2 \\ 12 & 3 & 0 \end{bmatrix}$

c $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

d $\begin{bmatrix} 0.5 \\ 2.0 \\ 3.7 \end{bmatrix}$

e $\begin{bmatrix} 0 & 2 & 4 \end{bmatrix}$

f $\begin{bmatrix} 2 & -4 & -3 \end{bmatrix}$

g $\begin{bmatrix} 2 & -1 \\ 3 & -3 \\ -5 & 0 \\ 6 & 3 \end{bmatrix}$

h $\begin{bmatrix} 0.2 & 0.5 & 3.1 & 2.9 \\ 3.5 & 2.1 & 0.1 & 0.8 \end{bmatrix}$

8 For each of the matrices given in question **7**, state the ones in which x_{32} exists.

9 The value of a_{21} in the matrix is:

A 3.6
B 1.6
C –0.5
D 0
E 2.4

$$\begin{bmatrix} 2.4 & 3.6 \\ -0.5 & 1.6 \\ 1.6 & 0 \\ -2.5 & 2.4 \end{bmatrix}$$

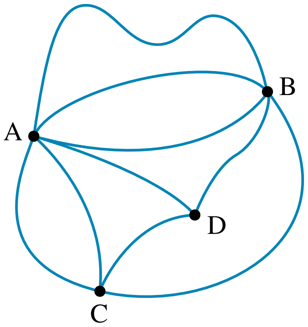
10 The number 3 in the following matrix can be represented using the notation:

A a_{23}
B a_{32}
C a_{22}
D a_{12}
E none of these

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 3 & 2 \\ 2 & 5 & 4 \end{bmatrix}$$

11 Using the values 0 and 1 only, state the 4 × 4 diagonal matrix.

12 The figure shown represents the number of routes between four towns.



Represent the possible routes as a 4 × 4 matrix in the following form.

$$\begin{array}{cc} & \text{Number of routes to} \\ \text{Number of routes from} & \begin{bmatrix} \qquad\qquad & \\ & \qquad\qquad \end{bmatrix} \end{array}$$

13 Represent the following contingency table as a 3 × 3 matrix.

Rent preference	Men	Women	Total
Live independently	12	23	35
Share with friends	9	16	25
Total	21	39	60

14 Represent the final score at an AFL match as a matrix.

	Goals	Behinds	Points
Geelong	15	10	100
Carlton	12	15	87

MASTER

15 The following information represents the goods to be delivered from a warehouse to individual suppliers.

To supplier A:	5 bicycles	12 helmets	6 tyres
To supplier B:	7 bicycles	2 helmets	15 tyres
To supplier C:	15 bicycles	7 helmets	0 tyres

Present the information as a matrix with the suppliers placed in the columns.

16 Represent the following coordinates of Cartesian points as a 4 × 2 matrix.

(3, 2), (–4, –1), (4, –1), (4, 1)

8.3 Addition, subtraction and scalar operations with matrices

Addition and subtraction

eBook*plus*

Interactivity
Adding and subtracting matrices
int-6463

Matrices can be added or subtracted by applying the usual rules of arithmetic on corresponding elements of the matrices. It follows that:

1. **Addition and subtraction of matrices can be performed only if the matrices are of the same order.**
2. **Addition and subtraction of matrices is performed by adding or subtracting corresponding elements.**

WORKED EXAMPLE 3

Given the following matrices

$$A = \begin{bmatrix}2 & 4\\3 & -2\end{bmatrix},\ B = \begin{bmatrix}1 & 6\\8 & 2\end{bmatrix},\ C = \begin{bmatrix}2 & 3 & -1\\4 & -2 & 5\end{bmatrix},\ D = \begin{bmatrix}5 & 2 & 3\\4 & -2 & -2\end{bmatrix},$$

find, if possible:

a $A + B$

b $B + A$

c $B - C$

d $B - A$

e $D - C$.

THINK	WRITE
a 1 A and B are both of the same order 2×2, so they can be added.	**a** $A + B = \begin{bmatrix}2 & 4\\3 & -2\end{bmatrix} + \begin{bmatrix}1 & 6\\8 & 2\end{bmatrix}$
2 Add the numbers in the corresponding positions of each matrix.	$= \begin{bmatrix}2+1 & 4+6\\3+8 & -2+2\end{bmatrix}$ $= \begin{bmatrix}3 & 10\\11 & 0\end{bmatrix}$
b 1 B and A are both of the same order 2×2, so they can be added.	**b** $B + A = \begin{bmatrix}1 & 6\\8 & 2\end{bmatrix} + \begin{bmatrix}2 & 4\\3 & -2\end{bmatrix}$
2 Add the numbers in the corresponding positions of each matrix.	$= \begin{bmatrix}1+2 & 6+4\\8+3 & 2+-2\end{bmatrix}$ $= \begin{bmatrix}3 & 10\\11 & 0\end{bmatrix}$
c The subtraction cannot be performed since the order of B is 2×2 and the order of C is 2×3.	**c** $B - C$ cannot be calculated since B and C are of different orders.
d Subtract the numbers in the corresponding positions of each matrix as B and A are of the same order.	**d** $B - A = \begin{bmatrix}1 & 6\\8 & 2\end{bmatrix} - \begin{bmatrix}2 & 4\\3 & -2\end{bmatrix}$ $= \begin{bmatrix}1-2 & 6-4\\8-3 & 2--2\end{bmatrix}$ $= \begin{bmatrix}-1 & 2\\5 & 4\end{bmatrix}$
e 1 D and C are both of the same order 2×3, so they can be subtracted.	**e** $D - C = \begin{bmatrix}5 & 2 & 3\\4 & -2 & -2\end{bmatrix} - \begin{bmatrix}2 & 3 & -1\\4 & -2 & 5\end{bmatrix}$
2 Subtract the numbers in the corresponding positions of each matrix.	$= \begin{bmatrix}5-2 & 2-3 & 3--1\\4-4 & -2--2 & -2-5\end{bmatrix}$ $= \begin{bmatrix}3 & -1 & 4\\0 & 0 & -7\end{bmatrix}$

Notice in parts **a** and **b** that the resulting matrices are both the same. In general, $A + B = B + A$ when A and B are of the same order; that is, matrix addition is **commutative**. Try this for other matrices. Using the matrices from above, try $A - B$. What do you notice about $B - A$ and $A - B$?

Scalar multiplication

studyon

Unit 4
AOS M1
Topic 1
Concept 4

Addition, subtraction and scalar multiples
Concept summary
Practice questions

As we have seen in arithmetic, repeated addition can be more efficiently calculated using multiplication. For example, $2 + 2 + 2 = 3 \times 2$.

A similar approach applies to matrices. Consider the matrix $A = \begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix}$.

To find the addition of $A + A + A$, the simplest approach is to find $3A$. This is achieved by multiplying each element of A by 3.

$$A + A + A = \begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix} \qquad 3A = 3\begin{bmatrix} 1 & 5 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 5+5+5 \\ 4+4+4 & -2+-2+-2 \end{bmatrix} \qquad = \begin{bmatrix} 3\times 1 & 3\times 5 \\ 3\times 4 & 3\times -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 15 \\ 12 & -6 \end{bmatrix} \qquad = \begin{bmatrix} 3 & 15 \\ 12 & -6 \end{bmatrix}$$

In the term $3A$, the number 3 is called a **scalar**, and the term $3A$ is an example of **scalar multiplication** of matrices.

Scalar multiplication applies to matrices of any order. The scalar quantity can be any number positive or negative, fraction or decimal, real or imaginary. This can be generalised as follows.

$$c \times \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ \cdot & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix} = \begin{bmatrix} c\times a_{11} & c\times a_{12} & c\times a_{13} & c\times a_{14} & \cdots \\ c\times a_{21} & c\times a_{22} & c\times a_{23} & c\times a_{24} & \cdots \\ c\times a_{31} & c\times a_{32} & c\times a_{33} & c\times a_{34} & \cdots \\ \cdot & \cdot & \cdot & \cdot & c\times a_{mn} \end{bmatrix}$$

When a matrix is multiplied by a scalar, each element in the matrix is multiplied by the scalar. The order of the matrix remains unchanged.

WORKED EXAMPLE 4

Given the following two matrices

$A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$,

calculate:

a $2A$

b $0.4B$

c $3A + 4A$

d $A + \frac{1}{3}B$

e $3(A + B)$.

THINK	WRITE
a Multiply each element of A by 2.	**a** $2A = 2\begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$ $= \begin{bmatrix} 2\times 2 & 2\times 6 \\ 2\times -1 & 2\times 3 \end{bmatrix}$ $= \begin{bmatrix} 4 & 12 \\ -2 & 6 \end{bmatrix}$

b Multiply each element of B by 0.4.

b
$$\begin{aligned}0.4B &= 0.4\begin{bmatrix}3 & 6\\1 & 3\end{bmatrix}\\ &= \begin{bmatrix}0.4\times 3 & 0.4\times 6\\0.4\times 1 & 0.4\times 3\end{bmatrix}\\ &= \begin{bmatrix}1.2 & 2.4\\0.4 & 1.2\end{bmatrix}\end{aligned}$$

c **1** $3A + 4A$ simplifies to $7A$.

c $3A + 4A = 7A$

2 Multiply each element of A by 7.

$$\begin{aligned}7A &= 7\begin{bmatrix}2 & 6\\-1 & 3\end{bmatrix}\\ &= \begin{bmatrix}14 & 42\\-7 & 21\end{bmatrix}\end{aligned}$$

d **1** Find $\frac{1}{3}B$ by multiplying each element of B by $\frac{1}{3}$.

d
$$\begin{aligned}\frac{1}{3}B &= \frac{1}{3}\begin{bmatrix}3 & 6\\1 & 3\end{bmatrix}\\ &= \begin{bmatrix}\frac{3}{3} & \frac{6}{3}\\\frac{1}{3} & \frac{3}{3}\end{bmatrix}\\ &= \begin{bmatrix}1 & 2\\\frac{1}{3} & 1\end{bmatrix}\end{aligned}$$

2 Complete the addition by adding A to $\frac{1}{3}B$.

$$\begin{aligned}A + \frac{1}{3}B &= \begin{bmatrix}2 & 6\\-1 & 3\end{bmatrix} + \begin{bmatrix}1 & 2\\\frac{1}{3} & 1\end{bmatrix}\\ &= \begin{bmatrix}3 & 8\\-\frac{2}{3} & 4\end{bmatrix}\end{aligned}$$

e **1** Find the sum of A and B.

e
$$\begin{aligned}A + B &= \begin{bmatrix}2 & 6\\-1 & 3\end{bmatrix} + \begin{bmatrix}3 & 6\\1 & 3\end{bmatrix}\\ &= \begin{bmatrix}5 & 12\\0 & 6\end{bmatrix}\end{aligned}$$

2 Multiply this matrix by 3.

$$\begin{aligned}3(A + B) &= 3\begin{bmatrix}5 & 12\\0 & 6\end{bmatrix}\\ &= \begin{bmatrix}15 & 36\\0 & 18\end{bmatrix}\end{aligned}$$

Properties of addition of matrices

The following list is a summary of properties of addition of matrices. These properties hold true when A, B and C are $m \times n$ matrices, k and c are constants and O is a zero matrix (a matrix with all elements equal to zero).

Property	Example
Commutative (does not matter which order the matrices are operated on)	$A + B = B + A$
Associative (does not matter where the brackets are placed)	$(A + B) + C = A + (B + C)$ $(kc)A = k(cA)$
Identity	$A + O = A = O + A$
Inverse	$A + -A = O = -A + A$
Distributive	$kA + kB = k(A + B)$ $kA + cA = (k + c)A$

Note: For subtraction, matrices *do not* obey the **associative** or the commutative laws. For example, $A - B \neq B - A$ and $(A - B) - C \neq A - (B - C)$.

Simple matrix equations

To solve an algebraic equation such as $4x - 3 = 5$

1. add 3 to both sides to obtain $4x - 3 + 3 = 5 + 3$ or $4x = 8$
2. divide both sides by 4 $\left(\text{or multiply by } \frac{1}{4}\right)$ to obtain $x = 2$.

Simple matrix equations that require the addition or subtraction of a matrix or multiplication of a scalar can be solved in a similar way.

WORKED EXAMPLE 5

Solve the following matrix equations.

a $5E = \begin{bmatrix} 20 & 15 & -5 \\ 5 & 0 & 12 \end{bmatrix}$

b $D + \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix}$

c **If** $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$**, find** C **if** $2C + A = 3B$**.**

THINK	WRITE
a **1** To get E by itself, multiply both sides by $\frac{1}{5}$.	**a** $5E = \begin{bmatrix} 20 & 15 & -5 \\ 5 & 0 & 12 \end{bmatrix}$ $E = \frac{1}{5}\begin{bmatrix} 20 & 15 & -5 \\ 5 & 0 & 12 \end{bmatrix}$
2 Simplify the matrix E.	$E = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 0 & 2.4 \end{bmatrix}$
b **1** To get D by itself, subtract $\begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$ from both sides.	**b** $D + \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix}$ $D = \begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$
2 Simplify the matrix D.	$D = \begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix}$

c 1 First solve algebraically to get C by itself.

c $C + A = 3B$

$2C = 3B - A$

$C = \frac{1}{2}(3B - A)$

2 Find the value of $3B - A$.

$$3B - A = \begin{bmatrix} 9 & 18 \\ 3 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 6 \end{bmatrix}$$

3 Multiply this by $\frac{1}{2}$ to solve for C.

$$C = \frac{1}{2}\begin{bmatrix} 7 & 12 \\ 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3.5 & 6 \\ 2 & 3 \end{bmatrix}$$

The matrix operations involving addition, subtraction and scalar multiplication can be applied to practical situations such as stock inventory, price discounting and marking up of store prices.

WORKED EXAMPLE 6

A retail chain of three stores has an inventory of three models each of televisions and DVD players, represented as matrices as follows. The first column represents the televisions and the second column represents the DVD players. Each row represents a different model.

$$\text{Store } A = \begin{bmatrix} 12 & 21 \\ 5 & 12 \\ 3 & 7 \end{bmatrix}, \text{ Store } B = \begin{bmatrix} 23 & 32 \\ 8 & 15 \\ 1 & 11 \end{bmatrix}, \text{ Store } C = \begin{bmatrix} 5 & 17 \\ 2 & 12 \\ 0 & 14 \end{bmatrix}$$

a **If the third row represents the most expensive models, which store has the most models of expensive televisions?**

b **Give the matrix that would represent the total stock of televisions and DVD players for all three stores.**

The wholesale price (in dollars) of each model of television and DVD player is presented in the following matrix.

$$\begin{bmatrix} 100 & 30 \\ 250 & 80 \\ 400 & 200 \end{bmatrix}$$

c **If the wholesale prices are marked up by 50%, calculate the recommended retail prices.**

d **Store C wishes to have a sale. If it discounts all retail prices by 10%, represent the discounted prices as a matrix.**

THINK	WRITE
a The first column represents the televisions and the third row represents the most expensive model (a_{31}). Store *A* has 3 expensive televisions, Store *B* has 1 expensive television and Store *C* has 0 expensive televisions.	**a** Store *A* has the greatest number of expensive televisions.
b The total stock of televisions and DVD players for all three stores is given by the sum of the three matrices. Add the matrices and simplify.	**b** Total inventory $= A + B + C$ $= \begin{bmatrix} 12 & 21 \\ 5 & 12 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 23 & 32 \\ 8 & 15 \\ 1 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 17 \\ 2 & 12 \\ 0 & 14 \end{bmatrix}$ $= \begin{bmatrix} 12 + 23 + 5 & 21 + 32 + 17 \\ 5 + 8 + 2 & 12 + 15 + 12 \\ 3 + 1 + 0 & 7 + 11 + 14 \end{bmatrix}$ $= \begin{bmatrix} 40 & 70 \\ 15 & 39 \\ 4 & 32 \end{bmatrix}$
c **1** A mark-up of 50% represents 150% of the wholesale prices.	**c** $50\% + 100\% = 150\%$
2 Multiply each element of the wholesale price matrix by 1.5 (150%).	150% of $\begin{bmatrix} 100 & 30 \\ 250 & 80 \\ 400 & 200 \end{bmatrix} = 1.5 \times \begin{bmatrix} 100 & 30 \\ 250 & 80 \\ 400 & 200 \end{bmatrix}$ $= \begin{bmatrix} 150 & 45 \\ 375 & 120 \\ 600 & 300 \end{bmatrix}$
d **1** A discount of 10% represents 90% of the retail price.	**d** $100\% - 10\% = 90\%$
2 Multiply each element of the retail price matrix by 0.9 (90%).	90% of $\begin{bmatrix} 150 & 45 \\ 375 & 120 \\ 600 & 300 \end{bmatrix} = 0.9 \times \begin{bmatrix} 150 & 45 \\ 375 & 120 \\ 600 & 300 \end{bmatrix}$ $= \begin{bmatrix} 135 & 40.5 \\ 337.5 & 108 \\ 540 & 270 \end{bmatrix}$

EXERCISE 8.3 Addition, subtraction and scalar operations with matrices

PRACTISE

1 **WE3** Given the following matrices

$$A = \begin{bmatrix} 4 & 7 \\ 3 & -4 \end{bmatrix}, B = \begin{bmatrix} 0 & -3 \\ 5 & 2 \end{bmatrix}, C = \begin{bmatrix} 5 & 7 & -2 \\ 1 & 6 & -5 \end{bmatrix}, D = \begin{bmatrix} -7 & 7 & -2 \\ -5 & 6 & -8 \end{bmatrix}$$

find, if possible:

a $A + B$ **b** $B + A$ **c** $B - C$ **d** $B - A$ **e** $D - C$.

2 Given the following matrices

$$A = \begin{bmatrix} 12 & 9 \\ -8 & -10 \end{bmatrix}, B = \begin{bmatrix} 15 & 20 & -21 \\ 18 & 10 & -13 \end{bmatrix}, C = \begin{bmatrix} 8 & 17 & -4 \\ 1 & 12 & -9 \end{bmatrix}, D = \begin{bmatrix} -9 & 17 & -12 \\ -12 & 5 & -11 \end{bmatrix}$$

find, if possible:

a $A + B$ **b** $B + C$ **c** $B - C$ **d** $B - D$ **e** $D - C$.

3 **WE4** Matrix $A = \begin{bmatrix} -6 & 8 \\ 6 & -4 \end{bmatrix}$.

Find:

a $3A$ **b** $-2A$ **c** $\frac{1}{2}A$ **d** $0.4A$.

4 Given $A = \begin{bmatrix} 2 & -6 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -8 & 0 \\ 7 & -3 \end{bmatrix}$,

find:

a $A + 2B$ **b** $2A - B$.

5 **WE5** Solve the following matrix equations.

a $4E = \begin{bmatrix} 24 & 40 & -36 \\ 48 & -8 & -80 \end{bmatrix}$ **b** $D + \begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 2 & -5 \end{bmatrix}$

c If $A = \begin{bmatrix} 3 & 7 \\ -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix}$, find C if $2C + A = 4B$.

6 If $\begin{bmatrix} a & 2 \\ 6 & -6 \end{bmatrix} + \begin{bmatrix} -3 & b \\ c & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -2 & d \end{bmatrix}$, find the values of a, b, c and d.

7 **WE6** A chain of three car yards has an inventory of front-wheel drive and 4WD cars. They have new and used vehicles of both types. The inventory is represented in matrix form, where the new vehicles are placed in the first column and the front-wheel drive vehicles are placed in the first row.

Store $A = \begin{bmatrix} 6 & 4 \\ 4 & 7 \end{bmatrix}$ Store $B = \begin{bmatrix} 5 & 6 \\ 2 & 5 \end{bmatrix}$ Store $C = \begin{bmatrix} 3 & 3 \\ 3 & 0 \end{bmatrix}$

a Which store has the most new 4WDs? State the label of the element.

b Give the matrix that would represent the total stock of each type of vehicle for all three stores combined.

8 A student has kept records of her test results in matrix form. In semester 1, for English tests she got 72%, 76% and 81% and for Further Maths tests she got 84%,

68% and 82%. In semester 2, for English tests she got 78%, 76% and 89% and for Further Maths tests she got 74%, 77% and 85%.

a Write the semester results in two separate 3 × 2 matrices.

b State the matrix equation that would give the average for each test of the two subjects.

c Calculate the average result from the two semesters and present it as a 3 × 2 matrix.

CONSOLIDATE

9 Given the following matrices

$$A = \begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 5 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 5 & 0 \\ 3 & -3 & 3 \end{bmatrix}, D = \begin{bmatrix} 4 & 2 & 2 \\ 3 & -2 & -1 \end{bmatrix},$$

calculate, if possible:

a $A + B$ **b** $A + A$ **c** $B - C$ **d** $B - A$ **e** $D - C$.

10 The solution to $F - E$, given the matrices

$$E = \begin{bmatrix} 1.2 & -0.5 \\ 3.6 & 5.0 \\ -3.5 & 2.2 \end{bmatrix} \text{ and } F = \begin{bmatrix} 0.2 & -0.5 \\ 2.4 & 2.5 \\ 0 & 1.1 \end{bmatrix} \text{ is:}$$

A $\begin{bmatrix} 1.0 & 0 \\ 1.2 & 2.5 \\ -3.5 & 1.1 \end{bmatrix}$ **B** $\begin{bmatrix} 1.4 & -1.0 \\ 6.0 & 7.5 \\ -3.5 & 3.3 \end{bmatrix}$ **C** $\begin{bmatrix} 1.0 & -1.0 \\ 1.2 & 2.5 \\ 3.5 & 1.1 \end{bmatrix}$

D $\begin{bmatrix} 1.0 & 0 \\ 1.2 & 2.5 \\ -3.5 & 1.1 \end{bmatrix}$ **E** $\begin{bmatrix} 1.0 & -1.0 \\ 1.2 & 2.5 \\ -3.5 & 1.1 \end{bmatrix}$

11 The cost price (C) and sale price (S) of four items at an electrical appliance store are given as follows.

$$C = \begin{bmatrix} 23.50 & 45.00 & 87.50 & 140.00 \end{bmatrix} \text{ and } S = \begin{bmatrix} 49.50 & 135.00 & 169.00 & 299.95 \end{bmatrix}$$

Calculate and show as a matrix, the profit made on the sale of these four items.

12 Given the following two matrices

$$A = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix},$$

calculate:

a $3A$ **b** $0.1B$ **c** $2A + 3A$ **d** $A + \frac{1}{4}B$ **e** $2(A + B)$ **f** $\frac{1}{2}(3A - B)$.

Check your answers using CAS.

13 Given the following two matrices

$$C = \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 2 & 2 & 4 & 1 & 3 \\ 1 & 0 & -2 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & 1 & 5 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 6 & 0 & 1 & -2 \\ 2 & 3 & -4 & 1 & 3 \\ 3 & 0 & 2 & 1 & 1 \\ 2 & 2 & -3 & 1 & 10 \\ 5 & 1 & 8 & 1 & 0 \end{bmatrix},$$

use CAS to evaluate the following:

a $4C$ **b** $-0.1D$ **c** $3C - 3D$ **d** $2(D + 3C)$ **e** $\frac{1}{2}(C - 3D)$.

14 Solve the following matrix equations.

a $6E = \begin{bmatrix} 24 & 42 & -6 \\ 6 & 0 & 12 \end{bmatrix}$

b $D + \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 8 & 5 \end{bmatrix}$

c If $A = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix}$, calculate C if $2C + A = 4B$.

15 The order of five matrices is given as follows

$$A = 3 \times 2,\ B = 2 \times 3,\ C = 3 \times 2,\ D = 3 \times 3 \text{ and } E = 3 \times 2.$$

Which of the following cannot be calculated?

a $-0.3C$

b $D + A$

c $2C - 3A$

d $2(E + 3C)$

e $\frac{1}{2}(A - 3B)$

f $A + C - E$

16 A car dealership has three new car sales centres where they stock three models each of 4-wheel-drives and sedans, represented as matrices as follows. The first column represents the 4-wheel-drives and the second column represents the sedans. Each row represents a different model.

$$\text{Centre } A = \begin{bmatrix} 7 & 16 \\ 5 & 8 \\ 2 & 4 \end{bmatrix}, \text{ Centre } B = \begin{bmatrix} 13 & 12 \\ 8 & 15 \\ 3 & 1 \end{bmatrix}, \text{ Centre } C = \begin{bmatrix} 14 & 7 \\ 12 & 12 \\ 9 & 4 \end{bmatrix}$$

a If the third row represents the most expensive models, which centre has the most models of expensive sedans?

b Give the matrix that would represent the total stock of 4-wheel-drives and sedans for all three centres.

The wholesale price (in dollars) of each model of 4-wheel-drives and sedans is presented in the following matrix.

$$\begin{bmatrix} 20\,000 & 13\,000 \\ 25\,000 & 18\,000 \\ 40\,000 & 28\,000 \end{bmatrix}$$

c If the wholesale prices are marked up by 100%, calculate the recommended retail prices.

d Centre B wishes to have a clearance sale. If it discounts all retail prices by 10%, represent the discounted prices as a matrix.

17 The percentages of households in a township that own no pets is 25%, households with one pet is 40%, those with two pets is 20% and more than two pets is 15%.

a Set up a 1×4 matrix to represent the percentage ownership of pets.

b Write an equation that will enable you to calculate the number of households for each category, given that there are 800 households in the town.

c Evaluate the number of households for each category as a 1×4 matrix.

18 a Place the following football team's total season score in a suitable matrix format.

Away games total score:
154 goals 214 behinds 1138 points
Home games total score:
207 goals 180 behinds 1422 points

b Present the average game score for away games and home games as a suitable matrix. There are 11 away games and 11 home games.

MASTER

19 The stock on hand (S) of items in a warehouse is summarised as a matrix as follows:

$$S = \begin{bmatrix} 23 & 150 & 112 & 5 & 1090 \\ 35 & 210 & 145 & 4 & 800 \\ 45 & 230 & 130 & 8 & 1200 \\ 32 & 700 & 230 & 4 & 400 \\ 77 & 420 & 160 & 7 & 850 \end{bmatrix}$$

a For the company's plans to reduce costs, complete the following matrix equation if the new stock levels are to be set at 40% of the current stocks.

New stock level (N) = __________ of current stock levels (matrix __________)
= __________ × __________.

b Evaluate the new stock levels and display as a matrix. (Round answers to the nearest whole number.)

20 Use CAS to store three 3 × 2 matrices as A, B and C (they can contain any numbers). Check to see whether the following are true.

a $A + B = B + A$

b $A - B = B - A$

c $(A - B) - C = A - (B - C)$

d $3A + 3B = 3(A + B)$

8.4 Multiplying matrices

In the previous section, we discussed the multiplication of a matrix by a scalar. In this situation, each element of the matrix is multiplied by the scalar, and the order of the matrix remains unchanged. Now we wish to investigate the multiplication of a matrix by another matrix.

The multiplication rule

studyon

Unit 4
AOS M1
Topic 1
Concept 5

Product of matrices — matrix multiplication
Concept summary
Practice questions

Two matrices can be multiplied if the number of columns in the first matrix is equal to the number of rows in the second matrix. Consider the following matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 6 \\ 3 & -4 & 0 & 8 \end{bmatrix}$$

The number of columns in the first matrix is 3 and the number of rows in the second matrix is 3. It is possible for these two matrices to be multiplied. A way to remember this is to write the order of each matrix beside each other as shown.

$$2 \times \mathbf{3} \times \mathbf{3} \times 4$$

The order of the first matrix is 2 × 3 and the order of the second matrix is 3 × 4. If the two inner numbers (shown in blue) are the same, it is possible to multiply the matrices. If they are not the same, then it will not be possible to multiply the matrices.

The resulting matrix will be of order 2 × 4 — the outer two numbers in the multiplication shown on the previous page.

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Interactivity
Matrix multiplication
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If matrix A is of order $m \times n$ and matrix B is of order $n \times p$, then $A \times B$ exists and its order is $m \times p$.

The following matrix multiplication is not possible because the number of columns in the first matrix is not equal to the number of rows in the second matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & -4 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$$3 \times \mathbf{3} \times \mathbf{2} \times 3$$

The two inner numbers are not the same.

The following example shows how two 2 × 2 matrices are multiplied. (The resulting matrix will be of order 2 × 2.)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$$

Each row of the first matrix is multiplied by each column of the second matrix:

$$\begin{bmatrix} r_1c_1 & r_1c_2 \\ r_2c_1 & r_2c_2 \end{bmatrix} \text{ gives } \begin{bmatrix} 1 \times 5 + 2 \times 6 & 1 \times 7 + 2 \times 8 \\ 3 \times 5 + 4 \times 6 & 3 \times 7 + 4 \times 8 \end{bmatrix}$$

and simplified as follows: $\begin{bmatrix} 5 + 12 & 7 + 16 \\ 15 + 24 & 21 + 32 \end{bmatrix}$

$$= \begin{bmatrix} 17 & 23 \\ 39 & 53 \end{bmatrix}$$

As long as the number of columns in the first matrix is equal to the number of rows in the second matrix, the method highlighted above can be used to multiply the matrices.

The procedure is repeated for two 3 × 3 matrices as shown.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$, then

$$A \times B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31} & a_{11} \times b_{12} + a_{12} \times b_{22} + a_{13} \times b_{32} & a_{11} \times b_{13} + a_{12} \times b_{23} + a_{13} \times b_{33} \\ a_{21} \times b_{11} + a_{22} \times b_{21} + a_{23} \times b_{31} & a_{21} \times b_{12} + a_{22} \times b_{22} + a_{23} \times b_{32} & a_{21} \times b_{13} + a_{22} \times b_{23} + a_{23} \times b_{33} \\ a_{31} \times b_{11} + a_{32} \times b_{21} + a_{33} \times b_{31} & a_{31} \times b_{12} + a_{32} \times b_{22} + a_{33} \times b_{32} & a_{31} \times b_{13} + a_{32} \times b_{23} + a_{33} \times b_{33} \end{bmatrix}$$

The rows of the first matrix are multiplied by the columns of the second matrix.

When preparing to multiply two matrices, the order of the resultant matrix should be established first.

WORKED EXAMPLE 7

Given three matrices

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix},\ B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix},\ C = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix},$$

a write the order of the three matrices

b find which of the following products exist

i AB ii AC

iii BA iv CA

c write the order for the products that exist

d calculate the products that exist.

THINK	WRITE
a **1** Matrix A has 2 rows and 2 columns.	**a** Matrix A is a 2×2 matrix.
2 Matrix B has 3 rows and 2 columns.	Matrix B is a 3×2 matrix.
3 Matrix C has 2 rows and 2 columns.	Matrix C is a 2×2 matrix.
b **i** The number of columns in matrix $A \neq$ the number of rows in matrix B, so AB does not exist.	**b** **i** AB does not exist.
ii The number of columns in matrix $A =$ the number of rows in matrix C, so AC does exist.	**ii** AC exists.
iii The number of columns in matrix $B =$ the number of rows in matrix A, so BA exists.	**iii** BA exists.
iv The number of columns in matrix $C =$ the number of rows in matrix A, so CA exists.	**iv** CA exists.
c Identify the order of the products that exist by looking at the two outer numbers.	**c** The order of AC is 2×2. The order of BA is 3×2. The order of CA is 2×2.
d **1** Multiply the rows of matrix A by the columns of matrix C.	**d** $AC = \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ $= \begin{bmatrix} 1 \times 3 + 4 \times 4 & 1 \times 2 + 4 \times 1 \\ 3 \times 3 + 0 \times 4 & 3 \times 2 + 0 \times 1 \end{bmatrix}$
2 Simplify AC.	$= \begin{bmatrix} 19 & 6 \\ 9 & 6 \end{bmatrix}$
3 Multiply the rows of matrix B by the columns of matrix A.	$BA = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix}$

4 Simplify *BA*.

$$= \begin{bmatrix} 2\times 1+1\times 3 & 2\times 4+1\times 0 \\ 1\times 1+-2\times 3 & 1\times 4+-2\times 0 \\ 3\times 1+4\times 3 & 3\times 4+4\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 \\ -5 & 4 \\ 15 & 12 \end{bmatrix}$$

5 Multiply the rows of matrix *C* by the columns of matrix *A*.

$$CA = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 1+2\times 3 & 3\times 4+2\times 0 \\ 4\times 1+1\times 3 & 4\times 4+1\times 0 \end{bmatrix}$$

6 Simplify *CA*.

$$= \begin{bmatrix} 9 & 12 \\ 7 & 16 \end{bmatrix}$$

The identity matrix

In Worked example 7, you would have noticed that the results of the multiplication of *AC* and *CA* did not produce the same answer. It can be said that matrix multiplication is not commutative.

In general, for two matrices A and B, $AB \neq BA$.

However, there is one exception where a matrix multiplication is commutative. The following situation demonstrates this.

Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1+4\times 0 & 1\times 0+4\times 1 \\ 3\times 1+5\times 0 & 3\times 0+5\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1+0\times 3 & 1\times 4+0\times 5 \\ 0\times 1+1\times 3 & 0\times 4+1\times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$$

Look closely at matrix *B*. A square matrix with the number 1 for all the elements on the main diagonal and 0 for all the other elements is called an **identity matrix**. The identity matrix can only be defined for square matrices; that is, for matrices of order 1 × 1, 2 × 2, 3 × 3, 4 × 4 and so on. The identity matrix is commonly referred to as *I*. Note that the identity matrix is a specific type of diagonal matrix.

From this, it can be stated that $AI = IA = A$. The identity matrix I acts in a similar way to the number 1 when numbers are being multiplied (for example, $2 \times 1 = 1 \times 2 = 2$).

$AI = IA = A$ where A is a square matrix and I is the identity matrix of the same order as A.

Permutation matrix

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Unit 4
AOS M1
Topic 2
Concept 4

Allocating tasks, binary and permutation matrices
Concept summary
Practice questions

We mentioned at the start of the topic that a binary matrix is a matrix that consists of elements that are all either 0 or 1. Just like the identity matrix being a special type of a binary matrix, a **permutation matrix** is another special type. A permutation matrix is an $n \times n$ matrix that is a row or column permutation of the identity matrix. Each row and column of a permutation matrix contains the digit 1 exactly once. Permutation matrices reorder the rows or columns of another matrix via multiplication.

If you have a matrix Q and a permutation matrix P, then:

- QP is a column permutation of Q.
- PQ is a row permutation of Q.

Given $Q = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then:

- $QP = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 6 & 4 \\ 8 & 9 & 7 \end{bmatrix}$, which is a **column permutation** of Q.

  This is because the permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has 1s in the positions a_{13}, a_{21} and a_{32}, thus column 1→3, column 2→1 and column 3→2.

- $PQ = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, which is a **row permutation** of Q.

  This is because the permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has 1s in the position a_{13}, a_{21} and a_{32}, thus row 3→1, row 1→2 and row 2→3.

WORKED EXAMPLE 8

Given the matrix $R = \begin{bmatrix} 7 & 3 & -1 \\ 6 & 5 & 2 \\ -3 & 9 & 8 \end{bmatrix}$ and the permutation matrix,

$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, answer the following:

a What type permutation is RP?

b Which column does not change when performing a column permutation of R?

c When performing a row permutation of R, which row does row 2 go to?

THINK	WRITE
a Look at the order of the matrix multiplication.	a Since the permutation matrix is the second matrix in the multiplication of RP, it is a column permutation of R.

b **1** Look at the positions of the 1s in the permutation matrix to determine the change in columns.	**b** The position of the 1s are a_{12}, a_{21} and a_{33}, therefore column 1→2, 2→1 and 3→3.
2 Write the answer.	The column that doesn't change is column 3.
c **1** Look at the positions of the 1s in the permutation matrix to determine the change in rows.	**c** The position of the 1s are a_{12}, a_{21} and a_{33}, therefore row 2→1, 1→2 and 3→3.
2 Write the answer.	Row 2 goes to row 1.

Properties of multiplication of matrices

The following list is a summary of properties of multiplication of matrices. These properties hold true when A, B and C are $m \times n$ matrices, I is an identity matrix and O is a zero matrix.

Property	Example
Associative	$(AB)C = A(BC)$
Identity	$AI = A = IA$
Distributive	$(A + B)C = AC + BC$ $C(A + B) = CA + CB$
Zero matrix	$AO = O = OA$

WORKED EXAMPLE 9

'Soundsmart' has three types of televisions priced at \$350, \$650 and \$890 and three types of DVD players priced at \$69, \$120 and \$250. The store owner wishes to mark up the prices of the televisions by 12% and mark down the prices of the DVD players by 10%.

a **Show the prices of the televisions and the DVD players as a suitable matrix.**

b **Show the matrix obtained by marking up the prices of the televisions by 12% and marking down the prices of the DVD players by 10%.**

c **Use matrix multiplication to calculate the new prices (correct to the nearest dollar).**

THINK	WRITE
a Place the televisions and the DVD players in columns and the prices in the rows. This results in a 3×2 matrix.	**a** $\begin{bmatrix} 350 & 69 \\ 650 & 120 \\ 890 & 250 \end{bmatrix}$
b **1** A 12% mark-up is equivalent to 112% or 1.12. A markdown of 10% is equivalent to 90% or 0.9.	**b** 12% + 100% = 112%, 100% – 10% = 90%

2 We need to multiply the first column of the price matrix from **a** by 1.12 and multiply the second column by 0.9. This is a diagonal matrix.

$$\begin{bmatrix} 1.12 & 0 \\ 0 & 0.90 \end{bmatrix}$$

c **1** Multiply the two matrices to calculate the new prices. The order of the first matrix is 3 × **2** and the order of the second matrix is **2** × 2. The resulting matrix will be of order 3 × 2.

c New prices $= \begin{bmatrix} 350 & 69 \\ 650 & 120 \\ 890 & 250 \end{bmatrix} \times \begin{bmatrix} 1.12 & 0 \\ 0 & 0.90 \end{bmatrix}$

$$= \begin{bmatrix} 350 \times 1.12 + 69 \times 0 & 350 \times 0 + 69 \times 0.9 \\ 650 \times 1.12 + 120 \times 0 & 650 \times 0 + 120 \times 0.9 \\ 890 \times 1.12 + 250 \times 0 & 890 \times 0 + 250 \times 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 392 & 62.1 \\ 728 & 108 \\ 996.8 & 225 \end{bmatrix}$$

2 Round the answers to the nearest dollar.

The marked-up prices for the three types of televisions will be $392, $728 and $997. The marked-down prices for the three types of DVD players will be $62, $108 and $225.

WORKED EXAMPLE 10

The number of desktop and notebook computers sold by four stores is given in the table.

	Desktop	**Notebook**
Store A	10	4
Store B	4	5
Store C	5	10
Store D	3	2

If the desktop computers were priced at $1500 each and the notebook computers at $2300 each, use matrix operations to find:

a **the total sales figures of each computer at each store**

b **the total sales figures for each store**

c **the store that had the highest sales figures for**

  i **desktop computers**

  ii **total sales.**

THINK	WRITE
a **1** Set up a 4 × 2 matrix to represent the sales figures and a 2 × 2 matrix to enable us to determine the total sales of each computer at each store.	**a** Sales matrix $= \begin{bmatrix} 10 & 4 \\ 4 & 5 \\ 5 & 10 \\ 3 & 2 \end{bmatrix}$, price matrix $= \begin{bmatrix} 1500 & 0 \\ 0 & 2300 \end{bmatrix}$
2 Multiply the two matrices. The resultant matrix displays the total sales of each computer at each store.	Total sales figures at each store $= \begin{bmatrix} 10 & 4 \\ 4 & 5 \\ 5 & 10 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1500 & 0 \\ 0 & 2300 \end{bmatrix}$ $= \begin{bmatrix} 15\,000 & 9\,200 \\ 6\,000 & 11\,500 \\ 7\,500 & 23\,000 \\ 4\,500 & 4\,600 \end{bmatrix}$
b **1** Use the 4 × 2 sales matrix from part **a** and a 2 × 1 column matrix to calculate the total sales figure for each store.	**b** Matrix to determine the total sales $= \begin{bmatrix} 1500 \\ 2300 \end{bmatrix}$
2 Multiply the matrices. The resultant matrix displays the total sales at each store. *Note:* CAS can be used to perform the multiplication.	Total sales $= \begin{bmatrix} 10 & 4 \\ 4 & 5 \\ 5 & 10 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1500 \\ 2300 \end{bmatrix}$ $= \begin{bmatrix} 24\,200 \\ 17\,500 \\ 30\,500 \\ 9\,100 \end{bmatrix}$ The total sales figure for store A was \$24 200, store B was \$17 500, store C was \$30 500 and store D was \$9100.
c **i** Examine the matrix from part **a**. The highest sales figure for desktop computers is the highest number in the first column.	**c** **i** Store A has the highest sales figure for desktop computers: \$15 000.
ii Examine the matrix from part **b**. The store with the highest total sales figure is the row with the highest number.	**ii** Store C has the highest total sales figure: \$30 500.

EXERCISE 8.4 Multiplying matrices

PRACTISE

1 **WE7** Given $A = \begin{bmatrix} 8 & 2 \\ -6 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, find:

a AB **b** BA **c** B^2.

2 For the three matrices

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}:$$

a write the order of the three matrices

b find which of these products exist.

i AB **ii** AC **iii** BA **iv** CA **v** BC

3 **WE8** Given the matrix $R = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 5 & -7 \\ -4 & 9 & 6 \end{bmatrix}$ and the permutation matrix

$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, answer the following:

a What type of permutation is RP?

b Which column does not change when performing a column permutation of R?

c When performing a row permutation of R, which row does row 1 go to?

4 Given the matrix $R = \begin{bmatrix} 2 & 3 & -7 \\ 4 & 6 & -2 \\ -1 & 9 & 8 \end{bmatrix}$ and the permutation matrix, $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$,

answer the following:

a What type of permutation is PR?

b With a row permutation of R, where does row 1 move to?

c If a row permutation of R resulted in $R' = \begin{bmatrix} -1 & 9 & 8 \\ 2 & 3 & -7 \\ 4 & 6 & -2 \end{bmatrix}$, then find the permutation matrix that achieved this.

5 **WE9** The 'Whiteguys Store' has three types of washing machines, priced at \$550, \$750 and \$990, and three types of dryers, priced at \$160, \$220 and \$350. The owner of the store wishes to mark up the prices of the washing machines by 8% and mark down the prices of the dryers by 8%.

a Show the prices of washing machines and dryers as a suitable matrix.

b Show the diagonal matrix that would mark up washing machines by 8% and mark down the dryers by 8%.

c Calculate the new prices (correct to the nearest dollar) using the matrices from **a** and **b**.

6 A supermarket has three types of lettuces priced at \$1.50, \$2.00 and \$2.75 each and three types of potatoes priced at \$2.50, \$3.00 and \$3.25 per kilogram.

The manager wants to mark up the prices of lettuces by 15% and mark down the prices of potatoes by 12%.

a Show the prices of the lettuces and potatoes as a suitable matrix.

b Show the matrix obtained by marking down the prices of potatoes by 12% and marking up the price of lettuces by 15%.

c Use matrix multiplication to calculate the new prices (correct to the nearest cent).

7 **WE10** The number of pies and cans of soft drinks sold to four year-level groups at the canteen on a particular day is given in the table below.

Year	Pies	Soft drinks
Year 12	10	26
Year 10/11	25	45
Year 8/9	22	30
Year 7	5	22

If the pies were priced at \$2.50 each and the soft drinks at \$1.00 each, use matrix operations to find:

a the total sales figures of each food item for each year-level group

b the total sales figures for each year-level group

c the year-level group that had the highest sales figures for

  i pies

  ii total sales.

8 The number of iPads and iPad minis sold by four stores is given in the table shown.

	iPad	iPad mini
Store A	14	5
Store B	9	7
Store C	10	8
Store D	7	6

If the iPads were priced at \$550 each and the iPad minis at \$320 each, use matrix operations to find:

a the total sales figures of iPads at each store

b the total sales figures for each store

c the store that had the highest sales figures for:

  i iPads

  ii total sales.

CONSOLIDATE

9 Given three matrices

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 3 \end{bmatrix},\ B = \begin{bmatrix} 2 & -2 & 0 \\ 3 & 1 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & -2 \\ 3 & 5 \end{bmatrix}$$

a write the order of the three matrices

b find which of the following products exist

i AB ii BA iii AC iv CA v BC vi CB

c write the order of the products that exist

d calculate the products that exist.

10 a For $M = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, show all your working and find the products:

i MN **ii** NM.

b For $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show all your working and find the products:

i MN **ii** NM.

c Explain why $MN = NM$ (from parts **a** and **b** above), but $MN \neq NM$ in question **10a**.

11 If $A = B \times C$, then the element a_{31} is the result of:

A multiplying the third row by the first column
B multiplying the first row by the third column
C multiplying the third column by the first row
D multiplying the first column by the third row
E multiplying the third row by the first row

12 Use the following matrices to answer question **12**.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 3 & 0 \\ 2 & -1 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 5 & 2 \\ 3 & 6 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

a Which one of the following products does not exist?

A AB **B** BC **C** CA **D** DE **E** CE

b The order of the matrix BE is:

A 3 × 4 **B** 2 × 2 **C** 4 × 3 **D** 3 × 2 **E** 2 × 4

c Which of the following products gives a matrix of order 2 × 2?

A BA **B** BC **C** CA **D** AB **E** BD

d The matrix CE is:

A $\begin{bmatrix} 1 & 5 & 0 & 16 \\ 5 & 12 & 13 & 28 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 9 & 1 & 28 \\ 3 & 7 & 8 & 16 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 2 \\ 7 & 9 \\ 8 & 1 \\ 16 & 28 \end{bmatrix}$

D $\begin{bmatrix} 2 & 3 \\ 9 & 7 \\ 1 & 8 \\ 28 & 16 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & 7 & 8 & 16 \\ 2 & 9 & 1 & 28 \end{bmatrix}$

e The matrix D^2 is:

A $\begin{bmatrix} 4 & 30 & 12 \\ 6 & 37 & 16 \\ 12 & 60 & 24 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & 30 & 6 \\ 6 & 37 & 4 \\ -6 & 24 & 24 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & 30 & 6 \\ 6 & 37 & 4 \\ 0 & 24 & 24 \end{bmatrix}$

D $\begin{bmatrix} 4 & 30 & 6 \\ 6 & 37 & 4 \\ 6 & 24 & 24 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 2 & 30 & 30 & 6 & 12 \\ 6 & 6 & 37 & 24 & 4 & 8 \\ -6 & 12 & 24 & 36 & 24 & 12 \end{bmatrix}$

13 Simplify the expressions below for the matrices

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 3 \end{bmatrix},\ B = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix},\ C = \begin{bmatrix} 0 & -2 \\ 3 & 5 \end{bmatrix},\ \text{and } D = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

a $2A + AB$ **b** $A(B + C)$ **c** $AB + CD$ **d** $BA + DC$ **e** $2DB - D$.

14 A matrix of 3 × 3 order is the product of three matrices S, E and N in that order ($S \times E \times N$). If matrix $E = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ then the order of matrix N is:

A 3 × 2 **B** 2 × 3 **C** 3 × 3 **D** 2 × 2
E Not enough information. Also need the order of matrix S.

15 A supermarket has three types of apples priced at \$2.50, \$3.50 and \$4.00 per kilogram and three types of avocados priced at \$0.90, \$1.90 and \$2.50. The manager wishes to mark up the prices of the avocados by 15% and mark down the prices of the apples by 15%.

a Show the prices of the apples and avocados as a suitable matrix.
b Show the matrix obtained by marking down the prices of the apples by 15% and marking up the prices of the avocados by 15%.
c Use matrix multiplication to calculate the new prices (correct to the nearest cent).

16 A sports store has four types of tennis racquets priced at \$25.00, \$35.00, \$95.00 and \$140.00 and four types of footballs priced at \$9.90, \$19.90, \$75.00 and \$128.00. The manager wishes to mark down the prices of all items by 20% in preparation for a sale.

a Show the prices of the tennis racquets and footballs as a suitable matrix.
b Show the matrix obtained by marking down the prices by 20%.
c Use matrix multiplication to calculate the sale prices (correct to the nearest ten cents).

17 Four golf shops, A, B, C and D are supplied with boxes of different brands of golf balls (Longer, Higher, Further and Straighter) as shown in the table.

Shop	Longer	Higher	Further	Straighter
Shop A	12	10	10	12
Shop B	15	25	15	25
Shop C	10	10	10	10
Shop D	8	20	5	18

The cost of the golf balls, per box, is:

Longer: \$15 Higher: \$20 Further: \$30 Straighter: \$32

a Present the stock of golf balls for the four shops as a 4×4 matrix.
b Write the costs as a matrix.
c Use matrix multiplication to determine the total value of each golf ball brand at each shop.
d Calculate the total value of golf balls at each shop.

18 Given the matrix $D = \begin{bmatrix} 1 & 3 & -3 \\ 5 & 7 & 8 \\ -6 & 4 & 2 \end{bmatrix}$ and the permutation matrix, $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, answer the following questions.

a What type of permutation is PD?
b With a row permutation of D, where does row 1 move to?
c With a column permutation of D, where does column 3 move to?
d Which of the following is not a permutation matrix?

A $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e If a row permutation of D resulted in $D' = \begin{bmatrix} -6 & 4 & 2 \\ 1 & 3 & -3 \\ 5 & 7 & 8 \end{bmatrix}$, find the permutation matrix that achieved this.

MASTER

19 The sales figures (in dollars) for three months, at three stores, of three brands of mobile phones are shown in the following tables.

February	Mobile A (\$)	Mobile B (\$)	Mobile C (\$)
Store A	600	500	0
Store B	480	750	840
Store C	240	1000	0

March	Mobile A (\$)	Mobile B (\$)	Mobile C (\$)
Store A	1200	1000	420
Store B	840	1500	1260
Store C	1200	1750	0

April	Mobile A (\$)	Mobile B (\$)	Mobile C (\$)
Store A	1440	750	1680
Store B	600	1500	2100
Store C	1560	250	420

a Write a matrix to represent the sales figures for each month.
b Use a suitable matrix operation to show the total sales figures (\$) for the three months by store and mobile phone brand.

c If brand A mobile phones cost \$120, brand B mobile phones cost \$250 and brand C mobile phones cost \$420, use an appropriate matrix operation to calculate how many mobile phones of each brand were sold in total by each store, for the three months.

d Calculate the total sales for each store for the three months.

20 The Fibonacci numbers are a part of an interesting sequence of numbers that have been the focus of a great deal of study over the years. The first 12 numbers of the sequence are presented in the table.

Term	1	2	3	4	5	6	7	8	9	10	11	12
Number	1	1	2	3	5	8	13	21	34	55	89	144

The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ can be used to find terms of the Fibonacci sequence.

a Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and complete the following calculations using CAS.

i $A \times A$ (that is, A^2)

ii A^3

iii A^4

iv A^5

b Study each of the answers found in part a carefully. How do the elements in each answer and the power in the question relate to the Fibonacci numbers?

c Using your finding from part b (that is, without performing any calculations) what will be the elements in the matrix A^8?

d Use CAS and matrix methods to find the 30th Fibonacci number.

8.5 Multiplicative inverse and solving matrix equations

Recall from arithmetic that any number multiplied by its reciprocal (multiplicative inverse) results in 1. For example, $8 \times \frac{1}{8} = 1$.

Now, consider the matrix multiplication below.

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 3 \times -3 & 2 \times -3 + 3 \times 2 \\ 3 \times 5 + 5 \times -3 & 3 \times -3 + 5 \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that the answer is the identity matrix (I).

This means that one matrix is the **multiplicative inverse** of the other. In matrices, we use the symbol A^{-1} to denote the multiplicative inverse of A. A must be a square matrix.

If $AA^{-1} = A^{-1}A = I$, then A^{-1} is called the multiplicative inverse of A.

Finding the inverse of a square 2 × 2 matrix

In matrices, we are often given a square 2 × 2 matrix and are asked to determine its inverse. The following notes display how this is achieved.

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Determinants and inverses of matrices
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Let A represent a matrix of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

1. Swap the elements on the main diagonal and multiply the elements on the other diagonal by –1. This results in the matrix $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
2. Multiply this matrix by $\dfrac{1}{ad - bc}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

Note: The value of $ad - bc$ is known as the **determinant** of matrix A. It is commonly written as det A or $|A|$.

WORKED EXAMPLE 11

Calculate the determinants of the following matrices.

$A = \begin{bmatrix} 3 & 4 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 3 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}$

THINK	WRITE
1 For a matrix of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is given by $ad - bc$.	det $A = ad - bc$
2 Calculate the determinant for each of the three matrices.	det $A = 3 \times -3 - 4 \times 2$ $= -17$ det $B = 6 \times 2 - 3 \times 2$ $= 6$ det $C = 4 \times 6 - 8 \times 3$ $= 0$

Singular and regular matrices

Notice that the determinant of matrix C in Worked example 11 was 0. For any matrix that has a determinant of 0, it is impossible for an inverse to exist. This is because $\dfrac{1}{0}$ is undefined. A matrix with a determinant equal to 0 is called a **singular matrix**.

If the determinant of a matrix is not 0, it is called a regular matrix.

WORKED EXAMPLE 12

If $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, find its inverse, A^{-1}.

THINK	WRITE
1 Calculate the determinant of A. (If the determinant is equal to 0, the inverse will not exist.)	det $A = ad - bc$ $= 3 \times 4 - 5 \times 2$ $= 2$

THINK	WRITE
2 Use the rule to find the inverse. That is, $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Swap the elements on the main diagonal and multiply the elements on the other diagonal by –1. Multiply this matrix by $\frac{1}{ad - bc}$.	$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$ *Note:* Fractional scalars can be left outside the matrix unless they give whole numbers when multiplied by each element.

For matrices of a higher order than 2 × 2, for example, 3 × 3, 4 × 4, and so on, finding the inverse (and determinant) is more difficult and CAS is required.

Further matrix equations

studyon

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Solving matrix equations
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Recall from algebra that to solve an equation in the form $4x = 9$, we need to divide both sides by 4 $\left(\text{or multiply both sides by } \frac{1}{4}\right)$ to obtain the solution $x = \frac{9}{4}$.

A matrix equation of the type $AX = B$ is solved in a similar manner. Both sides of the equation are multiplied by A^{-1}. Since the order of multiplying matrices is important, we must be careful of the position of the inverse (remember that the products of AX and XA are different).

Solving for X in the following situations:

1. For $AX = B$

Pre-multiply by A^{-1}

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B && \text{since } A^{-1}A = I \\ X &= A^{-1}B && \text{since } IX = X \end{aligned}$$

2. For $XA = B$

Post-multiply by A^{-1}

$$\begin{aligned} XAA^{-1} &= BA^{-1} \\ XI &= BA^{-1} && \text{since } AA^{-1} = I \\ X &= BA^{-1} && \text{since } XI = X \end{aligned}$$

In summary:
1. **If $AX = B$, then $X = A^{-1}B$**
2. **If $XA = B$, then $X = BA^{-1}$**

WORKED EXAMPLE 13

For the given matrices

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 16 & -3 \\ 3 & 7 \end{bmatrix}$$

solve for the unknown matrix X if:

a $AX = B$

b $AX = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$.

THINK	WRITE
a 1 We are required to pre-multiply by A^{-1} to get matrix X by itself.	**a** $\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$

2 Find A^{-1}. First calculate the determinant. Then swap the elements on the leading diagonal of A and multiply the elements on the other diagonal by –1.

$$\begin{aligned}\det A &= ad - bc\\ &= 2 \times 3 - 5 \times -1\\ &= 11\end{aligned}$$

$$\begin{aligned}A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b\\ -c & a\end{bmatrix}\\ &= \frac{1}{11}\begin{bmatrix} 3 & -5\\ 1 & 2\end{bmatrix}\end{aligned}$$

3 Write the equation to be solved and substitute in the matrices.

$$\begin{aligned}X &= A^{-1}B\\ &= \frac{1}{11}\begin{bmatrix} 3 & -5\\ 1 & 2\end{bmatrix}\begin{bmatrix} 16 & -3\\ 3 & 7\end{bmatrix}\end{aligned}$$

4 Calculate the product of A^{-1} and B. Multiply each element by the fractional scalar as they will all result in whole numbers.

$$\begin{aligned}&= \frac{1}{11}\begin{bmatrix} 33 & -44\\ 22 & 11\end{bmatrix}\\ &= \begin{bmatrix} 3 & -4\\ 2 & 1\end{bmatrix}\end{aligned}$$

b **1** We are required to pre-multiply by A^{-1} to get matrix X by itself.

b

$$\begin{aligned}AX &= \begin{bmatrix} 13\\ -1\end{bmatrix}\\ A^{-1}AX &= \begin{bmatrix} 13\\ -1\end{bmatrix}\\ X &= A^{-1}\begin{bmatrix} 13\\ -1\end{bmatrix}\end{aligned}$$

2 Use A^{-1} from part **a**. Calculate the product of A^{-1} and $\begin{bmatrix} 13\\ -1\end{bmatrix}$. Multiply each element by the fractional scalar.

$$\begin{aligned}&= \frac{1}{11}\begin{bmatrix} 3 & -5\\ 1 & 2\end{bmatrix}\begin{bmatrix} 13\\ -1\end{bmatrix}\\ &= \frac{1}{11}\begin{bmatrix} 44\\ 11\end{bmatrix}\\ &= \begin{bmatrix} 4\\ 1\end{bmatrix}\end{aligned}$$

EXERCISE 8.5 Multiplicative inverse and solving matrix equations

PRACTISE

1 **WE11** Calculate the determinants of the following matrices.

$$A = \begin{bmatrix} 2 & 2\\ 4 & 7\end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 4\\ 3 & -5\end{bmatrix}$$

2 Calculate the determinants of the following matrices.

$$A = \begin{bmatrix} -6 & -8\\ 15 & 12\end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 8\\ 3 & 4\end{bmatrix}$$

3 **WE12** If $A = \begin{bmatrix} 5 & 6 \\ 6 & 8 \end{bmatrix}$, find its inverse, A^{-1}.

4 For the matrix $C = \begin{bmatrix} -5 & -1 \\ 10 & 4 \end{bmatrix}$, find:

a det C

b C^{-1}, the inverse of C.

5 **WE13** For the matrices $A = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -3 \\ 3 & 7 \end{bmatrix}$, solve for the unknown matrix X, given:

a $AX = B$

b $AX = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$.

6 For the matrices

$$A = \begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -4 & 8 \\ -5 & 7 \end{bmatrix}$$

solve for the unknown matrix X, given:

a $AX = B$

b $AX = \begin{bmatrix} 12 \\ -2 \end{bmatrix}$.

CONSOLIDATE

7 Calculate the determinants of the following matrices.

a $A = \begin{bmatrix} 9 & 8 \\ 7 & 5 \end{bmatrix}$

b $B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ 1 & \frac{1}{3} \end{bmatrix}$

c $C = \begin{bmatrix} 5 & 9 \\ 8 & 3 \end{bmatrix}$

d $D = \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix}$

8 State which matrices from question **7** will not have an inverse. Explain your answer.

9 Calculate the inverse matrix for each matrix (where possible) in question **7**.

10 If $C = \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix}$, find its inverse, C^{-1}.

11 Given the two matrices below, show that A and B are inverses of each other.

$$A = \begin{bmatrix} 11 & 12 \\ 10 & 11 \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 & -12 \\ -10 & 11 \end{bmatrix}$$

12 If $T = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix}$, then T^{-1} is equal to:

A $-\frac{1}{4}\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$

B $\frac{1}{4}\begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$

C $-\frac{1}{4}\begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}$

D $-\frac{1}{4}\begin{bmatrix} 2 & -2 \\ 3 & -5 \end{bmatrix}$

E $\frac{1}{4}\begin{bmatrix} 5 & 3 \\ 2 & 2 \end{bmatrix}$

13 If $P = \begin{bmatrix} 12 & 4 \\ -12 & -6 \end{bmatrix}$, then P^{-1} could be:

A $\begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

B $\begin{bmatrix} -\frac{1}{4} & -\frac{1}{6} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

C $\begin{bmatrix} \frac{1}{20} & \frac{1}{30} \\ -\frac{1}{10} & -\frac{1}{10} \end{bmatrix}$

D $\begin{bmatrix} 0.3 & 0.2 \\ -0.5 & -0.5 \end{bmatrix}$

E $\begin{bmatrix} \frac{1}{4} & \frac{1}{6} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

14 For the matrix $\begin{bmatrix} 0.1 & 0.2 \\ 0.25 & 0.45 \end{bmatrix}$:

a calculate the determinant

b state the inverse in the form $\begin{bmatrix} \dfrac{d}{ad - bc} & -\dfrac{b}{ad - bc} \\ -\dfrac{c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$.

15 Use CAS to find:

i the determinant

ii the inverse matrix

for each of the following matrices.

a $A = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}$

b $B = \begin{bmatrix} -3 & 4 \\ 3 & -6 \end{bmatrix}$

c $C = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$

d $D = \begin{bmatrix} 0.4 & 1.0 \\ 0.2 & 0.25 \end{bmatrix}$

e $E = \begin{bmatrix} 5 & 2 \\ -10 & -4 \end{bmatrix}$

f $F = \begin{bmatrix} 0.3 & 0.48 \\ 0.5 & 0.8 \end{bmatrix}$

g $G = \begin{bmatrix} 4 & 6 & 8 \\ 6 & 4 & 6 \\ 8 & 6 & 4 \end{bmatrix}$

h $H = \begin{bmatrix} 12 & 8 & 0 & 4 \\ 8 & 4 & 8 & 8 \\ 4 & 12 & 8 & 12 \\ 8 & 4 & 4 & 8 \end{bmatrix}$

i $I = \begin{bmatrix} 0 & 1 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 2 & 1 & 0 & 4 \\ 2 & 1 & 3 & 0 \end{bmatrix}$

j $J = \begin{bmatrix} 1 & 4 \\ 5 & 3 \\ 6 & 3 \end{bmatrix}$

16 For the given matrices

$A = \begin{bmatrix} 5 & 5 \\ 0 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -3 \\ 3 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 0 \\ 3 & 6 \end{bmatrix}$

solve for the unknown matrix X if:

a $AX = B$

b $XA = B$

c $XC = A$

d $CX = \begin{bmatrix} 100 \\ 120 \end{bmatrix}$.

MASTER

17 If $F = \begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}$ and $G = \begin{bmatrix} 0.2 & -1.2 \\ 0.1 & 2.8 \end{bmatrix}$, solve the following matrix equations.

a $FX = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ **b** $GX = \begin{bmatrix} -1.0 \\ 2.9 \end{bmatrix}$

18 Solve the following matrix equation for X.

$$\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} X = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$$

8.6 Dominance and communication matrices

A **directed graph** (or digraph) is a graph or network where every edge has a direction. Directed graphs can be used to represent many situations, such as traffic flow, competitions between teams or the order of activities in a production line.

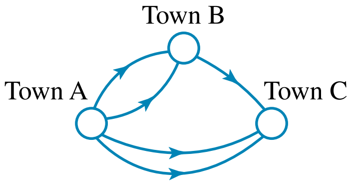
Reachability

study on

Unit 4
AOS M2
Topic 2
Concept 2

Matrix representation of directed graphs
Concept summary
Practice questions

As the name suggests, **reachability** is the concept of how it is possible to go from one vertex in a directed network to another. The different pathways that link the vertices are analysed.



Consider the directed network shown, representing possible pathways (routes) from town A to town C. By inspection, it can be seen that there are two pathways that go directly from A to C, without passing through B. That is, there are two **one-stage pathways** from A to C. A one-stage pathway is one that includes one edge only.

There are also two pathways that go from A to C via B. These are called **two-stage pathways**. A two-stage pathway is one that contains two edges only.

Notice that there are no routes entering town A but there are four leaving it. We say that the **indegree** of A is zero, while its **outdegree** is four. The indegree is the number of edges moving into a vertex and the outdegree is the number of edges moving away from a vertex. The indegree of B is two and its outdegree is one. The indegree of C is three and its outdegree is zero. A is the **source** and C is the **sink** of the network.

Matrix representation

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Interactivity
The adjacency matrix
int-6466

The one-stage and two-stage pathways for a directed network can be represented in matrix form. The matrix here displays all of the possible one-stage pathways for the previous network. It is commonly known as the **adjacency matrix** and is denoted by A.

$$\text{To}$$
$$\text{From} \quad \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

The matrix shows that there are two one-stage pathways from A to B and two one-stage pathways from A to C. There is also a one-stage pathway from B to C. Notice that the sum of each row is equal to the outdegree of each vertex and the sum of each column is equal to the indegree of each vertex. This can be a useful tip to ensure you have completed the adjacency matrix correctly.

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Adjacency matrix

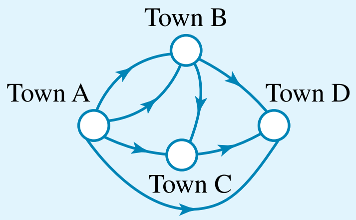
The matrix here displays all of the possible two-stage pathways of the network. This matrix is denoted by A^2. There are two two-stage pathways from A to C (via B).

Note that it is also possible to represent other stage pathways in matrix form; for example, three-stage is denoted by A^3, four-stage is denoted by A^4 and so on. This text will only concentrate on representing up to two-stage pathways in matrix form.

$$\begin{array}{ccc} & & \text{To} \\ & & \begin{array}{ccc} A & B & C \end{array} \\ \text{From} & \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad A^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

WORKED EXAMPLE 14

Wendy is a businessperson working in Town A and wishes to meet with a colleague in Town D. She also needs to pick up some documents from Town B to take to the meeting.



- **a** **Find the number of two-stage paths that Wendy could take and name them.**
- **b** **Find the number of three-stage paths that Wendy could take and name them.**
- **c** **Represent the one-stage and two-stage pathways of the directed network in matrix form.**

THINK	WRITE
a **1** Find the number of pathways from A to B (first stage).	**a** There are two paths from A to B.
2 Find the number of pathways from B to D (second of the two stages).	There is only one path from B to D.
3 Write the pathways.	There are 2 two-stage paths. A–B–D and A–B–D
b **1** What two-stage pathways are possible from B to D?	**b** The only two-stage path from B to D is B–C–D.
2 Write the pathways.	The only three-stage paths are A–B–C–D and A–B–C–D. There are two possible routes.
c **1** Draw a matrix to show all the possible one-stage pathways throughout the network.	**c** $\begin{array}{ccc} & & \text{To} \\ & & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{From} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

2 The matrix can be represented without the labels along the side and the top.

$$A=\begin{bmatrix}0&2&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

3 Repeat steps 1 and 2 and display all the two-stage pathways throughout the network.

$$A^2=\begin{bmatrix}0&0&2&3\\0&0&0&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Dominance

studyon

Unit 4
AOS M1
Topic 2
Concept 5

Dominance matrix
Concept summary
Practice questions

If an edge in a directed network moves from A to B, then it can be said that A is dominant, or has a greater influence, over B. If an edge moves from B to C, then B is dominant over C. However, we often wish to find the dominant vertex in a network; that is, the vertex that holds the most influence over all the other vertices. This may be clearly seen by inspection, by examining the pathways between the vertices. It may be the vertex that has the most edges moving away from it. Generally speaking, if there are more ways to go from A to B than there are to go from B to A, then A is the dominant vertex. In Worked example 14, town A is dominant over all the other vertices (towns) as it has edges moving to each of the other vertices. Similarly, B has edges moving to C and D, so B is dominant over C and D and C is dominant over D. Using this inspection technique, we can list the vertices in order of dominance from A then B then C and finally D.

However, the dominant vertex in a directed network may not be easily determined by inspection. There may be an edge moving from A to B and another one from B to A. What is the most dominant then? A more formal approach to determine a dominant vertex can be taken using matrix representation. Using the matrices from Worked example 14, this approach is outlined below.

Take the matrices that represent the one-stage pathways (the adjacency matrix, A) and two-stage pathways (A^2) and add them together. (When adding matrices, simply add the numbers in the corresponding positions.)

$$\underset{A}{\begin{bmatrix}0&2&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}}+\underset{A^2}{\begin{bmatrix}0&0&2&3\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}}=\begin{bmatrix}0&2&3&4\\0&0&1&2\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

The resulting matrix, which we will call the **dominance matrix**, consists of all the possible one- and two-stage pathways in the network. By taking the sum of each row in this matrix, we can determine the **dominant vertex**. The dominant vertex belongs to the row that has the highest sum.

The first row corresponds with vertex A and has a sum of 9. Row 2 (vertex B) has a sum of 3, row 3 (vertex C) has a sum of 1 and row 4 (vertex D) has a sum of 0. The highest sum is 9, so the dominant vertex is A. The order of dominance is the same as for the inspection technique described earlier.

This formal approach just described is not the only technique used to determine dominance in a network. Other approaches are possible, but this section will concentrate only on the inspection technique and summing the rows of the matrix that results from $A + A^2$.

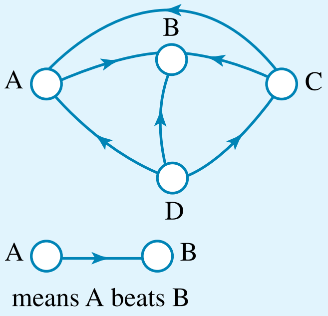
The concept of dominance can be applied to various situations such as transportation problems, competition problems and situations involving relative positions.

WORKED EXAMPLE 15

The results of a round robin (each competitor plays each other once) tennis competition are represented by the directed graph at right.

a **By inspection, determine the dominant vertex (dominant competitor); that is, the winner. Rank the competitors in finishing order.**

b **Confirm your answer to part a by finding the matrix, $A + A^2$, and summing the rows of this matrix.**



THINK

a Vertex D has arrows moving to all other vertices, so D is the dominant vertex. Vertex C is dominant over B and A and vertex A is dominant over B. List the competitors (vertices) in finishing order. (It is clear that B is the loser as all the arrows lead into it.)

b **1** Find the adjacency matrix, A (representing all of the one-stage pathways in the directed graph).

2 Find A^2, the matrix representing all the two-stage pathways in the matrix.

3 Find the resultant matrix when A and A^2 are added (add the numbers in the corresponding positions).

4 Determine the sum of each row of the resultant matrix.

WRITE

a The finishing order from first to fourth is D, C, A and B.

b $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

The resultant matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \end{bmatrix}.$$

The sum of the first row (vertex A) is 1, the second row (vertex B) is 0, the third row (vertex C) is 3 and the final row (vertex D) is 6.

5 Write a statement. The row with the highest sum is the dominant vertex and the row with the lowest sum is the least dominant.	D is the winner, followed by C, then A and finally B.

study on

Unit 4
AOS M1
Topic 2
Concept 6

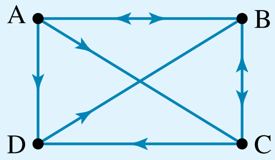
Communication matrix
Concept summary
Practice questions

Communication

From Worked example 15 we looked at results of a round robin tennis competition. This is where one competitor either wins or loses to another competitor. Thus, the network has arrows going only one way. A **communication network** has the ability of the arrows to travel both ways. A communication network contains a set of people where they can have a one-way or two-way communication link. A two-way communication link could be via phone. We can set up **communication matrices** from communication networks.

WORKED EXAMPLE 16

Find the communication matrix from the following communication network.



THINK / **WRITE**

1 Set up the communication matrix with 'send call' being the vertical and 'receive call' being the horizontal.

$$\begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

2 Place 1s where A can communicate, that is with B, C and D.

$$\begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

3 Place 1s where B can communicate, that is with A and C.

$$\begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ & & & \\ & & & \end{bmatrix} \end{array}$$

4 Place 1s where C can communicate, that is with B and D.

$$\begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ & & & \end{bmatrix} \end{array}$$

5 Place 1s where D can communicate, that is B.

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

6 Write the answer.

The communication matrix for the communication network shown is:

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

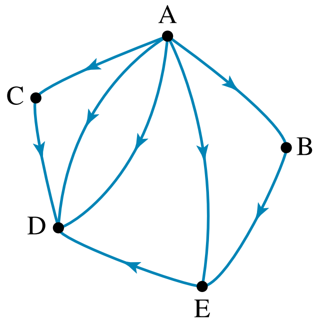
EXERCISE 8.6 Dominance and communication matrices

PRACTISE

1 **WE14** For the directed network at right determine the number and name of the:

a **i** one-stage paths
ii two-stage paths
iii three-stage paths
from A to D.

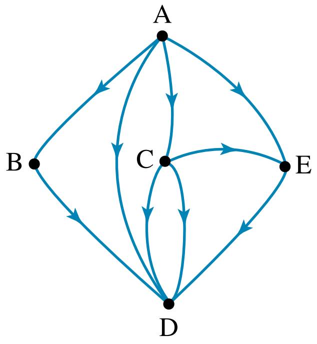
b Represent the one-stage and two-stage pathways of the directed network in matrix form.



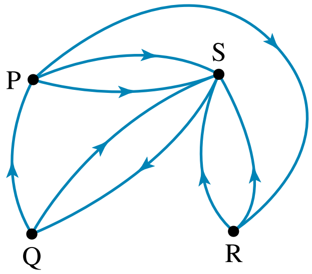
2 For the directed network at right determine the number and name of the:

a **i** one-stage paths
ii two-stage paths
iii three-stage paths
from A to D.

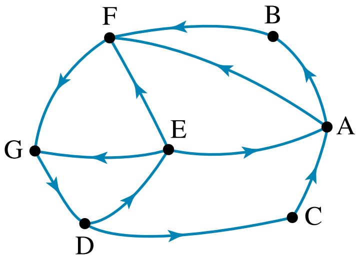
b Represent the one-stage and two-stage pathways of the directed network in matrix form.



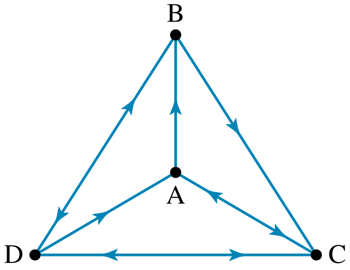
3 **WE15** Determine the dominant vertex for the following directed graph.



4 Determine the dominant vertex for the following directed graph.



5 **WE16** Find the communication matrix from the following communication network.



6 Given the following communication matrix, answer the following questions.

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

a Who can C talk to?
b Who can B receive calls from?
c Why is the main diagonal all zeroes?
d Who can D not call?

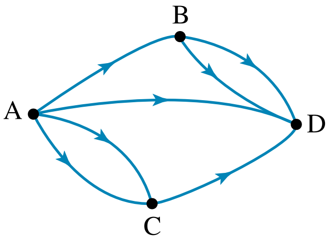
CONSOLIDATE

7 For each directed network shown determine the number and name of the:

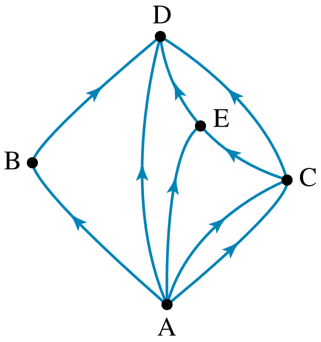
i one-stage paths
ii two-stage paths
iii three-stage paths

from A to D.

a

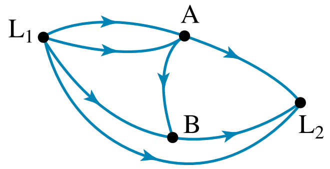


b



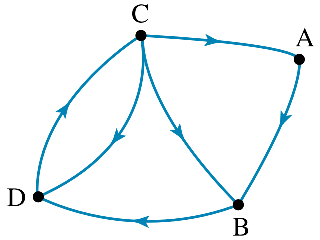
8 Represent the one-stage and two-stage pathways of the directed networks in question **7** in matrix form.

9 The directed graph represents part of a river system where the water flows from the lake, L_1, to another lake, L_2. If fish eggs flow from L_1 to L_2, via how many different routes is it possible for the eggs to go? Name all the routes.



10 The bus routes between certain landmarks are shown in the diagram. Name all the different routes by which it is possible to reach:

a D from A
b A from D
c B from D.

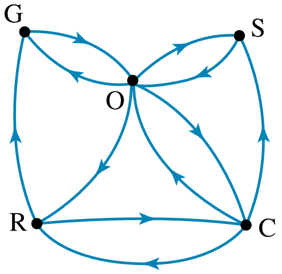


11 The directed network represents the pathways available to students as they move around their school. Name the different pathways by which the students can get from the:

a office to the gym
b common room to the science block
c science block to the common room

given that they wish to make a:

i one-stage trip
ii two-stage trip
iii three-stage trip.

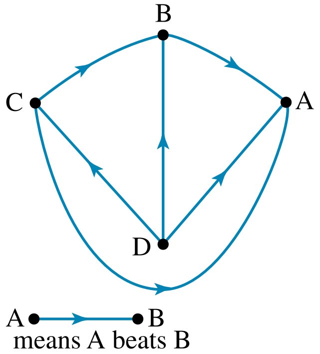


G = gym
S = science block
O = office
C = cafeteria
R = common room

12 The results of a round robin chess competition are represented by the directed graph.

a By inspection, determine the dominant vertex (dominant competitor); that is, the winner. Rank the competitors in finishing order.

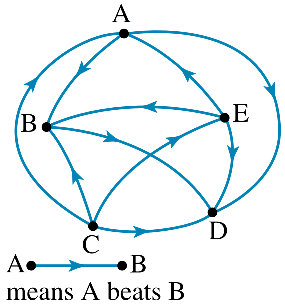
b Confirm your answer to part **a** by finding the matrix, $A + A^2$, and summing the rows of this resultant matrix.



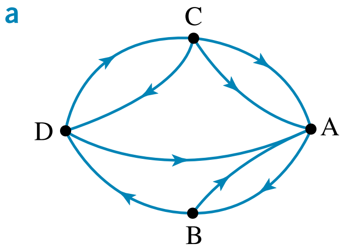
13 The results of a round robin basketball competition are represented by the directed graph.

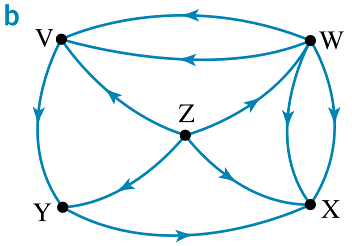
a By inspection, determine the dominant vertex (dominant team); that is, the winner. Rank the teams in finishing order.

b Confirm your answer to part **a** by finding the matrix, $A + A^2$, and summing the rows of this resultant matrix.



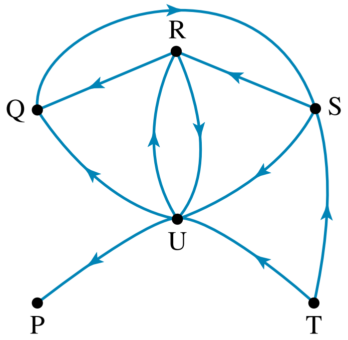
14 Determine the dominant vertex for each of the following directed graphs.



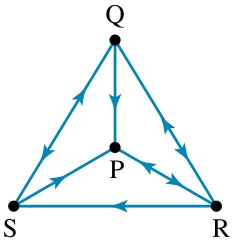


15 In the directed graph shown, the dominant vertex (by inspection) is:

A Q **B** R **C** S **D** T **E** U



16 Find the communication matrix from the following communication network.



MASTER

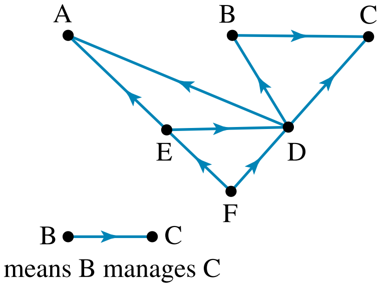
17 Given the following communication matrix, answer the following questions.

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a Who can C talk to?
b Who can B receive calls from?
c Who can call only one person?
d Who can D not call?
e Who can receive calls from everyone?

18 The personnel management roles of six employees are shown in the directed graph.

a Which employee(s) exerts most influence in this group?
b Which employee(s) exerts least influence?
c Determine the order of influence of all six employees.



means B manages C

8.7 Application of matrices to simultaneous equations

studyon

Unit 4

AOS M1

Topic 2

Concept 3

Simultaneous equations and matrices

Concept summary

Practice questions

eBook*plus*

Interactivity

Using matrices to solve simultaneous equations

int-6291

When solving equations containing one unknown, only one equation is needed. The equation is transposed to find the value of the unknown. In the case where an equation contains two unknowns, two equations are required to solve the unknowns. These equations are known as **simultaneous equations**. You may recall the algebraic methods of substitution and elimination used in previous years to solve simultaneous equations.

Matrices may also be used to solve linear simultaneous equations. The following technique demonstrates how to use matrices to solve simultaneous equations involving two unknowns.

Consider a pair of simultaneous equations in the form:

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

The equations can be expressed as a matrix equation in the form $AX = B$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called the **coefficient matrix**, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} e \\ f \end{bmatrix}$.

Notes

1. A is the matrix of the coefficients of x and y in the simultaneous equations.
2. X is the matrix of the pronumerals used in the simultaneous equations.
3. B is the matrix of the numbers on the right-hand side of the simultaneous equations.

As we have seen from Exercise 8.5, an equation in the form $AX = B$ can be solved by pre-multiplying both sides by A^{-1}.

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Simultaneous equations are not just limited to two equations and two unknowns. It is possible to have equations with three or more unknowns. To solve for these unknowns, one equation for each unknown is needed.

Simultaneous equations involving more than two unknowns can be converted to matrix equations in a similar manner to the methods described previously.

Let us consider an ancient Chinese problem that dates back to one of the oldest Chinese mathematics books, *The Nine Chapters on the Mathematical Art*.

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?

This information can be converted to equations, using the pronumerals x, y and z to represent the three types of corn, as follows:

$$\begin{aligned} 3x + 2y + 1z &= 39 \\ 2x + 3y + 1z &= 34 \\ 1x + 2y + 3z &= 26 \end{aligned}$$

(Note the importance of lining up the pronumerals on the left side and the numbers on the right side.)

As was the case earlier with two simultaneous equations, this system of equations can also be written as a matrix equation in the form $AX = B$ as follows:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 39 \\ 34 \\ 26 \end{bmatrix}$$

X can be solved by pre-multiplying both sides of the equation by A^{-1}. As the order of A is greater than (2×2), CAS should be used to find the inverse (A^{-1}). Try to solve this problem for yourself after reading the following worked example.

WORKED EXAMPLE 17

a **Solve the two simultaneous linear equations by matrix methods.**

$$\begin{aligned} 2x + 3y &= 13 \\ 5x + 2y &= 16 \end{aligned}$$

b **Use matrix methods to solve the following system of equations.**

$$\begin{aligned} x - 2y + z &= -2 \\ -2x + 3y &= -3 \\ 2x - z &= 4 \end{aligned}$$

THINK	WRITE
a **1** Write the simultaneous equations as a matrix equation in the form $AX = B$. Matrix A is the matrix of the coefficients of x and y in the simultaneous equations, X is the matrix of the pronumerals and B is the matrix of the numbers on the right-hand side of the simultaneous equations.	**a** $AX = B$ $\begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$
2 Matrix X is found by pre-multiplying both sides by A^{-1}.	$X = A^{-1}B$
3 Calculate the inverse of A.	$\det A = 2 \times 2 - 3 \times 5$ $= -11$ The inverse (A^{-1}) is $-\frac{1}{11}\begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}$.
4 Solve the matrix equation by calculating the product of A^{-1} and B and simplify.	$X = A^{-1}B$ $= -\frac{1}{11}\begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ 16 \end{bmatrix}$ $= -\frac{1}{11}\begin{bmatrix} -22 \\ -33 \end{bmatrix}$ $= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
5 Equate the two matrices and solve for x and y.	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

6 Write the answers.	The solution to the simultaneous equations is $x = 2$ and $y = 3$.
b 1 Use the information from the equations to construct a matrix equation. Insert a 0 in the coefficient matrix where the pronumeral is 'missing'.	**b** $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$
2 To solve, pre-multiply both sides by the inverse of the coefficient matrix.	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$
3 Use CAS to evaluate the right-hand side.	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$
4 Interpret the results and answer the question. You can double-check your answer by substituting these values into the original equations.	The values of the pronumerals are $x = 0$, $y = -1$ and $z = -4$.

Remember that if the determinant of a matrix equals zero, the inverse will not exist. When this happens there will not be a unique solution (i.e. one solution) for the system of simultaneous equations. It means there is either no solution (i.e. the graphs of the equations are parallel) or there are infinite solutions (i.e. the graphs of the equations draw the same line).

Dependent systems of equations

If the graphs of two or more equations coincide, that is if they are the same line, then there is no unique solution to the equations and we say that the equations are dependent.

If we try to solve **dependent equations** using matrix methods then we would find that there would be no solution as the determinant of the coefficient matrix is 0.

Inconsistent systems of equations

If the graphs of two or more equations do not meet, that is if the lines are parallel, then there is no solution to the equations and we say that the equations are inconsistent.

If we try to solve **inconsistent equations** using matrix methods then we would find that there would be no solution as the determinant of the coefficient matrix is 0.

WORKED EXAMPLE 18

Use a matrix method to decide if the simultaneous equations have a unique solution.

i $2x - y = 8$
$x + y = 1$

ii $2x + 4y = 12$
$3x + 6y = 8$

THINK	**WRITE**
i 1 Write the simultaneous equations as a matrix equation in the form $AX = B$.	**i** $AX = B$ $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

2 Find the determinant of the matrix A.	$\det A = ad - bc$ $= (2 \times 1) - (-1 \times 1)$ $= 2 - -1$ $= 3$
3 Write the answer.	Since $\det A \neq 0$, a unique solution can be found.
ii 1 Write the simultaneous equations as a matrix equation in the form $AX = B$.	ii $AX = B$ $\begin{bmatrix}2 & 4\\3 & 6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}12\\8\end{bmatrix}$
2 Find the determinant of the matrix A.	$\det A = ad - bc$ $= (2 \times 6) - (3 \times 4)$ $= 12 - 12$ $= 0$
3 Write the answer.	Since $\det A = 0$, a unique solution cannot be found and the equations are either dependent or inconsistent.

Matrices are a very efficient tool for solving problems with two or more unknowns. As a result, matrices are used in many areas such as engineering, computer graphics and economics. Matrices may also be applied to solving problems from other modules of the Further Mathematics course, such as break-even analysis, finding the first term and the common difference in arithmetic sequences and linear programming.

When answering problems of this type, take care to follow these steps:

1. Read the problem several times to ensure you fully understand it.
2. Identify the unknowns and assign suitable pronumerals. (Remember that the number of equations needed is the same as the number of unknowns.)
3. Identify statements that define the equations and write the equations using the chosen pronumerals.
4. Use matrix methods to solve the equations. (Remember, for matrices of order 3 × 3 and higher, use CAS.)

WORKED EXAMPLE 19

A bakery produces two types of bread, wholemeal and rye. The respective processing times for each batch on the dough-making machine are 12 minutes and 15 minutes, while the oven baking times are 16 minutes and 12 minutes respectively. How many batches of each type of bread should be processed in an 8-hour shift so that both the dough-making machine and the oven are fully occupied?

THINK	WRITE
1 Identify the unknowns and choose a suitable pronumeral for each unknown.	We need to determine the number of batches of wholemeal bread and the number of batches of rye bread. Let x = the number of batches of wholemeal bread. Let y = the number of batches of rye bread.

2 Write two algebraic equations from the given statements. All times must be expressed in the same units. (8 hours = 480 minutes)

$$12x + 15y = 480$$
$$16x + 12y = 480$$

3 Write the simultaneous equations as a matrix equation in the form $AX = B$.

$$AX = B$$
$$\begin{bmatrix} 12 & 15 \\ 16 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 480 \\ 480 \end{bmatrix}$$

4 Solve the matrix equation to find the values for x and y.

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{96}\begin{bmatrix} 12 & -15 \\ -16 & 12 \end{bmatrix}\begin{bmatrix} 480 \\ 480 \end{bmatrix}$$
$$= -\frac{1}{96}\begin{bmatrix} -1440 \\ -1920 \end{bmatrix}$$
$$= \begin{bmatrix} 15 \\ 20 \end{bmatrix}$$

5 Write your answer, relating the pronumerals to the original problem.

$x = 15$ and $y = 20$. To fully utilise the dough-making machine and the oven during an 8-hour shift, 15 batches of wholemeal bread and 20 batches of rye bread should be processed.

EXERCISE 8.7 Application of matrices to simultaneous equations

PRACTISE

1 **WE17** Solve these simultaneous linear equations by matrix methods.

$$x + y = 12$$
$$x + 4y = 36$$

2 Solve these simultaneous linear equations by matrix methods.

$$6x + 2y = 30$$
$$2x - y = 10$$

3 **WE18** Use a matrix method to decide whether the simultaneous equations have a unique solution.

$$6x + 9y = 17$$
$$2x + 3y = 6$$

4 Use a matrix method to decide whether the simultaneous equations have a unique solution.

$$x + 2y = -5$$
$$-2x + y = 2$$

5 **WE19** At a car spray-painting company, each car receives two coats of paint, which have to be completed within one day. There are two types of cars that this company spray paints — sedans and utilities.

The times are displayed in the following table.

	Stage of painting	
	1st coat	2nd coat
Sedan	5 minutes	9 minutes
Utility	7 minutes	8 minutes
Total time available for each stage	140 minutes	183 minutes

To fully utilise the company's time, how many sedans and utilities should be planned for in a day?

6 The cost of manufacturing jeans ($\$C$) is related to the number of pairs of jeans produced (n), by the formula

$$C = 1000 + 20n.$$

The revenue ($\$R$) made from selling n pairs of jeans is

$$R = 45n - 800.$$

The break-even point is when the cost and revenue are the same; that is, when $C = R$.

Using matrix methods, find out the number of pairs of jeans that need to be manufactured, and the amount that was invested in production if the break-even point was reached.

CONSOLIDATE

7 Solve each of the following matrix equations.

a $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 15 \end{bmatrix}$

c $\begin{bmatrix} 3 & 7 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -21 \\ 15 \end{bmatrix}$

d $\begin{bmatrix} 1 & -4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -10 \\ 21 \end{bmatrix}$

8 Solve the following simultaneous linear equations by matrix methods.

a $-3x + 7y = 65$
$-9x + 6y = -15$

b $3x + 2y = 9$
$6x + 4y = 22$

c $-x + 2y = 0$
$-6x + 14y = 2$

d $4x - y = -3$
$3x - y = -1$

e Use CAS and matrix methods to solve the following system of equations.

$$\begin{aligned} 2x + y + 4z &= 17 \\ 3x - y &= -3 \\ x + 4y + 5z &= 7 \end{aligned}$$

9 **a** Use a matrix method to decide whether the simultaneous equations have a unique solution.

i $2x + 2y = -4$
$-3x + y = 14$

ii $x - 3y = 4$
$-2x + 6y = 12$

iii $3x + 5y = -11$
$-6x - 10y = 22$

iv $4x + y = 7$
$6x + 2y = 14$

b If there is no unique solution, draw graphs to determine whether they are dependent or inconsistent systems of equations.

10 Consider the following two pairs of simultaneous linear equations.

i $6x + 2y = 4$
$9x + 3y = 14$

ii $\frac{1}{2}x - y = 3$
$\frac{3}{2}x - 3y = 9$

a Write each pair of simultaneous equations as a matrix equation in the form $AX – B$.

b Calculate the determinant for both coefficient matrices.

c Find the solution for each pair of simultaneous equations. What do you notice? Suggest a reason for this.

d Transpose the two equations in **i** into $y = mx + c$ form and graph them both using CAS. How do these graphs relate to your answer from part **c**?

e Transpose the two equations in **ii** into $y = mx + c$ form and graph them both using CAS. How do these graphs relate to your answer from part **c**?

11 Consider the pair of simultaneous equations:

$$y = 2x + 3$$
$$y = x + 1$$

- **a** Transpose the equations so that they are in the form $ax + by = c$.
- **b** Write the simultaneous equations as a matrix equation in the form $AX = B$.
- **c** Solve the matrix equation, writing the solution in coordinate form.

12 Solve the following set of simultaneous equations using matrix methods and CAS.

$$\begin{aligned} 2a - 3b + 6c + 2d &= 16 \\ 2b + 4c - d &= -3 \\ -a - b - c - d &= -4.5 \\ 0.1a + 0.4b - 0.6c + 1.2d &= 3.1 \end{aligned}$$

13 Consider the following problem studied by the Babylonians. (*Note:* We have substituted square metres, instead of square yards, as the units of area.)

There are two fields whose total area is 1800 square metres. One produces grain at the rate of $\frac{2}{3}$ of a bushel per square metre while the other produces grain at the rate of $\frac{1}{2}$ a bushel per square metre. If the total yield is 1100 bushels, what is the size of each field?

Use matrix methods to solve the problem.

14 The cost (in dollars) of manufacturing electronic components, d, is related to the number of components produced, n, by the formula $d = 6000 + 2.5n$. The revenue, d (in dollars), generated from selling n components is given by the formula $d = 4.5n - 8000$. Use matrix methods to calculate the number of components that need to be manufactured so that the manufacturing cost and revenue are equal.

15 The table shown displays the attendance numbers and the box-office takings for the first three shows of a new stage play.

Show	Adults	Children	Pensioners	Box-office takings ($)
First	40	20	5	945
Second	50	15	15	1165
Third	30	0	40	800

Use matrix methods and CAS to calculate the ticket prices for adults, children and pensioners.

16 Use matrix methods to find two numbers, where twice a number plus three times another number is 166 and the sum of the two numbers is 58.

MASTER

17 A factory produces two different models of transistor radios. Each model requires two workers to assemble it. The time taken by each worker varies according to the following table.

	Worker 1	**Worker 2**
Model A	5 minutes	5 minutes
Model B	18 minutes	4 minutes
Maximum time available for each worker	360 minutes	150 minutes

a Use matrix methods to calculate how many of each model should be produced so that each worker is used for the total time available.

b If the company makes \$2.50 on each model A sold and \$4.00 on each model B sold, what is the maximum amount of revenue from the sales?

18 The sum of the first 15 terms in an arithmetic sequence is 633 and the 30th term is 187.4. To find the first term, a, of this sequence and the common difference, d, the following two equations can be used

$$15a + 105d = 633$$
$$a + 29d = 187.4$$

Use matrix methods to find the first term and the common difference for this arithmetic sequence.

8.8 Transition matrices

Andrei Markov was a Russian mathematician whose name is given to a technique that calculates probability associated with the state of various transitions (which can be represented in matrix form). It answers questions such as, 'What is the probability that it will rain today given that it rained yesterday?' or 'What can be said about the long-term prospect of rainy days?'

Andrei Markov, 1856–1922

Powers of matrices

Throughout this section, it will be necessary to evaluate a matrix raised to the power of a particular number, for example M^3. Only square matrices can be raised to a power, as the order of a non-square matrix does not allow for repeated matrix multiplication. For example, a 2 × 3 matrix cannot be squared, because using the multiplication rule, we see the inner two numbers are not the same ($2 \times \mathbf{3} \times \mathbf{2} \times 3$).

WORKED EXAMPLE 20

For the given matrices

$$A = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}, B = \begin{bmatrix} 0.81 & 0.6 \\ 0.19 & 0.4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$$

use CAS to:

a evaluate i A^3 ii B^2 iii C^0

b evaluate C^{40}, expressing the matrix in whole numbers multiplied by a fractional scalar.

THINK	WRITE
a i Enter matrix A in CAS and raise it to the power of 3.	**a i** $A^3 = \begin{bmatrix}0.7 & 0.4\\0.3 & 0.6\end{bmatrix}^3$ $= \begin{bmatrix}0.583 & 0.556\\0.417 & 0.444\end{bmatrix}$
ii Enter matrix B in CAS and square it.	**ii** $B^2 = \begin{bmatrix}0.81 & 0.6\\0.19 & 0.4\end{bmatrix}^2$ $= \begin{bmatrix}0.7701 & 0.726\\0.2299 & 0.274\end{bmatrix}$
iii Enter matrix C in CAS and raise it to the power of 0.	**iii** $C^0 = \begin{bmatrix}5.6 & 0.1 & 0.3\\0.3 & 0.6 & 0.1\\0.1 & 0.3 & 0.6\end{bmatrix}^0$ $= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$
b 1 Raise matrix C to the power of 40.	**b** $C^{40} = \begin{bmatrix}0.6 & 0.1 & 0.3\\0.3 & 0.6 & 0.1\\0.1 & 0.3 & 0.6\end{bmatrix}^{40}$ $= \begin{bmatrix}0.333333 & 0.333333 & 0.333333\\0.333333 & 0.333333 & 0.333333\\0.333333 & 0.333333 & 0.333333\end{bmatrix}$
2 Identify the fractional scalar common to each element and place it outside the matrix to give your answer using whole number elements.	$C^{40} = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$

Markov systems and transition matrices

A **Markov system** (or Markov chain) is a system that investigates estimating the distribution of states of an event, given information about the current states. It also investigates the manner in which these states change from one state (condition or location) to the next, according to fixed probabilities. Matrices can be used to model such situations where:

- there are defined sets of conditions or *states*
- there is a *transition* from one state to the next, where the next state's probability is *conditional* on the result of the preceding outcome
- the conditional probabilities for each outcome are the same on each occasion; that is, the same matrix is used for each transition
- information about an **initial state** is given.

studyon

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Description of transition matrices
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Initial state and transitions
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A Markov system can be illustrated by means of a state transition statement, a table or a diagram. The following transition statements describe the movement of delivery trucks between two locations.

A group of delivery trucks transfer goods between two warehouses A and B. They start the day at either warehouse and finish the day parked at one of them.

70% of the trucks that start at A will park at A that night and 30% will park at B.

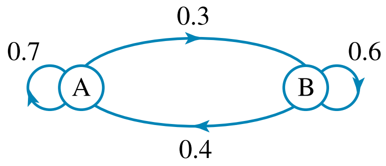
60% of the trucks that start at B will park at B that night and 40% will park at A.

This is how the statements are represented as a table and as a diagram. (In both, the values have been expressed as probabilities.)

Transition table

		Transition FROM	
		Warehouse A	**Warehouse B**
Transition TO	**Warehouse A**	0.7	0.4
	Warehouse B	0.3	0.6

Transition diagram



The statements, the table and the diagram all represent the same information. They can all be summarised as a **transition matrix** as shown. Throughout this section, the transition matrix will be denoted as T.

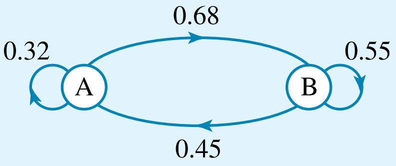
$$T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$$

Note that each of the columns of a transition matrix must add up to 1.

WORKED EXAMPLE 21

Represent each of the following as a transition matrix.

a



b **There are a number of train carriages operating between two depots, North depot and South depot. At the end of each week, 40% of the carriages that started at North depot end up at South depot and 25% of the carriages that started at South depot end up at North depot.**

THINK	WRITE
a **1** Identify there are two states, A and B. Enter the values in the correct manner; that is, *from* the column *to* the row. 0.32 from A to A and 0.68 from A to B. 0.55 from B to B and 0.45 from B to A.	**a** $\begin{array}{cc} & \text{From} \\ & \begin{array}{cc} \text{A} & \text{B} \end{array} \\ \text{To} \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{bmatrix} 0.32 & 0.45 \\ 0.68 & 0.55 \end{bmatrix} \end{array}$
2 Check that each column adds up to 1 and remove any labels.	$T = \begin{bmatrix} 0.32 & 0.45 \\ 0.68 & 0.55 \end{bmatrix}$

b **1** Identify there are two states, North depot and South depot. Enter the given percentage probabilities in decimal form.

b

$$\begin{array}{ccc} & & \text{From} \\ & & \begin{array}{cc}\text{North} & \text{South}\end{array} \\ \text{To} & \begin{array}{c}\text{North} \\ \text{South}\end{array} & \begin{bmatrix} & 0.25 \\ 0.40 & \end{bmatrix} \end{array}$$

2 The missing values can be calculated knowing the columns must add up to 1. Remove any labels.

$$T = \begin{bmatrix} 0.60 & 0.25 \\ 0.40 & 0.75 \end{bmatrix}$$

Distribution vector and powers of the transition matrix

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A distribution vector is a column vector with an entry for each state of the system. It is often referred to as the **initial state matrix** and is denoted by S_0.

If S_0 is an $n \times 1$ initial distribution vector state matrix involving n components and T is the transition matrix, then the distribution vector after 1 transition is the matrix product $T \times S_0$.

Distribution after 1 transition: $S_1 = T \times S_0$

The distribution one stage later is given by

Distribution after 2 transitions:
$$\begin{aligned} S_2 &= T \times S_1 \\ &= T \times (T \times S_0) \\ &= (T \times T) \times S_0 \\ &= T^2 \times S_0 \end{aligned}$$

We can continue this pattern to create a matrix recurrence relation.

The matrix recurrence relation $S_{n+1} = TS_n$, where T is a transition matrix and S_n is a column state matrix will generate a sequence of state matrices.

The distribution after n transitions can be obtained by premultiplying S_0 by T, n times or by multiplying T^n by S_0.

Distribution after n transitions:
$$\begin{aligned} S_n &= T \times S_{n-1} \\ &= T^n \times S_0 \end{aligned}$$

The sequence of states, S_0, S_1, S_2, ..., S_n is called a Markov chain.

Applications to marketing

One common use for the above approach is in marketing, where organisations can predict their share of the market at any given moment. Marketing records show that when consumers are able to purchase certain goods — for example, groceries — from competing stores A and B, we can associate conditional probabilities with the likelihood that they will purchase from a given store, or its competitor, depending on the store from which they had made their previous purchases over a set period, such as a month.

The following worked example highlights the application of transition matrices to marketing.

WORKED EXAMPLE 22

A survey shows that 75% of the time, customers will continue to purchase their groceries from store A if they purchased their groceries from store A in the previous month, while 25% of the time consumers will change to purchasing their groceries from store B if they purchased their groceries

from store A in the previous month. Similarly, the records show that 80% of the time, consumers will continue to purchase their groceries from store B if they purchased their groceries from store B in the previous month.

a **Use a matrix recurrence relation to determine how many customers are still purchasing their groceries from A and B at the end of two months, if 300 customers started at A and 300 started at B?**

b **What percentage (in whole numbers) of customers are purchasing their groceries at A and B at the end of 6 months, if 50% of the customers started at A?**

THINK	WRITE
a **1** Set up the transition matrix using the correct methods. Complete any missing probabilities knowing the columns must add up to 1.	**a** $T = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{bmatrix} \end{array}$ $T = \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix}$
2 Set up the initial state matrix. The elements for this matrix are the initial number of customers for each store.	$S_0 = \begin{bmatrix} 300 \\ 300 \end{bmatrix}$
3 The number of transitions is 2. Calculate the number of customers still purchasing at A and B by using the recurrence relation $S_{n+1} = TS_n$ and evaluating S_1 and S_2.	$S_1 = TS_0$ $= \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix} \begin{bmatrix} 300 \\ 300 \end{bmatrix}$ $= \begin{bmatrix} 285 \\ 315 \end{bmatrix}$ $S_2 = TS_1$ $= \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix} \begin{bmatrix} 285 \\ 315 \end{bmatrix}$ $= \begin{bmatrix} 276.75 \\ 323.25 \end{bmatrix}$
4 Interpret the answer represented by S_2.	After 2 months, 277 customers will be purchasing their groceries from store A and 323 will be purchasing their groceries from store B.
b **1** Identify that after 6 months, $n = 6$. The initial state matrix contains the percentage of customers for each store at the start.	**b** $S_0 = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$
2 Use the formula to set up the matrix calculation.	$S_n = T^n \times S_0$ $S_6 = T^6 \times S_0$ $= \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix}^6 \times \begin{bmatrix} 50 \\ 50 \end{bmatrix}$

3 Use CAS to evaluate S_6.

$S_6 = \begin{bmatrix} 44.5982 \\ 55.4018 \end{bmatrix}$

4 Interpret the answer represented by S_6. (Round to the nearest per cent.)

After 6 months, 45% of the customers will be purchasing their groceries from store *A* and 55% of the customers will be purchasing their groceries from store *B*.

New state matrix with culling and restocking

The new state matrix $S_n = T^n S_0$ can be extended to include culling and restocking. This can be done by adding (restocking) or subtracting (culling) a matrix to our original new state matrix.

The matrix recurrence relation $S_{n+1} = TS_n + B$, where B is a matrix (usually a column matrix), represents a transition situation including culling and restocking.

WORKED EXAMPLE 23

Betta Health Centres run concurrent Lift and Cycle fitness classes at all 3 of their gyms in FitTown.

A study shows that 80% of the clients who attended a Lift class one week will attend the Lift class the next week, while the other 20% will move to the Cycle class the next week.

Similarly, 70% of the clients who attended a Cycle class one week will attend the Cycle class the next week, while the other 30% will move to the Lift class the next week.

The numbers are also affected by people joining and leaving the gym, with 2 additional people joining the Lift classes each week and 3 additional people joining the Cycle classes each week.

In the first week 55 people attended the Lift classes and 62 people attended the Cycle classes.

Set up the matrix recurrence relation that would be used to find how many people attended the Lift and Cycle classes in week 2.

THINK

1 Set up a transition matrix to represent the changing numbers between the classes.

Let L represent the Lift class and C represent the Cycle class.

WRITE

$$\begin{array}{cc} & \begin{array}{cc} L & C \end{array} \\ \begin{array}{c} L \\ C \end{array} & \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \end{array}$$

2 Determine the value of matrix B (represented by the additional people joining the classes).

$B = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{array}{c} L \\ C \end{array}$

3 Set up the initial state matrix.

$S_1 = \begin{bmatrix} 55 \\ 62 \end{bmatrix} \begin{array}{c} L \\ C \end{array}$

4 Enter the information into the matrix recurrence relation $S_2 = TS_1 + B$

$S_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 55 \\ 62 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

WORKED EXAMPLE 24

A school was running extra maths and English classes each week, with students being able to choose which extra classes they would attend. A matrix equation used to determine the number of students expected to attend extra classes is given by

$$S_{n+1} = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix} S_n - \begin{bmatrix} 7 \\ 9 \end{bmatrix},$$

where S_n is the column matrix that lists the number of students attending in week n.

The attendance matrix for the first week is given by

$$S_1 = \begin{array}{c c} \begin{bmatrix} 104 \\ 92 \end{bmatrix} & \begin{array}{l} \text{Maths} \\ \text{English} \end{array} \end{array}$$

a Calculate the number of students who are expected to attend extra English lessons in week 3.

b Of the students who attended extra classes in week 3, how many are not expected to return for extra classes in week 4?

THINK	WRITE
a 1 First calculate week 2 by using S_1.	**a** $S_2 = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix}\begin{bmatrix} 104 \\ 92 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ $= \begin{bmatrix} 109 \\ 71 \end{bmatrix}$
2 Find week 3 by using S_2 and then round off.	$S_3 = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix}\begin{bmatrix} 109 \\ 71 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ $= \begin{bmatrix} 106.95 \\ 57.05 \end{bmatrix}$ $= \begin{bmatrix} 107 \\ 57 \end{bmatrix}$
3 Write the answer.	In week 3 it is expected that 57 attend extra English classes.
b 1 Find week 4 using S_3, not using rounded numbers, but round off at the end.	**b** $S_4 = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix}\begin{bmatrix} 106.95 \\ 57.05 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ $= \begin{bmatrix} 101.0225 \\ 46.9775 \end{bmatrix}$ $= \begin{bmatrix} 101 \\ 47 \end{bmatrix}$
2 Find total students doing extra classes in week 3 and subtract total student doing extra classes in week 4.	Total in week 3 = 107 + 57 = 164 students. Total in week 4 = 101 + 47 = 148 students. Difference = 164 – 148 = 16 students.
3 Write the answer.	It is estimated that there will be 16 students that attended extra classes in week 3 that don't attend in week 4.

Steady state

study on

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Equilibrium or steady state
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Practice questions

As higher and higher powers of T are taken, we find the values of the elements in the transition matrix show no noticeable difference, and approach a fixed matrix T^{∞}. We refer to T^{∞} as the **steady state** or equilibrium state matrix. To test for steady state, a suitable value of n to test is 50. Then test $n = 51$. If the elements in the matrix haven't changed, then a steady state has been reached.

When there is no noticeable change from one state matrix to the next, the system is said to have reached its steady state.

If a Markov system is regular, then its long-term transition matrix is given by the square matrix, whose columns are the same and equal to, the steady state probability vector. This occurs as long as the transition matrix squared, T^2, has no zeros.

If T^2 contains any zeros, then it is not possible to reach a steady state.

T	T^2	T^3
$\begin{bmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{bmatrix}$	$\begin{bmatrix} 0.6125 & 0.31 \\ 0.3875 & 0.69 \end{bmatrix}$	$\begin{bmatrix} 0.536\,875 & 0.3705 \\ 0.463\,125 & 0.6295 \end{bmatrix}$
T^4	T^8	T^{16}
$\begin{bmatrix} 0.495\,281\,25 & 0.403\,775 \\ 0.504\,718\,75 & 0.596\,225 \end{bmatrix}$	$\begin{bmatrix} 0.449\,096... & 0.440\,722... \\ 0.550\,903... & 0.559\,277... \end{bmatrix}$	$\begin{bmatrix} 0.444\,48... & 0.444\,41... \\ 0.555\,51... & 0.555\,58... \end{bmatrix}$
T^{50}	T^{51}	
$\begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{bmatrix}$	$\begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{bmatrix}$	

These probabilities can be easily expressed as fractions if n is very large.

Applications to weather

study on

Unit 4
AOS M1
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Concept 5

Modelling and applications
Concept summary
Practice questions

Predicting the long-term weather forecast is important to insurance companies who insure event organisers against losses if the event is 'rained on'. To do this they need to predict the long-term probability of there being rain. Suppose that for a 'Melbourne spring', long run data suggest that there is a 65% chance that if today is dry, then the next day will also be dry. Conversely, if today is wet, there is an 82% chance that the next day will also be wet. What is the long-term probability for it being a wet day if the initial day was dry? This style of problem is highlighted in the following worked example.

WORKED EXAMPLE 25

An insurance company needs to measure its risk if it is to underwrite a policy for a major outdoor event planned. The company used the following information about the region.

Long run data gathered about the region's weather suggests that there is a 75% chance that if today is dry, then the next day will also be dry. Conversely, if today is wet, there is a 72% chance that the next day will also be wet. This information is given in the table.

	Today is dry	Today is wet
Next day dry	0.75	0.28
Next day wet	0.25	0.72

a Find the probability it will rain in three days time if initially the day is dry.

b Find the long-term probability of rain if initially the day is wet.

c If the company insures only if they have the odds in their favour, will they insure this event?

THINK	WRITE
a **1** Set up the transition matrix.	**a** $T = \begin{array}{ccc} & \text{dry} & \text{wet} \\ \text{dry} & 0.75 & 0.28 \\ \text{wet} & 0.25 & 0.72 \end{array}$ $= \begin{bmatrix} 0.75 & 0.28 \\ 0.25 & 0.72 \end{bmatrix}$
2 Set up the initial state matrix. For the initial day being dry, set dry as 1 and wet as 0.	$S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
3 Identify that in three days time, $n = 3$. Substitute the matrices and use CAS to evaluate S_3.	$S_3 = T^3 \times S_0$ $= \begin{bmatrix} 0.75 & 0.28 \\ 0.25 & 0.72 \end{bmatrix}^3 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0.577\,275 \\ 0.422\,725 \end{bmatrix}$
4 Interpret the answer represented by S_3.	The probability of a dry day, three days after a dry day, is 57.7%.
b **1** Identify that for a long-term steady state, n needs to be large, say $n = 50$. (Test that the same result is obtained when $n = 51$.)	**b** $S_{50} = T^{50} \times S_0$

2 Set up the initial state matrix. For the initial day being wet, set dry as 0 and wet as 1.

$$S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

3 Substitute the matrices and use CAS to evaluate S_{50} and S_{51}.

$$\begin{aligned} S_{50} &= T^{50} \times S_0 \\ &= \begin{bmatrix} 0.75 & 0.28 \\ 0.25 & 0.72 \end{bmatrix}^{50} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.528\,302 \\ 0.471\,698 \end{bmatrix} \\ S_{51} &= T^{51} \times S_0 \\ &= \begin{bmatrix} 0.75 & 0.28 \\ 0.25 & 0.72 \end{bmatrix}^{51} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.528\,302 \\ 0.471\,698 \end{bmatrix} \end{aligned}$$

4 As there is no change from S_{50} to S_{51}, a steady state has been reached. Interpret the answer.

In the long run, there is a 47.2% chance it will be a wet day (or 25 in 53 chance) if the initial day is wet.

c For it to be in the insurer's favour, there must be more than a 50% chance of it being dry.

c In the long run, there is a 52.8% chance it will be a dry day. The odds are slightly in favour of the insurance company. Therefore, they will insure this event.

Other well-known examples of the application of transition matrices are to population studies, stock inventory and sport.

EXERCISE 8.8 Transition matrices

PRACTISE

The following matrices are required to answer questions **1** and **2**. Define them in CAS to assist with your calculations.

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}, C = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}$$

1 **WE20** Use CAS to:

a evaluate (correct to 2 decimal places)

i A^3 **ii** B^2 **iii** C^4 **iv** C^8

b evaluate C^{46}, expressing the elements of the matrix as whole numbers.

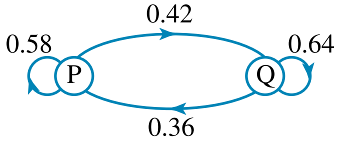
2 **a** Use CAS to perform the following matrix operations.

i A^0 **ii** B^0 **iii** C^0 **iv** D^0

b Describe the type of matrices produced in question **2a**.

3 **WE21** Represent the following as a transition matrix.

a



b There are a number of delivery trucks operating between two warehouses, warehouse A and warehouse B. At the end of each week, 31% of the trucks that started at warehouse A end up at warehouse B and 27% of trucks that started at warehouse B end up at warehouse A.

4 The missing elements in the following transition matrix are:

$$T = \begin{bmatrix} 0.3 & 0.2 & \\ 0.3 & 0.7 & 0.3 \\ & 0.1 & 0.6 \end{bmatrix}$$

A $t_{31} = 0.3$ and $t_{13} = 0.5$

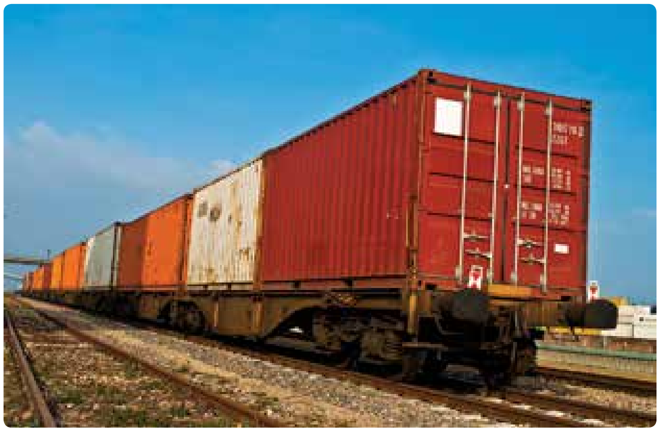
B $t_{13} = 0.3$ and $t_{31} = 0.5$

C $t_{13} = 0.5$ and $t_{31} = 0.4$

D $t_{31} = 0.4$ and $t_{13} = 0.1$

E $t_{13} = 0.4$ and $t_{13} = 0.1$

5 **WE22** A railway knows that 250 goods wagons will be needed to carry goods from point A to point B. At the end of each week, it finds that 10% of the wagons that started the week at point A ended at point B, and 8% of the wagons that started at point B ended at point A.



a Write a transition matrix to represent this situation.

b Use a matrix recurrence relation to determine how many wagons are located at point A and point B at the end of two weeks, if 125 wagons started at point A and 125 wagons started at point B.

c What percentage (in whole numbers) of wagons will be at point A and point B at the end of 6 weeks, if 40% of the wagons started at A?

6 A transition analysis of the movement of a population of people at the Tamworth Country Festival is given by $S_4 = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}^4 \begin{bmatrix} 10\,000 \\ 12\,000 \end{bmatrix}$. The most correct answer for S_4 is:

A $\begin{bmatrix} 6300 \\ 15\,700 \end{bmatrix}$

B $\begin{bmatrix} 6288 \\ 15\,711 \end{bmatrix}$

C $\begin{bmatrix} 6320 \\ 15\,690 \end{bmatrix}$

D $\begin{bmatrix} 6315.8 \\ 15\,684.2 \end{bmatrix}$

E $\begin{bmatrix} 28.7 \\ 71.3 \end{bmatrix}$

7 **WE23** Top Climbers run concurrent Agility and Climb classes at all 3 of their centres in FitTown.

A study shows that 65% of the clients who attended an Agility class one week will attend the Agility class the next week, while the other 35% will move to the Climb class the next week.

Similarly, 77% of the clients who attended a Climb class one week will attend the Climb class the next week, while the other 23% will move to the Agility class the next week.

The numbers are also affected by people joining and leaving the gym, with 5 additional people joining the Agility classes each week and 2 people leaving the Climb classes each week.

In the first week 48 people attended the Agility classes and 41 people attended the Climb classes.

Set up the matrix recurrence relation that would be used to find how many people attended the Agility and Climb classes in week 2.

8 A matrix recurrence relation representing the changing number of people who attend two different supermarkets on a weekly basis is given by

$$S_{n+1} = \begin{bmatrix} 0.61 & 0.18 \\ 0.39 & 0.82 \end{bmatrix} S_n + \begin{bmatrix} 16 \\ 7 \end{bmatrix}.$$

The first row in the state matrices represents Supermarket A and the second row in the state matrices represents Supermarket B.

a What percentage of people who attend Supermarket A in week 1 go back to Supermarket A in week 2?

b What percentage of people who attend Supermarket B in week 1 go back to Supermarket B in week 2?

c How many additional people go to each supermarket each week?

9 **WE24** A school was running extra music lessons for the piano and guitar each week, with students being able to choose which extra lessons they would attend. A matrix recurrence relation used to determine the number of students expected to attend extra lessons is given by

$$S_{n+1} = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix} S_n - \begin{bmatrix} 5 \\ 3 \end{bmatrix},$$

where S_n is the column matrix that lists the number of students attending in week n.

The attendance matrix for the first week is given by

$$S_1 = \begin{array}{c} \begin{bmatrix} 34 \\ 26 \end{bmatrix} \end{array} \begin{array}{l} \text{Piano} \\ \text{Guitar} \end{array}$$

a Calculate the number of students who are expected to attend extra guitar lessons in week 2.

b Of the students who attended extra lessons in week 2, how many are not expected to return for extra lessons in week 3?

10 A farmer has his sheep moving between two paddocks, paddock A and paddock B, each week. A matrix recurrence relation used to determine the number of sheep expected to be in each of the paddocks (including the loss of sheep) is given by

$$S_{n+1} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} S_n - \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$

where S_n is the column matrix that lists the number of sheep in week n.

The number of sheep in each paddock in the first week is given by the matrix

$$S_1 = \begin{bmatrix} 136 \\ 108 \end{bmatrix} \begin{array}{l} \text{Paddock A} \\ \text{Paddock B} \end{array}$$

a Calculate the number of sheep who are expected to be in paddock B in week 4.

b How many sheep have been lost between week 4 and week 5?

11 **WE25** An insurance company needs to measure its risk if it is to underwrite a policy for a major outdoor event planned. The company used the following information about the region where the event was to take place.

Long run data gathered about the region's weather suggests that there is a 95% chance that if today is dry, then the next day will also be dry. Conversely, if today is wet, there is a 45% chance that the next day will also be wet. This information is given in the table below.

	Today is dry	Today is wet
Next day is dry	0.95	0.55
Next day is wet	0.05	0.45

a Find the probability it will rain in three days time if initially the day is dry.

b Find the long-term probability if initially the day is wet.

c If the company insures only if the odds are in its favour, will the company insure this event?

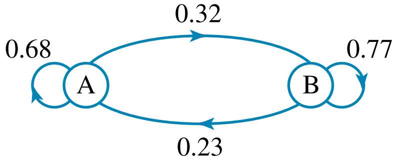
12 For $T = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 100 \\ 50 \\ 80 \end{bmatrix}$, the steady state distribution vector is:

A $\begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix}$ **B** $\begin{bmatrix} 69 \\ 138 \\ 23 \end{bmatrix}$ **C** $\begin{bmatrix} 30 \\ 60 \\ 100 \end{bmatrix}$ **D** $\begin{bmatrix} 70 \\ 135 \\ 25 \end{bmatrix}$ **E** $\begin{bmatrix} 100 \\ 50 \\ 80 \end{bmatrix}$

CONSOLIDATE

13 Represent each of the following as a transition matrix.

a



b There are a number of train carriages operating between two depots, North depot and South depot. At the end of each week, 20% of the carriages that started at North depot end up at South depot and 15% of the carriages that started at South depot end up at North depot.

14 Using the following table of events and their transition probabilities, find the correct transition matrix.

Percentage of customers and their choice of shopping the next time			
	Shop A	**Shop B**	**Shop C**
Shop A	85%		20%
Shop B	10%	60%	
Shop C	5%	20%	75%

A $\begin{bmatrix} 0.85 & 0.2 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.2 & 0.75 \end{bmatrix}$ **B** $\begin{bmatrix} 0.85 & 0.2 & 0.2 \\ 0.1 & 0.6 & 0.05 \\ 0.5 & 0.2 & 0.75 \end{bmatrix}$ **C** $\begin{bmatrix} 0.85 & 0.2 & 0.2 \\ 0.1 & 0.6 & 0.05 \\ 0.05 & 0.2 & 0.75 \end{bmatrix}$

D $\begin{bmatrix} 0.85 & -0.05 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.2 & 0.75 \end{bmatrix}$ **E** $\begin{bmatrix} 85 & 20 & 20 \\ 10 & 60 & 5 \\ 5 & 20 & 75 \end{bmatrix}$

15 Write the transition matrices to represent the following tables of events and their transition probabilities. Complete any missing information.

a

		From	
		Dry	Wet
To	Dry	85%	70%
	Wet	15%	30%

b

		From		
	Shoe brand	Gulius	Grasby	Warlow
To	Gulius	0.65		0.03
	Grasby	0.15	0.68	
	Warlow		0.12	0.97

16 The following information relates to a survey conducted in October 2012 on supermarket shopping.

- 12% of store A customers will shop at store B the following month.
- 36% of store A customers will shop at store C the following month.
- 40% of store B customers will shop at store B the following month.
- 44% of store B customers will shop at store C the following month.
- 14% of store C customers will shop at store A the following month.
- 7% of store C customers will shop at store B the following month.

a Represent this information as a transition matrix.

b If the market share at the time of the survey showed that 1200 customers shopped at store A, 800 customers shopped at store B and 1000 customers shopped at store C, find the number of customers expected to be shopping at each store in March 2013.

c Calculate the long-term share of the customers shopping at each store as a percentage (correct to 1 decimal place).

d Express the answer for part **c** as a fraction of the customers shopping at each store.

17 If a train is late on one day, there is a 15% probability that the same train will be late the next day. If the train is on time one day, there is a 40% chance that it will be late the next day.

If, in a given week, a train arrives on time on Monday, calculate:

a the probability that the train will be on time on the following Friday

b the probability that the train will be late on the following Monday (assume that there are no trains operating on weekends).

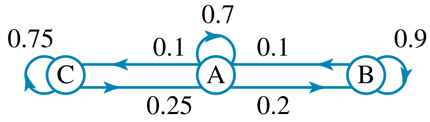
18 For the transition matrix

$$T = \begin{bmatrix} 0.6 & 0.1 & 0 & 0.05 \\ 0.2 & 0.8 & 0 & 0 \\ 0.1 & 0.1 & 1 & 0 \\ 0.1 & 0 & 0 & 0.95 \end{bmatrix}$$

a State the matrix T^2.

b State why this matrix (T) has no steady state.

c The transition matrix that matches the following transition diagram is:



A $\begin{bmatrix} 0.7 & 0.1 & 0.25 \\ 0.2 & 0.9 & 0 \\ 0.1 & 0 & 0.75 \end{bmatrix}$
B $\begin{bmatrix} 0.7 & 0.1 & 0.25 \\ 0.2 & 0.9 & 0.45 \\ 0.1 & 0.2 & 0.75 \end{bmatrix}$
C $\begin{bmatrix} 0.7 & 0.1 & 0.25 \\ 0.1 & 0.9 & 0 \\ 0.2 & 0 & 0.75 \end{bmatrix}$
D $\begin{bmatrix} 0.7 & 0 & 0.25 \\ 0.2 & 0.9 & 0 \\ 0.1 & 0.1 & 0.75 \end{bmatrix}$
E $\begin{bmatrix} 0.7 & 0.1 & 0.75 \\ 0.2 & 0.9 & 0 \\ 0.1 & 0 & 0.25 \end{bmatrix}$

19 The following transition table describes the proportion of customers that purchase fuel from four nearby petrol stations each week.

	Petro	**Texcal**	**Oilmart**	**CP**
Petro	0.6	0.1	0	0.05
Texcal	0.2	0.8	0.05	0
Oilmart	0.1	0.1	0.9	0
CP	0.1	0	0.05	0.95

a Represent the information as a transition matrix.

b If the initial customer base is represented by the matrix $\begin{bmatrix} 4500 \\ 4000 \\ 2000 \\ 1500 \end{bmatrix}$, which petrol station has the largest customer base (include how many):

- **i** after 2 weeks
- **ii** after 8 weeks
- **iii** in the long term? (*Hint:* Investigate large values of n.)

20 A farmer has two types of cows that he milks each day, the Holstein Friesian and the Jersey. He also breeds them to increase his herd. A matrix recurrence relation used to determine the expected number of cows to be milked is given by

$$S_{n+1} = \begin{bmatrix} 0.60 & 0.35 \\ 0.40 & 0.65 \end{bmatrix} S_n + \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where S_n is the column matrix that lists the number of cows on day n.

The number of cows milked on the first day is given by the matrix

$$S_1 = \begin{bmatrix} 106 \\ 72 \end{bmatrix} \begin{array}{l} \text{Holstein Friesian} \\ \text{Jersey} \end{array}$$

a Calculate the number of Jersey cows that are expected to be milked on day 4.

b How many more cows does the farmer have to milk on day 4 compared to the first day?

21 The Fine Drink Co. manufacture two drinks, Sweet Cola and Tangy Lemon. The manufacturing of the two products is determined by the matrix recurrence relation:

$$S_{n+1} = \begin{bmatrix} 0.84 & 0.22 \\ 0.16 & 0.78 \end{bmatrix} S_n + \begin{bmatrix} 110 \\ 71 \end{bmatrix},$$

where S_n is the column matrix that lists the number of drinks produced in week n.

The number of drinks produced in the first week is given by the matrix

$$S_1 = \begin{bmatrix} 950 \\ 1050 \end{bmatrix} \begin{array}{l} \text{Sweet Cola} \\ \text{Tangy Lemon} \end{array}$$

a Calculate the number of each type of drink produced in weeks 2, 3 and 4. Round all answers correct to whole numbers.

b How many more drinks in total were produced in week 4 than week 1?

22 The population of the two main cities on an island is determined by the matrix recurrence relation

$$S_{n+1} = \begin{bmatrix} 0.95 & 0.08 \\ 0.05 & 0.92 \end{bmatrix} S_n + \begin{bmatrix} 1200 \\ 955 \end{bmatrix}$$

where S_n is the column matrix that lists the population produced in year n.

The population in 2012 is given by the matrix

$$S_1 = \begin{bmatrix} 55\,000 \\ 72\,000 \end{bmatrix} \begin{array}{l} \text{City A} \\ \text{City B} \end{array}$$

Calculate the population of each city in 2013, 2014 and 2015. Round all answers correct to whole numbers.

MASTER

23 There are two supermarkets to choose from in Matrixtown, and as the town grows more shoppers shop at the supermarkets each week. A matrix recurrence relation used to determine the number of shoppers expected to shop in each of the supermarkets is given by

$$S_{n+1} = \begin{bmatrix} 0.80 & 0.25 \\ 0.20 & 0.75 \end{bmatrix} S_n + \begin{bmatrix} 6 \\ 4 \end{bmatrix},$$

where S_n is the column matrix that lists the number of shoppers in week n.

The number of shoppers who shopped at each supermarket in the first week is given by the matrix

$$S_1 = \begin{bmatrix} 256 \\ 194 \end{bmatrix} \begin{array}{l} \text{Supermarket A} \\ \text{Supermarket B} \end{array}$$

a Calculate the number of shoppers who are expected to shop in supermarket A in week 3.

b What is the difference in shoppers at supermarket A compared to supermarket B after 4 weeks?

24 Many Victorians have been retiring and moving to Queensland, yet only some have come from Queensland to retire in Victoria. The government of Victoria wants to investigate the expected long-term transition using the following information.

- 6000 Victorians moved to Queensland in 2009.
- 400 Queenslanders moved to Victoria in 2009.
- 73% of Victorians who moved to Queensland stayed.
- 27% of Victorians who moved to Queensland eventually returned to Victoria.
- 10% of Queenslanders who moved to Victoria stayed.
- 90% of Queenslanders who moved to Victoria returned to Queensland.

The government wants to budget for its proposed aged care program and needs to find out the long-term impact on numbers the 2009 transition will have. Calculate the net gain or loss of retirees to the state of Victoria.

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8.9 Review

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- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
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- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

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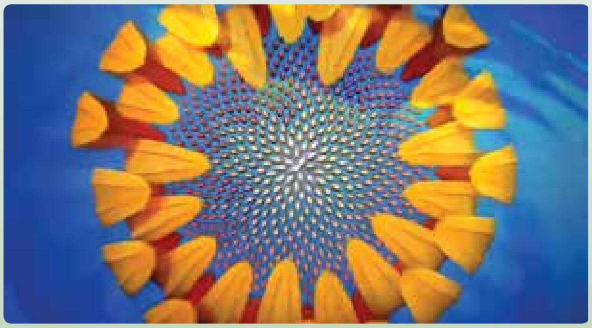

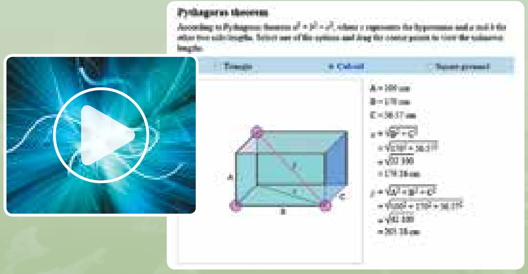

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8 Answers

EXERCISE 8.2

1 A is a 3×3 diagonal matrix; 2,1 is 0; 3 is represented by a_{33}.

B is a 2×2 square matrix; 2,1 is 3; 3 is represented by b_{21}.

C is a 2×2 binary matrix; 2,1 is 0; 3 is not in the matrix.

D is a 3×3 symmetrical matrix; 2,1 is 1; 3 is represented by d_{11}.

E is a 3×3 lower triangular matrix; 2,1 position is 5; 3 is represented by e_{32}.

2 A is a 3×3 upper triangular matrix; 1,2 is −5; 3 is represented by a_{11}.

B is a 1×3 row matrix; 1,2 is −3; 3 is not in the matrix.

C is a 2×2 square matrix; 1,2 is 10; 3 is represented by c_{12}.

D is a 3×3 symmetrical matrix; 1,2 is 7; 3 is represented by d_{32} and d_{23}.

3
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

Vertex A is connected once to vertices B, C and D.

Vertex B is connected once to vertices A and C and connected twice to vertex D.

Vertex C is connected once to vertices A and B.

Vertex D is connected once to vertex A and twice to vertex B.

4
$$\begin{bmatrix} 26 & 22 \\ 15 & 9 \\ 41 & 31 \end{bmatrix}$$

5 **a** **i** 4×2 rectangular matrix

ii 2

iii a_{22}

b **i** 3×3 square matrix

ii 0

iii b_{33}

c Not a matrix

d **i** 3×1 column matrix

ii 3

iii d_{21}

e **i** 1×3 row matrix

ii Does not exist

iii e_{11}

f **i** 3×3 symmetrical matrix

ii −4

iii f_{32} and f_{23}

6 A is a (3×3) lower triangular matrix. The element in the 2,2 position is 4 and the number 7 is represented by a_{33}.

B is a (1×3) row matrix. The element in the 2,2 position is 7 and the number 7 is represented by b_{12}.

C is a (2×2) identity matrix. The element in the 2,2 position is 1 and the number 7 is not in the matrix.

D is a (3×3) symmetrical matrix. The element in the 2,2 position is 0 and the number 7 is represented by d_{33}.

7 **a** 1 **b** −5

c 2 **d** 2.1

e Does not exist **f** Does not exist

g 3 **h** 3.5

8 **a**, **b** and **g**

9 C

10 B

11
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

12
$$\begin{bmatrix} 0 & 3 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

13
$$\begin{bmatrix} 12 & 23 & 35 \\ 9 & 16 & 25 \\ 21 & 39 & 60 \end{bmatrix}$$

14
$$\begin{bmatrix} 15 & 10 & 100 \\ 12 & 15 & 87 \end{bmatrix}$$

15
$$\begin{bmatrix} 5 & 7 & 15 \\ 12 & 2 & 7 \\ 6 & 15 & 0 \end{bmatrix}$$

16
$$\begin{bmatrix} 3 & 2 \\ -4 & -1 \\ 4 & -1 \\ 4 & 1 \end{bmatrix}$$

EXERCISE 8.3

1 **a** $\begin{bmatrix} 4 & 4 \\ 8 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 4 \\ 8 & -2 \end{bmatrix}$

c This cannot be done.

d $\begin{bmatrix} -4 & -10 \\ 2 & 6 \end{bmatrix}$ **e** $\begin{bmatrix} -12 & 0 & 0 \\ -6 & 0 & -3 \end{bmatrix}$

2 a This cannot be done. **b** $\begin{bmatrix} 23 & 37 & -25 \\ 19 & 22 & -22 \end{bmatrix}$

c $\begin{bmatrix} 7 & 3 & -17 \\ 17 & -2 & -4 \end{bmatrix}$ **d** $\begin{bmatrix} 24 & 3 & -9 \\ 30 & 5 & -2 \end{bmatrix}$

e $\begin{bmatrix} -17 & 0 & -8 \\ -13 & -7 & -2 \end{bmatrix}$

3 a $\begin{bmatrix} -18 & 24 \\ 18 & -12 \end{bmatrix}$ **b** $\begin{bmatrix} 12 & -16 \\ -12 & 8 \end{bmatrix}$

c $\begin{bmatrix} -3 & 4 \\ 3 & -2 \end{bmatrix}$ **d** $\begin{bmatrix} -2.4 & 3.2 \\ 2.4 & -1.6 \end{bmatrix}$

4 a $\begin{bmatrix} -14 & -6 \\ 19 & -6 \end{bmatrix}$ **b** $\begin{bmatrix} 12 & -12 \\ 3 & 3 \end{bmatrix}$

5 a $E = \begin{bmatrix} 6 & 10 & -9 \\ 12 & -2 & -20 \end{bmatrix}$ **b** $D = \begin{bmatrix} 6 & 0 \\ 5 & 2 \end{bmatrix}$

c $C = \begin{bmatrix} \frac{13}{2} & \frac{13}{2} \\ 5 & 11 \end{bmatrix}$

6 $a = 9, b = -5, c = -8, d = -4$

7 a $a_{21} = 4$, $b_{21} = 2$ and $c_{21} = 3$

Store A

b $\begin{bmatrix} 14 & 13 \\ 9 & 12 \end{bmatrix}$

8 a Semester 1 $\begin{bmatrix} 72 & 84 \\ 76 & 68 \\ 81 & 82 \end{bmatrix}$,

Semester 2 $\begin{bmatrix} 78 & 74 \\ 76 & 77 \\ 89 & 85 \end{bmatrix}$

b $C = \frac{1}{2}(A + B)$

c $\begin{bmatrix} 75 & 79 \\ 76 & 72.5 \\ 85 & 83.5 \end{bmatrix}$

9 a $\begin{bmatrix} 4 & 5 \\ 2 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 10 \\ -6 & 4 \end{bmatrix}$

c Not possible **d** $\begin{bmatrix} 2 & -5 \\ 8 & -4 \end{bmatrix}$

e $\begin{bmatrix} 3 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}$

10 D

11 [26.00 90.00 81.50 159.95]

12 a $\begin{bmatrix} 12 & 15 \\ 3 & 6 \end{bmatrix}$ **b** $\begin{bmatrix} 0.4 & 0.8 \\ 0.2 & 0.6 \end{bmatrix}$ **c** $\begin{bmatrix} 20 & 25 \\ 5 & 10 \end{bmatrix}$

d $\begin{bmatrix} 5 & 7 \\ 1.5 & 3.5 \end{bmatrix}$ **e** $\begin{bmatrix} 16 & 26 \\ 6 & 16 \end{bmatrix}$ **f** $\begin{bmatrix} 4 & 3.5 \\ 0.5 & 0 \end{bmatrix}$

13 a $\begin{bmatrix} 4 & 12 & 0 & 8 & 8 \\ 8 & 8 & 16 & 4 & 12 \\ 4 & 0 & -8 & 4 & -4 \\ 8 & 4 & 12 & 4 & 0 \\ 4 & 4 & 20 & 4 & 4 \end{bmatrix}$

b $\begin{bmatrix} -0.4 & -0.6 & 0 & -0.1 & 0.2 \\ -0.2 & -0.3 & 0.4 & -0.1 & -0.3 \\ -0.3 & 0 & -0.2 & -0.1 & -0.1 \\ -0.2 & -0.2 & 0.3 & -0.1 & -1 \\ -0.5 & -0.1 & -0.8 & -0.1 & 0 \end{bmatrix}$

c $\begin{bmatrix} -9 & -9 & 0 & 3 & 12 \\ 0 & -3 & 24 & 0 & 0 \\ -6 & 0 & -12 & 0 & -6 \\ 0 & -3 & 18 & 0 & -30 \\ -12 & 0 & -9 & 0 & 3 \end{bmatrix}$

d $\begin{bmatrix} 14 & 30 & 0 & 14 & 8 \\ 16 & 18 & 16 & 8 & 24 \\ 12 & 0 & -8 & 8 & -4 \\ 16 & 10 & 12 & 8 & 20 \\ 16 & 8 & 46 & 8 & 6 \end{bmatrix}$

e $\begin{bmatrix} -5.5 & -7.5 & 0 & -0.5 & 4 \\ -2 & -3.5 & 8 & -1 & -3 \\ -4 & 0 & -4 & -1 & -2 \\ -2 & -2.5 & 6 & -1 & -15 \\ -7 & -1 & -9.5 & -1 & -0.5 \end{bmatrix}$

14 a $E = \begin{bmatrix} 4 & 7 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ **b** $D = \begin{bmatrix} 5 & -3 \\ 4 & 3 \end{bmatrix}$

c $C = \begin{bmatrix} 6 & 13.5 \\ 3.5 & 11 \end{bmatrix}$

15 b, e

16 a Centre A and Centre C **b** $\begin{bmatrix} 34 & 35 \\ 25 & 35 \\ 14 & 9 \end{bmatrix}$

c $\begin{bmatrix} 40\,000 & 26\,000 \\ 50\,000 & 36\,000 \\ 80\,000 & 56\,000 \end{bmatrix}$ **d** $\begin{bmatrix} 36\,000 & 23\,400 \\ 45\,000 & 32\,400 \\ 72\,000 & 50\,400 \end{bmatrix}$

17 a [0.25 0.40 0.20 0.15]

b $A = 800B$

c [200 320 160 120]

18 a $\begin{bmatrix} 154 & 214 & 1138 \\ 207 & 180 & 1422 \end{bmatrix}$ **b** $\begin{bmatrix} 14 & 19\,\frac{5}{11} & 103\,\frac{5}{11} \\ 18\,\frac{9}{11} & 16\,\frac{4}{11} & 129\,\frac{3}{11} \end{bmatrix}$

19 a

New stock level (N) = 40% of current stock levels (matrix S)

$$= 0.4 \times S$$

b $\begin{bmatrix} 9 & 60 & 45 & 2 & 436 \\ 14 & 84 & 58 & 2 & 320 \\ 18 & 92 & 52 & 3 & 480 \\ 13 & 280 & 92 & 2 & 160 \\ 31 & 180 & 64 & 3 & 340 \end{bmatrix}$

20 a True **b** Not true

c Not true **d** True

EXERCISE 8.4

1 a $\begin{bmatrix} 14 & 14 \\ 12 & -18 \end{bmatrix}$ **b** $\begin{bmatrix} -4 & 14 \\ 30 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

2 a A is a 2×3 matrix.

B is a 3×2 matrix.

C is a 2×2 matrix.

b **i** AB does exist.

ii AC does not exist.

iii BA does exist.

iv CA does exist.

v BC does exist.

3 a Column permutation of R

b They all change

c Row 1 is now Row 3

4 a Row permutation of R

b Row 1 moves to Row 2

c $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

5 a $\begin{bmatrix} 550 & 160 \\ 750 & 220 \\ 990 & 350 \end{bmatrix}$ **b** $\begin{bmatrix} 1.08 & 0 \\ 0 & 0.92 \end{bmatrix}$

c The washing machines are priced at \$594, \$810 and \$1069, and the dryers are priced at \$147, \$202 and \$322.

6 a $\begin{bmatrix} 1.50 & 2.50 \\ 2.00 & 3.00 \\ 2.75 & 3.75 \end{bmatrix}$ **b** $\begin{bmatrix} 1.15 & 0 \\ 0 & 0.88 \end{bmatrix}$

c New price for lettuce is \$1.73, \$2.30 and \$3.16.

New price for potatoes (per kg) is \$2.20, \$2.64 and \$2.86.

7 a $\begin{bmatrix} 25.00 & 26.00 \\ 62.50 & 45.00 \\ 55.00 & 30.00 \\ 12.50 & 22.00 \end{bmatrix}$ **b** $\begin{bmatrix} 51.00 \\ 107.50 \\ 85.00 \\ 34.50 \end{bmatrix}$

c **i** Year 10/11

ii Year 10/11

8 a

Total sales at each store = $\begin{bmatrix} 7700 & 1600 \\ 4950 & 2240 \\ 5500 & 2560 \\ 3850 & 1920 \end{bmatrix}$

b Store A = \$9300, Store B = \$7190, Store C = \$8060 and Store D = \$5770

c **i** Store A has the highest sales for iPads at \$7700.

ii Store A has the highest total sales at \$9300.

9 a $A = 2 \times 2 \quad B = 2 \times 3 \quad C = 2 \times 2$

b **i**, **iii**, **iv** and **vi**

c $AB = 2 \times 3 \quad AC = 2 \times 2 \quad CA = 2 \times 2 \quad CB = 2 \times 3$

d $AB = \begin{bmatrix} 16 & 0 & 16 \\ 3 & 9 & 12 \end{bmatrix} \; AC = \begin{bmatrix} 12 & 16 \\ 9 & 21 \end{bmatrix}$

$CA = \begin{bmatrix} 6 & -6 \\ -9 & 27 \end{bmatrix} \; CB = \begin{bmatrix} -6 & -2 & -8 \\ 21 & -1 & 20 \end{bmatrix}$

10 a **i** $\begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$ **ii** $\begin{bmatrix} 13 & 11 \\ 5 & 8 \end{bmatrix}$

b **i** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **ii** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

c N in part **b** is the identity matrix

11 A

12 a D **b** A **c** D **d** E **e** B

13 a $\begin{bmatrix} 24 & 26 \\ -9 & -3 \end{bmatrix}$ **b** $\begin{bmatrix} 32 & 34 \\ 6 & 12 \end{bmatrix}$ **c** $\begin{bmatrix} 16 & 20 \\ 10 & -5 \end{bmatrix}$

d $\begin{bmatrix} 2 & 44 \\ -3 & 9 \end{bmatrix}$ **e** $\begin{bmatrix} 25 & 19 \\ 8 & 17 \end{bmatrix}$

14 C

15 a $\begin{bmatrix} 2.50 & 0.90 \\ 3.50 & 1.90 \\ 4.00 & 2.50 \end{bmatrix}$ **b** $\begin{bmatrix} 0.85 & 0 \\ 0 & 1.15 \end{bmatrix}$ **c** $\begin{bmatrix} 2.13 & 1.04 \\ 2.98 & 2.19 \\ 3.40 & 2.88 \end{bmatrix}$

16 a $\begin{bmatrix} 25.00 & 9.90 \\ 35.00 & 19.90 \\ 95.00 & 75.00 \\ 140.00 & 128.00 \end{bmatrix}$ **b** $\begin{bmatrix} 0.80 & 0 \\ 0 & 0.80 \end{bmatrix}$

c $\begin{bmatrix} 20.00 & 7.90 \\ 28.00 & 15.90 \\ 76.00 & 60.00 \\ 112.00 & 102.40 \end{bmatrix}$

17 a $\begin{bmatrix} 12 & 10 & 10 & 12 \\ 15 & 25 & 15 & 25 \\ 10 & 10 & 10 & 10 \\ 8 & 20 & 5 & 18 \end{bmatrix}$ **b** $\begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 32 \end{bmatrix}$

c $\begin{bmatrix} 180 & 200 & 300 & 384 \\ 225 & 500 & 450 & 800 \\ 150 & 200 & 300 & 320 \\ 120 & 400 & 150 & 576 \end{bmatrix}$ **d** $\begin{bmatrix} 1064 \\ 1975 \\ 970 \\ 1246 \end{bmatrix}$

18 a A row permutation of D.

b Row 1 is moved to row 3.

c Column 3 is moved to column 1.

d D is not a permutation matrix since it can only have a single 1 in each column and row.

e $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

19 a $\text{February} = \begin{bmatrix} 600 & 500 & 0 \\ 480 & 750 & 840 \\ 240 & 1000 & 0 \end{bmatrix}$

$\text{March} = \begin{bmatrix} 1200 & 1000 & 420 \\ 840 & 1500 & 1260 \\ 1200 & 1750 & 0 \end{bmatrix}$

$\text{April} = \begin{bmatrix} 1440 & 750 & 1680 \\ 600 & 1500 & 2100 \\ 1560 & 250 & 420 \end{bmatrix}$

b $\begin{bmatrix} 3240 & 2250 & 2100 \\ 1920 & 3750 & 4200 \\ 3000 & 3000 & 420 \end{bmatrix}$

c $\begin{bmatrix} 27 & 9 & 5 \\ 16 & 15 & 10 \\ 25 & 12 & 1 \end{bmatrix}$

d $\begin{bmatrix} 7590 \\ 9870 \\ 6420 \end{bmatrix}$ Store A sold \$7590
Store B sold \$9870
Store C sold \$6420

20 a **i** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

iii $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ **iv** $\begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}$

b The numbers produced in the solution matrix are Fibonacci numbers. If we let F_n represent the nth Fibonacci number, then the situation can be represented as follows.

For a matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then A^n can be written as $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

c
$$A^8 = \begin{bmatrix} F_9 & F_8 \\ F_8 & F_7 \end{bmatrix} = \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix}$$

d
$$A^{30} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{30} = \begin{bmatrix} 1\,346\,269 & 832\,040 \\ 832\,040 & 514\,229 \end{bmatrix}$$

Given that $A_{30} = \begin{bmatrix} F_{31} & F_{30} \\ F_{30} & F_{29} \end{bmatrix}$, the 30th Fibonacci number, F_{30} is 832 040.

EXERCISE 8.5

1 $\det A = 6$
$\det B = -42$

2 $\det A = 48$
$\det B = 0$

3 $A^{-1} = \dfrac{1}{4}\begin{bmatrix} 8 & -6 \\ -6 & 5 \end{bmatrix}$

4 a $|C| = -10$

b $C^{-1} = \dfrac{1}{-10}\begin{bmatrix} 4 & 1 \\ -10 & -5 \end{bmatrix}$ or $\begin{bmatrix} -0.4 & -0.1 \\ 1 & 0.5 \end{bmatrix}$

5 a $X = \begin{bmatrix} 3 & -44 \\ -3 & 61 \end{bmatrix}$ **b** $X = \begin{bmatrix} 44 \\ -59 \end{bmatrix}$

6 a $X = \begin{bmatrix} 9 & -3 \\ -17 & 7 \end{bmatrix}$ **b** $X = \begin{bmatrix} 58 \\ -102 \end{bmatrix}$

7 a -11 **b** $\frac{1}{3}$ **c** 13 **d** 0

8 **a**, **b** and **c** will have an inverse; **d** has no inverse because it is a singular matrix, that is, its determinant is equal to 0.

9 a $-\dfrac{1}{11}\begin{bmatrix} 5 & -8 \\ -7 & 9 \end{bmatrix}$ **b** $3\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ -1 & \frac{1}{2} \end{bmatrix}$

c $\dfrac{1}{13}\begin{bmatrix} 8 & -9 \\ -3 & 5 \end{bmatrix}$ **d** No inverse exists.

10 $\dfrac{1}{2}\begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix}$

11 $AB = BA = I$

12 A

13 E

14 a -0.005 **b** $\begin{bmatrix} -90 & 40 \\ 50 & -20 \end{bmatrix}$

15 a **i** -28 **ii** $\begin{bmatrix} \frac{1}{14} & \frac{3}{28} \\ \frac{3}{14} & -\frac{5}{28} \end{bmatrix}$

b **i** 6 **ii** $\begin{bmatrix} -1 & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

c **i** -1 **ii** $\begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$

d **i** -0.1 **ii** $\begin{bmatrix} -2.5 & 10 \\ 2 & -4 \end{bmatrix}$

e **i** 0

ii Does not exist — singular matrix

f **i** 0

ii Does not exist — singular matrix

g **i** 96 **ii** $\begin{bmatrix} -\frac{5}{24} & \frac{1}{4} & \frac{1}{24} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{24} & \frac{1}{4} & -\frac{5}{24} \end{bmatrix}$

h **i** -3072 **ii** $\begin{bmatrix} \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 \\ \frac{1}{12} & 0 & \frac{1}{12} & -\frac{1}{6} \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{5}{48} & -\frac{3}{16} & \frac{1}{48} & \frac{1}{3} \end{bmatrix}$

i **i** -72 **ii** $\begin{bmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{9} & -\frac{2}{9} & \frac{1}{9} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{6} \end{bmatrix}$

j **i** and **ii** Not possible — *J* is not a square matrix.

16 a
$$X = \frac{1}{30}\begin{bmatrix} 21 & -33 \\ 15 & 15 \end{bmatrix} = \begin{bmatrix} 0.7 & -1.1 \\ 0.5 & 0.5 \end{bmatrix}$$

b
$$X = \frac{1}{30}\begin{bmatrix} 36 & -45 \\ 18 & 0 \end{bmatrix} = \begin{bmatrix} 1.2 & -1.5 \\ 0.6 & 0 \end{bmatrix}$$

c
$$X = \frac{1}{30}\begin{bmatrix} 15 & 25 \\ -18 & 30 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.33 \\ -0.6 & 1 \end{bmatrix}$$

d
$$X = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

17 a
$$X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

b
$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

18
$$X = \begin{bmatrix} 5 & 6 \\ -3 & -\frac{8}{3} \end{bmatrix}$$

EXERCISE 8.6

1 a **i** A–D, A–D (2)

ii A–C–D, A–E–D (2)

iii A–B–E–D (1)

b
$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2 a **i** A–D (1)

ii A–B–D, A–C–D, A–C–D, A–E–D (4)

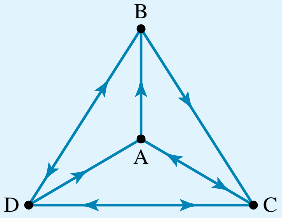
iii A–C–E–D (1)

b
$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 P

4 E

5



$$\text{Communication matrix} = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 0 \\ B & 0 & 0 & 1 & 1 \\ C & 1 & 0 & 0 & 1 \\ D & 1 & 1 & 1 & 0 \end{array}$$

6 a ‘Talk to’ (vertical) so *C* talks to *A*, *B* and *D*.

b ‘Receive calls’ (horizontal) so *B* receives calls from *A*, *C* and *D*.

c Since they cannot call themselves.

d ‘Call’ or ‘talk to’ (vertical) so *D* cannot call *C*.

7 a **i** A–D (1)

ii A–B–D, A–B–D, A–C–D, A–C–D (4)

iii none possible

b **i** A–D (1)

ii A–B–D, A–E–D, A–C–D, A–C–D (4)

iii A–C–E–D, A–C–E–D (2)

8 a
$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b
$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

9 One-stage routes: L_1– L_2 (1)

Two-stage routes: L_1–B–L_2, L_1–A–L_2, L1–A–L2 (3)

Three-stage routes: L_1–A–B–L_2, L1–A–B–L2 (2)

6 routes overall.

10 a A–B–D

b D–C–A

c D–C–B, D–C–A–B

11 a i O–G

ii O–R–G

iii O–C–R–G, O–C–O–G, O–S–O–G

b i None possible

ii R–C–S

iii R–G–O–S, R–C–O–S

c i None possible

ii S–O–R

iii S–O–C–R

12 a First: D, second: C, third: B and fourth: A

b

$$A + A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Sum of the first row is 0, A is placed fourth.

Sum of the second row is 1, B is placed third.

Sum of the third row is 3, C is placed second.

Sum of the fourth row is 6, D is placed first (as it has the highest sum).

13 a First: C, second: E, third: A, fourth: B and fifth: D

b

$$A + A^2 = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 & 0 \end{bmatrix}$$

Sum of the first row is 3, A is placed third.

Sum of the second row is 1, B is placed fourth.

Sum of the third row is 10, C is placed first (as it has the highest sum).

Sum of the fourth row is 0, D is placed fifth.

Sum of the fifth row is 6, E is placed second.

14 a C **b** Z

15 D

16

$$\text{Communication matrix} = \begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

17 a ‘Talk to’ (vertical) so *C* talks to *A* and *D*.

b ‘Receive calls’ (horizontal) so *B* receives calls from *D*.

c ‘Call/talk to’ (vertical) *B* can only call *D*.

d *D* cannot call *A*.

e *D* can receive calls from *A*, *B* and *C*.

18 a F **b** A and C

c F, E, D, B and A/C

EXERCISE 8.7

1 $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

2 $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$

3 No unique solution

4 Unique solution

5 7 sedans and 15 utilities

6 72 jeans, \$2440

7 a $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ **b** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

c $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ **d** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

8 a $\begin{bmatrix} 11 \\ 14 \end{bmatrix}$ **b** Cannot be solved

c $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **d** $\begin{bmatrix} -2 \\ -5 \end{bmatrix}$

e $\begin{bmatrix} 3\frac{1}{3} \\ -7 \\ 7\frac{2}{3} \end{bmatrix}$

9 a i Unique solution **ii** No unique solution

iii No unique solution **iv** Unique solution

b System **i** is inconsistent and System **ii** is dependent.

10 a $\begin{bmatrix} 6 & 2 \\ 9 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{3}{2} & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

b The determinant for both coefficient matrices is 0.

c No solutions. As the determinant is 0, (singular matrix), the inverse does not exist.

d The two graphs are parallel and do not intersect. Therefore, there is no solution.

e The two graphs are the same line.

11 a $2x - y = -3$ or $-2x + y = 3$
$x - y = -1 \qquad -x + y = 1$

b $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

c (–2, –1)

12 $a = 2$, $b = -1$, $c = 0.5$ and $d = 3$

13 Field 1: 1200 square metres; field 2: 600 square metres

14 The costs and revenue are equal (\$23 500) when 7000 components are manufactured.

15 Adults: \$20, Children: \$6, Pensioners: \$5

16 8 and 50

17 a Model A: 18, Model B: 15

b \$105

18 $a = -4$ and $d = 6.6$

EXERCISE 8.8

1 a i $\begin{bmatrix}0.5 & 0.5\\0.5 & 0.5\end{bmatrix}$ **ii** $\begin{bmatrix}0.63 & 0.38\\0.38 & 0.63\end{bmatrix}$

iii $\begin{bmatrix}0.56 & 0.44\\0.44 & 0.56\end{bmatrix}$ **iv** $\begin{bmatrix}0.51 & 0.49\\0.49 & 0.51\end{bmatrix}$

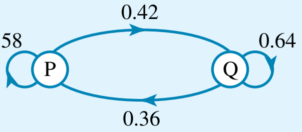
b $\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

2 a i $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ **ii** $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

iii $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ **iv** $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

b The identity matrix

3 a



From

$$\begin{array}{ccc} & & \begin{array}{cc}P & Q\end{array}\\ \text{To} & \begin{array}{c}P\\Q\end{array} & \begin{bmatrix}0.58 & 0.36\\0.42 & 0.64\end{bmatrix}\end{array}$$

b $\begin{bmatrix}0.69 & 0.27\\0.31 & 0.73\end{bmatrix}$

4 D

5 a $\begin{bmatrix}0.9 & 0.08\\0.1 & 0.92\end{bmatrix}$

b 120 wagons at point A and 130 wagons at point B

c 43% of the wagons at point A and 57% of the wagons at point B

6 A

7
$$S_2 = \begin{bmatrix}0.65 & 0.23\\0.35 & 0.77\end{bmatrix}\begin{bmatrix}48\\41\end{bmatrix} + \begin{bmatrix}5\\-2\end{bmatrix}$$

8 a 61% **b** 82%

c 16 additional people go to Supermarket A and 7 additional people go to Supermarket B.

9 a 21 students

b 8 students

10 a 86 sheep

b Difference = 220 – 212 = 8 sheep

11 a 0.078

b Dry: $\frac{11}{12}$, wet: $\frac{1}{12}$

c They should insure the event as there is a very good chance that the day will be dry.

12 B

13 a $\begin{array}{cc} & \begin{array}{cc}A & B\end{array}\\ \begin{array}{c}A\\B\end{array} & \begin{bmatrix}0.68 & 0.23\\0.32 & 0.77\end{bmatrix}\end{array}$ **b** $\begin{array}{cc} & \begin{array}{cc}\text{North} & \text{South}\end{array}\\ \begin{array}{c}\text{North}\\\text{South}\end{array} & \begin{bmatrix}0.80 & 0.15\\0.20 & 0.85\end{bmatrix}\end{array}$

14 C

15 a $\begin{bmatrix}0.85 & 0.7\\0.15 & 0.3\end{bmatrix}$ **b** $\begin{bmatrix}0.65 & 0.20 & 0.03\\0.15 & 0.68 & 0\\0.20 & 0.12 & 0.97\end{bmatrix}$

16 a
$$T = \begin{bmatrix}0.52 & 0.16 & 0.14\\0.12 & 0.40 & 0.07\\0.36 & 0.44 & 0.79\end{bmatrix}$$

b Store A will have 694 customers, store B will have 369 customers and store C will have 1937 customers.

c Store A: 23.0%, store B: 12.2%, store C: 64.9%

d Store A: $\frac{17}{74}$, store B: $\frac{9}{74}$, store C: $\frac{24}{37}$

17 a 68.1%

b 32.0%

18 a $\begin{bmatrix}0.385 & 0.14 & 0 & 0.0775\\0.28 & 0.66 & 0 & 0.01\\0.18 & 0.19 & 1 & 0.005\\0.155 & 0.01 & 0 & 0.9075\end{bmatrix}$

b T has no steady state because the matrix T^2 contains zero elements

c A

19 a $\begin{bmatrix}0.6 & 0.1 & 0 & 0.05\\0.2 & 0.8 & 0.05 & 0\\0.1 & 0.1 & 0.9 & 0\\0.1 & 0 & 0.05 & 0.95\end{bmatrix}$

b i Texcal, 4128 customers

ii Oilmart, 4116 customers

iii CP, 5600 customers

20 a There are 100 jersey cows expected to be milked on day 4.

b Nine more cows were milked on day 4 compared to day 1.

21 a Week 2: Sweet Cola: 1139, Tangy Lemon: 1042

Week 3: Sweet Cola: 1296, Tangy Lemon: 1066

Week 4: Sweet Cola: 1433, Tangy Lemon: 1110

b 543

22 2013: City A: 59 210, City B: 69 945

2014: City A: 63 045, City B: 68 265

2015: City A: 66 554, City B: 66 911

23 a 264 shoppers

b 56 shoppers

24 Loss of 5229 (6000 − 771) Victorians to Queensland

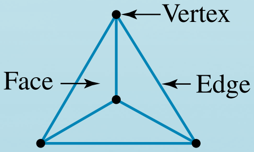
9

Undirected graphs and networks

9.1 Kick off with CAS

Planar graphs and Euler's formula

In graph theory, graphs are made up of vertices, with edges connecting the vertices.



Planar graphs are graphs in which there are no intersecting edges. The edges and vertices in a planar graph divide the graph into a number of faces, as shown in the following diagram. When counting the number of faces remember to include the infinite face — the region surrounding the graph.

Euler's formula links the number of vertices (v), edges (e) and faces (f) in a planar graph with the rule $v - e + f = 2$.

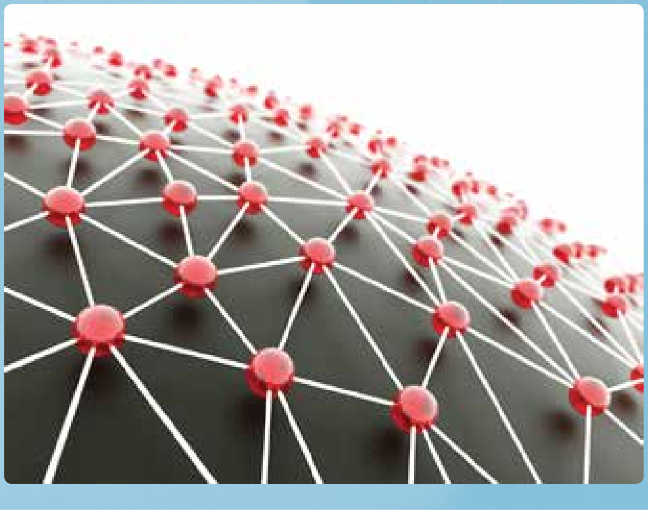
1 Using CAS, define and save Euler's formula for planar graphs.

2 Use CAS to solve Euler's formula for
  - a the number of faces (f)
  - b the number of edges (e)
  - c the number of vertices (v).

3 Use your answer from 2a to calculate the number of faces in a graph with:
  - a 6 vertices and 8 edges
  - b 5 vertices and 5 edges.

4 Use your answer from 2b to calculate the number of edges in a graph with:
  - a 5 vertices and 4 faces
  - b 7 vertices and 5 faces.

5 Use your answer from 2c to calculate the number of vertices in a graph with:
  - a 6 edges and 4 faces
  - b 9 edges and 5 faces.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

9.2 Basic concepts of a network

Definition of a network

studyon

Unit 4
AOS M2
Topic 1
Concept 1

Graphs of undirected networks
Concept summary
Practice questions

What do the telephone system, the Australian Army, your family tree and the internet have in common? The answer is that they can all be considered **networks**.

The simplest possible definition of a network, which will suit our purposes throughout this topic, is:

> **A network is a collection of objects connected to each other in some specific way.**

In the case of the telephone system, the objects are telephones (and exchanges, satellites, ...). In the case of the Australian Army, the objects are units (platoons, companies, regiments, divisions, ...). In the case of the internet, the objects are computers; while your family tree is made up of parents, grandparents, cousins, aunts, ...

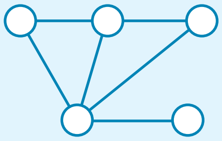
The mathematical term for these objects is a vertex. Consider the network represented in the diagram. This is perhaps the simplest possible network. It consists of two vertices (circles labelled 1 and 2) and one *connection* between them. This connection is called an **edge**.

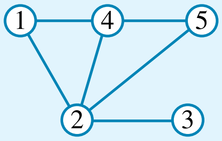
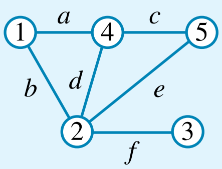
In the case of the telephone system, the edges are the cables connecting homes and exchanges; in the Australian Army they are the commanding officers of various ranks; while in the family tree the links between the generations and between husband and wife can be considered as edges.

The first distinguishing features of a network are the total number of vertices and total number of edges.

WORKED EXAMPLE 1

Count the number of vertices and edges in the network shown.



THINK	WRITE/DRAW
1 Count the vertices by labelling them with numbers.	 Thus, there are 5 vertices.
2 Count the edges by labelling them with letters.	 Thus, there are 6 edges.

There are two things worth noting about this classification of a network:

1. the vertices and edges can be labelled in any order, using any suitable labelling system
2. vertices may have different numbers of edges connected to them. How many edges are connected to vertex 2 in Worked example 1?

The degree of a vertex

Each vertex may have a number of edges connecting it with the rest of the network. This number is called the **degree**. To determine the degree of a vertex, simply count its edges. The following table shows the degree of each vertex in Worked example 1.

Vertex	1	2	3	4	5
Degree	2	4	1	3	2

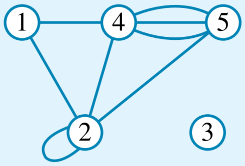
A vertex with degree 0 is *not connected* to any other vertex, and is called an **isolated vertex**.

An edge which connects a vertex to itself is called a **loop** and contributes 2 towards the degree.

If two (or more) edges connect the *same pair* of vertices they are called **parallel edges** (or *multiple edges*) and all count towards the degree. Otherwise, if there is only *one* connection between two vertices, the connection is called a *simple*, or *single*, connection.

WORKED EXAMPLE 2

Determine the degree of each vertex in the figure shown.



THINK	WRITE
1 Node 1: Has 2 simple edges.	Degree of node 1 = 2
2 Node 2: Has 3 simple edges and 1 loop.	Degree of node 2 = 3 + 2 = 5
3 Node 3: Has no edges — an isolated node.	Degree of node 3 = 0
4 Node 4: Has 2 simple edges and 3 parallel edges.	Degree of node 4 = 2 + 3 = 5
5 Node 5: Has 1 simple edge and 3 parallel edges.	Degree of node 5 = 1 + 3 = 4

Representations of networks

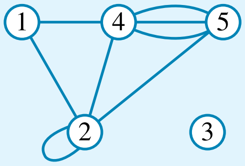
So far we have seen the graphical representation of a network as a two-dimensional collection of vertices and edges. Hence, networks are sometimes called *graphs*. There are other ways to represent the network without losing any of its essential features:

1. labelling vertices and labelling edges according to their vertices
2. matrix representation.

To label vertices, simply list them. If there are three vertices labelled A, B, and C write $V = \{A, B, C\}$. To label edges according to their vertices, identify the vertices that the edge connects. If an edge connects vertex 1 with vertex 3, we represent the edge as (1, 3). If there is a loop at vertex 4, its edge is (4, 4). If there are 2 parallel edges between vertices 2 and 4, we write (2, 4), (2, 4).

WORKED EXAMPLE 3

Label the vertices and edges for the figure shown, as in Worked example 2.



THINK	WRITE
1 Label the vertices.	$V = \{1, 2, 3, 4, 5\}$
2 Examine each edge, in turn.	
Vertex 1–vertex 4	(1, 4)
Vertex 1–vertex 2	(1, 2)
Vertex 2–vertex 2 (loop)	(2, 2)
Vertex 2–vertex 4	(2, 4)
Vertex 2–vertex 5	(2, 5)
Vertex 4–vertex 5 (3 parallel edges)	(4, 5), (4, 5), (4, 5)
3 Combine vertices and edges into a list.	$V = \{1, 2, 3, 4, 5\}$ $E = \{(1, 4), (1, 2), (2, 2), (2, 4), (2, 5), (4, 5), (4, 5), (4, 5)\}$

There are several points to note about this representation:

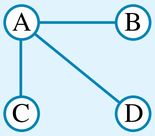
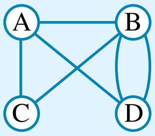
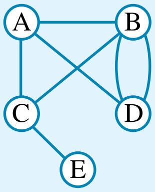
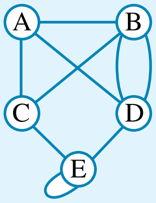
1. there is no '3' in the list of edges (E). This implies it is an isolated vertex.
2. the number of pairs in E {(1, 4), (1, 2), ...} = 8 which must be the same as the number of edges
3. the number of times a vertex appears anywhere inside E equals the degree of the vertex. For example, the digit 4 appears 5 times, so the degree of vertex 4 = 5.
4. from this representation of V and E we can construct (or reconstruct) the original graph.

WORKED EXAMPLE 4

Construct a graph (network) from the following list of vertices and edges.

$V = \{A, B, C, D, E\}$

$E = \{(A, B), (A, C), (A, D), (B, C), (B, D), (B, D), (C, E), (D, E), (E, E)\}$

THINK	WRITE/DRAW
1 Start with a single vertex, say vertex A, and list the vertices to which it is connected.	Vertex A is connected to B, C and D.
2 Construct a graph showing these connections.	
3 Take the next vertex, say B, and list the vertices to which it is connected.	Vertex B is connected to A (already done), C and D (twice: parallel edge).
4 Add the edges from step 3.	
5 Repeat steps 3 and 4 for vertex C.	Vertex C is connected to A (already done), B (already done) and E. 
6 Repeat steps 3 and 4 for vertex D and, finally, add the loop (E, E). As a check, count the edges in the list *E* (9) and compare it with the number of edges in your final graph.	Vertex D is connected to A (already done), B (already done) and E. Vertex E is connected to C (already done), D (already done) and E (loop). 

There may be other geometric configurations which can be drawn from the same vertex and edge lists, but they would be **isomorphic** (or equivalent) to this one.

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Unit 4

AOS M2

Topic 1

Concept 2

Matrix representation of networks

Concept summary

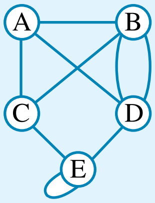
Practice questions

Matrix representation of networks

A method of representing a network in concise form is through the use of a *matrix*. Recall that a matrix is a rectangular collection, or 'grid' of numbers. To represent the network, write the names of the vertices above the columns of the matrix and to the left side of the rows of the matrix. The number of edges connecting vertices is placed at the intersection of the corresponding row and column. This is best shown with an example.

WORKED EXAMPLE 5

Represent the network shown (from Worked example 4) as a matrix.



THINK	WRITE
1 Set up a blank matrix, putting the vertex names across the top and down the side. Thus there are 25 possible entries inside the matrix.	$\begin{array}{c\|ccccc\|} & A & B & C & D & E \\ A & & & & & \\ B & & & & & \\ C & & & & & \\ D & & & & & \\ E & & & & & \end{array}$
2 Consider vertex A. It is connected to vertices B, C, and D once each, so put a 1 in the corresponding columns of row 1 and in the corresponding row of column 1.	$\begin{array}{c\|ccccc\|} & A & B & C & D & E \\ A & & 1 & 1 & 1 & \\ B & 1 & & & & \\ C & 1 & & & & \\ D & 1 & & & & \\ E & & & & & \end{array}$
3 Consider vertex B. It is connected to C once and D twice. Put 1 and 2 in the corresponding columns of row 2 and in the corresponding rows of column 2.	$\begin{array}{c\|ccccc\|} & A & B & C & D & E \\ A & & 1 & 1 & 1 & \\ B & 1 & & 1 & 2 & \\ C & 1 & 1 & & & \\ D & 1 & 2 & & & \\ E & & & & & \end{array}$
4 Repeat for vertices C, D and E.Vertex C is connected to vertex E once, so put a 1 in the corresponding column of row 3 and in the corresponding row of column 3 (shown in red). Vertex D is connected to vertex E once, so put a 1 in the corresponding column of row 4 and in the corresponding row of column 4 (shown in black). Vertex E is connected to itself once (loop), so put a 1 in the corresponding column 5, row 5. *Note:* Only one entry is needed for loops (shown in green). A value of 1 in the leading diagonal denotes a loop in the network, connecting a vertex to itself. This is important to understand when calculating the degree.	$\begin{array}{c\|ccccc\|} & A & B & C & D & E \\ A & & 1 & 1 & 1 & \\ B & 1 & & 1 & 2 & \\ C & 1 & 1 & & & 1 \\ D & 1 & 2 & & & 1 \\ E & & & 1 & 1 & 1 \end{array}$
5 Complete the matrix by placing a 0 in all unoccupied places.	$\begin{array}{c\|ccccc\|} & A & B & C & D & E \\ A & 0 & 1 & 1 & 1 & 0 \\ B & 1 & 0 & 1 & 2 & 0 \\ C & 1 & 1 & 0 & 0 & 1 \\ D & 1 & 2 & 0 & 0 & 1 \\ E & 0 & 0 & 1 & 1 & 1 \end{array}$

6 Check your result by comparing the entries in the matrix with the original network representation. This is best done on a vertex-by-vertex basis.

In matrix representation:

1. the sum of a row (or a column) gives a degree of that vertex, except where a loop is present. Where a loop is present (denoted by a 1 in the leading diagonal), add 1 to the sum of the row or column.
2. if an entire row and its corresponding entire column has only 0s then that vertex is isolated
3. the matrix is diagonally symmetric.

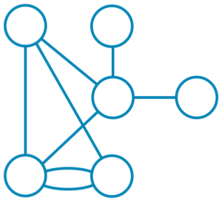
$$\begin{array}{c|ccccc|l} & A & B & C & D & E & \\ A & 0 & 1 & 1 & 1 & 0 & \\ B & 1+ & 0+ & 1+ & 2+ & 0 & =4 \\ C & 1 & 1 & 0 & 0 & 1 & \\ D & 1 & 2 & 0 & 0 & 1 & \\ E & 0 & \underline{0} & 1 & 1 & \underline{1} & 3+1=4 \\ & & \underline{4} & & & \underline{3+1=4} & \end{array}$$

EXERCISE 9.2 Basic concepts of a network

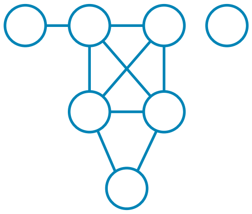
PRACTISE

For questions **1** and **2**, count the number of vertices and edges in the following networks.

1 **WE1**

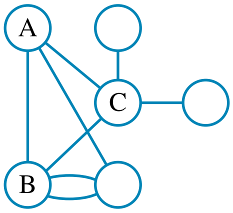


2

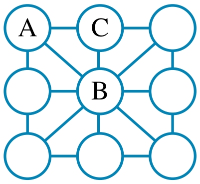


For questions **3** and **4**, determine the degree of the labelled vertices in each diagram.

3 **WE2**

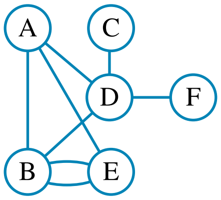


4

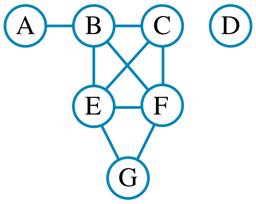


For questions **5** and **6**, list the vertices and label *all* the edges, according to their vertices, in each of these diagrams.

5 **WE3**



6



7 **WE4** Construct a network from the following list of vertices and edges.

$V = \{1, 2, 3, 4, 5, 6\}$ $E = \{(1, 2), (1, 4), (1, 6), (2, 3), (2, 6), (3, 4), (4, 6)\}$

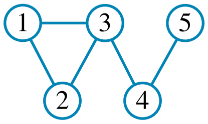
8 Construct a network from the following list of vertices and edges.

$V = \{\text{A, B, C, D, E, F, G}\}$

$E = \{(\text{A, B}), (\text{A, C}), (\text{A, F}), (\text{B, F}), (\text{D, E}), (\text{D, F}), (\text{E, F}), (\text{F, G})\}$

9 **WE5** Represent the network from question **7** as a matrix.

10 Copy and complete the matrix representation of the network at right. The first few entries are shown.

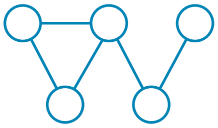


$$\begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4 & 5\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\end{matrix} & \begin{bmatrix}0 & 1 & 1 & 0 & 0\\1 & & & & \\1 & & & & \\0 & & & & \\0 & & & & \end{bmatrix}\end{array}$$

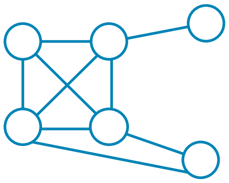
CONSOLIDATE

11 Count the number of vertices and edges in the following networks.

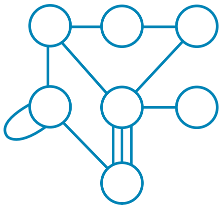
a



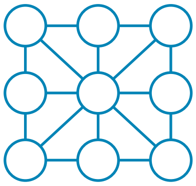
b



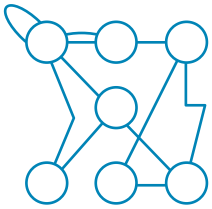
c



d



12 The number of vertices and edges in the figure is:



A Vertices = 7, edges = 7

B Vertices = 7, edges = 10

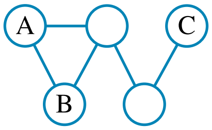
C Vertices = 7, edges = 11

D Vertices = 11, edges = 11

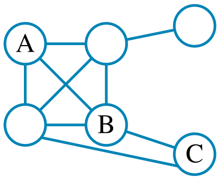
E Vertices = 11, edges = 7

13 Determine the degree of the labelled vertices in each diagram.

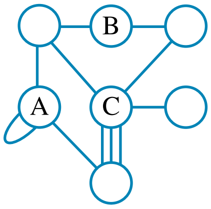
a



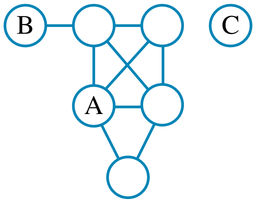
b



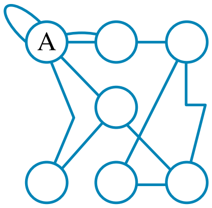
c



d

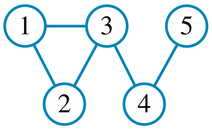


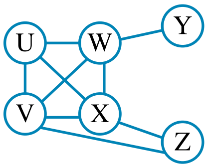
14 The degree of vertex A in the figure is:

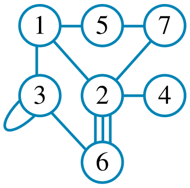


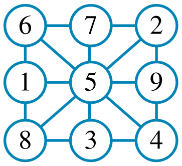
A 3
B 4
C 5
D 6
E 7

15 List the vertices and label *all* the edges, according to their vertices, in each of these diagrams.

a


b


c


d


16 Construct a network from the following list of vertices and edges.

a $V = \{1, 2, 3, 4\}$ $E = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$
b $V = \{A, B, C, D, E\}$ $E = \{(A, B), (A, C), (A, C), (B, B), (B, C), (B, D), (C, D)\}$

17 Consider a network of 4 vertices, where each vertex is connected to each of the other 3 vertices with a single edge (no loops, isolated vertices or parallel edges).

a List the vertices and edges.
b Construct a diagram of the network.
c List the degree of each vertex.

18 Repeat question **17** for:

i a network of 5 vertices
ii a network of 8 vertices.

19 Using the results from questions **17** and **18**, predict the number of edges for a similar network of:

a 10 vertices **b** 20 vertices **c** 100 vertices.

Note that the increase in the number of edges is one of the problems that had to be overcome in the design of computer networks.

20 Construct a network representing the following family tree. Use a single node to represent each married couple.

Allan and Betty had 3 children: Charles, Doris and Earl.

Charles married Frances and had 2 children, George and Harriet.

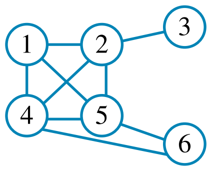
Doris married Ian and had 1 child, John.

Earl married Karen and had 3 children, Louise, Mary and Neil.

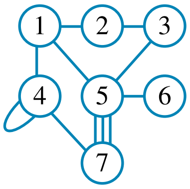
MASTER

21 Represent the following networks by matrices.

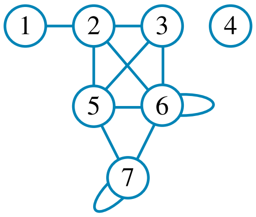
a



b



c



22 Construct networks from the following matrix representations.

Note that the number of rows = number of columns = number of vertices. Watch out for loops.

a $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$

9.3

Planar graphs and Euler's formula

A **planar graph** is a special kind of network or graph. The additional properties of planar graphs will allow us to map two-dimensional and even three-dimensional objects into graphs.

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Unit 4

AOS M2

Topic 1

Concept 3

Types of graphs (networks)

Concept summary
Practice questions

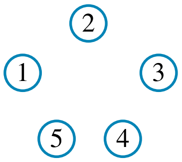
Degenerate graph

A graph with *no* edges is called a **degenerate graph** (or *null* graph).

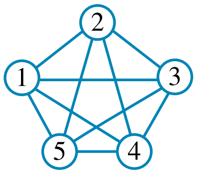
Complete graph

A graph where all vertices are connected directly to all other vertices without parallel edges or loops is called a **complete graph**.

The figure on the left is degenerate; the one on the right is complete. How many edges would there be in a complete graph of 6 vertices?



A degenerate graph



A complete graph

For a complete graph, if E = number of edges and V = number of vertices then $E = \dfrac{V(V-1)}{2}$.

Planar graphs

A planar graph can be defined as follows:

If a graph has no edges which cross, then it is a planar graph.

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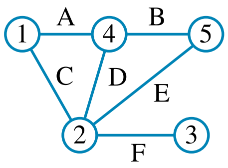
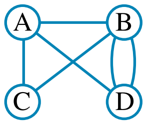
Interactivity
Planar graphs
int-6467

Consider the following graphs.

a



b



c

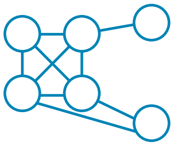


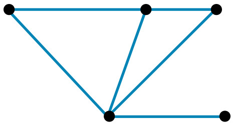
Figure **a** is a planar graph because none of the paths {A, B, C, D, E, F} cross each other.

Figure **b** is apparently not a planar graph because the path (A, D) crosses the path (B, C).

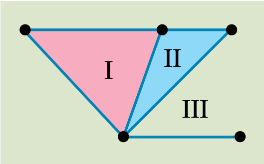
Is figure **c** a planar graph?

The regions of a planar graph

Consider a simplified version of the graph in figure **a**, as shown at right. Note that the large circular vertices have been replaced by small black circles. Otherwise this is the identical network to figure **a**.



Now, observe how this planar graph can be divided into 3 *regions*: region I, region II and region III.



Note also that one of the regions (III) will always be *infinite*, because it continues beyond the bounds of the diagram. All the other regions have a *finite* area. These regions are also called **faces**, for a reason which will soon become apparent.

The reason one region becomes infinite can be seen by considering the fact that when you look at three-dimensional objects, you can't see all the faces at the same time, no matter from which angle you look.

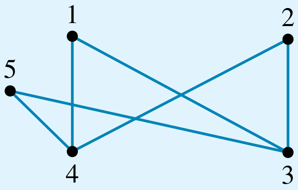
Converting non-planar graphs

Although it may *appear* that a graph is not planar, by modifying the graph it may become clearly planar.

There is no specific method, but by trial and error it may be possible to remove all the crossing paths. (It may also help to imagine the nodes as nails in a board and the edges as flexible rubber bands.) Alternatively, it may be possible to move the vertices so that the connecting edges don't cross. If there are no crossings left, the graph is planar.

WORKED EXAMPLE 6

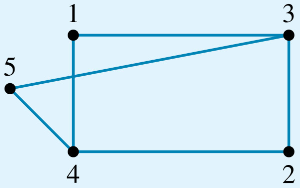
Convert the graph below to a planar graph. Indicate the faces (regions) of the planar graph.



THINK	WRITE/DRAW
1 Confirm that the graph is non-planar. $E = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 5), (4, 5)\}$	Edge (1, 3) crosses (2, 4). Edge (3, 5) crosses (1, 4) and (2, 4).

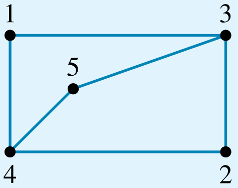
2 Two crossings could be eliminated if vertex 2 were exchanged with vertex 3. Redraw the modified graph.

Check that all the edges are connected to the same vertices.

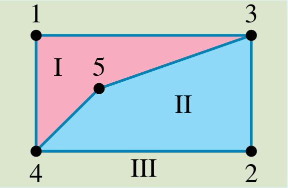


$E = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 5), (4, 5)\}$

3 Placing node 5 inside the rectangle is one way of eliminating all crossings. Note that this planar graph is only one of several possible answers.



4 Define the faces (regions).



The *degree* of each face is the number of edges defining that region. Consider the last figure in Worked example 6.

Face I is defined by edges (1, 3), (1, 4), (4, 5) and (5, 3), so its degree = 4.

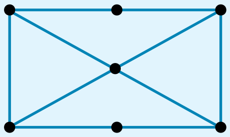
Face II is defined by edges (3, 5), (5, 4), (4, 2) and (2, 3), so its degree = 4.

Face III is defined by edges (1, 3), (1, 4), (4, 2) and (2, 3), so its degree = 4.

In almost all cases, each region will have a degree of *at least* 3. Why? Can you think of exceptions?

WORKED EXAMPLE 7

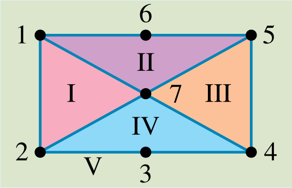
Find the degree of each face of the graph shown in the figure.



THINK

Define the edges and faces of the graph.

WRITE/DRAW



$E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1), (1, 7), (5, 7), (4, 7), (2, 7)\}$

Count the edges for each face.
For example:
Face I — edges (1, 2), (2, 7), (7, 1)
Face II — edges (1, 6), (6, 5), (5, 7), (7, 1).

Face I — degree = 3
Face II — degree = 4
Face III — degree = 3
Face IV — degree = 4
Face V — degree = 6

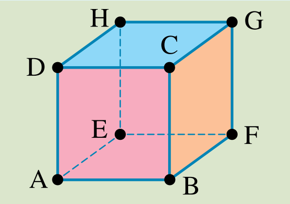
Converting three-dimensional solids to planar graphs

Another application of planar graphs is the conversion of the graph representing a three-dimensional solid (with flat faces) to a planar graph.

WORKED EXAMPLE 8

The figure at right shows a cube with vertices, $V = \{A, B, C, D, E, F, G, H\}$.

Convert this to a planar graph.



THINK

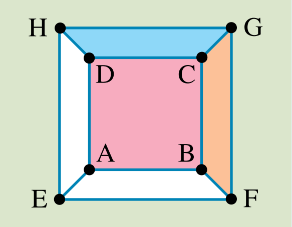
1 List the edges (12 in all).

WRITE/DRAW

$E = \{(A, B), (A, D), (A, E), (B, C), (B, F), (C, D), (C, G), (D, H), (E, F), (E, H), (F, G), (G, H)\}$

2 Imagine the three-dimensional cube 'collapsing' to a two-dimensional graph.

Try collapsing the face A–B–C–D into the face E–F–G–H.



3 Check the edges to see that they are the same as in step 1. Note also the edges (A, E), (B, F), (C, G) and (D, H) which link the 'collapsed' faces.

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There are some other interesting features of this planar graph:

1. the planar graph is, in a sense, a two-dimensional 'projection' of the original cube
2. the original 'base' of the cube (A–B–F–E) has become the *infinite* region of the planar graph.

Euler's formula

By now it may be clear that there is a mathematical relationship between the vertices, edges and faces of planar graphs.

In fact, it is the same relationship, known as **Euler's formula**, that you may have learned when studying solid geometry:

$$\text{Vertices} = \text{edges} - \text{faces} + 2$$
$$V = E - F + 2$$

WORKED EXAMPLE 9

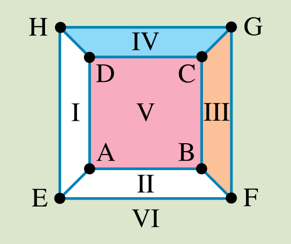
Verify Euler's formula for the 'cube' of the last figure in Worked example 8.

THINK

1 List the vertices.

WRITE/DRAW

$V = \{A, B, C, D, E, F, G, H\}$

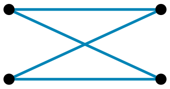
2 Count the vertices.	$V = 8$
3 List the edges.	$E = \{(A, B), (A, D), (A, E), (B, C), (B, F), (C, D), (C, G), (D, H), (E, F), (E, H), (F, G), (G, H)\}$
4 Count the edges.	$E = 12$
5 Define the faces (regions).	 There are 6 faces in all: {I, II, III, IV, V, VI} $F = 6$
6 Confirm Euler's formula by substitution.	$V = E - F + 2$ $8 = 12 - 6 + 2$ $8 = 8$ Therefore, Euler's formula is verified.

Note that the cube is a form of prism (an object with a uniform cross-section), and all prisms can be converted to planar graphs using the above technique of one face 'collapsing' into another.

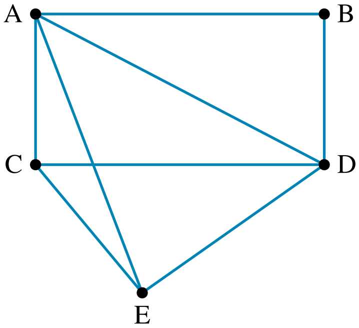
EXERCISE 9.3 Planar graphs and Euler's formula

PRACTISE

1 **WE6** Convert the following graph to a planar graph.

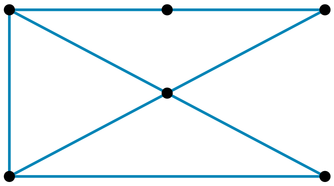


2 Redraw the following network diagram so that it is a planar graph.

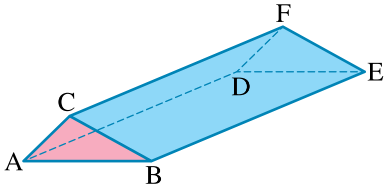


3 **WE7** Find the degree of each face of the graph in question **2**.

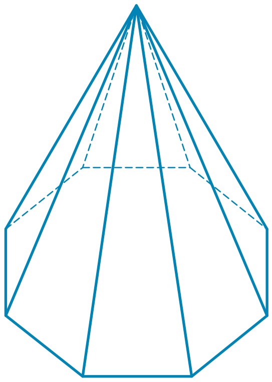
4 Find the degree of each face of the graph shown.



5 **WE8** Convert the three-dimensional triangular prism to a planar graph.



6 Consider a ‘pyramid’ with an octagon for a base.



Convert the representation to a planar graph.

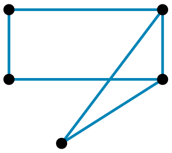
7 **WE9** Verify Euler’s formula for the graph in question **5**.

8 Convert a triangular pyramid to a planar graph. Verify Euler’s formula for your graph.

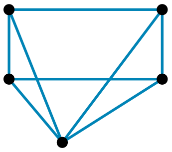
CONSOLIDATE

9 Modify the following graphs so that their representations are planar.

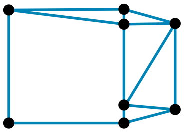
a



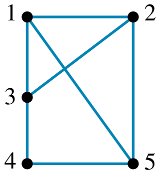
b



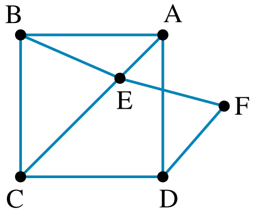
c



d



10 The graph represented in the figure is apparently not planar because:



- **A** edge (A, C) crosses edge (B, E)
- **B** edge (A, D) crosses edge (E, F)
- **C** edges (A, E), (F, E), (C, E) and (B, E) intersect
- **D** vertex E has a degree of 4
- **E** none of the above

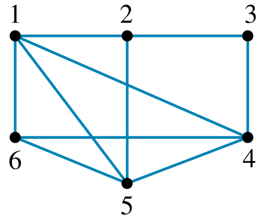
11 A complete graph with 7 vertices would have:

- **A** 7 edges
- **B** 14 edges
- **C** 21 edges
- **D** 28 edges
- **E** 42 edges

12 **a** By moving vertex F only, modify the graph in question **10** so that it is clearly planar.

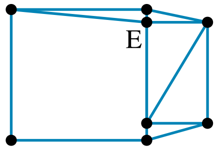
b How many faces are there in your planar graph?

c Find the degree of each face.



13 **a** By moving vertex 5 only, modify the graph so that it is clearly planar.

b How many faces are there in your planar graph? (*Hint:* You may have to use curved edges to connect all the vertices.)



14 The degree of vertex E in the figure is:

A 1 **B** 2 **C** 3
D 4 **E** 5

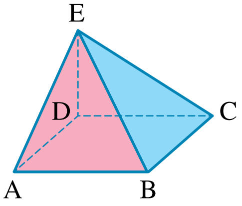
15 Verify Euler's formula for the figure in question **14**.

16 Show that the sum of the degrees of *all* the vertices of *any* planar graph is always an even number. Also show that if S = sum of the degrees, and E = number of edges, that $S = 2E$. (This is known as the handshaking lemma.)

17 In a planar graph, the number of edges = 5, the number of vertices = 4, therefore the number of faces is:

A 1 **B** 3 **C** 9 **D** 11
E unable to be determined from the given information

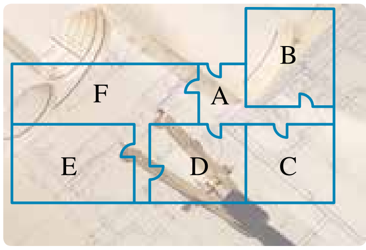
18 Convert the rectangular pyramid shown into a planar graph.



MASTER

19 The diagram shown is a crude floor plan for a small house with 6 rooms, labelled A, B, . . ., F. Convert this plan to a planar graph where rooms are considered as vertices.

(*Hint:* What should the edges be?)



20 Eight people in a room shake hands with each other once.

a How many handshakes are there?

b Represent the handshakes as a complete graph.

c Represent the handshakes as a matrix.

9.4 Walks, trails, paths, cycles and circuits

In planar graphs we can define a **walk** as a sequence of edges and look at various sequences or pathways through the network. Sometimes you may wish to have a walk that goes through *all* nodes only once, for example, for a travelling salesperson who wishes to visit each town once. Sometimes you may wish to use all edges only once, such as for a road repair gang repairing all the roads in a shire.

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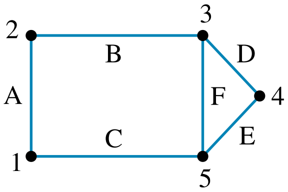
Walks

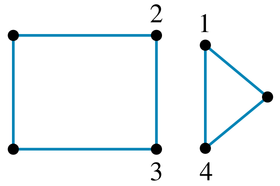
There are different ways of naming a walk. For example, consider travelling from node 1 to node 3 in the figure. A walk could be specified via node 2, namely A–B, or by specifying the vertices, 1–2–3. Alternatively one could take the walk C–E–D, or C–F. Each of these routes is a walk.

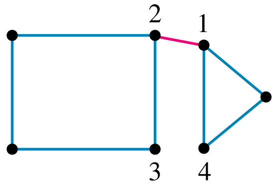


Connected graphs and bridges

If there is a walk between all possible pairs of vertices, then it is a connected graph.

For example, in the figure on the left, there is no walk between vertices 1 and 2, nor between vertices 3 and 4, so it is not a connected graph. However, if we add a single edge, as in the figure on the right, between vertices 1 and 2, the entire graph becomes connected.





A bridge is an edge in a connected graph whose deletion will no longer cause the graph to be connected.

In the above example on the right, the edge between vertices 1 and 2 is a bridge. When this edge is removed, the graph is no longer connected.

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Euler trails

A trail is a walk in which no edges are repeated.

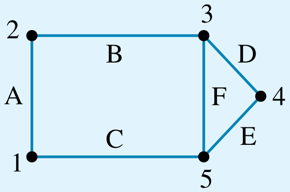
Consider a trail where every edge is used only once, as in our road repair gang example.

An Euler trail is one which uses every edge exactly once.

1. For an Euler trail to exist, all vertices must be of an even degree *or* there must be exactly two vertices of odd degree.
2. If the degrees of *all* the vertices are even numbers, start with *any* vertex. In this case the starting vertex and ending vertex are the same.
3. If there are two vertices whose degree is an odd number use either as a starting point. The other vertex of odd degree must be the ending point.

WORKED EXAMPLE 10

Using the following figure, identify an Euler trail.



THINK	WRITE
1 Determine a starting vertex. Since there are vertices (3 and 5) whose degree is an odd number, use one of these to start.	Use vertex 3 as the start.

2 Attempt to find a path which uses each edge exactly once. You are allowed to visit a vertex more than once; it is only edges that are restricted to one use.	B–A–C–F–D–E or B–A–C–E–D–F or D–E–F–B–A–C are all Euler trails. There are several other possible Euler trails.

Note that the starting vertex and ending vertex are not the same, but we started and ended on the vertices with odd degree (vertices 3 and 5).

Euler circuits

A circuit is a trail beginning and ending at the same vertex.

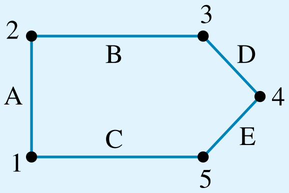
With our road repair gang example, it would be desirable that the Euler trail started and finished at the same point. This kind of Euler trail is called an **Euler circuit**.

An Euler circuit is an Euler trail where the starting and ending vertices are the same.

It is important to note that an Euler circuit *cannot* exist for planar graphs that have *any* vertices whose degree is odd. In such graphs there is no Euler circuit. Therefore, the planar graph of Worked example 10 does not contain an Euler circuit because vertices 3 and 5 were of odd degree.

WORKED EXAMPLE 11

Find an Euler circuit for the planar graph shown.



THINK	WRITE
1 Confirm that all vertices have *even* degree.	Vertex 1 — degree = 2 Vertex 2 — degree = 2 Vertex 3 — degree = 2 Vertex 4 — degree = 2 Vertex 5 — degree = 2
2 Pick any vertex to start and determine a trail that uses each edge and ends at the same vertex.	Start with vertex 1. Then the Euler circuit could be: A–B–D–E–C or C–E–D–B–A

An Euler circuit algorithm

For some networks, it may be difficult to determine an Euler circuit, even after determining that all vertices have even degree. Here is an algorithm that ‘guarantees’ an Euler circuit.

Consider a network where all vertices are of even degree. Let $V = \{1, 2, 3, ...\}$ be the list of vertices.

Step 1. Choose a starting vertex from the list V. Call this vertex A.

Step 2. From vertex A, find the *smallest possible* path which returns to vertex A. This is a ‘*subcircuit*’ of the original network. Let S be the list of vertices in this subcircuit.

Step 3. For each vertex in S, choose a single vertex in turn as the starting vertex of a different subcircuit. It should also be as small as possible, and not use any previously used edge.

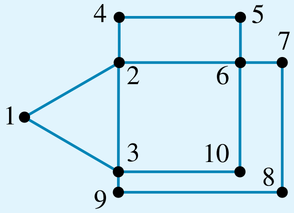
Step 4. For each of these new subcircuits (if there are any), add any new vertices to the list in S.

Step 5. Repeat steps 3 and 4 until there are no more new vertices, edges or subcircuits left; that is, the lists S and V are the same.

Step 6. Join the subcircuits at their intersection points.

WORKED EXAMPLE 12

Find one possible Euler circuit for the network shown using the Euler circuit algorithm.



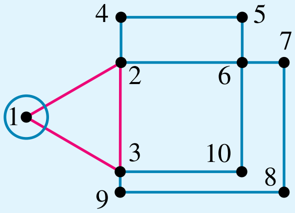
THINK

WRITE/DRAW

1 Choose a starting vertex, and find its smallest subcircuit.

The subcircuit is marked in pink.

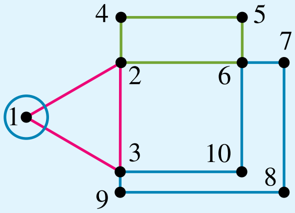
Vertex 1 is selected.
The path 1–2–3–1 is the smallest possible.



2 Create the list S from the first subcircuit. Find new subcircuits, not using any edges already used (apply step 3 of the algorithm).

The new subcircuit is marked in green. Note that this subcircuit does not use any previously used edges.

$S = \{1, 2, 3\}$
From vertex 2, there is a subcircuit 2–4–5–6–2.



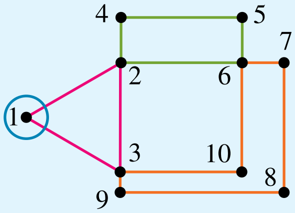
3 Add to the list S (apply step 4 of the algorithm).

$S = \{1, 2, 3, 4, 5, 6\}$

4 Find the new subcircuits (re-apply step 3).

The new subcircuit is marked in orange.

From vertex 3, there is a subcircuit 3–9–8–7–6–10–3.



5 Add to list S (re-apply step 4).
Check that all vertices are in the list (step 5).

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$= V$, so stop.

6 Apply step 6. Form the Euler trail, starting with the first subcircuit, and proceeding through all the other subcircuits at their intersections. Note that the second subcircuit is in the 1st set of square brackets [ ] and the next subcircuit is in the 2nd set of square brackets [ ].	The red circuit and the blue circuit are connected at vertex **2**. The red circuit and the green circuit are connected at vertex **3**. Euler circuit = 1– [**2**–4–5–6–**2**] – [**3**–9–8–7–6–10–**3**] – 1
7 List the Euler circuit.	1–2–4–5–6–2–3–9–8–7–6–10–3–1

Paths and cycles

A path is a walk in which no vertices are repeated (except possibly the start and finish).

A cycle is a path beginning and ending at the same vertex.

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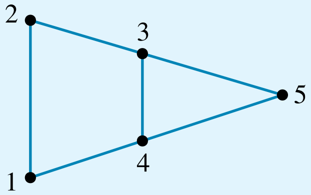
In Euler trails and circuits each edge was used *exactly once*, while vertices could be re-used. Now, consider the case where it is desirable to use each vertex *exactly once*, as in our travelling salesman problem mentioned at the start of this section.

A Hamiltonian path uses every vertex exactly once.

It is important to note that not all edges need to be used. Furthermore, there can be only up to 2 vertices with degree 1 (dead ends). In this case these would be the start and/or the finishing vertices.

WORKED EXAMPLE **13**

Determine a Hamiltonian path in the planar graph shown.



THINK

1 Choose a starting node. If there is a vertex with degree = 1, then use it to start.

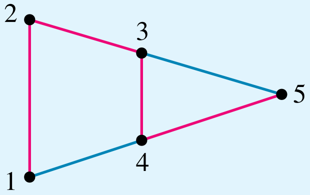
WRITE

Since there are no vertices with degree 1, choose any node to start. Choose vertex 1.

2 Attempt to visit each vertex. This will work for all feasible planar graphs.

The Hamiltonian path found is shown in red.

The path connecting nodes 1–2–3–4–5 was chosen as one of the Hamiltonian paths.



Note that there are several possible Hamiltonian paths for the planar graph in Worked example 13 and there are several paths which will not result in a Hamiltonian path. Can you find such a path?

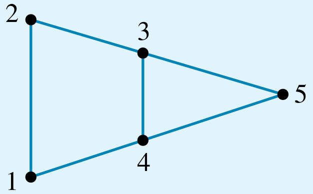
Hamiltonian cycles

When determining a Hamiltonian path, sometimes it is desirable to start and finish with the same vertex. For example, our travelling salesperson may live in one of the towns (vertices) she visits and would like to start and finish at her home town after visiting all the other towns once. This is similar to the concept of an Euler circuit.

A Hamiltonian cycle is a Hamiltonian path which starts and finishes at the same vertex.

WORKED EXAMPLE 14

Determine a Hamiltonian cycle in the planar graph shown. (This is the same graph used in Worked example 13.)



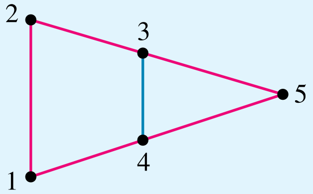
THINK

1 Choose a starting (and finishing) vertex.

2 Attempt to visit each vertex and return to vertex 1. The Hamiltonian cycle found is shown in pink.

WRITE

Choose vertex 1.

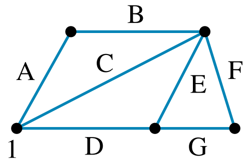


The path connecting nodes 1–2–3–5–4–1 was chosen as one of the Hamiltonian cycles.

EXERCISE 9.4 Walks, trails, paths, cycles and circuits

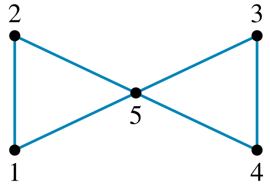
PRACTISE

1 **WE10** Using the figure shown at right and starting at vertex 1, identify an Euler trail.



2 Choosing the other vertex of degree 3 in the figure used for question 1, identify another Euler trail.

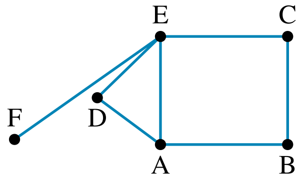
3 **WE11** Using the figure shown and starting at vertex 1, identify an Euler circuit.



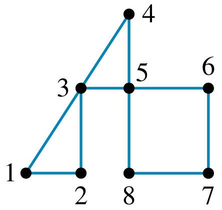
4 a What additional edge should be added to the planar graph at right so that it could be possible to define an Euler circuit?

A CA
B FA
C FD
D EB
E None of the above

b Which edge in the original planar graph is a bridge?

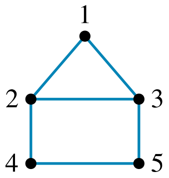


5 **WE12** Find an Euler circuit for the graph shown, using the Euler circuit algorithm.

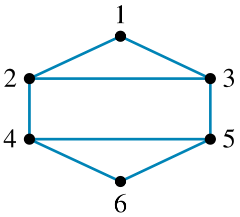


6 In which of the following does an Euler circuit exist?

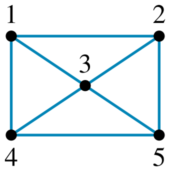
A



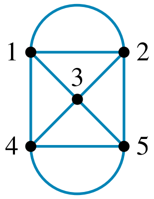
B



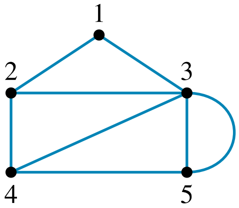
C



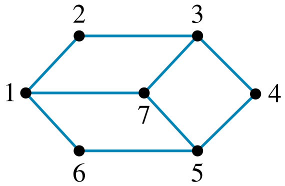
D



E



7 **WE13** Starting at vertex 2, determine a Hamiltonian path for the graph shown.



8 The path shown in pink in the figure is:

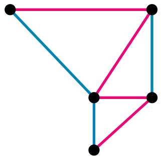
A an Euler trail

B an Euler circuit

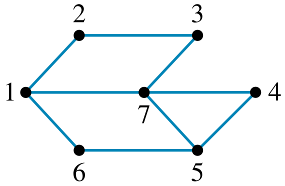
C a Hamiltonian path

D a Hamiltonian cycle

E none of the above



9 **WE14** Starting at vertex 2, determine a Hamiltonian cycle for the graph shown.



10 The network shown has:

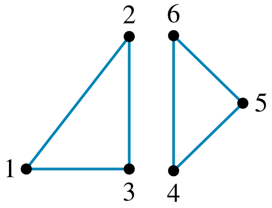
A an Euler circuit and a Hamiltonian path

B a Hamiltonian path and cycle

C an Euler trail and a Hamiltonian cycle

D an Euler trail only

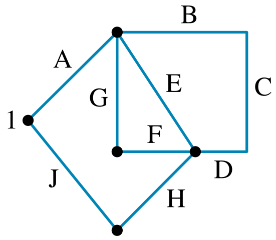
E none of the above



CONSOLIDATE

11 Using the figure shown at right and starting at vertex 1, identify an Euler trail.

12 Starting with a vertex of degree 4 from the figure in question **11**, identify another Euler trail.



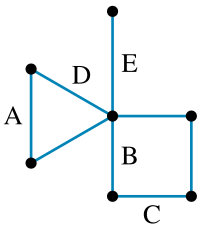
13 Which edge in the following connected graph is a bridge?

A A

B B

C C

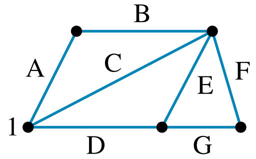
D D



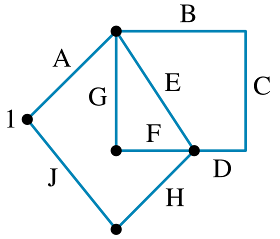
14 Considering the networks as shown, which have Euler circuits?

A Both
B Neither
C Figure **a** only
D Figure **b** only
E None of the above

a



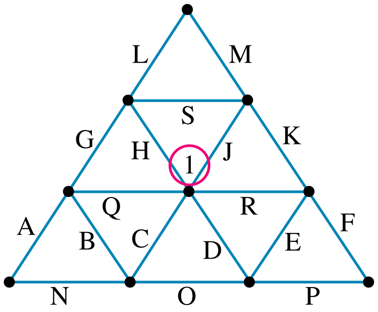
b



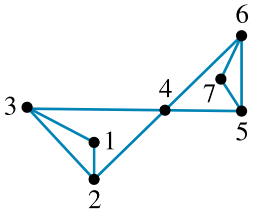
15 Using the graph shown at right, determine:

a an Euler trail

b an Euler circuit starting at vertex 1 (circled in pink).

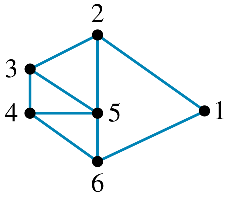


16 Starting at vertex 7, determine a Hamiltonian path for the graph shown.



17 Which of the following paths is a Hamiltonian cycle for the figure at right?

A 2–3–4–5–6–1–2
B 2–3–4–5–2
C 2–5–3–4–5–6–1–2
D 2–5–6–4–3–2
E 2–1–6–4–5–2



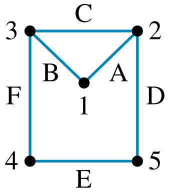
18 Using the network shown at right:

a determine an Euler trail

b determine an Euler circuit

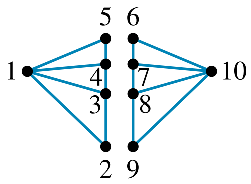
c determine a Hamiltonian path

d determine a Hamiltonian cycle.

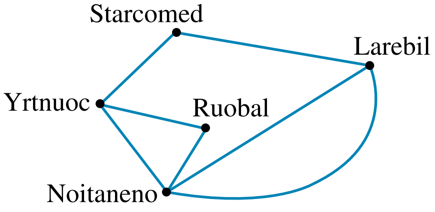


19 **a** Using the network shown at right, what two edges should be added to the network so that it has both an Euler circuit and a Hamiltonian cycle?

b Determine the Euler circuit and Hamiltonian cycle.



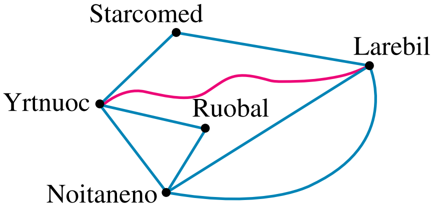
20 The Police Commissioner wishes to give the impression of an increased police presence on the roads. The roads that the commissioner has to cover are depicted in the network diagram. A speed camera is set up once during the day on each of the roads.



a Determine a walk that the police officer could follow so that she does not travel more than once on any road.

b What type of walk is this?

21 The police officer knows that there is a dirt track linking the towns of Larebil and Yrtnuoc. She wishes to meet the commissioner's instructions from question **20** but also wishes to start and finish in her home town, Larebil.



a Determine a walk that would meet both the police officer's and the commissioner's requirements.

b What type of walk is this?

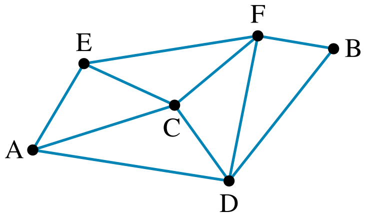
MASTER

22 A security company, 'Wotchemclose', is responsible for patrolling stores in the towns from question **20**. The company wants a patrol car to visit each town once each night without resorting to using the dirt road.

a If the security guard starts at Ruobal, determine a walk that will meet the company's requirements.

b What type of walk is this?

23 A physical education teacher, I. M. Grate, wishes to plan an orienteering course through a forest following marked tracks. She has placed checkpoints at the points shown in the diagram. The object of any orienteer is to visit each of the checkpoints once to collect a mark.



a What walk could an efficient orienteer follow if the course starts at C and finishes at B?

b What walk should the orienteer follow if starting and finishing at C?

c What type of walks are these?

9.5 Trees and their applications

study on

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Topic 1
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Weighted graphs
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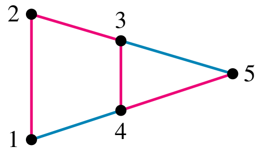
There are many applications where only part of the network is required as a solution to a problem. This section will look at such problems involving **subgraphs** and **trees**. To begin with we need a few more definitions.

Graphs and subgraphs

Until now we have used the term network to refer to a collection of vertices and edges. This network can also be called a graph. In practice, a graph should have at least 2 vertices and 1 edge. All or part of this graph can be considered as a *subgraph*.

For example, in the figure, the entire network can be considered as a graph, while the path in pink can be considered a subgraph.



Another subgraph could be defined by the path 1–2–3–4–1.

A 'minimum' subgraph could be defined by the path 1–2.

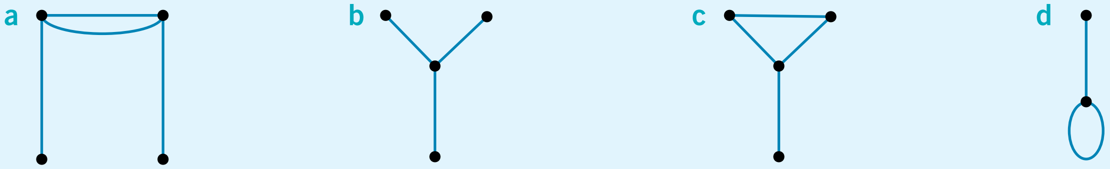
Often the edges in a graph are not just simply connectors, but could be assigned some quantity, such as distance, time or cost. For example, in the figure the distance between vertex 1 and vertex 2 could be assigned a distance of 40 metres. If the graph contains such quantities, then it is called a **weighted graph**.

Trees

A *tree* is a connected subgraph which cannot contain any:
1. loops
2. parallel (or multiple) edges
3. cycles.

WORKED EXAMPLE 15

Determine whether each of the figures is a tree, and if not, explain why not.



THINK	WRITE
1 Examine each figure in turn, looking for loops, parallel edges or cycles.	
2 Examine figure **a**.	Figure **a** has parallel edges (at the top) so it is not a tree.
3 Examine figure **b**.	Figure **b** has no loops, parallel edges or cycles so it is a tree.
4 Examine figure **c**.	Figure **c** has a cycle (at the top) so it is not a tree.
5 Examine figure **d**.	Figure **d** has a loop (at the bottom) so it is not a tree.

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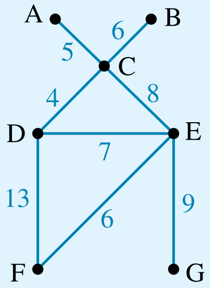
The advantage of trees within a network is that the tree could determine an 'efficient' connection between vertices in the sense that there is a minimum distance, cost or time.

Shortest paths

Sometimes it may be useful to determine the **shortest path** between 2 selected vertices of a graph. For example, when going shopping, a person may leave his home and travel east via the playground, or north via the parking lot and still end up at the same shop. In one case the distance travelled may be the minimum.

WORKED EXAMPLE 16

Determine the shortest path between nodes A and F in the figure shown. Nodes are labelled A, ..., G and distances (in metres) between them are labelled in blue.



THINK	**WRITE**
1 From the starting node, by inspection, determine the possible trees between A and F.	(1) A–C–E–D–F (2) A–C–D–F (3) A–C–D–E–F (4) A–C–E–F
2 For each tree, calculate the total distance travelled.	(1) $5 + 8 + 7 + 13 = 33$ (2) $5 + 4 + 13 = 22$ (3) $5 + 4 + 7 + 6 = 22$ (4) $5 + 8 + 6 = 19$
3 Choose the path with the shortest distance.	Path A–C–E–F is the shortest path with a distance of 19 m.

When choosing the possible paths in step 1, there is no point in finding paths that are *not* trees. There will *always* be a tree which covers the same vertices in less distance. Non-tree paths will include cycles and loops, which only add to the total distance.

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Interactivity
A shortest path algorithm
int-6284

A shortest path algorithm

Sometimes it can be difficult to list all the paths between the starting and ending vertex. Here is an algorithm which 'guarantees' the shortest path — assuming that the starting vertex is already chosen.

Step 1. From the starting vertex, find the *shortest* path to all other *directly connected* vertices. Include all such vertices, including the starting one in the list $S = \{A, B, ...\}$.

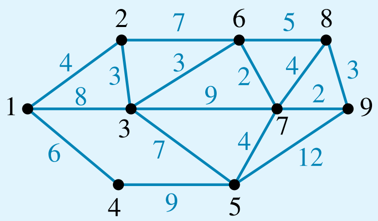
Step 2. Choose a vertex (V) directly connected to those in S and find the shortest path to the starting vertex. Generally, there is one possible path for each degree of V, although some obvious paths can be eliminated immediately.

Step 3. Add the new vertex, V, to the list S.

Step 4. Repeat steps 2 and 3 until all vertices are in S. Find the shortest path to the vertex you want.

WORKED EXAMPLE 17

Find the shortest path from vertex 1 to vertex 9. Vertices are labelled in black, distances in blue. (*Note:* Lines are not to scale.)



THINK

WRITE

1 From the starting vertex (1), find the shortest path to each of the vertices directly connected to it.

From	To	Via	Distance	
1	2	—	4	shortest path to 2
1	3	—	8	
1	3	2	$4 + 3 = 7$	shortest path to 3
1	4	—	6	shortest path to 4

2 Determine the set of vertices, S.

$S = \{1, 2, 3, 4\}$

3 Apply step 2 of the algorithm for a vertex connected directly to one in S.

Select vertex 5.

From	**To**	**Via**	**Distance**	
1	5	3	7 + 7 = 14	shortest path to 5
1	5	4	6 + 9 = 15	

4 Apply step 3 of the algorithm and add to the set of vertices, S.

$S = \{1, 2, 3, 4, 5\}$

5 Re-apply step 2 of the algorithm for another vertex directly connected to one in S.

Select vertex 6.

From	**To**	**Via**	**Distance**	
1	6	2	4 + 7 = 11	
1	6	3	7 + 3 = 10	shortest path to 6
1	6	7	7 + 9 + 2 = 18	

6 Re-apply step 3 of the algorithm and add to the set of vertices, S.

$S = \{1, 2, 3, 4, 5, 6\}$

7 Re-apply step 2 for another vertex directly connected to one in S.

Select vertex 7.

From	**To**	**Via**	**Distance**	
1	7	6	10 + 2 = 12	shortest path to 7
1	7	3	7 + 9 = 16	
1	7	5	14 + 4 = 18	

8 Re-apply step 3 and add to the set of vertices, S.

$S = \{1, 2, 3, 4, 5, 6, 7\}$

9 Re-apply step 2 for another vertex directly connected to one in S.

Select vertex 8.

From	**To**	**Via**	**Distance**	
1	8	6	10 + 5 = 15	shortest path to 8
1	8	7	12 + 4 = 16	

10 Re-apply step 3 and add to the set of vertices, S.

$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

11 Re-apply step 2 for another vertex directly connected to one in S.

Select vertex 9.

From	**To**	**Via**	**Distance**	
1	9	8	15 + 3 = 18	
1	9	7	12 + 2 = 14	shortest path to 9
1	9	5	14 + 12 = 26	

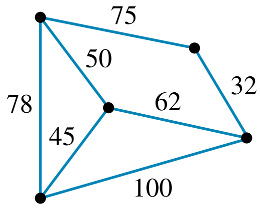
12 Re-apply step 3 and add to the set of vertices, S.

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ Stop, because all vertices are in the list.

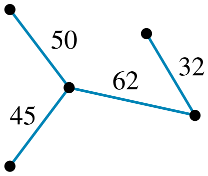
13 Apply step 4 and determine the shortest path.

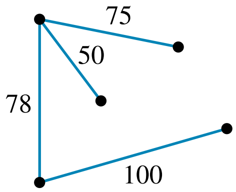
1 to 9 via 7 (see step 11)
1 to 7 via 6 (see step 7)
1 to 6 via 3 (see step 5)
1 to 3 via 2 (see step 1)
1 to 2 (see step 1)
Path = 1–2–3–6–7–9
Distance = 4 + 3 + 3 + 2 + 2
= 14.

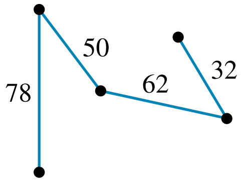
Spanning trees



In the network shown, the vertices represent school buildings and the edges represent footpaths. The numbers represent the distance, in metres, between the buildings. The school council has decided to cover some of the footpaths so that the students can access any building during rainy weather without getting wet. Three possible trees which would accomplish this are shown in the figures.







Note that each of these trees included all the vertices of the original network. These trees are called **spanning trees** because of this property. In practice, the school council would like to make the total distance of covered footpaths as small as possible, in order to minimise cost. In this case, they would have the **minimum spanning tree**. Can you determine which of the figures is the minimum spanning tree?

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Interactivity
Minimum spanning trees and Prim's algorithm
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Minimum spanning trees and Prim's algorithm

One method of determining the minimum spanning tree is called **Prim's algorithm**. The steps are as follows:

Step 1. Choose the edge in the network which has the smallest value. If 2 or more edges are the smallest, choose any of these.

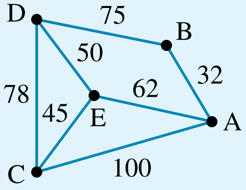
Step 2. Inspect the 2 vertices included so far and select the smallest edge leading from either vertex. Again, if there is a 'tie', arbitrarily choose any one.

Step 3. Inspect all vertices included so far and select the smallest edge leading from any included vertex. If there is a 'tie', choose any, arbitrarily.

Step 4. Repeat step 3 until all vertices in the graph are included in the tree.

WORKED EXAMPLE 18

Determine the minimum spanning tree for the network representing footpaths in a school campus.

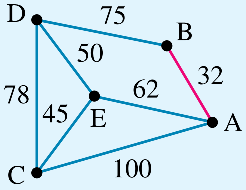


THINK

1 Find the edge with the smallest distance. This can be done by listing all the edges and choosing the smallest.

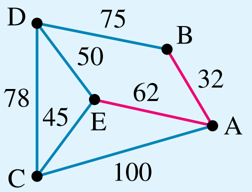
WRITE

A–B = 32, by inspection



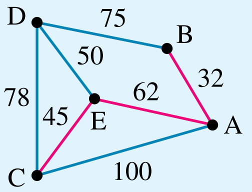
2 Inspect A and B and find the shortest edge connecting one of these to a third vertex.

A–E = 62 — choose this
A–C = 100
B–D = 75



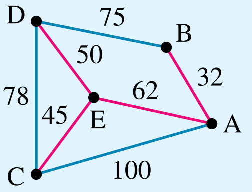
3 Inspect A, B and E and find the shortest edge connecting one of these to another vertex.

A–C = 100
E–C = 45 — choose this
E–D = 50
B–D = 75



4 Continue until all vertices have been connected. In this case only vertex D remains.

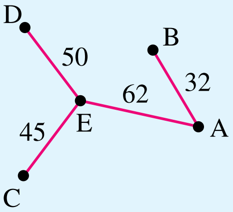
B–D = 75
E–D = 50 — choose this
C–D = 78



5 Since all vertices have been connected, this is the minimum spanning tree.

Calculate the total distance of the minimum spanning tree.

Total distance
$= 32 + 62 + 45 + 50$
$= 189$ m



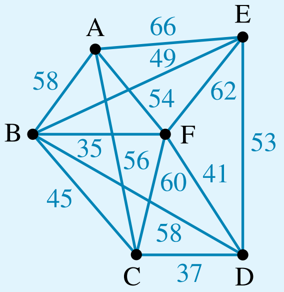
Maximum spanning tree

In some cases you may be required to find the **maximum spanning tree** instead of the minimum spanning tree. In this case, Prim's algorithm works by finding the largest edges at each stage instead of the smallest edges.

WORKED EXAMPLE 19

The figure shown represents a telephone network connecting 6 towns, A, B, ..., F. The numbered edges represent the 'capacity' of the telephone connection between the towns connected, that is, the maximum number of calls that can be made at the same time along that edge.

A telephone engineer wishes to determine the maximum capacity of the system in terms of a tree connecting all the towns so that calls can be routed along that tree.

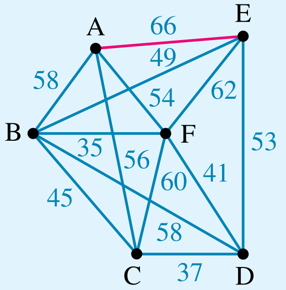


THINK

WRITE

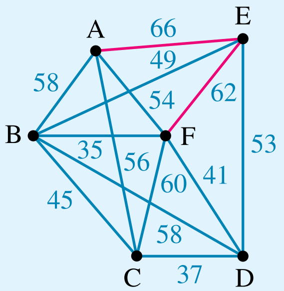
1 Because this is a maximum spanning tree, find the edge with the largest capacity.

This is edge A–E (66).



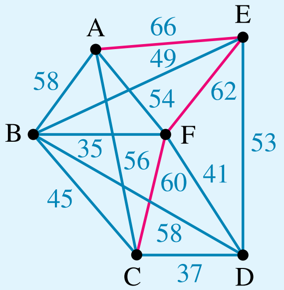
2 Inspect edges from vertices A and E, find the edge with the largest capacity.

This is edge E–F (62).



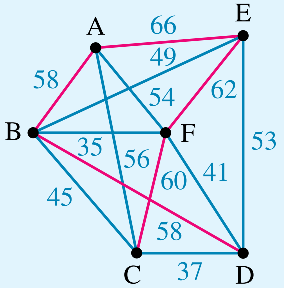
3 Inspecting edges from vertices, A, E and F, find the edge with the largest capacity.

This is edge F–C (60).



4 Repeat until all towns are connected.

A–B (58) and B–D (58) are the edges selected.



5 Determine maximum capacity.

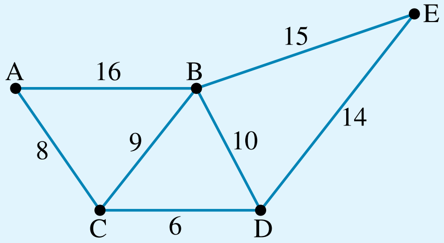
Maximum capacity
= 66 + 62 + 60 + 58 + 58
= 304 telephone calls at the same time.

Dijkstra's algorithm

Another method for determining the shortest path between a given vertex and each of the other vertices is by using **Dijkstra's algorithm**. An effective way to use this algorithm is using a tabular method, as shown in Worked example 20.

WORKED EXAMPLE 20

Determine the shortest path from A to E, where the distances are in kilometres, by using Dijkstra's algorithm in tabular form.



THINK

1 Construct a table representing the network. Where there is no value, you are unable to take that path.

WRITE

	A	B	C	D	E
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

2 Starting at E, place a 0 above it. The options are B (15 km) and D (14 km). Write these above the letters.

		15		14	0
	A	B	C	D	~~E~~
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

3 Go to the shortest distance – D and go down vertically.

B: Add 14 and 10 (= 24); this is greater than the 15 already above B so not a choice.

C: Add 14 and 6 (= 20); since nothing above C write this above C.

E: We started at E so don’t want to go back there.

		15	20	14	0
	A	B	C	~~D~~	~~E~~
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

4 Now E and D are done, the next lowest is B; again work vertically.

A: Add 15 and 16 (= 31); since nothing above A write this above A.

C: Add 15 and 9 (= 34); this is greater than 20 so leave it as 20.

D: Already been to D.

E: Already been to E.

	31	15	20	14	0
	A	~~B~~	C	~~D~~	~~E~~
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

5 Now E, D and B are done, the next lowest is C, again work vertically.

A: Add 20 and 8 (= 28), since this is lower than the 31 that is currently there so replace it with the 28.

B: Already been to B.

D: Already been to D.

	28				
	~~31~~	15	20	14	0
	A	~~B~~	~~C~~	~~D~~	~~E~~
A	X	16	8		
B	16	X	9	10	15
C	8	9	X	6	
D		10	6	X	14
E		15		14	X

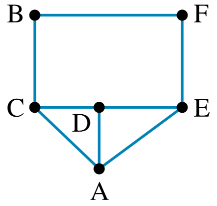
6 The shortest distance from A to E is then the number above E, in this case 28.

The shortest distance from A to E is 28 km.

EXERCISE 9.5 Trees and their applications

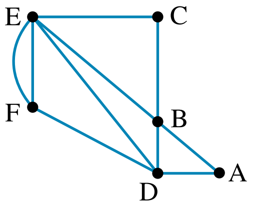
PRACTISE

1 **WE15** Determine *all* the trees connecting vertices A and B, without going through vertex F.



2 Consider the following paths:

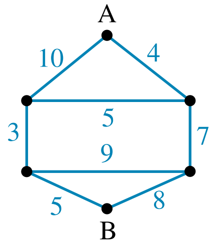
i A–B–C–E–F–D–A
ii A–D–F–E–C–B
iii A–B–D–A
iv A–B–D–F–E–C



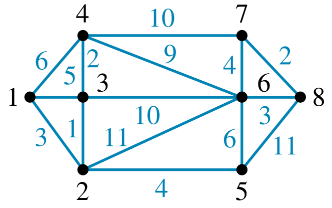
Which (if any) are trees?

A All are trees
B **i** and **iv** only
C **ii** and **iv** only
D **iv** only
E None are trees

3 **WE16** Determine the shortest path from A to B, where the distances (in blue) are in kilometres.

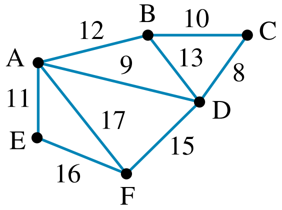


4 The network shown at right represents the time (blue numerals in minutes) that it takes to walk along pathways connecting 8 features in a botanical garden. Vertices 1 and 8 are entrances. Find the minimum time to walk between the entrances, along pathways only.



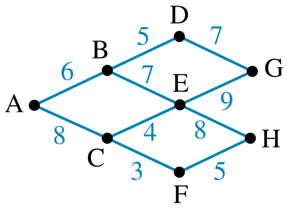
5 **WE17** Referring to the network shown at right, where distances are in km, find:

a the shortest path from B to F
b the shortest path from A to C.



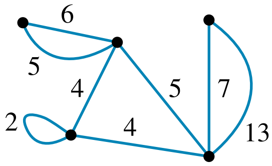
6 Referring to the network in question **5**, where distances are in km, find the shortest path from E to C.

7 **WE18** Find the minimum spanning tree for the network shown at right.



8 The total length of the minimum spanning tree in question **7** is:

A 20
B 37
C 40
D 51
E 66

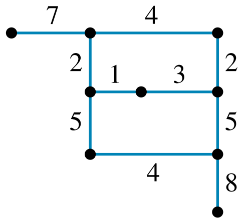


9 **WE19** In the figure shown, the calculations using Prim's algorithm for the minimum spanning tree would be:

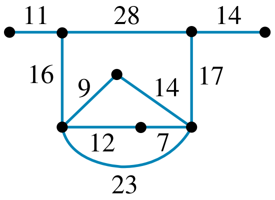
A 2 + 4 + 4 + 5 + 7
B 4 + 4 + 5 + 7
C 4 + 4 + 6 + 7
D 4 + 5 + 7 + 4
E 20

10 Draw the minimum spanning tree in each of the following graphs and calculate the total length:

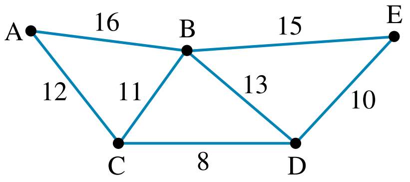
a



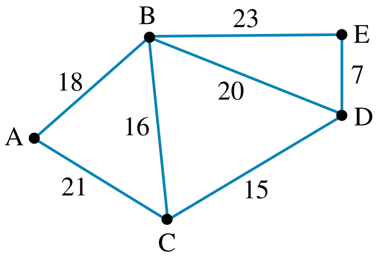
b



11 **WE20** Determine the shortest path from A to E, where the distances are in kilometres, by using Dijkstra's algorithm in tabular form.

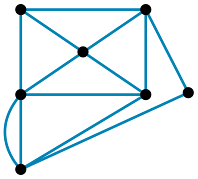


12 Determine the shortest path from A to E, where the distances are in kilometres, by using Dijkstra's algorithm in tabular form.

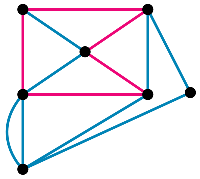


CONSOLIDATE

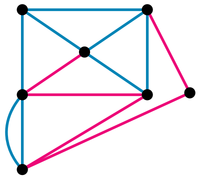
13 Consider the figure shown at right. Which of the paths marked in pink on the figures are trees?



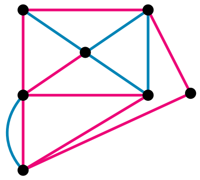
a



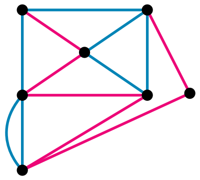
b



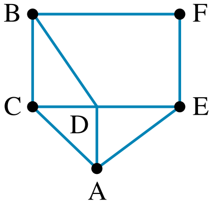
c



d



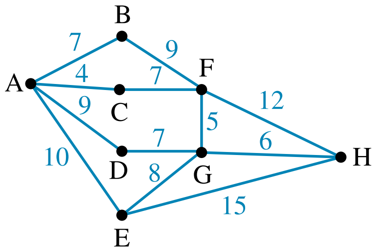
14 Determine *all* the trees connecting vertices A and B, without going through vertex F.



15 Which of the following statements is true?

A A Hamiltonian cycle is a tree.
B A tree can contain multiple edges or loops.
C A Hamiltonian path is not a tree.
D A tree can visit the same vertex more than once.
E A tree can have one edge.

Questions **16** to **18** refer to the network shown in the figure. Vertices are labelled, A, B, ..., H and the *time* it takes to travel between them, in minutes, is given by the numbers in blue.



16 a List all possible trees connecting A and H, passing through B.

b Use Dijkstra's algorithm to determine the shortest time for A–H.

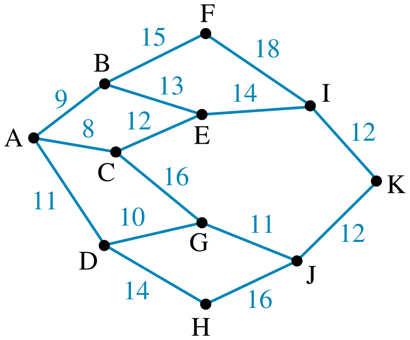
17 The total number of possible trees connecting A and H is given by:

A 8
B 10
C 12
D 15
E 28

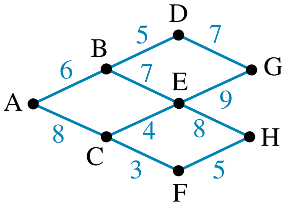
18 The shortest time it would take to travel between A and H is given by the tree:

A A–B–F–H
B A–B–F–G–H
C A–C–F–H
D A–C–F–G–H
E A–E–H

19 Use Dijkstra's algorithm to determine the shortest path from A to K, where the distances are in km.



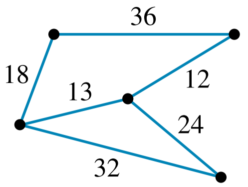
20 The figure shows a network connecting vertices A, ..., H



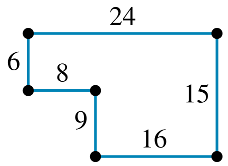
a How many different trees are there connecting A to G?
b Find the shortest path connecting A to G.
c How many different trees are there connecting D to F?
d Find the shortest path connecting D to F.

21 Using Prim's algorithm, determine the length of the minimum spanning tree in each of the graphs shown.

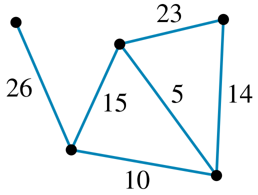
a



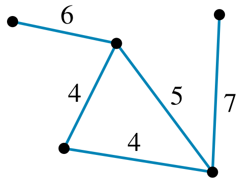
b



c

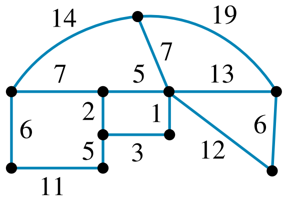


d

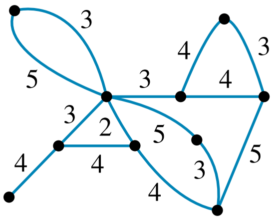


22 Draw the minimum spanning tree in each of the following graphs and calculate the total length:

a

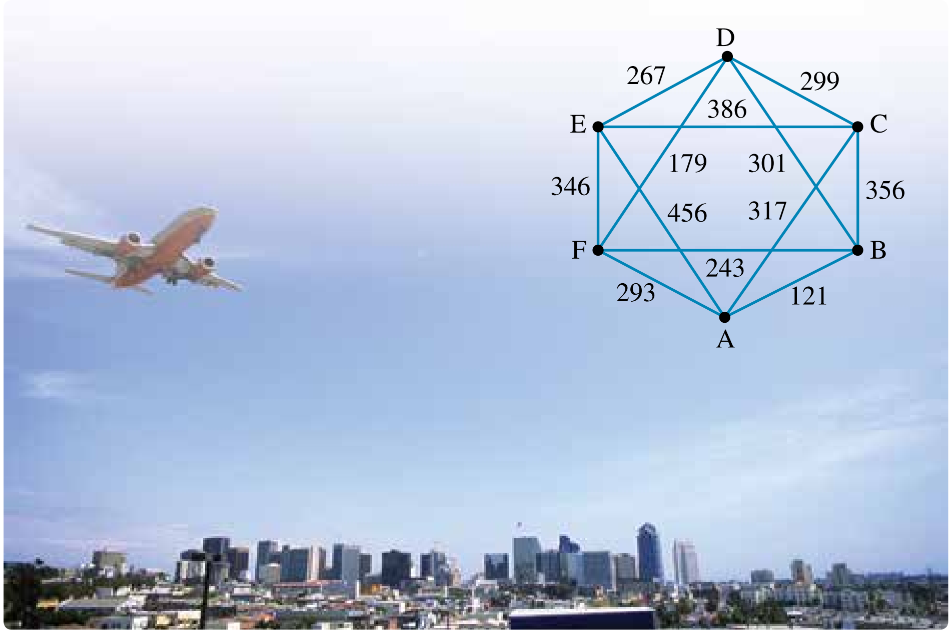


b

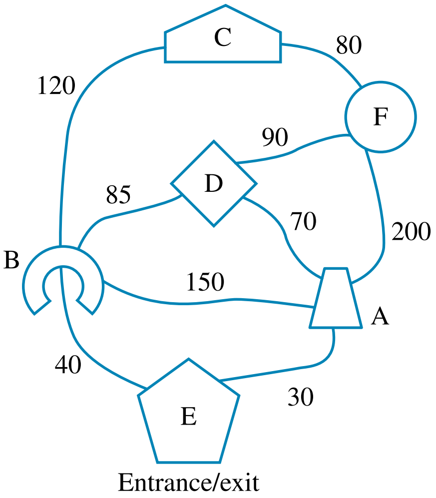


23 Flyemsafe Airlines wish to service six cities. The directors have decided that it is too costly to have direct flights between all the cities. The airline needs to minimise the number of routes which they open yet maximise the total number of passengers that they can carry. The network diagram shown has edges representing routes and vertices representing cities. The numbers on the edges are projected capacities. Find:

a the maximum spanning tree that will meet the airline's requirements
b the total carrying capacity of this tree.



24 A fairground has 5 main attractions which are joined by paths to the entrance/exit gate. The numbers show the distance along the paths in metres.



a Draw an undirected graph to represent the fairground and then write down:

  i the number of edges

  ii the number of vertices

  iii the degree of each vertex.

b What is the minimum distance a person would have to walk to visit every attraction, beginning and ending at the entrance/exit?

c If each attraction needs to be able to communicate via a phone line, draw the minimum possible tree to represent this.

d Complete a matrix for the graph shown.

e Following a Hamiltonian cycle would be an efficient way to visit every attraction in the fairground. Suggest a route a visitor could follow in order to create a Hamiltonian cycle, beginning and ending at the entrance/exit.

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9.6 Review

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- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
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- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

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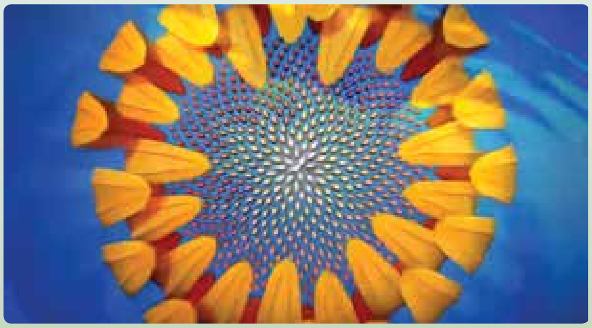

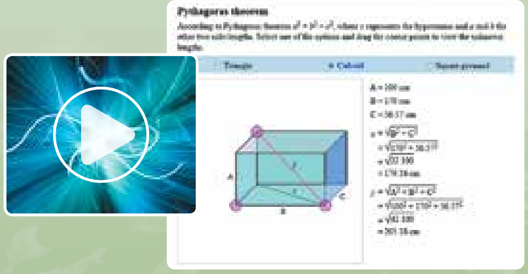

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9 Answers

EXERCISE 9.2

1 Vertices = 6, edges = 8

2 Vertices = 7, edges = 9

3 deg(A) = 3, deg(B) = 4, deg(C) = 4

4 deg(A) = 3, deg(B) = 8, deg(C) = 3

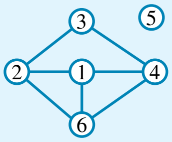
5 $V = \{A, B, C, D, E, F\}$

$E = \{(A, B), (A, D), (A, E), (B, D), (B, E), (B, E), (C, D), (D, F)\}$

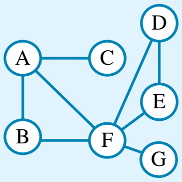
6 $V = \{A, B, C, D, E, F, G\}$

$E = \{(A, B), (B, C), (B, E), (B, F), (C, E), (C, F), (E, F), (E, G), (F, G)\}$

7



8



9

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}$$

10

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

11 a Vertices = 5, edges = 5

b Vertices = 6, edges = 9

c Vertices = 7, edges = 11

d Vertices = 9, edges = 16

12 C

13 a deg(A) = 2, deg(B) = 2, deg(C) = 1

b deg(A) = 3, deg(B) = 4, deg(C) = 2

c deg(A) = 4, deg(B) = 2, deg(C) = 6

d deg(A) = 4, deg(B) = 1, deg(C) = 0

14 D

15 a $V = \{1, 2, 3, 4, 5\}$

$E = \{(1, 2), (1, 3), (2, 3), (3, 4), (4, 5)\}$

b $V = \{U, V, W, X, Y, Z\}$

$E = \{(U, V), (U, W), (U, X), (V, W), (V, X), (V, Z), (W, X), (W, Y), (X, Z)\}$

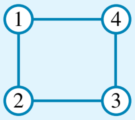
c $V = \{1, 2, 3, 4, 5, 6, 7\}$

$E = \{(1, 2), (1, 3), (1, 5), (2, 4), (2, 6), (2, 6), (2, 6), (2, 7), (3, 3), (3, 6), (5, 7)\}$

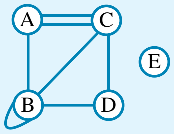
d $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$E = \{(1, 5), (1, 6), (1, 8), (2, 5), (2, 7), (2, 9), (3, 4), (3, 5), (3, 8), (4, 5), (4, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7)\}$

16 a



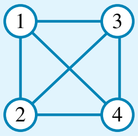
b



17 a $V = \{1, 2, 3, 4\}$

$E = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

b

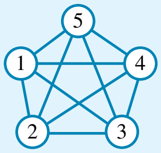


c Degree of each vertex = 3

18 i a $V = \{1, 2, 3, 4, 5\}$

$E = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

b

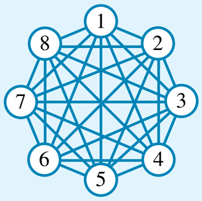


c Degree of each vertex = 4

ii a $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$E = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8)\}$

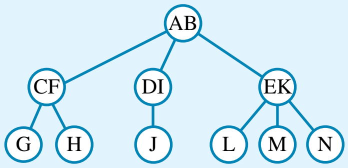
b



c Degree of each vertex = 7

19 a 45 b 190 c 4950

20



21 a

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

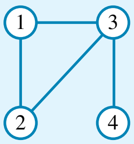
b

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 \end{bmatrix}$$

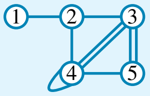
c

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

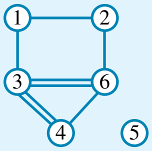
22 a



b

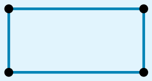


c

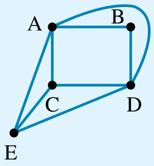


EXERCISE 9.3

1



2



3 FACE 1: degree = 3

FACE 2: degree = 3

FACE 3: degree = 3

FACE 4: degree = 3

FACE 5: degree = 4

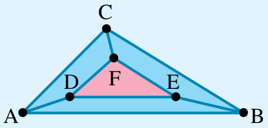
4 FACE 1: degree = 4

FACE 2: degree = 3

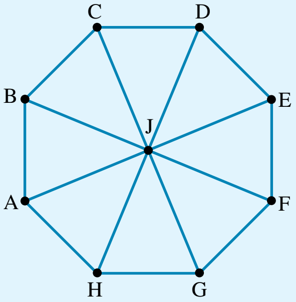
FACE 3: degree = 3

FACE 4: degree = 6

5

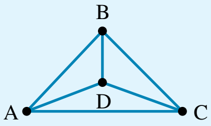


6



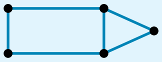
7 $V = 9$, $E = 16$, $F = 9$, so $9 = 16 - 9 + 2$

8 $V = 4$, $E = 6$, $F = 4$, so $4 = 6 - 4 + 2$.

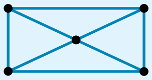


9 Answers may vary. Below are suggested solutions.

a

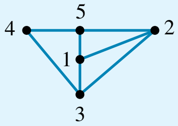


b



c Already planar

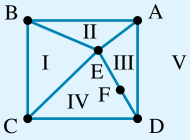
d



10 B

11 C

12 a



b 5

c Face I: degree = 3

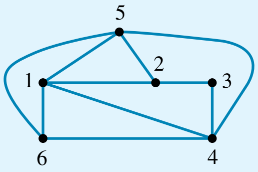
Face II: degree = 3

Face III: degree = 4

Face IV: degree = 4

Face V: degree = 4

13 a



b 6

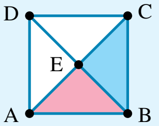
14 D

15 $V = 8$, $E = 13$, $F = 7$, so $8 = 13 - 7 + 2$

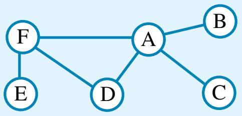
16 Since the degree of a single node is determined by the number of edges (E) 'leaving' it, and each such edge must be the 'entering' edge of another node, each edge is counted twice in the sum of degrees (S). Thus the sum must be an even number. And since each edge is counted twice $S = 2E$.

17 B

18



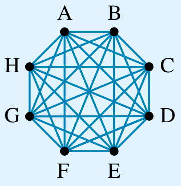
19



The edges should represent the doorways between rooms. (*Note:* The hall space between rooms E and D belongs to room F. Similarly, the hall space between rooms B and C belongs to room A.)

20 a 28

b



c

$$\begin{array}{c|cccccccc} & A & B & C & D & E & F & G & H \\ \hline A & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ B & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ C & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ D & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ E & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ F & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ G & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ H & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

EXERCISE 9.4

1 A–B–C–D–E–F–G

2 E–F–G–D–C–B–A

3 1–2–5–3–4–5–1

4 a B **b** FE

5 1–3–4–5–6–7–8–5–3–2–1

6 D

7 2–1–6–5–4–3–7

8 C

9 2–1–6–5–4–7–3–2

10 E

11 A–B–C–D–E–G–F–H–J

12 A–J–H–E–G–F–D–C–B

13 E

14 D

15 a A–G–L–M–S–H–J–K–F–P–E–R–D–O–C–Q–B–N

b H–S–M–L–G–A–N–B–Q–C–O–P–F–E–D–R–K–J

16 7–6–5–4–3–2–1

17 A

18 a 2–1–3–4–5–2–3

b Cannot be done (odd vertices)

c 1–2–3–4–5 **d** 1–3–4–5–2–1

19 a Join 4 to 7 and 3 to 8.

b Euler circuit = 3–2–1–3–4–1–5–4–7–6–10–7–8–10–9–8–3;
Hamiltonian cycle = 3–2–1–5–4–7–6–10–9–8–3

20 a Y–R–N–Y–S–L–N–L

b An Euler trail

21 a L–S–Y–R–N–Y–L–N–L

b An Euler circuit

22 a R–Y–S–L–N

b A Hamiltonian path

23 a C–D–A–E–F–B

b C–A–E–F–B–D–C

c Hamiltonian path, Hamiltonian cycle

EXERCISE 9.5

1 A–C–B; A–D–C–B; A–E–D–C–B

2 C

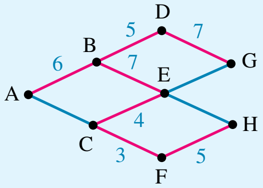
3 17 km

4 16 minutes

5 a 28 km **b** 17 km

6 28 km

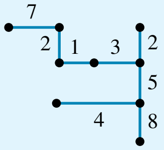
7

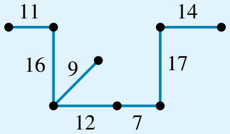


8 B

9 B

10 a 32 **b** 86





(1 of 2 possible answers)

11 30 km

12 41 km

13 **b**, **d**

14 A–C–B; A–D–B; A–C–D–B; A–D–C–B; A–E–D–B; A–E–D–C–B

15 E

16 a A–B–F–H; A–B–F–G–H; A–B–F–G–E–H;

b 22

17 C

18 D

19 44 km

20 a 5 **b** 18

c 7 **d** 19

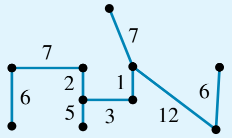
21 a $12 + 13 + 18 + 24 = 67$

b $6 + 8 + 9 + 16 + 15 = 54$

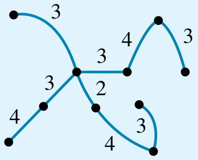
c $5 + 10 + 14 + 26 = 55$

d $4 + 4 + 6 + 7 = 21$

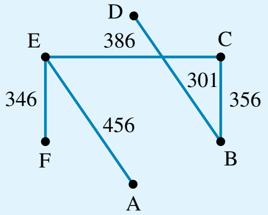
22 a 49



b 29

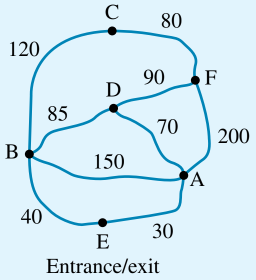


23 a



b 1845 passengers

24 a



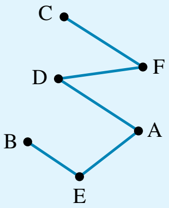
i 9

ii 6

iii A–4, B–4, C–2, D–3, E–2, F–3

b 430 m

c



d

$$\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

e E–A–D–F–C–B–E

10

Directed graphs and networks

10.1 Kick off with CAS

Directed graphs and networks

This topic does not lend itself to a CAS activity.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

10.2 Critical path analysis

Activity charts and networks

eBookplus

Interactivity
Critical path analysis
int-6290

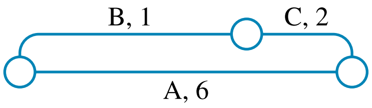
In any process, ranging from our daily schedule to major construction operations, tasks need to be completed within a certain period of time.

Consider Frieda's morning schedule, where she needs to eat her cooked breakfast, download her email and read her email. The first two tasks take 6 minutes and 1 minute respectively, while the last takes 2 minutes. Frieda needs to complete all these tasks in 7 minutes. How might she accomplish this?

Clearly, she needs to be able to do some tasks simultaneously. Although this seems like a simple problem, let us look at what might happen each minute.

Time	Activity	Activity
1st minute		Download email
2nd minute	Eat breakfast	
3rd minute	Eat breakfast	
4th minute	Eat breakfast	
5th minute	Eat breakfast	Read email
6th minute	Eat breakfast	Read email
7th minute	Eat breakfast	

More complex activities require more planning and analysis. A network diagram can be used to represent the 'flow' of activities.



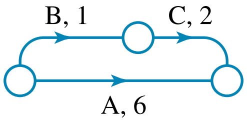
In the figure at right, the *edges* of our network represent the three activities of downloading (B), reading (C) and eating (A). The left **node** represents the start of all activity, the right node the end of all activity and the middle node indicates that activity B must occur before activity C can begin. In other words, activity B is the **immediate predecessor** of activity C.

Another way of representing this information is in an activity chart.

Activity letter	Activity	Predecessor	Time (min)
A	Eat breakfast	—	6
B	Download email	—	1
C	Read email	B	2

This chart also shows that activity B (downloading) is the immediate predecessor of activity C (reading), and that activities B and A have no predecessors.

An alternative network diagram is also shown.



The activities can be undertaken only in a certain sequence, so arrowheads are placed on the edges. Because of the implied direction, these networks are called **directed graphs** or *directed networks*. (The edges in a directed graph represent a *one-way path* between the nodes, as compared with undirected graphs where the edges represent a two-way path between the nodes.)

We can use the network diagram to help Frieda reduce the total time spent on the tasks. If the tasks were spread out in a straight line, so that no two tasks were

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completed at the same time, then they would take her 9 minutes. The diagram shows that some of Frieda's tasks can be carried out simultaneously. Let us investigate the time savings that can be made.

To determine the time saving, first determine the **earliest start time** for each activity.

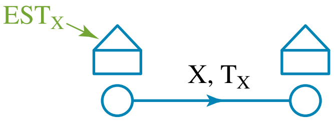
Forward scanning

By **forward scanning** through a network we can calculate the earliest start times for each activity and the earliest completion time for the whole project.

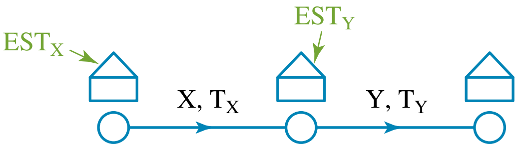
The *earliest start time* (EST) is the earliest that any activity can be started after all prior activities have been completed.

The EST is determined by looking at all the previous activities, starting with the immediate predecessors and working back to the start of the project. An activity can start no earlier than the *completion* of such predecessors. Obviously, the EST for the first activity is 0.

The EST can be recorded on a network diagram by using triangles and boxes, as shown in the diagram at right. The activity is represented by the edge between the nodes. The duration (T_X) of the activity is represented as the number above the edge. The earliest start time of activity X (EST_X) is recorded in the triangle preceding the edge.



When a network includes two or more activities, the same labelling process is used.



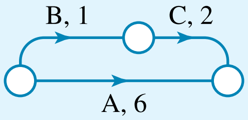
The purpose of the boxes beneath the triangles will be explained in a later section.

WORKED EXAMPLE 1 **Use forward scanning to determine the earliest completion time for Frieda's initial three tasks.**

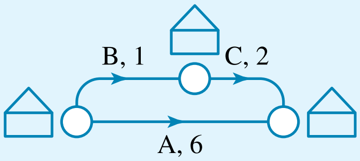
THINK

WRITE/DRAW

1 Begin with the network diagram.

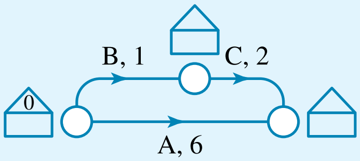


2 Add boxes and triangles near each of the nodes.

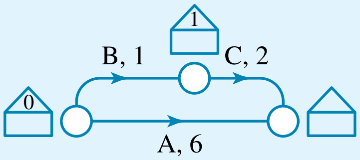


3 The earliest start time (EST) for each node is entered in the appropriate triangle. Nodes with no immediate predecessors are given the value of zero.

As activities B and A have no immediate predecessor then their earliest start time is zero.



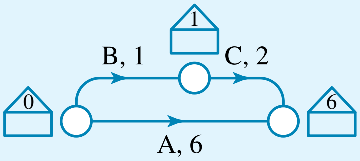
4 Move to another node and enter the earliest start time (EST) in its triangle. In the case of activity C, it must wait one minute while its immediate predecessor, B, is completed.



5 The last node's earliest start time is entered. When more than one edge joins at a node then the earliest start time is the largest value of the paths to this node. This is because all tasks along these paths must be completed before the job is finished.

There are two paths converging at the final node. The top path takes 3 minutes to complete and the bottom, 6 minutes. The larger value is entered in the triangle.

Path B–C = 1 + 2
= 3 minutes
Path A = 6 minutes



6 The earliest completion time is the value in the triangle next to the end node.

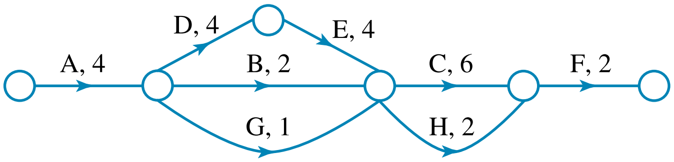
All tasks can be completed in 6 minutes.

It is important for anybody planning many tasks to know which tasks can be delayed and which tasks must be completed immediately. In Worked example 1, the eating must be commenced immediately if the 6-minute time is to be attained, whereas downloading the email could be delayed three minutes and there would still be enough time for it to be read while eating.

Let us now extend Frieda's activity chart to a more complex set of activities for her morning routine.

Activity letter	Activity	Predecessor	Time (min)
A	Prepare breakfast	—	4
B	Cook breakfast	A	2
C	Eat breakfast	B, E, G	6
D	Have shower	A	4
E	Get dressed	D	4
F	Brush teeth	C, H	2
G	Download email	A	1
H	Read email	B, E, G	2
Total time			25

The network diagram for these activities is shown below.

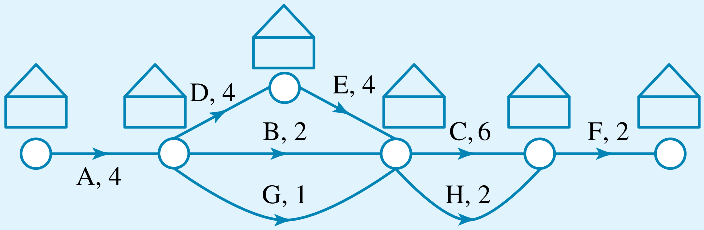


WORKED EXAMPLE 2

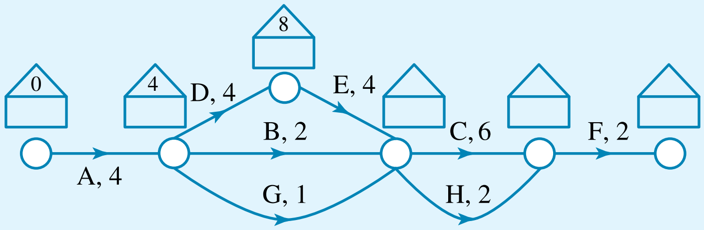
Using all the activities listed in Frieda's morning routine, find the earliest completion time and hence identify those tasks that may be delayed without extending the completion time.

THINK | **WRITE/DRAW**

1 Add the boxes and triangles to the directed network diagram.

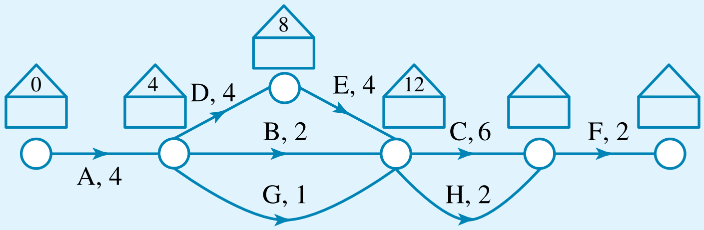


2 Begin forward scanning. The earliest start time (EST) for the first three nodes in the path can be entered immediately.



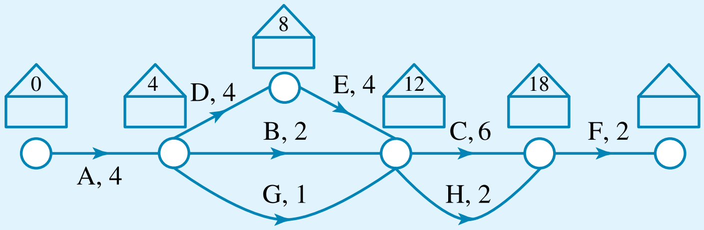
3 Calculate the time values for the paths to the fourth node. Enter the largest value (or longest time) into the appropriate triangle.

$$\begin{aligned} \text{A–D–E} &= 4 + 4 + 4 \\ &= 12 \text{ minutes} \\ \text{A–B} &= 4 + 2 \\ &= 6 \text{ minutes} \\ \text{A–G} &= 4 + 1 \\ &= 5 \text{ minutes} \end{aligned}$$



4 Repeat step 3 for the next node. Note that calculations begin by using the time from the previous node (12 minutes).

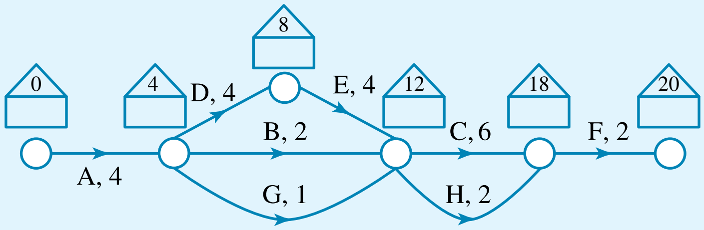
$$\begin{aligned} \text{A–E–C} &= 12 + 6 \\ &= 18 \text{ minutes} \\ \text{A–E–H} &= 12 + 2 \\ &= 14 \text{ minutes} \end{aligned}$$



5 There is only one path to the last activity (F). Add its time requirement to that of the previous node (18 minutes).

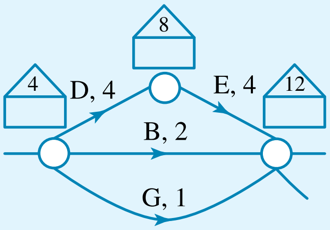
$$\begin{aligned} \text{A–C–F} &= 18 + 2 \\ &= 20 \text{ minutes} \end{aligned}$$

Earliest completion time is 20 minutes.



6 The time in the last triangle indicates the earliest completion time. | Earliest completion time = 20 minutes

7 Identify sections of the network where there was a choice of paths. There are two such sections of the network. Examine the first one (the 4th node).

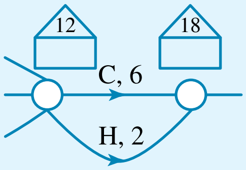


8 List and total the time for each path through this section of the network. Activities on the path with the largest value cannot be delayed.

D–E = 4 + 4 = 8 minutes
B = 2 minutes
G = 1 minute
Activities B and G can be delayed.

9 Repeat step 8 for the next section identified in step 7.

C = 6 minutes
H = 2 minutes
Activity H can be delayed.



studyon

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Critical paths

The path through the network which follows those activities that cannot be delayed without causing the entire project to be delayed is called the critical path.

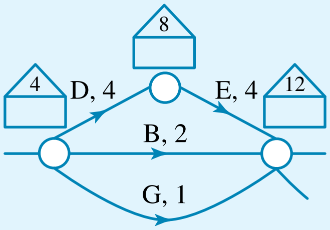
Therefore the critical path for the activities listed in Frieda's morning routine would be A–D–E–C–F. It is easily seen that this path takes the longest time (20 minutes).

Float time and latest start time

Float time is the difference in time between those paths that cannot be delayed and those that can. When planning projects, paths with float time are often delayed if there is a cost saving, otherwise, they are done as soon as possible if this is more appropriate. The **latest start time** for such activities is defined as the latest time they may begin without delaying the project.

WORKED EXAMPLE 3

Work out the float time for activities B and G in Worked example 2, and hence identify the latest starting time for these activities.



THINK	WRITE
1 List the alternative paths for the section containing activities B and G and the times for these alternatives.	D–E $= 4 + 4$ $= 8$ minutes B $= 2$ minutes G $= 1$ minute
2 Subtract the smaller times separately from the maximum time.	Float time for activity B $= 8 - 2$ $= 6$ minutes Float time for activity G $= 8 - 1$ $= 7$ minutes
3 Look up the earliest completion time for the activity on the critical path and subtract the activity times.	D–E is on the critical path. Earliest completion time = 12 minutes Latest start time for activity B $= 12 - 2$ $= 10$ minutes Latest start time for activity G $= 12 - 1$ $= 11$ minutes

The float times indicate the amount of time for which these activities can be delayed without delaying the completion of all tasks. Furthermore, activity B could begin up to 6 minutes (4 + 6) after the start of the critical activity (D), while G could begin up to 7 minutes (4 + 7) after the same critical activity (D). There will be a more formal treatment of float time in the next section.

Drawing network diagrams

Let us now look at how networks are prepared from activity charts. As an example, we shall see how Frieda's morning schedule network was prepared.

WORKED EXAMPLE 4

From the activity chart below, prepare a network diagram of Frieda's morning schedule.

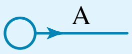
Activity letter	Activity	Predecessor	Time (min)
A	Prepare breakfast	—	4
B	Cook breakfast	A	2
C	Eat breakfast	B, E, G	6
D	Have shower	A	4
E	Get dressed	D	4
F	Brush teeth	C, H	2
G	Download email	A	1
H	Read email	B, E, G	2
Total time			25

THINK	WRITE/DRAW
1 Begin the diagram by drawing the starting node.	

2 (a) Examine the table looking for activities that have no predecessors. There must be at least one of these. Why?

(b) This activity becomes the first edge and is labelled with its activity letter and arrowhead.

Activity A has no predecessors.



3 (a) List all activities for which A is the immediate predecessor.

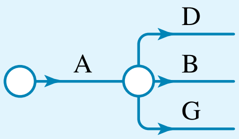
(b) Add a node to the end of the edge for activity A.

(c) Create one edge from this node for each of the listed activities.

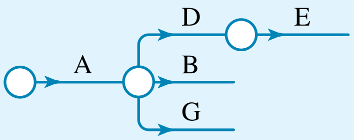
Label these edges.

Note: The end node for each of these activities is not drawn until either you are certain that it is not the immediate predecessor of any later activities, or all activities have been completed.

Activity B has A as an immediate predecessor.
Activity D has A as an immediate predecessor.
Activity G has A as an immediate predecessor.

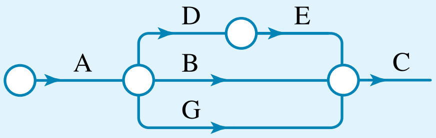


4 Repeat step 3 for activity D. Since it is the *only* immediate predecessor of activity E, this can be added to the diagram. Otherwise, activity E could not be added yet.

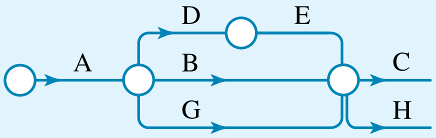


5 (a) Repeat step 3 for activities B and G. They have no activities for which they are the only predecessors. Since activity C is preceded by all of B, G and E, join all the edges at a single node.

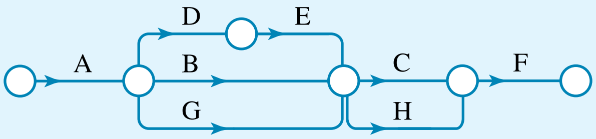
(b) Add activity C after this joining node. Note that activity H is also preceded by all of B, G and E but *not* by activity C.



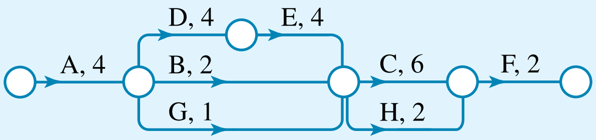
6 Determine whether activity C and H are independent of each other. Since they are independent, activity H starts from the same node as activity C.



7 The last activity is F, which has C and H as its immediate predecessors. Therefore join C and H with a node, then add an edge for F. Since F is the final activity, also add the end node.



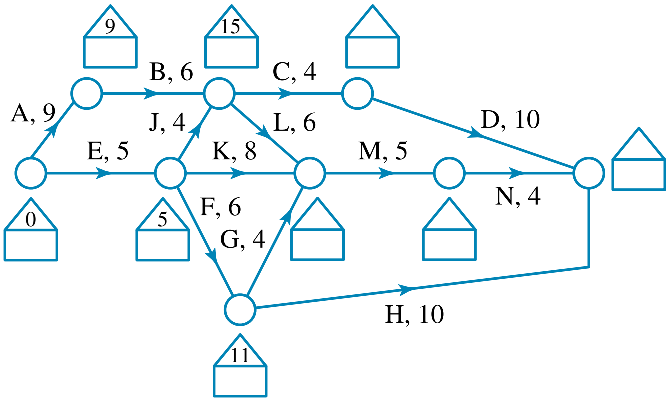
8 Add the time required for each activity next to its letter.



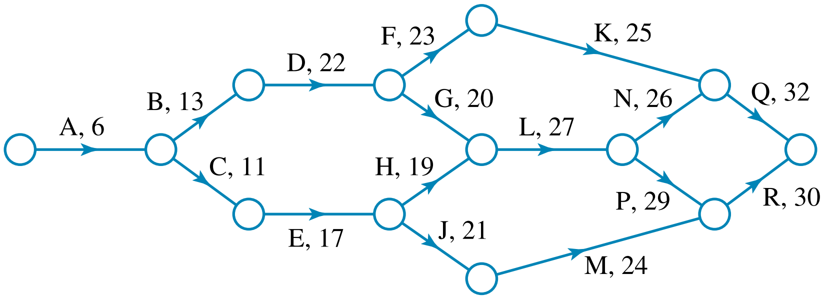
EXERCISE 10.2 Critical path analysis

PRACTISE

1 **WE1** Complete a forward scan for the critical path network shown. Determine the earliest completion time.



2 The project plan for a new computer software program is shown in the figure below. Time is measured in days. Determine the earliest completion time.



3 **WE2** From question **1**, what is the maximum time that path J can be delayed without increasing the earliest completion time?

4 From question **2**, what is the maximum time that path K can be delayed without increasing the earliest completion time?

5 **WE3** For the network in question **1**, complete a backward scan and determine:

a the critical path

b the float time for non-critical activities.

6 For the network in question **2**, determine the float time for non-critical activities.

7 **WE4** Prepare a network diagram from the activity chart.

Activity	Immediate predecessor
D	—
E	D
F	D
G	E, F

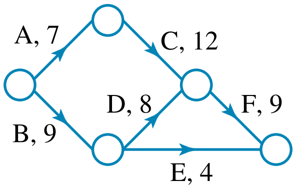
8 Prepare a network diagram from the activity chart.

Activity	Immediate predecessor
N	—
O	N
P	O, T
Q	P
R	—
S	N

Activity	Immediate predecessor
T	S, Y
U	O, T
V	O, T
W	V
X	Y
Y	R
Z	X

CONSOLIDATE

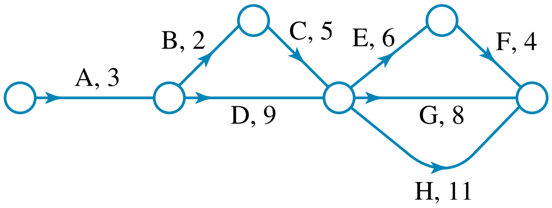
9 Consider the network diagram shown. Times shown are in minutes.



a Which of the following statements is true?
- **A** Activity A is an immediate predecessor of F.
- **B** Activity D is an immediate predecessor of F.
- **C** Activity F must be done before activity D.
- **D** Activity F must be done before activity E.
- **E** Activity D is an immediate predecessor of E.

b The minimum time taken to complete all activities is:
- **A** 19 minutes
- **B** 21 minutes
- **C** 23 minutes
- **D** 28 minutes
- **E** 49 minutes

10 Refer to the diagram shown.

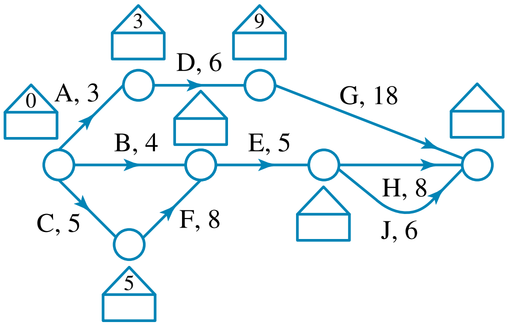


a Use forward scanning to determine the earliest completion time.

b Identify tasks that may be delayed without increasing the earliest completion time.

11 Determine the critical path for the network in question **9**.

12 Refer to the network diagram shown.

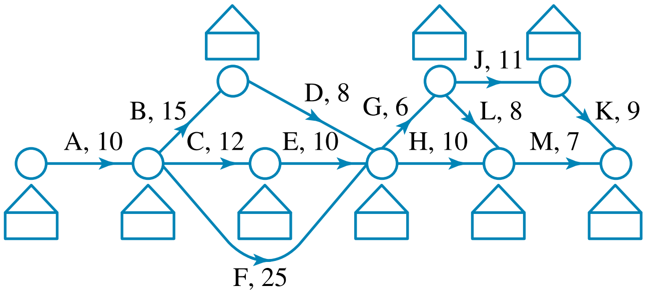


a The number required in the triangle above the node after activities B and F is:
- **A** 0
- **B** 4
- **C** 5
- **D** 8
- **E** 13

b The number required in the triangle above the node after activity E is:
- **A** 5
- **B** 9
- **C** 10
- **D** 18
- **E** none of these

c The earliest completion time for all tasks is:
- **A** 27
- **B** 24
- **C** 21
- **D** 18
- **E** 15

13 **a** Find the earliest start time for each node in the network shown.



b Hence, find the earliest completion time for the project.

14 From the network diagram in question **9**, produce an activity chart.

15 From the network diagram in question **12**, produce an activity chart.

16 From the network in question **13**, produce an activity chart.

17 For the network in question **12**:

a find the critical path

b determine which activities have float time and hence calculate their float times

c determine the latest start time for all non-critical activities.

18 For the network in question **13**:

a find the critical path

b determine which activities have float time.

MASTER

19 Prepare a network diagram from each of the activity charts.

a

Activity	Immediate predecessor
A	—
B	—
C	A

b

Activity	Immediate predecessor
A	—
B	A
C	A
D	C
E	B
F	B
G	F
H	D, E, G
I	J, H
J	D, E, G

20 When a personal computer is being assembled the following processes must be performed.

Activity letter	Activity	Predecessor	Time (min)
A	Install memory board	—	2
B	Install hard drive	A	20
C	Test hard drive	B, E	4
D	Install I/O ports	A	5
E	Install DVD drive	D	3
F	Test DVD drive	E	5
G	Install operating system	C, F	10
H	Test assembled computer	G	12
Total time			61

a Construct a network diagram.

b Determine the minimum time in which all tasks could be completed.

10.3 Critical path analysis with backward scanning and crashing

With more complex projects requiring the coordination of many activities, it is necessary to record more information on the network diagrams and to display the information more formally using charts.

In the previous section the float times and the critical path were worked out using somewhat informal methods. In this section a more formal method will be shown to enable float times to be calculated and the critical path to be determined. This method involves **backward scanning**.

Requirements for critical path analysis

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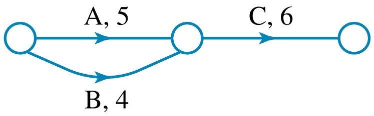
Along with the informal rules and techniques already developed in the previous section, we need to define two more rules that must be followed in order to successfully complete a critical path analysis of a network.

Rule 1. Two nodes can be connected directly by a maximum of one edge.

Consider the following activity table, and associated network diagram.

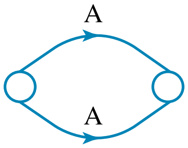
Activity letter	Immediate predecessor	Time (min)
A	—	5
B	—	4
C	A, B	6

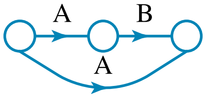


For activity C to have both A and B as predecessors, activities A and B must be drawn as parallel edges. Clearly this does not meet the requirement of Rule 1, which allows for only one edge (activity) connecting two nodes.

Violation of this rule does not affect forward scanning and the calculation of minimum completion time but will cause problems when identifying the critical path using the method of backward scanning described later in this section. A method for dealing with parallel edges will be suggested below.

Rule 2. An activity must be represented by exactly one edge. Consider the two network diagrams shown here.





The left-hand drawing indicates two separate flows along the same edge. If A were a water pipe, how could you keep the two flows separate? The right-hand example suggests that A can happen at the same time as B while still being its immediate predecessor.

Backward scanning

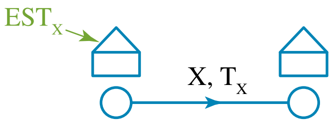
To complete critical path analysis, a procedure called *backward scanning* must be performed.

***Backward scanning* starts at the end node and moves backward through the network subtracting the time of each edge from the earliest start time of each succeeding node.**

When two or more paths are followed back to the same node the smallest such difference is recorded. The results of each backward scanning step yield the *latest start time* for each activity. Latest start time is the latest time an activity can start without delaying the project.

Earliest finish time (EFT) for an activity is equal to the earliest start time (EST) of the activity plus its duration (T).

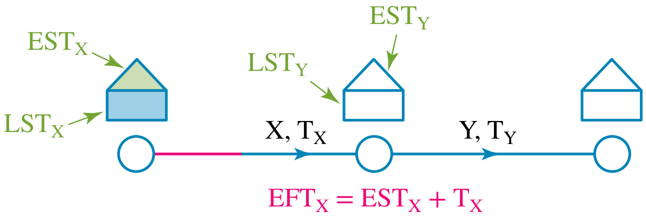
The following diagram illustrates the activity X.



The earliest finish time for activity X (EFT_X) will be the earliest start time for activity X (EST_X) plus the duration of activity X (T_X).

$$EFT_X = EST_X + T_X$$

The following diagram illustrates two activities X and Y, where activity Y directly follows activity X.



As previously established, EST_X is represented by the triangle preceding activity X (in pink on the previous diagram). The latest start time for activity X (LST_X) is represented by the blue box preceding the activity. The same applies for activity Y and all following activities.

Note that EFT cannot be read from the triangles or boxes; it must be calculated.

***Float time*, also called 'slack', is the maximum time that an activity can be delayed without delaying a subsequent activity on the critical path and thus affecting the earliest completion time.**

From the above, it can be seen that there is a relationship between float time and the other quantities, namely:

Float time for activity X = Latest start time for activity Y (LST_Y) – Earliest Start Time for activity X (EST_X) – duration of activity X (T_X)

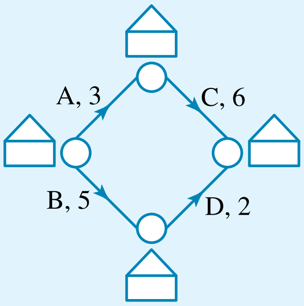
That is: $$\text{Float time}_X = LST_Y - EST_X - T_X$$

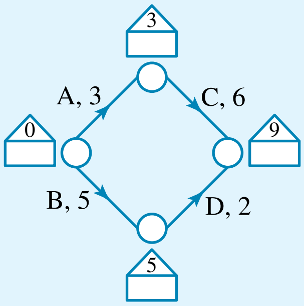
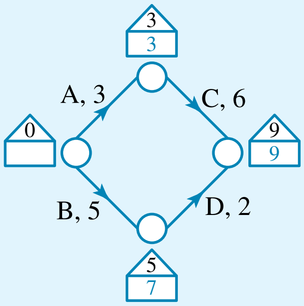
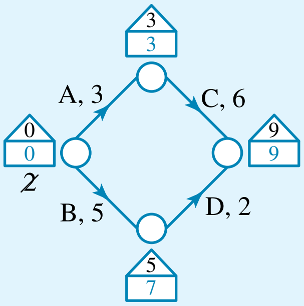
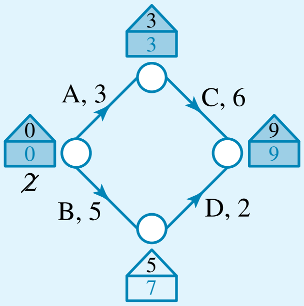
where activity Y directly follows activity X.

The technique of backward scanning is best explained with an example.

WORKED EXAMPLE 5

The network diagram shown has been constructed for a project manager. Use forward and backward scanning to clearly display the critical path and to list any float times.



THINK	WRITE/DRAW
1 Forward scan through the network and record the earliest start time (EST) for each activity in the appropriate triangle.	
2 Begin backward scanning. (a) Start at the end node and trace backwards along all paths from this node. (b) Subtract the times of the activities along each path from the earliest finish time (EST = 9) and record the value in the box at the previous node. These values are the latest start times (LST) for the activities leaving this node.	Along path C: 9 – 6 = 3 Along path D: 9 – 2 = 7 Latest start time for activity C (LST_C) = 3 Latest start time for activity D (LST_D) = 7 
3 Repeat the process backwards through the diagram. Where two (or more) paths come together (activities A and B), record the *smaller* value in the box.	Along path A: 3 – 3 = 0 Along path B: 7 – 5 = 2 Smallest value = 0. 
4 The critical path can now be clearly identified. It is the path that has the same numbers in both the triangles and boxes at any node. Remember to include *all* such nodes in the critical path.	Critical path is A–C (as shown in blue). 

5 (a) Float times are calculated now. Construct a table with the headings shown.

Activity X	Activity time (T_X)	Earliest start time of this activity (EST_X)	Latest start time of the following activity (LST_Y)	Float time
A	3	0	3	0
B	5	0	7	2
C	6	3	9	0
D	2	5	9	2

(b) Record the times from the triangles in the earliest start times (EST) column, the times in the boxes in the latest start time (LST_Y) column as well as the activity times (T_X). Calculate float times using the equation:

$$\text{Float}_X = LST_Y - EST_X - T_X$$

In this example the float times are also the differences between the corresponding times in the boxes and triangles. This is not the rule in the general case.

For activity D: Float $= 9 - 5 - 2$
$= 2$

For activity C: Float $= 9 - 3 - 6$
$= 0$

For activity B: Float $= 7 - 0 - 5$
$= 2$

For activity A: Float $= 3 - 0 - 3$
$= 0$

Worked example 5 is fairly simple as the critical path could easily be determined by direct inspection. There is only one path that is not on the critical path, therefore the calculation of float time is also simple. In the real world, the problems are more complicated and so require the use of the formal method. Float times are important for the efficient management of any project. They enable the manager to determine what delays can be tolerated in the project. For example, the manager in charge of a building site is able to tell sub-contractors that they have a time window in which they must work. The sub-contractors can then arrange their schedules to incorporate this time window.

WORKED EXAMPLE 6

The chart shown has been given to an operations manager. The activities have been simplified to letter names. The manager is required to find all critical activities and the earliest completion time for the project by:

a creating a network diagram

b completing a forward scan and determining the earliest completion time

c completing a backward scan and identifying the critical path

d calculating the float times for each activity.

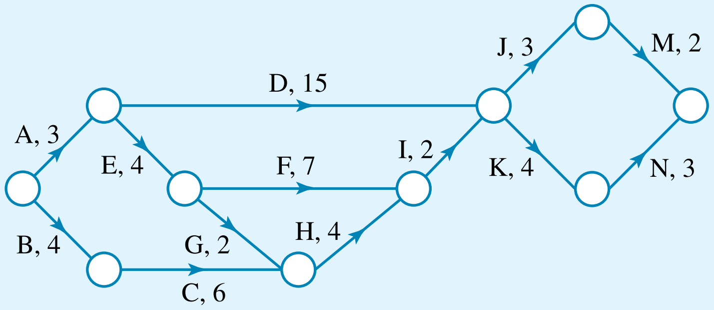
Activity letter	Immediate Predecessor	Time (days)
A	—	3
B	—	4
C	B	6
D	A	15
E	A	4
F	E	7
G	E	2
H	C, G	4
I	F, H	2
J	D, I	3
K	D, I	4
M	J	2
N	K	3

THINK

a Construct the network diagram from the table.

WRITE/DRAW

a



THINK

b 1 Draw boxes and triangles at each node. Forward scan through the network. Start at zero for the first node and then add the times taken for a path (T_X) and write it in the triangle at the next node (EST_Y).

Adding the times of the paths (T_X) to the times in the triangle at the previous node (EST_X) gives the next value to be entered (EST_Y).

When two paths converge at a node, the largest time value is entered as all immediate predecessors need to be completed before the next activity can begin.

WRITE/DRAW

b A = 3 days

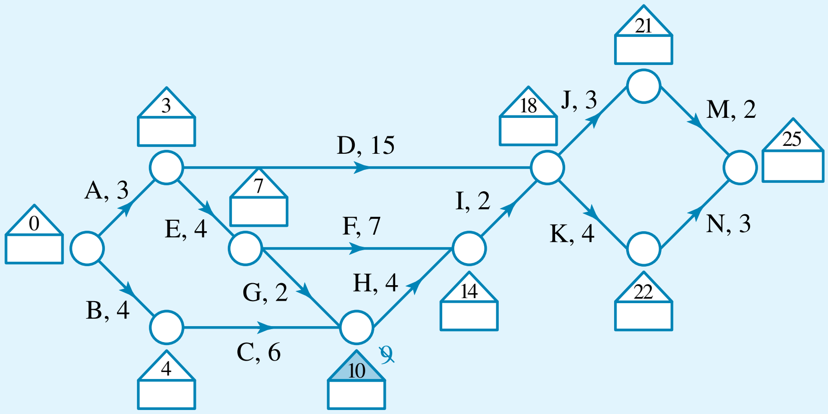
A–E = 3 + 4 = 7 days

B = 4 days

The blue triangle (EST_H) may be reached by following two paths:

A–E–G = 3 + 4 + 2
= 9 days

B–C = 4 + 6
= 10 days, thus the larger of the times, 10, is entered in the triangle.

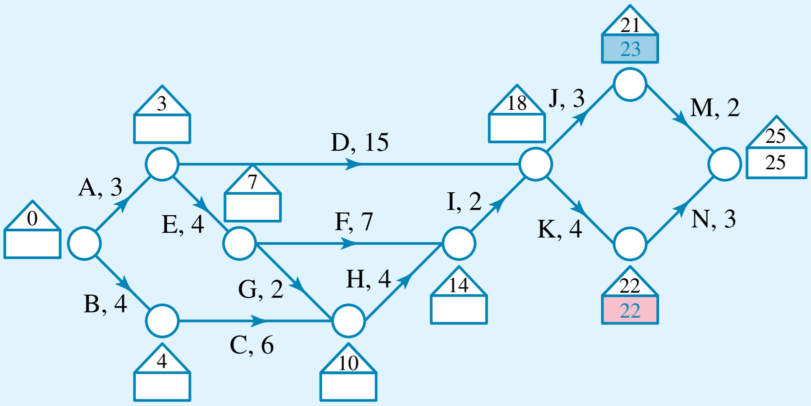


2 The earliest completion time can be read from the last triangle.

Earliest completion time = 25 days

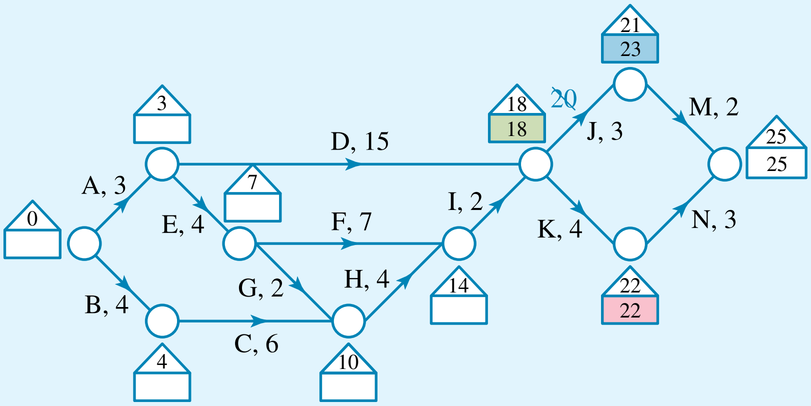
c 1 Starting at the end node, begin the backward scan. Enter the earliest completion time in the last box. Subtract the times of any paths ending at this node (M and N) from the value in the last box and enter the result in the appropriate boxes to calculate the latest start time (LST) of that activity.

c Blue box value (LST_M) $= 25 - 2$
$= 23$ days
Pink box value (LST_N) $= 25 - 3$
$= 22$ days

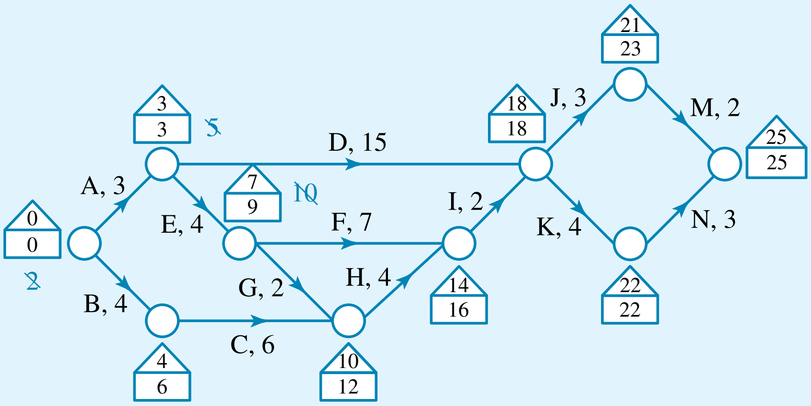


2 Repeat the process backwards through the network diagram. Where the paths converge the smallest value is entered in the box.

Backtracking to the green box (LST_J) via activity
J $= 23 - 3$
$= 20$
Backtracking to the green box (LST_K) via activity
K $= 23 - 4$
$= 18$
Enter the smaller of the values (18) in the green box.

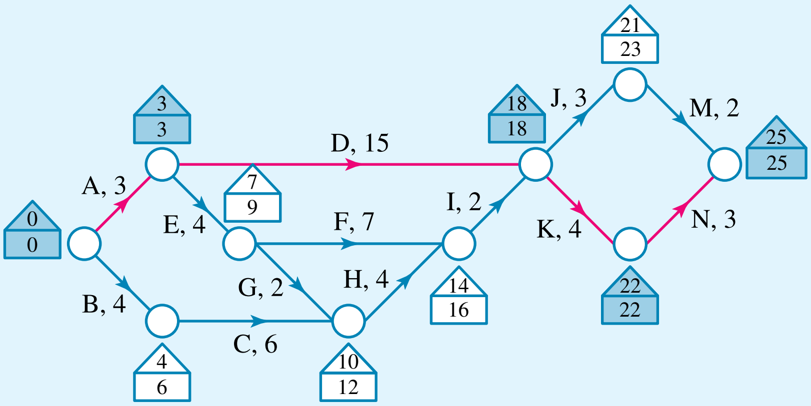


3 Repeat the process back through all paths.



4 Nodes which have the earliest start time (triangles) equal to the latest start times (boxes) are identified as being on the critical path.

A–D–K–N is the critical path.



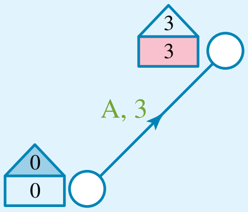
d **1** (a) The float times for each of the activities are calculated using the formula.

(b) The section showing activity A from part **c**, step 4 clearly shows the latest start time of the following activity (LST_Y) (pink), the earliest start time (EST_X) (blue) and the activity time (T_X) (green).

d Float time of X = $LST_Y - EST_X - T_X$

Float (A) = 3 – 0 – 3 = 0

A result of zero indicates that activity A is on the critical path with no float time available.



2 (a) The best way to keep organised and to calculate float times is to set up a table.

(b) Add columns for earliest start times (EST), latest start times (LST) and float times to the original table.

Repeat step 1 for all activities.

Activity letter	Immediate predecessor	T_X	EST_X	LST_Y	Float time$_X$ = LST_Y – EST_X – T_X
A	—	3	0	3	3 – 0 – 3 = 0
B	—	4	0	6	6 – 0 – 4 = 2
C	B	6	4	12	12 – 4 – 6 = 2
D	A	15	3	18	18 – 3 – 15 = 0
E	A	4	3	9	9 – 3 – 4 = 2
F	E	7	7	16	16 – 7 – 7 = 2
G	E	2	7	12	12 – 7 – 2 = 3
H	C, G	4	10	16	16 – 10 – 4 = 2
I	F, H	2	14	18	18 – 14 – 2 = 2
J	D, I	3	18	23	23 – 18 – 3 = 2
K	D, I	4	18	22	22 – 18 – 4 = 0
M	J	2	21	25	25 – 21 – 2 = 2
N	K	3	22	25	25 – 22 – 3 = 0

Note that all activities that were on the critical path have float times of zero. It is important to note that if even a single activity is 'floated' by having its start delayed, then the entire network diagram should be re-drawn and float times recalculated.

If the manager employed extra workers for a critical activity, its duration time could be reduced, hence reducing the completion time for the project. The reduction in the duration time of an activity is called **crashing**. Crashing may result in a different critical path. This will be explored further in a later section.

Dummy activities

A dummy activity is an edge that must be added to avoid a network with two or more activities having the same name or occurring in parallel.

Earlier in this section we set up two rules:

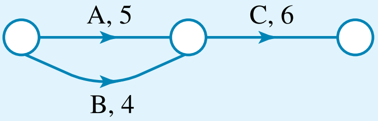
Rule 1. Two nodes can be connected directly by a maximum of one edge.

Rule 2. An activity must be represented by exactly one edge.

A table and a drawing were presented in which there were parallel edges (breaking Rule 1). A method, explained by example, will be given to overcome this problem.

WORKED EXAMPLE 7

Introduce a dummy edge to the network diagram shown.

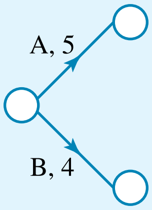


Activity letter	Immediate predecessor	Time (min)
A	—	5
B	—	4
C	A, B	6

THINK **WRITE/DRAW**

1 Construct edges A and B.

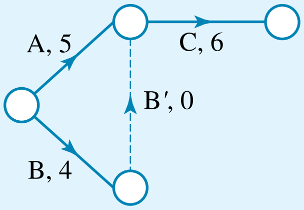
C must follow both A and B. This is clearly a problem causing parallel edges (violating Rule 1 for networks).



2 Construct C after activity A. Introduce a dummy activity (B′) label and allocate a time of zero.

Note: Use a dotted line to show dummy activity.

Therefore, not only is A the immediate predecessor of C, but B (via B′ with a time = 0) is also effectively the immediate predecessor of C.



The introduction of the dummy activity with a time value of zero enables scanning to take place along both edges, A and B. Additionally, the critical path can be shown more clearly.

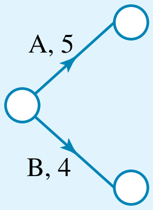
WORKED EXAMPLE 8

From the table shown construct a network diagram.

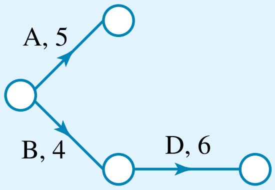
Activity letter	Immediate predecessor	Time (days)
A	—	5
B	—	4
C	A, B	4
D	B	6

THINK | **WRITE/DRAW**

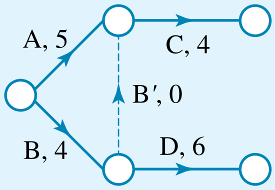
1 Construct edges A and B.



2 Construct edge D following B as it has only one immediate predecessor.



3 C must follow both A and B (clearly a problem). Construct C after activity A. Introduce a dummy activity (B′) from B to C and allocate it a time of zero.



Worked example 8 provides a method of not only avoiding parallel edges but also avoiding A being shown as the immediate predecessor of D, which (from the table) it clearly is not.

Once any required dummy activities have been defined, it is possible, using forward and backward scanning, to determine the earliest completion time for the project and float times for each non-critical activity, as per the methods of Worked example 6.

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Interactivity
Crashing
int-6286

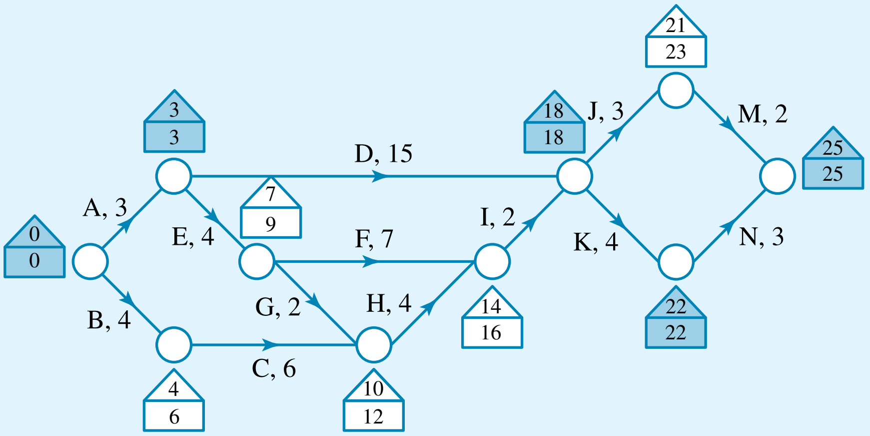
Crashing

As discussed earlier, crashing is a method of speeding up the completion time of a project by shortening the critical path. Follow the same method as in previous sections to calculate the new critical path and minimum completion time.

WORKED EXAMPLE 9

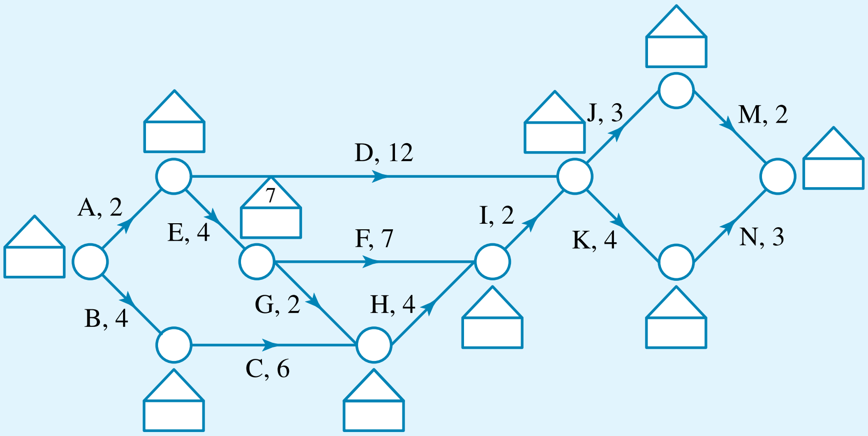
Take the critical path found in the network in Worked example 6 reproduced here.

To shorten the overall completion time of the project, activity A is to be shortened to 2 days and activity D is to be shortened to 12 days. Determine the new critical path and the new minimum completion time for the project.

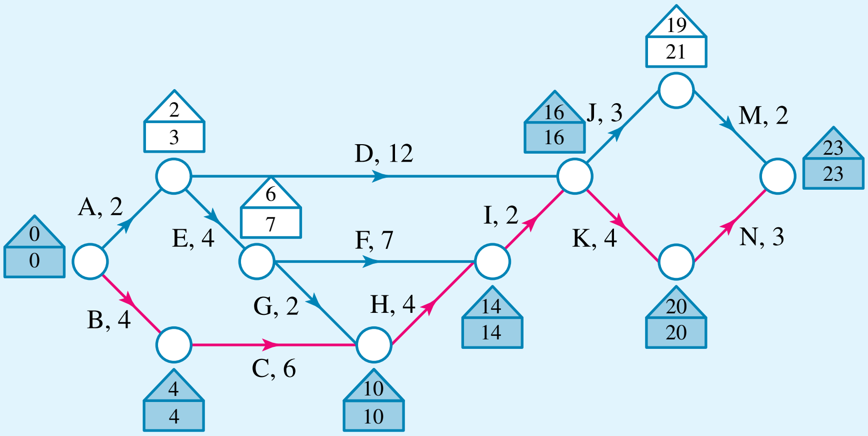


THINK | **WRITE/DRAW**

1 Redraw the network with the new completion times.



2 Recalculate the earliest start times (EST) and latest start times (LST) by completing forward and backward scanning.



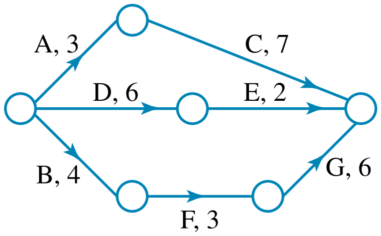
3 Write the answer.

The new critical path is B–C–H–I–K–N. The minimum finishing time is 23 days.

EXERCISE 10.3 Critical path analysis with backward scanning and crashing

PRACTISE

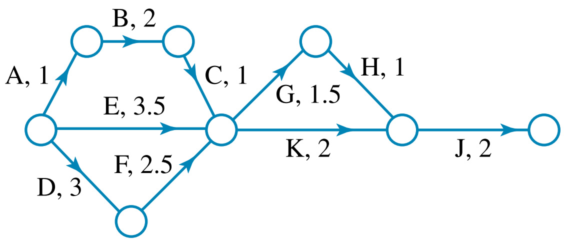
1 **WE5** For the network diagram shown, use forward and backward scanning to clearly display the critical path and to list any float times for non-critical activities. Times are in hours.



2 Perform a backward scan on the network shown. Determine:

a the critical path

b float times for non-critical activities.



3 **WE6** The manufacturing of bicycles can be considered as a 7-step process:

A — Collect all the parts — 12 minutes

B — Paint frame — 35 minutes (requires A to be completed first)

C — Assemble brakes — 16 minutes (requires A to be completed first)

D — Assemble gears — 20 minutes (requires B to be completed first)

E — Install brakes — 12 minutes (requires C to be completed first)

F — Install seat — 5 minutes (requires C to be completed first)

G — Final assembly — 18 minutes (requires D and E to be completed first)

a Construct an activity chart.

b Construct a network diagram.

c Determine the earliest completion time using forward and backward scanning.

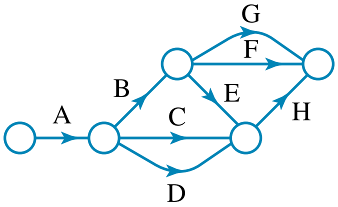
d Determine the critical path.

4 In the bicycle manufacturing system described in question **3**, activities with float time are:

A A, B, C, D, E, F, G **B** A, B, D, G **C** C, E, F

D C only **E** none

5 **WE7** Re-draw the network diagram shown, inserting any necessary dummy activities so that Rule 1 for critical path problems is not violated.



6 Consider the following activity table.

Activity	Immediate predecessor	Activity Time (min)
A	—	7
B	—	6
C	A, B	8
D	C	12
E	C	7
F	D, E	9

By creating one (or more) dummy activities, construct a proper critical path network diagram.

7 WE8 From the following activity table construct a network diagram and indicate the location and direction of the dummy activity.

Activity letter	Immediate predecessor	Time (h)
A	—	3
B	—	5
C	A	7
D	B	7
E	B, C	1
F	D, E	2

8 Convert the following activity table into a network diagram.

Activity letter	Immediate predecessor	Time (h)
A	—	1000
B	—	600
C	—	800
D	A, B	1100
E	B	400
F	C	100
G	C	600
H	D, E	1600

9 WE9 From your network diagram in question 6, determine:

a the earliest completion time

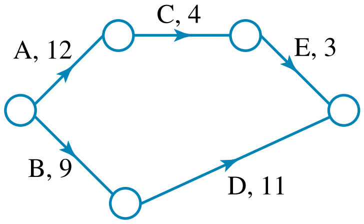
b the critical path

c float times for non-critical activities.

10 For the network in Exercise 10.2, question 2, determine the critical path.

CONSOLIDATE

11 For the network diagram shown, use forward and backward scanning to clearly display the critical path and to list any float times. Times are in minutes.



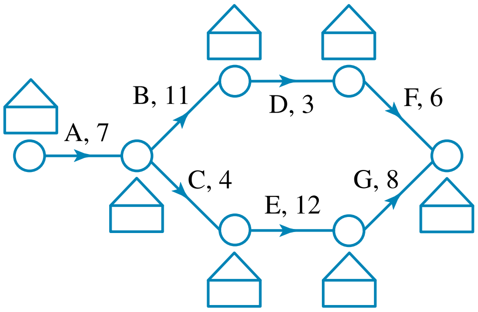
12 Complete the figure by forward and backward scanning and hence:

a determine the earliest completion time

b indicate the critical path.

Times are in days.

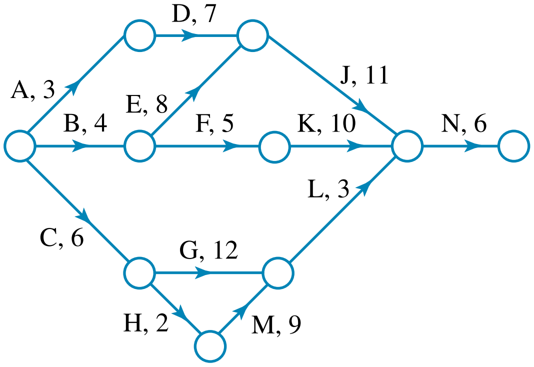
c Imagine now that activity E can be completed in 9 days. How does this affect the answers to parts a and b?



13 The float time for activity D in question **12** is:

A 1 day **B** 2 days **C** 3 days **D** 4 days **E** 7 days

14 From the network diagram shown:



a produce an activity table

b complete a forward scan and hence determine the earliest completion time

c complete a backward scan and hence determine the critical path

d use the activity table from **a** to calculate float times for all non-critical activities.

15 From the following activity table:

a construct a network diagram

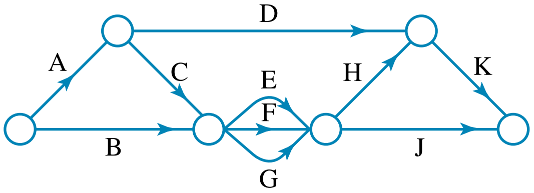
b determine the earliest completion time

c by forward and backward scanning, determine the critical path

d determine the float time for all non-critical activities.

Activity letter	Immediate predecessor	Time (h)
A	—	5
B	—	3
C	A	4
D	A	7
E	B	4
F	C	9
G	D, E	5
H	G	3
J	G	6
K	F, H	4

16 Re-draw the network diagram shown, inserting any necessary dummy activities so that rule 1 for critical path problems is not violated.



17 From the following activity table:

a construct a network diagram, adding any dummy activities that may be required

b determine the earliest completion time

c determine the critical path by forward and backward scanning

d determine the float time for all non-critical activities.

Activity letter	Immediate predecessor	Time (h)
A	—	11
B	—	9
C	A	2
D	A	5
E	B	12
F	C	3
G	D	3
H	E	4
J	E, F, G	7

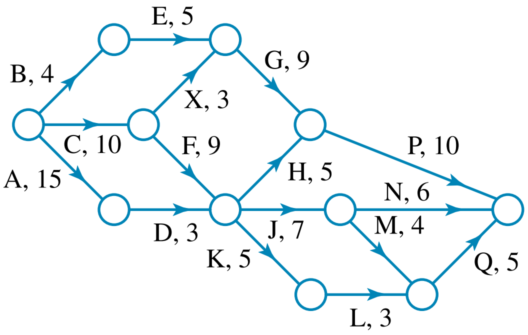
18 From the network diagram at right (activity times are recorded in days):

a forward scan to determine the earliest completion time

b backward scan to determine the critical path

c determine the float time for activity X

d imagine that crashing results in J being completed in 5 days. How does this affect the earliest completion time and the critical path?

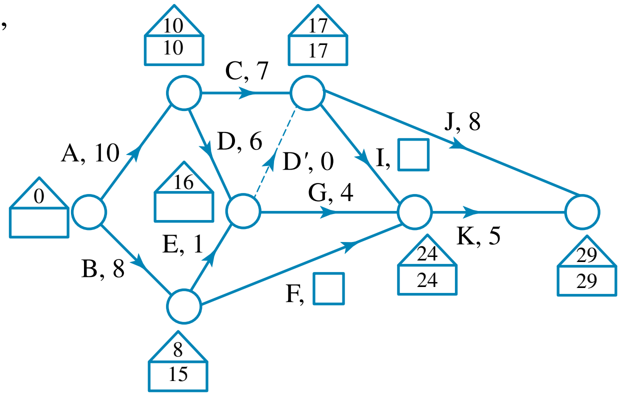


19 Given the network diagram at right, determine the:

a four missing values in the boxes

b critical path

c float time for activity F.

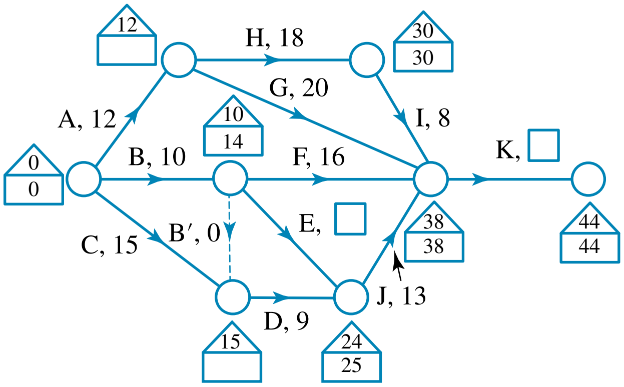


20 Given the network diagram at right, determine the:

a four missing values in the boxes

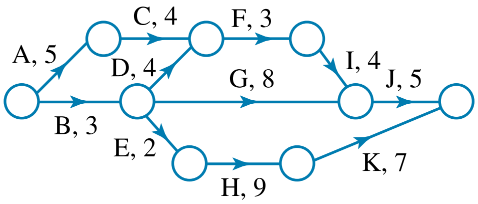
b critical path

c float time for activity F.



MASTER

21 The network diagram shows the activities, along with the times (in hours), needed to complete a particular project. Also drawn below is the corresponding activity chart showing earliest start times rather than activity times. There is an activity X, which is yet to be drawn on the network diagram.



Activity	Immediate predecessor(s)	EST
A	—	0
B	—	0
C		5
D	B	3
E	B	3
F		9
G	B	
H	E	5
I	F	12
J	G, I, X	17
K	H	
X	C, D	8

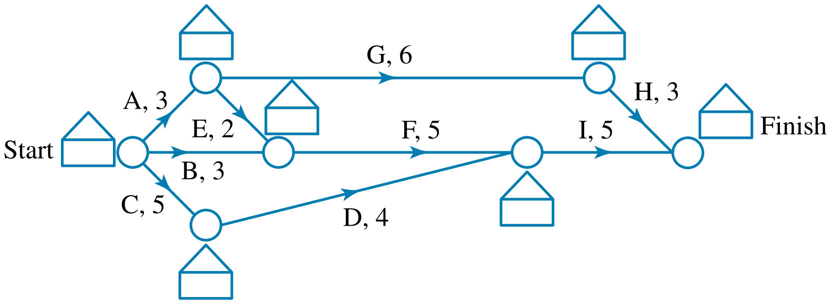
a Use the information in the network diagram to complete the table above by filling in the shaded boxes.

b Draw and label activity X on the network diagram, including its direction and time.

c What is the critical path?

d Determine the latest start time for activity H.

22 A school is building a new library. The separate stages required for construction and the number of weeks taken to complete them are shown on the directed network.



a What is the earliest start time for stage F?

b What is the minimum time it will take to build the library?

c What is the critical path?

d What is the slack time for stage H?

e If the overall time of the project was to be reduced, which stages could be shortened?

f If stage A was reduced to 2 weeks and stage F was also reduced to 2 weeks, what would be the new critical path and the new minimum time for completion of the library?

10.4 Network flow

There are many examples of networks throughout the modern world.

The road and rail systems that link all parts of our country, such as the one in the image, and the airline flight paths that not only link us to other places within the country but also to places overseas, are examples of networks that move people and products. Other networks such as the information superhighway (the internet), telephone lines and the postal system allow the transfer of information.

All these networks have common attributes. The attribute dealt with in this section is **network flow**. There are networks that allow for the flow to be in both directions along a path and those that allow for flow in one direction only. The *direction* of flow needs to be displayed clearly on any diagram: simple arrows on the path suffice. The *quantity* of flow is just as important.

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Concept 3

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Flow capacities and maximum flow

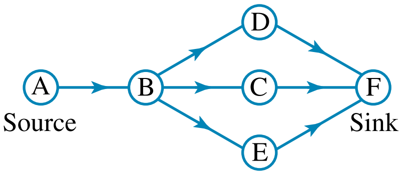
The network's starting node(s) is called the **source**. This is where all flows commence. The flow goes through the network to the end node(s) which is called the **sink**.

The **flow capacity** (capacity) of an edge is the amount of flow that an edge can allow through if it is not connected to any other edges.

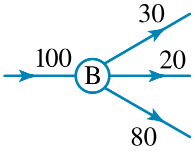
The **inflow** of a node is the total of the flows of all edges leading into the node.

The **outflow** of a node is the *minimum* value obtained when one compares the inflow to the sum of the capacities of all the edges leaving the node.

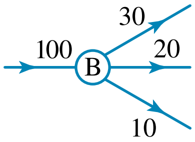
Consider the following figures.



All flow commences at A. It is therefore the source. All flow converges on F indicating it is the sink.



B still has an inflow of 100 but now the capacity of the edges leaving B is 130 (80 + 20 + 30). The outflow from B is now 100.



B has an inflow of 100. The flow capacity of the edges leaving B is 30 + 20 + 10 = 60. The outflow is the smaller of these two total values, which is 60.

WORKED EXAMPLE 10

Convert the information presented in the following table to a network diagram, clearly indicating the direction and quantity of the flow.

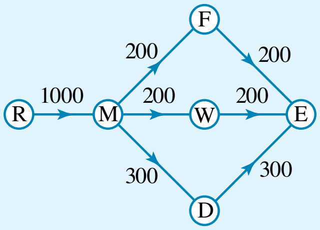
From	To	Quantity (litres per minute)	Demand (E)*
Rockybank Reservoir (R)	Marginal Dam (M)	1000	—
Marginal Dam (M)	Freerange (F)	200	200
Marginal Dam (M)	Waterlogged (W)	200	200
Marginal Dam (M)	Dervishville (D)	300	300

THINK

Construct and label the required number of nodes.

Note: The nodes are labelled with the names of the source of the flow and the corresponding quantities are recorded on the edges. Link the nodes with edges and record flow direction and quantity on these.

WRITE/DRAW



Note: In Worked example 10 there does not exist a location called Demand (E). It is preferable for a network diagram to have both a single source and a single sink, so the Demand (E) was included to simplify the diagram. The reason for this will become clear in the following worked examples.

Worked example 10 is a simple case of a network in which the direction and quantity of flow are evident. Such a network diagram allows for analysis of the flow in the network; it allows us to see if various edges in the network are capable of handling the required flow.

The *flow capacity of the network* is the total flow possible through the entire network.

WORKED EXAMPLE 11

Use the information from Worked example 10 to determine, by inspection:

a the flow capacity of the network

b whether the flow through the network is sufficient to meet the demand of all towns.

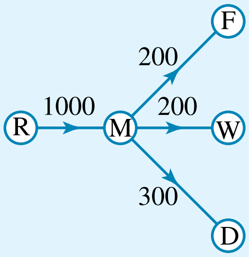
THINK

a 1 Examine the flow into and out of the Marginal Dam node. Record the smaller of the two at the node. This is the maximum flow through this point in the network.

2 In this case the maximum flow through Marginal Dam is also the maximum flow of the entire network.

WRITE/DRAW

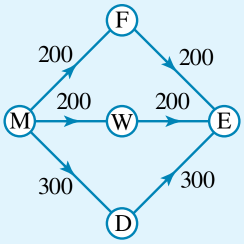
a Even though it is possible for the reservoir to send 1000 L/min (in theory), the maximum flow that the dam can pass on is 700 L/min (the minimum of the inflow and the sum of the capacities of the edges leaving the dam).



Maximum flow is 700 L/min.

b **1** Determine that the maximum flow through Marginal Dam meets the total flow demanded by the towns.

b



Flow through Marginal Dam = 700 L/min
Flow demanded = 200 + 300 + 200
= 700 L/min

2 If the requirements of step 1 are able to be met then determine that the flow into each town is equal to the flow demanded by them.

By inspection of the table in Worked example 10, all town inflows equal town demands (capacity of edges leaving the town nodes).

Excess flow capacity

Consider what would happen to the system if Rockybank Reservoir continually discharged 1000 L/min into Marginal Dam while its output remained at 700 L/min.

Such flow networks enable future planning. Future demand may change, the population may grow or a new industry that requires more water may come to one of the towns. Worked example 12 will examine such a case.

Excess flow capacity is the surplus of the capacity of an edge less the flow into the edge.

WORKED EXAMPLE 12

A new dairy factory (Creamydale (C)) is to be set up on the outskirts of Dervishville. The factory will require 250 L/min of water.

a **Determine whether the original flow to Dervishville is sufficient.**

b **If the answer to part a is no, is there sufficient flow capacity into Marginal Dam to allow for a new pipeline to be constructed directly to the factory to meet their demand?**

c **Determine the maximum flow through the network if the new pipeline is constructed.**

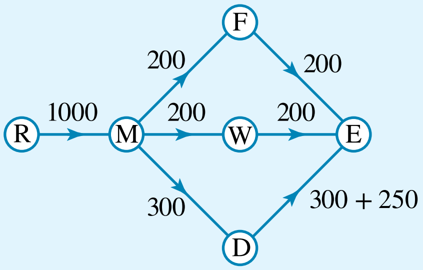
THINK

a **1** Add the demand of the new factory to Dervishville's original flow requirements. If this value exceeds the flow into Dervishville then the new demand cannot be met.

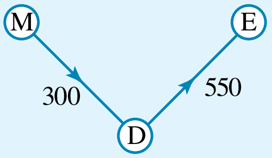
2 The new requirements exceed the flow.

WRITE/DRAW

a

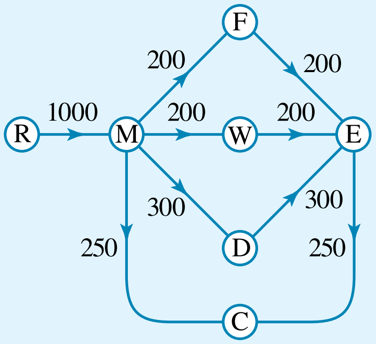


The present network is not capable of meeting the new demands.



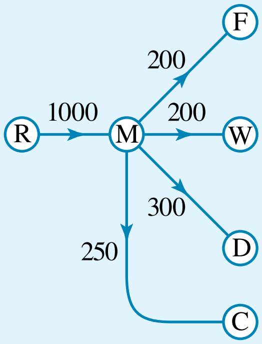
b **1** Reconstruct the network including a new edge for the factory after Marginal Dam.

b



2 Repeat step 1 from Worked example 11 to find the outflow of node M.

Marginal Dam inflow = 1000
Marginal Dam outflow
= 200 + 200 + 300 + 250
= 950



3 Determine if the flow is sufficient.

There is excess flow capacity of 300 into Marginal Dam which is greater than the 250 demanded by the new factory. The existing flow capacity to Marginal Dam is sufficient.

c This answer can be gained from part **b** step 2 above.

c The maximum flow through the new network is 950.

The maximum flow through most simple networks can be determined using these methods, but more complex networks require different methods.

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Interactivity
Network flow cuts
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Minimum cut–maximum flow method

To determine the maximum flow, the network first needs to be divided or **cut** into two parts.

A *cut* in a network diagram is a line drawn through a number of edges which stops *all* flow from the source to the sink.

The value of the cut is the total flow of the edges that are cut.

The *minimum cut* is the cut with the minimum value.

The *maximum flow* through a network is equal to the value of the minimum cut.

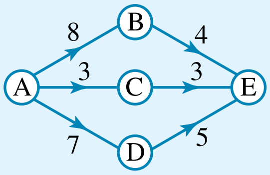
WORKED EXAMPLE 13

For the network diagram shown:

a make three cuts

b calculate the value of each cut

c determine the value of all possible cuts to give the value of the minimum cut and hence the maximum flow through the network.

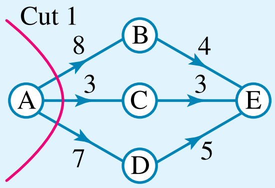


THINK **WRITE/DRAW**

a 1 Isolate the source, A, by cutting any edges leaving it.

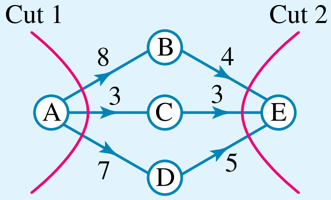
Cut 1 = {AB, AC, AD}

a



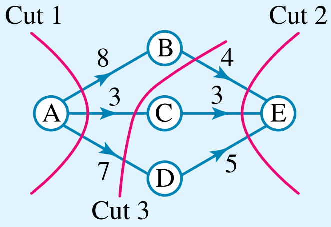
2 Isolate the sink, E, by cutting any edges leading into it.

Cut 2 = {BE, CE, DE}



3 Place a cut through each of the three paths going from A to E.

Cut 3 = {AD, AC, BE}



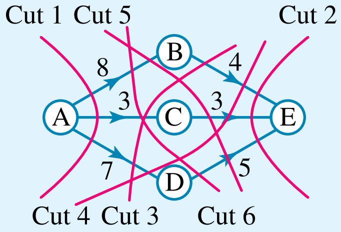
b Add the values of the edges crossed by each of the cuts in **a**.

b Value of cut 1 = 8 + 3 + 7
= 18
Value of cut 2 = 4 + 3 + 5
= 12
Value of cut 3 = 7 + 3 + 4
= 14

c 1 Identify the cuts in the diagram.

Cut 1 = {AB, AC, AD}
Cut 2 = {BE, CE, DE}
Cut 3 = {AD, AC, BE}
Cut 4 = {AD, CE, BE}
Cut 5 = {AB, CE, DE}
Cut 6 = {AB, AC, DE}

c



2 Calculate the value of all cuts.

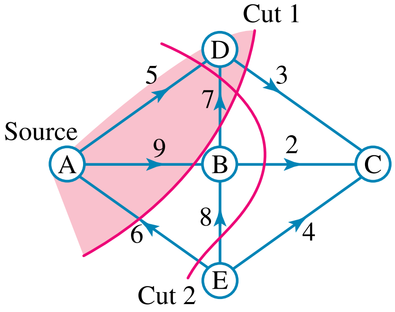
Value of cut 1 = 8 + 3 + 7
= 18
Value of cut 2 = 4 + 3 + 5
= 12
Value of cut 3 = 7 + 3 + 4
= 14
Value of cut 4 = 7 + 3 + 4
= 14
Value of cut 5 = 8 + 3 + 5
= 16
Value of cut 6 = 8 + 3 + 5
= 16

3 Select the minimum value as the maximum flow.	The value of the minimum cut is 12, therefore the maximum flow through the network has a value of 12.

As can be seen in Worked example 13, ensuring that all cuts have been made is a complicated procedure. The diagram becomes cluttered.

There are two cuts missing from the diagram. Can you find them?

Performing cuts at the source and sink first enables an upper limit for the value of the minimum cut to be set. In part **b**, it was clear that the value of the minimum cut in the diagram must be less than or equal to 12. In this case it was 12.

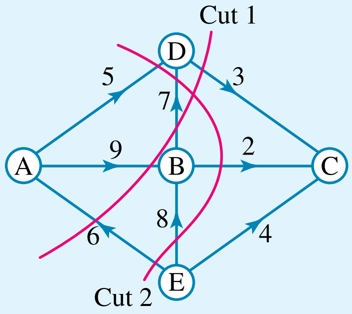


The shaded section represents the inside of cut 1.

In some networks it is possible to produce a cut in which an edge actually heads back *inside* the cut rather than being directed out of the cut, as is required and as all edges in Worked example 13 do. If an edge does this, then its flow value is ignored in the calculation of the cut value. The inside of a cut is the side on which the source node lies. Sometimes the inside of a cut is shaded.

WORKED EXAMPLE 14

Determine the values of the cuts made on the network diagram.



THINK	WRITE
1 Determine the overall direction of the flow.	The flow is from A to C, so the inside of the cut is on the left-hand side where the source node, A, is.
2 Determine the edges that are crossed by the cuts and identify any which head back into the cut.	Cut 1 = {AE, AB, BD, DC} BD heads back inside the cut, so ignore its flow value. Cut 2 = {AD, BD, BC, BE, AE} BE heads back inside the cut, so ignore its flow value. Note that for this cut, BD heads outside the cut.
3 Calculate the cut values.	Cut 1 = 6 + 9 + 3 = 18 Cut 2 = 5 + 7 + 2 + 6 = 20

EXERCISE 10.4 Network flow

PRACTISE

Convert the flow tables for questions **1** and **2** into network diagrams, clearly indicating the direction and quantity of the flow.

1 WE10

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	V	100
U	V	50

2

From	To	Flow capacity
E	F	8
E	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	K	8
H	K	8

3 WE11 From the network in question **1**, determine

a the flow capacity

b whether the flow rate through the network is sufficient to meet the demand.

4 From the network in question **2**, determine

a the flow capacity

b whether the flow rate through the network is sufficient to meet the demand.

For questions **5** and **6**

a introduce new edges, as shown, to the network diagrams produced in questions **1** and **2**

b calculate the new network flow capacities.

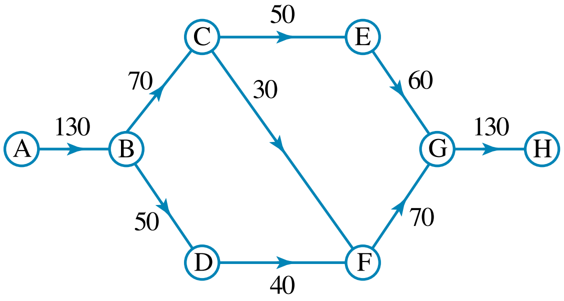
5 WE12

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	V	100
U	V	50
S	T	100

6

From	To	Flow capacity
E	F	8
E	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	K	8
H	K	8
E	K	10

7 WE13 Consider the network flow diagram shown, representing the flow of water between a reservoir (A) and a town (H). The other nodes (B, …, G) represent pumping stations.

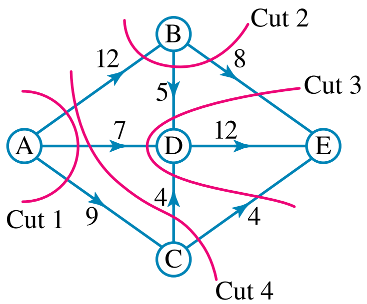


Using the minimum cut method, determine the maximum flow through the network.

8 **a** Which of the four cuts shown in the network at right are invalid?

b Determine the value of each cut.

c What is the minimum cut and hence the maximum flow?

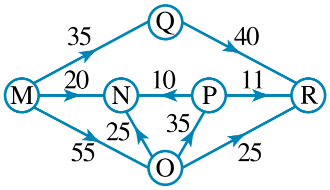


For the networks in questions **9** and **10**:

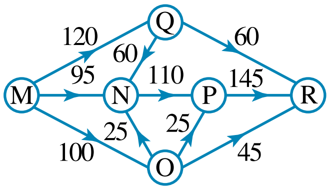
a draw in the line of minimum cut

b state the maximum flow.

9 **WE14**



10



CONSOLIDATE

11 Convert the following flow tables into network diagrams, clearly indicating the direction and quantity of the flow.

a

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300

b

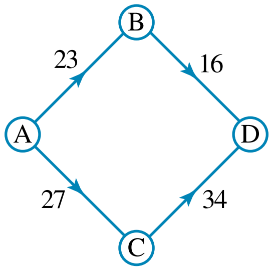
From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	P	12
R	P	12

12 For node B in the network, state:

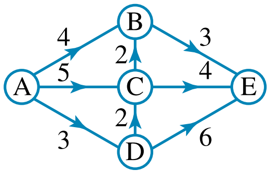
a the inflow at B

b the edge capacities flowing out of B

c the outflow from B.



13 Repeat question **12** for the network shown below.



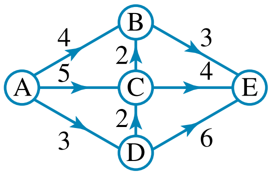
14 For each of the networks in question **11**, determine:

i the flow capacity

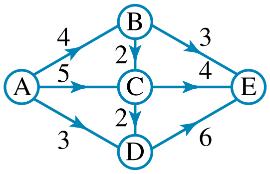
ii whether the flow through the network is sufficient to meet the demand.

15 Convert the following flow diagrams to tables as in question **11**.

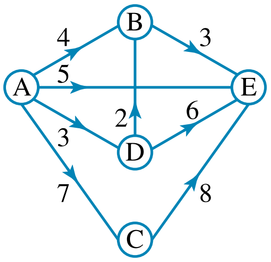
a



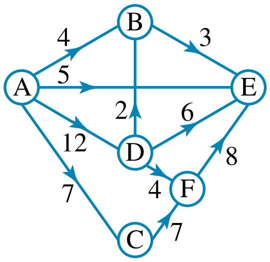
b



c



d



16 Calculate the capacity of each of the networks in question **15**.

17 **i** Introduce new edges, as shown, to each of the network diagrams produced in question **11**.

ii Calculate the new network flow capacities.

a

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300
B	E	100

b

From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	P	12
R	P	12
N	P	5

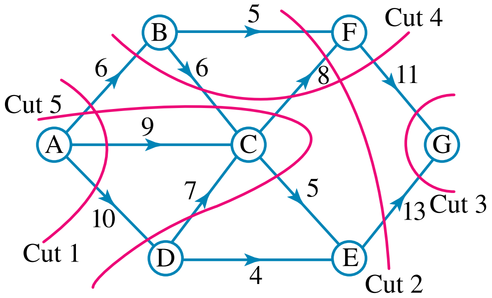
18 In question **17b** the outflow from N is:

A 5 **B** 20 **C** 15 **D** 25 **E** 18

19 **a** Which, if any, of the five cuts in the network diagram are invalid?

b Determine the value of each cut.

c What is the minimum cut and hence the maximum flow?

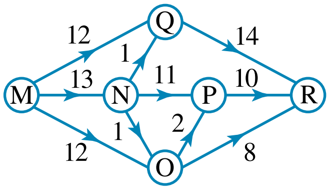


20 For each of the following networks:

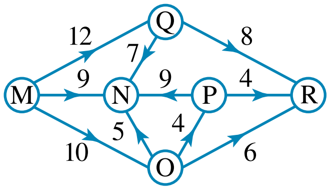
i draw in the line of minimum cut

ii state the maximum flow.

a

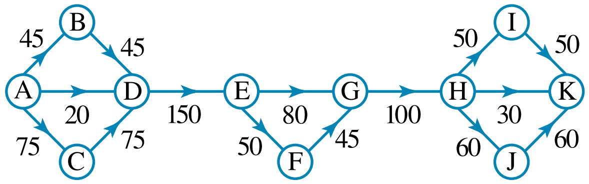


b



MASTER

21 A network diagram of streets and freeways flowing into a city at peak hour is shown. Copy the diagram and clearly label the freeways.



22 Refer to the diagram in question **21** and explain what would happen to the traffic at:

a node E

b node H.

c Add a new edge to the diagram in question **21** so that bottlenecks would not occur at peak hour.

10.5 Assignment problems and bipartite graphs

With network flow problems, the task was to find the maximum flow through a network. Another possible problem could be to determine the *exact* flows through each path in a network, so that as much of the capacity of each edge as possible is used. This is a class of situation called assignment (or allocation) problems. For example, we could be generating electricity at three power plants for distribution to five towns, so that each town gets the required amount of electricity, regardless of the plant it came from. To start this technique we need to define the concept of **bipartite graphs**.

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Bipartite graphs
Concept summary
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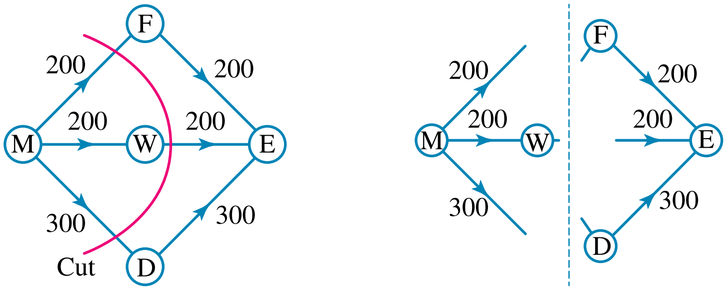
Bipartite graphs

In the previous section on network flow, the minimum cut method was introduced to help solve the flow problem. Consider the cut through the network as dividing the network into two parts: 'flow in and flow out', or 'supply and demand'. Imagine separating the graph along the cut into its two parts. This graph is known as a bipartite graph.

A bipartite graph is one where the nodes can be separated into two types of node — supply and demand.

Consider a typical network flow problem with a cut defined in the figure on the left. By separating the graph, as in the figure on the right, we have divided the graph into two parts. The supply nodes are M and W, while the demand nodes are F, D and E.

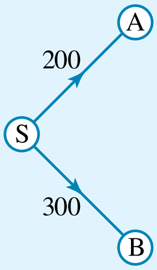
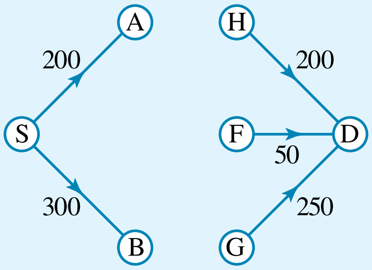


Representing information as bipartite graphs

In many practical situations not only flows, but goods, money and even people can be represented by bipartite graphs.

WORKED EXAMPLE 15

The Shiny Shoe Company make shoes at two factories, one at Alphaville producing 200 pairs per day, the other at Beta City producing 300 pairs per day. It ships them to three distributors: Fartown, who want 50 pairs per day, Giver River, who want 250 pairs per day and Hamlet, who want 200 pairs per day. Represent this information as a bipartite graph.

THINK	WRITE/DRAW
1 Determine all the supply nodes, starting with a single 'supply' node.	Supply nodes are S, A and B.
2 Represent the supply side of the graph.	
3 Determine all the demand nodes, ending with a single 'demand' node.	Demand nodes are F, G, H and D.
4 Add the demand side of the graph.	

The next section concerns the problem of allocating the 'flow' of shoes between the suppliers and the distributors.

Sometimes the supply and demand nodes have quite distinct types. For example, the supply nodes might be students, while the demand nodes could be the subjects that they study, or even the sports they play. The key to bipartite graphs is that there is a separation between the two sides and that there is some sort of flow from supply to demand. The separation does not always have to be literal, as in the next example.

WORKED EXAMPLE 16

The following table lists four students and the four subjects offered in the Science program in Year 12.

a Represent this information as a bipartite graph.

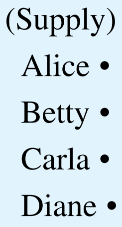
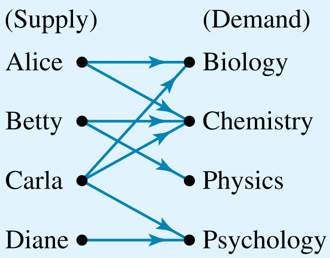
b Determine whether the following statements are true or false.

i Alice studies more Science subjects than Carla does.

ii Between Carla and Betty, all Science subjects are studied.

iii Between Alice and Betty, more subjects are studied than by Carla.

Student	Subjects taken
Alice	Biology, Chemistry
Betty	Chemistry, Physics
Carla	Biology, Chemistry, Psychology
Diane	Psychology

THINK	WRITE/DRAW
a 1 Determine the supply nodes and represent the supply side of the graph. In this case they are (arbitrarily) assigned to the students.	**a** Supply nodes are Alice, Betty, Carla and Diane. 
2 Determine the demand nodes. Represent the demand side and show the connections as arrows joining supply nodes to demand nodes. Note there is no quantity of flow between supply and demand.	Demand nodes are Biology, Chemistry, Physics and Psychology. 
b i 1 Determine the number of subjects studied by Alice and Carla.	**b i** Alice studies two subjects (Biology and Chemistry). Carla studies three subjects (Biology, Chemistry, Psychology).
2 Determine the truth of the statement.	Clearly, the statement is false.
ii 1 Determine the number of subjects studied by Betty and Carla.	**ii** Betty studies Chemistry and Physics. Carla studies Biology, Chemistry and Psychology.
2 Determine the truth of the statement.	Clearly, the statement is true.
iii 1 Determine the subjects studied between Alice and Betty, and Carla.	**iii** Alice and Betty between them study three subjects (Biology, Chemistry and Physics). Carla studies three subjects (Biology, Chemistry, Psychology).
2 Determine the truth of the statement.	Clearly the statement is false (as both groups study three subjects).

Note that in part **b iii**, the fact that Alice and Betty both studied Chemistry does not mean that two (distinct) subjects were studied.

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Topic 2
Concept 5

Optimal allocation
Concept summary
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The allocation problem

Consider a situation where there are a number of jobs to be done and the same number of people to do them. However, each person can do each job in a different amount of time (or at a different cost). How can the jobs be allocated: one per person, so that time (or cost) is minimised? This is known as **optimal allocation**. Sometimes the optimal allocation may be obvious, but in most cases it is not, and may even require some trial and error work.

In the general case the jobs and people can be put in an allocation matrix as in the following worked example.

	Job X	Job Y	Job Z
Alan	10	4	9
Bob	8	11	10
Carl	9	8	7

So, Alan would take 10 hours to do job X, 4 hours to do job Y and 9 hours to do job Z. Similarly, Bob would take 8 hours to do job X, … and so on.

The first step is to perform row reduction by subtracting the smallest value in each row from all the numbers in that row. This may produce the optimal allocation in one step.

This is the simplest possible case: the optimal allocation is almost obvious. Remember to calculate the total time (or cost) and ensure it is indeed the minimum.

WORKED EXAMPLE 17

A building site has three more jobs to be done by the three remaining workers, Alan, Bob and Carl. The times taken by each to do the three jobs are given in the following table. Determine the optimal allocation and hence state the minimum time.

	Job X	Job Y	Job Z
Alan	10	4	9
Bob	8	11	10
Carl	9	8	7

THINK	WRITE/DRAW
1 Generate or write the matrix of people against job times.	$\begin{array}{c\|ccc} & X & Y & Z \\ \hline A & 10 & 4 & 9 \\ B & 8 & 11 & 10 \\ C & 9 & 8 & 7 \end{array}$
2 Step 1a. Perform row reduction. Find the smallest value in each row, and subtract it from all numbers in that row.	The smallest number in row A is 4. The smallest number in row B is 8. The smallest number in row C is 7.
3 Subtract 4 from each number in row A. Subtract 8 from each number in row B. Subtract 7 from each number in row C.	$\begin{array}{c\|ccc} & X & Y & Z \\ \hline A & 6 & 0 & 5 \\ B & 0 & 3 & 2 \\ C & 2 & 1 & 0 \end{array}$

4 Step 1b. Attempt an allocation.

Cover all the zeroes with the smallest number of straight (horizontal/vertical, but not diagonal) lines. If the number of lines equals the number of jobs to be allocated, go to step 2.

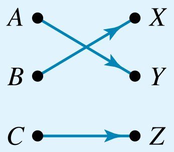
If not, further steps to be taken will be shown in another worked example.

	X	Y	Z
A	~~6~~	~~0~~	~~5~~
B	~~0~~	~~3~~	~~2~~
C	~~2~~	~~1~~	~~0~~

In this case there are three lines and three jobs.

5 Step 2a. Produce a bipartite graph.

Show *all* possible allocations, where there are zeroes connecting people to jobs.



6 Step 2b. List the possible allocations and determine the smallest total.

There is only 1 possible allocation:

$A \rightarrow Y$, $B \rightarrow X$, $C \rightarrow Z$

$$\begin{aligned}\text{Total time} &= 4 + 8 + 7\\ &= 19 \text{ hours}\end{aligned}$$

Hence, the minimum time is 19 hours.

study on

Unit 4

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Concept 6

Hungarian algorithm

Concept summary

Practice questions

The Hungarian algorithm

Unfortunately, there are two cases where the basic algorithm will not produce the optimal allocation:

1. when there are not enough zeros to draw the right number of straight lines in step 1b
2. when there are too many zeros so that one person may be allocated more than one job in step 2a.

In either case, we need to use the **Hungarian algorithm** which involves the following steps, replacing step 2 of the previous example.

Step 2. Perform column reduction (when there are not enough zeros). This is similar to row reduction and is performed when step 1 has not produced an optimal allocation. This may produce an optimal allocation using the method of steps 1b and 1c. If not, proceed to step 3.

Step 3. Modify the original matrix according to the Hungarian algorithm. This involves an addition and subtraction operation involving various quantities in the matrix from step 2 and will be detailed in the worked example below. This should produce an optimal allocation.

Step 4. Produce a bipartite graph and list all possible allocations.

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Interactivity

The allocation problem and the Hungarian algorithm

int-6289

WORKED EXAMPLE 18

Amy, Beth, Cate and Dana offer quotes on how much each of them will charge to complete four different jobs, *P*, *Q*, *R* and *S*. The table summarises these charges (in dollars). Use the Hungarian algorithm to minimise the total cost to complete the four jobs, one job per person.

	P	Q	R	S
Amy	17	24	42	21
Beth	25	18	19	20
Cate	29	14	31	22
Dana	11	20	17	14

THINK **WRITE/DRAW**

1 Step 1a. Perform row reduction.
Row A: Subtract 17
Row B: Subtract 18
Row C: Subtract 14
Row D: Subtract 11

$$\begin{array}{c|cccc} & P & Q & R & S \\ \hline A & 0 & 7 & 25 & 4 \\ B & 7 & 0 & 1 & 2 \\ C & 15 & 0 & 17 & 8 \\ D & 0 & 9 & 6 & 3 \end{array}$$

2 Step 1b. Attempt an allocation.
Draw the minimum number of straight lines to cover all the zeros.

$$\begin{array}{c|cccc} & P & Q & R & S \\ \hline A & 0 & 7 & 25 & 4 \\ B & 7 & 0 & 1 & 2 \\ C & 15 & 0 & 17 & 8 \\ D & 0 & 9 & 6 & 3 \end{array}$$

Only 2 lines — cannot continue allocation.

3 Step 2a. Perform column reduction.
Column P: Subtract 0 (do nothing)
Column Q: Subtract 0 (do nothing)
Column R: Subtract 1
Column S: Subtract 2

$$\begin{array}{c|cccc} & P & Q & R & S \\ \hline A & 0 & 7 & 24 & 2 \\ B & 7 & 0 & 0 & 0 \\ C & 15 & 0 & 16 & 6 \\ D & 0 & 9 & 5 & 1 \end{array}$$

4 Step 2b. Attempt an allocation.
Draw the minimum number of straight lines to cross out all the zeros.

$$\begin{array}{c|cccc} & P & Q & R & S \\ \hline A & 0 & 7 & 24 & 2 \\ B & 7 & 0 & 0 & 0 \\ C & 15 & 0 & 16 & 6 \\ D & 0 & 9 & 5 & 1 \end{array}$$

Only 3 lines are required — cannot continue allocation.

5 Step 3. Perform the Hungarian algorithm.

The smallest uncovered number = 1.

Step 3a. Find the smallest uncovered number from step 2b.

Step 3b. Add this number to all covered numbers in the matrix from step 2b. At the intersections of straight lines, add this number twice (circled at right).

$$\begin{array}{c|cccc} & P & Q & R & S \\ \hline A & 1 & 8 & 24 & 2 \\ B & \textcircled{9} & \textcircled{2} & 1 & 1 \\ C & 16 & 1 & 16 & 6 \\ D & 1 & 10 & 5 & 1 \end{array}$$

6 Step 3c. Subtract the overall smallest number from all the numbers in the matrix.

$$\begin{array}{c|cccc} & P & Q & R & S \\ \hline A & 0 & 7 & 23 & 1 \\ B & 8 & 1 & 0 & 0 \\ C & 15 & 0 & 15 & 5 \\ D & 0 & 9 & 4 & 0 \end{array}$$

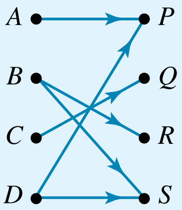
The smallest overall number = 1.

Step 3d. Attempt an allocation. Repeat step 3 until a possible allocation is found.

$$\begin{array}{c|cccc} & P & Q & R & S \\ \hline A & 0 & 7 & 23 & 1 \\ B & 8 & 1 & 0 & 0 \\ C & 15 & 0 & 15 & 5 \\ D & 0 & 9 & 4 & 0 \end{array}$$

In this case there are 4 lines and 4 jobs, so go to step 4.

Step 4a. Produce a bipartite graph. Show all possible allocations, where there are zeros connecting people to jobs.



7 Step 4b. List possible allocations and determine the smallest total.

- By inspection, Cate must be allocated *Q* (she's the only one at *Q*).
- Beth must be allocated *R* (she's the only one at *R*).
- Therefore Dana must be allocated *S* (she's the only one left at *S*).
- Therefore Amy must be allocated *P*.

Amy → *P*, Beth → *R*, Cate → *Q*, Dana → *S*

Minimum cost = 17 + 19 + 14 + 14
= \$64

(numbers from the original matrix in the question)

Sometimes the optimal allocation is carried out to maximise a quantity such as score. In this process, all elements in the matrix are subtracted from the largest one first, and in doing so, modifying the situation to a minimisation problem. From then on, the procedure is exactly the same as that set out in Worked example 18.

EXERCISE 10.5 Assignment problems and bipartite graphs

PRACTISE

1 **WE15** An electricity company produces 4000 kWh, 5000 kWh and 6000 kWh at its three hydroelectric plants. This electricity is supplied to four towns as follows: 20% to Town A, 25% to Town B, 15% to Town C and the rest to Town D. Represent this information with a bipartite graph.

2 An oil company supplies petrol to three towns, A, B and C. Within town A there are two sub-depots where petrol is stored temporarily before being delivered to service stations. The other two towns have only a single depot. Town A gets 30 000 and 40 000 litres per week for each of its sub-depots, while town B gets 10 000 litres and town C gets 25 000 litres for its depots. However, the demand for petrol is as follows; town A: 60 000, town B: 15 000 and town C: 30 000 litres.

a Represent this information in a bipartite graph.

b Suggest a delivery system so that all towns get the required amount of petrol each week.

3 **WE16** Five diners (Albert, Brian, Chris, David and Earl) go to the pub for dinner and place the orders as shown in the table. Represent this information as a bipartite graph.

Diner	Dishes
Albert	Soup, fish
Brian	Fish
Chris	Soup, beef, desert
David	Beef, dessert
Earl	Fish, dessert

4 From the information in question 3, which of the following statements is true?

- **A** Albert and Brian between them have more kinds of dishes than Chris and David.
- **B** Chris and David between them have tried all the dishes.
- **C** David and Earl between them have more kinds of dishes than Brian and Chris.
- **D** Brian and Chris between them have more kinds of dishes than David and Earl.
- **E** All the above statements are false.

For questions 5 and 6, perform row reduction on the matrices, which represent times (in hours), and attempt an optimal allocation for the minimum time. State the minimum time.

5 **WE17** $\begin{bmatrix} 6 & 3 & 7 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{bmatrix}$

6 $\begin{bmatrix} 4 & 3 & 7 & 3 \\ 9 & 4 & 6 & 5 \\ 5 & 6 & 7 & 8 \\ 4 & 8 & 3 & 5 \end{bmatrix}$

For questions 7 and 8, perform an optimal allocation on the matrices by:

1. row reduction
2. column reduction
3. the Hungarian algorithm.

State the minimum total allocation and at which stage each matrix was solved.

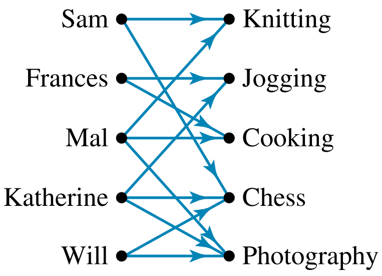
7 **WE18** $\begin{bmatrix} 6 & 9 & 9 & 4 \\ 10 & 9 & 9 & 7 \\ 4 & 9 & 6 & 3 \\ 5 & 8 & 8 & 6 \end{bmatrix}$

8 $\begin{bmatrix} 5 & 23 & 19 & 4 \\ 11 & 29 & 6 & 14 \\ 21 & 17 & 14 & 13 \\ 20 & 27 & 22 & 8 \end{bmatrix}$

CONSOLIDATE

9 A publisher produces 1000 copies per month of the latest bestseller by Wolf Thomas at two factories. The first factory produces 400 copies. The books are then distributed to two states (Queensland and Victoria), with Queensland requiring 350 copies and Victoria getting the rest. Represent this information as a bipartite graph.

10 Five students have various hobbies as indicated by the bipartite graph.



From this graph it can be said that:

- **A** Sam and Mal have all the hobbies between them
- **B** Mal and Frances, in total, have more hobbies than Sam and Will
- **C** Mal and Sam each have the same number of hobbies
- **D** Katherine and Sam, in total, have fewer hobbies than Frances and Mal
- **E** Will had fewer hobbies than all other students

11 Perform row reduction on the matrix, which represents times (in hours), and attempt an optimal allocation for the minimum time. State the minimum time.

$$\begin{bmatrix} 16 & 14 & 20 & 13 \\ 15 & 16 & 17 & 16 \\ 19 & 13 & 13 & 18 \\ 22 & 26 & 20 & 24 \end{bmatrix}$$

12 Draw the bipartite graph from question **11**.

13 A mother wishes to buy presents (a game, a doll and a toy truck) for her three children from three different stores. The game's cost is (\$30, \$45, \$40) from the three stores, the doll's cost is (\$50, \$50, \$60) while the truck's cost is (\$35, \$30, \$30). Show that the optimal allocation yields a total of \$110.

14 Given the matrix shown, the total value of the optimal allocation is:

A 9
B 11
C 15
D 16
E 19

$$\begin{bmatrix} 7 & 3 & 7 \\ 3 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix}$$

15 A government department wishes to purchase five different cars for its fleet. For reasons of fairness it must not purchase more than one car from any dealer. It receives quotes from the dealers according to the following table (cost in thousands of dollars).

a Perform row and column reduction.
b Draw the bipartite graph after reduction.
c i Find the optimal allocation.
ii State the minimum cost.

$$\left[\begin{array}{l|ccccc} & \text{D1} & \text{D2} & \text{D3} & \text{D4} & \text{D5} \\ \hline \text{Car 1} & 20 & 15 & 17 & 16 & 18 \\ \text{Car 2} & 17 & 15 & 19 & 17 & 16 \\ \text{Car 3} & 18 & 19 & 16 & 19 & 16 \\ \text{Car 4} & 19 & 19 & 17 & 21 & 17 \\ \text{Car 5} & 24 & 19 & 17 & 17 & 17 \end{array}\right]$$

16 Perform an optimal allocation on the following matrices by:

1. row reduction
2. column reduction
3. the Hungarian algorithm.

State the minimum total allocation and at which stage each matrix was solved.

a
$$\begin{bmatrix} 10 & 15 & 12 & 17 \\ 17 & 21 & 19 & 14 \\ 16 & 22 & 17 & 19 \\ 23 & 26 & 29 & 27 \end{bmatrix}$$

b
$$\begin{bmatrix} 12 & 10 & 11 & 13 & 11 \\ 11 & 11 & 13 & 12 & 12 \\ 12 & 16 & 13 & 16 & 12 \\ 9 & 10 & 9 & 11 & 9 \\ 14 & 11 & 11 & 11 & 11 \end{bmatrix}$$

17 A company wishes to hire four computer programmers (A, B, C and D) to develop four software packages. Because of time constraints each programmer can accept at most one job. The quotes (in weeks) given by the four programmers are shown in the following table.

a Perform row (and if necessary, column) reduction.

b Perform the Hungarian algorithm (if necessary).

c Display possible allocations using a bipartite graph.

d Determine:

i the optimal allocation

ii the total time required to complete the four tasks.

	Job 1	Job 2	Job 3	Job 4
A	30	40	50	60
B	70	30	40	70
C	60	50	60	30
D	20	80	50	70

18 Four workers (Tina, Ursula, Vicky, Wendy) need to be optimally allocated four tasks (one per worker). The times (in minutes) that each worker takes are shown in the table.

	Job 1	Job 2	Job 3	Job 4
T	100	50	35	55
U	60	45	70	55
V	40	70	50	30
W	70	50	70	70

Perform an optimal allocation stating:

a the tasks allocated to each worker

b the total time required to complete all tasks.

MASTER

19 David Lloyd George High School will be competing in a Mathematics competition. There are four categories: Algebra, Calculus, Functions and Geometry and four possible competitors: Ken, Louise, Mark and Nancy — one per category. To determine the best competitor for each category, a test is given and the results (as percentages) for each student recorded.

	Algebra	Calculus	Functions	Geometry
Ken	60	78	67	37
Louise	45	80	70	90
Mark	60	35	70	86
Nancy	42	66	54	72

a Determine the row-reduced matrix.

b Determine the column-reduced matrix.

c Determine the optimal allocation using a bipartite graph.

d Determine the average score for the team.

20 In a Meals-on-Wheels program in a remote region of Victoria, there are four elderly people (P1, P2, P3, P4) who require meals to be delivered. There are four volunteers (V1, V2, V3, V4) who live at varying distances (specified in kilometres) away.

	P1	P2	P3	P4
V1	13	17	14	23
V2	8	12	17	9
V3	9	17	14	11
V4	21	16	13	14

a Determine the optimum allocation.

b State the total distance travelled given that each volunteer delivers exactly one meal.

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10.6 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

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- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
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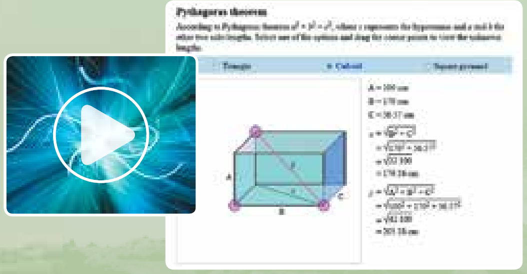

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10 Answers

EXERCISE 10.2

1 30 minutes

2 147 days

3 10

4 58 days

5 a Critical path = A–B–L–M–N

b Float(D) = 30 – 19 – 10 = 1

Float(C) = 20 – 15 – 4 = 1

Float(H) = 30 – 11 – 10 = 9

Float(F) = 17 – 5 – 6 = 6

Float(G) = 21 – 11 – 4 = 6

Float(E) = 11 – 0 – 5 = 6

Float(K) = 21 – 5 – 8 = 8

Float(J) = 15 – 5 – 4 = 6

6 Float(C) = 25 – 6 – 11 = 8 days

Float(E) = 42 – 17 – 17 = 8 days

Float(F) = 90 – 41 – 23 = 26 days

Float(H) = 61 – 34 – 19 = 8 days

Float(J) = 93 – 34 – 21 = 38 days

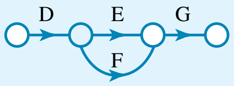
Float(K) = 115 – 64 – 25 = 26 days

Float(M) = 117 – 55 – 24 = 38 days

Float(N) = 115 – 88 – 26 = 1 day

Float(Q) = 147 – 114 – 32 = 1 day

7



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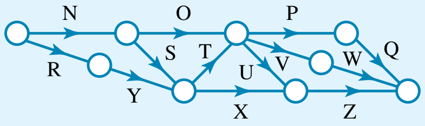


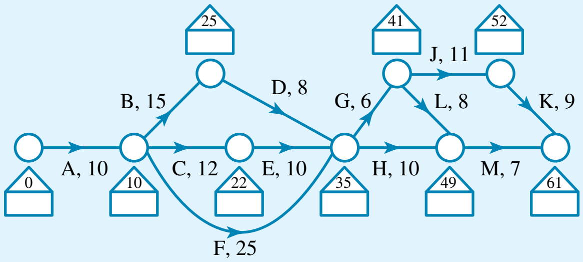
9 a B **b** D

10 a 23 minutes **b** B, C, E, F, G

11 A–C–F

12 a E **b** D **c** A

13 a



b 61 minutes

14

Activity letter	Immediate predecessor	Time
A	—	7
B	—	9
C	A	12
D	B	8
E	B	4
F	C, D	9

15

Activity letter	Immediate predecessor	Time
A	—	3
B	—	4
C	—	5
D	A	6
E	B, F	5
F	C	8
G	D	18
H	E	8
J	E	6

16

Activity letter	Immediate predecessor	Time
A	—	10
B	A	15
C	A	12
D	B	8
E	C	10
F	A	25
G	D, E, F	6
H	D, E, F	10
J	G	11
K	J	9
L	G	8
M	H, L	7

17 a A–D–G

b Float (H) = 1, Float (J) = 3,

Float (E) = 1,

Float (B) = 10, Float (C) = 1,

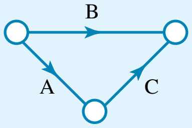
Float (F) = 1

c Activity B can be delayed 10 minutes,
activity C can be delayed 1 minute,
activity E can be delayed 1 minute,
activity F can be delayed 1 minute,
activity H can be delayed 1 minute,
activity J can be delayed 3 minutes.

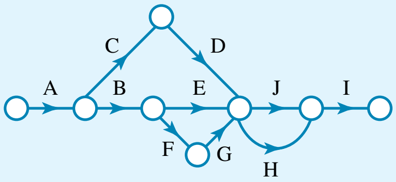
18 a A–F–G–J–K

b M, L, H, C, E, B, D

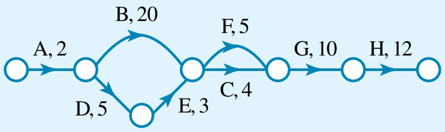
19 a



b



20 a



b 49 minutes

EXERCISE 10.3

1 Critical path = B–F–G; Float (C) = 3 h,
Float (E) = 5 h, Float (A) = 3 h,
Float (D) = 5 h

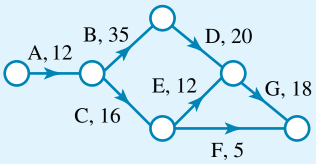
2 a The critical path is D–F–G–H–J.

b Float(K) = 8 – 5.5 – 2 = 0.5 hours
Float(C) = 5.5 – 3 – 1 = 1.5 hours
Float(B) = 4.5 – 1 – 2 = 1.5 hours
Float(A) = 2.5 – 0 – 1 = 1.5 hours
Float(E) = 5.5 – 0 – 3.5 = 2 hours

3 a

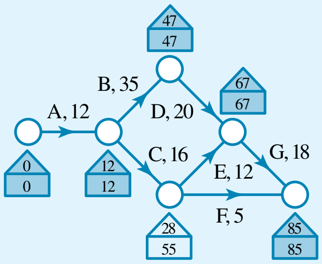
Activity letter	Activity	Immediate predecessor	Time
A	Collect parts	—	12
B	Paint frame	A	35
C	Assemble brakes	A	16
D	Assemble gears	B	20
E	Install brakes	C	12
F	Install seat	C	5
G	Final assembly	D, E	18

b



c 85 minutes

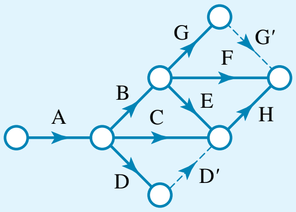
d



Critical path = A–B–D–G

4 C

5



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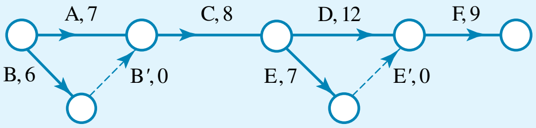

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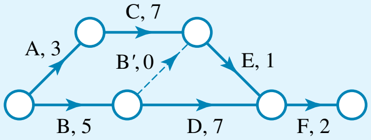

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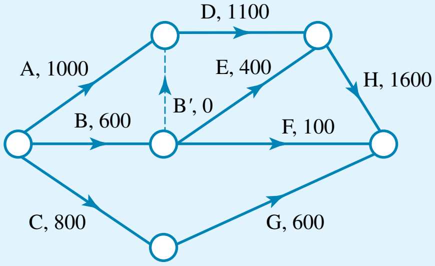


9 a 36 minutes

b A–C–D–F

c Float(E) = 27 – 15 – 7 = 5 minutes
Float(B) = 7 – 0 – 6 = 1 minute

10 A–B–D–G–L–P–R.

11 Critical path = B–D; Float (E) = 1 min,
Float (C) = 1 min, Float (A) = 1 min

12 a 31 days

b Critical path = A–C–E–G

c The completion time is reduced to 28 days.
The critical path is not affected.

13 D

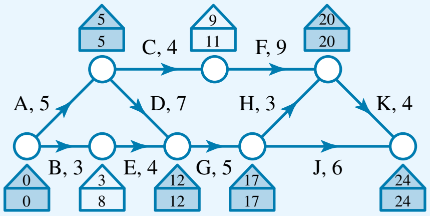
14 a, d

Activity letter	Immediate predecessor	Time	Float time
A	—	3	2
B	—	4	0
C	—	6	2
D	A	7	2
E	B	8	0
F	B	5	4
G	C	12	2
H	C	2	3
J	D, E	11	0
K	F	10	4
L	G, M	3	2
M	H	9	3
N	J, K, L	6	0

b 29

c B–E–J–N

15 a



b 24 hours

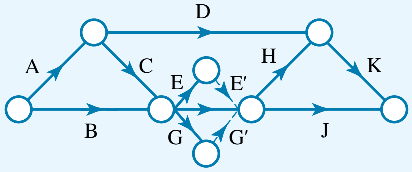
c A–D–G–H–K

d Float (B) = 5, Float (E) = 5,

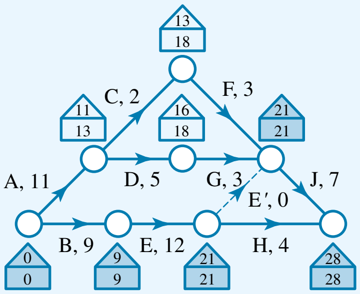
Float (C) = 2,

Float (F) = 2, Float (J) = 1

16



17 a



b 28 h

c B–E–E′–J

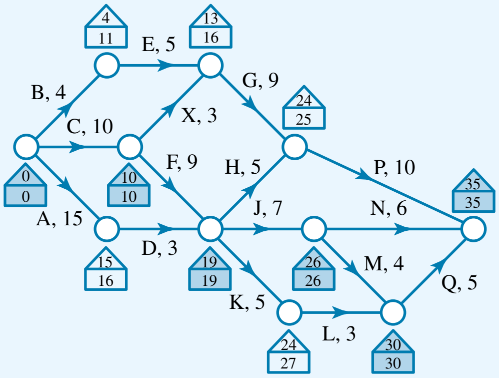
d Float (A) = 2, Float (C) = 5,

Float (F) = 5, Float (D) = 2,

Float (G) = 2, Float (H) = 3

18 a 35 days

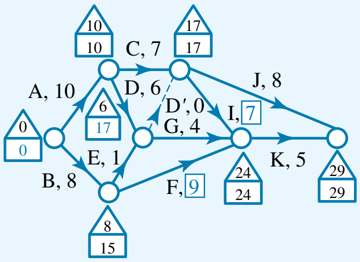
b C–F–J–M–Q



c 3 days

d When J is reduced to 5 days, the earliest completion time is reduced to 34 days. The critical path becomes C–F–H–P.

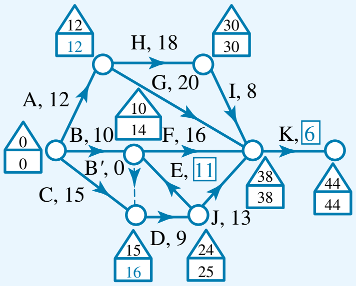
19 a



b A–C–I–K

c Float (F) = 7

20 a

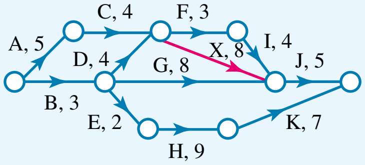


b A–H–I–K

c Float (F) = 12

21 a C: A; F: C, D; EST G = 3; EST K = 14

b



c A–C–X–J

d Latest start time for H = 6

22 a 5 weeks

b 15 weeks

c A–E–F–I

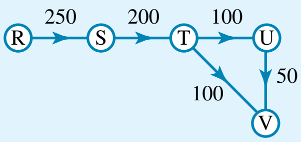
d 3 weeks

e A, E, F, I

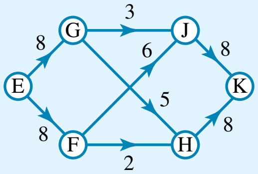
f C–D–I, 14 weeks

EXERCISE 10.4

1



2



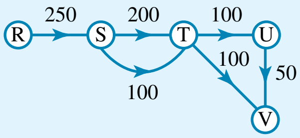
3 a 150

b Yes

4 a 15

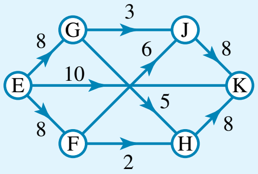
b No

5 a



b 150

6 a



b 25

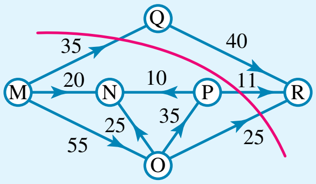
7 Minimum cut = maximum flow = 110

8 a Cut 2

b Cut 1 = 28, cut 3 = 28, cut 4 = 27

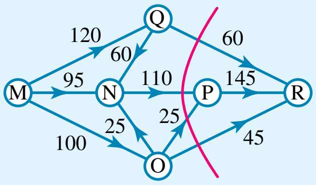
c Maximum flow = 24

9 a



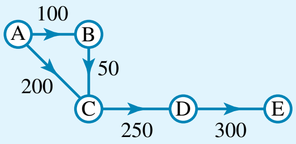
b Maximum flow = 71

10 a

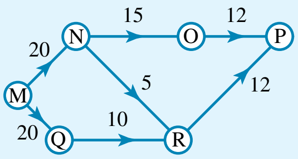


b Maximum flow = 240

11 a



b



12 a 23

b 16

c 16

13 a 6

b 3

c 3

14 a i 250

ii No

b i 24

ii Yes

15 a

From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
C	B	2
C	E	4
D	C	2
D	E	6

b

From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
B	C	2
C	E	4
D	C	2
D	E	6

c

From	To	Flow capacity
A	B	4
A	C	7
A	D	3
A	E	5
B	E	3
C	E	8
D	B	2
D	E	6

d

From	To	Flow capacity
A	B	4
A	C	7
A	D	12
A	E	5
C	F	7
D	B	2
D	E	6
D	F	4
F	E	8

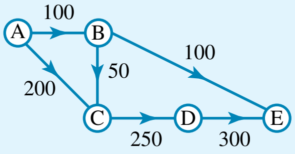
16 a 10

b 10

c 18

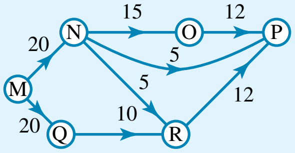
d 22

17 a i



ii 300

b i



ii 29

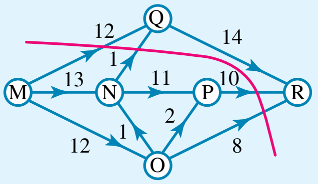
18 B

19 a Cut 4

b Cut 1 = 25, cut 2 = 26, cut 3 = 24, cut 5 = 29

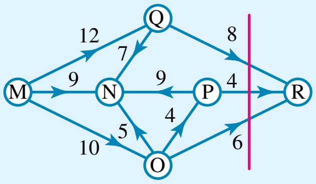
c Maximum flow = 20

20 a i



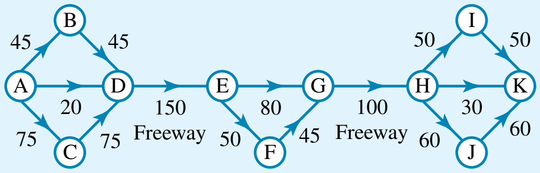
ii Maximum flow = 31

b i



ii Maximum flow = 18

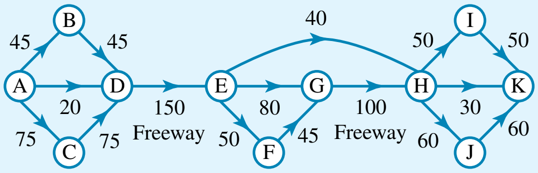
21



22 a There would be a traffic jam.

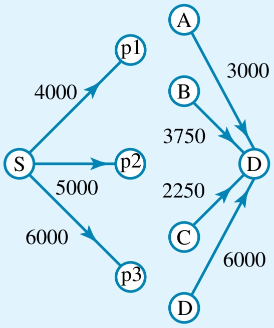
b The traffic should flow smoothly as the inflow is less than the capacity of flows leading from H.

c

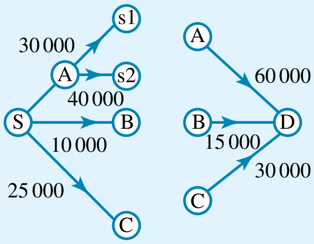


EXERCISE 10.5

1

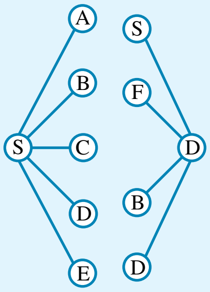


2 a



b Send 30 000 from S1 to A, 30 000 from S2 to A, 10 000 from S2 to B, 5000 from B to B, 5000 from B to C and 25 000 from C to C. (This may not be the cheapest method.)

3



4 D

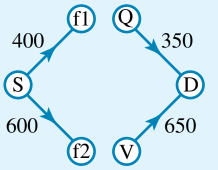
5 7

6 15

7 24 — Hungarian algorithm

8 36 — column reduction

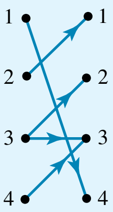
9



10 B

11 61

12

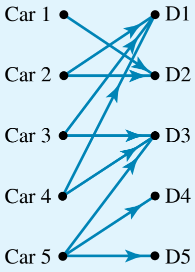


13 Answers will vary.

14 B

15 a $$\begin{bmatrix} 3 & 0 & 2 & 1 & 3 \\ 0 & 0 & 4 & 2 & 1 \\ 0 & 3 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 5 & 2 & 0 & 0 & 0 \end{bmatrix}$$

b



c **i** C1 → D2, C2 → D1, C5 → D4, C3 → D3, C4 → D5 or C1 → D2, C2 → D1, C5 → D4, C3 → D5, C4 → D3

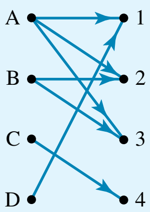
ii Total = \$82 000

16 a 67 — column reduction **b** 53 — row reduction

17 a $$\begin{bmatrix} 0 & 10 & 10 & 30 \\ 40 & 0 & 0 & 40 \\ 30 & 20 & 20 & 0 \\ 0 & 60 & 20 & 50 \end{bmatrix}$$

b $$\begin{bmatrix} 0 & 0 & 0 & 20 \\ 50 & 0 & 0 & 40 \\ 40 & 20 & 20 & 0 \\ 0 & 50 & 10 & 40 \end{bmatrix}$$ (one possible result)

c



d **i** A → 2, B → 3, C → 4, D → 1 or A → 3, B → 2, C → 4, D → 1

ii Total = 130 minutes

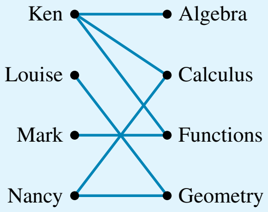
18 a T → 3, U → 1, V → 4, W → 2

b 175 minutes

19 a $$\begin{array}{c|cccc} & A & C & F & G \\ \hline K & 18 & 0 & 11 & 41 \\ L & 45 & 10 & 20 & 0 \\ M & 26 & 51 & 16 & 0 \\ N & 30 & 6 & 18 & 0 \end{array}$$

b $$\begin{array}{c|cccc} & A & C & F & G \\ \hline K & 0 & 0 & 0 & 41 \\ L & 27 & 10 & 9 & 0 \\ M & 8 & 51 & 5 & 0 \\ N & 12 & 6 & 7 & 0 \end{array}$$

c



Ken → Algebra, Louise → Geometry, Mark → Functions, Nancy → Calculus

d 71.5%

20 a V1 → P2, V2 → P4, V3 → P1, V4 → P3 or V1 → P3, V2 → P4, V3 → P1, V4 → P2

b Total = 48 km

11

Geometry: similarity and mensuration

11.1 Kick off with CAS

Calculating areas with CAS

We can use CAS to define formulas which allow us to quickly and efficiently calculate the areas of different shapes.

1 Use CAS to define and save the formula for calculating the area of a trapezium ($A = \frac{1}{2}(a + b) \times h$).

2 Use your formula to calculate the area of trapeziums with the following lengths and heights.

a $a = 7$ cm, $b = 15$ cm, $h = 11$ cm

b $a = 7.8$ cm, $b = 9.4$ cm, $h = 4$ cm

3 Use your formula to calculate the height of trapeziums with the following areas and lengths.

a $A = 63\text{ cm}^2$, $a = 6$ cm, $b = 12$ cm

b $A = 116.25\text{ mm}^2$, $a = 13$ mm, $b = 18$ mm

4 Use CAS to define and save the formula for calculating the area of a circle ($A = \pi r^2$).

5 Use your formula to calculate the area of circles with the following radii, correct to 2 decimal places.

a Radius = 4 cm

b Radius = 7.5 cm

6 Use your formula to calculate the radii of circles with the following areas, correct to 1 decimal place.

a Area = 181.5 cm^2

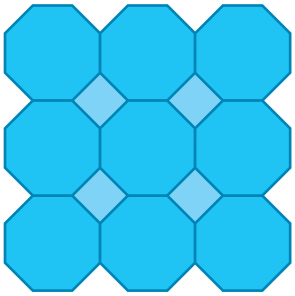
b Area = 47.8 m^2

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

11.2 Properties of angles, triangles and polygons

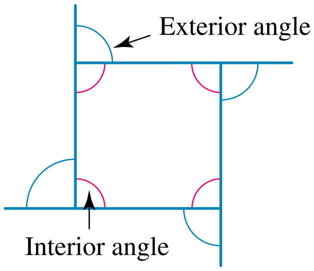
Geometry

Geometry is an important area of study. Many professions and tasks require and use geometrical concepts and techniques. Besides architects, surveyors and navigators, all of us use it in our daily lives — for example, to describe shapes of objects, directions on a car trip and space or position of a house. Much of this area of study is assumed knowledge gained from previous years of study.



In this module, we will often encounter problems in which some of the information we need is not clearly given. To solve the problems, some missing information will need to be deduced using the many common rules, definitions and laws of geometry. Some of the more important rules are presented in this topic.

Interior angles of polygons



For a regular polygon (all sides and angles are equal) of n sides, the interior angle is given by $180° - \frac{360°}{n}$.

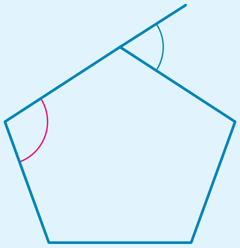
For example, for a square the interior angle is:

$$180° - \frac{360°}{4} = 180° - 90°$$
$$= 90°.$$

The exterior angle of a regular polygon is given by $\frac{360°}{n}$.

WORKED EXAMPLE 1

Calculate the interior and exterior angle of the regular polygon shown.

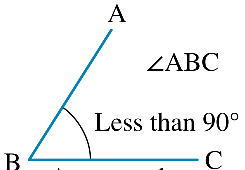


THINK	WRITE
1 This shape is a regular pentagon, a 5-sided figure. Substitute $n = 5$ into the interior angle formula.	Interior angle $= 180° - \frac{360°}{5}$ $= 180° - 72°$ $= 108°$
2 Substitute $n = 5$ into the exterior angle formula.	Exterior angle $= \frac{360°}{5}$ $= 72°$
3 Write your answer.	A regular pentagon has an interior angle of 108° and an exterior angle of 72°.

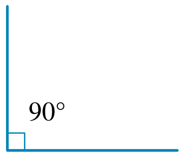
Geometry rules, definitions and notation rules

The following geometry rules and notation will be most valuable in establishing unknown values in the topics covered and revised in this module.

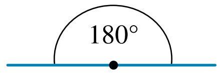
Definitions of common terms



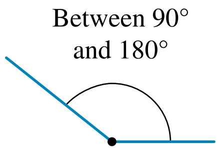
Acute angle



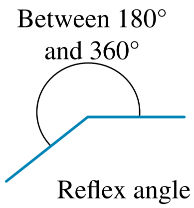
Right angle



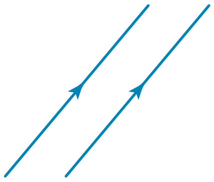
Straight angle



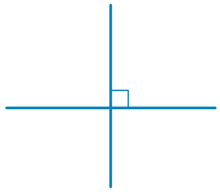
Obtuse angle



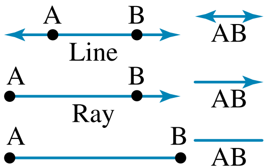
Reflex angle



Parallel lines

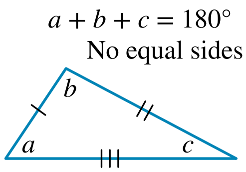


Perpendicular lines

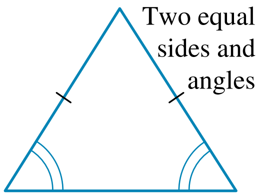


Line segment

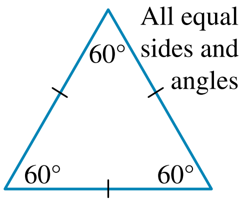
Some common notations and rules



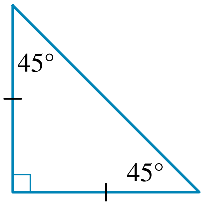
Scalene triangle



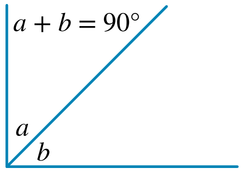
Isosceles triangle



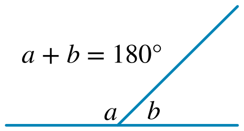
Equilateral triangle



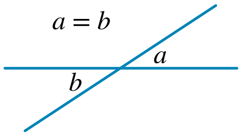
Right-angled isosceles triangle



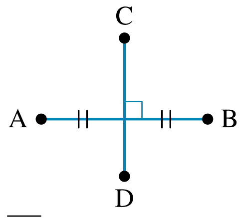
Complementary angles



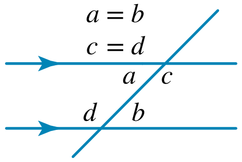
Supplementary angles



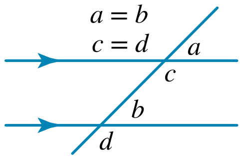
Vertically opposite angles



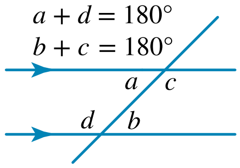
$\overline{CD}$ is a perpendicular bisector of $\overline{AB}$



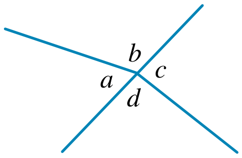
Alternate angles



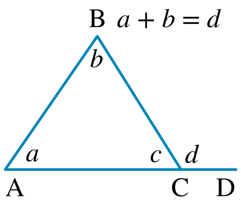
Corresponding angles



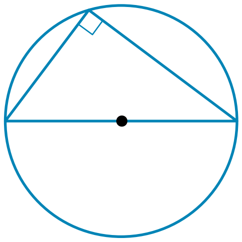
Co-interior angles



$a + b + c + d = 360°$



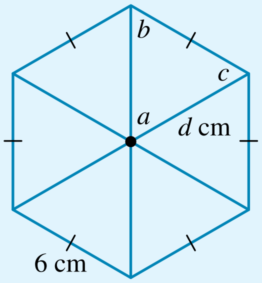
∠BCD is an exterior angle.



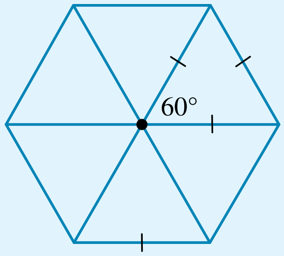
Right angle at the circumference in a semicircle

WORKED EXAMPLE 2

Calculate the values of the pronumerals in the polygon shown.

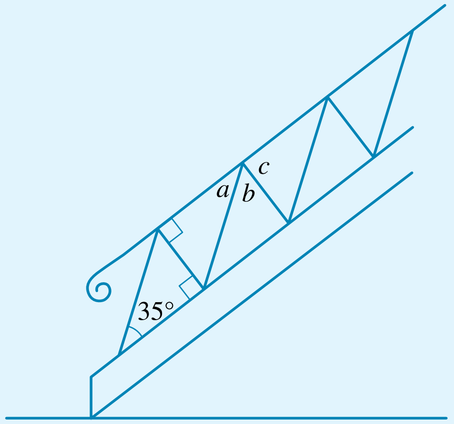


THINK	**WRITE**
1 This shape is a regular hexagon. The angles at the centre are all equal.	$a = \frac{360°}{6}$ $= 60°$
2 The other two angles in the triangle are equal.	$a + b + c = 180°$ $b = c$ So: $60 + 2b = 180°$ $b = 60°$ $c = 60°$
3 The 6 triangles are equilateral triangles; therefore, all sides are equal.	$d \text{ cm} = 6 \text{ cm}$

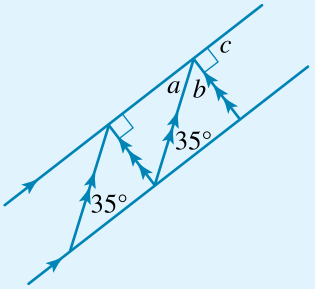


WORKED EXAMPLE 3

Calculate the missing pronumerals in the diagram of railings for the set of stairs shown.



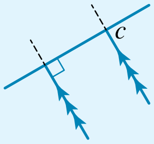
THINK	**WRITE/DRAW**
1 Recognise that the top and bottom of the stair rails are parallel lines.	



2 To find the unknown angle a, use the alternate angle law and the given angle.

Given angle = 35°
$a = 35°$

3 Using the corresponding angle law and the given right angle, recognise that the unknown angle c is a right angle.



$c = 90°$

4 Use the straight angle rule to find the unknown angle b.

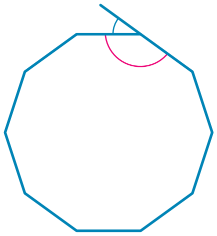
$$\begin{aligned} a + b + c &= 180° \\ 35° + b + 90° &= 180° \\ b &= 180° - 125° \\ &= 55° \end{aligned}$$

EXERCISE 11.2 Properties of angles, triangles and polygons

PRACTISE

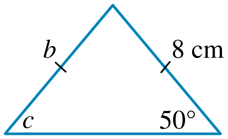
1 **WE1** Calculate the interior and exterior angles for a regular nonagon.

2 Calculate the interior and exterior angles for the following regular polygon.

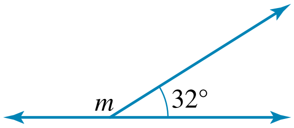


For questions **3**–**6**, calculate the value of the pronumeral in the following figures.

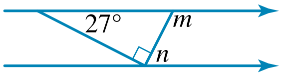
3 **WE2**



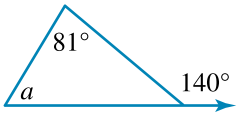
4



5 **WE3**



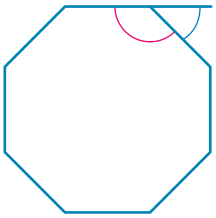
6



CONSOLIDATE

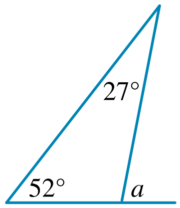
7 Calculate the interior and exterior angles for each of the following regular polygons.

- **a** Equilateral triangle
- **b** Regular quadrilateral
- **c** Hexagon
- **d**
- **e** Heptagon

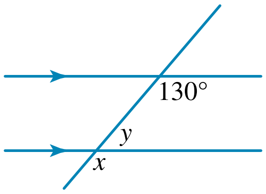


8 Calculate the value of the pronumerals in the following figures.

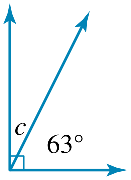
a



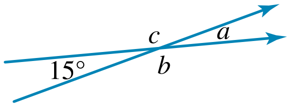
b



c

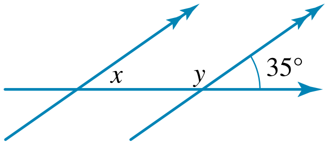


d

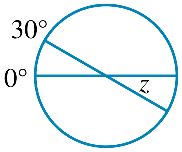


9 Calculate the value of the pronumerals in the following figures.

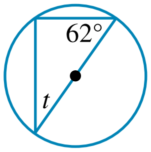
a



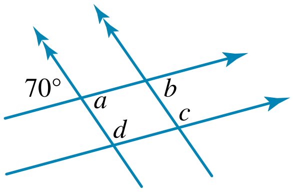
b



c



d

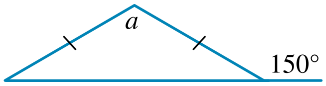


10 Name the regular polygon that has the given angle(s).

a Interior angle of 108°, exterior angle of 72°
b Interior angle of 150°, exterior angle of 30°
c Interior angle of 135°, exterior angle of 45°
d Interior angle of 120°
e Exterior angle of 120°

11 In the figure, the value of a is:

A 30° **B** 75° **C** 90°
D 120° **E** 150°

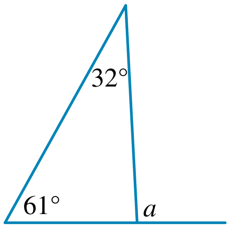


12 An isosceles triangle has a known angle of 50°. The largest possible angle for this triangle is:

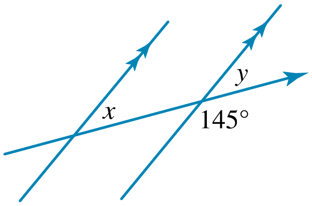
A 80° **B** 130° **C** 90°
D 65° **E** 50°

13 Find the values of the pronumerals in each of the following figures.

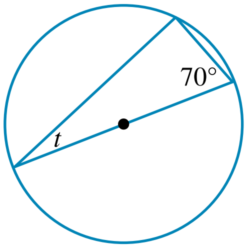
a



b

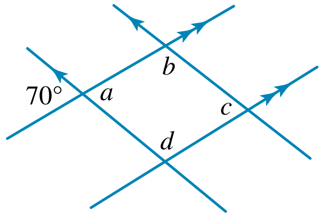


c

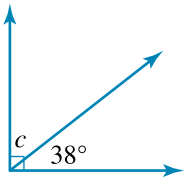


14 Find the values of the pronumerals in each of the following figures.

a

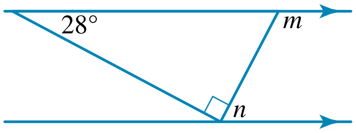


b

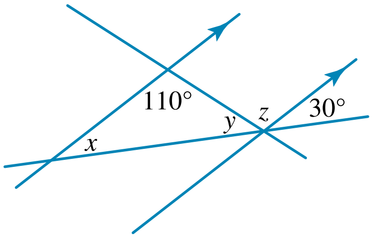


15 Find the values of the pronumerals in each of the following figures.

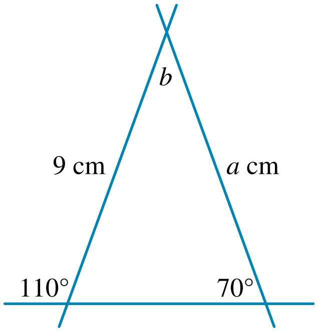
a



b

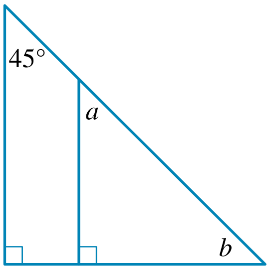


c

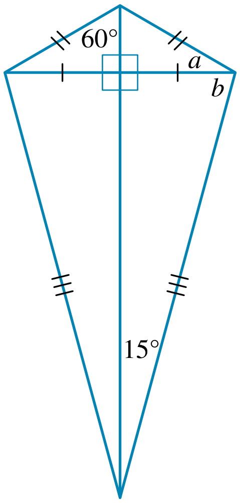


16 Find the values of the pronumerals in each of the following figures.

a



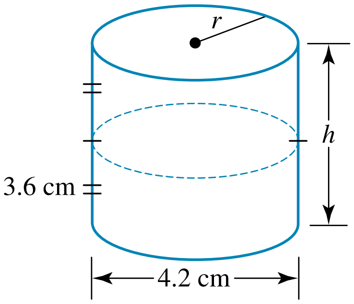
b



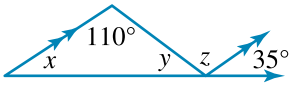
MASTER

17 Calculate the unknown pronumerals.

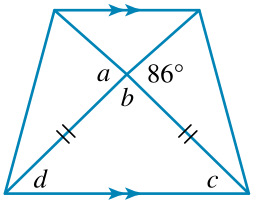
a



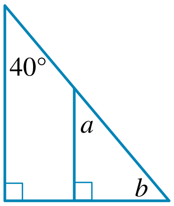
b



18 a



b



11.3 Area and perimeter I

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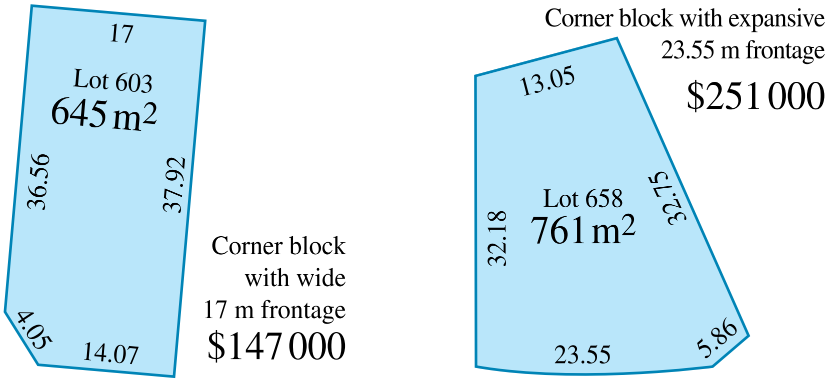
Interactivity
Area and perimeter
int-6474

Much of our world is described by **area** (the amount of space enclosed by a closed figure) and **perimeter** (the distance around a closed figure).



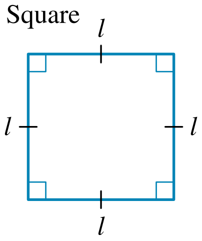
Some examples are the area of a house block, the fencing of a block of land, the size of a bedroom and the amount of paint required to cover an object. In this section we will review the more common shapes.

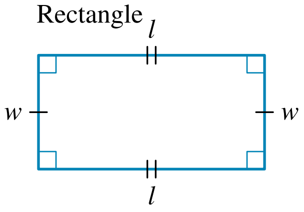
Perimeter

Perimeter is the distance around a closed figure.

Some common rules are:

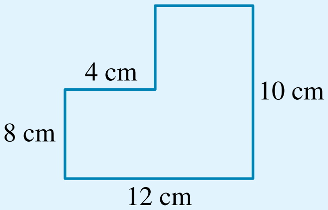
1. For squares, the perimeter = $4l$
2. For rectangles, the perimeter = $2(l + w)$





WORKED EXAMPLE 4

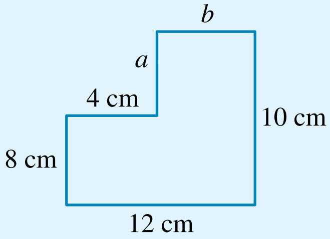
Find the perimeter of the closed figure shown.



THINK

1 To find the perimeter we need the lengths of each side around the closed figure. So we need to find a and b.

WRITE



2 Find the length of a.	$a = 10 - 8$ $= 2$ cm
3 Find the length of b.	$b = 12 - 4$ $= 8$ cm
4 Calculate the perimeter by adding all the lengths around the outside of the figure.	Perimeter $= 12 + 8 + 4 + 2 + 8 + 10$ $= 44$ cm
5 Write your answer.	The perimeter of the closed figure is 44 cm.

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Unit 4

AOS M3

Topic 1

Concept 6

Linear dimensions in 2D and 3D

Concept summary

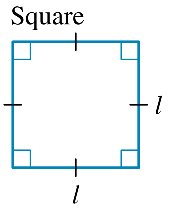
Practice questions

Area of common shapes

The areas of shapes commonly encountered are:

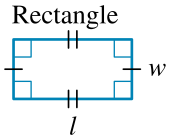
1. Area of a square:

$$A = \text{length}^2 = l^2$$



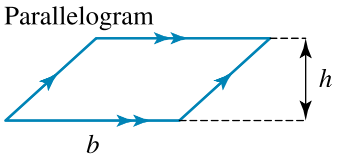
2. Area of a rectangle:

$$A = \text{length} \times \text{width} = l \times w$$



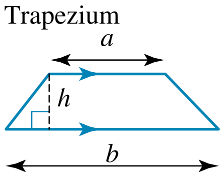
3. Area of a parallelogram:

$$A = \text{base} \times \text{height} = b \times h$$



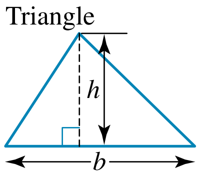
4. Area of a trapezium:

$$A = \frac{1}{2}(a + b) \times h$$



5. Area of a triangle:

$$A = \frac{1}{2} \times b \times h$$

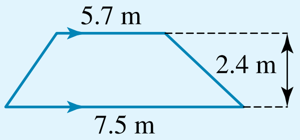


Area is measured in mm^2, cm^2, m^2, km^2 and hectares.

$$1 \text{ hectare} = 100 \text{ m} \times 100 \text{ m} = 10\,000 \text{ m}^2$$

WORKED EXAMPLE 5

Calculate the area of the garden bed given in the diagram (correct to the nearest square metre).



THINK | **WRITE**

1 The shape of the garden is a trapezium.

Use the formula for area of a trapezium. Remember that the lengths of the two parallel sides are a and b, and h is the perpendicular distance between the two parallel sides.

Area of a trapezium $= \frac{1}{2}(a + b) \times h$

$a = 7.5 \qquad b = 5.7 \qquad h = 2.4$

2 Substitute and evaluate.

$$\begin{aligned} A &= \frac{1}{2}(7.5 + 5.7) \times 2.4 \\ &= \frac{1}{2} \times 13.2 \times 2.4 \\ &= 15.84\ \text{m}^2 \end{aligned}$$

3 Write your answer.

The area of the garden bed is approximately 16 square metres.

Composite areas

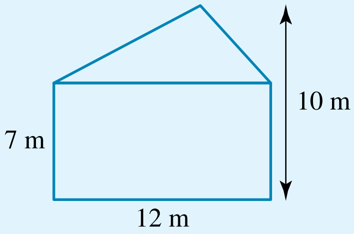
Often a closed figure can be identified as comprising two or more different common figures. Such figures are called **composite figures**. The area of a composite figure is the sum of the areas of the individual common figures.

Area of a composite figure = sum of the areas of the individual common figures

$$A_{\text{composite}} = A_1 + A_2 + A_3 + A_4 + \ldots$$

WORKED EXAMPLE 6

Calculate the area of the composite shape shown.



THINK | **WRITE**

1 Identify the two shapes.

The shapes are a rectangle and a triangle.

2 Calculate the area of the rectangle.

$$\begin{aligned} A_{\text{Rectangle}} &= l \times w \\ &= 12 \times 7 \\ &= 84 \end{aligned}$$

3 Calculate the area of the triangle.

$$\begin{aligned} A_{\text{Triangle}} &= \frac{1}{2}b \times h \\ &= \frac{1}{2} \times 12 \times (10 - 7) \\ &= 6 \times 3 \\ &= 18 \end{aligned}$$

4 Sum the two areas together.

$$\begin{aligned} A_{\text{Total}} &= A_{\text{Rectangle}} + A_{\text{Triangle}} \\ &= 84 + 18 \\ &= 102 \end{aligned}$$

5 Write your answer.

The total area of the composite shape is 102 m^2.

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Interactivity
Conversion of units of area
int-6269

Conversion of units of area

Often the units of area need to be converted, for example, from cm^2 to m^2 and vice versa.

> **1. To convert to smaller units, for example, m^2 to cm^2, multiply (×).**
> **2. To convert to larger units, for example, mm^2 to cm^2, divide (÷).**

Some examples are:

(a) $1 \text{ cm}^2 = 10 \text{ mm} \times 10 \text{ mm}$
$= 100 \text{ mm}^2$
(b) $1 \text{ m}^2 = 100 \text{ cm} \times 100 \text{ cm}$
$= 10\,000 \text{ cm}^2$
(c) $1 \text{ km}^2 = 1000 \text{ m} \times 1000 \text{ m}$
$= 1\,000\,000 \text{ m}^2$
(d) $1 \text{ hectare} = 10\,000 \text{ m}^2$

$$\begin{tikzcd}
\text{mm}^2 \arrow[r, bend left, "\div 10^2"] & \text{cm}^2 \arrow[l, bend left, "\times 10^2"] \arrow[r, bend left, "\div 100^2"] & \text{m}^2 \arrow[l, bend left, "\times 100^2"] \arrow[r, bend left, "\div 1000^2"] & \text{km}^2 \arrow[l, bend left, "\times 1000^2"]
\end{tikzcd}$$

WORKED EXAMPLE 7

Convert 1.12 m^2 to square centimetres (cm^2).

THINK

1 To convert from m^2 to cm^2, multiply by 100^2 or 10 000.

2 Write your answer.

WRITE

$1.12 \text{ m}^2 = 1.12 \times 10\,000 \text{ cm}^2$
$= 11\,200 \text{ cm}^2$

1.12 m^2 is equal to 11 200 square centimetres (cm^2).

WORKED EXAMPLE 8

Convert 156 000 metres^2 to:

a kilometres^2

b hectares.

THINK

a 1 To convert from metres^2 to kilometres^2 divide by 1000^2 or 1 000 000.

2 Write the answer in correct units.

b 1 10 000 m^2 equals 1 hectare. To convert from m^2 to hectares, divide by 10 000.

2 Write the answer.

WRITE

a $156\,000 \text{ m}^2 = 156\,000 \div 1\,000\,000 \text{ km}^2$
$= 0.156 \text{ km}^2$

156 000 m^2 = 0.156 square kilometres (km^2)

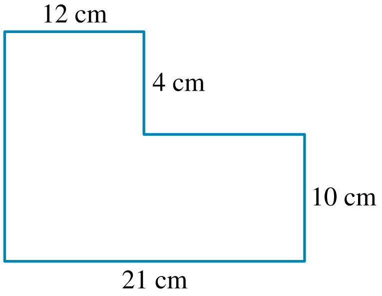
b $156\,000 \text{ m}^2 = \dfrac{156\,000}{10\,000} \text{ hectares}$
$= 15.6 \text{ hectares}$

156 000 m^2 = 15.6 hectares

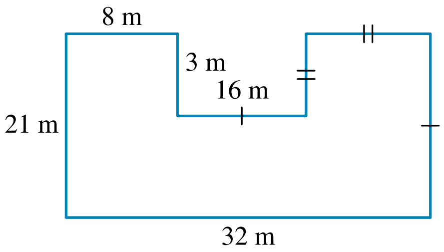
EXERCISE 11.3 Area and perimeter I

PRACTISE

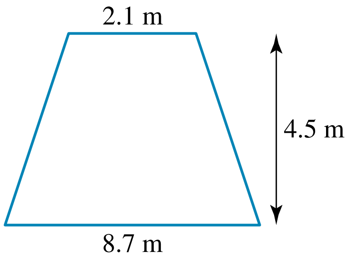
1 **WE4** Calculate the perimeter of the closed figure shown.



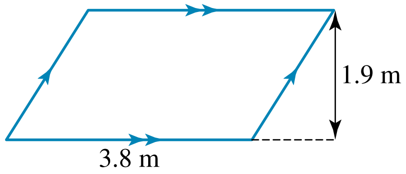
2 Calculate the perimeter of the closed figure shown.



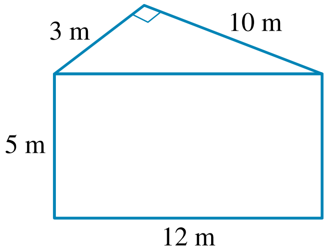
3 **WE5** Calculate the area of the backyard given in the diagram (correct to the nearest square metre).



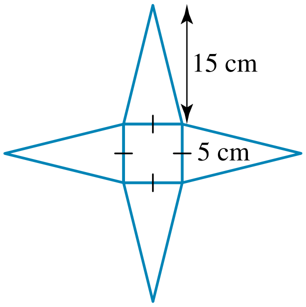
4 Calculate the area of the shape shown (correct to the nearest square metre).



5 **WE6** Calculate the area of the composite shape shown.



6 Calculate the area of the composite shape shown.



7 **WE7** Convert 2.59 m^2 to square centimetres (cm^2).

8 Convert 34.56 cm^2 to square millimetres (mm^2).

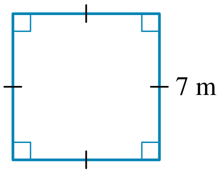
9 **WE8** Convert 2 500 000 m^2 to kilometres^2 (km^2).

10 Convert 0.0378 m^2 to millimetres^2 (mm^2).

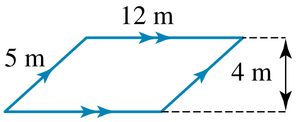
CONSOLIDATE

11 Calculate the perimeters of the following figures (correct to the nearest whole units).

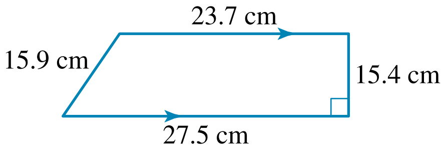
a



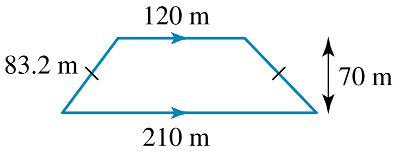
b



c



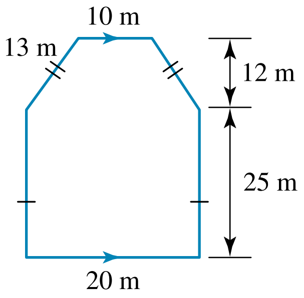
d



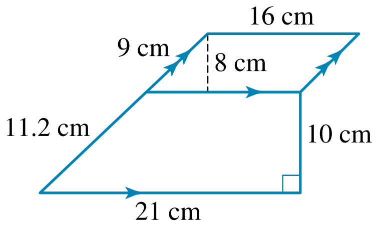
12 Calculate the areas of the closed figures in question **11**.

13 Calculate the areas of the following figures (correct to 1 decimal place).

a



b

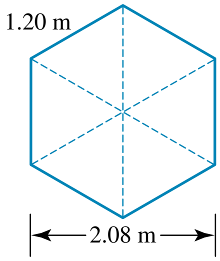


14 Calculate the perimeters of the closed figures in question **13**.

15 Convert the following areas to the units given in brackets.

a 20 000 mm^2 (cm^2)
b 320 000 cm^2 (m^2)
c 0.035 m^2 (cm^2)
d 0.035 m^2 (mm^2)
e 2 500 000 m^2 (km^2)
f 357 000 m^2 (hectares)
g 2 750 000 000 mm^2 (m^2)
h 0.000 06 km^2 (m^2)

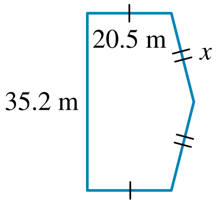
16 Find the area of the regular hexagon as shown in the diagram (correct to 2 decimal places, in m^2).



17 The perimeter of the enclosed figure shown is 156.6 metres.

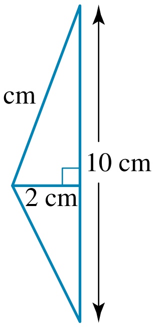
The unknown length, x, is closest to:

A 20.5 m
B 35.2 m
C 40.2 m
D 80.4 m
E 90.6 m

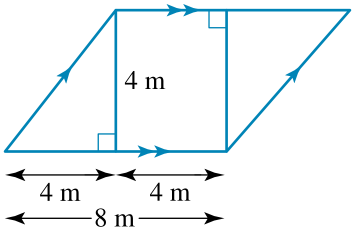


18 Find the area of each of the following figures.

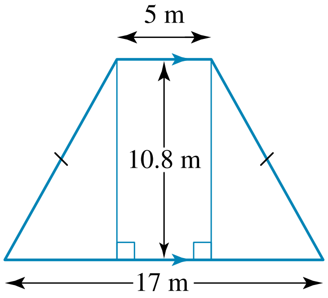
a



b



c



19 Calculate the perimeter of each of the shapes in question **18**. Give your answers correct to the nearest whole number.

20 Convert the following areas to the units given in the brackets.

a 45 000 cm^2 (m^2)

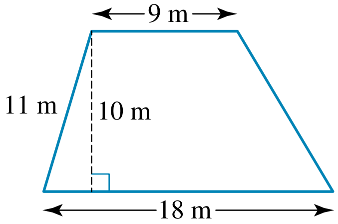
b 0.71 m^2 (mm^2)

c 216 000 m^2 (hectares)

d 0.0737 km^2 (m^2)

MASTER

21 Find the perimeter, in millimetres, of the following shape. Give your answer correct to the nearest whole number.



22 On the set of a western movie, a horse is tied to a railing outside a saloon bar. The railing is 2 metres long; the reins are also 2 metres long and are tied at one of the ends of the railing.

a Draw a diagram of this situation.

b To how much area does the horse now have access (correct to 1 decimal place)?

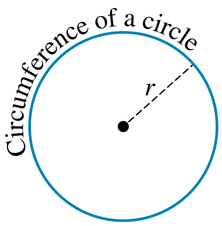
The reins are now tied to the centre of the railing.

c Draw a diagram of this situation.

d To how much area does the horse have access (correct to 1 decimal place)?

11.4 Area and perimeter II

There are more shapes we can calculate the perimeter and area of. We can also calculate the perimeter of a circle, known as the **circumference**, as well as the area of a circle.

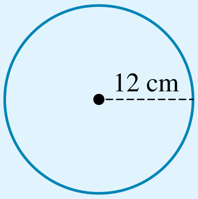


Circumference (C) is the perimeter of a circle.

$$C = 2 \times \pi \times \text{radius}$$
$$= 2\pi r$$

WORKED EXAMPLE 9

Calculate the circumference of the following shape correct to 2 decimal places.



THINK	WRITE
1 Use the formula for the circumference of a circle with radius in it.	$C = 2\pi r$
2 Substitute the radius into the equation, using the π key on your calculator.	$C = 2 \times \pi \times 12$ $= 75.398\,223\,69$ ≈ 75.40 cm
3 Write the answer correct to 2 decimal places.	The circumference is 75.40 cm.

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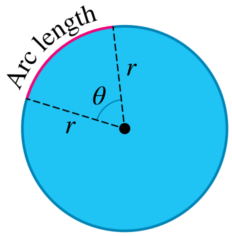
Arc length

- An **arc** is part of the circumference of a circle.
- The **arc length** of a circle is calculated by finding the circumference of a circle and multiplying by the fraction of the angle that it forms at the centre of the circle.

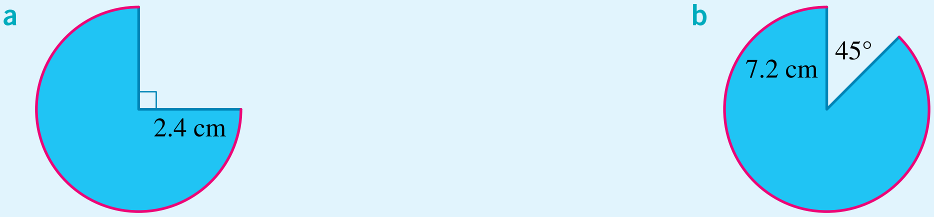


$$\boxed{\text{Arc length} = \frac{\theta}{360} \times 2\pi r}$$

where r = radius of the circle and θ = the angle the ends of the arc makes with the centre of the circle.

WORKED EXAMPLE 10

Calculate the arc lengths for each of the following shapes correct to 2 decimal places.



THINK	WRITE
a 1 Determine the angle the ends of the arcs make with the centre.	a $\theta = 360° - 90°$ $= 270°$
2 Substitute the values into the arc length equation.	$\text{Arc length} = \frac{\theta}{360} \times 2\pi r$ $= \frac{270}{360} \times 2 \times \pi \times 2.4$ $= 11.3097$ ≈ 11.31 cm

3 Write the answer.

Arc length = 11.31 cm

b **1** Determine the angle the ends of the arcs make with the centre.

b $\theta = 360° - 45°$
$= 315°$

2 Substitute the values into the arc length equation.

$$\begin{aligned}\text{Arc length} &= \frac{\theta}{360} \times 2\pi r \\ &= \frac{315}{360} \times 2 \times \pi \times 7.2 \\ &= 39.5841 \\ &\approx 39.58 \text{ cm}\end{aligned}$$

3 Write the answer.

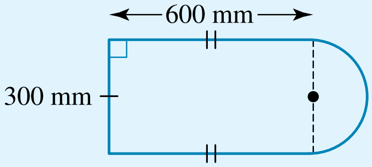
Arc length = 39.58 cm

Perimeter of composite shapes

The perimeter of a composite shape is the same as the regular perimeter; it is the total distance around the outside of a closed figure.

WORKED EXAMPLE 11

Calculate the perimeter of the closed figure given (correct to the nearest mm).



THINK

1 The shape is composed of a semicircle and three sides of a rectangle.

WRITE

Perimeter = $300 + 2 \times 600 + \frac{1}{2}$ circumference where

$$\begin{aligned}\frac{1}{2} \text{ of circumference} &= \frac{1}{2} \times 2\pi r \\ &= \pi \times 150 \\ &\approx 471.24\end{aligned}$$

2 Add together the three components of the perimeter.

$$\begin{aligned}\text{Perimeter} &= 300 + 2 \times 600 + 471.24 \\ &= 1971.24\end{aligned}$$

3 Write your answer.

The perimeter of the closed figure is 1971 mm, correct to the nearest millimetre.

Area of circles, sectors and segments

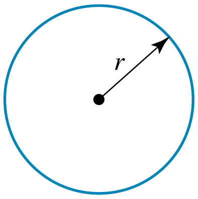
Circles

The area of a circle is calculated by multiplying π by the radius squared. This can be used to calculate the area of a sector, as shown.

Area of a circle:

$$\begin{aligned}A &= \pi \times \text{radius}^2 \\ &= \pi r^2\end{aligned}$$

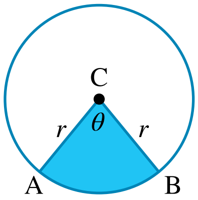
Circle



Sectors

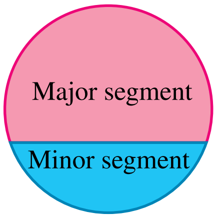
A **sector** is a part of the area of a circle, as shown by the shaded area in the diagram. The angle θ, at the centre of the circle, subtends the arc at AB. The area of a sector is calculated using:

$$A_{\text{Sector}} = \frac{\theta}{360}\pi r^2$$



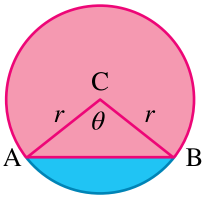
Segments

A **segment** is formed by joining two points on the circumference with a straight line, known as a chord. This forms two areas, with the smaller area being the minor segment and the larger area the major segment. This is shown by the blue (minor segment) and pink (major segment) sections in the diagram.



The area of the minor segment can be calculated by:

$$\begin{aligned} A_{\text{Minor segment}} &= A_{\text{sector}} - A_{\text{triangle}} \\ &= \frac{\theta}{360}\pi r^2 - \frac{1}{2}r^2\sin\theta \\ &= r^2\left(\frac{\theta}{360}\pi - \frac{1}{2}\sin\theta\right) \end{aligned}$$



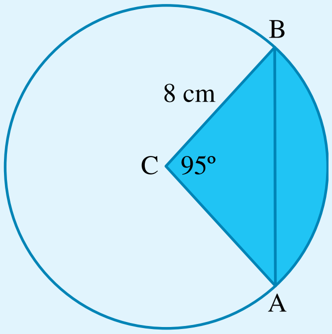
WORKED EXAMPLE 12

From the circle shown, which has a radius of 8 cm, calculate:

a the area of the minor sector

b the area of the minor segment of the circle.

Give your answers correct to 2 decimal places.



THINK	WRITE
a 1 Use the formula for the area of a sector in degrees.	a $A_{\text{Sector}} = \frac{\theta}{360} \times \pi r^2$ $r = 8$ $\theta = 95°$

2 Substitute the quantities into the formula and simplify.

$$A_{\text{Sector}} = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{95}{360} \times \pi(8)^2$$
$$= 53.0580$$

3 Write the answer correct to 2 decimal places.

$$A_{\text{Sector}} = 53.06 \text{ cm}^2$$

b **1** Use the formula for the area of a segment in degrees.

b $$A_{\text{Minor segment}} = r^2\left(\frac{\theta}{360}\pi - \frac{1}{2}\sin(\theta)\right)$$
$$r = 8$$
$$\theta = 95°$$

2 Substitute the quantities into the formula and simplify.

$$A_{\text{Minor segment}} = 8^2\left(\frac{95}{360}\pi - \frac{1}{2}\sin 95\right)$$
$$= 21.1798...$$

3 Write the answer correct to 2 decimal places.

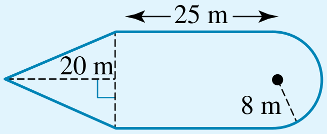
$$A_{\text{Minor segment}} = 21.18 \text{ cm}^2$$

Composite areas

As with composite areas, the area is the sum of the individual common figures, including circles or fractions of circles.

WORKED EXAMPLE 13

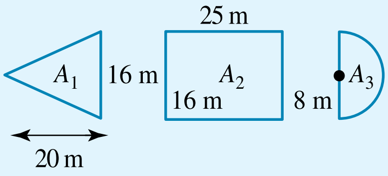
Calculate the area of the hotel foyer from the plan given (correct to the nearest square metre).



THINK

WRITE/DRAW

1 The shape is composite and needs to be separated into two or more common shapes: in this case, a rectangle, a triangle and half of a circle.



$$\text{Area of foyer} = A_1 + A_2 + A_3$$

2 Find the area of each shape. (The width of the rectangle and the base of the triangle is twice the radius of the circle, that is, 16 metres.)

$$A_1 = \text{area of triangle}$$
$$= \frac{1}{2} \times b \times h$$
$$= \frac{1}{2} \times 16 \times 20$$
$$= 160 \text{ m}^2$$
$$A_2 = \text{Area of rectangle}$$
$$= l \times w$$
$$= 25 \times 16$$
$$= 400 \text{ m}^2$$

$$A_3 = \text{Area of half of a circle}$$
$$= \frac{1}{2} \times \pi \times r^2$$
$$= \frac{1}{2} \times \pi \times 8^2$$
$$= 100.53 \text{ m}^2$$

3 Add together all three areas for the composite shape.

$$\text{Area of foyer} = A_1 + A_2 + A_3$$
$$= 160 + 400 + 100.53$$
$$= 660.53 \text{ m}^2$$

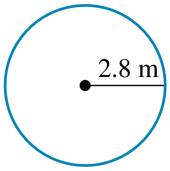
4 Write your answer.

The area of the hotel foyer is approximately 661 m^2.

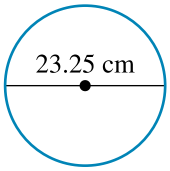
EXERCISE 11.4 Area and perimeter II

PRACTISE

1 **WE9** Calculate the circumference of the following correct to 2 decimal places.

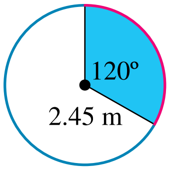


2 Calculate the circumference of the following correct to 2 decimal places.

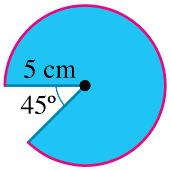


3 **WE10** Calculate the arc length of each of the following shapes correct to 2 decimal places.

a

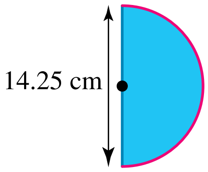


b

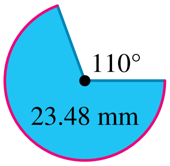


4 Calculate the arc length of each of the following shapes correct to 2 decimal places.

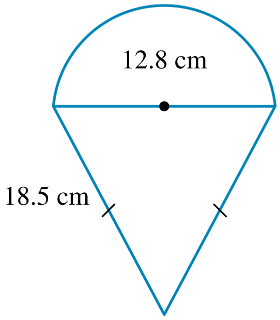
a



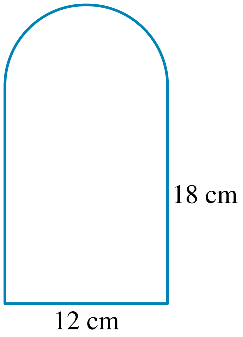
b



5 **WE11** Calculate the perimeter of the following shape, correct to 2 decimal places.



6 Calculate the perimeter of the following shape, correct to 1 decimal place.

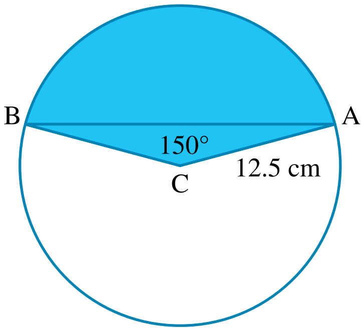


7 **WE12** From the circle shown, which has a radius of 12.5 cm, calculate:

a the area of the minor sector

b the area of the minor segment of the circle.

Give your answers correct to 2 decimal places.

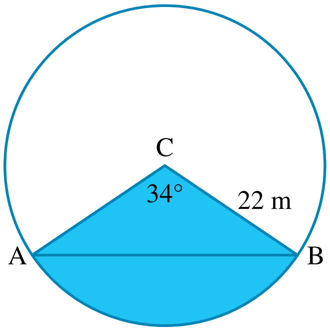


8 From the circle shown which has a radius of 22 m, calculate:

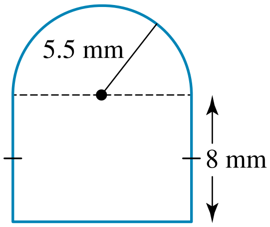
a the area of the minor sector

b the area of the minor segment of the circle.

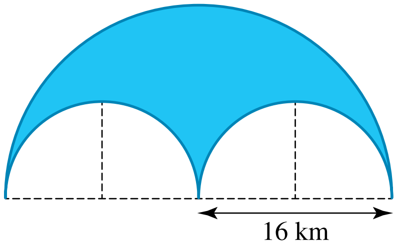
Give your answers correct to 2 decimal places.



9 **WE13** Find the area of the following figure (correct to the nearest whole unit).



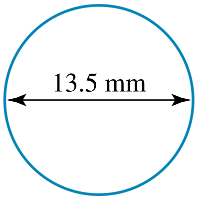
10 Calculate the area of the shaded shape shown, correct to 2 decimal places.



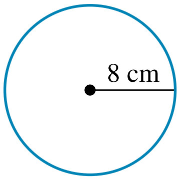
CONSOLIDATE

11 Calculate the perimeter and the area of the following figures (correct to the nearest whole unit).

a

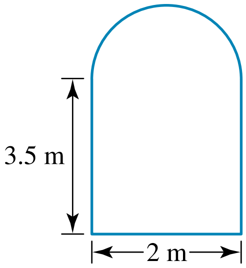


b

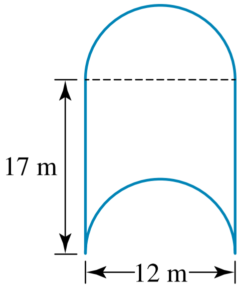


12 Calculate the areas of the following figures (where appropriate correct to 1 decimal place).

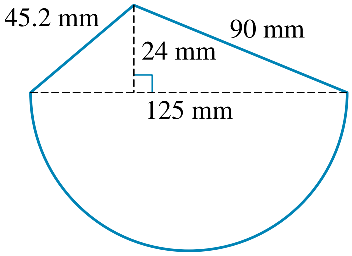
a



b



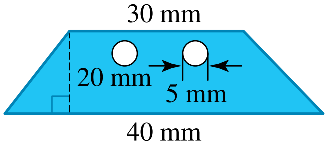
c



13 Calculate the perimeter of the closed figures in question **12** correct to 1 decimal place.

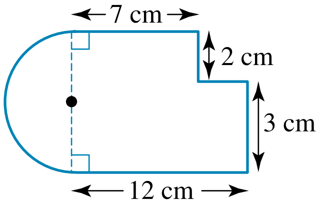
14 A cutting blade for a craft knife has the dimensions shown in the diagram.

What is the area of steel in the blade (correct to the nearest mm^2)?



15 The perimeter of the figure shown, in centimetres, is:

A 34
B $24 + 5\pi$
C $24 + 2.5\pi$
D $29 + 5\pi$
E $29 + 2.5\pi$

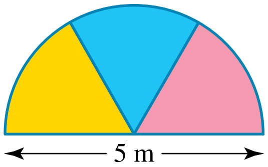


16 A minor sector is formed by an angle of 123° in a circle of diameter 9 cm. The area of the minor sector is closest to:

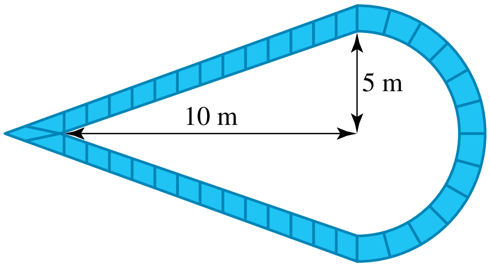
A 86.94 cm^2
B 63.62 cm^2
C 41.88 cm^2
D 15.17 cm^2
E 21.74 cm^2

17 Calculate the difference in area between a sector of a circle formed by an angle of 50° and a sector of a circle formed by an angle of 70° in a circle of radius 3 cm. Give your answer correct to 2 decimal places.

18 A church window is made in the shape of a semicircle with three equal stained glass sections. Calculate the area of one of the stained glass sections. Give your answer correct to 2 decimal places.

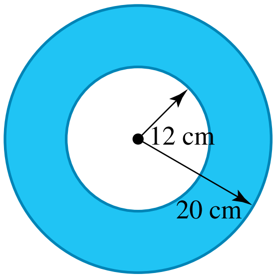


19 A pool is to have one row of caping tiles 500 mm wide around its perimeter. Calculate the inner perimeter of the tiles for the pool shape shown. Give your answer correct to 2 decimal places.

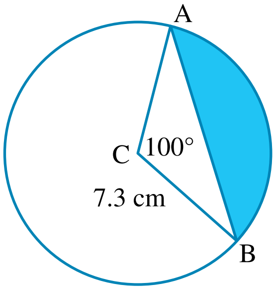


20 Calculate the areas of the shaded region in the following shapes, correct to 2 decimal places:

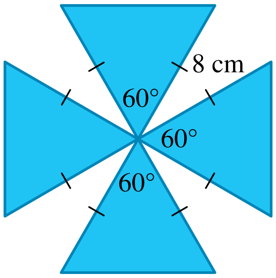
a



b

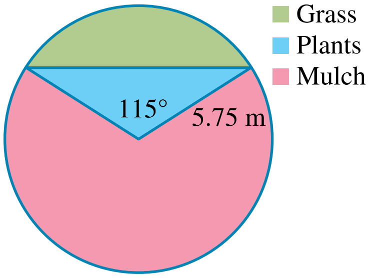


c



MASTER

21 A circular garden bed is to be landscaped with grass, plants and mulch. Given the circular garden is designed with the shown dimensions:



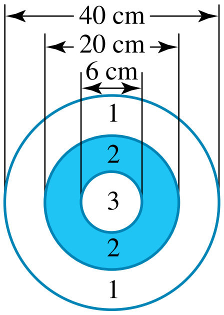
a calculate the area of the grassed area, correct to 2 decimal places

b calculate the area of the mulched area, correct to 2 decimal places

c calculate the area of the planted area, correct to 2 decimal places.

d If the cost of grass, plants and mulch are \$40, \$55 and \$15 per metre respectively, calculate the cost of landscaping the circular garden correct to the nearest cent.

22 A 3-ring dartboard has dimensions as shown. (Give all answers to 1 decimal place.)



a What is the total area of the dartboard?

b What is the area of the bullseye (inner circle)?

c What is the area of the 2-point middle ring?

d Express each area of the three rings as a percentage of the total area (where appropriate correct to 2 decimal places).

11.5 Great circles and small circles

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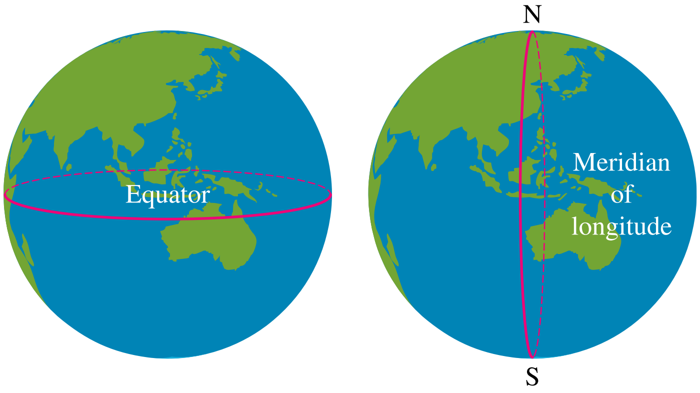
A **great circle** is a circle that divides a sphere into two equal hemispheres, where the centre of the great circle is the centre of the sphere.

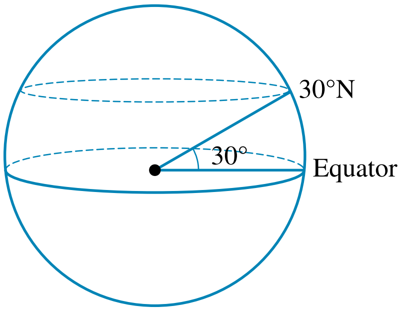
As with a circle the circumference of a great circle can be calculated by using $C = 2\pi r$.

The great circles that we will focus on are those that are related to Earth. These are the lines of **longitude** (also known as **meridians**) and the Equator.

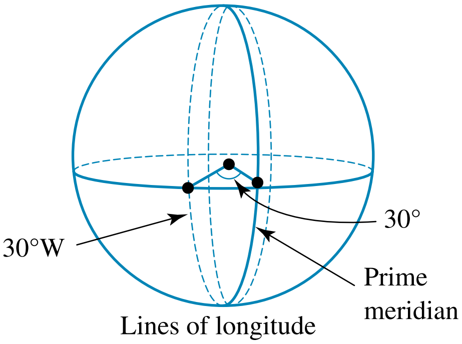


Latitude and longitude

Lines of **latitude** run from east to west and measure the angle north or south of the equator. The meridians, or lines of longitude, measure the angle east or west of the prime meridian, which passes through the Greenwich Observatory (0° longitude) in London, England.



Lines of latitude



Lines of longitude

Measuring distances with great circles

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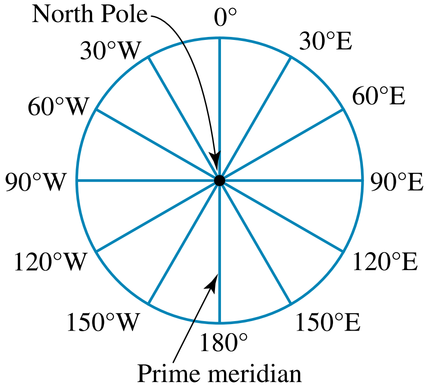
Distance between points on a sphere
Concept summary
Practice questions

The shortest distance between any two points on the Earth with the same longitude, or between any two points on the Equator, is along a great circle.

To conduct calculations in relation to great circles on Earth, consider the centre of the circle as the centre of the Earth, with the radius being 6400 km.

The shortest distance between two points on the Earth that have the same longitude, or which are both on the Equator, is the great circle arc length between those points.



WORKED EXAMPLE 14

Calculate the distance travelled in the following situations correct to the nearest kilometre:

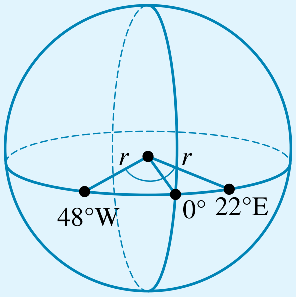
a an aeroplane that travels along the equator from a point of longitude 48°W to 22°E

b a boat that travels from a point M: 36°S, 138°W to point N: 40°S, 138°W.

THINK | **WRITE**

a 1 Both points lie on the equator, which is a great circle. The angle between the two points is made of 2 parts, from the starting point to 0°, then from 0° to the finishing point.

a



$$\theta = 48° + 22°$$
$$= 70°$$

2 The distance between the two points is the arc length. Substitute θ and r into the arc length formula and simplify.

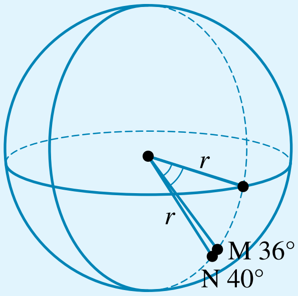
$$r = 6400 \text{ km}$$
$$\text{arc length} = \frac{\theta}{360°} \times 2\pi r$$
$$= \frac{70}{360} \times 2\pi \times 6400$$
$$= 7819.075...$$

3 Answer the question correct to the nearest kilometre.

The aeroplane travelled 7819 km.

b 1 Observing that M and N lie on the same longitude 138°W, path MN is along a great circle. Since both latitudes are S, the angle θ is formed by their difference.

b



$$\theta = 40° - 36°$$
$$= 4°$$

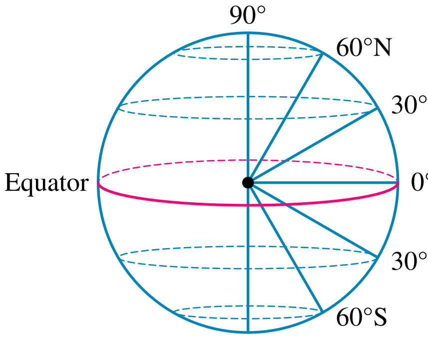
2 The distance between the two points is the arc length. Substitute θ and r into the arc length formula and simplify, using the radius of the Earth as 6400 km.

$$\text{arc length} = \frac{\theta}{360°} \times 2\pi r$$
$$= \frac{4}{360} \times 2\pi \times 6400$$
$$= 446.804...$$

3 Answer the question correct to the nearest kilometre.

The boat travelled 447 km.

Measuring distances with small circles



With the exception of the equator (which is a great circle), lines of latitude form small circles which have a radius smaller than 6400 km.

To calculate the distance along a parallel of latitude, first the radius of the small circle must be found using trigonometric ratios, and then the formula for arc length may be used.

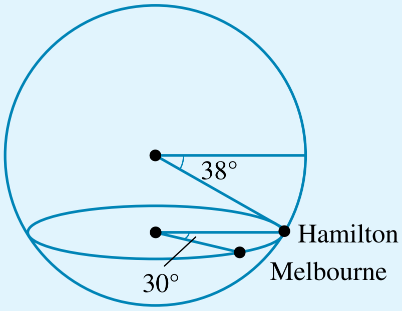
WORKED EXAMPLE 15 **Calculate the distance along the parallel of latitude between Melbourne, Australia 38°S, 145°E and Hamilton, New Zealand 38°S, 175°E. Give your answer correct to the nearest kilometre.**

THINK

WRITE

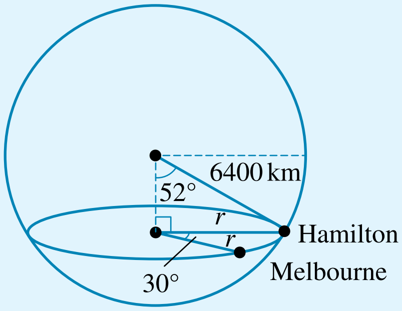
a Both points lie on the 38^{th} parallel south. The angle between the two points is the difference between the longitudes.

a



175° – 145° = 30°

b The radius of the Earth is 6400 km. The radius of the small circle is calculated using the right-angled triangle and the complementary angle.



$$\sin\theta = \frac{O}{H}$$

$$\sin 52° = \frac{r}{6400}$$

$$r = 6400 \sin 52°$$

$$= 5043.2688\ldots \text{ km}$$

c Using the radius of the small circle and the angle of 30°, calculate the distance using arc length.

$$l = \frac{\theta°}{180°} \times \pi r$$

$$= \frac{30}{180} \times \pi \times 5043.2688\ldots$$

$$= 2640.649\ldots$$

$$\approx 2641 \text{ km}$$

EXERCISE 11.5 Great circles

PRACTISE

1 **WE14** Calculate the distance travelled in the following situations correct to the nearest kilometre:

a an aeroplane that travels along the equator from a point of longitude 23°W to 35°E

b a boat that travels from a point M: 12°S, 100°W to point N: 58°S, 100°W.

2 Calculate the distance travelled in the following situations correct to the nearest kilometre:

a a plane that travels along the equator from a point of longitude 67°W to 15°E

b a yacht that travels from a point P: 36°S, 98°E to point Q: 81°S, 98°E.

3 Find the distance, correct to the nearest kilometre, between Moscow 56°N, 38°E and Copenhagen 56°N, 13°E along the parallel of latitude.

4 Find the distance, correct to the nearest kilometre, between Prague 50°N, 14°E and Krakow 50°N, 20°E along the parallel of latitude.

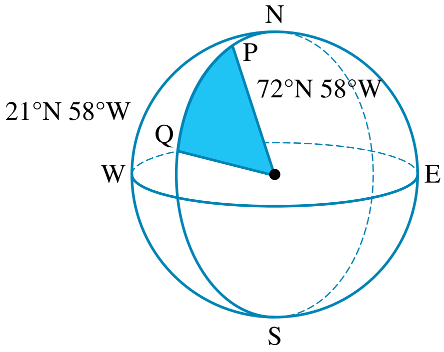
CONSOLIDATE

5 Calculate the distance travelled by an aeroplane travelling along the equator from a point of longitude 36°W to 123°E. Give your answer correct to the nearest km.

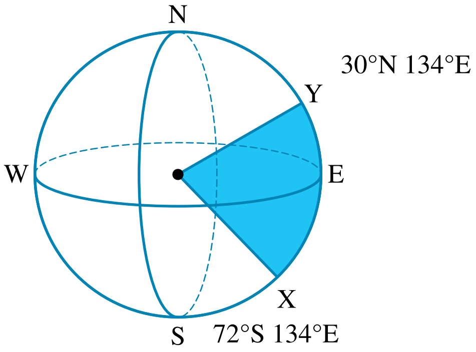
6 Calculate the distance travelled by a cruise ship that travels from point P: 18°N, 25°W to point Q: 30°S, 25°W. Give your answer correct to the nearest km.

7 Calculate the great circle distances shown in these figures. Give your answers correct to the nearest kilometre.

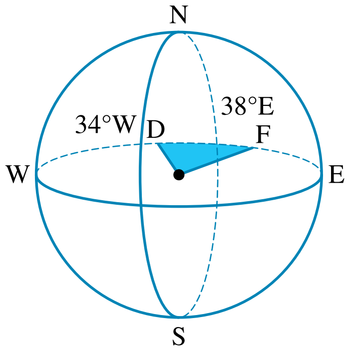
a



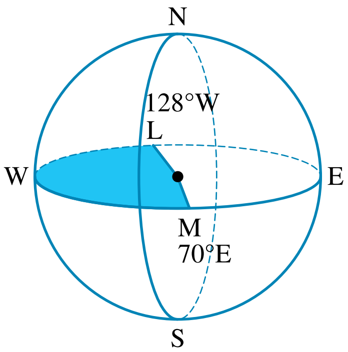
b



c



d



8 Calculate the shortest distance, correct to the nearest kilometre, between two points with the same longitude that form the following angles at the centre of the Earth.

a 12° **b** 3° **c** 0.5° **d** 0.08°

9 Calculate the distance between two points on the equator that form the following angles at the centre of the Earth. Give your answers correct to the nearest kilometre.

a 15.75° **b** 10.31°

10 Use meridians to calculate the shortest distance, correct to the nearest kilometre, between the following cities and the North Pole.

a London (51.51°N, 0.13°W)
b Rome (41.90°N, 12.50°E)
c Dallas (32.78°N, 96.80°W)

11 Melbourne (37.83°S, 145°E) and Port Douglas (16.48°S, 145°E) are on the same meridian. What is the shortest distance (correct to the nearest kilometre) between the two towns?

12 The cities of Washington in the USA and Lima in Peru are 5625 km apart. They both lie on the same meridian. What is the angle formed at the centre of the Earth by the two cities? Give your answer correct to 2 decimal places.

13 The cities of Port Hedland in Australia and Nanjing in China are 3795 km apart. They both lie on the same line of longitude. What is the angle formed at the centre of the Earth by the two cities? Give your answer correct to 2 decimal places.

14 Use meridians to calculate the shortest distance, correct to the nearest kilometre, between the following cities and the equator.

a Bangalore (12.97°N, 77.57°E)
b São Paulo (23.55°S, 46.63°W)
c Tokyo (35.68°N, 139.68°E)

Use the information in the table to answer questions **15** *to* **18**.

City	Latitude	Longitude
Sydney	34°S	151°E
New York	41°N	74°W
Cape Town	34°S	19°E
Naples	41°N	14°E
Budapest	47°N	19°E
Prague	50°N	14°E

15 a Which cities are on the same meridian?

b If two points represented these cities, what would be the angle subtended between them at the centre of the Earth?

16 a Which cities are on the same parallel of latitude?

b If two points represented these cities, what would be the angle subtended between them at the centre of the Earth?

17 **a** Calculate the radius, correct to 2 decimal places, of the circle which forms the 34°S parallel of latitude.

b Calculate the distance, correct to the nearest kilometre, between Sydney and Cape Town along this parallel of latitude.

18 **a** Calculate the radius of the circle, correct to 2 decimal places, which forms the 41°N parallel of latitude.

b Calculate the distance, correct to the nearest kilometre, between New York and Naples along this parallel of latitude.

19 **a** An aircraft flies from Tampa Florida 28°N, 82°W to Corpus Christi Texas 28°N, 97°W. How far, correct to the nearest kilometre, has it flown along the parallel of latitude?

b From Texas it then flies to Oklahoma City 35°N, 97°W. Calculate the total distance, correct to the nearest kilometre, travelled by the aircraft.

20 An aircraft begins at Melbourne Australia and travels non-stop around the 38°S parallel of latitude until it returns to Melbourne. How far has it travelled correct to the nearest kilometre?

MASTER

21 Melbourne and Hokkaido Island in Japan lie on the same line of longitude and are 9094 km apart. If Melbourne has a latitude of 37.83°S, find the latitude of Hokkaido Island, which is north of Melbourne. Give your answer correct to 2 decimal places.

22 The cities of Perth and Beijing lie on the same line of longitude and are 7985 km apart. If Perth has a latitude of 31.95°S, find the latitude of Beijing, which is north of Perth. Give your answer correct to 2 decimal places.

11.6 Total surface area

The **total surface area** (TSA) of a solid object is the sum of the areas of the surfaces.

In some cases we can use established formulas of very common everyday objects. In other situations we will need to derive a formula by using the net of an object.

Total surface area formulas of common objects

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Unit 4
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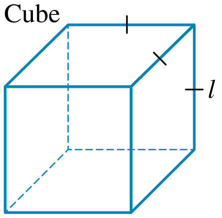
Concept 7

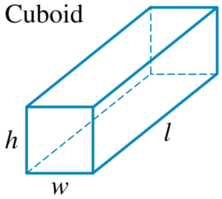
Surface area and volume
Concept summary
Practice questions

Cube



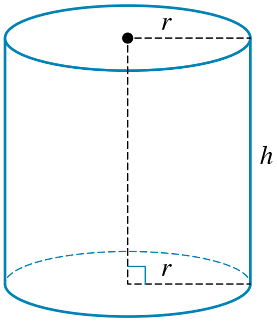
Cubes:
TSA = $6l^2$

Cuboid



Cuboids:
TSA = $2(lw + lh + wh)$

Cylinder



Cylinders:
TSA = $2\pi r\ (r + h)$

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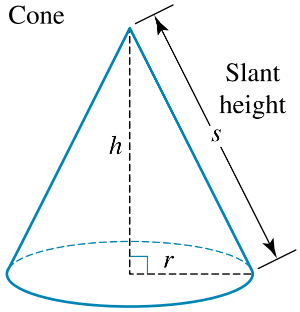
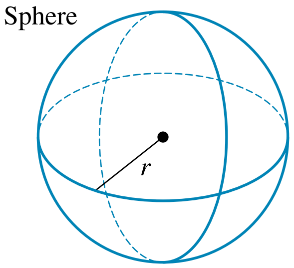
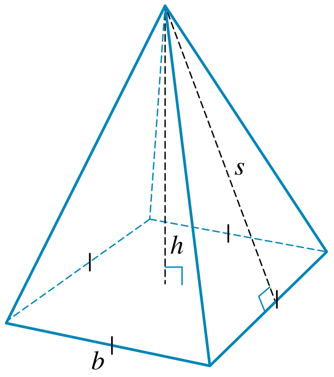
Interactivity
Surface area
int-6477



Cones:
TSA = $\pi r(r + s)$,
where s is the slant height



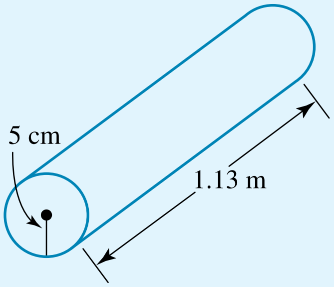
Spheres:
TSA = $4\pi r^2$



Square pyramid:
TSA = $b^2 + 2bs$,
where s is the slant height

WORKED EXAMPLE 16

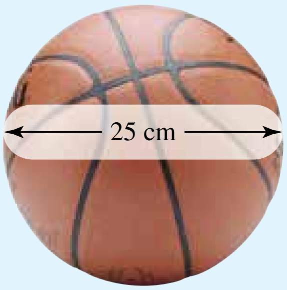
Calculate the total surface area of a poster tube with a length of 1.13 metres and a radius of 5 cm. Give your answer correct to the nearest 100 cm^2.



THINK	WRITE
1 A poster tube is a cylinder. Express all dimensions in centimetres. Remember 1 metre equals 100 centimetres.	TSA of a cylinder = $2\pi r(r + h)$ Radius, $r = 5$ cm Height, $h = 1.13$ m $= 113$ cm
2 Substitute and evaluate. Remember BODMAS.	TSA $= 2 \times \pi \times 5(5 + 113)$ $= 2 \times \pi \times 5 \times 118$ ≈ 3707.08
3 Write your answer.	The total surface area of a poster tube is approximately 3700 cm^2.

WORKED EXAMPLE 17

Calculate the total surface area of a size 7 basketball with a diameter of 25 cm. Give your answer correct to the nearest 10 cm^2.



THINK	WRITE
1 Use the formula for the total surface area of a sphere. Use the diameter to find the radius of the basketball and substitute into the formula.	TSA of sphere $= 4\pi r^2$ Diameter $= 25$ cm Radius $= 12.5$ cm $\text{TSA} = 4 \times \pi \times 12.5^2$ ≈ 1963.495
2 Write your answer.	Total surface area of the ball is approximately 1960 cm^2.

WORKED EXAMPLE 18

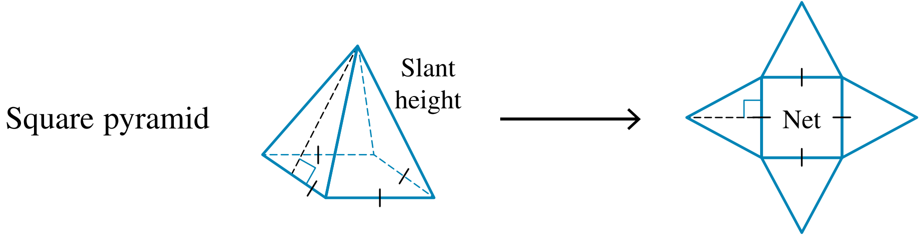
A die used in a board game has a total surface area of 1350 mm^2. Calculate the linear dimensions of the die (correct to the nearest millimetre).

THINK	WRITE
1 A die is a cube. We can substitute into the total surface area of a cube to determine the dimension of the cube. Divide both sides by 6.	$\text{TSA} = 6 \times l^2$ $= 1350 \text{ mm}^2$ $1350 = 6 \times l^2$ $l^2 = \dfrac{1350}{6}$ $= 225$
2 Take the square root of both sides to find l.	$l = \sqrt{225}$ $= 15$ mm
3 Write your answer.	The dimensions of the die are 15 mm × 15 mm × 15 mm.

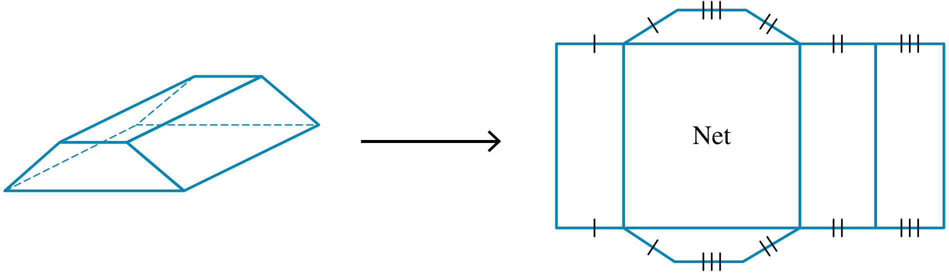
Total surface area using a net

If the object is not a common object or a variation of one, such as an open cylinder, then it is easier to generate the formula from first principles by constructing a net of the object.

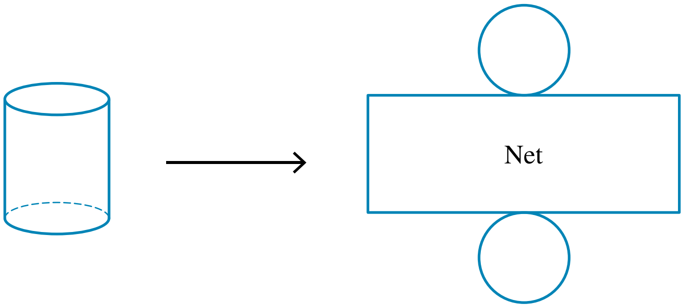
A net of an object is a plane figure that represents the surface of a 3-dimensional object.



Trapezoidal prism

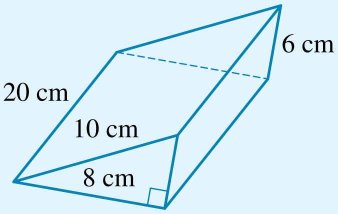


Cylinder



WORKED EXAMPLE 19

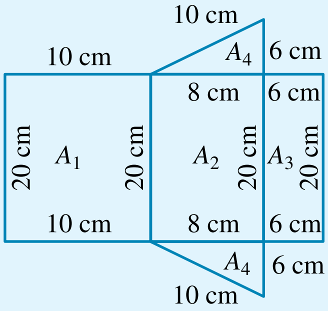
Calculate the total surface area of the triangular prism shown in the diagram.



THINK

WRITE/DRAW

1 Form a net of the triangular prism, transferring all the dimensions to each of the sides of the surfaces.



2 Identify the different-sized common figures and set up a sum of the surface areas. The two triangles are the same.

$$\text{TSA} = A_1 + A_2 + A_3 + 2 \times A_4$$

$$\begin{aligned} A_1 &= l_1 \times w_1 \\ &= 20 \times 10 \\ &= 200\text{ cm}^2 \\ A_2 &= l_2 \times w_2 \\ &= 20 \times 8 \\ &= 160\text{ cm}^2 \\ A_3 &= l_3 \times w_3 \\ &= 20 \times 6 \\ &= 120\text{ cm}^2 \\ A_4 &= \frac{1}{2} \times b \times h \\ &= \frac{1}{2} \times 8 \times 6 \\ &= 24\text{ cm}^2 \end{aligned}$$

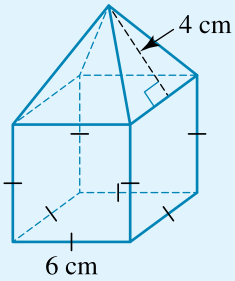
3 Sum the areas.	$\text{TSA} = A_1 + A_2 + A_3 + 2 \times A_4$ $= 200 + 160 + 120 + 2 \times 24$ $= 528 \text{ cm}^2$
4 Write your answer.	The total surface area of the triangular prism is 528 cm^2.

Total surface area of composite solids

When calculating the total surface area of composite solids, be careful to include only the surfaces that form the outer part of the solid.

WORKED EXAMPLE 20

Calculate the total surface area of the following solid.



THINK	WRITE
1 Identify the components of the composite solid.	The composite solid consists of a cube of length 6 cm and a square pyramid with base length 6 cm and slant height 4 cm.
2 Use the formulas to calculate the surface areas of the component parts. Remember to exclude internal surfaces.	TSA of a cube (minus top) $= 5l^2$ $= 5 \times 6^2$ $= 5 \times 36$ $= 180 \text{ cm}^2$ TSA of a pyramid (minus bottom) $= 2bs$ $= 2 \times 6 \times 4$ $= 48 \text{ cm}^2$
3 Add the areas together to obtain the total surface area.	$180 + 48 = 228 \text{ cm}^2$
4 Write your answer.	The total surface area of the solid is 228 cm^2.

EXERCISE 11.6 Total surface area

PRACTISE

1 WE16 Calculate the total surface area of a closed cylinder with a radius of 1.2 cm and a height of 6 cm. Give your answer correct to 1 decimal place.

2 Calculate the total surface area of a closed cone with a radius of 7 cm and a slant height of 11 cm. Give your answer correct to 1 decimal place.

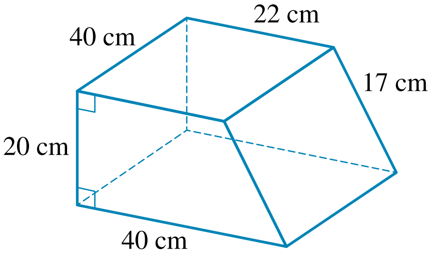
3 WE17 Find the total surface area of a sphere with a radius of 0.8 m. Give your answer correct to 1 decimal place.

4 Find the total surface area of a cube with side length 110 cm.

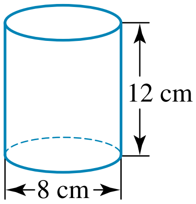
5 **WE18** Find the unknown length of a cube, given that the total surface area is 255 m^2. Give your answer correct to 1 decimal place.

6 Find the unknown radius of a sphere, given that the total surface area is 785.12 cm^2. Give your answer correct to 1 decimal place.

7 **WE19** Calculate the total surface area of this prism.

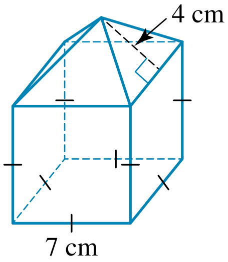


8 Calculate the surface area of an open cylindrical can that is 12 cm high and 8 cm in diameter (correct to 1 decimal place) by completing the following steps.

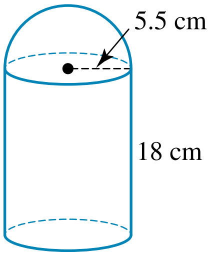


- **a** Form a net of the open cylinder, transferring all the dimensions to each of the surfaces.
- **b** Identify the different-sized common figures and set up a sum of the surface areas. The length of the rectangle is the circumference of the circle.
- **c** Add the areas.
- **d** Write your answer.

9 **WE20** Calculate the total surface area of the following solid



10 Calculate the total surface area of the following solid, correct to 2 decimal places.

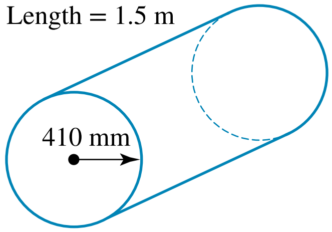


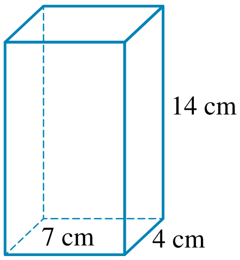
CONSOLIDATE

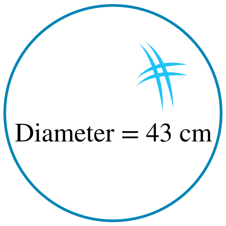
11 Calculate the total surface area for each of the solids **a** to **d** from the following information. Give answers correct to 1 decimal place where appropriate.

- **a** A cube with side lengths of 135 mm
- **b** A cuboid with dimensions of 12 m × 5 m × 8 m ($l \times w \times h$)
- **c** A sphere with a radius of 7.1 cm
- **d** An opened cylinder with a diameter of 100 mm and height of 30 mm

12 Calculate the total surface area of the objects given in the diagrams. Give answers correct to 1 decimal place.

a 

b 

c 

13 Calculate the unknown dimensions, given the total surface area of the objects. Give answers correct to 1 decimal place.

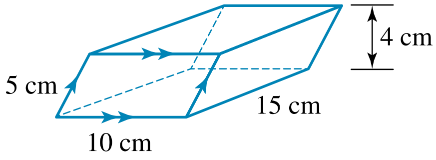
a Length of a cube with a total surface area of 24 m^2

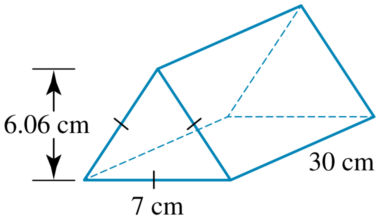
b The radius of a sphere with a total surface area of 633.5 cm^2

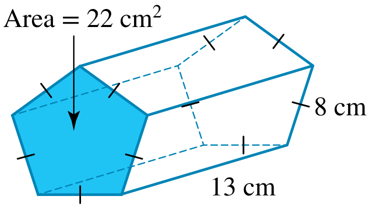
c Length of a cuboid with width of 12 mm, height of 6 cm and a total surface area of 468 cm^2

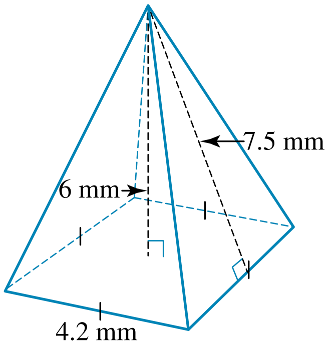
d Diameter of a playing ball with a total surface area of 157 630 cm^2

14 Calculate the total surface areas for the objects given in the diagrams. Give answers correct to 1 decimal place.

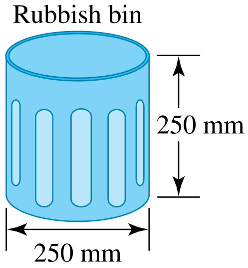
a 

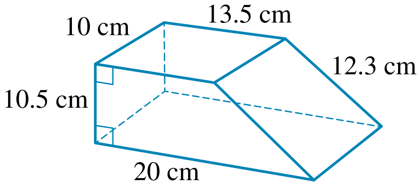
b 

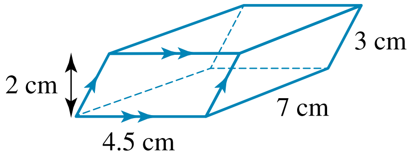
c 

d 

15 Calculate the total surface area of each of the objects in the diagrams. Give answers correct to 1 decimal place.

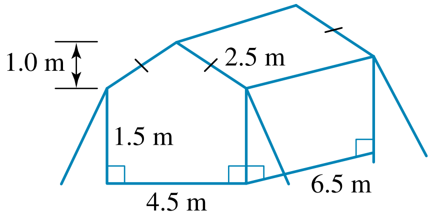
a 

b 

c 

16 A concrete swimming pool is a cuboid with the following dimensions: length of 6 metres, width of 4 metres and depth of 1.3 metres. What surface area of tiles is needed to line the inside of the pool? (Give your answer in m^2 and cm^2.)

17 What is the total area of canvas needed for the tent (including the base) shown in the diagram? Give your answer correct to 2 decimal places.



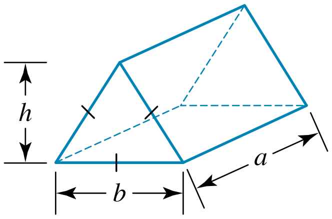
18 The total surface area of a 48 mm diameter ball used in a game of pool is closest to:

A 1810 mm^2
B 2300 mm^2
C 7240 mm^2
D 28 950 mm^2
E 115 800 mm^2

19 The total surface area of a square-based pyramid with a base length of 3.55 cm and a slant height of 6.71 cm is closest to:

A 107.88 cm^2
B 60.24 cm^2
C 60.25 cm^2
D 92.66 cm^2
E 92.67 cm^2

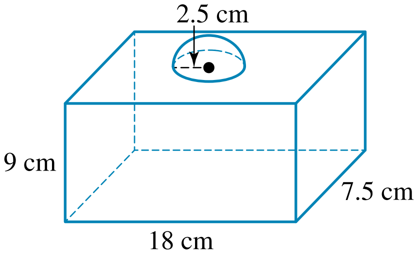
20 The formula for the total surface area for the object shown is:



A $\frac{1}{2}abh$
B $2 \times \frac{1}{2}bh + ab + 2 \times ah$
C $3\left(\frac{1}{2}bh + ab\right)$
D $\frac{1}{2}bh + 3ab$
E $bh + 3ab$

MASTER

21 Calculate the total surface area of the following solid, correct to 2 decimal places.



22 A baker is investigating the best shape for a loaf of bread. The shape with the smallest surface area stays freshest. The baker has come up with two shapes: a rectangular prism with a 12 cm square base and a cylinder with a round end that has a 14 cm diameter.

a Which shape stays fresher if they have the same overall length of 32 cm?
b What is the difference between the total surface areas of the two loaves of bread?

11.7 Volume of prisms, pyramids and spheres

The most common volumes considered in the real world are the volumes of prisms, pyramids, spheres and objects that are a combination of these. For example, people who rely on tank water need to know the capacity (volume) of water that the tank is holding.

eBook*plus*

Interactivity
Volume
int-6476

Volume is the amount of space occupied by a 3-dimensional object.

The units of volume are mm^3 (cubic millimetres), cm^3 (cubic centimetres or cc) and m^3 (cubic metres).

$$\mathbf{1000\ mm^3 = 1\ cm^3}$$
$$\mathbf{1\,000\,000\ cm^3 = 1\ m^3}$$

Another measure of volume is the litre, which is used primarily for quantities of liquids but also for capacity, such as the capacity of a refrigerator or the size of motor car engines.

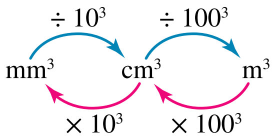
$$\mathbf{1\ litre = 1000\ cm^3}$$
$$\mathbf{1000\ litres = 1\ m^3}$$

eBook*plus*

Interactivity
Conversion of units of volume
int-6272

Conversion of units of volume

Often the units of volume need to be converted, for example, from cm^3 to m^3 and vice versa.



WORKED EXAMPLE 21

Convert 1.12 cm^3 to mm^3.

THINK	WRITE
1 To convert from cm^3 to mm^3 multiply by 10^3 or 1000. (That is, 1 cm^3 equals 1000 mm^3.)	$1.12\ \text{cm}^3 = 1.12 \times 1000\ \text{mm}^3$ $= 1120\ \text{mm}^3$
2 Write the answer in the correct units.	1.12 cm^3 is equal to 1120 mm^3.

WORKED EXAMPLE 22

Convert 156 000 cm^3 to:

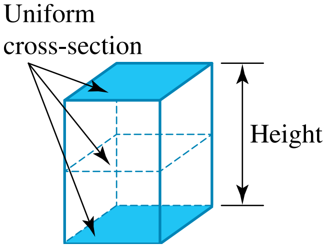
a m^3 **b** litres.

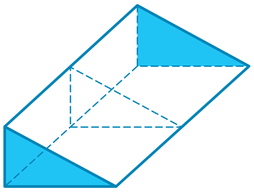
THINK	WRITE
a 1 To convert from cm^3 to m^3 divide by 100^3 or 1 000 000. (That is, 1 000 000 cm^3 equals 1 m^3.)	**a** $156\,000\ \text{cm}^3 = 156\,000 \div 1\,000\,000\ \text{m}^3$ $= 0.156\ \text{m}^3$
2 Write the answer in correct units.	$156\,000\ \text{cm}^3 = 0.156$ cubic metres (m^3)
b 1 1000 cm^3 is equivalent to 1 litre; therefore, to convert from cm^3 to litres, divide by 1000.	**b** $156\,000\ \text{cm}^3 = \dfrac{156\,000}{1000}$ litres $= 156$ litres
2 Write the answer.	$156\,000\ \text{cm}^3 = 156$ litres

Volume of prisms

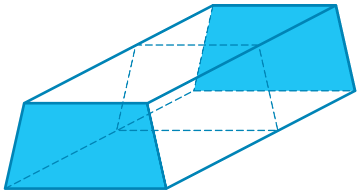
A prism is a **polyhedron** with a uniform cross-section.

It is named in accordance with its uniform cross-sectional area.

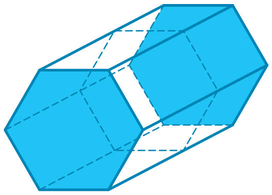




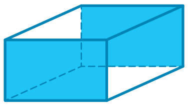
Triangular prism



Trapezoidal prism



Hexagonal prism



Rectangular prism

To find the volume of a prism we need to determine the area of the uniform cross-section (or base) and multiply by the height. This is the same for all prisms. Volume of a prism, V_{prism}, can be generalised by the formula:

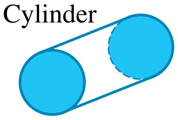
$$\boxed{\begin{aligned} V_{\text{prism}} &= \text{area of uniform cross-section} \times \text{height} \\ V &= AH \end{aligned}}$$

For example:

$$V_{\text{rect. prism}} = A_{\text{rect.}} \times H$$

$$V_{\text{triangular prism}} = A_{\text{triangle}} \times H$$

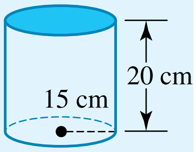
Note: Although cylinders are not prisms, they have a uniform cross-section (which is a circle) therefore, the same formula can be applied to find volume of a cylinder.



That is, $\boxed{\begin{aligned} V_{\text{cylinder}} &= A_{\text{circle}} \times H \\ &= \pi r^2 H \end{aligned}}$

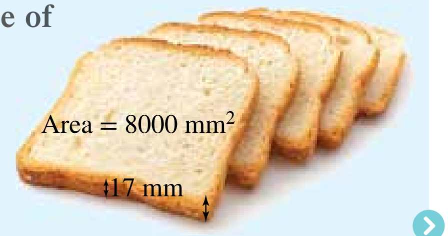
In fact, the formula $V = A \times H$ can be applied to any solid with a uniform cross-section of area A.

WORKED EXAMPLE 23 **Calculate the volume of the object shown. Give your answer correct to the nearest cm^3.**



THINK	WRITE
1 The object has a circle as a uniform cross-section. It is a cylinder. The area of the base is: area of a circle $= \pi r^2$. Volume is cross-sectional area times height.	$V_{\text{cylinder}} = A \times H$, where $A_{\text{circle}} = \pi r^2$ $= \pi \times r^2 \times H$ $= \pi \times 15^2 \times 20$ $= 4500\pi$ $\approx 14\,137.1669\text{cm}^3$
2 Write your answer.	The volume of the cylinder is approximately $14\,137\ \text{cm}^3$.

WORKED EXAMPLE 24 **Calculate (correct to the nearest mm^3) the volume of the slice of bread with a uniform cross-sectional area of $8000\ \text{mm}^2$ and a thickness of 17 mm.**

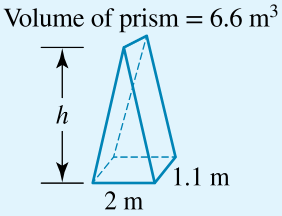


THINK	**WRITE**
1 The slice of bread has a uniform cross-section. The cross-section is not a common figure but its area has been given.	$V = A \times H$ where $A = 8000\ \text{mm}^2$ $V = 8000\ \text{mm}^2 \times 17\ \text{mm}$ $= 136\,000\ \text{mm}^3$
2 Write your answer.	The volume of the slice of bread is 136 000 mm^3.

Given the volume of an object, we can use the volume formula to find an unknown dimension of the object by transposing the formula.

WORKED EXAMPLE 25

Calculate the height of the triangle (correct to 1 decimal place) from the information provided in the diagram.

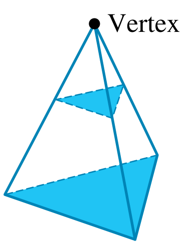


THINK	**WRITE**
1 The volume of the object is given, along with the width of the triangular cross-section and the height of the prism.	$V = 6.6\ \text{m}^3,\ H = 1.1\ \text{m},\ b = 2\ \text{m}$ $V = A \times H,$ where $A = \frac{1}{2} \times b \times h$ $V = \frac{1}{2} \times b \times h \times H$
2 Substitute the values into the formula and solve for h.	$6.6 = \frac{1}{2} \times 2 \times h \times 1.1$ $= 1.1\,h$ $h = \frac{6.6}{1.1}$ $= 6$
3 Write your answer.	The height of the triangle in the given prism is 6.0 metres.

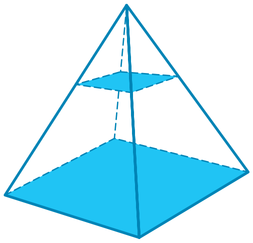
Volume of pyramids

A pyramid is a polyhedron, where the base is any polygon and all other faces are triangles meeting at the vertex.

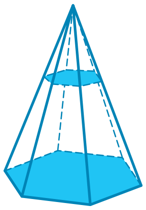
The name of the pyramid is related to the shape of the polygon at the base.



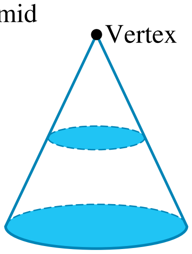
Triangular pyramid



Square-based pyramid



Hexagonal pyramid



Cone

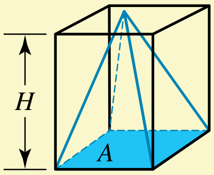
The shape of the cross-section of the pyramid remains unchanged, but its size reduces as it approaches the vertex.

Similarly, for cones, the shape of the cross-section is always the same (a circle), but its size reduces as we move from base towards the vertex.

The volume of a pyramid is always one-third of the volume of a prism with the same base and same height, H. This holds for all pyramids.

Volume of a pyramid, V_{pyramid}, can be found by using the formula:

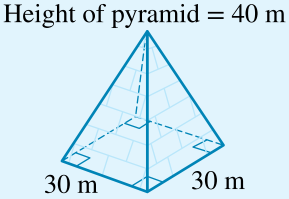
$$V_{\text{pyramid}} = \frac{1}{3} \times \text{area of the base} \times \text{height}$$
$$= \frac{1}{3}AH$$
$$V_{\text{cone}} = \frac{1}{3}\pi r^2 H$$



The height of a pyramid, H, is sometimes called the altitude.

WORKED EXAMPLE 26

Calculate the volume of the pyramid shown (correct to the nearest m^3).



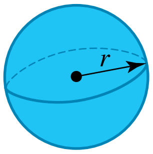
THINK	WRITE
1 The pyramid has a square base. The area of the base is: Area of a square = l^2.	$V_{\text{pyramid}} = \frac{1}{3} \times A \times H,\ \text{where} A_{\text{square}} = l^2$ $= \frac{1}{3} \times l^2 \times H$ $= \frac{1}{3} \times 30^2 \times 40$ $= 12\,000\,\text{m}^3$
2 Write your answer.	The volume of the pyramid is 12 000 m^3.

Volume of spheres and composite objects

Volume of a sphere

Spheres are unique (but common) objects that deserve special attention.

The formula for the volume of a sphere is:



$$V_{\text{sphere}} = \frac{4}{3}\pi r^3$$

where r is the radius of the sphere.

Volume of composite objects

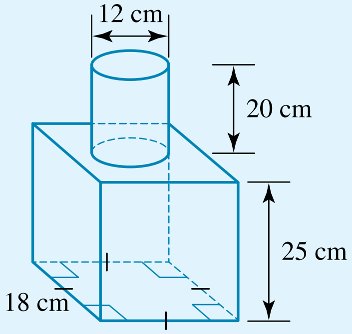
Often the object can be identified as comprising two or more different common prisms, pyramids or spheres. Such figures are called composite objects. The volume of a composite object is found by adding the volumes of the individual common figures or deducting volumes. For example, each of the structures on the right can be roughly modelled as the sum of a cylinder and a cone.

Volume of a composite object = the sum of the volumes of the individual components.

$$V_{\text{composite}} = V_1 + V_2 + V_3 + \ldots \text{ (or } V_{\text{composite}} = V_1 - V_2\text{)}$$

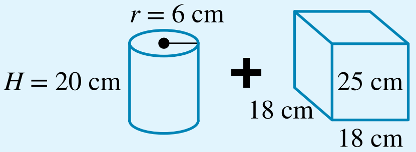
WORKED EXAMPLE 27

Calculate the capacity of the container shown. (Give your answer correct to the nearest litre.)



THINK | **WRITE/DRAW**

1 The object consists of a cylinder and a square-based prism.



2 The volume of the composite object is the sum of the volumes of the cylinder and the prism.

$$
\begin{aligned}
V_{\text{composite}} &= \text{volume of cylinder} + \text{volume of square-based prism}\\
&= A_{\text{circle}} \times H_{\text{cylinder}} + A_{\text{square}} \times H_{\text{prism}}\\
&= (\pi r^2 \times H_c) + (l^2 \times H_p)\\
&= (\pi \times 6^2 \times 20) + (18^2 \times 25)\\
&= 2261.946711 + 8100\\
&= 10361.946711\,\text{cm}^3
\end{aligned}
$$

3 Convert to litres using the conversion of 1000 cm^3 equals 1 litre.

10 362 cm^3 = 10.362 litres

4 Write your answer.

The capacity of the container is 10 litres, correct to the nearest litre.

EXERCISE 11.7 Volume of prisms, pyramids and spheres

PRACTISE

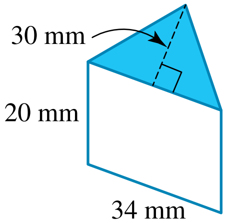
1 **WE21** Convert 0.35 cm^3 to mm^3.

2 Convert 4800 cm^3 to m^3.

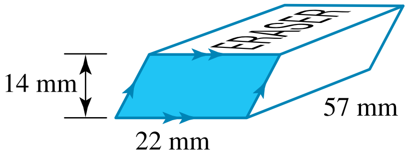
3 **WE22** Convert 1.6 m^3 to litres.

4 Convert 250 000 mm^3 to cm^3.

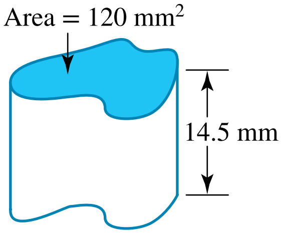
5 **WE23** Calculate the volume of the following solid correct to the nearest whole unit.



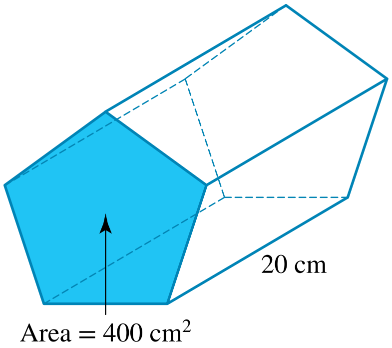
6 Calculate the volume of the following solid correct to the nearest whole unit.



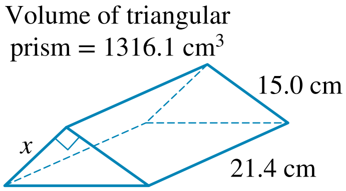
7 **WE24** Calculate the volume of the following object (correct to 2 decimal places).



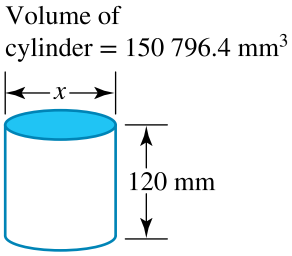
8 Calculate the volume of the following object.



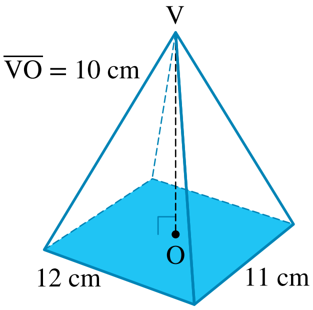
9 **WE25** Calculate the measurement of the unknown dimension (correct to 1 decimal place).



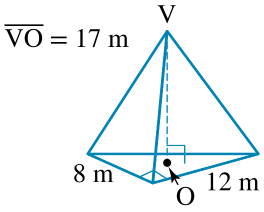
10 Calculate the measurement of the unknown dimension (correct to 1 decimal place).



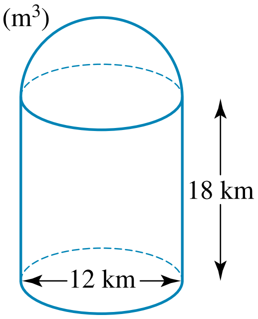
11 **WE26** Calculate the volume of the object (correct to the nearest whole unit).



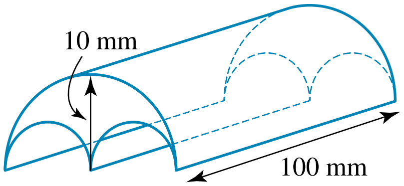
12 Calculate the volume of the object (correct to the nearest whole unit).



13 **WE27** Calculate the volume of the figure below. Give your answer in m^3 correct to 3 significant figures.



14 Calculate the volume of the figure. Give your answer in m^3 correct to 4 significant figures.



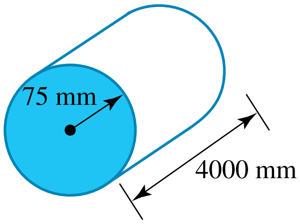
CONSOLIDATE

15 Convert the volumes to the units specified.

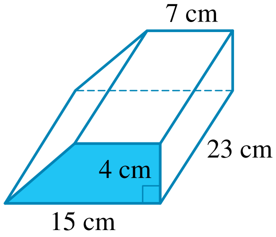
a 530 cm^3 to m^3
b 56 000 cm^3 to litres
c 15 litres to cm^3
d 72.1 m^3 to litres
e 0.0023 cm^3 to mm^3
f 0.000 57 m^3 to cm^3
g 140 000 mm^3 to litres
h 16 000 mm^3 to cm^3

16 Calculate the volume of the following solids correct to the nearest whole unit.

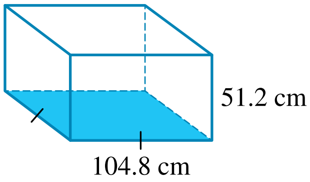
a



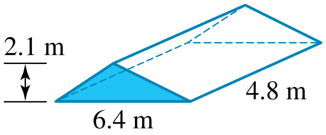
b



c

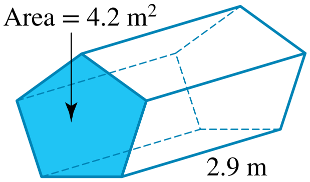


d

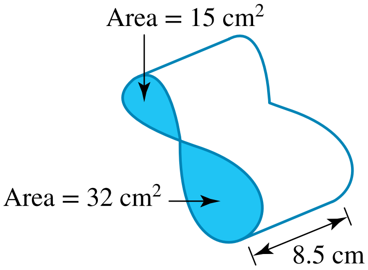


17 Calculate the volume of the following objects (correct to 2 decimal places).

a

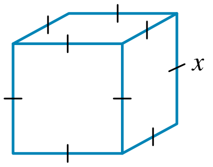


b

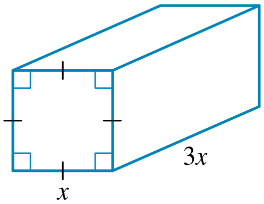


18 Calculate the measurement of the unknown dimension (correct to 1 decimal place).

a Volume of
cube = 1.728 m^3

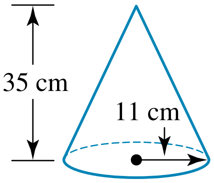


b Volume of prism = $10\frac{1}{8}$ litres

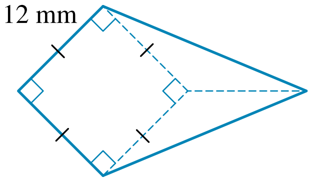


19 Calculate the volume of these objects (correct to the nearest whole unit).

a

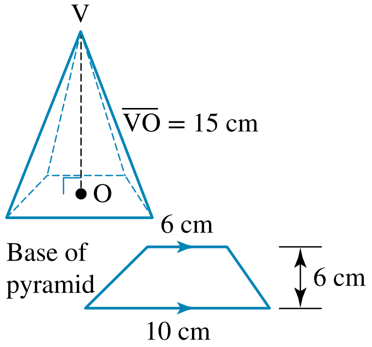


b



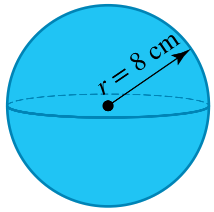
Altitude of square-based pyramid = 18 mm

c

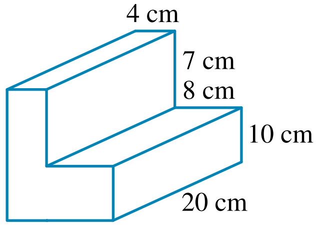


20 Calculate the volume of these objects (correct to the nearest whole unit).

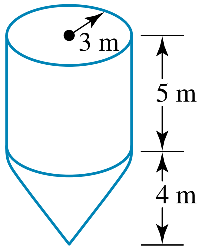
a



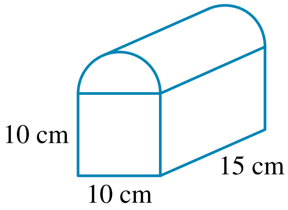
b



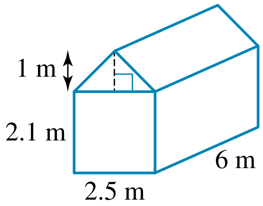
c



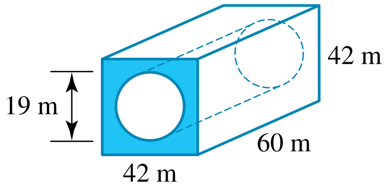
d



e



f



21 **a** Calculate the volume of a cube with sides 4.5 cm long.

b Calculate the volume of a room, 3.5 m by 3 m by 2.1 m high.

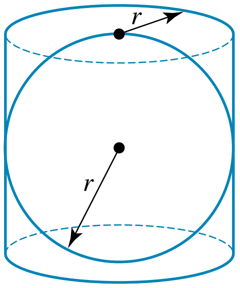
c Calculate the radius of a baseball that has a volume of 125 cm^3.

d Calculate the height of a cylinder that is 20 cm in diameter with a volume of 2.5 litres (correct to the nearest unit).

e Calculate the height of a triangular prism with a base area of 128 mm^2 and volume of 1024 mm^3.

f Calculate the depth of water in a swimming pool that has a capacity of 56 000 litres. The pool has rectangular dimensions of 8 metres by 5.25 metres.

22 The ratio of the volume of a sphere to that of a cylinder of similar dimensions, as shown in the diagram, is best expressed as:



A $\frac{4}{3}$

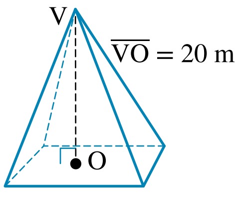
B $\frac{2}{3}$

C $\frac{\frac{4}{3}r}{h}$

D $\frac{3}{4}$

E $\frac{3}{2}$

23 If the volume of the square-based pyramid shown is 6000 m^3, then the perimeter of the base is closest to:



A 900 m

B 20 m

C 30 m

D 80 m

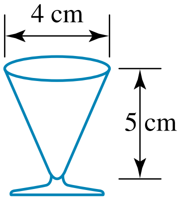
E 120 m

24 A tin of fruit is 13 cm high and 10 cm in diameter. Its volume, correct to 1 decimal place, is:

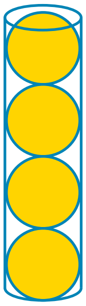
A 1021.0 cm^3 **B** 510.5 cm^3 **C** 1021.4 cm^3 **D** 1020.1 cm^3 **E** 4084.1 cm^3

MASTER

25 The medicine cup shown has the shape of a cone with a diameter of 4 cm and a height of 5 cm (not including the cup's base). Calculate the volume of the cone correct to the nearest millilitre, where 1 cm^3 = 1 mL.



26 Tennis balls have a diameter of 6.5 cm and are packaged in a cylinder that can hold four tennis balls. Assuming the balls just fit inside the cylinder, calculate:



- a the height of the cylindrical can
- b the volume of the can (correct to 1 decimal place)
- c the volume of the four tennis balls (correct to 1 decimal place)
- d the volume of the can occupied by air
- e the fraction of the can's volume occupied by the balls.

11.8 Similar figures

study on

- Unit 4
- AOS M3
- Topic 1
- Concept 5

Similarity
Concept summary
Practice questions

Objects that have the same shape but different size are said to be **similar**.

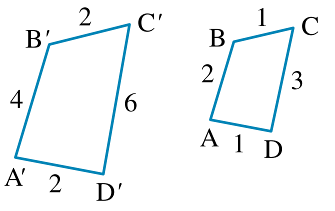
For two figures to be similar, they must have the following properties:

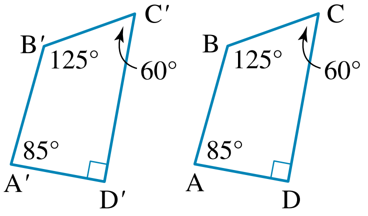
1. The ratios of the corresponding sides must be equal.

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{A'D'}{AD} = \text{common ratio}$$

2. The corresponding angles must be equal.

$\angle A = \angle A'$ $\angle B = \angle B'$ $\angle C = \angle C'$
$\angle D = \angle D'$





Scale factor, *k*

A measure of the relative size of the two similar figures is the **scale factor**. The scale factor is the common ratio of the corresponding sides and quantifies the amount of enlargement or reduction one figure undergoes to transform into the other figure. The starting shape is commonly referred to as the *original* and the transformed shape as the *image*.

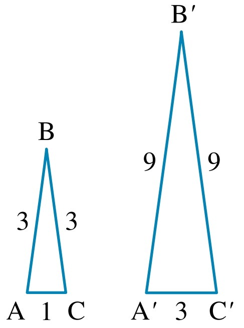
Scale factor, k, is the amount of enlargement or reduction and is expressed as integers, fraction or scale ratios.

For example, $k = 2$, $k = \frac{1}{12}$ or 1 : 10 000.

$$\textbf{Scale factor, } k = \frac{\textbf{length of image}}{\textbf{length of original}}$$

$$= \frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{C'A'}}{\overline{CA}}$$

where for enlargements k is greater than 1 and for reductions k is between 0 and 1.



For $k = 1$, the figures are exactly the same shape and size and are referred to as **congruent**.

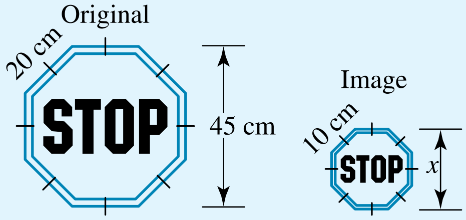
Enlargements and reductions are important in many aspects of photography, map making and modelling. Often, photographs are increased in size (enlarged) to examine fine detail without distortion, while house plans are an example of a reduction to a scale; for example, 1 : 25.

WORKED EXAMPLE 28

For the similar shapes shown:

a find the scale factor for the reduction of the shape

b find the unknown length in the smaller shape.

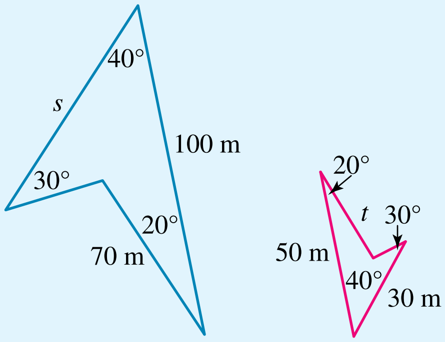


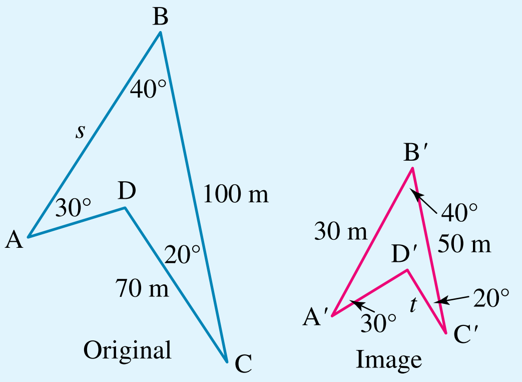
THINK	WRITE
a **1** As it is a reduction, the larger shape is the *original* and the smaller shape is the *image*.	**a** Scale factor, $k = \dfrac{\text{length of image}}{\text{length of original}}$
2 The two shapes have been stated as being similar, so set up the scale factor, k.	$= \dfrac{\overline{A'B'}}{\overline{AB}}$ $= \dfrac{10\text{ cm}}{20\text{ cm}}$ $= \dfrac{1}{2}$
b **1** Use the scale factor to determine the unknown length as all corresponding lengths are in the same ratio.	**b** Scale factor, $k = \dfrac{1}{2}$ $k = \dfrac{\text{length of image}}{\text{length of original}}$ $\dfrac{1}{2} = \dfrac{x}{45\text{ cm}}$ $x = \dfrac{1}{2} \times 45\text{ cm}$ $= 22.5\text{ cm}$
2 Write your answers.	The scale factor of reduction is $\dfrac{1}{2}$ and the unknown length on the smaller shape is 22.5 cm.

WORKED EXAMPLE 29

The figures given are similar.

Find the lengths of the two unknown sides *s* and *t*.



THINK	WRITE/DRAW
1 First, redraw the figures so that they are oriented the same way. (This will help to identify corresponding sides and angles.) Label the vertices of the 1st figure A, B, C and D and corresponding vertices of the 2nd figure A′, B′, C′ and D′ respectively.	

2 Since it is known that the figures are similar, ratios of the corresponding side lengths must be equal. Both length, $\overline{BC}$ and $\overline{B'C'}$ are known. These can be used to find the value of s, as the ratio of $\overline{BA}$ to $\overline{B'A'}$ must be the same.

$$\frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{B'A'}}{\overline{BA}}$$

$$\frac{50}{100} = \frac{30}{s}$$

$$s = \frac{100 \times 30}{50}$$

$$s = 60 \text{ m}$$

3 Now find the value of t. (The ratio of $\overline{DC}$ to $\overline{D'C'}$ must be the same as the ratio of $\overline{BC}$ to $\overline{B'C'}$.)

$$\frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{D'C'}}{\overline{DC}}$$

$$\frac{50}{100} = \frac{t}{70}$$

$$t = \frac{50 \times 70}{100}$$

$$t = 35 \text{ m}$$

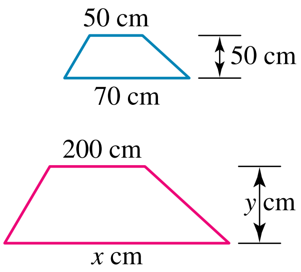
EXERCISE 11.8 Similar figures

PRACTISE

1 **WE28** For the pair of similar shapes, calculate:

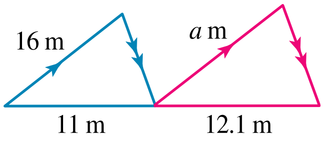
a the scale factor

b the value of x and y.

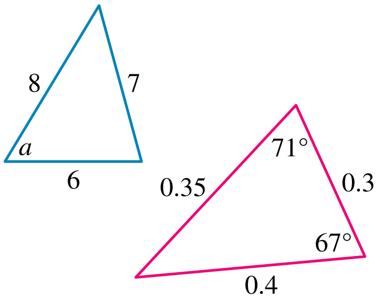


2 A tree is 7 mm long in a photograph, while it stands 3 m tall in real life. What is the scale factor?

3 **WE29** Determine the scale factor and calculate the pronumeral for the following figure.



4 Determine the scale factor and calculate the pronumeral for the following figure.



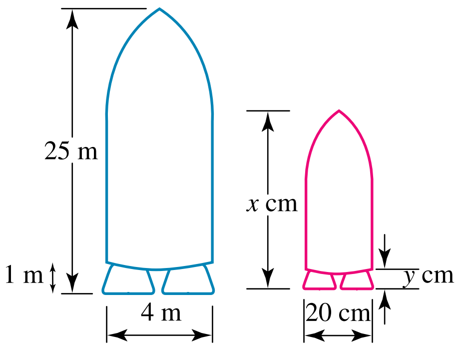
CONSOLIDATE

5 For each of these pairs of similar shapes, calculate:

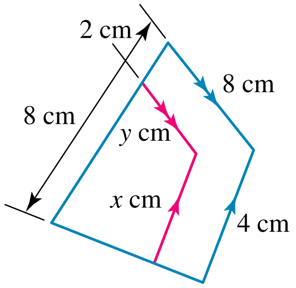
i the scale factor

ii the value of x and y.

a

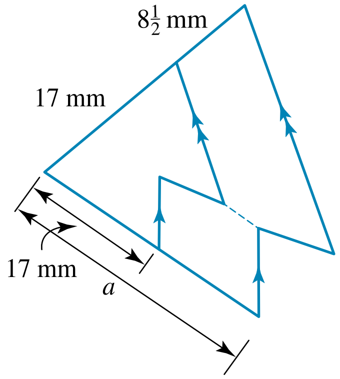


b

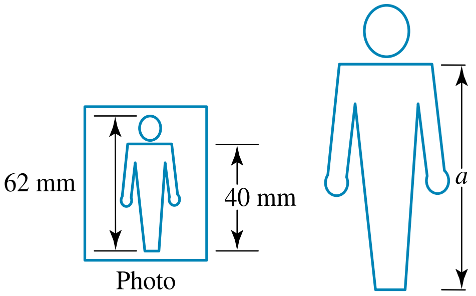


6 For each of the following pairs of similar figures, calculate the value of a.

a



b



Height of person = 186 cm

7 A photo has the dimensions 10 cm by 12 cm. The photo is enlarged by a factor of 2.5. Calculate the new dimensions of the photo.

8 A set of model cars is made using the scale ratio 1 : 12. Calculate:

a the length of a real car if the model is 20 cm long (in metres correct to 1 decimal place)

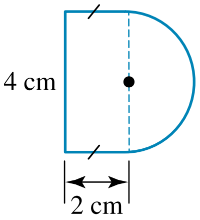
b the height of a real car if the model is 3 cm high (correct to the nearest centimetre)

c the length of a model if the real car is 3 metres long.

9 The dimensions of a student's room are 4300 mm by 3560 mm. A scale diagram of the room is to be drawn on an A4 sheet, using the scale ratio 1 : 20. Calculate the dimensions of the scale drawing of the room and state whether the drawing should be landscape or portrait on the A4 sheet.

10 The scale used to draw the diagram is 1 : 25. The perimeter of the real object is closest to:

A 464 cm
B 514 cm
C 357 cm
D 14.28 cm
E 150 cm



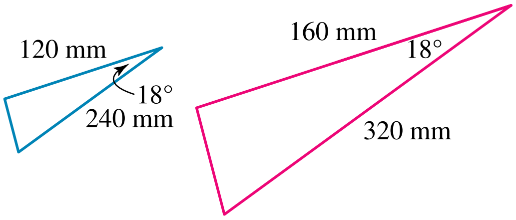
11 A 1 : 27 scale model of a truck is made from clay. What is the length of the tray on the original truck, if length of the tray on the model is 27 cm?

A 1 cm **B** 100 cm **C** 270 cm **D** 540 cm **E** 729 cm

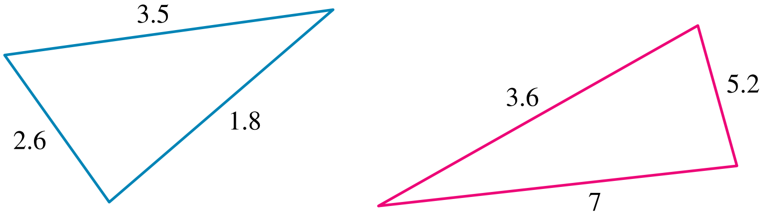
12 A scale factor of 0.2 is:

A a reduction with a scale of 1 cm = 2 cm
B an enlargement with a scale of 1 cm = 0.2 cm
C an enlargement with a scale of 1 cm = 5 cm
D a reduction with a scale of 1 cm = 5 cm
E a reduction with a scale of 1 cm = 20 cm

13 Determine the scale factor of the figure.



14 Determine the scale factor of the pair of triangles.



MASTER

15 Jane wants a larger clock in her house. The current clock has a radius of 20 cm, but Jane wants one 2.5 times its size. Calculate the circumference of the new clock.

16 The distance from Melbourne to Ballarat is 113 km. If a map used a scale ratio of 1 : 2 500 000, what is the distance on the map that represents this distance correct to one decimal place?

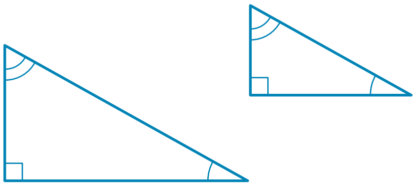
11.9 Similar triangles

eBook*plus*

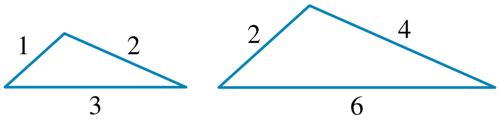
Interactivity
Similar triangles
int-6273

Similar triangles can be used in a variety of situations. For example, with the aid of similar triangles, we could find the heights of trees and buildings or the width of rivers and mountains. Two triangles are similar if one of the following is true:

1. All three corresponding angles are equal (AAA).

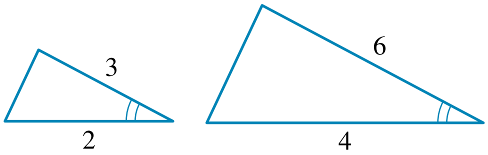


2. All three corresponding pairs of sides are in the same ratio (linear scale factor) (SSS).



$$\text{Scale factor} = \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = 0.5 = k$$

3. Two corresponding pairs of sides are in the same ratio and the included angles are equal (SAS).



$$\text{Scale factor} = \frac{6}{3} = \frac{4}{2} = 2 = k$$

As in the previous section, we use the known values of a pair of corresponding sides to determine the scale factor (sf) for the similar triangles.

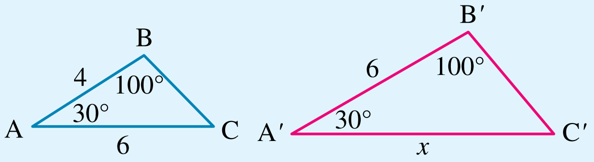
$$\textbf{Scale factor, } k = \frac{\textbf{length of side of image}}{\textbf{length of corresponding side of original}}$$

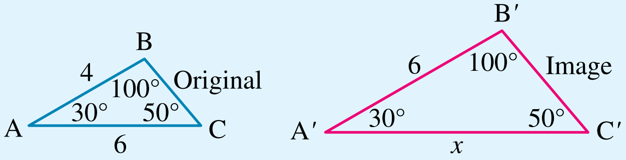
WORKED EXAMPLE 30

For the triangles shown:

a **State the rule that proves that the triangles are similar and determine the scale factor.**

b **Find the value of the pronumeral, x.**



THINK	WRITE/DRAW
a **1** Two corresponding angles are equal. The third angle is not given but can be easily found using the rule that all angles in a triangle add to 180°. State the rule that proves similarity.	**a** $\angle A = \angle A' = 30°$ $\angle B = \angle B' = 100°$ $\angle C = \angle C' = 180° - (30° + 100°)$ $= 50°$ $\therefore$ ΔABC is similar to $\Delta A'B'C'$ because all three corresponding angles are equal (AAA).
2 Always select the triangle with the unknown length, x, as the *image*. Evaluate the scale factor by selecting a pair of corresponding sides with both lengths known.	 $\text{Scale factor, } k = \frac{\text{length of side of image}}{\text{length of corresponding side of original}}$ $= \frac{\overline{A'B'}}{\overline{AB}}$ $= \frac{6}{4}$ $= 1.5$

b **1** Use the scale factor to find the unknown length, x.

b Scale factor, $k = 1.5$

$$1.5 = \frac{\overline{A'C'}}{\overline{AC}}$$

$$1.5 = \frac{x}{6}$$

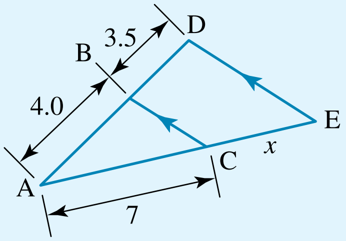
$$x = 1.5 \times 6$$

$$= 9$$

2 Write the answer.

The value of the pronumeral, x, is 9.

WORKED EXAMPLE 31 **For the given triangles, find the value of the pronumeral, x.**



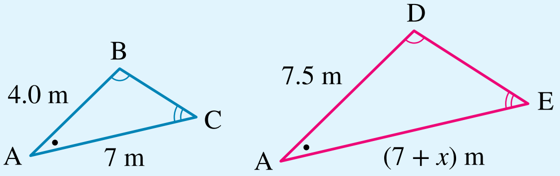
All measurements in metres

THINK

WRITE/DRAW

1 For clear analysis, separate the two triangles. *Note:* The lengths of the sides $\overline{AE}$ and $\overline{AD}$ are the sum of the given values.

Establish that the two triangles are similar using an appropriate rule.



$\overline{AD} = 4.0 + 3.5$
$= 7.5\,\text{m}$
$\overline{AE} = (7 + x)\,\text{m}$
$\angle A = \angle A$ (common)
$\angle B = \angle D$ (corresponding angles are equal)
$\angle C = \angle E$ (corresponding angles are equal)
$\therefore \Delta ABC$ is similar to ΔADE (AAA).

2 Since the triangles are similar, the ratios of the corresponding sides are equal.

$$\frac{\overline{AE}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AB}}$$

$$\frac{7 + x}{7} = \frac{7.5}{4}$$

3 Cross-multiply and solve for x.

$$4(7 + x) = 7 \times 7.5$$
$$28 + 4x = 52.5$$
$$4x = 24.5$$
$$x = 6.125$$

4 Write the answer including units.

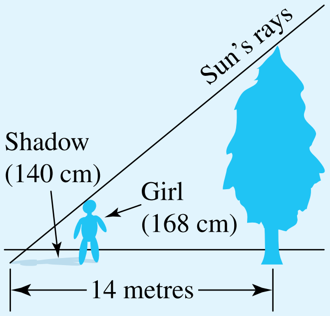
The value of x is 6.125 metres.

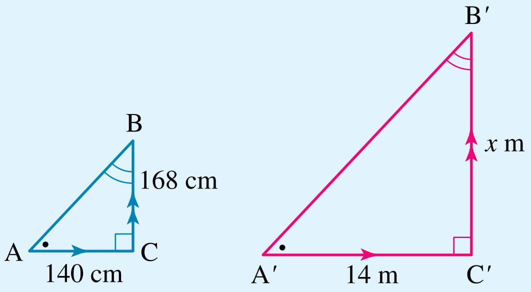
There are many practical applications of similar triangles in the real world. It is particularly useful for determining the lengths of inaccessible features, such as the height of tall trees or the width of rivers. This problem is overcome by setting up a triangle similar to the feature to be examined, as shown in the next example.

WORKED EXAMPLE 32

Find the height of the tree shown in the diagram.

Give the answer correct to 1 decimal place.

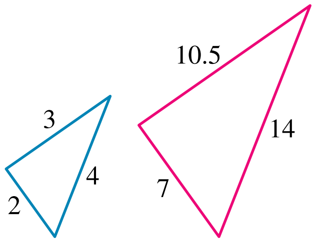


THINK	WRITE/DRAW
1 For clear analysis, separate the two triangles. Establish that the two triangles are similar. (Assuming both the tree and the girl are perpendicular to the ground, they form parallel lines.)	 $\angle A = \angle A'$ (common angles) $\angle B = \angle B'$ (corresponding) $\angle C = \angle C'$ (corresponding) $\therefore \Delta ABC$ is similar to $\Delta A'B'C'$ (AAA).
2 Since the triangles are similar, their corresponding sides must be in the same ratio. *Note:* Convert all measurements to metres, as it is the most appropriate unit for the height of a tree. So 140 cm = 1.4 m and 168 cm = 1.68 m.	$\frac{14}{1.4} = \frac{x}{1.68}$
3 Solve for x.	$10 = \frac{x}{1.68}$ $x = 10 \times 1.68$ $= 16.8$ m
4 Write the answer including units.	The height of the tree is 16.8 metres.

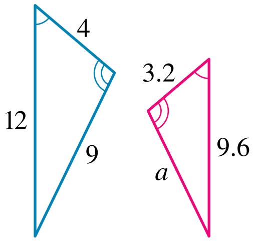
EXERCISE 11.9 Similar triangles

PRACTISE

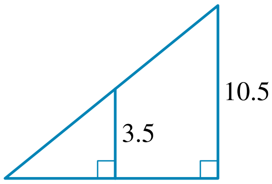
1 a WE30 State the rule (SSS or AAA or SAS) that proves that the triangles at right are similar and determine the scale factor (expressed as an enlargement $k > 1$).



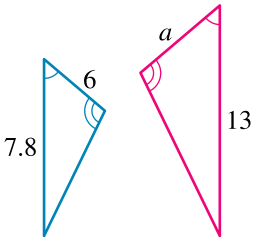
b Find the value of the pronumeral a.



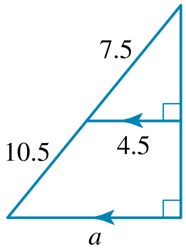
2 **a** State the rule (SSS or AAA or SAS) that proves that the triangles below are similar and determine the scale factor (expressed as an enlargement $k > 1$).



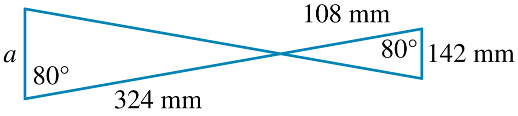
b Find the value of the pronumeral a.



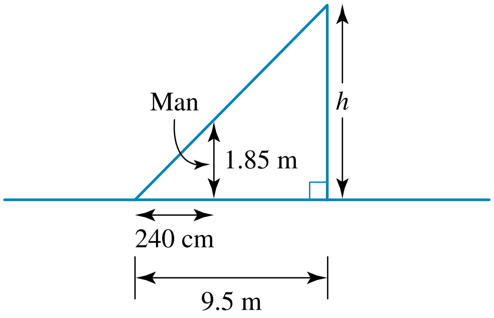
3 **WE31** Find the value of the pronumeral a.



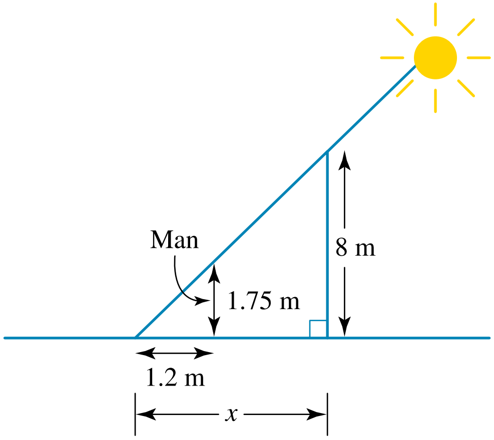
4 Find the value of the pronumeral a.



5 **WE32** Find the height h, to the nearest centimetre, of the billboard shown in the diagram.



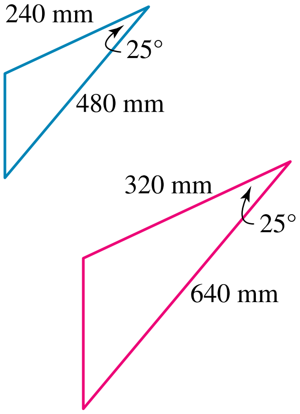
6 Find the length of the shadow cast by the telecommunications pole, correct to 1 decimal place.



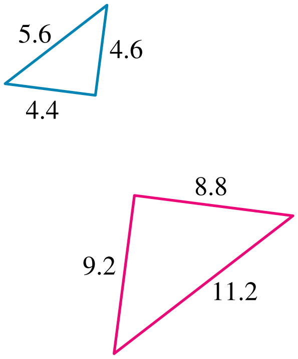
CONSOLIDATE

7 State the rule (SSS or AAA or SAS) that proves that the triangles in each pair are similar and determine the scale factor (expressed as an enlargement $k > 1$).

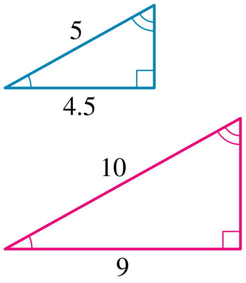
a



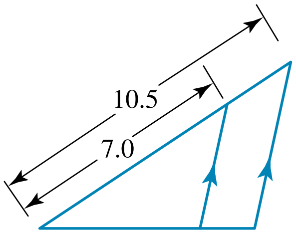
b



c

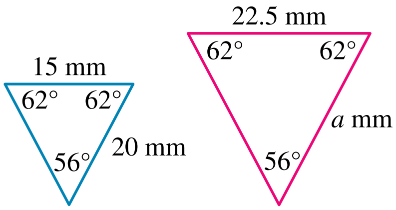


d

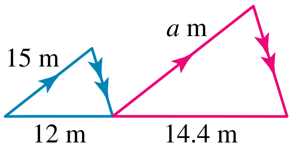


8 For the given pairs of similar triangles, find the value of the pronumeral a.

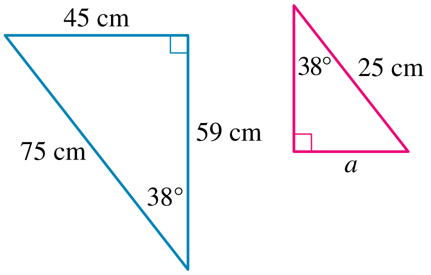
a



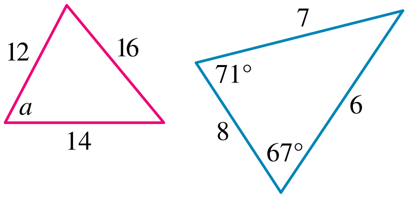
b



c

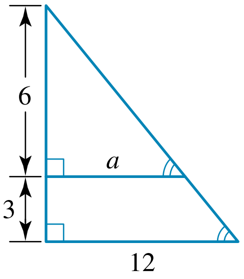


d

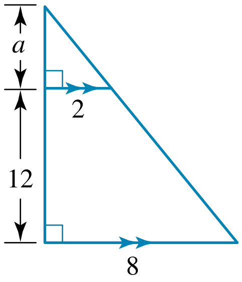


9 For the given pairs of triangles, find the value of the pronumeral a.

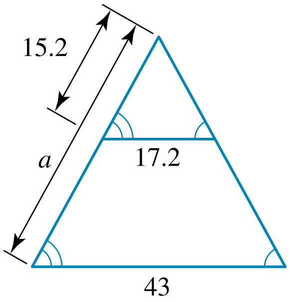
a



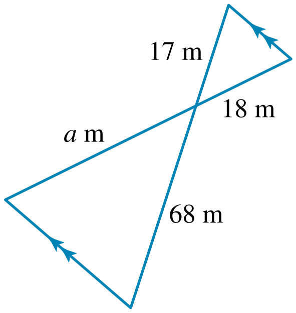
b



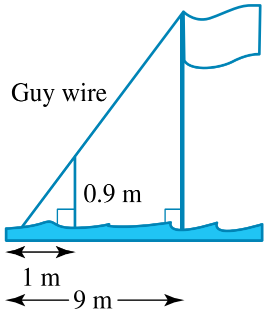
c



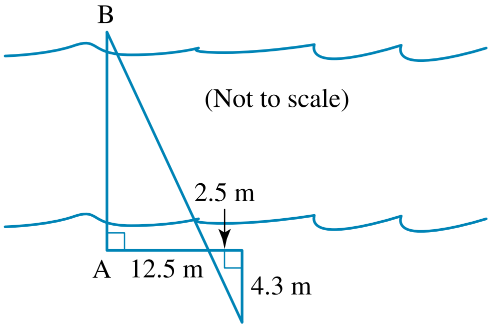
d



10 Find the height (correct to the nearest centimetre) of the flagpole shown in the diagram.

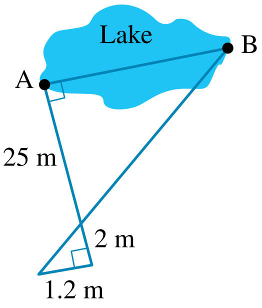


11 Find the length (correct to 1 decimal place) of the bridge, $\overline{AB}$, needed to span the river, using similar triangles as shown.



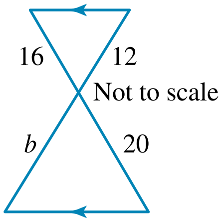
12 The shadow of a tree is 4 metres and at the same time the shadow of a 1-metre stick is 25 cm. Assuming both the tree and stick are perpendicular to the horizontal ground, what is the height of the tree?

13 Find the width of the lake (correct to the nearest metre) using the surveyor's notes.

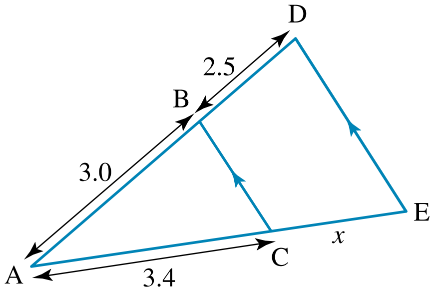


14 In the given diagram, the length of side b is closest to:

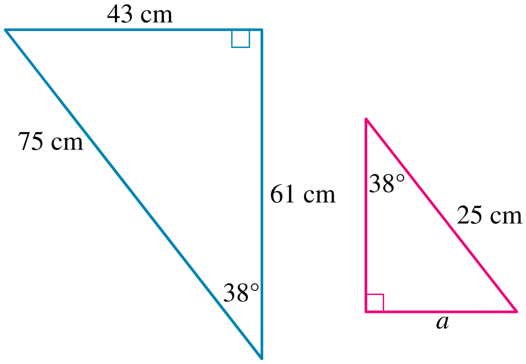
A 24 **B** 22 **C** 16 **D** 15 **E** 9.6



15 For the following triangles, state the rule (SSS, AAA or SAS) that proves that the pair of triangles is similar, and then calculate the value of the pronumeral.



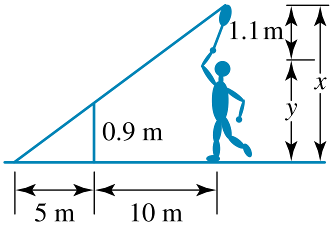
16 For the following triangles, state the rule (SSS, AAA or SAS) that proves that the pair of triangles is similar, and then calculate the value of the pronumeral.



MASTER

Questions **17** and **18** refer to the following information.

A young tennis player's serve is shown in the diagram.



Assume the ball travels in a straight line.

17 The height of the ball x, just as it is hit, is closest to:

A 3.6 m **B** 2.7 m **C** 2.5 m **D** 1.8 m **E** 1.6 m

18 The height of the player, y, as shown is closest to:

A 190 cm **B** 180 cm **C** 170 cm **D** 160 cm **E** 150 cm

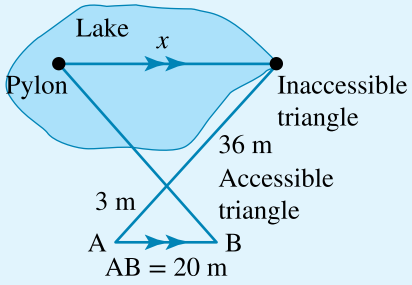
11.10 Triangulation — similarity

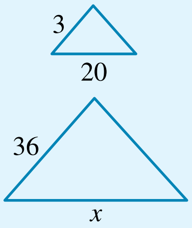
We can solve triangulation problems by using similar triangles. There are situations where a triangle can be constructed in an area that is accessible so as to determine the dimensions of a similar triangle in an inaccessible region.

1. We need two corresponding lengths to establish the scale factor between the two similar triangles. A second accessible side will be used to scale up or down to the corresponding inaccessible side.
2. For similar triangles use the following rules as proof:
   (a) AAA — all corresponding angles are the same
   (b) SSS — all corresponding sides are in the same ratio
   (c) SAS — two corresponding sides are in the same ratio with the same included angle.

WORKED EXAMPLE 33

Find the unknown length, x, from the pylon to the edge of the lake.

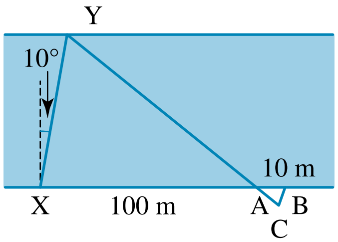


THINK	WRITE
1 Identify that the two triangles are similar (proof: AAA rule).	
2 Draw the triangles separately, highlighting the corresponding sides.	
3 In similar triangles, all corresponding sides are in the same ratio.	$\frac{3}{36} = \frac{20}{x}$
4 Transpose the equation to get the unknown by itself and evaluate.	$x = \frac{20 \times 36}{3}$ $= 240$ metres
5 Write the length and include units with the answer.	The distance from the edge of the lake to the pylon is 240 metres.

EXERCISE 11.10 Triangulation — similarity

PRACTISE

1 **WE33** A bridge is to be built across a river at an angle of 10 degrees to the river between points X and Y, as shown in the figure below.



A surveyor walks along the bank 100 m from X to A, then a further 10 m to B. He then walks to point C, so that a line of sight can be drawn through A and Y from C.

If the distance BC = 3.1 m, what will the length of the bridge be?

2 A young boy wishes to determine the height of a tall building. He stands 100 m away from the building and puts a 1-metre stick vertically in the ground. He then waits until the sun is in a position where the shadow of the stick and the shadow of the building end at the same point. This point is 1.7 m from the base of the stick. Determine the height of the building.

CONSOLIDATE

3 In Figure 1, find the length of the proposed bridge, AB.

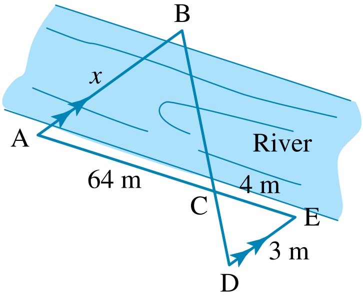


Figure 1

4 In Figure 2, find the length of the base of a hillside from C to D.

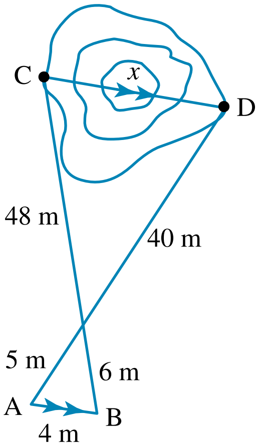


Figure 2

5 In Figure 3, find the perpendicular gap between the two city buildings.

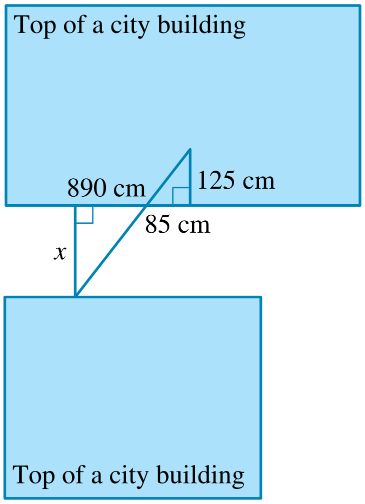


Figure 3

6 In Figure 4, find the distance between the two lighthouses.

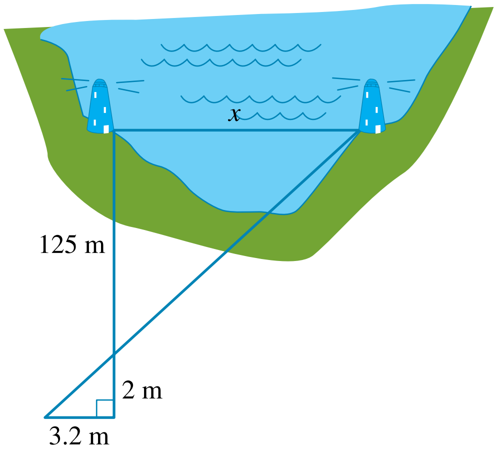


Figure 4

7 In Figure 5, find the distance across the lake.

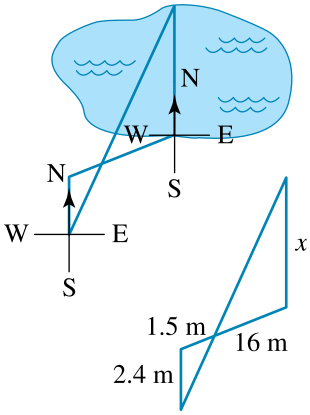


Figure 5

8 In Figure 6, find the height of the cyprus tree.

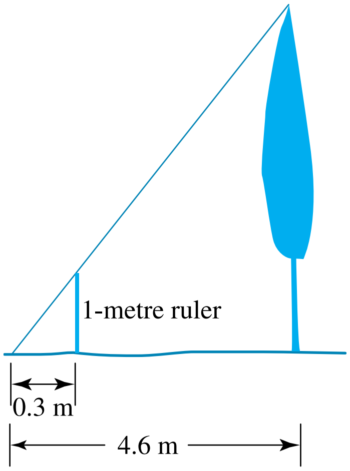
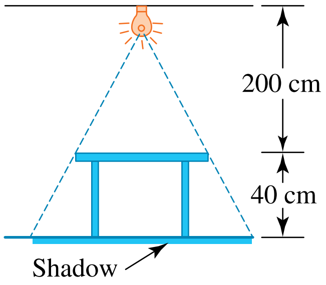
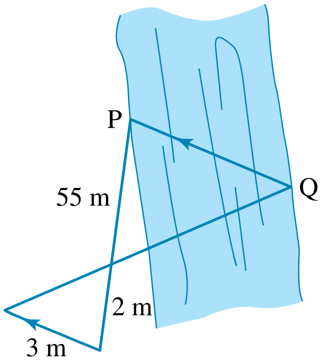


Figure 6

9 Find the width (correct to the nearest centimetre) of the shadow under the round table which has a diameter of 115 centimetres.



10 The distance across a river is to be determined using the measurements outlined.



The width from P to Q is closest to:

A 37 m **B** 60 m **C** 83 m **D** 113 m **E** 330 m

11 The shadow formed on the ground by a person who is 2 m in height was 5 m. At the same time a nearby tower formed a shadow 44 m long. The height of the tower, correct to the nearest metre, is:

A 18 m **B** 20 m **C** 51 m **D** 110 m **E** 123 m

12 In Figure 7, find the height (correct to the nearest centimetre) of the person being photographed.

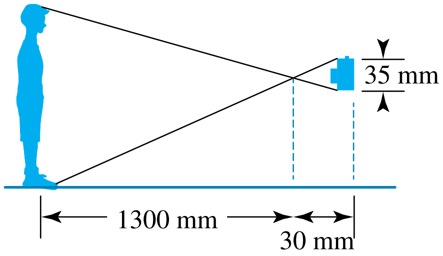


Figure 7

MASTER

13 In Figure 8, find the minimum distance from the tree to the camera, x metres, so that the tree is completely in the photo.

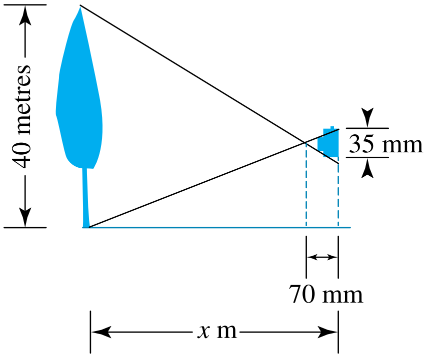


Figure 8

14 A girl is looking through her window.

a She is standing 2 metres from the window which is 2.4 metres wide. What is the width of her view:

- **i** 300 metres from the window (correct to the nearest metre)?
- **ii** 1.5 kilometres from the window (correct to the nearest 100 metres)?
- **iii** 6 kilometres from the window (correct to the nearest kilometre)?

b She is now standing 1 metre from the window. What is the width of her view:

- **i** 300 metres from the window (correct to the nearest metre)?
- **ii** 1.5 kilometres from the window (correct to the nearest 100 metres)?
- **iii** 6 kilometres from the window (correct to the nearest kilometre)?

11.11 Area and volume scale factors

An unknown area or volume of a figure can be found without the need to use known formulas such as in Exercises 11.3 and 11.7. We have seen that two figures that are similar have all corresponding lengths in the same ratio or (linear) scale factor, *k*. The same can be shown for the area and volume of two similar figures.

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Area scale factor
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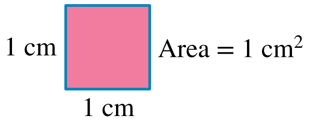
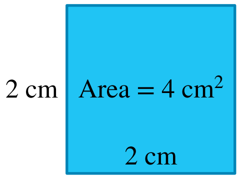
Area of similar figures

If the lengths of similar figures are in the ratio $a : b$ or k, then the areas of the similar shapes are in the ratio $a^2 : b^2$ or k^2. Following are investigations to support this relationship.

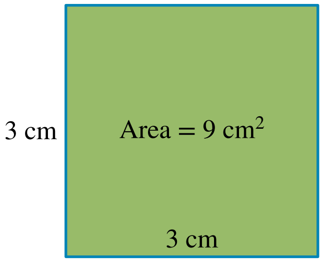
Different length ratios (or scale factors) of a square



$$\begin{aligned}\frac{\text{Length of blue square}}{\text{Length of pink square}} &= \frac{2\text{ cm}}{1\text{ cm}}\\ &= 2\\ &= k\end{aligned}$$



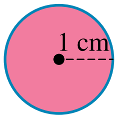
$$\begin{aligned}\frac{\text{Area of blue square}}{\text{Area of pink square}} &= \frac{4\text{ cm}^2}{1\text{ cm}^2}\\ &= 4\\ &= 2^2\\ &= k^2\end{aligned}$$



$$\begin{aligned}\frac{\text{Length of green square}}{\text{Length of pink square}} &= \frac{3\text{ cm}}{1\text{ cm}}\\ &= 3\\ &= k\end{aligned}$$

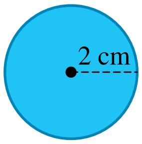
$$\begin{aligned}\frac{\text{Area of green square}}{\text{Area of pink square}} &= \frac{9\text{ cm}^2}{1\text{ cm}^2}\\ &= 9\\ &= 3^2\\ &= k^2\end{aligned}$$

Different length ratios (or scale factors) of a circle



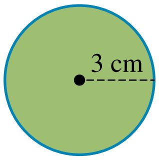
Area $= \pi r^2 = 1\pi\,\text{cm}^2$

$$\begin{aligned}\frac{\text{Radius length of blue circle}}{\text{Radius length of pink circle}} &= \frac{2\text{ cm}}{1\text{ cm}}\\ &= 2\\ &= k\end{aligned}$$



Area $= \pi r^2 = 4\pi\,\text{cm}^2$

$$\begin{aligned}\frac{\text{Area of blue circle}}{\text{Area of pink circle}} &= \frac{4\pi\,\text{cm}^2}{1\pi\,\text{cm}^2}\\ &= 4\\ &= 2^2\\ &= k^2\end{aligned}$$



Area $= \pi r^2 = 9\pi\,\text{cm}^2$

$$\begin{aligned}\frac{\text{Radius length of green circle}}{\text{Radius length of pink circle}} &= \frac{3\text{ cm}}{1\text{ cm}}\\ &= 3\\ &= k\end{aligned}$$

$$\begin{aligned}\frac{\text{Area of green circle}}{\text{Area of pink circle}} &= \frac{9\pi\,\text{cm}^2}{1\pi\,\text{cm}^2}\\ &= 9\\ &= 3^2\\ &= k^2\end{aligned}$$

As we can see, as long as two figures are similar, then the area ratio or scale factor is the square of the linear scale factor, k. The same applies for the total surface area.

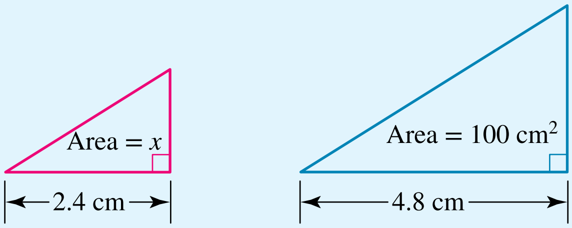
$$\begin{aligned}\textbf{Area scale ratio or factor (asf)} &= \frac{\textbf{area of image}}{\textbf{area of original}}\\ &= \textbf{square of linear scale factor (lsf)}\\ &= (\text{lsf})^2\\ &= k^2\end{aligned}$$

The steps required to solve for length, area or volume (investigated later) using similarity are:

1. Clearly identify the known corresponding measurements (length, area or volume) of the similar shape.
2. Establish a scale factor (linear, area or volume) using known measurements.
3. Convert to an appropriate scale factor to determine the unknown measurement.
4. Use the scale factor and ratio to evaluate the unknown.

WORKED EXAMPLE 34

For the two similar triangles shown, find the area, x cm^2, of the small triangle.



THINK

WRITE

1 Determine a scale factor, in this instance the linear scale factor, from the two corresponding lengths given. It is preferred that the unknown triangle is the image.

$$\text{Linear scale factor} = \frac{\text{length of small triangle (image)}}{\text{length of large triangle (original)}}$$

$$\begin{aligned}k &= \frac{2.4\text{ cm}}{4.8\text{ cm}}\\ &= \frac{1}{2}\end{aligned}$$

2 Determine the area scale factor.

$$\begin{aligned}\text{Area scale factor} &= k^2\\ &= \left(\frac{1}{2}\right)^2\\ &= \frac{1}{4}\end{aligned}$$

3 Use the area scale factor to find the unknown area.

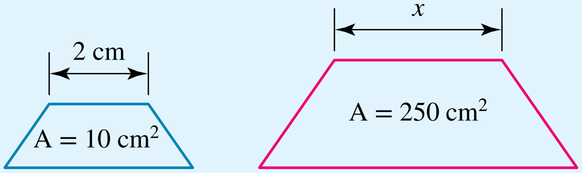
$$\text{Area scale factor} = \frac{\text{area of small triangle (image)}}{\text{area of large triangle (original)}}$$

$$\frac{1}{4} = \frac{x\text{ cm}^2}{100\text{ cm}^2}$$

4 Transpose the equation to get the unknown by itself.	$x = \frac{1}{4} \times 100$ $= 25$
5 Write your answer.	The area of the small triangle is 25 cm^2.

WORKED EXAMPLE 35

For the two similar shapes shown, find the unknown length, x cm.



THINK	WRITE
1 Determine a scale factor, in this instance the area scale factor, as both areas are known. It is preferred that the triangle with the unknown dimension is stated as the image.	$\text{Area scale factor} = \frac{\text{area of image (large trapezium)}}{\text{area of original (small trapezium)}}$ $k^2 = \frac{250\ \text{cm}^2}{10\ \text{cm}^2}$ $= 25$
2 Determine the linear scale factor.	$\text{Linear scale factor} = \sqrt{k^2}$ $k = \sqrt{25}$ $= 5$
3 Use the linear scale factor to find the unknown length.	$\text{Linear scale factor} = \frac{\text{length of image (large trapezium)}}{\text{length of original (small trapezium)}}$ $5 = \frac{x\ \text{cm}}{2\ \text{cm}}$
4 Transpose the equation to get the unknown by itself.	$x = 5 \times 2$ $= 10$
5 Write your answer.	The length, x, is 10 cm.

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Interactivity
Volume scale factor
int-6479

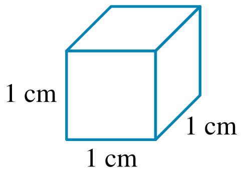
Volume of similar figures

If the lengths of similar figures are in the ratio $a : b$ or k, then the volume of the similar shapes are in the ratio $a^3 : b^3$ or k^3. The following is an investigation of two different objects, cubes and rectangular prisms.

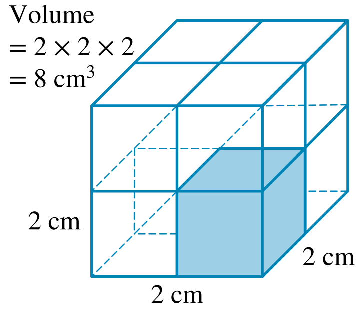
A cube

$$\frac{\text{Length of large cube}}{\text{Length of small cube}} = \frac{2\ \text{cm}}{1\ \text{cm}}$$
$$= 2$$
$$= k$$

$\text{Volume} = 1 \times 1 \times 1$
$= 1\ \text{cm}^3$

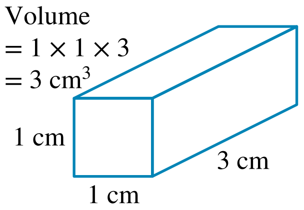


$$\begin{aligned}\frac{\text{Volume of large cube}}{\text{Volume of small cube}} &= \frac{8\text{ cm}^2}{1\text{ cm}^2}\\ &= 8\\ &= 2^3\\ &= k\end{aligned}$$

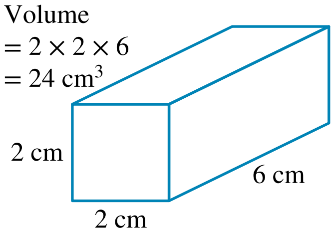


A rectangular prism

$$\begin{aligned}\frac{\text{Length of small prism}}{\text{Length of large prism}} &= \frac{3\text{ cm}}{6\text{ cm}}\\ &= \frac{1}{2}\\ &= k\end{aligned}$$



$$\begin{aligned}\frac{\text{Volume of small prism}}{\text{Volume of large prism}} &= \frac{3\text{ cm}^3}{24\text{ cm}^3}\\ &= \frac{1}{8}\\ &= \left(\frac{1}{2}\right)^3\\ &= k^3\end{aligned}$$

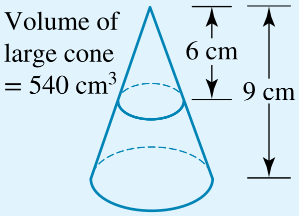


From above, as long as two figures are similar, then the volume ratio or scale factor is the cube of the linear scale factor, k.

$$\begin{aligned}\textbf{Volume scale factor (vsf)} &= \frac{\textbf{volume of image}}{\textbf{volume of original}}\\ &= \textbf{cube of linear scale factor (lsf)}\\ &= (\textbf{lsf})^3\\ &= k^3\end{aligned}$$

WORKED EXAMPLE 36

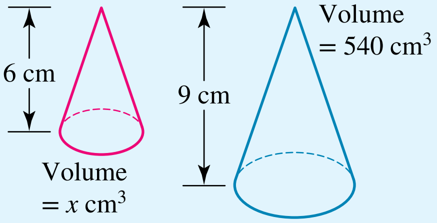
For the two similar figures shown, find the volume of the smaller cone.



THINK

WRITE/DRAW

1 Draw the two figures separately.



2 Determine a scale factor, in this instance the linear scale factor, from the two corresponding lengths given. It is preferred that the triangle with the unknown volume is the image.

$$\text{Linear scale factor} = \frac{\text{length of small triangle (image)}}{\text{length of large triangle (original)}}$$

$$k = \frac{6\text{ cm}}{9\text{ cm}}$$

$$= \frac{2}{3}$$

3 Determine the volume scale factor.

$$\text{Volume scale factor} = k^3$$

$$= \left(\frac{2}{3}\right)^3$$

$$= \frac{8}{27}$$

4 Use the volume scale factor to find the unknown length.

$$\text{Volume scale factor} = \frac{\text{volume of small cone (image)}}{\text{volume of large cone (original)}}$$

$$\frac{8}{27} = \frac{x\text{ cm}^3}{540\text{ cm}^3}$$

5 Transpose the equation to get the unknown by itself.

$$x = \frac{8}{27} \times 540$$

$$= 160$$

6 Write your answer.

The volume of the smaller cone is 160 cm^3.

We can use the relationship between linear, area and volume scale factors to find any unknown in any pair of similar figures as long as a scale factor can be established.

Given	Then	
Linear scale factor (lsf) $= k$ Example ($k = 2$) $= 2$	Area scale factor $= k^2$ $= 2^2$ $= 4$	Volume scale factor $= k^3$ $= 2^3$ $= 8$
Area scale factor (asf) $= k^2$ Example $= 4$	Linear scale factor $= \sqrt{k^2}$ $= \sqrt{4}$ $= 2$	Volume scale factor $= k^3$ $= 2^3$ $= 8$
Volume scale factor (vsf) $= k^3$ Example $= 8$	Linear scale factor $= \sqrt[3]{k^3}$ $= \sqrt[3]{8}$ $= 2$	Area scale factor $= k^2$ $= 2^2$ $= 4$

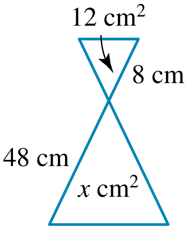
WORKED EXAMPLE 37

For two similar triangular prisms with volumes of 64 m^3 and 8 m^3, find the total surface area of the larger triangular prism, if the smaller prism has a total surface area of 2.5 m^2.

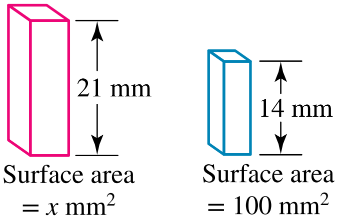
THINK	WRITE
1 Determine a scale factor, in this instance the volume scale factor, from the two known volumes. It is preferred that the larger unknown triangular prism is stated as the image.	$\text{Volume scale factor} = \dfrac{\text{volume of larger prism (image)}}{\text{volume of smaller prism (original)}}$ $k^3 = \dfrac{64\text{ m}^3}{8\text{ m}^3}$ $= 8$
2 Determine the area scale factor. For ease of calculation, change volume scale factor to linear scale factor first and then to area scale factor.	$\text{Linear scale factor} = k$ $= \sqrt[3]{k^3}$ $= \sqrt[3]{8}$ $= 2$ $\text{Area scale factor} = k^2$ $= 2^2$ $= 4$
3 Use the area scale factor to find the total surface area.	$\text{Area scale factor} = \dfrac{\text{area of larger prism (image)}}{\text{area of smaller prism (original)}}$ $4 = \dfrac{x\text{ m}^2}{2.5\text{ m}^2}$
4 Transpose the equation to get the unknown by itself.	$x = 4 \times 2.5$ $= 10$
5 Write your answer.	The total surface area of the larger triangular prism is 10 m^2.

EXERCISE 11.11 Area and volume scale factors

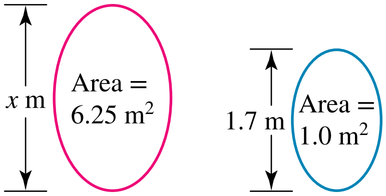
1 **WE34** Find the unknown area for the pair of similar figures at right.



2 Find the unknown area for the pair of similar figures at right.

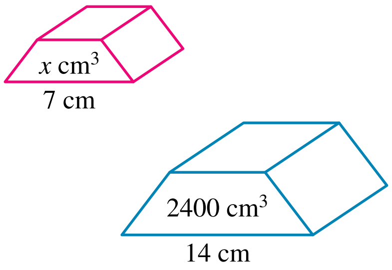


3 **WE35** Find the unknown length for the pair of similar figures at right.



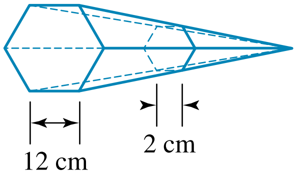
4 Two similar trapezium-shaped strips of land have an area of 0.5 hectares and 2 hectares. The larger block has a distance of 50 metres between the parallel sides. Find the same length in the smaller block.

5 **WE36** Find the unknown volume in the pair of similar objects at right.



6 Find the unknown volume in the pair of similar objects at right.

Volume of small pyramid = 40 cm³



7 **WE37** For a baseball with diameter of 10 cm and a basketball with a diameter of 25 cm, calculate the total surface area of the baseball if the basketball has a total surface area of 1963.5 cm^2.

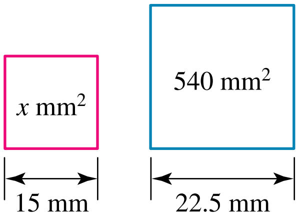
8 For a 14-inch car tyre and 20-inch truck tyre that are similar, calculate the volume (correct to the nearest litre) of the truck tyre if the car tyre has a volume of 70 litres.

CONSOLIDATE

9 Complete the following table of values.

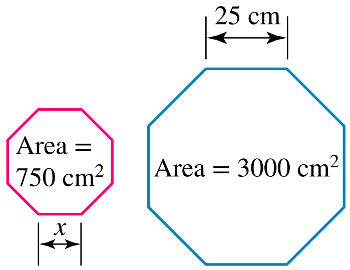
Linear scale factors k	Area scale factors k^2	Volume scale factors k^3
2		8
	16	
3		
		125
	100	
	64	
		0.027
	36	
0.1		
100		
	0.16	
	400	

10 Find the unknown area for the pair of similar figures at right.



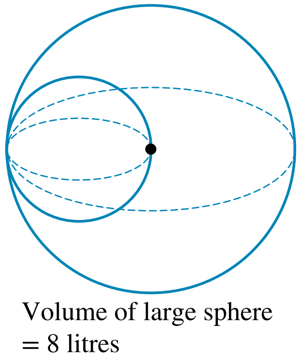
11 **a** Find the unknown length for the pair of similar figures below.

b Two photographs have areas of 48 cm^2 and 80 cm^2. The smaller photo has a width of 6 cm. Find the width of the larger photo.

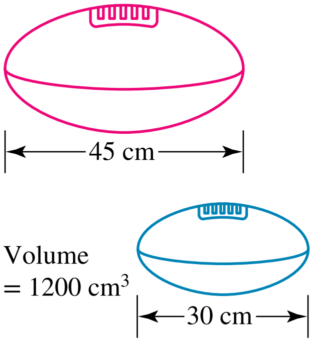


12 Find the unknown volume in the following pairs of similar objects.

a



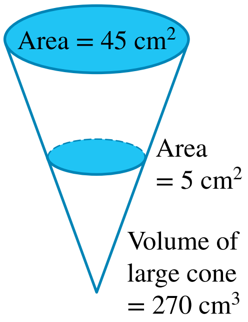
b



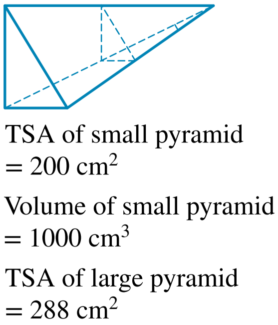
13 **a** For two similar triangular pyramids with volumes of 27 m^3 and 3 m^3, calculate the total surface area of the larger triangular prism if the smaller prism has a total surface area of 1.5 m^2.

b For two similar kitchen mixing bowls with total surface areas of 1500 cm^2 and 3375 cm^2, calculate the capacity of the larger bowl if the smaller bowl has a capacity of 1.25 litres (correct to the nearest quarter of a litre).

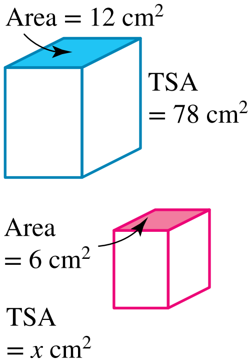
14 **a** Calculate the volume of the small cone.



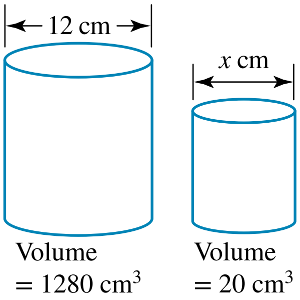
b Calculate the volume of the larger triangular pyramid.



c Calculate the total surface area of the small prism.



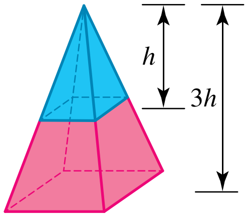
d Calculate the diameter of the small cylinder.



15 What is the area ratio of:

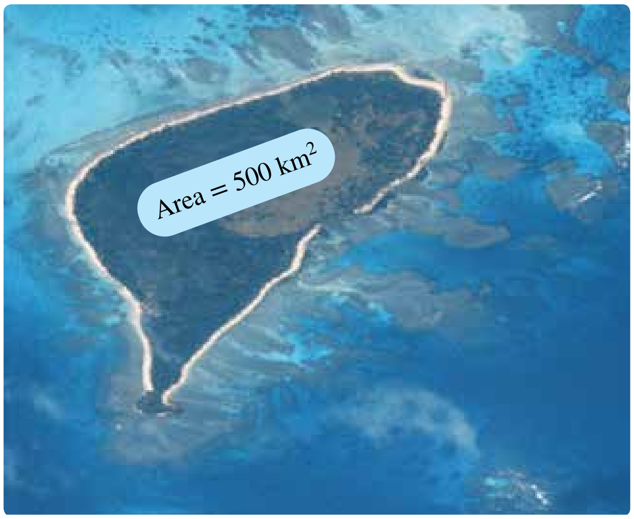
- **a** **i** two similar squares with side lengths of 3 cm and 12 cm
  - **ii** two similar circles with diameters of 9 m and 12 m
  - **iii** two similar regular pentagons with sides of 16 cm and 20 cm
  - **iv** two similar right-angled triangles with bases of 7.2 mm and 4.8 mm?
- **b** Calculate the volume ratios from the similar shapes given in part **a**.

16 The ratio of the volume of the blue portion to the volume of the pink portion is:



A 1 : 3 **B** 1 : 8 **C** 1 : 9
D 1 : 26 **E** 1 : 27

17 An island in the Pacific Ocean has an area of 500 km^2. What is the area of its representation on a map drawn to scale of 1 cm ⇔ 5 km?



18 Two statues of a famous person used 500 cm^3 and 1.5 litres of clay. The smaller statue stood 15 cm tall. What is the height of the other statue (correct to the nearest centimetre)?

MASTER

19 A 1 : 12 scale model of a car is created from plaster and painted.

- **a** If the actual car has a volume of 3.5 m^3, calculate the amount of plaster needed for the model to the nearest litre.
- **b** The model needed 25 millilitres of paint. How much paint would be needed for the actual car (correct in litres to 1 decimal place)?

20 **i** The ratio of the volume of two cubes is 27 : 8. What is the ratio of:

- **a** the lengths of their edges
- **b** the total surface area?

ii A cone is filled to half its height with ice-cream. What is the ratio of ice-cream to empty space?

11.12 Time zones

The world is divided into 24 different **time zones**.

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Unit 4

AOS M3

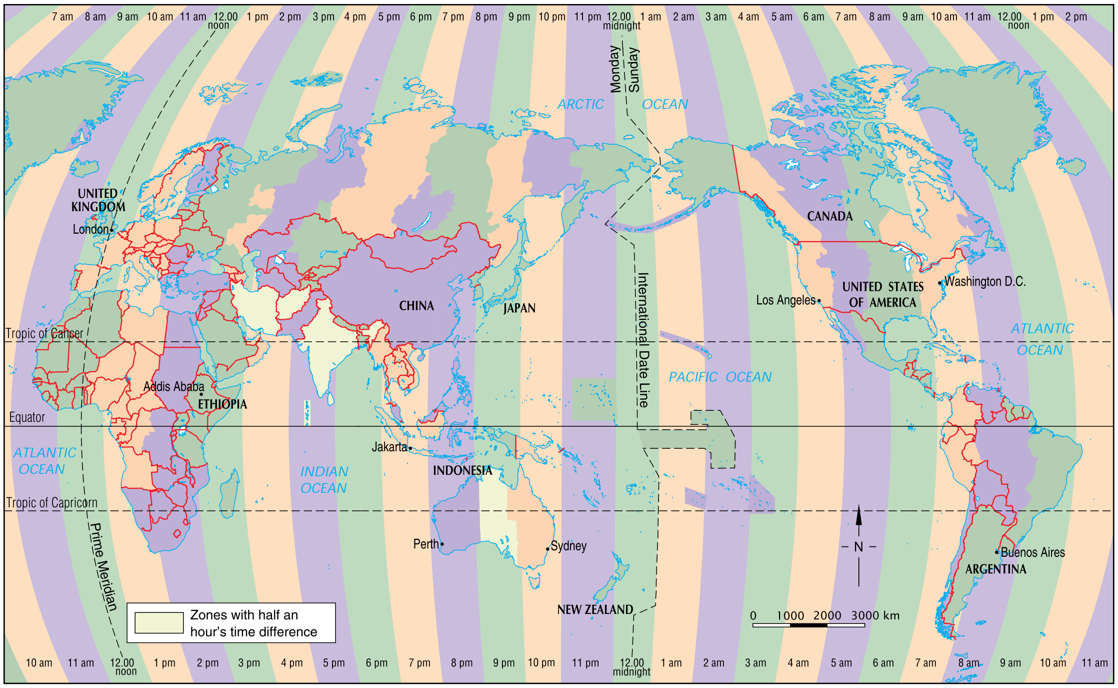
Topic 2

Concept 5

Time zones

Concept summary

Practice questions



If you move from one time zone to another in an easterly direction, you need to move your clock forwards. If you move from one time zone to another in a westerly direction, you need to move your clock backwards.

The **International Date Line** is the line on Earth where a calendar day officially starts and ends. If you cross the International Date Line in an easterly direction, your calendar goes back one day. If you cross the International Date Line in a westerly direction, your calendar goes forward one day.

The time in England used to be called Greenwich Mean Time (GMT), but is now known as Coordinated Universal Time (UTC). It is referred to as Zulu time in the military.

UTC is the reference point for every time zone in the world. Time zones are either ahead of or behind UTC.

For example, the time difference between Melbourne and Los Angeles is 18 hours, not taking into account Pacific Daylight Time (PDT). This is because Melbourne is 10 hours ahead of UTC (UTC + 10) and Los Angeles is 8 hours behind UTC (UTC – 8).

WORKED EXAMPLE 38

What time is it in Melbourne (UTC + 10) if the time in London is (UTC) 9.00 am?

THINK	WRITE
1 Melbourne is 10 hours ahead of London. Add 10 hours to the London time.	The time in London is 9.00 am. The time in Melbourne is 9.00 am + 10 hours.
2 Calculate the time in Melbourne.	The time in Melbourne is 7.00 pm.

Longitude and time difference

We spoke about longitude with great circles in section 11.5. Longitude is also linked to time differences around the world. As with the time zones map of the world shown at the start of this section, each 1 hour time difference is linked to a change of 15° in longitude. This enables us to calculate time differences between countries and cities if we know the difference in their longitude.

WORKED EXAMPLE 39

Calculate the time difference between Perth with a longitude of 116°E and Cairns with a longitude of 146°E.

THINK	WRITE
1 Find the difference in their longitudes.	$\begin{aligned}\text{Difference} &= 146° - 116° \\ &= 30°\end{aligned}$
2 Each 15° represents a time difference of 1 hour.	$\frac{30°}{15°} = 2$ Therefore passes through two 1 hour time differences.
3 Write the time difference.	The difference between Perth time and Cairns time is 2 hours.

EXERCISE 11.12 Time zones

PRACTISE

1 **WE38** What time is it in Melbourne (UTC + 10) if the time in London (UTC) is 11.45 am?

2 What time is it in Los Angeles (UTC – 8) if the time in Sydney (UTC + 10) is 3.35 pm?

3 **WE39** Calculate the time difference between Perth with a longitude of 116°E and Hamilton (NZ) with a longitude of 176°E.

4 Calculate the time difference between Memphis (USA) with a longitude of 90°W and Philadelphia (USA) with a longitude of 75°W.

CONSOLIDATE

5 a How does the local time change as you move around the world in a westerly direction?

b How does the local time change as you move around the world in an easterly direction?

6 What is the time in Melbourne (UTC + 10) if the time in London (UTC) is:

a 11.33 am b noon c 2.55 pm d 7.14 am?

7 What is the time in Los Angeles (UTC – 8) if the time in Melbourne (UTC + 10) is:

a 2.30 pm b noon c 6.05 am d midnight?

8 Your friend in New York wishes to call you. If you wish to receive the telephone call at 6.00 pm in Melbourne (UTC + 10), at what time in New York (UTC – 5) should your friend make the call?

9 To have the time zones work in Santa's favour:

a to which country should he deliver his presents first

b what direction should he travel from his first country?

10 A plane flew from Melbourne to London. Some of the details of the trip are listed.

- Departure date: Friday 5 August
- Departure time: 8.00 am
- Travel time: 25 hours

a What was the time and date in Melbourne (UTC + 10) when the plane arrived in London (UTC)?

b What was the local time (UTC) when the plane arrived?

11 The time difference between a city with a longitude of 42° compared to a city with a longitude of 87° is:

A 1.5 hours
B 2 hours
C 2.5 hours
D 3 hours
E 4 hours

The following information relates to questions **12** and **13**.

The city of Memphis, USA is positioned at longitude 90°W and latitude 35°N, compared to the city of Los Angeles, USA at longitude 120°W and latitude 35°N.

12 Determine the time difference between the two cities.

13 If the plane departed Memphis at 9.30 am local time and the journey took 3 hours and 32 minutes, at what time did it arrive at Los Angeles in Los Angeles time?

14 The city of Melbourne has a longitude of 145°E and the city of Hamburg (Germany) has a longitude of 10°E.

If it takes 22 hours and 45 minutes to fly from Melbourne to Hamburg (including stopovers), at what time would a tourist land in Hamburg if their flight left Melbourne at 5.45 pm?

MASTER

The following information relates to questions **15** and **16**.

A cargo ship travels from Perth (115°W longitude and latitude 33°S) to Cape Town, South Africa (20°E longitude and latitude 33°S). The ship set sail at 4.00 am on 3 April and travelled at an average speed of 15 m/s.

15 If the journey from Perth to Cape Town was 8700 km, calculate the duration of the trip correct to the nearest minute.

16 **a** Find the time in Perth when the ship arrived at Cape Town.

b Find the time difference between Perth and Cape Town.

c Find the local time of the ship's arrival.

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11.13 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

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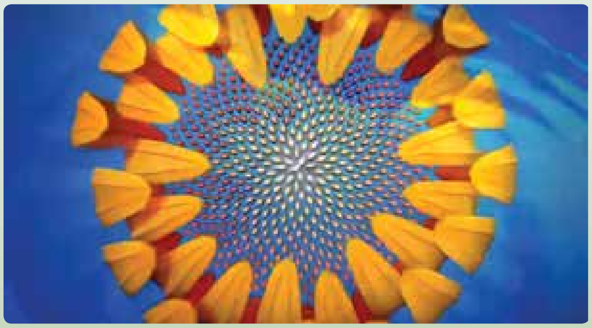
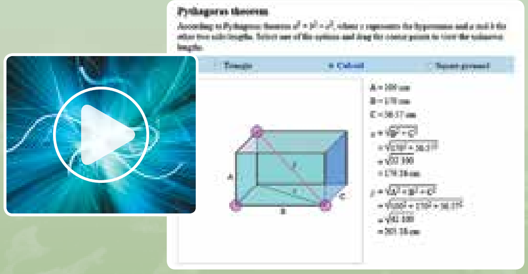
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

11 Answers

EXERCISE 11.2

1 140°, 40°

2 144°, 36°

3 $b = 8$ cm, $c = 50^\circ$

4 148°

5 $m = 117^\circ$, $n = 63^\circ$

6 59°

7 a 60°, 120° b 90°, 90° c 120°, 60°
d 135°, 45° e $128\frac{4}{7}^\circ, 51\frac{3}{7}^\circ$

8 a 79° b $x = 130^\circ$, $y = 50^\circ$
c 27° d $a = 15^\circ$, $b = 165^\circ$, $c = 165^\circ$

9 a $x = 35^\circ$, $y = 145^\circ$
b 30°
c 28°
d $a = 70^\circ$, $b = 110^\circ$, $c = 70^\circ$, $d = 110^\circ$

10 a Pentagon (5-sided)
b Dodecagon (12-sided)
c Octagon
d Hexagon
e Equilateral triangle

11 D

12 A

13 a $a = 93^\circ$ b $x = 35^\circ$ c $t = 20^\circ$
$y = 35^\circ$

14 a $a = 70^\circ$ b $c = 52^\circ$
$b = 110^\circ$
$c = 70^\circ$
$d = 110^\circ$

15 a $m = 118^\circ$ b $x = 30^\circ$
$n = 62^\circ$ $y = 40^\circ$
$z = 110^\circ$
c $a = 9$ cm (isosceles triangle)
$b = 40^\circ$

16 a $a = 45^\circ$ b $a = 30^\circ$
$b = 45^\circ$ $b = 75^\circ$

17 a $h = 7.2$ cm, $r = 2.1$ cm
b $x = 35^\circ$, $y = 35^\circ$, $z = 110^\circ$

18 a $a = 86^\circ$, $b = 94^\circ$, $c = d = 43^\circ$
b $a = 40^\circ$, $b = 50^\circ$

EXERCISE 11.3

1 70 cm

2 112 m

3 24 m^2

4 7 m^2

5 $A_{\text{total}} = 75\ m^2$

6 $A_{\text{total}} = 175\ cm^2$

7 25 900 cm^2

8 3456 mm^2

9 2.5 km^2

10 37 800 mm^2

11 a 28 m b 34 m
c 82.5 cm d 496.4 m

12 a 49 m^2 b 48 m^2
c 394 cm^2 d 11 550 m^2

13 a 680 m^2 b 313 cm^2

14 a 106 m b 76.2 cm

15 a 200 cm^2 b 32 m^2 c 350 cm^2
d 35 000 mm^2 e 2.5 km^2 f 35.7 hectares
g 2750 m^2 h 60 m^2

16 3.74 m^2

17 C

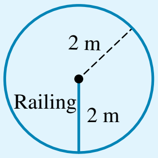
18 a 10 cm^2 b 32 m^2 c 118.8 m^2

19 a 21 cm b 27 m c 47 m

20 a 4.5 m^2 b 710 000 mm^2
c 21.6 hectares d 73 700 m^2

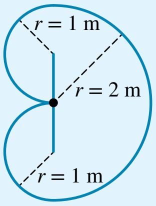
21 48 932 mm

22 a



b 12.6 m^2

c



d 9.4 m^2

EXERCISE 11.4

1 17.59 m

2 73.04 cm

3 a 5.13 m b 27.49 cm

4 a 22.38 cm b 102.45 mm

5 57.11 cm

6 66.85 cm

7 a 204.53 cm^2 b 165.47 cm^2

8 a 143.61 m^2 b 8.28 m^2

9 136 mm^2

10 201.06 km^2

11 a $P = 42$ mm, $A = 143$ mm^2

b $P = 50$ cm, $A = 201$ cm^2

12 a 8.6 m^2 **b** 204 m^2 **c** 7635.9 mm^2

13 a 12.1 m **b** 71.7 m **c** 331.5 mm

14 661 mm^2

15 E

16 E

17 1.57 cm^2

18 3.27 m^2

19 38.07 m

20 a 804.25 cm^2 **b** 20.26 cm^2 **c** 110.85 cm^2

21 a 18.20 m^2 **b** 70.69 m^2

c 14.98 m^2 **d** $2612.25

22 a 1256.6 cm^2 **b** 28.3 cm^2

c 285.9 cm^2

d 75%, 22.75%, 2.25%

EXERCISE 11.5

1 a 6479 km **b** 5138 km

2 a 9159 km **b** 5027 km

3 1562 km

4 431 km

5 17 760 km

6 5362 km

7 a 5697 km **b** 11 394 km

c 8042 km **d** 22 117 km

8 a 1340 km **b** 335 km

c 56 km **d** 9 km

9 a 1759 km **b** 1152 km

10 a 4299 km **b** 5373 km **c** 6392 km

11 2385 km

12 50.36°

13 33.97°

14 a 1499 km **b** 2631 km **c** 3985 km

15 a Cape Town and Budapest

Naples and Prague

b Cape Town and Budapest – 81°

Naples and Prague – 9°

16 a Sydney and Cape Town

New York and Naples

b Sydney and Cape Town – 132°

New York and Naples – 88°

17 a 5305.84 km **b** 12 224 km

18 a 4830.14 km **b** 7419 km

19 a 1479 km **b** 2261 km

20 31 688 km

21 43.58°N

22 39.54°N

EXERCISE 11.6

1 54.3 cm^2

2 395.8 cm^2

3 8.0 m^2

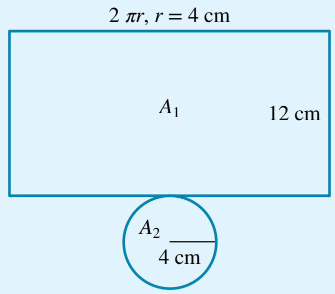
4 72 600 cm^2

5 6.5 m

6 7.9 cm

7 5200 cm^2

8 a



b $\text{TSA} = A_1 + A_2$

$$\begin{aligned} A_1 &= 2\pi r \times h \\ &= 2 \times \pi \times 4 \times 12 \\ &= 301.59 \text{ cm}^2 \\ A_2 &= \pi \times r^2 \\ &= \pi \times 4^2 \\ &= 50.27 \text{ cm}^2 \end{aligned}$$

c
$$\begin{aligned} \text{TSA} &= A_1 + A_2 \\ &= 301.59 + 50.27 \\ &= 351.86 \text{ cm}^2 \end{aligned}$$

d The total surface area of the open cylindrical can is 351.9 cm^2, correct to 1 decimal place.

9 301 cm^2

10 907.13 cm^2

11 a 109 350 mm^2

b 392 m^2

c 633.5 cm^2

d 17 278.8 mm^2

12 a 4.9 m^2 **b** 364 cm^2

c 5808.8 mm^2 or 58.0 cm^2

13 a 2 m **b** 7.1 cm

c 31.5 cm **d** 224.0 cm

14 a 530 cm^2 **b** 672.4 cm^2

c 564 cm^2 **d** 80.6 mm^2

15 a 245 436.9 mm^2 **b** 914.8 cm^2 **c** 123 cm^2

16 50 m^2, 500 000 cm^2

17 99.25 m^2

18 C

19 B

20 E

21 748.63 cm

22 a Cylinder loaf

b 108.7 cm^2

EXERCISE 11.7

1 350 mm^3

2 0.0048 m^3

3 1600 litres

4 250 cm^3

5 10 200 mm^3

6 17 556 mm^3

7 1740 mm^3

8 8000 cm^3

9 8.2 cm

10 40 mm

11 440 cm^3

12 272 m^3

13 2.49×10^{12} m^3

14 7.854×10^{-6} m^3

15 a 0.00 053 m^3 b 56 litres

c 15 000 cm^3 d 72 100 litres

e 2.3 mm^3 f 570 cm^3

g 0.14 litres h 16 cm^3

16 a 70 685 835 mm^3

b 1012 cm^3

c 562 332 cm^3

d 32 m^3

17 a 12.18 m^3 b 399.5 cm^3

18 a 1.2 m b 15 cm

19 a 4435 cm^3 b 864 mm^3 c 240 cm^3

20 a 2145 cm^3 b 2960 cm^3 c 179 m^3

d 2089 cm^3 e 39 m^3 f 88 828 m^3

21 a 91.125 cm^3 b 22.05 m^3 c 3.1 cm

d 8 cm e 8 mm f $1\frac{1}{3}$ m

22 B

23 E

24 A

25 21 mL

26 a 26 cm b 862.8 cm^3 c 575.2 cm^3

d 287.6 cm^3 e $\frac{2}{3}$

EXERCISE 11.8

1 a $k = 4$

b $x = 280, y = 200$

2 0.002 333

3 $k = 1.1, a = 17.6$

4 $k = 0.05, a = 67°$

5 a i 0.05

ii $x = 125, y = 5$

b i 0.8

ii $x = 3.2$ cm, $y = 6.4$ cm

6 a 25.5 mm

b 1.2 m

7 25 cm by 30 cm

8 a 2.4 m b 36 cm c 25 cm

9 215 mm by 178 mm, landscape

10 C

11 E

12 D

13 $\frac{4}{3}$

14 2

15 314.16 cm

16 4.52 cm

EXERCISE 11.9

1 a SSS, $k = 3.5$ b 7.2

2 a AAA, $k = 3$ b 10

3 10.8

4 426

5 732 cm

6 5.5 m

7 a SAS, $k = \frac{4}{3}$ or $1.\dot{3}$ b SSS, $k = 2$

c AAA or SAS, $k = 2$ d AAA, $k = 1.5$

8 a 30 b 18 c 15 d 42

9 a 8 b 4 c 38 d 72

10 810 cm

11 21.5 metres

12 16 m

13 15 m

14 D

15 AAA

$k = \frac{11}{6}$

$x = 2.83$

16 AAA

$k = \frac{1}{3}$

$a = 14.3$ cm

17 B

18 D

EXERCISE 11.10

1 31 m

2 59.82 m

3 48 m

4 32 m

5 13.09 m

6 200 m

7 25.6 m

8 15.33 m

9 138 cm^2

10 C

11 A

12 152 cm

13 80.07 m

14 a i 362 m ii 1800 m iii 7 km

b i 722 m ii 3600 m iii 14 km

EXERCISE 11.11

1 432 cm^2

2 225 mm^2

3 4.25 m

4 25 m

5 300 cm^3

6 8640 cm^3

7 314.2 cm^2

8 204 litres

9

Linear scale factors k	Area scale factors k^2	Volume scale factors k^3
2	4	8
4	16	64
3	9	27
5	25	125
10	100	1000
8	64	512
0.3	0.09	0.027
6	36	216
0.1	0.01	0.001
100	10 000	1 000 000
0.4	0.16	0.064
20	400	8000

10 240 mm^2

11 a 12.5 cm b 7.75 cm

12 a 1 litre b 4050 cm^3

13 a 6.5 m^2 b 4.25 litres

14 a 10 cm^3 b 1728 cm^3

c 39 cm^2 d 3 cm

15 a i 16 ii $\frac{16}{9}$ iii $\frac{25}{16}$ iv $\frac{4}{9}$

b i 64 ii $\frac{64}{27}$ iii $\frac{125}{64}$ iv $\frac{8}{27}$

16 D

17 20 cm^2

18 22 cm

19 a 2 litres b 3.6 litres

20 i a 3 : 2 b 9 : 4

ii 1 : 7

EXERCISE 11.12

1 Melbourne time is 9.45 pm

2 Los Angeles time is 9.35 pm the day before.

3 4 hours difference

4 1 hours difference

5 a Backwards b Forwards

6 a 9.33 pm b 10.00 pm

c 12.55 pm d 5.14 pm

7 a 8.30 pm the day before

b 6.00 pm the day before

c 12.05 pm the day before

d 6.00 am

8 3.00 am

9 a New Zealand

b Westerly direction

10 a 9.00 am Saturday 6 August (Melbourne time)

b 11.00 pm Friday 5 August

11 D

12 2 hours time difference

13 11.02 am

14 7.30 am the next day

15 6 days, 17 hours and 7 minutes

16 a 9.07 pm on 9 April

b 9 hours

c 12.07 pm on 9 April

12

Trigonometry

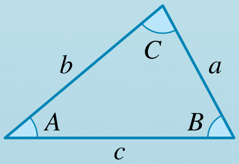
12.1 Kick off with CAS

Exploring the sine rule with CAS

The sine rule can be used to find unknown side lengths and angles in triangles.

For any triangle ABC, the sine rule states that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Before completing any questions involving angles on your CAS, ensure that your angle setting is in degrees mode.

1 Use CAS to determine the value of the following:

a **i** sin(40°) **ii** sin(140°)

b **i** sin(25°) **ii** sin(155°)

2 **a** What do you notice about your answers to questions **1a** and **1b**? Can you spot a relationship between the given angles?

b Use CAS to explore if this works for other pairs of angles with the same relationship.

3 Use the sine rule to determine the size of angle B in the domain $0° < 90°$ given:

$a = 16.2$, $b = 25.1$, $A = 33°$

4 Which angle between 90° and 180° gives the same sine value as your answer to question **3**?

5 Is your answer to question **4** a potential second solution to your triangle (from question **3**)? Try to draw a triangle with these given values.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

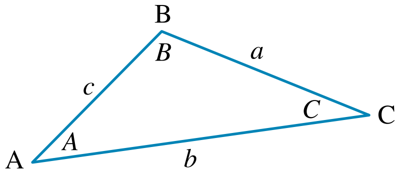
12.2 Trigonometry

Trigonometry is a branch of mathematics that is used to solve problems involving the relationships between the angles and sides of triangles.

Often the problem is a descriptive one and, to solve it confidently, you need to visualise the situation and draw an appropriate diagram or sketch.

Labelling conventions

When we use trigonometry to solve problems involving triangles, there are several labelling conventions that help us remain clear about the relationships between the vertices, angles and lines being used. These will be explained as they arise; however, the basic convention used in this book is shown in the figure below right. Note the use of italics.



The angle A is at vertex A, which is opposite line a.

The angle B is at vertex B, which is opposite line b.

The angle C is at vertex C, which is opposite line c.

To avoid cluttered diagrams, only the vertices (A, B, C) are usually shown; the associated angles (A, B, C) are assumed.

Note: Naturally, we do not need such labels in all diagrams, and sometimes we wish to label vertices, angles and lines in other ways, but these will always be clear from the diagram and its context.

Pythagoras' theorem

Before investigating the relationships between the angles and sides of a triangle, we should consider a problem-solving technique that involves only the sides of triangles: **Pythagoras' theorem**.

Pythagoras' theorem is attributed to the Greek mathematician and philosopher, Pythagoras, around 500 BC. (However, the principle was known much earlier, and it seems that even the pyramid builders of ancient Egypt used the theorem in constructing the pyramids.)

The theorem describes the relationship between the lengths of the sides of all **right-angled triangles**.

studyon

Unit 4

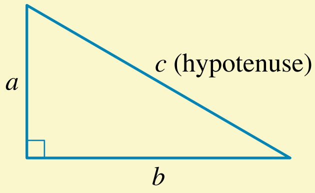
AOS M3

Topic 1

Concept 1

Right-angled triangles
Concept summary
Practice questions

Pythagoras' theorem states that the square of the hypotenuse is equal to the sum of the squares of the other two sides, or



$$c^2 = a^2 + b^2$$

and, therefore, to find c,

$$c = \sqrt{a^2 + b^2}$$

where c is the longest side or hypotenuse and a and b are the two shorter sides.

eBook*plus*

Interactivity
Pythagoras’ theorem
int-6473

Note: Because the equation $c^2 = a^2 + b^2$ has become a standard way of expressing Pythagoras’ theorem, we often adjust the labelling convention to use c for the hypotenuse no matter how the opposite (right) angle and vertex is labelled. However, this will always be clear from the diagram.

The longest side is always opposite the largest angle (90° for right-angled triangles) and similarly, the shortest side is opposite the smallest angle.

To find one of the shorter sides (for example, side a), the formula transposes to:

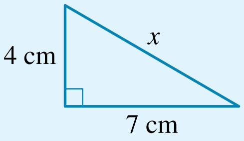
$$a^2 = c^2 - b^2$$

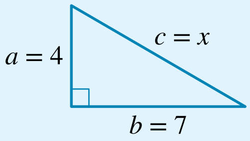
and so

$$a = \sqrt{c^2 - b^2}.$$

WORKED EXAMPLE 1

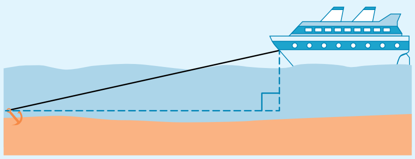
Find the length of the unknown side (correct to 1 decimal place) in the right-angled triangle shown.



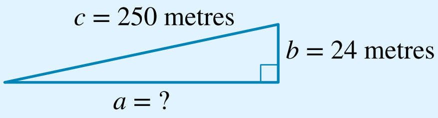
THINK	WRITE/DRAW
1 Note that the triangle is right-angled and we need to find the unknown *length*, given the other two lengths.	
2 Label the sides of the triangle, using the convention that c is the hypotenuse.	
3 Substitute the values into the appropriate formula.	$c^2 = a^2 + b^2$ $x^2 = 4^2 + 7^2$ $= 16 + 49$ $= 65$ $x = \sqrt{65}$ $= 8.0622$ *Alternatively,* $c = \sqrt{a^2 + b^2}$ $x = \sqrt{4^2 + 7^2}$ $= \sqrt{16 + 49}$ $= \sqrt{65}$
4 Write the answer using the correct units and to the appropriate degree of accuracy.	The unknown side’s length is 8.1 cm, correct to 1 decimal place.

WORKED EXAMPLE 2

Find the maximum horizontal distance (correct to the nearest metre) a ship could drift from its original anchored point, if the anchor line is 250 metres long and it is 24 metres to the bottom of the sea from the end of the anchor line on top of the ship’s deck.

THINK	WRITE/DRAW
1 Sketch a suitable diagram of the problem given. Note that the triangle is right-angled and we need to find the unknown length, given the other two lengths.	

2 Simplify the triangle, adding known lengths, and label the sides using the convention that c is the hypotenuse.



3 Substitute the values into the appropriate formula.

$$\begin{aligned}c^2 &= a^2 + b^2\\ 250^2 &= a^2 + 24^2\\ 62\,500 &= a^2 + 576\\ a^2 &= 62\,500 - 576\\ &= 61\,924\\ a &= \sqrt{61\,924}\\ &= 248.845\end{aligned}$$

Alternatively,

$$\begin{aligned}a &= \sqrt{c^2 - b^2}\\ &= \sqrt{250^2 - 24^2}\\ &= \sqrt{62\,500 - 576}\\ &= \sqrt{61\,924}\end{aligned}$$

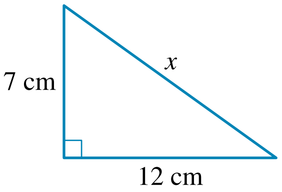
4 Write the answer using the correct units and to the required accuracy.

The ship can drift 249 metres, correct to the nearest metre.

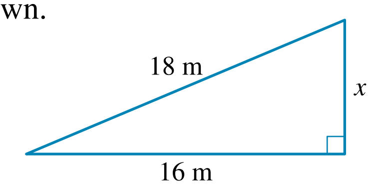
EXERCISE 12.2 Trigonometry

PRACTISE

1 **WE1** Find the length of the unknown side (correct to 1 decimal place) in the right-angled triangle shown.



2 Find the length of the unknown side (correct to 1 decimal place) in the right-angled triangle shown.



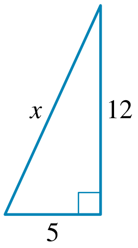
3 **WE2** Find the maximum horizontal distance (correct to the nearest metre) a ship could drift from its original anchored point, if the anchor line is 200 metres long and it is 50 metres to the bottom of the sea from the end of the anchor line on top of the ship's deck.

4 An extension ladder is used to paint windows on a multi-level house. The ladder extends fully to 6 metres and to be safe the base of the ladder is kept 1.5 metres from the house. At full extension how far vertically can the ladder reach to paint the windows (correct to 1 decimal place)?

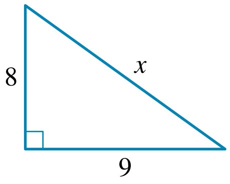
CONSOLIDATE

5 Find the length of the unknown side (correct to 1 decimal place) in each of the following right-angled triangles.

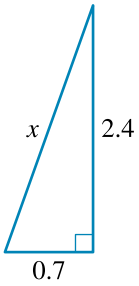
a



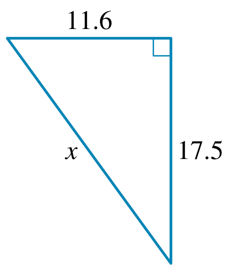
b



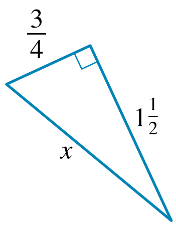
c



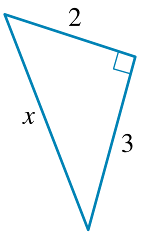
d



e



f

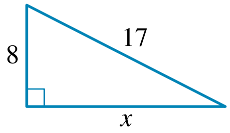


6 An aircraft is flying at an altitude of 5000 metres. If its horizontal distance from the airport is 3 kilometres, what is the distance (correct to the nearest metre) from the airport directly to the aircraft?

7 What is the length (correct to the nearest millimetre) of a diagonal brace on a rectangular gate that is 2600 mm wide and 1800 mm high?

8 Find the length of the unknown side (correct to 1 decimal place) in each of the following right-angled triangles.

a



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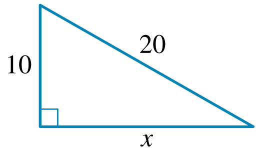

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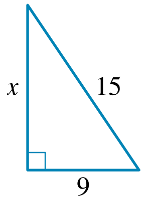

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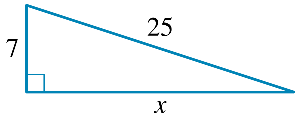

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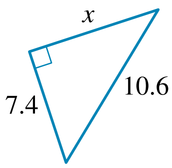

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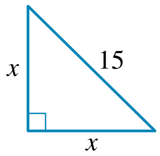
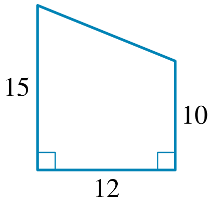
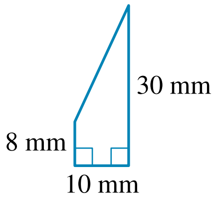


9 Calculate the lengths of the sloping sides in the following. (Remember to construct a suitable right-angled triangle.)

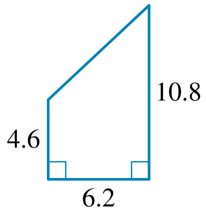
a



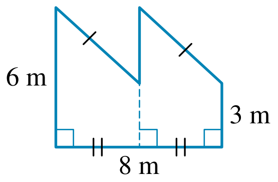
b



c



d



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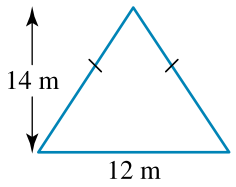

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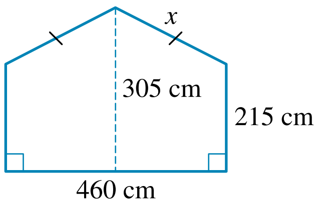
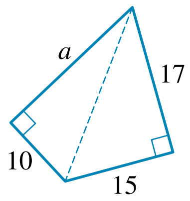


10 Calculate the value of the pronumerals.

a



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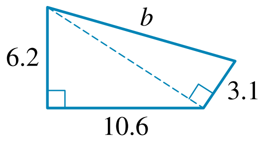

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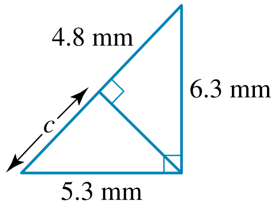

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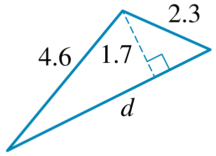
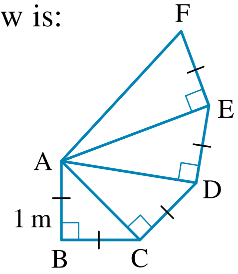


11 One of the smaller sides of a right-angled triangle is 16 metres long. The hypotenuse is 8 metres longer than the other unknown side.

- **a** Draw a suitable triangle to represent this situation.
- **b** Write an expression to show the relationship between the three sides.
- **c** State the lengths of all three sides.

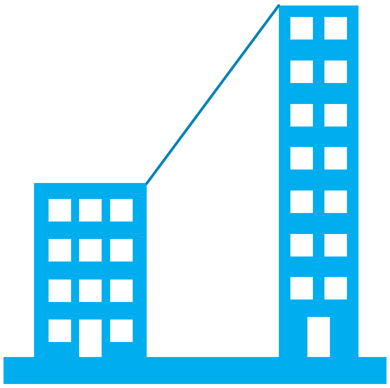
12 The length of side AF in the diagram below is:

- **A** $\sqrt{2}$
- **B** $\sqrt{3}$
- **C** 2
- **D** $\sqrt{5}$
- **E** $\sqrt{6}$



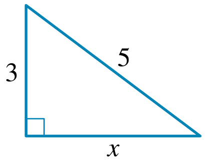
13 To the nearest metre, the length of cable that would connect the roofs of two buildings that are 40 metres and 80 metres high respectively and are 30 metres apart is:

- **A** 40 metres
- **B** 45 metres
- **C** 50 metres
- **D** 55 metres
- **E** none of these

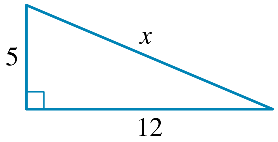


14 Find the length of the unknown side in each of the following right-angled triangles. Where necessary, give your answer correct to 3 decimal places.

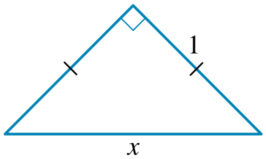
a



b



c



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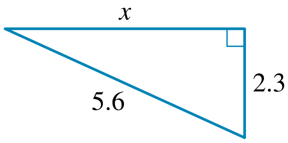
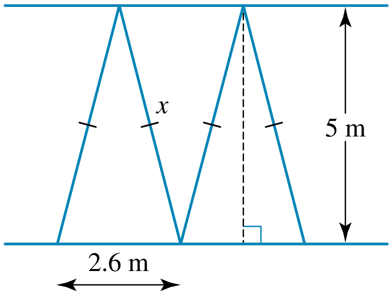


MASTER

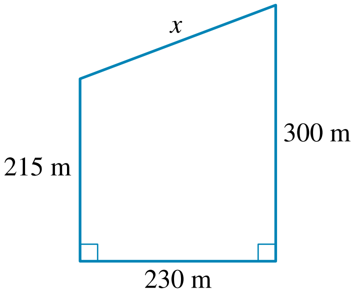
15 The sun shines above Guy's head such that the length of his shadow on the ground is 1.6 m. If the distance from the top of his head to the shadow of the top of his head on the ground is 2.0 m, how tall is Guy (in metres)?

16 Calculate the length of the pronumeral in each of the following. (Where applicable, construct a suitable right-angled triangle.) Give your answer correct to 2 decimal places.

a



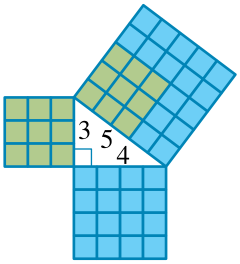
b



12.3 Pythagorean triads

A **Pythagorean triad** is a set of 3 numbers which satisfies Pythagoras' theorem. An example is the set of numbers 3, 4, 5 where $c^2 = a^2 + b^2$.

So,
$$5^2 = 3^2 + 4^2$$
$$25 = 9 + 16$$

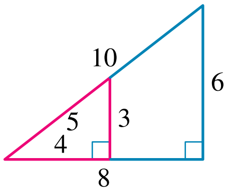


The diagram at right illustrates this relationship.

Another Pythagorean triad is the multiple (scale factor of 2) of the above set: 6, 8, 10.

Others are 5, 12, 13 and 0.5, 1.2, 1.3.

Prove these for yourself.



WORKED EXAMPLE 3

Is the set of numbers 4, 6, 7 a Pythagorean triad?

THINK	WRITE
1 Find the sum of the squares of the two smaller numbers.	$4^2 + 6^2 = 16 + 36$ $= 52$
2 Find the square of the largest number.	$7^2 = 49$
3 Compare the two results. The numbers form a Pythagorean triad if the results are the same.	$7^2 \neq 4^2 + 6^2$
4 Write your answer.	The set of numbers 4, 6, 7 is *not* a Pythagorean triad.

Another way to generate Pythagorean triads is by using the following rule:

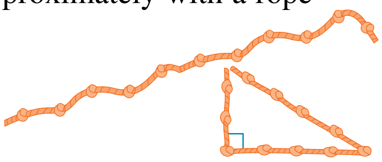
Step 1. Square an odd number ($5^2 = 25$).

Step 2. Find the two consecutive numbers that add up to the squared value ($12 + 13 = 25$).

Step 3. The triad is the odd number you started with together with the two consecutive numbers (5, 12, 13).

Try to find a triad for the odd number 9.

A triangle whose sides form a Pythagorean triad contains a right angle, which is opposite the longest side. This result can be illustrated approximately with a rope of any length, by tying 11 equally spaced knots and forming a triangle with sides equal to 3, 4 and 5 spaces, as shown at right. In doing this, a right angle is formed opposite the 5-space side.



WORKED EXAMPLE 4

A triangle has sides of length 8 cm, 15 cm and 17 cm. Is the triangle right-angled? If so, where is the right angle?

THINK	WRITE
1 The triangle is right-angled if its side lengths form a Pythagorean triad. Find the sum of the squares of the two smaller sides.	$8^2 + 15^2 = 64 + 225$ $= 289$

2 Find the square of the longest side and compare to the first result.

$17^2 = 289$
$17^2 = 8^2 + 15^2$
The triangle is right-angled.

3 The right angle is opposite the longest side.

The right angle is opposite the 17 cm side.

EXERCISE 12.3 Pythagorean triads

PRACTISE

1 **WE3** Is the set of numbers 5, 6, 8 a Pythagorean triad?

2 Is the set of numbers 5, 12, 13 a Pythagorean triad?

3 **WE4** A triangle has side lengths 3 cm, 4 cm and 5 cm. Is the triangle right-angled? If so, where is the right angle?

4 A triangle has side lengths 5 cm, 8 cm and 10 cm. Is the triangle right-angled? If so, where is the right angle?

CONSOLIDATE

5 Are the following sets of numbers Pythagorean triads?

a 9, 12, 15
b 4, 5, 6
c 30, 40, 50
d 3, 6, 9
e 0.6, 0.8, 1.0
f 7, 24, 25
g 6, 13, 14
h 14, 20, 30
i 11, 60, 61
j 10, 24, 26
k 12, 16, 20
l 2, 3, 4

6 Complete the following Pythagorean triads. Each set is written from smallest to largest.

a 9, __, 15
b __, 24, 25
c 1.5, 2.0, __
d 3, __, 5
e 11, 60, __
f 10, __, 26
g __, 40, 41
h 0.7, 2.4, __

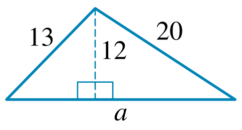
7 For each of the sets which were Pythagorean triads in question **5**, state which side the right angle is opposite.

8 A triangle has sides of length 16 cm, 30 cm and 34 cm. Is the triangle right-angled? If so, where is the right angle?

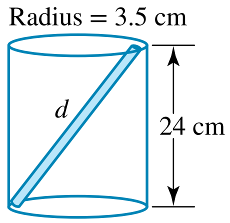
9 A triangle has sides of length 12 cm, 13 cm and 18 cm. Is the triangle right-angled? If so, where is the right angle?

10 Find the unknown length in each case below.

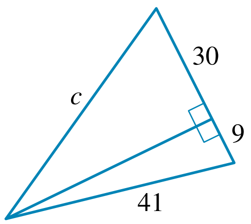
a



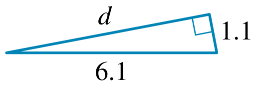
b



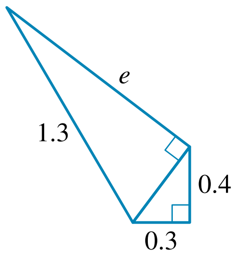
c



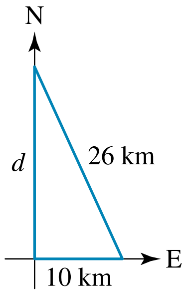
d



e

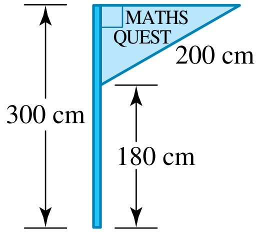


f



11 An athlete runs 700 m north and then 2.4 km west. How far away is the athlete from the starting point?

12 Find the perimeter of the flag (excluding the pole) as shown below.



13 Which of the following is a Pythagorean triad?

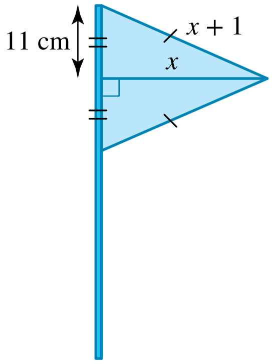
A 7, 14, 21
B 1.2, 1.5, 3.6
C 3, 6, 9
D 12, 13, 25
E 15, 20, 25

14 Which of the following is *not* a Pythagorean triad?

A 5, 4, 3
B 6, 9, 11
C 13, 84, 85
D 0.9, 4.0, 4.1
E 5, 12, 13

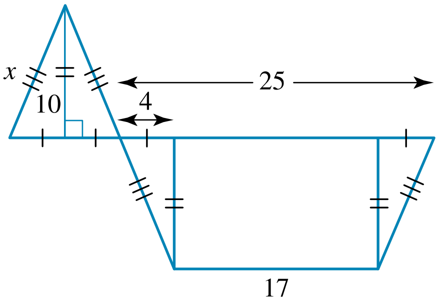
MASTER

15 Find the perimeter of the flag, excluding the pole, shown in the figure below.

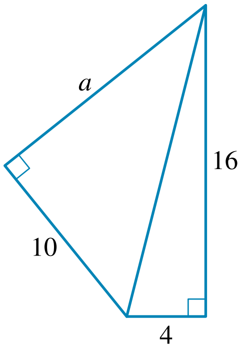


16 Calculate the value of the pronumeral in each of the following.

a



b



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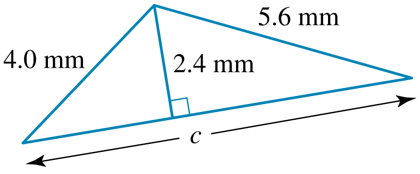
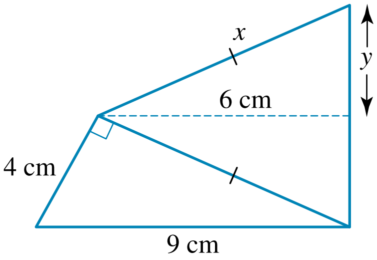


d

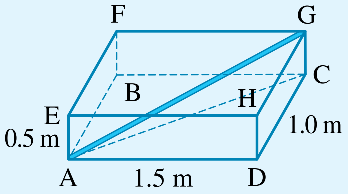
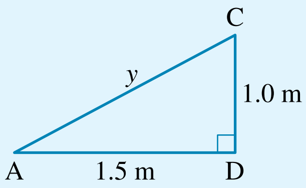
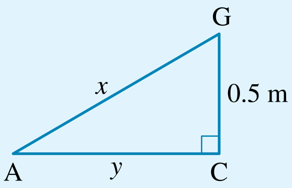


12.4 Three-dimensional Pythagoras' theorem

Many practical situations involve three-dimensional objects with perpendicular planes and therefore the application of Pythagoras' theorem. To solve three-dimensional problems, a carefully drawn and labelled diagram will help. It is also of benefit to identify right angles to see where Pythagoras' theorem can be applied. This enables you to progress from the known information to the unknown value(s).

WORKED EXAMPLE 5

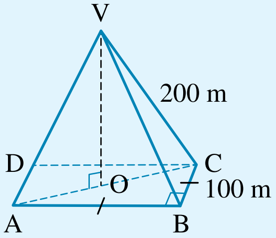
Correct to the nearest centimetre, what is the longest possible thin rod that could fit in the boot of a car? The boot can be modelled as a simple rectangular prism with the dimensions of 1.5 metres wide, 1 metre deep and 0.5 metres high.

THINK	WRITE/DRAW
1 Draw a diagram of the rectangular prism.	
2 Identify the orientation of the longest object — from one corner to the furthest diagonally opposite corner. In this case, it is AG.	
3 Identify the two right-angled triangles necessary to solve for the two unknown lengths.	 
4 Draw the triangles separately, identifying the lengths appropriately.	
5 Calculate the length of diagonal AC.	$c^2 = a^2 + b^2$ $y^2 = 1.5^2 + 1.0^2$ $= 2.25 + 1$ $= 3.25$ $y = \sqrt{3.25}$ $= 1.803$ (correct to 3 decimal places) The length of AC is 1.8 metres (correct to 1 decimal place).
6 Calculate the length of diagonal AG, using the calculated length for AC. *Note:* To avoid *rounding error*, use the most accurate form, which is the surd $\sqrt{3.25}$.	$c = \sqrt{a^2 + b^2}$ (alternative form) $x = \sqrt{0.5^2 + (\sqrt{3.25})^2}$ $= \sqrt{0.25 + 3.25}$ $= \sqrt{3.5}$ $= 1.8708$ (m)
7 Write the answer using the correct units and level of accuracy.	The longest rod that could fit in the car boot is 187 centimetres, correct to the nearest centimetre.

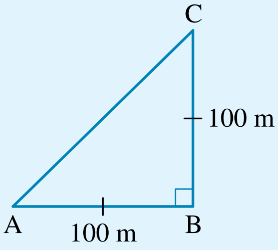
WORKED EXAMPLE 6

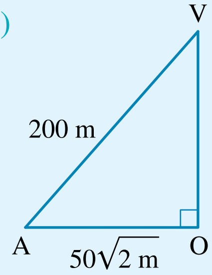
To find the height of a 100-metre square-based pyramid, with a slant height of 200 metres as shown, calculate the:

a length of AC (in surd form)
b length of AO (in surd form)
c height of the pyramid VO (correct to the nearest metre).



THINK	WRITE
a Calculate the length of diagonal AC in the right-angled triangle, ABC. Write surds in their simplest form.	**a** $c = \sqrt{a^2 + b^2}$ (alternative form) $AC = \sqrt{100^2 + 100^2}$ $= \sqrt{20000}$ $= \sqrt{10000} \times \sqrt{2}$ $= 100 \times \sqrt{2}$ $= 100\sqrt{2}$ The length of AC is $100\sqrt{2}$ metres.
b AO is half the length of AC.	**b** Length of AO is $\frac{100\sqrt{2}}{2}$ or $50\sqrt{2}$ metres.
c 1 Calculate the height of the pyramid, VO, in the right-angled triangle, VOA.	**c** $a = \sqrt{c^2 - b^2}$ (alternative form) $VO = \sqrt{200^2 + (50\sqrt{2})^2}$ $= \sqrt{40000 - 5000}$ $= \sqrt{35000}$ $= 187.0829$
2 Write the answer using the correct units and level of accuracy.	The height of the pyramid, VO, is 187 metres, correct to the nearest metre.

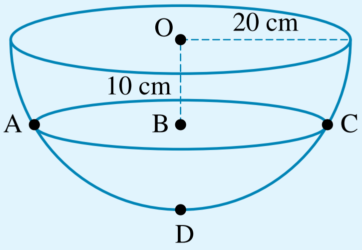


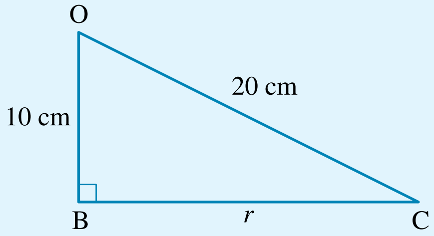
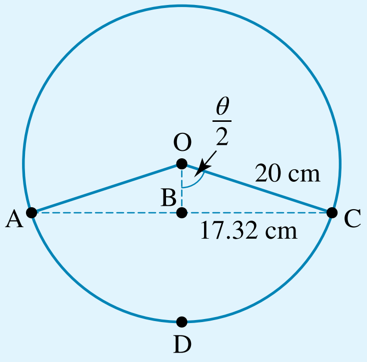


WORKED EXAMPLE 7

A hemispherical block of wood of radius 20 cm is sliced so that the distance OB is 10 cm

a Find the radius AB of the circular cut correct to 2 decimal places.
b Find the circumference of the new circle that passes through A and C.
c Find the area of this circular cut.
d Find the distance from A to C passing through D. Give your answer correct to 2 decimal places.

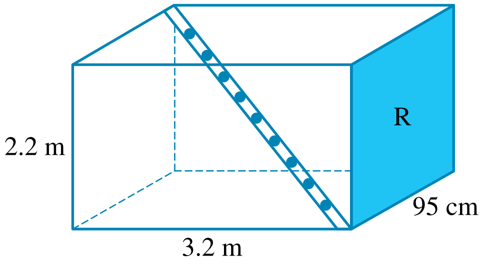


THINK	WRITE
a A right-angled triangle OBC is formed which has a hypotenuse of 20 cm and one side of 10 cm, so Pythagoras' theorem may be used.	 $r^2 + 10^2 = 20^2$ $r^2 = 400 - 100$ $r = \sqrt{300}$ ≈ 17.32 cm
b Use the rule for circumference of a circle, $C = 2\pi r$.	$C = 2 \times \pi \times 17.32$ $= 108.83$ cm
c Use the rule for area of a circle, $A = \pi r^2$.	$A = \pi \times 17.32^2$ $= 942.48\text{ cm}^2$
d Using the great circle through ADC, the angle subtending the minor arc AC is required.	 $\sin\left(\frac{\theta}{2}\right) = \frac{17.32}{20}$ $\frac{\theta}{2} = 60°$ $\theta = 120°$ length of minor arc AC $= \frac{\theta}{180} \times \pi \times r$ $l = \frac{120}{180} \times \pi \times 20$ ≈ 41.89 cm

EXERCISE 12.4 Three-dimensional Pythagoras' theorem

PRACTISE

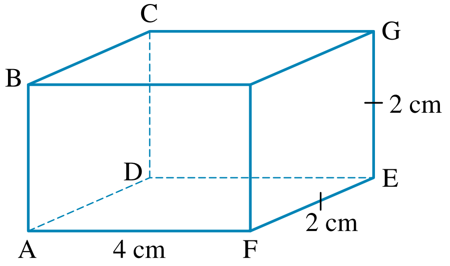
1 **WE5** A wooden plank of the greatest possible length is placed inside a garden shed. Use the diagram to calculate the length of the plank of wood correct to 1 decimal place.



2 Calculate the length of:

a AE

b AG.

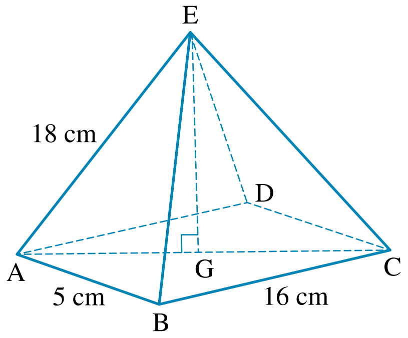


3 **WE6** Find:

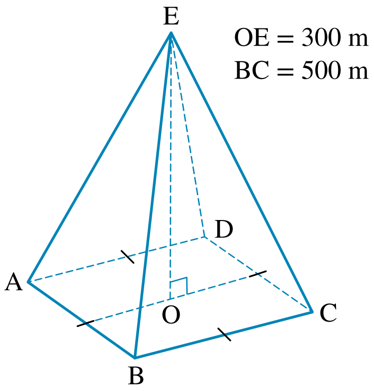
a the length AC (in surd form)

b the length AG (in surd form)

c the height of the pyramid EG (correct to the nearest cm).



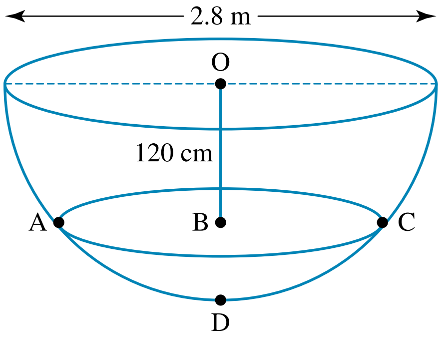
4 Use the diagram of the pyramid to answer this question.



a What is the length of the diagonal AC?

b What is the length of EB?

5 A hemispherical block of wood of diameter 2.8 m is sliced so that the distance OB is 120 cm.



a Find the radius AB of the circular cut correct to 1 decimal place.

b Find the circumference of the new circle that passes through A and C correct to 2 decimal places.

c Find the area of this circular cut correct to 2 decimal places.

d Find the distance from A to C passing through D correct to 1 decimal place.

6 Calculate the radius of the small circle formed when a circular cut is made 1.2 m from the centre of a sphere of diameter 4.6 m.

CONSOLIDATE

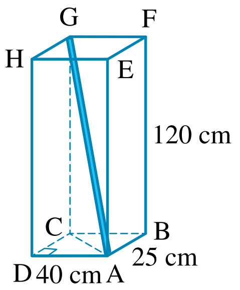
7 Correct to the nearest centimetre, what is the longest thin rod that could fit inside a cube with side length 2 m?

8 Correct to the nearest centimetre, what is the longest drumstick that could fit in a rectangular toy box whose dimensions are 80 cm long by 80 cm wide by 60 cm high?

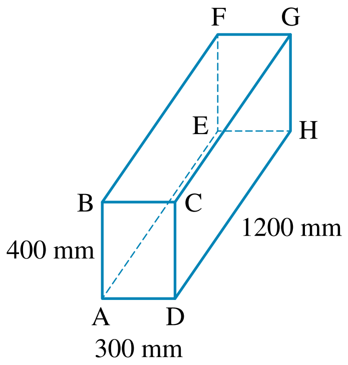
9 For each of the prisms shown, calculate:

i the length of AC **ii** the length of AG.

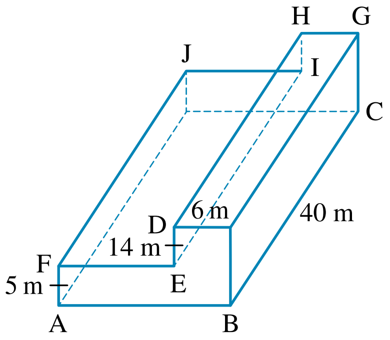
a



b



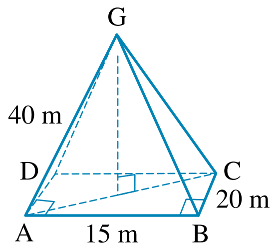
c



10 For each of the pyramids shown, calculate:

i the length of AC **ii** the perpendicular height.

a



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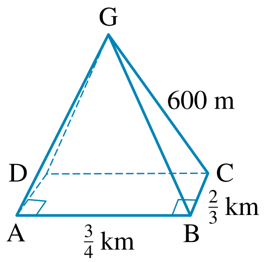
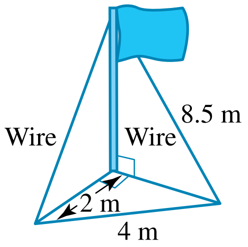


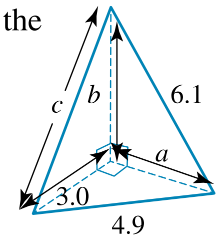
11 A 3.5-metre long ramp rises to a height of 1.2 metres. How long (correct to 1 decimal place) is the base of the ramp?

12 Two guide wires are used to support a flagpole as shown. The height of the flagpole would be closest to:

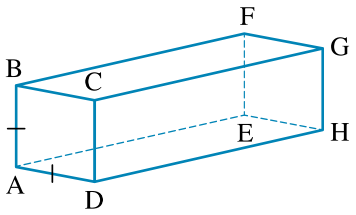
A 3 m
B 8 m
C 12 m
D 21 m
E 62 m



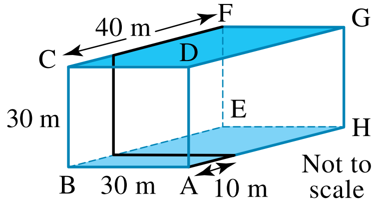
13 Find the values of the pronumerals (correct to 1 decimal place) in the pyramid at right.



14 Find the lengths of AB and DH (correct to 2 decimal places), where AC = 7.00 m and CH = 15.00 m.

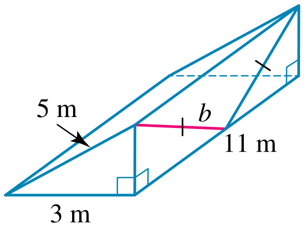


15 A man moves through a two-level maze by following the solid black line, as shown in the diagram. What is the *direct* distance from his starting point, A, to his end point, F (to the nearest metre)?

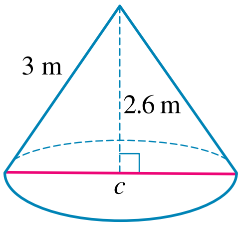


16 In each of the following typical building structures find the length of the unknown cross-brace shown in pink.

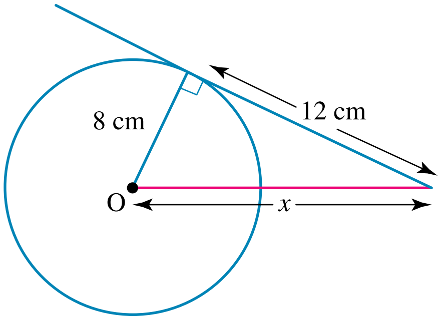
a



b



c



17 Calculate the circumference of the small circle formed 12 cm from the centre of a sphere of radius 18 cm. Give your answer correct to 3 significant figures.

18 A ping-pong ball of radius 2 cm has 2 holes made in it and a string is threaded through the holes. If the length of string needed to go from one hole to the other is 3 cm, find the distance from the centre of the ping-pong ball to the middle of the string. Give your answer correct to 2 decimal places.

19 A hemispherical fish bowl of radius 15 cm contains water to a depth of 12 cm. Find the surface area of the water correct to 1 decimal place.

MASTER

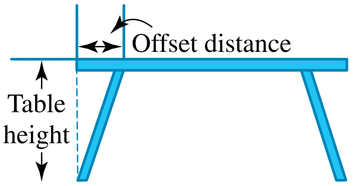
20 For the coffee table design at right, find the length of the legs (correct to the nearest millimetre) if the coffee table is to be:

a 500 mm off the ground

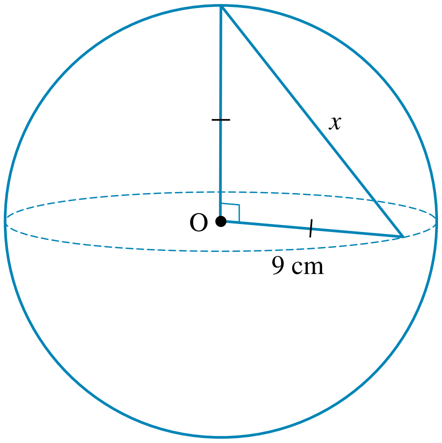
b 700 mm off the ground

and the legs are offset from the vertical by a distance of:

i 100 mm **ii** 150 mm.



21 Calculate the length of the pronumeral in the following sphere.



12.5 Trigonometric ratios

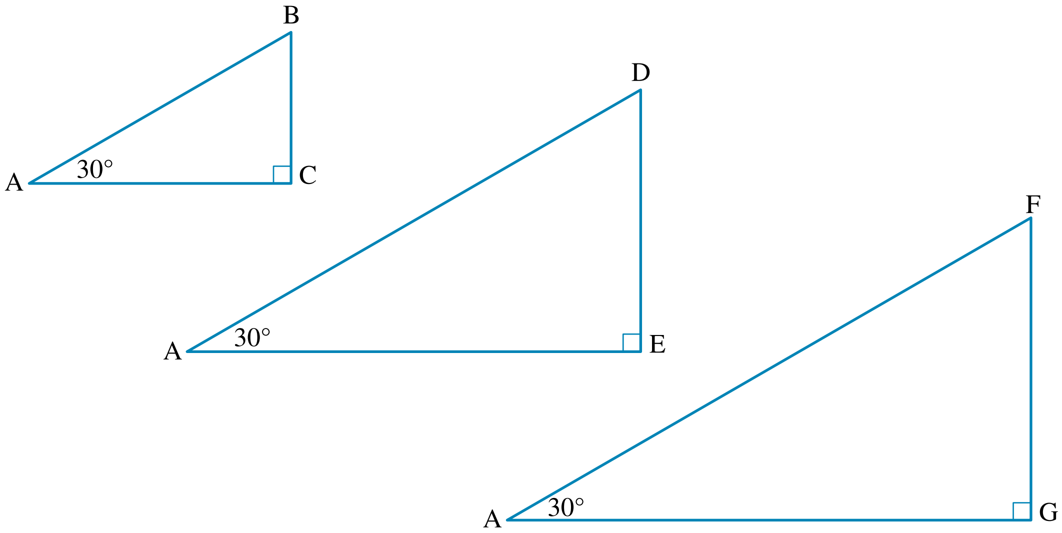
Trigonometric ratios include the sine ratio, the cosine ratio and the tangent ratio; three ratios of the lengths of sides of a right-angled triangle dependent on a given acute angle.

Labelling convention

For the trigonometric ratios the following labelling convention should be applied:

1. The hypotenuse is opposite the right angle (90°).
2. The **opposite side** is directly opposite the given angle, θ.
3. The **adjacent side** is next to the given angle, θ.

Consider the three triangles shown here. We know from the previous topic on similarity that ΔABC, ΔADE and ΔAFG are similar because the corresponding angles are the same. Therefore, the corresponding sides are in the same ratio (scale factor).



Ratio of lengths of sides

Copy and complete the table by identifying and measuring the lengths of the three sides for each of the three triangles shown. Evaluate the ratios of the sides.

	Length of side			Ratio of lengths of sides		
Triangle	**Opposite**	**Adjacent**	**Hypotenuse**	$\frac{\text{Opposite}}{\text{Hypotenuse}}$	$\frac{\text{Adjacent}}{\text{Hypotenuse}}$	$\frac{\text{Opposite}}{\text{Adjacent}}$
ABC						
ADE						
AFG						

Notice that for each of the ratios, for example $\frac{\text{opposite}}{\text{hypotenuse}}$, the value is the same for all three triangles. This is the same for all right-angled triangles with the same acute angle.

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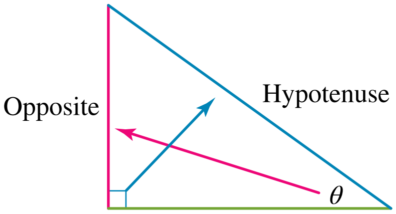
Interactivity
Trigonometric ratios
int-2577

Trigonometric ratios are used in right-angled triangles:
1. to find an unknown length, given an angle and a side
2. to find an unknown angle, given two lengths.

Sine ratio

The sine ratio is defined as follows:

The sine of an angle $= \dfrac{\text{length of opposite side}}{\text{length of hypotenuse side}}$.



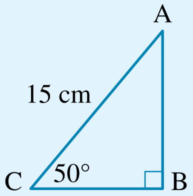
In short,

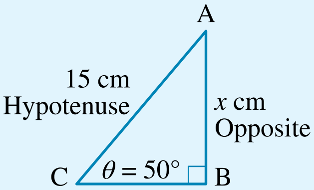
$$\sin(\theta) = \frac{\textbf{opposite}}{\textbf{hypotenuse}}$$

$$\sin(\theta) = \frac{\text{O}}{\text{H}} \qquad \text{[SOH]}$$

WORKED EXAMPLE 8

Find the length (correct to 1 decimal place) of the line joining the vertices A and B in the triangle.

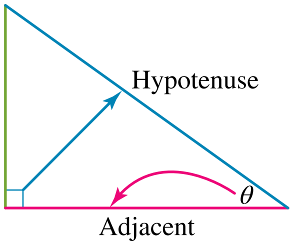


THINK	WRITE
1 Identify the shape as a right-angled triangle with a given length and angle. Label the sides as per the convention for trigonometric ratios.	
2 Identify the appropriate trigonometric ratio, namely the sine ratio, from the given information.	Angle $= 50°$ Opposite side $= x$ cm Hypotenuse $= 15$ cm [SOH]
3 Substitute into the formula.	$\sin(\theta) = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse side}}$ $\sin(\theta) = \dfrac{\text{O}}{\text{H}}$ $\sin(50°) = \dfrac{x}{15}$
4 Isolate x and evaluate.	$15 \times \sin(50°) = \dfrac{x}{15} \times 15$ $x = 15 \times \sin(50°)$ $= 15 \times 0.766$ $= 11.491$
5 Write the answer using the correct units and level of accuracy.	The length of the line joining vertices A and B is 11.5 centimetres, correct to 1 decimal place.

Cosine ratio

The cosine ratio is defined as follows:

$$\text{The cosine of an angle} = \frac{\text{length of adjacent side}}{\text{length of hypotenuse side}}.$$



In short,

$$\boldsymbol{\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}}$$

$$\boldsymbol{\cos(\theta) = \frac{\text{A}}{\text{H}}} \qquad \textbf{[CAH]}$$

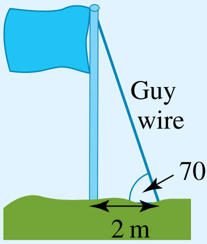
In Worked example 7 the sine ratio was used to find the unknown length. The cosine ratio can be used in the same way, if it is required.

WORKED EXAMPLE 9

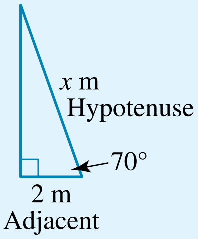
Find the length of the guy wire (correct to the nearest centimetre) supporting a flagpole, if the angle of the guy wire to the ground is 70° and it is anchored 2 metres from the base of the flagpole.

THINK | **WRITE/DRAW**

1 Draw a diagram to represent the situation and identify an appropriate triangle.



2 Label the diagram with the given angle and the given side to find an unknown side in a right-angled triangle.



3 Choose the appropriate trigonometric ratio, namely the cosine ratio.

Angle = 70°
Adjacent side = 2 m
Hypotenuse = x m [CAH]

4 Substitute into the formula.

$$\cos(\theta) = \frac{A}{H}$$

$$\cos(70°) = \frac{2}{x}$$

5 Isolate x and evaluate.

$$\frac{1}{\cos(70°)} = \frac{x}{2}$$

$$x = \frac{2}{\cos(70°)}$$

$$= 5.8476$$

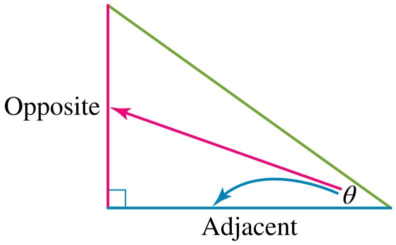
6 Write the answer using the correct units and level of accuracy.

The length of the guy wire is 5.85 metres or 585 centimetres, correct to the nearest centimetre.

Tangent ratio

The tangent ratio is defined as follows:

The tangent of an angle $= \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.

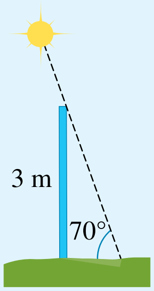
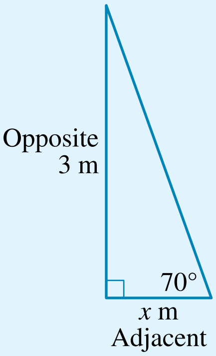


In short,

$$\mathbf{\tan(\theta) = \dfrac{opposite}{adjacent}}$$

$$\mathbf{\tan(\theta) = \dfrac{O}{A}} \qquad \textbf{[TOA]}$$

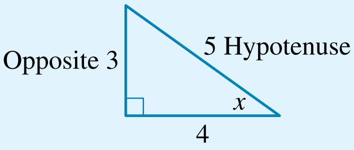
WORKED EXAMPLE 10 **Find the length of the shadow (correct to 1 decimal place) cast by a 3-metre tall pole when the angle of the sun to the horizontal is 70°.**

THINK	WRITE/DRAW
1 Draw a diagram to represent the situation and identify an appropriate triangle.	
2 Label the diagram with the given angle and the given side in order to find an unknown side in a right-angled triangle.	
3 Identify the appropriate trigonometric ratio, namely the tangent ratio.	Angle $= 70°$ Opposite side $= 3\,\text{m}$ Adjacent side $= x\,\text{m}$ [TOA]
4 Substitute into the formula.	$\tan(\theta) = \dfrac{O}{A}$ $\tan(70°) = \dfrac{3}{x}$
5 Isolate x and evaluate.	$\dfrac{1}{\tan(70°)} = \dfrac{x}{3}$ $x = \dfrac{3}{\tan(70°)}$ $= 1.0919$
6 Write the answer using the correct units and level of accuracy.	The length of the shadow is approximately 1.1 metres, correct to 1 decimal place.

Finding an unknown angle

If the lengths of the sides of a triangle are known, unknown angles within the triangle can be found.

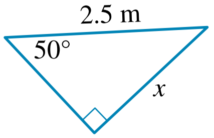
WORKED EXAMPLE 11 **Find the smallest angle (correct to the nearest degree) in a 3, 4, 5 Pythagorean triangle.**

THINK	WRITE/DRAW
1 The smallest angle is opposite the smallest side. Label the sides as given by convention for trigonometric ratios.	
2 All side lengths are known, therefore, any one of the 3 ratios can be used. Choose one ratio, for example, sine ratio.	Angle = x Opposite side = 3 Hypotenuse = 5 [SOH]
3 Substitute into the formula.	$\sin(\theta) = \frac{\text{O}}{\text{H}}$ $\sin(x) = \frac{3}{5}$
4 Convert the ratio to a decimal.	$= 0.6$
5 Evaluate x.	$x = \sin^{-1}(0.6).$ $= 36.87°$
6 Write the answer using the correct units and level of accuracy.	The smallest angle is 37°, correct to the nearest degree.

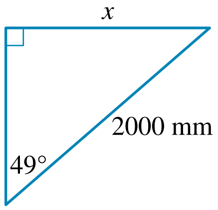
EXERCISE 12.5 Trigonometric ratios

PRACTISE

1 WE8 Find the length of the unknown side (correct to 1 decimal place) in the following triangle.



2 Find the length of the unknown side (correct to 1 decimal place) in the following triangle.



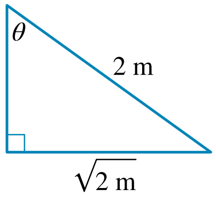
3 WE9 A boat is moored in calm waters with its depth sounder registering 14.5 m. If the anchor line makes an angle of 72° with the vertical, what is the length of line (correct to the nearest metre) that is out of the boat?

4 Find the length of the ramp (correct to the nearest centimetre), if the angle the ramp makes to the ground is 32° and the ramp covers 3.6 m horizontally.

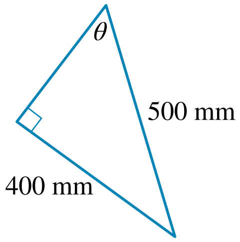
5 **WE10** Find the length of a shadow (correct to 1 decimal place) cast by a 4.5 m tall flag pole when the angle of the sun to the horizontal is 63°.

6 A person is hoping to swim directly across a straight river from point A to point B, a distance of 215 m. The river carries the swimmer downstream so that she actually reaches the other side at point C. If the line of her swim, AC, makes an angle of 67° with the river bank, find how far (correct to the nearest metre) downstream from point B she finished.

7 **WE11** Find the size of the unknown angle (correct to the nearest degree) in the triangle below.



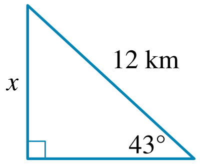
8 Find the size of the unknown angle (correct to the nearest degree) in the triangle below.



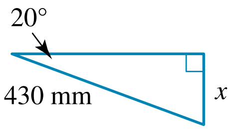
CONSOLIDATE

9 Find the length of the unknown side (correct to 1 decimal place) in each of the following triangles.

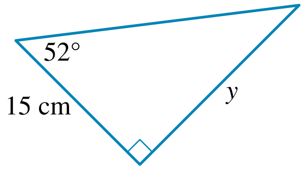
a



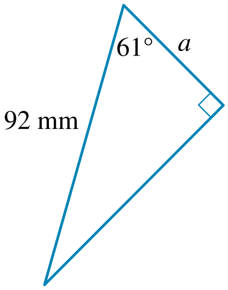
b



c

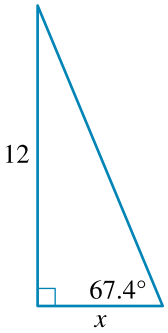


d

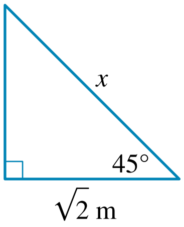


10 Find the value of the missing side (correct to 1 decimal place) of the following triangles.

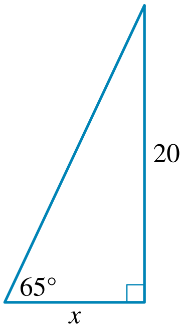
a



b

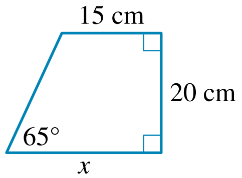


c

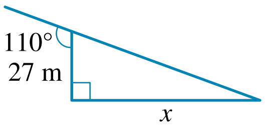


11 Find the value of the unknown sides (correct to 1 decimal place) of the following shapes.

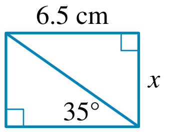
a



b

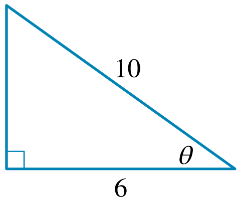


c

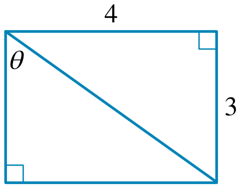


12 Find the size of the unknown angle (correct to the nearest degree) in each of the triangles.

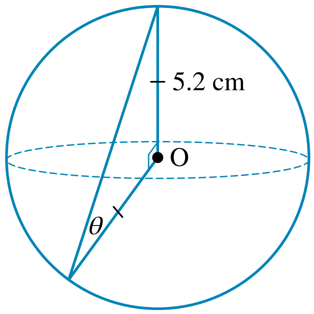
a



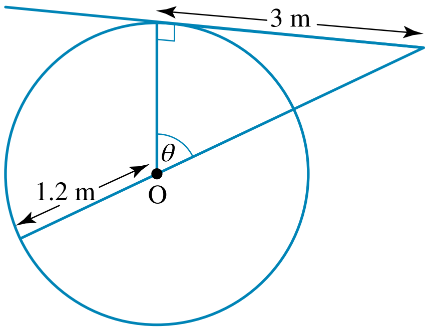
b



c

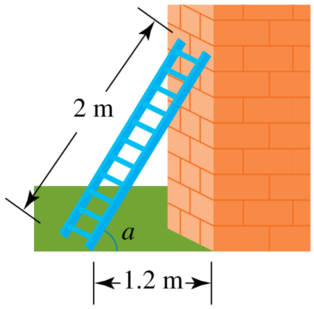


d

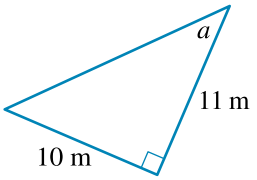


13 Find the values of the unknown angle, *a* (correct to the nearest degree).

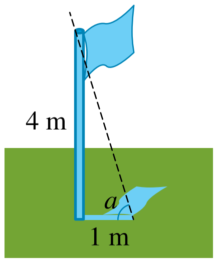
a



b

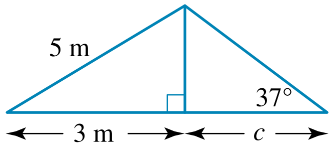


c



14 Find the sizes of the two acute angles in a 6, 8, 10 Pythagorean triangle.

15 The correct expression for the value of *c* in the figure at right is:



A $\dfrac{\tan(37°)}{4}$

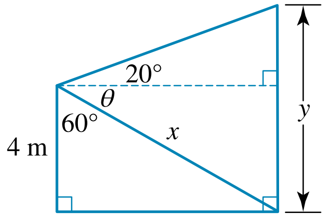
B $\dfrac{\cos(37°)}{4}$

C $\dfrac{5}{\tan(37°)}$

D $\dfrac{4}{\tan(37°)}$

E $\dfrac{4}{\sin(37°)}$

16 In the diagram below find *θ* (correct to the nearest degree), *x* metres and *y* metres (both correct to 1 decimal place).



17 A 1.9 m javelin is thrown so that 15 cm of its pointy end sticks into the ground. The sun is directly overhead, casting a shadow of 90 cm in length. Determine the angle (correct to the nearest degree) that the javelin makes with the ground.

18 A hot air balloon is hovering in strong winds, 10 m vertically above the ground. It is being held in place by a taut 12 m length of rope from the balloon to the ground. Find the angle (correct to the nearest degree) that the rope makes with the ground.

MASTER

19 A ramp joins two points, A and B. The horizontal distance between A and B is 1.2 m, and A is 25 cm vertically above the level of B.

a Find the length of the ramp (in metres correct to 2 decimal places).

b Find the angle that the ramp makes with the horizontal.

20 A cable car follows a direct line from a mountain peak (altitude 1250 m) to a ridge (altitude 840 m). If the horizontal distance between the peak and the ridge is 430 m, find the angle of descent (correct to the nearest degree) from one to the other.

12.6 The sine rule

Introduction — sine and cosine rules

Often the triangle that is apparent or identified in a given problem is *non-right-angled*. Thus, Pythagoras’ theorem or the trigonometric ratios are not as easily applied. The two rules that can be used to solve such problems are:

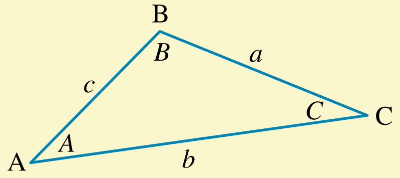
1. the **sine rule**
2. the **cosine rule**.

For the sine and cosine rules the following labelling convention should be used.

Angle A is opposite side a (at vertex A)

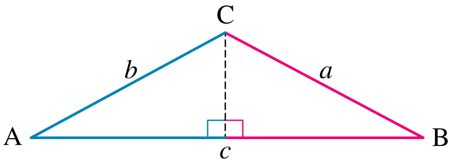
Angle B is opposite side b (at vertex B)

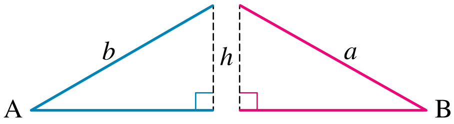
Angle C is opposite side c (at vertex C)



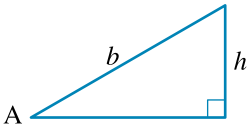
Note: To avoid cluttered diagrams, only the vertices (A, B and C) are usually shown and are used to represent the angles A, B and C.

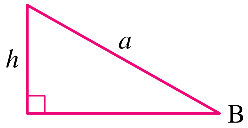
All triangles can be divided into two right-angled triangles.





Earlier, we saw that the new side, h, can be evaluated in two ways.





studyon

Unit 4

AOS M3

Topic 1

Concept 2

The sine rule
Concept summary
Practice questions

$$\sin(A) = \frac{h}{b} \qquad\qquad \sin(B) = \frac{h}{a}$$

$$h = b \times \sin(A) \qquad\qquad h = a \times \sin(B)$$

If we equate the two expressions for h:

$$b \times \sin(A) = a \times \sin(B).$$

and rearranging the equation, we obtain:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}.$$

Using a similar approach it can be shown that:

$$\boxed{\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}}$$

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Interactivity
The sine rule
int-6275

Similarly, if the triangle is labelled using other letters, for example STU, then:

$$\frac{s}{\sin(\text{S})} = \frac{t}{\sin(\text{T})} = \frac{u}{\sin(\text{U})}$$

This can be generalised as follows: in any triangle, the ratio of side length to the sine of the opposite angle is constant.

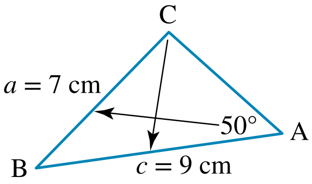
The sine rule is used if you are given:

1. two angles and one opposite side

or

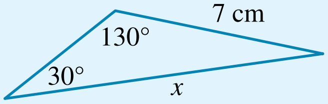
2. an angle and its opposite side length (a complete ratio) and one other side.

For example, in triangle ABC at right, $a = 7$ cm, $A = 50°$ and $c = 9$ cm. Angle C could then be found using the sine rule.



WORKED EXAMPLE 12

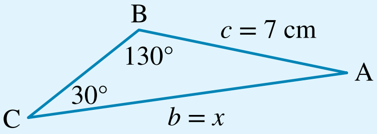
Find the unknown length, x cm, in the triangle (correct to 1 decimal place).



THINK

WRITE/DRAW

1 Label the triangle appropriately for the sine rule.



2 We have the angle opposite to the unknown side and a known $\frac{\text{side}}{\text{angle}}$ ratio, therefore, the sine rule can be used.

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

$b = x \qquad B = 130°$

$c = 7\text{ cm} \qquad C = 30°$

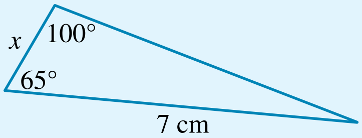
3 Substitute known values into the two ratios.

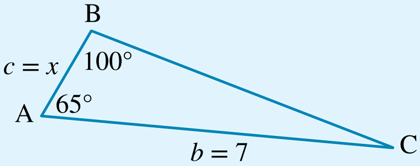
$$\frac{x}{\sin(130°)} = \frac{7}{\sin(30°)}$$

4 Isolate x and evaluate.	$x = \dfrac{7 \times \sin(130°)}{\sin(30°)}$ $= 10.7246$ ≈ 10.7
5 Write the answer.	The unknown length is 10.7 cm, correct to 1 decimal place.

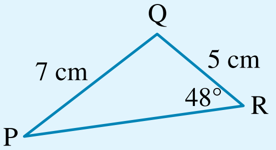
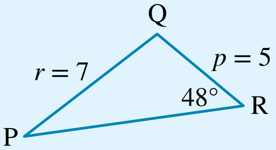
Sometimes it is necessary to find the third angle in a triangle in order to apply the sine rule.

WORKED EXAMPLE 13 **Find the unknown length, x cm (correct to 2 decimal places).**



THINK	WRITE/DRAW
1 Label the triangle appropriately for the sine rule.	
2 Calculate the third angle because it is opposite the unknown side.	$C = 180° - (65° + 100°)$ $= 15°$
3 Write the sine rule and identify the values of the pronumerals.	$\dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$ $c = x \qquad C = 15°$ $b = 7 \qquad B = 100°$
4 Substitute the known values into the rule.	$\dfrac{x}{\sin(15°)} = \dfrac{7}{\sin(100°)}$
5 Isolate x and evaluate.	$x = \dfrac{7 \times \sin(15°)}{\sin(100°)}$ $= 1.8397$
6 Write the answer.	The unknown length is 1.84 cm, correct to 2 decimal places.

WORKED EXAMPLE 14 **For a triangle PQR, find the unknown angle (correct to the nearest degree), P, given $p = 5$ cm, $r = 7$ cm and $R = 48°$.**

THINK	WRITE/DRAW
1 Draw the triangle and assume it is non-right-angled.	
2 Label the triangle appropriately for the sine rule (it is just as easy to use the given labels).	

3 Confirm that it is the sine rule that can be used as you have the angle opposite to the unknown angle and a known $\frac{\text{side}}{\text{angle}}$ ratio.

$$\frac{p}{\sin(P)} = \frac{r}{\sin(R)}$$

$$p = 5 \quad P = ?$$

$$r = 7 \quad R = 48°$$

4 Substitute known values into the two ratios.

$$\frac{5}{\sin(P)} = \frac{7}{\sin(48°)}$$

5 Isolate sin(*P*).

$$\frac{\sin(P)}{5} = \frac{\sin(48°)}{7}$$

$$\sin(P) = \frac{5 \times \sin(48°)}{7}$$

6 Evaluate the angle (inverse sine) and include units with the answer.

$$P = \sin^{-1}\left(\frac{5 \times \sin(48°)}{7}\right)$$

$$= 32.06°$$

$$\approx 32°$$

The unknown angle is 32°, correct to the nearest degree.

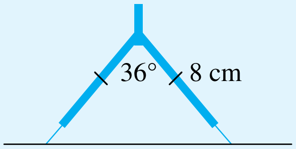
Sometimes the angle required for the sine rule is not given. In such cases simply subtract the two known angles from 180°, as was done in step 2 of Worked example 12.

WORKED EXAMPLE 15

A pair of compasses (often called a compass) used for drawing circles has two equal legs joined at the top. The legs are 8 centimetres long. If it is opened to an included angle of 36 degrees between the two legs, find the radius of the circle that would be drawn (correct to 1 decimal place).

THINK | **WRITE/DRAW**

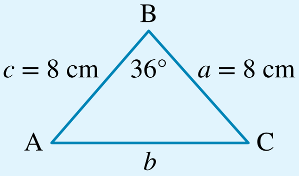
1 Draw the situation and identify that the triangle is non-right-angled.



2 Draw the triangle separately from the situation and label it appropriately.

The sine rule cannot be used straight away as we do not have both a known angle and known length opposite to the known angle. Therefore, we need to find either ∠*A* or ∠*C* first.

This is an *isosceles* triangle since $a = c$; therefore $\angle A = \angle C$. Using the fact that the angle sum of a triangle is 180°, find ∠*A* and ∠*C*.



$$180° = \angle A + \angle B + \angle C$$

$$= x + 36° + x$$

$$2x = 180° - 36°$$

$$= 144°$$

$$x = 72° \text{ and, therefore,}$$

$$\angle A = \angle C = 72°$$

3 Write the formula for the sine rule and identify the values of the pronumerals.

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

$$b = y \quad B = 36°$$

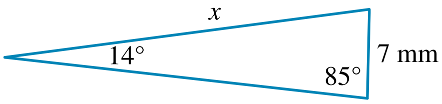
$$c = 8 \quad C = 72°$$

4 Substitute the known values into the formula.	$\frac{y}{\sin(36°)} = \frac{8}{\sin(72°)}$
5 Transpose the equation to get the unknown by itself.	$y = \frac{8 \times \sin(36°)}{\sin(72°)}$
6 Evaluate y correct to 1 decimal place and include the units.	$y \approx 4.9$ The radius of the circle is 4.9 cm, correct to 1 decimal place.

EXERCISE 12.6 The sine rule

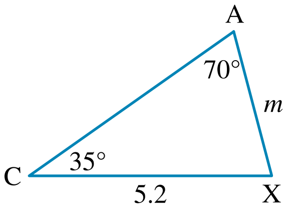
PRACTISE

1 **WE12** Find the unknown length, x.

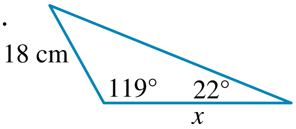


2 The length of side m is nearest to:

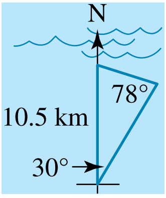
A 3.2
B 3.1
C 3.6
D 5.8
E 3.0



3 **WE13** Find the unknown length, x, correct to 1 decimal place.



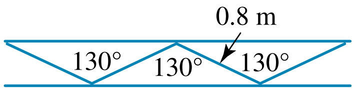
4 A sailing expedition followed a triangular course as shown. Find the total distance covered in the round trip.



5 **WE14** In ΔPQR, find the unknown angle, R, given $p = 48$, $q = 21$ and $\angle P = 110°$, correct to the nearest degree.

6 Construct a suitable triangle from the following instructions and find all unknown sides and angles. One of the sides is 23 cm; the smallest side is 15 cm; the smallest angle is 28°.

7 **WE15** Steel trusses are used to support the roof of a commercial building. The struts in the truss shown are each made from 0.8 m steel lengths and are welded at the contact points with the upper and lower sections of the truss.



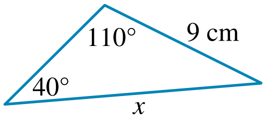
a On the lower section of the truss, what is the distance (correct to the nearest centimetre) between each pair of consecutive welds?
b What is the height (correct to the nearest centimetre) of the truss?

8 A logo is in the shape of an isosceles triangle with the equal sides being 6.5 cm long and the equal angles 68°. Use the sine rule to find the length (correct to 1 decimal place) of the unknown side.

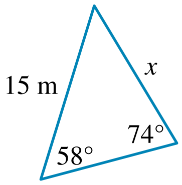
CONSOLIDATE

9 Find the unknown length, x, in each of the following.

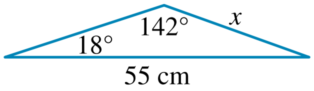
a



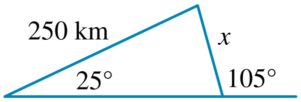
b



c

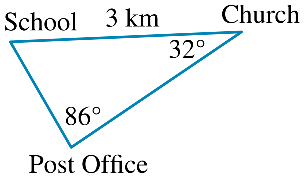


d



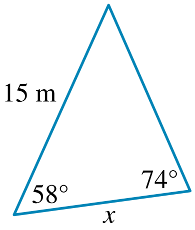
10 The relative positions of the school, church and post office in a small town are shown at the vertices of the triangle.

Find the straight-line distance between the school and the post office (correct to 1 decimal place).

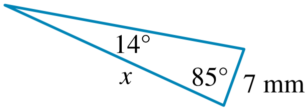


11 Find the unknown length, x (correct to 1 decimal place) in each case.

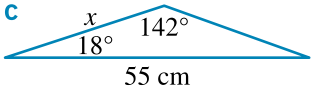
a



b



c



12 For the following questions give answers correct to the nearest degree.

a In ΔABC, find the unknown angle, B, given $b = 6$, $c = 6$ and $\angle C = 52°$.

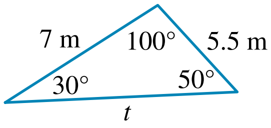
b In ΔLMN, find the unknown angle, M, given $m = 14.1$, $n = 27.2$ and $\angle N = 128°$.

c In ΔSTU, find the unknown angle, S, given $s = 12.7$, $t = 16.3$ and $\angle T = 45°$.

d In ΔPQR, find the unknown angle, P, given $p = 2$, $r = 3.5$ and $\angle R = 128°$.

e In ΔABC, find the unknown angle, A, given $b = 10$, $c = 8$ and $\angle B = 80°$.

13 The correct expression for the value of t in the given triangle is:



A $\dfrac{7 \times \sin(100°)}{\sin(30°)}$

B $\dfrac{5.5 \times \sin(100°)}{\sin(30°)}$

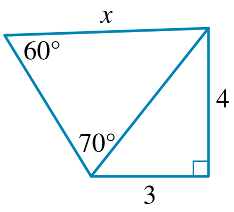
C $\dfrac{5.5 \times \sin(30°)}{\sin(100°)}$

D $\dfrac{5.5 \times \sin(100°)}{\sin(50°)}$

E $\dfrac{7 \times \sin(50°)}{\sin(100°)}$

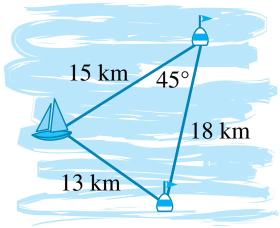
14 The value of x (correct to 1 decimal place) in the given figure is:

A 4.3
B 4.6
C 5.4
D 3.3
E 3.6

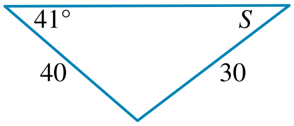


15 A yacht sails the three-leg course shown. The largest angle between any two legs within the course, to the nearest degree, is:

A 34°
B 55°
C 45°
D 78°
E 90°



16 The correct expression for angle S in the given triangle is:



A $\sin^{-1}\left(\dfrac{40 \times \sin(41^\circ)}{30}\right)$

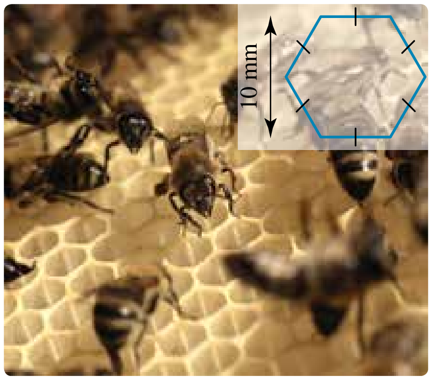
B $\cos^{-1}\left(\dfrac{40 \times \cos(41^\circ)}{30}\right)$

C $\sin^{-1}\left(\dfrac{30 \times \sin(41^\circ)}{40}\right)$

D $\sin^{-1}\left(\dfrac{41 \times \sin(41^\circ)}{30}\right)$

E $\sin^{-1}\left(\dfrac{30}{40 \times \sin(41^\circ)}\right)$

17 Find the perimeter of the beehive cell shown.



18 A rope is pegged at one end into the ground, pulled tightly up over a branch and pegged into the ground at the other end. It is known that one peg-to-branch length of rope is 8 m and it makes an angle of 39° with the ground. The other end of the rope makes an angle of 48° with the ground. Find (correct to 1 decimal place):

a the length of the rope

b the distance between the two pegs.

MASTER

19 A playground swing, which is 2.3 m long, makes an angle of 74°, at its swing point, in one complete swing. Determine the horizontal distance (in metres correct to 1 decimal place) between the extreme positions of the swing seat.

20 A scenic flight leaves Geelong and flies west of north for the 80 km direct journey to Ballarat. At Ballarat the plane turns 92° to the right to fly east of north to Kyneton. From here the plane again turns to the right and flies the 103 km straight back to Geelong.

a Determine the angle (in degrees correct to 1 decimal place) through which the plane turned at Kyneton.

b Find the distance (correct to the nearest km) of the direct flight from Ballarat to Kyneton.

12.7 Ambiguous case of the sine rule

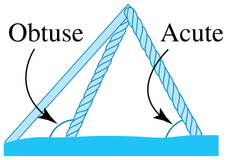
Using CAS, investigate the values for each of these pairs of sine ratios:

- sin(30°) and sin(150°)
- sin(110°) and sin(70°).

You should obtain the same number for each value in a pair.

Similarly, sin(60°) and sin(120°) give an identical value of 0.8660.

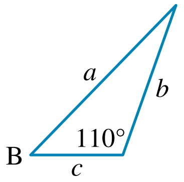
Now try to find the inverse sine of these values; for example, $\sin^{-1}(0.8660)$ is 60°. The obtuse (greater than 90°) angle is not given by the calculator. When using the inverse sine function on your calculator, *the calculator will give only the acute angle*.

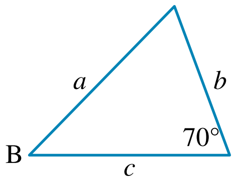


A rope attached to a pole can be anchored in two possible positions.

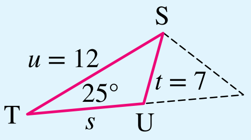
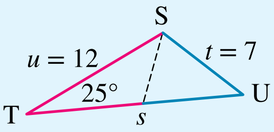
The situation is illustrated practically in the diagram above where the sine of the acute angle equals the sine of the obtuse angle.

Therefore always *check your diagram* to see if the *unknown angle* is the *acute* or *obtuse angle* or perhaps *either*. This situation is illustrated in the two diagrams on the next page. The triangles have two corresponding sides equal, *a* and *b*, as well as angle *B*. The sine of 110° also equals the sine of 70°; however, the side *c* is quite different. It is worth noting that this ambiguity occurs when the smaller known side is opposite the known acute angle. That is, an ambiguous case occurs if $\angle B < 90°$ and $a\sin B \times \angle b < \angle a$:

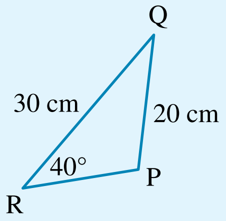


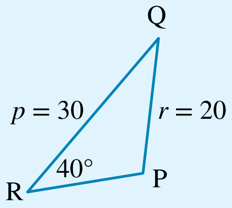


WORKED EXAMPLE 16 **Correct to the nearest degree, find the angle, *U*, in a triangle, given *t* = 7, *u* = 12 and angle *T* is 25°.**

THINK	WRITE/DRAW
1 Draw a suitable sketch of the triangle given. As the length of *s* is not given, side *t* can be drawn two different ways. Therefore angle *U* could be either *an acute or an obtuse angle*. Label the triangles appropriately for the sine rule. (It is just as easy to use the given labels.)	 
2 Identify that it is the sine rule that can be used as you have the side opposite to the unknown angle and a known $\frac{\text{side}}{\text{angle}}$ ratio.	$\frac{t}{\sin(T)} = \frac{u}{\sin(U)}$ $t = 7 \qquad T = 25°$ $u = 12 \qquad U = ?$
3 Substitute the known values into the two ratios.	$\frac{7}{\sin(25°)} = \frac{12}{\sin(U)}$
4 Transpose the equation to get the unknown by itself.	$\frac{\sin(U)}{12} = \frac{\sin(25°)}{7}$ $\sin(U) = \frac{12 \times \sin(25°)}{7}$
5 Evaluate the angle (inverse sine). Note that the value is an *acute* angle but it may also be an *obtuse* angle.	$\sin(U) = 0.724\,488$ $U = 46.43°$
6 Calculate the obtuse angle.	$U = 180° - 46.43°$ $= 133.57°$
7 Write the answer, giving both the acute and obtuse angles, as not enough information was given (the information was ambiguous) to precisely position side *t*.	The angle *U* is either 46° or 134°, correct to the nearest degree.

WORKED EXAMPLE 17 **In the obtuse-angled triangle PQR, find the unknown angle (correct to the nearest degree), *P*.**



THINK	WRITE/DRAW
1 Label the triangle appropriately for the sine rule. (It is just as easy to use the given labels.)	
2 Identify that the sine rule is used as you have the side opposite to the unknown angle and a known $\frac{\text{side}}{\text{angle}}$ ratio.	$\frac{p}{\sin(P)} = \frac{r}{\sin(R)}$ $p = 30 \quad P = ?$ $r = 20 \quad R = 40°$
3 Substitute the known values into the two ratios.	$\frac{30}{\sin(P)} = \frac{20}{\sin(40°)}$
4 Transpose the equation to get the unknown by itself.	$\frac{\sin(P)}{30} = \frac{\sin(40°)}{20}$ $\sin(P) = \frac{30 \times \sin(40°)}{20}$
5 Evaluate the angle (inverse sine). Note that the value is an *acute* angle while in the diagram given it is an *obtuse* angle.	$\sin(P) = 0.964\,18$ $P = 74.62°$
6 Calculate the obtuse angle.	$P = 180° - 74.62°$ $= 105.38°$ The angle P is 105°, correct to the nearest degree.

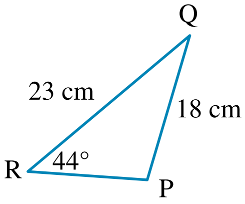
EXERCISE 12.7 Ambiguous case of the sine rule

PRACTISE

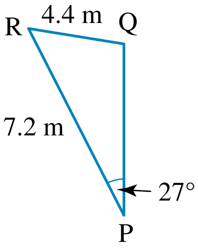
1 **WE16** To the nearest degree, find the angle, U, in a triangle, given $t = 12$, $u = 16$ and angle T is 33°.

2 To the nearest degree, find the angle, U, in a triangle, given $t = 7$, $u = 8$ and angle T is 27°.

3 **WE17** In the obtuse-angled triangle PQR shown, find the unknown angle (correct to the nearest degree), P.



4 In the obtuse-angled triangle PQR shown, find the unknown angle (correct to the nearest degree), Q.



CONSOLIDATE

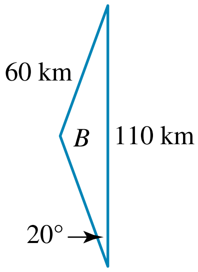
5 Find both the acute and obtuse angles correct to one decimal place.
In ΔABC, find the unknown angle, B, given $b = 10.8$, $c = 6$ and $\angle C = 26°$.

6 Find both the acute and obtuse angles correct to one decimal place.
In ΔSTU, find the unknown angle, S, given $t = 12.7$, $s = 16.3$ and $\angle T = 45°$.

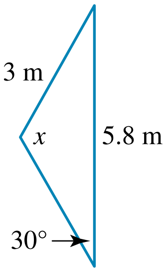
7 Find both the acute and obtuse angles correct to one decimal place.
In ΔPQR, find the unknown angle, P, given $p = 3.5$, $r = 2$ and $\angle R = 12°$.

8 Find both the acute and obtuse angles correct to one decimal place.
In ΔLMN, find the unknown angle, M, given $n = 0.22$ km, $m = 0.5$ km and $\angle N = 18°$.

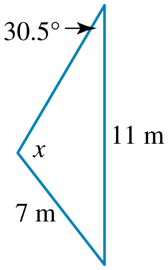
9 Find the unknown angle (correct to the nearest degree) in the following obtuse-angled triangle.



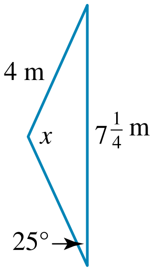
10 Find the unknown angle (correct to the nearest degree) in the following obtuse-angled triangle.



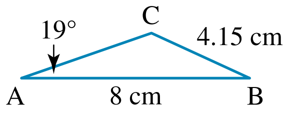
11 Find the unknown angle (correct to the nearest degree) in the following obtuse-angled triangle.



12 Find the unknown angle (correct to the nearest degree) in the following obtuse-angled triangle.

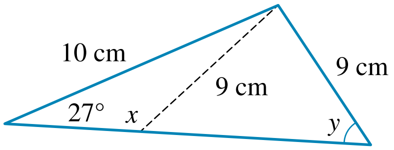


13 In the triangle given, angle C is (correct to the nearest degree):



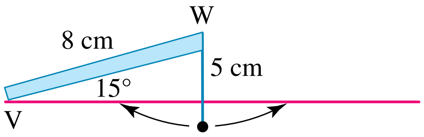
A 38° **B** 39° **C** 78°
D 141° **E** 142°

14 Find the two unknown angles shown in the diagram (correct to 1 decimal place).



MASTER

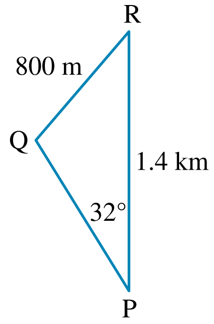
15 Look at the swinging pendulum shown.



- **a** Draw the two possible positions of the bob at the level of the horizontal line.
- **b** Find the value of the angle, W, at these two extreme positions.
- **c** Find the smallest and largest distances between vertex V and the bob.

16 If a boat travelled the path shown:

- **a** what is the obtuse angle between the PQ and QR legs of the trip?
- **b** what is the distance travelled from P to Q?



12.8

The cosine rule

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Concept 3

The cosine rule
Concept summary
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The cosine rule
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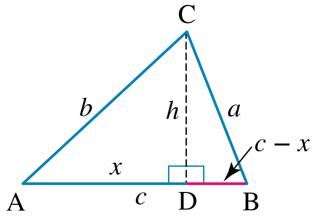
The **cosine rule** is derived from a non-right-angled triangle divided into two right-angled triangles in a similar way to the derivation of the sine rule. The difference is that, in this case, Pythagoras' theorem and the cosine ratio are used to develop it.



The triangle ABC in the figure has been divided into two right-angled triangles with base sides equal to x and $(c - x)$.

In ΔACD, $h^2 = b^2 - x^2$

and in ΔBCD, $h^2 = a^2 - (c - x)^2$ (Pythagoras' theorem)

Equating expressions for h^2,

$$\begin{aligned} b^2 - x^2 &= a^2 - (c - x)^2 \\ a^2 &= b^2 - x^2 + (c - x)^2 \\ &= b^2 - x^2 + c^2 - 2cx + x^2 \\ a^2 &= b^2 + c^2 - 2cx \qquad [1] \end{aligned}$$

Now, from ΔACD, $\cos(A) = \dfrac{x}{b}$

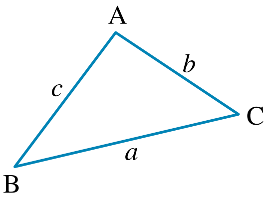
$$x = b\cos(A)$$

Substitute this value of x into [1] above.

$$a^2 = b^2 + c^2 - 2c[b\cos(A)]$$

So, the cosine rule can be written as:

$$\boxed{a^2 = b^2 + c^2 - 2bc \times \cos(A).}$$



In a similar way to that above, it can be shown that:

$$b^2 = a^2 + c^2 - 2ac \times \cos(B)$$
$$c^2 = a^2 + b^2 - 2ab \times \cos(C).$$

Also, if the triangle is labelled using other letters, for example STU, then:

$$s^2 = t^2 + u^2 - 2tu \times \cos(S).$$

The formula may be transposed in order to find an unknown angle.

$$\boxed{\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}}$$

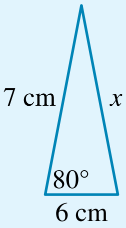
or alternatively, $\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$ and $\cos(C) = \dfrac{a^2 + b^2 - c^2}{2ab}$.

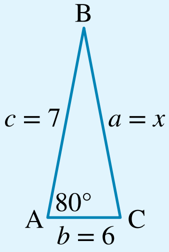
The cosine rule is used to find:

1. **an unknown length when you have the lengths of two sides and the angle in between**
2. **an unknown angle when you have the lengths of all three sides.**

WORKED EXAMPLE 18

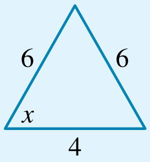
Find the unknown length (correct to 2 decimal places), x, in the triangle.

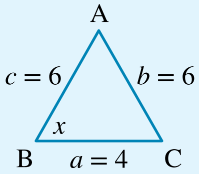


THINK	WRITE/DRAW
1 Identify the triangle as non-right-angled.	
2 Label the triangle appropriately for the sine rule or cosine rule.	
3 Identify that it is the cosine rule that is required as you have the two sides and the angle in between.	$b = 6 \quad A = 80°$ $c = 7 \quad a = x$

4	Substitute the known values into the cosine rule formula and evaluate the right-hand side.	$a^2 = b^2 + c^2 - 2bc \times \cos(A)$ $x^2 = 6^2 + 7^2 - 2 \times 6 \times 7 \times \cos(80^\circ)$ $= 36 + 49 - 84 \times \cos(80^\circ)$ $= 70.4136$
5	Remember to get the square root value, x.	$x = \sqrt{70.4136}$ $= 8.391$
6	Write the answer, rounding off to the required number of decimal places and including the units.	$x = 8.39$ The unknown length is 8.39 cm, correct to 2 decimal places.

WORKED EXAMPLE 19 **Find the size of angle x in the triangle, correct to the nearest degree.**

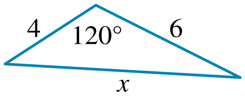


	THINK	WRITE/DRAW
1	Identify the triangle as non-right-angled.	
2	Label the triangle appropriately for the sine rule or cosine rule.	
3	As all three sides are given, the cosine rule should be used. Write the rule and identify the values of the pronumerals.	$\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$ $a = 4,\ b = 6,\ c = 6,\ B = x$
4	Substitute the known values into the formula and simplify.	$\cos(x) = \dfrac{4^2 + 6^2 - 6^2}{2 \times 4 \times 6}$ $\cos(x) = \dfrac{16}{48}$ $\cos(x) = 0.3333$
5	Evaluate x [$x = \cos^{-1}(0.3333)$].	$x = 70.53^\circ$ $x \approx 71^\circ$
6	Round to the nearest degree and state your answer.	The angle x is 71°, correct to the nearest degree.

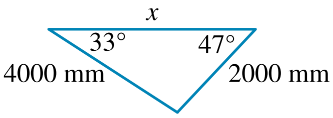
EXERCISE 12.8 The cosine rule

PRACTISE

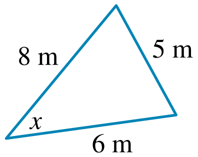
1 **WE18** Find the unknown length correct to 2 decimal places.



2 Find the unknown length correct to 2 decimal places.



3 **WE19** Find the size of the unknown angle (correct to the nearest degree).

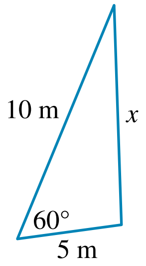


4 Construct a suitable triangle from the following instructions and find all unknown sides and angles. Two sides are 23 cm and 15 cm and the angle in between is 28°.

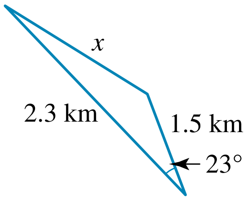
CONSOLIDATE

5 Find the unknown length in each of the following (correct to 2 decimal places).

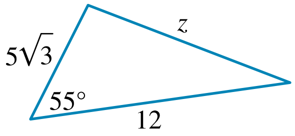
a



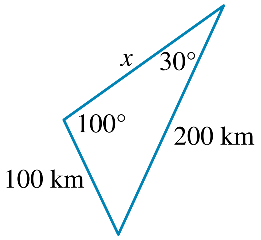
b



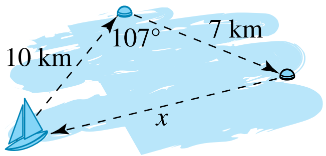
c



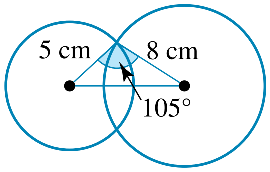
d



6 During a sailing race, the boats followed a triangular course as shown. Find the length, x, of the third leg (correct to 1 decimal place).

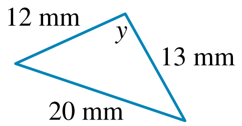


7 Two circles, with radii 5 cm and 8 cm, overlap as shown. If the angle between the two radii that meet at the point of intersection of the circumferences is 105°, find the distance between the centres of the circles (correct to 1 decimal place).

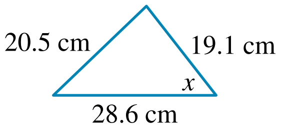


8 Find the size of the unknown angle in each of the following (correct to the nearest degree).

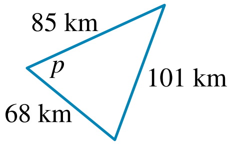
a



b

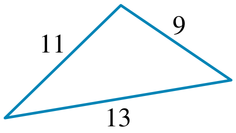


c



9 Consider the sailing expedition course in question **6**. Find the two unknown angles (correct to the nearest degree) in the triangular course.

10 For the triangle shown, find all three unknown angles (correct to the nearest degree).

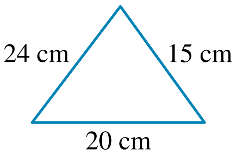


11 For the following questions, give answers correct to 1 decimal place.

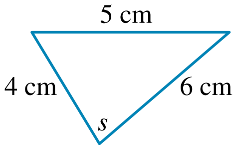
- **a** For ΔABC, find the unknown side, b, given $a = 10$ km, $c = 8$ km and $\angle B = 30°$.
- **b** For ΔABC, find the unknown angle, B, given $a = b = 10$ and $c = 6$.
- **c** For ΔABC, find the unknown side, c, given $a = 7$ m, $b = 3$ m and $\angle C = 80°$.
- **d** For ΔSTU, find the unknown angle, S, given $t = 12.7$, $s = 16.3$ and $u = 24.5$.
- **e** For ΔPQR, find the unknown angle, P, given $p = 2$, $q = 3.5$ and $r = 2.5$.
- **f** For ΔABC, find the unknown side, a, given $b = 260$, $c = 120$ and $\angle A = 115°$.

12 In the triangle given, the largest angle is:

A 39° **B** 45° **C** 56°
D 85° **E** 141°



13 The correct expression for angle s is:

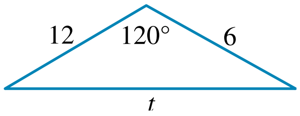


A $\cos^{-1}\left(\dfrac{6^2 + 4^2 - 5^2}{2 \times 6 \times 4}\right)$ **B** $\cos^{-1}\left(\dfrac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5}\right)$

C $\cos^{-1}\left(\dfrac{4^2 - 6^2 + 5^2}{2 \times 4 \times 6}\right)$ **D** $\cos^{-1}\left(\dfrac{4^2 + 6^2 - 5^2}{2 \times 4 \times 5}\right)$

E $\cos^{-1}\left(\dfrac{5^2 + 6^2 - 4^2}{2 \times 5 \times 6}\right)$

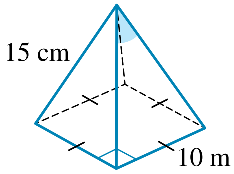
14 The correct expression for the value of t is:



A $\sqrt{180 + 144\cos(120°)}$ **B** $\sqrt{180 - 120}$
C $\sqrt{180 - 144 \times 0.5}$ **D** $\sqrt{180 - 72}$
E $\sqrt{180 + 72}$

MASTER

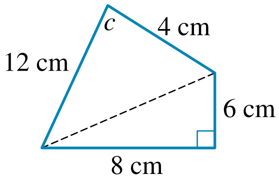
15 The 4 surface angles at the vertex of a regular square-based pyramid are all the same. The magnitude of these angles for the pyramid shown at right (correct to the nearest degree) is:



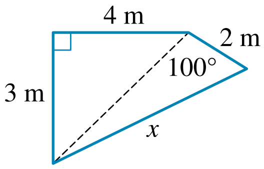
A 1° **B** 34° **C** 38° **D** 39° **E** 71°

16 Find the unknown values.

a



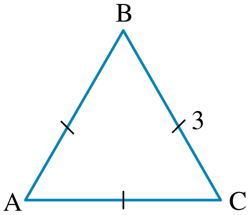
b



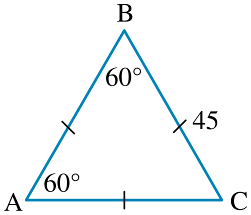
12.9 Special triangles

Often, the triangles encountered in problem solving are either *equilateral* or *right-angled isosceles* triangles. They exhibit some unique features that, when recognised, can be very useful in solving problems.

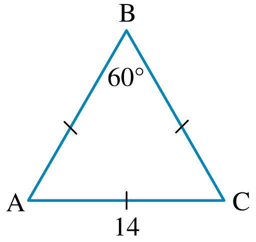
Equilateral triangles have three equal sides and three equal angles. Therefore, when given the length of one side, all sides are known. The three angles are always equal to 60°.



$a = b = c = 3$
$A = B = C = 60°$

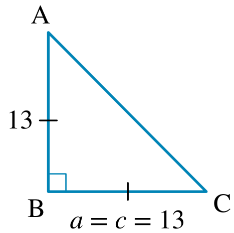


$a = b = c = 45$
$C = 60°$

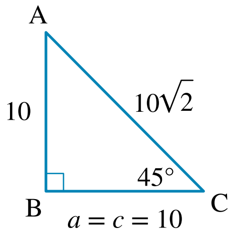


$b = a = c = 14$
$B = A = C = 60°$

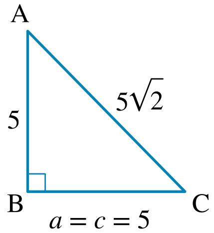
Right-angled isosceles triangles have one right angle (90°) opposite the longest side (hypotenuse) and two equal sides and angles. The two other angles are always 45°.



$a = c = 13$
$b = 13\sqrt{2}$
$A = C = 45°$
$B = 90°$



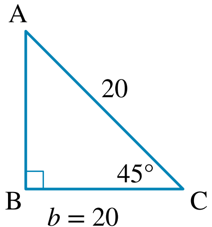
$a = c = 10$
$b = 10\sqrt{2}$
$A = C = 45°$
$B = 90°$



$a = c = 5$
$b = 5\sqrt{2}$
$A = C = 45°$
$B = 90°$

Also, the hypotenuse is *always* $\sqrt{2}$ times the length of the smaller sides.

Check for yourself using Pythagoras' theorem.

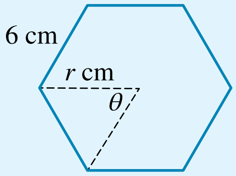


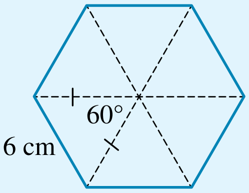
$b = 20$
$a = c = \dfrac{20}{\sqrt{2}}$
$A = C = 45°$
$B = 90°$

WORKED EXAMPLE 20

Find the values of r and angle θ in the hexagon shown.

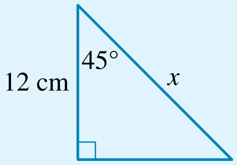
Regular hexagon

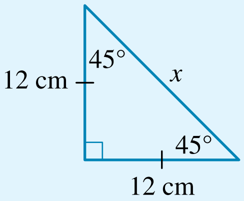


THINK	WRITE/DRAW
1 Triangles in a regular hexagon are all identical. The six angles at the centre are equal. The magnitude of each is one revolution divided by 6.	 $\theta = 360° \div 6$ $= 60°$
2 Furthermore, the two sides that form the triangle are equal. Thus the two equal angles on the shape's perimeter are also 60°. All three angles are the same; therefore, all three sides are equal. Therefore, the triangles in a regular hexagon are all equilateral triangles.	$r = 6$ cm

WORKED EXAMPLE 21

Find the value of the pronumeral (correct to 1 decimal place) in the figure.

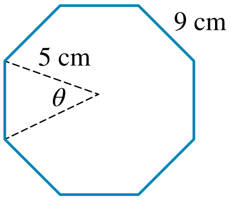


THINK	WRITE/DRAW
1 The triangle is a right-angled isosceles triangle. Two angles are 45° and the third angle is 90°.	
2 Two sides are equal and the longer side opposite the right angle is $\sqrt{2}$ times longer than these equal sides.	$c = a \times \sqrt{2}$ $x = 12 \times \sqrt{2}$ $= 16.97056$
3 Write your answer using the required accuracy and include units.	The value of x is 17.0 cm, correct to 1 decimal place.

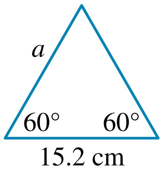
EXERCISE 12.9 Special triangles

PRACTISE

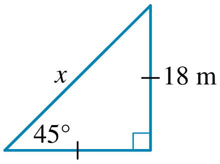
1 **WE20** Find the value of θ in the regular octagon shown.



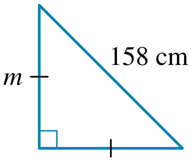
2 Find the value of the unknown length.



3 **WE21** Find the value of the pronumeral (correct to 1 decimal place) in the figure shown.

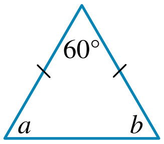


4 Find the value of the unknown length.

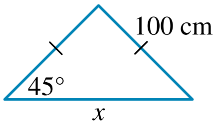


CONSOLIDATE

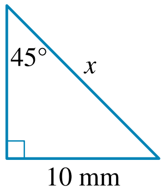
5 Find the value of the unknowns.



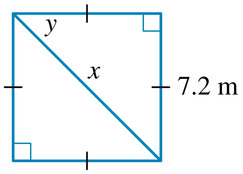
6 Find the value of the unknown length.



7 Find the value of the unknown length.



8 Find the value of the unknowns.



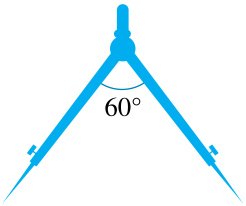
9 In ΔABC, find the unknown angle, B, given $b = 10$, $c = 10\sqrt{2}$ and $\angle C = 90°$.

10 In ΔSTU, find the unknown side, s, given $t = 12.7$, $\angle S = 45°$ and $\angle T = 45°$.

11 In ΔPQR, find the unknown angle, P, given $p = 3.5$, $r = 3.5$ and $\angle R = 60°$.

12 In ΔLMN, find the unknown side, m, given $n = 0.22$, $\angle L = 60°$ and $\angle N = 60°$.

13 A pair of compasses used for drawing circles has legs that are 6 cm long. If it is opened as shown in the diagram, what is the radius of the circle that could be drawn?

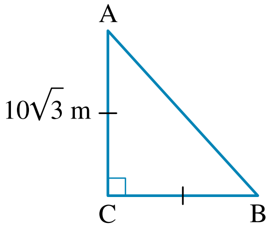


14 What is the height of a tree if its shadow, on horizontal ground, is 12 metres long when the sun's rays striking the tree are at 45° to the ground?

MASTER

15 In the triangle given, the length of side AB (in metres) is:

A $20\sqrt{2}$
B 10
C 20
D $\sqrt{20}$
E $\sqrt{40}$



16 A 40 cm square serviette is prepared for presentation by completing three folds — firstly, by taking a corner and placing it on top of the opposite corner; secondly, by taking one of the two corners on the crease that has been made and placing it on the other one; and finally, by placing the two corners at the ends of the longest side on top of each other.

a Find the length of the crease made after the:
  i first fold
  ii second fold
  iii third fold.

b With the final serviette lying flat, what angles are produced at the corners?

12.10 Area of triangles

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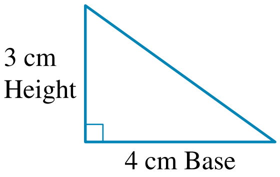
Interactivity
Area of triangles
int-6483

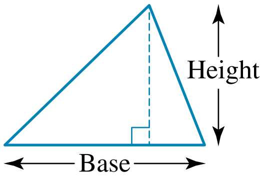
Three possible methods can be used to find the area of a triangle:

Method 1. When the *two known lengths are perpendicular* to each other we would use:

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times \text{Base} \times \text{Height}$$

$$A = \frac{1}{2}bh$$





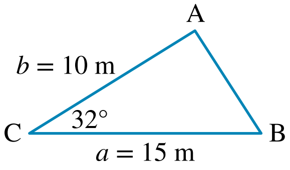
studyon

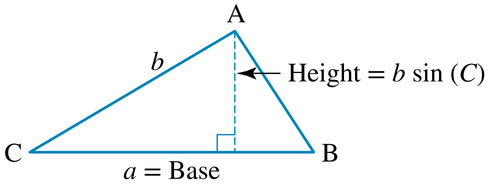
Unit 4
AOS M3
Topic 1
Concept 4

Area of a triangle
Concept summary
Practice questions

Method 2. When we are given *two lengths and the angle in between* we would use:

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times a \times b \times \sin(C)$$
$$A = \frac{1}{2} ab \sin(C)$$





$$\text{Area} = \tfrac{1}{2} \times \text{Base} \times \text{Height}$$
$$= \tfrac{1}{2} \times a \times b \sin(C)$$

eBook*plus*

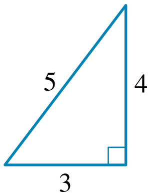
Interactivity
Using Heron's formula to find the area of a triangle
int-6475

Method 3. When *all three sides are known* we would use:

$\text{Area}_{\text{triangle}} = \sqrt{s(s-a)(s-b)(s-c)}$ **where the semi-perimeter,** $s = \dfrac{(a+b+c)}{2}$.

This formula is known as *Heron's formula*. It was developed by Heron (or Hero) of Alexandria, a Greek mathematician and engineer who lived around CE 62.

Let us find the area of the triangle below to demonstrate that all three formulas provide the same result.



For the 3, 4, 5 triangle, the most appropriate method is method 1 because it is a right-angled triangle.

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times \text{Base} \times \text{Height}$$
$$A = \frac{1}{2} \times 3 \times 4$$
$$= 6$$

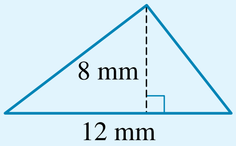
The other two methods may also be used.

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times a \times b \times \sin(C)$$
$$A = \frac{1}{2} \times 3 \times 4 \times \sin(90°)$$
$$= 6 \times 1$$
$$= 6$$

$$\begin{array}{rlrl}
\text{Area}_{\text{triangle}} & = \sqrt{s(s-a)(s-b)(s-c)} & s & = \dfrac{(a+b+c)}{2} \\
A & = \sqrt{6(6-3)(6-4)(6-5)} & & = \dfrac{(3+4+5)}{2} \\
 & = \sqrt{6 \times 3 \times 2 \times 1} & & = \dfrac{12}{2} \\
 & = \sqrt{36} & & = 6 \\
 & = 6 & &
\end{array}$$

WORKED EXAMPLE 22

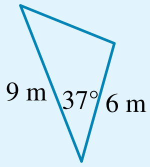
Find the area of the triangle shown.

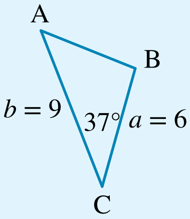


THINK	WRITE
1 The two given lengths are perpendicular. Write the most appropriate formula for finding the area.	$\text{Area}_{\text{triangle}} = \frac{1}{2} \times \text{Base} \times \text{Height}$
2 Substitute the known values into the formula.	$= \frac{1}{2} \times 12 \times 8$ $= 48$
3 Write the answer using correct units.	The area of the triangle is 48 mm^2.

WORKED EXAMPLE 23

Find the area of the triangle (correct to 2 decimal places).

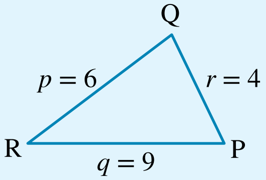


THINK	WRITE/DRAW
1 Identify the shape as a triangle with two known sides and the angle in between.	
2 Identify and write down the values of the two sides, a and b, and the angle in between them, C.	$a = 6$ $b = 9$ $C = 37°$
3 Identify the appropriate formula and substitute the known values into it.	$\text{Area}_{\text{triangle}} = \frac{1}{2}ab\sin(C)$ $= \frac{1}{2} \times 6 \times 9 \times \sin(37°)$ $= 16.249$
4 Write the answer using correct units.	The area of the triangle is 16.25 m^2, correct to 2 decimal places.

WORKED EXAMPLE 24 **Find the area of a triangle PQR (correct to 1 decimal place), given $p = 6$, $q = 9$ and $r = 4$, with measurements in centimetres.**

THINK | **WRITE/DRAW**

1 All three sides of the triangle have been given; therefore, Heron's formula can be used to find the area.



2 Write the values of the three sides, a, b and c, and calculate the semi-perimeter value, s.

$$a = p = 6,\ b = q = 9,\ c = r = 4$$

$$s = \frac{(a + b + c)}{2} = \frac{(6 + 9 + 4)}{2} = 9.5$$

3 Substitute the known values into Heron's formula and evaluate.

$$\begin{aligned}\text{Area}_{\text{triangle}} &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{9.5(9.5-6)(9.5-9)(9.5-4)}\\ &= \sqrt{9.5 \times 3.5 \times 0.5 \times 5.5}\\ &= \sqrt{91.4375}\\ \text{Area} &= 9.5623\end{aligned}$$

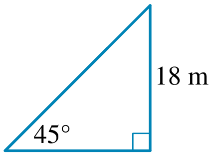
4 Write the answer, using the correct units.

The area of triangle PQR is 9.6 cm^2, correct to 1 decimal place.

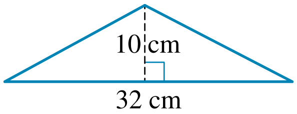
EXERCISE 12.10 Area of triangles

PRACTISE

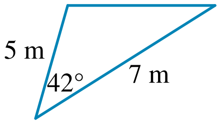
1 **WE22** Find the area of the triangle shown.



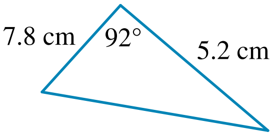
2 Find the area of the triangle shown.



3 **WE23** Find the area of the triangle shown (correct to 2 decimal places).



4 Find the area of the triangle shown (correct to 2 decimal places).



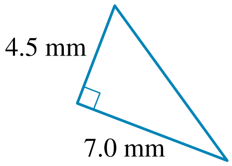
5 **WE24** Find the area of ΔABC (correct to 1 decimal place) given $a = b = 10$ cm and $c = 6$ cm.

6 Find the area of an equilateral triangle with side lengths of 10 cm.

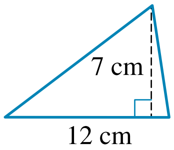
CONSOLIDATE

7 Find the areas of the following triangles (correct to 1 decimal place).

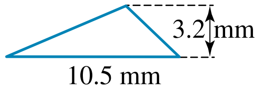
a



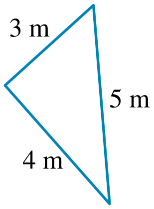
b



c

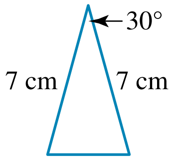


d

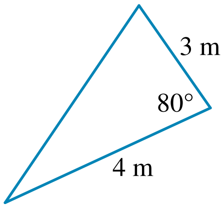


8 Find the areas of the following triangles (correct to 1 decimal place).

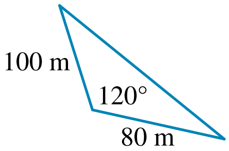
a



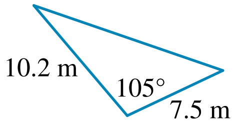
b



c

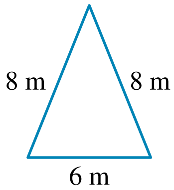


d

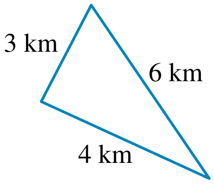


9 Find the areas of the following triangles (correct to 1 decimal place).

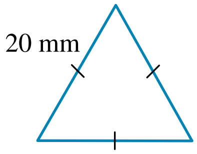
a



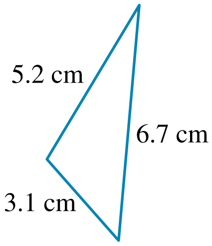
b



c

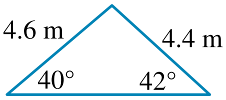


d

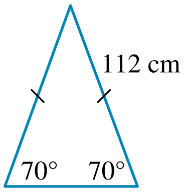


10 Find the areas of the following triangles (correct to 1 decimal place).

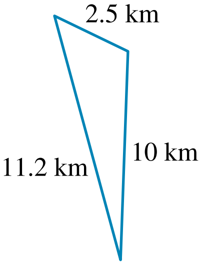
a



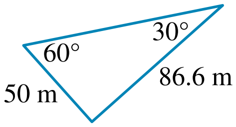
b



c



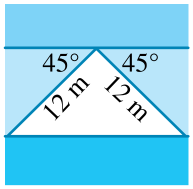
d



11 Find the area of each of the following triangles. (Give all answers correct to 1 decimal place.)

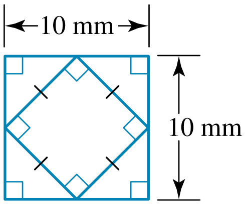
- **a** ΔABC, given $a = 10$ km, $c = 8$ km and $\angle B = 30°$
- **b** ΔABC, given $a = 7$ m, $b = 3$ m, $c = 8.42$ m and $\angle C = 108°$
- **c** ΔSTU, given $t = 12.7$ m, $s = 16.3$ m and $u = 24.5$ m
- **d** ΔPQR, given $p = 2$ units, $q = 3.5$ units and $r = 2.5$ units
- **e** ΔABC, given $b = 260$ cm, $c = 120$ cm and $\angle A = 90°$

12 A triangular arch has supporting legs of equal length of 12 metres as shown in the diagram. What is its area?

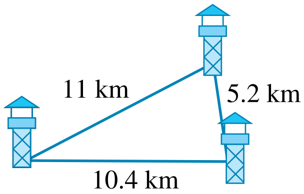


13 From the diagram given at right,

- **a** find the area of:
  - **i** one of the triangles
  - **ii** all of the triangles.
- **b** Use another technique to verify your answer in **a i**.



14 Find the area of the state forest as defined by the three fire-spotting towers on the corners of its boundary.

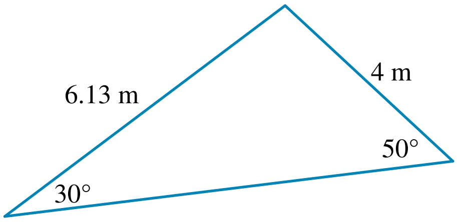


15 If the perimeter of an equilateral triangle is 210 metres, its area is closest to:

A 2100 m^2 **B** 2450 m^2 **C** 4800 m^2 **D** 5500 m^2 **E** 1700 m^2

16 The correct expression for the area of the shape shown is:

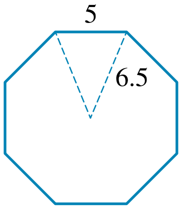
- **A** $\frac{1}{2} \times 6.13 \times 4 \times \sin(80°)$
- **B** $\frac{1}{2} \times 6.13 \times 4 \times \cos(100°)$
- **C** $\frac{1}{2} \times 6.13 \times 4 \times \sin(100°)$
- **D** $\frac{1}{2} \times 6.13 \times 4$
- **E** none of the above



MASTER

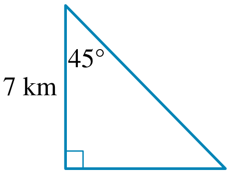
17 The correct expression for the area of the octagon at right is:

- **A** $195 \times \sin(45°)$
- **B** $169 \times \sin(45°)$
- **C** $195 \times \sin(60°)$
- **D** $338 \times \sin(60°)$
- **E** $5 \times 6.5 \times \sin(67.5°)$

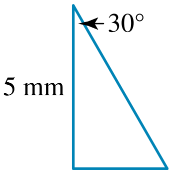


18 Find the area of the following triangles.

a



b



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12.11 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

To access eBookPLUS activities, log on to

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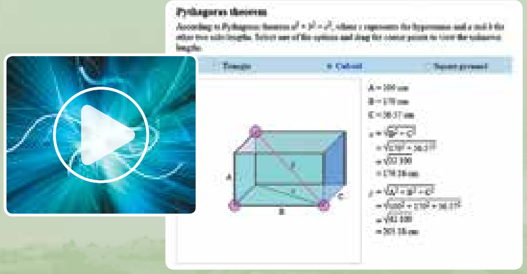
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.



studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

12 Answers

EXERCISE 12.2

1 13.9 cm

2 8.2 m

3 194 m

4 5.8 m

5 a 13.0 b 12.0 c 2.5
d 21.0 e 1.7 f 3.6

6 5831 m

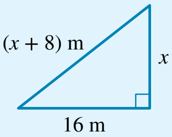
7 3162 mm

8 a 15.0 b 17.3 c 12.0
d 24.0 e 7.6 f 10.6

9 a 13 b 24.17 mm c 8.77
d 5 m e 15.23 m f 246.98 cm

10 a 20.3 b 12.7
c 3.4 mm d 5.8

11 a



b $(x + 8)^2 = x^2 + 16^2$

c 12 m, 16 m, 20 m

12 D

13 C

14 a $x = 4$ b $x = 13$
c $x = 1.414$ d $x = 5.106$

15 1.2 m

16 a $x = 5.17$ m b $x = 245.20$ m

EXERCISE 12.3

1 Not a Pythagorean triad

2 Pythagorean triad

3 Yes, the right angle is opposite the 5 cm side.

4 Not right-angled

5 a Yes b No c Yes d No
e Yes f Yes g No h No
i Yes j Yes k Yes l No

6 a 9, **12**, 15 b **7**, 24, 25 c 1.5, 2.0, **2.5**
d 3, **4**, 5 e 11, 60, **61** f 10, **24**, 26
g **9**, 40, 41 h 0.7, 2.4, **2.5**

7 a 15 c 50 e 1.0 f 25
i 61 j 26 k 20

8 Yes, opposite the 34-cm side

9 No

10 a 21 b 25 cm c 50 d 6.0
e 1.2 f 24 km

11 2.5 km

12 480 cm

13 E

14 B

15 144 cm

16 a $x = 10.77$ b $a = 13.11$
c $c = 8.26$ mm
d $x = 8.06$ cm, $y = 5.39$ cm

EXERCISE 12.4

1 4.0 m

2 a 4.47 cm b 4.90 cm

3 a $\sqrt{281}$ cm b $\frac{\sqrt{281}}{2}$ cm c 16 cm

4 a 707.11 m b 463.68 m

5 a 72.1 cm b 453.02 cm
c 16 331.29 cm^2 d 151.5 cm

6 1.96 cm 7 346 cm 8 128 cm

9 a i $\sqrt{2225}$ cm or 47.2 cm
ii $\sqrt{16625}$ cm or 128.9 cm
b i 500 mm ii 1300 mm
c i 44.72 m ii 45.83 m

10 a i 25 m ii 38 m
b i 1.00 km ii 332 m

11 3.3 m

12 B

13 $a = 3.9$, $b = 4.7$, $c = 5.6$

14 AB = 4.95 m, DH = 14.16 m

15 58 m

16 a 6.8 m b 3.0 m c 14.42 cm

17 84.3 cm

18 1.32 cm

19 678.6 cm^2

20 a i 510 mm ii 522 mm
b i 707 mm ii 716 mm

21 $x = 12.73$ cm

EXERCISE 12.5

1 1.9 m 2 1509.4 mm 3 47 m
4 425 cm 5 2.3 m 6 91 m

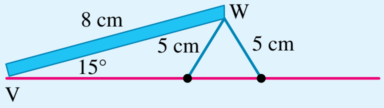
7 45°
8 53°
9 a 8.2 km b 147.1 mm
c 19.2 cm d 44.6 mm
10 a 5.0 b 2 m c 9.3
11 a 24.3 cm b 74.2 m c 4.6 cm
12 a 53° b 53°
c $\theta = 45^\circ$ d $\theta = 68.20^\circ$
13 a 53° b 42° c 76°
14 37°, 53°
15 D
16 $\theta = 30^\circ$, $x = 8$ m, $y = 6.5$ m
17 59°
18 56°
19 a 1.23 m b 11.8°
20 44°

EXERCISE 12.6

1 28.8 mm
2 A
3 30.2 cm
4 26.1 km
5 46°
6 46°, 106°, 30.7 cm
7 a 145 cm b 34 cm
8 4.9 cm
9 a 13.2 cm b 13.2 m
c 27.6 cm d 109.4 km
10 1.6 km
11 a 11.6 m b 28.6 mm c 30.6 cm
12 a 52° b 24° c 33°
d 27° e 48°
13 B
14 C
15 D
16 A
17 34.6 mm
18 a 14.8 m b 10.8 m
19 2.8 m
20 a 129.1° b 68 km

EXERCISE 12.7

1 47° or 133°
2 31° or 149°
3 117°
4 132°
5 52.1° or 127.9°
6 65.2° or 114.8°
7 21.3° or 158.7°
8 44.6° or 135.4°
9 141°
10 105°
11 127°
12 130°
13 D
14 $y = 30.3^\circ$ and $x = 149.7^\circ$
15 a



b 9.5° or 140.5°
c 3.2 cm or 12.3 cm
16 a 112°
b 0.89 km or 890 m

EXERCISE 12.8

1 8.72
2 4772.67 mm
3 39°
4 12.03 cm, 116.2°, 35.8°
5 a 8.66 m b 1.09 km
c 9.99 units d 155.85 km
6 13.8 km
7 10.5 cm
8 a 106° b 46° c 82°
9 44°, 29°
10 43°, 80°, 57°
11 a 5.0 km b 72.5° c 7.1 m
d 37.2° e 34.0° f 329.2
12 D
13 A
14 E
15 D
16 a 51.3° b 5.7 m

EXERCISE 12.9

1 $\theta = 45^\circ$
2 15.2 cm
3 $x = 25.5$
4 111.7 cm
5 60°, 60°
6 $100\sqrt{2}$ cm or 141.4 cm
7 $10\sqrt{2}$ mm or 14.1 mm
8 10.2 m, 45°

9 45°

10 12.7

11 60°

12 0.22

13 6 cm

14 12 m

15 C

16 **a** **i** $40\sqrt{2}$ cm or 56.6 cm
  ii $20\sqrt{2}$ cm or 28.3 cm
  iii 20 cm
 b 45° 45° 90°

EXERCISE 12.10

1 162 m^2

2 160 cm^2

3 11.71 m^2

4 20.27 m^2

5 28.6 cm^2

6 43.3 cm^2

7 **a** 15.8 mm^2 **b** 42.0 cm^2
 c 16.8 mm^2 **d** 6.0 m^2

8 **a** 12.3 cm^2 **b** 5.9 m^2
 c 3464.1 m^2 **d** 36.9 m^2

9 **a** 22.2 m^2 **b** 5.3 km^2
 c 173.2 mm^2 **d** 7.8 cm^2

10 **a** 10.0 m^2 **b** 4031.6 cm^2
 c 11.5 km^2 **d** 2165 m^2

11 **a** 20.0 km^2 **b** 10.0 m^2 **c** 94.0 m^2
 d 2.4 $units^2$ **e** 15 600 cm^2

12 72 m^2

13 **a** **i** 12.5 mm^2 **ii** 50 mm^2

14 26.8 km^2

15 A

16 C

17 B

18 **a** 24.5 km^2 **b** 7.22 mm^2

13

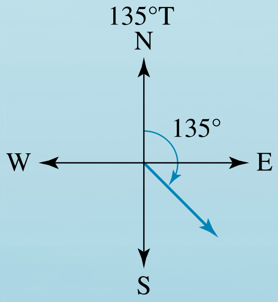
Applications of geometry and trigonometry

13.1 Kick off with CAS

Exploring bearings with CAS

True bearings allow us to state the direction from one object to another object, with respect to the north direction.

All true bearings should be stated as 3-digit figures, with the angle measure taken from north, as in the diagram at right.



True bearings can be used to solve navigation and triangulation problems, allowing us to accurately determine the location of objects if given limited information.

A cruise ship set off for sail on a bearing of 135°T and travelled in this direction for 6 kilometres.

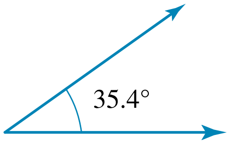
1. Determine how far east the ship is from its starting position by completing the following steps:
   a. Copy the above diagram and mark the blue direction line as having a length of 6 km.
   b. Determine the angle between the 135° line and the east direction on the compass.
   c. Use the east direction (adjacent side), the 135° line (hypotenuse) and the angle calculated in part **b** to draw a right-angled triangle to represent the situation.
   d. Use CAS and the cosine ratio to calculate the length of the adjacent side length, which will tell you how far east the ship is from its starting position.
2. Use CAS and the sine ratio to calculate how far south the ship is from its starting position.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

13.2 Angles

An angle represents the space between two intersecting lines or surfaces at the point where they meet.

Angles are measured in degrees (°). In navigation, accuracy can be critical, so fractions of a degree are also used. For example, a cruise ship travelling 1000 kilometres on a course that is out by half a degree would miss its destination by almost 9 kilometres.



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Interactivity
Adding and subtracting angles
int-6278

Adding and subtracting angles

Angles can be added and subtracted using basic arithmetic. When using CAS or other technology to solve problems involving angles, make sure you are in degrees mode.

WORKED EXAMPLE 1

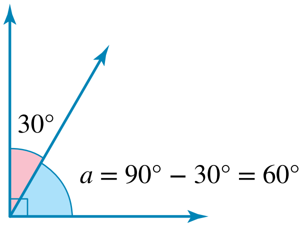
a **Add 46.37° and 65.49°.**

b **Subtract 16.55° from 40.21°.**

THINK	WRITE
a **1** Add the angles together, remembering to include the decimal points.	**a** $\begin{array}{r} 46.37° \\ +\ 65.49° \\ \hline 111.86° \end{array}$
2 Write the answer.	111.86°
b **1** Subtract 16.55° from 40.21°, remembering to include the decimal points.	**b** $\begin{array}{r} 40.21° \\ -\ 16.55° \\ \hline 23.66° \end{array}$
2 Write the answer.	23.66°

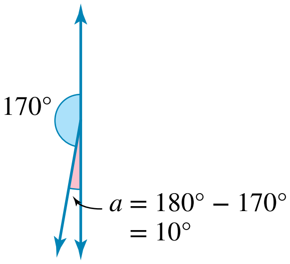
Some geometry (angle) laws

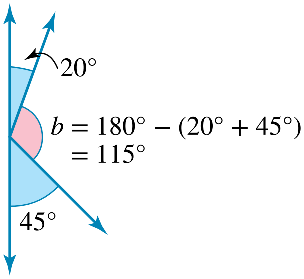
The following angle laws will be valuable when finding unknown values in the applications to be examined in this topic. These laws were dealt with at the start of topic 11, but we will review them here. Often we will need the laws to convert given directional bearings into an angle in a triangle.



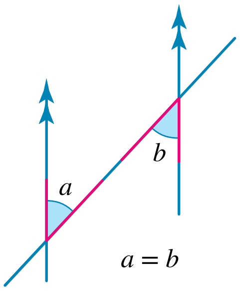
Two or more angles are **complementary** if they add up to 90°.

Two or more angles are **supplementary** if they add up to 180°. An angle of 180° is also called a *straight angle*.

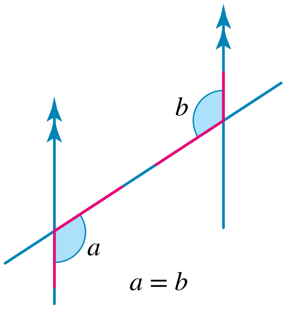




For **alternate angles** to exist we need a minimum of one pair of parallel lines and one transverse line. Alternate angles are *equal*.

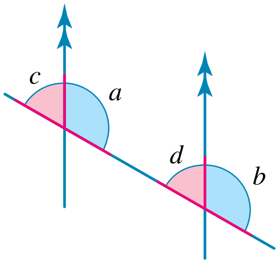


$a = b$

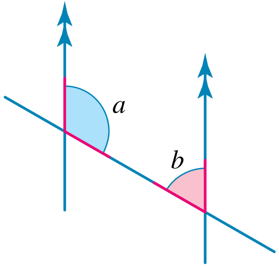


$a = b$

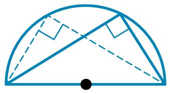
Other types of angles to be considered are **corresponding angles**, **co-interior angles**, *triangles in a semicircle* and **vertically opposite angles**.



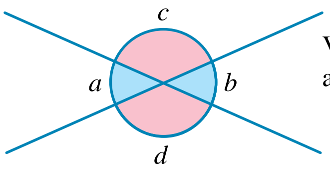
Corresponding angles are equal:
$a = b$
$c = d$



Co-interior angles are supplementary:
$a + b = 180°$

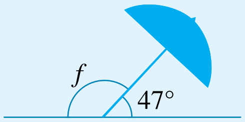


A triangle in a semicircle always gives a right-angled triangle.



Vertically opposite angles are equal:
$a = b$
$c = d$

WORKED EXAMPLE 2 **Find the value of the pronumeral, f, the angle a beach umbrella makes with the ground.**

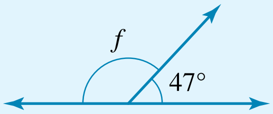


THINK

1 Recognise that the horizontal line is a straight angle, or 180°.

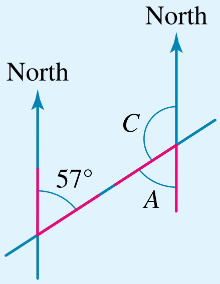
2 To find the unknown angle, use the supplementary or straight angle law.

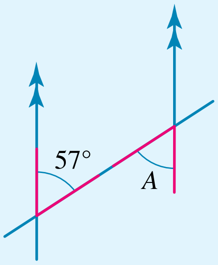
WRITE/DRAW



$$180° = 47° + f$$
$$f = 180° - 47°$$
$$= 133°$$

WORKED EXAMPLE 3 **Find the value of the pronumerals A and C in the directions shown.**



THINK	WRITE/DRAW
1 Recognise that the two vertical lines are parallel lines.	
2 To find the unknown angle A, use the alternate angle law.	$A = 57°$
3 To find the unknown angle C, use the straight angle law. Alternatively, the co-interior angle law could be used with the same solution.	$180° = 57° + C$ $C = 180° - 57°$ $= 123°$

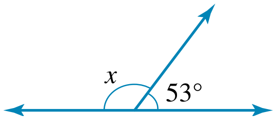
EXERCISE 13.2 Angles

PRACTISE

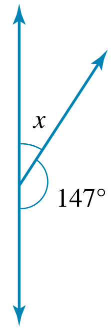
1 **WE1** **a** Add 28.29° and 42.12°. **b** Subtract 21.38° from 74.51°.

2 **a** Add 51.33° and 18.74°. **b** Subtract 35.17° from 80.13°.

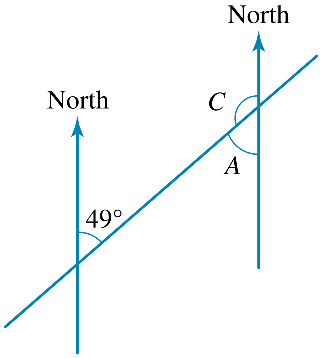
3 **WE2** Find the value of the pronumeral.



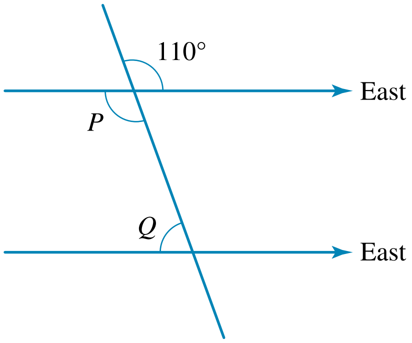
4 Find the value of the pronumeral.



5 **WE3** Find the values of the pronumerals A and C in the diagram shown.



6 Find the values of the pronumerals P and Q in the diagram shown.



CONSOLIDATE

7 Which of the following angle calculations gives the greatest result?

A 27.16° + 33.83°
B 13.84° + 42.69°
C 83.09° − 24.55°
D 9.12° + 50.84°
E 90.11° − 29.25°

8 a What is the supplementary angle of 45.56°?

b What is the complementary angle of 72.18°?

9 Use your calculator to find the values of the following trigonometric ratios correct to 3 decimal places.

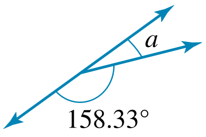
a sin(40.25°)
b cos(122.33°)
c tan(82.10°)
d cos(16.82°)
e sin(147.50°)
f tan(27.46°)

10 Add and then subtract the pairs of angles.

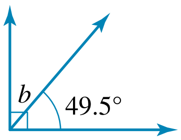
a 40.25°, 28.08°
b 122.21°, 79.35°
c 82.06°, 100.83°
d 247.52°, 140.58°
e 346.37°, 176.84°
f 212.33°, 6.64°

For questions **11**, **12** and **13**, find the values of the pronumerals.

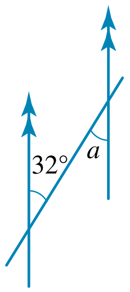
11 a



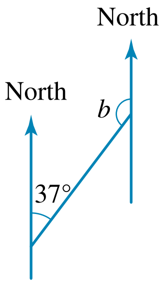
b



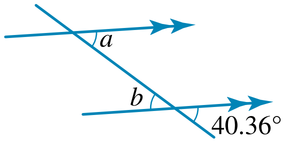
12 a



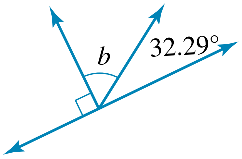
b



13 a



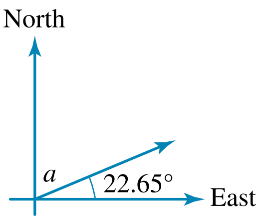
b



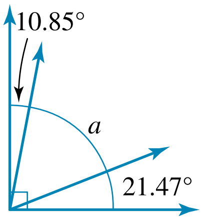
For questions **14**, **15** and **16**, find the values of the pronumerals.

14 a

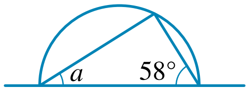
b



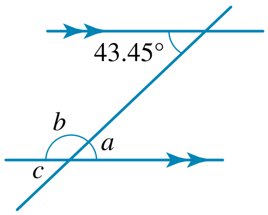
15 a



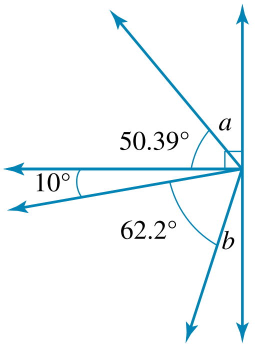
b



16 a

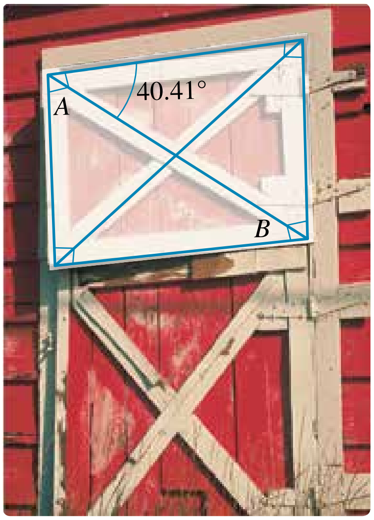


b



MASTER

17 A barn door is shown here.



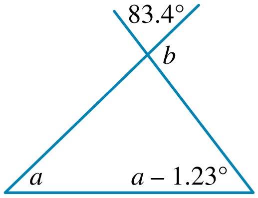
a The value of angle *A* is:

A 40.41° **B** 149.59° **C** 49.41° **D** 50 **E** 90°

b The value of angle *B* is:

A 40.41° **B** 49.59° **C** 49.41° **D** 50° **E** 139.59°

18 Find the values of the pronumerals in the figure.

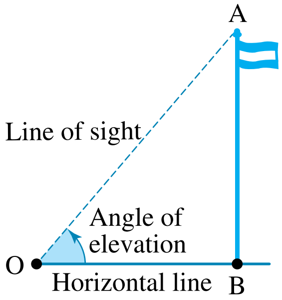


13.3 Angles of elevation and depression

One method for locating an object in the real world is by its position above or below a horizontal plane or reference line.

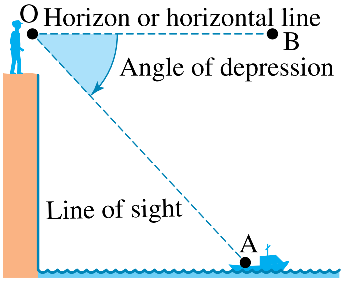
The **angle of elevation** is the angle *above* the horizon or horizontal line.

Looking up at the top of the flagpole from position O, the angle of elevation, ∠AOB, is the angle between the horizontal line OB and the line of sight OA.



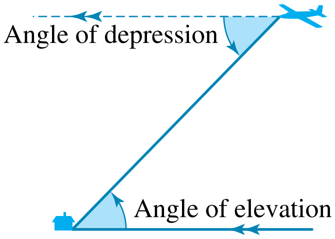
The **angle of depression** is the angle *below* the horizon or horizontal line.

Looking down at the boat from position O, the angle of depression, ∠AOB, is the angle between the horizontal line, OB, and the line of sight, OA.



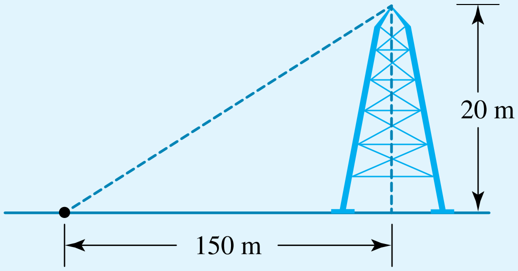
studyon
Unit 4
AOS M3
Topic 1
Concept 8
Angles in the horizontal and vertical planes
Concept summary
Practice questions

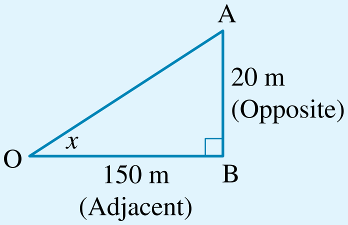
Angles of elevation and depression are in a *vertical plane.*



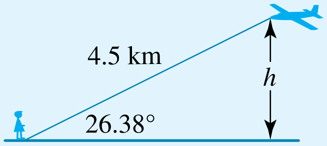
We can see from the diagram the angle of depression given from one location can give us the angle of elevation from the other position using the alternate angle law.

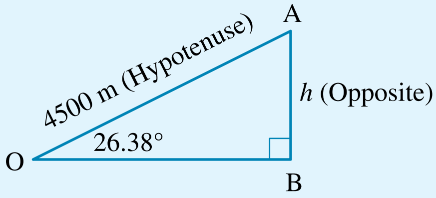
WORKED EXAMPLE 4 **Find the angle of elevation (correct 2 decimal places) of the tower measured from the road as given in the diagram.**



THINK	WRITE/DRAW
1 The angle of elevation is ∠AOB. Use ΔAOB and trigonometry to solve the problem.	
2 The problem requires the tangent ratio. Substitute the values and simplify.	$\tan(\theta) = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$ $= \frac{\text{opposite}}{\text{adjacent}}$ $\tan(x) = \frac{20}{150} = 0.133\,33$
3 Evaluate x and round correct to 2 decimal places.	$x = \tan^{-1}(0.133\,33)$ $= 7.5946°$ $\approx 7.59°$
4 Write the answer.	From the road the angle of elevation to the tower is 7.59°, correct to 2 decimal places.

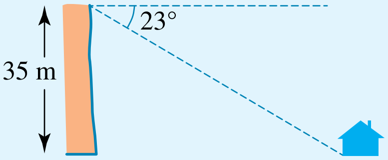
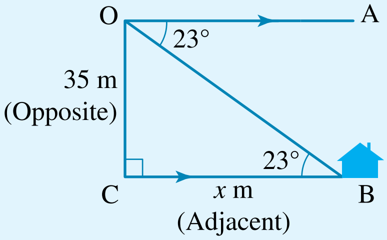
WORKED EXAMPLE 5 **Find the altitude of a plane (correct to the nearest metre) if the plane is sighted 4.5 km directly away from an observer who measures its angle of elevation as 26.38°.**



THINK	WRITE/DRAW
1 Draw a suitable diagram. Change distance to metres.	

2 Use the sine ratio and simplify.	$\sin(\theta) = \frac{\text{length of opposite side}}{\text{length of hypotenuse side}}$ $= \frac{\text{opposite}}{\text{hypotenuse}}$ $\sin(26.38°) = \frac{h}{4500}$ $h = 4500 \times \sin(26.38°)$
3 Evaluate.	$= 1999.45$
4 Write the answer in correct units.	The plane is flying at an altitude of 2000 m, correct to the nearest metre.

WORKED EXAMPLE 6 **The angle of depression from the top of a 35-metre cliff to a house at the bottom is 23°. How far from the base of the cliff is the house (correct to the nearest metre)?**

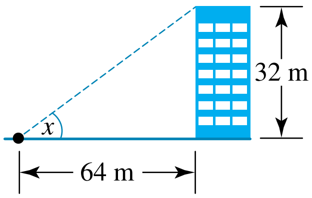
THINK	WRITE/DRAW
1 Draw a suitable diagram.	
2 Angle of depression is ∠AOB. Use the alternate angle law to give the angle of elevation ∠CBO.	
3 Use the tangent ratio. Substitute into the formula and evaluate.	$\tan(\theta) = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$ $= \frac{\text{opposite}}{\text{adjacent}}$ $\tan(23°) = \frac{35}{x}$ $\frac{1}{\tan(23°)} = \frac{x}{35}$ $x = \frac{35}{\tan(23°)}$ $= 82.4548...$
4 Write the answer in correct units.	The distance from the house to the base of the cliff is 82 metres, correct to the nearest metre.

EXERCISE 13.3 Angles of elevation and depression

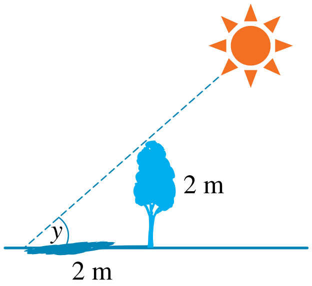
PRACTISE

For questions **1**–**6**, find the values of the pronumerals.

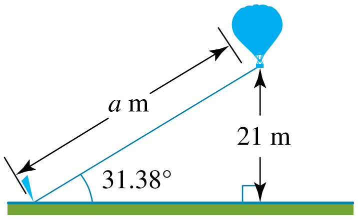
1 **WE4**



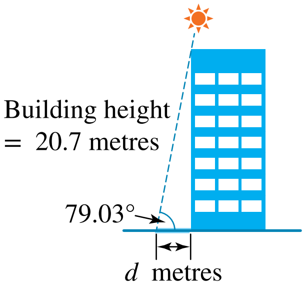
2



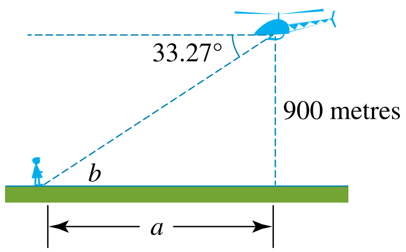
3 **WE5**



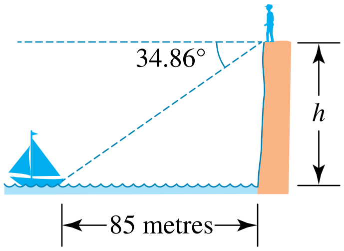
4



5 **WE6**



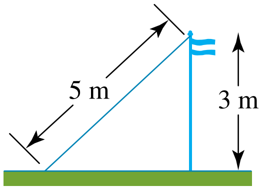
6



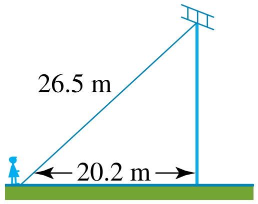
CONSOLIDATE

7 Find the angle of elevation (correct to 2 decimal places) in the following situations.

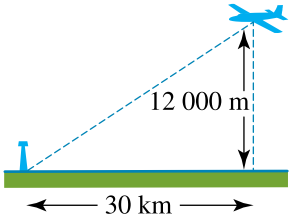
a



b

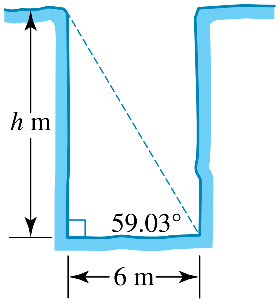


c

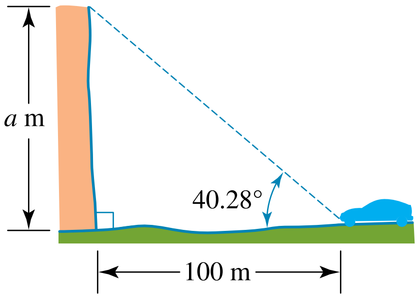


8 Find the values of the pronumerals (correct to the nearest metre).

a



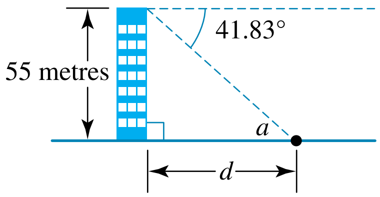
b



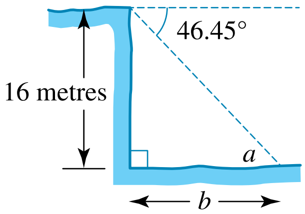
9 The angle of elevation of the sun at a particular time of the day was 49°. What was the length of a shadow cast by a 30-metre tall tower at that time?

10 Find the values of these pronumerals (correct to 2 decimal places).

a

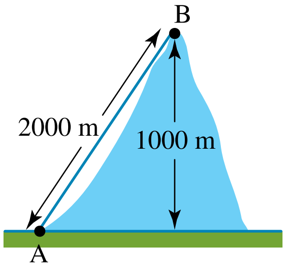


b

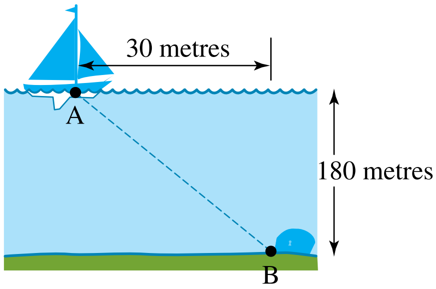


11 Find the angle of elevation or depression from an observer positioned at point A to the object at point B in each situation shown below, correct to two decimal places. State clearly whether it is an angle of depression or elevation.

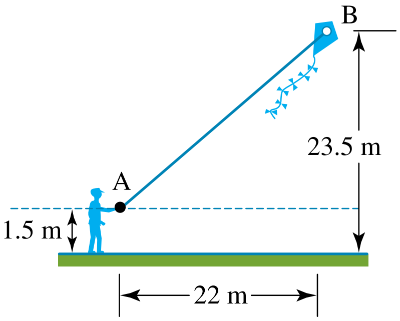
a



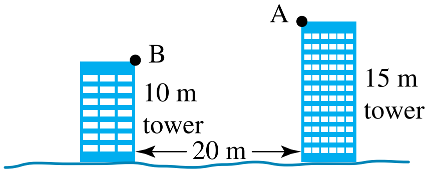
b



c



d



12 A hole has a diameter of 4 metres and is 3.5 metres deep. What is the angle of depression from the top of the hole to the bottom opposite side of the hole?

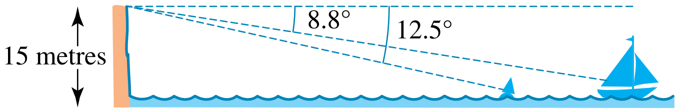
13 The angle of elevation of the top of a tree from a point 15.2 m from the base of the tree is 52.2°. The height of the tree is closest to:

A 12 m **B** 15 m **C** 19 m **D** 20 m **E** 25 m

14 A supporting wire for a 16-m high radio tower is 23.3 m long and is attached at ground level and to the top of the tower. The angle of depression of the wire from the top of the tower is:

A 34.48° **B** 43.37° **C** 46.63° **D** 55.52° **E** 45.16°

15 The angle of depression to a buoy from the top of a 15-metre cliff is 12.5°. A boat is observed to be directly behind but with an angle of depression of 8.8°.



Find (correct to the nearest metre):

a the distance to the buoy from the base of the cliff

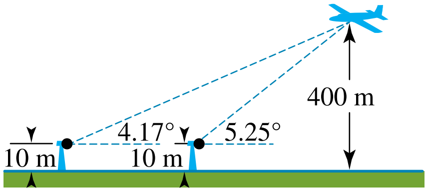
b the distance between the boat and the buoy.

16 Two buildings are 50 metres apart. Building A is 110 metres high. Building B is 40 metres high.

- a Find the angle of elevation from the bottom of building A to the top of building B.
- b Find the angle of depression from the top of building A to the bottom of building B.
- c Find the angle of depression from the top of building B to the bottom of building A.

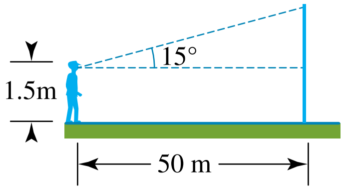
MASTER

17 Watchers in two 10-metre observation towers each spot an aircraft at an altitude of 400 metres. The angles of elevation from the two towers are shown in the diagram.
(Assume all three objects are in a direct line.)



- a What is the horizontal distance between the nearest tower and the aircraft (correct to the nearest 10 metres)?
- b How far apart are the two towers from each other (correct to the nearest 100 metres)?

18 A boy standing 1.5 metres tall measures the angle of elevation to the top of the goalpost using an inclinometer.



- a If the angle was 15° when measured 50 metres from the base of the goalpost, how tall is the goalpost?
- b If the angle of elevation to the top of the goalpost is now 55.5°, how far is the boy from the base of the goalpost?
- c The angle of elevation is measured at ground level and is found to be 45°. Find the distance from the base of the goalpost to where the measurement was made.
- d The result in part c is the same as the height of the goalpost. Explain why.

13.4 Bearings

Bearings are used to locate the position of objects or the direction of a journey on a two-dimensional horizontal plane. Bearings or directions are straight lines from one point to another. To find bearings, a compass rose (a diagram, as shown below, showing N, S, E and W) should be drawn centred on the point from where the bearing measurement is taken.

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Bearings
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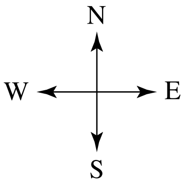
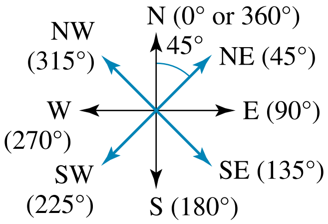
There are three main ways of specifying bearings or direction:

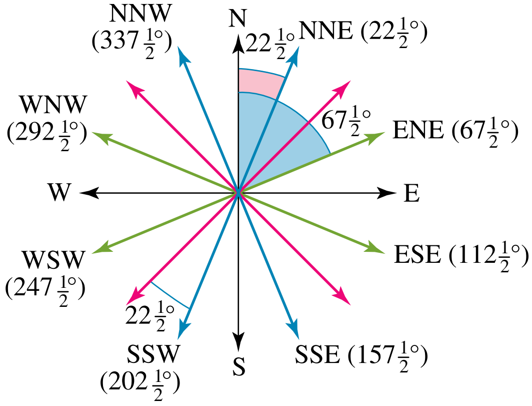
1. standard compass bearings (for example, N, SW, NE)
2. other compass bearings (for example, N10°W, S30°E, N45.37°E)
3. **true bearings** (for example, 100°T, 297°T, 045°T, 056°T)

Standard compass bearings

There are 16 main standard bearings as shown in the diagrams. The N, S, E and W standard bearings are called **cardinal points**.

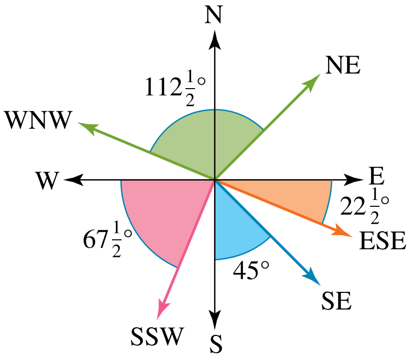






It is important to consider the angles between any two bearings. For example, the angles from north (N) to all 16 bearings are shown in brackets in the diagrams above and at the end of the opposite page.

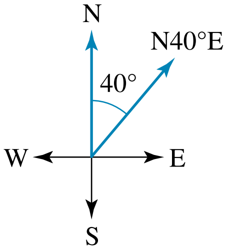
It can be seen that the angle between two adjacent bearings is 22$\frac{1}{2}$°. Some other angles that will need to be considered are shown in the diagram.



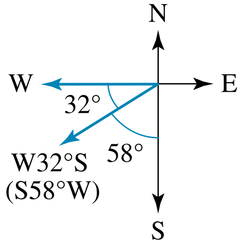
Other compass bearings

Often the direction required is not one of the 16 standard bearings. To specify other bearings the following approach is taken.

1. Start from north (N) or south (S).
2. Turn through the angle specified towards east (E) or west (W).

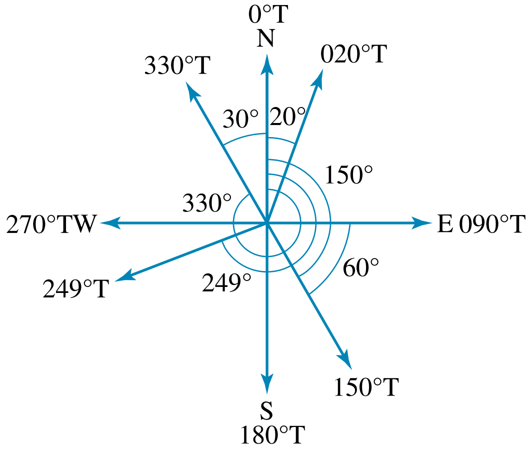


Sometimes the direction may be specified unconventionally, for example, starting from east or west as given by the example W32°S. This bearing is equivalent to S58°W.



True bearings

True bearings is another method for specifying directions and is commonly used in navigation.



To specify true bearings, first consider the following:

1. **the angle is measured from north**
2. **the angle is measured in a clockwise direction to the bearing line**
3. **the angle of rotation may take any value from 0° to 360°**
4. **the symbol T is used to indicate it is a true bearing, for example, 125°T, 270°T**
5. **for bearings less than 100°T, use three digits with the first digit being a zero to indicate it is a bearing, for example, 045°T, 078°T.**

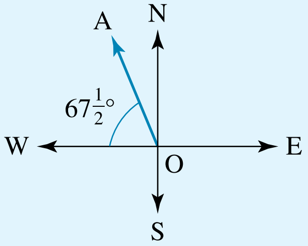
WORKED EXAMPLE 7

Specify the direction in the figure as:

a a standard compass bearing

b a compass bearing

c a true bearing.



THINK

a 1 Find the angle between the bearing line and north, that is, ∠AON.

2 Since the angle is $22\frac{1}{2}°$, the bearing is a standard bearing. Refer to the standard bearing diagram.

b The bearing lies between the north and the west. The angle between north and the bearing line is $22\frac{1}{2}°$.

c Find the angle from north to the bearing line in a clockwise direction. The bearing of west is 270°T.

WRITE

a $\angle AON = 90° - 67\frac{1}{2}°$
$= 22\frac{1}{2}°$

The standard bearing is NNW.

b The compass bearing is N$22\frac{1}{2}°$W.

c Angle required $= 270° + 67\frac{1}{2}°$
$= 337\frac{1}{2}°$

The true bearing is $337\frac{1}{2}°$T.

WORKED EXAMPLE 8

Draw a suitable diagram to represent the following directions.

a S17°E

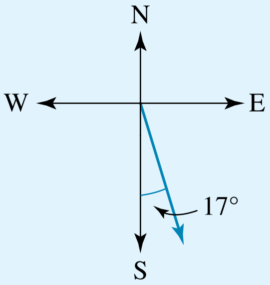
b 252°T

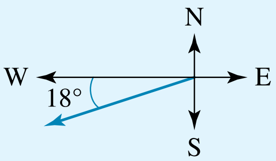
THINK

a Draw the 4 main standard bearings. A compass bearing of S17°E means start from south; turn 17° towards east. Draw a bearing line at 17°. Mark in an angle of 17°.

WRITE/DRAW

a



b A true bearing of 252°T is more than 180° and less than 270°, so the direction lies between south and west. Find the difference between the bearing and west (or south). Draw the 4 main standard bearings and add the bearing line. Add the angle from west (or south).	**b** Difference from west $= 270^\circ - 252^\circ$ $= 18^\circ$ 

WORKED EXAMPLE 9

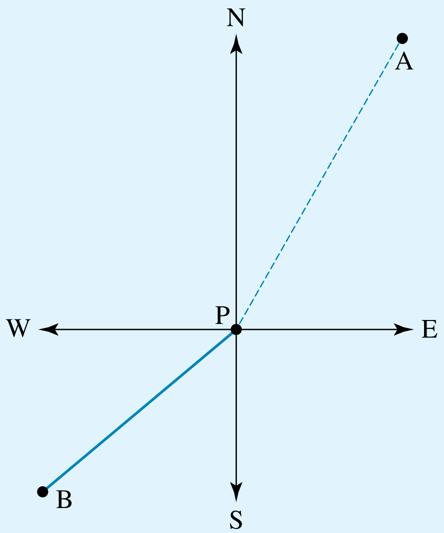
Convert:

a **the true bearing, 137°T, to a compass bearing**
b **the compass bearing, N25°W, to a true bearing.**

THINK	WRITE
a 1 The true bearing 137°T means the direction is between south and east. Find the angle from south to the bearing line.	**a** Angle required $= 180^\circ - 137^\circ$ $= 43^\circ$
2 Write the compass bearing.	Compass bearing is S43°E.
b 1 State the angle between the bearing line and north.	**b** Angle from north $= 25^\circ$
2 Find the angle from north to the bearing line in a clockwise direction.	Angle required $= 360^\circ - 25^\circ$ $= 335^\circ$
3 Write the true bearing.	True bearing is 335°T.

WORKED EXAMPLE 10

Use your protractor to find the bearing of points A and B from location P. State the directions as true bearings and as compass bearings.



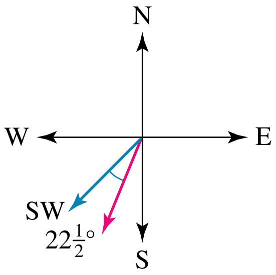
THINK	WRITE
1 Find ∠NPA and write as a true bearing and as a compass bearing.	$\angle NPA = 30^\circ$ True bearing of A from P is 030°T. Compass bearing is N30°E.
2 Repeat for location B, this time with reference to south.	$\angle SPB = 50^\circ$ True bearing of B from P is $180^\circ + 50^\circ = 230^\circ$T. Compass bearing is S50°W.

EXERCISE 13.4 Bearings

PRACTISE

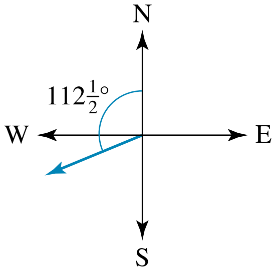
1 **WE7** Specify the direction of the pink arrow as:

- **a** a standard compass bearing
- **b** a compass bearing
- **c** a true bearing.



2 Specify the direction of the figure shown as:

- **a** a standard compass bearing
- **b** a compass bearing
- **c** a true bearing.



3 **WE8** Draw a suitable diagram to represent 080°T direction.

4 Draw a suitable diagram to represent N70°W direction.

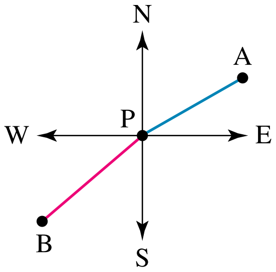
5 **WE9** Convert:

- **a** the true bearing, 152°T, to a compass bearing
- **b** the compass bearing, N37°W, to a true bearing.

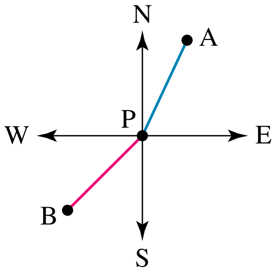
6 Convert:

- **a** the true bearing, 239°T, to a compass bearing
- **b** the compass bearing, S69°E, to a true bearing.

7 **WE10** Use your protractor to find the bearing of points A and B from location P. State the directions as true bearings and as compass bearings.



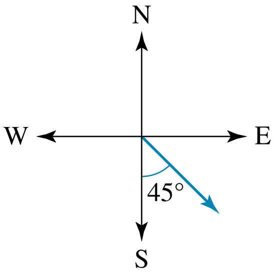
8 Use your protractor to find the bearing of points A and B from location P. State the directions as true bearings and as compass bearings.



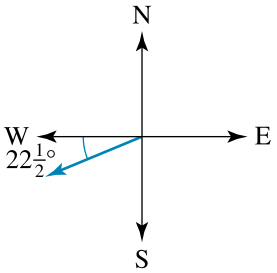
CONSOLIDATE

9 Specify the following directions as standard compass bearings.

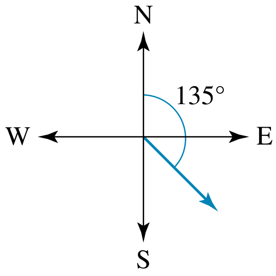
a



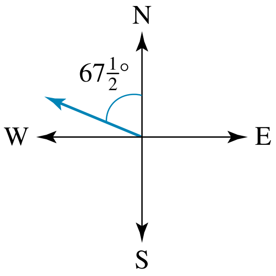
b



c

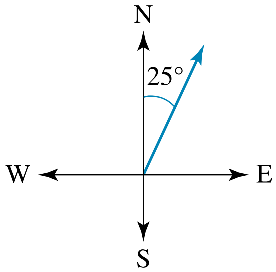


d

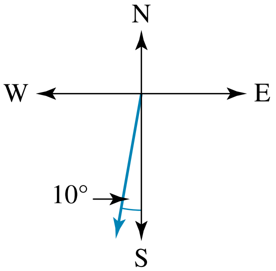


10 Specify the following directions as true bearings.

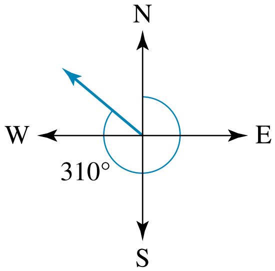
a



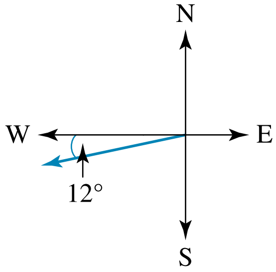
b



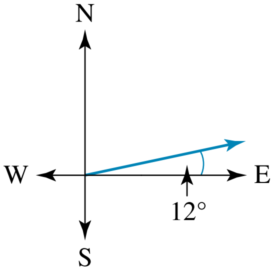
c



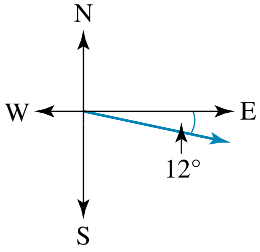
d



e



f



11 Draw suitable diagrams to represent the following directions.

a 045°T **b** 200°T **c** 028°T **d** 106°T **e** 270°T **f** 120°T

12 Convert the following true bearings to compass bearings.

a 040°T **b** 022$\frac{1}{2}$°T **c** 180°T **d** 350°T

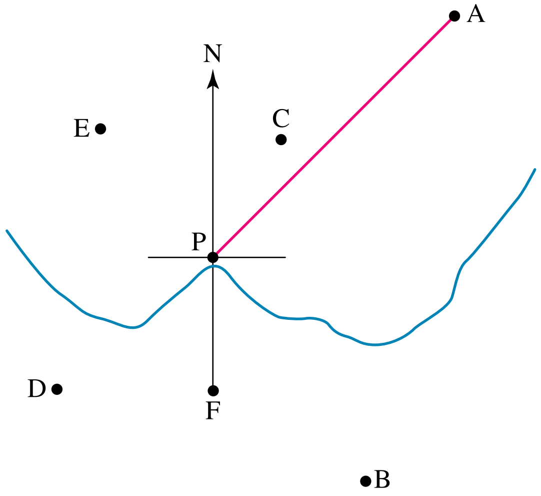
e 147°T **f** 67.5°T **g** 120°T **h** 135°T

13 Convert the following compass bearings to true bearings.

a N45°W **b** S40$\frac{1}{2}$°W **c** S **d** S35°E

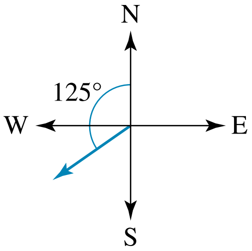
e N47°E **f** S67.5°W **g** NNW **h** S5°E

14 Use your protractor to find the bearing of each of the points from location P. State the directions as true bearings.



15 Now find the bearing of each of the points in the diagram from question **14** from location B (as true bearings). Also include the bearing from B to P and compare it to the direction from P to B.

16 The direction shown in the diagram is:



A 225°T **B** 305°T **C** 145°T
D 235°T **E** 125°T

17 An unknown direction — given that a second direction, 335°T, makes a straight angle with it — is:

A 165°T **B** 25°T **C** 155°T **D** 235°T **E** 135°T

18 The direction of a boat trip from Sydney directly to Auckland was 160°T. The direction of the return trip would be:

A 200°T **B** 250°T **C** 020°T **D** 235°T **E** 340°T

MASTER

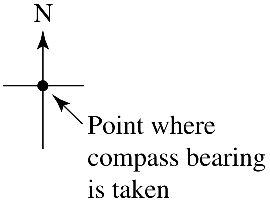
19 The direction of the first leg of a hiking trip was 220°T. For the second leg, the hikers turn 40° right. The new direction for the second leg of the hike is:

A 270°T **B** 180°T **C** 260°T **D** 040°T **E** 280°T

20 A hiker heads out on the direction 018°T. After walking 5 km they adjust their bearing 12° to their right and on the last leg of their hike they adjust their bearing 5° to their left. Calculate the bearing of their last leg.

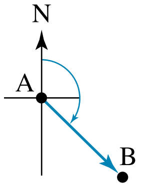
13.5 Navigation and specification of locations

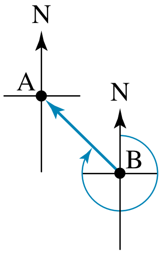
In most cases when you are asked to solve problems, a carefully drawn sketch of the situation will be given. When a problem is described in words only, very careful sketches of the situation are required. Furthermore, these sketches of the situation need to be converted to triangles with angles and lengths of sides included. This is so that Pythagoras' theorem, trigonometric ratios, areas of triangles, similarity and sine or cosine rules may be used.



Hints

1. Carefully follow given instructions.
2. Always draw the compass rose at the starting point of the direction requested.
3. Key words are *from* and *to*. For example:
   The bearing *from* A *to* B (see diagram left) is very different from the bearing *from* B *to* A (see diagram right).

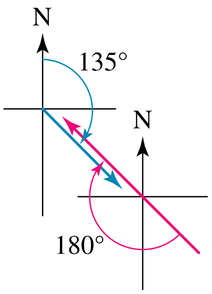


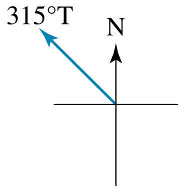


4. When you are asked to determine the direction to return directly back to an initial starting point, it is a 180° rotation or difference. For example, to return directly back after heading north, we need to change the direction to head south.
   Other examples are:

   Returning directly back after heading 135°T

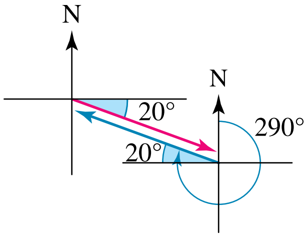
$$\begin{aligned}\text{New bearing} &= 135^\circ + 180^\circ \\ &= 315^\circ\text{T}\end{aligned}$$

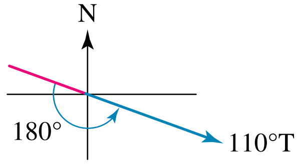




   Returning directly back after heading 290°T

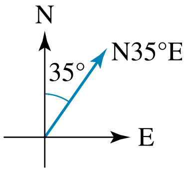
$$\begin{aligned}\text{New bearing} &= 290^\circ - 180^\circ \\ &= 110^\circ\text{T}\end{aligned}$$

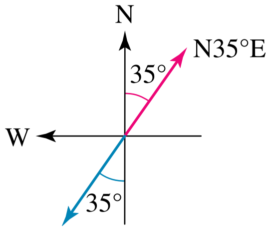




   Returning directly back after heading N35°E

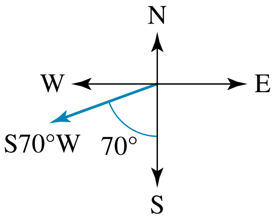
$$\begin{aligned}\text{New bearing} &= \text{N}35^\circ\text{E} + 180^\circ \\ &= \text{S}35^\circ\text{W}\end{aligned}$$

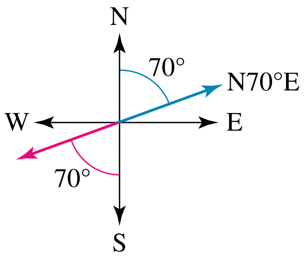




   Returning directly back after heading S70°W

$$\text{New bearing} = \text{N}70^\circ\text{E}$$





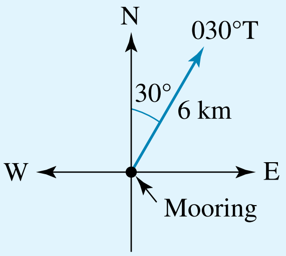
   or simply use the opposite compass direction. North becomes south and east becomes west and vice versa.

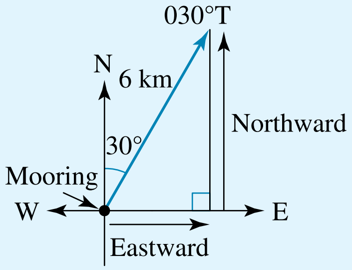
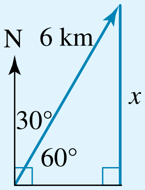
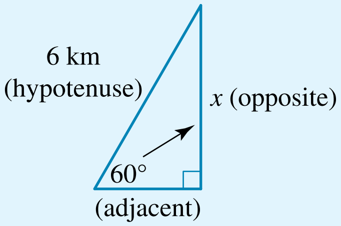
WORKED EXAMPLE 11

A ship leaves port, heading 030°T for 6 kilometres as shown.

a How far north or south is the ship from its starting point (correct to 1 decimal place)?

b How far east or west is the ship from its starting point (correct to 1 decimal place)?



THINK	WRITE/DRAW
a 1 Draw a diagram of the journey and indicate or superimpose a suitable triangle.	**a** 
2 Identify the side of the triangle to be found. Redraw a simple triangle with the most important information provided. Use the bearing given to establish the angle in the triangle, that is, use the complementary angle law.	 $90° - 30° = 60°$
3 As the triangle is right-angled, the sine ratio can be used to find the distance north.	
4 Substitute and evaluate.	$\sin(\theta) = \frac{\text{length of opposite side}}{\text{length of hypotenuse side}}$ $= \frac{\text{opposite}}{\text{hypotenuse}}$ $\sin(60°) = \frac{y}{6}$ $x = 6 \times \sin(60°)$ $= 6 \times 0.8660$ $= 5.196$
5 State the answer to the required number of decimal places.	The ship is 5.2 km north of its starting point, correct to 1 decimal place.

b **1** Use the same approach as in part **a**. This time the trigonometric ratio is cosine to find the distance east, using the same angle.

b
$$\begin{aligned}\cos(\theta) &= \frac{\text{length of adjacent side}}{\text{length of hypotenuse side}}\\ &= \frac{\text{adjacent}}{\text{hypotenuse}}\\ \cos(60°) &= \frac{y}{6}\\ y &= 6 \times \cos(60°)\\ &= 6 \times 0.5\\ &= 3.0\end{aligned}$$

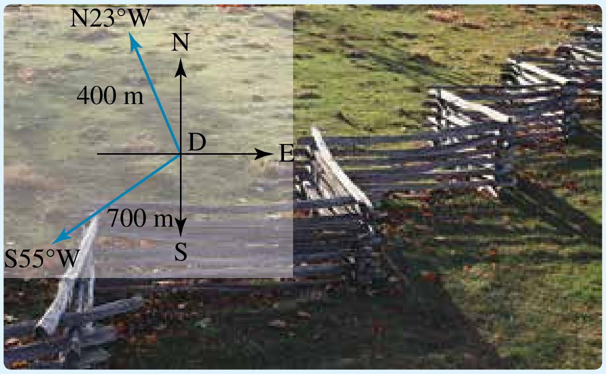
2 Answer in correct units and to the required level of accuracy.

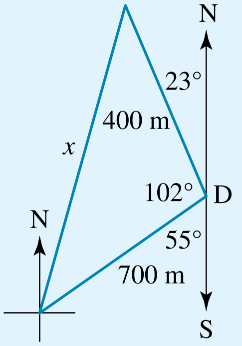
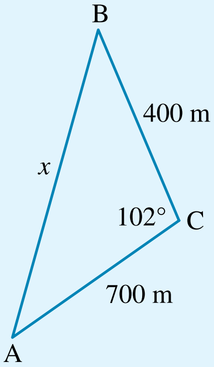
The ship is 3.0 km east of its starting point, correct to 1 decimal place.

WORKED EXAMPLE 12

A triangular paddock has two complete fences. From location D, one fence line is on a bearing of 337°T for 400 metres. The other fence line is 235°T for 700 metres.

Find the length of fencing (correct to the nearest metre) required to complete the enclosure of the triangular paddock.



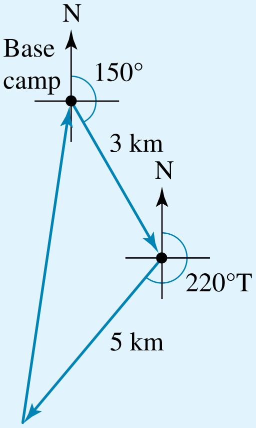
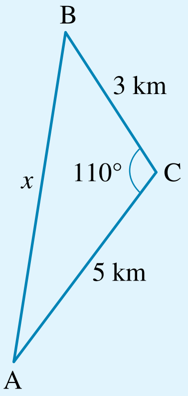
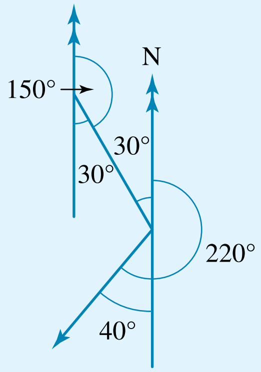
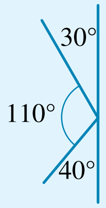
THINK	**WRITE/DRAW**
1 Redraw a simple triangle with the most important information provided. Identify the side of the triangle to be found.	
2 Use the bearings given to establish the angle in the triangle; that is, use the supplementary angle law.	In the triangle, $\angle D = 180° - 23° - 55°$ $= 102°$
3 The cosine rule can be used, as two sides and the included angle are given. State the values of the pronumerals and write the formula for finding the unknown side length using cosine rule.	 $a = 400$ m, $b = 700$ m, $C = 102°$, $c = x$ m

4 Substitute the values of the pronumerals into the formula and evaluate.	$c^2 = a^2 + b^2 - 2ab \times \cos(C)$ $x^2 = 400^2 + 700^2 - 2 \times 400 \times 700 \times \cos(102°)$ $= 650000 - 560000 \times\ - 0.20791$ $= 766430.55$ $x = \sqrt{766430.55}$ $= 875.46$
5 Answer in correct units and to the required level of accuracy.	The new fence section is to be 875 metres long, correct to the nearest metre.

WORKED EXAMPLE 13

Soldiers on a reconnaissance set off on a return journey from their base camp. The journey consists of three legs. The first leg is on a bearing of 150°T for 3 km; the second is on a bearing of 220°T for 5 km.

Find the direction (correct to the nearest degree) and distance (correct to 2 decimal places) of the third leg by which the group returns to its base camp.

THINK	WRITE/DRAW
1 Draw a diagram of the journey and indicate or superimpose a suitable triangle.	
2 Identify the side of the triangle to be found. Redraw a simple triangle with the most important information provided. (∠BCA = 180° − 30° − 40°. ∴ ∠BCA = 110°)	  

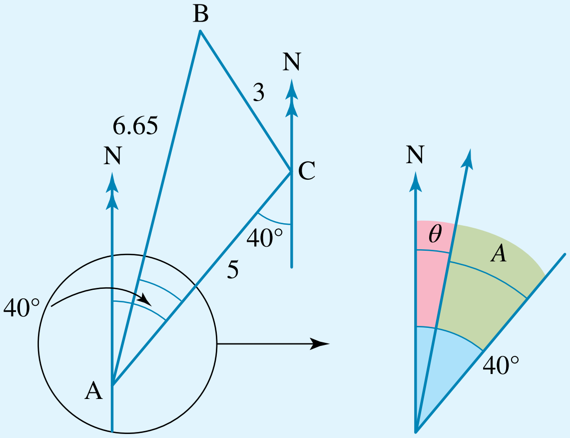
3 The cosine rule can be used to find the length of the side AB, as we are given two sides and the angle in between.

$a = 3$ km, $b = 5$ km, $C = 110°$, $c = x$ km

4 Substitute the known values into the cosine rule and evaluate.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \times \cos(C) \\ x^2 &= 3^2 + 5^2 - 2 \times 3 \times 5 \times \cos(110°) \\ &= 44.260\,604 \\ x &= \sqrt{44.260\,604} \\ &= 6.65 \end{aligned}$$

5 For direction, we need to find the angle between the direction of the second and third legs first, that is, ∠BAC. Once the size of ∠BAC (or simply, ∠A) is established, it can be subtracted from 40° to find angle θ. This will give the bearing for the third leg of the journey. Since all 3 side lengths in ΔABC are now known, use the cosine rule to find ∠A.



$a = 3$, $b = 5$, $c = 6.65$ or $\sqrt{44.260604}$

6 Substitute the known values into the rearranged cosine rule.

Note: Use the most accurate form of the length of side c.

$$\begin{aligned} \cos(A) &= \frac{b^2 + c^2 - a^2}{2 \times b \times c} \\ &= \frac{5^2 + 44.260604 - 3^2}{2 \times 5 \times \sqrt{44.260604}} \\ &= 0.9058 \\ A &= 25.07° \\ &= 25°\text{(correct to the nearest degree)} \end{aligned}$$

7 Calculate the angle of the turn from the north bearing.

$$\begin{aligned} \theta &= 40° - 25° \\ &= 15° \end{aligned}$$

Bearing is 015°T.

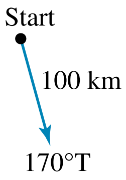
8 Write the answer in correct units and to the required level of accuracy.

The distance covered in the final leg is 6.65 km, correct to 2 decimal places, on a bearing of 015°T, correct to the nearest degree.

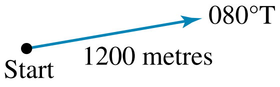
EXERCISE 13.5 Navigation and specification of locations

PRACTISE

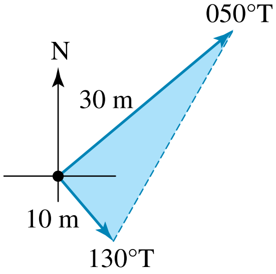
1 **WE11** For the diagram at right, find how far north or south and east or west the end point is from the starting point (correct to 1 decimal place).



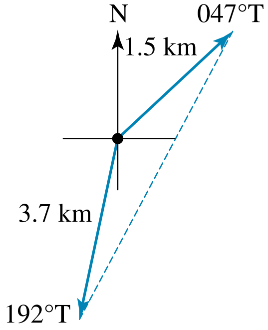
2 For the diagram at right, find how far north or south and east or west the end point is from the starting point (correct to 1 decimal place).



3 **WE12** Find the length of the unknown side for the triangle (correct to the nearest unit).



4 Find the length of the unknown side for the triangle (correct to the nearest unit).

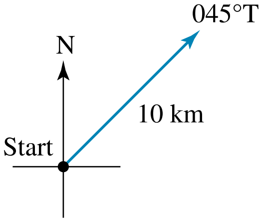


5 **WE13** A scout troop travels on a hike of 2 km with a bearing of 048°T. They then turn and travel on a bearing of 125°T for 5 km, then camp for lunch. How far is the camp from the starting point? Give the answer correct to 2 decimal places.

6 Find the bearing required in question 5 to get the scout troop back to their starting point. Give your answer correct to the nearest degree.

CONSOLIDATE

7 For the following, find how far north or south and east or west the end point is from the starting point (correct to 1 decimal place).

a


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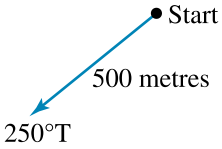
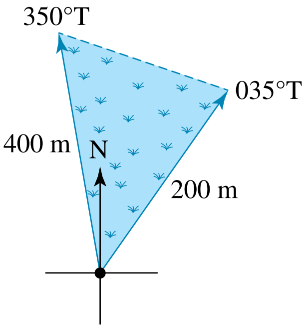


8 Find the length of the unknown side for each of the triangles (correct to the nearest unit).

a


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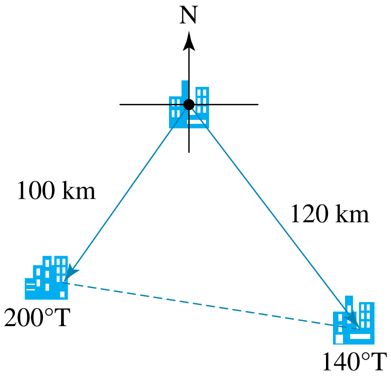
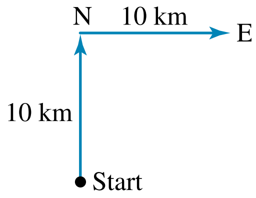
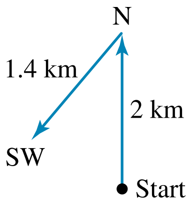


9 In each of the following diagrams, the first two legs of a journey are shown. Find the direction and distance of the third leg of the journey which returns to the start.

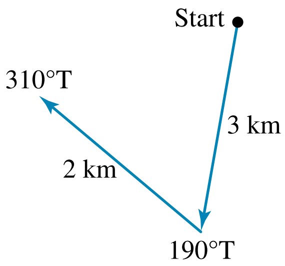
a



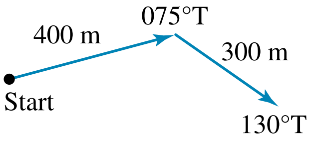
b



c



d



10 Draw a diagram to represent each of the directions specified below and give the direction required *to return* to the starting point:

- **a** from A to B on a bearing of 320°T
- **b** from C to E on a bearing of 157°T
- **c** to F from G on a bearing of 215°T
- **d** from B to A on a bearing of 237°T.

11 A boat sails from port A for 15 km on a bearing of N15°E before turning and sailing for 21 km in a direction of S75°E to port B.

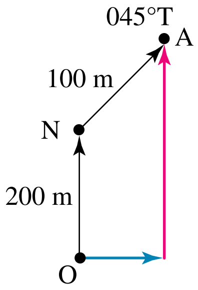
- **a** The distance between ports A and B is closest to:
  A 15 km **B** 18 km **C** 21 km **D** 26 km **E** 36 km
- **b** The bearing of port B from port A is closest to:
  A N69.5°E **B** N54.5°E **C** N20.5°E **D** S20.5°E **E** 54.5°T

12 In a pigeon race, the birds start from the same place. In one race, pigeon A flew 35 km on a bearing of 295°T to get home, while pigeon B flew 26 km on a bearing of 174°T.

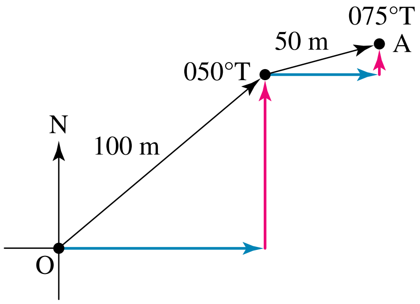
- **a** The distance between the two pigeons' homes is closest to:
  A 13 km **B** 18 km **C** 44 km **D** 50 km **E** 53 km
- **b** The bearing of pigeon A's home from pigeon B's home is closest to:
  A 332°T **B** 326°T **C** 320°T **D** 208°T **E** 220°T

13 For each of the following, find how far north/south and east/west position A is from position O.

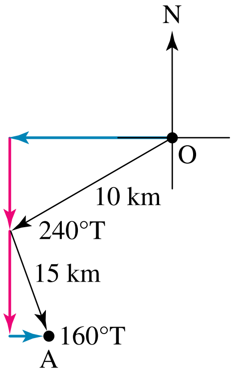
a



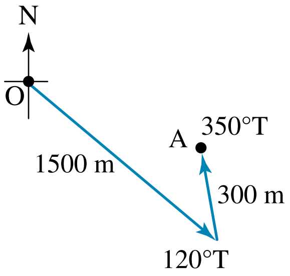
b



c

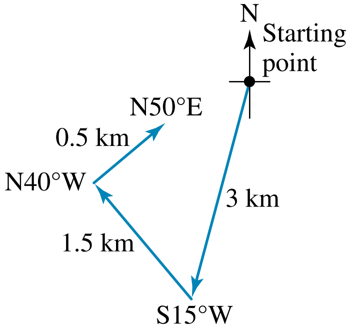


d



14 For the hiking trip shown in the diagram, find (correct to the nearest metre):

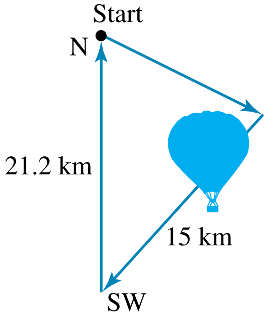
a how far south the hiker is from the starting point
b how far west the hiker is from the starting point
c the distance from the starting point
d the direction of the final leg to return to the starting point.



15 Captain Cook sailed from Cook Island on a bearing of 010°T for 100 km. He then changed direction and sailed for a further 50 km on a bearing of SE to reach a deserted island.

a How far from Cook Island is Captain Cook's ship (correct to the nearest kilometre)?
b Which direction would have been the most direct route from Cook Island to the deserted island (correct to 2 decimal places)?
c How much shorter would the trip have been using the direct route?

16 A journey by a hot-air balloon is as shown. The balloonist did not initially record the first leg of the journey. Find the direction and distance for the first leg of the balloonist's journey.



MASTER

17 A golfer is teeing off on the 1st hole. The distance and direction to the green is 450 metres on a bearing of 190°T. If the tee shot of the player was 210 metres on a bearing of 220°T, how far away from the green is the ball and in what direction should she aim to land the ball on the green with her second shot? (Give the distance correct to the nearest metre and the direction correct to the nearest degree.)

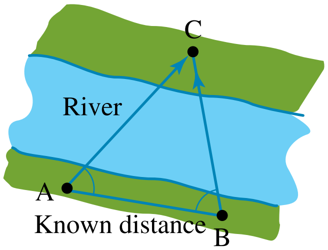
18 A boat begins a journey on a bearing of 063.28°T and travels for 20 km.

- **a** How far east of its starting point is it?
  It then changes to a bearing of 172.43°T and travels for a further 35 km.
- **b** Through what angle did the boat turn?
- **c** How far is it now from its starting point?
- **d** What is the bearing of its end point from the starting point?

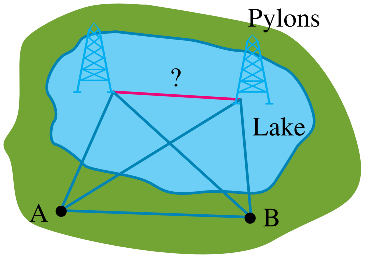
13.6 Triangulation — cosine and sine rules

In many situations, certain geographical or topographical features are not accessible to a survey. To find important locations or features, triangulation is used. This technique requires the coordination of bearings from two known locations, for example, fire spotting towers, to a third inaccessible location, the fire (see Worked example 14).

1. Triangulation should be used when:
   - a. the distance between two locations is given and
   - b. the direction from each of these two locations to the third inaccessible location is known.

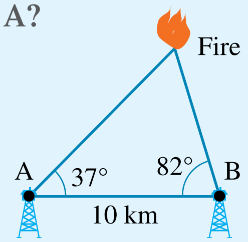


2. For triangulation:
   - a. the sine rule is used to find distances from the known locations to the inaccessible one
   - b. the cosine rule may be used occasionally for locating a fourth inaccessible location.



WORKED EXAMPLE 14

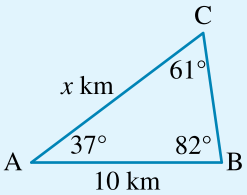
How far (correct to 1 decimal place) is the fire from Tower A?



THINK

1 Draw a triangle and identify it as a non-right-angled triangle with a given length and two known angles. Determine the value of the third angle and label appropriately for the sine rule.

WRITE/DRAW



$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

where

$c = 10\text{km}$ $\quad C = 180° - (37° + 82°)$

$\quad = 61°$

$b = x$ $\quad B = 82°$

2 Substitute into the formula and evaluate.

$$\frac{x}{\sin(82)^\circ} = \frac{10}{\sin(61)^\circ}$$

$$x = \frac{10 \times \sin(82)^\circ}{\sin(61)^\circ}$$

$$= 11.322$$

3 Write the answer using correct units and to the required level of accuracy.

The fire is 11.3 km from Tower A, correct to 1 decimal place.

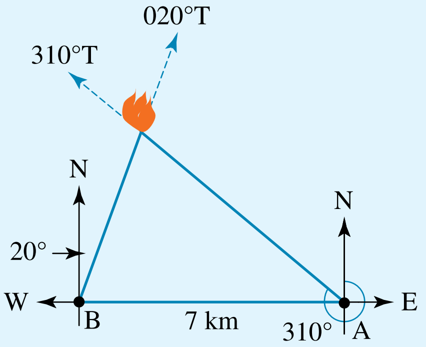
WORKED EXAMPLE 15

Two fire-spotting towers are 7 kilometres apart on an east–west line. From Tower A a fire is seen on a bearing of 310°T. From Tower B the same fire is spotted on a bearing of 020°T. Which tower is closest to the fire and how far is that tower from the fire (correct to 1 decimal place)?

THINK

WRITE/DRAW

1 Draw a suitable sketch of the situation described. It is necessary to determine whether Tower A is east or west of Tower B.



2 Identify the known values of the triangle and label appropriately for the sine rule.

The *shortest side* of a triangle is *opposite* the *smallest angle*. Therefore, Tower B is closest to the fire. Use the sine rule to find the distance of Tower B from the fire.

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

where

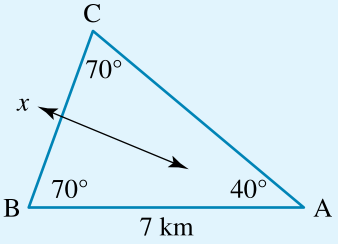
$a = x$

$c = 7$ km

$A = 40°$

$C = 180° - (70° + 40°)$

$= 70°$



3 Substitute into the formula and evaluate.

Note: ΔABC is an isosceles triangle, so Tower A is 7 km from the fire.

$$\frac{x}{\sin(40)^\circ} = \frac{7}{\sin(70)^\circ}$$

$$x = \frac{7 \times \sin(40)^\circ}{\sin(70)^\circ}$$

$$= 4.788\,282 \text{ km}$$

4 Write the answer in the correct units.

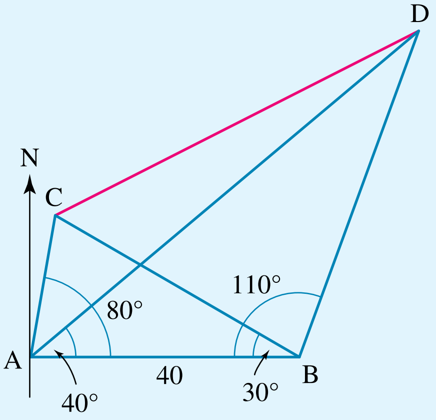
Tower B is closest to the fire at a distance of 4.8 km, correct to 1 decimal place.

WORKED EXAMPLE 16

From the diagram find:

a the length of CD (correct to 1 decimal place)

b the bearing from C to D (correct to 1 decimal place).

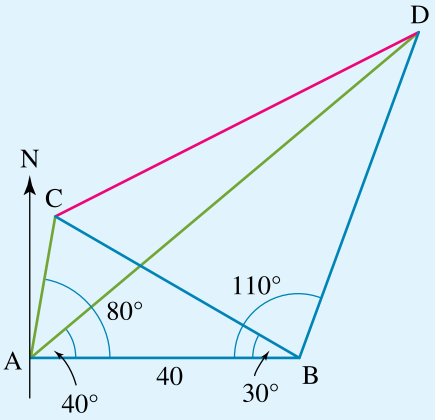


THINK

WRITE/DRAW

a 1 To evaluate the length of CD, we need to first determine the lengths of AC and AD. (Alternatively, we can find the lengths of BC and BD.)

a



2 Label ΔABC for the sine rule and evaluate the length of AC.
(∠C = 180° − 30° − 80°
= 70°)

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

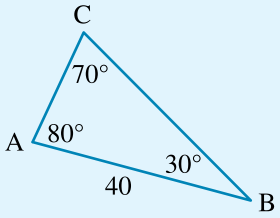
$b = \overline{AC},\ B = 30°$

$c = 40,\quad C = 70°$

$$\frac{\overline{AC}}{\sin(30)°} = \frac{40}{\sin(70)°}$$

$$\overline{AC} = \frac{40 \times \sin(30)°}{\sin(70)°}$$

$$= 21.283\,555$$



3 Label ΔABD for the sine rule and evaluate the length of AD.
(∠D = 180° − 40° − 110°
= 30°)

$$\frac{b}{\sin(B)} = \frac{d}{\sin(D)}$$

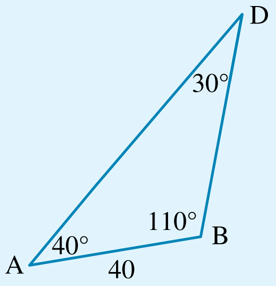
$b = \overline{AD},\ B = 110°$

$d = 40,\quad D = 30°$

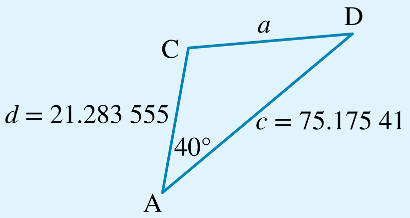
$$\frac{\overline{AD}}{\sin(110)°} = \frac{40}{\sin(30)°}$$

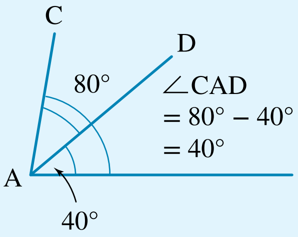
$$\overline{AD} = \frac{40 \times \sin(110)°}{\sin(30)°}$$

$$= 75.175\,41$$



4 Draw ΔACD, which is needed to find the length of CD. Use the two given angles to find ∠CAD. Now label it appropriately for the cosine rule.





$a = \overline{\text{CD}}$
$d = 21.283\,55$
$A = 40°$
$c = 75.175\,41$

5 Substitute into the formula and evaluate.

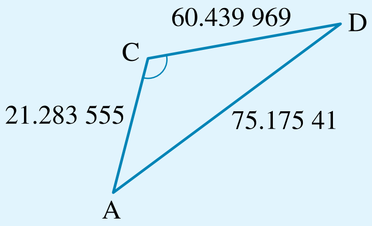
$$a^2 = c^2 + d^2 - 2cd \times \cos(A)$$
$$\overline{\text{CD}}^2 = 75.175\,41^2 + 21.283\,555^2 - 2 \times 75.175\,41 \times 21.283\,555 \times \cos(40°)$$
$$\overline{\text{CD}} = \sqrt{3652.9898}$$
$$= 60.439\,969$$

6 Write the answer in the correct units.

The length of $\overline{\text{CD}}$ is 60.4 units, correct to 1 decimal place.

b **1** Redraw ΔACD and label it with the known information.

b



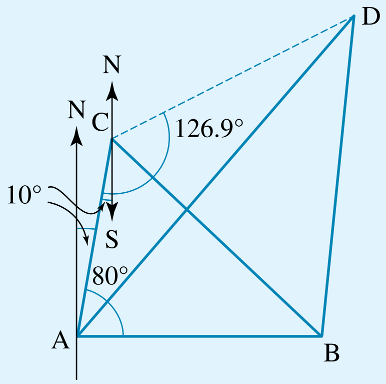
2 Bearing required is taken *from* C, so find ∠ACD by using the cosine rule first.

$a = 60.439\,969$
$c = 75.175\,41$
$d = 21.283\,555$

3 Substitute into the rearranged cosine rule and evaluate *C*.

$$\cos(C) = \frac{a^2 + d^2 - c^2}{2 \times a \times d}$$
$$= \frac{(60.439\,969)^2 + (21.283\,555)^2 - (75.175\,41)^2}{2 \times 60.439\,969 \times 21.283\,555}$$
$$= -0.6007$$
$$C = 126.9°$$

4 Redraw the initial diagram (from the question) with known angles at point C in order to find the actual bearing angle.



5 Determine the angle from south to the line CD.

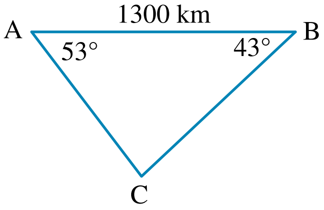
$$\angle SCD = 126.9° - 10°$$
$$= 116.9°$$

6 Determine the bearing angle.	$\angle NCD = 180° - 116.9°$ $= 63.1°$
7 Write the bearing of D from C.	The bearing of D from C is N63.1°E, correct to 1 decimal place.

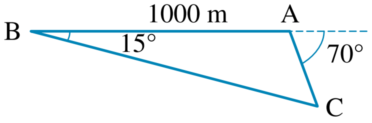
EXERCISE 13.6 Triangulation — cosine and sine rules

PRACTISE

1 WE14 Find the distance from A to C (correct to 1 decimal place).

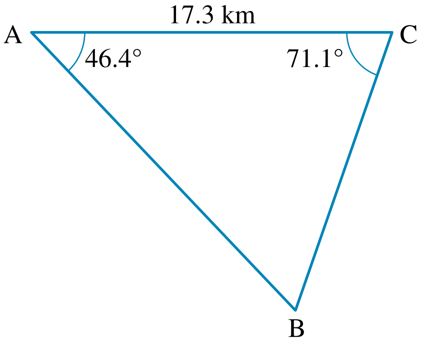


2 Find the distance from A to C (correct to 1 decimal place).



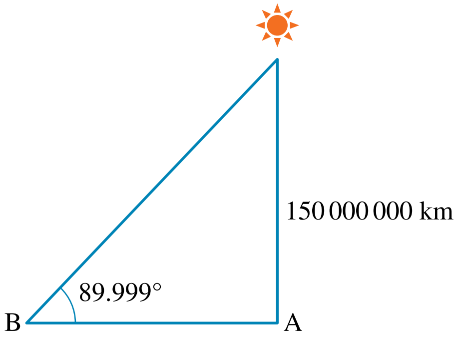
3 WE15 A boat (B) is spotted by 2 lighthouses (A and C), which are 17.3 km apart. The angles measured by each lighthouse are shown in the figure at right.

Determine the distance from each lighthouse to the boat.

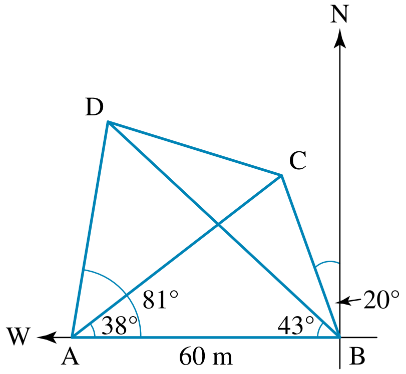


4 Two ancient astronomers lived in different towns (A and B) and wished to know the distance between the towns. At the same time they measured the angle of the sun, as shown in the diagram at right. Assuming the distance of the astronomer from town A to the sun was 150 000 000 km, find the distance between the towns (i.e. the distance AB).

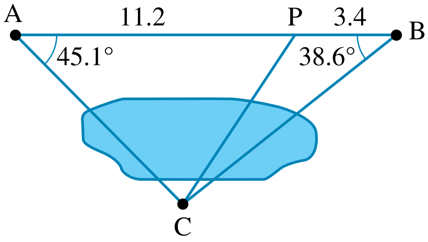
(Note that due to the nature of the angles, this is not an accurate way to measure this distance.)



5 WE16 Given the information in the diagram below, find the length of CD.



6 Three towns, A, B and C, are situated as shown in the diagram below.



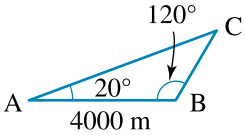
A mountain range separates town C from the other two, but a road is to be built, starting at P and tunnelling through the mountain in a straight line to C. From geographic information, the angles (45.1°, 38.6°) and distances (11.2 km, 3.4 km) between points are determined as shown in the figure.

Find the angle the road PC makes with the line AB (∠APC).

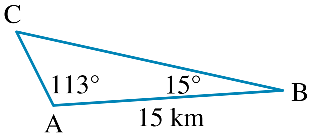
CONSOLIDATE

7 Find the distance from A to C in each case (correct to 1 decimal place).

a



b



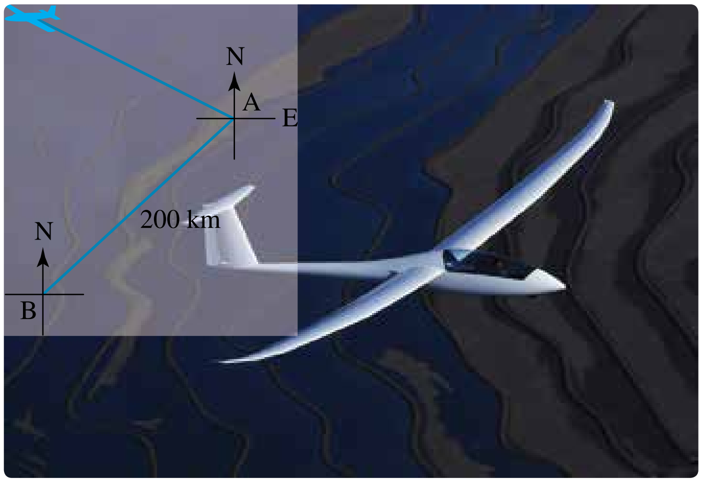
8 **a** Two fire-spotting towers are 17 kilometres apart on an east–west line. From Tower A, a fire is seen on a bearing of 130°T. From Tower B, the same fire is spotted on a bearing of 200°T. Which tower is closest to the fire and how far is that tower from the fire?

b Two fire-spotting towers are 25 kilometres apart on a north–south line. From Tower A, a fire is reported on a bearing of 082°T. Spotters in Tower B see the same fire on a bearing of 165°T. Which tower is closest to the fire and how far is that tower from the fire?

9 Two lighthouses are 20 km apart on a north–south line. The northern lighthouse spots a ship on a bearing of 100°T. The southern lighthouse spots the same ship on a bearing of 040°T.

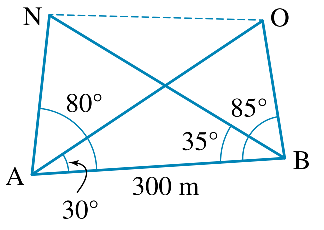
a Find the distance from the northern lighthouse to the ship.
b Find the distance from the southern lighthouse to the ship.

10 Two air traffic control towers detect a glider that has strayed into a major air corridor. Tower A has the glider on a bearing of 315°T. Tower B has the glider on a bearing of north. The two towers are 200 kilometres apart on a NE line as shown. To which tower is the glider closer? What is the distance?

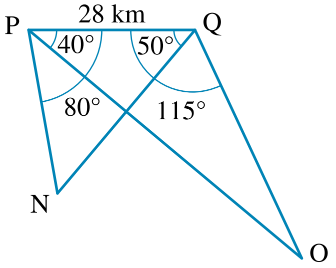


11 Find the value of line segment NO in each case (correct to 1 decimal place).

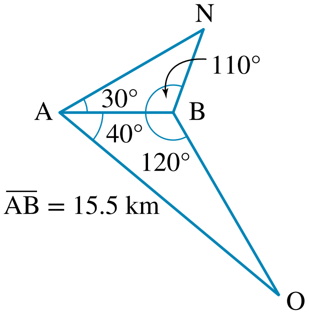
a



b

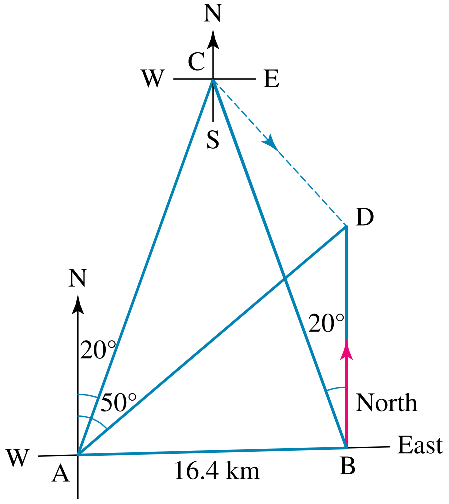


c

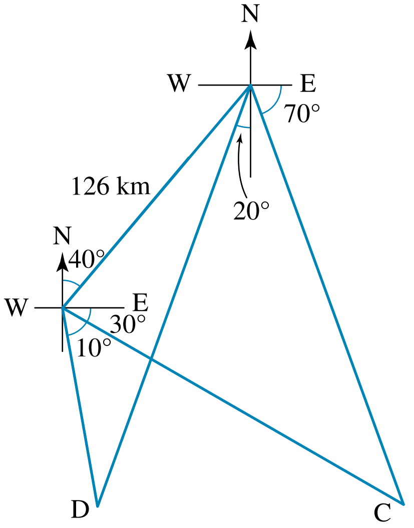


12 Find the distance and bearing from C to D (both correct to 1 decimal place).

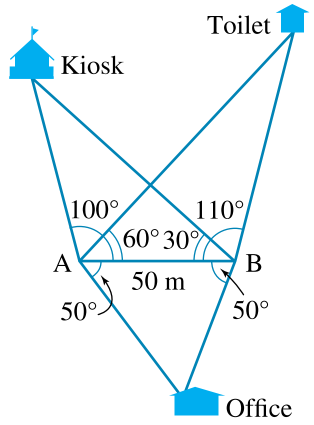
a



b



13 A student surveys her school grounds and makes the necessary measurements to 3 key locations as shown in the diagram.



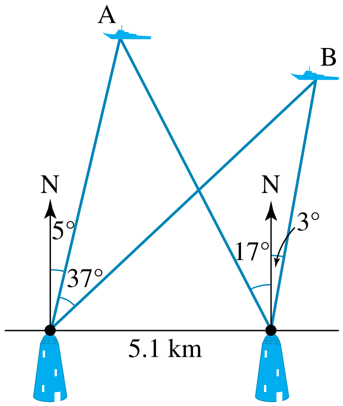
a Find the distance to the kiosk from:
- **i** location A
- **ii** location B.

b Find the distance to the toilets from:
- **i** location A
- **ii** location B.

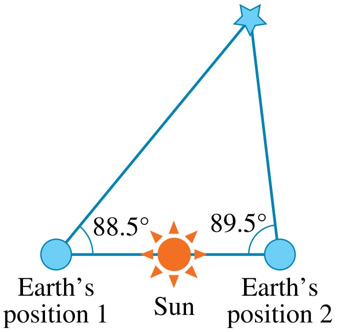
c Find the distance from the toilet to the kiosk.

d Find the distance from the office to location A.

14 From the diagram below, find the distance between the two ships and the bearing from ship A to ship B.



15 An astronomer uses direction measurements to a distant star taken 6 months apart, as seen in the diagram below (which is not drawn to scale). The known diameter of the Earth's orbit around the Sun is 300 million kilometres. Find the closest distance from Earth to the star (correct to the nearest million kilometres).



16 Two girls walk 100 metres from a landmark. One girl heads off on a bearing of 136°T, while the other is on a bearing of 032°T. After their walk, the distance between the two girls, correct to the nearest metre, is closest to:

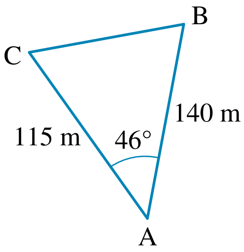
A 123 m **B** 158 m **C** 126 m **D** 185 m **E** 200 m

MASTER

17 Two ships leave the same port and sail the same distance, one ship on a bearing of NW and the other on a bearing of SSE. If they are 200 kilometres apart, what was the distance sailed by each ship?

A 100 km **B** 101 km **C** 102 km **D** 202 km **E** 204 km

18 In the swimming component of a triathlon, competitors have to swim around the buoys A, B and C, as marked in the figure, to end at point A.



- **a** What is the shortest leg of the swimming component?
- **b** What is the angle at buoy B?
- **c** What is the total distance travelled?

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13.7 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

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- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
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- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

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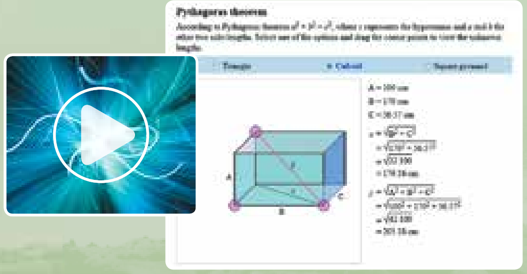
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.



studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

13 ANSWERS

EXERCISE 13.2

1 a 70.41° **b** 53.13°

2 a 70.07° **b** 44.96°

3 $x = 127^\circ$

4 $x = 33^\circ$

5 $A = 49^\circ$, $C = 131^\circ$

6 $P = 110^\circ$, $Q = 70^\circ$

7 A

8 a 134.44° **b** 17.82°

9 a 0.646 **b** −0.535

c 7.207 **d** 0.957

e 0.537 **f** 0.520

10 a 68.33°, 12.17° **b** 201.56°, 42.86°

c 182.89°, 18.77° **d** 388.1°, 106.94°

e 523.21°, 169.53° **f** 218.97°, 205.69°

11 a 21.67° **b** 40.5°

12 a 32° **b** 143°

13 a $a = b = 40.36^\circ$ **b** 57.71°

14 a 162.5° **b** 67.35°

15 a 57.68° **b** 32°

16 a $a = c = 43.45^\circ$, $b = 136.55^\circ$ **b** $a = 39.61^\circ$, $b = 17.8^\circ$

17 a B **b** A

18 $a = 48.915^\circ$, $b = 96.6^\circ$

EXERCISE 13.3

1 26.57°

2 45°

3 40.33 m

4 4.01 m

5 $b = 33.27^\circ$, $a = 1371.68$ m

6 59.21 m

7 a 36.87° **b** 40.34° **c** 21.80°

8 a 10 m **b** 85 m

9 26 metres

10 a $a = 41.83^\circ$ $d = 61.45$ m

b $a = 46.45^\circ$ $b = 15.21$ m

11 a Elevation 30° **b** Depression 80.54°

c Elevation 45° **d** Depression 14.04°

12 41.19°

13 D

14 B

15 a 68 m **b** 29 m

16 a 38.66° **b** 65.56° **c** 38.66°

17 a 4240 m **b** 1100 m

18 a 14.9 m **b** 9.2 m

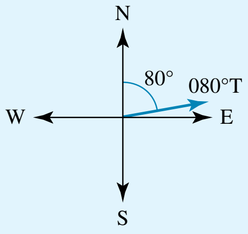
c 14.9 m **d** A right-angled isosceles triangle

EXERCISE 13.4

1 a SSW **b** S $22\frac{1}{2}^\circ$ W **c** $202\frac{1}{2}$T°

2 a WSW **b** S $67\frac{1}{2}^\circ$ W **c** $247\frac{1}{2}$T°

3



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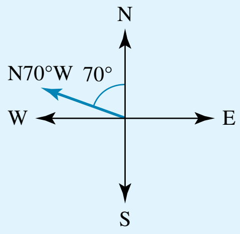


5 a S28°E **b** 323°T

6 a S59°W **b** 111°T

7 Point A: 060°T, N60°E Point B: 230°T, S50°W

8 Point A: 025°T, N25°E Point B: 225°T, S45°W

9 a SE **b** WSW

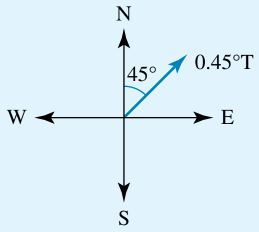
c SE **d** WNW

10 a 025°T **b** 190°T

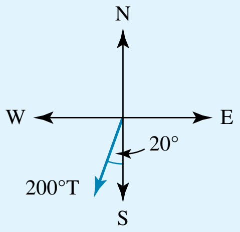
c 310°T **d** 258°T

e 078°T **f** 102°T

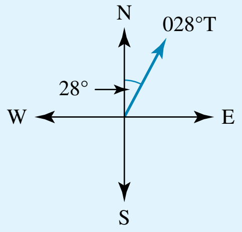
11 a



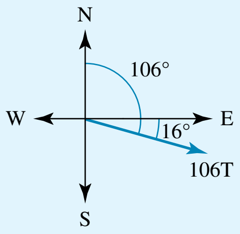
b



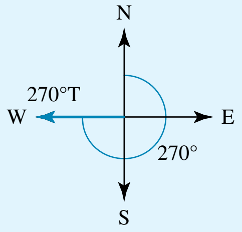
c



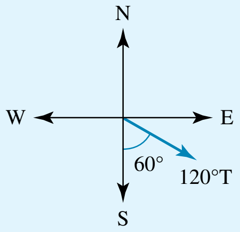
d



e



f



12 a N40°E b N22.5°E
c S d N10°W
e S33°E f N67.5°E
g S60°E h SE

13 a 315°T b 220.5°T
c 180°T d 145°T
e 047°T f 247.5°T
g 337$\frac{1}{2}$°T h 175°T

14 A 045°T B 150°T
C 030°T D 230°T
E 318°T F 180°T

15 A 010°T C 347°T D 293°T
E 326°T F 310°T
From B to P: 330°T; From P to B: 150°T

16 D

17 C

18 E

19 C

20 025°T

EXERCISE 13.5

1 98.5 km south, 17.4 km east

2 208.4 m north, 1181.8 m east

3 30 m

4 5 km

5 5.79 km

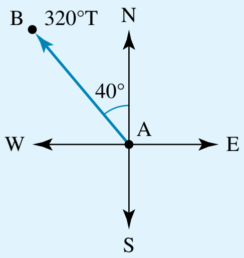
6 286°T

7 a 7.1 km north, 7.1 km east
b 171.0 m south, 469.8 m west

8 a 295 m b 111 km

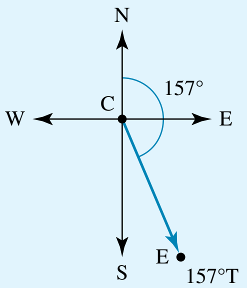
9 a 14.1 km SW or 225°T b 1.4 km SE or 135°T
c 2.65 km, 050.9°T d 622.6 m, 278.25°T

10 a



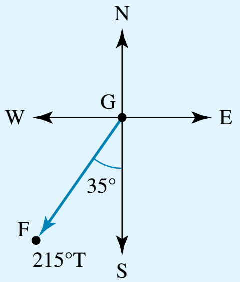
Return = 140°T

b



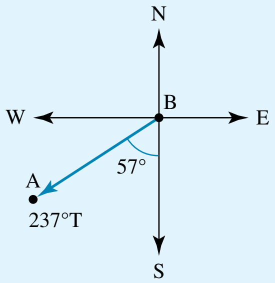
Return = 337°T

c



Return = 035°T

d



Return = 057°T

11 a D b A

12 a E b C

13 a 271 m north, 71 m east
b 77.2 m north, 124.9 m east
c 19.1 km south, 3.5 km west
d 454.6 m south, 1246.9 m east

14 a 1427 m b 1358 m
c 1970 m d N43°34′E

15 a 82 km b 039.87°T c 68 km

16 15 km, SE

17 288 m, 169°T

18 a 17.86 km b 109.15°
c 34.14 km d 119.02°T

EXERCISE 13.6

1 891.5 km

2 316.0 m

3 CB = 14.12 km, AB = 18.45 km

4 2618 km

5 40.87 m

6 53.9°

7 a 5389.2 m b 4.9 km

8 a Tower B, 11.6 km b Tower A, 6.5 km

9 a 14.8 km b 22.7 km

10 Tower A (200 km)

11 a 253.6 m b 42.6 km c 38.0 km

12 a 12.0 km, 136.1°T b 801.8 km, 222.6°T

13 a i 33 m
ii 64 m
b i 271 m
ii 249 m
c 247 m
d 39 m

14 8.1 km, 148.53°T

15 8593 million km

16 B

17 C

18 a BC
b 54°
c 357.26 m

14

Construction and interpretation of graphs

14.1 Kick off with CAS

Solving simultaneous equations with CAS

Simultaneous equations are a set of two (or more) equations that contain the same variables. By solving simultaneous equations we are finding the solution(s) which satisfy all of the equations within the set.

To solve simultaneous equations graphically we can sketch graphs of the equations on the same set of axes. The intersection of the two graphs is the solution to the set of simultaneous equations.

1. Using CAS, sketch the graph of $y = 4x - 3$.
2. Using CAS, sketch the graph of $y = 6 - 3x$ on the same set of axes as the graph in question 1.
3. Use CAS to find the point of intersection of both of the graphs you have sketched.
4. Confirm your solution to question 3 by using CAS to solve the pair of simultaneous equations ($y = 4x - 3$ and $y = 6 - 3x$) algebraically on a calculator page.
5. Use CAS to sketch the graphs of $y = 2x + 5$ and $y = 2x - 1$ on the same set of axes.
6. Is there a point of intersection between the two graphs you plotted in question 5? What is the relationship between the two equations?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

14.2 Constructing and interpreting straight-line graphs

studyon

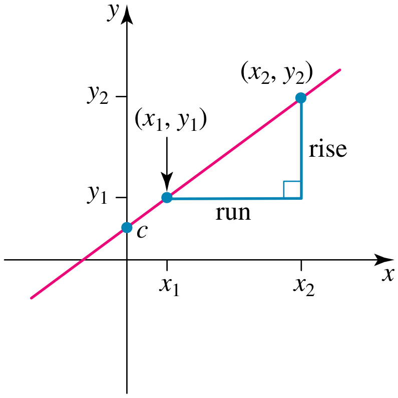
Unit 4
AOS M4
Topic 1
Concept 1

Straight-line graphs
Concept summary
Practice questions

Equations and graphs are used to study the relationship between two variables such as distance and speed, tax payable and income, or radioactivity and time.

If a relationship exists between the variables, one can be said to be a **function** of the other. A function can be described by a table, a rule or a graph. A function whose graph is a straight line is called a **linear function**.

The general equation of a straight line is $y = mx + c$, where m is the gradient and c is the y-intercept.



The **y-intercept** is the value of y where the graph cuts the y-axis.

The **gradient** (or slope) of a straight line is denoted by m where

$$m = \frac{\text{rise}}{\text{run}} \quad \text{or} \quad m = \frac{y_2 - y_1}{x_2 - x_1}.$$

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Equations of straight lines
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Interactivity
Linear graphs
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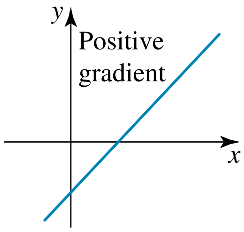
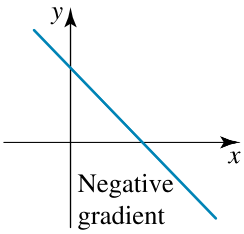


A graph that rises as x increases will have a positive gradient.



A graph that falls as x increases will have a negative gradient.

WORKED EXAMPLE 1

Consider the equation $y = 3x + 4$.

a What is the value of y when $x = 2$?

b What is the value of x when $y = 13$?

c Does the point (2, 4) lie on the graph of $y = 3x + 4$?

d Does the point (5, 19) lie on the graph of $y = 3x + 4$?

e State the value of the gradient and the y-intercept.

THINK	WRITE
a Substitute $x = 2$ into the equation.	a When $x = 2$, $\quad y = 3 \times 2 + 4$ $= 10$
b Substitute $y = 13$ into the equation and solve for x.	b When $y = 13$, $\quad 13 = 3x + 4$ $3x = 9$ $x = 3$

c Substitute 2 for x and 4 for y. If this makes the equation true then the point lies on the line; otherwise it does not lie on the line.	**c** Does $4 = 3 \times 2 + 4$? 4 does not equal 10. The equation is not true, therefore the point (2, 4) does not lie on the line.
d Substitute 5 for x and 19 for y. If this makes the equation true then the point lies on the line; otherwise it does not lie on the line.	**d** Does $19 = 3 \times 5 + 4$? The equation is true, therefore the point (5, 19) does lie on the line.
e Compare the given equation with that of the general equation of a straight line, $y = mx + c$. The gradient is the coefficient of the x-term, m, and the y-intercept is the constant term, c.	**e** The gradient is 3 and the y-intercept is 4.

WORKED EXAMPLE 2

Two points A (1, 7) and B (3, 13) lie on the same line. Use the points A and B to calculate the gradient of the line.

THINK	WRITE
1 Write the gradient formula and match points A and B with (x_1, y_1) and (x_2, y_2).	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ where $(x_1, y_1) = (1, 7)$ and $(x_2, y_2) = (3, 13)$
2 Substitute the values into the formula and evaluate.	$= \dfrac{13 - 7}{3 - 1}$ $= \dfrac{6}{2}$ $= 3$ The gradient is 3.

Sketching straight-line graphs

eBook*plus*

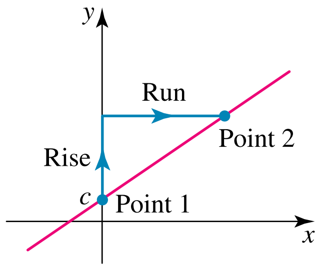
Interactivity
Constructing and interpreting straight-line graphs
int-6280

In previous years of study you have learned that only one straight line can be drawn through two distinct points. Therefore, the graph of a straight line can be obtained by plotting and joining together any two points on the line. Thus, to sketch a straight-line graph by hand, we first need to find the coordinates of any two points on the line. This can be done in a number of different ways. The three most common methods are outlined as follows.

Gradient–intercept method

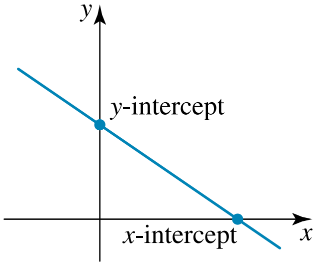
This method is used if the equation is in $y = mx + c$ form.

1. The first point plotted is the y-intercept, given by the value of c. Plot it on a set of axes.
2. The gradient is given by $m = \dfrac{\text{rise}}{\text{run}}$. Write the gradient as a fraction and identify the values of the rise and the run.
3. To obtain the second point, start from the y-intercept and move up (or down) and across, as suggested by the gradient.
4. Join the two points together with a straight line and label the graph.



x- and y-intercept method

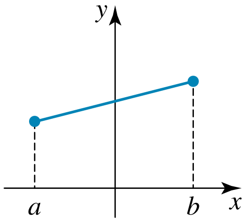
This method is used if the equation is in $ax + by = c$ form, or if you are required to show both the x- and y-intercepts.



1. If a point is on the y-axis, its x-coordinate is 0. To find the y-intercept, substitute 0 for x and solve the resultant equation.
2. If a point is on the x-axis, its y-coordinate is 0. To find the x-intercept, substitute 0 for y and solve the resultant equation.
3. Plot the x-intercept and the y-intercept on a set of axes.
4. Join the two points together with a straight line and label the graph.

Sketching a line over a required interval

If a graph needs to be sketched between two given x-values, its end points must be shown. Since only two points are needed to sketch a line, we can obtain the coordinates of the end points and join them together. To construct a graph of a straight line between a and b, follow these steps:



1. Rearrange the equation to make y the subject.
2. Substitute each of the two given x-values (that is, a and b) into the equation and find corresponding values of y.
3. Plot the end points on a set of axes.
4. Join the two points together with a straight line and label the graph.

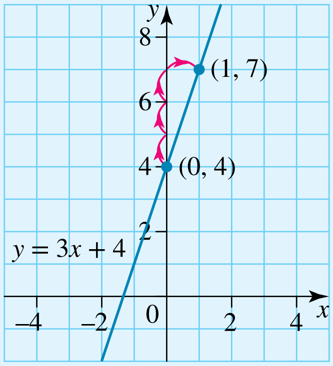
Worked example 3 illustrates the use of these three methods.

WORKED EXAMPLE 3

Sketch the graph of each of the following equations.

a $y = 3x + 4$ **b** $y = -\frac{2}{3}x$ **c** $2x + 3y = 6$

d $x - y = 3$ **between** $x = -2$ **and** $x = 6$.

THINK	WRITE/DRAW
a 1 The equation is in $y = mx + c$ form, so use the gradient-intercept method. State the value of the y-intercept.	**a** $y = 3x + 4$ $c = 4$, so the y-intercept is $(0, 4)$
2 Write the gradient as a fraction and identify the values of the rise and the run.	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{3}{1}$
3 Draw a set of axes and plot the y-intercept. To obtain the second point, start from the y-intercept and move up and across as suggested by the positive gradient; that is, 3 units up and 1 unit right. The second point is (1, 7). **4** Join the two points with a straight line and label the graph.	

b **1** The equation is in $y = mx + c$ form, so use the gradient-intercept method. State the value of the y-intercept. Since the value of c is 0, the line passes through the origin.

b $y = -\frac{2}{3}x$

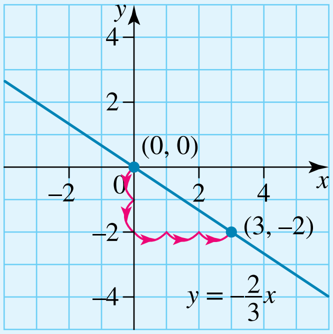
$c = 0$, so the y-intercept is (0, 0)

2 Write the gradient as a fraction and identify the values of the rise and the run. Note that the negative sign always belongs to the rise.

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= -\frac{2}{3} \end{aligned}$$

3 Draw a set of axes and plot the y-intercept. To obtain the second point, start from the origin and move down and across as suggested by the negative gradient; that is, 2 units down and 3 units right. The second point is (3, −2).

4 Join the two points with a straight line and label the graph.



c **1** The equation is in $ax + by = c$ form, so find the x- and y-intercepts.

c $2x + 3y = 6$

2 Find the y-intercept by substituting $x = 0$ into the equation and solving.

$$\begin{aligned} 2(0) + 3y &= 6 \\ 3y &= 6 \\ y &= 2 \end{aligned}$$

3 State the coordinates of the y-intercept.

The coordinates of the y-intercept are (0, 2).

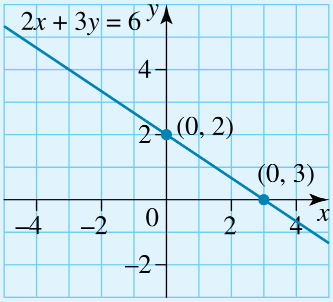
4 Find the x-intercept by substituting $y = 0$ into the equation and solving.

$$\begin{aligned} 2x + 3y &= 6 \\ 2x + 3(0) &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

5 State the coordinates of the x-intercept.

The coordinates of the x-intercept are (3, 0).

6 Plot the intercepts on a set of axes and join them together with a straight line. Label the graph.



d **1** The graph needs to be sketched between two x-values. Rearrange the equation to make y the subject.

d
$$\begin{aligned} x - y &= 3 \\ -y &= -x + 3 \\ y &= x - 3 \end{aligned}$$

2 Substitute each of the two given x-values (that is, −2 and 6) into the equation and find corresponding values of y.

When $x = -2$:
$$\begin{aligned} y &= x - 3 \\ &= -2 - 3 \\ &= -5 \end{aligned}$$

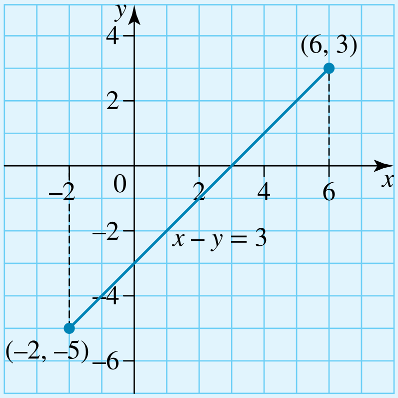
When $x = 6$:
$$\begin{aligned} y &= x - 3 \\ &= 6 - 3 \\ &= 3 \end{aligned}$$

3 State the coordinates of the end points.

The end points are (−2, −5) and (6, 3).

4 Plot the end points on a set of axes and join them with a straight line. Label the graph.

Note: Do not extend the line beyond the end points.



Applications of straight-line graphs

Many real-life situations involve variables whose relationship is linear and hence can be described by a linear rule and represented graphically by a straight line.

When modelling a linear relationship, remember that the *y*-intercept represents the value of the function when $x = 0$. In most situations, it represents the initial (or original) value of something, a fixed cost or flag fall.

The gradient represents the rate of change of the *y*-value with respect to *x*. It shows the change (increase or decrease) in *y*, as *x* increases by 1 unit. For example, let the equation $V = 1000 - 200t$ describe the volume (in litres) of water in a tub *t* minutes after a plug is removed. The *y*-intercept shows that the initial amount of water in the tub is 1000 litres. The gradient means that the volume of water decreases (since the gradient is negative) at a rate of 200 litres every minute.

WORKED EXAMPLE 4

Mikaela works as a car salesperson. She is paid a retainer of \$150 a week and receives 2% commission on her sales.

a How much does Mikaela earn in a week in which she sold cars worth \$40 000 in total?

b Construct a table to describe the relationship between earnings and sales using sales of \$0, \$10 000, \$20 000 and \$80 000.

c Write an equation that relates earnings (*E*) to sales (*S*).

d State the value of the gradient and the *y*-intercept and interpret their meaning.

e Draw a graph of the relationship between earnings and sales. That is, draw a graph of *E* versus *S*.

THINK	WRITE/DRAW
a Calculate earnings by finding the sum of the retainer and the commission. $2\% = \frac{2}{100}$ $= 0.02$	a Earnings = retainer + commission = retainer + 2% of sales $= 150 + 2\% \times 40000$ $= 150 + 0.02 \times 40000$ $= 150 + 800$ $= 950$ Mikaela earns \$950.

b For each of the four values of sales, repeat the working of **a** opposite.

b

Sales	Earnings
0	150
10 000	350
20 000	550
80 000	1750

c Use the pattern seen in **a** and **b** to relate E and S. Writing the relationship in words may assist in finding the equation.

c Earnings = retainer + 2% of sales

$$E = 150 + 0.02 \times S$$

$$\text{or } E = 0.02S + 150$$

d **1** Compare the equation relating earnings to sales with that of the general equation of a straight line, $y = mx + c$. The gradient is the coefficient of the x-term, m, and the y-intercept is the constant term, c.

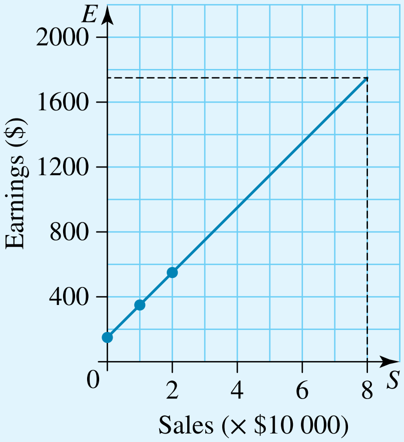
d The gradient is 0.02 and the y-intercept is 150.

2 Interpret the meaning of the gradient and the y-intercept.

The gradient represents the commission rate. It shows that for every $100 worth of sales, Mikaela earns $2 in commission. The y-intercept represents the value of the retainer; that is, the amount Mikaela is paid if no sales are made.

e Earnings depends on sales. Thus sales is the *explanatory* variable and goes on the horizontal axis. Do not crowd the axes by writing the sales values in thousands — use a legend and write only the number of multiples of ten thousand.

e



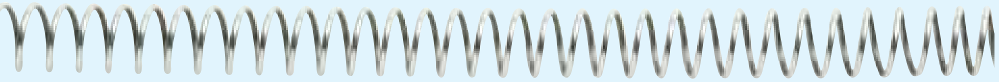
Extrapolation and interpolation

Two terms used widely in interpreting data are **extrapolation** and **interpolation**. Extrapolation means to examine the relationship between the variables by extending it beyond the data. Interpolation means to infer the relationship between distinct data points. Worked example 5 should make these meanings clear.

WORKED EXAMPLE 5

The tension (measured in Newtons) in a spring is linearly related to the extension of the spring (measured in centimetres). Some values relating tension (T) and extension (x) are given:

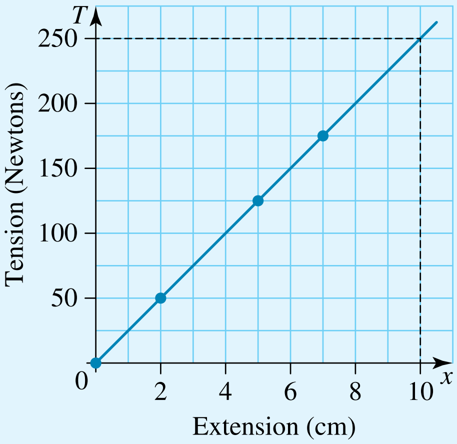
x	2	5	7
T	50	125	175



a Plot these data.

b By extrapolating from the data, predict the tension when the extension is 10 cm.

c By interpolating, predict the tension when the extension is 4 cm.

THINK	WRITE/DRAW
a Plot the data on a set of axes that shows x-values up to 10. Draw a line through these points, extending it beyond the given values.	**a** 
b Extrapolation involves inference beyond the data range. Use the graph to find T when x is 10.	**b** When $x = 10$, $T = 250$ When the spring is stretched 10 cm, the tension is 250 newtons.
c Interpolation involves inference between the data points. Use the graph to find T when x is 4.	**c** When $x = 4$, $T = 100$ When the spring is stretched 4 cm, the tension is 100 newtons.

EXERCISE 14.2 Constructing and interpreting straight-line graphs

PRACTISE

1 **WE1** Consider the equation of the line $y = 2x + 5$.

- a What is the value of y when $x = 4$?
- b What is the value of x when $y = 17$?
- c Does the point (3, 12) lie on the line?
- d State the value of the gradient and the y-intercept.

2 Consider the equation $5x - 2y = 10$.

- a Find the value of y when $x = 2$.
- b Find the value of x when $y = 5$.
- c Does the point $(-8, -25)$ lie on the line?

3 **WE2** Two points P (2, 5) and Q (4, 13) lie on the same line.

Calculate the gradient of the line.

4 The points $(5, -3)$ and $(-2, 4)$ lie on a line. Calculate the gradient of this line.

5 WE3 Sketch the graphs of the following functions.

a $y = 3x - 4$
b $y - 3x = 12$
c $y = \frac{3}{4}x$
d $4x - 3y = 12$ between $x = -2$ and $x = 4$

6 Sketch the graph of the following functions.

a $5y + x = 10$
b $y = -x + 3$
c $y = 3x$
d $y = 3x - 2$, between $x = -2$ and $x = 2$

7 WE4 Greta sells boats and she is paid \$275 per week plus 3% of her weekly sales.

a How much does Greta receive if the value of her weekly sales is \$12 000?
b Construct a table to show Greta's weekly pay for the following values of weekly sales: \$5000, \$10 000, \$15 000 and \$20 000.
c Write an equation that relates her weekly pay (P) and the value of her weekly sales (S).
d Draw a graph of the relationship between weekly pay and weekly sales. That is, draw a graph of P versus S. Place S on the horizontal axis.
e State the value of the gradient and the y-intercept, and interpret their meaning.

8 Sheela, who works as a car salesperson, is paid a retainer of \$200 a week and receives 3% commission on her sales.

a Copy and complete this table.

Sales	0	\$20 000	\$40 000	\$60 000
Earnings				

b Write an equation that relates earnings (E) to sales (S).
c Draw the graph of earnings (E) versus sales (S) using the information in a.

studyon

Unit 4
AOS M4
Topic 1
Concept 8

Interpolation and extrapolation
Concept summary
Practice questions

9 WE5 For a farm water tank, the volume of water in the tank (V) is related linearly to the depth of water in the tank (D) for values of D greater than 450 mm.

Depth (mm)	Volume (L)
500	2500
700	4500
1000	7500

a Plot these data using suitable axes.
b By extrapolating from the data, predict the depth needed to give a volume of 13 000 litres.
c By interpolating from the data, predict the volume when the depth of the water is 600 mm.

10 The distance covered (in metres) of an object over a period of time (in seconds) is shown in the table.

Time (s)	2	4	5	7
Distance (m)	8	16	20	28

a Plot these data.
b By extrapolating from the data, predict the distance covered when 9 seconds have elapsed.
c By interpolating, predict the distance covered when the time elapsed is 6 seconds.

CONSOLIDATE

11 Consider the equation of the line $y = -2x + 4$.

a What is the value of y when $x = 2$?
b What is the value of x when $y = -4$?
c Does the point (0, 4) lie on the line?
d State the value of the gradient and the y-intercept.

12 Consider the equation of the line $2y + 3x = 12$.

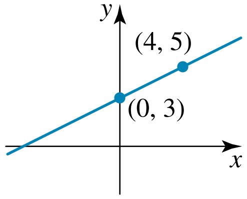
a What is the value of y when $x = 2$?
b What is the value of x when $y = 6$?
c Does the point (3, 2) lie on the line?
d State the value of the gradient and the y-intercept.

13 Consider the equation of the line $2y + x = 8$.

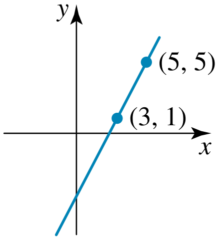
a What is the value of y when $x = 2$?
b What is the value of x when $y = 1$?
c Does the point (3, 2) lie on the line?
d Using the two points whose coordinates were found in **a** and **b**, calculate the gradient.

14 Calculate the gradient of the line shown in each of the following.

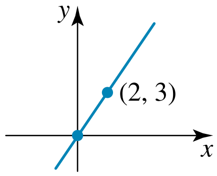
a



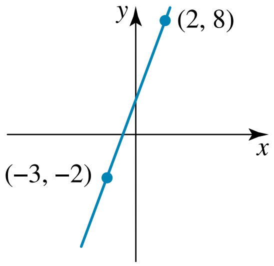
b



c



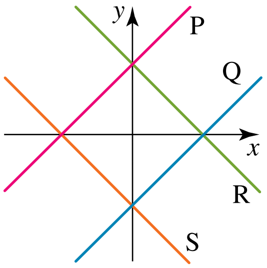
d



15 Sketch the graphs of the following functions.

a $3x + y = 12$ **b** $y = -2x + 8$ **c** $y = -2x$
d $y = 2x + 1$ between $x = -2$ and $x = 2$

16 The graphs of the following lines are shown at right.



i $y + x = 4$ **ii** $y = x + 4$
iii $y + x + 4 = 0$ **iv** $y = x - 4$

The best match between graphs P to S and equations **i** to **iv** is:

A P → **iv** Q → **i** R → **iii** S → **ii**
B P → **i** Q → **iii** R → **ii** S → **iv**
C P → **iv** Q → **i** R → **ii** S → **iii**
D P → **iv** Q → **iii** R → **i** S → **ii**
E P → **ii** Q → **iv** R → **i** S → **iii**

17 An object that falls freely due to gravity increases its speed (S) by 9.8 m/s each second. Assume that at the start the speed of the object was 0.

a How fast is the object travelling after 3 seconds?
b How long has the object been falling when its speed is 78.4 metres per second?
c If t stands for the number of seconds the object has been falling, write a formula that relates S to t.
d Draw a graph of S versus t.

18 Visa car rentals charge $75 per day plus $15 per hundred kilometres.

a How much would it cost to rent a car for one day if the car travelled 345 km?
b The bill for one day's rental came to $142.50. How many kilometres did the car travel?
c Sketch a graph of the cost of renting the car for one day (C) versus the number of kilometres travelled (d).

19 A psychology experiment is testing the relationship between the number of errors (n) made by a child on a task and the time taken to complete the task in seconds (t). The results of five trials of the experiment are given in the table below.

If the number of errors made is linearly related to the time taken, which experiment does not fit the pattern?

Experiment	Time taken (seconds)	Number of errors
A	30	10
B	45	6
C	36	9
D	48	7
E	42	8

20 Janine sells cosmetics at a department store. She knows she is paid a retainer plus commission on sales but she is not sure of the exact rates. For three weeks she records her wages and the value of product she sold during that week.

Week	Sales	Wages
1	$1200	$494
2	$750	$440
3	$880	$455.60

a In week 4 her sales totalled $1000. Predict her wage for that week.
b Write a formula to calculate Janine's weekly wage in terms of her weekly sales.

MASTER

21 In a one-day international cricket match, each team bats for 50 overs. After 20 overs Australia's score had reached 108 runs. If the target for Australia is 250 runs at the end of 50 overs:

a how many runs per over does the team need to score for the remaining 30 overs?
b Write an equation to relate the score (S) to the number of overs completed (n).
c The graph of S versus n could be a straight line. What would be the gradient of that straight line?

22 Taxi hire charges are shown in the diagram.

a Calculate the cost of travelling 15 km in a taxi.

b If the taxi fare was $20.80, how far did the taxi travel?

c Write an equation relating the fare (F) to the distance (d).

d Draw a graph relating F to d.

e What is the slope of this graph?

14.3 Line segments and step functions

In this section we consider graphs which are not straight lines but are made from straight lines. We also consider graphs which are not straight lines but discrete sets of points.

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Interactivity
Piecewise linear graphs
int-6486

WORKED EXAMPLE 6

When a real estate agent sells a property, he earns commission at the following rate:

1.5% on the first $20 000
0.9% on the remainder.

a Calculate the commission earned on sales of:

i $10 000 **ii $20 000**
iii $30 000 **iv $40 000.**

b Draw a line segment graph of commission (C) versus the value of the sales (S) up to a sales value of $40 000.

c Give a reason for the difference of the slopes of the two segments.

THINK	WRITE/DRAW
a i For any amount up to and including $20 000, the commission is 1.5% of the amount.	a i Commission on $10 000 = 1.5% of 10 000 = 0.015 × 10 000 = $150
ii The commission is 1.5% of the amount.	ii Commission on $20 000 = 1.5% of 20 000 = 0.015 × 20 000 = $300
iii $30 000 is more than $20 000, so calculate 1.5% on $20 000 plus 0.9% on the remainder over $20 000; that is, $10 000.	iii Commission on $30 000 = 1.5% of $20 000 + 0.9% of $10 000 = 0.015 × 20 000 + 0.009 × 10 000 = 300 + 90 = $390

iv $40 000 is more than $20 000, so calculate 1.5% on $20 000 plus 0.9% on the remainder over $20 000; that is, $20 000.

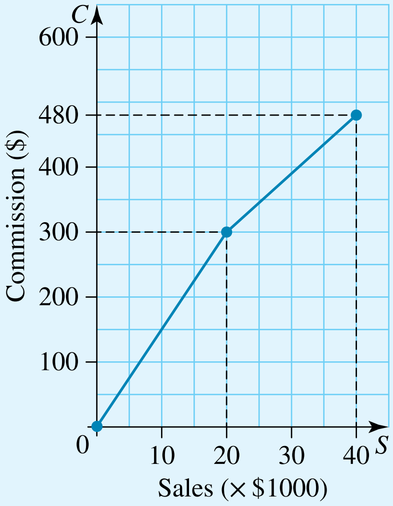
iv Commission on $40 000
= 1.5% of 20 000 + 0.9% of 20 000
= 0.015 × 20 000 + 0.009 × 20 000
= 300 + 180
= $480

b **1** Since the commission rate changes when sales exceed $20 000, the graph will consist of two segments. One segment for all sales up to and including $20 000 and the other for sales of more than $20 000 and up to and including $40 000.

2 The starting point of the first segment is (0, 0) (since there is no commission when there are no sales). The end point of this segment is (20 000, 300) (as was found in part **a** (**ii**)). Plot these points and join them with a straight line.

3 The second segment starts where the first segment ends. The end point of the second segment is (40 000, 480) (as was found in part **a** (**iv**)). Plot this end point and join it to the first segment.

b



c The slope of the line is determined by its gradient. In this case, the gradient is the rate of commission. When the sales exceed $20 000, the commission rate decreases — this results in a line segment that is less steep.

c The gradient of each line segment represents the commission rates. Since the rates on values lower than $20 000 differ from the rates on values greater than $20 000, the gradients also differ.

WORKED EXAMPLE 7

An electrician charges the rates shown.

a **Calculate the charges for service calls of the following durations:**
10 minutes, 25 minutes, 40 minutes,
55 minutes, 70 minutes and 85 minutes.

b **Draw a step graph of charges (C) versus the time of the service call in minutes (t) for calls of up to 120 minutes.**


$35 call-out fee plus
$30 per half hour
or part thereof.

THINK

a ‘Part thereof’ means that whether the repairs take 3 minutes, 10 minutes or 30 minutes, you are charged for 30 minutes. For a service call up to 30 minutes, the electrician charges $35 call-out fee plus $30. For a service call over 30 minutes long and up to 60 minutes, the charge is $35 call-out fee plus 2 lots of $30. Finally, for a service call over 60 minutes and up to 90 minutes, the charge is $3 call-out fee plus 3 lots of $30.

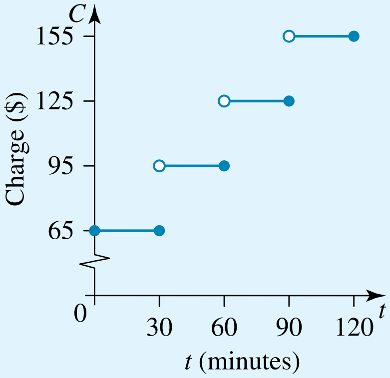
WRITE/DRAW

a

Time	Charge
10 min	35 + 30 = $65
25 min	35 + 30 = $65
40 min	35 + 2 × 30 = $95
55 min	35 + 2 × 30 = $95
70 min	35 + 3 × 30 = $125
85 min	35 + 3 × 30 = $125

b Draw the graph. Place t along the horizontal axis as it is the explanatory variable. The graph consists of four horizontal sections (steps). This reflects the fact that the fixed fee is charged for any amount of time up to and including 30 minutes; then a different fixed fee for any amount of time above 30 minutes and up to and including 60 minutes, and so on. Note that where the end point of the step is included in the step, it is shown as a full circle (closed circle), whereas if the end point is not included, it is shown as an empty circle (open circle).

b



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Interactivity
Step functions
int-6281

In the previous example, the explanatory variable, time (t), was a **continuous variable**; that is, t can take any values — 2, 15.5, 87 — and so on. In the next example we consider a situation where the explanatory variable can take whole number values only. A variable that is not continuous is called a **discrete variable**. If the data are discrete, the points on the graph are not joined together.

WORKED EXAMPLE 8

A bakery sells bread rolls for 50 cents each or $2.50 for 6.

a Calculate the cost of 4, 5, 6, 7, 11, 12, 13, 17, 18 and 19 bread rolls.

b Sketch this information on a graph for up to 24 rolls.

THINK

a Calculate the cost for each number of bread rolls. If the number of rolls is above 5, it can be formed by combining pack(s) of 6 rolls and single rolls. For example, 7 rolls can be bought as a pack of 6 plus 1 single roll; 17 rolls as 2 packs of 6 plus 5 single rolls etc.

WRITE/DRAW

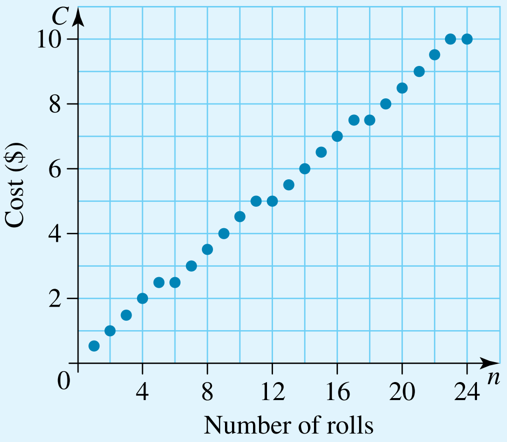
a

Number of rolls	Cost
4	4 × 50c = $2.00
5	5 × 50c = $2.50
6	$2.50
7	$2.50 + 1 × 50c = $3.00
11	$2.50 + 5 × 50c = $5.00
12	2 × $2.50 = $5.00
13	2 × $2.50 + 1 × 50c = $5.50
17	2 × $2.50 + 5 × 50c = $7.50
18	3 × $2.50 = $7.50
19	3 × $2.50 + 1 × 50c = $8.00

b Sketch the information using the number of bread rolls as the explanatory variable.

Note that the points on the graph are not joined together as the data are discrete.

b



EXERCISE 14.3 Line segments and step functions

PRACTISE

studyon
Unit 4
AOS M4
Topic 1
Concept 2
Segment graphs
Concept summary
Practice questions

1 **WE6** Suppose a real estate agent is paid commission at the following rates: 1.5% on the first $40 000 and 1% on the remainder.

- **a** Calculate the commission due on sales of $20 000, $30 000, $40 000, $50 000 and $60 000.
- **b** Draw a graph of commission (C) versus sales (S).
- **c** Give a reason for the difference of the slopes of the two segments.

2 Beena, who runs a real estate agency, sells a property and earns commission at the following rate:

2% on the first $25 000

0.8% on the remainder.

Calculate the commission earned by Beena on sales of:

a $10 000 **b** $25 000 **c** $35 000 **d** $45 000.

e Construct a graph of commission (C) versus the value of sales (S).

studyon
Unit 4
AOS M4
Topic 1
Concept 3
Step graphs
Concept summary
Practice questions

3 **WE7** A plumber charges these rates:

$55 call out fee

$40 per half hour or part thereof.

- **a** Using this information, construct the graph of charges (C) versus time of the service call, in minutes (t), for calls of up to 120 minutes.
- **b** Calculate the charges for these service calls:

  12 minutes, 23 minutes, 44 minutes,

  56 minutes, 73 minutes, 87 minutes.

4 A telephone company charges users at a rate of 25 cents for each completed 30 seconds. This implies a call of less than 30 seconds is free.

- **a** Copy and complete this table for the calls shown.
- **b** Construct the graph of cost versus length of call for calls of up to 120 seconds.

Length of call (seconds)	Cost (cents)
15	
30	
45	
60	
75	
90	
105	
120	

5 **WE8** A country bakery sells buns for 40 cents each or 6 for $2.00.

a Using this information, construct a graph for buying up to 20 buns using the number of buns as the independent variable.
b Calculate the cost of 2, 4, 6, 8 and 10 buns.

6 The Explorindo Travel Company specialises in surfing tours of remote islands in Indonesia. They will take individuals but prefer to deal with groups of people. They have the following charges for a holiday package:

1 person	$900
2 people	$1650
Each extra person	$600

a Draw a table with the following headings and complete it for costs for 1, 2, 3, 6, 8 and 10 people.

Number of people	Total cost ($)

b Draw the graph of total cost versus number of people. (Include only the number of people discussed in part **a**.)

CONSOLIDATE

This information relates to questions **7** and **8**. The amount of electricity used around the home is measured in kilowatt hours (kWh). A light bulb left on for 10 hours will consume about 1 kWh of power. The local power supplier charges at the following rates.

Power	Cost per kWh ($)
First 400 kWh	0.20
Next 1000 kWh	0.15
Remaining kWh	0.10

7 Copy and complete the following table by calculating the cost due for each of the consumptions.

Consumption (kWh)	Power bill ($)
200	
400	
600	
1000	
1500	

8 Using the data in the table, draw a graph of power bill versus consumption.

9 An electrician charges at the following rates:

$45 call out fee plus
$35 per half hour or part thereof.

a Calculate the charges for the following service calls:
20 minutes, 30 minutes, 45 minutes, 60 minutes, 80 minutes and 90 minutes.
b Draw a graph of charges (C) versus time of the service call in minutes (t) for calls of up to 90 minutes.

10 A mobile phone company charges users a rate of 15 cents for each completed 20 seconds of the call. This means a call of less than 20 seconds is free.

a Draw a table with the following headings and complete it for calls of 10, 20, 30, 40, 50, 60, 70, 80, and 90 seconds.

Length of call (seconds)	Cost (cents)

b Draw the graph of cost versus length of call for calls of up to 90 seconds.

11 Which of the following is clearly a discrete variable?

A the time taken for a phone call
B the number of CDs in a collection
C the commission earned on sales
D the power consumed by a hot-water jug
E a person's weight

12 On a particular visit, a tradesperson charges the householder $100. It was noted he arrived at 11.30 am. Which of the following departure times could be correct?

A 11.38 am **B** 11.48 am **C** 12.05 pm **D** 12.20 pm **E** 12.35 pm

$50 call-out fee
$25 per 15 minutes or part thereof

13 This is the 2011–12 tax table.

Taxable income	Tax on this income
0–$6000	Nil
$6001–$37 000	15c for each $1 over $6000
$37 001–$80 000	$4650 plus 30c for each $1 over $37 000
$80 001–$180 000	$17 550 plus 37c for each $1 over $80 000
$180 001 and over	$54 550 plus 45c for each $1 over $180 000

Use this tax table to calculate the amount of tax paid in 2012 by people with taxable incomes of:

a $4000 **b** $7000 **c** $20 000 **d** $35 000
e $40 000 **f** $60 000 **g** $100 000 **h** $200 000.

14 **a** Using the values calculated in question **13** and any other values, draw the graph of income tax versus taxable income.

b Use the graph to estimate the income tax payable on a taxable income of:

i \$24 000 **ii** \$95 000.

Confirm your answers with calculations.

15 This is the 2014–15 tax table.

Taxable income	Tax on this income
0–\$18 200	Nil
\$18 201–\$37 000	19c for each \$1 over \$18 200
\$37 001–\$80 000	\$3572 plus 32.5c for each \$1 over \$37 000
\$80 001–\$180 000	\$17 547 plus 37c for each \$1 over \$80 000
\$180 001 and over	\$54 547 plus 45c for each \$1 over \$180 000

Use this tax table to calculate the amount of tax paid in 2015 by people with taxable incomes of:

a \$4000 **b** \$7000 **c** \$20 000 **d** \$35 000
e \$40 000 **f** \$60 000 **g** \$100 000 **h** \$200 000.

16 **a** Use the values calculated in question **15** to draw a graph of income tax versus taxable income.

b Use the graph to estimate the income tax payable on a taxable income of:

i \$24 000 **ii** \$95 000.

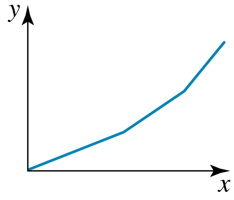
Confirm your answers with calculations.

c Compare your answers to questions **14b** and **16b**.

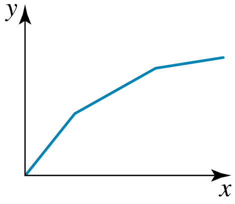
MASTER

17 What situation could be described by the following graphs? In your response, clearly identify the variables used on both axes and explain how the graph represents that situation. For example, the first graph could measure the profit versus the number of units of production.

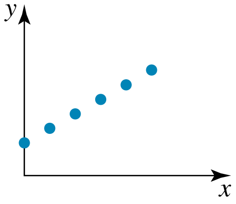
a



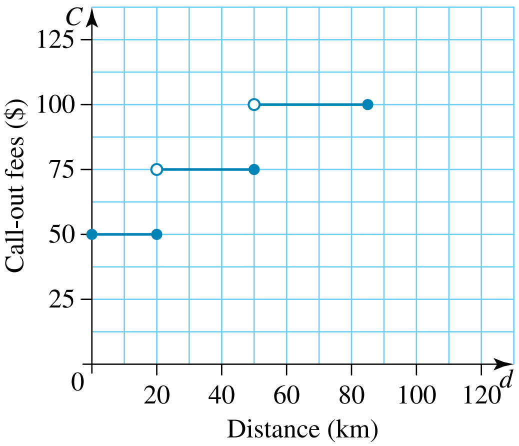
b



c



18 'Fix-it-fast' is a photocopier repair service for schools in North-central Victoria. The call-out charge depends on the distance the repair person has to travel. The call-out fees for distances up to 85 km are shown on the following graph.



a **i** What is the call-out fee for a distance of 40 km?

ii What is the maximum distance travelled for a call-out fee of \$50?

A call-out fee of \$125 is charged for schools at a distance of more than 85 km but less than 120 km.

b Copy the shown graph and add this information to it.

14.4 Simultaneous equations and break-even point

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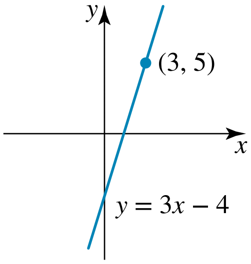
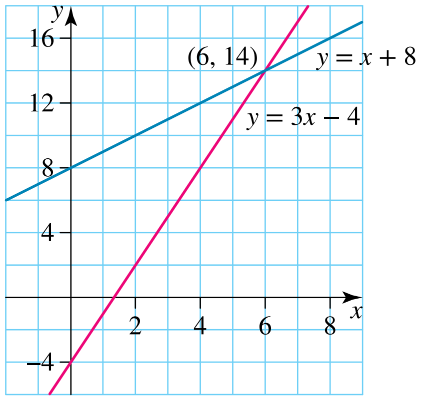
Interactivity
Solving simultaneous equations using substitution
int-6453

In this topic, we have seen that when an equation such as $y = 3x - 4$ is graphed it forms a straight line. Each point on the line will have an x- and a y-coordinate and these values of x and y satisfy the equation. The reverse is also true. Each pair, x and y, which satisfy the equation will be the coordinates of a point on the line.



Solving linear simultaneous equations

If we draw the graph of $y = 3x - 4$ and $y = x + 8$ on the same set of axes the result is shown at right.



The point where the lines intersect (6, 14) is called the **simultaneous solution** because $x = 6$ and $y = 14$ satisfies both equations.

In this example we found the simultaneous solution graphically; that is, from the graph.
It is also possible to find the simultaneous solution algebraically, either by the substitution method or the elimination method.

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Unit 4
AOS M4
Topic 1
Concept 5
Algebraic solution to simultaneous equations
Concept summary
Practice questions

WORKED EXAMPLE 9

Find the simultaneous solutions to these equations using algebraic methods.

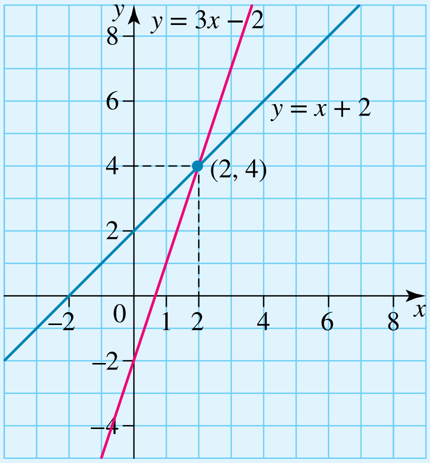
a $y = 3x - 4$ and $y = x + 8$

b $2x + 3y = 13$ and $x - 4y = 1$

THINK	WRITE
a 1 Since both equations start with $y =$, solve using the substitution method. Use equation [1] to substitute for y in equation [2].	**a** $y = 3x - 4$ [1] $y = x + 8$ [2] Substitute [1] into [2] $3x - 4 = x + 8$ [3]
2 Solve equation [3].	$3x - x = 8 + 4$ $2x = 12$ $x = 6$
3 Find y by substituting $x = 6$ into one of the original equations.	Substitute $x = 6$ into [2]: $y = 6 + 8$ $= 14$
4 State the solution.	The solution is (6, 14).
b 1 If one of the equations was rearranged to a $y = mx + c$ format, the substitution method could be used. However, it will be more efficient to use the elimination method in this case.	**b** $2x + 3y = 13$ [1] $x - 4y = 1$ [2]
2 Multiply equation [2] by 2 so that the x-coefficients of both equations are the same.	[2] × 2: $2x - 8y = 2$ [3]

3 Eliminate x by subtracting [3] from [1].	[1] − [3]: $11y = 11$
4 Solve to find y.	$y = 1$
5 Find x by substituting $y = 1$ into one of the equations (into equation [2] is easiest).	Substitute $y = 1$ into [2]: $x - 4 \times 1 = 1$ $x - 4 = 1$ $x = 5$
6 State the solution.	The solution is (5, 1).

WORKED EXAMPLE 10 **Find the simultaneous solution of $y = 3x - 2$ and $y = x + 2$ using a graphical method.**

THINK	WRITE/DRAW
1 Draw both graphs on the one set of axes.	
2 The simultaneous solution is the point where the graphs intersect.	The point of intersection is (2, 4) so the simultaneous solution is (2, 4) or $x = 2$, $y = 4$.

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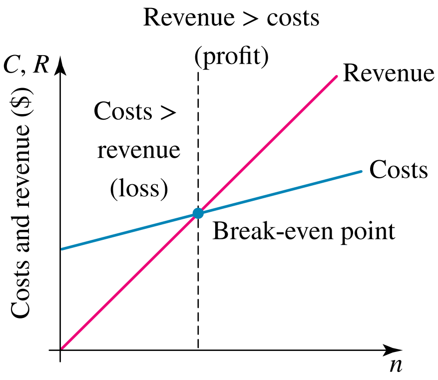
Break-even analysis

The aim of most businesses is to make a profit. The profit depends on the costs associated with the business (labour, raw materials and plant) and its revenue (the money it earns through sales). It represents the difference between the revenue and the costs.

Profit = revenue − costs

eBook*plus*
Interactivity
Break-even points
int-6454



It is evident that a profit will occur if the revenue exceeds the costs. However, if the costs exceed the revenue, a loss will result. Finally, if the costs equal the revenue, there will be neither a profit nor a loss. This is referred to as a **break-even point**.

The diagram shows the graph of a cost function and a revenue function, drawn on the same set of axes. The point of intersection of the two lines represents the point at which costs and revenue are equal; that is, the break-even point. To the left of the break-even point, the cost line is above the revenue line. This means that the costs are higher than the revenue and will result in a loss. To the right of the break-even point, the cost line is below the revenue line. This means that costs are lower than the revenue and will result in a profit.

WORKED EXAMPLE 11

The cost associated with publishing a particular maths book (C) is given by

$$C = 12n + 24\,000$$

where n represents the number of books. The revenue (R) made from selling n books is given by $R = 28n$. Both C and R are in dollars.

a **Copy and complete the following table.**

Number of books (n)	Costs ($)	Revenue ($)
500		
1000		
1500		
2000		

b **Sketch the graph of the costs (C) versus number (n), and the graph of revenue (R) versus number (n) on the same set of axes.**

c **How many books need to be published and sold so that the revenue equals the costs?**

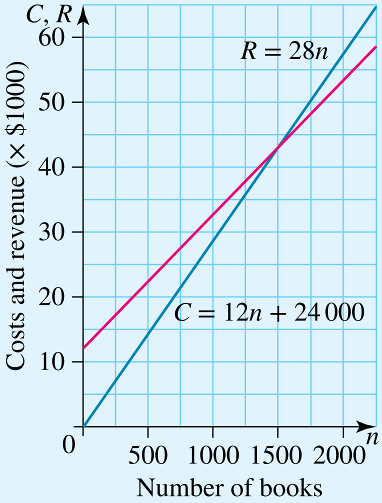
d **State the coordinates of the break-even point and interpret its meaning.**

THINK	WRITE/DRAW
a **1** Write the equations for the costs (C) and the revenue (R).	**a** $C = 12n + 24\,000$, $\quad R = 28n$
2 Calculate the value of C and R for each value of n given in the table.	When $n = 500$, $C = 12 \times 500 + 24\,000 = \$30\,000$ $\quad R = 28 \times 500 = \$14\,000$ When $n = 1000$, $C = 12 \times 1000 + 24\,000 = \$36\,000$ $\quad R = 28 \times 1000 = \$28\,000$ When $n = 1500$, $C = 12 \times 1500 + 24\,000 = \$42\,000$ $\quad R = 28 \times 1500 = \$42\,000$ When $n = 2000$, $C = 12 \times 2000 + 24\,000 = \$48\,000$ $\quad R = 28 \times 2000 = \$56\,000$
3 Use the results from the calculations in step 2 to complete the table.	

Number of books (n)	Costs ($)	Revenue ($)
500	30 000	14 000
1000	36 000	28 000
1500	42 000	42 000
2000	48 000	56 000

b The number of books (n) is the explanatory variable, so place it on the horizontal axis. The graphs of the cost and revenue functions are straight lines, so they can be constructed by plotting any two points from the table then joining them with a straight line.

b



c From the table (or the graph), we can see that when $n = 1500$, the costs and the revenue are both the same.

c 1500 books need to be published and sold so that the revenue equals the costs.

d The break-even point is the point where the costs equal the revenue. On the graph, it is the point of intersection of the two lines.

d The coordinates of the break-even point are (1500, 42 000). The cost associated with publishing and the revenue made from selling 1500 books is $42 000. If less than 1500 books are sold, the costs are higher than the revenue (loss). If more than 1500 books are sold, the revenue is higher than the costs (profit).

Note that the break-even point in the previous worked example could have been found using CAS or algebraically.

EXERCISE 14.4 Simultaneous equations and break-even point

PRACTISE

1 **WE9** Solve the following simultaneous equations.

$$y = 10x - 7$$
$$y = 2x + 1$$

2 Solve the following simultaneous equations.

$$6x - 11y = 2$$
$$5x - 9y = 1$$

3 **WE10** Find the simultaneous solution of $y = 2x - 3$ and $y = -x$ using a graphical method.

4 Find the simultaneous solution of $y = -2x - 3$ and $y = x + 3$ using a graphical method.

5 **WE11** The cost of manufacturing toys (C) is related to the number of toys produced (n), by the formula $C = 600 + 3n$. The revenue (R) made from selling n toys is $R = 7n$. Both C and R are in dollars.

studyon

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a Copy and complete the following table.

Number of toys	Cost ($)	Revenue ($)
50		
100		
150		
200		

b Sketch the graph of cost (C) versus number (n) and the graph of revenue (R) versus number (n) on the same set of axes.

c How many toys need to be produced before revenue equals cost?

d State the coordinates of the break-even point and interpret its meaning.

6 The cost of manufacturing basketballs (C) is related to the number of basketballs produced (n) by the formula $C = 2800 + 4n$. The revenue (R) made from selling n basketballs is $R = 14n$.

a Copy and complete the table below.

Number of basketballs	Cost ($)	Revenue ($)
150		
200		
250		
300		
350		

b Using the information supplied, construct the graph of cost (C) versus number (n), and the graph of revenue (R) versus number (n), on the same set of axes.

c Using the graph, write the number of basketballs produced before revenue equals cost to 'break even'.

CONSOLIDATE

7 Find the simultaneous solution, algebraically, to:

a $y = 4x - 4$
$y = 2x$

b $y = 3x - 4$
$y = 2x - 2$

c $2y + x = 5$
$3y - 2x = 4$

8 Find the simultaneous solution of each pair of linear equations in question **7** using a graphical method.

9 Find the simultaneous solution of $y = 3x - 10$ and $x = 2y + 5$ using the substitution method.

10 Find the simultaneous solution of $3x - y = -8$ and $2x + 2y = 0$ using the elimination method.

11 Find the simultaneous solution of $y = x$ and $y = -x$ using a graphical method.

12 Find the simultaneous solution of $-3x + 2y = 1$ and $-2x + 3y = 9$ using the elimination method.

Plan A: $15 monthly fee plus 60 cents a minute for calls.

Plan B: $25 monthly fee plus 30 cents a minute for calls.

13 Consider the phone connection plans shown at left.

a Copy and complete the following table.

Call time (minutes)	Cost ($) — Plan A	Cost ($) — Plan B
10		
20		
30		
40		

b Sketch the graph of *cost* versus *time* for each of the two plans on the same set of axes.

c How many minutes of calls would you need to make for both plans to cost the same?

14 The cost of manufacturing electronic components (C) is related to the number of components produced (n), by the formula $C = 6000 + 2.5n$. The revenue (R) made from selling n components is $R = 4.5n - 8000$. Both C and R are in dollars.

a Copy and complete the following table.

Number of components	Cost ($)	Revenue ($)
5000		
10000		
15000		
20000		

b Sketch the graph of cost (C) versus number (n) and the graph of revenue (R) versus number (n) on the same set of axes.

c How many components need to be produced before revenue equals cost?

15 A new employee, whose job it is to sell a software package, is offered two different salary plans by Minitech:

Plan A: $400 per week plus $25 for each package sold

Plan B: $150 per week plus $45 for each package sold.

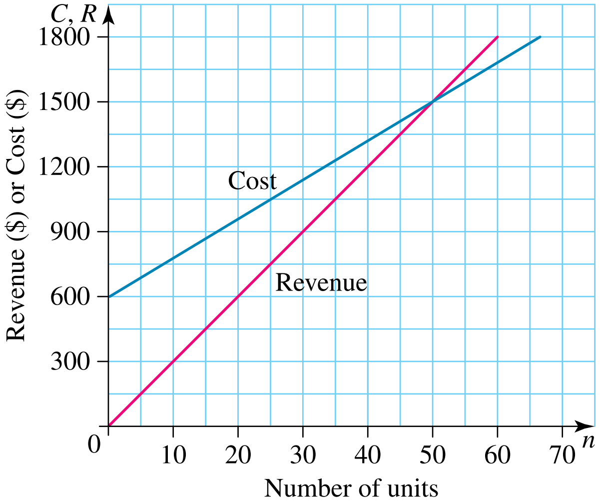
a Copy and complete the following table.

Number of packages sold	Salary ($) Plan A	Salary ($) Plan B
5		
10		
15		
20		

b Sketch the graph of salary versus number of packages sold for both Plan A and Plan B on the same set of axes.

c How many packages need to be sold before Plan B is the better choice?

16 The statement that best matches the graph is:



A revenue exceeds costs when more than 50 units are sold

B revenue exceeds costs when less than 50 units are sold

C revenue exceeds costs when more than 1500 units are sold

D revenue exceeds costs when less than 1500 units are sold

E revenue exceeds costs when exactly 50 units are sold.

MASTER

17 A factory producing mattresses finds that the equation linking cost in dollars (C) and the number of mattresses produced (m) is $C = 800m$.

a What is the cost of producing 10 mattresses?
b How many mattresses could be produced for $124 000?

18 From question 17, if all mattresses are sold, the revenue in dollars (R) from the sale of m mattresses is $R = 1800m$.

a Draw graphs of C and R on the same set of axes for $0 \leq m \leq 200$.
b Determine the number of mattresses which would need to be sold for the factory to break even.
c From the two equations given, write a profit equation (P).
d Calculate the profit if 175 mattresses were produced and sold.

14.5 Interpreting non-linear graphs

study on

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Topic 1
Concept 7

Non-linear graphs
Concept summary
Practice questions

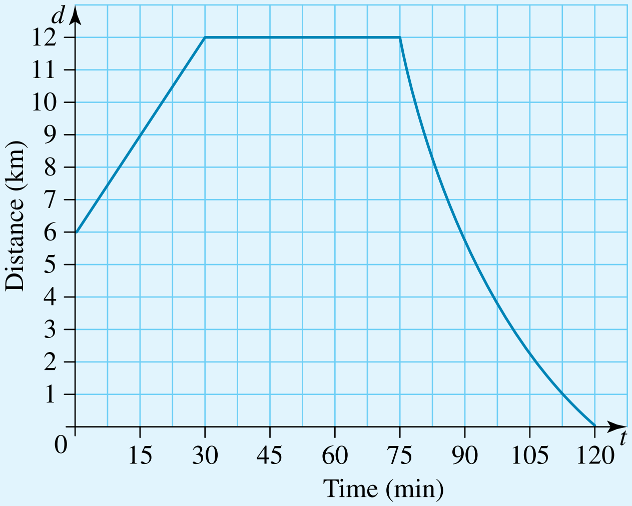
Why are graphs used so widely in papers, magazines, journals, in education, in government, and in sales and marketing? Graphs are used because they have the capacity to convey a significant amount of information effectively.

To get the most from a graph, a user needs to learn to read and interpret information presented graphically. In this section, we will look at **non-linear graphs**. A non-linear graph is a graph which is not a straight line.

WORKED EXAMPLE 12

The graph shows the distance a cyclist is from her home over a period of 120 minutes.

a How far from her home was the cyclist at the start of the time period?
b At what speed did the cyclist travel for the first 30 minutes?
c Describe the motion of the cyclist after 45 minutes.
d What was the cyclist's furthest distance from home?
e When did the cyclist turn for home?
f How long did the cyclist take to get home on the way back?



THINK

a The start of the time period means $t = 0$ min. The graph shows a distance of 6 km when $t = 0$.

b Use the relationship: speed $= \dfrac{\text{distance}}{\text{time}}$.

The cyclist travels from 6 km to 12 km in 30 minutes or 0.5 hour.

WRITE

a The cyclist was 6 km from home at the start of the time period.

b $$\text{Speed} = \frac{\text{distance}}{\text{time}}$$
$$= \frac{6 \text{ km}}{0.5 \text{ h}}$$
$$= 12 \text{ km/h}$$

c Between $t = 30$ min and $t = 75$ min the distance from home remains at 12 km.	**c** At 45 minutes into the time period, the cyclist is not moving.
d Observe the furthest distance from home from the graph.	**d** The furthest distance from home is 12 km.
e From the graph it can be seen that the distance the cyclist is from home decreases when $t = 75$ min.	**e** The cyclist turns for home after 75 min.
f The cyclist began travelling home when $t = 75$ min. The cyclist reached home when $t = 120$ min.	**f** Time taken to travel home $= (120 - 75)$ min $= 45$ min

WORKED EXAMPLE 13

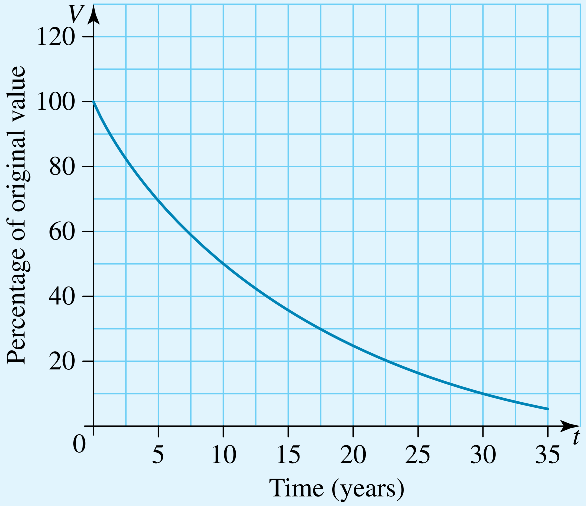
The value of a car originally worth $30 000 decreases over time. The graph describes the value of the car as a percentage of its starting price.

a What is the value of the car, in percentage terms, after 5 years?

b The ‘half-life’ is the time taken for the value to decrease by half. How long does it take for the car to lose half its original value?

c How long does it take for the car to fall in value from 50% of its original value to 25% of its original value?

d Estimate, in dollar terms, the value of the car after 30 years.

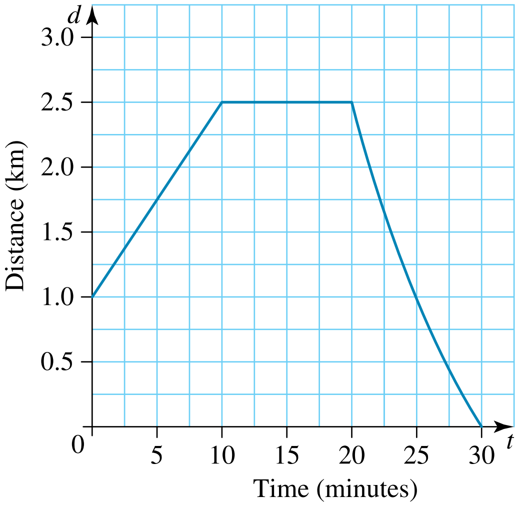


THINK	WRITE
a Read directly from the graph.	**a** The value of the car after 5 years is approximately 70% of its original value.
b ‘Half’ means 50%. Reading from the graph, it takes 10 years for the value of the car to fall to 50%.	**b** The time taken to fall in value by 50% (half-life) is 10 years.
c From the graph, the car is worth 50% of the original price at $t = 10$ and it is worth 25% at $t = 20$. Note that the half-life is also the time taken to fall from 50% to 25%.	**c** Time taken to fall from 50% to 25% of original value $= 20 - 10$ years $= 10$ years.
d As the half-life is 10 years, use this to calculate the value after 30 years. That is, after 3 half-lives. Calculate 12.5% of $30 000 to obtain the value of the car in dollar terms.	**d** Half-life = 10 years Therefore, the value after: 10 years is 50% 20 years is 25% 30 years is 12.5%. The value after 30 years = 12.5% of $30 000 $= 0.125 \times \$30\,000$ = $3750

EXERCISE 14.5 Interpreting non-linear graphs

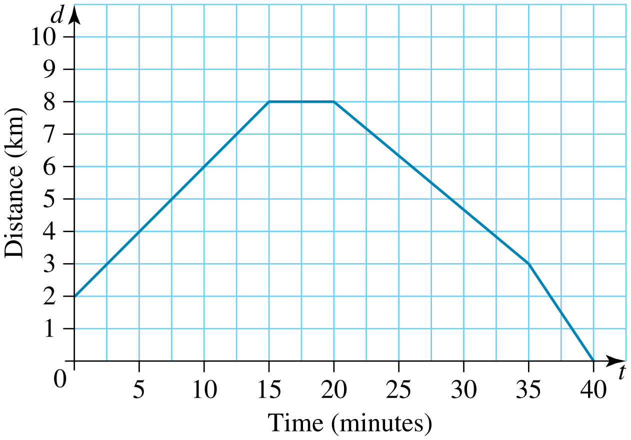
PRACTISE

1 **WE12** The graph shows the distance a skateboarder is from the skate park over a period of 30 minutes.



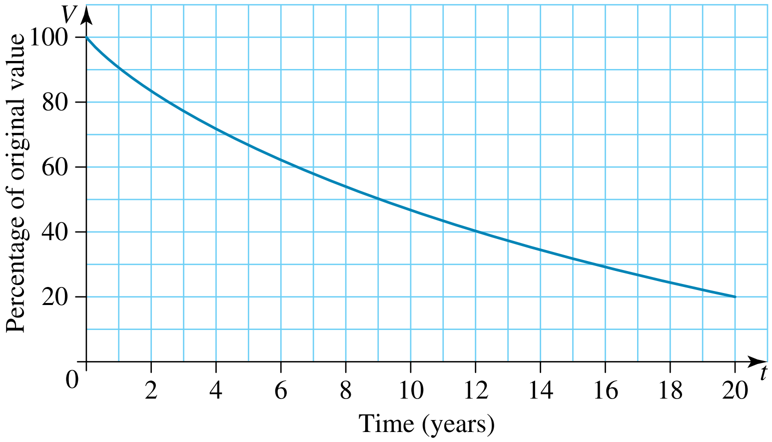
- **a** How far from the skate park was the boarder at the start of the time period?
- **b** At what speed did the boarder travel for the first 10 minutes?
- **c** Describe the motion of the boarder after 15 minutes.
- **d** What was the boarder's furthest distance from the skate park?
- **e** When did the boarder start travelling towards for the skate park?
- **f** How long did it take the boarder to get back to the skate park on the way back?

2 The graph at right shows the distance a runner is from the finish line over a period of 40 minutes.



- **a** How far from the finish line was the runner at the start of the time period?
- **b** At what speed did the runner travel for the first 15 minutes?
- **c** Describe the motion of the runner after 25 minutes.
- **d** What was the runner's furthest distance from the finish line?
- **e** When did the runner start travelling towards the finish line?
- **f** How long did it take the runner to get back to the finish line on the way back?

3 **WE13** The value of a ute originally worth \$42 000 decreases over time. The graph describes the value of the car as a percentage of its starting price.

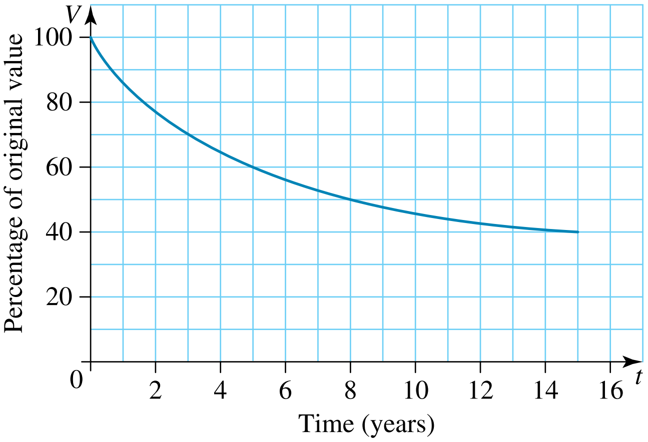


- **a** What is the value of the ute, in percentage terms, after 5 years?
- **b** The 'half-life' is the time taken for the value to decrease by half. How long does it take for the car to lose half its original value?

c How long does it take for the car to fall in value from 50% of its original value to 25% of its original value?

d Estimate, in dollar terms, the value of the car after 20 years.

4 The value of a boat originally worth $37 500 decreases over time. The graph drawn describes the value of the boat as a percentage of its starting price.



a What is the value of the boat, in percentage terms, after 3 years?

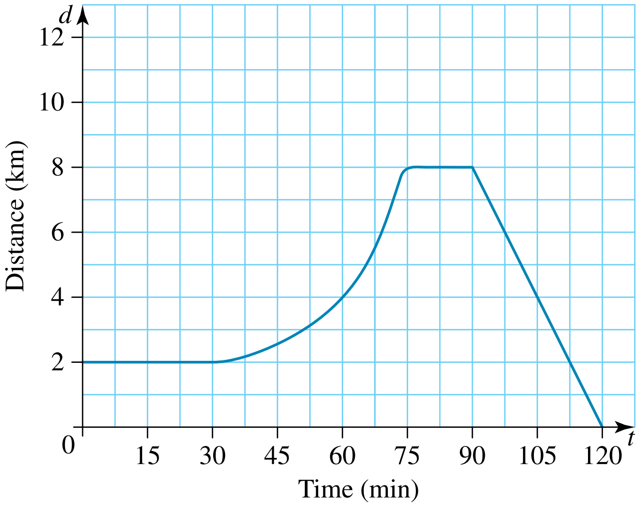
b The 'half-life' is the time taken for the value to decrease by half. How long does it take for the boat to lose half its original value?

c How long does it take for the boat to fall in value from 50% of its original value to 40% of its original value?

d Estimate, in dollar terms, the value of the boat after 15 years.

CONSOLIDATE

5 The graph shown gives the distance of a cyclist from her home over a period of 120 minutes.



a How far from her home was the cyclist at the start of the time period?

b At what speed did the cyclist travel for the first 30 minutes?

c Describe the motion of the cyclist after 7 minutes.

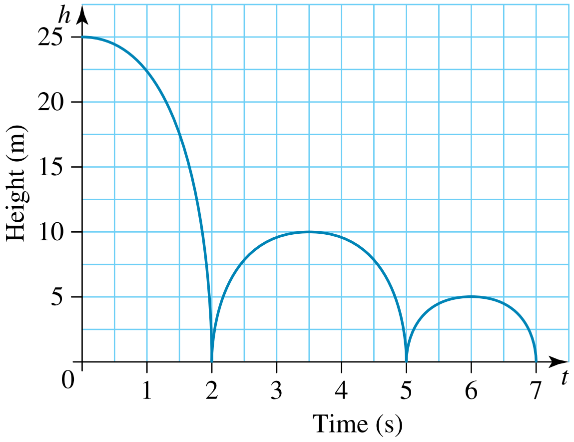
d What was the cyclist's furthest distance from home?

e When did the cyclist begin to travel home?

f How long did the cyclist take to get home?

g What was the average speed of the cyclist on the journey home?

6 In an experiment, a ball is dropped from a height of 20 m. The height of the ball above ground level, as it bounces, is given in the graph.



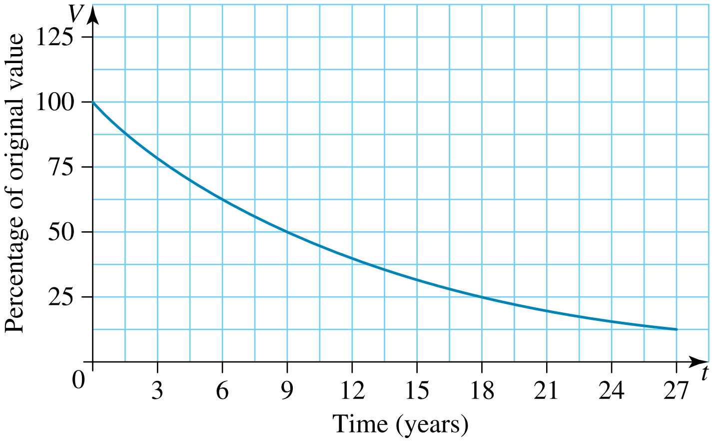
a How long did it take the ball to reach the ground for the first time?

b How long was the ball in the air between the second and third bounce?

c What total distance is travelled by the ball in 7 seconds?

d Calculate the average speed of the ball for the first 2 seconds.

7 The value of a car, originally worth \$32 000, decreases over time. The graph describes the value of the car as a percentage of its original price.



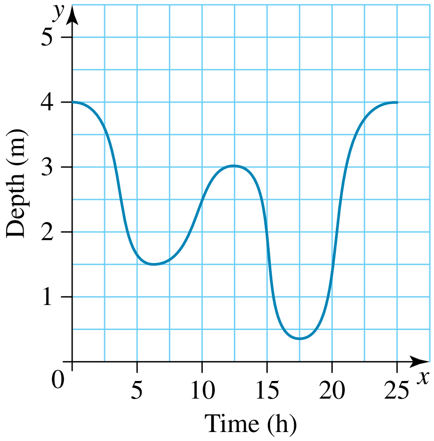
a What is the value of the car, as a percentage of its original price, after 6 years?

b The 'half-life' is the time taken for the value to decrease by half. How long does it take for the car to lose half its original value?

c How long does it take the car to fall in value from 50% of its original value to 25% of its original value?

d Estimate, in dollars, the value of the car after 27 years.

8 The depth of water across a sand bar varies according to the tide. This is shown in the graph.



a Approximately how long after high tide is the next low tide?

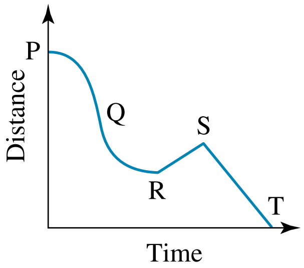
b How deep is the water at the first low tide?

c What is the lowest value in the depth of the water across the bar?

d What is the difference in the heights of the first and second high tide?

e If the first high tide is at 9.40 am Tuesday, at what time approximately will the first high tide occur on Wednesday?

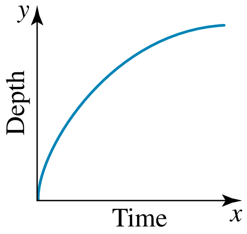
f The water must be at least 1.5 m deep for Judy's boat to travel across the bar. Between what times, after the first high tide, can the boat *not* travel across the bar?



9 The graph shown describes the distance from home of an object as it travels.

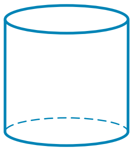
The point at which the object is moving with the greatest speed is:

A P **B** Q **C** R **D** S **E** T

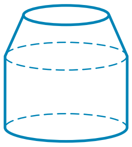


10 Water is poured into a container at a constant rate. The depth of water in the vessel is described in the graph. From the options following, the vessel that best matches the rate at which the vessel fills with water is:

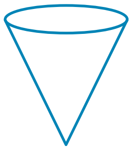
A



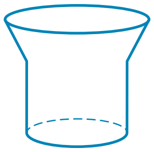
B



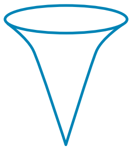
C



D



E



11 Carbon-14 is a radioactive element which breaks down over time. It has a half-life of about 6000 years; that is, every 6000 years, half of the material present breaks down.

a Copy and complete the table for the amount of Carbon-14 (C-14) present.

Time (years)	0	6000	12 000	18 000
Amount of Carbon-14	800 units			

b Plot the graph using these data.

c Use the graph to estimate the time taken for the amount of C-14 to fall to 600 units.

d Use extrapolation to estimate the amount of C-14 present after 24 000 years.

12 The following table displays the profit a company makes from selling x units of toothpaste.

Units of toothpaste sold (×1000)	0	1	2	3	4	5
Profit (× \$1000)	–5	–6	–5	–2	1	9

a Plot the graph using these data.

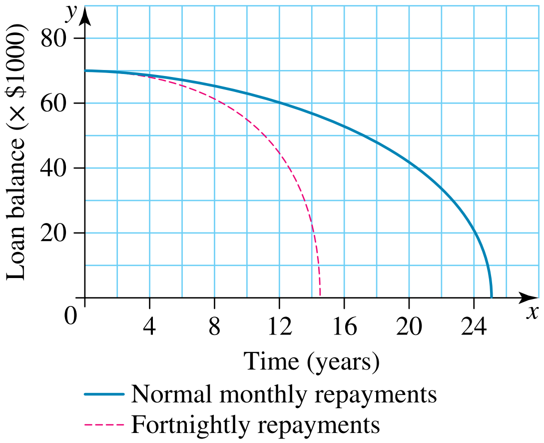
b Use your graph to estimate how many units of toothpaste need to be sold in order to break even.

13 Yasmin borrows \$70 000. She can pay a certain amount each month or half that value each fortnight. The graph shows the progress of the loan under both systems of payment.

a How much time does it take to pay off the loan using monthly repayments?

b How long does it take to pay off the loan using fortnightly repayments?

c Using monthly repayments, after how many years has the balance fallen to 50% of its original value?

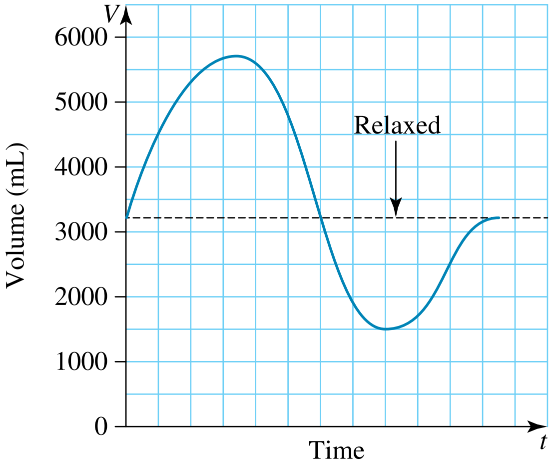


14 The graph shows the volume of air in a person's lungs during a cycle of breathing.

a What is the maximum volume of the lungs?

b When relaxed, how much air is contained in the lungs?

c How much air is exhaled during the breathing cycle?

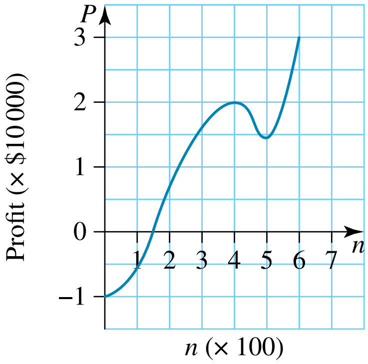


MASTER

15 The graph at right shows the profits made by a manufacturing firm versus the number of units produced.

a What profit is made when 400 units are produced?

b What is the result if 0 units are produced?



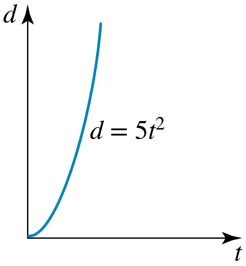
16 **a** Using the graph from question **15**, how many units need to be produced to break even?

b Can you suggest an explanation for the dip in profits in the region $n = 500$?

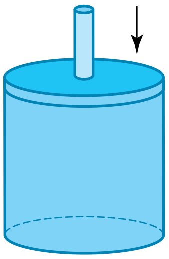
14.6 Constructing non-linear relations and graphs

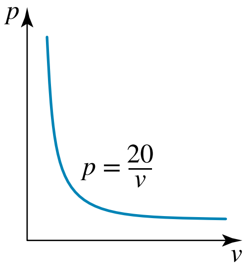
Linear relationships between two variables are the simplest. However, there are many important non-linear relationships where the graphs are not a straight line. Some examples follow.

A rivet is accidentally dropped from a tall pylon under construction. The relationship between the distance the rivet has fallen (d), and the time taken (t) is $d = 5t^2$.

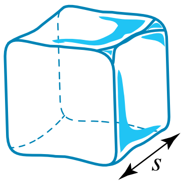


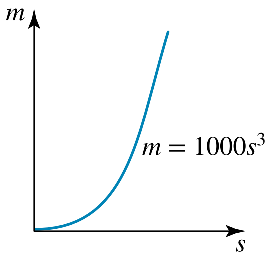
The gas in a piston is compressed. The relationship between the pressure (p) and the volume (v), is $p = 20v^{-1}$ or $p = \dfrac{20}{v}$.





The mass of a cubic block of ice (m) is related to the length of the side of the block (s) by the formula $m = 1000s^3$.





Graphical representation of relations of the form $y = kx^n$

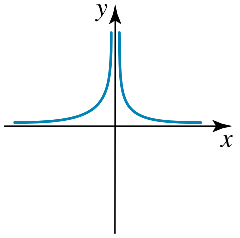
Given a non-linear equation of the form $y = kx^n$, the easiest way to graph it is to use CAS. Alternatively, from the equation a table of values can be produced and this can be used to sketch a graph of the relationship. The general shape of the graphs of $y = kx^n$ for $n = -2, -1, 1, 2, 3$ are given here.

studyon

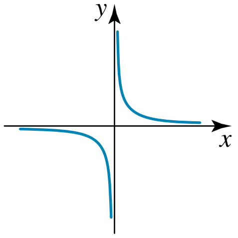
Unit 4
AOS M4
Topic 1
Concept 9

$y = kx^n$
Concept summary
Practice questions

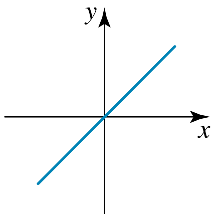
$y = \dfrac{k}{x^2}\ (n = -2)$



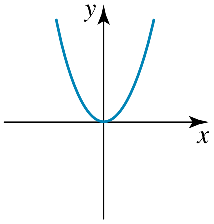
$y = \dfrac{k}{x}\ (n = -1)$



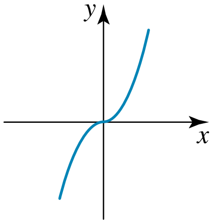
$y = kx\ (n = 1)$



$y = kx^2\ (n = 2)$



$y = kx^3\ (n = 3)$



The value of k in these relationships is known as the **constant of proportionality**.

studyon

Unit 4
AOS M4
Topic 1
Concept 10

Constant of proportionality
Concept summary
Practice questions

WORKED EXAMPLE 14

A research scientist has discovered that her data are related according to the equation $y = 5x^2$. Construct a table of values to draw the graph of the equation $y = 5x^2$ for x between 0 and 10.

THINK

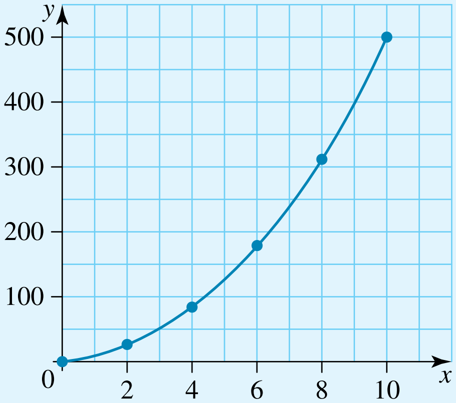
1 Substitute values of x into $y = 5x^2$ to obtain corresponding y-values. For example, if $x = 4$ then

$$\begin{aligned} y &= 5 \times 4^2 \\ &= 5 \times 16 \\ &= 80. \end{aligned}$$

WRITE/DRAW

x	0	2	4	6	8	10
y	0	20	80	180	320	500

2 Plot the points and draw a smooth curve through these points.



Finding a non-linear relationship using linear graphs

If you know the equation, it is reasonably straightforward to produce a table of values. The reverse process (that is, discovering a formula for a non-linear relationship from a table of values) is more difficult.

If the relationship between x and y is of the form $y = kx^n$, then plotting y against x^n will produce a straight line from the origin. The gradient of this line will equal the value of k, the constant of proportionality. The following algorithm can be used for finding the rule of the relationship, if the value of n is known.

1. Plot y against x^n. The result must be a straight line, coming from the origin.
2. Select any two points on the line to calculate the value of the gradient.
3. Since the gradient represents the value of k, substitute it into $y = kx^n$ to give the rule for the relationship.

Note that if the value of n is not known, plot y against x first. The shape of the graph will indicate the possible value(s) of n. Test this value by plotting y against x^n. If a straight line is produced, proceed as above; otherwise try a different value of n.

WORKED EXAMPLE 15

By graphing y against x^2 find the equation for the relationship between x and y if the equation is of the form $y = kx^2$.

x	0	2	4	6	8	10
y	0	12	48	108	192	300

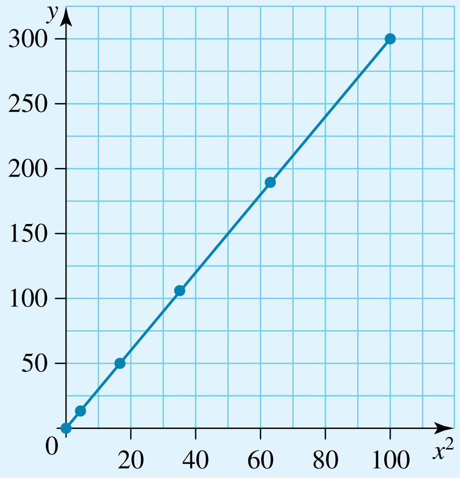
THINK

1 Since it is known that the relationship is of the form $y = kx^2$, draw up a table of values for x^2 and y.

WRITE/DRAW

x^2	0	4	16	36	64	100
y	0	12	48	108	192	300

2 Plot the values of x^2 on the horizontal axis and y on the vertical axis and join with a smooth line. (A straight line from the origin confirms that the relationship is of the form $y = kx^2$.)



3 Choose two points on the line, say (0, 0) and (100, 300), and calculate the gradient.

$$\text{Gradient} = \frac{300 - 0}{100 - 0} = \frac{300}{100} = 3$$

4 The gradient represents the value of k in $y = kx^2$. Replace k with 3 to state the equation for the relationship.

$y = kx^2$

$k = 3$

Therefore, $y = 3x^2$.

EXERCISE 14.6 Constructing non-linear relations and graphs

PRACTISE

1 **WE14** A statistician found that his data are related according to the equation $y = 3x^2$. Construct a table of values and draw the graph of the equation $y = 3x^2$ for values of x between 0 and 10.

2 A researcher found that her data are related according to the equation $y = \frac{2}{x}$. Construct a table of values and draw the graph of the equation $y = \frac{2}{x}$ for values of x between 2 and 8.

3 **WE15** By graphing y against x^2 find the equation for the relationship between x and y if the equation is of the form $y = kx^2$.

x	0	2	4	6	8	10
y	0	8	32	72	128	200

4 By graphing y against x^2, find the equation for the relationship between x and y if the equation is of the form $y = kx^2$.

x	0	2	4	6	8	10
y	0	10	40	90	160	250

CONSOLIDATE

5 Construct a table of values to draw the graphs of the following equations. (Use values of x from −5 to 5.)

a $y = x^2$

b $y = 2x^2$

c $y = 2x^3$

d $y = 0.5x^3$

e $y = \frac{0.5}{x}$

f $y = \frac{1}{x^2}$

Use CAS to check your answers.

6 Construct a table of values to draw the graphs of the following equations. (Use values of x from −5 to 5.)

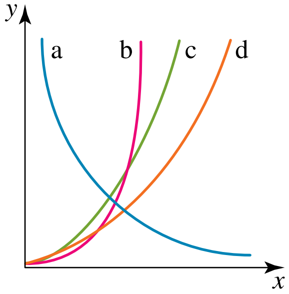
a $y = 0.5x^2$ **b** $y = x^3$ **c** $y = \frac{1}{x}$

d $y = \frac{2}{x}$ **e** $y = \frac{2}{x^2}$ **f** $y = \frac{0.5}{x^2}$

Use CAS to check your answers.

7 Write the letters **a** to **d** in your book. Against each letter that identifies a graph, write the equation which best matches that graph.

$y = x^2$
$y = 2x^2$
$y = x^3$
$y = \frac{3}{x}$



8 By graphing y against x^3, find the equation of the relationship between x and y, if the equation is of the form $y = kx^3$.

x	0	2	4	6	8
y	0	12	96	324	768

9 The Safety Council conducted research on the braking distance of vehicles and its relationship to the speed of the vehicle. The following data were obtained.

Speed (s) (km/h)	30	45	60	80	100
Braking distance (d) (metres)	7.5	16.9	30	53.3	83.3

a Plot d versus s^2.
b What is the equation relating d and s?

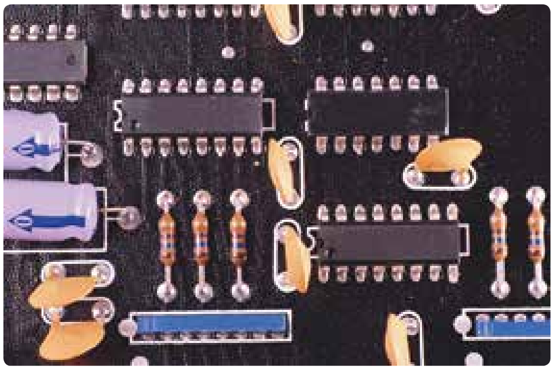
10

x	0.1	0.4	1	2	5
y	200	50	20	10	4

a Plot these values on a set of axes.
b Plot y versus $\frac{1}{x}$ and draw a straight line through the points.
c What is the slope of the line?
d Deduce the relationship between y and x.

11 In a physics experiment a student measured the current (I) flowing through a resistor for different values of the resistance (R) and obtained the following data.

Resistance (R) (ohms)	100	200	1000	1500
Current (I) (milliamps)	300	150	30	20



a Plot values of I versus $\frac{1}{R}$.
b Deduce a relationship between I and R.

12 Examine the following table of values.

x	0	2	4	6	8
y	0	1	8	27	64

- **a** Plot these values on a set of axes.
- **b** Plot y versus x^3 and draw a straight line through the points.
- **c** What is the slope of the line?
- **d** Deduce the relationship between y and x.

13 It is suspected that for male adults, mass (m) is related to height (h) by a formula like $m = kh^3$. Use the data in the table to find a relationship between m and h.

Mass (kg)	45	55	71	84
Height (cm)	150	160	175	185

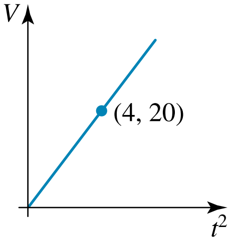
14 The intensity of light drops off as you move away from the source. The relationship is of the form $I = \dfrac{k}{r^2}$. If the intensity (I) is 50 when $r = 20$, find:

- **a** the value of k
- **b** the distance at which the intensity of light falls below 35.

MASTER

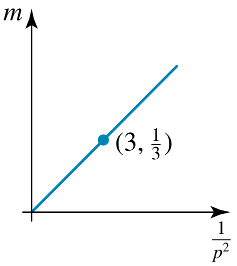
15 Which of the following equations describes the relationship shown on the graph at right?

- **A** $V = 10t^2$
- **B** $V = 10t$
- **C** $V = 5t^2$
- **D** $V = 5t$
- **E** $V^2 = 5t$

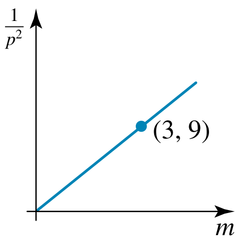


16 Which of the graphs shows the relationship $m = \dfrac{3}{p^2}$?

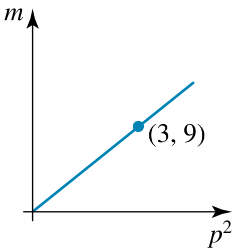
A



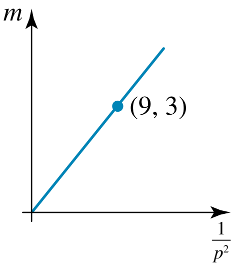
B



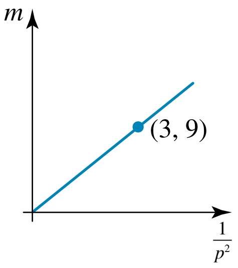
C



D



E



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14.7 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

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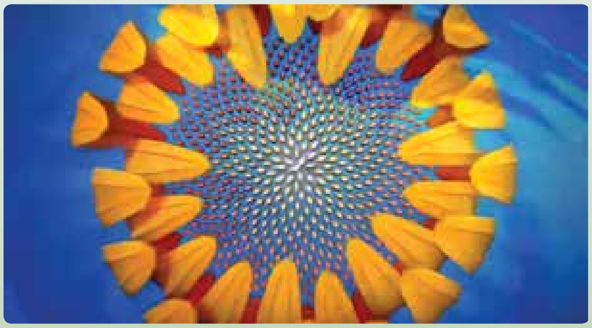
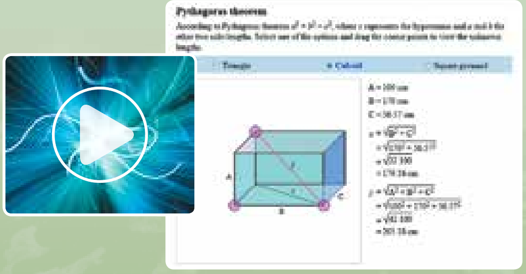
A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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14 Answers

EXERCISE 14.2

1 a 13 **b** 6

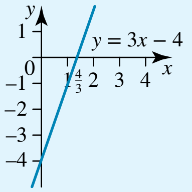
c No **d** $m = 2, c = 5$

2 a $y = 0$ **b** $x = 4$ **c** Yes

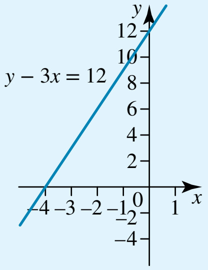
3 4

4 −1

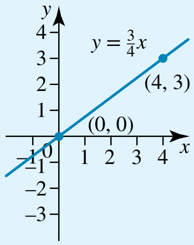
5 a



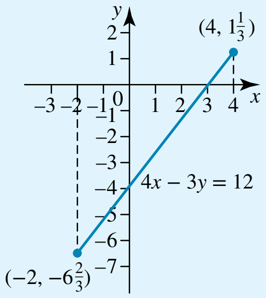
b



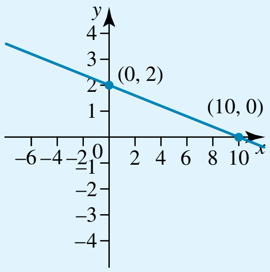
c



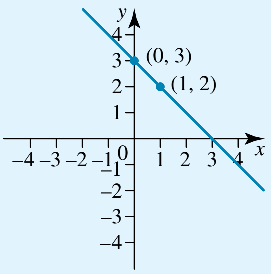
d



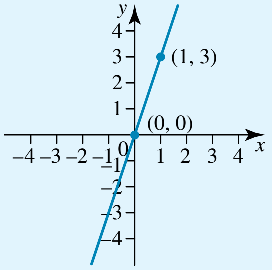
6 a



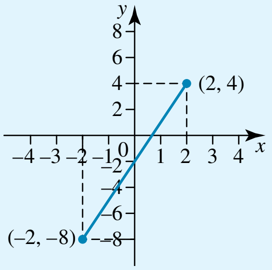
b



c



d



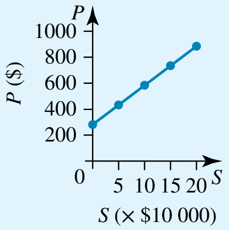
7 a $635

b

Weekly sales	Weekly pay
5 000	$425
10 000	$575
15 000	$725
20 000	$875

c $P = 275 + 0.03S$

d



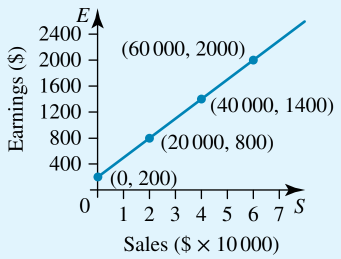
e $m = 0.03$; y-intercept = 275; retainer amount

8 a

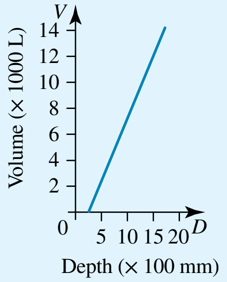
Sales	0	$20 000	$40 000	$60 000
Earnings	$200	$800	$1400	$2000

b $E = 200 + 0.03S$

c



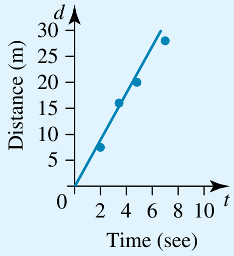
9 a



b 1550 mm

c 3500 L

10 a



b $d = 36$ m

c $d = 24$ m

11 a 0 **b** 4

c Yes **d** $m = -2, c = 4$

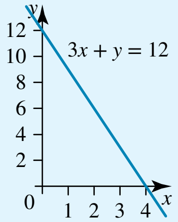
12 a 3 **b** 0

c No **d** $m = \frac{-3}{2}, c = 6$

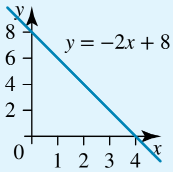
13 a 3 **b** 6 **c** No **d** −0.5

14 a 0.5 **b** 2 **c** 1.5 **d** 2

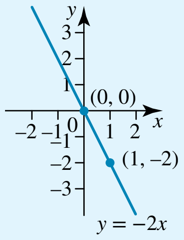
15 a



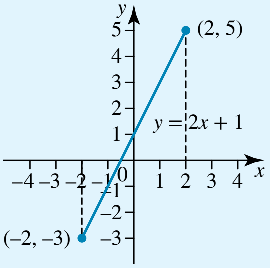
b



c



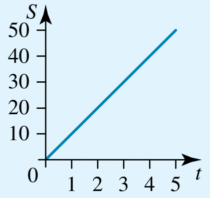
d



16 E

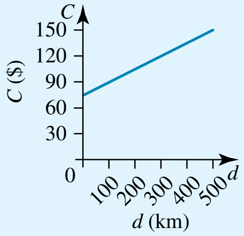
17 a 29.4 m/s **b** 8 s **c** $S = 9.8t$

d



18 a $126.75 **b** 450 km

c



19 B

20 a $470 **b** $W = 350 + \frac{3S}{25}$

21 a 4.73

b $S = 108 + (n - 20) \times 4.73$

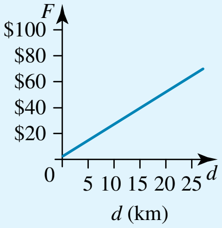
c 4.73

22 a $40.30

b 7.2 km

c $F = 2.8 + 2.5d$

d



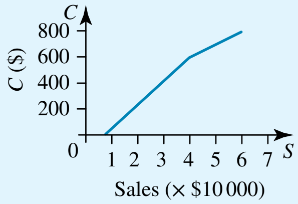
e 2.5

EXERCISE 14.3

1 a

Sales ($)	Commission ($)
20 000	300
30 000	450
40 000	600
50 000	700
60 000	800

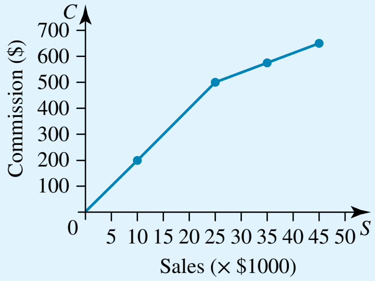
b



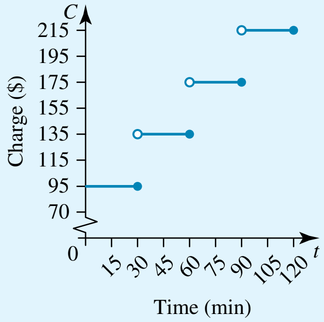
c Different commission rates mean different gradients.

2 a $200 **b** $500 **c** $580 **d** $660

e



3 a



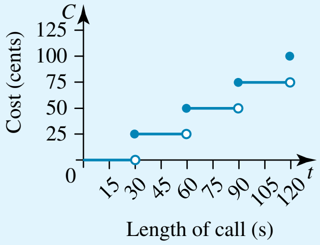
b

Time	Charge
12 min	$55 + $40 = $95
23 min	$55 + $40 = $95
44 min	$55 + 2 × $40 = $135
56 min	$55 + 2 × $40 = $135
73 min	$55 + 3 × $40 = $175
87 min	$55 + 3 × $40 = $175

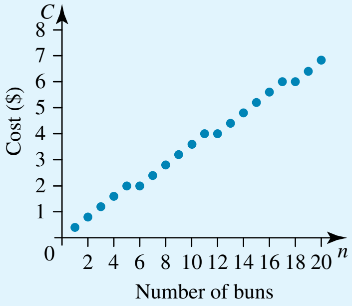
4 a

Length of call (seconds)	cost (cents)
15	0
30	25
45	25
60	50
75	50
90	75
105	75
120	100

b



5 a



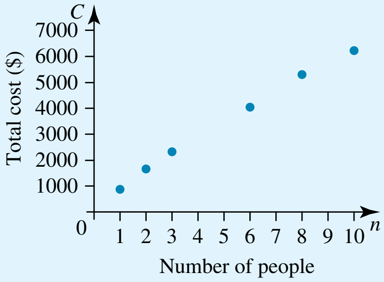
b

Number of buns	Cost
2	2 × 40c = $0.80
4	4 × 40c = $1.60
6	$2.00
8	$2.00 + 2 × 40c = $2.80
10	$2.00 + 4 × 40c = $3.60

6 a

Number of people	Total cost
1	$900
2	$1650
3	$2250
6	$4050
8	$5250
10	$6450

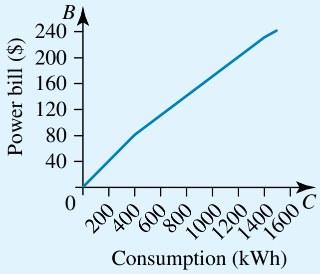
b



7

Consumption (kWh)	Power bill
200	$40
400	$80
600	$110
1000	$170
1500	$240

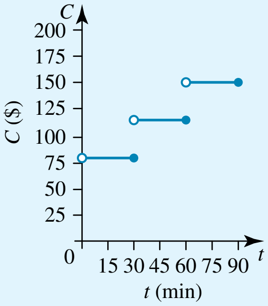
8



9 a

Time of service (minutes)	Cost
20	$80
30	$80
45	$115
60	$115
80	$150
90	$150

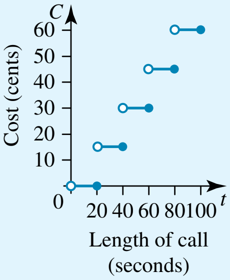
b



10 a

Length of call (seconds)	Cost (cents)
10	0
20	15
30	15
40	30
50	30
60	45
70	45
80	60
90	60

b



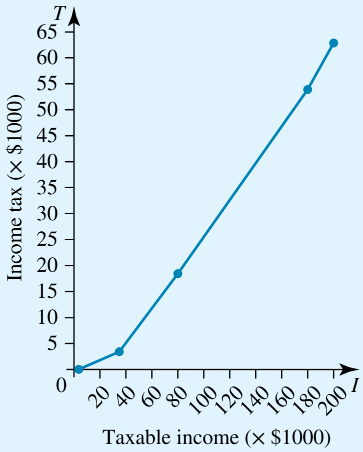
11 B

12 B

13 a $0 **b** $150 **c** $2100 **d** $4350

e $5550 **f** $11 550 **g** $24 950 **h** $63 550

14 a

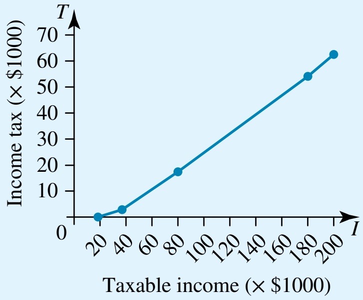


b i $2700 **ii** $23 100

15 a $0 **b** $0 **c** $342 **d** $3192

e $4547 **f** $11 047 **g** $24 947 **h** $63 547

16 a



b i Approx $1100

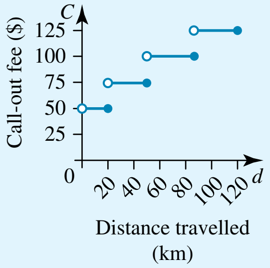
ii Approx $23 000

c Answers will vary.

17 Answers will vary.

18 a i $75 **ii** 20 km

b



EXERCISE 14.4

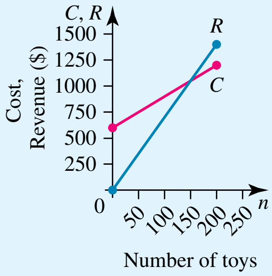
1 (1, 3) **2** (–7, –4)

3 (1, –1) **4** (–2, 1)

5 a

Number of toys	Cost ($)	Revenue ($)
50	750	350
100	900	700
150	1050	1050
200	1200	1400

b



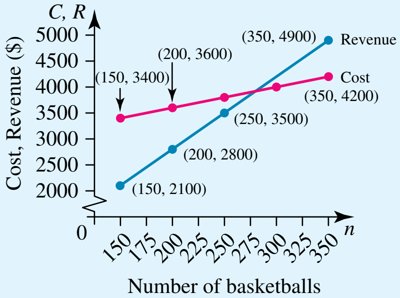
c 150

d (150, 1050). If 150 toys are produced, the cost and revenue is equal to $1050. No profit nor loss is made.

6 a

Number of basketballs	Cost ($)	Revenue ($)
150	$3400	$2100
200	$3600	$2800
250	$3800	$3500
300	$4000	$4200
350	$4200	$4900

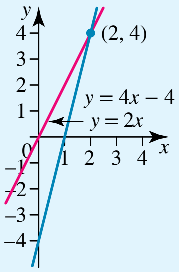
b



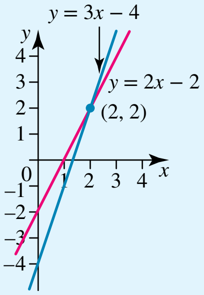
c Number of basketballs (n) = 280 required to break even

7 a (2, 4) **b** (2, 2) **c** (1, 2)

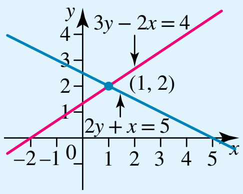
8 a



b



c



9 (3, –1)

10 (–2, 2)

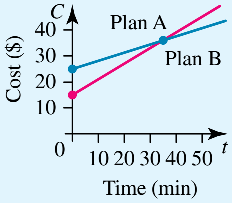
11 (0, 0)

12 (3, 5)

13 a

Call time (minutes)	Cost Plan A	Cost Plan B
10	$21	$28
20	$27	$31
30	$33	$34
40	$39	$37

b

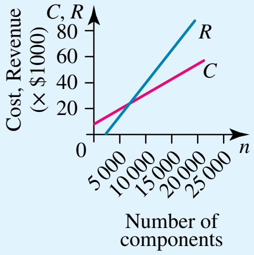


c Approx. 33

14 a

Number of components	Cost ($)	Revenue ($)
5 000	18 500	14 500
10 000	31 000	37 000
15 000	43 500	59 500
20 000	56 000	82 000

b

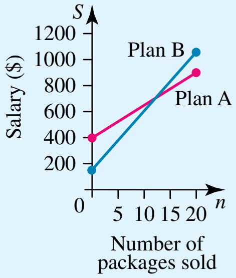


c 7000

15 a

Number of packages sold	Salary Plan A	Salary Plan B
5	$525	$375
10	$650	$600
15	$775	$825
20	$900	$1050

b

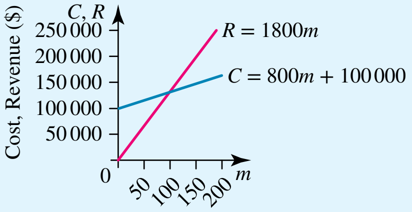


c 13

16 A

17 a \$108 000 **b** 30

18 a



b 100

c $P = 1000m - 100\,000$

d \$75 000

EXERCISE 14.5

1 a 1.0 km

b 0.15 km/min

c The boarder is not moving.

d 2.5 km **e** 20 mins

f 10 mins

2 a 2 km **b** 0.4 km/min

c The runner is moving towards the finish line.

d 8 km

e 20 mins

f 20 mins

3 a ≈ 70% **b** 9 years

c 9 years **d** \$8400

4 a ≈ 70% **b** 8 years

c 7 years **d** \$15 000

5 a 2 km **b** 0 km/h

c Stationary (not moving)

d 8 km **e** 90 min

f 30 min **g** 16 km/h

6 a 2 s **b** 2 s **c** 50 m **d** 10 m/s

7 a ≈ 62%

b 9 years **c** 9 years **d** \$4000

8 a About 6 hours **b** 1.5 m

c 0.5 m **d** 1 m

e 10.40 am

f Between about 15 hours and 21 hours after the first high tide

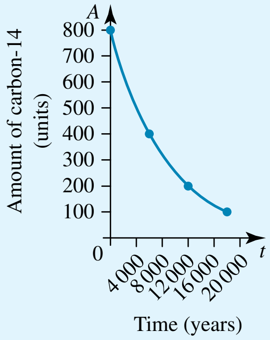
9 B

10 C

11 a

Time (number of years)	Amount of Carbon-14 (units)
0	800
6 000	400
12 000	200
18 000	100

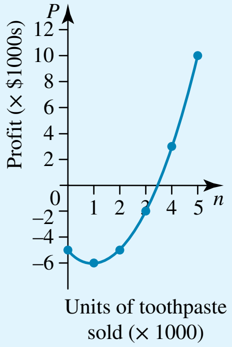
b



c 2500 years

d 50 units

12 a



b 3450

13 a 25 years **b** 14.5 years

c Approximately 21 years

14 a 5750 mL **b** 3100 mL **c** 4250 mL

15 a \$20 000

b Loss of \$10 000

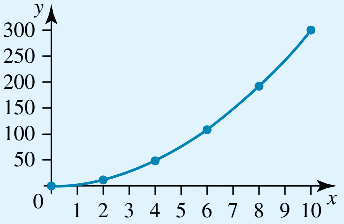
16 a 150

b May need to employ extra staff or install new plant.

EXERCISE 14.6

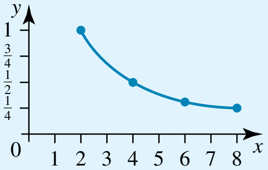
1 $y = 3x^2$

x	0	2	4	6	8	10
y	0	12	48	108	192	300



2

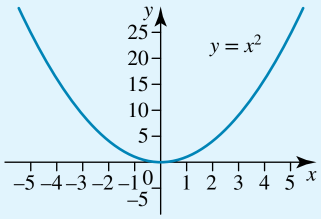
x	2	4	6	8
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



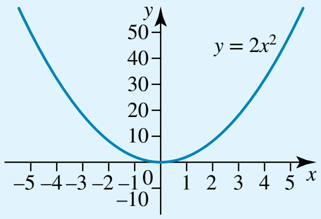
3 $y = 2x^2$

4 $y = 2.5x^2$

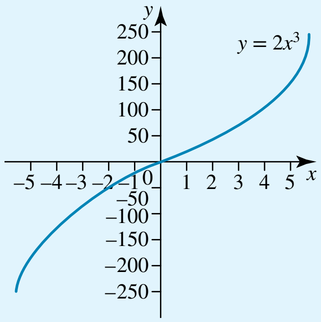
5 a



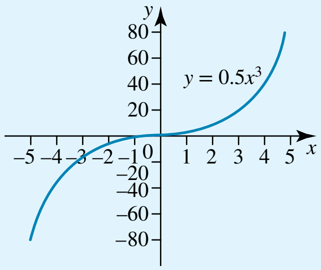
b



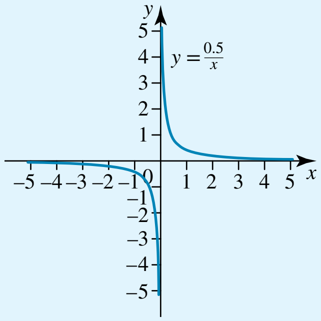
c



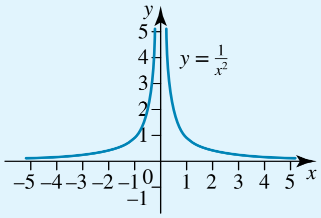
d



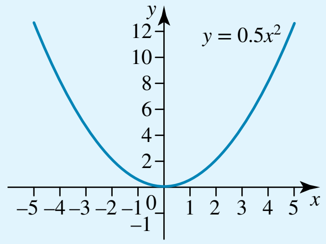
e



f



6 a



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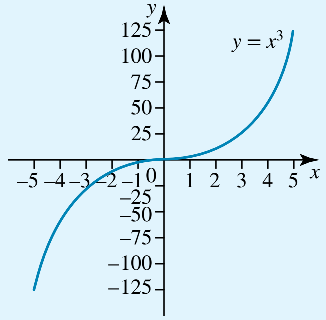

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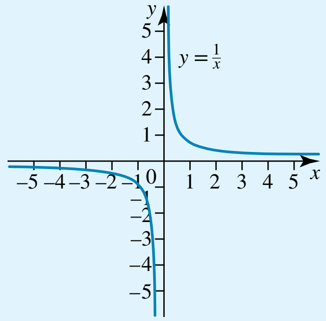

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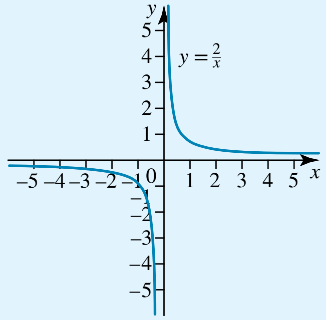

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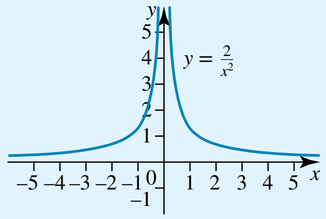
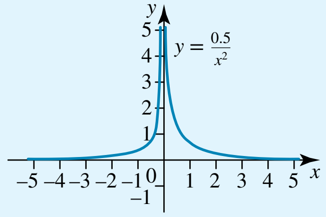


f



7 **a** $y = \frac{3}{x}$

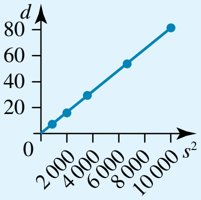
b $y = x^3$

c $y = 2x^2$

d $y = x^2$

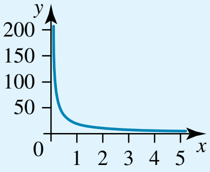
8 $y = 1.5x^3$

9 **a**

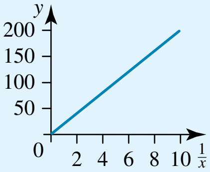


b $d = 0.0083s^2$

10 **a**



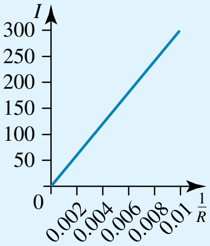
b



c 20

d $y = \frac{20}{x}$

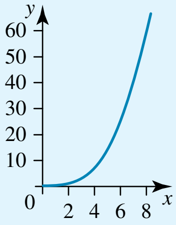
11 **a**



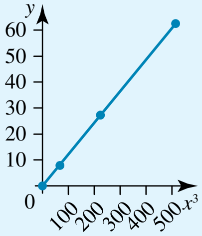
R	100	200	1000	1500
$\frac{1}{R}$	0.01	0.005	0.001	0.000 67
I	300	150	30	20

b $I = \frac{30\,000}{R}$

12 **a**



b



x	0	2	4	6	8
x^3	0	8	64	216	512
y	0	1	8	27	64

c 0.125

d $y = 0.125x^3$

13 $m = \frac{h^3}{75\,000}$

14 **a** $k = 20\,000$

b 23.9

15 C

16 E

15

Linear inequalities and linear programming

15.1 Kick off with CAS

Shading regions with CAS

When graphing a linear equation, the Cartesian plane is split into two regions (one on either side of the line). Linear inequalities replace the equals sign in an equation with an inequality sign ($\leq$, $\geq$, $<$ or $>$). This means that one of the regions the Cartesian plane is split into satisfies the inequality, whereas the other region doesn't satisfy the inequality.

1 Using CAS, sketch the graph of $y < 12 - 3x$.

2 Using any appropriate method, determine which region of the graph (shaded or unshaded) satisfies the inequality, i.e. on which side of the line do the points make the inequality true?

3 Using CAS, sketch the graphs of $y < 12 - 3x$ and $y < \dfrac{12 - x}{3}$ on the same set of axes.

4 Determine the point of intersection of the two lines sketched in question 3.

5 Determine which region of the graphs plotted in question 3 is the required region; that is, which region satisfies both of the inequalities.

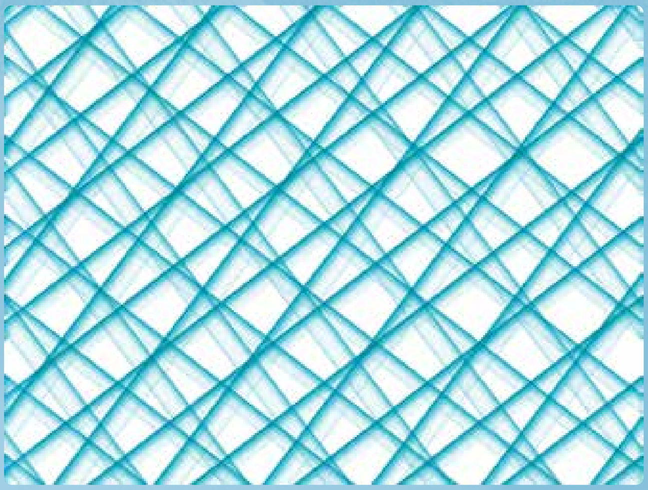
6 Use CAS to plot the following system of linear inequalities:

$y < 10 - x$

$y < 4x$

$y \geq 0$

7 Use CAS to find all the corner points of the required region for the system of linear inequalities plotted in question 6.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

15.2 Linear inequalities

To begin our work in linear programming, we first have to develop some skills with **linear inequalities**. Recall that linear *equations* come in two forms:

$$y = mx + c$$

and

$$ax + by = c.$$

For example, $y = 3x - 4$ is a linear equation, as is $3x + 4y = -9$.

Linear *inequalities* also have two forms:

$$y < mx + c$$

and

$$ax + by < c.$$

For example, $y < 3x - 4$ is a linear inequality, as is $3x + 4y \geq -9$. It is the second form of equation that we will be chiefly working with in this topic. The only difference between equations and inequalities is that the equals sign (=) is replaced with an 'inequality' sign. The possible inequality signs are:

≥ greater than or equal to
> greater than
< less than
≤ less than or equal to.

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Topic 2
Concept 1

Variables, constraints and inequalities
Concept summary
Practice questions

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Linear inequalities in two variables
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Graphing a linear inequality

The steps for graphing a linear inequality are the same as those for linear equations, with one extra step. The graph of an inequality is a **region** or *half-plane* while the graph of a linear equation is a straight line.

Step 1. Temporarily treat the inequality as an equation to find the boundary of the region.

Step 2. Find two points using the equation to assist in graphing the boundary. Pick any value of x and find the corresponding value of y. Pick another value of x and find the corresponding value of y. This value of x should be *reasonably far away* from the first one chosen.

Step 3. Plot the points from step 2 and join them with a straight line. Extend the straight line to the extent of your graph paper. A dashed line should be used for a $<$ or $>$ inequality and a solid line for a $\leq$ or $\geq$ inequality.

Step 4. Shade in a region, either above the graph or below the graph, depending upon the inequality sign given. The convention is to shade the region *not* required. Any point inside the shaded region is *not a solution* to the inequation. Thus, the unshaded (or clear) area is called the *solution region* or the *feasible region*. *Note:* Some CAS calculators *shade* the solution region.

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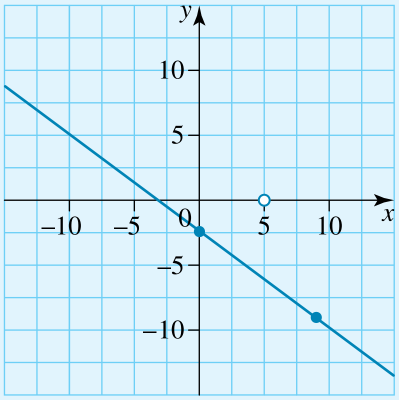
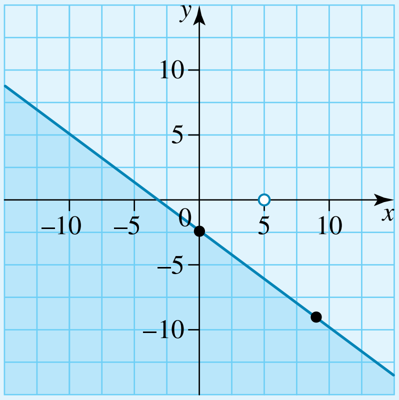
Concept 2

Graphing inequalities
Concept summary
Practice questions

WORKED EXAMPLE 1

Graph the linear inequality $3x + 4y \geq -9$.

THINK	WRITE/DRAW
1 Find the boundary of the region.	
(a) Write the inequality as an equation.	The boundary is $3x + 4y = -9$.
(b) Pick any value for x and find the corresponding y-value.	Let $x = 0$ $3(0) + 4y = -9$

(c) Substitute the *x*-value into the equation and solve for *y*.	$4y = -9$ $y = -\frac{9}{4}$ $= -2.25.$ First point is (0, –2.25)
(d) Pick another value for *x* and find the corresponding *y*-value.	Let $x = 9$ $3(9) + 4y = -9$
(e) Substitute the *x*-value into the equation and solve for *y*.	$27 + 4y = -9$ $4y = -36$ $y = -9.$ Second point is (9, −9)
(f) Plot the two points on a graph and join them with a straight line. A solid line is used since the inequality contains a ≥ sign. Observe how the line is extended the full extent of the graph paper. This line divides the coordinate plane into two 'half-planes'.	
2 Find the required region.	
(a) Select a test point above the line to work out the region or half-plane required.	Test point: (5, 0) $3x + 4y \geq -9$ $3(5) + 4(0) \geq -9?$
(b) Substitute the *x*- and *y*-values into the inequality. If the inequality is satisfied (or true) then the half-plane containing the test point is required. If not, the other half-plane is required.	$15 \geq -9$ is true, so the region required is above the line.
3 Shade in the region not required (that is, the half-plane below the line) and add the legend.	 ☐ Region required

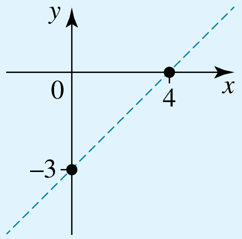
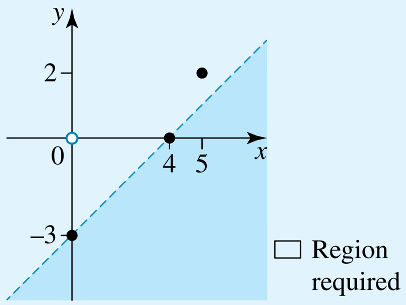
Note that in Worked example 1, because the inequality was 'greater than or equal to', the solution region also included the straight line itself (shown with a solid line).

If the inequality had been just 'greater than' then the region would not have included the straight line points (a dashed line would have been used).

However, most problems in this topic will be of the '... equal to' type.

A faster alternative to steps 1(b) to 1(e) in Worked example 1 is to find the *x*- and *y*-intercepts to the boundary line. First substitute $x = 0$ and find the corresponding value of y and then substitute $y = 0$ and find the corresponding value of x. Plot these two points to get the straight line. This is called the x- and y-*intercept method*.

WORKED EXAMPLE 2 **Graph the inequality $3x - 4y < 12$ using the x- and y-intercept method and show that (5, 2) is a solution.**

THINK	WRITE/DRAW
1 Find the boundary.	
(a) Write the equation of the boundary.	The boundary of the region is $3x - 4y = 12$.
(b) Find the y-intercept by substituting $x = 0$ into the equation and finding the corresponding value of y.	When $x = 0$, $3(0) - 4y = 12$ $-4y = 12$ $y = -3$ Point is (0, −3)
(c) Find the x-intercept by substituting $y = 0$ into the equation and finding the corresponding value of x.	When $y = 0$, $3x - 4(0) = 12$ $3x = 12$ $x = 4$ Point is (4, 0)
(d) Plot the two points on a graph and join them with a straight, dashed line.	
2 Find the required region. (a) Select a test point, say (0, 0), to work out the half-plane required.	Test point: (0, 0) $3x - 4y < 12$ $3(0) + 4(0) < 12?$
(b) Substitute the x- and y-values into the inequality. If the inequality is satisfied (or true) then the half-plane containing the test point is required. If not, the other half-plane is required.	$0 < 12$ is true, so the region required is above the line.
3 Shade in the region not required, that is, the half-plane below the line.	
4 Plot the given point (5, 2) to see whether it is within the required region.	The point (5, 2) lies inside the unshaded region. Therefore, it is a solution to the inequality.
Alternatively: Substitute the co-ordinates of the given point (5, 2) into the inequality to see if it will make a true statement.	Test point (5, 2): $3(5) - 4(2) < 12?$ $15 - 8 < 12?$ $7 < 12$ is true, therefore (5, 2) is a solution to the inequality.

EXERCISE 15.2 Linear inequalities

PRACTISE

1 **WE1** Graph the solution to the linear inequality $4x + 7y \leq 28$.

2 Graph the solution to the linear inequality $50x - 65y \geq 650$.

3 **WE2** Graph the inequality $7x - 12y \leq 84$ and show that $(4, -3)$ is a solution.

4 Graph the inequality $-5x - 3y \geq 20$ and show that $(-10, -5)$ is a solution.

CONSOLIDATE

5 Graph the solution to the linear inequality $4x - 5y \geq 10$.

6 Graph the solutions to the following linear inequalities.

a $6x + 2y > 12$ **b** $4x + y \leq 5$

c $3x + 2y \geq 0$ **d** $x - y > 20$

7 Graph the inequality $4x + 5y \leq 20$ and show that $(1, 1)$ is a solution.

8 Graph the inequality $6x - 4y + 20 > 0$ and show that $(10, 5)$ is a solution.

9 Graph the inequality $2x + 3y \geq 6$ and show that $(7, 0)$ is a solution.

10 Graph the inequality $1.5x - 3.5y < 2.7$.

11 Which one or more of the points below is/are solutions to the inequality in question **10**?

A $(5, -5)$ **B** $(5, 5)$ **C** $(10, 2)$ **D** $(-1, -2)$ **E** $(10, 6)$

12 Graph the inequality $20x + 30y \geq 300$.

13 The point which is *not* a solution to the inequality in question **12** is:

A $(10, 5)$ **B** $(5, 8)$ **C** $(5, 5)$ **D** $(1, 10)$ **E** $(2, 10)$

14 Graph the solution to the linear inequality $4x - 5y \geq 40$.

MASTER

15 Graph the linear inequality $2x + 7y \leq 0$ and show that $(-5, 1)$ is inside the solution region.

16 Graph the solution to the linear inequality $-1.8x + 0.4y \leq 14.4$.

15.3 Simultaneous linear inequalities

As with linear equations, it is possible to solve groups of linear inequalities *simultaneously*; that is, to find the solution that satisfies more than one inequality *at the same time*. You may recall that there are several methods for solving simultaneous linear equations; some of these techniques will be useful for inequations. Note that simultaneous linear inequalities are also called *systems of linear inequalities*.

Graphical solution of simultaneous linear inequalities

To solve for two linear inequalities, follow these steps.

Step 1. Graph the solution for the first linear inequality.

Step 2. Graph the solution for the second linear inequality.

Step 3. The solution to both inequalities is the intersection of the solution regions from Steps 1 and 2.

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Graphing a linear inequality (simultaneous inequalities)
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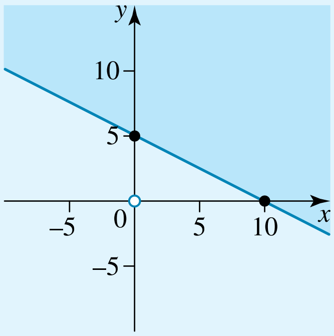
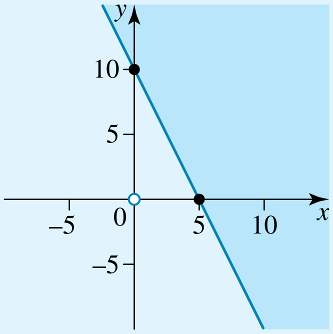
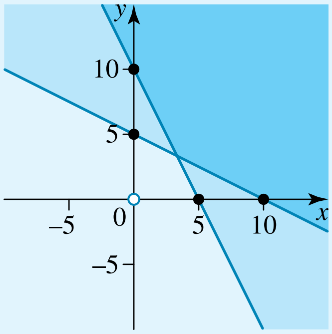
WORKED EXAMPLE 3 Find the solution to the simultaneous linear inequalities:

$$\begin{aligned} x + 2y &\le 10 \qquad [1] \\ 2x + y &\le 10 \qquad [2] \end{aligned}$$

THINK	WRITE/DRAW
1 Write the boundary equation for region [1].	The boundary for region [1] is $x + 2y = 10$.
2 Find the x- and y-intercepts of the boundary line.	When $x = 0$, $0 + 2y = 10$ $y = 5$ (0, 5) When $y = 0$, $x + 2(0) = 10$ $x = 10$ (10, 0)
3 Plot the straight line and determine the region of the solution by using a test point.	Test point: (0, 0) $x + 2y \le 10$ $0 + 2(0) \le 10$ is true, so (0, 0) is inside the solution region.  ☐ Region required
4 Repeat steps 1–3 for region [2].	The boundary for region [2] is $2x + y = 10$. When $x = 0$, $2(0) + y = 10$ $y = 10$ (0, 10) When $y = 0$, $2x + 0 = 10$ $2x = 10$ $x = 5$ (5, 0) Test point: (0, 0) $2x + y \le 10$ $2(0) + 0 \le 10$ is true, so (0, 0) is inside the region.  ☐ Region required
5 Determine the overlapping areas of the two regions. This can be done graphically, by showing the two regions on the one set of axes. The clear area is the overlap of the two solution regions and represents the solution to the simultaneous linear inequalities.	 ☐ Region required

Note that in Worked example 3, there were three *non-solution regions*:

1. the region above and to the left of the first line,
2. the region above and to the right of the second line, and
3. the region where the two exclusion zones overlap.

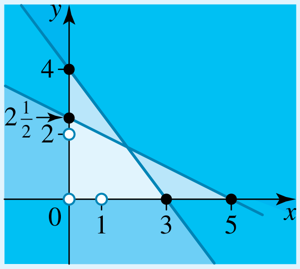
Notes

1. The convention of shading the region that is *not* required makes it easier to identify overlapping exclusion zones; however, CAS may shade the *required* region. Use test points to confirm which style is being used.
2. In this topic, where more exclusion zones overlap, the regions will be shown in increasingly darker shades of blue.
3. In subsequent sections, we examine systems of three or more linear inequalities, but the methods are identical.
4. Examination questions usually ask for the *required* region to be shaded. Always read the question carefully and include a legend with your graphical solutions.

WORKED EXAMPLE 4

Find the solution to the system of linear inequalities at right.

$x + 2y \leq 5$ [1]
$4x + 3y \leq 12$ [2]
$x \geq 0$ [3]
$y \geq 0$ [4]

THINK	**WRITE/DRAW**
1 Write the boundary equations for each region and find the intercepts or relevant information to sketch each line.	The boundary for region [1] is $x + 2y = 5$. When $x = 0, y = 2.5$ (0, 2.5) When $y = 0, x = 5$ (5, 0) The boundary for region [2] is $4x + 3y = 12$. When $x = 0, y = 4$ (0, 4) When $y = 0, x = 3$ (3, 0) The boundary for region [3] is $x = 0$. This is a vertical line through $x = 0$. The boundary for region [4] is $y = 0$. This is a horizontal line through $y = 0$.
2 Use test points to identify the required region for each inequality. Note that (0, 0) cannot be used for inequalities [3] and [4] as this point lies on the boundary for each.	For $x + 2y \leq 5$: Test point (0, 0) $(0) + 2(0) \leq 5$ is true, so (0, 0) is inside the region. For $4x + 3y \leq 12$: Test point (0, 0) $4(0) + 3(0) \leq 12$ is true, so (0, 0) is inside the region. For $x \geq 0$: Test point (1, 0) $1 \geq 0$, so (1, 0) is inside the region. For $y \geq 0$: Test point (0, 2) $2 \geq 0$, so (0, 2) is inside the region.
3 Show each region on the one set of axes. Remember to shade the region not required in each case. The clear area is the solution for this system of inequalities.	 ☐ Region required

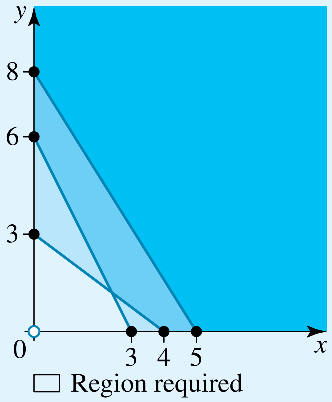
Observe that the solution region is a polygon (in this case a quadrilateral). This is a feature of all of the subsequent problems we will study. Generally speaking, if there are n linear inequalities, we can get a polygon of up to n sides. The solution region is also called the **feasible region**.

Sometimes the overlap between solution regions is not so obvious.

WORKED EXAMPLE 5

Find the solution to the following system of linear inequalities.

$$\begin{array}{ll} 8x + 5y \le 40 & [1] \\ 3x + 4y \le 12 & [2] \\ 2x + y \le 6 & [3] \\ x \ge 0 & [4] \\ y \ge 0 & [5] \end{array}$$

THINK	WRITE/DRAW
1 Find the intercepts of the boundary lines and use a test point to determine which side of the boundary to shade. Note that inequations [4] and [5] indicate that we are interested only in positive values of x and y, as well as when x or y equal zero; that is, non-negative values of x and y.	Region [1]: $8x + 5y = 40$ has intercepts of (0, 8) and (5, 0) with a test point of (0, 0) inside the feasible region. Region [2]: $3x + 4y = 12$ has intercepts (0, 3) and (4, 0) with a test point of (0, 0) inside the feasible region. Region [3]: $2x + y = 6$ has intercepts (0, 6) and (3, 0) with a test point of (0, 0) inside the feasible region.
2 Graph the solution to each of the inequalities on the same set of axes. Remember to shade the regions not required. The clear area is the solution region for this system of inequalities. *Note:* When we are interested in only positive values of x and y, it is usual not to extend lines beyond the axes, or to shade in the negative quadrants.	

What Worked example 5 indicates is that *not all* the inequalities are part of the solution region. In Worked example 5, inequality [1] has no role to play. However, this is usually not obvious until all the inequalities have been graphed.

Note: In the exercises that follow, you will notice that the majority of the systems contain inequalities $x \ge 0$ and $y \ge 0$. These two inequalities describe the first quadrant of the Cartesian plane, including the origin and the positive sections of the x- and y-axis.

EXERCISE 15.3 Simultaneous linear inequalities

PRACTISE

1 **WE3** Graph the solution to the following system of inequalities.

$2x - 3y \leq 12$
$5x + 4y \leq 40$

2 Graph the solution to the following system of inequalities.

$-4x + y \geq 12$
$x + 3y \leq 21$

3 **WE4** Graph the solution to the following system of inequalities.

$2x + 3y \leq 24$
$3x + 2y \leq 18$
$x \geq 0$
$y \geq 0$

4 Graph the solution to the following system of inequalities.

$5x + 12y \leq 120$
$6x + 5y \leq 60$
$x \geq 0$
$y \geq 0$

5 **WE5** Graph the solution to the following system of inequalities.

$2x + y \leq 2$
$x + 3y \leq 3$
$3x + 2y \leq 6$
$x \geq 0$
$y \geq 0$

6 Graph the solution to the following system of inequalities.

$3x + 5y \leq 75$
$x + y \leq 20$
$10y + 4x \leq 100$
$x \geq 0$
$y \geq 0$

CONSOLIDATE

For questions 7–15, graph the solution to the given system of linear inequalities.

7 $5x + 3y \leq 15$
$3x + 5y \leq 15$

8 $5x + 3y \leq 15$
$-x + y \leq 2$

9 $2x + 3y \leq 6$
$2x - 5y \geq 10$

10 $6x + 8y \geq 0$
$3x + 4y \leq 12$

11 $8x + 3y \leq 24$
$5x + 7y \leq 35$
$x \geq 0$
$y \geq 0$

12 $8x + 2y \leq 16$
$4x + 5y \leq 20$
$x \geq 0$
$y \geq 0$

13 $2x + 5y \leq 10$
$7x + 3y \leq 21$
$x \geq 0$
$y \geq 0$

14 $3x + 2y \geq 6$
$x + y \leq 6$
$-2x + 6y \leq 6$
$x \geq 0$
$y \geq 0$

15 $10x + 3y \leq 30$
$7x + 5y \leq 35$
$3x + 7y \leq 21$
$x \geq 0$
$y \geq 0$

16 Modify the first and third inequalities of question 15 so that ‘less than or equal to’ becomes ‘greater than or equal to’ and re-solve.

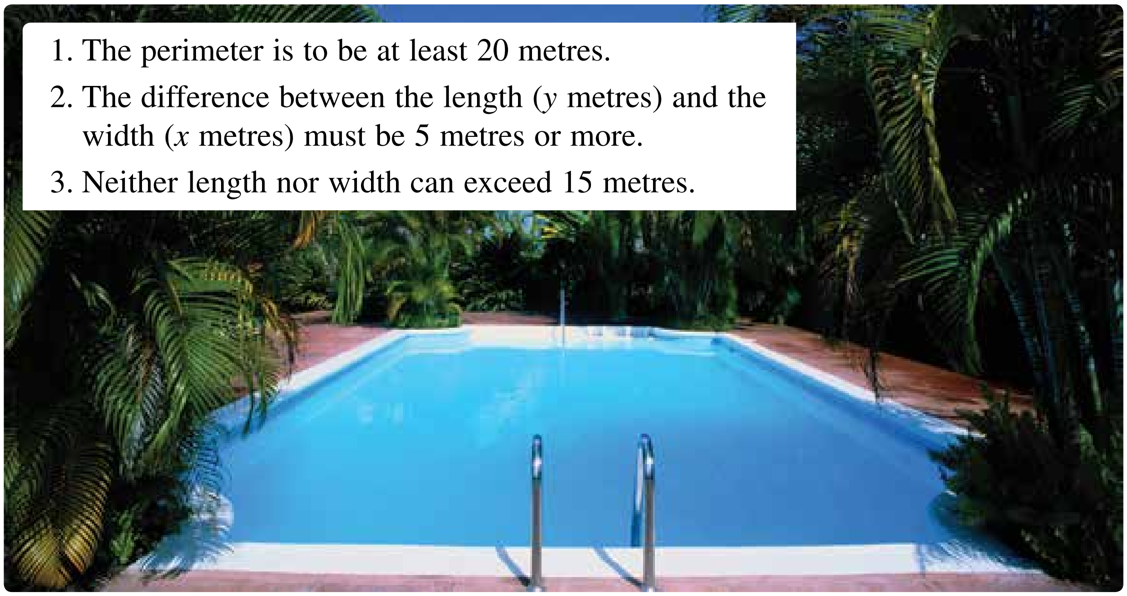
MASTER

17 $3x + y \geq 3$ $5x + 4y \leq 20$ $14x + 4y \leq 28$ $x \geq 0$ $y \geq 0$

The point that is part of the solution to the system of linear inequalities above is:

A (1, 5) **B** (2, 1) **C** (3, 0) **D** (1, 3) **E** (0, 6)

18 A rectangular swimming pool is constructed with the following restrictions in mind:



1. The perimeter is to be at least 20 metres.
2. The difference between the length (y metres) and the width (x metres) must be 5 metres or more.
3. Neither length nor width can exceed 15 metres.

a Write the restrictions in terms of algebraic inequalities.
b Graph the solution to the system of linear inequalities.
c Show that a pool of length of 10 m and a width of 10 m is not possible.

15.4 Linear programming

Now that you have the skills necessary for graphing inequations, you are in a position to tackle the main purpose of this topic: solving **linear programming** problems.

What is a linear programming problem? It consists of 3 vital components:

1. a set of *variables* (in this topic we shall always have exactly two) called **decision variables**
2. a set of restrictions or **constraints** on the values of these variables
3. a *function* (called the **objective function**) of these two variables that we wish to make either as large as possible (maximise) or as small as possible (minimise).

Generally, by convention, x and y are used to represent the two variables.

The decision variables

In the general linear programming problem there are many variables. In the Further Mathematics course we are restricted to only two, so that Cartesian graphical methods can be employed. In real-life engineering situations (linear programming is part of a study known as industrial engineering), linear programming problems with up to 100 variables have been solved. In some cases the variables must be integers (whole numbers), while in other problems they can take on any real value; sometimes it may not be obvious which case applies.

The constraints

The constraints are expressed as a set of inequalities, similar to those we encountered in the previous section. Let us consider a scenario which we will be working on in the next few examples. In this case there will be three constraints; but the procedures outlined in Worked examples 6, 7 and 8 would be exactly the same as for only two constraints.

WORKED EXAMPLE 6

A company produces clothes washers and dryers.

Each product requires assembly in three different factories before it can be sold.

Each factory requires different amounts of time to work on each washer or dryer, according to the following table.

Use of time	Factory A	Factory B	Factory C
Time spent on washer	2 hours	4 hours	2 hours
Time spent on dryer	1 hour	2 hours	2 hours
Maximum time the factory is available each day	24 hours	16 hours	12 hours

From this table we conclude that it takes 2 + 4 + 2 = 8 hours to produce a single washer and 1 + 2 + 2 = 5 hours to produce a single dryer. Convert these times into a series of constraints. The aim is to investigate the number of washers and dryers produced each day.

THINK	WRITE
1 Identify the decision variables. (a) What is the number of washers produced each day?	Let x = number of washers made daily
(b) What is the number of dryers produced each day? (You could just as easily choose x for dryers and y for washers.)	Let y = number of dryers made daily
2 Consider the daily capacity of factory A.	For factory A: Time spent on x washers = $2x$ hours Time spent on y dryers = $1y$ hours Maximum time spent on items = 24 hours
3 Turn this into an inequality. (Because 24 hours is a maximum, we could use less time, therefore the 'less than or equal sign' is used.)	$2x + 1y \leq 24$
4 Repeat steps 2 and 3 for factory B.	For factory B: Time spent on x washers = $4x$ hours Time spent on y dryers = $2y$ hours Maximum time spent on items = 16 hours $4x + 2y \leq 16$
5 Repeat steps 2 and 3 for factory C.	For factory C: Time spent on x washers = $2x$ hours Time spent on y dryers = $2y$ hours Maximum time spent on items = 12 hours $2x + 2y \leq 12$
6 Consider other (physical) constraints. Obviously, both x and y cannot be negative, since they represent a *number* of washers and dryers.	$x \geq 0$ $y \geq 0$

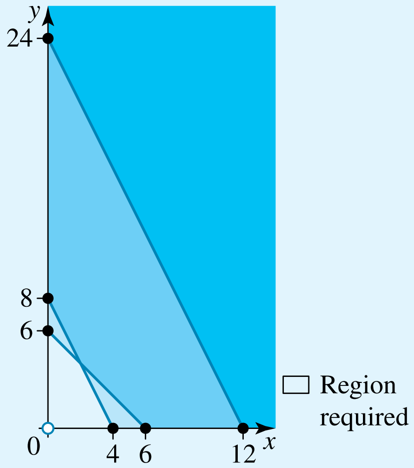
7 List all the constraints together.	$2x + y \leq 24$ $4x + 2y \leq 16$ $2x + 2y \leq 12$ $x \geq 0$ $y \geq 0$

It may or may not be reasonable to insist that x and y be integers. Is it possible to make 2.4 washers in a day? The answer to this is not necessarily obvious from the question, so for now, assume that x and y can take any positive value.

Graphing the constraints

The next stage in the process is to graph the constraints as we did in the previous section on simultaneous linear inequalities, and determine the overlapping solution region.

WORKED EXAMPLE 7 **Graph the constraints from Worked example 6.**

THINK	WRITE/DRAW
1 List the constraints.	$2x + y \leq 24$ $4x + 2y \leq 16$ $2x + 2y \leq 12$ $x \geq 0$ $y \geq 0$
2 (a) Since $x \geq 0$ and $y \geq 0$, we need to graph only in the first quadrant. (b) Find intersection points with axes. (c) Demonstrate that (0, 0) is in the solution region of all three inequalities.	$2x + y = 24$ has intercepts (0, 24) and (12, 0). Since $2(0) + 0 \leq 24$, then (0, 0) is inside the solution region. $4x + 2y = 16$ has intercepts (0, 8) and (4, 0). Since $4(0) + 2(0) \leq 16$, then (0, 0) is inside the solution region. $2x + 2y = 12$ has intercepts (0, 6) and (6, 0). Since $2(0) + 2(0) \leq 12$, then (0, 0) is inside the solution region.
3 Graph each equation and shade the non-solution region for each inequality. (It is the 2nd and 3rd inequalities that ultimately determine the overlapped solution region, shown as a clear area.)	

Note the characteristic polygonal shape of the solution region. This will play a crucial role in the next stage.

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The objective function

As mentioned at the start of this section, the third component of a linear programming problem is the *objective function*. This is a function of the two decision variables usually relating to a cost, distance or some other relationship between them. The object in linear programming is to maximise (or in some cases minimise) the value of this function within the solution region. In theory, we would have to check *every point* in our solution region to see which one gave us the maximum (or minimum) value required. Even if we restricted ourselves to integers only, this could take a long time. Fortunately there is a theorem (which will not be proved) that states:

> **The maximum (or minimum) of the objective function occurs at one of the vertices (corner points) of the solution region polygon.**

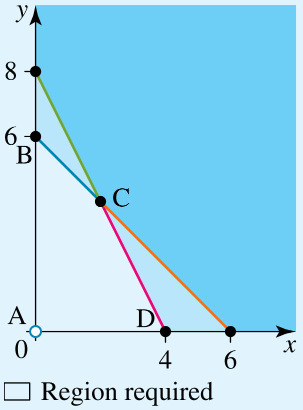
Sliding line method: By equating your objective function to different values, it will create a number of parallel lines on your graph. If this line touches or passes through the feasible region then it is a possible solution. As shown, the maximum (or minimum) of the objective function occurs at one of the corners. So this line can slide up and down (keeping the same gradient) to find the required solution.

Corner point methods: Since the maximum (or minimum) of the objective function occurs at one of the corners, the corner point method is an efficient process to use. To do this, evaluate the objective function at each of the corner points to find the maximum (or minimum).

WORKED EXAMPLE 8

When selling its clothes washers and dryers, the company can make a profit of \$150 on each washer and \$100 on each dryer.

How many of each should be made to maximise the company's daily profit?

THINK	**WRITE/DRAW**
1 List the decision variables.	Let x be the number of washers, and y the number of dryers.
2 Identify the objective function. \$150 is made on each washer, \$100 on each dryer.	Profit $= 150 \times$ number of washers $+ 100 \times$ number of dryers $= 150x + 100y$
3 Label the vertices (corner points) of the solution region polygon.	

4 Find the coordinates of the vertices.

(a) The coordinates of the Points A, B and D can be read directly from the graph.

Point A: (0, 0)
Point B: (0, 6)
Point D: (4, 0)

(b) Point C is the point of intersection of the two lines, whose equations are $4x + 2y = 16$ and $2x + 2y = 12$.

Equation 1: $4x + 2y = 16$ [1]
Equation 2: $2x + 2y = 12$ [2]

Therefore, to find the coordinates of point C, solve these equations simultaneously.

(c) Eliminate one of the variables (y) by subtraction and solve for the other variable (x).

$$\begin{aligned} 2x &= 4 \qquad [1] - [2] \\ x &= 2 \end{aligned}$$

(d) Substitute back into one of the original equations [1] or [2], to find the other variable (y).

Substitute into [1]:
$$\begin{aligned} 4(2) + 2y &= 16 \\ 8 + 2y &= 16 \\ 2y &= 8 \\ y &= 4 \end{aligned}$$

Point C: (2, 4)

5 Since the maximum profit will occur at one of the vertices, evaluate the objective function for each vertex in the solution region.

Profit $= 150x + 100y$

Point A (0, 0): Profit $= 150(0) + 100(0)$
$= 0$

Point B (0, 6): Profit $= 150(0) + 100(6)$
$= 600$

Point C (2, 4): Profit $= 150(2) + 100(4)$
$= 700$

Point D (4, 0): Profit $= 150(4) + 100(0)$
$= 600$

6 Find the maximum profit.

Maximum profit ($700) occurs when $x = 2$, $y = 4$. Thus, the company should make 2 washers and 4 dryers per day.

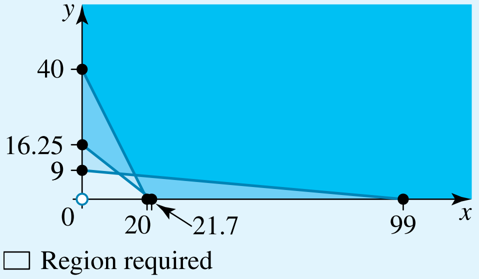
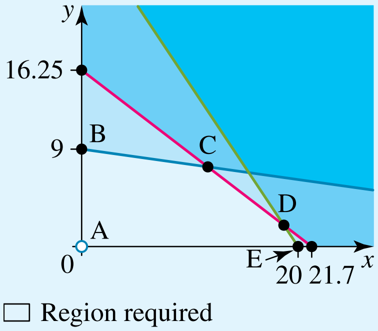
Let us now combine all the stages into a single example.

WORKED EXAMPLE 9

A company is deciding how much to invest in two new calculator models, graphics (x) and CAS (y). They will base their decision on the capacity of their workforce to design, assemble and ship the devices. They wish to ship as many calculators as possible. The workforce restrictions are outlined in the following table. (All restrictions are per thousand calculators.)

	Design phase	Assembly phase	Shipping
Graphics	1 worker	3 workers	2 workers
CAS	11 workers	4 workers	1 worker
Total capacity	99 workers	65 workers	40 workers

Use linear programming methods to determine the number of calculators (in thousands) that should be built.

THINK	WRITE/DRAW
1 Identify the decision variables.	Let x = number of graphics calculators (in thousands) Let y = number of CAS calculators (in thousands)
2 State the constraints. Clearly both x and y must not be negative.	Design: $1x + 11y \le 99$ [1] Assembly: $3x + 4y \le 65$ [2] Shipping: $2x + 1y \le 40$ [3] Non-negative values: $x \ge 0$ $y \ge 0$
3 Identify the objective function. The question stated that the company wishes to make *as many calculators* as possible.	Objective function: Number = $x + y$
4 Graph the constraint inequations and identify the feasible region. Because of non-negative restriction on x and y, draw the graph in the 1st quadrant only.	
The second graph shows an enlarged version of the feasible region.	
5 Find the coordinates of the vertices of the feasible region. (Coordinates of points A, B and E can be read directly from the graph.)	Point A: (0, 0) Point B: (0, 9) Point E: (20, 0) Point C: Intersection of $x + 11y = 99$ [4] and $3x + 4y = 65$ [5]
(a) Solve simultaneously $x + 11y = 99$ and $3x + 4y = 65$ to find the coordinates of point C.	
(b) Multiply [4] by 3 to have the same x-coefficient as [5].	3 × [4]: $3x + 33y = 297$
(c) Subtract the 2 equations.	3 × [4] – [5]: $29y = 232$
(d) Solve for y.	$y = 8$
(e) Substitute the value of y into equation [4].	Substitute into [4]: $x + 11(8) = 99$
(f) Solve for x.	$x + 88 = 99$ $x = 11$ Point C: (11, 8)
(g) Similarly, solve simultaneously $3x + 4y = 65$ and $2x + y = 40$ to find coordinates of point D.	Point D: Intersection of $3x + 4y = 65$ [6] and $2x + y = 40$ [7]

(h) Multiply [7] by 4 to have the same y-coefficient as [6].	4 × [7]: $8x + 4y = 160$
(i) Subtract the two equations.	4 × [7] – [6]: $5x = 95$
(j) Solve for x.	$x = 19$
(k) Substitute the value of x into equation [6].	Substitute into [6]: $3(19) + 4y = 65$
(l) Solve for y.	$57 + 4y = 65$ $4y = 8$ $y = 2$ Point D: (19, 2)
6 Substitute coordinates of each corner point into the objective function and choose the maximum. (Remember that x and y are in thousands.)	Objective function: Number $= x + y$ Point A: (0, 0), number $= 0 + 0$ $= 0$ Point B: (0, 9), number $= 0 + 9$ $= 9$ Point C: (11, 8), number $= 11 + 8$ $= 19$ Point D: (19, 2), number $= 19 + 2$ $= 21$* *Maximum Point E: (20, 0), number $= 20 + 0$ $= 20$ So, the company should make 19 000 graphics and 2000 CAS calculators.

Sometimes the constraints are related to each other, rather than a constant as in the previous examples.

WORKED EXAMPLE 10

Jenny spends some of her spare time outside school hours playing netball and working a part-time job. Each week she has a maximum of 20 hours available for the two activities.

The number of hours that she spends playing sport is less than or equal to four times the hours she spends working.

The hours spent at sport are greater than or equal to twice the hours spent at work.

Find the time spent on each activity during a week in order to maximise Jenny's earnings if she earns $2.50 per hour playing netball and $6 per hour working. Also find her maximum earnings.

THINK	**WRITE/DRAW**
1 Identify the decision variables.	x = number of hours spent at sport y = number of hours spent at work
2 State the constraints. Maximum of 20 hours available. (Clearly both x and y are non-negative.)	$x + y \le 20$ [1] $x \le 4y$ [2] $x \ge 2y$ [3] $x \ge 0$ $y \ge 0$

3 Identify the objective function. (The objective is to maximise earning.)

Objective function:

Earnings $= 2.5x + 6y$

4 Graph the constraints.

$x + y = 20$ has intercepts (0, 20) and (20, 0), since $0 + 0 \le 20$ is true, (0, 0) is inside the solution region.

The boundary lines for constraints [2] and [3] pass through the origin. Therefore, a second point is required to sketch each line. Also a test point other than (0, 0) must be selected for these constraints.

$x = 4y$ has an intercept of (0, 0)

When $x = 20$, $y = 5$ (20, 5)

Consider (20, 0) as a test point.

Since $20 \le 4(0)$ is untrue, then (20, 0) is outside the solution region.

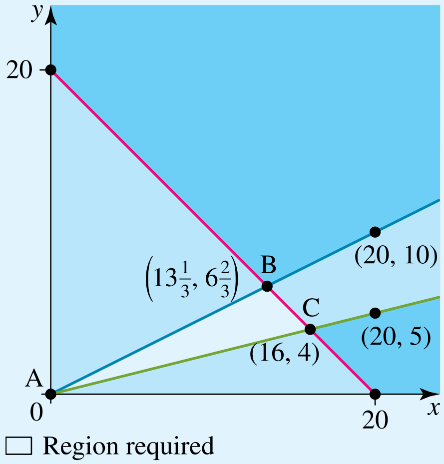
$x = 2y$ has an intercept of (0, 0)

When $x = 20$, $y = 10$ (20, 10)

Consider (20, 0) as a test point.

Since $20 \ge 2(0)$, is true, then (20, 0) is inside the solution region.

Identify the feasible region overall. We are dealing with the 1st quadrant only.



5 Identify the coordinates of the vertices of the feasible region.

(a) To find coordinates of point B, solve $x + y = 20$ and $x = 2y$ simultaneously.

Point A: (0, 0)

Point B: Intersection of $x + y = 20$ [4]

and $x = 2y$ [5]

[5] becomes $x - 2y = 0$ [6]

[4] – [6]: $3y = 20$

$y = 6\frac{2}{3}$

Substitute into [5]: $x = 2 \times 6\frac{2}{3}$

$x = 13\frac{1}{3}$

Point B: $\left(13\frac{1}{3}, 6\frac{2}{3}\right)$

(b) To find coordinates of point C, solve $x + y = 20$ and $x = 4y$ simultaneously.

Point C: Intersection of $x + y = 20$ [7]

and $x = 4y$ [8]

[8] becomes: $x - 4y = 0$ [9]

[7] – [9]: $5y = 20$

$y = 4$

Substitute into [8]: $x = 4(4)$

$x = 16$

Point C: (16, 4)

6 Substitute coordinates of each point into the objective function to find the maximum earnings.	Objective function: Earnings $= 2.5x + 6y$ Point A: (0, 0) earnings $= 2.5(0) + 6(0) = 0$ Point B: $\left(13\frac{1}{3}, 6\frac{2}{3}\right)$ Earnings $= 2.5\left(13\frac{1}{3}\right) + 6\left(6\frac{2}{3}\right) = 73.33$ Point C: (16, 4) earnings $= 2.5(16) + 6(4) = 64$ So, Jenny should play netball for $13\frac{1}{3}$ hours and work for $6\frac{2}{3}$ hours for maximum earnings of \$73.33.

EXERCISE 15.4 Linear programming

PRACTISE

1 **WE6** The Danish–Australian Club is organising a camping trip to the Snowy Mountains. A large campsite has 2-person cabins available at \$40 per night and 3-person cabins at \$45 per night. The club has a budget of \$3000 to accommodate up to 180 people.

a Define the decision variables.
b Determine the constraint based upon the total club budget.
c Determine the constraint based upon the number of members of the club.

2 An investment banker has a maximum of \$1 000 000 to invest on behalf of a group of doctors. She can either purchase bonds or invest in real estate mortgages. However, the doctors, being conservative, insist that the amount invested in bonds be *at least* twice as much as that invested in mortgages.

Based on past experience, the fund manager knows that bonds will return 6% profit and real estate will return 11%.

a State the decision variables. **b** State the constraints.

3 **WE7** The objective function from question **1** is based upon the total number of cabins used. Graph the constraints from question **1**.

4 State the objective function from question **2** and graph the constraints.

5 **WE8** Use linear programming techniques (sliding line or corner point) to find the values of the decision variables that minimise the total number of cabins rented from question **1**.

6 Use linear programming techniques (sliding line or corner point), to determine the optimal 'mix' of investments from question **2**; that is, determine maximum return.

7 **WE9** A company is deciding how much to invest in two new TV models, flat screen (x) and curved screen (y). They will base their decision on the capacity of their workforce to design, assemble and ship the devices. They wish to ship as many TVs as possible. The workforce restrictions are shown in the following table (all restrictions are per thousand TVs).

	Design	Assembly	Shipping
Flat	2	5	3
Curved	15	7	3
Total capacity	120	75	50

Use linear programming (shifting line or corner point) methods to determine the number of TVs (in thousands) that should be built.

8 Two teachers work together to produce a Mathematics text consisting of *text material* and *tests*. For each chapter of text material and for each test, the teachers devote a certain amount of time (in hours) to the task. Furthermore, each teacher can devote time only up to a maximum number of hours. The data is summarised in the table below.

	Teacher A	Teacher B
Text chapter	20	10
Test	8	2
Maximum time available	320	100

Let x be the number of text chapters and y be the number of tests.

If the publisher pays \$1500 for each chapter and \$250 for each test, how many of each should the teachers produce in order to maximise their income?

9 WE10 Kyle spends his leisure time playing basketball and football. Each week he has a maximum of 30 hours available for his two sports.

The number of hours he spends playing football is less than or equal to three times the hours he spends on basketball.

The hours spent at football are greater than or equal to two times the hours spent at basketball.

Find the time spent on each activity during a week in order to maximise Kyle's earning if he earns \$6.50 per hour playing football and \$10 per hour playing basketball. Also find his maximum earnings.

10 A furniture manufacturer makes 2 kinds of sofas: leather and cloth. Each leather sofa requires 9 workers and each cloth sofa requires 8 workers. A maximum of 207 workers are available to make sofas each day. Furthermore, shipping and packaging cloth sofas requires 5 workers, while leather sofas require only 2 workers. There are 75 workers available for this task each day.

Let x = number of leather sofas.

Let y = number of cloth sofas.

If the profit for a leather sofa is \$400 and the profit for a cloth sofa is \$370, how many of each should be produced each day?

CONSOLIDATE

11 A factory produces two different models of transistor radio. Each model requires two workers to assemble it. The time taken by each worker varies according to the following table.

	Worker 1	Worker 2
Model A	5 minutes	5 minutes
Model B	18 minutes	4 minutes
Maximum time available for each worker	360 minutes	150 minutes

The company makes \$2.50 on each of Model A sold and \$4.00 on each of Model B sold.

a Define the decision variables.
b Write the constraints as linear inequalities.
c Graph the constraints and indicate the solution region.
d State the objective function.
e Use linear programming methods to find how many of each radio should be made to maximise takings.

12 At a car spray-painting workshop, each car receives three coats of paint, all of which have to be completed within one day. There are two types of car: sedan and utility. The times taken for each are listed in the following table.

	Stage of painting		
	1st stage	2nd stage	3rd stage
Sedan	5 minutes	9 minutes	6 minutes
Utility	7 minutes	8 minutes	5 minutes
Total time available for each stage	140 minutes	183 minutes	120 minutes

a Define the decision variables.
b Write the constraints as linear inequalities.

13 From the data in question 12,

a graph the constraints and indicate the feasible region
b state the objective function
c use linear programming methods to work out how many of each vehicle type should be sprayed to maximise the total number of vehicles sprayed.

14 A large electronics firm must decide on the number of different models of computer monitor and models of hard disk drive that it will manufacture. For each model of each type there are the following budgetary constraints for different departments. (All budgetary numbers are in thousands of dollars.)

	Department			
	Marketing	Development	Design	Production
Computer monitor	7	11	22	11
Disk drive	8	7	12	3
Total budget available	320	319	594	264

a Define the decision variables.
b Write the constraints as linear inequalities.

15 From the data in question 14,

a graph the constraints and indicate the solution region
b use linear programming methods to find how many models of monitor and disk drive should be made to maximise profit, given that each monitor model provides a profit of \$150 000 and each disk drive model provides a profit of \$120 000.

16 Consider the washer and dryer problem in Worked examples 6–8. How would the solution change if the profit per washer was modified to $120 and the profit per dryer was modified to $130?

17 The Swiss Army Diet requires that a person eat only two foods: apples and cheese. A dieter is told that they may eat as much food as they wish, subject to the dietary constraints listed in the table.

	Amount of vitamin X	**Amount of vitamin Y**
An apple	3 units	3 units
A piece of cheese	10 units	4 units
Number of units	Maximum of 98	Minimum of 50

Each apple also supplies 100 calories, while each piece of cheese also supplies 85 calories. Use linear programming techniques to determine the number of apples and pieces of cheese needed for the *minimum* calorie intake.

18 A paint shop mixes two kinds of paint — indoor paint and outdoor paint — subject to the following constraints:

1 *At least* 30 litres of paint in total must be mixed.
2 Indoor paint requires 3 units of dye, while outdoor paint requires 8 units of dye and there is a *maximum* of 100 units of dye available.

Indoor paint yields a profit of $2.50 per litre, while outdoor paint yields $3.25 per litre. Use linear programming techniques to determine the volume of indoor and outdoor paint to be mixed to maximise the shop's profit.

19 A flower grower sells roses and tulips. She has enough land to grow no more than 3000 of these plants altogether, subject to the following constraints:

1 The number of roses that she grows is less than or equal to 3 times the number of tulips grown.
2 The number of roses is greater than or equal to twice the number of tulips grown.

If she sells roses for $15 each and tulips for $6 each, find the number of each that would maximise her income from the sale of these plants.

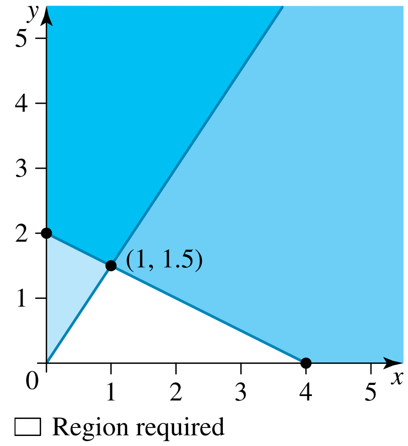
20 An investment fund manager is given a maximum of $100 000 to invest. She invests in shares and gold according to the following constraints:

1 The value of the shares must be less than or equal to 2 times the value of gold.
2 The value of the shares must be greater than or equal to $\frac{1}{2}$ the value of gold.

Historically, shares have returned 10% (they increase in value by 10%) per year, while gold has returned 16% per year. Use linear programming techniques to determine the manager's investment strategy to maximise profit (or return).

MASTER

21 The required region shown in the following graph is best described by which set of inequalities?



A $2x + 4y \geq 8$
$2y - 3x \geq 0$
$x \geq 0$
$y \geq 0$

B $4x + 2y \leq 0$
$3y - 2x \leq 0$
$x \geq 0$
$y \geq 0$

C $2x + 4y \leq 8$
$2y - 3x \leq 0$
$x \geq 0$
$y \geq 0$

D $4x + 2y \geq 0$
$3y - 2x \geq 0$
$x \geq 0$
$y \geq 0$

E $2x + 4y < 8$
$2y - 3x < 0$
$x > 0$
$y > 0$

22 A small, used-car yard stocks sedans and wagons. Due to limited space in the yard, they cannot have more than 50 cars at any one time. Since they know that sedans are more popular than wagons, they decide never to have more wagons than sedans in stock. On average, they can purchase a wagon for \$12 000 and a sedan for \$12 500. To minimise insurance costs, they cannot have more than \$490 000 worth of stock at any given time.
Let x = the number of wagons and y = the number of sedans in the car yard.

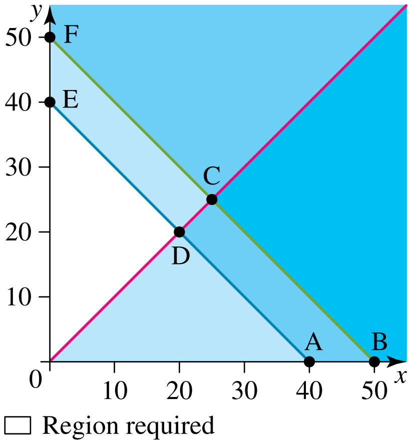
Use a linear programming method (sliding line or corner point) to answer the following.

a Write down the 5 constraints involved.

b The constraints are sketched on the graph as shown. Identify the inequalities with the boundary lines OC, BF and AE.

c Write down the coordinates of each of the points C and D.

d If sedans make an average profit of \$3500 and wagons make an average profit of \$4000, optimise the number of each to stock to maximise profit.



15.5 Applications

study on

Unit 4
AOS M4
Topic 2
Concept 4

Solving linear programming
Concept summary
Practice questions

The applications in this section are particularly suited for the case where there are only two decision variables (x and y). In practice, when more decision variables are allowed, these problems can be much more sophisticated, but this is beyond the scope of Further Mathematics.

Blending problems

Blending consists of combining raw material from several different sources into a single composite. The raw materials contain one or more chemicals (or other components) in varying amounts, which are then restricted in some way in the blended composite.

WORKED EXAMPLE 11

A miller can buy wheat from three suppliers: Airy Farm, Berry Farm and Cherry Farm. In each case the wheat is contaminated with two things — bran and husks. When combined, the wheat must contain no more than 5% bran and no more than 4% husks. The miller wishes to make 50 tonnes of wheat in total, by purchasing from each farm. Each farm's wheat contains the following amounts of bran and husks.

	Bran	Husks	Cost per tonne
Airy Farm	3%	5%	\$70
Berry Farm	5%	2%	\$60
Cherry Farm	7%	6%	\$40
Maximum amount of contaminant allowed	5%	4%	

Airy Farm charges \$70 per tonne, Berry Farm charges \$60 per tonne and Cherry Farm charges \$40 per tonne.

Use linear programming techniques (sliding line or corner point) to find the amount of wheat to purchase from each supplier in order to keep costs to a minimum. That is:

a identify the decision variables

b define the constraint inequality caused by the restriction on bran

c define the constraint inequality caused by the restriction on husks

d define the remaining constraints

e graph the solution region

f define the objective function

g find the amount of wheat purchased from each supplier.

THINK	WRITE/DRAW
a Identify the decision variables. Note that there are exactly 50 tonnes of wheat needed so if x and y come from Airy and Berry farms, then the rest must come from Cherry Farm.	a Let x = tonnes of wheat purchased from Airy Farm. Let y = tonnes of wheat purchased from Berry Farm. Let $(50 - x - y)$ = tonnes of wheat purchased from Cherry Farm.
b (a) Define the constraint due to bran.	b 3% of x + 5% of y + 7% of $(50 - x - y) \leq$ 5% of 50 tonnes
(b) Multiply the inequality by 100.	$0.03x + 0.05y + 0.07(50 - x - y) \leq 0.05(50)$
(c) Expand and collect like terms.	$3x + 5y + 7(50 - x - y) \leq 5(50)$ $3x + 5y + 350 - 7x - 7y \leq 250$ $-4x - 2y + 350 \leq 250$
(d) Multiply the inequality by –1, so the sign changes.	$-4x - 2y \leq -100$ $4x + 2y \geq 100$

c Define the constraint due to husks.

c 5% of x + 2% of y + 6% of $(50 - x - y) \leq$ 4% of 50 tonnes

$$\begin{aligned} 0.05x + 0.02y + 0.06(50 - x - y) &\leq 0.04(50) \\ 5x + 2y + 6(50 - x - y) &\leq 4(50) \\ 5x + 2y + 300 - 6x - 6y &\leq 200 \\ -x - 4y + 300 &\leq 200 \\ -x - 4y &\leq -100 \\ x + 4y &\geq 100 \end{aligned}$$

d Complete the list of constraints. Note also that we need to look in the 1st quadrant only, since all supplies of wheat must be non-negative.

d $x \geq 0$

$y \geq 0$

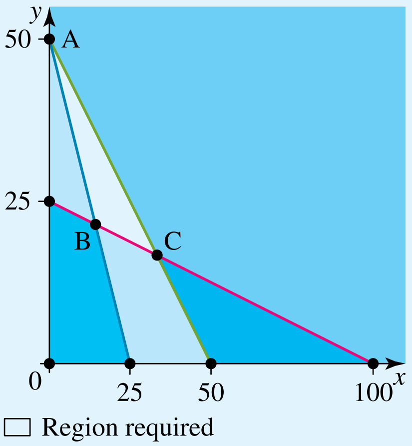
Supply of wheat from Cherry Farm must be non-negative too.

$50 - x - y \geq 0$ or $-x - y \geq -50$ or

$x + y \leq 50$

e **1** Graph the solution region.

e



2 Find the coordinates of the vertices of the solution region.

Point A is (0, 50).

(a) Point B is the intersection of $4x + 2y = 100$ and $x + 4y = 100$.

$$\begin{aligned} 4x + 2y &= 100 \quad [1] \\ x + 4y &= 100 \quad [2] \end{aligned}$$

(b) Multiply [2] by 4 to eliminate x from the simultaneous equations.

$4 \times [2]$: $\quad 4x + 16y = 400$

$4 \times [2] - [1]$: $\quad 14y = 300$

(c) Solve for y.

$y = 21.43$

(d) Substitute the value of y into [1] and solve for x.

Substitute into [1]: $\quad 4x + 2(21.43) = 100$

$$\begin{aligned} 4x + 42.86 &= 100 \\ x &= 14.29 \end{aligned}$$

Thus, point B is (14.29, 21.43)

(e) Point C is the intersection of $x + 4y = 100$ and $x + y = 50$.

$$\begin{aligned} x + 4y &= 100 \quad [2] \\ x + y &= 50 \quad [3] \end{aligned}$$

(f) Subtract equations to eliminate x and solve for y.

[2] – [3]: $\quad 3y = 50$

$y = 16.67$

(g) Substitute the value of y into [3] and solve for x.

Substitute into [3]: $\quad x + 16.67 = 50$

$x = 33.33$

Thus, point C is (33.33, 16.67)

f Define the objective function. The desired result is the values of x and y which produce the minimum cost.	**f** Cost $= \$70x + \$60y + \$40\,(50 - x - y)$ $= 70x + 60y + 2000 - 40x - 40y$ Cost $= 30x + 20y + 2000$
g **1** (a) Substitute the coordinates of each vertex into the objective function.	**g** Point A: $x = 0$, $y = 50$ Cost $= 30(0) + 20(50) + 2000$ $= \$3000$
(b) Determine the minimum cost.	Point B: $x = 14.29$, $y = 21.43$ Cost $= 30(14.29) + 20(21.43) + \$2000$ $= \$2857.30$ Point C: $x = 33.33$, $y = 16.67$ Cost $= 30(33.33) + 20(16.67) + 2000$ $= \$3333.30$
2 Determine the amounts from each supplier.	Since point B gives minimum costs, purchase: 14.29 tonnes from Airy (x) 21.43 tonnes from Berry (y) 14.28 tonnes from Cherry ($50 - x - y$).

In order for there to be exactly two decision variables, there must be exactly three sources of material. In Worked example 11 these were the three farms. If only two sources were available to provide the blended composite, then the solution would be a line segment, not a feasible region. There is *no limit* on the number of constraints (bran and husks); there can be as many of these as necessary.

Transportation problems

Consider a company which produces a single item in a number of factories for a number of showrooms. Transportation problems are concerned with how many *from* each factory should be sent *to* each showroom, so that each showroom's demand for the product is fully met. Furthermore, there are different shipping costs between the factories and the showrooms. In Further Mathematics we are restricted to two factories and two distributors.

WORKED EXAMPLE 12

Birmingham Bicycle Builders makes bicycles at two factories and ships them to two different distributors.

	Factory A	Factory B
Distributor X	\$6	\$7
Distributor Y	\$3	\$5

Factory A produces 100 bicycles; Factory B produces 150.

Distributor X wants 70 bicycles; Distributor Y wants 90.

Furthermore, the shipping costs between factories (in dollars per bicycle) and distributors are shown in the table above right.

Use linear programming techniques to determine the number of bicycles shipped between factories and distributors in order to minimise total shipping cost. That is:

- **a** **define the decision variables and summarise in a table**
- **b** **define the constraints based on the fact that all shipments must be non-negative**

c graph the solution region

d define the objective function

e determine the best way to ship to minimise cost.

THINK | **WRITE/DRAW**

a **1** Define the decision variables. (The controlling influence is what the distributors want.)

(a) Distributor X needs 70 bicycles. Find how many it gets from each factory.

(b) Proceed similarly for distributor Y.

a Let x = number of bicycles shipped to distributor X from factory A.
Therefore $70 - x$ is the number shipped to X from factory B.
Let y = number of bicycles shipped to distributor Y from factory A.
Therefore $90 - y$ is the number shipped to Y from factory B.

2 Summarise the decision variables, supply and demand data in a table.

	Factory A	**Factory B**	**Total demand from each distributor**
Distributor X	x	$70 - x$	70
Distributor Y	y	$90 - y$	90
Total supply from each factory	100	150	

b **1** Define constraints based on the fact that all shipments must be non-negative amounts.

b $x \geq 0$
$y \geq 0$
$70 - x \geq 0$ or $x \leq 70$
$90 - y \geq 0$ or $y \leq 90$

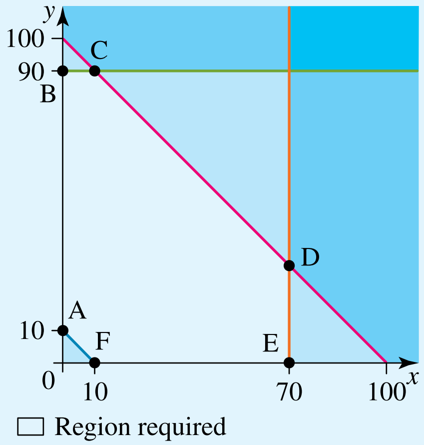
2 Define the remaining constraints on each factory's supply.

Factory A: $x + y \leq 100$
Factory B: $70 - x + 90 - y \leq 150$
$160 - x - y \leq 150$
$-x - y \leq -10$
$x + y \geq 10$

c **1** (a) Graph the solution region. Note that there are six constraints, but that the first two force us to look at the 1st quadrant only.

(b) Label the vertices of the feasible region.

c



2 Determine the coordinates of the vertices. The intersection of a diagonal line with a vertical or horizontal line is done easily without using simultaneous linear equations as follows.

For point C:
substitute $y = 90$ into $x + y = 100$.

For point D:
substitute $x = 70$ into $x + y = 100$.

Point A: (0, 10)
Point B: (0, 90)
Point C: (10, 90)
Point D: (70, 30)
Point E: (70, 0)
Point F: (10, 0)

d Determine the objective function from the shipping cost table.

d Factory A to distributor X:
x bicycles at \$6 per bicycle

Factory A to distributor Y:
y bicycles at \$3 per bicycle

Factory B to distributor X:
$(70 - x)$ at \$7 per bicycle

Factory B to distributor Y:
$(90 - y)$ at \$5 per bicycle

$$\begin{aligned}\text{Total cost} &= 6x + 3y + 7(70 - x) + 5(90 - y)\\ &= 6x + 3y + 490 - 7x + 450 - 5y\\ &= -x - 2y + 940\end{aligned}$$

e 1 Determine the cost for each vertex and find the minimum cost.

e Point A: Cost $= -(0) - 2(10) + 940 = 920$
Point B: Cost $= -(0) - 2(90) + 940 = 760$
Point C: Cost $= -(10) - 2(90) + 940 = 750$*
*Minimum
Point D: Cost $= -(70) - 2(30) + 940 = 810$
Point E: Cost $= -(70) - 2(0) + 940 = 870$
Point F: Cost $= -(10) - 2(0) + 940 = 930$

2 Determine the value of each variable, and thus, the amount to ship from each factory to each distributor.

Minimum costs occur when:
$x = 10$, $y = 90$.
Ship 10 from factory A to distributor X.
Ship 90 from factory A to distributor Y.
Ship $(70 - 10) = 60$ from factory B to distributor X.
Ship $(90 - 90) = 0$ from factory B to distributor Y.

Note that the demand from each distributor is *always* met. In Worked example 12, distributor X received $10 + 60 = 70$ bicycles, while distributor Y received $90 + 0 = 90$. However, there is a surplus in factory B of 90 bicycles. This surplus could be shipped to a third distributor. Under what conditions will there be no surplus?

Manufacturing problems

Manufacturing problems involve manufacturing more than one item (in Further Mathematics, exactly two items) and having to share resources such as money, staff, parts or time. Worked examples 6 to 9 in the previous section demonstrated such problems, as did questions **11** to **14** of Exercise 15.4.

Limitations of linear programming techniques

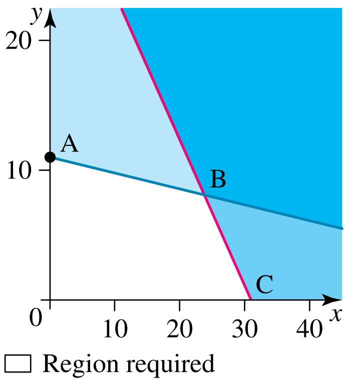
Many of the linear programming problems we have encountered required that the decision variables be positive integers; for example, they referred to quantities like the number of bicycles shipped from a factory or the number of calculators manufactured. However, what happens if the solution region for such a problem has vertices that are not integers? Consider the following example.

A manufacturer makes two products — blankets and cookware sets — that need to be shipped in special containers. Each blanket weighs 3.7 kg and has a volume of 6300 cm^3. Each cookware set weighs 25 kg and has a volume of 5000 cm^3. The shipping containers can hold 250 kg and have a maximum volume of 200 000 cm^3. Use linear programming techniques to determine the largest number of each item that can be shipped in one container for optimal efficiency.

The decision variables are: let x = number of blankets shipped
let y = number of cookware sets shipped.

The constraints are: $x \geq 0$
$y \geq 0$
$3.7x + 25y \leq 250$
$6300x + 5000y \leq 200\,000$ or
$6.3x + 5y \leq 200$ (dividing by 1000)

The solution region is shown in the diagram.



The vertices are found as follows. The intersection of the two lines shown (Point B) is found by solving the simultaneous linear equations:

$3.7x + 25y = 250$
$6.3x + 5y = 200$

The solution is $x = 26.98$, $y = 6.01$.

The other vertices are (0, 10) and (31.7, 0).

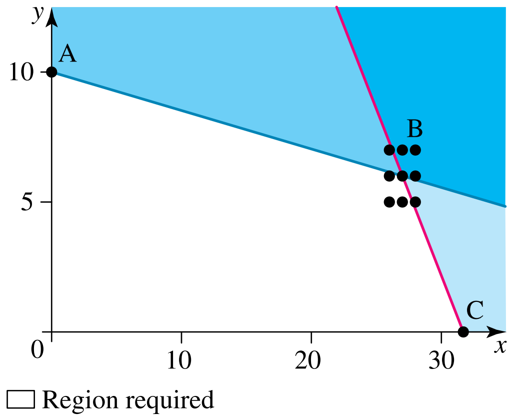
The objective function, which we wish to maximise, is:

Objective function: number = $x + y$.

The values of the objective function are:

Point A: (0, 10); number = 10
Point B: (26.9, 6.01); number = 32.91* *Maximum
Point C: (31.7, 0); number = 31.7

Note that the 'optimal' solution (26.9, 6.01) has non-integer values for x and y. How is it possible to ship 26.9 blankets and 6.01 cooking sets? This is a class of problem known as *integer programming*, which is beyond the scope of Further Mathematics; however, a possible solution is given as follows.



Consider the integer points closest to the optimal solution: (27, 6) and its nearest 8 neighbours, (26, 7), (27, 7), (28, 7), ..., (28, 5).

Check *each one* in turn to see if they satisfy the constraints and choose the one that has the largest objective function value.

The completion of this problem is left as question **16** in Exercise 15.5.

EXERCISE 15.5 Applications

PRACTISE

1 **WE11** A gardener can buy grass blends from three suppliers: Greener Grass, Green Lawns and Green Turf. In each case the grass contains a number of grasses; two of these are tall fescue and ryegrass. When combined, the grass can contain no more than 25% tall fescue and no more than 15% ryegrass. The gardener needs to use 2 tonnes of grass in total, by purchasing from each supplier. Each supplier contains the following amounts of tall fescue and ryegrass.

	Tall fescue	**Ryegrass**	**Cost per tonne**
Greener Grass	20%	18%	$100
Green Lawns	25%	12%	$90
Green Turf	30%	16%	$75
Maximum allowed	25%	15%	

Use linear programming techniques (sliding line or corner point) to find the amount of grass to purchase from each supplier in order to keep costs to a minimum. That is:

a define the decision variables
b define the constraints
c graph the solution region
d define the objective function
e determine the amount of grass purchased from each nursery.

2 A smoothie store buys orange juice from suppliers A, B and C. In each case the juice contains pulp and seeds. When combined, the juice must contain no more than 7% pulp and no more than 4% seeds. The owner wishes to purchase a total of 150 litres of juice from three suppliers. Each supplier's juice contains the following amounts of pulp and seeds.

	Pulp	Seeds	Cost per litre (c)
Supplier A	10%	4%	45
Supplier B	8%	6%	60
Supplier C	5%	3%	75
Maximum allowed	7%	4%	

Use linear programming techniques (sliding line or corner point) to find the amount of juice to purchase from each supplier in order to keep costs to a minimum. That is:

a define the decision variables
b define the constraints
c graph the solution region
d define the objective function
e determine the amount of juice purchased from each supplier.

3 **WE12** Fast Grow nurseries grow plants at two nurseries and ship them to two different distributors.

Nursery A produces 750 plants; Nursery B produces 900 plants.

Distributor X wants 600 plants while Distributor Y wants 700 plants.

The shipping costs between nurseries (in dollars per plant) and distributors are shown in the following table.

	Nursery A	Nursery B
Distributor X	\$0.70	\$0.80
Distributor Y	\$0.60	\$0.65

Use linear programming techniques (sliding line or corner point) to determine the number of plants shipped between nurseries and distributors in order to minimise total shipping cost. That is:

a define the decision variables and summarise in a table
b define the constraints based on the fact that all shipments must be non-negative
c graph the solution region
d define the objective function
e determine the best way to ship to minimise cost.

4 Digi Electronics produces digital radios at two factories and ships them to two different distributors. Each factory produces 27 radios per day.

Distributor X requires 23 radios while Distributor Y requires 19 radios.

The shipping costs (in dollars per radio) between the factories and the distributors are shown in the following table:

	Factory A	Factory B
Distributor X	\$7	\$12
Distributor Y	\$5	\$10

Use linear programming techniques (sliding line or corner point) to determine the number of radios shipped between factories and distributors in order to minimise total shipping cost. That is:

a define the decision variables
b define the constraints based on the fact that all shipments must be non-negative
c graph the solution region
d define the objective function
e determine the best way to ship to minimise cost.

CONSOLIDATE

5 A local restaurant owner buys his orange juice from three suppliers: A, B and C. In each case the juice contains unwanted pulp and seeds. When combined, the juice must contain no more than 5% pulp and no more than 3% seeds. The owner wishes to purchase a total of 100 litres of juice from among the three suppliers. Each supplier's juice contains the following amounts of pulp and seeds:

	Pulp	Seeds	Cost per litre (¢)
Supplier A	7%	5%	35
Supplier B	6%	3%	45
Supplier C	3%	2%	55
Maximum amount of contaminant allowed	5%	3%	

Supplier A charges $0.35 per litre, supplier B charges $0.45 per litre and supplier C charges $0.55 per litre. Use linear programming methods to find the amount of juice to buy from each supplier in order to minimise the cost. That is:

a identify the decision variables
b define the constraint inequation caused by the restriction on pulp
c define the constraint inequation caused by the restriction on seeds
d define the remaining constraints.

6 Using the information provided and the answers calculated in question 5,

a graph the solution region
b define the objective function
c find the amount of juice purchased from each supplier.

7 A canner of fruit salad can buy from three different suppliers, A, B and C, subject to the following restrictions on the minimum amount of peaches and cherries in the salad.

	Ingredient		Cost per kg ($)
	Peaches	Cherries	
Supplier A	23%	5%	1.10
Supplier B	20%	3%	0.90
Supplier C	30%	7%	1.40
Minimum amount of ingredient allowed	22%	4%	

Use linear programming techniques (sliding line or corner point) to find the amount of salad to buy from each supplier to make up a total of 100 kg with the minimum cost.

8 Kumquat Computers produces computers at its two factories and ships them to two distributors. Each factory produces 35 computers per day.
Distributor X requires 30 computers; distributor Y requires 25.
The shipping costs (in dollars per computer) between the factories and the distributors are shown in the table:

	Factory 1	**Factory 2**
Distributor X	\$25	\$55
Distributor Y	\$20	\$35

Use linear programming methods (sliding line or corner point) to find the number of computers shipped between factories and distributors in order to minimise the cost. That is:

a define the decision variables
b define the constraints based on the fact that all shipments must be non-negative
c graph the solution region
d define the objective function
e determine the best way to ship to minimise cost.

9 A manufacturer of refrigerators has two factories: the first produces 60 refrigerators per week, the second produces 25 per week. Their two distributors require 30 and 45 refrigerators respectively. The shipping costs (in dollars per refrigerator) are given in the following table.

	Factory 1	**Factory 2**
Distributor X	\$18	\$24
Distributor Y	\$20	\$27

Use linear programming techniques (sliding line or corner point) to determine the best way to ship to minimise cost.

10 A shoemaker has two factories which produce 50 and 80 dozen pairs each. These are shipped to two factory outlets, Outlet X and Outlet Y, which require 50 and 30 dozen pairs each respectively. The shipping costs (in dollars per dozen pairs) are given in the following table.

	Factory 1	**Factory 2**
Outlet X	\$14	\$11
Outlet Y	\$16	\$8

Use linear programming techniques (sliding line or corner point) to determine the best way to ship to minimise cost.

The following information relates to question 11, 12 and 13.

A cereal retailer obtains a particular type of cereal from three different suppliers, P, Q and R, subject to the following restrictions on the amounts of sultanas and dried apricots.

	Sultanas	Apricots	Cost per kg ($)
Supplier P	15%	12%	1.20
Supplier Q	19%	10%	0.90
Supplier R	12%	15%	1.10
Minimum amount of ingredient needed	14%	11%	

The total amount of cereal required is 100 kg.

11 If x = the number of kilograms of cereal from Supplier P and

y = the number of kilograms of cereal from Supplier Q,

then the constraint inequality caused by the restriction on sultanas is:

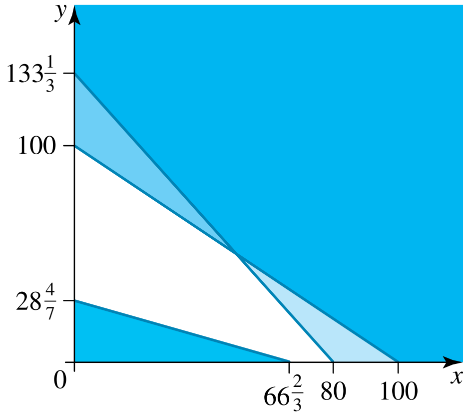
A $15x + 12y \leq 1.20$
B $3x + 7y \geq 200$
C $3x + 7y \leq 200$
D $3x + 5y \geq 400$
E $3x + 5y \leq 400$

12 The inequality which is not a constraint for this situation is:

A $3x + 5y \leq 400$
B $x \geq 0$
C $y \geq 0$
D $x + y \geq 100$
E $x + y \leq 100$

13 The feasible region for this situation is:

A



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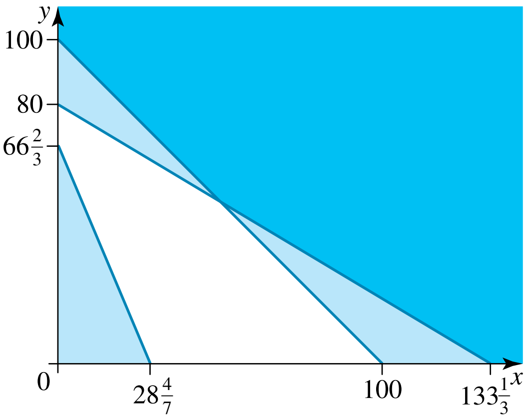
B



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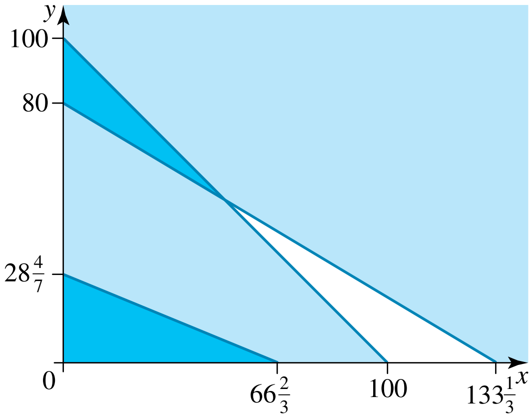
C



☐ Region required

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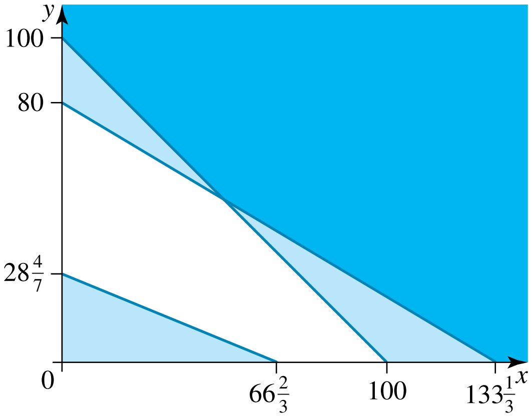
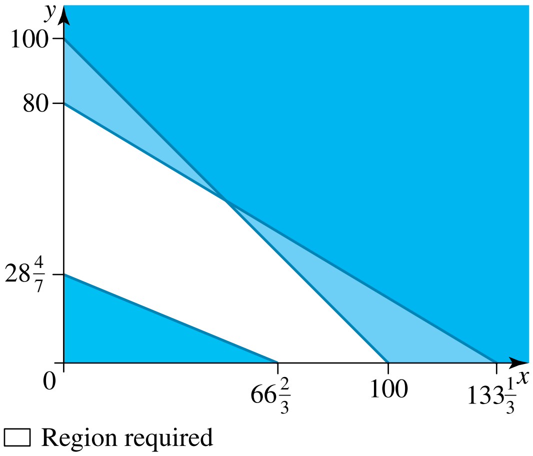


☐ Region required

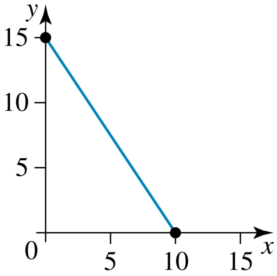
E



14 The line shown below has the equation of the form $2y + 3x = 6c$.

The value of c must therefore be:

A 2
B 5
C 6
D –6
E –5



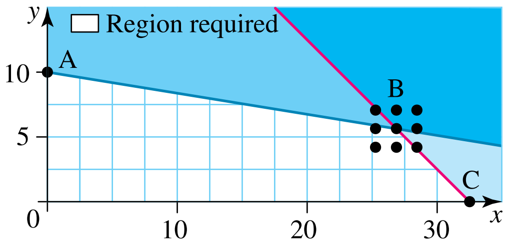
MASTER

15 A refinery gets its petrol from three oil wells. Each well provides oil with a certain amount of lead and iso-octane according to the following table. The blended product must contain a *maximum* of 3.5% lead and a *minimum* of 65% iso-octane.

	Ingredient		
	Lead	**Iso-octane**	**Cost per litre ($)**
Oil well A	4%	70%	0.24
Oil well B	2%	60%	0.22
Oil well C	6%	80%	0.26
Amount of ingredient allowed	3.5% maximum	65% minimum	

Use linear programming techniques (sliding line or corner point) to determine how much to buy, at minimum cost, from each well per 100 litres of petrol refined.

16 Refer back to the text example on pages 772–3, which illustrated a limitation of the linear programming techniques used in this topic. The optimal solution involved non-integers but the situation required integer solutions. The 9 integer points surrounding the optimal solution are shown in the diagram {(26, 7), (27, 7), (28, 7), ..., (28, 5)}.



The constraints are: $x \geq 0$

$$\begin{aligned} y &\geq 0 \\ 3.7x + 25y &\leq 250 \\ 6.3x + 5y &\leq 200 \end{aligned}$$

The objective function is: number $= x + y$.

Check each integer point to see if the values satisfy the constraints and determine which one maximises the objective function. That is, find the maximum numbers of blankets and cookware sets that can be shipped.

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15.6 Review

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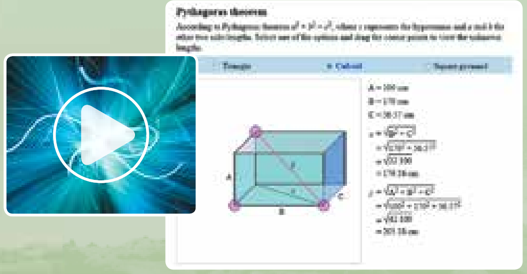

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15 Answers

Unless stated otherwise, for all graphical answers to topic 15, ▭ indicates the region required.

EXERCISE 15.2

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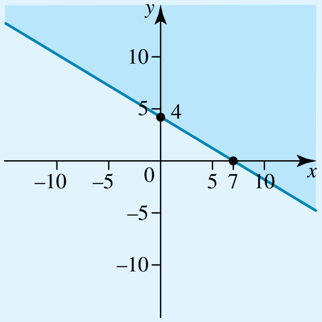

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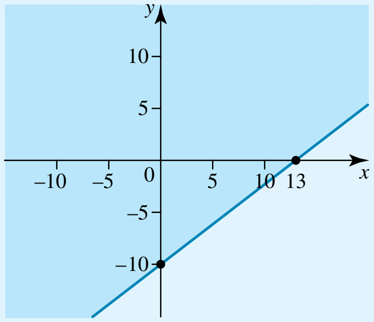

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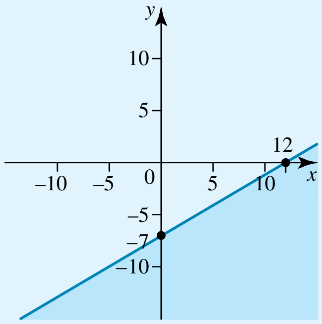

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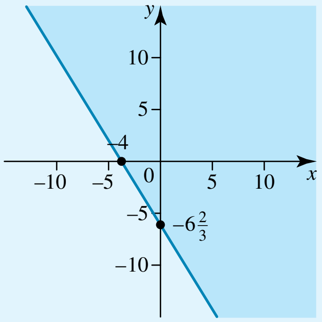

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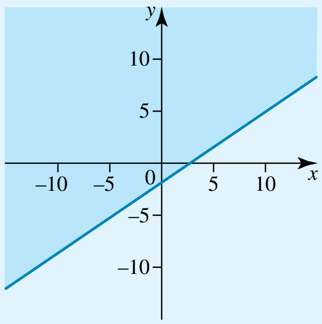

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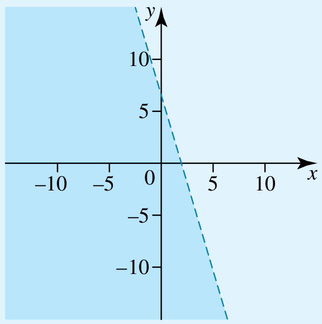

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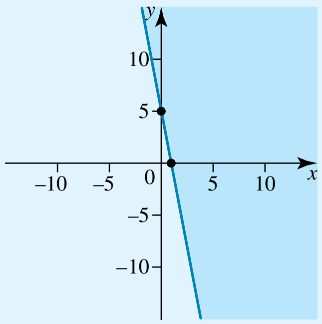

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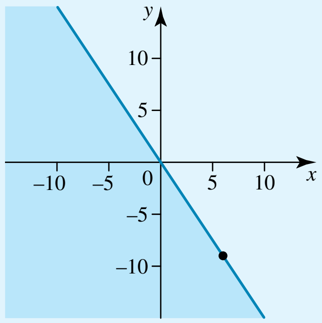

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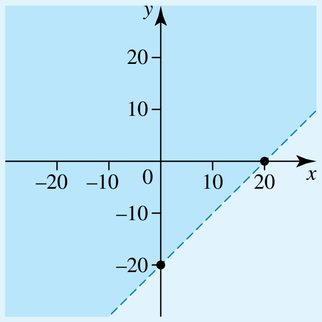

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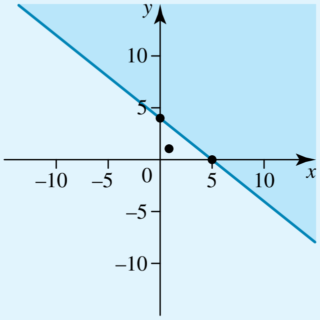

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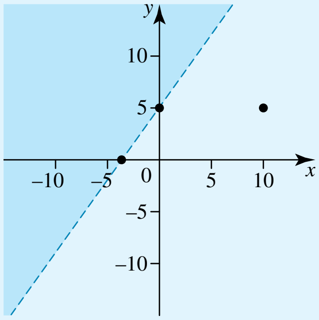

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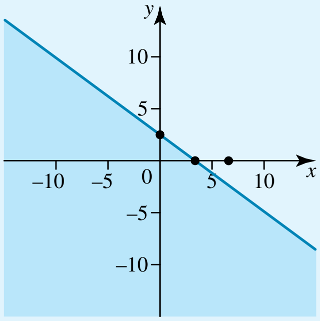

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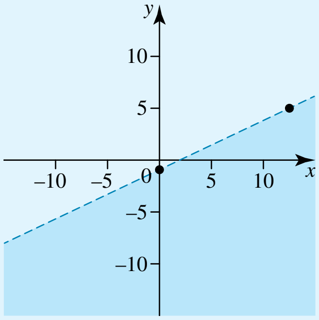

11 B, E

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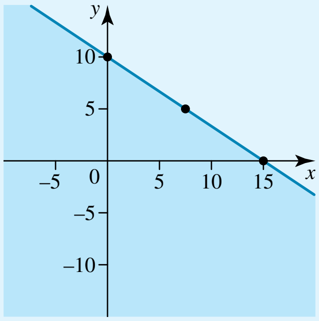

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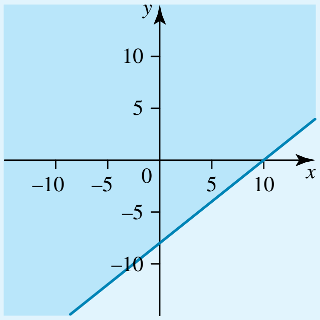

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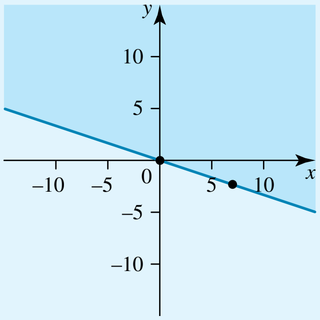

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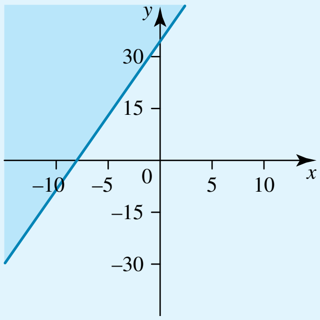

EXERCISE 15.3

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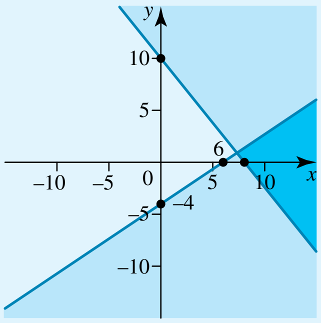

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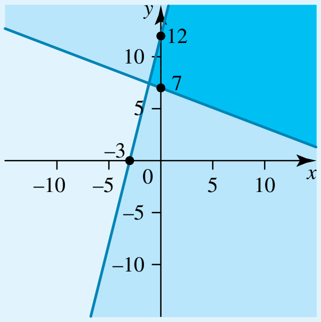

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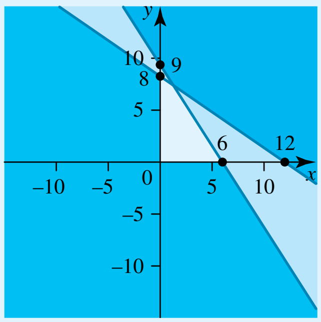

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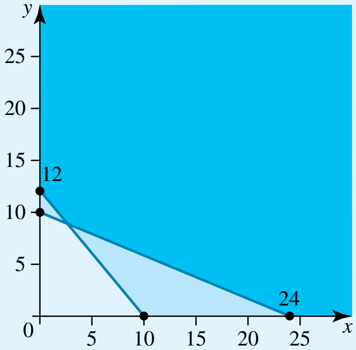

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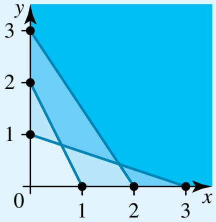

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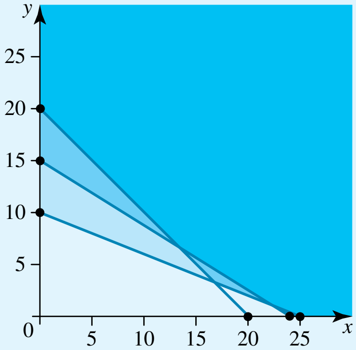

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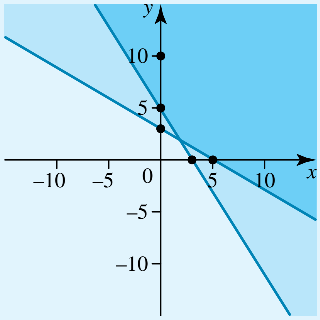

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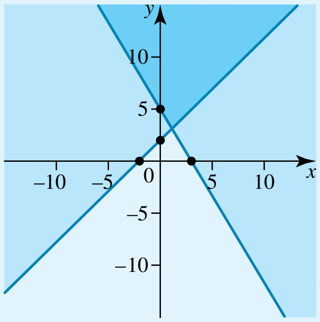

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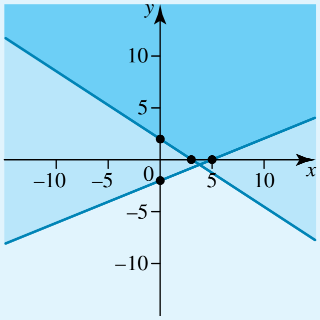

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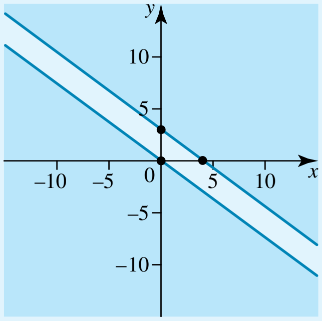

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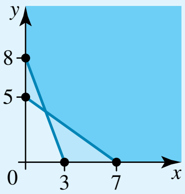

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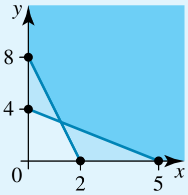

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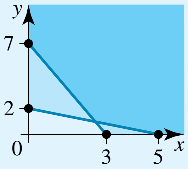

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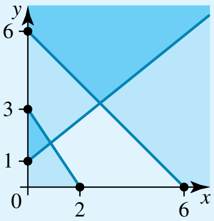

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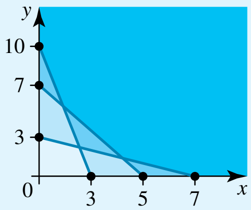

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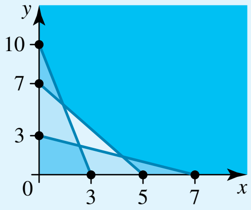
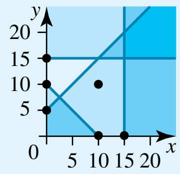


17 D

18 a $2x + 2y \geq 20, -x + y \geq 5, x \leq 15, y \leq 15, x \geq 0, y \geq 0$

b



c (10, 10) does not lie in the solution region.

EXERCISE 15.4

1 a x = the number of 2-person cabins rented,
y = the number of 3-person cabins rented.

b $40x + 45y \leq 3000$

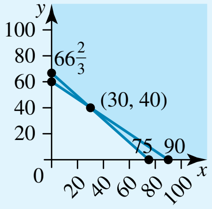
c $2x + 3y \leq 180$

2 x = amount invested in bonds,
y = amount invested in real estate mortgages.

a $x + y \leq 1\,000\,000$

b $x \geq 2y$ or $x - 2y \geq 0$

3



4 Total return $= 0.06x + 0.11y$

5 The club should rent 60 3-person cabins (and not rent any 2-person cabins).

6 The optimal 'mix' is to invest one-third in real estate (at 11%) and the rest in bonds (at 6%) in order to maximise return.

7 x = number of flat screen TVs,
y = number of curved screen TVs

Objective function: Number $= x + y$

The company should make 15 000 flat screen TVs and 0 curved screen TVs.

8 Teacher A: $20x + 8y \leq 320$

Teacher B: $10x + 2y \leq 100$

10 text chapters and 0 tests

9 x = number of hours on football,
y = number of hours on basketball

Objective function: Earnings $= 6.5\,x + 10\,y$

Kyle should play football for 20 hours and basketball for 10 hours to maximise his earnings at $230.

10 Staff to build the sofas: $9x + 8y \leq 207$

Shipping and packaging constraint: $2x + 5y \leq 75$

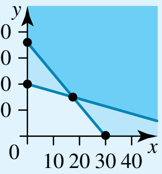
The optimal solution is when 15 leather and 9 cloth sofas are produced.

11 a x = number of model A made

y = number of model B made

b $x \geq 0, y \geq 0, 5x + 18y \leq 360, 5x + 4y \leq 150$

c



d Profit $= 2.5x + 4y$

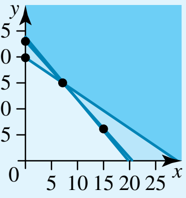
e $x = 18, y = 15$, Profit = $105

12 a x = number of sedans sprayed

y = number of utilities sprayed

b $x \geq 0, y \geq 0, 5x + 7y \leq 140, 9x + 8y \leq 183,$
$6x + 5y \leq 120$

13 a



b Number $= x + y$

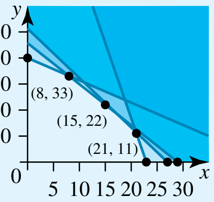
c $x = 7, y = 15$, Vehicles = 22

14 a x = models of monitor

y = models of disk drive

b $x \geq 0, y \geq 0, 7x + 8y \leq 320, 11x + 7y \leq 319,$
$22x + 12y \leq 594, 11x + 3y \leq 264$

15 a



b Profit $= 150\,000x + 120\,000y$

$x = 8, y = 33$, Profit = $5 160 000

16 The solution would be $x = 0, y = 6$, with a profit of $780.

17 x = number of apples

y = number of pieces of cheese

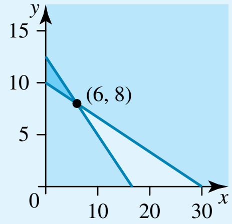
Constraints: $x \geq 0, y \geq 0,$

$3x + 10y \leq 98, 3x + 4y \geq 50$

Objective function:

Calorie intake $= 100x + 85y$

Solution: $x = 6$, $y = 8$, Calories $= 1280$



18 x = indoor paint

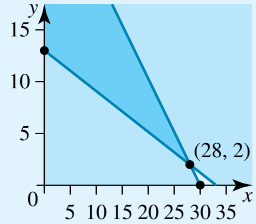
y = outdoor paint

Constraints: $x \geq 0$, $y \geq 0$, $3x + 8y \leq 100$, $x + y \geq 30$

Objective function:

Profit $= 2.50x + 3.25y$

Solution: $x = 33.33$, $y = 0$, Profit = \$83.33



19 x = number of roses

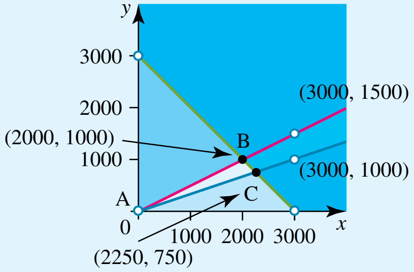
y = number of tulips

Constraints: $x \geq 0$, $y \geq 0$, $x + y \leq 3000$, $x \leq 3y$, $x \geq 2y$

Objective function: Income $= 15x + 6y$

Solution: $x = 2250$, $y = 750$,

Income = \$38 250



20 x = value of shares

y = value of gold

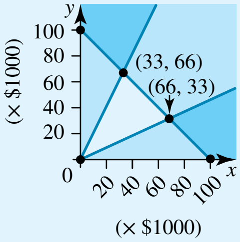
Constraints: $x \geq 0$, $y \geq 0$, $x + y \leq 100\,000$, $x - 2y \leq 0$, $x - 0.5y \geq 0$

Objective function:

Profit $= 0.1x + 0.16y$

Solution: $x = 33\,333.33$,

$y = 66\,666.67$, profit = \$14 000



21 C

22 **a** $x + y \leq 50$

$12\,000x + 12\,500y \leq 490\,000$

$y \geq x$

$x \geq 0$

$y \geq 0$

b OC is $y \geq x$

BF is $x + y \leq 50$

AE is $12\,000x + 12\,500y \leq 490\,000$

c C (25, 25)

D (20, 20)

d 20 sedans and 20 wagons

EXERCISE 15.5

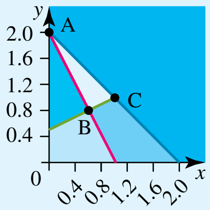
1 **a** Let x = tonnes of grass from Greener Grass

Let y = tonnes of grass from Green Lawns

Let $(2 - x - y)$= tonnes of grass from Green Turf

b $10x + 5y \geq 10$, $4y - 2x \geq 2$, $x \geq 0$, $y \geq 0$, $2 - x - y \geq 0$ or $x + y \leq 2$

c



Point A: $(0, 2)$, Point B: $\left(\frac{3}{5}, \frac{4}{5}\right)$, Point C: $(1, 1)$

d Cost $= 25x + 15y + 150$

e $\frac{3}{5}$ of a tonne from Greener Grass, $\frac{4}{5}$ of a tonne from Green Lawns, $\frac{3}{5}$ of a tonne from Green Turf

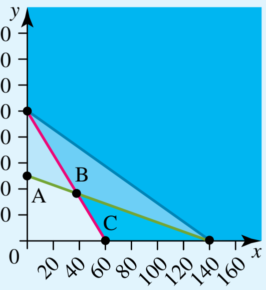
2 **a** x = litres of orange juice from Supplier A

y = litres of orange juice from Supplier B

$(150 - x - y)$ = litres of orange juice from Supplier C

b $5x + 3y \leq 300$, $x + 3y \leq 150$, $x \geq 0$, $y \geq 0$, $150 - x - y \geq 0$ or $x + y \leq 150$

c



Point A: (0, 50)

Point B: (37.5, 37.5)

Point C: (60, 0)

d Cost $= 11\,250 - 30x - 15y$

e 60 litres from Supplier A

0 litres from Supplier B

90 litres from Supplier C

3 a

	Nursery A	Nursery B	Total
Distributor X	x	$(600 - x)$	600
Distributor Y	y	$(700 - y)$	700
Total	750	900	

b $x \geq 0$

$y \geq 0$

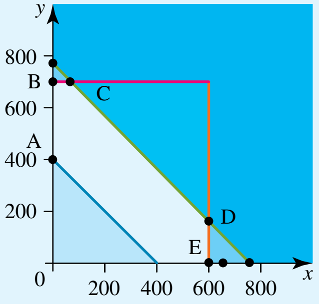
$600 - x \geq 0$ or $x \leq 600$

$700 - y \geq 0$ or $y \leq 700$

Nursery A: $x + y \leq 750$

Nursery B: $x + y \geq 400$

c



Point A: (0, 400)

Point B: (0, 700)

Point C: (50, 700)

Point D: (600, 150)

Point E: (600, 0)

Point F: (400, 0)

d Total cost $= 935 - 0.1x - 0.05y$

e Minimum cost is at Point D

Ship 600 from nursery A to distributor X

Ship 150 from nursery A to distributor Y

Ship 0 from nursery B to distributor X

Ship 550 from nursery B to distributor Y

4 a

	Factory A	Factory B	Total
Distributor X	x	$(23 - x)$	23
Distributor Y	y	$(19 - y)$	19
Total	27	27	

b $x \geq 0$

$y \geq 0$

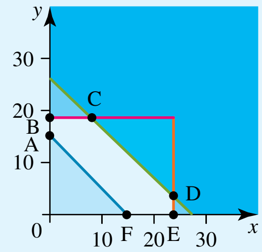
$23 - x \geq 0$ or $x \leq 23$

$19 - y \geq 0$ or $y \leq 19$

Factory A: $x + y \leq 27$

Factory B:$x + y \geq 15$

c



Point A: (0, 15)

Point B: (0, 19)

Point C: (8, 19)

Point D: (23, 4)

Point E: (23, 0)

Point F: (15, 0)

d Total cost $= 466 - 5x - 5y$

e Minimum cost is at Point C or D: (8, 19) or (23, 4).

Ship 8 or 23 from factory A to distributor X

Ship 19 or 4 from factory A to distributor Y

Ship 15 or 0 from factory B to distributor X

Ship 0 or 15 from factory B to distributor Y

5 a x = litres purchased from supplier A

y = litres purchased from supplier B

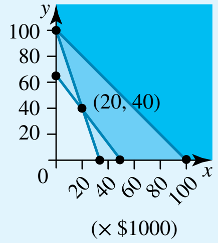
$100 - x - y$ = litres purchased from supplier C

b $4x + 3y \leq 200$

c $3x + y \leq 100$

d $x \geq 0, y \geq 0, x + y \leq 100$

6 a



b Cost $= -20x - 10y + 5500$ (in cents)

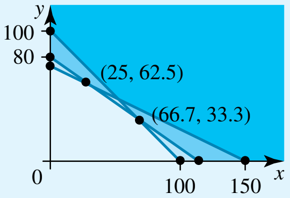
c $x = 20$, $y = 40$. Cost = $47.00. Purchase 20 litres from A, 40 litres from each of B and C.

7 x = kg purchased from supplier A

y = kg purchased from supplier B

$100 - x - y$ = kg purchased from supplier C

Constraints: $7x + 10y \leq 800$, $2x + 4y \leq 300$, $x \geq 0$, $y \geq 0$, $x + y \leq 100$



Objective function:

Cost = $-0.3x - 0.5y + 140$

$x = 25$, $y = 62.5$, Cost = \$101.25

Purchase 25 kg from A, 62.5 kg from B and 12.5 kg from C

8 a x = computers from factory 1 to distributor X

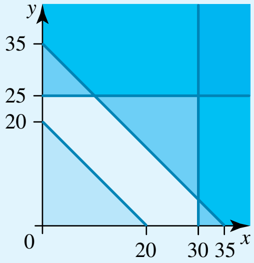
y = computers from factory 1 to distributor Y

$(30 - x)$ = computers from factory 2 to distributor X

$(25 - y)$ = computers from factory 2 to distributor Y

b $x \geq 0, y \geq 0, x \leq 30, y \leq 25, x + y \leq 35, x + y \geq 20$

c



d Cost = $-30x - 15y + 2525$

e $x = 30$, $y = 5$, Cost = \$1550

Ship 30 from factory 1 to distributor X, 5 from factory 1 to distributor Y, 20 from factory 2 to distributor Y.

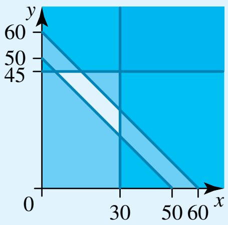
9 Variables:

x = refrigerators from factory 1 to distributor X

y = refrigerators from factory 1 to distributor Y

$(30 - x)$ = refrigerators from factory 2 to distributor X

$(45 - y)$ = refrigerators from factory 2 to distributor Y



$x \geq 0, y \geq 0, x \leq 30, y \leq 45,$

$x + y \leq 60, x + y \geq 50$

Cost = $-6x - 7y + 1935$

$x = 15$, $y = 45$, Cost = \$1530

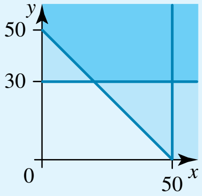
Ship 15 from factory 1 to distributor X, 45 from factory 1 to distributor Y, 15 from factory 2 to distributor X.

10 Variables:

x = dozen pairs of shoes from factory 1 to distributor X

y = dozen pairs of shoes from factory 1 to distributor Y

$(50 - x)$ = dozen pairs of shoes from factory 2 to distributor X



$(30 - y)$ = dozen pairs of shoes from factory 2 to distributor Y $x \geq 0, y \geq 0, x \leq 50, y \leq 30,$ $x + y \leq 50, x + y \geq 0$

Cost = $3x + 8y + 790$ $x = 0, y = 0.$

Cost = \$790 Ship 50 from factory 2 to distributor X, 30 from factory 2 to distributor Y.

11 B

12 D

13 D

14 B

15 x = litres purchased from supplier A

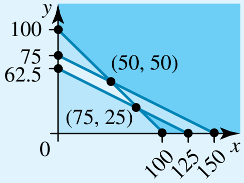
y = litres purchased from supplier B

$100 - x - y$ = litres purchased from supplier C

Constraints: $2x + 4y \geq 250,$

$x + 2y \leq 150, x \geq 0, y \geq 0,$

$x + y \leq 100$



Objective function:

Cost = $-0.02x - 0.04y + 26$

$x = 0$, $y = 75$. Cost = \$23

$x = 50$, $y = 50$. Cost = \$23

Purchase 75 L from B and 25 L from C (none from A) or purchase 50 L from A and 50 L from B (none from C)

16 Feasible solutions are (26, 5), (27, 5), (26, 6). Choose either of the last 2 points ($x + y = 32$).

The maximum number of Blankets and cookware sets that can be shipped is 32.

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