JACARANDA MATHS QUEST
GENERAL MATHEMATICS 11

VCE UNITS 1 AND 2 | SECOND EDITION

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VCE UNITS 1 AND 2 | SECOND EDITION

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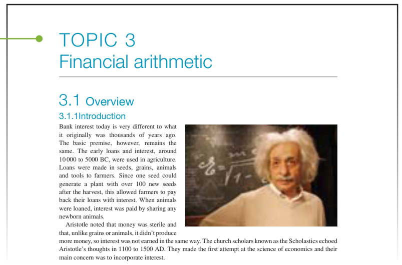
ABOUT THIS RESOURCE

Jacaranda Maths Quest 11 General Mathematics VCE Units 1 & 2 Second Edition has been updated to enhance our in-depth coverage of the Study Design, with hundreds of new questions to expand students' understanding and improve learning outcomes. The *Jacaranda Maths Quest* series provides easy-to-follow text and is supported by a bank of resources for both teachers and students. At Jacaranda we believe that every student should experience success and build confidence, while those who want to be challenged are supported as they progress to more difficult concepts and questions.

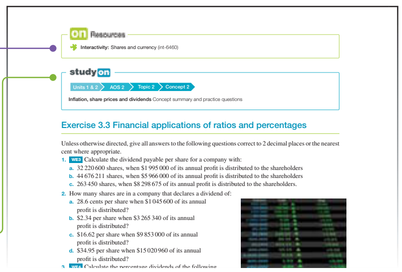
Preparing students for exam success

Topic openers place mathematics in real-world contexts to drive engagement.



FREE access to studyON — our exam, study, revision and practice tool — is included with every title. studyON allows you to revise at the concept, topic, area of study or unit level.

Each subtopic concludes with carefully graded technology free and technology active questions.



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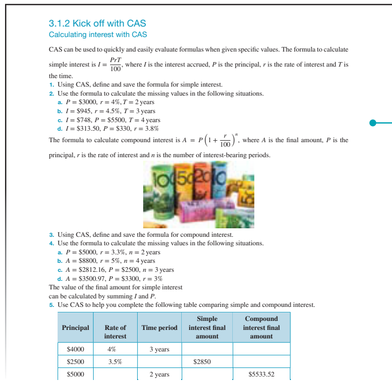
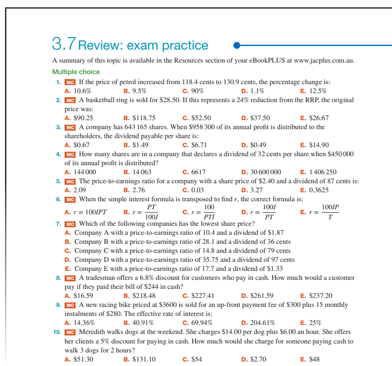


Each topic opens with an engaging *Kick off with CAS* activity designed to stimulate students' interest and curiosity and to highlight the important applications of CAS technology in developing deep understanding of the mathematical concepts presented.

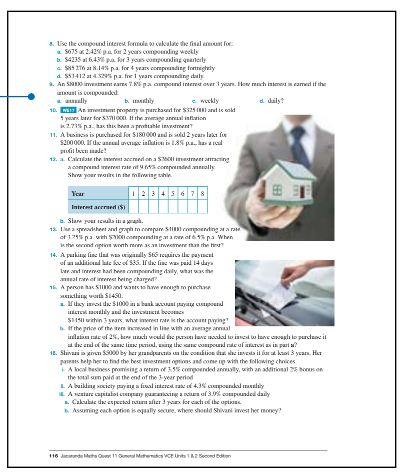


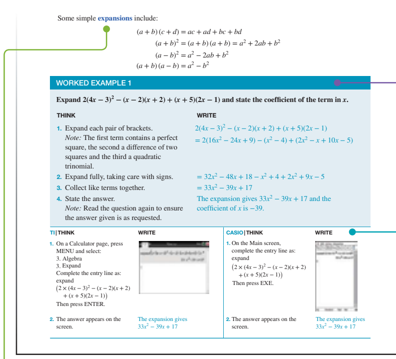
At the conclusion of each topic, Review questions allow you to revise the entire topic. Short-answer technology free questions allow you to consolidate your learning. Technology active questions are provided through multiple-choice and extended-response questions.

Features of the *Maths Quest* series

Questions and topics are sequenced from lower to higher levels of complexity; ideas and concepts are logically developed and questions are carefully graded, allowing every student to achieve success.

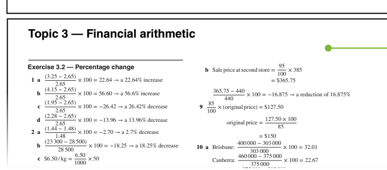
An extensive glossary of mathematical terms is provided in print and as a hover-over feature in the eBookPLUS.





Fully worked examples in the Think/Write format provide guidance and are linked to questions.

Selected worked examples demonstrate the use of CAS calculators.



FREE fully worked solutions are provided, enabling students to get help where they need it, whether at home or in the classroom — help at the point of learning is critical. Answers are provided at the end of each topic in the print and offline PDF.

eBookPLUS features

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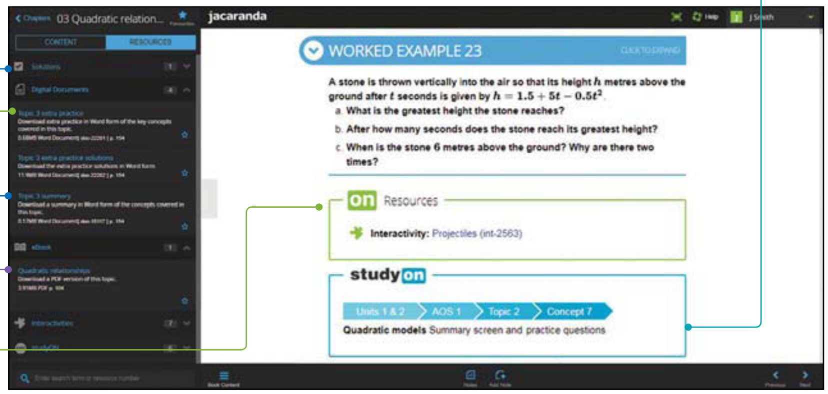
Interactivities and video eLessons placed at the point of learning to enhance understanding and correct common misconceptions

Concept summary links to studyON for study, revision and exam practice



In the Prelims section of your eBookPLUS

A downloadable PDF of the entire solutions manual, containing fully worked solutions for every question in the text

FREE copies of the *Maths Quest Manual for the TI-Nspire CAS calculator* and the *Maths Quest Manual for the Casio ClassPad II calculator*

Downloadable fully worked solutions, with screenshots, to every question in the Casio manual's problem-solving chapter.

A downloadable PDF of the entire print text

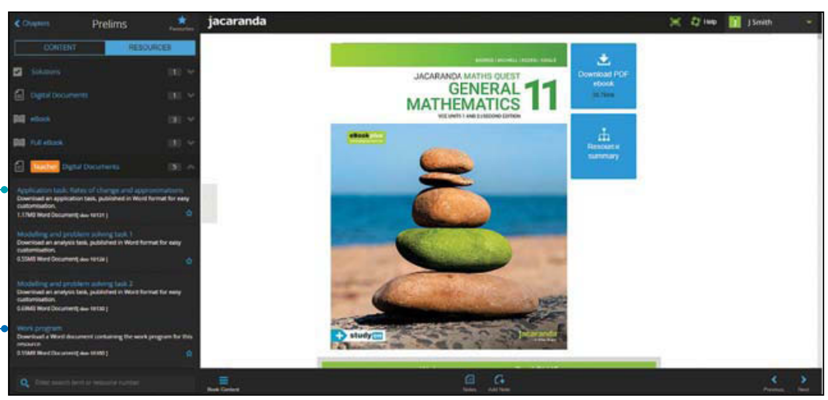
Additional resources for teachers available in the eGuidePLUS

In the Resources tab of every topic there are two topic tests in downloadable, customisable Word format with worked solutions.

In the Prelims section of the eGuidePLUS

Additional resources for teachers include an application task and two problem solving and modelling tasks.

A work program is provided to assist with classroom planning.



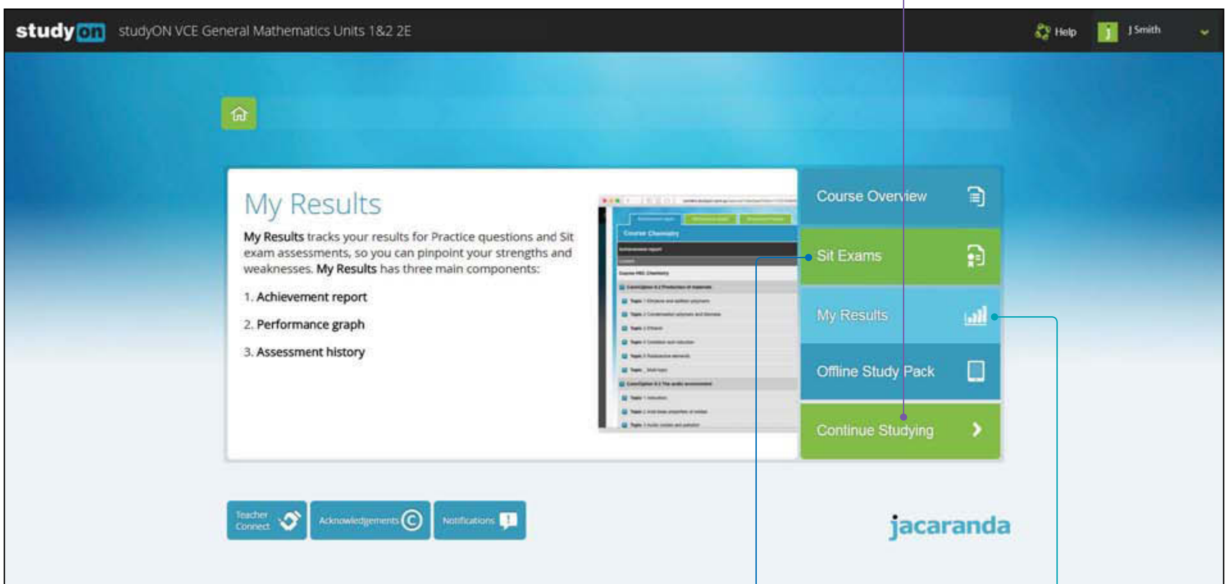
studyON — an invaluable exam preparation tool

studyON provides a complete study solution. An interactive and highly visual online study, revision and exam practice tool, it is designed to help students and teachers maximise exam results.

Concept summary screens and interactivities summarise key concepts and help prevent misconceptions.

Direct links from the eBookPLUS help scaffold students' understanding and study practices.

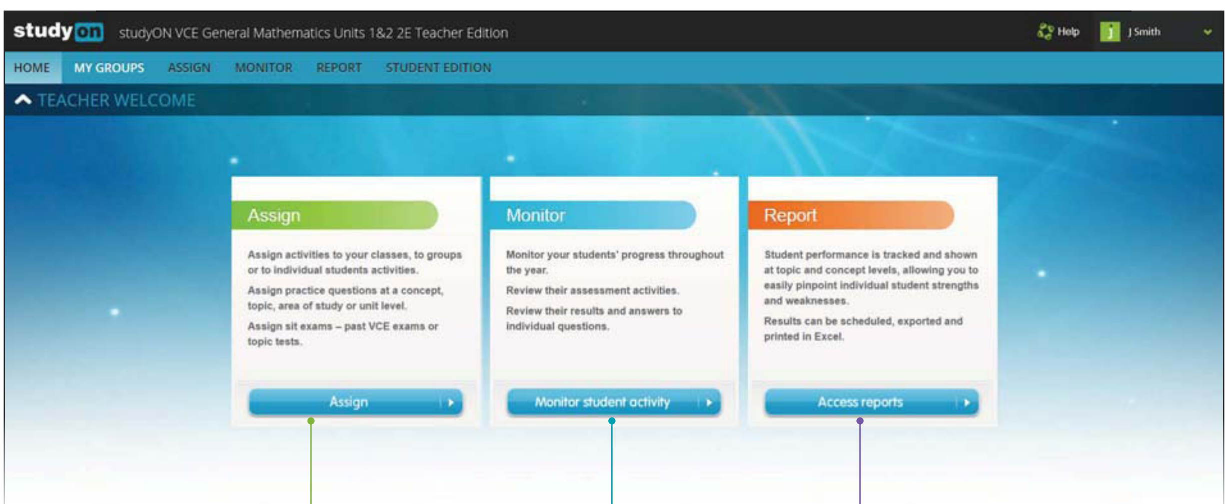
The studyON question hierarchy allows students in the *Continue Studying* feature to revise across the entire course, or to drill down to concept level for a more granular set of questions.



- The Sit Exams feature allows students to sit past VCAA exams (Units 3 & 4) or topic tests (Units 1 & 2) in timed, exam-like situations.
- Students have access to 1000+ past VCAA questions or custom-authored practice questions at a concept, topic or entire course level, and receive immediate feedback for every question.

studyON's built-in progress tracker enables self-diagnosis of strengths and weaknesses at a topic and concept level, so students know exactly what needs extra revision and can sit their exams with confidence.

studyON Teacher edition is a powerful diagnostic tool



Enables teachers to assign activities for extra revision and practice, and track progress at an individual, group and classroom level

Allows teachers to monitor students' activities and results to pinpoint strengths and weaknesses. Armed with evidence-based insights, teachers can intervene at the right time.

Alignment with the Jacaranda text helps with planning, and instant feedback saves marking time. Built-in reporting functionality lets teachers easily schedule, export and print reports in Excel.

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Access your online Jacaranda resources anywhere, anytime, from any device in three easy steps:

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eBookPLUS is an electronic version of the textbook, together with a targeted range of supporting multimedia resources.

eBookPLUS features:

- **eBook** — the entire textbook in electronic format
- **Digital documents** designed for easy customisation and editing
- **Interactivities** to reinforce and enhance students' learning
- **eLessons** — engaging video clips and supporting material
- **Weblinks** to relevant support material on the internet

eGuidePLUS features assessment and curriculum material to support teachers.

studyON is an interactive and highly visual online study, revision and exam practice tool designed to help students and teachers maximise exam results.

studyON features:

- **Concept summary screens** provide concise explanations of key concepts, with relevant examples.
- **Access 1000+ past VCAA questions** or custom-authored practice questions at a concept, topic or entire course level, and receive immediate feedback.
- **Sit past VCAA exams** (Units 3 & 4) or **topic tests** (Units 1 & 2) in exam-like situations.
- **Video animations and interactivities** demonstrate concepts to provide a deep understanding (Units 3 & 4 only).
- **All results and performance in practice and sit questions** are tracked to a concept level to pinpoint strengths and weaknesses.

NEED HELP? Go to www.jacplus.com.au and select the Help link.

- Visit the JacarandaPLUS Support Centre at **http://jacplus.desk.com** to access a range of step-by-step user guides, ask questions or search for information.
- **Contact** John Wiley & Sons Australia, Ltd.
Email: support@jacplus.com.au
Phone: 1800 JAC PLUS (1800 522 7587)

Minimum requirements

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com**.

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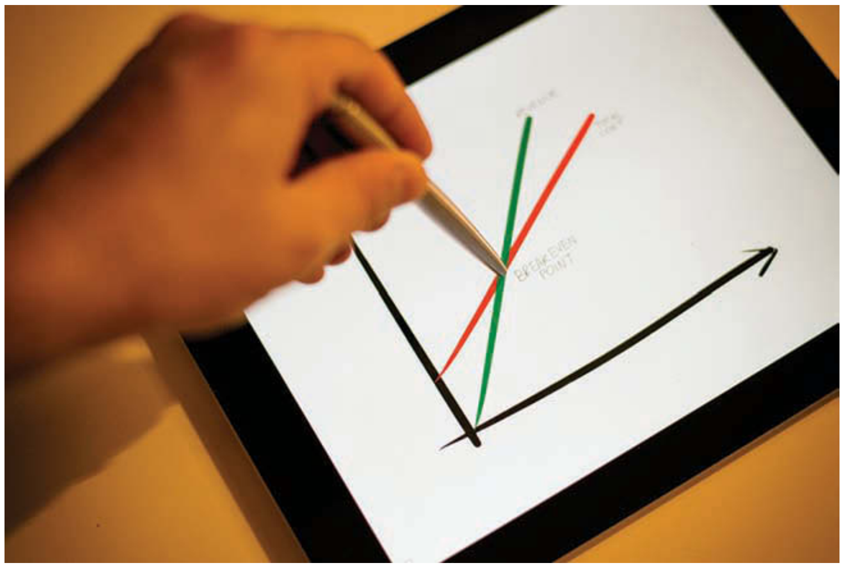
TOPIC 1
Linear relations and equations

1.1 Overview

1.1.1 Introduction



Linear equations have been around for over 4000 years. A simple 2×2 linear equation system with two unknowns was first solved by the Babylonians. Around 200 BC, the Chinese demonstrated the ability to solve a 3×3 system of equations. However, it wasn't until the 17th century that progress was made in linear algebra by the inventor of calculus, Leibniz. This was followed by work by Cramer, and later Gauss. Linear equations themselves were invented in 1843 by Irish mathematician Sir William Rowan Hamilton. He made important contributions to mathematics and his work was also used in quantum mechanics. Sir William Hamilton was reputedly a genius: at the young age of 13, he reportedly spoke 13 languages, and at 22 he was a professor at the University of Dublin. His work has been applied in many fields, given that there are many situations in which there is a direct relationship between two variables. Classic examples are of water being added to a tank at a constant rate or a taxi trip being charged at a constant rate per kilometre. A linear equation to model the cost of a taxi trip can be used to compare one taxi company to another. The break-even point refers to the point at which the cost is the same for each taxi company. This is the point at which the two linear graph intersect and it can be found using a graphical technique, or using substitution or elimination techniques of simultaneous equations.

LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au

1.1.2 Kick off with CAS

Linear equations with CAS

Linear equations link two variables in a linear way such that as one variable increases or decreases, the other variable increases or decreases at a constant rate. We can use CAS to quickly and easily solve linear equations.

1. Use CAS to solve the following linear equations.

a. $6x = 24$

b. $0.2y - 3 = 7$

c. $5 - 3p = -17$

CAS can also be used to solve linear equations involving fractions and brackets.

2. Use CAS to solve the following linear equations.

a. $\dfrac{5x + 3}{2} = 14$

b. $0.5(y + 6) = 9$

c. $\dfrac{3(t + 1)}{2} = 12$

A literal equation is an equation containing several pronumerals or variables. We can solve literal equations by expressing the answer in terms of the variable we are looking to solve for.

3. Use CAS to solve the following literal equations for a.

a. $ax + b = 2m$

b. $m(a - 3b) = 2t$

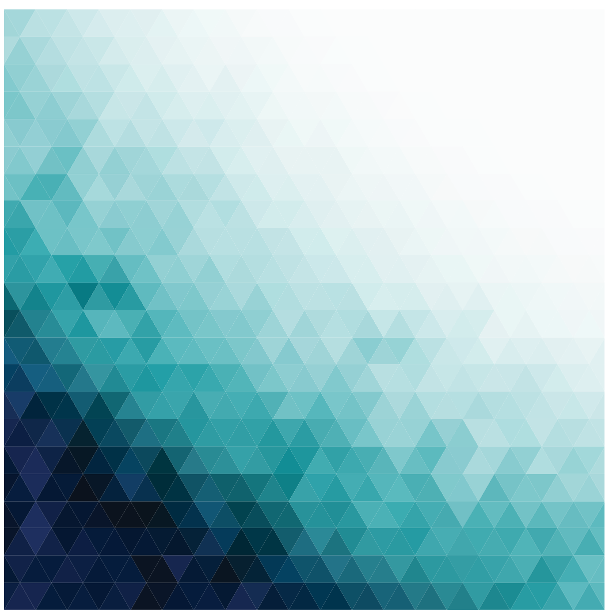
c. $\dfrac{3am - 4}{t} = cd$

4. a. The equation $A = \frac{1}{2}bh$ is used to find the area of a triangle given the base length and the height. Use CAS to solve $A = \frac{1}{2}bh$ for b.

b. Use your answer to part **a** to find the base lengths of triangles with the following heights and areas.

i. Height = 5 cm, area = 20 cm^2

ii. Height = 6.5 cm, area = 58.5 cm^2



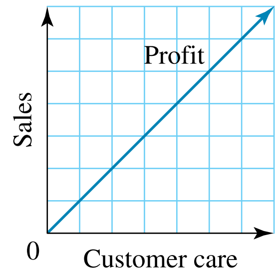
on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

1.2 Linear relations

1.2.1 Identifying linear relations

A **linear relation** is a relationship between two **variables** that when plotted give a straight line. Many real-life situations can be described by linear relations, such as water being added to a tank at a constant rate, or money being saved when the same amount of money is deposited into a bank at regular time intervals.



When a linear relation is expressed as an equation, the highest power of both variables in the equation is 1.

WORKED EXAMPLE 1

Identify which of the following equations are linear.

a. $y = 4x + 1$ **b.** $b = c^2 - 5c + 6$ **c.** $y = \sqrt{x}$

d. $m^2 = 6(n - 10)$ **e.** $d = \dfrac{3t + 8}{7}$ **f.** $y = 5^x$

THINK	WRITE
a. 1. Identify the variables.	**a.** y and x
2. Write the power of each variable.	y has a power of 1. x has a power of 1.
3. Check if the equation is linear.	Since both variables have a power of 1, this is a linear equation.
b. 1. Identify the two variables.	**b.** b and c
2. Write the power of each variable.	b has a power of 1. c has a power of 2.
3. Check if the equation is linear.	c has a power of 2, so this is not a linear equation.
c. 1. Identify the two variables.	**c.** y and x
2. Write the power of each variable. *Note:* A square root is a power of $\frac{1}{2}$.	y has a power of 1. x has a power of $\frac{1}{2}$.
3. Check if the equation is linear.	x has a power of $\frac{1}{2}$, so this is not a linear equation.
d. 1. Identify the two variables.	**d.** m and n
2. Write the power of each variable.	m has a power of 2. n has a power of 1.
3. Check if the equation is linear.	m has a power of 2, so this is not a linear equation.

e. 1. Identify the two variables. — **e.** d and t

2. Write the power of each variable. — d has a power of 1.
t has a power of 1.

3. Check if the equation is linear. — Since both variables have a power of 1, this is a linear equation.

f. 1. Identify the two variables. — **f.** y and x

2. Write the power of each variable. — y has a power of 1.
x is the power.

3. Check if the equation is linear. — Since x is the power, this is not a linear equation.

1.2.2Rules for linear relations

Rules define or describe relationships between two or more variables. Rules for linear relations can be found by determining the **common difference** between consecutive terms of the pattern formed by the rule.

Consider the number pattern 4, 7, 10 and 13. This pattern is formed by adding 3 (the common difference is 3). If each number in the pattern is assigned a term number as shown in the table, then the expression to represent the common difference is $3n$ (i.e. $3 \times n$).

Term number, *n*	1	2	3	4
3*n*	3	6	9	12

Each term in the number pattern is 1 greater than $3n$, so the rule for this number pattern is $3n + 1$.

If a rule has an equals sign, it is described as an equation. For example, $3n+1$ is referred to as an expression, but if we define the term number as t, then $t = 3n + 1$ is an equation.

WORKED EXAMPLE 2

Find the equations for the linear relations formed by the following number patterns.

a. 3, 7, 11, 15 **b. 8, 5, 2, −1**

THINK	WRITE
a. 1. Determine the common difference.	**a.** $7 - 3 = 4$ $15 - 11 = 4$
2. Write the common difference as an expression using the term number n.	$4n$
3. Substitute any term number into $4n$ and evaluate.	$n = 3$ $4 \times 3 = 12$
4. Check the actual term number against the one found.	The actual 3rd term is 11.
5. Add or subtract a number that would result in the actual term number.	$12 - 1 = 11$
6. Write the equation for the linear relation.	$t = 4n - 1$
b. 1. Determine the common difference.	**b.** $5 - 8 = -3$ $2 - 5 = -3$
2. Write the common difference as an expression using the term number n.	$-3n$

3. Substitute any term number into $-3n$ and evaluate. | $n = 2$
$-3 \times 2 = -6$

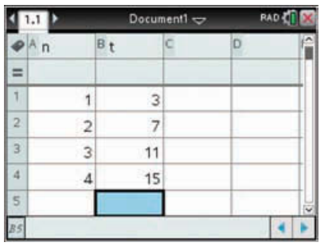
4. Check the actual term number against the one found. | The actual 2nd term is 5.

5. Add or subtract a number that would result in the actual term number. | $-6 + 11 = 5$

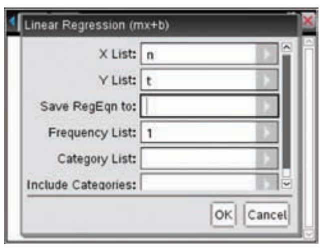
6. Write the equation for the linear relation. | $t = -3n + 11$

TI | THINK / **WRITE**

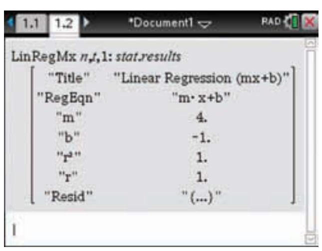
a. 1. On a Lists & Spreadsheet page, label the first column n for the term number, and the second column t for the term value.
As there are four terms given in this sequence, enter the numbers 1 to 4 in the first column.
Enter the terms of the given sequence in the second column.



2. On a Calculator page, press MENU and select:
6: Statistics
1: Stat Calculations
3: Linear Regression (mx + b) …
Select n as the X List and t as the Y List, then select OK.



3. Interpret the output on the screen.

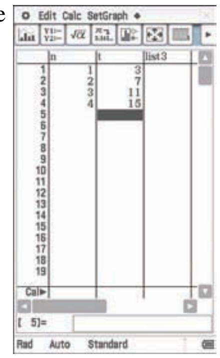


The equation is given in the form $y = mx + b$, where $y = t_n$, $m = 4$, $x = n$ and $b = -1$.

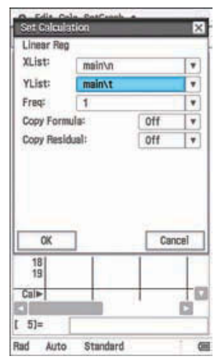
4. State the answer. | The equation is $t_n = 4n - 1$.

CASIO | THINK / **WRITE**

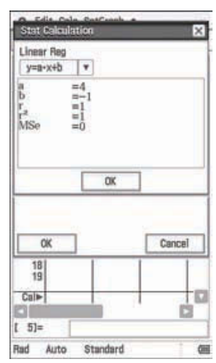
a. 1. On a Statistics screen, change the label of "list 1" to n and that of "list 2" to t.
As there are four terms given in this sequence, enter the numbers 1 to 4 in the first column.
Enter the terms of the given sequence in the second column.



2. On the Statistics screen, select:
- Calc
- Regression
- Linear Reg
Select main\n as the XList and main\t as the YList, then select OK.



3. Interpret the output on the screen.



The equation is given in the form $y = ax + b$, where $y = t_n$, $a = 4$, $x = n$, and $b = -1$.

4. State the answer. | The equation is $t_n = 4n - 1$.

Note: It is good practice to substitute a second term number into your equation to check that your answer is correct.

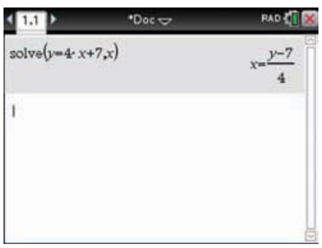
1.2.3 Transposing linear equations

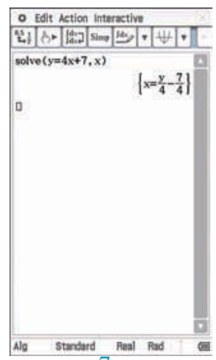
If we are given a **linear equation** between two variables, we are able to **transpose** this relationship. That is, we can change the equation so that the variable on the right-hand side of the equation becomes the stand-alone variable on the left-hand side of the equation.

WORKED EXAMPLE 3

Transpose the linear equation $y = 4x + 7$ to make x the subject of the equation.

THINK	WRITE
1. Isolate the variable on the right-hand side of the equation (by subtracting 7 from both sides).	$y - 7 = 4x + 7 - 7$ $y - 7 = 4x$
2. Divide both sides of the equation by the coefficient of the variable on the right-hand side (in this case 4).	$\frac{y-7}{4} = \frac{4x}{4}$ $\frac{y-7}{4} = x$
3. Transpose the relation by interchanging the left-hand side and the right-hand side.	$x = \frac{y-7}{4}$

TI | THINK	WRITE
1. On a Calculator page, press MENU and select: 3: Algebra 1: Solve Complete the entry line as solve $(y = 4 \cdot x + 7, x)$ then press ENTER.	
2. The answer appears on the screen.	$x = \frac{y-7}{4}$

CASIO | THINK	WRITE
1. On the Main screen, complete the entry line as: solve $(y = 4x + 7, x)$ then press EXE.	
2. The answer appears on the screen.	$x = \frac{y}{4} - \frac{7}{4}$

on Resources

Interactivity: Transposing linear equations (int-6449)

studyon

Units 1 & 2 > AOS 1 > Topic 1 > Concept 3 > **Transposition** Concept summary and practice questions

Exercise 1.2 Linear relations

1. **WE1** Identify which of the following equations are linear.
   a. $y^2 = 7x + 1$
   b. $t = 7x^3 - 6x$
   c. $y = 3(x + 2)$
   d. $m = 2^{x+1}$
   e. $4x + 5y - 9 = 0$

2. Bethany was asked to identify which equations from a list were linear. The following table shows her responses.

Equation	Bethany's response
$y = 4x + 1$	Yes
$y^2 = 5x - 2$	Yes
$y + 6x = 7$	Yes
$y = x^2 - 5x$	No
$t = 6d^2 - 9$	No
$m^3 = n + 8$	Yes

a. Insert another column into the table and add your responses identifying which of the equations are linear.
b. Provide advice to Bethany to help her to correctly identify linear equations.

3. Identify which of the following are linear equations.
a. $y = 2t + 5$
b. $x^2 = 2y + 5$
c. $m = 3(n + 5)$
d. $d = 80t + 25$
e. $y^2 = 7x + 12$
f. $\sqrt{y} = x + 5$
g. $s = \dfrac{100}{t}$

4. Samson was asked to identify which of the following were linear equations. His responses are shown in the table.

Equation	Samson's response
$y = 5x + 6$	Yes, linear
$y^2 = 6x - 1$	Yes, linear
$y = x^2 + 4$	Not linear
$y^3 = 7(x + 3)$	Yes, linear
$y = \frac{1}{2}x + 6$	Yes, linear
$\sqrt{y} = 4x + 2$	Yes, linear
$y^2 + 5x^3 + 9 = 0$	Not linear
$10y - 11x = 12$	Yes, linear

a. Based on Samson's responses, would he state that $6y^2 + 7x = 9$ is linear? Justify your answer.
b. What advice would you give to Samson to ensure that he can correctly identify linear equations?

5. **WE2** Find the equations for the linear relations formed by the following number patterns.
a. $2, 6, 10, 14, 18, \ldots$
b. $4, 4.5, 5, 5.5, 6, \ldots$

6. Jars of vegetables are stacked in ten rows. There are 8 jars in the third row and 5 jars in the sixth row. The number of jars in any row can be represented by a linear relation.
a. Find the common difference.
b. Find an equation that will express the number of jars in any of the ten rows.
c. Determine the total number of jars of vegetables.

7. A number pattern is formed by multiplying the previous term by 1.5. The first term is 2.
   a. Find the next four terms in the number pattern.
   b. Could this number pattern be represented by a linear equation? Justify your answer.
8. Find equations for the linear relations formed by the following number patterns.
   a. $3, 7, 11, 15, 19, \ldots$
   b. $7, 10, 13, 16, 19, \ldots$
   c. $12, 9, 6, 3, 0, -3, \ldots$
   d. $13, 7, 1, -5, -11, \ldots$
   e. $-12, -14, -16, -18, -20, \ldots$
9. Consider the following number pattern: $1.2, 2.0, 2.8, 3.6, 4.4, \ldots$
   a. Find the first common difference.
   b. Could this number pattern be represented by a linear equation? Justify your answer.
10. **WE3** Transpose the linear equation $y = 6x - 3$ to make x the subject of the equation.
11. Transpose the linear equation $6y = 3x + 1$ to make x the subject of the equation.
12. Transpose the following linear equations to make x the subject.
   a. $y = 2x + 5$
   b. $3y = 6x + 8$
   c. $p = 5x - 6$
13. Water is leaking from a water tank at a linear rate. The amount of water, in litres, is measured at the start of each day. At the end of the first day there are 950 litres in the tank, and at the end of the third day there are 850 litres in the tank.
   a. Complete the following table.

Day	1	2	3	4	5
Amount of water (L)	950		850		

   b. Determine the amount of water that was initially in the tank (i.e. at day 0).
   c. Determine an equation that finds the amount of water, w, in litres, at the end of any day, d.

14. At the start of the year Yolanda has $1 500 in her bank account. At the end of each month she deposits an additional $250.
   a. How much, in dollars, does Yolanda have in her bank account at the end of March?
   b. Find an equation that determines the amount of money, A, Yolanda has in her bank account at the end of each month, m.
   c. At the start of the following year, Yolanda deposits an additional $100 each month. How does this change the equation found in part b?

15. On the first day of Sal's hiking trip, she walks halfway into a forest. On each day after the first, she walks exactly half the distance she walked the previous day. Could the distance travelled by Sal each day be described by a linear equation? Justify your answer.

16. Catalina is a runner whose goal is to run a total of 350 km over 5 weeks to raise money for charity.
    a. If each week she runs 10 km more than she did on the previous week, how far does she run in week 3?
    b. Find an equation that determines the distance Catalina runs each week.

17. Using CAS or otherwise, determine an equation that describes the number pattern shown in the table below.

Term number	1	2	3	4	5
Value	–4	–2	0	2	4

18. The terms in a number sequence are found by multiplying the term number, n, by 4 and then subtracting 1. The first term of the sequence is 3.
    a. Find an equation that determines the terms in the sequence.
    b. Using CAS or otherwise, find the first 10 terms of the sequence.
    c. Show that the common difference is 4.

1.3 Solving linear equations

1.3.1 Solving linear equations with one variable

To solve linear equations with one variable, all operations performed on the variable need to be identified in order, and then the opposite operations need to be performed in reverse order.

In practical problems, solving linear equations can answer everyday questions such as the time required to have a certain amount in the bank, the time taken to travel a certain distance, or the number of participants needed to raise a certain amount of money for charity.

WORKED EXAMPLE 4

Solve the following linear equations to find the unknowns.

a. $5x = 12$ b. $8t + 11 = 20$ c. $12 = 4(n - 3)$ d. $\dfrac{4x-2}{3} = 5$

THINK	WRITE
a. 1. Identify the operations performed on the unknown.	a. $5x = 5 \times x$ So the operation is $\times 5$.
2. Write the opposite operation.	The opposite operation is $\div 5$.
3. Perform the opposite operation on both sides of the equation.	Step 1 ($\div 5$): $5x = 12$ $\dfrac{5x}{5} = \dfrac{12}{5}$ $x = \dfrac{12}{5}$
4. Write the answer in its simplest form.	$x = \dfrac{12}{5}$

b. 1. Identify the operations performed in order on the unknown.

b. $8t + 11$

The operations are $\times 8, +11$.

2. Write the opposite operations.

$\div 8, -11$

3. Perform the opposite operations in reverse order on both sides of the equation, one operation at a time.

Step 1 (-11):

$$8t + 11 = 20$$
$$8t + 11 - 11 = 20 - 11$$
$$8t = 9$$

Step 2 $(\div 8)$:

$$8t = 9$$
$$\frac{8t}{8} = \frac{9}{8}$$
$$t = \frac{9}{8}$$

4. Write the answer in its simplest form.

$$t = \frac{9}{8}$$

c. 1. Identify the operations performed in order on the unknown. (Remember operations in brackets are performed first.)

c. $4(n - 3)$

The operations are $-3, \times 4$.

2. Write the opposite operations.

$+3, \div 4$

3. Perform the opposite operations on both sides of the equation in reverse order, one operation at a time.

Step 1 $(\div 4)$:

$$12 = 4(n - 3)$$
$$\frac{12}{4} = \frac{4(n-3)}{4}$$
$$3 = n - 3$$

Step 2 $(+3)$:

$$3 = n - 3$$
$$3 + 3 = n - 3 + 3$$
$$6 = n$$

4. Write the answer in its simplest form.

$$n = 6$$

d. 1. Identify the operations performed in order on the unknown.

d. $\dfrac{4x - 2}{3}$

The operations are $\times 4, -2, \div 3$.

2. Write the opposite operations.

$\div 4, +2, \times 3$

3. Perform the opposite operations on both sides of the equation in reverse order, one operation at a time.

Step 1 (× 3):

$$\frac{4x-2}{3}=5$$

$$3\times\frac{4x-2}{3}=5\times 3$$

$$4x-2=15$$

Step 2 (+ 2):

$$4x-2=15$$

$$4x-2+2=15+2$$

$$4x=17$$

Step 3 (÷ 4):

$$4x=17$$

$$\frac{4x}{4}=\frac{17}{4}$$

$$x=\frac{17}{4}$$

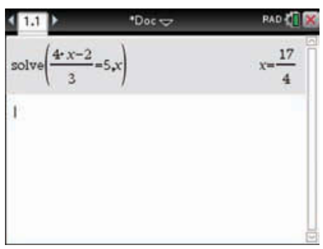
4. Write the answer in its simplest form.

$$x=\frac{17}{4}$$

TI | THINK

d. 1. On a Calculator page, press MENU and select:
3: Algebra
1: Solve
Complete the entry line as
solve $\left(\frac{4\cdot x-2}{3}=5,x\right)$
then press ENTER.

WRITE



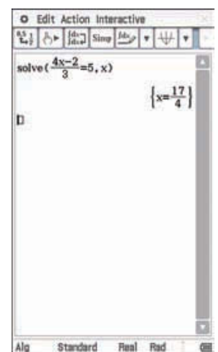
2. The answer appears on the screen.

$x=\frac{17}{4}$

CASIO | THINK

d. 1. On the Main screen, complete the entry line as:
solve $\left(\frac{4x-2}{3}=5,x\right)$
then press EXE.

WRITE



2. The answer appears on the screen.

$x=\frac{17}{4}$

1.3.2 Substituting into linear equations

If we are given a linear equation between two variables and we are given the value of one of the variables, we can **substitute** this into the equation to determine the other value.

WORKED EXAMPLE 5

Substitute $x = 3$ into the linear equation $y = 2x + 5$ to determine the value of y.

THINK

1. Substitute the variable (x) with the given value.

2. Equate the right-hand side of the equation.

WRITE

$y=2(3)+5$

$y=6+5$

$y=11$

TI | THINK

WRITE

1. On a Calculator page, complete the entry line as:
$y = 2 \cdot x + 5|x = 3$
then press ENTER.
Note: the 'given' or 'with' symbol (|) is found by pressing CTRL then the = button, then selecting |.

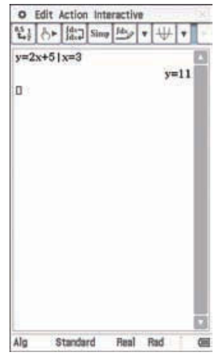
2. The answer appears on the screen.

$y = 11$

CASIO | THINK

WRITE

1. On the Main screen, complete the entry line as:
$y = 2x + 5|x = 3$
then press EXE.
Note: the 'given' or 'with' symbol (|) is found in the "Math3" menu on the Keyboard menu.



2. The answer appears on the screen.

$y = 11$

1.3.3 Literal linear equations

A **literal equation** is an equation that includes several pronumerals or variables. Literal equations often represent real-life situations.

The equation $y = mx + c$ is an example of a literal linear equation that represents the general form of a straight line.

To solve literal linear equations, you need to isolate the variable you are trying to solve for.

WORKED EXAMPLE 6

Solve the linear literal equation $y = mx + c$ for x.

THINK

WRITE

1. Isolate the terms containing the variable you want to solve on one side of the equation.

$y - c = mx$

2. Divide by the coefficient of the variable you want to solve for.

$\dfrac{y - c}{m} = x$

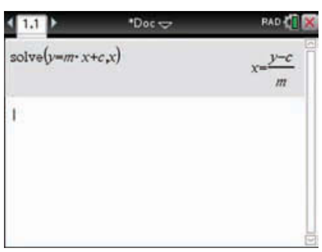
3. Transpose the equation.

$x = \dfrac{y - c}{m}$

TI | THINK

WRITE

1. On a Calculator page, press MENU and select:
3: Algebra
1: Solve
Complete the entry line as
solve $(y = m \cdot x + c, x)$
then press ENTER.
Note: be sure to enter the multiplication operation between the variables m and x.



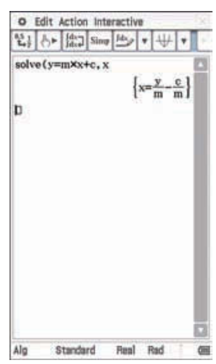
2. The answer appears on the screen.

$x = \dfrac{y - c}{m}$

CASIO | THINK

WRITE

1. On the Main screen, complete the entry line as:
solve $(y = m \times x + c, x)$
then press EXE.
Note: be sure to enter the multiplication operation between the variables m and x.



2. The answer appears on the screen.

$x = \dfrac{y}{m} - \dfrac{c}{m}$

on Resources

Interactivity: Solving linear equations (int-6450)

studyon

Units 1 & 2 > AOS 1 > Topic 1 > Concepts 1, 4 and 5

Formulas and substitution Concept summary and practice questions

Solution of numeric linear equations Concept summary and practice questions

Solution of literal linear equations Concept summary and practice questions

Exercise 1.3 Solving linear equations

1. **WE4** Solve the following linear equations to find the unknowns.
   a. $2(x + 1) = 8$ b. $n - 12 = -2$ c. $4d - 7 = 11$ d. $\frac{x + 1}{2} = 9$
2. a. Write the operations in order that have been performed on the unknowns in the following linear equations.
      i. $10 = 4a + 3$ ii. $3(x + 2) = 12$ iii. $\frac{s + 1}{2} = 7$ iv. $16 = 2(3c - 9)$
   b. Find the exact values of the unknowns in part **a** by solving the equations. Show all of the steps.
3. Find the exact values of the unknowns in the following linear equations.
   a. $14 = 5 - x$ b. $\frac{4(3y - 1)}{5} = -2$ c. $\frac{2(3 - x)}{3} = 5$
4. Solve the following literal linear equations for the pronumerals given in brackets.
   a. $v = u + at$ (a) b. $xy - k = m$ (x) c. $\frac{x}{p} - r = s$ (x)
5. **WE5** Substitute $x = 5$ into the equation $y = 5 - 6x$ to determine the value of y.
6. Substitute $x = -3$ into the equation $y = 3x + 3$ to determine the value of y.
7. The equation $w = 10t + 120$ represents the amount of water in a tank, w (in litres), at any time, t (in minutes). Find the time, in minutes, that it takes for the tank to have the following amounts of water.
   a. 450 litres b. 1200 litres

8. **WE6** Solve the literal linear equation $px - q = r$ for x.
9. Solve the literal linear equation $C = \pi d$ for d.
10. Yorx was asked to solve the linear equation $5w - 13 = 12$. His solution is shown.

    Step 1: $\times 5, -13$

    Step 2: Opposite operations $\div 5, +13$

    Step 3:
    $$5w - 13 = 12$$
    $$\frac{5w - 13}{5} = \frac{12}{5}$$
    $$w - 13 = 2.4$$

    Step 4:
    $$w - 13 + 13 = 2.4 + 13$$
    $$w = 15.4$$

    a. Show that Yorx's answer is incorrect by finding the value of w.
    b. What advice would you give to Yorx so that he can solve linear equations correctly?
11. The literal linear equation $F = 1.8(K - 273) + 32$ converts the temperature in Kelvin (K) to Fahrenheit (F). Solve the equation for K to give the formula for converting the temperature in Fahrenheit to Kelvin.

12. Consider the linear equation $y = \frac{3x+1}{4}$. Find the value of x for the following y-values.

a. 2 b. −3 c. $\frac{1}{2}$ d. 10

13. The distance travelled, d (in kilometres), at any time t (in hours) can be found using the equation $d = 95t$. Find the time in hours that it takes to travel the following distances. Give your answers correct to the nearest minute.

a. 190 km b. 250 km c. 65 km d. 356.5 km e. 50 000 m

14. The amount, A, in dollars in a bank account at the end of any month, m, can found using the equation $A = 150m + 400$.

a. How many months would it take to have the following amounts of money in the bank account?

i. \$1750 ii. \$3200

b. How many years would it take to have \$10 000 in the bank account? Give your answer correct to the nearest month.

15. The temperature, C, in degrees Celsius can be found using the equation $C = \frac{5(F-32)}{9}$, where F is the temperature in degrees Fahrenheit. Nora needs to set her oven at 190°C, but her oven's temperature is measured in Fahrenheit.

a. Write the operations performed on the variable F.

b. Write the order in which the operations need to be performed to find the value of F.

c. Determine the temperature in Fahrenheit that Nora should set her oven to.

16. The equation that determines the surface area of a cylinder with a radius of 3.5 cm is $A = 3.5\pi(3.5 + h)$. Determine the height in cm of cylinders with radii of 3.5 cm and the following surface areas. Give your answers correct to 2 decimal places.

a. 200 cm^2 b. 240 cm^2 c. 270 cm^2

17. Using CAS or otherwise, solve the following equations to find the unknowns. Express your answer in exact form.

a. $\frac{2-5x}{8} = \frac{3}{5}$ b. $\frac{6(3y-2)}{11} = \frac{5}{9}$

c. $\left(\frac{4x}{5} - \frac{3}{7}\right) + 8 = 2$ d. $\frac{7x+6}{9} + \frac{3x}{10} = \frac{4}{5}$

18. The height of a plant can be found using the equation $h = \frac{2(3t+15)}{3}$, where h is the height in cm and t is time in weeks.

a. Using CAS or otherwise, determine the time the plant takes to grow to the following heights. Give your answers correct to the nearest week.

i. 20 cm ii. 30 cm

iii. 35 cm iv. 50 cm

b. How high is the plant initially?

When the plant reaches 60 cm it is given additional plant food. The plant's growth each week for the next 4 weeks is found using the equation $g = t + 2$, where g is the growth each week in cm and t is the time in weeks since additional plant food was given.

c. Determine the height of the plant in cm for the next 4 weeks.

1.4 Developing linear equations

1.4.1 Developing linear equations from word descriptions

To write a worded statement as a linear equation, we must first identify the unknown and choose a **pronumeral** to represent it. We can then use the information given in the statement to write a linear equation in terms of the pronumeral.

The linear equation can then be solved as before, and we can use the result to answer the original question.

WORKED EXAMPLE 7

Cans of soft drinks are sold at SupaSave in packs of 12 costing $ 5.40. Form and solve a linear equation to determine the price of 1 can of soft drink.

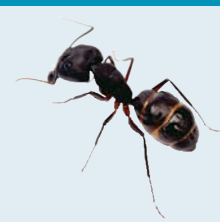
THINK	WRITE
1. Identify the unknown and choose a pronumeral to represent it.	S = price of a can of soft drink
2. Use the given information to write an equation in terms of the pronumeral.	$12S = 5.4$
3. Solve the equation.	$\frac{12S}{12} = \frac{5.4}{12}$ $S = 0.45$
4. Interpret the solution in terms of the original problem.	The price of 1 can of soft drink is $0.45 or 45 cents.

1.4.2 Word problems with more than one unknown

In some instances a word problem might contain more than one unknown. If we are able to express both unknowns in terms of the same pronumeral, we can create a linear equation as before and solve it to determine the value of both unknowns.

WORKED EXAMPLE 8

Georgina is counting the number of insects and spiders she can find in her back garden. All insects have 6 legs and all spiders have 8 legs. In total, Georgina finds 43 bugs with a total of 290 legs. Form a linear equation to determine exactly how many insects and spiders Georgina found.



THINK	WRITE
1. Identify one of the unknowns and choose a pronumeral to represent it.	Let s = the number of spiders.
2. Define the other unknown in terms of this pronumeral.	Let $43 - s$ = the number of insects.

3. Write expressions for the total numbers of spiders' legs and insects' legs.

Total number of spiders' legs $= 8s$
Total number of insects' legs $= 6(43 - s)$
$= 258 - 6s$

4. Create an equation for the total number of legs of both types of creature.

$8s + (258 - 6s) = 290$

5. Solve the equation.

$$\begin{aligned} 8s + 258 - 6s &= 290 \\ 8s - 6s &= 290 - 258 \\ 2s &= 32 \\ s &= 16 \end{aligned}$$

6. Substitute this value back into the second equation to determine the other unknown.

The number of insects $= 43 - 16$
$= 27$

7. Answer the question using words.

Georgina found 27 insects and 16 spiders.

1.4.3 Tables of values

Tables of values can be generated from formulas by entering given values of one variable into the formula.

Tables of values can be used to solve problems and to draw graphs representing situations (as covered in more detail in Topic 10).

WORKED EXAMPLE 9

The amount of water that is filling a tank is found by the rule $W = 100t + 20$, where W is the amount of water in the tank in litres and t is the time in hours.

a. Generate a table of values that shows the amount of water, W, in the tank every hour for the first 8 hours (i.e. $t = 0, 1, 2, 3, \ldots, 8$).

b. Using your table, how long in hours will it take for there to be over 700 litres in the tank?

THINK

a. 1. Enter the required values of t into the formula to calculate the values of W.

WRITE

a. $t = 0$:
$W = 100(0) + 20$
$= 20$

$t = 1$:
$W = 100(1) + 20$
$= 120$

$t = 2$:
$W = 100(2) + 20$
$= 220$

$t = 3$:
$W = 100(3) + 20$
$= 320$

$t = 4$:
$W = 100(4) + 20$
$= 420$

$t = 5$:
$W = 100(5) + 20$
$= 520$

$t = 6$:
$W = 100(6) + 20$
$= 620$

$t = 7$:
$W = 100(7) + 20$
$= 720$

$t = 8$:

$W = 100(8) + 20$
$= 820$

2. Enter the calculated values into a table of values.

t	0	1	2	3	4	5	6	7	8
W	20	120	220	320	420	520	620	720	820

b. 1. Using your table of values, locate the required column.

b.

t	0	1	2	3	4	5	6	7	8
W	20	120	220	320	420	520	620	720	820

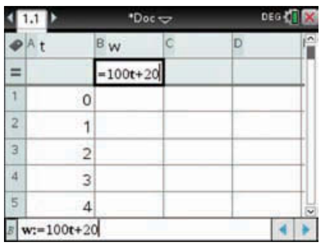
2. Read the corresponding values from your table and answer the question.

$t = 7$

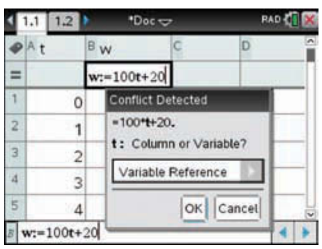
It will take 7 hours for there to be over 700 litres of water in the tank.

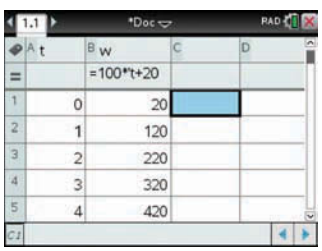
TI | THINK **WRITE**

a. 1. In a Lists & Spreadsheet page, label the first column t and the second column w. Enter the values 0 to 8 in the first column. In the formula cell underneath the label w, enter the rule for w starting with an = sign, then press ENTER.



2. When prompted, select Variable Reference for *t* and select OK.



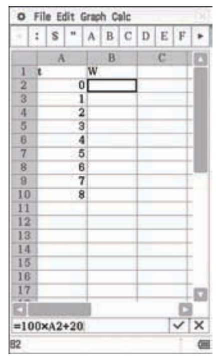


3. The table of values can be read from the screen.

t	0	1	2	3	4	5	6	7	8
w	20	120	220	320	420	520	620	720	820

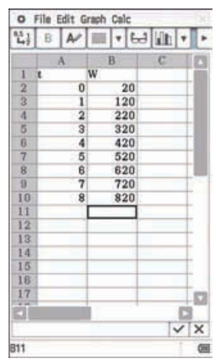
CASIO | THINK **WRITE**

a. 1. On a Spreadsheet screen, type t into cell A1 and w into cell B1. Enter the values 0 to 8 in cells A2 to A10. In cell B2, complete the entry line as: $= 100 \times A2 + 20$ then press EXE.



2. Click on cell B2 and select:
- Edit
- Copy

Highlight cells B3 to B10, then select:
- Edit
- Paste



3. The table of values can be read from the screen.

t	0	1	2	3	4	5	6	7	8
w	20	120	220	320	420	520	620	720	820

1.4.4 Linear relations defined recursively

Many sequences of numbers are obtained by following rules that define a relationship between any one term and the previous term. Such a relationship is known as a **recurrence relation**.

A term in such a sequence is defined as t_n, with n denoting the place in the sequence. The term t_{n-1} is the previous term in the sequence.

If a recurrence relation is of a linear nature — that is, there is a common difference (d) between each term in the sequence — then we can define the recurrence relation as:

$$t_n = t_{n-1} + d,\ t_1 = a$$

This means that the first term in the sequence is a, and each subsequent term is found by adding d to the previous term.

WORKED EXAMPLE 10

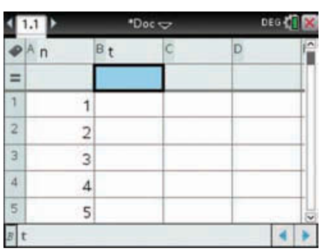
A linear recurrence relation is given by the formula $t_n = t_{n-1} + 6$, $t_1 = 5$. Write the first six terms of the sequence.

THINK	WRITE
1. Calculate the value for t_2 by substituting the value for t_1 into the formula. Then use this to calculate the value for t_3 and so on.	$t_2 = t_1 + 6 = 5 + 6 = 11$ $t_3 = t_2 + 6 = 11 + 6 = 17$ $t_4 = t_3 + 6 = 17 + 6 = 23$ $t_5 = t_4 + 6 = 23 + 6 = 29$ $t_6 = t_5 + 6 = 29 + 6 = 35$
2. State the answer.	The first six values are 5, 11, 17, 23, 29 and 35.

TI | THINK

1. In a Lists & Spreadsheet page, label the first column n and the second column t. Enter the values 1 to 6 in the first column.

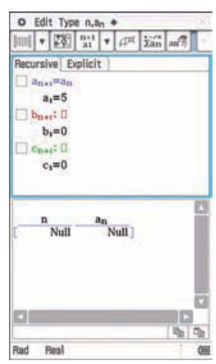
WRITE



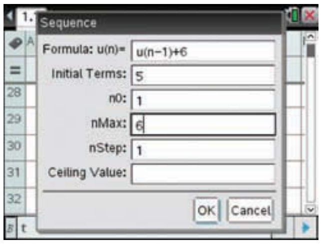
CASIO | THINK

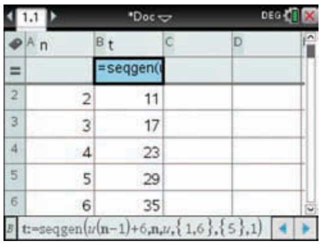
1. On the Sequence screen, select the Recursive tab then complete the entry line for a_{n+1} as:
   $a_{n+1} = a_n + 6$
   $a_1 = 5$
   then click the tick box. *Note:* To change the format so that the first term is labelled a_1 rather than a_0, select the & icon.

WRITE



2. Click on the formula cell underneath the label t, then press MENU and select:
3: Data
1: Generate Sequence
Complete the entry line for the Formula as u(n) = u(n − 1) + 6 and set the Initial Terms as 5, n0 as 1 and nMax as 6, then select OK.

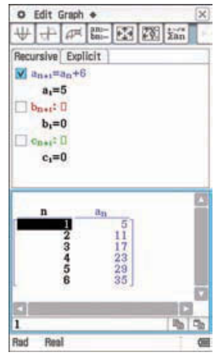




3. The first six terms can be read from the screen.

The first six terms are 5, 11, 17, 23, 29 and 35.

2. Click the # icon to view the terms in the sequence. Click the 8 icon to specify the number of terms to be displayed.



3. The first six terms can be read from the screen.

The first six terms are 5, 11, 17, 23, 29 and 35 .

WORKED EXAMPLE 11

The weekly rent on an inner-city apartment increases by \$10 every year. In a certain year the weekly rent is \$310.

a. Model this situation by setting up a linear recurrence relation between the weekly rental prices in consecutive years.

b. Find the weekly rent for the first six years.

c. Find an expression for the weekly rent (*r*) in the *n*th year.

THINK	WRITE
a. 1. Write the values of a and d in the generalised linear recurrence relation formula.	a. $a = 310,\ d = 10$
2. Substitute these values into the generalised linear recurrence relation formula.	$t_n = t_{n-1} + 10,\ t_1 = 310$
b. 1. Substitute $n = 2$, $n = 3$, $n = 4$, $n = 5$ and $n = 6$ into the recurrence relation.	b. $t_2 = t_1 + 10 = 310 + 10 = 320$ $t_3 = t_2 + 10 = 320 + 10 = 330$ $t_4 = t_3 + 10 = 330 + 10 = 340$ $t_5 = t_4 + 10 = 340 + 10 = 350$ $t_6 = t_5 + 10 = 350 + 10 = 360$
2. State the answer.	The weekly rent for the first 6 years will be: \$310, \$320, \$330, \$340, \$350, \$360

c. 1. Take a look at the answers obtained in part **b** and observe that the weekly went is found by adding 300 to 10 times the term's number.

c. $t_1 = 310 = 300 + 10 \times 1$
$t_2 = 320 = 300 + 10 \times 2$
$t_3 = 330 = 300 + 10 \times 3$
$t_4 = 340 = 300 + 10 \times 4$
$t_5 = 350 = 300 + 10 \times 5$
$t_6 = 360 = 300 + 10 \times 6$

2. Extend this pattern to the *n*th term.

$t_n = 300 + 10 \times n$
$= 300 + 10n$

3. Answer the question.

The expression for the weekly rent in the *n*th year is $r = 300 + 10n$.

on Resources

Interactivity: Linear relations defined recursively (int-6451)

studyon

Units 1 & 2 > AOS 1 > Topic 1 > Concepts 6, 7 and 8

Mathematical modelling Concept summary and practice questions

Tables of values Concept summary and practice questions

Defining a linear model recursively Concept summary and practice questions

Exercise 1.4 Developing linear equations

1. **WE7** Art pencils at the local art supply store sell in packets of 8 for \$17.92. Form and solve a linear equation to determine the price of 1 art pencil.

2. Natasha is trying to determine which type of cupcake is the best value for money. The three options Natasha is considering are:

- 4 red velvet cupcakes for \$9.36
- 3 chocolate delight cupcakes for \$7.41
- 5 caramel surprise cupcakes for \$11.80.

Form and solve linear equations for each type of cupcake to determine which has the cheapest price per cupcake.

3. Three is added to a number and the result is then divided by four, giving an answer of nine. Find the number.

4. The sides in one pair of sides of a parallelogram are each 3 times the length of a side in the other pair. Find the side lengths if the perimeter of the parallelogram is 84 cm.

5. **WE8** Fredo is buying a large bunch of flowers for his mother in advance of Mother's Day. He picks out a bunch of roses and lilies, with each rose costing \$6.20 and each lily costing \$4.70. In total he picks out 19 flowers and pays \$98.30. Form a linear equation to determine exactly how many roses and lilies Fredo bought.
6. Miriam has a sweet tooth, and her favourite sweets are strawberry twists and chocolate ripples. The local sweet shop sells both as part of their pick and mix selection, so Miriam fills a bag with them. Each strawberry twist weighs 5 g and each chocolate ripple weighs 9 g. In Miriam's bag there are 28 sweets, weighing a total of 188 g. Determine the number of each type of sweet that Miriam bought by forming and solving a linear equation.

7. One week Jordan bought a bag of his favourite fruit and nut mix at the local market. The next week he saw that the bag was on sale for 20% off the previously marked price. Jordan purchased two more bags at the reduced price. Jordan spent \$20.54 in total for the three bags. Find the original price of a bag of fruit and nut mix.
8. Six times the sum of four plus a number is equal to one hundred and twenty-six. Find the number.
9. **WE9** Libby enjoys riding along Beach Road on a Sunday morning. She rides at a constant speed of 0.4 kilometres per minute.
   a. Generate a table of values that shows how far Libby has travelled for each of the first 10 minutes of her journey.
   b. One Sunday Libby meets a friend 3 kilometres into her journey. Between which minutes does Libby meet her friend and keep riding?

10. Tommy is saving for a remote-controlled car that is priced at \$49. He has \$20 in his piggy bank. Tommy saves \$3 of his pocket money every week and puts it in his piggy bank. The amount of money in dollars, M, in his piggy bank after w weeks can be found using the rule $M = 3w + 20$.
    a. Generate a table of values that shows the amount of money, M, in Tommy's piggy bank every week for the 12 weeks (i.e. $w = 0, 1, 2, 3, \ldots, 12$).
    b. Using your table, how many weeks will it take for Tommy to have saved enough money to purchase the remote-controlled car?
11. **WE10** A linear recurrence relation is given by the formula $t_n = t_{n-1} - 6$, $t_1 = 12$. Write the first six terms of the sequence.
12. A linear recurrence relation is given by the formula $t_n = t_{n-1} + 3.2$, $t_1 = -5.8$. Write the first six terms of the sequence.
13. **WE11** Jake is a stamp collector. He notices that the value of the rarest stamp in his collection increases by \$25 each year. Jake purchased the stamp for \$450.
    a. Set up a recurrence relation between the yearly values of Jake's rarest stamp.
    b. Find the value of the stamp for each year over the first 8 years.
    c. Find an expression for the stamp's value (v) in the nth year.

14. Juliet is a zoologist and has been monitoring the population of a species of wild lemur in Madagascar over a number of years. Much to her dismay, she finds that on average the population decreases by 13 each year. In her first year of monitoring, the population was 313.
  a. Set up a recurrence relation between the yearly populations of the lemurs.
  b. Find the population of the lemurs for each year over the first 7 years.
  c. Find an expression for the population of lemurs (l) in the nth year.
15. Fred is saving for a holiday and decides to deposit $40 in his bank account each week. At the start of his saving scheme he has $150 in his account.
  a. Set up a recurrence relation between the amounts in Fred's account on consecutive weeks.
  b. Use the recurrence relation to construct a table of values detailing how much Fred will have in his account after each of the first 8 weeks.
  c. The holiday Fred wants to go on will cost $720 dollars. How many weeks will it take Fred to save up enough money to pay for his holiday?
16. Sabrina is a landscape gardener and has been commissioned to work on a rectangular piece of garden. The length of the garden is 6 metres longer than the width, and the perimeter of the garden is 64 m. Find the parameters of the garden.
17. Yuri is doing his weekly grocery shop and is buying both carrots and potatoes. He calculates that the average weight of a carrot is 60 g and the average weight of a potato is 125 g. Furthermore, he calculates that the average weight of the carrots and potatoes that he purchases is 86 g. If Yuri's shopping weighed 1.29 kg in total, how many of each did he purchase?
18. Ho has a water tank in his back garden that can hold up to 750 L in water. At the start of a rainy day (at 0:00) there is 165 L in the tank, and after a heavy day's rain (at 24:00) there is 201 L in the tank.
  a. Assuming that the rain fell consistently during the 24-hour period, set up a linear equation to represent the amount of rain in the tank at any point during the day.
  b. Generate a table of values that shows how much water is in the tank after every 2 hours of the 24-hour period.
  c. At what time of day did the amount of water in the tank reach 192 L?
19. A large fish tank is being filled with water. After 1 minute the height of the water is 2 cm and after 4 minutes the height of the water is 6 cm. The height of the water, h, in cm after t minutes can be modelled by a linear equation.
  a. Construct a recurrence relation between consecutive minutes of the height of water in the fish tank.
  b. Determine the height of the water in the fish tank after each of the first five minutes.
  c. Was the fish tank empty of water before being filled? Justify your answer by using calculations.

20. Michelle and Lydia live 325 km apart. On a Sunday they decide to drive to each other's respective towns. They pass each other after 2.5 hours. If Michelle drives an average of 10 km/h faster than Lydia, calculate the speed at which they are both travelling.
21. Jett is starting up a small business selling handmade surfboard covers online. The start-up cost is $250. He calculates that each cover will cost $14.50 to make. The rule that finds the cost, C, to make n covers is $C = 14.50n + 250$.
  a. Using CAS or otherwise, generate a table of values to determine the cost of producing 10 to 20 surfboard covers.

b. If Jett sells the covers for \$32.95, construct a table of values to determine the revenue for selling 10 to 20 surfboard covers.
c. The profit Jett makes is the difference between his selling price and the cost price. Explain how the profit Jett makes can be calculated using the tables of values constructed in parts **a** and **b**.
d. Using your explanation in part **c** and your table of values, determine the profits made by Jett if he sells 10 to 20 surfboard covers.

22. Benito decides to set up a market stall selling fruit based energy drinks. He has to pay \$300 for his stall on a particular day. The ingredients for each energy drink total \$1.35, and he sells each energy drink for \$4.50.
    a. If the cost of making m energy drinks is c_m, write a recurrence relation for the cost of making the energy drinks.
    b. If the income from selling n energy drinks is s_n, write a recurrence relation for the income from selling the energy drinks.
    c. Using CAS or otherwise, determine the minimum number of energy drinks Benito needs to sell to make a profit.
    d. Generate a table of values showing the profit/loss for selling up to 120 energy drinks in multiples of 10 (i.e. 0, 10, 20, … , 120).

1.5 Simultaneous linear equations

1.5.1 Solving simultaneous equations graphically

Simultaneous equations are sets of equations that can be solved at the same time. They often represent practical problems that have two or more unknowns. For example, you can use simultaneous equations to find the costs of individual apples and oranges when different amounts of each are bought.

Solving simultaneous equations gives the set of values that is common to all of the equations. If these equations are presented graphically, then the set of values common to all equations is the point of intersection.

To solve a set of simultaneous equations graphically, the equations must be sketched on the same set of axes and the point of intersection must be found. If the equations do not intersect then there is no solution for the simultaneous equations.

WORKED EXAMPLE 12

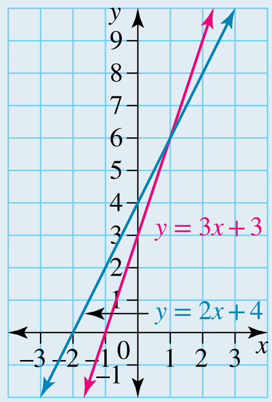
The following equations represent a pair of simultaneous equations.

$$y = 2x + 4 \text{ and } y = 3x + 3$$

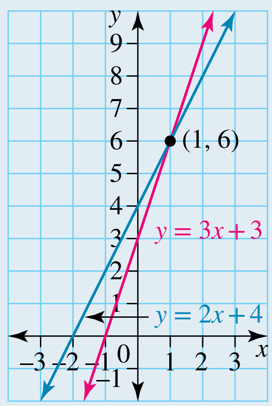
Using CAS or otherwise, sketch both graphs on the same set of axes and solve the equations.

THINK | **WRITE/DRAW**

1. Use CAS or another method to sketch the graphs $y = 2x + 4$ and $y = 3x + 3$.



2. Locate the point where the graphs intersect (or cross over).

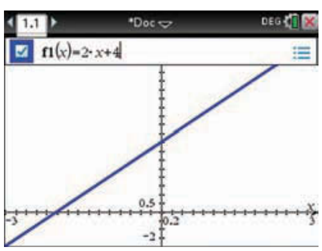


3. Using the graph, find the x- and y-values of the point of intersection.

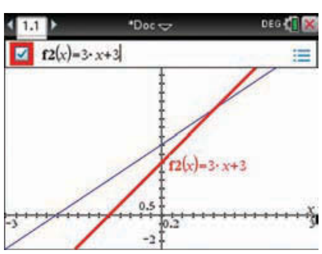
The solution is (1, 6), or $x = 1$ and $y = 6$.

TI | THINK | **WRITE**

1. On a Graphs page, complete the entry line for function 1 as $f1(x) = 2 \cdot x + 4$ then press ENTER.

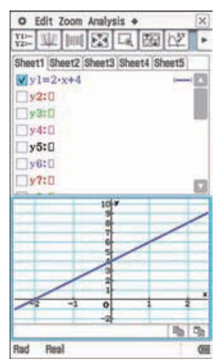


2. Complete the entry line for function 2 as $f2(x) = 3 \cdot x + 3$ then press ENTER.
Note: press the e button to bring up the entry line.

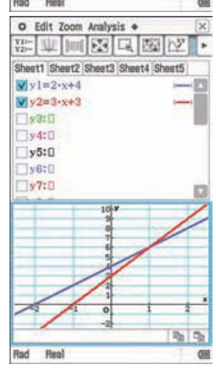


CASIO | THINK | **WRITE**

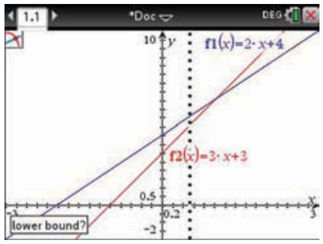
1. On a Graph & Table screen, complete the entry line for function 1 as $y1 = 2 \cdot x + 4$ then click the tick box. Tap the $ icon.



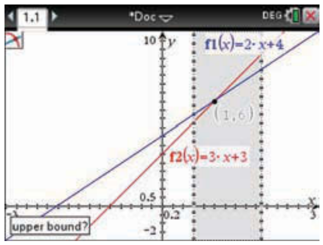
2. Complete the entry line for function 2 as $y2 = 3 \cdot x + 3$ then click the tick box. Tap the $ icon.



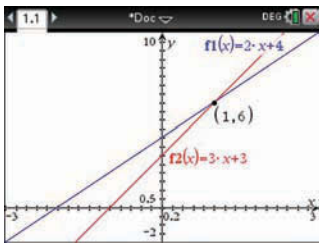
3. Press MENU and select:
6: Analyze Graph
4: Intersection
Move to the left of the point of intersection and press the CLICK button.



4. Move to the right of the point of intersection and press the CLICK button.



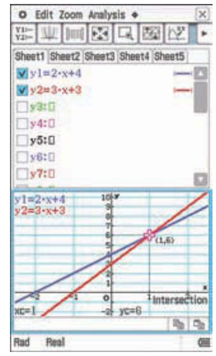
5. The point of intersection appears on the screen.



6. State the answer.

The solution is (1, 6), or $x = 1$ and $y = 6$.

3. Select:
- Analysis
- G
-Solve
- Intersection
The point of intersection appears on the screen.



4. State the answer.

The solution is (1, 6), or $x = 1$ and $y = 6$.

1.5.2 Solving simultaneous equations using substitution

Simultaneous equations can also be solved algebraically. One algebraic method is known as substitution. This method requires one of the equations to be substituted into the other by replacing one of the variables. The second equation is then solved and the value of one of the variables is found. The substitution method is often used when one or both of the equations are written with variables on either side of the equals sign; for example, $c = 12b - 15$ and $2c + 3b = -3$,
or $y = 4x + 6$ and $y = 6x + 2$.

WORKED EXAMPLE 13

Solve the following pairs of simultaneous equations using substitution.

a. $c = 12b - 15$ **and** $2c + 3b = -3$ **b.** $y = 4x + 6$ **and** $y = 6x + 2$
c. $3x + 2y = -1$ **and** $y = x - 8$

THINK	WRITE
a. 1. Identify which variable will be substituted into the other equation.	**a.** $c = 12b - 15$
2. Substitute the variable into the equation.	$2c + 3b = -3$ $2(12b - 15) + 3b = -3$
3. Expand and simplify the left-hand side, and solve the equation for the unknown variable.	$24b - 30 + 3b = -3$ $27b - 30 = -3$ $27b = -3 + 30$ $27b = 27$ $b = 1$

4. Substitute the value for the unknown back into one of the equations.	$c = 12b - 15$ $= 12(1) - 15$ $= -3$
5. Answer the question.	The solution is $b = 1$ and $c = -3$.
b. 1. Both equations are in the form $y =$, so let them equal each other.	**b.** $4x + 6 = 6x + 2$
2. Move all of the pronumerals to one side.	$4x - 4x + 6 = 6x - 4x + 2$ $6 = 6x - 4x + 2$ $6 = 2x + 2$
3. Solve for the unknown.	$6 - 2 = 2x + 2 - 2$ $4 = 2x$ $\frac{4}{2} = \frac{2x}{2}$ $2 = x$
4. Substitute the value found into either of the original equations.	$y = 4x + 6$ $= 4 \times 2 + 6$ $= 8 + 6$ $= 14$
5. Answer the question.	The solution is $x = 2$ and $y = 14$.
c. 1. One equation is not in the form $y =$, so substitute this equation into the other.	**c.** $3x + 2y = -1$ $3x + 2(x - 8) = -1$
2. Expand and simplify the equation.	$3x + 2x - 16 = -1$ $5x - 16 = -1$
3. Solve for the unknown.	$5x - 16 + 16 = -1 + 16$ $5x = 15$ $x = 3$
4. Substitute the value found into either of the original equations.	$y = x - 8$ $= 3 - 8$ $= -5$
5. Answer the question.	The solution is $x = 3$ and $y = -5$.

1.5.3 Solving simultaneous equations using elimination

Solving simultaneous equations using **elimination** requires the equations to be added or subtracted so that one of the pronumerals is eliminated or removed. Simultaneous equations that have both pronumerals on the same side are often solved using elimination. For example, $3x + y = 5$ and $4x - y = 2$ both have x and y on the same side of the equation, so they can be solved with this method.

WORKED EXAMPLE 14

Solve the following pairs of simultaneous equations using elimination:

a. $3x + y = 5$ **and** $4x - y = 2$
b. $2a + b = 7$ **and** $a + b = 5$
c. $3c + 4d = 5$ **and** $2c + 3d = 4$

THINK	WRITE
a. 1. Write the simultaneous equation with one on top of the other.	**a.** $3x + y = 5$ [1] $4x - y = 2$ [2]
2. Select one pronumeral to be eliminated.	Select y.
3. Check the coefficients of the pronumeral being eliminated.	The coefficients of y are 1 and −1.
4. If the coefficients are the same number but with different signs, add the equations together.	[1] + [2]: $3x + 4x + y - y = 5 + 2$ $7x = 7$
5. Solve the equation for the unknown pronumeral.	$7x = 7$ $\frac{7x}{7} = \frac{7}{7}$ $x = 1$
6. Substitute the pronumeral back into one of the equations.	$3x + y = 5$ $3(1) + y = 5$
7. Solve the equation to find the value of the other pronumeral.	$3 + y = 5$ $3 - 3 + y = 5 - 3$ $y = 2$
8. Answer the question.	The solution is $x = 1$ and $y = 2$.
b. 1. Write the simultaneous with one on top of the other.	**b.** $2a + b = 7$ [1] $a + b = 5$ [2]
2. Select one pronumeral to be eliminated.	Select b.
3. Check the coefficients of the pronumeral being eliminated.	The coefficients of b are both 1.
4 If the coefficients are the same number with the same sign, subtract one equation from the other.	[1] − [2]: $2a - a + b - b = 7 - 5$
5. Solve the equation for the unknown pronumeral.	$a = 2$
6. Substitute the pronumeral back into one of the equations.	$a + b = 5$ $2 + b = 5$
7. Solve the equation to find the value of the other pronumeral.	$b = 5 - 2$ $b = 3$
8. Answer the question.	The solution is $a = 2$ and $b = 3$.
c. 1. Write the simultaneous equations with one on top of the other.	**c.** $3c + 4d = 5$ [1] $2c + 3d = 4$ [2]
2. Select one pronumeral to be eliminated.	Select c.
3. Check the coefficients of the pronumeral being eliminated.	The coefficients of c are 3 and 2.
4. If the coefficients are different numbers, then multiply them both by another number, so they both have the same coefficient value.	$3 \times 2 = 6$ $2 \times 3 = 6$
5. Multiply the equations (all terms in each equation) by the numbers selected in step 4.	[1] × 2: $6c + 8d = 10$ [2] × 3: $6c + 9d = 12$

6. Check the sign of each coefficient for the selected pronumeral.

$6c + 8d = 10$ [3]
$6c + 9d = 12$ [4]
Both coefficients of c are positive 6.

7. If the signs are the same, subtract one equation from the other and simplify.

[3] − [4]:
$6c - 6c + 8d - 9d = 10 - 12$
$-d = -2$

8. Solve the equation for the unknown.

$d = 2$

9. Substitute the pronumeral back into one of the equations.

$2c + 3d = 4$
$2c + 3(2) = 4$

10. Solve the equation to find the value of the other pronumeral.

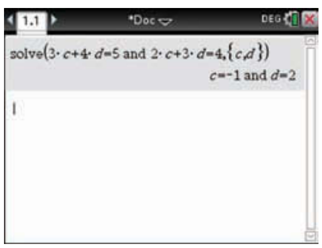
$2c + 6 = 4$
$2c + 6 - 6 = 4 - 6$
$2c = -2$
$$\frac{2c}{2} = \frac{-2}{2}$$
$c = -1$

11. Answer the question.

The solution is $c = -1$ and $d = 2$.

TI | THINK / **WRITE**

c. 1. On a Calculator page, press MENU and select:
3: Algebra
1: Solve
Complete the entry line assolve then press ENTER.



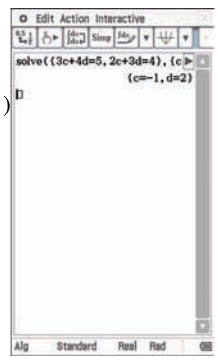
$(3 \cdot c + 4 \cdot d = 5 \text{ and } 2 \cdot c + 3 \cdot d = 4, \{c, d\})$

2. The answer appears on the screen.

$c = -1$ and $d = 2$.

CASIO | THINK / **WRITE**

c. 1. On the Main screen, complete the entry line as:
solve
$(\{3c + 4d = 5, 2c + 3d = 4\}, \{c, d\})$
then press EXE.



2. The answer appears on the screen.

$c = -1$ and $d = 2$.

on Resources

Interactivity: Solving simultaneous equations graphically (int-6452)
Interactivity: Solving simultaneous equations using substitution (int-6453)

studyon

Units 1 & 2 › AOS 1 › Topic 2 › Concepts 1, 2 and 3

Graphical solutions Concept summary and practice questions

Algebraic solutions — substitution method Concept summary and practice questions

Algebraic solutions — elimination method Concept summary and practice questions

Exercise 1.5 Simultaneous linear equations

1. **WE12** The following equations represent a pair of simultaneous equations.

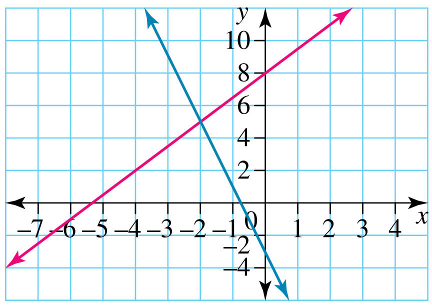
$$y = 5x + 1 \text{ and } y = 2x - 5$$

Using CAS or otherwise, sketch both graphs on the same set of axes and solve the equations.

2. Using CAS or otherwise, sketch and solve the following three simultaneous equations.

$$y = 3x + 7,\ y = 2x + 8 \text{ and } y = -2x + 12$$

3. A pair of simultaneous equations is solved graphically as shown in the diagram. From the diagram, determine the solution for this pair of simultaneous equations.



4. Using CAS or otherwise, solve the following groups of simultaneous equations graphically.

a. $y = 4x + 1$ and $y = 3x - 1$

b. $y = x - 5$ and $y = -3x + 3$

c. $y = 3(x - 1)$ and $y = 2(2x + 1)$

d. $y = \frac{x}{2} - 1$ and $y = \frac{x}{2} + 4$

e. $y = 3x + 4,\ y = 2x + 3$ and $y = -x$

5. **WE13** Solve the following pairs of simultaneous equations using substitution.

a. $y = 2x + 1$ and $2y - x = -1$

b. $m = 2n + 5$ and $m = 4n - 1$

c. $2x - y = 5$ and $y = x + 1$

6. Find the solutions to the following pairs of simultaneous equations using substitution.

a. $2(x + 1) + y = 5$ and $y = x - 6$

b. $\frac{x + 5}{2} + 2y = 11$ and $y = 6x - 2$

7. **MC** Which one of the following pairs of simultaneous equations would best be solved using the substitution method?

A. $4y - 5x = 7$ and $3x + 2y = 1$

B. $3c + 8d = 19$ and $2c - d = 6$

C. $12x + 6y = 15$ and $9x - y = 13$

D. $3(t + 3) - 4s = 14$ and $2(s - 4) + t = -11$

E. $n = 9m + 12$ and $3m + 2n = 7$

8. Using the substitution method, solve the following pairs of simultaneous equations.

a. $y = 2x + 5$ and $y = 3x - 2$

b. $y = 5x - 2$ and $y = 7x + 2$

c. $y = 2(3x + 1)$ and $y = 4(2x - 3)$

d. $y = 5x - 9$ and $3x - 5y = 1$

e. $3(2x + 1) + y = -19$ and $y = x - 1$

f. $\frac{3x + 5}{2} + 2y = 2$ and $y = x - 2$

9. A student chose to solve the following pair of simultaneous equations using the elimination method.

$$3x + y = 8 \text{ and } 2x - y = 7$$

a. Explain why this student's method would be the most appropriate method for this pair of simultaneous equations.

b. Show how these equations would be solved using this method.

10. **WE14** Solve the following pairs of simultaneous equations using the method of elimination.

a. $3x + y = 5$ and $4x - y = 2$

b. $2a + b = 7$ and $a + b = 5$

c. $3c + 4d = 5$ and $2c + 3d = 4$

11. Consider the following pair of simultaneous equations:

$$ax - 3y = -16 \text{ and } 3x + y = -2.$$

If $y = 4$, find the values of a and x.

12. Using the elimination method, solve the following pairs of simultaneous equations.

a. $4x + y = 6$ and $x - y = 4$

b. $x + y = 7$ and $x - 2y = -5$

c. $2x - y = -5$ and $x - 3y = -10$

d. $4x + 3y = 29$ and $2x + y = 13$

e. $5x - 7y = -33$ and $4x + 3y = 8$

f. $\frac{x}{2} + y = 7$ and $3x + \frac{y}{2} = 20$

13. MC The first step when solving the following pair of simultaneous equations using the elimination method is:

$$2x + y = 3 \qquad [1]$$
$$3x - y = 2 \qquad [2]$$

A. Equations [1] and [2] should be added together.
B. Both equations should be multiplied by 2.
C. Equation [1] should be subtracted from equation [2].
D. Equation [1] should be multiplied by 2 and equation [2] should be multiplied by 3.
E. Equation [2] should be subtracted from equation [1].

14. Brendon and Marcia were each asked to solve the following pair of simultaneous equations.

$$3x + 4y = 17 \qquad [1]$$
$$4x - 2y = 19 \qquad [2]$$

Marcia decided to use the elimination method. Her solution steps were:

Step 1: [1] × 4:

$12x + 16y = 68 \qquad [3]$

[2] × 3:

$12x - 6y = 57 \qquad [4]$

Step 2: [3] + [4]:

$10y = 125$

Step 3: $y = 12.5$

Step 4: Substitute $y = 12.5$ into [1]:

$3x + 4(12.5) = 17$

Step 5: Solve for x:

$3x = 17 - 50 \qquad 3x = -33 \qquad x = -11$

Step 6: The solution is $x = -11$ and $y = 12.5$.

a. Marcia has made an error in step 2. Explain where she has made her error, and hence correct her mistake.

b. Using the correction you made in part a, find the correct solution to this pair of simultaneous equations. Brendon decided to eliminate y instead of x.

c. Using Brendon's method of eliminating y first, show all the appropriate steps involved to reach a solution.

15. In a ball game, a player can kick the ball into the net to score a goal or place the ball over the line to score a behind. The scores in a game between the Rockets and the Comets were:

Rockets: 6 goals 12 behinds, total score 54

Comets: 7 goals 5 behinds, total score 45

The two simultaneous equations that can represent this information are shown.

Rockets: $6x + 12y = 54$

Comets: $7x + 5y = 45$

a. By solving the two simultaneous equations, determine the number of points that are awarded for a goal and a behind.

b. Using the results from part **a**, determine the scores for the game between the Jetts, who scored 4 goals and 10 behinds, and the Meteorites, who scored 6 goals and 9 behinds.

16. Mick and Minnie both work part time at an ice-cream shop. The simultaneous equations shown represent the number of hours Mick (x) and Minnie (y) work each week.

Equation 1: Total number of hours worked by Minnie and Mick: $x + y = 15$

Equation 2: Number of hours worked by Minnie in terms of Mick's hours: $y = 2x$

a. Explain why substitution would be the best method to use to solve these equations.

b. Using substitution, determine the number of hours worked by Mick and Minnie each week. To ensure that he has time to do his Mathematics homework, Mick changes the number of hours he works each week. He now works $\frac{1}{3}$ of the number of hours worked by Minnie. An equation that can be used to represent this information is $x = \dfrac{y}{3}$.

c. Find the number of hours worked by Mick, given that the total number of hours that Mick and Minnie work does not change.

17. Using CAS or otherwise, solve the following groups of simultaneous equations. Write your answers correct to 2 decimal places.

a. $y = 5x + 6$ and $3x + 2y = 7$

b. $4(x + 6) = y - 6$ and $2(y + 3) = x - 9$

c. $6x + 5y = 8.95$, $y = 3x - 1.36$ and $2x + 3y = 4.17$

18. Consider the following groups of graphs.

i. $y_1 = 5x - 4$ and $y_2 = 6x + 8$

ii. $y_1 = -3x - 5$ and $y_2 = 3x + 1$

iii. $y_1 = 2x + 6$ and $y_2 = 2x - 4$

iv. $y_1 = -x + 3$, $y_2 = x + 5$ and $y_3 = 2x + 6$

a. Where possible, find the point of intersection for each group of graphs using any method.

b. Are there solutions for all of these groups of graphs? If not, for which group of graphs is there no solution, and why is this?

1.6 Problem solving with simultaneous equations

1.6.1 Setting up simultaneous equations

The solutions to a set of simultaneous equations satisfy all equations that were used. Simultaneous equations can be used to solve problems involving two or more variables or unknowns, such as the cost of 1 kilogram of apples and bananas, or the number of adults and children attending a show.

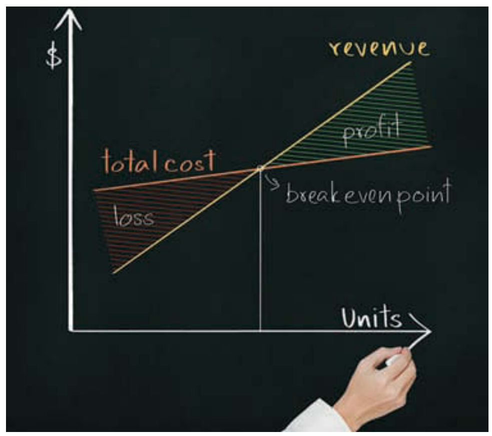
WORKED EXAMPLE 15

At a fruit shop, 2 kg of apples and 3 kg of bananas cost \$13.16, and 3 kg of apples and 2 kg of bananas cost \$13.74. Represent this information in the form of a pair of simultaneous equations.

THINK	WRITE
1. Identify the two variables.	The cost of 1 kg of apples and the cost of 1 kg of bananas
2. Select two pronumerals to represent these variables. Define the variables.	a = cost of 1 kg of apples b = cost of 1 kg of bananas
3. Identify the key information and rewrite it using the pronumerals selected.	2 kg of apples can be written as $2a$. 3 kg of bananas can be written as $3b$.
4. Construct two equations using the information.	$2a + 3b = 13.16$ $3a + 2b = 13.74$

1.6.2 Break-even points

A **break-even point** is a point where the costs equal the selling price. It is also the point where there is zero profit. For example, if the equation $C = 45 + 3t$ represents the production cost to sell t shirts, and the equation $R = 14t$ represents the revenue from selling the shirts for \$14 each, then the break-even point is the number of shirts that need to be sold to cover all costs.



To find the break-even point, the equations for cost and revenue are solved simultaneously.

WORKED EXAMPLE 16

Santo sells shirts for \$25. The revenue, R, for selling n shirts is represented by the equation $R = 25n$. The cost to make n shirts is represented by the equation $C = 2200 + 3n$.

a. Solve the equations simultaneously to determine the break-even point.

b. Determine the profit or loss, in dollars, for the following shirt orders.

i. 75 shirts

ii. 220 shirts

THINK			WRITE
a.	**1.**	Write the two equations.	**a.** $C = 2200 + 3n$ $R = 25n$
	2.	Equate the equations ($R = C$).	$2200 + 3n = 25n$
	3.	Solve for the unknown.	$2200 + 3n - 3n = 25n - 3n$ $2200 = 22n$ $\frac{2200}{22} = n$ $n = 100$
	4.	Substitute back into either equation to determine the values of C and R.	$R = 25n$ $= 25 \times 100$ $= 2500$
	5.	Answer the question in the context of the problem.	The break-even point is (100, 2500). Therefore 100 shirts need to be sold to cover the production cost, which is \$2500.
b. i.	**1.**	Write the two equations.	**b. i.** $C = 2200 + 3n$ $R = 25n$
	2.	Substitute the given value into both equations.	$n = 75$ $C = 2200 + 3 \times 75$ $= 2425$ $R = 25 \times 75$ $= 1875$
	3.	Determine the profit/loss.	Profit/loss $= R - C$ $= 1875 - 2425$ $= -550$ Since the answer is negative, it means that Santo lost \$550 (i.e. selling 75 shirts did not cover the cost to produce the shirts).
ii.	**1.**	Write the two equations.	**ii.** $C = 2200 + 3n$ $R = 25n$
	2.	Substitute the given value into both equations.	$n = 220$ $C = 2200 + 3 \times 220$ $= 2860$ $R = 25 \times 220$ $= 5500$
	3.	Determine the profit/loss.	Profit/loss $= R - C$ $= 5500 - 2860$ $= 2640$ Since the answer is positive, it means that Santo made \$2640 profit from selling 220 shirts.

on Resources

Interactivity: Break-even points (int-6454)

studyon

Units 1 & 2 > AOS 1 > Topic 2 > Concept 4

Solving problems with simultaneous equations Concept summary and practice questions

Exercise 1.6 Problem solving with simultaneous equations

1. **WE15** Mary bought 4 donuts and 3 cupcakes for \$10.55, and Sharon bought 2 donuts and 4 cupcakes for \$9.90. Letting d represent the cost of a donut and c represent the cost of a cupcake, set up a pair of simultaneous equations to represent this information.

2. A pair of simultaneous equations representing the number of adults and children attending the zoo is shown below.
   Equation 1: $a + c = 350$
   Equation 2: $25a + 15c = 6650$
   a. By solving the pair of simultaneous equations, determine the total number of adults and children attending the zoo.
   b. In the context of this problem, what does equation 2 represent?

3. **WE16** Yolanda sells handmade bracelets at a market for \$12.50. The revenue, R, for selling n bracelets is represented by the equation $R = 12.50n$.
   The cost to make n bracelets is represented by the equation $C = 80 + 4.50n$.
   a. i. By solving the equations simultaneously, determine the break-even point.
      ii. In the context of this problem, what does the break-even point mean?
   b. Determine the profit or loss, in dollars, if Yolanda sells:
      i. 8 bracelets
      ii. 13 bracelets.

4. The entry fee for a charity fun run event is \$18. It costs event organisers \$2550 for the hire of the tent and \$3 per entry for administration. Any profit will be donated to local charities.
   An equation to represent the revenue for the entry fee is $R = an$, where R is the total amount collected in entry fees, in dollars, and n is the number of entries.
   a. Write an equation for the value of a.
   The equation that represents the cost for the event is $C = 2550 + bn$.
   b. Write an equation for the value of b.
   c. By solving the equations simultaneously, determine the number of entries needed to break even.
   d. A total of 310 entries are received for this charity event. Show that the organisers will be able to donate \$2100 to local charities.
   e. Determine the number of entries needed to donate \$5010 to local charities.

5. A school group travelled to the city by bus and returned by train. The two equations show the adult, a, and student, s, ticket prices to travel on the bus and train.
   Bus: $3.5a + 1.5s = 42.50$
   Train: $4.75a + 2.25s = 61.75$
   a. Write the cost of a student bus ticket, s, and an adult bus ticket, a.
   b. What is the most suitable method to solve these two simultaneous equations?
   c. Using your method in part **b**, solve the simultaneous equations and hence determine the number of adults and the number of students in the school group.
6. The following pair of simultaneous equations represents the number of adult and concession tickets sold and the respective ticket prices for the premier screening of the blockbuster *Aliens Attack*.

   Equation 1: $a + c = 544$
   Equation 2: $19.50a + 14.50c = 9013$
   a. What are the costs, in dollars, of an adult ticket, a, and a concession ticket, c?
   b. In the context of this problem, what does equation 1 represent?
   c. By solving the simultaneous equations, determine how many adult and concession tickets were sold for the premier screening.
7. Charlotte has a babysitting service and charges $12.50 per hour. After Charlotte calculated her set-up and travel costs, she constructed the cost equation $C = 45 + 2.50h$, where C represents the cost in dollars per job and h represents the hours Charlotte babysits for.

   a. Write an equation that represents the revenue, R, earned by Charlotte in terms of number of hours, h.
   b. By solving the equations simultaneously, determine the number of hours Charlotte needs to babysit to cover her costs (that is, the break-even point).
   c. In one week, Charlotte had four babysitting jobs as shown in the table.

Babysitting job	1	2	3	4
Number of hours (h)	5.0	3.5	4.0	7.0

      i. Determine whether Charlotte made a profit or loss for each individual babysitting job.
      ii. Did Charlotte make a profit this week? Justify your answer using calculations.
   d. Charlotte made a $50 profit on one job. Determine the total number of hours she babysat for.
8. Trudi and Mia work part time at the local supermarket after school. The following table shows the number of hours worked for both Trudi and Mia and the total wages, in dollars, paid over two weeks.

Week	**Trudi's hours worked**	**Mia's hours worked**	**Total wages**
Week 1	15	12	$400.50
Week 2	9	13	$328.75

a. Construct two equations to represent the number of hours worked by Trudi and Mia and the total wages paid for each week. Write your equations using the pronumerals t for Trudi and m for Mia.
b. In the context of this problem, what do t and m represent?
c. By solving the pair of simultaneous equations, find the values of t and m.

9. Brendan uses carrots and apples to make his special homemade fruit juice. One week he buys 5 kg of carrots and 4 kg of apples for $31.55. The next week he buys 4 kg of carrots and 3 kg of apples for $24.65.

a. Set up two simultaneous equations to represent the cost of carrots, x, in dollars per kilogram, and the cost of apples, y, in dollars per kilogram.
b. By solving the simultaneous equations, determine how much Brendan spends on 1 kg each of carrots and apples.
c. Determine the amount Brendan spends the following week when he buys 2 kg of carrots and 1.5 kg of apples. Give your answer correct to the nearest 5 cents.

10. The table shows the number of 100-g serves of strawberries and grapes and the total kilojoule intake.

Fruit	**100-g serves**	
Strawberries, s	3	4
Grapes, g	2	3
Total kilojoules	**1000**	**1430**

a. Construct two equations to represent the number of serves of strawberries, s, and grapes, g, and the total kilojoules using the pronumerals shown.
b. By solving the pair of simultaneous equations constructed in part a, determine the number of kilojoules (kJ) for a 100-g serve of strawberries.

11. Two budget car hire companies offer the following deals for hiring a medium size family car.

Car company	**Deal**
FreeWheels	$75 plus $1.10 per kilometre travelled
GetThere	$90 plus $0.90 per kilometre travelled

a. Construct two equations to represent the deals for each car hire company. Write your equations in terms of cost, C, and kilometres travelled, k.
b. By solving the two equations simultaneously, determine the value of k at which the cost of hiring a car will be the same.
c. Rex and Jan hire a car for the weekend. They expect to travel a distance of 250 km over the weekend. Which car hire company should they use and why? Justify your answer using calculations.

12. The following table shows the number of boxes of three types of cereal bought each week for a school camp, as well as the total cost for each week.

Cereal	Week 1	Week 2	Week 3
Corn Pops, c	2	1	3
Rice Crunch, r	3	2	4
Muesli, m	1	2	1
Total cost, $	**27.45**	**24.25**	**36.35**

Wen is the cook at the camp. She decides to work out the cost of each box of cereal using simultaneous equations. She incorrectly sets up the following equations:

$$2c + c + 3c = 27.45$$
$$3r + 2r + 4r = 24.25$$
$$m + 2m + m = 36.35$$

a. Explain why these simultaneous equations will not determine the cost of each box of cereal.
b. Write the correct simultaneous equations.
c. Using CAS or otherwise, solve the three simultaneous equations, and hence write the total cost for cereal for week 4's order of 3 boxes of Corn Pops, 2 boxes of Rice Crunch and 2 boxes of muesli.

13. Sally and Nem decide to sell cups of lemonade from their front yard to the neighbourhood children. The cost to make the lemonade using their own lemons can be represented using the equation $C = 0.25n + 2$, where C is the cost in dollars and n is the number of cups of lemonade sold.

a. If they sell cups of lemonade for 50 cents, write an equation to represent the selling price, S, for n number of cups of lemonade.
b. By solving two simultaneous equations, determine the number of cups of lemonade Sally and Nem need to sell in order to break even (i.e. cover their costs).
c. Sally and Nem increase their selling price. If they make a $7 profit for selling 20 cups of lemonade, what is the new selling price?

14. The CotX T-Shirt Company produces T-shirts at a cost of $7.50 each after an initial set-up cost of $810.
a. Determine the cost to produce 100 T-shirts.
b. Using CAS or otherwise, complete the following table that shows the cost of producing T-shirts.

n	0	20	30	40	50	60	80	100	120	140
C										

c. Write an equation that represents the cost, C, to produce n T-shirts.

d. CotX sells each T-shirt for \$25.50. Write an equation that represent the amount of sales, S, in dollars for selling n T-shirts.

e. By solving two simultaneous equations, determine the number of T-shirts that must be sold for CotX to break even.

f. If CotX needs to make a profit of at least \$5000, determine the minimum number of T-shirts they will need to sell to achieve this outcome.

15. There are three types of fruit for sale at the market: starfruit, s, mango, m, and papaya, p. The following table shows the amount of fruit bought and the total cost in dollars.

Starfruit, s	**Mango, m**	**Papaya, p**	**Total cost, \$**
5	3	4	19.40
4	2	5	17.50
3	5	6	24.60

a. Using the pronumerals s, m and p, represent this information with three equations.

b. Using CAS or otherwise, find the cost of one starfruit, one mango and one papaya.

c. Using your answer from part **b**, determine the cost of 2 starfruit, 4 mangoes and 4 papayas.

16. The Comet Cinema offers four types of tickets to the movies: adult, concession, senior and member. The table below shows the number and types of tickets bought to see four different movies and the total amount of tickets sales in dollars.

Movie	**Adult, a**	**Concession, c**	**Seniors, s**	**Members, m**	**Total sales, \$**
Wizard Boy	24	52	12	15	1071.00
Champions	35	8	45	27	1105.50
Pixies on Ice	20	55	9	6	961.50
Horror Nite	35	15	7	13	777.00

a. Represent this information in four simultaneous equations, using the pronumerals given in the table.

b. Using CAS or otherwise, determine the cost, in dollars, for each of the four different movie tickets.

c. The blockbuster movie *Love Hurts* took the following tickets sales: 77 adults, 30 concessions, 15 seniors and 45 members. Using your values from part **b**:

  i. write the expression that represents this information

  ii. determine the total ticket sales in dollars and cents.

1.7 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. **MC** Which one of the following number patterns can't be represented by a linear expression?

   A. 5, 8, 12, 17, … **B.** $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$ **C.** 13, 10.5, 8, 5.5, …
   D. 21.5, 23, 24.5, 26, … **E.** $2a, 4a, 6a, 8a, \ldots$

2. **MC** Which of the following is the correctly transposed version of $y = 3x - 6$ that makes x the subject of the equation?

   A. $x = \frac{y-6}{3}$ **B.** $x = \frac{y+6}{3}$ **C.** $x = 3y - 6$ **D.** $x = 3y + 6$
   E. $x = 3y + 2$

3. **MC** The value of x in the linear equation $3(2x + 5) = 12$ is:

   A. −1.5 **B.** −1 **C.** −0.5 **D.** 2
   E. 4.5

4. **MC** The literal linear equation $v = u + at$ is an equation for motion, given an initial velocity, a rate of acceleration and a period of time. The correction solution to this equation for a is:

   A. $a = \frac{v-u}{t}$ **B.** $a = \frac{u-v}{t}$ **C.** $a = v - tu$ **D.** $a = u - tv$
   E. $a = \frac{v}{t} - u$

The following information relates to questions 5 and 6.
Juliana and Alyssa live in two towns 232 km apart. One weekday they both have to drive to each other's town for a business meeting, leaving at the same time in the morning. Alyssa drives an average of 12 km/h faster than Juliana, and they pass each other after 2 hours.

5. **MC** If Juliana drives at an average speed of j km/h, what is Alyssa's average driving speed?

   A. $(232 - j)$ km/h **B.** $(232 + j)$ km/h **C.** $(j - 12)$ km/h
   D. $(j + 12)$ km/h **E.** $\frac{232}{j}$ km/h

6. **MC** What is Juliana's average driving speed?

   A. 50 km/h **B.** 52 km/h **C.** 60 km/h
   D. 64 km/h **E.** 72 km/h

7. **MC** At the start of the year Theodore had $1000 in his savings account. He then deposited $250 each month into his savings account. A linear recurrence relation to model this situation is:

   A. $t_n = t_{n-1} + 1000, t_1 = 250$ **B.** $t_n = t_{n-1} - 1000, t_1 = 250$ **C.** $t_n = t_{n-1} + 250, t_1 = 1000$
   D. $t_n = t_{n-1} - 250, t_1 = 1000$ **E.** $t_n = t_{n-1} + 1250, t_1 = 0$

8. **MC** The solution to the following pair of simultaneous equations is:

$$x + 5y = 7$$
$$y = 5 - 2x$$

   A. $x = -8, y = 3$ **B.** $x = 2.25, y = 0.5$ **C.** $x = 2, y = 1$
   D. $x = 1, y = 2$ **E.** $x = 3, y = -8$

9. MC Two adults and four children went to the circus. They paid a total of \$55.00 for their tickets. One adult and three children paid \$35.00 to enter the same circus. Which one of the following sets of simultaneous equations represents this situation, where a is the cost for an adult and c is the cost of a child's entry?

A. $a + c = 55$, $a + c = 35$

B. $2a + c = 55$, $a + 3c = 35$

C. $2a + 3c = 55$, $a + 4c = 35$

D. $2a + 4c = 35$, $a + 3c = 55$

E. $2a + 4c = 55$, $a + 3c = 35$

10. MC Bertha knits teddy bears and sells them at the local farmers' market. Bertha spends \$120 in wool, and it costs her an additional \$4.50 to make each teddy bear. She sells the bears for \$14.50 each. How many teddy bears does Bertha need to sell to break even?

A. 6 B. 8 C. 9 D. 10 E. 12

Short answer

1. a. Complete the following table of values for the linear equation $p = 5(2n + 4)$.

n	1	2	3	4	5	6
p	30	40				

b. Write down the common difference.
c. What is the value of p when $n = 12$?

2. Solve each of the following equations for the unknown.
a. $3x + 5 = 14$
b. $7(y + 4) = 35$
c. $\frac{5 - 2m}{3} = -2$
d. $\frac{3s}{4} + 6 = 10$
e. $\frac{-2(3t + 1)}{5} + 3 = 9$

3. Solve the following simultaneous equations by using either substitution or elimination.
a. $y = 3x - 5$ and $3x + 2y = 17$
b. $2x + y = 7$ and $3x - y = 3$
c. $3x - 2y = -16$ and $y = 2(x + 9)$
d. $4x + 3y = 17$ and $3x + 2y = 13$

4. Determine the break-even points for the following cost and revenue equations. Where appropriate, give your answers correct to 2 decimal places.
a. $C = 150 + 2x$ and $R = 7.5x$
b. $C = 13.5x + 25$ and $R = 19.7x$
c. $C = 3500 + 22.5x$ and $R = 35x$

5. Find equations for the linear relations formed by the following number patterns.
a. 5, 9, 13, 17, …
b. 6, 2, −2, −6, …
c. 1.2, 1.7, 2.2, 2.7, …
d. 103, 106, 109, 112, …

6. Petra is doing a survey of how many humans and pets are in her extended family. She gives the following information to her friend Juliana:
- There are 33 humans and pets (combined).
- The combined number of legs (pets and owners) is 94.
- Each human has 2 legs and each pet has 4 legs.

a. If h is the number of humans, express the number of pets in terms of h.
b. Write expressions for the total number of human legs and the total number of pet legs in terms of h.
c. How many humans and pets are there in Petra's extended family?

Extended response

1. A study of the homework habits of a group of students showed that the amount of weekly homework, in hours, completed by the students had an effect on their performance on the weekly assessment tasks. The table shown represents the number of weekly homework hours spent by the group of students and the average percentage mark they achieved on their weekly assessment tasks.

Hours of homework, *h*	1	2	3	4	5	6	7
Average percentage mark, *m*	22	31	40	49	58	67	76

   a. On average, how many marks are gained for each additional hour spent doing homework?
   b. Using any appropriate method, determine the rule that finds the average percentage mark, m, for each hour spent doing homework, h.
   c. Using the rule you found in part **b**, determine the average percentage mark for the following numbers of hours spent doing homework:
      i. 9 hours
      ii. 5.5 hours
   d. Seth scored 51.25% on the assessment task. Determine how many hours he spent doing homework during the week according to the rule.
   e. Nerada did not do any homework during the week. What will be her expected percentage mark on the assessment task according to the rule?
   f. Freda argues that the rule does not apply for students who spend more than 10 hours each week doing their homework. Find the average percentage mark for a student who spends 10 hours doing homework. Does Freda have a valid argument?

2. Hank is cooking a Sunday dinner of roast lamb and roast beef for 20 guests. He has a 2.5–kg leg of lamb and a 4.2–kg cut of beef. The recommended cooking time for the lamb is 62.5 minutes; the recommended cooking time for the beef is 105 minutes. Hank's cookbook recommends that the meat be left to rest for 15 minutes before carving.
   The cooking time is the same per kilogram for both cuts of meats and increases at a constant rate per kilogram.
   a. Find the cooking time in minutes, t, per kilogram of meat, k.
   b. Construct an equation that finds the cooking time, in minutes, including the resting time per kilogram of meat (lamb and beef).
   c. Using a spreadsheet and your equation from part **b**, complete the following table for different-sized cuts of meat. Write your answers correct to the nearest whole number.

Weight (g)	Cooking time (minutes)	Weight (g)	Cooking time (minutes)
500		2250	
750		2500	
1000		2750	
1250		3000	
1500		3250	
1750		3500	
2000		3750	

Marcia uses the equation from part **b** to help her with the cooking time of her Christmas turkey, which weighs 5.5 kg.

d. Using the equation you found in part **b**, determine the cooking time in hours and minutes for the turkey.

e. Marcia finds that the cooking time is incorrect. Explain why the equation did not help her to accurately determine the cooking time.

Marcia finds a cookbook which suggests that the cooking time for a turkey is $\frac{3}{4}$ of an hour per kilogram.

f. If the resting time for roast turkey is 30 minutes, construct an equation that finds the cooking time per kilogram of turkey.

g. Using the equation you found in part **f**, determine the recommended cooking time, in hours and minutes, for Marcia's turkey.

3. Suzanne is starting a business selling homemade cupcakes. It will cost her \$250 to buy all of the equipment, and each cupcake will cost \$2.25 to make. Suzanne models her costs, C, in dollars, to make n cupcakes using the following equation:

$$C = 2.25n + b$$

a. What is the value of b?

b. How much will it cost Suzanne to make 50 cupcakes?

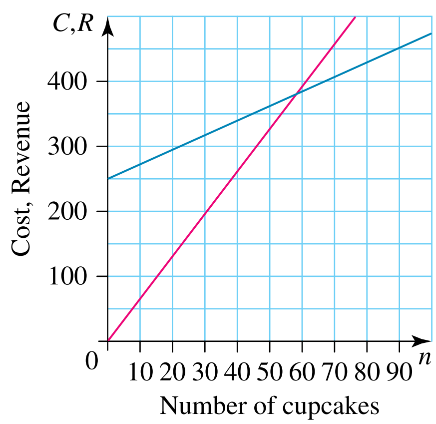
c. Suzanne receives an order to supply cupcakes for an afternoon tea. If it costs Suzanne \$373.75 to make the order, how many cupcakes did she make?

d. Suzanne sells each cupcake for \$6.50. Write an equation to represent the revenue Suzanne earns, R, from selling n cupcakes.

e. One week, Suzanne sells 150 cupcakes. Determine the total profit, in dollars, that Suzanne makes for that week.

f. By solving a pair of simultaneous equations, find the total number of cupcakes Suzanne needs to sell to break even. Give your answer correct to the nearest whole number.

g. The graph shows Suzanne's cost, C, and revenue, R, for making and selling n cupcakes.



On the graph, add labels for:

i. the line that represents the cost, C

ii. the line that represents the revenue, R

iii. the break-even point.

Suzanne uses flour and eggs in her cupcakes. One week she buys 5 kg of flour and 3 dozen eggs; these purchases cost her \$20.60. The next week she buys 3 kg of flour and 2 dozen eggs, which cost her \$13.15.

The equation $5f + 3e = 20.6$ represents the amount of flour and eggs Suzanne bought one week, where f is the cost of 1 kg of flour and e is the cost of 1 dozen eggs.

h. Write the equation that represents the amount of flour and eggs Suzanne bought the next week.

i. By solving a pair of simultaneous equations, determine how much it will cost Suzanne to buy 4 kg of flour and 4 dozen eggs.

4. There are two types of tickets available for the GoodFeel Festival: full price and concession. On the first day, 6473 people attended the festival. The total ticket revenue, in dollars, was \$109 559.80. A full-price ticket cost \$19.80 and a concession ticket cost \$14.60.

a. Construct two equations that represent the number of people attending and the ticket sales in terms of full price tickets, f and concession tickets, c.

b. By solving your equations from part **a**, determine the number of each type of ticket sold on the first day.

It costs the organisers \$43 500 plus \$5.90 for each ticket sold to run the festival.

c. Construct an equation to represent the cost to the organisers, C, for the festival in terms of the number of tickets sold, x.

d. Construct an equation that shows the relationship between variables x, f and c.

e. Construct an equation to represent the revenue from ticket sales, R, in terms of f and c.

f. Using the ticket sales for the first day of the festival, determine if the organisers made a profit.

studyon

Units 1 & 2 Sit topic test

Answers

Topic 1 Linear relations and equations

Exercise 1.2 Linear relations

1. a. Non-linear b. Non-linear
   c. Linear d. Non-linear
   e. Linear
2. a.

Equation	Bethany's response	Correct response
$y = 4x + 1$	Yes	Yes
$y^2 = 5x - 2$	Yes	No
$y + 6x = 7$	Yes	Yes
$y = x^2 - 5x$	No	No
$t = 6d^2 - 9$	No	No
$m^3 = n + 8$	Yes	No

   b. Bethany should look at both variables (pronumerals or letters). Both variables need to have a highest power of 1.
3. a. Linear b. Non-linear
   c. Linear d. Linear
   e. Non-linear f. Non-linear
   g. Non-linear
4. a. Yes, as the power of x is 1. (Note that this equation is not linear:)
   b. The power of both variables in a linear relation must be 1.
5. a. $4n - 2$ b. $0.5n + 3.5$
6. a. -1 b. $-n + 11$ c. 55 jars
7. a. 3, 4.5, 6.75, 10.125
   b. No, as there is no common difference.
8. a. $4n - 1$ b. $3n + 4$ c. $-3n + 15$
   d. $-6n + 19$ e. $-2n - 10$
9. a. 0.8
   b. Yes, as it has a common difference.
10. $x = \dfrac{y+3}{6}$
11. $x = 2y - \dfrac{1}{3}$
12. a. $x = \dfrac{y-5}{2}$ b. $x = \dfrac{3y-8}{6}$
    c. $x = \dfrac{p+6}{5}$
13. a.

Day	1	2	3	4	5
Amount of water (L)	950	900	850	800	750

    b. 1000 L
    c. $w = -50d + 1000$
14. a. \$2250
    b. $A = 1500 + 250m$
    c. This changes the equation to $A = 4500 + 350m$.
15. No, because her distance each day is half the previous distance, so there is no common difference.
16. a. 70 km
    b. $10w + 40$, where w = number of weeks
17. $T_n = 2n - 6$
18. a. $t = 4n - 1$
    b. 3, 7, 11, 15, 19, 23, 27, 31, 35, 39
    c. $7 - 3 = 4$
    $11 - 7 = 4$
    $15 - 11 = 4$
    and so on …

Exercise 1.3 Solving linear equations

1. a. $x = 3$ b. $n = 10$
   c. $d = 4.5$ d. $x = 17$
2. a. i. $\times 4, +3$ ii. $+2, \times 3$
   iii. $+1, \div 2$ iv. $\times 3, -9, \times 2$
   b. i. $a = \frac{7}{4}$ ii. $x = 2$
   iii. $s = 13$ iv. $c = \frac{17}{3}$
3. a. $x = -9$ b. $y = -0.5$ c. $x = -4.5$
4. a. $a = \dfrac{v-u}{t}$ b. $x = \dfrac{m+k}{y}$ c. $x = p(r+s)$
5. $y = -25$
6. $y = -6$
7. a. 33 minutes b. 108 minutes
8. $x = \dfrac{r+q}{p}$
9. $d = \dfrac{C}{\pi}$
10. a. $w = 5$
    b. Operations need to be performed in reverse order.
11. $K = \dfrac{F-32}{1.8} + 273$
12. a. $x = \frac{7}{3}$ b. $x = \frac{-13}{3}$
    c. $x = \frac{1}{3}$ d. $x = 13$
13. a. 2 hours b. 2 hours 38 minutes
    c. 41 mins d. 3 hours 45 minutes
    e. 32 minutes
14. a. i. 9 months b. 5 years, 4 months
    ii. 19 months (= 18.67 months)
15. a. $-32, \times 5, \div 9$
    b. $\times 9, \div 5, +32$
    c. 374°F
16. a. 14.69 cm
    b. 18.33 cm
    c. 21.06 cm
17. a. $x = \frac{-14}{25}$ b. $y = \frac{163}{162}$
    c. $x = \frac{-195}{28}$ d. $x = \frac{12}{97}$
18. a. i. 5 weeks ii. 10 weeks
    iii. 13 weeks iv. 20 weeks
    b. 10 cm
    c. 63 cm, 67 cm, 72 cm, 78 cm

Exercise 1.4 Developing linear equations

1. $2.24
2. The red velvet cupcakes are the cheapest per cupcake.
3. 33
4. 10.5 cm and 31.5 cm
5. 6 roses and 13 lilies
6. 16 strawberry twists and 12 chocolate ripples
7. $7.90
8. 17
9. a. See table at the foot of the page*
   b. Between the 7th and 8th minute
10. a. See table at the foot of the page*
    b. 10 weeks
11. 12, 6, 0, −6, −12, −18
12. −5.8, −2.6, 0.6, 3.8, 7, 10.2
13. a. $t_n = t_{n-1} + 25,\ t_1 = 450$
    b. $450, $475, $500, $525, $550, $575, $600, $625
    c. $v = 425 + 25n$
14. a. $t_n = t_{n-1} - 13,\ t_1 = 313$
    b. 313, 300, 287, 274, 261, 248, 235
    c. $l = 326 - 13n$
15. a. $t_n = t_{n-1} + 40,\ t_1 = 150$
    b. See table at the foot of the page*
    c. 16 weeks
16. 13 m by 19 m
17. 9 carrots and 6 potatoes
18. a. $w = 165 + 1.5h$
    b. See table at the foot of the page*
    c. 6:00 pm
19. a. $t_n = t_{n-1} + \frac{4}{3},\ t_1 = 2$
    b. 2 cm, $3\frac{1}{3}$ cm, $4\frac{2}{3}$ cm, 6 cm, $7\frac{1}{3}$ cm
    c. No, there was $\frac{2}{3}$ cm of water in the tank before being filled.
20. Michelle: 70 km/h, Lydia: 60 km/h
21. a. See table at the foot of the page*
    b. See table at the foot of the page*
    c. Subtract the values in the first table from the values in the second table.
    d. See table at the foot of the page*

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9. a. *

Minute	1	2	3	4	5	6	7	8	9	10
Distance (km)	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	3.6	4

10. a. *

Week	0	1	2	3	4	5	6	7	8	9	10	11	12
Money ($)	20	23	26	29	32	35	38	41	44	47	50	53	56

15. b. *

Week	1	2	3	4	5	6	7	8
Amount ($)	150	190	230	270	310	350	390	430

12. *

Hour	2	4	6	8	10	12	14	16	18	20	22	24
Water in tank (L)	168	171	174	177	180	183	186	189	192	195	198	201

21. a. *

Number of boards	10	11	12	13	14	15	16	17	18	19	20
Cost ($)	395	409.50	424	438.50	453	467.50	482	496.50	511	525.50	540

b. *

Number of boards	10	11	12	13	14	15	16	17	18	19	20
Revenue ($)	329.50	362.45	395.40	428.35	461.30	494.25	527.20	560.15	593.10	626.05	659

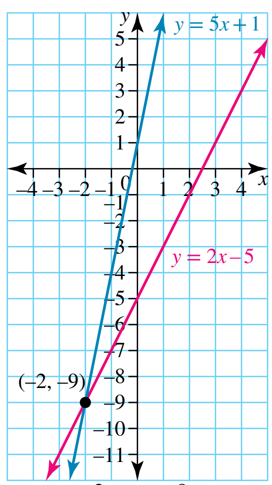
d. *

Number of boards	10	11	12	13	14	15	16	17	18	19	20
Profit ($)	−65.50	−47.05	−28.60	−10.15	8.30	26.75	45.20	63.65	82.10	100.55	119

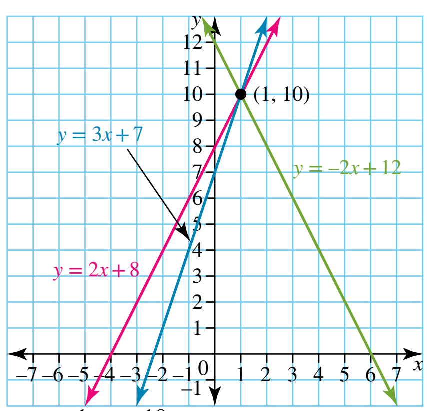
22. a. $c_m = c_{m-1} + 1.35,\ c_1 = 301.35$
b. $s_n = s_{n-1} + 4.5,\ s_1 = 4.5$
c. 96
d.

Number of drinks sold	($)
0	−300.00
10	−268.50
20	−237.00
30	−205.50
40	−174.00
50	−142.50
60	−111.00
70	−79.50
80	−48.00
90	−16.50
100	15.00
110	46.50
120	78.00

Exercise 1.5 Simultaneous linear equations

1.


$x = -2,\ y = -9$

2.


$x = 1,\ y = 10$

3. $x = -2,\ y = 5$

4. a. $x = -2,\ y = -7$
b. $x = 2,\ y = -3$
c. $x = -5,\ y = -18$
d. No solution
e. $x = -1,\ y = 1$

5. a. $x = -1,\ y = -1$
b. $m = 11,\ n = 3$
c. $x = 6,\ y = 7$

6. a. $x = 3,\ y = -3$
b. $x = 1,\ y = 4$

7. E

8. a. $x = 7,\ y = 19$
b. $x = -2,\ y = -12$
c. $x = 7,\ y = 44$
d. $x = 2,\ y = 1$
e. $x = -3,\ y = -4$
f. $x = 1,\ y = -1$

9. a. Both unknowns are on the same side.
b. Add the two equations and solve for x, then substitute x into one of the equations to solve for y. $x = 3,\ y = -1$

10. a. $x = 1,\ y = 2$
b. $a = 2,\ b = 3$
c. $c = -1,\ d = 2$

11. $a = 2$ and $x = -2$

12. a. $x = 2,\ y = -2$
b. $x = 3,\ y = 4$
c. $x = -1,\ y = 3$
d. $x = 5,\ y = 3$
e. $x = -1,\ y = 4$
f. $x = 6,\ y = 4$

13. A

14. a. Marcia added the equations together instead of subtracting (and did not perform the addition correctly). The correct result for step 2 is [1] − [2]: $22y = 11$.
b. $x = 5,\ y = \frac{1}{2}$
c. $x = 5,\ y = \frac{1}{2}$

15. a. Goal = 5 points, behind = 2 points
b. Jetts 40 points, Meteorites 48 points

16. a. The equation has unknowns on each side of the equals sign.
b. Mick works 5 hours and Minnie works 10 hours.
c. 3 hours 45 minutes (3.75 hours)

17. a. $x = -0.38,\ y = 4.08$
b. $x = -10.71,\ y = -12.86$
c. $x = 0.75,\ y = 0.89$

18. a. i. $(-12, -64)$
ii. $(-1, -2)$
iii. No solution
iv. $(-1, 4)$
b. No, the graphs in part iii are parallel (they have the same gradient).

Exercise 1.6 Problem solving with simultaneous equations

1. $4d + 3c = 10.55$ and $2d + 4c = 9.90$

2. a. 140 adults and 210 children
b. The cost of an adult's ticket is \$25, the cost of a children's ticket costs \$15, and the total ticket sales is \$6650.

3. a. i. 10
ii. Yolanda needs to sell 10 bracelets to cover her costs.
b. i. \$16 loss
ii. \$24 profit

4. a. $a = 18$
b. $b = 3$
c. 170 entries
d. $R = \$5580,\ C = \$3480,\ P = \$2100$
e. 504 entries

5. a. $s = \$1.50,\ a = \$3.50$
   b. Elimination method
   c. 4 adults and 19 students

6. a. $a = \$19.50,\ c = \$14.50$
   b. The total number of tickets sold (both adult and concession)
   c. 225 adult tickets and 319 concession tickets

7. a. $R = 12.50h$
   b. 4.5 hours
   c. i. Charlotte made a profit for jobs 1 and 4, and a loss for jobs 2 and 3.
      ii. Yes, she made \$15 profit.
      $(25 + 5 - (10 + 5)) = 30 - 15 = \$15$
   d. 9.5 hours

8. a. $15t + 12m = 400.50$ and $9t + 13m = 328.75$
   b. t represents the hourly rate earned by Trudi and m represents the hourly rate earned by Mia.
   c. $t = \$14.50,\ m = \$15.25$

9. a. $5x + 4y = 31.55$ and $4x + 3y = 24.65$
   b. $x = \$3.95,\ y = \$2.95$
   c. \$12.35

10. a. $3s + 2g = 1000$ and $4s + 3g = 1430$
    b. 140 kJ

11. a. $C = 75 + 1.10k$ and $C = 90 + 0.90k$
    b. $k = 75$ km
    c. $C_{\text{FreeWheels}} = \$350,\ C_{\text{GetThere}} = \$315$. They should use GetThere.

12. a. The cost is for the three different types of cereal, but the equations only include one type of cereal.
    b. $2c + 3r + m = 27.45$
    $c + 2r + 2m = 24.25$
    $3c + 4r + m = 36.35$
    c. \$34.15

13. a. $S = 0.5n$
    b. 8 cups of lemonade
    c. 70 cents

14. a. \$1560
    b. See table at the foot of the page*
    c. $C = 810 + 7.5n$
    d. $S = 25.50n$
    e. 45 T-shirts
    f. 323 T-shirts

15. a. $5s + 3m + 4p = 19.4$
    $4s + 2m + 5p = 17.5$
    $3s + 5m + 6p = 24.6$
    b. $s = \$1.25,\ m = \$2.25,\ p = \$1.60$
    c. \$17.90

16. a. $24a + 52c + 12s + 15m = 1071$
    $35a + 8c + 45s + 27m = 1105.5$
    $20a + 55c + 9s + 6m = 961.5$
    $35a + 15c + 7s + 13m = 777$
    b. Adult ticket = \$13.50, concession = \$10.50, seniors = \$8.00, members = \$7.00
    c. i. $77 \times 13.50 + 30 \times 10.50 + 15 \times 8.00 + 45 \times 7.00$
       ii. \$1789.50

Review: exam practice — answers

Multiple choice

1. A 2. B 3. C 4. A 5. D
6. B 7. C 8. C 9. E 10. E

Short answer

1. a.

n	1	2	3	4	5	6
p	30	40	50	60	70	80

   b. 10
   c. 140

2. a. $x = 3$ b. $y = 1$ c. $m = 5.5$
   d. $s = 5\frac{1}{3}$ e. $t = -5\frac{1}{3}$

3. a. $x = 3, y = 4$ b. $x = 2, y = 3$
   c. $x = -20, y = -22$ d. $x = 5, y = -1$

4. a. $x = 27.27$ b. $x = 4.03$
   c. $x = 280$

5. a. $t = 4n + 1$ b. $t = -4n + 10$
   c. $t = 0.5n + 0.7$ d. $t = 3n + 100$

6. a. $33 - h$
   b. Total number of human legs = $2h$; total number of pet legs = $4(33 - h)$
   c. 19 humans and 14 pets

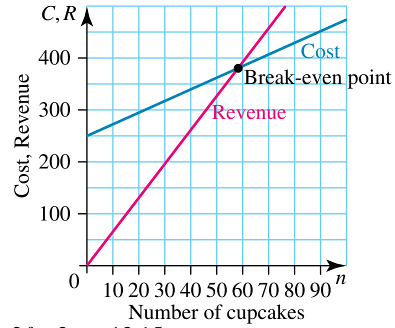
Extended response

1. a. 9 marks
   b. $m = 9h + 13$
   c. i. 94% ii. 62.5%
   d. 4.25 hours
   e. 13%
   f. According to the equation, the average percentage mark for a student spending 10 hours on their homework would be 103%, so Freda has a valid argument.

14. b. *

n	0	20	30	40	50	60	80	100	120	140
C	810	960	1035	1110	1185	1260	1410	1560	1710	1860

2. a. 25 minutes per kilogram
   b. $t = 25k + 15$
   c. See table at the foot of the page*
   d. 2 hours, 32.5 minutes
   e. This equation will probably varry for different tyes of meat.
   f. $t = 45k + 30$
   g. 4 hours, 37.5 minutes

3. a. 250
   b. $362.50
   c. 55
   d. $R = 6.5n$
   e. $387.50
   f. 59
   g. 

   h. $3f + 2e = 13.15$
   i. $22.80

4. a. $f + c = 6473$, $19.8f + 14.6c = 109\,559.8$
   b. 2895 full price tickets and 3578 concession tickets
   c. $C = 5.9x + 43\,500$
   d. $x = f + c$
   e. $R = 19.8f + 14.6c$
   f. Yes, they made a profit of $27 869.10.

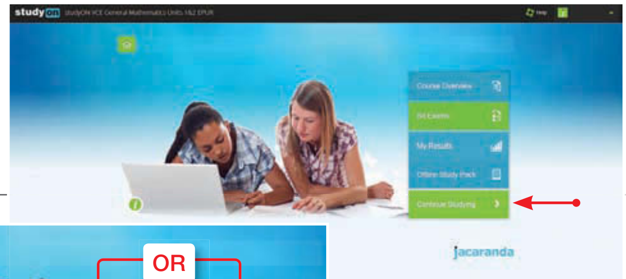
2. c. *

Weight (g)	Cooking time (minutes)	Weight (g)	Cooking time (minutes)
500	27.5	2250	71.25
750	33.75	2500	77.5
1000	40	2750	83.75
1250	46.25	3000	90
1500	52.5	3250	96.25
1750	58.75	3500	102.5
2000	65	3750	108.75

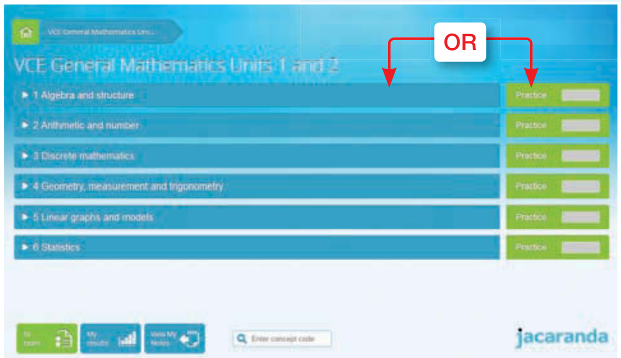
REVISION: AREA OF STUDY 1 Arithmetic and number

TOPIC 1

- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

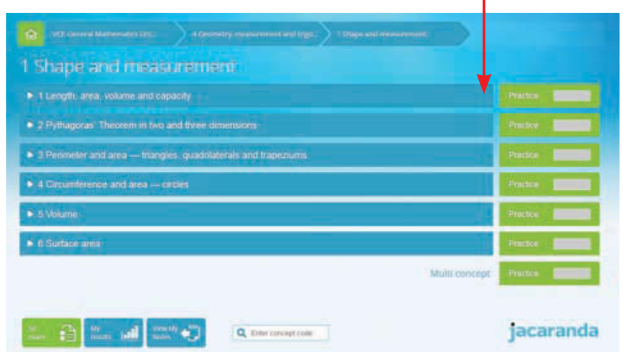


- Select your **course** *VCE General Mathematics Units 1 & 2* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.

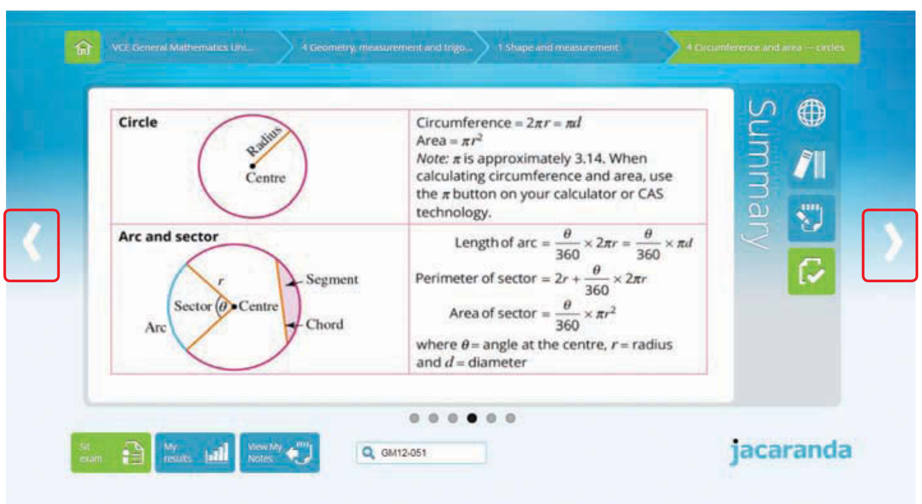


- Select **Practice** at the topic level to access all questions in the topic.

- At **topic level**, drill down to concept level.



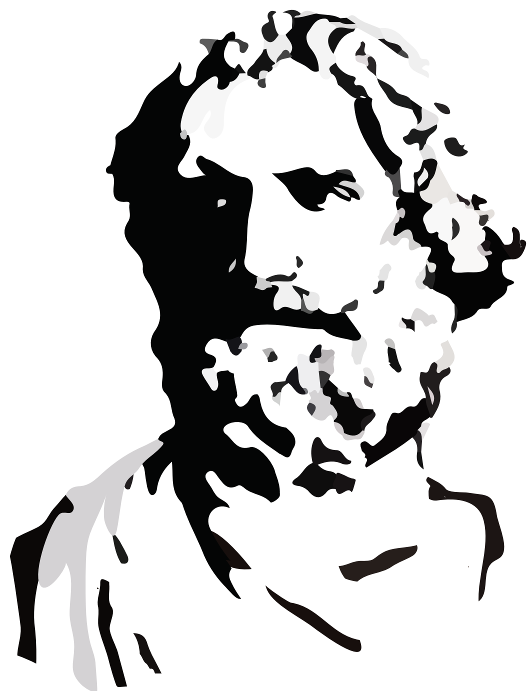
- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 2 Computation and practical arithmetic

2.1 Overview

2.1.1 Introduction



Scientific notation is used in mathematics to simplify either very large numbers (mass of the Earth = 5.972×10^{24} kg) or very small numbers (mass of an electron = 9.109×10^{-31} kg).

Scientific notation was originally thought of by Archimedes. Archimedes was born in Syracuse of Sicily in 287 BC and lived to 211 BC. He studied in Alexandria (Egypt), then the chief centre of Greek learning. Archimedes made many discoveries in mathematics and geometry, including finding the centre of gravity of an object, the initial introduction to calculus and a more accurate estimation of π. But it was his work on developing a system to express large numbers in a more simplified way that was some of his more impressive work. This started when he calculated the number of grains of sand in the universe for King Gelon. Archimedes' universe was different from what it is now and he calculated with Greek letter numerals, since the current number system and scientific notation had not yet been invented. Even though it is impossible to calculate the number of grains of sand in the universe, the fact that Archimedes tried highlights how long mathematicians have been interested in very large and small values. In ancient Greece, the capital letter M referred to the number 10 000, and Archimedes developed a system of writing lowercase Greek letters over M to denote multiples of 10 000. It wasn't until Rene Descartes' work centuries later that the superscript method for notating exponents was introduced, which led to the modern use of scientific notation.

LEARNING SEQUENCE

- **2.1** Overview
- **2.2** Computation methods
- **2.3** Orders of magnitude
- **2.4** Ratio, rates and percentages
- **2.5** Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.

2.1.2 Kick off with CAS

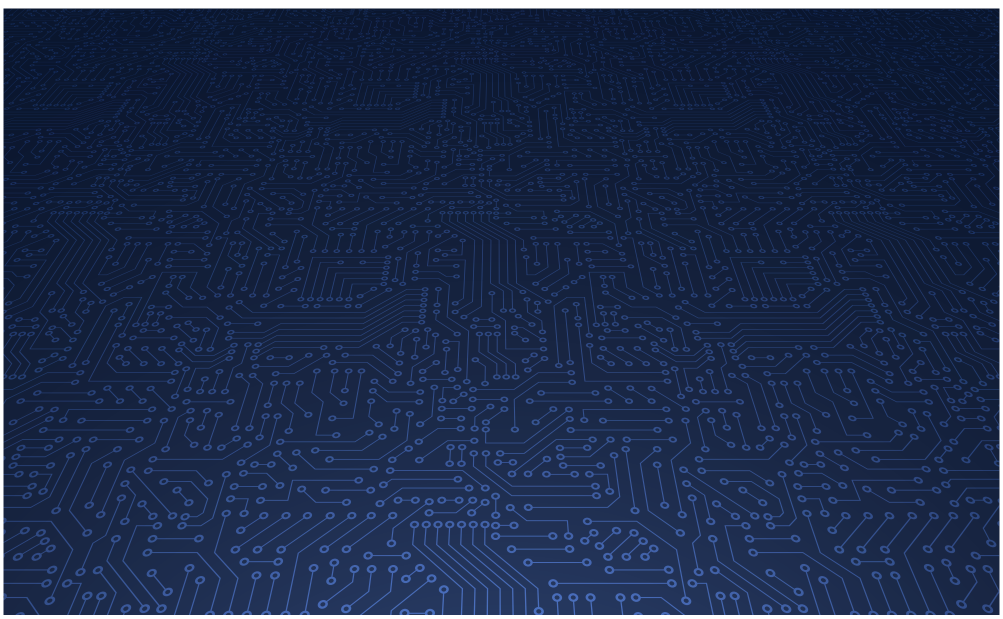
Computation with CAS

CAS can be used to simply and efficiently perform a wide range of mathematical operations.

1. Use CAS to determine the values of the following.
   a. $\frac{54}{186}$ (give the fraction in simplest form)
   b. $(-9.5)^2$
   c. $\left(\frac{1}{3}\right)^{-2}$

The order of operations defines the procedures that need to be carried out first when determining the value of a mathematical expression or equation. CAS will automatically calculate values according to the order of operations rules.

2. Compare the incorrect with the correct values when the following expressions are evaluated in two different ways. Calculate the value of each of the following expressions by completing the operations from left to right.
   a. $15 + 2 \times 5 - 8 \div 2$
   b. $(11 + 7) \div 2 + (6 \times 5) \div 3$
   c. $\frac{25 - 4^2}{3 + 10 \div 2}$
3. Use CAS to determine the true value of each of the expressions in question 2. Scientific notation allows us to express extremely large and small numbers in an easy-to-digest form.
4. Use CAS to complete the following calculations.
   a. $352\,000\,000 \times 189\,000\,000$
   b. $0.000\,000\,245 \div 591\,000\,000$
5. Interpret the notation given on your CAS when completing these calculations.



on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology

2.2 Computation methods

Review of computation

2.2.1 Order of operations

Mathematics plays an essential part in our daily lives, from calculating shopping bills to the password encryption algorithms on your computer and the gear ratios of a bike or car. As with any language, mathematics follows rules and restrictions to ensure the meanings of mathematical sentences are interpreted in a consistent way.

Just as there are road rules to help traffic move safely and reliably, mathematics has rules to ensure equations are solved in a consistent manner. When presented with a mathematical situation, it is important to complete each operation in the correct order.

A common acronym for recalling the **order of operations** is BODMAS.

Step	Acronym letter	Meaning
1st	B	Brackets
2nd	O	Order (powers and roots)
3rd	D and M	Division and multiplication (working left to right)
4th	A and S	Addition and subtraction (working left to right)

WORKED EXAMPLE 1

Calculate the following expressions by correctly applying the order of operations rules.

a. $(27 - 12) + 180 \div 3^2$

b. $\dfrac{(5 \times 8 + 4^2)}{(2 + \sqrt{144})}$

THINK	WRITE
a. 1. Approach the brackets first.	**a.** $(27 - 12) + 180 \div 3^2$ $= 15 + 180 \div 3^2$
2. Resolve the exponent (power).	$= 15 + 180 \div 9$
3. Complete the division.	$= 15 + 20$
4. Add the remaining values.	$= 35$
b. 1. Begin by addressing the exponent within the top bracket.	**b.** $\dfrac{(5 \times 8 + 4^2)}{(2 + \sqrt{144})}$ $= \dfrac{(5 \times 8 + 16)}{(2 + \sqrt{144})}$
2. Complete the multiplication component of the top bracket.	$= \dfrac{(40 + 16)}{(2 + \sqrt{144})}$

3.	Finalise the numerator by completing the addition.	$= \dfrac{56}{(2+\sqrt{144})}$
4.	Moving onto the bottom bracket, resolve the square root as part of the O, for order, in BODMAS.	$= \dfrac{56}{(2+12)}$
5.	Finalise the denominator.	$= \dfrac{56}{14}$
6.	Complete the division to calculate the final answer.	$= 4$

TI | THINK

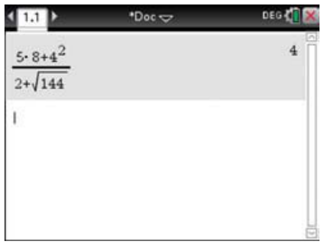
b. 1. On a Calculator page, complete the entry line as:

$$\frac{5 \times 8 + 4^2}{2 + \sqrt{144}}$$

then press ENTER.

Note: The fraction template is located above the p button.

WRITE



CASIO | THINK

b. 1. On the Main screen, complete the entry line as:

$$\frac{5 \times 8 + 4\hat{}2}{2 + \sqrt{144}}$$

then press EXE.

Note: The fraction template is located in the Math1 tab in the keyboard.

WRITE

Just for a comparison, complete the first expression by working from left to right. The resulting answer will be significantly different. This highlights the need to follow consistent mathematical rules.

2.2.2 Directed numbers

Integers are numbers that can be found on either side of zero on a number line. Due to their location, they are classified as either positive (+) or negative (−). They are called **directed numbers** as their values provide both a size, in the form of a numerical value, and a direction relative to zero, either positive or negative.

When evaluating equations involving positive and negative integers it is important to consider the effect of the directional information.

Addition and subtraction of directed numbers

When a direction sign (positive or negative) follows a plus or minus operation sign, the two signs can be combined to simplify the expression.

Instructions	Operation and direction sign	Resulting operation sign	Example
Adding a negative number	(+ −)	Minus operation	10 + −6 = 10 − 6 = 4
Adding a positive number	(+ +)	Plus operation	10 + +6 = 10 + 6 = 16
Subtracting a positive number	(− +)	Minus operation	10 − +6 = 10 − 6 = 4
Subtracting a negative number	(− −)	Plus operation	10 − −6 = 10 + 6 = 16

Like signs $(+\ +)$ or $(-\ -)$ make a plus operation.
Different signs $(+\ -)$ or $(-\ +)$ make a minus operation.

A useful way to look at these expressions is to think in terms of borrowing or lending money.

For example: Yesterday you lent Francesco \$5 (-5), and today he asks to borrow another \$12 (-12). To record this new loan, you need to add $(+)$ Francesco's additional debt to yesterday's total. The corresponding equation would be $-5 + -12 = ?$ How much does he owe you?

The combination of the plus and minus signs can be simplified:

$$\begin{aligned} -5 + -12 &= ? \\ -5 - 12 &= ? \\ &= -17 \end{aligned}$$

Therefore, Francesco owes you \$17.

WORKED EXAMPLE 2

Evaluate these expressions by first simplifying the mathematical symbols.

a. $-73 - -42 + 19$ **b.** $150 + -85 - -96$

THINK	WRITE
a. 1. Begin by combining the negative direction and minus operation signs into a single plus operation sign.	**a.** $-73 - -42 + 19$ $= -73 + 42 + 19$
2. As there are only addition and subtraction operations, complete the sum by working left to right.	$= -12$
b. 1. Combine the direction and operation signs. In this example there are two sets that need to be simplified.	**b.** $150 + -85 - -96$ $= 150 - 85 + 96$
2. Complete the sum by working left to right.	$= 161$

Multiplication and division of directed numbers

When multiplying or dividing an **even** number of negative values, the resulting solution will be positive. However, multiplying or dividing an **odd** number of negative values will produce a negative solution.

WORKED EXAMPLE 3

Determine the value of each of the following expressions.

a. $-3 \times -5 \times -10$ **b.** $(-4 \times -9) \div (-7 \times 6)$ **c.** $(-4)^2 + -2^3$

THINK	WRITE
a. 1. Complete the first multiplication. Recall that an even number of negative signs will produce a positive answer.	**a.** $-3 \times -5 \times -10$ $= +15 \times -10$
2. Now the remaining equation has an odd number of negative values, which produces a negative answer.	$= -150$

b. 1. Apply the appropriate order of operation rules and solve the brackets first. Remember a negative sign multiplied by another negative sign will produce a positive answer, while a positive sign multiplied by a negative sign will produce a negative answer.

b. $(-4 \times -9) \div (-7 \times 6)$
$= +36 \div -42$

2. As there are an odd number of negative signs in the remaining equation, the resulting answer will be negative.

$36 \div -42 = -\frac{6}{7}$

c. 1. Consider the effect of the exponents and their position relative to the brackets. Rewrite the expanded equation.

c. $(-4)^2 + -2^3$
$= -4 \times -4 + -2 \times -2 \times -2$

2. Simplify the combination plus/minus signs.

$= -4 \times -4 - 2 \times -2 \times -2$

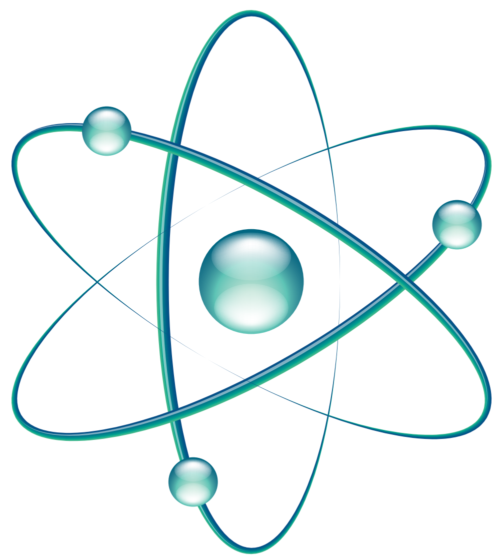
3. Complete the calculations by following the order of operations rules.

$= +16 - 8$
$= 8$

2.2.3 Scientific notation

Scientific notation is used to simplify very large numbers, such as the mass of the Earth (5.972×10^{24} kg), or very small numbers, such as the mass of an electron ($9.109\,382\,91 \times 10^{-31}$ kg).

A number written in scientific notation is in the form $a \times 10^b$, where a is a real number between 1 and 10 and b is an integer.



Scientific notation uses multiplications of 10. For example, look at the following pattern:

$$5 \times 10 = 50$$
$$5 \times 10 \times 10 = 500$$
$$5 \times 10 \times 10 \times 10 = 5000$$

Notice how the number of zeros in the answer increases in proportion to the number of times 5 is multiplied by 10. Scientific notation can be used here to simplify the repetitive multiplication.

$$5 \times 10^1 = 50$$
$$5 \times 10^2 = 500$$
$$5 \times 10^3 = 5000$$

To write a basic numeral using scientific notation there are four key steps.

Step	Instructions	Example 1: 7256	Example 2: 0.008 923
1	Identify the first non-zero value of the original number.	7	8
2	Write that digit, followed by a decimal point and all remaining digits.	7.256	8.923
3	Multiply the decimal by 10.	7.256×10	8.923×10
4	Count the number of places the decimal point is moved. The exponent of the base value 10 will reflect the movement of the decimal point. If the decimal point is moved to the left, the exponent will be positive. If moved to the right, the exponent will be negative.	7.256×10^3	8.923×10^{-3}

To convert a scientific notation value to basic numerals, use the following steps:

Step	Instructions	Example 1: 2.007×10^5	Example 2: 9.71×10^{-4}
1	Look to see if the exponent is positive or negative.	Positive	Negative
2a	If positive, rewrite the number without the decimal point, adding zeros **behind** the last number to fill in the necessary number of place values. In this case, move the decimal point 5 to the right as the exponent is +5.	200 700	
2b	If negative, rewrite the number without the decimal point, adding zeros **in front of** the last number to fill in the necessary number of place values. In this case, move the decimal point 4 to the left as the exponent is −4.		0.000 971
3	Double check by counting the number of places the decimal point has moved. This should match the value of the exponent.		

Note: When using CAS or a scientific calculator, you may be presented with an answer such as 3.19E– 4. This is an alternative form of scientific notation; 3.19E– 4 means 3.19×10^{-4}.

WORKED EXAMPLE 4

Rewrite these numbers using scientific notation:

a. **640 783** **b.** **0.000 005 293**.

THINK	WRITE
a. 1. Identify the first digit (6). Rewrite the full number with the decimal place moved directly after this digit.	**a.** 6.407 843

2. Following the new decimal number, multiply by a base of 10. — $6.407\,843 \times 10$

3. Count the number of places the decimal point has moved, with this becoming the exponent of the base 10. Here the decimal point has moved 5 places. As the decimal point has moved to the left, the exponent will be positive. — $6.407\,843 \times 10^5$

b. 1. Identify the first non-zero digit (5). Rewrite the number with the decimal place moved to directly after this digit. — b. 5.293

2. Place the decimal point after the 5 and multiply by a base of 10. — 5.293×10

3. Identify the exponent value by counting the number of places the decimal point has moved. Here the decimal point has moved 6 places. As the decimal point has moved to the right, the exponent will be negative. — 5.293×10^{-6}

WORKED EXAMPLE 5

Rewrite these numbers as basic numerals:

a. 2.5×10^{-11} m (the size of a helium atom)

b. 3.844×10^8 m (the distance from Earth to the Moon)

THINK	WRITE
a. 1. First, note that the exponent is negative, indicating the number will begin with a zero. In front of the 2, record 11 zeros.	a. 0 000 000 000 025
2. Put a decimal point between the first two zeros.	0.000 000 000 025 m
b. 1. The exponent of this example is positive. Rewrite the number without the decimal point.	b. 3844
2. Add the necessary number of zeros to move the decimal point 8 places to the right, as indicated by the exponent.	384 400 000 m

2.2.4 Significant figures and rounding

Significant figures are a method of simplifying a number by rounding it to a base 10 value. Questions relating to significant figures will require a number to be written correct to x number of significant figures. In order to complete this rounding, the relevant significant figure(s) needs to be identified.

Let's have a look at an example. Consider the number 123.456 789.

This value has 9 significant figures, as there are nine numbers that tell us something about the particular place value in which they are located. The most significant of these values is the number 1, as it indicates the overall value of this number is in the hundreds. If asked to round this value to 1 significant figure, the number would be rounded to the nearest hundred, which in this case would be 100. If rounding to 2 significant figures, the answer would be rounded to the nearest 10, which is 120.

Rounding this value to 6 significant figures means the first 6 significant figures need to be acknowledged, 123.456. However, as the number following the 6th significant figure is 5 or more, the corresponding value needs to round up, therefore making the final answer 123.457.

Rounding hint:

If the number after the required number of significant figures is 5 or more, round up. If this number is 4 or below, leave it as is.

Zeros

Zeros present an interesting challenging when evaluating significant figures and are best explained using examples.

4056 contains 4 significant figures. The zero is considered a significant figure as there are numbers on either side of it.

4000 contains 1 significant figure. The zeros are ignored as they are place holders and may have been rounded.

4000.0 contains 5 significant figures. In this situation the zeros are considered important due to the zero after the decimal point. A zero after the decimal point indicates the numbers before it are precise.

0.004 contains 1 significant figure. As with 4000, the zeros are place holders.

0.0040 contains 2 significant figures. The zero following the 4 implies the value is accurate to this degree.

WORKED EXAMPLE 6

With reference to the following values:

i. **identify the number of significant figures**

ii. **round correct to 3 significant figures.**

a. **19 080** b. **0.000 076 214**

THINK	WRITE
a. i. In this number, the 1, 9 and 8 are considered significant, as well as the first zero. The final zero is not significant, as it gives no specific information about the units place value.	a. 19 080 has 4 significant figures.
ii. Round the number to the third significant figure. It is important to consider the number that follows it. In this case, as the following number is above 5, the value of the third significant figure needs to be rounded up by 1.	Rounded to 3 significant figures, 19 080 = 19 100.
b. i. The first significant figure is the 7. The zeros before the 7 are not considered significant as they are place holders.	b. 0.000 076 214 has 5 significant figures.
ii. The third significant figure is 2; however, it is important to consider the next value as it may require additional rounding. In this case the number following the 3rd significant number is below 5, meaning no additional rounding needs to occur.	Rounded to 3 significant figures, 0.000 076 214 = 0.000 076 2.

2.2.5 Exact and approximate answers

More often than not it is necessary to provide exact answers in mathematics. However, there are times when the use of rounding or significant figures is needed, even though this reduces the accuracy of the answers. At other times it is reasonable to provide an estimate of an answer by simplifying the original numbers. This can be achieved by rounding the number of decimal places or rounding to a number of significant figures.

When rounding to a specified number of decimal places, it is important to consider the value that directly follows the final digit. If the following number is 4 or less, no additional rounding needs to occur; however, if the following number is 5 or more, then the final value needs to be rounded up by 1.

For example, Gemma has organised a concert at the local hall. She is charging \$18.50 a ticket and on the night 210 people purchased tickets. To get an idea of her revenue, Gemma does a quick estimate of the money made on ticket sales by rounding the values to 1 significant figure.

$$\begin{aligned} 18.50 \times 210 &\approx 20 \times 200 \\ &\approx \$4000 \end{aligned}$$

Note: The use of the approximate equals sign ≈ indicates the values used are no longer exact. Therefore, the resulting answer will be an approximation.

When compared to the exact answer, the approximate answer gives a reasonable evaluation of her revenue.

$$18.50 \times 210 = \$3885$$

WORKED EXAMPLE 7

a. **Calculate 42.6 × 59.7 × 2.2, rounding the answer correct to 1 decimal place.**
b. **Redo the calculation by rounding the original values correct to 1 significant figure.**
c. **Comparing your two answers, would the approximate value be considered a reasonable result?**

THINK	WRITE
a. 1. Complete the calculation. The answer contains 3 decimal places; however, the required answer only needs 1.	**a.** $42.6 \times 59.7 \times 2.2$ $= 5595.084$
2. Look at the number following the first decimal place (8). As it is above 5, additional rounding needs to occur, rounding the 0 up to a 1.	≈ 5595.1
b. 1. Round each value correct to 1 significant figure. Remember to indicate the rounding by using the approximate equals sign (≈).	**b.** $42.6 \times 59.7 \times 2.2$ $\approx 40 \times 60 \times 2$
2. Complete the calculation.	≈ 4800
c. There is a sizeable difference between the approximate and exact answers, so in this instance the approximate answer would not be considered a reasonable result.	**c.** No

on Resources

Interactivity: Addition and subtraction of directed numbers (int-6455)
Interactivity: Scientific notation (int-6456)

studyon

Units 1 & 2 > AOS 2 > Topic 1 > Concept 1 > **Arithmetic** Concept summary and practice questions

Exercise 2.2 Computation methods

1. **WE1** Manually calculate $4 + 2 \times 14 - \sqrt{81}$ by correctly applying the order of operations rules. For a comparison, complete the question working from left to right, ignoring the order of operations rules. Do your answers match?
2. Evaluate $\dfrac{\sqrt[3]{125}}{(6^2 + 2 \times 7)}$.
3. Solve the following expressions by applying the correct order of operations.
   a. $24 - 4 \times 5 + 10$
   b. $(9 - 8 + 15 \times 4) \div \sqrt[3]{64}$
   c. $(9^2 + 2 \times 10 - 2) \div (16 + 17)$
   d. $\dfrac{3 \times 12 + 10^2}{2^4 + 1}$
4. **WE2** Evaluate $95 - -12 - +45$ by first combining operation and direction signs where appropriate.
5. Julie-Ann evaluated the expression $-13 + -12 - +11$ and came up with an answer of -36. Conrad insisted that the expression was equal to -14. Determine whether Julie-Ann or Conrad was correct, and give advice to the other person to ensure they don't make a mistake when evaluating similar expressions in the future.
6. Determine the value of the following expressions.
   a. $17 - -12 + -6$
   b. $-222 - -64$
   c. $430 + -35 - +40$
   d. $-28 - +43 + +15$
   e. $-4 \times -7 \times -3$
   f. $-8 \times -6 + 50 \div -10$
   g. $-3^2 + (-5)^2$
   h. $-4^2 \div \sqrt{64}$
   i. $\dfrac{-8 + -5 \times -12}{-3 + 5^2 \times -3}$
7. **WE3** Evaluate $(2 \times -15) \div (-5)^2$.
8. Evaluate $(-3)^2 + -3^2 - (-4 \times 2)$.
9. **WE4** The Great Barrier Reef stretches 2 600 000 m in length. Rewrite this distance using scientific notation.

10. The wavelength of red light is approximately 0.000 000 55 metres. Rewrite this number using scientific notation.

11. Rewrite the following values using scientific notation.

a. 7319 b. 0.080 425 c. 13 000 438

d. 0.000 260 e. 92 630 051 f. 0.000 569 2

12. WE5 The thickness of a DNA strand is approximately 3×10^{-9} m. Convert this value to a basic numeral.

13. The universe is thought to be approximately 1.38×10^{10} years old. Convert this value to a basic numeral.

14. Rewrite the following values as basic numerals.

a. 1.64×10^{-4} b. 2.3994×10^{-8} c. 1.4003×10^{9} d. 8.6×10^{5}

15. WE6 The distance around the Earth's equator is approximately 40 075 000 metres.

a. How many significant figures are in this value?

b. Round this value correct to 4 significant figures.

16. The average width of a human hair is 1.2×10^{-3} cm.

a. Rewrite this value as a basic numeral.

b. Identify the number of significant figures in the basic numeral.

c. Round your answer to part **a** correct to 1 significant figure.

17. For each of the following values:

i. identify the number of significant figures

ii. round the value correct to the number of significant figures specified in the brackets.

a. 1901 (2) b. 0.001 47 (2)

c. 21 400 (1) d. 0.094 250 (3)

e. 1.080 731 (4) f. 400.5 (3)

18. WE7 a. Calculate 235.47 + 1952.99 − 489.73, rounding the answer correct to 1 decimal place.

b. Repeat the calculation by rounding the original numbers correct to 1 significant figure.

c. Comparing your two answers, would the approximate answer be considered a reasonable result?

19. Bruce is planning a flight around Victoria in his light aircraft. On average the plane burns 102 litres of fuel an hour, and Bruce estimates the trip will require 37 hours of flying time.

a. Manually calculate the amount of fuel required by rounding each value correct to 1 significant figure.

b. Using a calculator, determine the exact amount of fuel required.

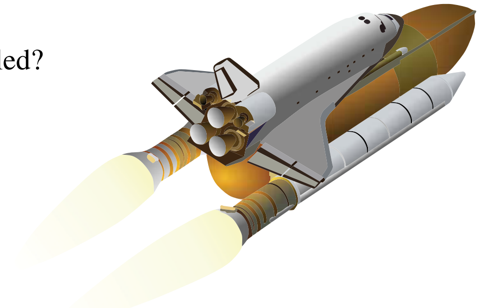
20. Elia and Lisa were each asked to calculate $\frac{3 + 17 \times -2}{3^2 - 1}$. When they revealed their answers, they realised they did not match.

Elia's working steps	Lisa's working steps
$\frac{3 + 17 \times -2}{3^2 - 1}$ $= \frac{20 \times -2}{2^2}$ $= \frac{-40}{4}$ $= -10$	$\frac{3 + 17 \times -2}{3^2 - 1}$ $= \frac{3 - 34}{9 + 1}$ $= \frac{-31}{8}$ $= -3\frac{7}{8}$

a. Review Elia's and Lisa's working steps to identify who has the correct answer.
b. Explain the error(s) made in the other person's working and what should have been done to correctly complete the equation.

21. Zoe borrowed \$50 from Emilio on Friday. On Monday she repaid \$35, and then asked to borrow another \$23 on Tuesday.
a. Write a mathematical sentence to reflect Emilio's situation.
b. How much does Zoe owe Emilio?

22. The space shuttle *Discovery* completed 39 space missions in its lifetime, travelling a total distance of 238 539 663 km.
a. How many significant figures are in the total distance travelled?
b. Round this value correct to 2 significant figures.
c. Convert your answer to part **b** to scientific notation.



23. Harrison tracked his finances for a day. Firstly he purchased two chocolate bars, each costing \$1.10, before buying a tram ticket for \$4.80. He caught up with four of his friends for lunch and the final bill was \$36.80, of which Harrison paid a quarter. After lunch, Chris repaid the \$47 he borrowed from Harrison last week. On the way home, Harrison purchased three new T-shirts for \$21.35 each.
a. Write a mathematical sentence to reflect Harrison's financial situation for the day.
b. Using the correct orders of operation, determine Harrison's financial position at the end of the day.

24. The diameter of the Melbourne Star Observation Wheel is 110 m.
a. Using the equation $C = \pi d$, determine the circumference of the wheel correct to 2 decimal places.
b. Redo the calculation by rounding the diameter and the value of π correct to 2 significant figures. How does this change your answer? Would it be considered a reasonable approximation?

25. The Robinson family want to lay instant lawn in their backyard. The dimensions of the rectangular backyard are 12.6 m by 7.8 m.
a. If each square metre of lawn costs \$12.70, estimate the cost for the lawn by rounding each number correct to the nearest whole number before completing your calculations.
b. Compare your answer to part **a** to the actual cost of the lawn.
c. Could you have used another rounding strategy to improve your estimate?

26. An animal park uses a variety of ventilated boxes to safely transport their animals. The lid of the box used for the small animals measures 76 cm long by 20 cm wide. Along the lid of the box are 162 ventilation holes, of which each has a radius of 1.2 cm.
a. Using the formula $A = \pi r^2$, calculate the area of one ventilation hole, providing the full answer.
b. Round your answer to part **a** correct to 2 decimal places.
c. Determine the amount of surface area that remains on the lid after the ventilation holes are removed.
d. The company that manufacture the boxes prefers to work in millimetre measurements. Convert your remaining surface area to millimetres squared. (*Note:* $1\ \text{cm}^2 = 100\ \text{mm}^2$.)
e. Record your answer to part **d** in scientific notation.

2.3 Orders of magnitude

2.3.1 What are orders of magnitude?

An **order of magnitude** uses factors of 10 to give generalised estimates and relative scale to numbers.

To use orders of magnitude we need to be familiar with powers of 10.

Power of 10	Basic numeral	Order of magnitude
10^{-4}	0.0001	−4
10^{-3}	0.001	−3
10^{-2}	0.01	−2
10^{-1}	0.1	−1
10^{0}	1	0
10^{1}	10	1
10^{2}	100	2
10^{3}	1000	3
10^{4}	10 000	4

As you can see from the table, the exponents of the powers of 10 are equal to the orders of magnitude.

An increase of 1 order of magnitude means an increase in the basic numeral by a multiple of 10. Similarly, a decrease of 1 order of magnitude means a decrease in the basic numeral by a multiple of 10.

2.3.2 Using orders of magnitude

We use orders of magnitude to compare different values and to check that estimates we make are reasonable. For example, if we are given the mass of two objects as 1 kg and 10 kg, we can say that the difference in mass between the two objects is 1 order of magnitude, as one of the objects has a mass 10 times greater than the other.

WORKED EXAMPLE 8

Identify the order of magnitude that expresses the difference in distance between 3.5 km and 350 km.

THINK	WRITE
1. Calculate the difference in size between the two distances by dividing the larger distance by the smaller distance.	$\frac{350}{3.5} = 100$
2. Express this number as a power of 10.	$100 = 10^2$
3. The exponent of the power of 10 is equal to the order of magnitude. Write the answer.	The order of magnitude that expresses the difference in distance is 2.

2.3.3 Scientific notation and orders of magnitude

When working with orders of magnitude, it can be helpful to express numbers in scientific notation. If two numbers in scientific notation have the same coefficient, that is, if the numbers in the first part of the scientific notation (between 1 and 10) are the same, then we can easily determine the order of magnitude by finding the difference in value between the exponents of the powers of 10.

WORKED EXAMPLE 9

By how many orders of magnitude do the following distances differ?
Distance A: 2.6×10^{-3} km
Distance B: 2.6×10^{2} km

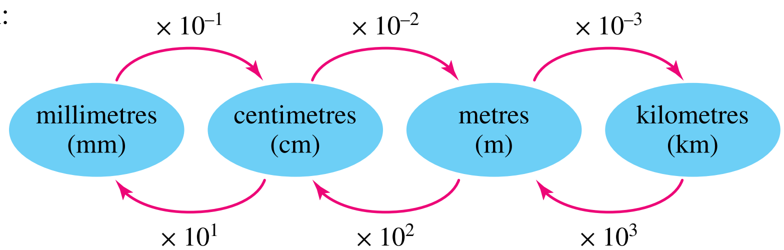
THINK	WRITE
1. Check that the coefficients of both numbers are the same in scientific notation.	In scientific notation, both numbers have a coefficient of 2.6.
2. Determine the order of magnitude difference between the numbers by subtracting the exponent of the smaller power of 10 (-3) from the exponent of the larger power of 10 (2).	$2 - -3 = 5$
3. Write the answer.	The distances differ by an order of magnitude of 5.

2.3.4 Units of measure

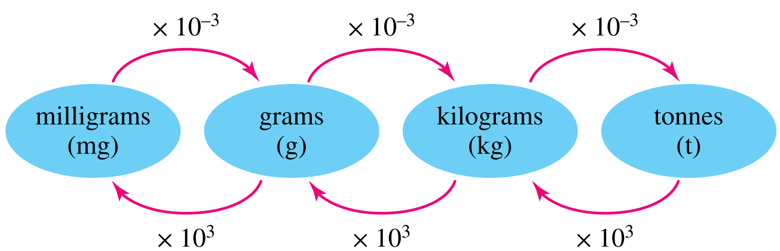
When using orders of magnitude to compare values it is important to factor in the units used. For example, if the weight of a fully-grown male giraffe is 1.5×10^3 kilograms and a full bottle of milk weighs 1.5×10^3 grams and the units were not considered, it would appear that the order of magnitude between the weight of the milk and the giraffe is 0. When converted into the same units (e.g. kilograms), the weight of the bottle of milk becomes 1.5×10^0 kg and the weight of the giraffe remains 1.5×10^3 kg, which is 3 orders of magnitude larger than the weight of the milk.

We can convert units of length and mass by using the following charts.

Converting length:



Converting mass:



Note: You can see from the charts that multiplying by 10^{-1} is the same as dividing by 10^1, multiplying by 10^{-3} is the same as dividing by 10^3, etc.

Our conversion charts show that there is a difference of 1 order of magnitude between millimetres and centimetres, and 3 orders of magnitude between grams and kilograms.

WORKED EXAMPLE 10

The mass of a single raindrop is approximately 1×10^{-4} grams and the mass of an average apple is approximately 1×10^{-1} kilograms.

a. By how many orders of magnitude do these masses differ?

b. Expressed as a basic numeral, how many times larger is the mass of the apple than that of the raindrop?

THINK	WRITE
a. 1. In this situation it will be easier to work in grams. To convert 1×10^{-1} kilograms to grams, multiply by 10^3.	**a.** $1 \times 10^{-1} \times 10^3 = 1 \times 10^2$ g
2. Compare the orders of magnitude by subtracting the smaller exponent (−4) from the larger exponent (2).	Raindrop: 1×10^{-4} g Apple: 1×10^2 g $2 - -4 = 6$ The masses differ by 6 orders of magnitude.
b. 1. Each order of magnitude indicates a power of 10.	**b.** $10^6 = 1\,000\,000$
2. Write the answer.	The mass of the apple is 1 000 000 times larger than the mass of the raindrop.

2.3.5 Logarithmic scales

Earthquakes are measured by seismometers, which record the amplitude of the seismic waves of the earthquake. There are large discrepancies in the size of earthquakes, so rather than using a traditional scale to measure their amplitude, a **logarithmic scale** is used.

A logarithmic scale represents numbers using a log (base 10) scale. This means that if we express all of the numbers in the form 10^a, the logarithmic scale will represent these numbers as a.

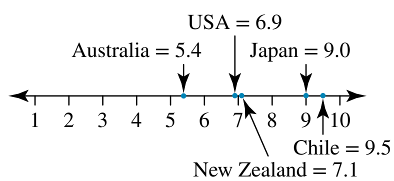
This means that for every increase of 1 in the magnitude in the scale, the amplitude or power of the earthquake is increasing by a multiple of 10. This allows us to plot earthquakes of differing sizes on the same scale.

The Richter Scale was designed in 1934 by Charles Richter and is the most widely used method for measuring the magnitude of earthquakes.

Let's consider the following table of historical earthquake data.

World earthquake data			
Year	**Location**	**Magnitude on Richter scale**	**Amplitude of earthquake**
2012	Australia (Moe, Victoria)	5.4	$10^{5.4}$
2011	Japan (Tohoku)	9	10^{9}
2010	New Zealand (Christchurch)	7.1	$10^{7.1}$
1989	USA (San Francisco)	6.9	$10^{6.9}$
1960	Chile (Valdivia)	9.5	$10^{9.5}$

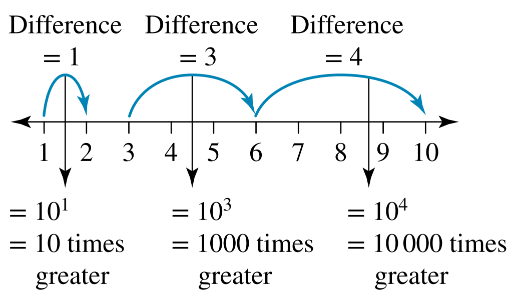
If we were to plot this data on a linear scale, the amplitude of the largest earthquake would be 12 589 times bigger ($10^{9.5}$ compared to $10^{5.4}$) than the size of the smallest earthquake. This would create an almost unreadable graph. By using a logarithmic scale the graph becomes easier for us to interpret.



However, this scale doesn't highlight the real difference between the amplitudes of the earthquakes, which only becomes clear when these values are calculated.

The difference in amplitude between an earthquake of magnitude 1 and an earthquake of magnitude 2 on the Richter scale is 1 order of magnitude, or $10^1 = 10$ times.

Let's consider the difference between the 2012 Australian earthquake and the 2010 New Zealand earthquake. According to the Richter scale the difference in magnitude is 1.7, which means that the real difference in the amplitude of the two earthquakes is $10^{1.7}$. When evaluated, $10^{1.7} = 50.12$, which indicates that New Zealand earthquake was more than 50 times more powerful than the Australian earthquake.



WORKED EXAMPLE 11

Using the information from the World Earthquake Data, compare the real amplitude of the earthquake to that of the Japanese earthquake, which resulted in a tsunami that damaged the Fukushima power plant.

THINK	WRITE
1. First identify the order of magnitude difference between Japan's earthquake and Australia's earthquake.	$9 - 5.4$ $= 3.6$
2. Express the order of magnitude difference in real terms by displaying it as a power of 10.	$10^{3.6}$
3. Evaluate to express the difference in amplitude.	$= 3981.87$
4. Write the answer.	Japan experienced an earthquake that was nearly 4000 times larger than the earthquake in Australia.

2.3.6 Why use a logarithmic scale instead of a linear scale?

As you can see from Worked example 11, very large numbers are involved when dealing with magnitudes of earthquakes. It would be challenging to represent an increase in amplitude of 4000 times while also having a scale accurate enough to accommodate smaller changes. Using the logarithmic scale enables such diversity in numbers to be represented on the same plane with a functioning scale.

Logarithmic scales are also used in measuring pH levels. On the pH scale 7 is considered neutral, while values from 6 to 0 indicate an increase in acidity levels and values from 8 to 14 indicate an increase in alkalinity.

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Units 1 & 2 > AOS 2 > Topic 1 > Concepts 2 & 3

Orders of magnitude Concept summary and practice questions

Logarithms—base 10 Concept summary and practice questions

Exercise 2.3 Orders of magnitude

1. **WE8** Identify the order of magnitude that expresses the difference between 0.3 metres and 3000 metres.
2. The Big Lobster in South Australia is a 4000 kg sculpture of a lobster. If a normal lobster has a mass of 4 kg, by what order of magnitude is the mass of the Big Lobster greater than the mass of a normal lobster?
3. Convert the values shown to the units in the brackets, using scientific form.
   a. 1×10^3 m (km)
   b. 1×10^4 g (kg)
   c. 9×10^5 mm (cm)
   d. 5.4×10^2 t (kg)
   e. 1.2×10^{-5} kg (mg)
   f. 6.3×10^{12} mm (km)

4. By how many orders of magnitude do the following pairs of values differ?
(*Note:* 1 kilotonne (kt) = 1000 tonnes (t).)
   a. 1.15×10^5 mm and 1.15×10^{-2} m
   b. 3.67×10^{-12} km and 3.67×10^7 cm
   c. 2.5×10^{17} km and 2.5×10^{26} mm
   d. 4.12×10^5 kt and 4.12×10^{15} t
   e. 5.4×10^{14} kg and 5.4×10^{20} mg
   f. 4.01×10^{-10} kt and 4.01×10^{10} mg
5. **WE9** The weight of a brown bear in the wild is 3.2×10^2 kg, while the weight of a cuddly teddy bear is 3.2×10^{-1} kg. By how many orders of magnitudes do the weights of the bears differ?
6. A cheetah covers 100 m in 7.2 seconds. It takes a snail a time 3 orders of magnitude greater than the cheetah to cover this distance. Express the time it takes the snail to cover 100 m in seconds.
7. **MC** Which of the following values is smallest?
   A. 1.4×10^{-5} km
   B. 1.4×10^{-6} m
   C. 1.4×10^{-3} cm
   D. 1.4×10^{-2} mm
   E. 1.4×10^4 cm

8. **MC** A virus has an approximate mass of 1×10^{-20} kg. Which of the following items has a mass 10 000 000 000 times smaller than that of the virus?
   A. A hydrogen atom (mass = 1×10^{-27} kg)
   B. An electron (mass = 1×10^{-30} kg)
   C. A bacterium (mass = 1×10^{-15} kg)
   D. An ant (mass = 1×10^{-6} kg)
   E. A grain of fine sand (mass = 1×10^{-9} kg)
9. **WE10** The length of paddock A is 2×10^3 m, while the length of the adjoining paddock B is 2×10^5 km. By how many orders of magnitude is paddock B longer than A?
10. The mass of an amoeba is approximately 1×10^{-5} grams, while the mass of a one-year-old child is approximately 1×10^1 kilograms.
   a. By how many orders of magnitude do these masses differ?
   b. Express as a basic numeral the number of times lighter the amoeba is than the child.

11. In Tran's backyard the average height of a blade of grass is 6 cm. Tran has a tree that has grown to a height 2 orders of magnitude taller than the average blade of grass.
   a. Express the height of the grass and the tree in scientific notation.
   b. State the height of the tree.
12. **WE11** Many of the earthquakes experienced in Australia are between 3 and 5 in magnitude on the Richter scale. Compare the experience of a magnitude 3 earthquake to that of a magnitude 5 earthquake, expressing the difference in amplitude as a basic numeral.
13. A soft drink has a pH of 5, while lemon juice has a pH of 2.
   a. Identify the order of magnitude difference between the acidity of the soft drink and the lemon juice.
   b. How many times more acidic is the juice than the soft drink?
14. **MC** Water is considered neutral and has a pH of 7. Which of the following liquids is either 10 000 times more acidic or alkaline than water?
   A. Soapy water, pH 12
   B. Detergent, pH 10
   C. Orange juice, pH 3
   D. Vinegar, pH 2
   E. Battery acid, pH 1
15. The largest ever recorded earthquake was in Chile in 1960 and had a magnitude of 9.5 on the Richter scale. In 2011, an earthquake of magnitude 9.0 occurred in Japan.
   a. Determine the difference in magnitude between the two earthquakes.
   b. Reflect this difference in real terms by calculating the difference in amplitude between the two earthquakes, giving your answer as a basic numeral correct to 2 decimal places.

16. Diluted sulfuric acid has a pH of 1, making it extremely acidic. If its acidity was reduced by 1000 times, what pH would it register?
17. Andrew has a tennis ball with a mass of 5×10^1 grams. At practice his coach sets him a series of exercise using a 5×10^3 gram medicine ball.
    a. What is the order of magnitude difference between the mass of the two balls?
    b. What is the mass of each ball written as basic numerals?
18. The size of a hydrogen atom is 1×10^{-10} m and the size of a nucleus is 1×10^{-15} m.
    a. By how many orders of magnitude do the atom and nucleus differ in size?
    b. How many times larger is the atom than the nucleus?
19. The volume of a 5000 mL container was reduced by 1 order of magnitude.
    a. Express this statement in scientific notation.
    b. Find the new volume of the container.
20. The distance between Zoe's house and Gwendolyn's house is 25 km, which is 2 orders of magnitude greater than the distance from Zoe's house to her school.
    a. Express the distance from Zoe's house to her school in scientific notation.
    b. Determine the distance from Zoe's house to her school.

2.4 Ratio, rates and percentages

Ratio, rates and **percentages** are all methods of comparison. Percentages represent a portion out of 100, ratios are used to compare quantities of the same units, and rates compare quantities of different units of measurement.

2.4.1 Percentages

Recall that:
- **percentages are fractions of 100**
- **the percentage of a given value is calculated by multiplying it by the percentage expressed as a fraction or decimal**
- **you can write a value as a percentage of another value by expressing it as a fraction and multiplying by 100.**

WORKED EXAMPLE 12

A teacher finds that 12% of students in their class obtain an A^+ for a test. To get an A^+, students need to score at least 28 marks. If there were 25 students in the class and the test was out of 32 marks:
a. what was the minimum percentage needed to obtain an A^+
b. how many students received an A^+?

THINK | WRITE

a. 1. Write the minimum number of marks needed as a fraction of the total number of marks.

a. $\frac{28}{32}$

2. Multiply the fraction by 100 and simplify where possible.

$$\frac{28}{32} \times 100 = \frac{{}^{7}\cancel{28}}{{}^{8}\cancel{32}} \times \frac{100}{1}$$
$$= \frac{7}{\cancel{2}} \times \frac{{}^{25}\cancel{100}}{1}$$
$$= \frac{175}{2}$$
$$= 87.5\%$$

3. State the final answer.

Students had to obtain a minimum of 87.5%, to receive an A^+.

b. 1. Write the percentage as a fraction.

b. $12\% = \frac{12}{100}$

2. Multiply the fraction by the total number in the class and simplify.

$$\frac{12}{100} \times 25 = \frac{12}{{}^{4}\cancel{100}} \times \frac{{}^{1}\cancel{25}}{1}$$
$$= \frac{{}^{3}\cancel{12}}{{}^{1}\cancel{4}} \times \frac{1}{1}$$
$$= 3$$

3. State the final answer.

3 students obtained an A^+.

2.4.2 Ratios

Ratios are used to compare quantities that are measured in the same units. For example, if the ratio of bicycles to cars on a particular road during rush hour is 1 : 4, there are 4 times as many cars on the road as bicycles.

Ratios compare two or more quantities, and are in their simplest form when all parts are expressed using whole numbers and the highest common factor (HCF) of all the numbers is 1. A simplified ratio is an equivalent ratio to the original ratio.

WORKED EXAMPLE 13

Simplify the following ratios by first finding the highest common factor.

a. **14 : 6**

b. **1.5 : 2 : 3.5**

THINK | WRITE

a. 1. Consider factors of each quantity. Both values are divisible by 2.

2. Divide both values by 2.

a. 14 : 6

÷2 ÷2

↓ ↓

= 7 : 3

b. 1. To work with whole numbers, multiply all quantities by 10. (We multiply by 10 as there is 1 decimal place value. If there were 2 decimal places, then we would multiply by 100.)

2. Express the whole number quantities as a ratio.

3. Identify the highest common factor (HCF) for the three parts (in this case 5). Simplify by dividing each part by the HCF.

b.
$$\begin{array}{ccccc} 1.5 & : & 2 & : & 3.5 \\ \times 10 & & \times 10 & & \times 10 \\ \downarrow & & \downarrow & & \downarrow \\ 15 & : & 20 & : & 35 \end{array}$$
$$\begin{array}{ccccc} 15 & : & 20 & : & 35 \\ \div 5 & & \div 5 & & \div 5 \\ \downarrow & & \downarrow & & \downarrow \\ = 3 & : & 4 & : & 7 \end{array}$$

2.4.3 Ratios of a given quantity

We can use ratios to find required proportions of a given quantity. This can be useful when splitting a total between different shares.

WORKED EXAMPLE 14

Carlos, Maggie and Gary purchased a winning lotto ticket; however, they did not each contribute to the ticket in equal amounts. Carlos paid $6, Maggie $10 and Gary $4. They agree to divide the $845 winnings according to their contributions.

a. Express the purchase contributions as a ratio.

b. How much of the winnings is each person entitled to?

THINK	WRITE
a. Write the purchase contributions as a ratio, remembering to simplify by identifying the highest common factor.	**a.** 6 : 10 : 4 HCF = 2 Ratio = 3 : 5 : 2
b. Multiply the winning amount by the fraction representing each person's contribution.	**b.** $\text{Carlos} = 845 \times \frac{3}{10}$ $= \$253.50$ $\text{Maggie} = 845 \times \frac{5}{10}$ $= \$422.50$ $\text{Gary} = 845 \times \frac{2}{10}$ $= \$169.00$

2.4.4 Rates

A rate is a measure of change between two variables of different units.

Common examples of rates include speed in kilometres per hour (km/h) or metres per second (m/s), costs and charges in dollars per hour ($ /h), and electricity usage in kilowatts per hour (kW/h).

Rates are usually expressed in terms of how much the first quantity changes with one unit of change in the second quantity.

WORKED EXAMPLE 15

At what rate (in km/h) are you moving if you are on a bus that travels 11.5 km in 12 minutes?

THINK	WRITE
1. Identify the two measurements: distance and time. As speed is commonly expressed in km/h, convert the time quantity units from minutes to hours.	The quantities are 11.5 km and 12 minutes. $\frac{12}{60}=\frac{1}{5}$ or 0.2 hours
2. Write the rate as a fraction and express in terms of one unit of the second value.	$\frac{11.5\text{ km}}{0.2\text{ hrs}}=\frac{11.5}{0.2}\times\frac{5}{5}$ $=\frac{57.5}{1}$ $=57.5$
3. State the final answer.	You are travelling at 57.5 km/h.

2.4.5 Unit cost calculations

In order to make accurate comparisons between the costs of differently priced and sized items, we need to identify how much a single unit of the item would be. This is known as the **unit cost**. For example, in supermarkets similar cleaning products may be packaged in different sizes, making it difficult to tell which option is cheaper.

2.4.6 The unitary method

Unit-cost calculations are an application of the unitary method, which is the same mathematical process we follow when simplifying a rate. If x items cost $\$y$, divide the cost by x to find the price of one item:

$$x\text{ items} = \$y$$

$$1\text{ item} = \$\frac{y}{x}$$

WORKED EXAMPLE 16

Calculate the cost per 100 grams of pet food if a 1.25 kg box costs $7.50.

THINK	WRITE
1. Identify the cost and the weight. As the final answer is to be referenced in grams, convert the weight from kilograms to grams.	Cost: \$7.50 Weight: 1.250 kg = 1250 g
2. Find the unit cost for 1 gram by dividing the cost by the weight.	$\frac{7.50}{1250} = 0.006$
3. Find the cost for 100 g by multiplying the unit cost by 100.	$0.006 \times 100 = 0.60$ Therefore the cost per 100 grams is \$0.60.

on Resources

interactivity: Percentages (int-6458)
interactivity: Speed (int-6457)

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Units 1 & 2 > AOS 2 > Topic 1 > Concepts 4, 5 & 6

Ratio Concept summary and practice questions

Percentages Concept summary and practice questions

Unitary method Concept summary and practice questions

Exercise 2.4 Ratio, rates and percentages

1. **WE12** A teacher finds that 15% of students in one class obtain a B^+ for a test. To get a B^+, students needed to score at least 62 marks. If there were 20 students in the class and the test was out of 80 marks:
   a. what was the minimum percentage needed to obtain a B^+
   b. how many students received a B^+?
2. A salesman is paid according to how much he sells in a week. He receives 3.5% of the total sales up to \$10 000 and 6.5% for amounts over \$10 000.
   a. How much is his monthly pay if his total sales in four consecutive weeks are \$8900, \$11 300, \$13 450 and \$14 200?
   b. What percentage of his total pay for this time period does each week represent? Give your answers correct to 2 decimal places.
3. A real-estate agent is paid 4.25% of the sale price of any property she sells. How much is she paid for selling properties costing:
   a. \$250 000
   b. \$310 500
   c. \$454 755
   d. \$879 256?

   Where necessary, give your answers correct to the nearest cent.

4. A student's test results in Mathematics are shown in the table.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
Mark	$\frac{16}{20}$	$\frac{14}{21}$	$\frac{26}{34}$	$\frac{36}{45}$	$\frac{14.5}{20}$	$\frac{13}{39}$	$\frac{42}{60}$	$\frac{26}{35}$
Percentage								

a. Complete the table by calculating the percentage for each test, giving values correct to 2 decimal places where necessary.
b. What is the student's overall result from all eight tests as a percentage correct to 2 decimal places?

5. A person has to pay the following bills out of their weekly income of $1100.

Food	$280
Electricity	$105
Telephone	$50
Petrol	$85
Rent	$320

Giving answers correct to 2 decimal places where necessary:
a. express each bill as a percentage of the total bills
b. express each bill as a percentage of the weekly income.

6. **WE13** Simplify the following ratios.
a. 81 : 27 : 12
b. 4.8 : 9.6

7. A recipe for Mars Bar slice requires 195 grams of chopped Mars Bar pieces and 0.2 kilograms of milk chocolate. Express the weight of the Mars Bar pieces to the milk chocolate as a ratio in simplest form.

8. Simplify the following ratios by first converting to the same units where necessary.
a. 36 : 84
b. 49 : 77 : 105
c. 3.225 kg : 1875 g
d. 2.4 kg : 960 g : 1.2 kg

9. **WE14** In a bouquet of flowers the ratio of red, yellow and orange flowers was 5 : 8 : 3. If there were 48 flowers in the bouquet, how many of each colour were included?

10. The Murphys are driving from Melbourne to Adelaide for a holiday. They plan to have two stops before arriving in Adelaide. First they will drive from Melbourne to Ballarat, then to Horsham, and finally to Adelaide. The total driving time, excluding stops, is estimated to be 7 hours and 53 minutes. If the distance between the locations is in the ratio 44 : 67 : 136, determine the driving time between each location correct to the nearest minute.

11. Mark, Henry, Dale and Ben all put in to buy a racecar. The cost of the car was $18 000 and they contributed in the ratio of 3 : 1 : 4 : 2.
   a. How much did each person contribute?
   b. The boys also purchased a trailer to tow the car. Mark put in $750, Henry $200, Dale $345 and Ben $615. Express these amounts as a ratio in simplest form.
12. WE15 At what rate (in km/h) are you moving if you are in a passenger aircraft that travels 1770 km in 100 minutes?

For questions 13–15, give answers correct to 2 decimal places where appropriate.

13. Calculate the rates in the units stated for:
   a. a yacht that travels 1.375 km in 165 minutes expressed in km/h
   b. a tank that loses 1320 mL of water in $2\frac{1}{3}$ hours expressed in mL/min
   c. a 3.6-metre-long carpet that costs $67.14 expressed in $ /m
   d. a basketball player who has scored a total of 833 points in 68 games expressed in points/game.
14. WE16 Calculate the cost in dollars per 100 grams for:
   a. a 650-g box of cereal costing $6.25
   b. a 350-g packet of biscuits costing $3.25
   c. a 425-g jar of hazelnut spread costing $3.98
   d. a 550-g container of yoghurt costing $3.69.
15. Calculate the cost:
   a. per litre if a box of 24 cans that each contains 375 mL costs $18.00
   b. per 100 mL if a 4-litre bottle of cooking oil costs $16.75
   c. per kilogram if a 400-g frozen chicken dinner costs $7.38
   d. per kilogram if a 250-g pack of cheese slices costs $5.66.

16. In a game of cricket, batsman A scored 48 runs from 66 deliveries, while batsman B scored 34 runs from 42 deliveries.
   a. Which batsman is scoring at the fastest rate (runs per delivery)?
   b. What is the combined scoring rate of the two batsmen in runs per 100 deliveries correct to 2 decimal places?

17. Change the following rates to the units as indicated. Where necessary, give answers correct to 2 decimal places.
   a. 1.5 metres per second to kilometres per hour
   b. 60 kilometres per hour to metres per second
   c. 65 cents per gram to dollars per kilogram
   d. $5.65 per kilogram to cents per gram
18. Calculate the amount paid per hour for the following incomes, giving all answers correct to the nearest cent.
   a. $75 000 per annum for a 38 hour week
   b. $90 000 per annum for a 40 hour week
   c. $64 000 per annum for a 35 hour week
   d. $48 000 per annum for a 30 hour week
19. A particular car part is shipped in containers that hold 2054 items. Give answers to the following questions correct to the nearest cent.
   a. If each container costs the receiver $8000, what is the cost of each item?
   b. If the car parts are sold for a profit of 15%, how much is charged for each?
   c. The shipping company also has smaller containers that cost the receiver $7000, but only hold 1770 items. If the smaller containers are the only ones available, how much must the car part seller charge to make the same percentage profit?
20. A butcher has the following pre-packed meat specials.

BBQ lamb chops in packs of 12 for $15.50
Steak in packs of 5 for $13.80
Chicken drumsticks in packs of 11 for $11.33

   a. Calculate the price per individual piece of meat for each of the specials, correct to the nearest cent.
   b. The weights of two packages of meat are shown in the table below.

Meat	Package 1	Package 2
BBQ lamb chops	2535 grams	2602 grams
Steak	1045 grams	1068 grams
Chicken drumsticks	1441 grams	1453 grams

   Calculate the price per kilogram for each package correct to the nearest cent.

21. The ladder for the top four teams in the A-League is shown in the table below.

Team		Win	Loss	Draw	Goals for	Goals against
1.	Western Sydney Wanderers	18	6	3	41	21
2.	Central Coast Mariners	16	5	6	48	22
3.	Melbourne Victory	13	9	5	48	45
4.	Adelaide United	12	10	5	38	37

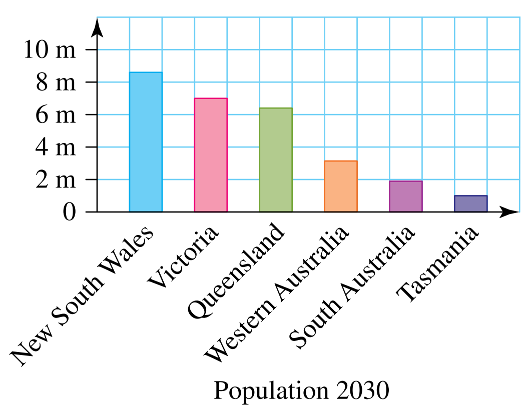
Use CAS to:

a. express the win, loss and draw columns as a percentage of the total games played, correct to 2 decimal places.

b. express the goals for as a percentage of the goals against, correct to 2 decimal places.

22. The actual and projected population figures for Australia are shown in the table below.

Year	NSW	VIC	QLD	WA	SA	TAS
2006	6 816 087	5 126 540	4 090 908	2 059 381	1 567 888	489 951
2013	7 362 207	5 669 525	4 757 385	2 385 445	1 681 525	515 380
2020	7 925 029	6 208 869	5 447 734	2 717 055	1 793 296	537 188
2030	8 690 331	6 951 030	6 424 193	3 185 288	1 940 032	559 757



a. Use CAS to express the populations of each state as a percentage of the total for each year, giving your answers correct to 2 decimal places.

b. Use CAS to calculate, to the nearest whole number, the average growth rate (population/year) of each state from:
   i. 2006–13
   ii. 2013–20
   iii. 2020–30.

c. Express your answers from part **b iii** as a percentage of the 2006 population for each state correct to 2 decimal places.

d. Which state is growing at the fastest rate during these time periods?

e. Which state is growing at the slowest rate during these time periods?

2.5 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC The correct answer to $3 + \sqrt{144} \div 6 - 3^2$ is:
   A. −6.5 B. −4 C. −5 D. 14 E. 0.25

2. MC When expressed in scientific notation, 490 000 becomes:
   A. 4.9×10^{-5} B. 4.9×10^{1} C. 4.9×10^{6}
   D. 4.9×10^{4} E. 4.9×10^{5}

3. MC Which of the following values has 4 significant figures?
   A. 0.365 B. 5000 C. 2090
   D. 4002 E. 0.209 15

4. MC Expressed as a basic numeral, 7.432×10^{-3} is:
   A. 7432 B. 743.2 C. 0.074 32
   D. 0.007 432 E. 0.000 743 2

5. MC By how many orders of magnitude do the distances 3×10^{-4} km and 3×10^{1} m differ?
   A. 2 B. −4 C. 100 D. −2 E. 5

6. MC The difference in magnitude between two earthquakes measuring 5.9 and 8.2 on the Richter scale is:
   A. 199.5 B. 1.1 C. 2.3 D. 9.2 E. 20.0

7. **MC** A liquid with a pH level of 9 is diluted to a pH level of 4. How many times less acidic is the liquid now?

A. 1 000 000 000 **B.** 100 000 **C.** 10 000
D. 1000 **E.** 100

8. **MC** If a salesman earns 3.7% from the sale of each car, how much would he be paid for a \$14 990 sale?

A. \$5546.30 **B.** \$4051.35 **C.** \$554.63
D. \$550.12 **E.** \$405.14

9. **MC** When fully simplified, $1\frac{4}{5} : 4\frac{1}{5}$ becomes:

A. 18 : 42 **B.** 1 : 4 **C.** 1 : 3 **D.** 8 : 2 **E.** 3 : 7

10. **MC** The unit cost (per gram) of a 120-gram tube of toothpaste sold for \$3.70 is:

A. \$32.43 **B.** \$0.03 **C.** \$0.44 **D.** \$0.05 **E.** \$0.31

Short answer

1. Using the appropriate order of operations, evaluate the following.

a. $-12 \div -2.5 + 17 \times -3$
b. $73 + 8^2 \div (-3 + 19)$
c. $(97 - 6^2) - 12 \times 7$
d. $\dfrac{12 + 9 \times 10}{\sqrt{16} + 2}$

2. For each of the following values:

i. identify the number of significant figures
ii. round each to the number of significant figures listed in the brackets
iii. express your answer to part **ii** in scientific notation.

a. 0.007 15 (2) b. 14 736 (1) c. 110 008 (3) d. 0.02 (1)

3. Convert each of the following values to the units shown in brackets.

a. 2×10^4 km (m)
b. 9×10^{-2} t (kg)
c. 5×10^{-7} kg (mg)
d. 7×10^{12} m (km)

4. The winning times for four races in an athletics championship are shown in the following table.

Distance	100 metres	200 metres	400 metres	800 metres
Winning time	9.8 s	19.6 s	43.52 s	1 min, 44 s

a. Express, correct to 2 decimal places, each winning time as a rate in:

i. m/s
ii. km/h.

b. If it was possible for the winner of the 100 metres to continue at the same speed for 400 metres, how far ahead of the actual winner would they be when they finished? Give your answer correct to the nearest metre.

5. While camping, the Blake family use powdered milk. They mix the powder with water in the ratio of 1 : 24. How much of each ingredient would they need to make up:

a. 600 mL
b. 1.2 L?

6. For each of the following, find the unit price for the quantity shown in brackets.

a. 750 g of Weetbix for \$4.99 (per 100 g)
b. \$16.80 for 900 g of jelly beans (per 100 g)
c. \$4.50 for 1.5 L of milk (per 100 mL)
d. \$126.95 for 15 L of paint (per L)

Extended response

1. The distance from Earth to the Sun is approximately 1×10^{11} m, whereas the distance to the nearest star is 1×10^{16} m.

a. By how many orders of magnitude do these distances differ?
b. Convert the units to km.

c. Express the difference in distance as a basic numeral.
d. The diameter of the Sun is 1 391 684 km, and the Earth has a diameter of 12 742 km. Express the diameter of the Earth as a percentage of the diameter of the Sun. Give your answer correct to 3 significant figures.

2. The monthly repayment for a $250 000 property loan is $1230.
a. What is the monthly repayment as a percentage of the loan?
b. What is the overall yearly repayment as a percentage of the loan?
c. If the repayments have to be made every month for 25 years, how much extra has to be paid back compared to the amount that was borrowed?
d. Express the extra amount to be paid back over the 25 years as a percentage of the amount borrowed.

3. John is comparing different brands of lollies at the local supermarket. A packet of Brand A lollies costs $7.25 and weighs 250 g. A packet of Brand B lollies weighs 1.2 kg and costs $22.50.
a. Which brand is the best value for money? Provide mathematical evidence to support your answer.
b. If the more expensive brand was to reconsider their price, what price for their lollies would match the unit price of the cheaper brand?

4. For a main course at a local restaurant, guests can select from a chicken, fish or vegetarian dish. On Friday night the kitchen served 72 chicken plates, 56 fish plates and 48 vegetarian plates.
a. Express the number of dishes served as a ratio in the simplest form.
b. On a Saturday night the restaurant can cater for 250 people. If the restaurant was full, how many people would be expected to order a non-vegetarian dish?
c. The Elmir family of five and the Cann family of three dine together. The total bill for the table was $268.
i. Calculate the cost of dinner per head.
ii. If the bill is split according to family size, what proportion of the bill will the Elmir family pay?

studyon

Units 1&2 **Sit topic test**

Answers

Topic 2 Computation and practical arithmetic

Exercise 2.2 Computational methods

1. 23 (from left to right without using the order of operations rules, the answer is 75)
2. $\frac{1}{10}$
3. a. 14 b. 15.25
   c. 3 d. 8
4. 62
5. Julie-Ann was right. Combine the signs before carrying out any calculations.
6. a. 23 b. −158 c. 355
   d. −56 e. −84 f. 43
   g. 16 h. −2 i. $-\frac{2}{3}$
7. $-1\frac{1}{5}$
8. 8
9. 2.6×10^6
10. 5.5×10^{-7}
11. a. 7.319×10^3 b. 8.0425×10^{-2}
    c. $1.300\,043\,8 \times 10^7$ d. 2.6×10^{-4}
    e. $9.263\,005\,1 \times 10^7$ f. 5.692×10^{-4}
12. 0.000 000 003 m
13. 13 800 000 000
14. a. 0.000 164 b. 0.000 000 023 994
    c. 1 400 300 000 d. 860 000
15. a. 5 b. 40 080 000 m
16. a. 0.0012 cm b. 2 c. 0.001 cm
17. a. i. 4 ii. 1900
    b. i. 3 ii. 0.0015
    c. i. 3 ii. 20 000
    d. i. 5 ii. 0.0943
    e. i. 7 ii. 1.081
    f. i. 4 ii. 401
18. a. 1698.7
    b. 1700
    c. Yes, the answers are very close.
19. a. 4000 litres b. 3774 litres
20. a. Lisa
    b. In the numerator, Elia added 3 + 17 first rather than completing the multiplication 17 × −2. Also, in the denominator Elia calculated the subtraction first rather than expanding the exponent.
21. a. −50 + 35 − 23 b. $38
22. a. 9 b. 240 000 000 km
    c. 2.4×10^8 km
23. a. $2 \times -1.10 + -4.80 + \frac{-36.80}{4} + 47 + 3 \times -21.35$
    b. −$33.25
24. a. 345.58 m
    b. 341 m, this is a reasonable approximation.
25. a. $1352
    b. Actual cost = $1248.16; there is a significant difference in the cost.
    c. Multiply then round.
26. a. 4.523 893 421 cm^2 b. 4.52 cm^2
    c. 791 cm^2 d. 79 100 mm^2
    e. 7.91×10^4 mm^2

Exercise 2.3 Orders of magnitude

1. 4
2. 3
3. a. 1×10^0 km b. 1×10^1 kg
   c. 9×10^4 cm d. 5.4×10^5 kg
   e. 1.2×10^1 mg f. 6.3×10^6 km
4. a. 4 b. 14
   c. 3 d. 7
   e. 0 f. 8
5. 3
6. 7200 seconds
7. B
8. B
9. 5
10. a. 9 b. 1 000 000 000
11. a. Grass: 6×10^0 cm; Tree: 6×10^2 cm
    b. 600 cm or 6 m
12. An earthquake of magnitude 5 is 100 times stronger than an earthquake of magnitude 3.
13. a. 3 b. 1000
14. C
15. a. 0.5 b. 3.16
16. 4
17. a. 2
    b. Tennis ball: 50 g; Medicine ball: 5000 g
18. a. 5 b. 100 000
19. a. $5 \times 10^3 \times 10^{-1}$ b. 500 mL
20. a. 2.5×10^{-1} km b. 0.25 km or 250 m

Exercise 2.4 Ratio, rates and percentages

1. a. 77.5% b. 3
2. a. $1943.25
   b. Week 1: 16.03%; Week 2: 22.36%; Week 3: 29.55%; Week 4: 32.06%
3. a. $10 625 b. $13 196.25
   c. $19 327.09 d. $37 368.38
4. a.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
Percentage	80%	66.67%	76.47%	80%	72.5%	33.33%	70%	74.29%

   b. 68.43%

5. a. Food: 33.33%; electricity: 12.5%; telephone: 5.95%; petrol: 10.12%; rent: 38.10%
   b. Food: 25.45%; electricity: 9.55%; telephone: 4.55%; petrol: 7.73%; rent: 29.09%
6. a. 27 : 9 : 4 b. 1 : 2
7. 39 : 40
8. a. 3 : 7 b. 7 : 11 : 15
   c. 43 : 25 d. 10 : 4 : 5
9. 15 red, 24 yellow, 9 orange
10. Melbourne to Ballarat: 1 hour, 24 minutes; Ballarat to Horsham: 2 hours, 8 minutes; Horsham to Adelaide: 4 hours, 20 minutes
11. a. Mark: $5400; Henry: $1800; Dale: $7200; Ben: $3600
   b. 150 : 40 : 69 : 123
12. 1062 km/h
13. a. 0.5 km/h b. 9.43 mL/min
   c. $18.65/min d. 12.25 points/game
14. a. $0.96 b. $0.93
   c. $0.94 d. $0.67
15. a. $2 b. $0.42
   c. $18.45 d. $22.64
16. a. Batsman B
   b. 75.93 runs/100 deliveries
17. a. 5.4 km/h b. 16.67 m/s
   c. $650/kg d. 0.57 cents/gram
18. a. $37.96 b. $43.27
   c. $35.16 d. $30.77
19. a. $3.89 b. $4.47
   c. $4.54
20. a. BBQ lamb chops: $1.29
   Steak: $2.76
   Chicken drumsticks: $1.03
   b. Pack A: $16.18/kg
   Pack B: $15.86/kg
21. a.

	Team	Win	Loss	Draw
1.	Western Sydney Wanderers	66.67%	22.22%	11.11%
2.	Central Coast Mariners	59.26%	18.52%	22.22%
3.	Melbourne Victory	48.15%	33.33%	18.52%
4.	Adelaide United	44.44%	37.04%	18.52%

b.

	Team	Goal percentage
1.	Western Sydney Wanderers	195.24%
2.	Central Coast Mariners	218.18%
3.	Melbourne Victory	106.67%
4.	Adelaide United	102.70%

22. a. See table at the foot of the page*
   b. See table at the foot of the page*
   c. NSW: 1.12%, VIC: 1.45%, QLD: 2.39%, WA: 2.27%, SA: 0.94%, TAS: 0.46%
   d. Western Australia
   e. Tasmania

22. a. *

Year	NSW	VIC	QLD	WA	SA	TAS
2006	33.83%	25.44%	20.30%	10.22%	7.78%	2.43%
2013	32.91%	25.34%	21.27%	10.66%	7.52%	2.30%
2020	32.18%	25.21%	22.12%	11.03%	7.28%	2.18%
2030	31.32%	25.05%	23.15%	11.48%	6.99%	2.02%

b. *

	Years	NSW	VIC	QLD	WA	SA	TAS
i	2006–13	78 017	77 569	95 211	46 581	16 234	3633
ii	2013–20	80 403	77 049	98 621	47 373	15 967	3115
iii	2020–30	76 530	74 216	97 645	46 823	14 674	2257

2.5 Review: exam practice

Multiple choice

1. B 2. E 3. D 4. D 5. A
6. C 7. B 8. C 9. E 10. B

Short answer

1. a. −46.2 b. 77
   c. −23 d. 17
2. a. i. 3
   ii. 0.0072
   iii. 7.2×10^{-3}
   b. i. 5
   ii. 10 000
   iii. 1×10^{4}
   c. i. 6
   ii. 110 000
   iii. 1.10×10^{5}
   d. i. 1
   ii. 0.02
   iii. 2×10^{-2}
3. a. 2×10^{7} m b. 9×10^{1} kg
   c. 5×10^{-1} mg d. 7×10^{9} km
4. a. See table at the foot of the page*
   b. 44 m
5. a. 24 g of powder and 576 mL of water
   b. 48 g of powder and 1152 mL of water
6. a. \$0.67 b. \$1.87
   c. \$0.30 d. \$8.46

Extended response

1. a. 5
   b. 1×10^{8} km and 1×10^{13} km
   c. 100 000
   d. 0.916%
2. a. 0.492% b. 5.904%
   c. \$119 000 d. 47.6%
3. a. Brand B (\$1.88 compared to \$2.90 per 100 g)
   b. \$4.70
4. a. 9 : 7 : 6
   b. 182
   c. i. \$33.50 ii. 62.5%

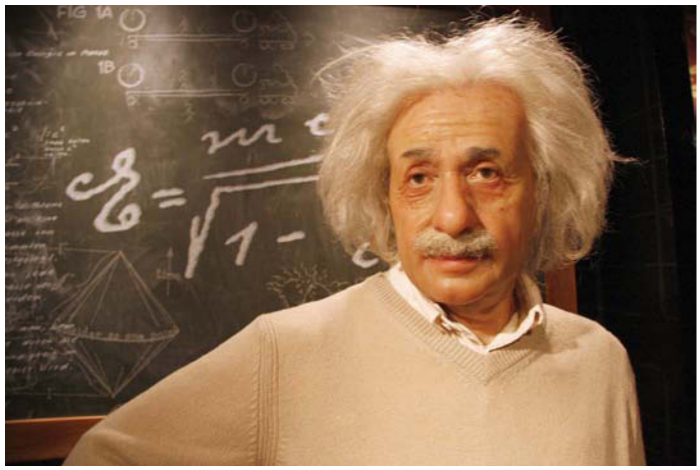
4. a. *

	Distance	100 metres	200 metres	400 metres	800 metres
i	**Winning rate (m/s)**	10.20 m/s	10.20 m/s	9.19 m/s	7.69 m/s
ii	**Winning rate (km/h)**	36.73 km/h	36.73 km/h	33.09 km/h	27.69 km/h

TOPIC 3
Financial arithmetic

3.1 Overview

3.1.1 Introduction



Bank interest today is very different to what it originally was thousands of years ago. The basic premise, however, remains the same. The early loans and interest, around 10 000 to 5000 BC, were used in agriculture. Loans were made in seeds, grains, animals and tools to farmers. Since one seed could generate a plant with over 100 new seeds after the harvest, this allowed farmers to pay back their loans with interest. When animals were loaned, interest was paid by sharing any newborn animals.

Aristotle noted that money was sterile and that, unlike grains or animals, it didn't produce more money, so interest was not earned in the same way. The church scholars known as the Scholastics echoed Aristotle's thoughts in 1100 to 1500 AD. They made the first attempt at the science of economics and their main concern was to incorporate interest.

Compound interest was first implemented in ancient Babylon. Compounding was the adding of accumulated interest back to the principal, so that interest was earned on interest. This is very different to simple interest, where the principal remains separate from the interest.

The *Wall Street Journal* in 1976 published an opinion article that attributed to Einstein the statement that compound interest was man's greatest invention. There is no proof that the renowned academic ever said this, but it's worth thinking about the importance of the statement nevertheless.

LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.

3.1.2 Kick off with CAS

Calculating interest with CAS

CAS can be used to quickly and easily evaluate formulas when given specific values. The formula to calculate simple interest is $I = \frac{PrT}{100}$, where I is the interest accrued, P is the principal, r is the rate of interest and T is the time.

1. Using CAS, define and save the formula for simple interest.
2. Use the formula to calculate the missing values in the following situations.
   a. $P = \$3000,\ r = 4\%, T = 2$ years
   b. $I = \$945,\ r = 4.5\%,\ T = 3$ years
   c. $I = \$748,\ P = \$5500,\ T = 4$ years
   d. $I = \$313.50,\ P = \$330,\ r = 3.8\%$

The formula to calculate compound interest is $A = P\left(1 + \frac{r}{100}\right)^n$, where A is the final amount, P is the principal, r is the rate of interest and n is the number of interest-bearing periods.

3. Using CAS, define and save the formula for compound interest.
4. Use the formula to calculate the missing values in the following situations.
   a. $P = \$5000,\ r = 3.3\%,\ n = 2$ years
   b. $A = \$8800,\ r = 5\%,\ n = 4$ years
   c. $A = \$2812.16,\ P = \$2500,\ n = 3$ years
   d. $A = \$3500.97,\ P = \$3300,\ r = 3\%$

The value of the final amount for simple interest can be calculated by summing I and P.

5. Use CAS to help you complete the following table comparing simple and compound interest.

Principal	Rate of interest	Time period	Simple interest final amount	Compound interest final amount
\$4000	4%	3 years		
\$2500	3.5%		\$2850	
\$5000		2 years		\$5533.52
	2.7%	5 years	\$7207.25	

6. Repeat question 4 using the Finance/Financial Solver on CAS.

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology

3.2 Percentage change

Percentages can be used to give an indication of the amount of change that has taken place, which makes them very useful for comparison purposes. Percentages are frequently used in comments in the media. For example, a company might report that its profits have fallen by 6% over the previous year.

The percentage change is found by taking the actual amount of change that has occurred and expressing it as a percentage of the starting value.

$$\text{Percentage change} = \frac{\text{finishing value} - \text{starting value}}{\text{starting value}} \times 100$$

WORKED EXAMPLE 1

The price of petrol was \$1.40 per litre but has now risen to \$1.65 per litre. What is the percentage change in the price of petrol, correct to 2 decimal places?

THINK

1. Identify the amount of change.

WRITE

$1.65 - 1.40 = 0.25$

The price of petrol has increased by 25 cents per litre.

2. Express the change as a fraction of the starting point, and simplify the fraction if possible.

$$\frac{0.25}{1.40} = \frac{25}{140} = \frac{5}{28}$$

3. Convert the fraction to a percentage by multiplying by 100.

$$\frac{5}{28} \times 100 = \frac{5}{7} \times \frac{25}{1} = \frac{125}{7} \approx 17.86$$

4. State the final answer.

The price of petrol has increased by approximately 17.86%.

3.2.1 Calculating percentage change

In the business world percentages are often used to determine the final selling value of an item. For example, during a sale period a store might decide to advertise '25% off everything' rather than specify actual prices in a brochure. At other times, when decisions are being made about the financial returns

needed in order for a business to remain viable, the total production cost plus a percentage might be used. In either case, the required selling price can be obtained through multiplying by an appropriate percentage.

Consider the situation of reducing the price of an item by 18% when it would normally sell for \$500. The reduced selling price can be found by evaluating the amount of the reduction and then subtracting it from the original value as shown in the following calculations:

Reduction of 18%: $\frac{18}{100} \times 500 = \$90$

Reduced selling price: $\$500 - \$90 = \$410$

The selling price can also be obtained with a one-step calculation of $\frac{82}{100} \times 500 = \$410$.

In other words, reducing the price by 18% is the same as multiplying by 82% or $(100 - 18)\%$.

To reduce something by $x\%$, multiply by $(100 - x)\%$.
To increase something by $x\%$, multiply by $(100 + x)\%$.

WORKED EXAMPLE 2

Increase \$160 by 15%.

THINK	WRITE
1. Add the percentage increase to 100.	$100 + 15 = 115$
2. Express the result as a percentage (by dividing by 100) and multiply by the value to be increased.	$\frac{115}{100} \times \$160 = \frac{115}{100} \times \frac{160}{1}$ $= \frac{23}{2} \times \frac{16}{1}$ $= 23 \times 8$ $= \$184$
3. State the final answer.	Increasing \$160 by 15% gives \$184.

When a large number of values are being considered in a problem involving percentages, spreadsheets or other technologies can be useful to help carry out most of the associated calculations. For example, a spreadsheet can be set up so that entering the original price of an item will automatically calculate several different percentage increases for comparison.

	A	B	C	D	E
1	Original		Increase by:		
2	price	+5%	+8%	+12%	+15%
3	\$ 100.00	\$ 105.00	\$ 108.00	\$ 112.00	\$ 115.00
4	\$ 150.00	\$ 157.50	\$ 162.00	\$ 168.00	\$ 172.50
5	\$ 200.00	\$ 210.00	\$ 216.00	\$ 224.00	\$ 230.00
6	\$ 300.00	\$ 315.00	\$ 324.00	\$ 336.00	\$ 345.00
7	\$ 450.00	\$ 472.50	\$ 486.00	\$ 504.00	\$ 517.50

on Resources

Interactivity: Calculating percentage change (int-6459)

studyon

Units 1 & 2 > AOS 2 > Topic 2 > Concept 1

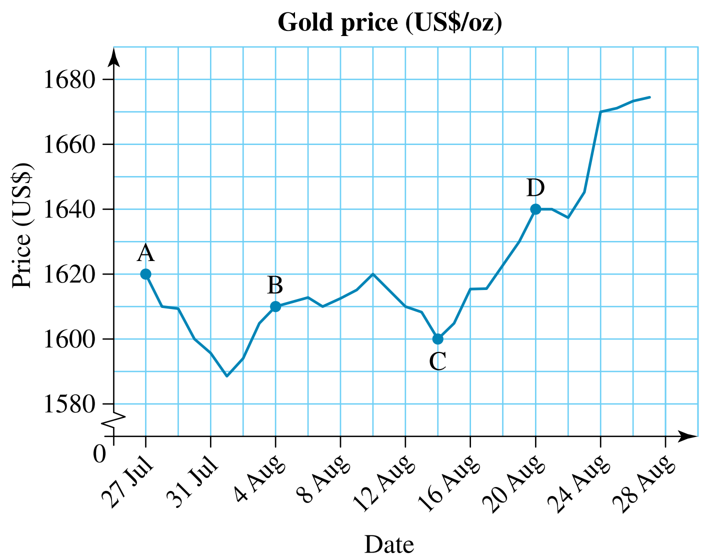
Percentage change Concept summary and practice questions

Exercise 3.2 Percentage change

Unless directed otherwise, give all answers to the following questions correct to 2 decimal places or the nearest cent where appropriate.

1. **WE1** If the price of bananas was \$2.65 per kg, calculate the percentage change (increase or decrease) if the price is now:
   a. \$3.25 per kg
   b. \$4.15 per kg
   c. \$1.95 per kg
   d. \$2.28 per kg.

2. Calculate the percentage change in the following situations.
   a. A discount voucher of 4 cents per litre was used on petrol advertised at \$1.48 per litre.
   b. A trade-in of \$5200 was applied to a car originally selling for \$28 500.
   c. A shop owner purchases confectionary from the manufacturer for \$6.50 per kg and sells it for 75 cents per 50 grams.
   d. A piece of silverware has a price tag of \$168 at a market, but the seller is bartered down and sells it for \$147.
3. The following graph shows the change in the price of gold (in US dollars per ounce) from 27 July to 27 August in 2012.



   a. Calculate the percentage change from:
      i. the point marked A to the point marked B
      ii. the point marked C to the point marked D.
   b. What is the percentage change from the point marked A to the point marked D?
4. A car yard offers three different vehicles for sale. The first car was originally priced at \$18 750 and is now on sale for \$14 991. The second car was originally priced at \$12 250 and is now priced \$9999, and the third car was originally priced at \$23 990 and is now priced \$19 888. Which represents the largest percentage reduction?

5. WE2 Increase:

a. \$35 by 8%

b. \$96 by 12.5%

c. \$142.85 by 22.15%

d. \$42 184 by 0.285%.

6. Decrease:

a. \$54 by 16%

b. \$7.65 by 3.2%

c. \$102.15 by 32.15%

d. \$12 043 by 0.0455%.

7. The price of a bottle of wine was originally \$19.95. After it received an award for wine of the year, the price was increased by 12.25%. Twelve months later the price was reduced by 15.5%.

a. What is the final price of a bottle of this wine?

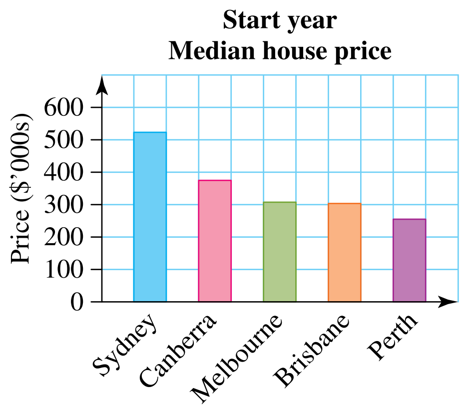
b. What is the percentage change of the final price from the original price?

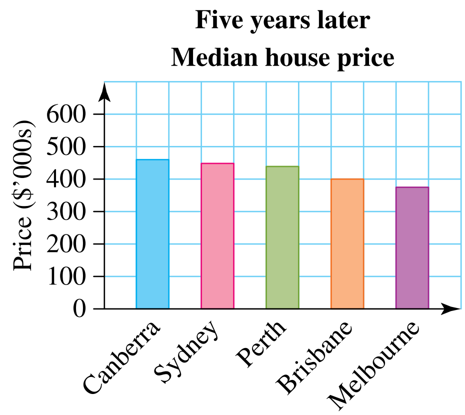
8. a. An advertisement for bedroom furniture states that you save \$55 off the recommended retail price when you buy it for \$385. By what percentage has the price been reduced?

b. If another store was advertising the same furniture for 5% less than the sale price of the first store, by what percentage has this been reduced from the recommended retail price?

9. A bracelet is sold for \$127.50. If this represents a 15% reduction from the RRP, what was the original price?

10. The following graphs show the changes in property prices in the major capital cities of Australia over a five-year period.





a. When comparing the median house prices for the five capital cities over this time period, which city had the largest percentage change and by how much?
b. In the same time period, which city had the smallest percentage change?

11. Over a period of time prices in a store increased by 15%, then decreased by 10%, and finally increased by a further 5%. What is the overall percentage change over this time period, correct to the nearest whole percentage?

12. A power company claims that if you install solar panels for $1800, you will make this money back in savings on your electricity bill in 2 years. If you usually pay $250 per month, by what percentage will your bill be reduced if their claims are correct?

13. A house originally purchased for $320 000 is sold to a new buyer at a later date for $377 600.
a. What is the percentage change in the value of the house over this time period?
b. The new buyer pays a deposit of 15% and borrows the rest from a bank. They are required to pay the bank 5% of the total amount borrowed each year. If they purchased the house as an investment, how much should they charge in rent per month in order to fully cover their bank payments?

14. The following table shows the number of participants in selected non-organised physical activities in Australia over a ten-year period.

Activity	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10
Walking	4283	4625	5787	6099	5875	5724	5309	6417	6110	6181
Aerobics	1104	1273	1340	1551	1623	1959	1876	2788	2855	3126
Swimming	2170	2042	2066	2295	2070	1955	1738	2158	2219	2153
Cycling	1361	1342	1400	1591	1576	1571	1532	1850	1809	1985
Running	989	1067	1094	1242	1143	1125	1171	1554	1771	1748
Bushwalking	737	787	824	731	837	693	862	984	803	772
Golf	695	733	690	680	654	631	488	752	703	744
Tennis	927	818	884	819	792	752	602	791	714	736
Weight training	313	230	274	304	233	355	257	478	402	421
Fishing	335	337	387	349	312	335	252	356	367	383

a. What is the percentage change in the number of participants swimming from year 1 to year 10?
b. What is the percentage change in the number of participants walking from year 1 to year 10?
c. What is the overall percentage change in the number of participants swimming and walking combined during the time period?

15. The following table shows the changes in an individual's salary over several years.

Year	Annual salary	Percentage change
2013	\$34 000	
2014	\$35 750	
2015	\$38 545	
2016	\$42 280	
2017	\$46 000	

Use CAS or a spreadsheet to answer these questions.
a. Evaluate the percentage change of each salary from the previous year.
b. In which year did the individual receive the biggest percentage increase in salary?

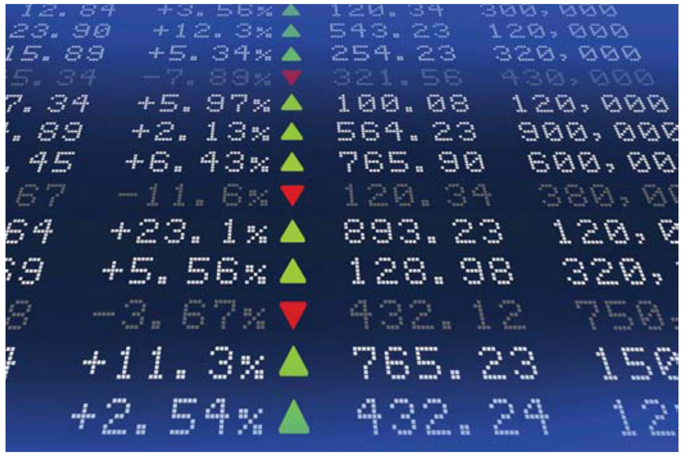
16. A population of possums in a particular area, N, changes every month according to the rule $N = 1.55M - 18$, where M is the number of possums the previous month. The number of possums at the end of December is 65.
Use CAS to:
a. construct a table and draw a graph of the number of possums over the next 6 months (rounding to the nearest number of possums)
b. evaluate the percentage change each month, correct to 1 decimal place.

3.3 Financial applications of ratios and percentages

Shares and currency

3.3.1Share dividends

Many people earn a second income through investments such as **shares**, seeking to make a profit through buying and selling shares in the stock market. Speculators attempt to buy shares when they are low in value and sell them when they are high in value, whereas other investors will keep their shares in a company for a longer period of time in the hope that they continue to gradually rise in value. When you purchase shares you are effectively becoming a part-owner of a company, which means you are entitled to a portion of any profits that are made. This is known as a **dividend**.



To calculate a dividend, the profit shared is divided by the total number of shares in the company.

WORKED EXAMPLE 3

Calculate the dividend payable for a company with 2 500 000 shares when $525 000 of its annual profit is distributed to the shareholders?

THINK	WRITE
1. Divide the profit by the number of shares.	$525\,000 \div 2\,500\,000 = 0.21$
2. State the final answer.	The dividend payable will be 21 cents per share.

3.3.2 Percentage dividends

Shares in different companies can vary drastically in price, from cents up to hundreds of dollars for a single share. As a company becomes more successful the share price will rise, and as a company becomes less successful the share price will fall.

An important factor that investors look at when deciding where to invest is the **percentage dividend** of a company. The percentage dividend is calculated by dividing the dividend per share by the share price per share.

$$\text{Percentage dividend} = \frac{\text{dividend per share}}{\text{share price per share}}$$

WORKED EXAMPLE 4

Calculate the percentage dividend of a share that costs \$13.45 with a dividend per share of \$0.45. Give your answer correct to 2 decimal places.

THINK	WRITE
1. Divide the dividend per share by the price of a share.	$\frac{0.45}{13.45} = 0.033\,457...$
2. Express the result as a percentage (by multiplying by 100).	$= 0.033\,457... \times 100\%$ $= 3.3457...\%$
3. Round your answer to 2 decimal places and state the answer.	$= 3.35\%$ (to 2 decimal places) The percentage dividend per share is 3.35%.

3.3.3 The price-to-earnings ratio

The **price-to-earnings ratio (P/E ratio)** is another way of comparing shares by looking at the current share price and the annual dividend. It is calculated by dividing the current share price by the dividend per share, giving an indication of how much shares cost per dollar of profit.

$$\textbf{P/E ratio} = \frac{\textbf{share price per share}}{\textbf{dividend per share}}$$

WORKED EXAMPLE 5

Calculate the price-to-earnings ratio for a company whose current share price is \$3.25 and has a dividend of 15 cents. Give your answer correct to 2 decimal places.

THINK	WRITE
1. Divide the current share price by the dividend.	$3.25 \div 0.15 = 21.67$ (to 2 decimal places)
2. State the answer.	The price-to-earnings ratio is 21.67.

3.3.4 Mark-ups and discounts

In the business world profits need to be made, otherwise companies may be unable to continue their operations. When deciding on how much to charge customers, businesses have to take into account all of the costs they incur in providing their services. If their costs increase, they must mark up their own charges in order to remain viable. For example, any businesses that rely on the delivery of materials by road transport are susceptible to fluctuations in fuel prices, and they will take these into account when pricing their services. If fuel prices increase, they will need to increase their charges, but if fuel prices decrease, they might consider introducing discounts.

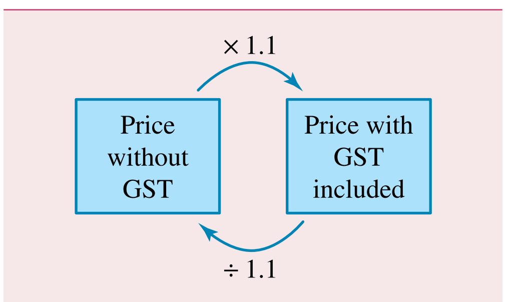
WORKED EXAMPLE 6

A transport company adjusts their charges as the price of petrol changes. By what percentage, correct to 2 decimal places, do their fuel costs change if the price per litre of petrol increases from \$1.36 to \$1.42?

THINK	WRITE
1. Calculate the amount of change.	\$1.42 – \$1.36 = \$0.06
2. Express the change as a fraction of the starting point.	$\frac{0.06}{1.36}$
3. Simplify the fraction where possible and then multiply by 100 to calculate the percentage change.	$\frac{0.06}{1.36} = \frac{6}{136}$ $= \frac{3}{68}$ $\frac{3}{68} \times 100 = \frac{3}{17} \times \frac{25}{1}$ $= \frac{75}{17}$ ≈ 4.41
4. State the answer.	The company's fuel costs increase by 4.41%.

3.3.5 Goods and services tax (GST)

In Australia we have a 10% tax that is charged on most purchases, known as a **goods and services tax** (or **GST**). Some essential items, such as medicine, education and certain types of food, are exempt from GST, but for all other goods GST is added to the cost of items bought or services paid for. If a price is quoted as being 'inclusive of GST', the amount of GST paid can be evaluated by dividing the price by 11.



WORKED EXAMPLE 7

Calculate the amount of GST included in an item purchased for a total of \$280.50.

THINK	WRITE
1. Does the price include the GST already?	Yes, GST is included.
2. If GST is not included, calculate 10% of the value. If GST is included, divide the value by 11.	GST is included, so divide \$280.50 by 11. $280.50 \div 11 = 25.5$
3. State the final answer.	The amount of GST is \$25.50.

on Resources

Interactivity: Shares and currency (int-6460)

studyon

Units 1 & 2 > AOS 2 > Topic 2 > Concept 2

Inflation, share prices and dividends Concept summary and practice questions

Exercise 3.3 Financial applications of ratios and percentages

Unless otherwise directed, give all answers to the following questions correct to 2 decimal places or the nearest cent where appropriate.

1. **WE3** Calculate the dividend payable per share for a company with:
   a. 32 220 600 shares, when $1 995 000 of its annual profit is distributed to the shareholders
   b. 44 676 211 shares, when $5 966 000 of its annual profit is distributed to the shareholders
   c. 263 450 shares, when $8 298 675 of its annual profit is distributed to the shareholders.
2. How many shares are in a company that declares a dividend of:
   a. 28.6 cents per share when $1 045 600 of its annual profit is distributed?
   b. $2.34 per share when $3 265 340 of its annual profit is distributed?
   c. $16.62 per share when $9 853 000 of its annual profit is distributed?
   d. $34.95 per share when $15 020 960 of its annual profit is distributed?

3. **WE4** Calculate the percentage dividends of the following shares.
   a. A share price of $14.60 with a dividend of 93 cents
   b. A share price of $22.34 with a dividend of 87 cents
   c. A share price of $45.50 with a dividend of $2.34
   d. A share price of $33.41 with a dividend of $2.88
4. Alexandra is having trouble deciding which of the following companies to invest in. She wants to choose the company with the highest percentage dividend. Calculate the percentage dividend for each company to find out which Alexandra should choose.
   a. A clothing company with a share price of $9.45 and a dividend of 45 cents
   b. A mining company with a share price of $53.20 and a dividend of $1.55
   c. A financial company with a share price of $33.47 and a dividend of $1.22
   d. A technology company with a share price of $7.22 and a dividend of 41 cents
   e. An electrical company with a share price of $28.50 and a dividend of $1.13
5. Calculate the dividend per share for a company with:
   a. a price-to-earnings ratio of 25.5 and a current share price of $8.75
   b. a price-to-earnings ratio of 20.3 and a current share price of $24.35
   c. a price-to-earnings ratio of 12.2 and a current share price of $10.10
   d. a price-to-earnings ratio of 26 and a current share price of $102.

6. **WE5** Calculate the price-to-earnings ratio for a company with:
   a. a current share price of $12.50 and a dividend of 25 cents
   b. a current share price of $43.25 and a dividend of $1.24
   c. a current share price of $79.92 and a dividend of $3.32
   d. a current share price of $116.46 and a dividend of $7.64.
7. Calculate the current share price for a company with:
   a. a price-to-earnings ratio of 22.4 and a dividend of 68 cents
   b. a price-to-earnings ratio of 36.8 and a dividend of 81 cents
   c. a price-to-earnings ratio of 17.6 and a dividend of $1.56
   d. a price-to-earnings ratio of 11.9 and a dividend of $3.42.
8. **WE6** A coffee shop adjusts its charges as the price of electricity changes. By what percentage does its power cost change if the price of electricity increases from:
   a. 88 cents to 94 cents per kWh
   b. 92 cents to $1.06 per kWh?
9. An electrical goods department store charges $50 plus n cents per km for delivery of its products, where n = the number of cents over $1.20 of the price per litre of petrol. What will be the percentage increase in the total delivery charge for a distance of 25 km when the petrol price changes from $1.45 to $1.52 per litre?
10. **WE7** Calculate the amount of GST included in an item purchased for a total of:
   a. $34.98
   b. $586.85
   c. $56 367.85
   d. $2.31.
11. Two companies are competing for the same job. Company A quotes a total of $5575 inclusive of GST. Company B quotes $5800 plus GST, but offers a 10% reduction on the total price for payment in cash. Which is the cheaper offer, and by how much?
12. A plumber quotes his clients the cost of any parts required plus $74.50 per hour for his labour, and then adds on the required GST.

   a. How much does he quote for a job that requires $250 in parts (excluding GST) and should take 4 hours to complete?
   b. If the job ends up being faster than he first thought, and he ends up charging the client for only 3 hours labour, what percentage discount on the original quote does this represent?
13. A company that has 350 000 shares declares an annual gross profit of $2 450 665, pays 18.5% of this in tax, and reinvests 25% of the net profit.
   a. What is the dividend per share payable to the shareholders?
   b. What is the price-to-earnings ratio if the current share price is $43.36?
14. A boat is purchased during a sale for a cash payment of $2698.
   a. If it had been discounted by 15%, and then a further $895 was taken off for a trade-in, what was the original price correct to the nearest dollar?
   b. What is the percentage change between the original price and the cash payment?
15. The details of two companies are shown in the following table.

Company	Share price	Net profit	Total shares
Company A	$34.50	$8 600 000	650 000
Company B	$1.48	$1 224 000	555 000

a. What is the dividend per share payable for shareholders in each company if each of the companies re-invests 12.5% of the net profit?
b. What is the price-to-earnings ratio for each company?
c. If a shareholder has 500 shares in Company A and 1000 shares in Company B, how much will they receive from their dividends?
d. Which company represents the best investment?

16. A South African company with a share price of 49.6 rand and 3 456 000 shares declares a dividend of 3.04 rand per share.
a. What is the total dividend payment in rand?
b. What is the price-to-earnings ratio of this company?

17. Jules is shopping for groceries and buys the following items.
Bread — $3.30*
Fruit juice — $5.50*
Meat pies — $5.80
Ice-cream — $6.90
Breakfast cereal — $5.00*
Biscuits — $2.90
All prices are listed before GST has been added-on.
a. The items marked with an asterisk (*) are exempt from GST. Calculate the total amount of GST Jules has to pay for his shopping.
b. Calculate the additional amount Jules would have to pay if all of the items were eligible for GST.
c. Jules has a voucher that gives him a 10% discount from this shop. Use your answer from part **a** to calculate how much Jules pays for his groceries.

18. A carpet company offers a trade discount of 12.5% to a builder for supplying the floor coverings on a new housing estate.
a. If the builder spends $32 250, how much was the carpet before the discount was applied? Round your answer to the nearest 5 cents.
b. If the builder charges his customers a total of $35 000, what percentage discount have they received compared to buying direct from the carpet company?

19. The share price of a mining company over several years is shown in the following table.

Year	2012	2013	2014	2015	2016
Share price	$44.50	$39.80	$41.20	$31.80	$29.60
Dividend per share	$1.73	$3.25	$2.74	$3.15	$3.42

a. If there are a total of 10 000 000 shares in the company, and 35% of the net profit was reinvested each year, use CAS or other technology to calculate the net profit for each of the years listed.
b. What are the price-to-earnings ratios for each of the years listed?
c. Which was the best year to purchase shares in the company?

20. The Australian government is considering raising the GST tax from 10% to 12.5% in order to raise funds and cut the budget deficit.
The following shopping bill lists all items exclusive of GST. Calculate the amount by which this shopping bill would increase if the rise in GST did go through.
Note: GST must be paid on all of the items in this bill.
1 litre of soft drink — $2.80
Large bag of pretzels — $5.30
Frozen lasagne — $6.15
Bottle of shampoo — $7.60
Box of chocolate — $8.35
2 tins of dog food — $3.50

21. Use CAS to answer the following questions about the companies shown in this table.

Company	Company A	Company B	Company C	Company D	Company E
Currency	Australian dollars	USA dollars	European euros	Chinese yuan	Indian rupees
Share price	$23.35	$26.80	€16.20	¥133.5	₹1288
Dividend	$1.46	$1.69	€0.94	¥8.7	₹65.5

a. Calculate the price-to-earnings ratio for each company.
b. Calculate the percentage dividend for each company.

22. The following table shows the mark-ups and discounts ple 1applied by a clothing store.

Item	Cost price	Normal retail price (255% mark-up)	Standard discount (12.5% mark-down of normal retail price)	January sale (32.25% mark-down of normal retail price)	Stocktake sale (55% mark-down of normal retail price)
Socks	\$1.85				
Shirts	\$12.35				
Trousers	\$22.25				
Skirts	\$24.45				
Jackets	\$32.05				
Ties	\$5.65				
Jumpers	\$19.95				

Use CAS or a spreadsheet to answer these questions.

a. Enter the information in your CAS or spreadsheet and use it to evaluate the normal retail prices and discount prices for each column as indicated.
b. What calculation is required in order to determine the stocktake sale price?
c. What would be the percentage change between the standard discount price and the stocktake sale price of a jacket?

3.4 Simple interest applications

3.4.1 The simple interest formula

When you invest money and receive a return on your investment,the amount of money you receive on top of your original investment is known as the interest. Similarly, when you take out a loan, the additional amount that you pay back on top of the loan value is known as the interest.

Interest is usually calculated as a percentage of the amount that is borrowed or invested, which is known as the **principal**. **Simple interest** involves a calculation based on the original amount borrowed or invested; as a result, the amount of simple interest for a particular loan is constant. For this reason simple interest is often called ‘flat rate’ interest.

The formula to calculate simple interest is $I = \frac{PrT}{100}$, where I is the amount of interest earned, P is the principal (initial amount invested or borrowed), r is the interest rate and T the time period.

It is important to remember that the rate and the time must be compatible. For example, if the rate is per annum (yearly, abbreviated ‘p.a.’), the time must also be in years.

The value of a simple interest investment can be evaluated by adding the total interest earned to the value of the principal.

WORKED EXAMPLE 8

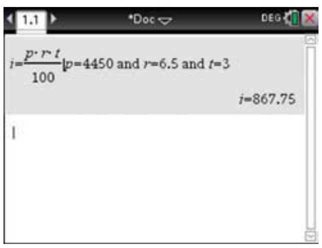
Calculate the amount of simple interest earned on an investment of \$4450 that returns 6.5% per annum for 3 years.

THINK	WRITE
1. Identify the components of the simple interest formula.	$P = \$4450$ $I = 6.5\%$ $T = 3$
2. Substitute the values into the formula and evaluate the amount of interest.	$I = \frac{PrT}{100}$ $= \frac{4450 \times 6.5 \times 3}{100}$ $= 867.75$
3. State the answer.	The amount of simple interest earned is \$867.75.

TI | THINK

1. On the Calculator page, complete the entry line as: $i = \frac{prt}{100}|p = 4450$ $r = 6.5$ and $t = 3$ then press ENTER.
Note: Be sure to enter the multiplication operation between the variables p, r and t.

WRITE



2. The answer can be read from the screen.

The amount of simple interest is \$867.75.

CASIO | THINK

1. On the Financial screen, select:
- Calc(1)
- Simple Interest
Complete the fields as:
Days: 1095
I%: 6.5
PV: −4450 then click the SI icon.
Note: Be sure to include a negative sign with the \$4550 for the present value to indicate that this money has been paid, not received.

WRITE

2. The answer can be read from the screen.
Note: The value for the simple interest is positive because money is being received.

The amount of simple interest is \$867.75.

3.4.2 Calculating the principal, rate or time

The simple interest formula can be transposed (rearranged) to find other missing values in problems. For example, we might want to know how long it will take for a simple interest investment of \$1500 to grow to \$2000 if we are being offered a rate of 7.5% per annum, or the interest rate needed for an investment to grow from \$4000 to \$6000 in 3 years.

The following formulas are derived from transposing the simple interest formula.

To find the time: $T = \frac{100I}{Pr}$

To find the interest rate: $r = \frac{100I}{PT}$

To find the principal: $P = \frac{100I}{rT}$

WORKED EXAMPLE 9

How long will it take an investment of \$2500 to earn \$1100 with a simple interest rate of 5.5% p.a.?

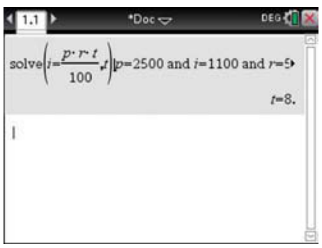
THINK	WRITE
1. Identify the components of the simple interest formula.	$P = 2500$ $I = 1100$ $r = 5.5$
2. Substitute the values into the formula and evaluate for T.	$T = \dfrac{100I}{Pr}$ $= \dfrac{100 \times 1100}{2500 \times 5.5}$ $= \dfrac{110\,000}{13\,750}$ $= 8$
3. State the answer.	It will take 8 years for the investment to earn \$1100.

TI | THINK

1. On the Calculator page, press MENU then select:

   3: Algebra
   1: Solve
   then complete the entry line as: solve $\left(i = \dfrac{p \times r \times t}{100}, t\right)$ $|p = 2500\,|i = 1100|$ $r = 5.5$. Press ENTER.
   Note: Be sure to enter the multiplication operation between the variables p, r and t.

WRITE



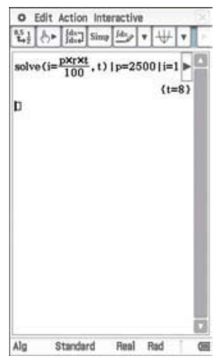
2. The answer can be read from the screen.

WRITE: It will take 8 years.

CASIO | THINK

1. On the Main screen, complete the entry line as: solve $\left(i = \dfrac{p \times r \times t}{100}, t\right)|p = 2500$ $|i = 1100\,|r = 5.5$, then press EXE.
   Note: Be sure to enter the multiplication operation between the variables p, r and t.

WRITE



2. The answer can be read from the screen.

WRITE: It will take 8 years.

3.4.3 Simple interest loans

The amount of a simple interest investment can be found by adding the simple interest to the principal. This can be expressed as $A = P + I$, where A represents the total amount of the investment.

For a simple interest loan, the total interest to be paid is usually calculated when the loan is taken out, and repayments are calculated from the total amount to be paid back (i.e. the principal plus the interest). For example, if a loan is for \$3000 and the total interest after 2 years is \$1800, the total to be paid back will be \$4800. Monthly repayments on this loan would therefore be $\$4800 \div 24 = \$200$.

WORKED EXAMPLE 10

Calculate the monthly repayments for a $14 000 loan that is charged simple interest at a rate of 8.45% p.a. for 4 years.

THINK | **WRITE**

1. Calculate the amount of interest charged.

$$I = \frac{PrT}{100}$$
$$= \frac{14\,000 \times 8.45 \times 4}{100}$$
$$= 4732$$

2. Add the interest to the principal to evaluate the total amount to be paid back.

$$A = P + I$$
$$= 14\,000 + 4732$$
$$= 18\,732$$

3. Divide by the number of months.

$$\frac{18\,732}{48} = 390.25$$

4. State the answer.

The monthly repayments will be $390.25.

3.4.4 Cash flow

Non-annual interest calculations

Although interest rates on investments and loans are frequently quoted in terms of an annual rate, in reality calculations on interest rates are made more frequently throughout a year. Quarterly, monthly, weekly and even daily calculations are not uncommon.

For example, a bank may offer 5% per annum on the amount its customers have in their savings accounts, but calculate the interest on a monthly basis (i.e. $\frac{5}{12}\%$).

WORKED EXAMPLE 11

How much interest is paid on a monthly balance of $665 with a simple interest rate of 7.2% p.a.?

THINK | **WRITE**

1. Express the interest as a monthly rate.

$$7.2\% \text{ p.a.} = \frac{7.2}{12}$$
$$= 0.6\% \text{ per month}$$

2. Use the simple interest formula to calculate the interest.

$$I = \frac{PrT}{100}$$
$$= \frac{665 \times 0.6 \times 1}{100}$$
$$= 3.99$$

3. State the answer.

$3.99 interest will be paid.

3.4.5 Minimum balance calculations

Banks and financial institutions need to make decisions about when to apply interest rate calculations on accounts of their customers.

For investment accounts it is common practice to use the minimum balance in the account over a set period of time. An example of this is a minimum monthly balance. The following bank account calculates interest at a rate of 5.5% per annum on the minimum monthly balance.

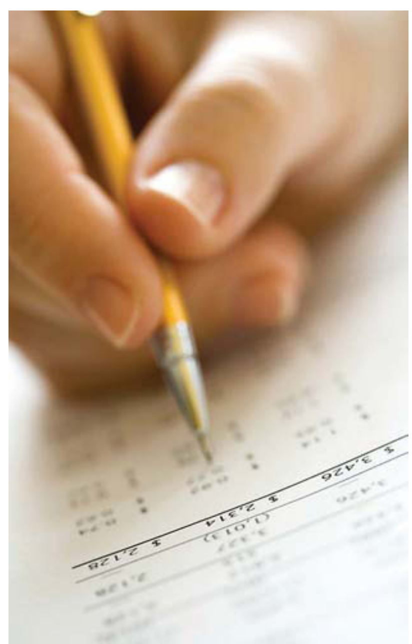
Date	Details	Withdrawal ($)	Deposit ($)	Balance ($)
June 1	Opening balance			4101.00
June 2	EFTPOS purchase	45.50		4055.50
June 8	Deposit — Salary		550.00	4605.50
June 12	EFTPOS purchase	56.00		4549.50
June 14	ATM withdrawal	220.00		4329.50
June 15	Deposit — Salary		550.00	4879.50
June 22	Deposit — Salary		550.00	5429.50
June 23	ATM withdrawal	285.00		5144.50
June 25	Payment —Direct debit insurance	350.00		4794.50
June 28	EFTPOS purchase	189.50		4605.00
June 29	Deposit — Salary		550.00	5155.00
June 30	Interest		18.59	5173.59

The minimum balance during the month was $4055.50, so the interest calculation is made on this amount.

$$I = \frac{4055.50 \times \left(5.5 \times \frac{1}{12}\right) \times 1}{100}$$

$$= \$18.59 \text{ (correct to 2 decimal places)}$$

Notice that the balance for the month was below $4101.00 for only six days, so the timing of withdrawals and deposits is very important for customers looking to maximise the interest they receive.



WORKED EXAMPLE 12

Interest on a savings account is earned at a simple rate of 7.5% p.a. and is calculated on the minimum monthly balance. How much interest is earned for the month of June if the opening balance is $1200 and the following transactions are made? Give your answer correct to the nearest cent.

Date	Details	Amount
June 2	Deposit	$500
June 4	Withdrawal	$150
June 12	Withdrawal	$620
June 18	Deposit	$220
June 22	Withdrawal	$500
June 29	Deposit	$120

THINK

1. Set up a balance sheet that places the transactions in chronological order. Use it to calculate the balance following each consecutive transaction.

WRITE

Date	Details	Withdrawal	Deposit	Balance
June 1	Opening balance			1200.00
June 2	Deposit		500.00	1700.00
June 4	Withdrawal	150.00		1550.00
June 12	Withdrawal	620.00		930.00
June 18	Deposit		220.00	1150.00
June 22	Withdrawal	500.00		650.00
June 29	Deposit		120.00	770.00

2. Identify the smallest balance and use this to calculate the monthly interest.

The smallest balance is $650.00.

The monthly interest is: $I = \dfrac{650 \times \left(7.5 \times \frac{1}{12}\right) \times 1}{100}$

$= \$4.06$

3. State the answer.

The interest earned for the month of June is $4.06.

on Resources

Interactivity: Simple interest (int-6461)

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Units 1 & 2 > AOS 2 > Topic 2 > Concept 3 > **Simple Interest** Concept summary and practice questions

Exercise 3.4 Simple interest applications

Unless otherwise directed, give all answers to the following questions correct to 2 decimal places or the nearest cent where appropriate.

1. **WE8** Calculate the amount of simple interest earned on an investment of:
   a. \$2575, returning 8.25% per annum for 4 years
   b. \$12 250, returning 5.15% per annum for $6\frac{1}{2}$ years
   c. \$43 500, returning 12.325% per annum for 8 years and 3 months
   d. \$103 995, returning 2.015% per annum for 105 months.
2. Calculate the value of a simple interest investment of:
   a. \$500, after returning 3.55% per annum for 3 years
   b. \$2054, after returning 4.22% per annum for $7\frac{3}{4}$ years
   c. \$3500, after returning 11.025% per annum for 9 years and 3 months
   d. \$10 201, after returning 1.008% per annum for 63 months.
3. **WE9** How long will it take an investment of:
   a. \$675 to earn \$216 with a simple interest rate of 3.2% p.a.?
   b. \$1000 to earn \$850 with a simple interest rate of 4.25% p.a.?
   c. \$5000 to earn \$2100 with a simple interest rate of 5.25% p.a.?
   d. \$2500 to earn \$775 with a simple interest rate of 7.75% p.a.?
4. a. If \$2000 earns \$590 in 5 years, what is the simple interest rate?
   b. If \$1800 earns \$648 in 3 years, what is the simple interest rate?
   c. If \$408 is earned in 6 years with a simple interest rate of 4.25%, how much was invested?
   d. If \$3750 is earned in 12 years with a simple interest rate of 3.125%, how much was invested?
5. **WE10** Calculate the monthly repayments for:
   a. a \$8000 loan that is charged simple interest at a rate of 12.25% p.a. for 3 years
   b. a \$23 000 loan that is charged simple interest at a rate of 15.35% p.a. for 6 years
   c. a \$21 050 loan that is charged simple interest at a rate of 11.734% p.a. for 6.25 years
   d. a \$33 224 loan that is charged simple interest at a rate of 23.105% p.a. for 54

6. Calculate the monthly repayments for:
   a. a \$6225 loan that is charged simple interest at a rate of 7.025% p.a. for 130 weeks
   b. a \$13 328 loan that is charged simple interest at a rate of 9.135% p.a. for 1095 days.
7. A savings account with a minimum monthly balance of \$800 earns \$3.60 interest in a month. What is the annual rate of simple interest?
8. **WE11** How much interest is paid on a monthly balance of:
   a. \$1224 with a simple interest rate of 3.6% p.a.
   b. \$955 with a simple interest rate of 6.024% p.a.
   c. \$2445.50 with a simple interest rate of 4.8% p.a.
   d. \$13 728.34 with a simple interest rate of 9.612% p.a.?
9. How much interest is paid on:
   a. a weekly balance of \$1020 with a simple interest rate of 18% p.a.
   b. a quarterly balance of \$12 340 with a simple interest rate of 23% p.a.
   c. a fortnightly balance of \$22 765 with a simple interest rate of 9.5% p.a.
   d. a daily balance of \$225 358 with a simple interest rate of 6.7% p.a.?

10. WE12 Interest on a savings account is earned at a simple rate and is calculated on the minimum monthly balance. How much interest is earned for the month of:

a. January, if the rate is 4.2% p.a., the opening balance is $200 and the following transactions are made

Date	Details	Amount
3 January	Deposit	$135
6 January	Deposit	$84
14 January	Withdrawal	$44
19 January	Withdrawal	$175
25 January	Deposit	$53
30 January	Deposit	$118

b. September, if the rate is 3.6% p.a., the opening balance is $885 and the following transactions are made?

Date	Details	Amount
2 September	Withdrawal	$225
4 September	Withdrawal	$150
12 September	Withdrawal	$73
18 September	Deposit	$220
22 September	Withdrawal	$568
29 September	Withdrawal	$36

11. Interest on a savings account is earned at a simple rate and is calculated on the minimum monthly balance. How much interest is earned for the month of:

a. May, if the rate is 5.8% p.a., the opening balance is $465 and the following transactions are made

Date	Details	Amount
2 May	Deposit	$111
4 May	Deposit	$150
12 May	Withdrawal	$620
18 May	Deposit	$135
22 May	Deposit	$203
29 May	Deposit	$45

b. October, if the rate is 2.85% p.a., the opening balance is $2240 and the following transactions are made?

Date	Details	Amount
2 October	Deposit	$300
4 October	Withdrawal	$683
12 October	Deposit	$220
18 October	Deposit	$304
22 October	Deposit	$164
29 October	Withdrawal	$736

12. Investment 1 is $1000 growing at a simple interest rate of 4.5% p.a., and investment 2 is $800 growing at a simple interest rate of 8.8%. When will investment 2 be greater than investment 1? Give your answer correct to the nearest year.

13. $25 000 is invested for 5 years in an account that pays 6.36% p.a. simple interest.

a. How much interest is earned each year?

b. What will be the value of the investment after 5 years?

c. If the money was reinvested for a further 2 years, what simple interest rate would result in the investment amounting to $35 000 by the end of that time?

14. A bank account pays simple interest at a rate of 0.085% on the minimum weekly balance.

a. What is the annual rate of interest? (Assume 52 weeks in a year.)

b. If $3.50 interest was earned, what was the minimum balance for that week?

c. How much interest was earned if the opening balance for a week was $3030 and the transactions in the table at right took place?

Day	Details
Day 1	Withdrawal $250
Day 2	Deposit $750
Day 3	Withdrawal $445
Day 4	Deposit $180
Day 5	Deposit $230
Day 7	Withdrawal $650

15. An overdraft account requires a minimum payment of 5% of the outstanding balance at the end of each month. Interest on the account is calculated at a simple rate of 15.5% p.a. calculated monthly.

a. What is the minimum monthly payment if the overdraft balance before the interest was charged was $10 000?

b. What is the percentage change (relative to the initial $10 000) in the balance of the account once the minimum payment has been made?

16. A borrower has to pay 7.8% p.a. simple interest on a 6-year loan. If the total interest paid is $3744:

a. how much was borrowed

b. what are the repayments if they have to be made fortnightly?

17. $19 245 is invested in a fund that pays a simple interest rate of 7.8% p.a. for 42 months.
  a. How much simple interest is earned on this investment?
  b. The investor considers an alternative investment with a bank that offers a simple interest rate of 0.625% per month for the first 2.5 years and 0.665% per month after that. Which is the best investment?
18. A bank offers a simple interest loan of $35 000 with monthly repayments of $545.
  a. What is the rate of simple interest if the loan is paid in full in 15 years?
  b. After 5 years of payments the bank offers to reduce the total time of the loan to 12 years if the monthly payments are increased to $650. How much interest would be paid over the life of the loan under this arrangement?
  c. What would be the average rate of simple interest over the 12 years under the new arrangement?
19. At the start of the financial year (1 July) a company opens a new account at a bank with a deposit of $25 000. The account pays simple interest at a rate of 7.2% p.a. payable on the minimum monthly balance and credited quarterly.
  a. Calculate the total amount of interest payable if the following transactions took place.

Date	**Details**	**Amount**	**Date**	**Details**	**Amount**	**Date**	**Details**	**Amount**
July 3	Withdraw	$8340	Aug 5	Withdraw	$1167	Sep 8	Withdraw	$750
July 13	Deposit	$6206	Aug 12	Deposit	$5449	Sep 17	Deposit	$2950
July 23	Withdraw	$3754	Aug 18	Deposit	$1003	Sep 24	Withdraw	$7821
July 29	Withdraw	$4241	Aug 23	Withdraw	$5775	Sep 29	Deposit	$1057

  b. What is the overall percentage change in the account for each month over this time period?
20. An account pays simple interest at a rate of 7.2% p.a. on the minimum daily balance and credits it to the account half-yearly.
  a. What is the daily rate of interest?
  b. Calculate the daily interest payable after 6 months if the account was opened with a deposit of $250 on 1 July, followed by further deposits of $350 on the first day of each subsequent month.
21. The table shows the transactions for a savings account over a 6-month period. Simple interest of 4.5% p.a. is calculated on the minimum daily balance of the account and credited to the account every 6 months.
   Use CAS or a spreadsheet to answer the following questions.

Date	**Details**	**Amount**
1/01/2016	Opening balance	$1200.00
12/01/2016	Deposit	$250.00
3/03/2016	Withdrawal	$420.00
14/04/2016	Withdrawal	$105.00
25/05/2016	Deposit	$265.00
9/06/2016	Deposit	$125.00

  a. Enter the details for the account into your CAS or spreadsheet and use it to calculate the balance in the account at each date.
  b. Use your CAS or a spreadsheet to calculate the amount of interest earned after 6 months.
22. $100 is invested in an account that earns $28 of simple interest in 8 months.
  a. Evaluate the annual rate of simple interest.
  b. Calculate the amount of interest that would have been earned in the 8 months if the annual interest rate was increased by 0.75%.

3.5 Compound interest applications

3.5.1 Step-by-step compounding

Simple interest rates calculate interest on the starting value. However, it is more common for interest to be calculated on the changing value throughout the time period of a loan or investment. This is known as compounding.

In compounding, the interest is added to the balance, and then the next interest calculation is made on the new value.

For example, consider an investment of \$5000 that earns 5% p.a. compounding annually. At the end of the first year, the interest amounts to $\frac{5}{100} \times 5000 = \$250$, so the total investment will become \$5250. At the end of the second year, the interest now amounts to $\frac{5}{100} \times 5250 = \$262.50$. As time progresses, the amount of interest becomes larger at each calculation. In contrast, a simple interest rate calculation on this balance would be a constant, unchanging amount of \$250 each year.

WORKED EXAMPLE 13

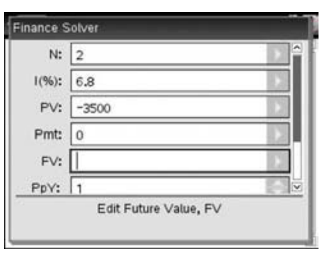
A bank offers its customers a compound interest rate of 6.8% p.a. on term deposits for amounts over \$3000, as long as the balance remains in the account for a minimum of 2 years. Calculate the amount of compound interest accumulated after 2 years on a term deposit of \$3500 correct to the nearest cent.

THINK	WRITE
1. Calculate the interest at the end of the first year.	$\frac{6.8}{100} \times 3500 = 238$
2. Add the interest after the first year to the principal.	$3500 + 238 = 3738$
3. Use the principal plus the first year's interest to calculate the interest at the end of the second year.	$\frac{6.8}{100} \times 3738 = 254.18$ (to 2 decimal places)
4. Add the interest after the first year to the interest after the second year.	$238 + 254.18 = 492.18$
5. State the answer.	After 2 years the amount of compound interest accumulated is \$492.18.

TI | THINK

1. On the Calculator page, press MENU then select:
8: Finance
1: Finance Solver …
Complete the fields as:
N: 2I(%): 6.8
PV: −3500
Pmt: 0
PpY: 1
CpY: 1
then move the cursor to the FV field and press ENTER.

WRITE

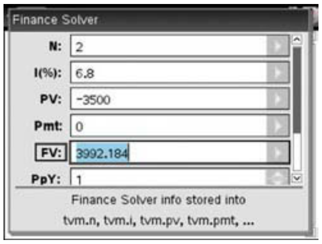


CASIO | THINK

1. On the Financial screen, select:
- Calc(1)
- Compound Interest
Complete the fields as:
N: 2
I%: 6.8
PV: −3500
PMT: 0
P/Y: 1
C/Y: 1 then click the FV icon.

WRITE

2 The value of the investment after 2 years can be read from the screen: FV = $3992.18



3 The compound interest accumulated in 2 years can be found by subtracting the initial value of the term deposit from the investment value after 2 years.

Interest
= $3992.18 – $3500
= $492.18
After 2 years the amount of compound interest accumulated is $492.18.

2. The value of the investment after 2 years can be read from
the screen: FV = $3992.18

3. The compound interest accumulated in 2 years can be found by subtracting the initial value of the term deposit from the investment value after 2 years.

Interest
=$ 3992.18- $ 3500
= $492.18
After 2 years the amount of compound interest accumulated is $492.18.

3.5.2 The compound interest formula

Although **compound interest** can be calculated step-by-step as shown above, it is usually easier to calculate compound interest by using the following formula:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where A is the final amount, P is the principal, r is the rate of interest per period and n is the number of compounding periods.

As with the simple interest formula, we need to ensure that the rate of interest and the number of compounding periods are compatible.

If we want to find the amount of compound interest, we need to subtract the principal from the final amount at the end of the compounding periods.

$$I = A - P$$

WORKED EXAMPLE 14

Use the compound interest formula to calculate the amount of interest on an investment of $2500 at 3.5% p.a. compounded annually for 4 years, correct to the nearest cent.

THINK	WRITE
1. Identify the components of the compound interest formula.	$P = 2500$ $r = 3.5$ $n = 4$
2. Substitute the values into the formula and evaluate the amount of the investment.	$A = P\left(1 + \frac{r}{100}\right)^n$ $= 2500\left(1 + \frac{3.5}{100}\right)^4$ $= 2868.81$ (to 2 decimal places)

3. Subtract the principal from the final amount of the investment to calculate the interest.

$I = A - P$
$= 2868.81 - 2500$
$= 368.81$

4. State the answer.

The amount of compound interest is \$368.81.

3.5.3 Calculating the interest rate or principal

As with the simple interest formula, the compound interest formula can be transposed if we need to find the interest rate or principal required to answer a particular problem. Transposing the compound interest formula gives the following formulas.

To find the interest rate:

$$r = 100\left(\left(\frac{A}{P}\right)^{\frac{1}{n}} - 1\right)$$

To find the principal:

$$P = \frac{A}{\left(1 + \frac{r}{100}\right)^n}$$

WORKED EXAMPLE 15

Use the compound interest formula to calculate the principal required, correct to the nearest cent, to have a final amount of \$10 000 after compounding at a rate of 4.5% p.a. for 6 years.

THINK | **WRITE**

1. Identify the components of the compound interest formula.

$A = \$10\,000$
$r = 4.5$
$n = 6$

2. Substitute the values into the formula to evaluate the principal.

$$P = \frac{A}{\left(1 + \frac{i}{100}\right)^n}$$
$$= \frac{10\,000}{\left(1 + \frac{4.5}{100}\right)^6}$$
$$= 7678.96 \text{ (to 2 decimal places)}$$

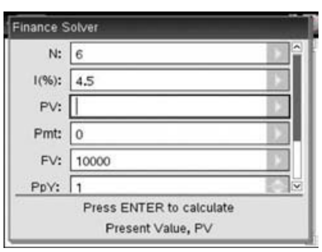
3. State the final answer.

The principal required is \$7678.96.

TI | THINK

1. On the Calculator page, press MENU then select:
8: Finance
1: Finance Solver …
Complete the fields as:
N: 6
I(%): 4.5
Pmt: 0
FV: 10 000
PpY: 1
CpY: 1 then move the cursor to the PV field and press ENTER.

WRITE

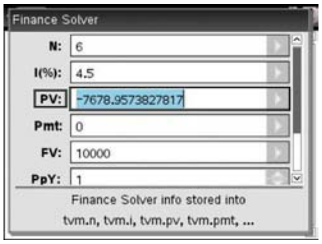


CASIO | THINK

1. On the Financial screen, select:
- Calc(1)
- Compound Interest
Complete the fields as:
N: 6
I%: 4.5
PMT: 0
FV: 10 000
P/Y: 1
C/Y: 1
then click the PV icon.

WRITE

2. The principal can be read from the screen: PV = \$7678.96. *Note:* The negative sign implies that the principal was paid, but does not need to be included in the answer.



The principal can be read from the screen: PV = \$7678.96. *Note:* The negative sign implies that the principal was paid, but does not need to be included in the answer.

Note: It is also possible to transpose the compound interest formula to find the number of compounding periods (n), but this requires logarithms and is outside the scope of this course.

3.5.4 Non-annual compounding

Interest rates are usually expressed per annum (yearly), but compounding often takes place at more regular intervals, such as quarterly, monthly or weekly. When this happens, adjustments need to be made when applying the formula to ensure that the rate is expressed in the same period of time. For example:

Compounding monthly: $A = P\left(1 + \frac{r}{1200}\right)^n$

Compounding weekly: $A = P\left(1 + \frac{r}{5200}\right)^n$

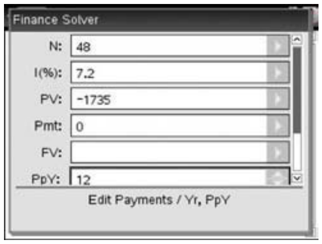
WORKED EXAMPLE 16

Use the compound interest formula to calculate the amount of interest accumulated on \$1735 at 7.2% p.a. for 4 years if the compounding occurs monthly. Give your answer correct to the nearest cent.

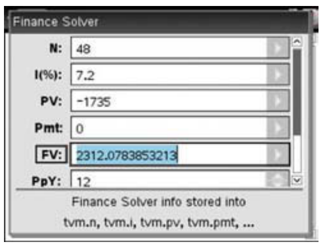
THINK	WRITE
1. Identify the components of the compound interest formula.	$P = \$1735$ $r = 7.2$ $n = 48$ (monthly periods)
2. Substitute the values into the formula and evaluate the amount.	$A = P\left(1 + \frac{r}{1200}\right)^n$ $= 1735\left(1 + \frac{7.2}{1200}\right)^{48}$ $= 2312.08$ (to 2 decimal places)
3. Subtract the principal from the amount of the investment.	$I = A - P$ $= 2312.08 - 1735$ $= 577.08$
4. State the answer.	The amount of interest accumulated is \$577.08.

TI | THINK **WRITE**

1. On the Calculator page, press MENU then select:
8: Finance
1: Finance Solver …
Complete the fields as:
N: 48
I(%): 7.2
PV: −1735
Pmt: 0
PpY: 12
CpY: 12 then move the curser to the FV field and press ENTER.



2. The value of the investment after 4 years can be read from the screen: FV = \$2312.08.

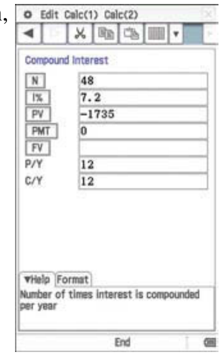


3. The compound interest accumulated in 4 years can be found by subtracting the initial value of the term deposit from the investment value after 4 years.

Interest
= \$2312.08 − \$1735
=\$ 577.08 After 4 years the amount of compound interest accumulated is \$577.08.

CASIO | THINK **WRITE**

1. On the Financial screen, select:
- Calc(1)
- Compound Interest
Complete the fields as:
N: 48
I%: 7.2
PV: −1735
PMT: 0
P/Y: 12
C/Y: 12 then click the FV icon.



2. The value of the investment after 4 years can be read from the screen:
FV = \$2312.08.

3. The compound interest accumulated in 4 years can be found by subtracting the initial value of the term deposit from the investment value after 4 years.

Interest
= \$2312.08 − \$1735
=\$ 577.08 After 4 years the amount of compound interest accumulated is \$577.08.

3.5.5 Inflation

Inflation is a term used to describe a general increase in prices over time that effectively decreases the purchasing power of a currency. Inflation can be measured by the inflation rate, which is an annual percentage change of the **Consumer Price Index (CPI)**.

Inflation needs to be taken into account when analysing profits and losses over a period of time. It can be analysed by using the compound interest formula.

3.5.6 Spending power

As inflation increases, the **spending power** of a set amount of money will decrease. For example, if the cost of a loaf of bread was \$4.00 and rose with the rate of inflation, in 5 years it might cost \$4.50. As inflation gradually decreases the spending of the dollar, people's salaries often increase in line with inflation. This counterbalances the decreasing spending power of money.

WORKED EXAMPLE 17

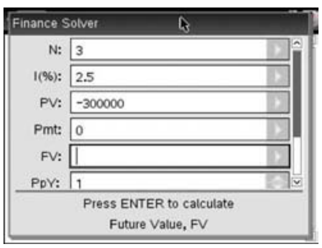
An investment property is purchased for \$300 000 and is sold 3 years later for \$320 000. If the average annual inflation is 2.5% p.a., has this been a profitable investment?

THINK	WRITE
1. Recall that inflation is an application of compound interest and identify the components of the formula.	$P = 300\,000$ $r = 2.5$ $n = 3$
2. Substitute the values into the formula and evaluate the amount.	$A = P\left(1 + \frac{r}{100}\right)^n$ $= 300\,000\left(1 + \frac{2.5}{100}\right)^3$ $= 323\,067.19$ (to 2 decimal places)
3. Compare the inflated amount to the selling price.	Inflated amount: \$323 067.19 Selling price: \$320 000
4. State the answer.	This has not been a profitable investment, as the selling price is less than the inflated purchase price.

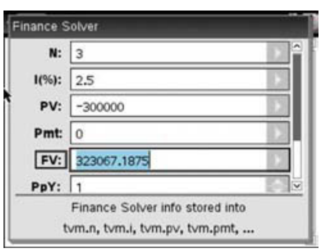
TI | THINK

1. On the Calculator page, press MENU then select:
8: Finance
1: Finance Solver …
Complete the fields as:
N: 3
I(%): 2.5
PV: −300 000
Pmt: 0
PpY: 1
CpY: 1 then move the cursor to the FV field and press ENTER.

WRITE



2. The value of the investment property after 3 years (the inflated price) can be read from the screen:
FV = \$323 067.19



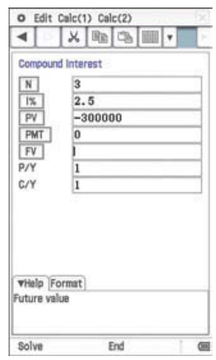
3. Answer the question.

This has not been a profitable investment, as the selling price (\$320 000) is less than the inflated price (\$323 067.19).

CASIO | THINK

1. On the Financial screen, select:
- Calc(1)
- Compound Interest
Complete the fields as:
N: 3
I%: 2.5
PV: - 300 000
PMT: 0
P/Y: 1
C/Y: 1 then click the FV icon.

WRITE



2. The value of the investment property after 3 years, the inflated price, can be read from the screen: FV = \$323 067.19

3. Answer the question.

This has not been a profitable investment, as the selling price (\$320 000) is less than the inflated price (\$323 067.19).

on Resources

Interactivity: Simple and compound interest (int-6265)
Interactivity: Non-annual compounding (int-6462)

studyon

Units 1 & 2 > AOS 2 > Topic 2 > Concept 4

Compound interest Concept summary and practice questions

Exercise 3.5 Compound interest applications

Unless otherwise directed, where appropriate give all answers to the following questions correct to 2 decimal places or the nearest cent.

1. **WE13** A bank offers its customers a compound interest rate on term deposits for amounts over \$3000 as long as the balance remains in the account for a minimum of 2 years. Calculate the amount of compound interest after:
   a. 3 years on a term deposit of \$5372 at 7.32% p.a.
   b. 4 years on a term deposit of \$9550 at 2.025% p.a.
   c. 5 years on a term deposit of \$10 099 at 1.045% p.a.
2. Calculate the value of an investment of \$1500 after 3 years at a compound interest rate of 2.85% p.a.
3. **WE14** Use the compound interest formula to calculate the amount of compound interest on an investment of:
   a. \$4655 at 4.55% p.a. for 3 years
   b. \$12 344 at 6.35% p.a. for 6 years
   c. \$3465 at 2.015% p.a. for 8 years
   d. \$365 000 at 7.65% p.a. for 20 years.
4. Use the compound interest formula to find the future amount of:
   a. \$358 invested at 1.22% p.a. for 6 years
   b. \$1276 invested at 2.41% p.a. for 4 years
   c. \$4362 invested at 4.204% p.a. for 3 years
   d. \$275 950 invested at 6.18% p.a. for 16 years.
5. **WE15** Use the compound interest formula to calculate the principal required to yield a final amount of:
   a. \$15 000 after compounding at a rate of 5.25% p.a. for 8 years
   b. \$22 500 after compounding at a rate of 7.15% p.a. for 10 years
   c. \$1000 after compounding at a rate of 1.25% p.a. for 2 years
   d. \$80 000 after compounding at a rate of 6.18% p.a. for 15 years.
6. Use the compound interest formula to calculate the compound interest rate p.a. that would be required to grow:
   a. \$500 to \$1000 in 2 years
   b. \$850 to \$2500 in 3 years
   c. \$1600 to \$2900 in 4 years
   d. \$3490 to \$9000 in 3 years.
7. **WE16** Use the compound interest formula to calculate the amount of interest accumulated on:
   a. \$2876 at 3.12% p.a. for 2 years, if the compounding occurs monthly
   b. \$23 560 at 6.17% p.a. for 3 years, if the compounding occurs monthly
   c. \$85.50 at 2.108% p.a. for 2 years, if the compounding occurs monthly
   d. \$12 345 at 5.218% p.a. for 6 years, if the compounding occurs monthly.

8. Use the compound interest formula to calculate the final amount for:
   a. $675 at 2.42% p.a. for 2 years compounding weekly
   b. $4235 at 6.43% p.a. for 3 years compounding quarterly
   c. $85 276 at 8.14% p.a. for 4 years compounding fortnightly
   d. $53 412 at 4.329% p.a. for 1 years compounding daily.
9. An $8000 investment earns 7.8% p.a. compound interest over 3 years. How much interest is earned if the amount is compounded:
   a. annually b. monthly c. weekly d. daily?
10. WE17 An investment property is purchased for $325 000 and is sold 5 years later for $370 000. If the average annual inflation is 2.73% p.a., has this been a profitable investment?

11. A business is purchased for $180 000 and is sold 2 years later for $200 000. If the annual average inflation is 1.8% p.a., has a real profit been made?
12. a. Calculate the interest accrued on a $2600 investment attracting a compound interest rate of 9.65% compounded annually. Show your results in the following table.

Year	1	2	3	4	5	6	7	8
Interest accrued ($)								

   b. Show your results in a graph.
13. Use a spreadsheet and graph to compare $4000 compounding at a rate of 3.25% p.a. with $2000 compounding at a rate of 6.5% p.a. When is the second option worth more as an investment than the first?
14. A parking fine that was originally $65 requires the payment of an additional late fee of $35. If the fine was paid 14 days late and interest had been compounding daily, what was the annual rate of interest being charged?

15. A person has $1000 and wants to have enough to purchase something worth $1450.
   a. If they invest the $1000 in a bank account paying compound interest monthly and the investment becomes $1450 within 3 years, what interest rate is the account paying?
   b. If the price of the item increased in line with an average annual inflation rate of 2%, how much would the person have needed to invest to have enough to purchase it at the end of the same time period, using the same compound rate of interest as in part **a**?
16. Shivani is given $5000 by her grandparents on the condition that she invests it for at least 3 years. Her parents help her to find the best investment options and come up with the following choices.
   i. A local business promising a return of 3.5% compounded annually, with an additional 2% bonus on the total sum paid at the end of the 3-year period
   ii. A building society paying a fixed interest rate of 4.3% compounded monthly
   iii. A venture capitalist company guaranteeing a return of 3.9% compounded daily
   a. Calculate the expected return after 3 years for each of the options.
   b. Assuming each option is equally secure, where should Shivani invest her money?

17. The costs of manufacturing a smart watch decrease by 10% each year.
   a. If the watch initially retails at $200 and the makers decrease the price in line with the manufacturing costs, how much will it cost at the end of the first 3 years?
   b. Inflation is at a steady rate of 3% over each of these years, and the price of the watch also rises with the rate of inflation. Recalculate the cost of the watch for each of the 3 years according to inflation. (*Note:* Apply the manufacturing cost decrease before the inflation price increase.)
18. In 2006 Matthew earned approximately $45 000 after tax and deductions. In 2016 he earned approximately $61 000 after tax and deductions. If inflation over the 10-year period from 2006 to 2016 averaged 3%, was Matthew earning comparatively more in 2006 or 2016?
19. Francisco is a purchaser of fine art, and his two favourite pieces are a sculpture he purchased in 1998 for $12 000 and a series of prints he purchased in 2007 for $17 000.
   a. If inflation averaged 3.3% for the period between 1998 and 2007, which item cost more in real terms?
   b. The value of the sculpture has appreciated at a rate of 7.5% since 1998, and the value of the prints has appreciated at a rate of 6.8% since 2007. How much were they both worth in 2015? Round your answers correct to the nearest dollar.

20. Use the compound interest formula to complete the following table. Assume that all interest is compounded annually.

Principal ($)	Final amount ($)	Interest earned	Interest rate (p.a.)	Number of years
11 000	12 012.28			2
14 000			3.25	3
22 050	25 561.99	3511.99		5
		2700.00	2.5	1

21. a. Using CAS, tabulate and graph an investment of $200 compounding at rate of 6.1% p.a. over 25 years.
   b. Evaluate, giving your answers to the nearest year, how long it will take the investment to:
      i. double
      ii. triple
      iii. quadruple.
22. Using CAS, compare compounding annually with compounding quarterly for $1000 at a rate of 12% p.a. over 5 years.
   a. Show the information in a graph or a table.
   b. What is the effect of compounding at regular intervals during the year while keeping the annual rate the same?

3.6 Purchasing options

3.6.1 Cash purchases

Buying goods with cash is the most straightforward type of purchase you can make. The buyer owns the goods outright and no further payments are necessary. Some retailers or services offer a discount if you pay with cash.

WORKED EXAMPLE 18

A plumber offers a 5% discount if his customers pay with cash. How much would a customer be charged if they paid in cash and the fee before the discount was $139?

THINK	WRITE
1. Determine the percentage of the fee that the customer will pay after the discount is taken into account.	$100\% - 5\% = 95\%$
2. Multiply the fee before the discount by the percentage the customer will pay. Turn the percentage into a fraction.	$139 \times 95\% = 139 \times \frac{95}{100}$
3. Evaluate the amount to be paid.	$= 132.05$
4. Write the answer.	The customer will be charged $132.05.

3.6.2 Credit and debit cards

Credit cards

A **credit card** is an agreement between a financial institution (usually a bank) and an individual to loan an amount of money up to a pre-approved limit. Credit cards can be used to pay for transactions until the amount of debt on the credit card reaches the agreed limit of the credit card.

If a customer pays off the debt on their credit card within a set period of time after purchases are made, known as an interest-free period, they will pay no interest on the debt. Otherwise they will pay a high interest rate on the debt (usually 20–30% p.a.), with the interest calculated monthly. Customers are obliged to pay at least a minimum monthly amount off the debt, for example 3% of the balance.

Credit cards often charge an annual fee, but customers can also earn rewards from using credit cards, such as frequent flyer points for major airlines or discounts at certain retailers.

3.6.3 Debit cards

Debit cards are usually linked to bank accounts, although they can also be pre-loaded with set amounts of money. When a customer uses a debit card the money is debited directly from their bank account or from the pre-loaded amount.

If a customer tries to make a transaction with a debit card that exceeds the balance in their bank account, then either their account will become overdrawn (which typically incurs a fee from the banking facility), or the transaction will be declined.

WORKED EXAMPLE 19

Heather has a credit card that charges an interest rate of 19.79% p.a. She tries to ensure that she always pays off the full amount at the end of the interest-free period, but an expensive few months over the Christmas holidays leaves the outstanding balance on her card at $635, $427 and $155 for three consecutive months. Calculate the total amount of interest Heather has to pay over the three-month period. Give your answer correct to the nearest cent.

THINK	WRITE
1. Use the simple interest formula to determine the amount of interest charged each month.	1st month: $I = \frac{PrT}{100} = \frac{635 \times 19.79 \times \frac{1}{12}}{100} \approx 10.47$ 2nd month: $I = \frac{PrT}{100} = \frac{427 \times 19.79 \times \frac{1}{12}}{100} \approx 7.04$ 3rd month: $I = \frac{PrT}{100} = \frac{155 \times 19.79 \times \frac{1}{12}}{100} \approx 2.56$
2. Calculate the sum of the interest for the three months.	$10.47 + 7.04 + 2.56 = 20.07$
3. Write the answer.	Heather has to pay $20.07 in interest over the three-month period.

3.6.4 Personal loans

A **personal loan** is a loan made by a lending institution to an individual. A personal loan will usually have a fixed interest rate attached to it, with the interest paid by the customer calculated on a reduced balance. This means that the interest for each period will be calculated on the amount still owing, rather than the original amount of the loan.

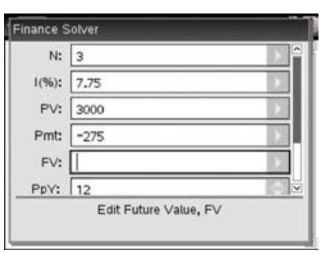
WORKED EXAMPLE 20

Francis takes out a loan of \$3000 to help pay for a business management course. The loan has a fixed interest rate of 7.75% p.a. and Francis agrees to pay back \$275 a month. Assuming that the interest is calculated before Francis's payments, calculate the outstanding balance on the loan after Francis's third payment. Give your answer correct to the nearest cent.

THINK	WRITE
1. Calculate the interest payable for the first month of the loan.	$I = \frac{PrT}{100}$ $= \frac{3000 \times 7.75 \times \frac{1}{12}}{100}$ ≈ 19.38
2. Calculate the total value of the loan before Francis's first payment.	\$3000 + \$19.38 = \$3019.38
3. Calculate the total value of the loan after Francis's first payment.	\$3019.38 − \$275 = \$2744.38
4. Calculate the interest payable for the second month of the loan.	$I = \frac{PrT}{100}$ $= \frac{2744.38 \times 7.75 \times \frac{1}{12}}{100}$ ≈ 17.72
5. Calculate the total value of the loan before Francis's second payment.	\$2744.38 + \$17.72 = \$2762.10
6. Calculate the total value of the loan after Francis's second payment.	\$2762.10 − \$275 = \$2487.10
7. Calculate the interest payable for the third month of the loan.	$I = \frac{PrT}{100}$ $= \frac{2487.1 \times 7.75 \times \frac{1}{12}}{100}$ ≈ 16.06
8. Calculate the total value of the loan before Francis's third payment.	\$2487.10 + \$16.06 = \$2503.16
9. Calculate the total value of the loan after Francis's third payment.	\$2503.16 − \$275 = \$2228.16
10. Write the answer.	The outstanding balance of the loan after Francis's third payment is \$2228.16.

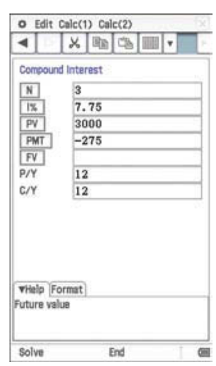
TI | THINK **WRITE**

1. On the Calculator page, press MENU then select:
8: Finance
1: Finance Solver …
Complete the fields as:
N: 3
I(%): 7.75
PV: 3000
Pmt: −275
PpY: 12
CpY: 12 then move the cursor to the FV field and press ENTER.

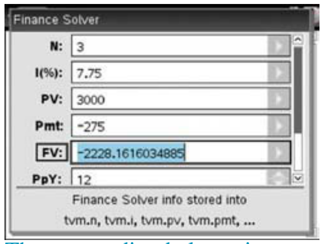


CASIO | THINK **WRITE**

1. On the Financial screen, select:
- Calc(1)
- Compound Interest
Complete the fields as:
N: 3
I%: 7.75
PV: 3000
PMT: −275
P/Y: 12
C/Y: 12 then click the FV icon.



2. The outstanding balance can be read from the screen.



The outstanding balance is \$2228.16.

2. The outstanding balance can be read from the screen.

The outstanding balance is \$2228.16.

3.6.5 Time payments (hire purchase)

A **time payment**, or hire purchase, can be used when a customer wants to make a large purchase but doesn't have the means to pay up front. Time payments usually work by paying a small amount up front, and then paying weekly or monthly instalments.

3.6.6 The effective rate of interest

Note: In the VCE Further Mathematics course you will learn a different, unrelated effective annual interest rate formula. That formula does not apply here, and this formula is not for use in the VCE Further Mathematics course.

The interest rate of a time payment can be determined by using the simple interest formula. However, the actual interest rate will be higher than that calculated, as these calculations don't take into account the reducing balance owing after each payment has been made.

The **effective rate of interest** can be used to give a more accurate picture of how much interest is actually charged on time payments. To determine this we can use the following formula:

$$R_{ef} = \frac{2400I}{P(m+1)}$$

where R_{ef} is the effective rate of interest, I is the total interest paid, P is the principal (the cash price minus the deposit) and m is the number of monthly payments.

WORKED EXAMPLE 21

A furniture store offers its customers the option of purchasing a \$2999 bed and mattress by paying \$500 up front, followed by 12 monthly payments of \$230.

a. How much does a customer pay in total if they choose the offered time payment plan?

b. What is the effective rate of interest for the time payment plan correct to 2 decimal places?

THINK	WRITE
a. 1. Determine the total amount to be paid under the time payment plan.	a. $\text{Total payment} = 500 + 12 \times 230$ $= 500 + 2760$ $= 3260$

2. Write the answer.	The total amount paid under the time payment plan is \$3260.
b. 1. Calculate the total amount of interest paid.	**b.** $I = 3260 - 2999$ $= 261$
2. Calculate the principal (the cash price minus the deposit).	$P = 2999 - 500$ $= 2499$
3. Identify the components of the formula for the effective rate of interest.	$I = 261$ $P = 2499$ $m = 12$
4. Substitute the information into the formula and determine the effective rate of interest.	$R_{ef} = \dfrac{2400I}{2499(m+1)}$ $= \dfrac{2400 \times 261}{2499(12+1)}$ $= 19.28\%$ (to 2 decimal places)
5. Write the answer.	The effective rate of interest for the time purchase plan is 19.28%.

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Units 1 & 2 > AOS 2 > Topic 2 > Concept 6

Purchasing options Concept summary and practice questions

Exercise 3.6 Purchasing options

Unless otherwise directed, where appropriate give all answers to the following questions correct to 2 decimal places or the nearest cent.

1. **WE18** An electrician offers a discount of 7.5% if his customers pay by cash. How much will his customers pay in cash if the charge before the discount being applied is:

a. \$200 **b.** \$312 **c.** \$126?

2. George runs a pet-care service where he looks after cats and dogs on weekend afternoons. He charges a fee of \$20 per pet plus \$9 per hour. He also gives his customers a 6% discount if they pay in cash.
Charlene asks George to look after her two cats between 1 pm and 5 pm on a Saturday afternoon. How much would she have to pay if she paid in cash?

a. \$33.85 **b.** \$52.65 **c.** \$71.45
d. \$72.95 **e.** \$73.85

3. **WE19** Barney is struggling to keep control of his finances and starts to use his credit card to pay for purchases. At the end of three consecutive months his outstanding credit card balance is \$311.55, \$494.44 and \$639.70 respectively. If the interest rate on Barney's credit card is 22.75% p.a., calculate how much interest he is charged for the three-month period.
4. Dawn uses her credit card while on an overseas trip and returns with an outstanding balance of \$2365.24 on it. Dawn can only afford to pay the minimum monthly balance of \$70.96 off her credit card before the interest-free period expires.
   a. Dawn's credit card charges an interest rate of 24.28% p.a. How much will Dawn be charged in interest for the next month?
   b. If Dawn spent \$500 less on her overseas trip, by how much would the interest she would be charged on her credit card be reduced? (*Note:* Assume that Dawn still pays \$70.96 off her credit card.)

5. **WE20** Petra takes out a loan of \$5500 to help pay for a business management course. The loan has a fixed interest rate of 6.85% p.a. and Petra agrees to pay back \$425 a month. Assuming that the interest is calculated before Petra's payments, calculate the outstanding balance on the loan after her third payment.
6. Calculate the total amount of interest paid on a \$2500 personal loan if the rate is 5.5% p.a. and \$450 is paid off the loan each month. (Assume that the interest is calculated before the monthly payments.)
7. Shawna takes out a personal loan of \$4000 to help support her brother in a new business venture. The loan has a fixed interest rate of 9.15% calculated on the reduced monthly balance, and Shawna agrees to pay \$400 back per month.
   a. How much interest will Shawna pay over the lifetime of the loan?
   Shawna's brother's business goes well, and he is able to pay her back the \$4000 after 1 year with 30% interest.
   b. How much does in total does Shawna earn from taking out the loan?
8. **WE21** A car dealership offers its customers the option of purchasing a \$13 500 car by paying \$2500 up front, followed by 36 monthly payments of \$360.
   a. How much does a customer pay in total if they choose the time payment plan?
   b. What is the effective rate of interest for the time payment plan?
9. Georgie is comparing purchasing plans for the latest 4K television. The recommended retail price of the television is \$3500. She goes to three stores and they offer her the following time payment plans.
   - Store 1: \$250 up front + 12 monthly payments of \$300
   - Store 2: 24 monthly payments of \$165
   - Store 3: \$500 up front + 6 monthly payment of \$540

   a. Calculate the total amount payable for each purchase plan.
   b. Which purchase plan has the lowest effective rate of interest?

10. A new outdoor furniture set normally priced at $1599 is sold for an up-front fee of $300 plus 6 monthly instalments of $240. The effective rate of interest is:

a. 27.78% **b.** 30.23% **c.** 31.51% **d.** 37.22% **e.** 43.42%

11. A car is purchased with a deposit of $1500, which is 10% of the cash purchase price, followed by three annual instalments of $6000.

a. What is the total interest that is charged over the 3 years to purchase the car this way?

b. Ignoring the effect of the annual payments on the balance owed, use the total interest for the 3 years, the total instalments and the amount that was borrowed (i.e. the cash price less the deposit) to calculate the annual rate of compound interest.

12. Drew has a leak in his water system and gets quotes from 5 different plumbers to try to find the best price for the job. From previous experience he believes it will take a plumber 90 minutes to fix his system. Calculate approximately how much each plumber will charge to help Drew decide which to go with.

- Plumber A: A call-out fee of $100 plus an hourly charge of $80, with a 5% discount for payment in cash
- Plumber B: A flat fee of $200 with no discount
- Plumber C: An hourly fee of $130, with a 10% discount for payment in cash
- Plumber D: A call-out fee of $70 plus an hourly fee of $90, with an 8% discount for payment in cash
- Plumber E: An hourly fee of $120 with no discount

13. Items in an online store advertised for more than $100 can be purchased for a 12.5% deposit, with the balance payable 9 months later at a rate of 7.5% p.a. compounding monthly. How much do the following items cost the purchaser under this arrangement?

a. A sewing machine advertised at $150

b. A portable air conditioner advertised at $550

c. A treadmill advertised at $285

d. A BBQ advertised at $675

14. Divya's credit card has a low interest rate of 13.55% p.a. but has no interest-free period on purchases. Calculate the total interest she has to pay after making the following purchases.

- New sound system — $499 — paid back after 7 days
- 3 Blu-Ray films — $39 — paid back after 12 days
- Food shopping — $56 — paid back after 2 days
- Coffee machine — $85 — paid back after 18 days

15. An electrical goods store allows purchasers to buy any item priced at $1000 or more for a 10% deposit, with the balance payable 6 months later at a simple interest rate of 7.64% p.a. Find the final cost of each of the following items under this arrangement.
   a. An entertainment system priced at $1265
   b. An oven priced at $1450
   c. A refrigerator priced at $2018
   d. A washing machine priced at $3124

16. Elise gets a new credit card that has an annual fee of $100 and earns 1 frequent flyer point per $1 spent. In her first year using the card she spends $27 500 and has to pay $163 in interest on the card. Elise exchanged the frequent flyer points for a gift card to her favourite store which values each, point as being worth 0.8 cents. Was using the credit card over the year a profitable investment?
17. Michelle uses all of the $12 000 in her savings account to buy a new car worth $25 000 on a time payment scheme. The purchase also requires 24 monthly payments of $750.
   a. How much does Michelle pay in total for the car?
   Michelle gets a credit card to help with her cash flow during this 24-month period, and over this time her credit card balance averages $215 per month. The credit card has an interest rate of 23.75% p.a.
   b. How much in interest does Michelle pay on her credit card over this period?
   c. In another 18 months Michelle could have saved the additional $13 000 she needed to buy the car outright. How much would she have saved by choosing to save this money first?
18. Javier purchases a new kitchen on a time payment plan. The kitchen usually retails for $24 500, but instead Javier pays an up-front fee of $5000 plus 30 monthly instalments of $820.

   a. How much does Javier pay in total?
   b. What is the effective rate of interest of the time payment plan?
   If Javier paid an up-front fee of $10 000, he would only have to make 24 monthly instalments of $710.
   c. How much would Javier save by going for the second plan?
   d. What is the effective rate of interest of the second plan?
19. a. Using CAS, calculate the time it will take to pay back a $10 000 loan with an interest rate of 6.55% p.a. on a reducing monthly balance when paying back $560 per month.
   b. How much interest is payable over the lifetime of the loan?
20. Kara takes out a $12 000 loan to invest in the stock of a friend's company. The loan has an interest rate of 7.24% on a reducing monthly balance. Kara pays $720 per month.
   a. Using CAS, calculate the total interest that Kara has to pay over the lifetime of the loan.
   b. The stock that Kara invests in grows at a rate of 9.35% p.a. for the first 3 years of Kara's investment. How much did she earn in 3 years, taking into account the interest payable on the loan she took out?

3.7 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. **MC** If the price of petrol increased from 118.4 cents to 130.9 cents, the percentage change is:
   A. 10.6% **B.** 9.5% **C.** 90% **D.** 1.1% **E.** 12.5%
2. **MC** A basketball ring is sold for \$28.50. If this represents a 24% reduction from the RRP, the original price was:
   A. \$90.25 **B.** \$118.75 **C.** \$52.50 **D.** \$37.50 **E.** \$26.67
3. **MC** A company has 643 165 shares. When \$958 300 of its annual profit is distributed to the shareholders, the dividend payable per share is:
   A. \$0.67 **B.** \$1.49 **C.** \$6.71 **D.** \$0.49 **E.** \$14.90
4. **MC** How many shares are in a company that declares a dividend of 32 cents per share when \$450 000 of its annual profit is distributed?
   A. 144 000 **B.** 14 063 **C.** 6617 **D.** 30 600 000 **E.** 1 406 250
5. **MC** The price-to-earnings ratio for a company with a share price of \$2.40 and a dividend of 87 cents is:
   A. 2.09 **B.** 2.76 **C.** 0.03 **D.** 3.27 **E.** 0.3625
6. **MC** When the simple interest formula is transposed to find r, the correct formula is:
   A. $r = 100IPT$ **B.** $r = \frac{PT}{100I}$ **C.** $r = \frac{100}{PTI}$ **D.** $r = \frac{100I}{PT}$ **E.** $r = \frac{100IP}{T}$
7. **MC** Which of the following companies has the lowest share price?
   A. Company A with a price-to-earnings ratio of 10.4 and a dividend of \$1.87
   B. Company B with a price-to-earnings ratio of 28.1 and a dividend of 36 cents
   C. Company C with a price-to-earnings ratio of 14.8 and a dividend of 79 cents
   D. Company D with a price-to-earnings ratio of 35.75 and a dividend of 97 cents
   E. Company E with a price-to-earnings ratio of 17.7 and a dividend of \$1.33
8. **MC** A tradesman offers a 6.8% discount for customers who pay in cash. How much would a customer pay if they paid their bill of \$244 in cash?
   A. \$16.59 **B.** \$218.48 **C.** \$227.41 **D.** \$261.59 **E.** \$237.20
9. **MC** A new racing bike priced at \$3600 is sold for an up-front payment fee of \$300 plus 15 monthly instalments of \$280. The effective rate of interest is:
   A. 14.36% **B.** 40.91% **C.** 69.94% **D.** 204.61% **E.** 25%
10. **MC** Meredith walks dogs at the weekend. She charges \$14.00 per dog plus \$6.00 an hour. She offers her clients a 5% discount for paying in cash. How much would she charge for someone paying cash to walk 3 dogs for 2 hours?
    A. \$51.30 **B.** \$131.10 **C.** \$54 **D.** \$2.70 **E.** \$48

Short answer

1. Complete the following percentage changes.
   a. Increase \$65.85 by 12.6%. **b.** Decrease \$14.56 by 23.4%.
   c. Increase \$150.50 by 2.83%. **d.** Decrease \$453.25 by 0.65%.
2. Calculate the percentage dividend for the following shares, rounding your answer correct to 2 decimal places.
   a. A share price of \$16.88 with a dividend of \$1.15
   b. A share price of \$21.26 with a dividend of 38 cents
   c. A share price of \$34.78 with a dividend of \$2.45
3. Determine the GST that's included in or needs to be added to the price for these amounts.
   a. \$45.50 with GST included **b.** \$109.00 plus GST
   c. \$448.75 with GST included **d.** \$13.25 plus GST

4. Determine the unknown variable for each of the following scenarios.
   a. Calculate the amount of simple interest earned on an investment of $4500 that returns 6.87% per annum for 5.5 years.
   b. How long will it take an investment of $1260 to earn $350 with a simple interest rate of 4.08%?
   c. What is the simple interest rate on an investment that earns $645 in 3 years when the initial principal was $5300?
   d. What are the monthly repayments for a $6250 loan that is charged simple interest at a rate of 9.32% per annum for 7.25 years?
5. Use the compound interest formula to find:
   a. the amount of interest on an investment of $3655 at 6.54% per annum for 2.5 years
   b. the future amount of $478 invested at 2.27% per annum for 10 years
   c. the compound interest rate per annum required to grow $1640 to $3550 in 3.25 years
   d. the principal required to yield a final amount of $22 000 after compounding at a rate of 11.2% per annum for 15 years.
6. Sophie bought an investment property for $250 000, and 4 years later she sold it for $275 000. If the average annual inflation was 2.82% per annum, was this a profitable investment for Sophie?

Extended response

1. Four years ago a business was for sale at $130 000. Amanda and Callan had the money to purchase the business but missed out at the auction. Four years later the business is again for sale, but now at $185 000.
   a. Determine the percentage increase in the price over the 4 years.
   b. Amanda and Callan will now need to borrow the increase in the price amount. How much interest will they have to pay on a loan compounded annually over 5 years with a rate of 12.75%?
2. Manny has received three quotes for the painting of her house:
   - Painter 1: $290 per day (4 days) plus $20 per litre of paint (120 litres needed), no further discounts
   - Painter 2: A flat fee of $4000 with a 10% discount for cash
   - Painter 3: An hourly fee of $80 (7 hours a day for 3 days) plus $25 per litre of paint (120 litres needed) with a 5% discount for cash.
   a. Calculate the cost of each painter.
   b. Based on price, which painter should Manny select?
   To pay for the painter, Manny withdraws $4000 from her bank account. The account had an opening balance of $6000 and earns a simple interest rate of 12.4%. The interest is calculated on the minimum monthly balance.

Date	Details	Amount
1st	Withdrawal	$4000
10th	Deposit	$151
15th	Withdrawal	$220
22nd	Deposit	$1500
29th	Withdrawal	$50
30th	Deposit	$250

   c. Use the account information to calculate the amount of interest earned on the account this month.
   d. Determine the percentage change of the account balance from the start to the end of the month.
3. Lorna is purchasing $1500 worth of furniture for her new house. She can make the purchase using the store's time payment plan with a deposit of $400 and 12 monthly repayments of $105.
   a. How much will Lorna end up paying for the furniture?

b. What is the effective rate of interest for the time payment plan?
   Alternatively, Lorna could put the purchase on her credit card with an interest rate of 15.66% p.a. and no interest-free period. She will repay $450 per month.
c. How long would it take her to repay the full amount?
d. Based on the amount of interest payable, which payment option should Lorna select?

4. Adam purchased 1600 shares in a company that recently announced it had achieved an annual profit of $890 500. The company has 200 000 shareholders.
   a. Determine the dividend payable to Adam.
   b. He originally purchased the shares for $2.17 each. Calculate the percentage increase in the value of the shares.
   c. Adam decides to reinvest the dividend payment in a compound interest account at 14.4% p.a. compounded weekly. How much interest will he earn if he invests for 1.5 years?

studyon

Units 1&2 **Sit topic test**

Answers

Topic 3 Financial arithmetic

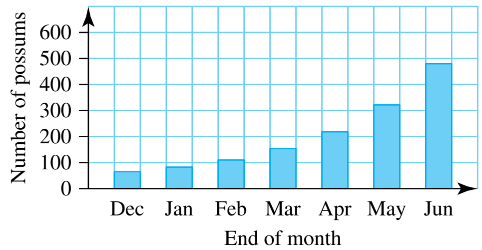
Exercise 3.2 Percentage change

1. a. 22.64% increase b. 56.60% increase
 c. 26.42% decrease d. 13.96% decrease
2. a. 2.70% decrease b. 18.25% decrease
 c. 130.77% increase d. 12.5% decrease
3. a. i. 0.62% ii. 2.5% b. 1.23%
4. The first car has the largest percentage reduction at 20.05%.
5. a. $37.80 b. $108
 c. $174.49 d. $42 304.22
6. a. $45.36 b. $7.41
 c. $69.31 d. $12 037.52
7. a. $18.92 b. 5.16% decrease
8. a. 12.5% b. 16.88%
9. $150
10. a. Perth has the largest increase of 72.16%.
 b. Sydney has the smallest percentage change with a decrease of 14.34%.
11. An overall increase of 9%
12. 30%
13. a. 18% b. $1337.33
14. a. 0.78% decrease b. 44.31% increase
 c. 29.15% increase
15. a.

Year	Annual salary	Percentage change
2013	$34 000	
2014	$35 750	5.15%
2015	$38 545	7.82%
2016	$42 280	9.69%
2017	$46 000	8.80%

 b. 2016
16. a.

End of month	Number of possums
Dec	65
Jan	83
Feb	110
Mar	153
Apr	219
May	321
Jun	480



 b.

End of month	Number of possums	Percentage change
Dec	65	
Jan	83	27.7%
Feb	110	32.5%
Mar	153	39.1%
Apr	219	43.1%
May	321	46.6%
Jun	480	49.5%

Exercise 3.3 Financial applications of ratios and percentages

1. a. 6 cents/share b. 13 cents/share
 c. $31.50/share
2. a. 3 655 944 shares b. 1 395 444 shares
 c. 592 840 shares d. 429 784 shares
3. a. 6.37% b. 3.89%
 c. 5.14% d. 8.62%
4. D
5. a. 34 cents/share b. $1.20/share
 c. 83 cents/share d. $3.92/share
6. a. 50 b. 34.88
 c. 24.07 d. 15.24
7. a. $15.23 b. $29.81
 c. $27.46 d. $40.70
8. a. 6.82% b. 15.22%
9. 3.11%

10. a. $3.18 b. $53.35
c. $5124.35 d. $0.21

11. Company A by $167

12. a. $602.80 b. 13.59%

13. a. $4.28 b. 10.13

14. a. $4227 b. 36.17%

15. a. Company A: $1.16; Company B: $0.19 b. Company A: 29.74; Company B: 7.79
c. $770 d. Company B

16. a. 10 506 240 rand b. 16.32

17. a. $1.56 b. $1.38
c. $27.86

18. a. $36 857.15 b. 5.04%

19. a., b.

Year	Net profit	Price-to-earnings ratio
2012	$26 615 384.62	25.72
2013	$50 000 000.00	12.25
2014	$42 153 846.15	15.04
2015	$48 461 538.46	10.10
2016	$52 615 384.62	8.65

b. 2016

20. $0.84

21. a. Company A: 15.99, Company B: 15.86, Company C: 17.23, Company D: 15.34, Company E: 19.66
b. Company A: 6.25%, Company B: 6.31%, Company C: 5.80%, Company D: 6.52%, Company E: 5.09%

22. a. See table at the foot of the page*
b. Normal retail price $\times 0.45$
c. 48.57%

Exercise 3.4 Simple interest applications

1. a. $849.75 b. $4100.69
c. $44 231.34 d. $18 335.62

2. a. $553.25 b. $2725.76
c. $7069.34 d. $10 740.84

3. a. 10 years b. 20 years
c. 8 years d. 4 years

4. a. 5.9% b. 12%
c. $1600 d. $10 000

5. a. $303.89 b. $613.65
c. $486.50 d. $1254.96

6. a. $243.94 b. $471.68

7. 5.4%

8. a. $3.67 b. $4.79
c. $9.78 d. $109.96

9. a. $3.53 b. $709.55
c. $83.18 d. $41.37

10. a. $0.70 b. $0.16

11. a. $0.51 b. $4.30

12. In the 8th year

13. a. $1590 b. $32 950
c. 3.11%

14. a. 4.42% b. $4117.65
c. $2.36

15. a. $506.46 b. 3.77%

16. a. $8000 b. $75.28

17. a. $5253.89 b. The first investment is the best (7.8% p.a.).

18. a. 12.02% b. $52 300
c. 12.45%

19. a. $224.01 b. July: 40.52% reduction, August: 3.30% reduction, September: 31.74% reduction

20. a. 0.02% per day b. $41.33

21. a.

Date	Balance
01/01/16	$1200
12/01/16	$1450
03/03/16	$1030
14/04/16	$925
25/05/16	$1190
09/09/16	$1315

b. $26.53

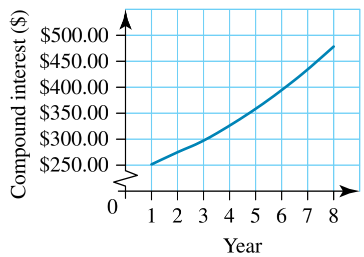
22. a. 42% p.a. b. $28.50

22. a. *

Item	Cost price	Normal retail price (255% mark-up)	Standard discount (12.5% mark-down of normal retail price)	January sale (32.25% mark-down of normal retail price)	Stocktake sale (55% mark-down of normal retail price)
Socks	$1.85	$6.57	$5.75	$4.45	$2.96
Shirts	$12.35	$43.84	$38.36	$29.70	$19.73
Trousers	$22.25	$78.99	$69.12	$53.52	$35.55
Skirts	$24.45	$86.80	$75.95	$58.81	$39.06
Jackets	$32.05	$113.78	$99.56	$77.09	$51.20
Ties	$5.65	$20.06	$17.55	$13.59	$9.03
Jumpers	$19.95	$70.82	$61.97	$47.98	$31.87

Exercise 3.5 Compound interest applications

1. a. \$1268.15 b. \$797.37
   c. \$538.82 p
2. \$1631.94
3. a. \$664.76 b. \$5515.98
   c. \$599.58 d. \$1 229 312.85
4. a. \$385.02 b. \$1403.52
   c. \$4935.59 d. \$720 300.86
5. a. \$9961.26 b. \$11 278.74
   c. \$975.46 d. \$32 542.37
6. a. 41.42% b. 43.28%
   c. 16.03% d. 37.13%
7. a. \$184.93 b. \$4777.22
   c. \$3.68 d. \$4526.95
8. a. \$708.47 b. \$5128.17
   c. \$118 035.38 d. \$55 774.84
9. a. \$2021.81 b. \$2101.50
   c. \$2107.38 d. \$2108.90
10. The inflated value is \$371 851.73, so it was not profitable.
11. The inflated value is \$186 538.32, so it is profitable.
12. a. See table at the foot of the page*
    b.



13. The second option will be worth more after 23 years.
14. 1140.57% p.a.
15. a. 12.45% p.a. b. \$1061.19
16. a. i. \$5654.46
       ii. \$5687.14
       iii. \$5620.56
    b. Shivani should invest her money with the building society.
17. a. Year 1: \$180, Year 2: \$162, Year 3: \$145.80
    b. Year 1: \$185.40, Year 2: \$171.86, Year 3: \$159.32
18. 2016
19. a. The series of prints
    b. Sculpture: \$41 032, prints: \$28 775
20. See table at the foot of the page*
21. a. See table at the foot of the page*
    b. i. 12 years ii. 19 years iii. 24 years

---

12. a. *

Year	1	2	3	4	5	6	7	8
Interest accrued (\$)	250.90	275.11	301.66	330.77	362.69	397.69	436.07	478.15

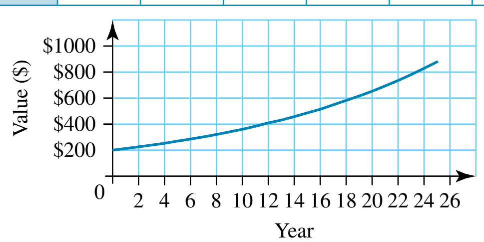
20. *

Principal (\$)	Final amount (\$)	Interest earned (\$)	Interest rat (p.a.)	Number of years
11 000	12 012.28	**1012.28**	**4.5**	2
14 000	**15 409.84**	**1409.84**	3.25	3
22 050	25 561.99	3511.99	**3**	5
108 000	**110 070.00**	2700.00	2.5	1

21. a. *

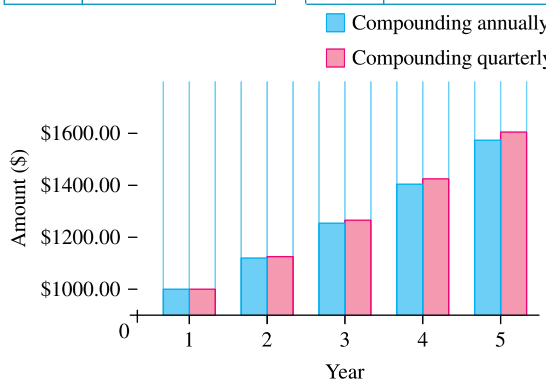
Year	0	1	2	3	4	5	6	7	8	9	10	11	12
Value (\$)	200.00	212.20	225.14	238.88	253.45	268.91	285.31	302.72	321.18	340.78	361.56	383.62	407.02

Year	13	14	15	16	17	18	19	20	21	22	23	24	25
Value (\$)	431.85	458.19	486.14	515.79	547.26	580.64	616.06	653.64	693.51	735.81	780.70	828.32	878.85



22. a.

Compounding annually		Compounding quarterly	
Year	**Amount**	**Year**	**Amount**
1	$1000.00	1	$1000.00
		2	$1030.00
		3	$1060.90
		4	$1092.73
2	$1120.00	5	$1125.51
		6	$1159.27
		7	$1194.05
		8	$1229.87
3	$1254.40	9	$1266.77
		10	$1304.77
		11	$1343.92
		12	$1384.23
4	$1404.93	13	$1425.76
		14	$1468.53
		15	$1512.59
		16	$1557.97
5	$1573.52	17	$1604.71
		18	$1652.85
		19	$1702.43
		20	$1753.51



Note: The graph shows the amounts at the beginning of each year.

b. Compounding at regular intervals during the year accumulates more interest than compounding only once a year.

Exercise 3.6 Purchasing options

1. a. $185 b. $288.60 c. $116.55
2. C
3. $27.41
4. a. $46.42 b. $10.12
5. $4312.44
6. $38.42
7. a. $176.94 b. $1023.06
8. a. $15 460 b. 11.56%
9. a. Store 1: $3850; Store 2: $3960; Store 3: $3740 b. Store 2
10. D
11. a. $4500 b. 10.06%
12. Plumber C
13. a. $157.57 b. $577.76 c. $299.38 d. $709.07
14. $2.08
15. a. $1308.49 b. $1499.85 c. $2087.38 d. $3231.40
16. No, Elise loses $43.
17. a. $30 000 b. $102.13 c. $5102.13
18. a. $29 600 b. 20.25% c. $2560 d. 16.82%
19. a. 19 months b. $540.42
20. a. $685.72 b. $3004.81

3.7 Review: exam practice

Multiple choice

1. A	2. D	3. B	4. E	5. B
6. D	7. B	8. C	9. B	10. A

Short answer

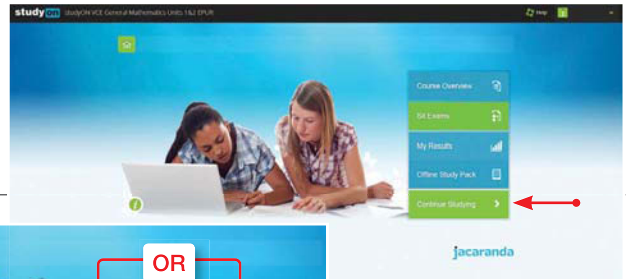
1. a. $74.15 b. $11.15 c. $154.76 d. $450.30
2. a. 6.81% b. 1.79% c. 7.04%
3. a. $4.14 b. $10.90 c. $40.80 d. $1.33
4. a. $1700.33 b. 7 years c. 4.06% d. $120.38
5. a. $627.22 b. $598.29 c. 26.82% d. $4475.60
6. In real terms she made a loss of $4415.44 when inflation is taken into account.

Extended response

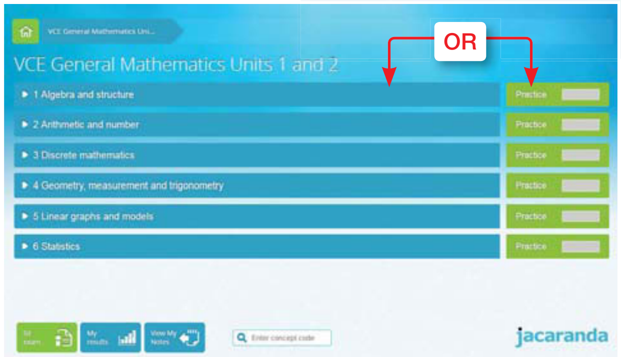
1. a. 42.31% b. $45 217.93
2. a. Painter 1: $3270; Painter 2: $3600; Painter 3: $4446 b. Painter 1 c. $19.95 d. 39.48%
3. a. $1660 b. 26.85% c. 4 months d. Her credit card
4. a. $7120 b. 205.07% c. $8834.01

TOPICS 2 and 3

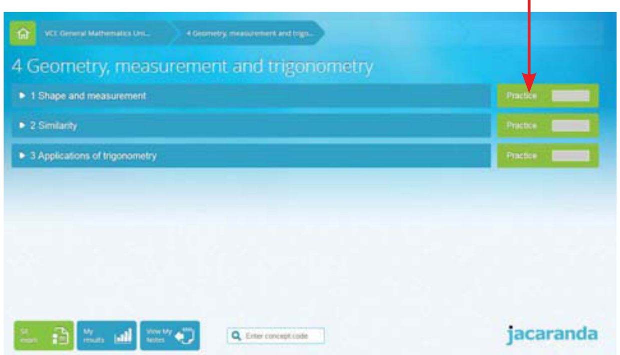
- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



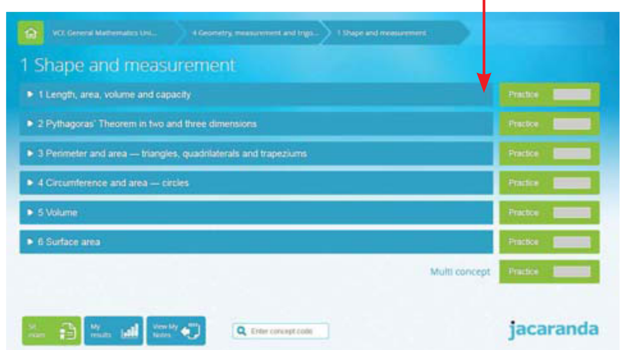
- Select your **course** *VCE General Mathematics Units 1 & 2* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.



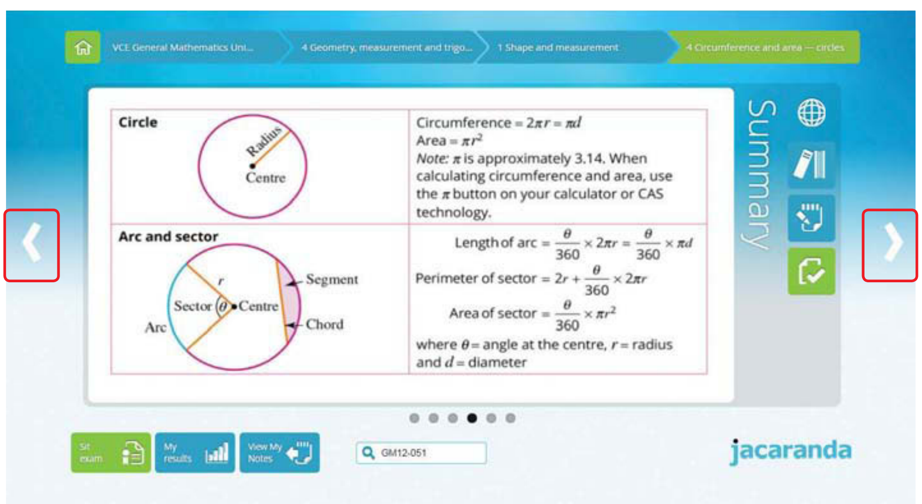
- Select **Practice** at the topic level to access all questions in the topic.



- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 4
Matrices

4.1 Overview

4.1.1 Introduction

Matrices are used in the study of solutions of linear simultaneous systems. For example, if you have two equations with two unknowns, you can use matrices to solve the two unknowns.

The Chinese were the first to demonstrate the use of matrices in the second century BC, with the publication of the *Nine Chapters on the Mathematical Art*. There was further development, but it was not until 1683 that the idea of a determinant appeared in Japan with Seki Takakazu's *Method of Solving the Dissimulated Problems*. This work used matrix methods in tables in the same way as the earlier work of the Chinese. The determinant first appeared in Europe only 10 years later, in work by Leibniz.

The actual word *determinant* was first introduced by Gauss in 1801 while discussing quadratic forms, but Cauchy was the first to use it in the modern sense in 1812.

The term *matrix* was used for the first time in 1850, by Sylvester. He defined the matrix as an oblong arrangement of terms which led to various determinants. Sylvester worked with Cayley, who published *A Memoir on the Theory of Matrices* in 1858.

There has been continual development in the field of matrices and today they are used in a vast array of important fields. It was Olga Taussky-Todd who described herself as a torchbearer for matrix theory after her important work using matrices to analyse vibrations of airplanes during World War II, at the National Physical Laboratory in the United Kingdom. More recently, matrices are used to describe the quantum mechanics of atomic structure, in designing the graphics for computer games, like the image above, and even in plotting complicated dance steps.

LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

4.1.2 Kick off with CAS

Using CAS to work with matrices

1. a. Define each of the following matrices using CAS.

$$A = \begin{bmatrix} -2 & 3 \\ -2 & 4 \end{bmatrix},\ B = \begin{bmatrix} 3 & -2 \\ 1 & -2 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   b. Using the matrices defined in part a, calculate:

   i. $5A$ ii. $2B$ iii. $2A + 3B$

   iv. $\det A$ v. B^{-1} vi. BI.

2. a. Define the following matrices using CAS.

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

   b. Using the matrices defined in part a, find X if:

   i. $AX = B$ ii. $XA = B$.

3. Solve the simultaneous equations

$$\begin{aligned} 9x + 10y &= 153 \\ 3x - y &= 12 \end{aligned}$$

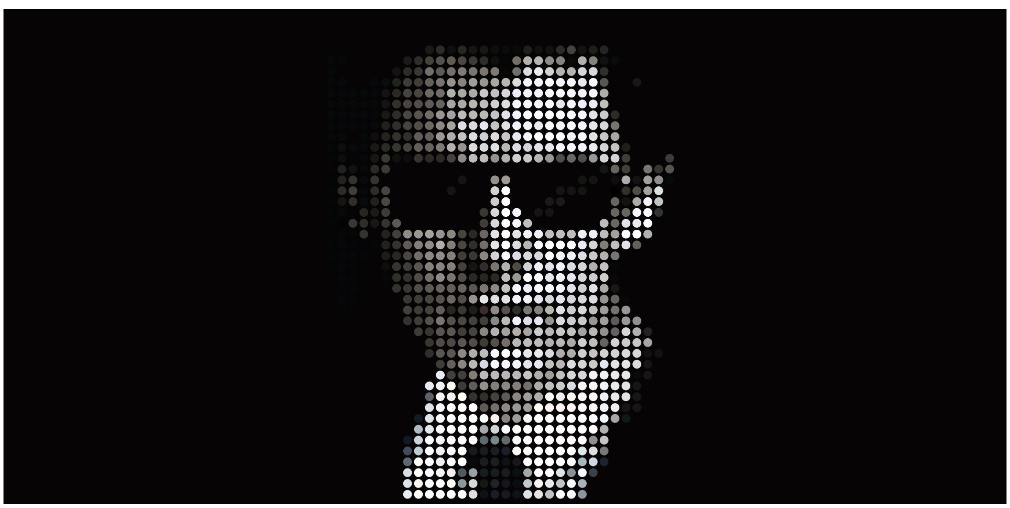
by setting up the simultaneous equations in the form

$$\begin{bmatrix} 9 & 10 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 153 \\ 12 \end{bmatrix}$$

and completing the following steps.

   a. Define $A = \begin{bmatrix} 9 & 10 \\ 3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 153 \\ 12 \end{bmatrix}$ using CAS.

   b. Using the matrices defined in part a, solve the equation $AX = B$ for X.



on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

4.2 Types of matrices

4.2.1 Matrices

A **matrix** is a rectangular array of rows and columns that is used to store and display information. Matrices can be used to represent many different types of information, such as the models of cars sold in different car dealerships, the migration of people to different countries and the shopping habits of customers at different department stores. Matrices also play an important role in encryption. Before sending important information, programmers encrypt or code messages using matrices; the people receiving the information will then use inverse matrices as the key to decode the message. Engineers, scientists and project managers also use matrices to help them to perform various everyday tasks.

4.2.2 Describing matrices

A matrix is usually displayed in square brackets with no borders between the rows and columns.

The table below left shows the number of participants attending three different dance classes (hip-hop, salsa and bachata) over the two days of a weekend. The matrix below displays the information presented in the table.

Number of participants attending the dance classes:

	Saturday	Sunday
Hip-hop	9	13
Salsa	12	8
Bachata	16	14

Matrix displaying the number of participants attending the dance classes:

$$\begin{bmatrix} 9 & 13 \\ 12 & 8 \\ 16 & 14 \end{bmatrix}$$

WORKED EXAMPLE 1

The table below shows the number of adults and children who attended three different events over the school holidays. Construct a matrix to represent this information.

	Circus	Zoo	Show
Adults	140	58	85
Children	200	125	150

THINK

WRITE

1. A matrix is like a table that stores information. What information needs to be displayed?

The information to be displayed is the number of adults and children attending the three events: circus, zoo and show.

2. Write down how many adults and children attend each of the three events.

	Circus	Zoo	Show
Adults	140	58	85
Children	200	125	150

3. Write this information in a matrix. Remember to use square brackets.

$$\begin{bmatrix} 140 & 58 & 85 \\ 200 & 125 & 150 \end{bmatrix}$$

4.2.3 Networks

Matrices can also be used to display information about various types of **networks**, including road systems and social networks. The following matrix shows the links between a group of schoolmates on Facebook, with a 1 indicating that the two people are friends on Facebook and a 0 indicating that the two people aren't friends on Facebook.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

From this matrix you can see that the following people are friends with each other on Facebook:

- person A and person B
- person A and person C
- person B and person D
- person C and person D.

WORKED EXAMPLE 2

The distances, in kilometres, along three major roads between the Tasmanian towns Launceston (L), Hobart (H) and Devonport (D) are displayed in the matrix below.

$$\begin{array}{c} \\ \mathbf{H} \\ \mathbf{D} \\ \mathbf{L} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{H} & \mathbf{D} & \mathbf{L} \end{array} \\ \begin{bmatrix} \mathbf{0} & \mathbf{207} & \mathbf{160} \\ \mathbf{207} & \mathbf{0} & \mathbf{75} \\ \mathbf{160} & \mathbf{75} & \mathbf{0} \end{bmatrix} \end{array}$$

a. **What is the distance, in kilometres, between Devonport and Hobart?**

b. **Victor drove 75 km directly between two of the Tasmanian towns. Which two towns did he drive between?**

c. **The Goldstein family would like to drive from Hobart to Launceston, and then to Devonport. Determine the total distance in kilometres that they will travel.**

THINK	WRITE
a. 1. Reading the matrix, locate the first city or town, i.e. Devonport (D), on the top of the matrix.	**a.** $\begin{array}{c} \\ H \\ D \\ L \end{array}\begin{array}{c} \begin{array}{ccc} H & \downarrow D & L \end{array} \\ \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & 75 \\ 160 & 75 & 0 \end{bmatrix} \end{array}$
2. Locate the second city or town, i.e. Hobart (H), on the side of the matrix.	$\begin{array}{c} \\ \rightarrow H \\ D \\ L \end{array}\begin{array}{c} \begin{array}{ccc} H & \downarrow D & L \end{array} \\ \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & 75 \\ 160 & 75 & 0 \end{bmatrix} \end{array}$
3. The point where both arrows meet gives you the distance between the two towns.	207 km
b. 1. Locate the entry '75' in the matrix.	**b.** $\begin{array}{c} \\ H \\ D \\ L \end{array}\begin{array}{c} \begin{array}{ccc} H & D & L \end{array} \\ \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & \textcircled{75} \\ 160 & 75 & 0 \end{bmatrix} \end{array}$
2. Locate the column and row 'titles' (L and D) for that entry.	$\begin{array}{c} \\ H \\ D \leftarrow \\ L \end{array}\begin{array}{c} \begin{array}{ccc} H & D & L \uparrow \end{array} \\ \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & \textcircled{75} \\ 160 & 75 & 0 \end{bmatrix} \end{array}$
3. Refer to the title headings in the question.	Victor drove between Launceston and Devonport.
c. 1. Locate the first city or town, i.e. Hobart (H), on the top of the matrix and the second city or town, i.e. Launceston (L), on the side of the matrix.	**c.** $\begin{array}{c} \\ H \\ D \\ \rightarrow L \end{array}\begin{array}{c} \begin{array}{ccc} \downarrow H & D & L \end{array} \\ \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & 75 \\ 160 & 75 & 0 \end{bmatrix} \end{array}$
2. Where the row and column meet gives the distance between the two towns.	160 km
3. Determine the distance between the second city or town, i.e. Launceston, and the third city or town, i.e. Devonport.	$\begin{array}{c} \\ H \\ \rightarrow D \\ L \end{array}\begin{array}{c} \begin{array}{ccc} H & D & \downarrow L \end{array} \\ \begin{bmatrix} 0 & 207 & 160 \\ 207 & 0 & 75 \\ 160 & 75 & 0 \end{bmatrix} \end{array}$
4. Where the row and column meet gives the distance between the two towns.	75 km
5. Add the two distances together.	$160 + 75 = 235$ km

4.2.4 Defining matrices

The **order** of a matrix is defined by the number of rows, m, and number of columns, n, in the matrix.

Consider the following matrix, A.

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & -6 & 5 \end{bmatrix}$$

Matrix A has two rows and three columns, and its order is 2×3 (read as a 'two by three' matrix).

A matrix that has the same number of rows and columns is called a **square matrix**.

$$B = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$$

Matrix B has two rows and two columns and is a 2×2 square matrix.

A **row matrix** has only one row.

$$C = \begin{bmatrix} 3 & 7 & -4 \end{bmatrix}$$

Matrix C has only one row and is called a row matrix.

A **column matrix** has only one column.

$$D = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

Matrix D has only one column and is called a column matrix.

WORKED EXAMPLE 3

At High Vale College, 150 students are studying General Mathematics and 85 students are studying Mathematical Methods. Construct a column matrix to represent the number of students studying General Mathematics and Mathematical Methods, and state the order of the matrix.

THINK	WRITE
1. Read the question and highlight the key information.	150 students study General Mathematics. 85 students study Mathematical Methods.
2. Display this information in a column matrix.	$\begin{bmatrix} 150 \\ 85 \end{bmatrix}$
3. How many rows and columns are there in this matrix?	The order of the matrix is 2×1.

4.2.5 Elements of matrices

The entries in a matrix are called **elements**. The position of an element is described by the corresponding row and column. For example, a_{21} means the entry in the 2nd row and 1st column of matrix A, as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \circled{a_{21}} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

WORKED EXAMPLE 4

Write the element a_{23} for the matrix $A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & -6 & 5 \end{bmatrix}$.

THINK	WRITE
1. The element a_{23} means the element in the 2nd row and 3rd column. Draw lines through the 2nd row and 3rd column to help you identify this element.	$A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & -6 & 5 \end{bmatrix}$
2. Identify the number at which the lines cross over.	$a_{23} = 5$

4.2.6 Identity matrices

An **identity matrix**, I, is a square matrix in which all of the elements on the diagonal line from the top left to bottom right are 1s and all of the other elements are 0s.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are both identity matrices.

As you will see later in this topic, identity matrices are used to find inverse matrices, which help solve matrix equations.

4.2.7 The zero matrix

A **zero matrix**, 0, is a square matrix that consists entirely of '0' elements.

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an example of a zero matrix.

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Units 1 & 2 > AOS 3 > Topic 1 > Concept 1 & 2

Definition of a matrix Concept summary and practice questions

Naming and uses of matrices Concept summary and practice questions

Exercise 4.2 Types of matrices

1. **WE1** Cheap Auto sells three types of vehicles: cars, vans and motorbikes. They have two outlets at Valley Heights and Hill Vale. The number of vehicles in stock at each of the two outlets is shown in the table.

	Cars	Vans	Motorbikes
Valley Heights	18	12	8
Hill Vale	13	10	11

Construct a matrix to represent this information.

2. Newton and Isaacs played a match of tennis. Newton won the match in five sets with a final score of 6–2, 4–6, 7–6, 3–6, 6–4. Construct a matrix to represent this information.

3. WE2 The distance in kilometres between the towns Port Augusta (P), Coober Pedy (C) and Alice Springs (A) are displayed in the following matrix.

$$\begin{array}{c} \\ P \\ C \\ A \end{array}\begin{array}{c} \begin{array}{ccc} P & C & A \end{array} \\ \begin{bmatrix} 0 & 545 & 1225 \\ 545 & 0 & 688 \\ 1225 & 688 & 0 \end{bmatrix} \end{array}$$

a. Determine the distance in kilometres between Port Augusta and Coober Pedy.
b. Greg drove 688 km between two towns. Which two towns did he travel between?
c. A truck driver travels from Port Augusta to Coober Pedy, then onto Alice Springs. He then drives from Alice Springs directly to Port Augusta. Determine the total distance in kilometres that the truck driver travelled.

4. A one-way economy train fare between Melbourne Southern Cross Station and Canberra Kingston Station is $91.13. A one-way economy train fare between Sydney Central Station and Melbourne Southern Cross Station is $110.72, and a one-way economy train fare between Sydney Central Station and Canberra Kingston Station is $48.02.
a. Represent this information in a matrix.
b. Drew travelled from Sydney Central to Canberra Kingston Station, and then onto Melbourne Southern Cross. Determine how much, in dollars, he paid for the train fare.

5. WE3 An energy-saving store stocks shower water savers and energy-saving light globes. In one month they sold 45 shower water savers and 30 energy-saving light globes. Construct a column matrix to represent the number of shower water savers and energy-saving light globes sold during this month, and state the order of the matrix.

6. Happy Greens Golf Club held a three-day competition from Friday to Sunday. Participants were grouped into three different categories: experienced, beginner and club member. The table shows the total entries for each type of participant on each of the days of the competition.

Category	Friday	Saturday	Sunday
Experienced	19	23	30
Beginner	12	17	18
Club member	25	33	36

a. How many entries were received for the competition on Friday?
b. Calculate the total number of entries for the three day competition.
c. Construct a row matrix to represent the number of beginners participating in the competition for each of the three days.

7. Write the order of matrices A, B and C.

$$A = [3],\ B = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix},\ C = \begin{bmatrix} 4 & -2 \end{bmatrix}$$

8. Which of the following represent matrices? Justify your answers.

a. $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ b. $\begin{bmatrix} 4 & 0 \\ & 3 \end{bmatrix}$ c. $\begin{bmatrix} 4 & 5 \\ & 7 \end{bmatrix}$ d. $\begin{bmatrix} a & c & e & g \\ b & d & f & h \end{bmatrix}$

9. **WE4** Write down the value of the following elements for matrix D.

$$D = \begin{bmatrix} 4 & 5 & 0 \\ 2 & -1 & -3 \\ 1 & -2 & 6 \\ 0 & 3 & 7 \end{bmatrix}$$

a. d_{12} b. d_{33} c. d_{43}

10. Consider the matrix $E = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{4} \\ -1 & -\frac{1}{2} & -3 \end{bmatrix}$.

a. Explain why the element e_{24} does not exist.
b. Which element has a value of -3?
c. Nadia was asked to write down the value of element e_{12} and wrote -1. Explain Nadia's mistake and state the correct value of element e_{12}.

11. Matrices D and E are shown. Write down the value of the following elements.

$$D = \begin{bmatrix} 5 & 0 & 2 & -1 \\ 8 & 1 & 3 & 6 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.5 & 0.3 \\ 1.2 & 1.1 \\ 0.4 & 0.9 \end{bmatrix}$$

a. d_{23} b. d_{14} c. d_{22} d. e_{11} e. e_{32}

12. a. The following matrix represents an incomplete 3×3 *identity* matrix. Complete the matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & & 0 \\ & 0 & \end{bmatrix}$$

b. Construct a 2×2 zero matrix.

13. The elements in matrix H are shown below.

$h_{12} = 3$
$h_{11} = 4$
$h_{21} = -1$
$h_{31} = -4$
$h_{32} = 6$
$h_{22} = 7$

a. State the order of matrix H.
b. Construct matrix H.

14. The land area and population of each Australian state and territory were recorded and summarised in the table below.

State/territory	Land area (km^2)	Population (millions)
Australian Capital Territory	2 358	0.4
Queensland	1 727 200	4.2
New South Wales	801 428	6.8
Northern Territory	1 346 200	0.2
South Australia	984 000	1.6
Western Australia	2 529 875	2.1
Tasmanian	68 330	0.5
Victoria	227 600	5.2

a. Construct an 8×1 matrix that displays the population, in millions, of each state and territory in the order shown in the table.
b. Construct a row matrix that represents the land area of each of the states in ascending order.
c. Town planners place the information on land area, in km^2, and population, in millions, for the states New South Wales, Victoria and Queensland respectively in a matrix.
   i. State the order of this matrix.
   ii. Construct this matrix.

15. The estimated number of Indigenous Australians living in each state and territory in Australia in 2006 is shown in the following table.

State and territory	Number of Indigenous persons	% of population that is Indigenous
New South Wales	148 178	2.2
Victoria	30 839	0.6
Queensland	146 429	3.6
South Australia	26 044	1.7
Western Australia	77 928	3.8
Tasmania	16 900	3.4
Northern Territory	66 582	31.6
Australian Capital Territory	4 043	1.2

a. Construct an 8×2 matrix to represent this information.
b. Determine the total number of Indigenous persons living in the following states and territories in 2006:
   i. Northern Territory
   ii. Tasmania
   iii. Queensland, New South Wales and Victoria (combined).
c. Determine the total number of Indigenous persons who were estimated to be living in Australia in 2006.

16. AeroWings is a budget airline specialising in flights between four mining towns: Olympic Dam (O), Broken Hill (B), Dampier (D) and Mount Isa (M).
The cost of airfares (in dollars) to fly from the towns in the top row to the towns in the first column is shown in the matrix below.

$$
\text{To}\;
\begin{matrix} \\ O \\ B \\ D \\ M \end{matrix}
\overset{\text{From}}{\begin{matrix} O & B & D & M \\ \end{matrix}}
$$

$$
\text{To} \begin{matrix} O \\ B \\ D \\ M \end{matrix} \begin{bmatrix} 0 & 70 & 150 & 190 \\ 89 & 0 & 85 & 75 \\ 175 & 205 & 0 & 285 \\ 307 & 90 & 101 & 0 \end{bmatrix}
$$

a. In the context of this problem, explain the meaning of the zero entries.
b. Find the cost, in dollars, to fly from Olympic Dam to Dampier.
c. Yen paid \$101 for his airfare with AeroWings. At which town did he arrive?
d. AeroWings offers a 25% discount for passengers flying between Dampier and Mount Isa, and a 15% discount for passengers flying from Broken Hill to Olympic Dam. Construct another matrix that includes the discounted airfares (in dollars) between the four mining towns.

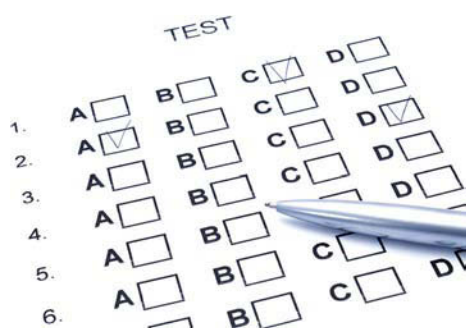
17. The matrix below displays the number of roads connecting five towns: Ross (R), Stanley (S), Thomastown (T), Edenhope (E) and Fairhaven (F).

$$
\begin{matrix} & R & S & T & E & F \end{matrix}
$$

$$
N = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} R \\ S \\ T \\ E \\ F \end{matrix}
$$

a. Construct a road map using the information shown.
b. Determine whether the following statements are true or false.
   i. There is a road loop at Stanley.
   ii. You can travel directly between Edenhope and Stanley.
   iii. There are two roads connecting Thomastown and Edenhope.
   iv. There are only three different ways to travel between Ross and Fairhaven.
c. A major flood washes away part of the road connecting Ross and Thomastown. Which elements in matrix N will need to be changed to reflect the new road conditions between the towns?

18. Mackenzie is sitting a Mathematics multiple choice test with ten questions. There are five possible responses for each question: A, B, C, D and E. She selects A for the first question and then determines the answers to the remaining questions using the following matrix.



$$\begin{array}{ccc} & & \text{This question} \\ & & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \text{Next question} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

a. Using the matrix above, what is Mackenzie's answer to question 2 on the test?

b. Write Mackenzie's responses to the remaining eight questions.

c. Explain why it is impossible for Mackenzie to have more than one answer with response A.

Mackenzie used another matrix to help her answer the multiple choice test. Her responses using this matrix are shown in this grid.

Question	1	2	3	4	5	6	7	8	9	10
Response	A	D	C	B	E	A	D	C	B	E

d. Complete the matrix that Mackenzie used for the test by finding the values of the missing elements.

$$\begin{array}{ccc} & & \text{This question} \\ & & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \text{Next question} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & \\ 0 & & & 0 & 0 \\ 0 & & & & 0 \\ 1 & 0 & 0 & & 1 \\ 0 & & 0 & 0 & \end{bmatrix} \end{array}$$

19. State the steps involved in constructing a matrix using CAS.

20. Matrix A was constructed using a spreadsheet.

$$A = \begin{bmatrix} 75 & 80 & 55 \\ 120 & 65 & 82 \\ 95 & 105 & 71 \end{bmatrix}$$

a. State the cell number for each of the following elements.

i. a_{13} ii. a_{22} iii. a_{32} iv. a_{21}

b. i. Explain how the element position can be used to locate the corresponding cell number in the spreadsheet.

ii. Using your response to bi, write down the cell number for any element e_{nm}.

4.3 Operations with matrices

4.3.1 Matrix addition and subtraction

Matrices can be added and subtracted using the same rules as in regular arithmetic. However, matrices can only be added and subtracted if they are the same order (that is, if they have the same number of rows and columns).

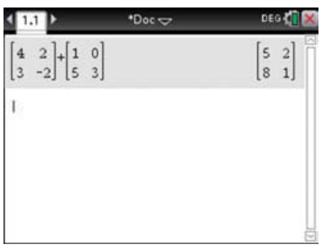
4.3.2 Adding matrices

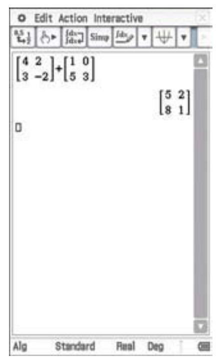
To add matrices, you need to add the corresponding elements of each matrix together (that is, the numbers in the same position).

WORKED EXAMPLE 5

If $A = \begin{bmatrix} 4 & 2 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix}$, find the value of $A + B$.

THINK	WRITE
1. Write down the two matrices in a sum.	$\begin{bmatrix} 4 & 2 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix}$
2. Identify the elements in the same position. For example, 4 and 1 are both in the first row and first column. Add the elements in the same positions together.	$\begin{bmatrix} 4+1 & 2+0 \\ 3+5 & -2+3 \end{bmatrix}$
3. Work out the sums and write the answer.	$\begin{bmatrix} 5 & 2 \\ 8 & 1 \end{bmatrix}$

TI | THINK	WRITE
1. On a Calculator page, complete the entry line as: $\begin{bmatrix} 4 & 2 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix}$ then press ENTER. *Note:* The matrix templates can be found by pressing the t button.	
2. The answer appears on the screen.	$A + B = \begin{bmatrix} 5 & 2 \\ 8 & 1 \end{bmatrix}$

CASIO | THINK	WRITE
1. On the Main screen, complete the entry line as: $\begin{bmatrix} 4 & 2 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix}$ then press EXE. *Note:* The matrix templates can be found in the Math2 tab in the Keyboard menu.	
2. The answer appears on the screen.	$A + B = \begin{bmatrix} 5 & 2 \\ 8 & 1 \end{bmatrix}$

4.3.3 Subtracting matrices

To subtract matrices, you need to subtract the corresponding elements in the same order as presented in the question.

WORKED EXAMPLE 6

If $A = \begin{bmatrix} 6 & 0 \\ 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, find the value of $A - B$.

THINK	WRITE
1. Write the two matrices.	$\begin{bmatrix} 6 & 0 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

2. Subtract the elements in the same position together. $\begin{bmatrix} 6-4 & 0-2 \\ 2-1 & -2-3 \end{bmatrix}$

3. Work out the subtractions and write the answer. $\begin{bmatrix} 2 & -2 \\ 1 & -5 \end{bmatrix}$

on Resources

Interactivity: Adding and subtracting matrices (int-6463)

study on

Units 1 & 2 > AOS 3 > Topic 1 > Concept 3

Equality, addition and subtraction Concept summary and practice questions

Exercise 4.3 Operations with matrices

1. a. **WE5** If $A = \begin{bmatrix} 2 & -3 \\ -1 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 9 \\ 0 & 11 \end{bmatrix}$, find the value of $A + B$.

   b. If $A = \begin{bmatrix} 0.5 \\ 0.1 \\ 1.2 \end{bmatrix}$, $B = \begin{bmatrix} -0.5 \\ 2.2 \\ 0.9 \end{bmatrix}$ and $C = \begin{bmatrix} -0.1 \\ -0.8 \\ 2.1 \end{bmatrix}$, find the matrix sum $A + B + C$.

2. Consider the matrices $C = \begin{bmatrix} 1 & -3 \\ 7 & 5 \\ b & 8 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & a \\ -5 & -4 \\ 2 & -9 \end{bmatrix}$.

   If $C + D = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -4 & -1 \end{bmatrix}$, find the values of a and b.

3. If $A = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$ and $C = \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$, calculate the following.

   a. $A + C$ b. $B + C$ c. $A - B$ d. $A + B - C$

4. **WE6** If $A = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -2 & 4 \end{bmatrix}$, find the value of $A - B$.

5. Consider the following.

$$B - A = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, A + B = \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

   a. Explain why matrix B must have an order of 3×1.
   b. Determine matrix B.

6. Evaluate the following.

a. $\begin{bmatrix} 0.5 & 0.25 & 1.2 \end{bmatrix} - \begin{bmatrix} 0.75 & 1.2 & 0.9 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 6 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 12 & 17 & 10 \\ 35 & 20 & 25 \\ 28 & 32 & 29 \end{bmatrix} - \begin{bmatrix} 13 & 12 & 9 \\ 31 & 22 & 22 \\ 25 & 35 & 31 \end{bmatrix}$

d. $\begin{bmatrix} 11 & 6 & 9 \\ 7 & 12 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 8 & 8 \\ 6 & 7 & 6 \end{bmatrix} - \begin{bmatrix} -2 & -1 & 10 \\ 4 & 9 & -3 \end{bmatrix}$

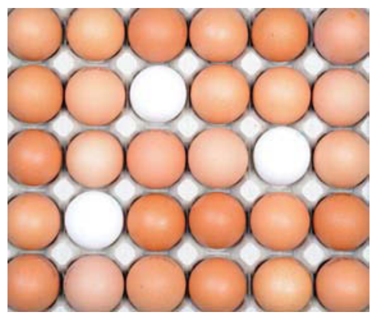
7. If $\begin{bmatrix} 3 & 0 \\ 5 & a \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ -b & 1 \end{bmatrix} = \begin{bmatrix} c & 2 \\ 3 & -4 \end{bmatrix}$, find the values of a, b and c.

8. If $\begin{bmatrix} 12 & 10 \\ 25 & 13 \\ 20 & a \end{bmatrix} - \begin{bmatrix} 9 & 11 \\ 26 & c \\ b & 9 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 8 \\ 21 & -3 \end{bmatrix}$, find the values of a, b and c.

9. By finding the order of each of the following matrices, identify which of the matrices can be added to and/or subtracted from each other and explain why.

$$A = \begin{bmatrix} 1 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -8 \end{bmatrix} \quad C = \begin{bmatrix} -1 \\ -9 \end{bmatrix} \quad D = [-4] \quad E = \begin{bmatrix} -3 & 6 \end{bmatrix}$$

10. Hard Eggs sells both free-range and barn-laid eggs in three different egg sizes (small, medium and large) to two shops, Appleton and Barntown. The number of cartons ordered for the Appleton shop is shown in the table below.



Eggs	Small	Medium	Large
Free range	2	3	5
Barn laid	4	6	3

a. Construct a 2×3 matrix to represent the egg order for the Appleton shop. The total orders for both shops are shown in the table below.

Eggs	Small	Medium	Large
Free range	3	4	8
Barn laid	6	8	5

b. i. Set up a matrix sum that would determine the order for the Barntown shop.

ii. Use the matrix sum from part bi to determine the order for the Barntown shop. Show the order in a table.

11. Marco was asked to complete the matrix sum $\begin{bmatrix} 8 & 126 & 59 \\ 17 & 102 & -13 \end{bmatrix} + \begin{bmatrix} 22 & 18 & 38 \\ 16 & 27 & 45 \end{bmatrix}$.

He gave $\begin{bmatrix} 271 \\ 194 \end{bmatrix}$ as his answer.

a. By referring to the order of matrices, explain why Marco's answer must be incorrect.

b. By explaining how to add matrices, write simple steps for Marco to follow so that he is able to add and subtract any matrices. Use the terms 'order of matrices' and 'elements' in your explanation.

12. Frederick, Harold, Mia and Petra are machinists who work for Stitch in Time. The table below shows the hours worked by each of the four employees and the number of garments completed each week for the last three weeks.

	Week 1		Week 2		Week 3	
Employee	**Hours worked**	**Number of garments**	**Hours worked**	**Number of garments**	**Hours worked**	**Number of garments**
Frederick	35	150	32	145	38	166
Harold	41	165	36	152	35	155
Mia	38	155	35	135	35	156
Petra	25	80	30	95	32	110

a. Construct a 4 × 1 matrix to represent the number of garments each employee made in week 1.

b. **i.** Create a matrix sum that would determine the total number of garments each employee made over the three weeks.

ii. Using your matrix sum from part **bi**, determine the total number of garments each employee made over the three weeks.

c. Nula is the manager of Stitch in Time. She uses the following matrix sum to determine the total number of hours worked by each of the four employees over the three weeks.

$$\begin{bmatrix} 35 \\ \\ 38 \\ 25 \end{bmatrix} + \begin{bmatrix} \\ 36 \\ \\ 30 \end{bmatrix} + \begin{bmatrix} 38 \\ 35 \\ 35 \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Complete the matrix sum by filling in the missing values.

13. There are three types of fish in a pond: speckles, googly eyes and fantails. At the beginning of the month there were 12 speckles, 9 googly eyes and 8 fantails in the pond. By the end of the month there were 9 speckles, 6 googly eyes and 8 fantails in the pond.

a. Construct a matrix sum to represent this information.

b. After six months, there were 12 speckles, 4 googly eyes and 10 fantails in the pond. Starting from the end of the first month, construct another matrix sum to represent this information.

14. Consider the following matrix sum: $A - C + B = D$. Matrix D has an order of 3×2.
   a. State the order of matrices A, B and C. Justify your answer.
      A has elements $a_{11} = x$, $a_{21} = 20$, $a_{31} = 3c_{31}$, $a_{12} = 7$, $a_{22} = y$ and $a_{32} = -8$.
      B has elements $b_{11} = x$, $b_{21} = 2x$, $b_{31} = 3x$, $b_{12} = y$, $b_{22} = 5$ and $b_{32} = 6$
      C has elements $c_{11} = 12$, $c_{21} = \frac{1}{2}a_{21}$, $c_{31} = 5$, $c_{12} = 9$, $c_{22} = 2y$ and $c_{32} = 2x$.
   b. Define the elements of D in terms of x and y.
   c. If $D = \begin{bmatrix} -8 & 1 \\ 14 & 2 \\ 16 & -8 \end{bmatrix}$, show that $x = 2$ and $y = 3$.
15. Using CAS or otherwise, evaluate the matrix sum

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{3}{5} & \frac{2}{7} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{4} & \frac{2}{9} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{1}{3} \\ \frac{1}{10} & \frac{3}{14} & \frac{4}{9} \\ \frac{1}{6} & \frac{1}{2} & \frac{2}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & \frac{7}{6} \\ \frac{2}{15} & \frac{8}{21} & \frac{3}{4} \\ \frac{2}{9} & \frac{5}{8} & \frac{10}{9} \end{bmatrix}.$$

16. Consider the matrices A and B.

$$A = \begin{bmatrix} 21 & 10 & 9 \\ 18 & 7 & 12 \end{bmatrix} \quad B = \begin{bmatrix} -10 & 19 & 11 \\ 36 & -2 & 15 \end{bmatrix}$$

   The matrix sum $A + B$ was performed using a spreadsheet. The elements for A were entered into a spreadsheet in the following cells: a_{11} was entered in cell A1, a_{21} into cell A2, a_{12} in cell B1, a_{22} in cell B2, a_{13} in cell C2 and a_{23} in cell C3.
   a. If the respective elements for B were entered into cells D1, D2, E1, E2, F1 and F2, write the formulas required to find the matrix sum $A + B$.
   b. Hence, using a spreadsheet, state the elements of $A + B$.

4.4 Matrix multiplication

4.4.1 Scalar multiplication

If $A = \begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 0 & 7 \end{bmatrix}$, then $A + A$ can be found by multiplying each element in matrix A by the scalar number 2, because $A + A = 2A$.

$$A + A = \begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 10 & 2 \\ 0 & 14 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 \times 3 & 2 \times 2 \\ 2 \times 5 & 2 \times 1 \\ 2 \times 0 & 2 \times 7 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 10 & 2 \\ 0 & 14 \end{bmatrix}$$

The number 2 is known as a scalar quantity, and the matrix $2A$ represents a **scalar multiplication**. Any matrix can be multiplied by any scalar quantity and the order of the matrix will remain the same. A scalar quantity can be any real number, such as negative or positive numbers, fractions or decimal numbers.

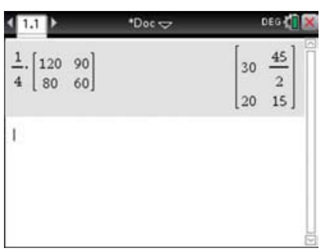
WORKED EXAMPLE 7

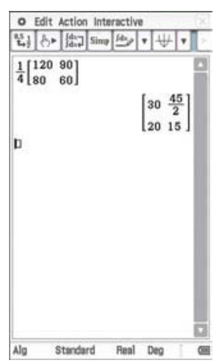
Consider the matrix $A = \begin{bmatrix} 120 & 90 \\ 80 & 60 \end{bmatrix}$.

Evaluate the following.

a. $\frac{1}{4}A$ **b.** $0.1A$

THINK	WRITE
a. 1. Identify the scalar for the matrix. In this case it is $\frac{1}{4}$, which means that each element in A is multiplied by $\frac{1}{4}$ (or divided by 4).	**a.** $\frac{1}{4}\begin{bmatrix} 120 & 90 \\ 80 & 60 \end{bmatrix}$
2. Multiply each element in A by the scalar.	$\begin{bmatrix} \frac{1}{4}\times 120 & \frac{1}{4}\times 90 \\ \frac{1}{4}\times 80 & \frac{1}{4}\times 60 \end{bmatrix}$
3. Simplify each multiplication by finding common factors and write the answer.	$\begin{bmatrix} \cancel{120}^{30}\times\frac{1}{\cancel{4}_1} & \cancel{90}^{45}\times\frac{1}{\cancel{4}_2} \\ \cancel{80}^{20}\times\frac{1}{\cancel{4}_1} & \cancel{60}^{15}\times\frac{1}{\cancel{4}_1} \end{bmatrix}$ $= \begin{bmatrix} 30 & \frac{45}{2} \\ 20 & 15 \end{bmatrix}$
b. 1. Identify the scalar. In this case it is 0.1, which means that each element in A is multiplied by 0.1 (or divided by 10).	**b.** $0.1\begin{bmatrix} 120 & 90 \\ 80 & 60 \end{bmatrix}$
2. Multiply each element in A by the scalar.	$\begin{bmatrix} 0.1\times 120 & 0.1\times 90 \\ 0.1\times 80 & 0.1\times 60 \end{bmatrix}$
3. Find the values for each element and write the answer.	$\begin{bmatrix} 12 & 9 \\ 8 & 6 \end{bmatrix}$

TI | THINK	WRITE
a 1. On a Calculator page, complete the entry line as: $\frac{1}{4}\begin{bmatrix} 120 & 90 \\ 80 & 60 \end{bmatrix}$ then press ENTER. *Note:* The matrix templates can be found by pressing the t button.	
2. The answer appears on the screen.	$\frac{1}{4}A = \begin{bmatrix} 30 & \frac{45}{2} \\ 20 & 15 \end{bmatrix}$

CASIO | THINK	WRITE
a 1. On the Main screen, complete the entry line as: $\frac{1}{4}\begin{bmatrix} 120 & 90 \\ 80 & 60 \end{bmatrix}$ then press EXE. *Note:* The matrix templates can be found in the Math2 tab in the Keyboard menu.	
2. The answer appears on the screen.	$\frac{1}{4}A = \begin{bmatrix} 30 & \frac{45}{2} \\ 20 & 15 \end{bmatrix}$

4.4.2 The product matrix and its order

Not all matrices can be multiplied together. However, unlike with addition and subtraction, matrices do not need to have the same order to be multiplied together.

For matrices to be able to be multiplied together (have a product), the *number of columns* in the first matrix must equal the *number of rows* in the second matrix.

For example, consider matrices A and B, with matrix A having an order of $m \times n$ (m rows and n columns) and matrix B having an order of $p \times r$ (p rows and r columns).

For A and B to be multiplied together, the number of columns in A must equal the number of rows in B; that is, n must equal p. If n does equal p, then the **product matrix** AB is said to exist, and the order of the **product matrix** AB will be $m \times r$.

Given matrix A with an order of $m \times n$ and matrix B with an order of $n \times r$, matrix AB will have an order of $m \times r$.

WORKED EXAMPLE 8

If $A = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \end{bmatrix}$, show that the product matrix AB exists and hence write down the order of AB.

THINK	WRITE
1. Write the order of each matrix.	A: 2×1 B: 1×2
2. Write the orders next to each other.	$2 \times 1\ 1 \times 2$
3. Circle the two middle numbers.	$2 \times (1\ 1) \times 2$
4. If the two numbers are the same, then the product matrix exists.	Number of columns in A = number of rows in B, therefore the product matrix AB exists.
5. The order of the resultant matrix (the product) will be the first and last number.	$(2) \times 1\ 1 \times (2)$ The order of AB is 2×2.

4.4.3 Multiplying matrices

To multiply matrices together, use the following steps.

Step 1: Confirm that the product matrix exists (that is, the number of columns in the first matrix equals the number of rows in the second matrix).

Step 2: Multiply the elements of each row of the first matrix by the elements of each column of the second matrix.

Step 3: Sum the products in each element of the product matrix.

Consider matrices A and B.

$$A = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

As previously stated, the order of the product matrix AB will be 2×2.

$$AB = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \end{bmatrix}$$

1st row × 1st column: 3×1
1st row × 2nd column: 3×2

2nd row × 1st column: 2 × 1
2nd row × 2nd column: 2 × 2

$$AB = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$

Unlike when multiplying with real numbers, when multiplying matrices together the order of the multiplication is important. This means that in most cases $AB \neq BA$.

Using matrices A and B as previously defined, the order of product matrix BA is 1×1.

$$BA = \begin{bmatrix} 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

As when calculating AB, to multiply the elements in these matrices you need to multiply the rows by the columns. Each element in the first row must be multiplied by the corresponding element in the first column, and the total sum of these will make up the element in the first row and first column of the product matrix.

For example, the element in the first row and first column of the product matrix BA is found by the sum $1 \times 3 + 2 \times 2 = 7$.

So the product matrix BA is [7].

WORKED EXAMPLE 9

If $A = \begin{bmatrix} 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, determine the product matrix AB.

THINK	WRITE
1. Set up the product matrix.	$\begin{bmatrix} 3 & 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 6 \end{bmatrix}$
2. Determine the order of product matrix AB by writing the order of each matrix A and B.	$A \times B$ $(1) \times 2 \times 2 \times (1)$ AB has an order of 1×1.
3. Multiply each element in the first row by the corresponding element in the first column; then calculate the sum of the results.	$3 \times 2 + 5 \times 6 = 36$
4. Write the answer as a matrix.	[36]

WORKED EXAMPLE 10

Determine the product matrix MN if $M = \begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$.

THINK	WRITE
1. Set up the product matrix.	$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$
2. Determine the order of product matrix MN by writing the order of each matrix M and N.	$M \times N$ $(2) \times 2 \times 2 \times (2)$ MN has an order of 2×2.
3. To find the element MN_{11}, multiply the corresponding elements in the first row and first column and calculate the sum of the results.	$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$ $3 \times 1 + 6 \times 5 = 33$

4. To find the element MN_{12}, multiply the corresponding elements in the first row and second column and calculate the sum of the results.

$$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$$
$$3 \times 8 + 6 \times 4 = 48$$

5. To find the element MN_{21}, multiply the corresponding elements in the second row and first column and calculate the sum of the results.

$$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$$
$$5 \times 1 + 2 \times 5 = 15$$

6. To find the element MN_{22}, multiply the corresponding elements in the second row and second column and calculate the sum of the results.

$$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$$
$$5 \times 8 + 2 \times 4 = 48$$

7. Construct the matrix MN by writing in each of the elements.

$$\begin{bmatrix} 33 & 48 \\ 15 & 48 \end{bmatrix}$$

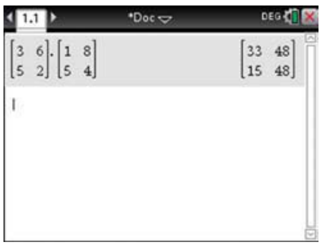
TI | THINK

1. On a Calculator page, complete the entry line as:

$$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$$

then press ENTER.
Note: The matrix templates can be found by pressing the t button.

WRITE



2. The answer appears on the screen.

$$MN = \begin{bmatrix} 33 & 48 \\ 15 & 48 \end{bmatrix}$$

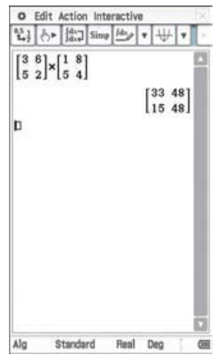
CASIO | THINK

1. On the Main screen, complete the entry line as:

$$\begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix}$$

then press EXE.
Note: The matrix templates can be found in the Math2 tab in the Keyboard menu.

WRITE



2. The answer appears on the screen.

$$MN = \begin{bmatrix} 33 & 48 \\ 15 & 48 \end{bmatrix}$$

4.4.4 Multiplying by the identity matrix

As previously stated, an identity matrix is a square matrix with 1s in the top left to bottom right diagonal and 0s for all other elements,

for example $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Just like multiplying by the number 1 in the real number system, multiplying by the identity matrix will not change a matrix.

If the matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$ is multiplied by the identity matrix on the left, that is IA, it will be multiplied by a 2×2 identity matrix (because A has 2 rows). If A is multiplied by the identity matrix on the right, that is AI, then it will be multiplied by a 3×3 identity matrix (because A has 3 columns).

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + 0 \times 3 & 1 \times 4 + 0 \times 5 & 1 \times 6 + 0 \times 7 \\ 0 \times 2 + 1 \times 3 & 0 \times 4 + 1 \times 5 & 0 \times 6 + 1 \times 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

$$= A$$

$$AI = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 1 + 4 \times 0 + 6 \times 0 & 2 \times 0 + 4 \times 1 + 6 \times 0 & 2 \times 0 + 4 \times 0 + 6 \times 1 \\ 3 \times 1 + 5 \times 0 + 7 \times 0 & 3 \times 0 + 5 \times 1 + 7 \times 0 & 3 \times 0 + 5 \times 0 + 7 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

$$= A$$

Therefore, $AI = IA = A$.

4.4.5 Powers of square matrices

When a square matrix is multiplied by itself, the order of the resultant matrix is equal to the order of the original square matrix. Because of this fact, whole number powers of square matrices always exist.

You can use CAS to quickly determine large powers of square matrices.

WORKED EXAMPLE 11

If $A = \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}$, calculate the value of A^3.

THINK	WRITE
1. Write the matrix multiplication in full.	$A^3 = AAA$ $= \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}$
2. Calculate the first matrix multiplication (AA).	$AA = \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}$ $AA_{11} = 3 \times 3 + 5 \times 5 = 34$ $AA_{21} = 5 \times 3 + 1 \times 5 = 20$ $AA_{12} = 3 \times 5 + 5 \times 1 = 20$ $AA_{22} = 5 \times 5 + 1 \times 1 = 26$ $AA = \begin{bmatrix} 34 & 20 \\ 20 & 26 \end{bmatrix}$
3. Rewrite the full matrix multiplication, substituting the answer found in the previous part.	$A^3 = AAA$ $= \begin{bmatrix} 34 & 20 \\ 20 & 26 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}$

4. Calculate the second matrix multiplication (AAA).

$AAA_{11} = 34 \times 3 + 20 \times 5 = 202$
$AAA_{21} = 34 \times 5 + 20 \times 1 = 190$
$AAA_{12} = 20 \times 3 + 26 \times 5 = 190$
$AAA_{22} = 20 \times 5 + 26 \times 1 = 126$

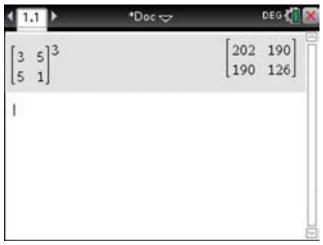
$$AAA = \begin{bmatrix} 202 & 190 \\ 190 & 126 \end{bmatrix}$$

5. Write the answer.

$$A^3 = \begin{bmatrix} 202 & 190 \\ 190 & 126 \end{bmatrix}$$

TI | THINK — **WRITE**

1. On a Calculator page, complete the entry line as:
$\begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}^3$
then press ENTER.
Note: The matrix templates can be found by pressing the t button.

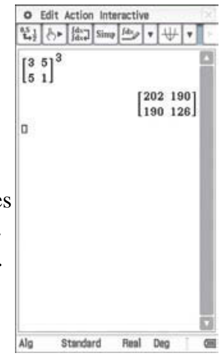


2. The answer appears on the screen.

$A^3 = \begin{bmatrix} 202 & 190 \\ 190 & 126 \end{bmatrix}$

CASIO | THINK — **WRITE**

1. On the Main screen, complete the entry line as:
$\begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}^3$
then press EXE.
Note: The matrix templates can be found in the Math2 tab in the Keyboard menu.



2. The answer appears on the screen.

$A^3 = \begin{bmatrix} 202 & 190 \\ 190 & 126 \end{bmatrix}$

on Resources

Interactivity: Matrix multiplication (int-6464)

studyon

Units 1 & 2 > AOS 3 > Topic 1 > Concepts 5 & 6

Multiplication by a scalar Concept summary and practice questions

Matrix multiplication Concept summary and practice questions

Matrix multiplication and powers Concept summary and practice questions

Exercise 4.4 Matrix multiplication

1. **WE7** Consider the matrix $C = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 4 & 6 \end{bmatrix}$. Evaluate the following.

   a. $4C$ b. $\frac{1}{5}C$ c. $0.3C$

2. Matrix D was multiplied by the scalar quantity x.

   If $3D = \begin{bmatrix} 15 & 0 \\ 21 & 12 \\ 33 & 9 \end{bmatrix}$ and $xD = \begin{bmatrix} 12.5 & 0 \\ 17.5 & 10 \\ 27.5 & 7.5 \end{bmatrix}$, find the value of x.

3. **MC** Consider the matrix $M = \begin{bmatrix} 12 & 9 & 15 \\ 36 & 6 & 21 \end{bmatrix}$. Which of the following is equal to the matrix M?

A. $0.1 \begin{bmatrix} 1.2 & 0.9 & 1.5 \\ 3.6 & 0.6 & 2.1 \end{bmatrix}$

B. $3 \begin{bmatrix} 3 & 3 & 5 \\ 9 & 2 & 7 \end{bmatrix}$

C. $3 \begin{bmatrix} 4 & 3 & 5 \\ 12 & 2 & 7 \end{bmatrix}$

D. $3 \begin{bmatrix} 36 & 27 & 45 \\ 108 & 18 & 63 \end{bmatrix}$

E. $10 \begin{bmatrix} 120 & 90 & 15 \\ 36 & 6 & 21 \end{bmatrix}$

4. **WE8** **a.** If $X = [\,3 \quad 5\,]$ and $Y = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, show that the product matrix XY exists and state the order of XY.

b. Determine which of the following matrices can be multiplied together and state the order of any product matrices that exist.

$$D = \begin{bmatrix} 7 & 4 \\ 3 & 5 \\ 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 5 & 7 \\ 8 & 9 \end{bmatrix} \text{ and } E = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 2 & 6 \end{bmatrix}$$

5. The product matrix ST has an order of 3×4. If matrix S has 2 columns, write down the order of matrices S and T.

6. Which of the following matrices can be multiplied together? Justify your answers by finding the order of the product matrices.

$$D = \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}, E = \begin{bmatrix} 5 & 8 \\ 7 & 1 \\ 9 & 3 \end{bmatrix}, F = \begin{bmatrix} 12 & 7 & 3 \\ 15 & 8 & 4 \end{bmatrix}, G = \begin{bmatrix} 13 & 15 \end{bmatrix}$$

7. **WE9** **a.** If $M = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $N = [\,7 \quad 12\,]$, determine the product matrix MN.

b. Does the product matrix NM exist? Justify your answer by finding the product matrix NM and stating its order.

8. Matrix $S = \begin{bmatrix} 1 & 4 & 3 \end{bmatrix}$, matrix $T = \begin{bmatrix} 2 \\ 3 \\ t \end{bmatrix}$ and the product matrix $ST = [5]$. Find the value of t.

9. **WE10** For matrices $P = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$, determine the product matrix PQ.

10. For a concert, three different types of tickets can be purchased; adult, senior and child. The cost of each type of ticket is \$12.50, \$8.50 and \$6.00 respectively. The number of people attending the concert is shown in the following table.

Ticket type	Number of people
Adult	65
Senior	40
Child	85

a. Construct a column matrix to represent the cost of the three different tickets in the order adult, senior and child.
If the number of people attending the concert is written as a row matrix, a matrix multiplication can be performed to determine the total amount in ticket sales for the concert.

b. By finding the orders of each matrix and then the product matrix, explain why this is the case.
c. By completing the matrix multiplication from part b, determine the total amount (in dollars) in ticket sales for the concert.

11. Find the product matrices when the following pairs of matrices are multiplied together.

a. $\begin{bmatrix} 6 & 9 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

b. $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 6 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 7 \\ 2 \\ 9 \end{bmatrix}$ and $\begin{bmatrix} 10 & 15 \end{bmatrix}$

d. $\begin{bmatrix} 6 & 5 \\ 8 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 9 \end{bmatrix}$

e. $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$

12. Evaluate the following matrix multiplications.

a. $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

c. Using your results from parts a and b, when will AB be equal to BA?
d. If A and B are not of the same order, is it possible for AB to be equal to BA?

13. WE11 If $P = \begin{bmatrix} 8 & 2 \\ 4 & 7 \end{bmatrix}$, calculate the value of P^2.

14. If $T = \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$, calculate the value of T^3.

15. The 3×3 identity matrix, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

a. Calculate the value of I_3^2.
b. Calculate the value of I_3^3.
c. Calculate the value of I_3^4.
d. Comment on your answers to parts a-c.

16. The table below shows the percentage of students who are expected to be awarded grades A–E on their final examinations for Mathematics and Physics.

Grade	A	B	C	D	E
Percentage of students	5	18	45	25	7

The number of students studying Mathematics and Physics is 250 and 185 respectively.

a. Construct a column matrix, S, to represent the number of students studying Mathematics and Physics.
b. Construct a 1×5 matrix, A, to represent the percentage of students expected to receive each grade, expressing each element in decimal form.
c. In the context of this problem, what does product matrix SA represent?
d. Determine the product matrix SA. Write your answers correct to the nearest whole numbers.
e. In the context of this problem, what does element SA_{12} represent?

17. A product matrix, $N = MPR$, has order 3×4. Matrix M has m rows and n columns, matrix P has order $1 \times q$, and matrix R has order $2 \times s$. Determine the values of m, n, s and q.

18. Dodgy Bros sell vans, utes and sedans. The average selling price for each type of vehicle is shown in the table below.

Type of vehicle	Monthly sales($)
Vans	$4000
Utes	$12 500
Sedans	$8500

The table below shows the total number of vans, utes and sedans sold at Dodgy Bros in one month.

Type of vehicle	Number of sales
Vans	5
Utes	8
Sedans	4

Stan is the owner of Dodgy Bros and wants to determine the total amount of monthly sales.

a. Explain how matrices could be used to help Stan determine the total amount, in dollars, of monthly sales.

b. Perform a matrix multiplication that finds the total amount of monthly sales.

c. Brian is Stan's brother and the accountant for Dodgy Bros. In finding the total amount of monthly sales, he performs the following matrix multiplication.

$$\begin{bmatrix} 5 \\ 8 \\ 4 \end{bmatrix} \begin{bmatrix} 4000 & 12\,500 & 8500 \end{bmatrix}$$

Explain why this matrix multiplication is not valid for this problem.

19. Rhonda was asked to perform the following matrix multiplication to determine the product matrix GH.

$$GH = \begin{bmatrix} 6 & 5 \\ 3 & 8 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

Rhonda's answer was $\begin{bmatrix} 60 & 65 \\ 30 & 104 \\ 50 & 117 \end{bmatrix}$.

a. By stating the order of product matrix GH, explain why Rhonda's answer is obviously incorrect.

b. Determine the product matrix GH.

c. Explain Rhonda's method of multiplying matrices and why this is the incorrect method.

d. Provide simple steps to help Rhonda multiply matrices.

20. In an AFL game of football, 6 points are awarded for a goal and 1 point is awarded for a behind. St Kilda and Collingwood played in two grand finals in 2010, with the two results given by the following matrix multiplication.

$$\begin{bmatrix} 9 & 14 \\ 10 & 8 \\ 16 & 12 \\ 7 & 10 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 \\ S_1 \\ C_2 \\ S_2 \end{bmatrix}$$

Complete the matrix multiplication to determine the scores in the two grand finals.

21. By using CAS or otherwise, calculate the following powers of square matrices.

a. $\begin{bmatrix} 4 & 8 \\ 7 & 2 \end{bmatrix}^4$

b. $\begin{bmatrix} \frac{2}{3} & \frac{5}{7} \\ \frac{1}{4} & 2 \end{bmatrix}^3$

c. $\begin{bmatrix} 3 & 1 & 7 \\ 4 & 2 & 8 \\ 5 & 6 & 9 \end{bmatrix}^3$

22. The number of adults, children and seniors attending the zoo over Friday, Saturday and Sunday is shown in the table.

Day	Adults	Children	Seniors
Friday	125	245	89
Saturday	350	456	128
Sunday	421	523	102

Entry prices for adults, children and seniors are \$35, \$25, \$20 respectively.

a. Using CAS or otherwise, perform a matrix multiplication that will find the entry fee collected for each of the three days.

b. Write the calculation that finds the entry fee collected for Saturday.

c. Is it possible to perform a matrix multiplication that would find the total for each type of entry fee (adults, children and seniors) over the three days? Explain your answer.

4.5 Inverse matrices and problem solving with matrices

4.5.1 Inverse matrices

In the real number system, a number multiplied by its reciprocal results in 1. For example, $3 \times \frac{1}{3} = 1$. In this case $\frac{1}{3}$ is the reciprocal or multiplicative inverse of 3.

In matrices, if the product matrix is the identity matrix, then one of the matrices is the multiplicative inverse of the other.

For example,

$$\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix} = \begin{bmatrix}2\times 3+5\times -1 & 2\times -5+5\times 2\\1\times 3+3\times -1 & 1\times -5+3\times 2\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \text{ (the } 2\times 2 \text{ identity matrix).}$$

If $A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$, then $\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ is the multiplicative inverse of A, which is denoted as A^{-1}.

Similarly, if $B = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$, then $\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix} = B^{-1}$.

Hence $AA^{-1} = I = A^{-1}A$.

WORKED EXAMPLE 12

By finding the product matrix *AB*, determine whether the following matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix} \text{ and } B = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$$

THINK	WRITE
1. Set up the product matrix.	$\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}\times\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$
2. Evaluate the product matrix.	$\begin{bmatrix}2\times 3+5\times -1 & 2\times -5+5\times 2\\1\times 3+3\times -1 & 1\times -5+3\times 2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
3. Check if the product matrix is the identity matrix.	The product matrix AB is the identity matrix. Therefore, A and B are multiplicative inverses of each other.

4.5.2 Finding inverse matrices

Inverse matrices only exist for square matrices and can be easily found for matrices of order 2×2. Inverses can also be found for larger square matrices; however, the processes to find these are more complicated, so technology is often used to find larger inverses.

Remember that the product of a matrix and its inverse is the identity matrix, I.

For matrix $A = \begin{bmatrix}a & c\\b & d\end{bmatrix}$, let $A^{-1} = \begin{bmatrix}e & g\\f & h\end{bmatrix}$.

$$\begin{bmatrix}a & c\\b & d\end{bmatrix}\begin{bmatrix}e & g\\f & h\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},$$

$$\text{so } \begin{bmatrix}a\times e+c\times f & a\times g+c\times h\\b\times e+d\times f & b\times g+d\times h\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

To find the values of e, f, g and h in terms of a, b, c and d would require four equations to be solved! However, there is a simpler method we can follow to find the inverse matrix A^{-1} in only three steps.

To determine the inverse for a matrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$:

Step 1: Swap the elements a_{11} and a_{22}: $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$

Step 2: Multiply elements a_{12} and a_{21} by -1: $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Step 3: Multiply by $\dfrac{1}{ad - bc}$:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

The expression $ad - bc$ is known as the **determinant** of matrix A. It is usually written as $\det A$ or $|A|$. If $\det A = 0$, then the inverse matrix A^{-1} does not exist, because the value $\frac{1}{0}$ is undefined.

Note: In practice it is best to check that the determinant does not equal 0 before proceeding with the other steps.

WORKED EXAMPLE 13

If $A = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}$, determine A^{-1}.

THINK	WRITE
1. Swap elements a_{11} and a_{22}.	$\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$
2. Multiply elements a_{12} and a_{21} by -1.	$\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ -4 & 7 \end{bmatrix}$
3. Find the determinant ($\det A = ad - bc$).	$a = 7, b = 2, c = 4, d = 1$ $ad - bc = 7 \times 1 - 2 \times 4$ $= -1$
4.. Multiply the matrix from step 2 by $\dfrac{1}{\det A}$.	$\frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -4 & 7 \end{bmatrix} = -1\begin{bmatrix} 1 & -2 \\ -4 & 7 \end{bmatrix}$ $= \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}$

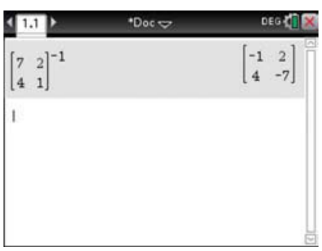
TI | THINK

1. On a Calculator page, complete the entry line as:

$$\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^{-1}$$

then press ENTER.
Note: The matrix templates can be found by pressing the t button.

WRITE



2. The answer appears on the screen.

$A^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}$

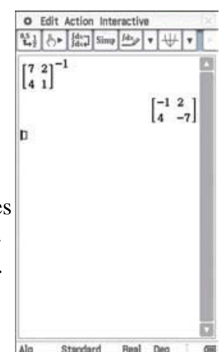
CASIO | THINK

1. On the Main screen, complete the entry line as:

$$\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^{-1}$$

then press EXE.
Note: The matrix templates can be found in the Math2 tab in the Keyboard menu.

WRITE



2. The answer appears on the screen.

$A^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}$

4.5.3 Using inverse matrices to solve problems

Unlike in the real number system, we can't divide one matrix by another matrix. However, we can use inverse matrices to help us solve matrix equations in the same way that division is used to help solve many linear equations.

Given the matrix equation $AX = B$, the inverse matrix can be used to find the matrix X as follows.

Step 1: (Multiply both sides of the equation by A^{-1}): $A^{-1}AX = A^{-1}B$

Step 2: ($A^{-1}A = I$, the identity matrix): $IX = A^{-1}B$

Step 3: ($IX = X$, as found in the previous section): $X = A^{-1}B$

Note: If we multiply the left-hand side of our equation by A^{-1} on the left, then we must also multiply the right-hand side of our equation by A^{-1} on the left.

Remember that when multiplying with matrices the order of the multiplication is important.

If the equation was $XA = B$ and matrix X needed to be found, then the inverse matrix multiplication would be:

Step 1: $XAA^{-1} = BA^{-1}$

Step 2: $XI = BA^{-1}$

Step 3: $X = BA^{-1}$

WORKED EXAMPLE 14

If $\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$, find the values of x and y.

THINK	WRITE
1. The matrix equation is in the form $AX = B$. Identify matrices A, B and X.	$A = \begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$
2. To determine X we need to multiply both sides of the equation by the inverse A^{-1}.	$A^{-1}AX = A^{-1}B$
3. Find the inverse A^{-1}.	$A^{-1} = \frac{1}{12 - 10}\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ $= \frac{1}{2}\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ $= \begin{bmatrix} 1.5 & -1 \\ -2.5 & 2 \end{bmatrix}$
4. Calculate $A^{-1}B$.	$A^{-1} = \begin{bmatrix} 1.5 & -1 \\ -2.5 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \end{bmatrix}$ $= \begin{bmatrix} 1.5 \times 8 + -1 \times 11 \\ -2.5 \times 8 + 2 \times 11 \end{bmatrix}$ $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

5. Solve for x and y.

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$x = 1$ and $y = 2$

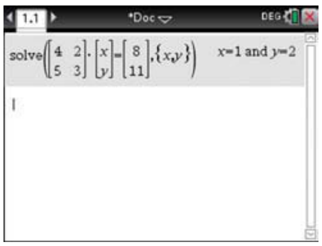
TI | THINK

1. On a Calculator page, press MENU then select:
3: Algebra
1: Solve
Complete the entry line as:

$$\text{solve}\left(\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}, \{x, y\}\right)$$

then press ENTER.
Note: The matrix templates can be found by pressing the t button.

WRITE



2. The answer appears on the screen.

$x = 1$ and $y = 2$

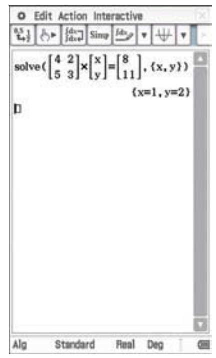
CASIO | THINK

1. On the Main screen, complete the entry line as:

$$\text{solve}\left(\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}, \{x, y\}\right)$$

then press EXE.
Note: The matrix templates can be found in the Math2 tab in the Keyboard menu.

WRITE



2. The answer appears on the screen.

$x = 1$ and $y = 2$

4.5.4 Using inverse matrices to solve a system of simultaneous equations

If you have a pair of simultaneous equations, they can be set up as a matrix equation and solved using inverse matrices.

Take the pair of simultaneous equations $ax + by = c$ and $dx + ey = f$.

These can be set up as the matrix equation

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}.$$

If we let $A = \begin{bmatrix} a & b \\ d & e \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c \\ f \end{bmatrix}$, this equation is of the form $AX = B$, which can be solved as $X = A^{-1}B$ (as determined previously).

WORKED EXAMPLE 15

Solve the following pair of simultaneous equations by using inverse matrices.
$2x + 3y = 6$
$4x - 6y = -4$

THINK

1. Set up the simultaneous equations as a matrix equation.

WRITE

$$\begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

2. Find the inverse of the matrix A, A^{-1}.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-12-12}\begin{bmatrix} -6 & -3 \\ -4 & 2 \end{bmatrix}$$

$$= \frac{1}{-24}\begin{bmatrix} -6 & -3 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{6} & -\frac{1}{12} \end{bmatrix}$$

3. Calculate $A^{-1}B$.

$$A^{-1}B = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{6} & -\frac{1}{12} \end{bmatrix}\begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} \times 6 + \frac{1}{8} \times -4 \\ \frac{1}{6} \times 6 + -\frac{1}{12} \times -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$$

4. State the answer.

$$x = 1,\ y = \frac{4}{3}$$

4.5.5 Using matrix equations to solve worded problems

To use matrices to solve worded problems, you must set up a matrix equation from the information provided. The matrix equation can then be solved using the skills you have previously learned.

WORKED EXAMPLE 16

On an excursion, a group of students and teachers travelled to the city by train and returned by bus. On the train. the cost of a student ticket was \$3 and the cost of a teacher ticket was \$4.50, with the total cost for the train tickets being \$148.50. On the bus, the cost of a student ticket was \$2.75 and the cost of a teacher ticket was \$3.95, with the total cost for the bus tickets being \$135.60.
By solving a matrix equation, determine how many students and teachers attended the excursion.

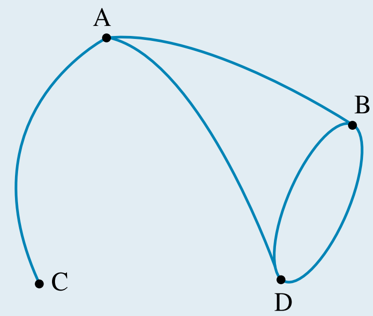
THINK	WRITE
1. Identify the two unknowns in the problem. Assign a pronumeral to represent each unknown.	Numbers of students = s Number of teachers = t
2. Construct a matrix to represent the unknowns.	$\begin{bmatrix} s \\ t \end{bmatrix}$
3. Highlight the key information, that is, how much the two different types of tickets were for students and teachers.	Student train ticket = \$3 Teacher train ticket = \$4.50 Student bus ticket = \$2.75 Teacher bus ticket = \$3.95
4. Construct a matrix to represent the information. Note that each row represents the two different types of travel.	$\begin{bmatrix} 3 & 4.50 \\ 2.75 & 3.95 \end{bmatrix} \begin{matrix} \text{Train} \\ \text{Bus} \end{matrix}$
5. Construct a matrix to represent the total cost in the same row order as in step 4.	$\begin{bmatrix} 148.50 \\ 135.60 \end{bmatrix}$
6. Set up a matrix equation in the form $AX = B$, remembering that X will represent the 'unknowns', that is, the values that need to be found.	$\begin{bmatrix} 3 & 4.50 \\ 2.75 & 3.95 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 148.50 \\ 135.60 \end{bmatrix}$
7. Solve the matrix equation by finding A^{-1} and multiplying it by B.	$A^{-1} = \frac{1}{3 \times 3.95 - 2.75 \times 4.50} \begin{bmatrix} 3.95 & -4.50 \\ -2.75 & 3 \end{bmatrix}$ $= \frac{1}{-0.525} \begin{bmatrix} 3.95 & -4.50 \\ -2.75 & 3 \end{bmatrix}$ $\begin{bmatrix} s \\ t \end{bmatrix} = A^{-1}B$ $= \frac{1}{-0.525} \begin{bmatrix} 3.95 & -4.50 \\ -2.75 & 3 \end{bmatrix} \begin{bmatrix} 148.50 \\ 135.60 \end{bmatrix}$ $= \frac{1}{-0.525} \begin{bmatrix} -23.625 \\ -1.575 \end{bmatrix}$ $= \begin{bmatrix} 45 \\ 3 \end{bmatrix}$
8. Answer the question.	There were 45 students and 3 teachers on the excursion.

4.5.6 Adjacency matrices

Matrices can be used to determine the number of different connections between objects, such as towns or people. They can also be used to represent tournament outcomes and determine overall winners. To determine the number of connections between objects, a matrix known as an **adjacency matrix** is set up to represent these connections.

WORKED EXAMPLE 17

The diagram at right shows the number of roads connecting between four towns, A, B, C and D.
Construct an adjacency matrix to represent this information.



THINK

1. Since there are four connecting towns, a 4 × 4 adjacency matrix needs to be constructed. Label the row and columns with the relevant towns A, B, C and D.

WRITE

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \end{array}$$

2. There is one road connecting town A to town B, so enter 1 in the cell from A to B.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} - & - & - & - \\ \textcircled{1} & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \end{array}$$

3. There is also only one road between town A and towns C and D; therefore, enter 1 in the appropriate matrix positions. There are no loops at town A (i.e. a road connecting A to A); therefore, enter 0 in this position.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & - & - & - \\ 1 & - & - & - \\ 1 & - & - & - \\ 1 & - & - & - \end{bmatrix} \end{array}$$

4. Repeat this process for towns B, C and D. Note that there are two roads connecting towns B and D, and that town C only connects to town A.

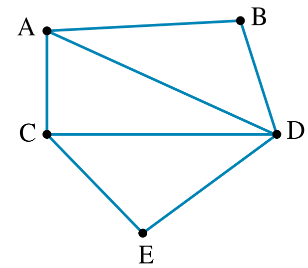
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

4.5.7 Determining the number of connections between objects

An adjacency matrix allows us to determine the number of connections either directly between or via objects. If a direct connection between two objects is denoted as one ‘step’, ‘two steps’ means a connection between two objects via a third object, for example the number of ways a person can travel between towns A and D via another town.

You can determine the number of connections of differing ‘steps’ by raising the adjacency matrix to the power that reflects the number of steps in the connection.

For example, the following diagram shows the number of roads connecting five towns, A, B, C, D and E. There are a number of ways to travel between towns A and D. There is one direct path between the towns; this is a one-step path. However, you can also travel between towns A and D via town C or B. These are considered two-step paths as there are two links (or roads) in these paths. The power on the adjacency matrix would therefore be 2 in this case.



WORKED EXAMPLE 18

The following adjacency matrix shows the number of pathways between four attractions at the zoo: lions (L), seals (S), monkeys (M) and elephants (E).

$$\begin{array}{c} \\ L \\ S \\ M \\ E \end{array}\begin{array}{c} \begin{array}{cccc} L & S & M & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Using CAS or otherwise, determine how many ways a family can travel from the lions to the monkeys via one of the other two attractions.

THINK	WRITE
1. Determine the link length.	The required path is between two attractions via a third attraction, so the link length is 2.
2. Using CAS or otherwise, evaluate the matrix.	$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & 2 & 3 & 3 \\ 2 & 3 & 3 & 3 \\ 3 & 3 & 6 & 2 \\ 3 & 3 & 2 & 6 \end{bmatrix}$
3. Interpret the information in the matrix and answer the question by locating the required value.	$\begin{bmatrix} 3 & 2 & 3 & 3 \\ 2 & 3 & 3 & 3 \\ \textcircled{3} & 3 & 6 & 2 \\ 3 & 3 & 2 & 6 \end{bmatrix}$ There are 3 ways in which a family can travel from the lions to the monkeys via one of the other two attractions.

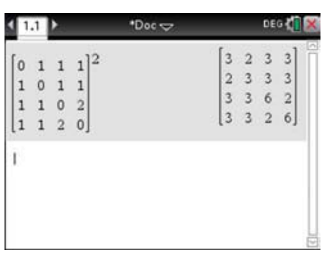
TI | THINK

1. On a Calculator page, complete the entry line as:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}^2$$

then press ENTER.
Note: The matrix templates can be found by pressing the t button.

WRITE



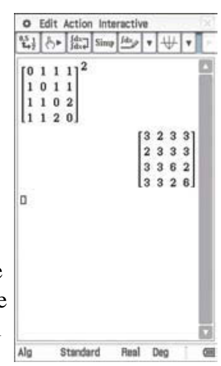
CASIO | THINK

1. On the Main screen, complete the entry line as:

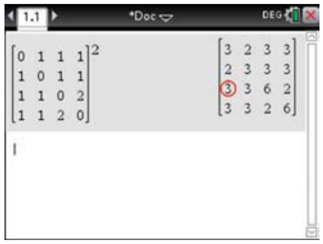
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}^2$$

then press EXE.
Note: The matrix templates can be found in the Math2 tab in the Keyboard menu. Clicking on the matrix icons multiple times will increase the order of the matrix.

WRITE

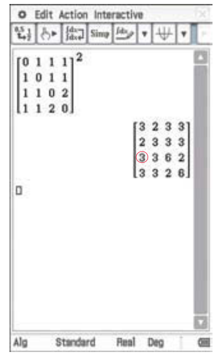


2. Read the answer from the third row and first column.



There are 3 ways in which a family can travel from the lions to the monkeys via one of the other two attractions.

2. Read the answer from the third row and first column.



There are 3 ways in which a family can travel from the lions to the monkeys via one of the other two attractions.

on Resources

Interactivity: Inverse matrices (int-6465)

Interactivity: Applications of matrices to simultaneous equations (int-6291)

studyon

Units 1 & 2 > AOS 3 > Topic 1 > Concepts 7, 8 & 9

The matrix multiplicative inverse Concept summary and practice questions

Applications of multiplicative inverses Concept summary and practice questions

Modelling and solving problems using matrices Concept summary and practice questions

Exercise 4.5 Inverse matrices and problem solving with matrices

1. **WE12** By finding the product matrix AB, determine whether the following matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1.5 & -2.5 \\ -1 & 2 \end{bmatrix}$$

2. Matrices A and B are multiplicative inverses of each other.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ a & 1 \end{bmatrix}$$

By finding the product matrix AB, show that the value of $a = -2$.

3. WE13 Find the inverses of the following matrices:

a. $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$

4. Consider the matrix $B = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$. By finding the value of the determinant, explain why B^{-1} does not exist.

5. a. Find the determinants of the following matrices to determine which of them have inverses.

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}, C = \begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix}, D = \begin{bmatrix} 0 & 1 \\ 3 & -5 \end{bmatrix}$$

b. For those matrices that have inverses, find the inverse matrices.

6. WE14 If $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$, find the values of x and y.

7. A matrix equation is represented by $XA = B$, where $B = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1.5 & 2 \end{bmatrix}$.

a. State the order of matrix X and hence find matrix X.

b. Find matrix A.

8. Consider the matrix equation $\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$.

a. Explain how this matrix equation can be solved using the inverse matrix.

b. State the inverse matrix used to solve this matrix equation.

c. Calculate the values of x and y, clearly showing your working.

9. Veronica bought two donuts and three cupcakes for \$14. The next week she bought three donuts and two cupcakes for \$12.25. This information is shown in the matrices below, where d and c represent the cost of a donut and cupcake respectively.

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} d \\ c \end{bmatrix} = \begin{bmatrix} 14.00 \\ 12.25 \end{bmatrix}$$

By solving the matrix equation, determine how much Veronica would pay for four donuts and three cupcakes.

10. WE15 Solve the following pair of simultaneous equations by using inverse matrices.

$$x + 2y = 4$$

$$3x - 5y = 1$$

11. Show that there is no solution to the following pair of simultaneous equations by attempting to solve them using inverse matrices.

$$3x + 5y = 4$$

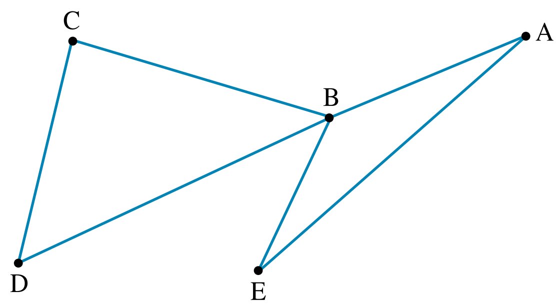
$$4.5x + 7.5y = 5$$

12. WE16 For his 8th birthday party, Ben and his friends went ice skating and ten-pin bowling. The price for ice skating was \$4.50 per child and \$6.50 per adult, with the total cost for the ice skating being \$51. For the ten-pin bowling, the children were charged \$3.25 each and the adults were charged \$4.95 each, with the total cost for the bowling being \$37.60. By solving a matrix equation, determine how many children (including Ben) attended the party.

13. At the cinema, Justine and her friends bought 5 drinks and 4 bags of popcorn, spending \$14. Sarah and her friends bought 4 drinks and 3 bags of popcorn, spending \$10.80. By solving a matrix equation, determine the price of 2 drinks and 2 bags of popcorn.

14. Jeremy has an interest in making jewellery, and he makes bracelets and necklaces which he sells to his friends. He charges the same amount for each bracelet and necklace, regardless of the quantity sold. Johanna buys 3 bracelets and 2 necklaces from Jeremy for \$31.80. Mystique buys 5 bracelets and 3 necklaces from Jeremy for \$49.80.
   a. Construct a pair of simultaneous equations and use an inverse matrix to help determine the prices that Jeremy charges for each bracelet and each necklace.
   b. How much would Jeremy charge for 7 bracelets and 4 necklaces?

15. **WE17** The diagram below shows the network cable between five main computers (A, B, C, D and E) in an office building.



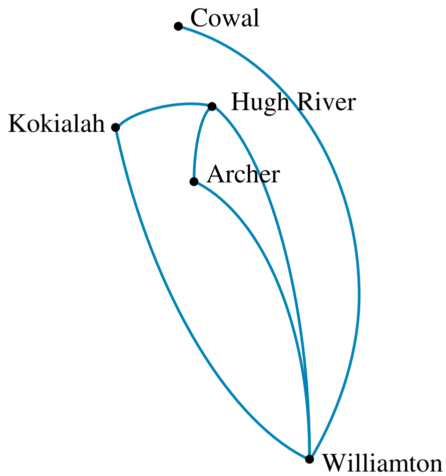
Construct an adjacency matrix to represent this information.

16. There are five friends on a social media site: Peta, Seth, Tran, Ned and Wen. The number of communications made between these friends in the last 24 hours is shown in the adjacency matrix below.

$$\begin{array}{c} \\ P \\ S \\ T \\ N \\ W \end{array}\begin{array}{c} \begin{array}{ccccc} P & S & T & N & W \end{array} \\ \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 & 4 \\ 3 & 0 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

   a. How many times did Peta and Tran communicate over the last 24 hours?
   b. Did Seth communicate with Ned at any time during the last 24 hours?
   c. In the context of this problem, explain the existence of the zeros along the diagonal.
   d. Using the adjacency matrix, construct a diagram that shows the number of communications between the five friends.

17. Airlink flies charter flights in the Cape Lancaster region. The direct flights between Williamton, Cowal, Hugh River, Kokialah and Archer are shown in the diagram.
   a. Using the diagram, construct an adjacency matrix that shows the number of direct flights between the five towns.
   b. How many ways can a person travel between Williamton and Kokialah via another town?
   c. Is it possible to fly between Cowal and Archer and stop over at two other towns? Justify your answer.



18. WE18 The adjacency matrix below shows the number of roads between three country towns, Glenorchy (G), St Arnaud (S) and Campbells Bridge (C).

$$\begin{array}{c} \\ G \\ S \\ C \end{array}\begin{array}{c} \begin{array}{ccc} G & S & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \end{array}$$

Using CAS or otherwise, determine the number of ways a person can travel from Glenorchy to St Arnaud via Campbell's Bridge.

19. The direct Cape Air flights between five cities, Boston (B), Hyannis (H), Martha's Vineyard (M), Nantucket (N) and Providence (P), are shown in the adjacency matrix.

$$\begin{array}{c} \\ B \\ H \\ M \\ N \\ P \end{array}\begin{array}{c} \begin{array}{ccccc} B & H & M & N & P \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a. Construct a diagram to represent the direct flights between the five cities.
b. Construct a matrix that determines the number of ways a person can fly between two cities via another city.
c. Explain how you would determine the number of ways a person can fly between two cities via two other cities.
d. Is it possible to fly from Boston and stop at every other city? Explain how you would answer this question.

20. The adjacency matrix below shows the number of text messages sent between three friends, Stacey (S), Ruth (R) and Toiya (T), immediately after school one day

$$\begin{array}{c} \\ S \\ R \\ T \end{array}\begin{array}{c} \begin{array}{ccc} S & R & T \end{array} \\ \begin{bmatrix} 0 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$$

a. State the number of text messages sent between Stacey and Ruth.
b. Determine the total number of text messages sent between all three friends.

21. Stefan was asked to solve the following matrix equation.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 9 \end{bmatrix}$$

His first step was to evaluate $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}^{-1}$. Stefan wrote $6\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, which was incorrect.

a. Explain one of the errors Stefan made in finding the inverse matrix.

b. Hence, find the correct inverse matrix $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}^{-1}$.

Stefan's next step was to perform the following matrix multiplication.

$$6\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 9 \end{bmatrix}$$

c. By finding the order of both matrices, explain why this multiplication is not possible.

d. Write the steps Stefan should have used to calculate this matrix multiplication.

e. Using your steps from part **d**, determine the values of x and y.

22. WholeFoods distribute two different types of apples, Sundowners and Pink Ladies, to two supermarkets, Foodsale and Betafoods. Foodsale orders 5 boxes of Sundowners and 7 boxes of Pink Ladies, with their order totalling \$156.80. Betafoods pays \$155.40 for 6 boxes each of Sundowners and Pink Ladies.

The matrix below represents part of this information, where s and p represent the price for a box of Sundowners and Pink Ladies respectively.

$$\begin{bmatrix} 5 & \\ 6 & 6 \end{bmatrix}\begin{bmatrix} s \\ p \end{bmatrix} = \begin{bmatrix} 156.80 \\ \end{bmatrix}$$

a. Complete the matrix equation.

b. By solving the matrix equation using CAS or otherwise, determine the cost of a box of Sundowner apples.

There are 5 kg of apples in each box. Betafoods sells Sundowner apples for \$3.49 per kilogram and Pink Ladies for \$4.50 per kilogram.

c. Construct a row matrix, K, to represent the number of kilograms of Sundowners and Pink Ladies in Betafood's order.

A matrix representing the selling price, S, of each type of apple is constructed.

A matrix multiplication is performed that determines the total selling price in dollars, for both types of apples.

d. Write the order of matrix S.

e. By performing the matrix multiplication, determine the total amount (in dollars) in revenue if all the apples are sold at the price stated.

f. Determine the profit, in dollars, made by Betafoods if all apples are sold at the stated selling prices.

23. Four hundred tickets were sold for the opening of the movie *The Robbit* at the Dendy Cinema. Two types of tickets were sold: adult and concession. Adult tickets were \$15.00 and concession tickets were \$9.50. The total revenue from the ticket sales was \$5422.50.

a. Identify the two unknowns and construct a pair of simultaneous equations to represent this information.

b. Set up a matrix equation representing this information.

c. Using CAS or otherwise, determine the number of adult tickets sold.

24. The senior school manager developed a matrix formula to determine the number of school jackets to order for Years 11 and 12 students. The column matrix, J_0, shows the number of jackets ordered last year.

$$J_0 = \begin{bmatrix} 250 \\ 295 \end{bmatrix}$$

J_1 is the column matrix that lists the number of Year 11 and 12 jackets to be ordered this year.
J_1 is given by the matrix formula

$$J_1 = AJ_0 + B, \text{ where } A = \begin{bmatrix} 0.65 & 0 \\ 0 & 0.82 \end{bmatrix} \text{ and } B = \begin{bmatrix} 13 \\ 19 \end{bmatrix}.$$

a. Using CAS or otherwise, determine J_1.

b. Using your value from part **a** and the same matrix formula, determine the jacket order for the next year. Write your answer to the nearest whole number.

25. Using CAS, find the inverse of the following matrices.

a. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \\ 2 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 1 & -1 & -2 \\ -1 & 2 & 0 & 2 \\ 0 & 3 & 5 & 3 \\ 1 & 1 & 4 & 1 \end{bmatrix}$

26. Using CAS, solve the following matrix equation to find the values of a, b, c and d.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & -1 & 3 \\ 3 & 1 & 1 & -2 \\ -1 & 2 & -4 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 12 \\ -14 \end{bmatrix}$$

4.6 Review: exam practice

A summary of this topic is avaiblable in the resources section of your eBook PLUS at www.jacplus.com.au.

Multiple choice

1. MC Consider the matrix equation $\begin{bmatrix} 6 & a \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$.

The values of a and b respectively are:

A. $a = 1, b = 1$ **B.** $a = 1, b = -1$ **C.** $a = 2, b = 1$ **D.** $a = 2, b = -1$
E. $a = 3, b = 1$

2. MC The order of the matrix $\begin{bmatrix} 6 & 1 & 2 \\ 7 & 3 & 5 \\ 9 & 5 & 0 \\ 0 & 7 & 2 \end{bmatrix}$ is:

A. 3×4 **B.** 4×2 **C.** 4×3 **D.** 8 **E.** 12

The following information relates to Questions 3 and 4.

The following matrix shows the airline ticket price between four cities: Perth, Melbourne, Hobart and Sydney.

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cccc} \text{P} & \text{M} & \text{H} & \text{S} \end{array} \\ \text{To} \begin{array}{c} \text{P} \\ \text{M} \\ \text{H} \\ \text{S} \end{array} & \begin{bmatrix} 0 & 140 & 450 & 190 \\ 180 & 0 & 90 & 50 \\ 350 & 80 & 0 & 80 \\ 240 & 60 & 110 & 0 \end{bmatrix} \end{array}$$

3. MC The cost to fly from Sydney to Hobart is:

A. \$50 **B.** \$80 **C.** \$90 **D.** \$110 **E.** \$190

4. MC A passengers flies from one city and then on to another, with the total cost of the flight being \$320. The order in which the plane flew between the three cities was:

A. Sydney to Melbourne to Perth
B. Hobart to Melbourne to Perth
C. Perth to Melbourne to Sydney
D. Perth to Sydney to Hobart
E. Hobart to Sydney to Perth

5. MC Sarah purchased 2 apples and 3 bananas from her local market for \$3.80. Later that week she went back to the market and purchased 4 more apples and 4 more bananas for \$6.20. A matrix equation that could be set up to determine the price of a single apple and banana is:

A. $\begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.8 \\ 6.2 \end{bmatrix}$ **B.** $\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 3.8 \\ 6.2 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.8 \\ 6.2 \end{bmatrix}$ **D.** $\begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 3.8 \\ 6.2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

E. $\begin{bmatrix} 3.8 \\ 6.2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

6. MC The matrix that represents the product $\begin{bmatrix} 5 & 1 & 2 \\ 3 & 7 & 8 \end{bmatrix} \begin{bmatrix} 12 \\ 10 \\ 9 \end{bmatrix}$ is:

A. $\begin{bmatrix} 88 \\ 178 \end{bmatrix}$ **B.** $\begin{bmatrix} 60 & 10 & 18 \\ 36 & 63 & 72 \end{bmatrix}$ **C.** $\begin{bmatrix} 88 & 178 \end{bmatrix}$

D. $\begin{bmatrix} 48 \\ 49 \end{bmatrix}$ **E.** Does not exist

7. MC The population age structure (in percentages) in 2010 for selected countries is shown in the following table.

	Percentage of population between the age groups		
Country	**0–14 years**	**15–64 years**	**Over 65 years**
Australia	18.9	67.6	13.5
China	16.4	69.5	14.1
Indonesia	27.0	67.4	5.6

A 3×1 matrix that could be used to represent the percentage of population across the three age groups for Indonesia is:

A. $\begin{bmatrix} 18.9 \\ 16.4 \\ 27.0 \end{bmatrix}$
B. $\begin{bmatrix} 27.0 & 67.4 & 5.6 \end{bmatrix}$
C. $\begin{bmatrix} 18.9 & 67.6 & 13.5 \\ 16.4 & 69.5 & 14.1 \\ 27.0 & 67.4 & 5.6 \end{bmatrix}$
D. $\begin{bmatrix} 16.4 \\ 69.5 \\ 14.1 \end{bmatrix}$
E. $\begin{bmatrix} 27.0 \\ 67.4 \\ 5.6 \end{bmatrix}$

8. MC Matrix A has an order of 3×2. Matrix B has an order of 1×3. Matrix C has an order of 2×1. Which one of the following matrix multiplications is not possible?
A. AC
B. BA
C. BC
D. CB
E. ACB

9. MC If matrix $A = \begin{bmatrix} 16 & 15 \\ 10 & 9 \end{bmatrix}$, then A^{-1} is which one of the following?

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
B. $\begin{bmatrix} -16 & 10 \\ 15 & -9 \end{bmatrix}$
C. $\begin{bmatrix} 9 & -15 \\ -10 & 10 \end{bmatrix}$
D. $\begin{bmatrix} -54 & 90 \\ 60 & -60 \end{bmatrix}$
E. $\begin{bmatrix} \frac{-3}{2} & \frac{5}{2} \\ \frac{5}{3} & \frac{-8}{3} \end{bmatrix}$

10. MC To help him answer his 10 multiple choice questions, Trei used the following matrix.

$$\text{Next question} \begin{matrix} \\ A \\ B \\ C \\ D \\ E \end{matrix} \overset{\text{This question}}{\begin{matrix} A & B & C & D & E \\ \end{matrix}} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Trei answered B to question 1 and then used the matrix to answer the remaining nine questions. What was Trei's answer to question 6?
A. C
B. D
C. E
D. A
E. B

Short answer

1. Given matrices $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, evaluate the following.
   a. $C + B$
   b. $B - 2C$
   c. $AB + C$
   d. $1.5A$
2. Matrix D has an order of 3×2, matrix E has an order of $1 \times p$ and matrix F has an order of 2×2.
   a. For what value of p would the product matrix ED exist?
   b. If the product matrix H exists and $H = EDF$, state the order of H.

3. The table below shows the three different ticket prices, in dollars, and the number of tickets sold for a school concert.

Ticket type	Ticket price	Number of tickets sold
Adult	\$12.50	140
Child/student	\$6.00	225
Teacher	\$10.00	90

a. Construct a column matrix to represent the ticket prices for adults, children/students and teachers respectively.

b. Perform a matrix multiplication to determine the total amount of ticket sales in dollars.

4. a. For each of the following pairs of matrices, state the order of the matrices, and hence state the order of the product matrix.

i. $\begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

ii. $\begin{bmatrix} 4 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$

iii. $\begin{bmatrix} -1 & 9 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 5 & 11 \end{bmatrix}$

iv. $\begin{bmatrix} 2 \\ 7 \\ 8 \end{bmatrix} \begin{bmatrix} 5 & 3 & 4 \end{bmatrix}$

b. Find the product matrices of the matrix multiplications given in part a.

5. a. Find the determinant of each of the following matrices.

i. $A = \begin{bmatrix} 6 & 3 \\ 5 & 3 \end{bmatrix}$

ii. $B = \begin{bmatrix} -2 & 2 \\ -1 & 3 \end{bmatrix}$

b. Find the inverse of each matrix given in part a.

c. Matrix $M = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$. Explain why M^{-1} does not exist.

6. Solve the following simultaneous equations using matrix methods.

a. $2x + 3y = 1$ and $4x - y = 9$

b. $3x - 2y = 12$ and $x + 2y = 19$

Extended response

1. The energy content and amounts of fat and protein contained in each slice of bread and cheese and one teaspoon of margarine is shown in the table below.

Food	Energy content (kilojoules)	Fat (grams)	Protein (grams)
Bread	410	0.95	3.7
Cheese	292	5.5	1.6
Margarine	120	3.3	0.5

Pedro made toasted cheese sandwiches for himself and his friends for lunch. The total amount of fat and protein (in grams) for each of the three foods — bread, cheese and margarine — in the prepared lunch were recorded in the following matrix.

$$\begin{bmatrix} 7.6 & 29.6 \\ 44.0 & x \\ 13.2 & 2 \end{bmatrix}$$

a. How many bread slices did Pedro use?

b. If each sandwich used two pieces of bread, how many cheese sandwiches did Pedro make?

c. Show that each sandwich had two slices of cheese.

d. Hence, find the exact value of x.

e. Construct a 1×3 matrix to represent the number of slices of bread and cheese and servings of margarine for each sandwich.

2. Tootin' Travel Agents sell three different types of train travel packages on the Midnight Express: Platinum, Gold and Red class. The price for each travel package is shown in the table.

Class	Price
Platinum	\$3890
Gold	\$2178
Red	\$868

a. Construct a column matrix, C, to represent the price of each of the three travel packages: Platinum, Gold and Red. State the order of C.

b. In the last month, Tootin' Travel Agents sold the following number of train travel packages.
- Platinum: 62
- Gold: 125
- Red: 270

Construct a row matrix, P, to represent the number of train travel packages sold over the last month

c. To determine the total amount in dollars for train travel packages in the month, a product matrix, PC is found. State the order of product matrix PC.

Travellers who book in a year in advance receive a 5% discount. To calculate the discounted price, matrix C is multiplied by a scalar product, d.

d. Write down the value of d.

e. Using your value for d, construct a new matrix, E, that represents the discounted travel prices. Write your answer correct to the nearest cent.

3. TruSport owns two stores, one at LeisureLand and one at SportLand shopping centres. The number of tennis racquets, baseball bats and soccer balls sold in the last week at the two stores is shown in the table below.

Store	Tennis racquets	Baseball bats	Soccer balls
LeisureLand	10	8	9
SportLand	9	12	11

The selling price of each item is shown in the table below.

	Tennis racquet	Baseball bat	Soccer ball
Selling price	\$45.95	\$25.50	\$18.60

a. Construct a 3×2 matrix to represent the number of tennis racquets, baseball bats and soccer balls sold at each of the two stores.

b. Construct a row matrix to represent the selling prices of each of the items.

c. **i.** Set up a matrix multiplication that finds the total amount, in dollars, that each store made in the last week.

ii. Hence find the total amount, in dollars, that each store made in the last week.

4. There are 472 students studying History and 424 studying Economics at a university. At the end of the academic year, 25% of students will be awarded a Pass grade, 38% will be awarded a Credit grade, 19% will be awarded Distinction grade, 8% will awarded a High Distinction grade and the remaining students will not pass.

The column matrix $N = \begin{bmatrix} 472 \\ 424 \end{bmatrix}$ represents the numplber of students studying History and Economics.

a. Write down the order of matrix N.
b. Construct a row matrix, G, to represent the percentages, in decimal form, of students who will be awarded one of the five grades
c. Evaluate the product matrix $A = NG$. Write your answer to the nearest whole number.
d. In the context of this problem, explain what the element A_{24} means.
   The cost for textbooks for a student studying History is \$125; for Economics, the textbooks cost \$235.
e. Construct a matrix calculation that will give the total cost for textbooks, C, paid in dollars by the students studying History and Economics.
f. Calculate the total cost for the textbooks.

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Units 1 & 2 Sit topic test

Answers

Topic 4 Matrices

Exercise 4.2 Types of matrices

1. $\begin{bmatrix} 18 & 12 & 8 \\ 13 & 10 & 11 \end{bmatrix}$

2. $\begin{bmatrix} 6 & 4 & 7 & 3 & 6 \\ 2 & 6 & 6 & 6 & 4 \end{bmatrix}$

3. **a.** 545 km
b. Coober Pedy and Alice Springs
c. 2458 km

4. **a.** $\begin{array}{c} \\ M \\ C \\ S \end{array}\begin{array}{c} \begin{array}{ccc} M & C & S \end{array} \\ \begin{bmatrix} 0 & 91.13 & 110.72 \\ 91.13 & 0 & 48.02 \\ 110.72 & 48.02 & 0 \end{bmatrix} \end{array}$
b. \$139.15

5. $\begin{bmatrix} 45 \\ 30 \end{bmatrix}$, order 2×1

6. **a.** 56 **b.** 213 **c.** $\begin{bmatrix} 12 & 17 & 18 \end{bmatrix}$

7. A: 1×1, B 3×1, C 1×2

8. **a.** **a** and **d** are matrices with orders of 2×1 and 2×4 respectively. The matrix shown in **b** is incomplete, and the matrix shown in **c** has a different number of rows in each column.

9. **a.** 5 **b.** 6 **c.** 7

10. **a.** There is no 4th column.
b. e_{23}
c. Nadia thought that e_{12} was read as 1st column, 2nd row. The correct value is 0.

11. **a.** 3 **b.** -1 **c.** 1 **d.** 0.5 **e.** 0.9

12. **a.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

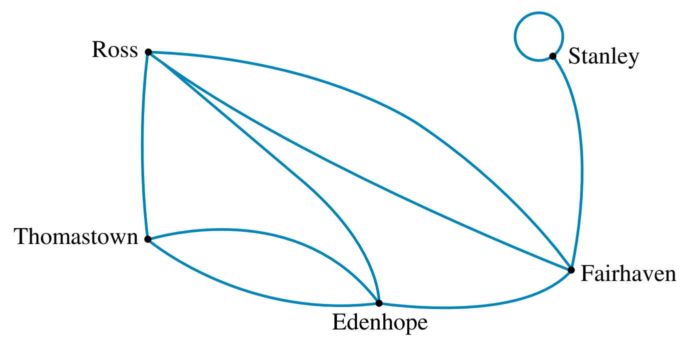
13. **a.** 3×2
b. $\begin{bmatrix} 4 & 3 \\ -1 & 7 \\ -4 & 6 \end{bmatrix}$

14. **a.** $\begin{bmatrix} 0.4 \\ 4.2 \\ 6.8 \\ 0.2 \\ 1.6 \\ 2.1 \\ 0.5 \\ 5.2 \end{bmatrix}$
b. $\begin{bmatrix} 2358 & 68\,330 & 227\,600 & 801\,428 & 984\,000 & 1\,346\,200 & 1\,727\,200 & 2\,529\,875 \end{bmatrix}$
c. **i.** 3×2
ii. $\begin{bmatrix} 801\,428 & 6.8 \\ 227\,600 & 5.2 \\ 1\,727\,200 & 4.2 \end{bmatrix}$

15. **a.** $\begin{bmatrix} 148\,178 & 2.2 \\ 30\,839 & 0.6 \\ 146\,429 & 3.6 \\ 26\,044 & 1.7 \\ 77\,928 & 3.8 \\ 16\,900 & 3.4 \\ 66\,582 & 31.6 \\ 4\,043 & 1.2 \end{bmatrix}$
b. **i.** 66 582 **ii.** 16 900 **iii.** 325 446
c. 516 943

16. **a.** The zeros mean they don't fly from one place back to the same place.
b. \$175
c. Mount Isa
d. $\begin{array}{c} \\ O \\ B \\ D \\ M \end{array}\begin{array}{c} \begin{array}{cccc} O & B & D & M \end{array} \\ \begin{bmatrix} 0 & 59.50 & 150 & 190 \\ 89 & 0 & 85 & 75 \\ 175 & 205 & 0 & 213.75 \\ 307 & 90 & 75.75 & 0 \end{bmatrix} \end{array}$

17. **a.**



b. **i.** True **ii.** False **iii.** True **iv.** False
c. N_{31} and N_{13}

18. **a.** D
b.

Question	1	2	3	4	5	6	7	8	9	10
Response	A	D	B	E	D	B	E	D	B	E

c. There are no 1s in row A, just 0s.
d. This question

$\text{Next question} \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$

19. Possible answers may include: Press enter to select a blank matrix, use the tab button to move between entries, etc.

20. a. Possible answers:
i. C1 ii. B2 iii. B3 iv. A2

b. i. The column position corresponds to the alphabetical order, e.g. column 2 would be B, the second letter in the alphabet. The row position corresponds to the cell row, e.g. row 4 would be cell row 4.

ii. The cell number would be the *m*th letter in the alphabet and the *n*th row.

Exercise 4.3 Operations with matrices

1. a. $\begin{bmatrix} 1 & 6 \\ -1 & 3 \end{bmatrix}$ b. $\begin{bmatrix} -0.1 \\ 1.5 \\ 4.2 \end{bmatrix}$

2. $a = 4$, $b = -6$

3. a. $\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$ b. $\begin{bmatrix} -5 \\ 3 \\ 6 \end{bmatrix}$ c. $\begin{bmatrix} 6 \\ 4 \\ -6 \end{bmatrix}$ d. $\begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix}$

4. $\begin{bmatrix} -1 & -2 \\ -1 & -5 \end{bmatrix}$

5. a. Both matrices must be of the same order for it to be possible to add and subtract them.

b. $B = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$

6. a. $\begin{bmatrix} -0.25 & -0.95 & 0.3 \end{bmatrix}$ b. $\begin{bmatrix} 3 & -1 \\ 9 & 1 \end{bmatrix}$

c. $\begin{bmatrix} -1 & 5 & 1 \\ 4 & -2 & 3 \\ 3 & -3 & -2 \end{bmatrix}$ d. $\begin{bmatrix} 15 & 15 & 7 \\ 9 & 10 & 8 \end{bmatrix}$

7. $a = -5$, $b = 2$, $c = 5$

8. $a = 6$, $b = -1$, $c = 5$

9. A and E have the same order, 1×2.
B and C have the same order, 2×1.

10. a. $\begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 3 \end{bmatrix}$

b. i. $\begin{bmatrix} 3 & 4 & 8 \\ 6 & 8 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 3 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \end{bmatrix}$

Eggs	Small	Medium	Large
Free range	1	1	3
Barn laid	2	2	2

11. a. Both matrices are of the order 2×3; therefore, the answer matrix must also be of the order 2×3. Marco's answer matrix is of the order 2×1, which is incorrect.

b. A possible response is:
Step 1: Check that all matrices are the same order.
Step 2: Add or subtract the corresponding elements.

12. a. $\begin{bmatrix} 150 \\ 165 \\ 155 \\ 80 \end{bmatrix}$

b. i. $\begin{bmatrix} 150 \\ 165 \\ 155 \\ 80 \end{bmatrix} + \begin{bmatrix} 145 \\ 152 \\ 135 \\ 95 \end{bmatrix} + \begin{bmatrix} 166 \\ 155 \\ 156 \\ 110 \end{bmatrix}$

ii. $\begin{bmatrix} 461 \\ 472 \\ 446 \\ 285 \end{bmatrix}$

c. $\begin{bmatrix} 35 \\ 41 \\ 38 \\ 25 \end{bmatrix} + \begin{bmatrix} 32 \\ 36 \\ 35 \\ 30 \end{bmatrix} + \begin{bmatrix} 38 \\ 35 \\ 35 \\ 32 \end{bmatrix} = \begin{bmatrix} 105 \\ 112 \\ 108 \\ 87 \end{bmatrix}$

13. a. $\begin{bmatrix} 12 \\ 9 \\ 8 \end{bmatrix} + \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$ or $\begin{bmatrix} 12 \\ 9 \\ 8 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \\ 10 \end{bmatrix}$ or $\begin{bmatrix} 12 \\ 4 \\ 10 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

14. a. To add or subtract matrices, all matrices must be of the same order. Since the resultant matrix D is of the order 3×2, all other matrices must also be of the order 3×2.

b. $\begin{bmatrix} 2x - 12 & y - 2 \\ 2x + 10 & 5 - y \\ 3x + 10 & -2 - 2x \end{bmatrix}$

c. When the equations $2x - 12 = -8$ and $y - 2 = 1$ are solved, the answer is $x = 2$, $y = 3$.

15. $\begin{bmatrix} \frac{5}{8} & \frac{5}{8} & 0 \\ \frac{17}{30} & \frac{5}{42} & \frac{-5}{9} \\ \frac{11}{18} & \frac{1}{8} & \frac{-2}{9} \end{bmatrix}$

16. a. Possible answer:
= sum (A1 + D1) = sum (B1 + E1) = sum (C1 + F1)
= sum (A2 + D2) = sum (B2 + E2) = sum (C2 + F2)

b. $\begin{bmatrix} 11 & 29 & 20 \\ 54 & 5 & 27 \end{bmatrix}$

Exercise 4.4 Matrix multiplication

1. a. $\begin{bmatrix} 8 & 12 & 28 \\ 4 & 16 & 24 \end{bmatrix}$ b. $\begin{bmatrix} \frac{2}{5} & \frac{3}{5} & \frac{7}{5} \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} \end{bmatrix}$

c. $\begin{bmatrix} 0.6 & 0.9 & 2.1 \\ 0.3 & 1.2 & 1.8 \end{bmatrix}$

2. $x = 2.5$

3. C

4. a. $1 \times 2\ 2 \times 1$
Number of columns = number of rows, therefore XY exists and is of order 1×1.

b. $DE{:}3 \times 3$, $DC{:}3 \times 2$, $ED{:}2 \times 2$, $CE{:}2 \times 3$

5. $S{:}3 \times 2$, $T{:}2 \times 4$

6. $DG{:}3 \times 2$, $FD{:}2 \times 1$, $FE{:}2 \times 2$, $EF{:}3 \times 3$, $GF{:}1 \times 3$

7. a. $MN = \begin{bmatrix} 28 & 48 \\ 21 & 36 \end{bmatrix}$ b. Yes, [64] is of the order 1×1.

8. $t = -3$

9. $PQ = \begin{bmatrix} 41 & 45 \\ 36 & 32 \end{bmatrix}$

10. a. $\begin{bmatrix} 12.50 \\ 8.50 \\ 6.00 \end{bmatrix}$

b. Total tickets requires an order of 1×1, and the order of the ticket price is 3×1. The number of people must be of order 1×3 to result in a product matrix of order 1×1. Therefore, the answer must be a row matrix.

c. \$1662.50

11. a. [66]   b. $\begin{bmatrix} 30 & 45 \\ 24 & 36 \end{bmatrix}$ c. $\begin{bmatrix} 70 & 105 \\ 20 & 30 \\ 90 & 135 \end{bmatrix}$

d. $\begin{bmatrix} 57 \\ 43 \end{bmatrix}$ e. $\begin{bmatrix} 38 & 19 \\ 14 & 7 \end{bmatrix}$

12. a. $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

c. When either A or B is the identity matrix

d. No. Consider matrix A with order of $m \times n$ and matrix B with order of $p \times q$, where $m \neq p$ and $n \neq q$. If AB exists, then it has order $m \times q$ and $n = p$. If BA exists, then it has order $p \times n$ and $q = m$. Therefore $AB \neq BA$, unless $m = p$ and $n = q$, which is not possible since they are of different orders.

13. $\begin{bmatrix} 72 & 30 \\ 60 & 57 \end{bmatrix}$

14. $\begin{bmatrix} 27 & 315 \\ 0 & 216 \end{bmatrix}$

15. a. $I_3{}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b. $I_3{}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $I_3{}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d. Whatever power you raise I to, the matrix stays the same.

16. a. $\begin{bmatrix} 250 \\ 185 \end{bmatrix}$

b. $\begin{bmatrix} 0.05 & 0.18 & 0.45 & 0.25 & 0.07 \end{bmatrix}$

c. The number of expected grades (A–E) for students studying Mathematics and Physics.

d. $\begin{bmatrix} 13 & 45 & 113 & 63 & 18 \\ 9 & 33 & 83 & 46 & 13 \end{bmatrix}$

e. 45 students studying maths are expected to be awarded a B grade.

17. $n = 1,\ m = 3,\ s = 4,\ q = 2$

18. a. Possible answer:
Represent the number of vehicles in a row matrix and the cost for each vehicle in a column matrix, then multiply the two matrices together. The product matrix will have an order of 1×1.

b. [154 000] or \$154 000

c. Possible answer:
In this multiplication each vehicle is multiplied by price of each type of vehicle, which is incorrect. For example, the ute is valued at \$12 500, but in this multiplication the eight utes sold are multiplied by \$4000, \$12 500 and \$8500 respectively.

19. a. Matrix G is of order 3×2 and matrix H is of order 2×1; therefore, GH is of order 3×1. Rhonda's matrix has an order of 3×2.

b. $\begin{bmatrix} 125 \\ 134 \\ 167 \end{bmatrix}$

c. Possible answer:
Rhonda multiplied the first column with the first row, and then the second column with the second row.

d. Possible answer:
Step 1: Find the order of the product matrix.
Step 2: Multiply the elements in the first row by the elements in the first column.

20. $C_1 = 68,\ S_1 = 68,\ C_2 = 108,\ S_2 = 52$. The two results were 68–68 and 108–52.

21. a. $\begin{bmatrix} 4 & 8 \\ 7 & 2 \end{bmatrix}^4 = \begin{bmatrix} 7200 & 6336 \\ 5544 & 5616 \end{bmatrix}$

b. $\begin{bmatrix} \frac{2}{3} & \frac{5}{7} \\ \frac{1}{4} & 2 \end{bmatrix}^3 = \begin{bmatrix} \frac{337}{378} & \frac{7505}{1764} \\ \frac{1501}{1008} & \frac{53}{6} \end{bmatrix}$

c. $\begin{bmatrix} 3 & 1 & 7 \\ 4 & 2 & 8 \\ 5 & 6 & 9 \end{bmatrix}^3 = \begin{bmatrix} 792 & 694 & 1540 \\ 984 & 868 & 1912 \\ 1356 & 1210 & 2632 \end{bmatrix}$

22. a. $\begin{bmatrix} 12\,280 \\ 26\,210 \\ 29\,850 \end{bmatrix}$; Friday \$12 280, Saturday \$26 210, Sunday \$29 850

b. $350 \times 35 + 456 \times 25 + 128 \times 20$

c. No, because you cannot multiply the entry price (3×1) by the number of people (3×3).

Exercise 4.5 Inverse matrices and problem solving with matrices

1. Yes, $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2. $AB = \begin{bmatrix} 3+a & 0 \\ 6+3a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3. a. $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -2 \\ -1.5 & 3.5 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$

4. det $B = 0$. We cannot divide by zero; therefore, B^{-1} does not exist.

5. a. $\det A = 1$, $\det B = 0$, $\det C = 0$, $\det D = -3$. Therefore, matrices A and D have inverses.

b. $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$, $D^{-1} = \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ 1 & 0 \end{bmatrix}$

6. $x = 2$, $y = -3$

7. a. Matrix X has an order of 1×2. $X = [-2\ 3]$ b. $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$

8. a. Find the inverse of $\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ and multiply by $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$, using $AX = B$ and $X = A^{-1}B$.
b. $\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$
c. $x = 2$, $y = 1$

9. $17.50

10. $x = 2$, $y = 1$

11. The determinant = 0, so no inverse exists. This means that there is no solution to the simultaneous equations.

12. 7 children and 3 adults

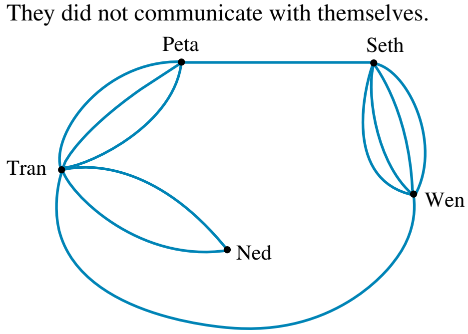
13. $6.40

14. a. Each bracelet costs $4.20 and each necklace costs $9.60.
b. $67.80

15.

	A	B	C	D	E
A	0	1	0	0	1
B	1	0	1	1	1
C	0	1	0	1	0
D	0	1	1	0	0
E	1	1	0	0	0

16. a. 3
b. No
c. They did not communicate with themselves.
d.



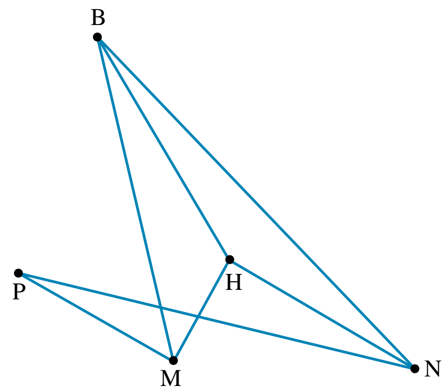
17. a.

	W	C	H	K	A
W	0	2	1	1	1
C	2	0	0	0	0
H	1	0	0	1	1
K	1	0	1	0	0
A	1	0	1	0	0

b. 1
c. Yes. The matrix raised to the power of 3 will provide the number of ways possible.

18. 2

19. a.



b.

	B	H	M	N	P
B	3	2	2	2	2
H	2	3	2	2	2
M	2	2	4	3	1
N	2	2	3	4	1
P	2	2	1	1	2

c. Raise the matrix to a power of 2.
d. Yes. Raise the matrix to a power of 4, as there are five cities in total.

20. a. 3 b. 6

21. a. Any one of: did not swap the elements on the diagonal; did not multiply the other elements by −1; or did not multiply the matrix by $\frac{1}{\det}$.
b. $\frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$
c. The respective order of matrices is 2×2 and 1×2. The number of columns in the first matrix does not equal the number of rows in the second matrix.
d. Possible answers:
Step 1: Find the correct inverse.
Step 2: Multiply $\begin{bmatrix} 5 & 9 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$.
e. $x = 1$, $y = 3$

22. a. $\begin{bmatrix} 5 & 7 \\ 6 & 6 \end{bmatrix}\begin{bmatrix} s \\ p \end{bmatrix} = \begin{bmatrix} 156.80 \\ 155.40 \end{bmatrix}$
b. $12.25
c. $K = \begin{bmatrix} 30 & 30 \end{bmatrix}$
d. 2×1
e. $239.70
f. $84.30

23. a. a = number of adult tickets sold
b = number of concession tickets sold
$a + c = 400$ and $15a + 9.5c = 5422.5$
b. $\begin{bmatrix} 1 & 1 \\ 15.00 & 9.50 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 400 \\ 5422.50 \end{bmatrix}$
c. 295

24. a. $J_1 = \begin{bmatrix} 175.5 \\ 260.9 \end{bmatrix}$
b. 127 Year 11 jackets and 233 Year 12 jackets

25. a. $\begin{bmatrix} \frac{-4}{19} & \frac{-1}{19} & \frac{13}{19} \\ \frac{-7}{19} & \frac{3}{19} & \frac{-1}{19} \\ \frac{8}{19} & \frac{2}{19} & \frac{-7}{19} \end{bmatrix}$

b. $\begin{bmatrix} 0 & \frac{7}{3} & \frac{-8}{3} & \frac{10}{3} \\ \frac{1}{3} & \frac{-7}{9} & \frac{11}{9} & \frac{-13}{9} \\ 0 & -1 & 1 & -1 \\ \frac{-1}{3} & \frac{22}{9} & \frac{-23}{9} & \frac{28}{9} \end{bmatrix}$

26. $a = 2, b = -1, c = 3, d = -2$

4.6 Review: exam practice

Multiple choice

1. A 2. C 3. B 4. D 5. A
6. A 7. E 8. C 9. E 10. C

Short Answer

1. a. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ b. $\begin{bmatrix} -5 \\ -4 \end{bmatrix}$ c. $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ d. $\begin{bmatrix} 3 & 4.5 \\ 1.5 & 6 \end{bmatrix}$

2. a. 3
b. 1×2

3. a. $\begin{bmatrix} 12.5 \\ 6 \\ 10 \end{bmatrix}$
b. $4000

4. a. i. 1×2 and 2×1; product matrix: 1×1
ii. 1×2 and 2×2; product matrix: 1×2
iii. 2×2 and 2×2; product matrix: 2×2
iv. 3×1 and 1×3; product matrix: 3×3
b. i. [36] ii. $\begin{bmatrix} 32 & 54 \end{bmatrix}$
iii. $\begin{bmatrix} 42 & 101 \\ 55 & 35 \end{bmatrix}$ iv. $\begin{bmatrix} 10 & 6 & 8 \\ 35 & 21 & 28 \\ 40 & 24 & 32 \end{bmatrix}$

5. a. i. 3 ii. -4
b. i. $\begin{bmatrix} 1 & -1 \\ \frac{-5}{3} & 2 \end{bmatrix}$ ii. $\begin{bmatrix} \frac{-3}{4} & \frac{1}{2} \\ \frac{-1}{4} & \frac{1}{2} \end{bmatrix}$
c. Matrix M is not a square matrix; therefore, it does not have an inverse.

6. a. $x = 2, y = -1$ b. $x = 7.75, y = 5.625$

Extended response

1. a. 8 slices of bread
b. 4 sandwiches
c. $44 \div 5.5 = 8, 8 \div 4 = 2$
d. 12.8
e. $\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}$

2. a. $\begin{bmatrix} 3890 \\ 2178 \\ 868 \end{bmatrix}$, 3×1
b. $\begin{bmatrix} 62 & 125 & 270 \end{bmatrix}$
c. 1×1
d. 0.95
e. $\begin{bmatrix} 3695.50 \\ 2069.10 \\ 824.60 \end{bmatrix}$

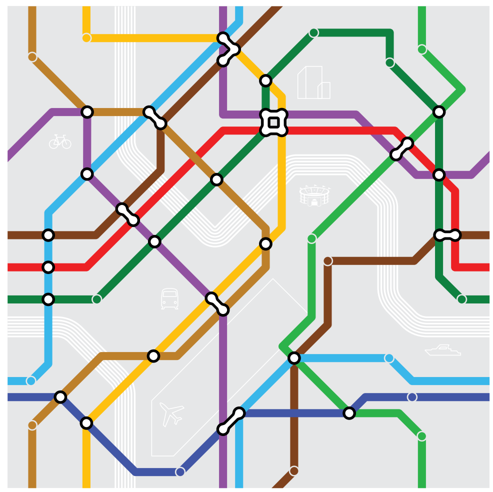
3. a. $\begin{bmatrix} 10 & 9 \\ 8 & 12 \\ 9 & 11 \end{bmatrix}$
b. $\begin{bmatrix} 45.95 & 25.50 & 18.60 \end{bmatrix}$
c. i. $\begin{bmatrix} 45.95 & 25.50 & 18.60 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 8 & 12 \\ 9 & 11 \end{bmatrix}$
ii. LeisureLand: $830.90, SportLand: $924.15

4. a. 2×1
b. $\begin{bmatrix} 0.1 & 0.25 & 0.38 & 0.19 & 0.08 \end{bmatrix}$
c. $\begin{bmatrix} 47 & 118 & 176 & 90 & 38 \\ 42 & 106 & 161 & 81 & 34 \end{bmatrix}$
d. 81 students studying Economics will receive Distinction grades.
e. $C = PN$, where $N = \begin{bmatrix} 472 \\ 424 \end{bmatrix}$ and $P = \begin{bmatrix} 125 & 235 \end{bmatrix}$
f. $158 640

TOPIC 5
Graphs and networks

5.1 Overview

5.1.1 Introduction



Just like matrices, networks are used to show how things are connected. The idea of networks and graph theory is usually credited to Leonhard Euler's 1736 work, *Seven Bridges of Königsberg*. This work carried on with the analysis situs initiated by Leibniz (mentioned for his work on matrices in chapter 4).

One of the most famous problems in graph theory is the 'four colour problem', which poses the question: 'Is it true that any map drawn in the plane may have its regions coloured with four colours, in such a way that any two regions having a common border have different colours?' This question was first posed by Francis Guthrie in 1852. There have been many failed attempts to prove this. The 'four colour problem' remained unsolved for more than a century, until in 1969 Heinrich Heesch published a method for solving the problem using a computer. Computers allowed networks to be used to solve problems that previously took too long due to the multitude of combinations.

Procedures called algorithms are applied to networks to find maximum and minimum values. This further study of constructed networks belongs to a field of mathematics called operational research. This developed rapidly during and after World War II, when mathematicians, industrial technicians and members of the armed services worked together to improve military operations.

In more recent times, graph theory and networks have been used to deliver mail around the neighbourhood, land people on the moon, organise train timetables and improve the flow of traffic. Graph theory and networks have also been applied to a wide range of disciplines, from social networks, used to examine the structure of relationships and social identities, to biological networks, which analyse molecular networks.

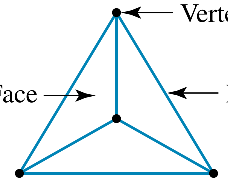
LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.

5.1.2 Kick off with CAS

Euler's formula

Leonhard Euler was a Swiss mathematician and physicist who is credited as the founder of graph theory. In graph theory, a graph is made up of vertices (nodes) and edges connecting the vertices.



Euler's formula is considered to be the first theorem of graph theory. It relates to planar graphs — graphs in which there are no intersecting edges. In all planar graphs, edges and vertices divide the graph into a number of faces, as shown in the diagram.

Euler's formula states that in a connected planar graph, $v - e + f = 2$, where v is the number of vertices, e is the number of edges and f is the number of faces in the graph.

1. Using CAS, define and save Euler's formula.
2. Use CAS to solve your formula for the pronumeral f.
3. Use your formula from question 2 to calculate how many faces a planar graph has if it consists of:
   a. 5 vertices and 7 edges
   b. 4 vertices and 9 edges
   c. 3 vertices and 3 edges.
4. Use CAS to solve Euler's formula for the pronumeral e.
5. Use your formula from question 4 to calculate how many edges a planar graph has if it consists of:
   a. 6 vertices and 4 faces
   b. 8 vertices and 8 faces
   c. 7 vertices and 3 faces.

Leonhard Euler

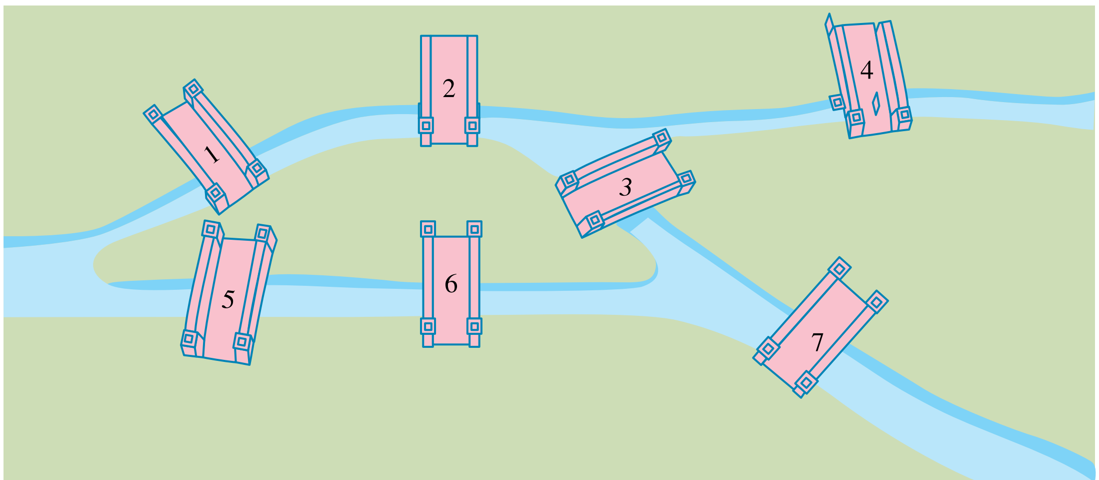
on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology

5.2 Definitions and terms

As you will have noticed in previous years, it is a common practice to draw diagrams and other visual and graphic representations when solving many mathematical problems. In the branch of mathematics known as graph theory, diagrams involving points and lines are used as a planning and analysis tool for systems and connections. Applications of graph theory include business efficiency, transportation systems, design projects, building and construction, food chains and communications networks.

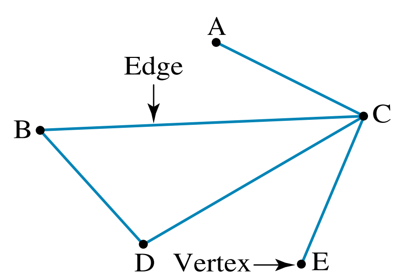
The mathematician Leonhard Euler (1707–83) is usually credited with being the founder of graph theory. He famously used it to solve a problem known as the 'Bridges of Königsberg'. For a long time it had been pondered whether it was possible to travel around the European city of Königsberg (now called Kaliningrad) in such a way that the seven bridges would only have to be crossed once each.



Bridges of Königsberg

5.2.1 Graphs

A **graph** is a series of points and lines that can be used to represent the connections that exist in various settings.

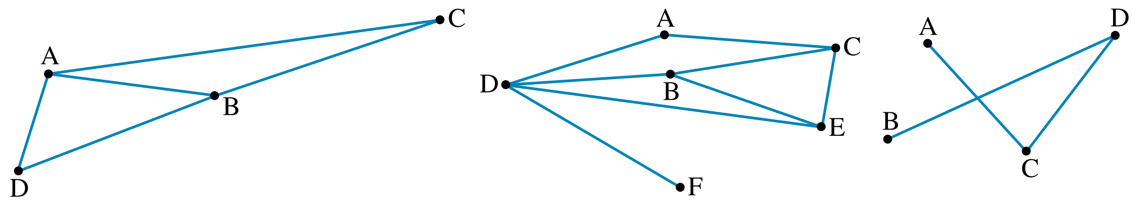


In a graph, the lines are called edges (sometimes referred to as 'arcs') and the points are called vertices (or 'nodes'), with each edge joining a pair of vertices.

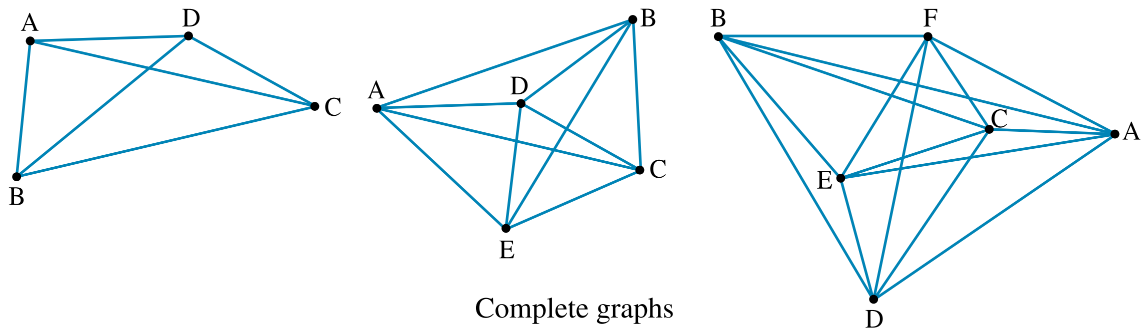
Although edges are often drawn as straight lines, they don't have to be.

When vertices are joined by an edge, they are known as 'adjacent' vertices. Note that the edges of a graph can intersect without there being a vertex. For example, the graph at right has five edges and five vertices.

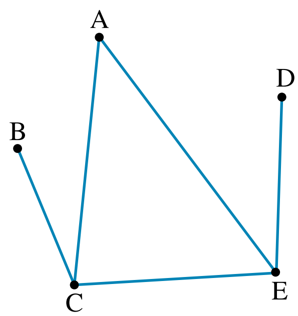
A simple graph is one in which pairs of vertices are connected by one edge at most.
If there is an edge connecting each vertex to all other vertices in the graph, it is called a complete graph.
If it is possible to reach every vertex of a graph by moving along the edges, it is called a connected graph; otherwise, it is a disconnected graph.



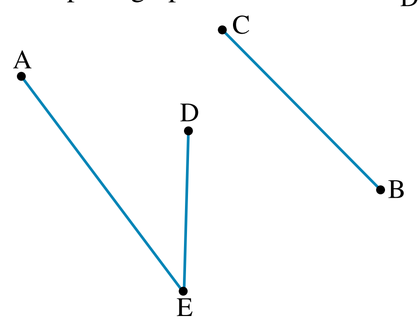
Simple graphs



Complete graphs

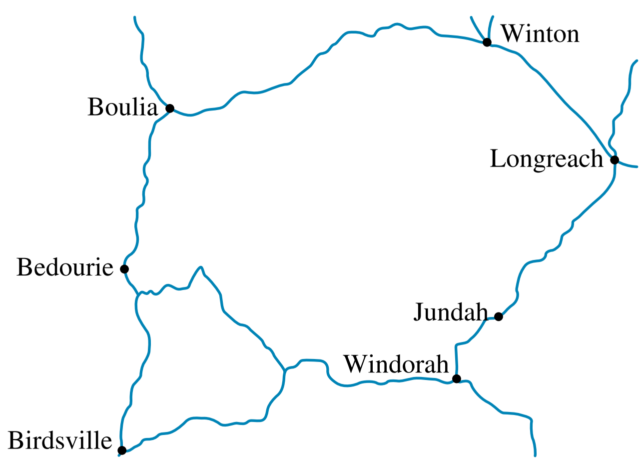


Connected graph

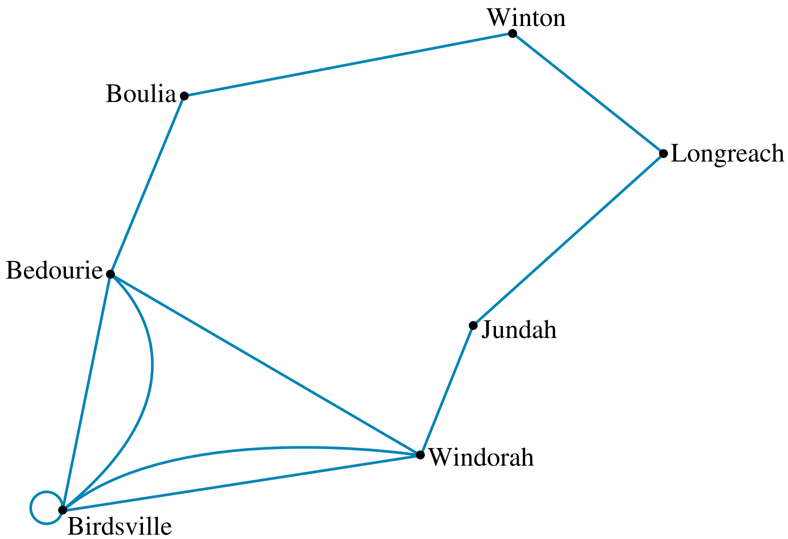


Disconnected graph

Consider the road map shown.



This map can be represented by the following graph.

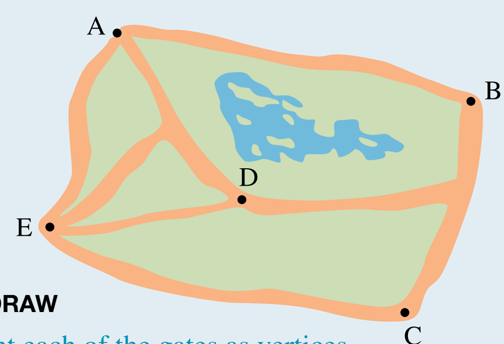


As there is more than one route connecting Birdsville to Windorah and Birdsville to Bedourie, they are each represented by an edge in the graph. In this case we say there are multiple edges. Also, as it is possible to travel along a road from Birdsville that returns without passing through another town, this is represented by an edge. When this happens, the edge is called a **loop**.

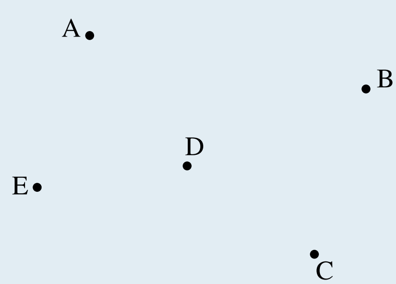
If it is only possible to move along the edges of a graph in one direction, the graph is called a directed graph and the edges are represented by arrows. Otherwise it is an undirected graph.

WORKED EXAMPLE 1

The diagram represents a system of paths and gates in a large park. Draw a graph to represent the possible ways of travelling to each gate in the park.

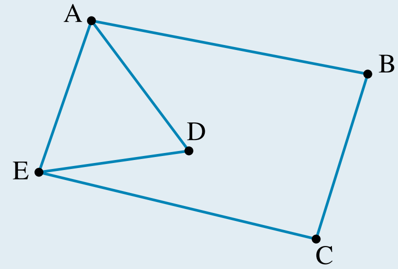


THINK	WRITE/DRAW
1. Identify, draw and label all possible vertices.	Represent each of the gates as vertices.



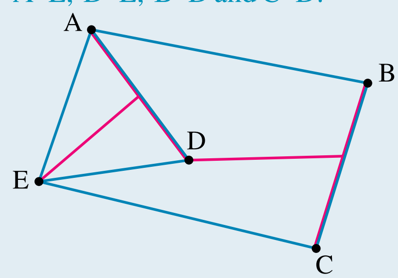
2. Draw edges to represent all the direct connections between the identified vertices.

Direct pathways exist for
A – B, A – D, A – E, B – C, C – E and D – E.

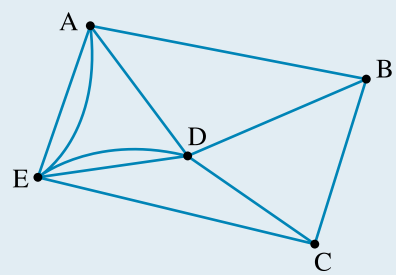


3. Identify all the other unique ways of connecting vertices.

Other unique pathways exist for
A–E, D–E, B–D and C–D.

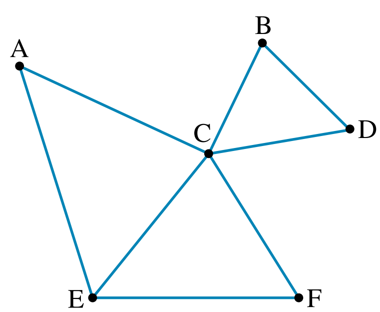


4. Draw the final graph.



5.2.2 The degree of a vertex

When analysing the situation that a graph is representing, it can often be useful to consider the number of edges that are directly connected to a particular vertex. This is referred to as the **degree** of the vertex and is given the notation deg(V), where V represents the vertex.



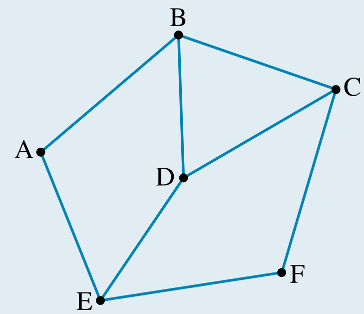
The degree of a vertex = the number of edges directly connected to that vertex.

In the diagram, deg(A) = 2, deg(B) = 2, deg(C) = 5, deg(D) = 2, deg(E) = 3 and deg(F) = 2.

Notice that the sum of the degrees in this graph is 16. The total number of edges in the graph should always be half of the sum of the degrees. In an undirected graph, a vertex with a loop counts as having a degree of 2.

WORKED EXAMPLE 2

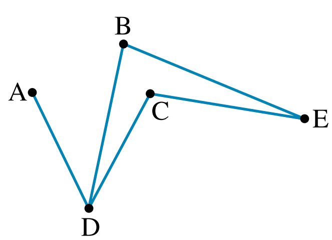
For the graph in the following diagram, show that the number of edges is equal to the half the sum of the degree of the vertices.

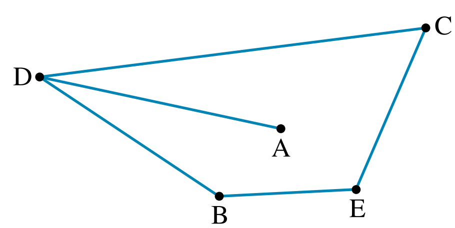


THINK	WRITE
1. Identify the degree of each vertex.	deg(A) = 2, deg(B) = 3, deg(C) = 3, deg(D) = 3, deg(E) = 3 and deg(F) = 2
2. Calculate the sum of the degrees for the graph.	The sum of the degrees for the graph $= 2 + 3 + 3 + 3 + 3 + 2$ $= 16$
3. Count the number of edges for the graph.	The graph has the following edges: A–B, A–E, B–C, B–D, C–D, C–F, D–E, E–F. The graph has 8 edges.
4. State the final answer.	The total number of edges in the graph is therefore half the sum of the degrees.

5.2.3 Isomorphic graphs

Consider the following graphs.





For the two graphs, the connections for each vertex can be summarised as shown in the table.

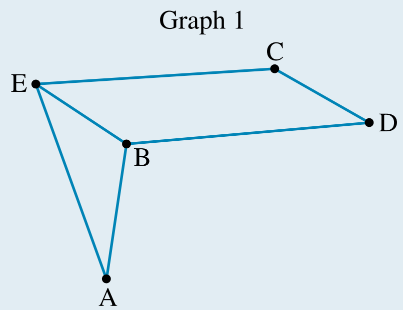
Although the graphs don't look exactly the same, they could be representing exactly the same information. Such graphs are known as **isomorphic graphs**.

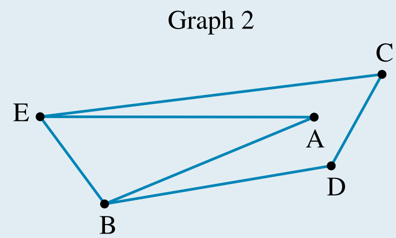
Vertex	Connections		
A	D		
B	D	E	
C	D	E	
D	A	B	C
E	B	C	

Isomorphic graphs have the same number of vertices and edges, with corresponding vertices having identical degrees and connections.

WORKED EXAMPLE 3

Confirm whether the following two graphs are isomorphic.





THINK

1. Identify the degree of the vertices for each graph.

WRITE

Graph	A	B	C	D	E
Graph 1	2	3	2	2	3
Graph 2	2	3	2	2	3

2. Identify the number of edges for each graph.

Graph	Edges
Graph 1	6
Graph 2	6

3. Identify the vertex connections for each graph.

Vertex	Connections		
A	B	E	
B	A	D	E
C	D	E	
D	B	C	
E	A	B	C

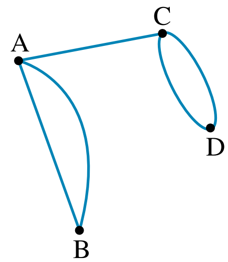
4. State the answer.

The two graphs are isomorphic as they have the same number of vertices and edges, with corresponding vertices having identical degrees and connections.

5.2.4 Adjacency matrices

Matrices are often used when working with graphs. A matrix that represents the number of edges that connect the vertices of a graph is known as an adjacency matrix.

Each column and row of an adjacency matrix corresponds to a vertex of the graph, and the numbers indicate how many edges are connecting them.



Graph

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Adjacency matrix

In the adjacency matrix, column 3 corresponds to vertex C and row 4 to vertex D. The '2' indicates the number of edges joining these two vertices.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

5.2.5 Characteristics of adjacency matrices

Adjacency matrices are square matrices with n rows and columns, where 'n' is equal to the number of vertices in the graph.

Column: 1 2 … $n-1$ n Row

$$\begin{bmatrix} 0 & 2 & \dots & 1 & 0 \\ 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 2 \\ 0 & 0 & \dots & 2 & 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{array}$$

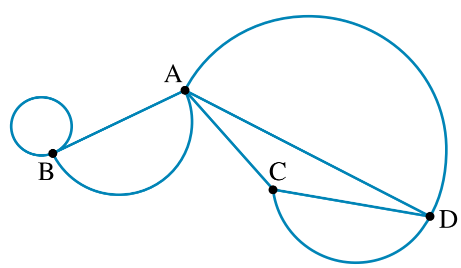
Adjacency matrices are symmetrical around the leading diagonal.

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Any non-zero value in the leading diagonal will indicate the existence of a loop.

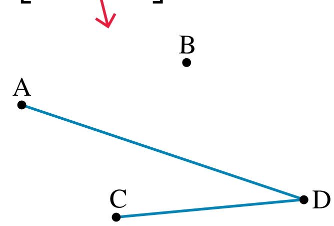
$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

The '1' indicates that a loop exists at vertex B:



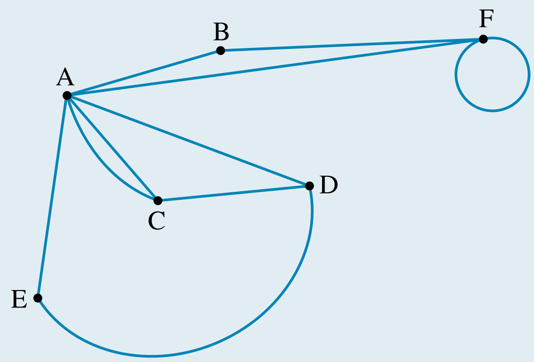
A row consisting of all zeros indicates an isolated vertex (a vertex that is not connected to any other vertex).

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$



WORKED EXAMPLE 4

Construct the adjacency matrix for the given graph.



THINK

1. Draw up a table with rows and columns for each vertex of the graph.

WRITE

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

2. Count the number of edges that connect vertex A to the other vertices and record these values in the corresponding space for the first row of the table.

	A	B	C	D	E	F
A	0	1	2	1	1	1

3. Repeat step 2 for all the other vertices.

	A	B	C	D	E	F
A	0	1	2	1	1	1
B	1	0	0	0	0	1
C	2	0	0	1	0	0
D	1	0	1	0	1	0
E	1	0	0	1	0	0
F	1	1	0	0	0	1

4. Display the numbers as a matrix.

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

on Resources

Interactivity: The adjacency matrix (int-6466)

studyon

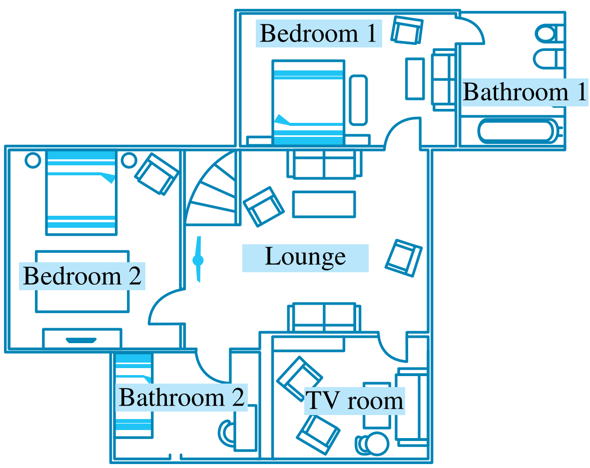
Units 1 & 2 > AOS 3 > Topic 2 > Concepts 1 & 2

Networks, vertices and edges Concept summary and practice questions

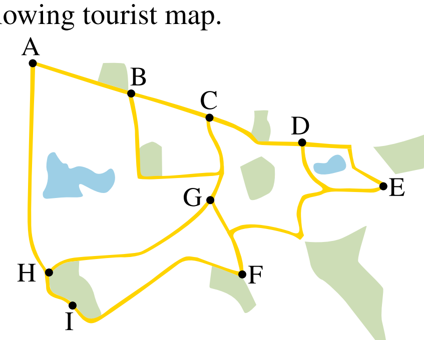
Isomorphic graphs and matrices Concept summary and practice questions

Exercise 5.2 Definitions and terms

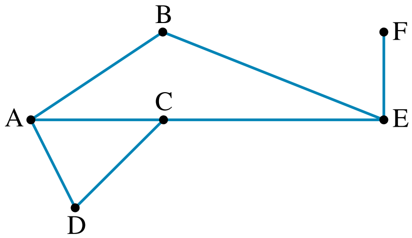
1. **WE1** The diagram shows the plan of a floor of a house. Draw a graph to represent the possible ways of travelling between each room of the floor.

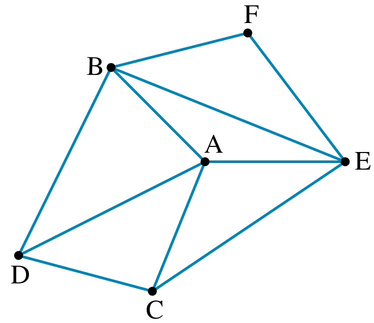


2. Draw a graph to represent the following tourist map.

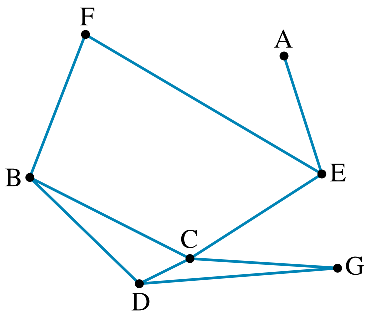


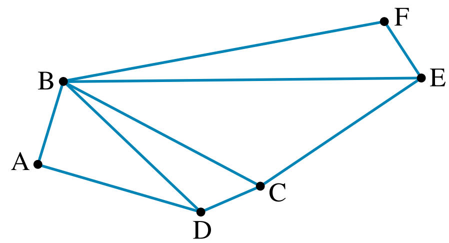
3. **WE2** For each of the following graphs, verify that the number of edges is equal to half the sum of the degree of the vertices.

a. 

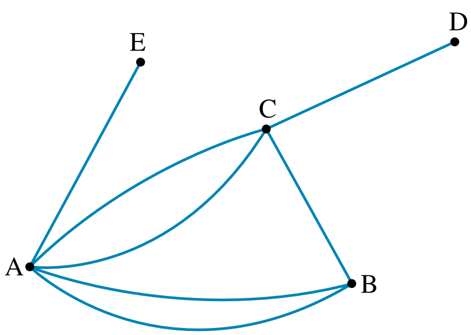
b. 

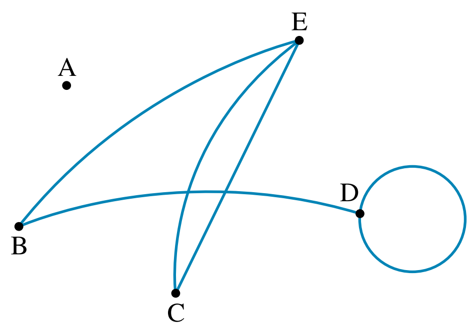
4. For each of the following graphs, verify that the number of edges is equal to half the sum of the degree of the vertices.

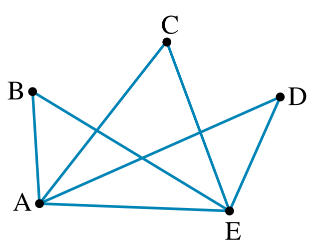
a. 

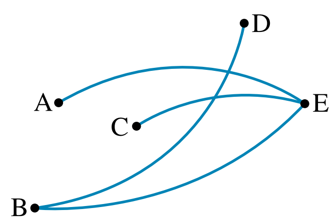
b. 

5. Identify the degree of each vertex in the following graphs.

a. 

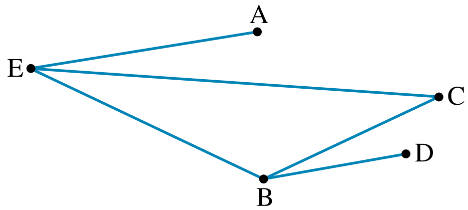
b. 

c. 

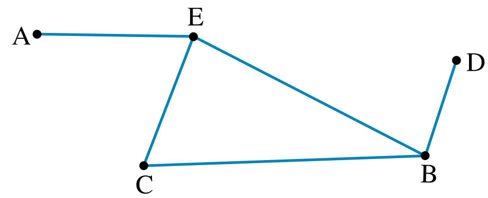
d. 

6. **WE3** Confirm whether the following pairs of graphs are isomorphic.

a.

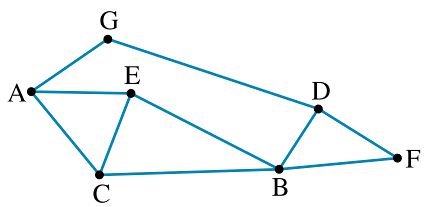


Graph 1

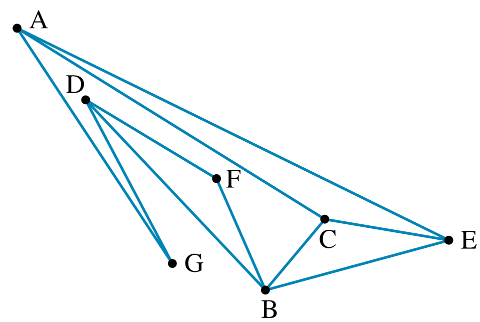


Graph 2

b.

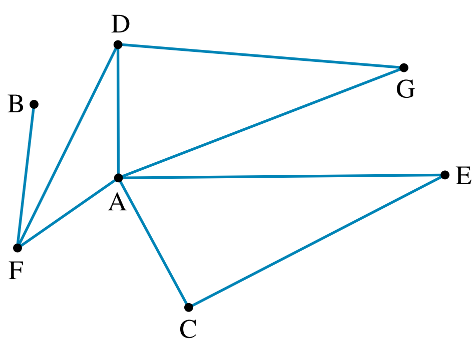


Graph 1

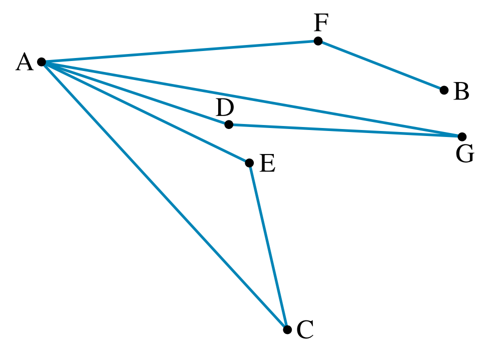


Graph 2

c.

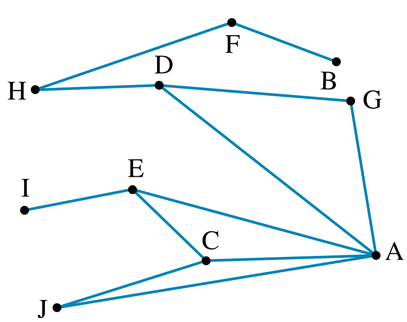


Graph 1

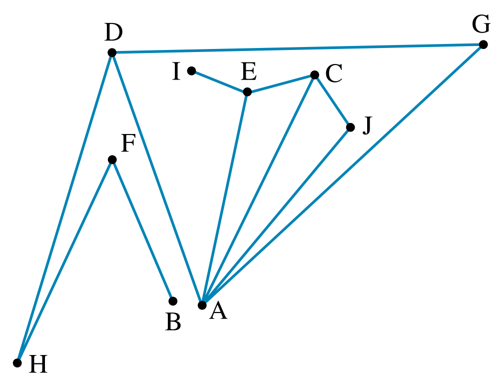


Graph 2

d.



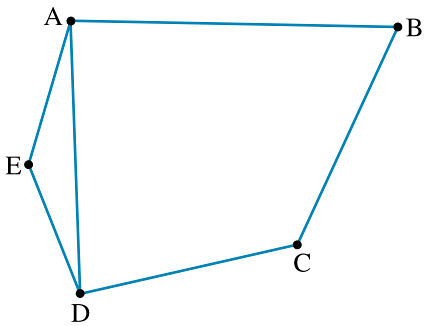
Graph 1



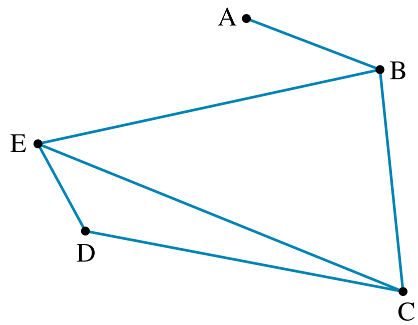
Graph 2

7. Explain why the following pairs of graphs are not isomorphic:

a.

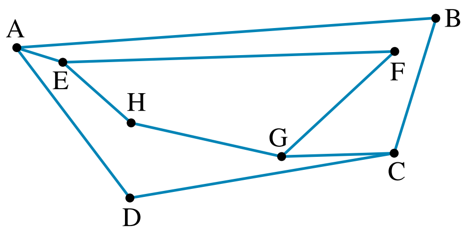


Graph 1

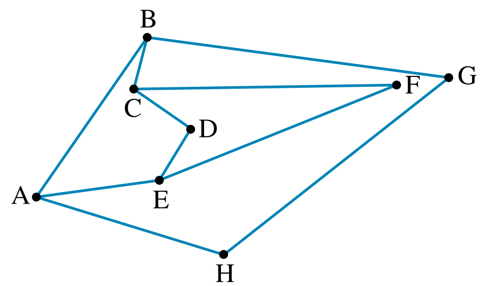


Graph 2

b.

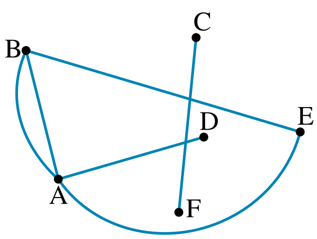


Graph 1

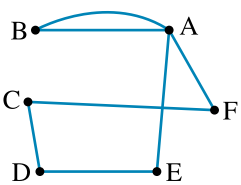


Graph 2

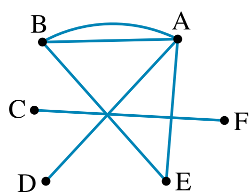
8. Identify pairs of isomorphic graphs from the following.



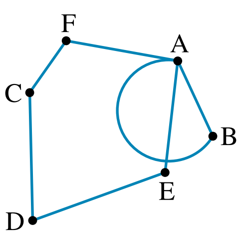
Graph 1



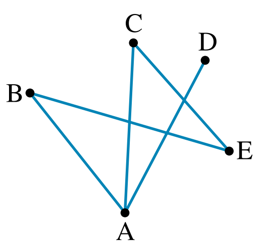
Graph 2



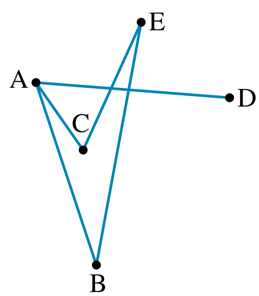
Graph 3



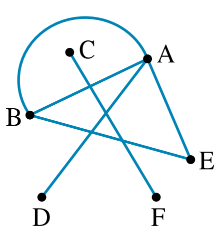
Graph 4



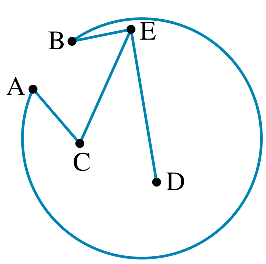
Graph 5



Graph 6

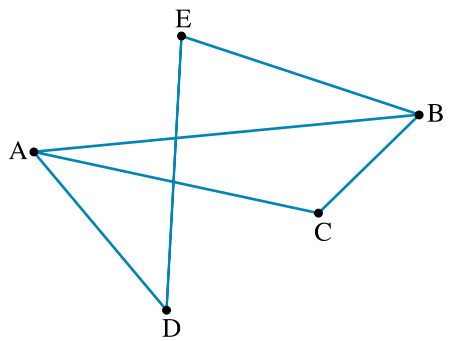


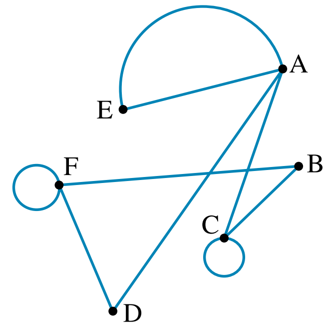
Graph 7

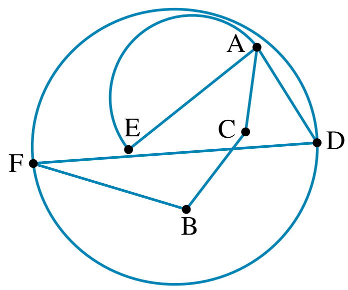


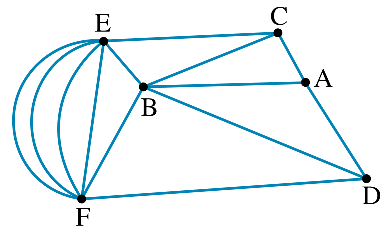
Graph 8

9. **WE4** Construct adjacency matrices for the following graphs.

a. 

b. 

c. 

d. 

10. Draw graphs to represent the following adjacency matrices.

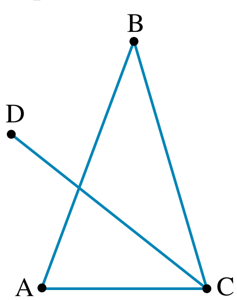
a. $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$

11. Construct the adjacency matrices for each of the graphs shown in question **10**.

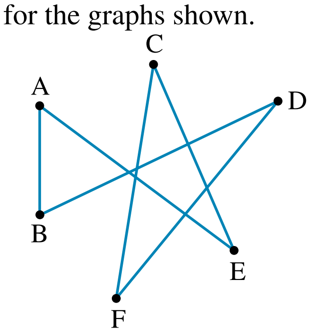
12. Complete the following adjacency matrices.

a. $\begin{bmatrix} 0 & 0 & \\ 0 & 2 & 2 \\ 1 & & 0 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 1 & & 0 \\ & 0 & & \\ 0 & 1 & 0 & 1 \\ & 2 & & 0 \end{bmatrix}$ c. $\begin{bmatrix} 0 & & 1 & & 0 \\ 0 & 0 & & & 0 \\ & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & & & 1 & 0 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \\ & & & 0 & 0 \\ 0 & & & 0 & 1 \end{bmatrix}$

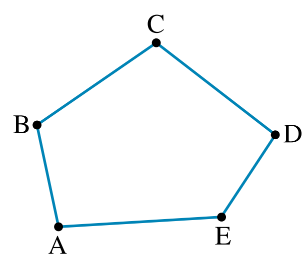
13. Complete the following table for the graphs shown.



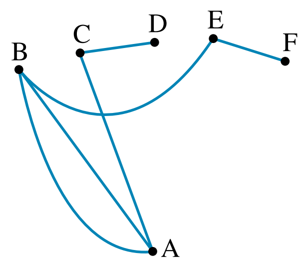
Graph 1



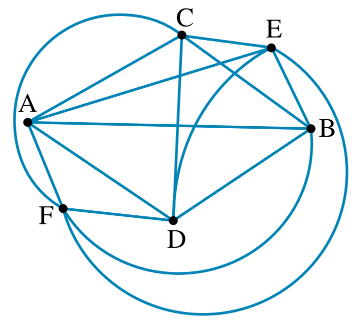
Graph 2



Graph 3



Graph 4



Graph 5

	Simple	Complete	Connected
Graph 1	No	No	Yes
Graph 2			
Graph 3			
Graph 4			
Graph 5			

14. Enter details for complete graphs in the following table.

Vertices	Edges
2	
3	
4	
5	
6	
n	

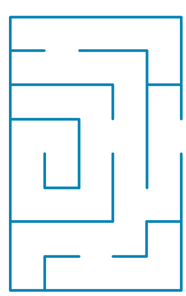
15. Draw a graph of:

a. a simple, connected graph with 6 vertices and 7 edges

b. a simple, connected graph with 7 vertices and 7 edges, where one vertex has degree 3 and five vertices have degree 2

c. a simple, connected graph with 9 vertices and 8 edges, where one vertex has degree 8.

16. By indicating the passages with edges and the intersections and passage endings with vertices, draw a graph to represent the maze shown in the diagram.


Maze

17. Five teams play a round robin competition.

a. Draw a graph to represent the games played.

b. What type of graph is this?

c. What does the total number of edges in the graph indicate?

18. The diagram shows the map of some of the main suburbs of Beijing.

a. Draw a graph to represent the shared boundaries between the suburbs.

b. Which suburb has the highest degree?

c. What type of graph is this?

19. The map shows some of the main highways connecting some of the states on the west coast of the USA.

a. Draw a graph to represent the highways connecting the states shown.
b. Use your graph to construct an adjacency matrix.
c. Which state has the highest degree?
d. Which state has the lowest degree?

20. Jetways Airlines operates flights in South East Asia.

The table indicates the number of direct flights per day between key cities.

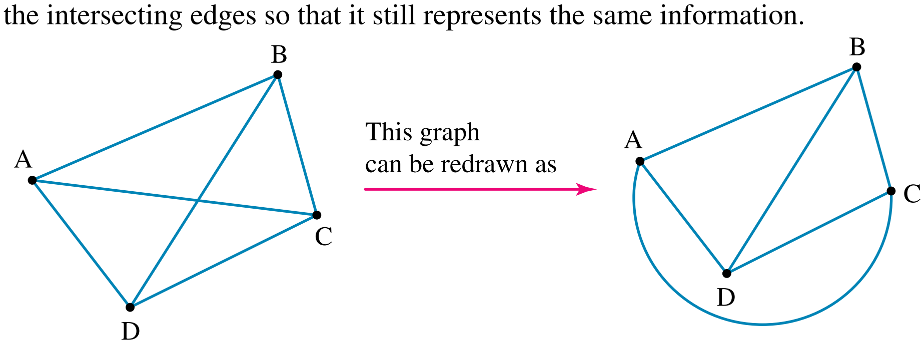
From: / To:	Bangkok	Manila	Singapore	Kuala Lumpur	Jakarta	Hanoi	Phnom Penh
Bangkok	0	2	5	3	1	1	1
Manila	2	0	4	1	1	0	0
Singapore	5	4	0	3	4	2	3
Kuala Lumpur	3	1	3	0	0	3	3
Jakarta	1	1	4	0	0	0	0
Hanoi	1	0	2	3	0	0	0
Phnom Penh	1	0	3	3	0	0	0

a. Draw a graph to represent the number of direct flights.
b. Would this graph be considered to be directed or undirected? Why?
c. In how many ways can you travel from:
   i. Phnom Penh to Manila
   ii. Hanoi to Bangkok?

5.3 Planar graphs

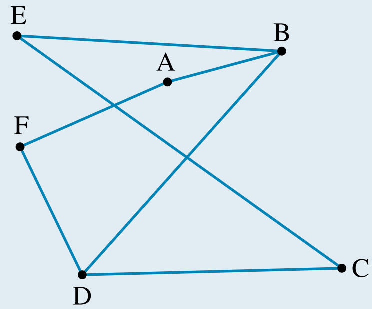
As indicated in Section 5.2, graphs can be drawn with intersecting edges. However, in many applications intersections may be undesirable. Consider a graph of an underground railway network. In this case intersecting edges would indicate the need for one rail line to be in a much deeper tunnel, which could add significantly to construction costs.

In some cases it is possible to redraw graphs so that they have no intersecting edges. When a graph can be redrawn in this way, it is known as a **planar graph**. For example, in the graph shown below, it is possible to redraw one of the intersecting edges so that it still represents the same information.



WORKED EXAMPLE 5

Redraw the graph so that it has no intersecting edges.



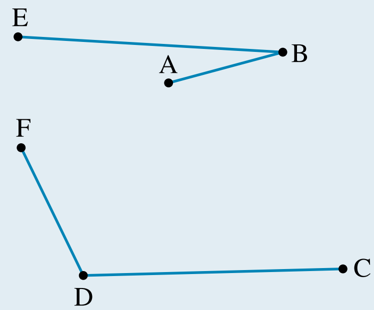
THINK

WRITE/DRAW

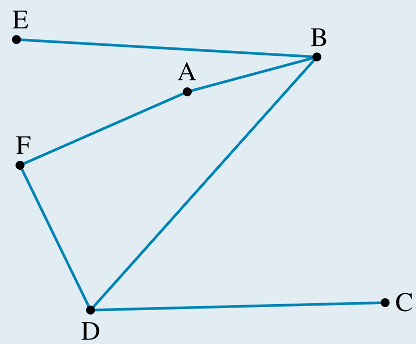
1. List all connections in the original graph.

Connections:
AB; AF; BD; BE; CD; CE; DF

2. Draw all vertices and any section(s) of the graph that have no intersecting edges.

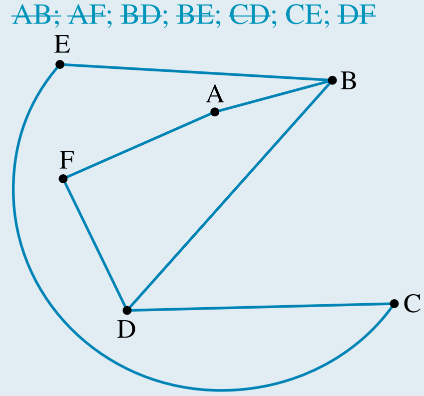


3. Draw any further edges that don't create intersections. Start with edges that have the fewest intersections in the original drawing.

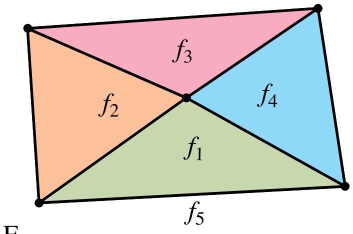


4. Identify any edges yet to be drawn and redraw so that they do not intersect with the other edges.

Connections:
~~AB~~; ~~AF~~; ~~BD~~; ~~BE~~; ~~CD~~; CE; ~~DF~~



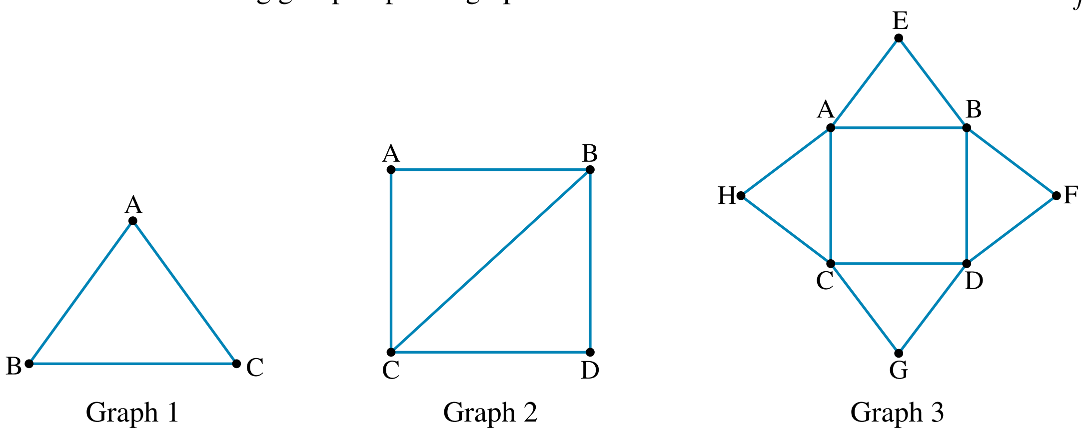
5.3.1 Euler's formula



In all planar graphs, the edges and vertices create distinct areas referred to as **faces**.

The planar graph shown in the diagram at right has five faces including the area around the outside.

Consider the following group of planar graphs.



The number of vertices, edges and faces for each graph is summarised in the following table.

Graph	Vertices	Edges	Faces
Graph 1	3	3	2
Graph 2	4	5	3
Graph 3	8	12	6

For each of these graphs, we can obtain a result that is well known for any planar graph: the difference between the vertices and edges added to the number of faces will always equal 2.

Graph 1: $3 - 3 + 2 = 2$

Graph 2: $4 - 5 + 3 = 2$

Graph 3: $8 - 12 + 6 = 2$

This is known as Euler's formula for connected planar graphs and can be summarised as:

$v - e + f = 2$, where v is the number of vertices, e is the number of edges and f is the number of faces.

WORKED EXAMPLE 6

How many faces will there be for a connected planar graph of 7 vertices and 10 edges?

THINK	WRITE
1. Substitute the given values into Euler's formula.	$v - e + f = 2$ $7 - 10 + f = 2$
2. Solve the equation for the unknown value.	$7 - 10 + f = 2$ $f = 2 - 7 + 10$ $f = 5$
3. State the final answer.	There will be 5 faces in a connected planar graph with 7 vertices and 10 edges.

on Resources

- **Interactivity:** Planar graphs (int-6467)
- **Interactivity:** Euler's formula (int-6468)

studyon

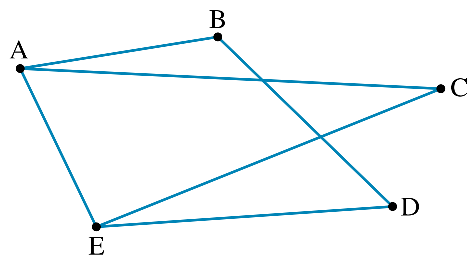
Units 1 & 2 > AOS 3 > Topic 2 > Concept 3

Faces, vertices, edges and Euler's formula Concept summary and practice questions

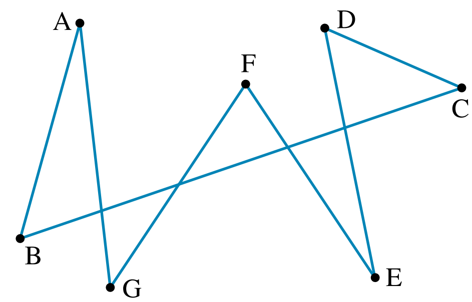
Exercise 5.3 Planar graphs

1. **WE5** Redraw the following graphs so that they have no intersecting edges.

a.

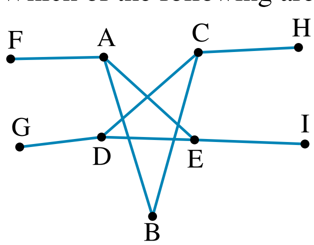


b.

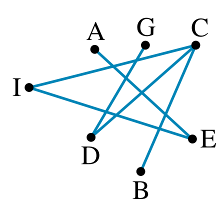


2. **WC** Which of the following are planar graphs?

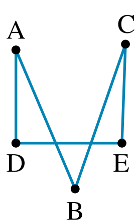
a. A.



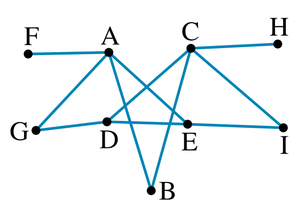
B.



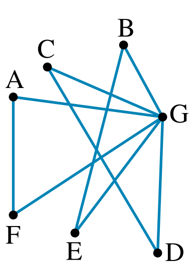
C.



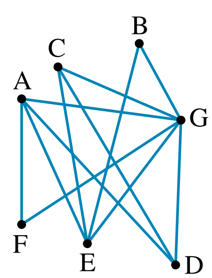
D.



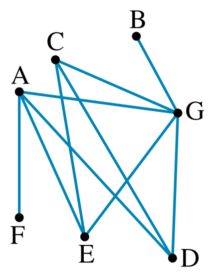
b. A.



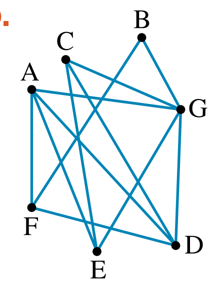
B.



C.

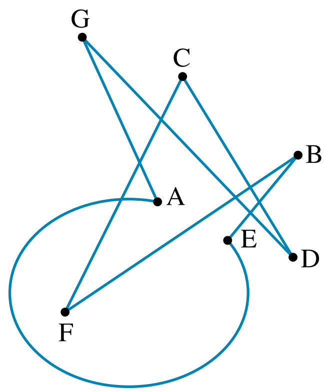


D.

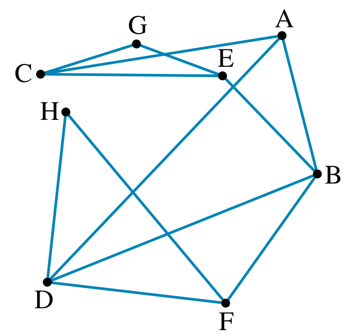


3. Redraw the following graphs to show that they are planar.

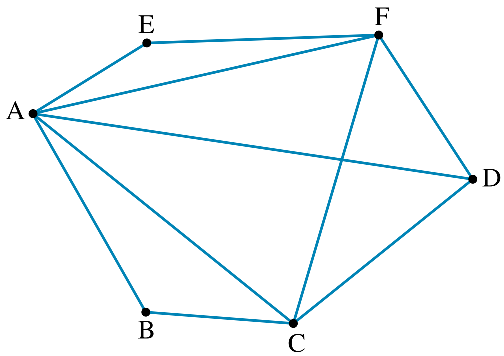
a.



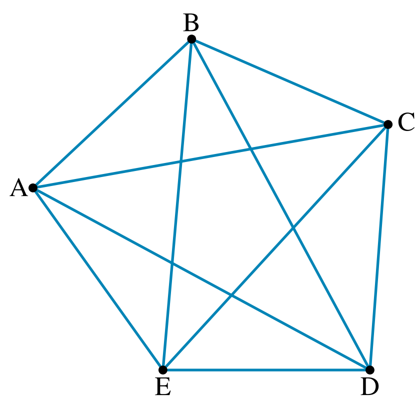
b.



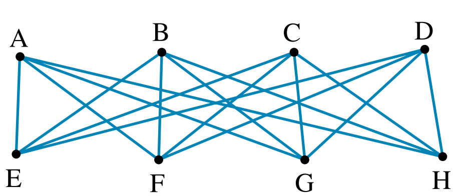
4. Which of the following graphs are not planar?



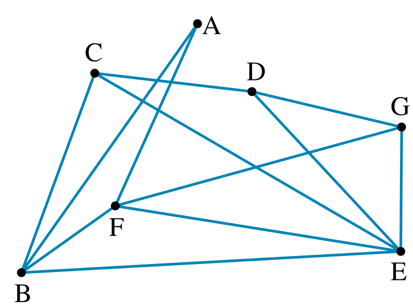
Graph 1



Graph 2



Graph 3



Graph 4

5. WE6 How many faces will there be for a connected planar graph of:

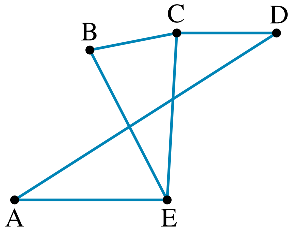
a. 8 vertices and 10 edges b. 11 vertices and 14 edges?

6. a. For a connected planar graph of 5 vertices and 3 faces, how many edges will there be?

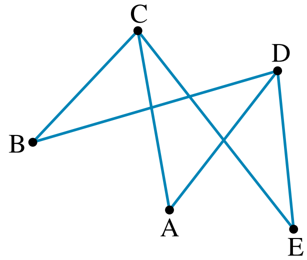
b. For a connected planar graph of 8 edges and 5 faces, how many vertices will there be?

7. For each of the following planar graphs, identify the number of faces:

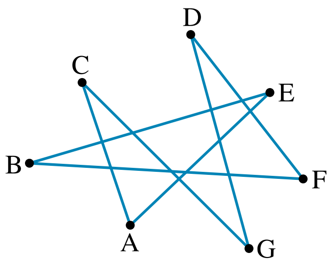
a.



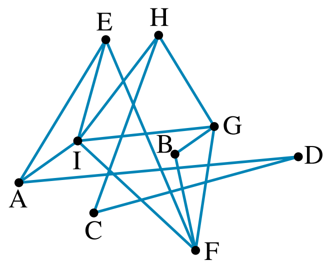
b.



c.



d.



8. Construct a connected planar graph with:

a. 6 vertices and 5 faces b. 11 edges and 9 faces.

9. Use the following adjacency matrices to draw graphs that have no intersecting edges.

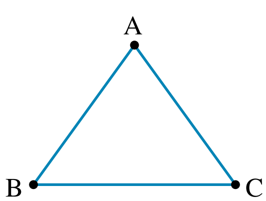
a. $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

10. For the graphs in question **9**:

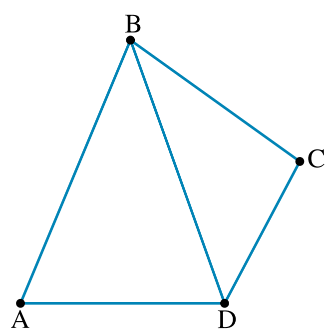
i. identify the number of enclosed faces

ii. identify the maximum number of additional edges that can be added to maintain a simple planar graph.

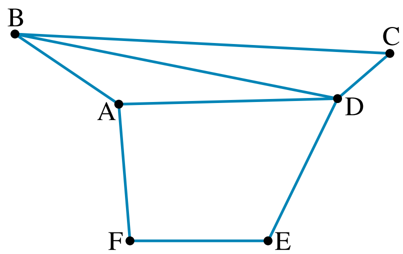
11. a. Use the planar graphs shown to complete the table.



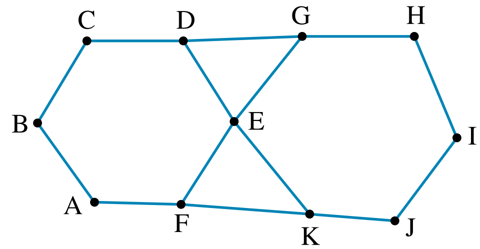
Graph 1



Graph 2



Graph 3

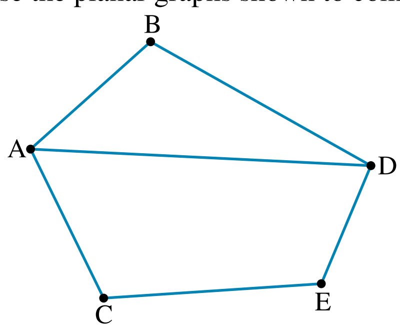


Graph 4

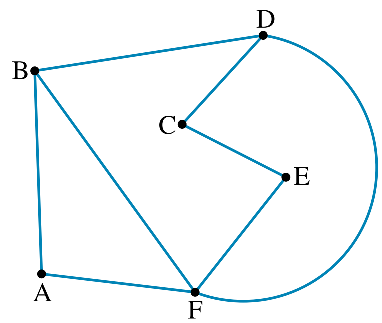
Graph	Total edges	Total degrees
Graph 1		
Graph 2		
Graph 3		
Graph 4		

b. What pattern is evident from the table?

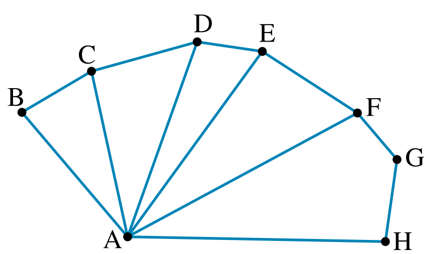
12. a. Use the planar graphs shown to complete the table.



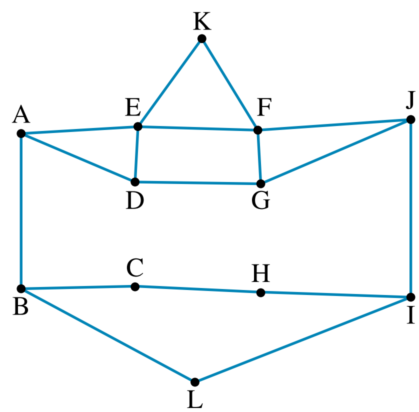
Graph 1



Graph 2



Graph 3

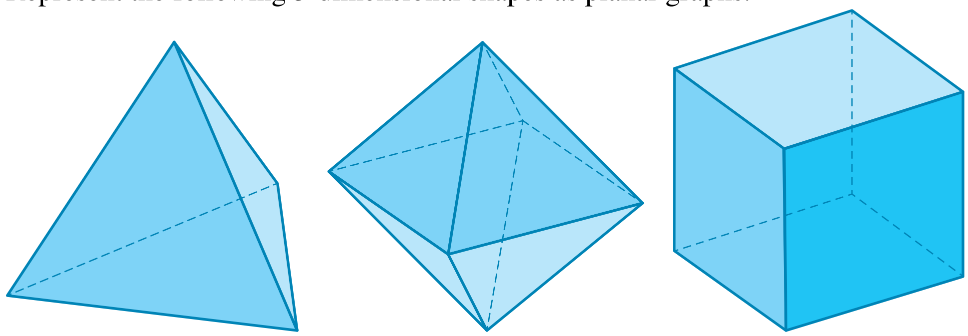


Graph 4

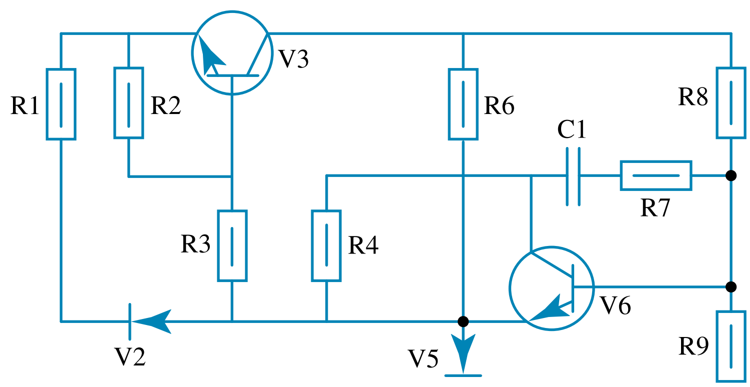
Graph	Total vertices of even degree	Total vertices of odd degree
Graph 1		
Graph 2		
Graph 3		
Graph 4		

b. Is there any pattern evident from this table?

13. Represent the following 3-dimensional shapes as planar graphs.



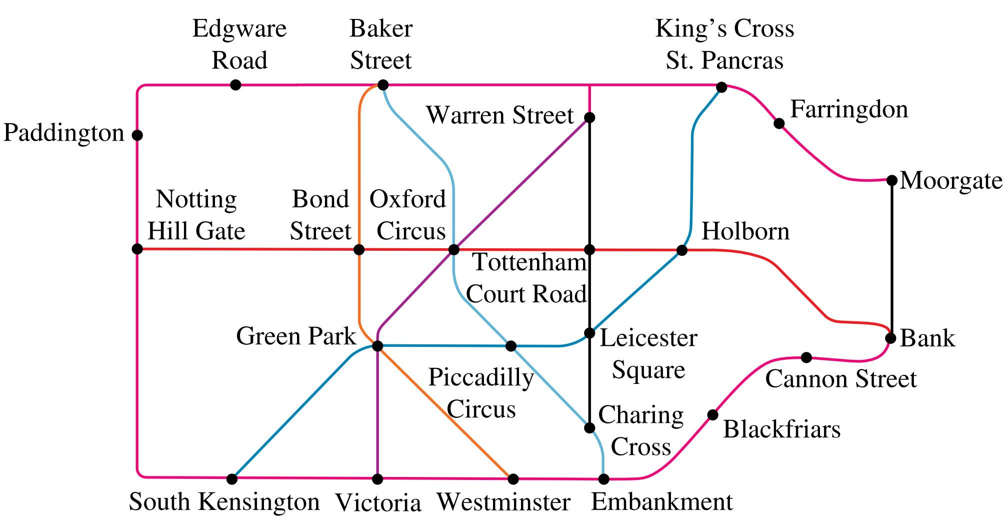
14. A section of an electric circuit board is shown in the diagram.



a. Draw a graph to represent the circuit board, using vertices to represent the labelled parts of the diagram.

b. Is it possible to represent the circuit board as a planar graph?

15. The diagram shows a section of the London railway system.



a. Display this information using an adjacency matrix.
b. What does the sum of the rows of this adjacency matrix indicate?

16. The table displays the most common methods of communication for a group of people.

	Email	**Facebook**	**SMS**
Adam	Ethan, Liam	Ethan, Liam	Ethan
Michelle		Sophie, Emma, Ethan	Sophie, Emma
Liam	Adam		
Sophie		Michelle, Chloe	Michelle, Chloe
Emma	Chloe	Chloe, Ethan, Michelle	Chloe, Ethan
Ethan		Emma, Adam, Michelle	Emma
Chloe	Emma, Sophie	Emma, Sophie	Emma, Sophie

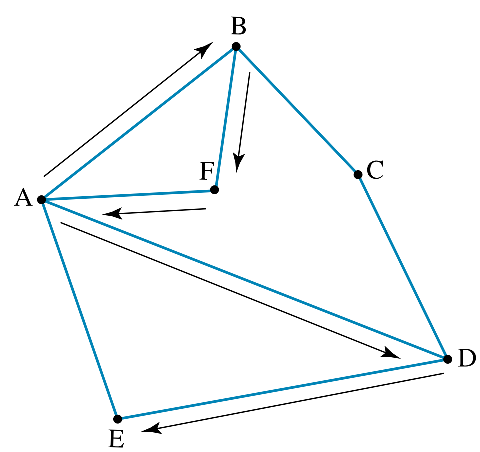
a. Display the information for the entire table in a graph.
b. Who would be the best person to introduce Chloe and Michelle?
c. Display the Facebook information in a separate graph.
d. If Liam and Sophie began communicating through Facebook, how many faces would the graph from part **c** then have?

5.4 Connected graphs

5.4.1 Traversing connected graphs

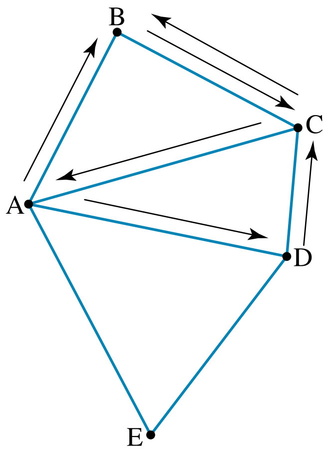
Many applications of graphs involve an analysis of movement around a network. These could include fields such as transport, communications or utilities, to name a few. Movement through a simple connected graph is described in terms of starting and finishing at specified vertices by travelling along the edges. This is usually done by listing the labels of the vertices visited in the correct order. In more complex graphs, edges may also have to be indicated, as there may be more than one connection between vertices.

The definitions of the main terms used when describing movement across a network are as follows.

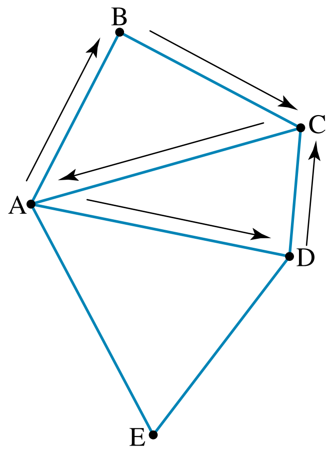


Route: ABFADE

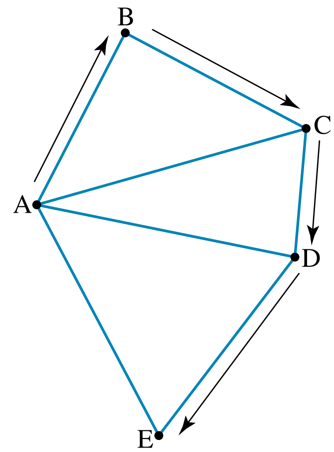
Walk: Any route taken through a network, including routes that repeat edges and vertices
Trail: A walk in which no edges are repeated
Path: A walk in which no vertices are repeated, except possibly the start and finish
Cycle: A path beginning and ending at the same vertex
Circuit: A trail beginning and ending at the same vertex



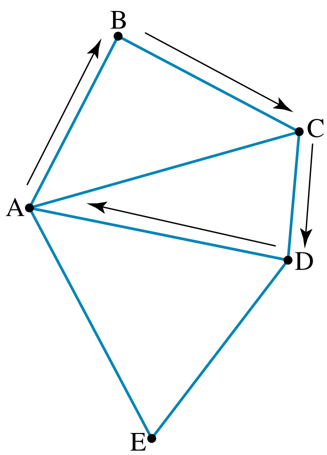
Walk: ABCADCB



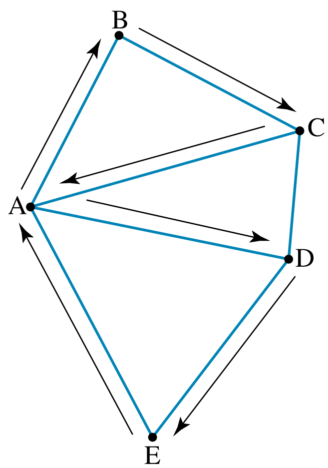
Trail: ABCADC



Path: ABCDE



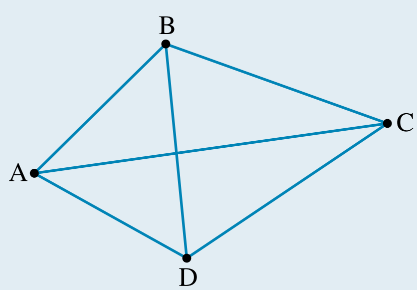
Cycle: ABCDA



Circuit: ABCADEA

WORKED EXAMPLE 7

In the following network, identify two different routes: one cycle and one circuit.

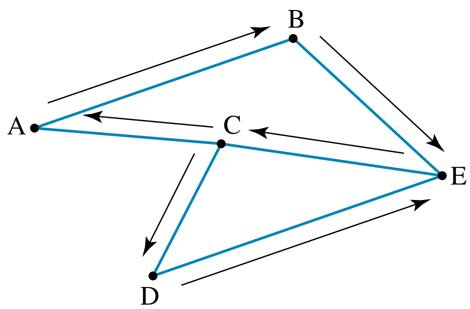


THINK	WRITE
1. For a cycle, identify a route that doesn't repeat a vertex apart from the start/finish.	Cycle: ABDCA
2. For a circuit, identify a route that doesn't repeat an edge and ends at the starting vertex.	Circuit: ADBCA

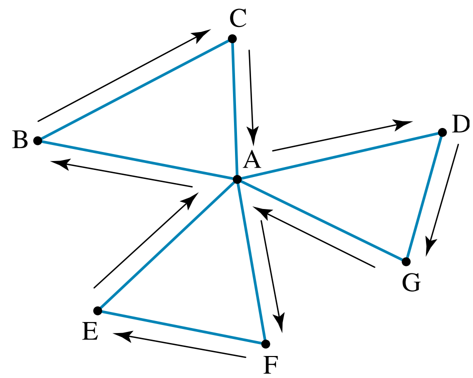
5.4.2 Euler trails and circuits

In some practical situations, it is most efficient if a route travels along each edge only once. Examples include parcel deliveries and council garbage collections. If it is possible to travel a network using each edge only once, the route is known as an **Euler trail** or **Euler circuit**.

> **An Euler trail is a trail in which every edge is used once.**
> **An Euler circuit is a circuit in which every edge is used once.**



Euler trail: CDECABE



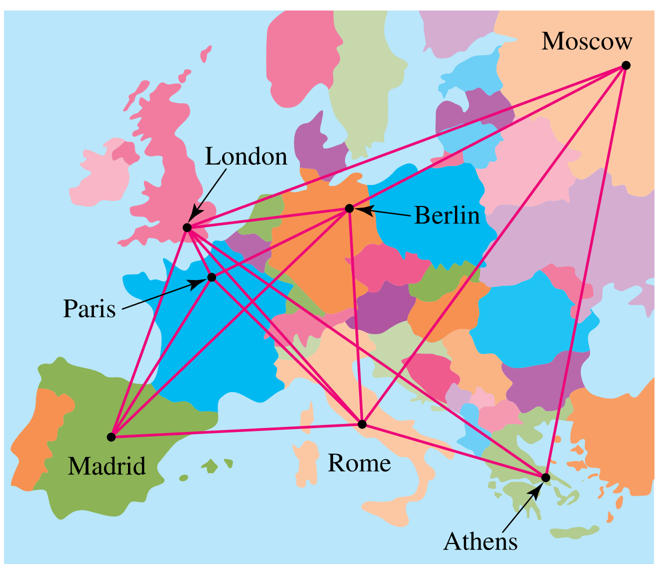
Euler circuit: ABCADGAFEA

Note that in the examples shown, the vertices for the Euler circuit are of even degree, and there are 2 vertices of odd degree for the Euler trail.

> **If all of the vertices of a connected graph are even, then an Euler circuit exists.**
> **If exactly 2 vertices of a connected graph are odd, then an Euler trail exists.**

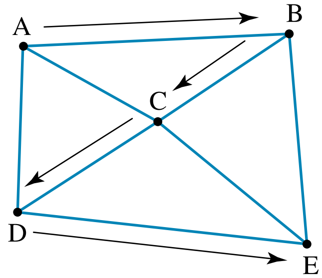
5.4.3 Hamiltonian pathsand cycles

In other situations it may be more practical if all vertices can be reached without using all of the edges of the graph. For example, if you wanted to visit a selection of the capital cities of Europe, you wouldn't need to use all the available flight routes shown in the diagram.

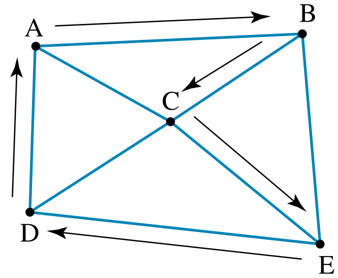


A Hamiltonian path is a path that reaches all vertices of a network.
A Hamiltonian cycle is a cycle that reaches all vertices of a network.

Hamiltonian paths and Hamiltonian cycles reach all vertices of a network once without necessarily using all of the available edges.



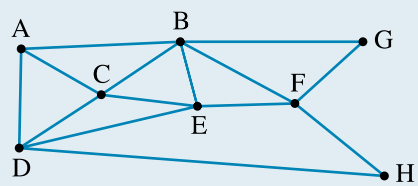
Hamiltonian path: ABCDE

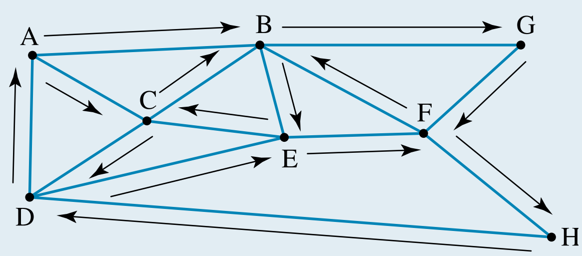
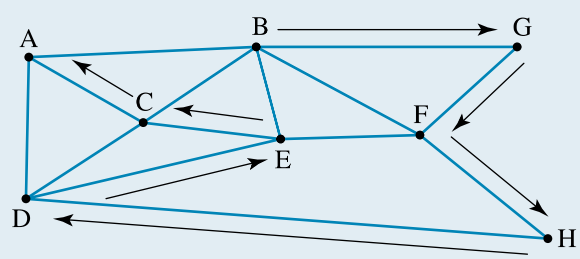


Hamiltonian cycle: ABCEDA

WORKED EXAMPLE 8

Identify an Euler trail and a Hamiltonian path in the following graph.



THINK	WRITE/DRAW
1. For an Euler trail to exist, there must be exactly 2 vertices with an odd-numbered degree.	Deg(A) = 3, deg(B) = 5, deg(C) = 4, deg(D) = 4, deg(E) = 4, deg(F) = 4, deg(G) = 2, deg(H) = 2 As there are only two odd-degree vertices, an Euler trail must exist.>5pt
2. Identify a route that uses each edge once.	 Euler trail: ABGFHDEFBECDACB >5pt
3. Identify a route that reaches each vertex once.	 Hamiltonian path: BGFHDECA
4. State the answer.	Euler trail: ABGFHDEFBECDACB Hamiltonian path: BGFHDECA

on Resources

Interactivity: Traversing connected graphs (int-6469)
Interactivity: Euler trails and Hamiltonian paths (int-6470)

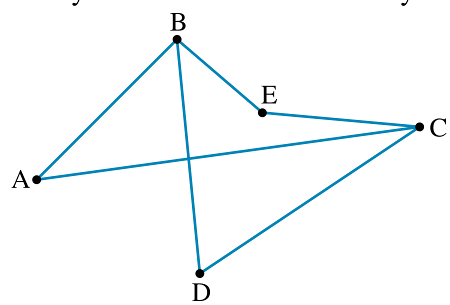
study on

Units 1 & 2 > AOS 3 > Topic 2 > Concept 4

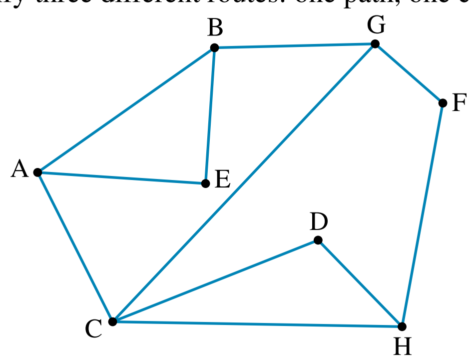
Euler trails and circuits Concept summary and practice questions

Exercise 5.4 Connected graphs

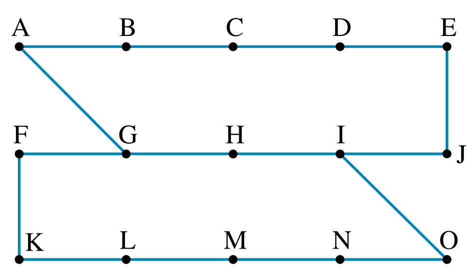
1. **WE7** In the following network, identify two different routes: one cycle and one circuit.



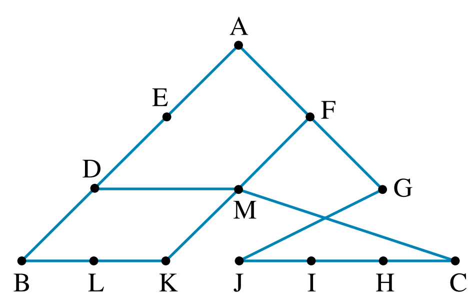
2. In the following network, identify three different routes: one path, one cycle and one circuit.



3. Which of the terms walk, trail, path, cycle and circuit could be used to describe the following routes on the graph shown?
   a. AGHIONMLKFGA
   b. IHGFKLMNO
   c. HIJEDCBAGH
   d. FGHIJEDCBAG

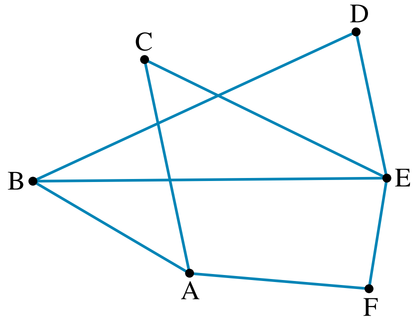


4. Use the following graph to identify the indicated routes.
   a. A path commencing at M, including at least 10 vertices and finishing at D
   b. A trail from A to C that includes exactly 7 edges
   c. A cycle commencing at M that includes 10 edges
   d. A circuit commencing at F that includes 7 vertices

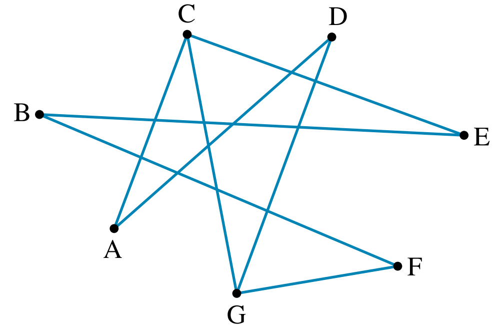


5. WE8 Identify an Euler trail and a Hamiltonian path in each of the following graphs.

a.

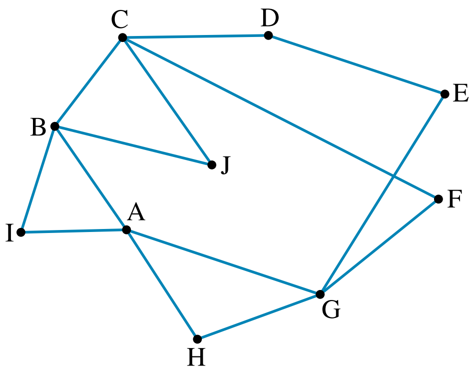


b.

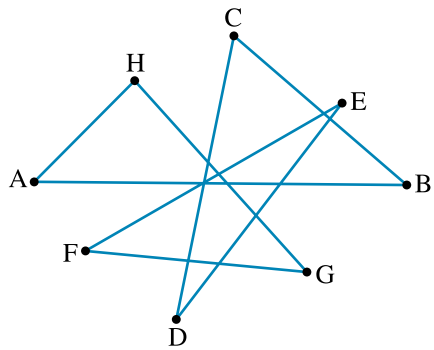


6. Identify an Euler circuit and a Hamiltonian cycle in each of the following graphs, if they exist.

a.

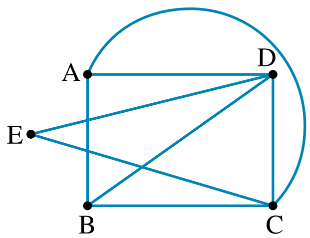


b.

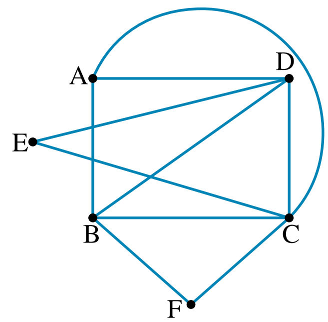


7. a. Identify which of the following graphs have an Euler trail.

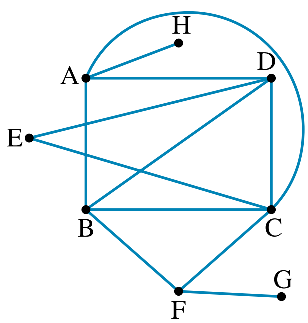
i.



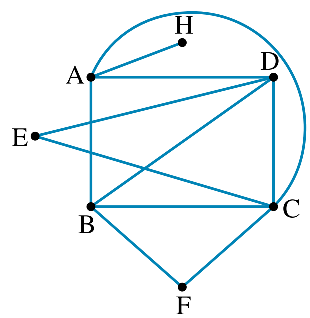
ii.



iii.



iv.



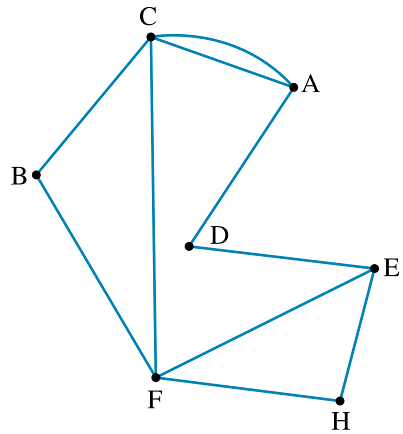
b. Identify the Euler trails found.

8. a. Identify which of the graphs from question **7** have a Hamiltonian cycle.

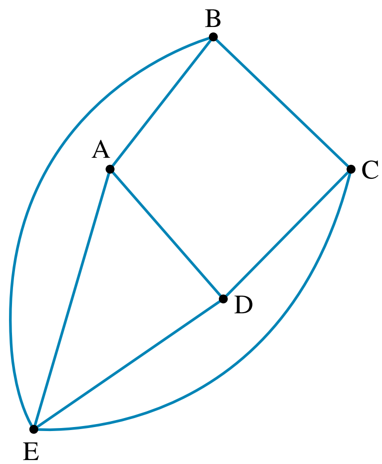
b. Identify the Hamiltonian cycles found.

9. a. Construct adjacency matrices for each of the graphs in question **7**.
   b. How might these assist with making decisions about the existence of Euler trails and circuits, and Hamiltonian paths and cycles?

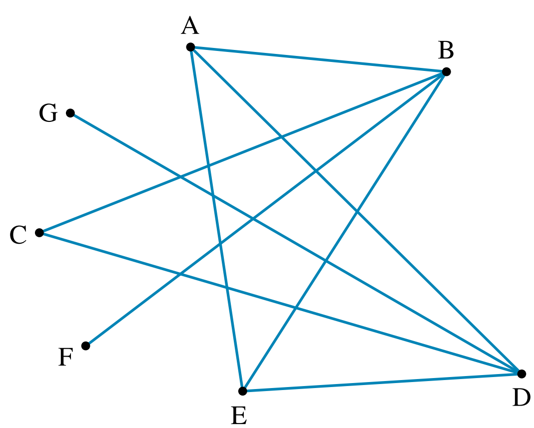
10. In the following graph, if an Euler trail commences at vertex A, at which vertices could it finish?



11. In the following graph, at which vertices could a Hamiltonian path finish if it commences by travelling from:
    a. B to E
    b. E to A?



12. In the following graph, other than from G to F, between which 2 vertices must you add an edge in order to create a Hamiltonian path that commences from vertex:
    a. G
    b. F?

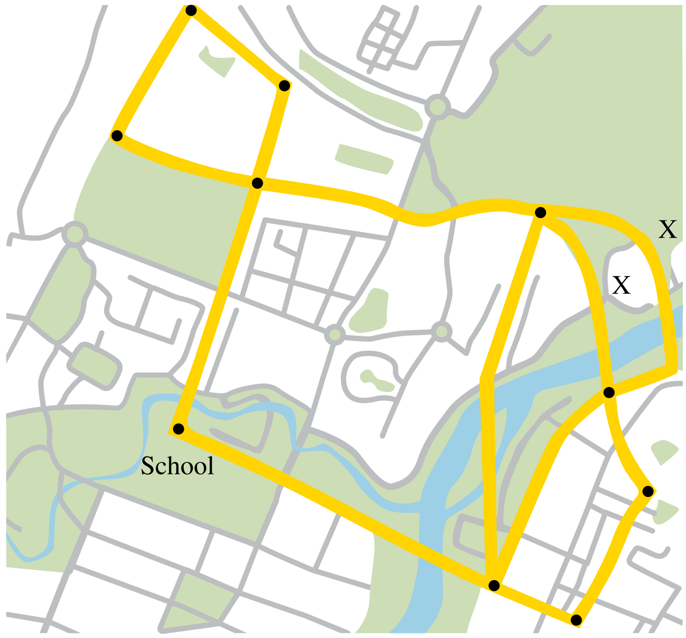


13. On the map shown, a school bus route is indicated in yellow. The bus route starts and ends at the school indicated.

a. Draw a graph to represent the bus route.

b. Students can catch the bus at stops that are located at the intersections of the roads marked in yellow. Is it possible for the bus to collect students by driving down each section of the route only once? Explain your answer.

c. If road works prevent the bus from travelling along the sections indicated by the Xs, will it be possible for the bus to still collect students on the remainder of the route by travelling each section only once? Explain your answer.

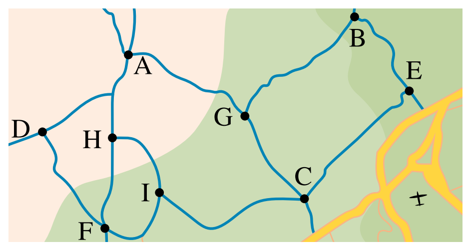


14. The map of an orienteering course is shown. Participants must travel to each of the nine checkpoints along any of the marked paths.

a. Draw a graph to represent the possible ways of travelling to each checkpoint.

b. What is the degree of checkpoint H?

c. If participants must start and finish at A and visit every other checkpoint only once, identify two possible routes they could take.

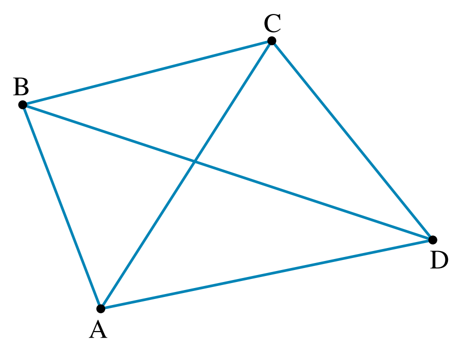


d. i. If participants can decide to start and finish at any checkpoint, and the paths connecting D and F, H and I, and A and G are no longer accessible, it is possible to travel the course by moving along each remaining path only once. Explain why.

ii. Identify the two possible starting points.

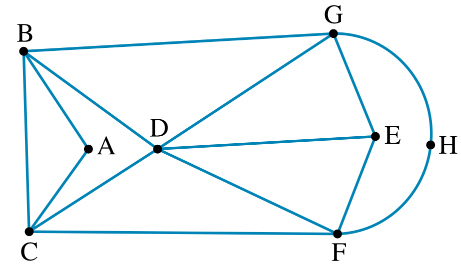
15. a. Use the following complete graph to complete the table to identify all of the Hamiltonian cycles commencing at vertex A.

	Hamiltonian circuit
1.	ABCDA
2.	
3.	
4.	
5.	
6.	



b. Are any other Hamiltonian cycles possible?

16. The graph shown outlines the possible ways a tourist bus can travel between eight locations.

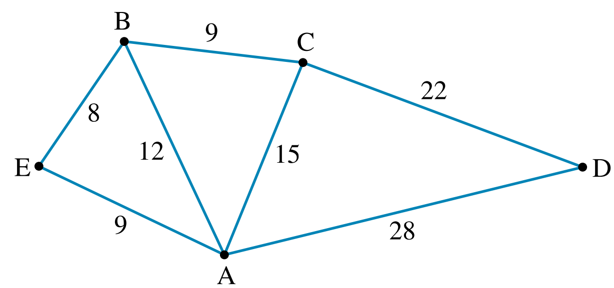


a. If vertex A represents the second location visited, list the possible starting points.
b. If the bus also visited each location only once, which of the starting points listed in part **a** could not be correct?
c. If the bus also needed to finish at vertex D, list the possible paths that could be taken.
d. If instead the bus company decides to operate a route that travelled to each connection only once, what are the possible starting and finishing points?
e. If instead the company wanted to travel to each connection only once and finish at the starting point, which edge of the graph would need to be removed?

5.5 Weighted graphs and trees

5.5.1 Weighted graphs

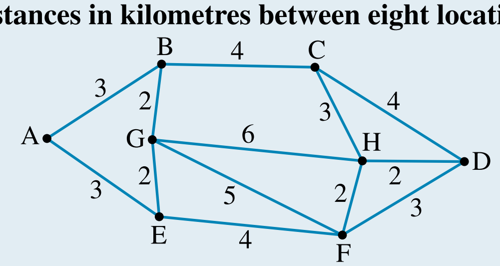
In many applications using graphs, it is useful to attach a value to the edges. These values could represent the length of the edge in terms of time or distance, or the costs involved with moving along that section of the path. Such graphs are known as **weighted graphs**.



Weighted graphs can be particularly useful as analysis tools. For example, they can help determine how to travel through a network in the shortest possible time.

WORKED EXAMPLE 9

The graph represents the distances in kilometres between eight locations.



Identify the shortest distance to travel from A to D that goes to all vertices.

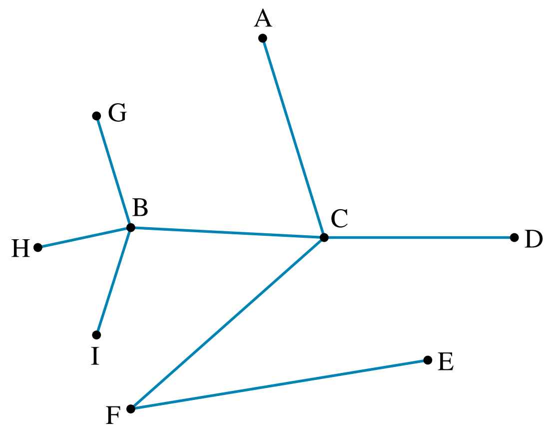
THINK	WRITE
1. Identify the Hamiltonian paths that connect the two vertices.	Possible paths: a. ABGEFHCD b. ABCHGEFD c. AEGBCHFD d. AEFGBCHD e. AEFHGBCD

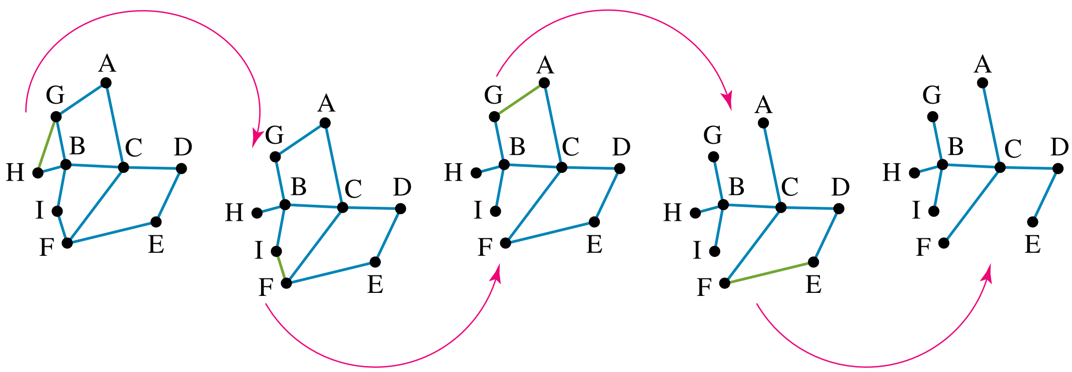
2. Calculate the total distances for each path to find the shortest.	**a.** $3 + 2 + 2 + 4 + 2 + 3 + 4 = 20$ **b.** $3 + 4 + 3 + 6 + 2 + 4 + 3 = 25$ **c.** $3 + 2 + 2 + 4 + 3 + 2 + 3 = 19$ **d.** $3 + 4 + 5 + 2 + 4 + 3 + 2 = 23$ **e.** $3 + 4 + 2 + 6 + 2 + 4 + 4 = 25$
3. State the final answer.	The shortest distance from A to D that travels to all vertices is 19 km.

5.5.2 Trees

A **tree** is a simple connected graph with no circuits. As such, any pairs of vertices in a tree are connected by a unique path, and the number of edges is always 1 less than the number of vertices.

Spanning trees are sub-graphs (graphs that are formed from part of a larger graph) that include all of the vertices of the original graph. In practical settings, they can be very useful in analysing network connections. For example a *minimum* spanning tree for a weighted graph can identify the lowest-cost connections. Spanning trees can be obtained by systematically removing any edges that form a circuit, one at a time.





5.5.3 Prim's algorithm

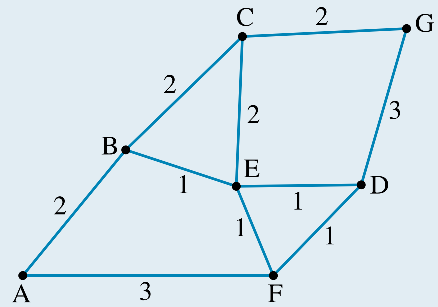
Prim's algorithm is a set of logical steps that can be used to identify the minimum spanning tree for a weighted connected graph.

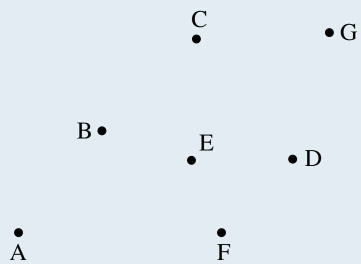
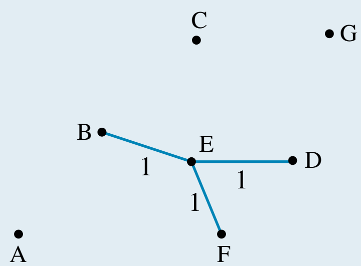
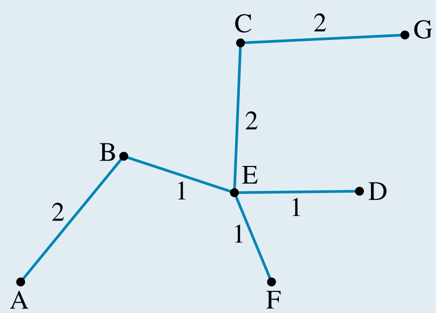
Steps for Prim's algorithm:

Step 1: Begin at a vertex with low weighted edges.
Step 2: Progressively select edges with the lowest weighting (unless they form a circuit).
Step 3: Continue until all vertices are selected.

WORKED EXAMPLE 10

Use Prim's algorithm to identify the minimum spanning tree of the graph shown.



THINK	DRAW
1. Draw the vertices of the graph.	
2. Draw in any edges with the lowest weighting that do not complete a circuit.	
3. Draw in any edges with the next lowest weighting that do not complete a circuit. Continue until all vertices are connected.	

on Resources

Interactivity: Minimum spanning trees and Prim's algorithm (int-6285)

studyon

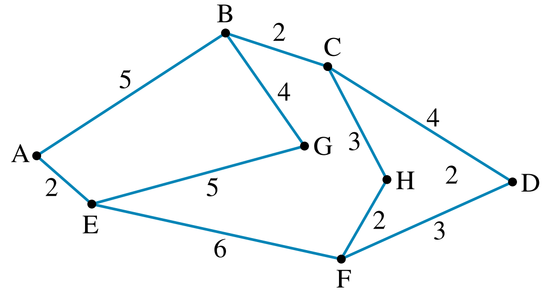
Units 1 & 2 > AOS 3 > Topic 2 > Concepts 5 & 6

Weighted graphs and minimum spanning trees Concept summary and practice questions

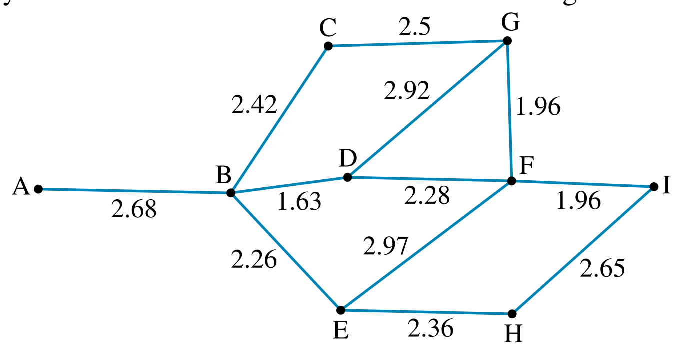
Prim's algorithm Concept summary and practice questions

Exercise 5.5 Weighted graphs and trees

1. **WE9** Use the graph to identify the shortest distance to travel from A to D that goes to all vertices.

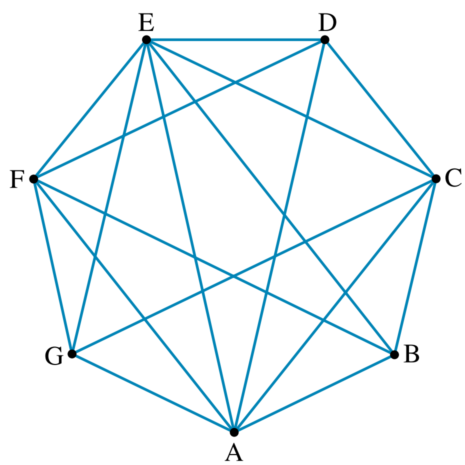


2. Use the graph to identify the shortest distance to travel from A to I that goes to all vertices.

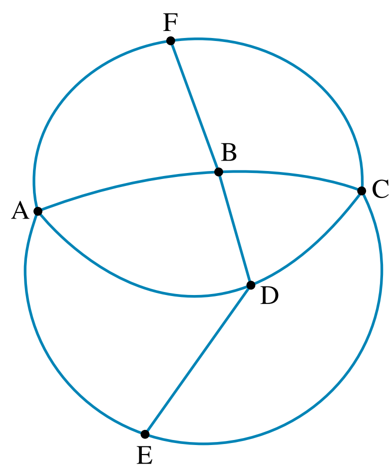


3. Draw three spanning trees for each of the following graphs.

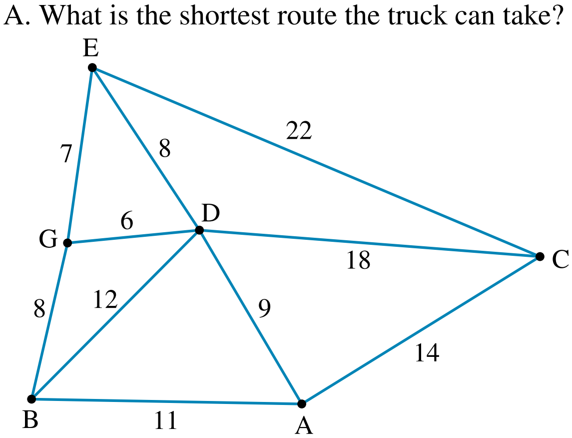
a.



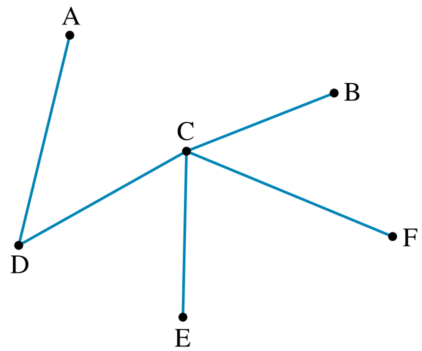
b.

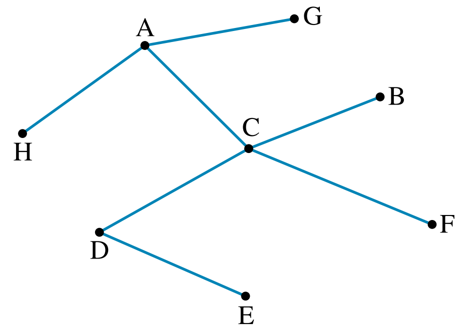


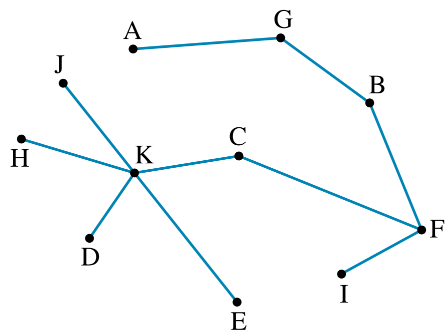
4. A truck starts from the main distribution point at vertex A and makes deliveries at each of the other vertices before returning to A. What is the shortest route the truck can take?

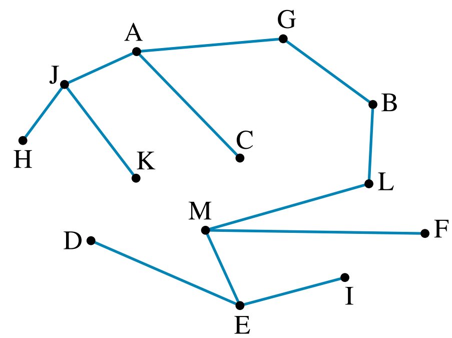


5. For the following trees:
   i. add the minimum number of edges to create an Euler trail
   ii. identify the Euler trail created.

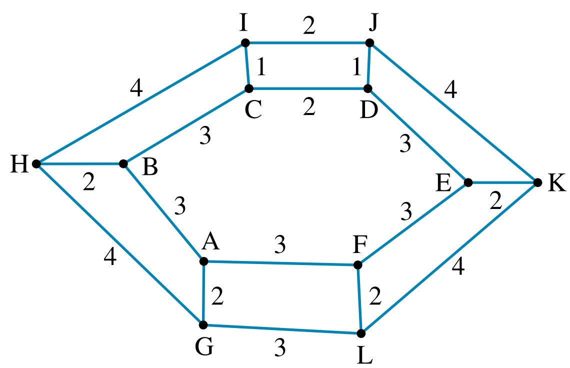
a. 

b. 

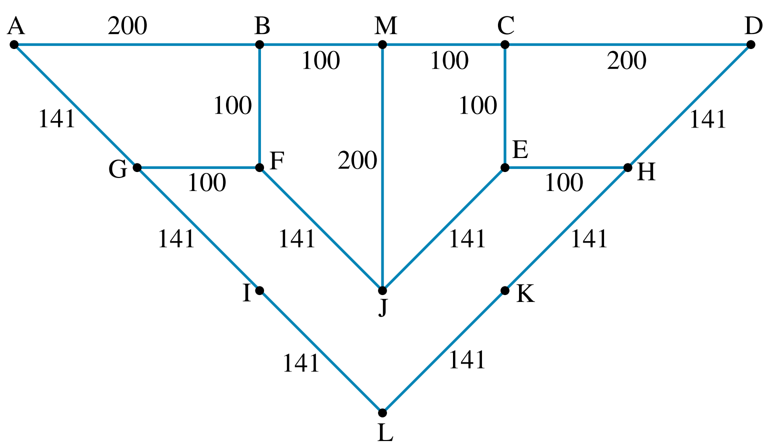
c. 

d. 

6. **WE10** Use Prim's algorithm to identify the minimum spanning tree of the graph shown.

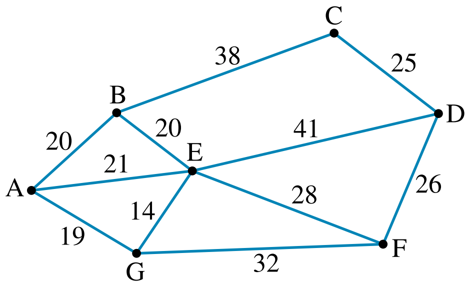


7. Use Prim's algorithm to identify the minimum spanning tree of the graph shown.

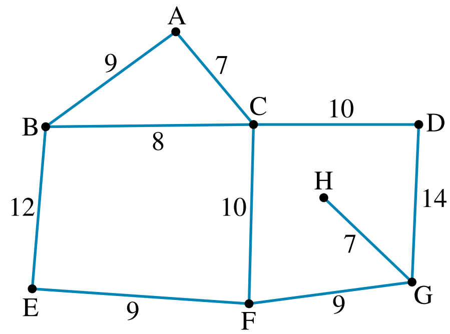


8. Identify the minimum spanning tree for each of the following graphs.

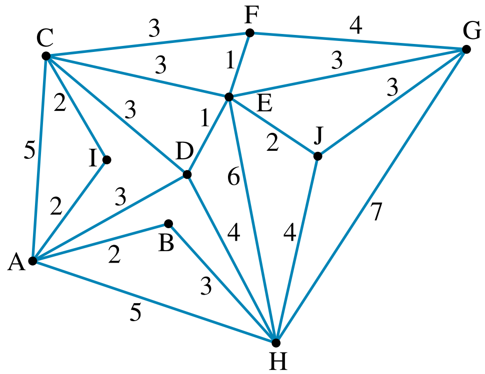
a.



b.



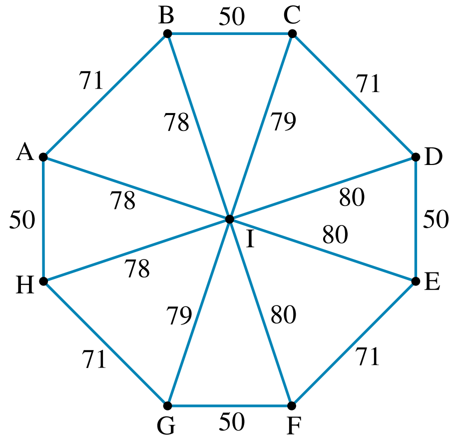
9. Draw diagrams to show the steps you would follow when using Prim's algorithm to identify the minimum spanning tree for the following graph.



10. Part of the timetable and description for a bus route is shown in the table. Draw a weighted graph to represent the bus route.

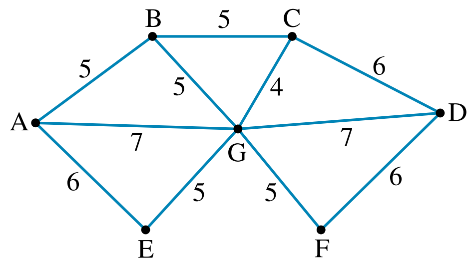
Bus stop	Description	Time
Bus depot	The northernmost point on the route	7:00 am
Northsea Shopping Town	Reached by travelling south-east along a highway from the bus depot	7:15 am
Highview Railway Station	Travel directly south along the road from Northsea Shopping Town.	7:35 am
Highview Primary School	Directly east along a road from the railway station	7:40 am
Eastend Medical Centre	Continue east along the road from the railway station.	7:55 am
Eastend Village	South-west along a road from the medical centre	8:05 am
Southpoint Hotel	Directly south along a road from Eastend Village	8:20 am
South Beach	Travel south-west along a road from the hotel.	8:30 am

11. Consider the graph shown.



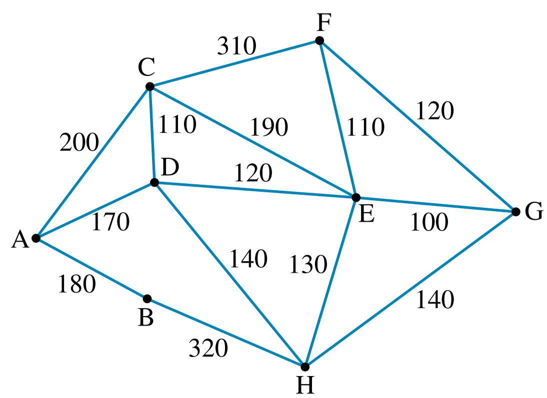
a. Identify the longest and shortest Hamiltonian paths.
b. What is the minimum spanning tree for this graph?

12. Consider the graph shown.



a. If an edge with the highest weighting is removed, identify the shortest Hamiltonian path.
b. If the edge with the lowest weighting is removed, identify the shortest Hamiltonian path.

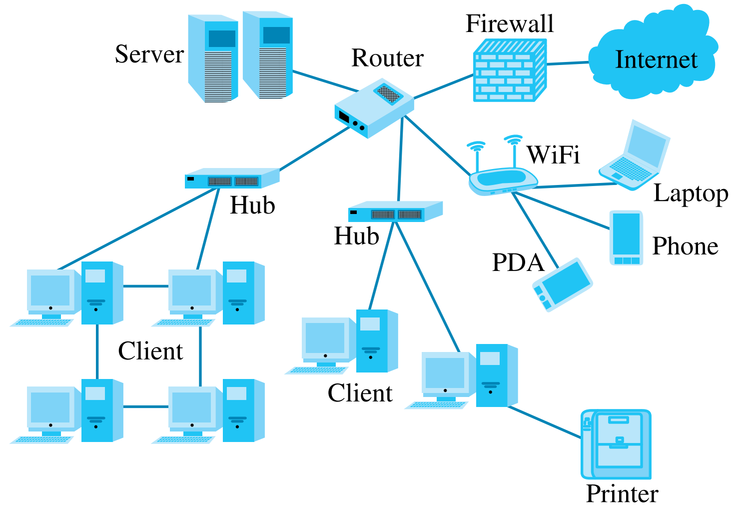
13. The weighted graph represents the costs incurred by a salesman when moving between the locations of various businesses.



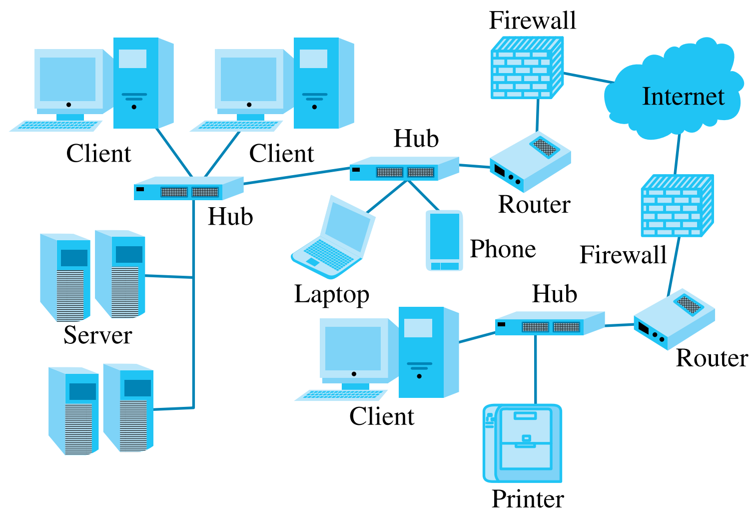
a. What is the cheapest way of travelling from A to G?
b. What is the cheapest way of travelling from B to G?
c. If the salesman starts and finishes at E, what is the cheapest way to travel to all vertices?

14. The diagrams show two options for the design of a computer network for a small business.

Option 1



Option 2



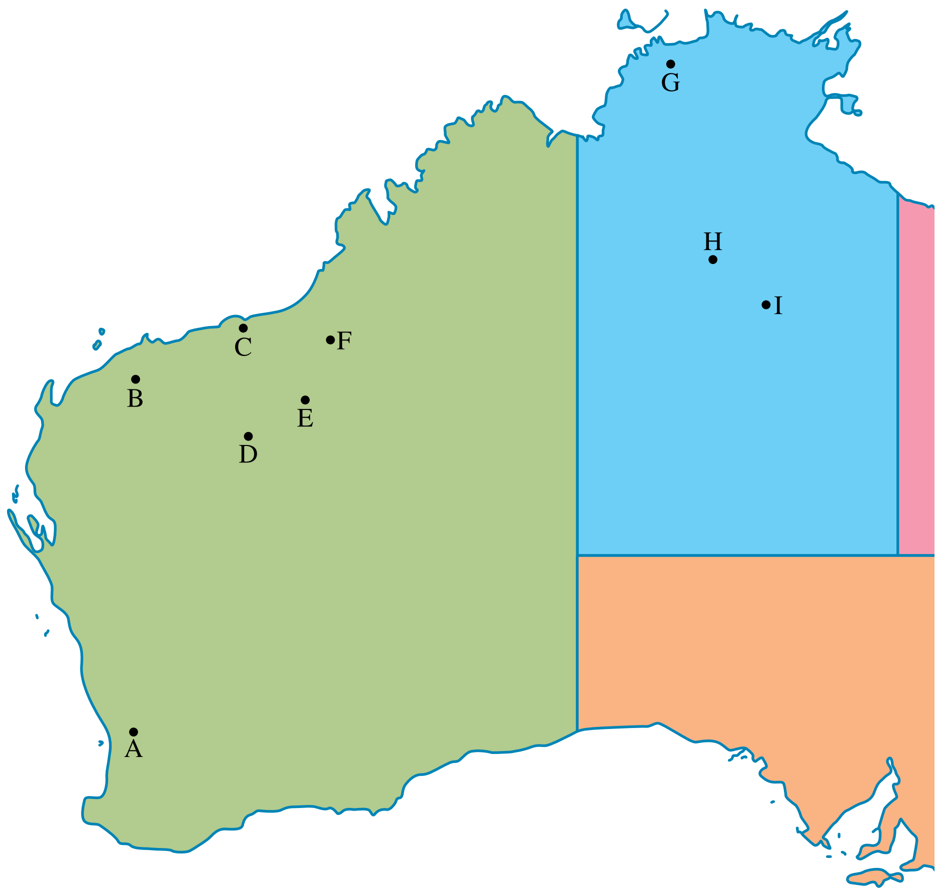
Information relating to the total costs of setting up the network is shown in the following table.

Connected to:	Server	Client	Hub	Router	Firewall	Wifi	Printer
Server			$995	$1050			
Client		$845	$355				$325
Hub			$365	$395			$395
Router	$1050		$395		$395	$395	
Laptop			$295			$325	
Phone			$295			$325	
PDA						$325	
Internet					$855		

a. Use this information to draw a weighted graph for each option.

b. Which is the cheapest option?

15. A mining company operates in several locations in Western Australia and the Northern Territory, as shown on the map.



Flights operate between selected locations, and the flight distances (in km) are shown in the following table.

	A	B	C	D	E	F	G	H	I
A		1090		960			2600		2200
B	1090		360	375	435				
C		360							
D	960	375							
E		435							
F							1590	1400	
G	2600					1590		730	
H						1400	730		220
I	2200							220	

a. Show this information as a weighted graph.

b. Does a Hamiltonian path exist? Explain your answer.

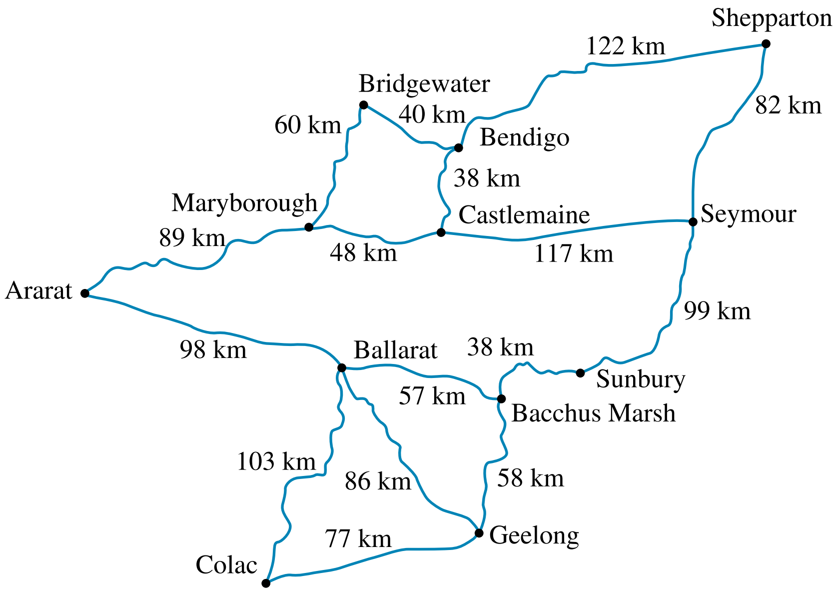
c. Identify the shortest distance possible for travelling to all sites the minimum number of times if you start and finish at:

i. A

ii. G.

d. Draw the minimum spanning tree for the graph.

16. The organisers of the 'Tour de Vic' bicycle race are using the following map to plan the event.



a. Draw a weighted graph to represent the map.
b. If they wish to start and finish in Geelong, what is the shortest route that can be taken that includes a total of nine other locations exactly once, two of which must be Ballarat and Bendigo?
c. Draw the minimal spanning tree for the graph.
d. If the organisers decide to use the minimum spanning tree as the course, what would the shortest possible distance be if each location had to be reached at least once?

5.6 Review: exam practice

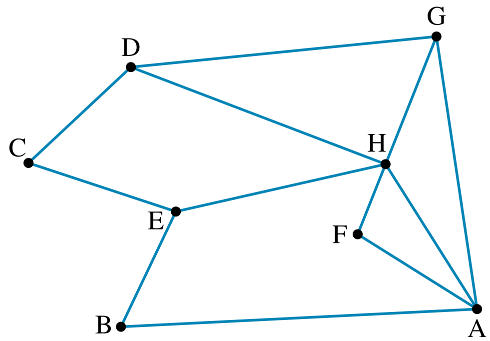
A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

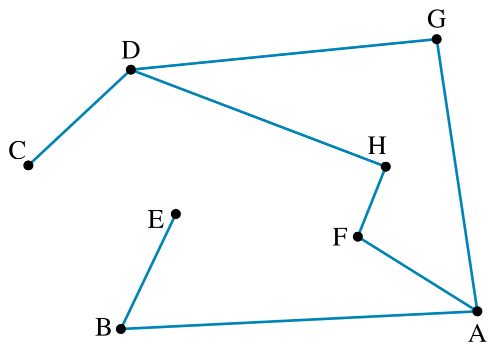
1. MC The minimum number of edges in a connected graph with eight vertices is:

A. 5 **B.** 6 **C.** 7 **D.** 8 **E.** 9

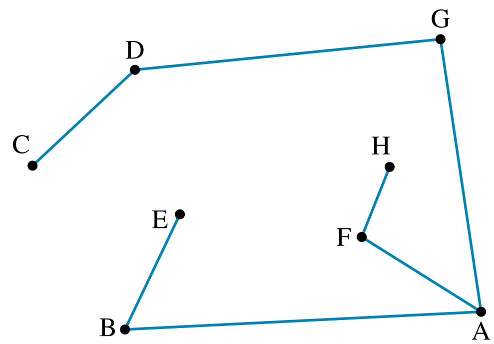
2. MC Which graph is a spanning tree for the following graph?



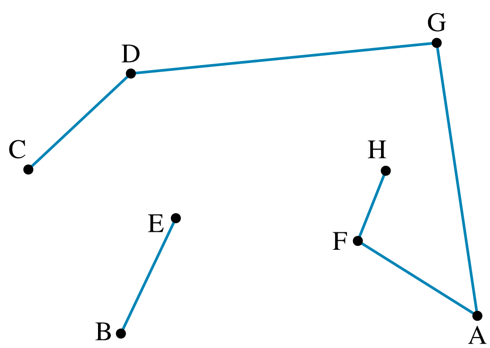
A.



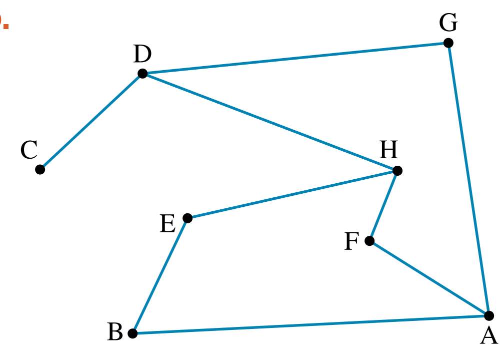
B.



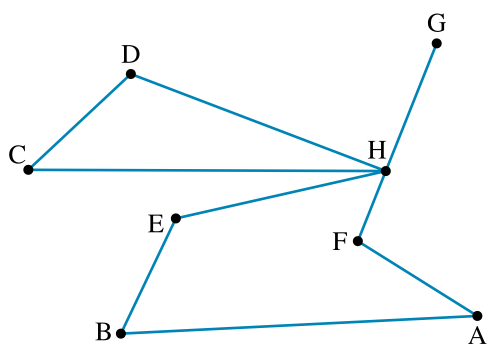
C.



D.



E.

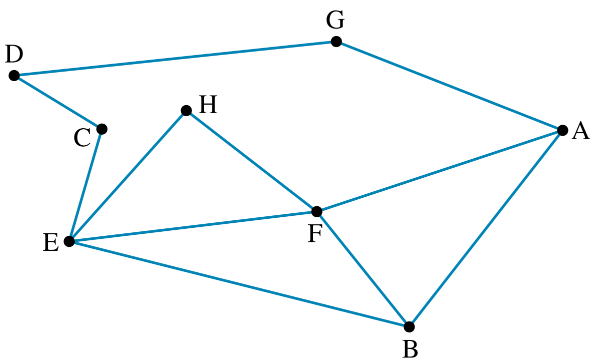


3. MC A connected graph with 9 vertices has 10 faces. The number of edges in the graph is:

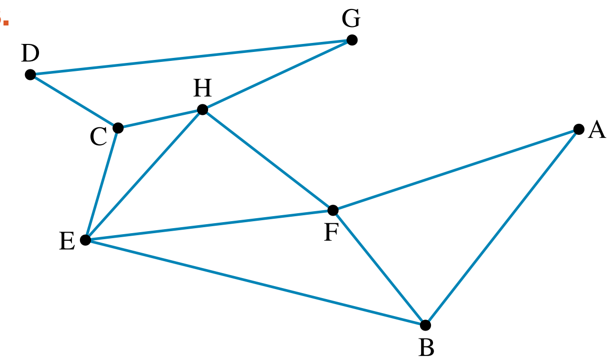
A. 15 **B.** 16 **C.** 17 **D.** 18 **E.** 19

4. MC Which of the following graphs will not have an Euler trail?

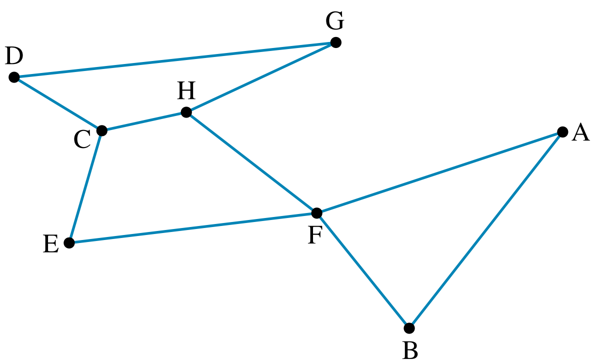
A.



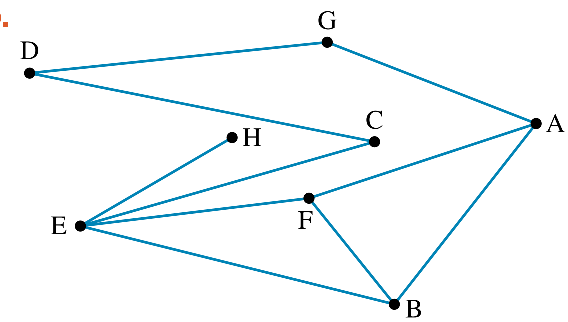
B.



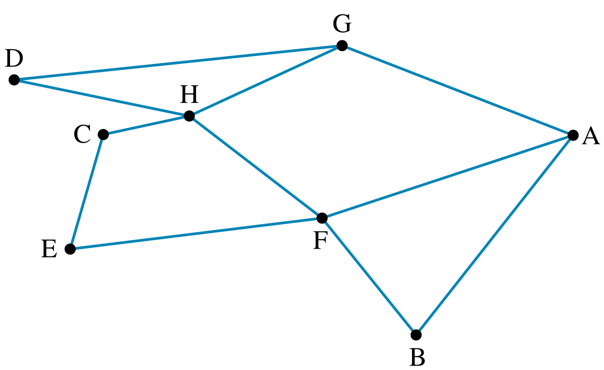
C.



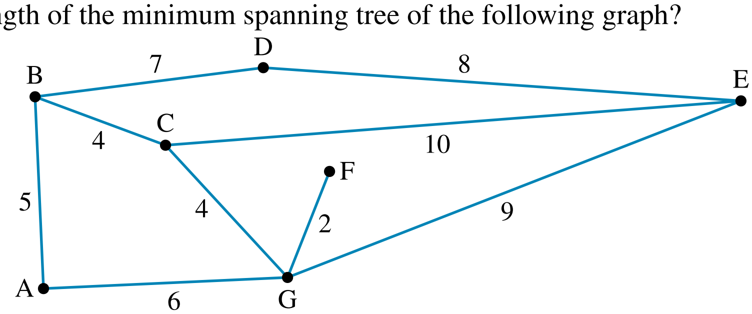
D.



E.

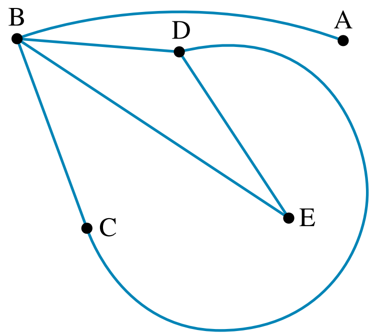


5. MC What is the length of the minimum spanning tree of the following graph?



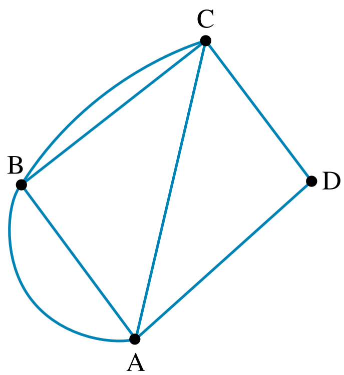
A. 33 **B.** 26 **C.** 34 **D.** 31 **E.** 30

6. **MC** An Euler circuit can be created in the following graph by adding an edge between the vertices:



A. A and D **B.** A and B **C.** A and C **D.** B and C **E.** A and E

7. **MC** The adjacency matrix that represents the following graph is:



A. $\begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}$

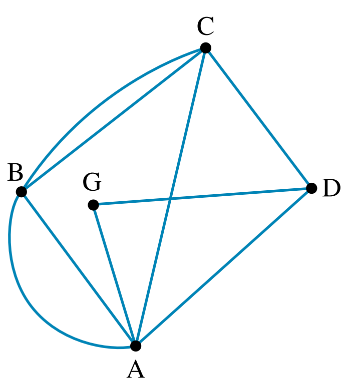
B. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

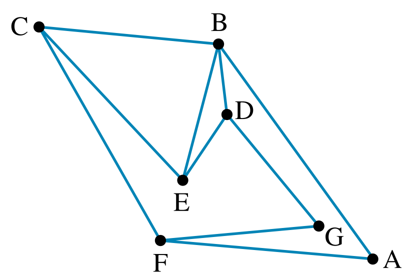
E. $\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

8. **MC** The number of faces in the following planar graph is:



A. 6 **B.** 7 **C.** 8 **D.** 9 **E.** 10

9. MC A Hamiltonian cycle for the following graph is:



A. ABCEDGFA B. ABDGFCEA C. ABDGFCEDEBCFA
D. ABDGFCECFA E. ABDGFA

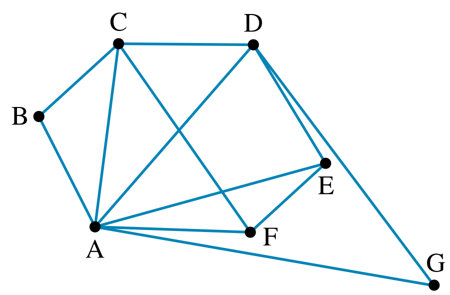
10. MC A complete graph with 7 vertices will have a total number of edges of:

A. 7 B. 8 C. 14 D. 42 E. 21

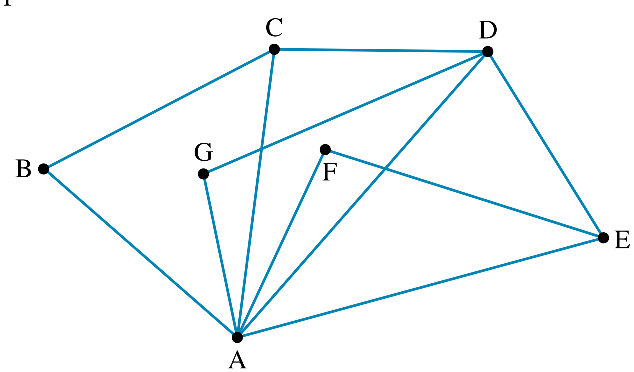
Short answer

1. a. Identify whether the following graphs are planar or not planar.

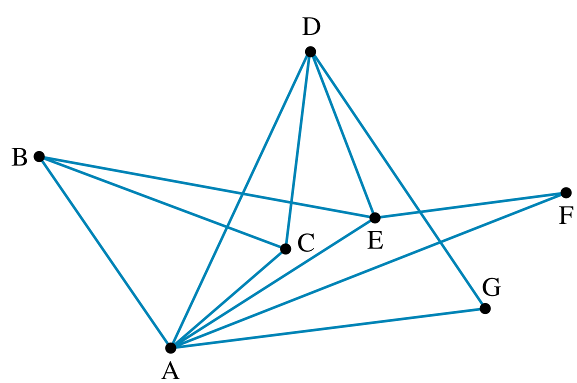
i.



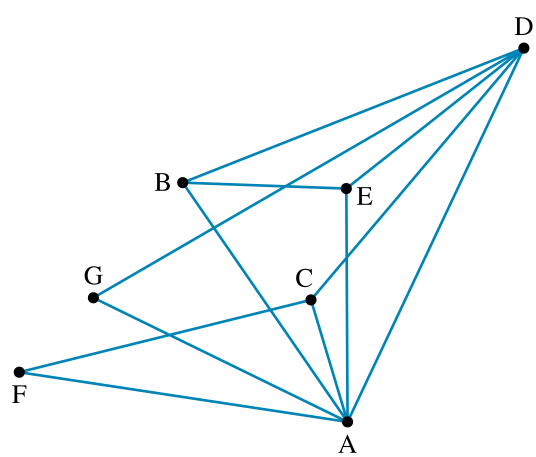
ii.



iii.



iv.



b. Redraw the graphs that are planar without any intersecting edges.

2. Complete the following adjacency matrices.

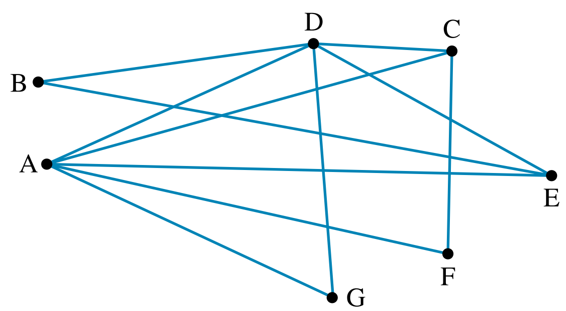
a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ & 0 & & 0 \\ & 3 & 1 & \\ & & 1 & 0 \end{bmatrix}$

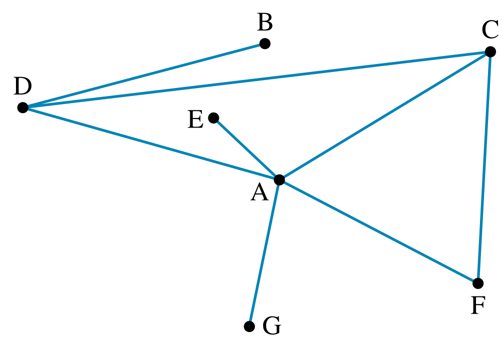
b. $\begin{bmatrix} 0 & 1 & 2 & 1 & \\ & 0 & & 0 & 1 \\ & 2 & 0 & 2 & \\ & & & 2 & 2 \\ 1 & & 3 & & 0 \end{bmatrix}$

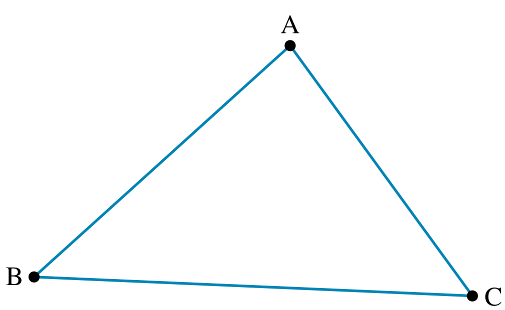
c. $\begin{bmatrix} 0 & & 1 & 3 & 1 & \\ 2 & 0 & & & 1 & \\ & 3 & 0 & 2 & & \\ & 1 & & 2 & 2 & 1 \\ & & 3 & & 3 & 1 \\ 2 & 0 & 1 & & & 0 \end{bmatrix}$

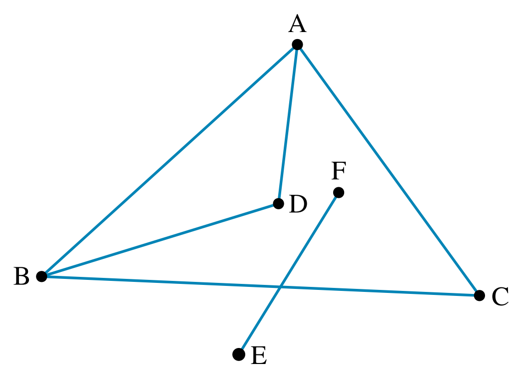
d. $\begin{bmatrix} 0 & & & & 0 & & \\ 2 & 0 & & 1 & & & \\ 1 & 2 & 0 & 1 & 1 & 0 & \\ 3 & & & 0 & & & 1 \\ & 2 & & 0 & 0 & & 3 \\ 1 & 1 & & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & & & & 0 \end{bmatrix}$

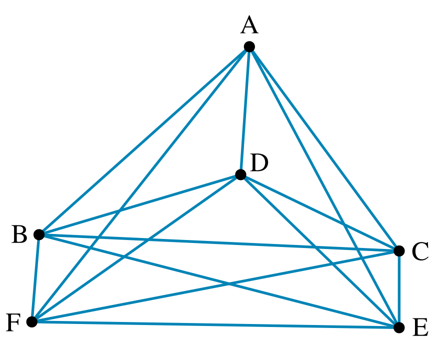
3. Identify which of the following graphs are:
   i. simple
   ii. complete
   iii. planar.

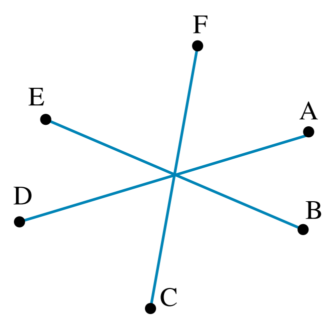
a.


b.


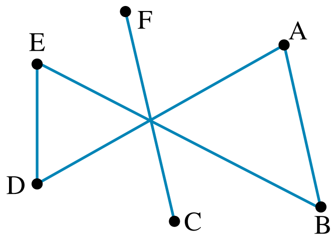
c.


d.


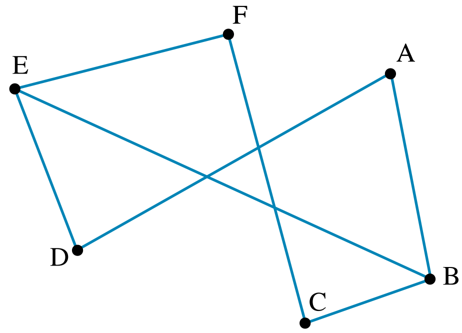
e.


f.


g.

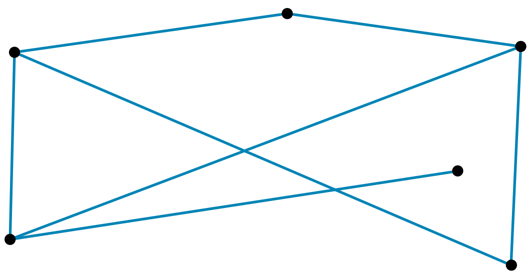


h.

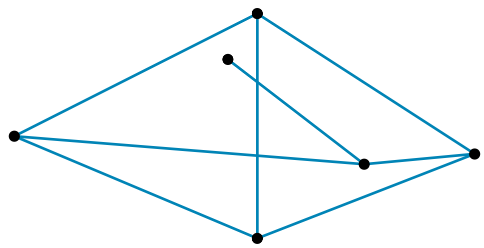


4. Which of the following graphs are isomorphic?

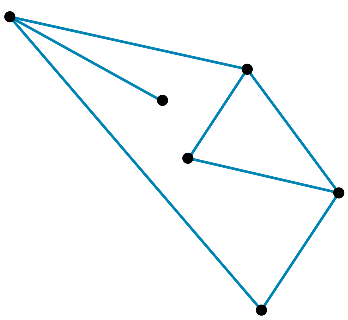
a.



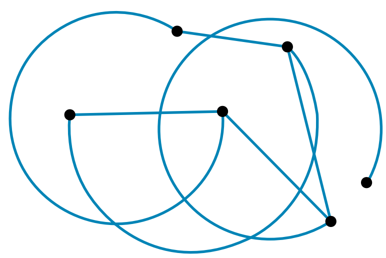
b.



c.



d.

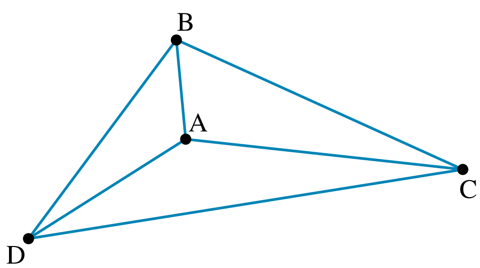


5. For each of the following graphs:

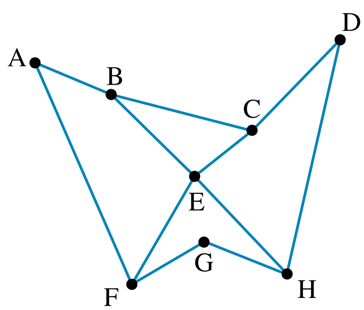
i. add the minimum number of edges to the following graphs in order to create an Euler trail

ii. state the Euler trail created.

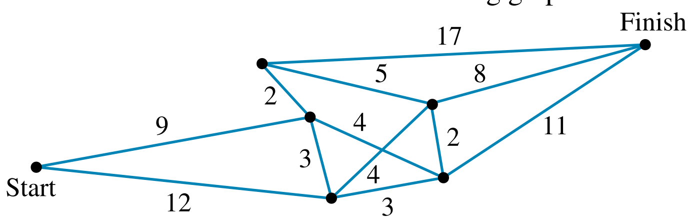
a.



b.



6. a. What is the shortest distance from start to finish in the following graph?



b. What is the total length of the shortest Hamiltonian path from start to finish?

c. Draw the minimum spanning tree for this graph.

Extended response

1. The flying distances between the capital cities of Australian mainland states and territories are listed in the following table.

	Adelaide	Brisbane	Canberra	Darwin	Melbourne	Perth	Sydney
Adelaide		2055	1198	3051	732	2716	1415
Brisbane	2055		1246	3429	1671	4289	982
Canberra	1198	1246		4003	658	3741	309
Darwin	3051	3429	4003		3789	4049	4301
Melbourne	732	1671	658	3789		3456	873
Perth	2716	4363	3741	4049	3456		3972
Sydney	1415	982	309	4301	873	3972	

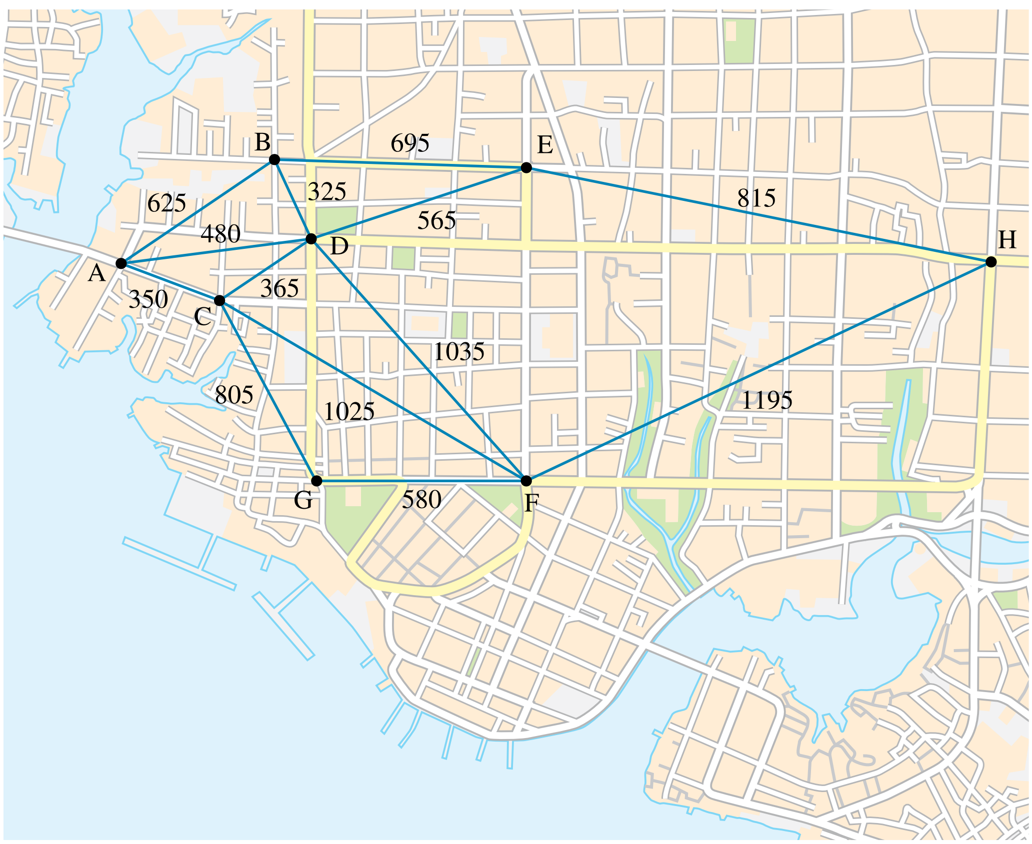
a. Draw a weighted graph to show this information.

b. If technical problems are preventing direct flights from Melbourne to Darwin and from Melbourne to Adelaide, what is the shortest way of flying from Melbourne to Darwin?

c. If no direct flights are available from Brisbane to Perth or from Brisbane to Adelaide, what is the shortest way of getting from Brisbane to Perth?

d. Draw the minimum spanning tree for the graph and state its total distance.

2. The diagram shows the streets in a suburb of a city with a section of underground tunnels shown in black. Weightings indicate distances in metres. The tunnels are used for utilities such as electricity, gas, water and drainage.



a. i. If the gas company wishes to run a pipeline that minimises its total length but reaches each vertex, what will be the total length required?

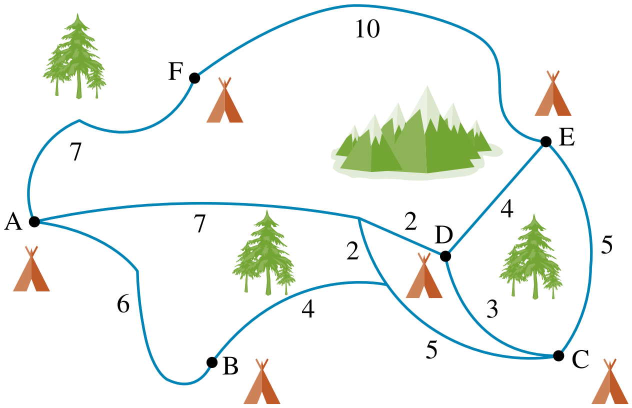
ii. Draw a graph to show the gas lines.

b. If drainage pipes need to run from H to A, what is the shortest path they can follow? How long will this path be in total?

c. A single line of cable for a computerised monitoring system needs to be placed so that it starts at D and reaches every vertex once. What is the minimum length possible, and what is the path it must follow?

d. If a power line has to run from D so that it reaches every vertex at least once and finishes back at the start, what path must it take to be a minimum?

3. A brochure for a national park includes a map showing the walking trails and available camping sites at the park.



a. Draw a weighted graph to represent all the possible ways of travelling to the camp sites.

b. Draw the adjacency matrix for the graph.

c. Is it possible to walk a route that travels along each edge exactly once? Explain your answer, and indicate the path if it is possible.

d. If the main entrance to the park is situated at A, what is the shortest way to travel to each campsite and return to A?

4. A cruise ship takes passengers around Tasmania between the seven locations marked on the map.

The sailing distances between locations are indicated in the table.

	Hobart	Bruny I.	Maria I.	Flinders I.	Devonport	Robbins I.	King I.
Hobart	–	65 km	145 km	595 km	625 km	–	–
Bruny I.	65 km	–	130 km	–	–	715 km	–
Maria I.	145 km	130 km	–	450 km	–	–	–
Flinders I.	595 km	–	450 km	–	330 km	405 km	465 km
Devonport	625 km	–	–	330 km	–	265 km	395 km
Robbins I.	–	715 km	–	405 km	265 km	–	120 km
King I.	–	–	–	465 km	395 km	120 km	–

a. Draw a weighted graph to represent all possible ways of travelling to the locations.
b. What is the shortest route from Hobart to Robbins Island?
c. What is the shortest way of travelling from Hobart to visit each location only once?
d. What is the shortest way of sailing from King Island, visiting each location once and returning to King Island?

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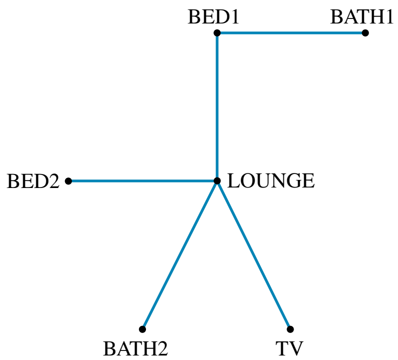
Answers

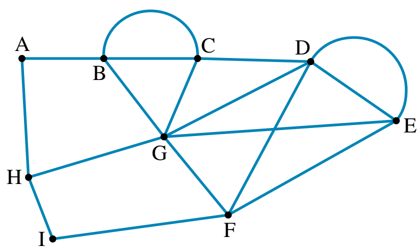
Topic 5 Graphs and networks

Exercise 5.2 Definitions and terms

1.



2.



3. a. Edges = 7; Degree sum = 14
b. Edges = 10; Degree sum = 20

4. a. Edges = 9; Degree sum = 18
b. Edges = 9; Degree sum = 18

5. a. deg(A) = 5; deg(B) = 3; deg(C) = 4; deg(D) = 1; deg(E) = 1
b. deg(A) = 0; deg(B) = 2; deg(C) = 2; deg(D) = 3; deg(E) = 3
c. deg(A) = 4; deg(B) = 2; deg(C) = 2; deg(D) = 2; deg(E) = 4
d. deg(A) = 1; deg(B) = 2; deg(C) = 1; deg(D) = 1; deg(E) = 3

6. a. The graphs are isomorphic.
b. The graphs are isomorphic.
c. The graphs are not isomorphic.
d. The graphs are isomorphic.

7. a. Different degrees and connections **b.** Different connections

8. The isomorphic pairs are graphs **2** and **4**, and graphs **5** and **6**.

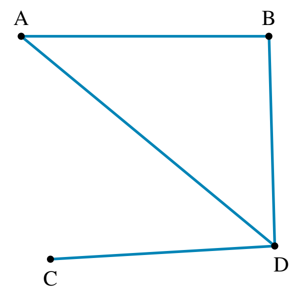
9. a. $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

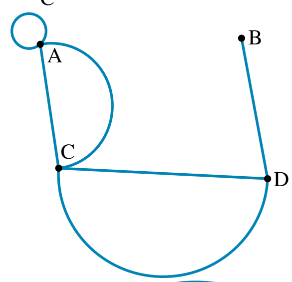
c. $\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 4 & 0 \end{bmatrix}$

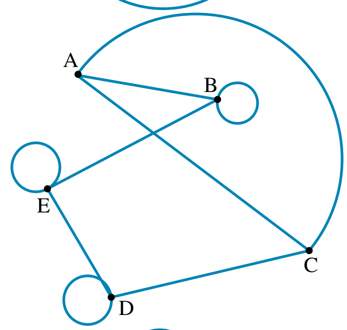
10. a.



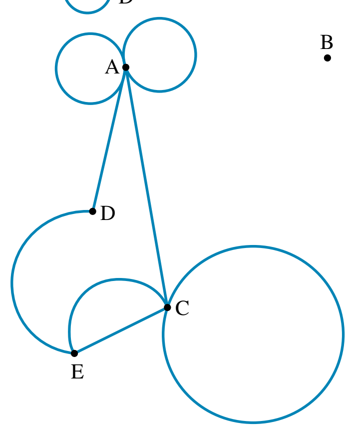
b.



c.



d.



11.

Graph 1:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Graph 2:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Graph 3:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Graph 4:

$$\begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Graph 5:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

12. a.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

b.

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

c.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$$

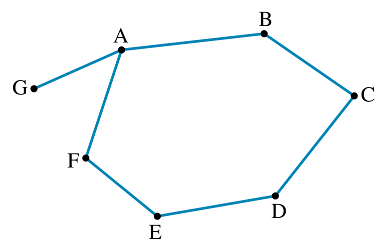
d.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

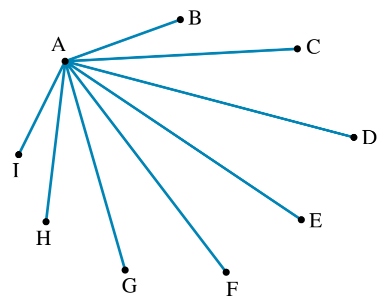
13.

Graph	Simple	Complete	Connected
Graph 1	Yes	No	Yes
Graph 2	Yes	No	Yes
Graph 3	Yes	No	Yes
Graph 4	No	No	Yes
Graph 5	No	Yes	Yes

b.



c.

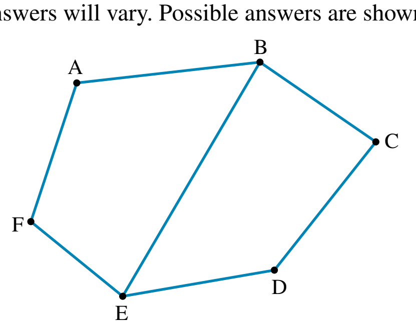


14.

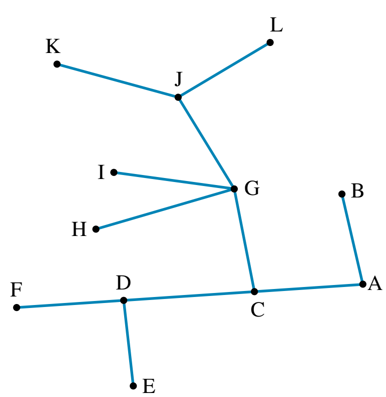
Vertices	Edges
2	1
3	3
4	6
5	10
6	15
n	$\dfrac{n(n-1)}{2}$

15. Answers will vary. Possible answers are shown.

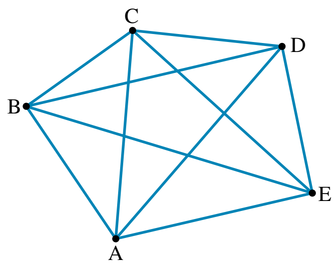
a.



16.



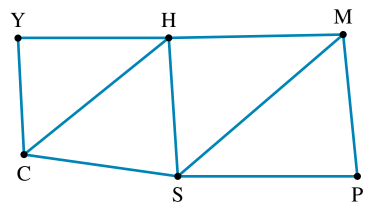
17. a.



b. Complete graph

c. Total number of games played

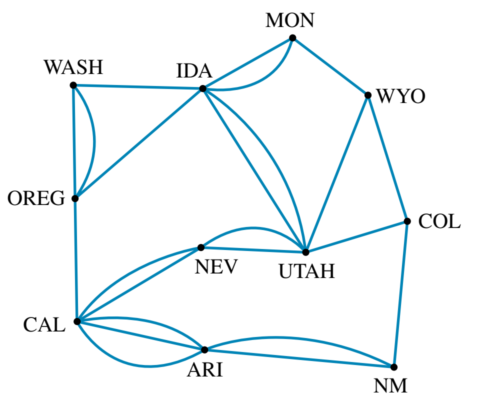
18. a.



b. Huairou and Shunyi

c. Simple connected graph

19. a.



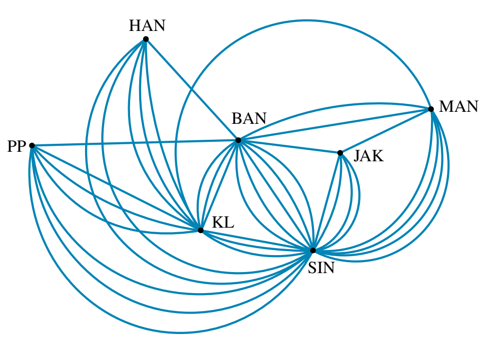
b.

	Wa	O	Ca	I	N	A	M	U	Wy	Co	NM
Wa	0	2	0	1	0	0	0	0	0	0	0
O	2	0	1	1	0	0	0	0	0	0	0
Ca	0	1	0	0	2	3	0	0	0	0	0
I	1	1	0	0	0	0	2	2	0	0	0
N	0	0	2	0	0	0	0	2	0	0	0
A	0	0	3	0	0	0	0	0	0	0	2
M	0	0	0	2	0	0	0	0	1	0	0
U	0	0	0	2	2	0	0	0	1	1	0
Wy	0	0	0	0	0	0	1	1	0	1	0
Co	0	0	0	0	0	0	0	1	1	0	1
NM	0	0	0	0	0	2	0	0	0	1	0

c. California, Idaho and Utah

d. Montana

20. a.

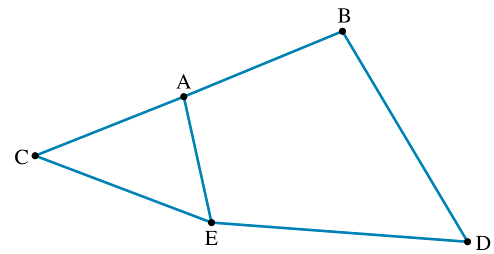


b. Directed, as it would be important to know the direction of the flight

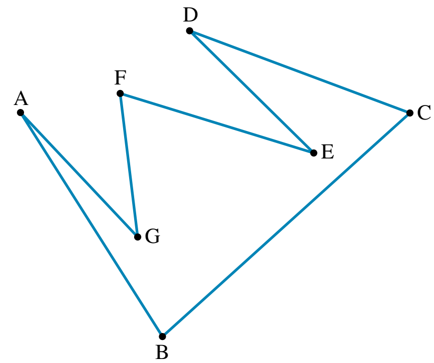
i. 10 **ii.** 7

Exercise 5.3 Planar graphs

1. a.

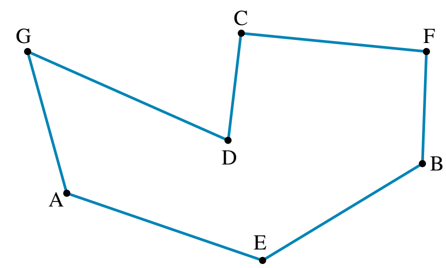


b.

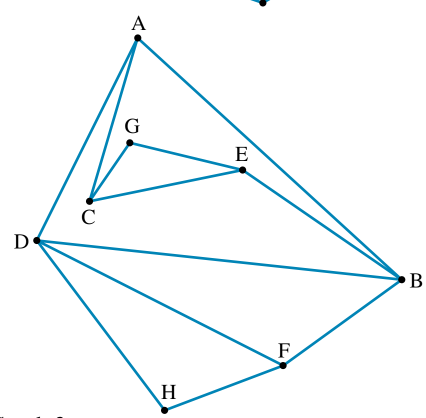


2. a. All of them **b.** All of them

3. a.



b.



4. Graph 3

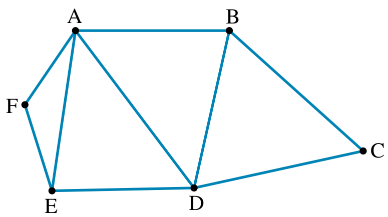
5. a. 4 **b.** 5

6. a. 6 **b.** 5

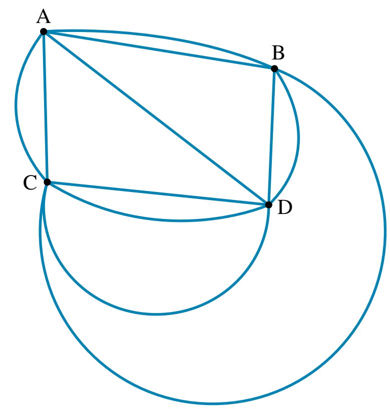
7. a. 3 **b.** 3

c. 2 **d.** 7

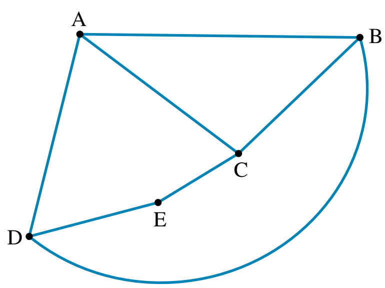
8. a.



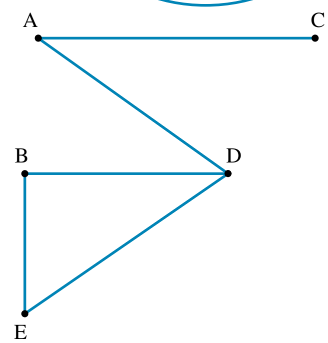
b.



9. a.



b.



10. a. **i.** 3 **ii.** 2 **b.** **i.** 1 **ii.** 4

11.

Graph	Total edges	Total degrees
Graph 1	3	6
Graph 2	5	10
Graph 3	8	16
Graph 4	14	28

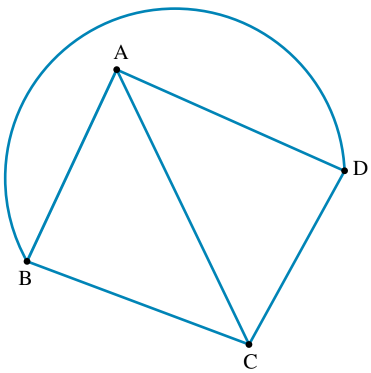
b. Total degrees $= 2 \times$ total edges

12.

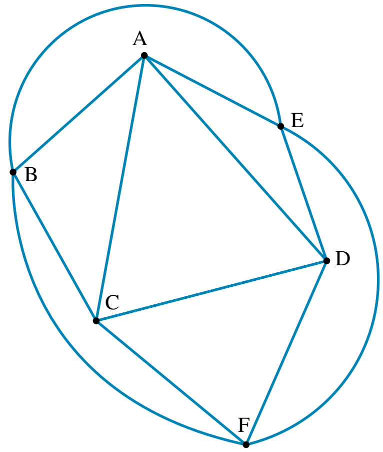
Graph	Total vertices of even degree	Total vertices of odd degree
Graph 1	3	2
Graph 2	4	2
Graph 3	4	6
Graph 4	6	6

b. No clear pattern evident.

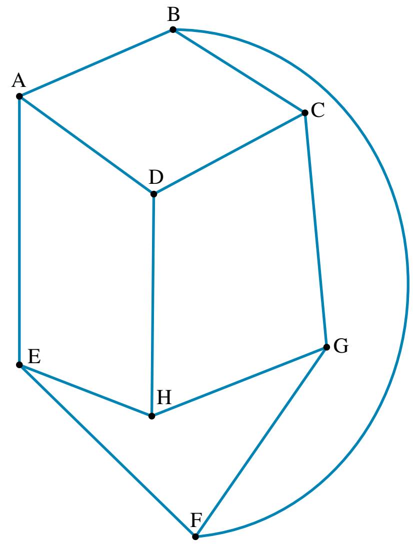
13.



Tetrahedron

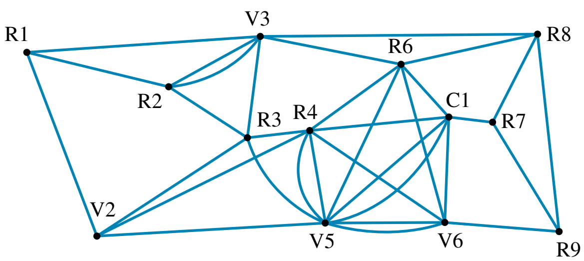


Octahedron



Cube

14. a.



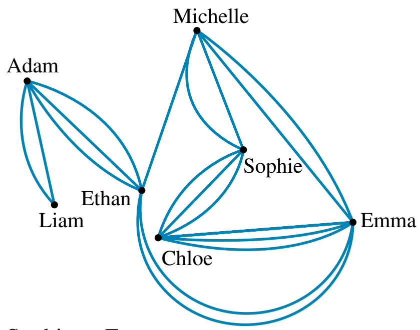
b. No

15. a.

	Pa	Ed	Bak	Wa	Ki	Fa	Mo	No	Bo	Ox	To	Ho	Ba	So	Vi	Gr	Pi	We	Em	Bl	Ca	Le	Ch
Pa	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Ed	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Bak	0	1	0	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
Wa	0	0	1	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
Ki	0	0	1	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
Fa	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Mo	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
No	1	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
Bo	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
Ox	0	0	1	1	0	0	0	0	1	0	1	0	0	0	0	1	1	0	0	0	0	0	0
To	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0
Ho	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0
Ba	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0
So	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
Vi	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0
Gr	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	1	1	0	0	0	0	0
Pi	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1
We	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	0	0	0	0
Em	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
Bl	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
Ca	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
Le	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	1
Ch	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0

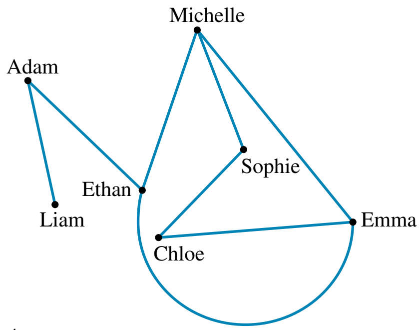
b. The sum of the rows represents the sum of the degree of the vertices, or twice the number of edges (connections).

16. a.



b. Sophie or Emma

c.



d. 4

Exercise 5.4 Connected graphs

1. Cycle: ABECA (others exist)
   Circuit: BECDA (others exist)
2. Path: ABGFHDC (others exist)
   Cycle: DCGFHD (others exist)
   Circuit: AEBGFHDCA (others exist)
3. a. Walk
   b. Walk, trail and path
   c. Walk, trail, path, cycle and circuit
   d. Walk and trail
4. a. MCHIJGFAED
   b. AEDBLKMC
   c. MDEAFGJIHCM
   d. FMCHIJGF
5. a. Euler trail: AFEDBECAB; Hamiltonian path: BDECAF
   b. Euler trail: GFBECGDAC; Hamiltonian path: BECADGF
6. a. Euler circuit: AIBAHGFCJBCDEGA; Hamiltonian cycle: none exist
   b. Euler circuit: ABCDEFGHA (others exist); Hamiltonian cycle: HABCDEFGH (others exist)
7. a. Graphs **i**, **ii** and **iv**
   b. Graph **i**: ACDABDECB (others exist)
   Graph **ii**: CFBCEDBADCA (others exist)
   Graph **iv**: CFBCEDCADBAH (others exist)
8. a. Graphs **i** and **ii**
   b. Graph **i**: CEDABC
   Graph **ii**: CEDABGC

9. a. i.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

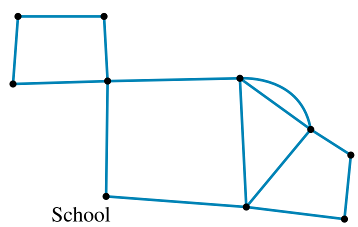
ii.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

iii.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

iv.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b. The presence of Euler trails and circuits can be identified by using the adjacency matrix to check the degree of the vertices. The presence of Hamiltonian paths and cycles can be identified by using the adjacency matrix to check the connections between vertices.

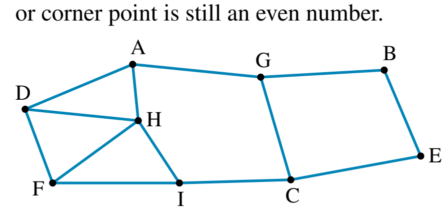
10. E
11. a. A or C
    b. B or D
12. a. G to C
    b. F to E
13. a.



b. Yes, because the degree of each intersection or corner point is an even number.

c. Yes, because the degree of each remaining intersection or corner point is still an even number.

14. a.



b. 4

c. i. ADHFICEBGA
   ii. AHDFICEBGA

d. i. Yes, because two of the checkpoints have odd degree.
   ii. H and C

15. a.

	Hamiltonian cycle
1.	ABCDA
2.	ABDCA
3.	ACBDA
4.	ACDBA
5.	ADBCA
6.	ADCBA

b. Yes, commencing on vertices other than A

16. a. B, C, D, F or G **b.** B or C **c.** None possible

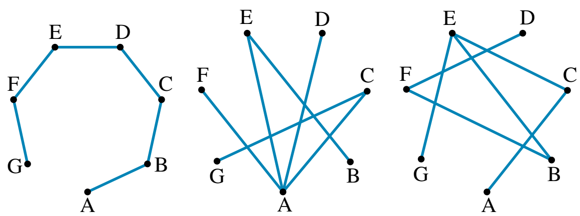
d. D or E **e.** D to E

Exercise 5.5 Weighted graphs and trees

1. 21

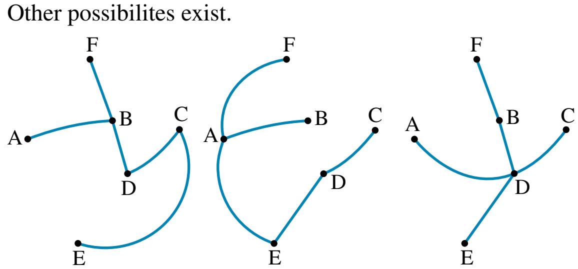
2. 20.78

3. a.



Other possibilites exist.

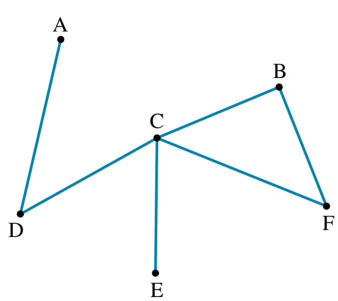
b.



Other possibilites exist.

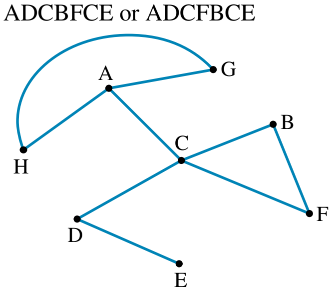
4. ABGEDCA or ACDEGBA (length 66)

5. a. i.



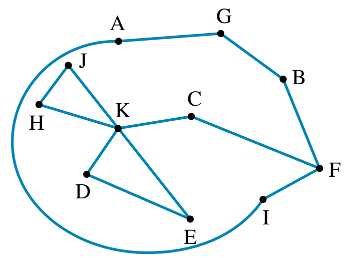
ii. ADCBFCE or ADCFBCE

b. i.



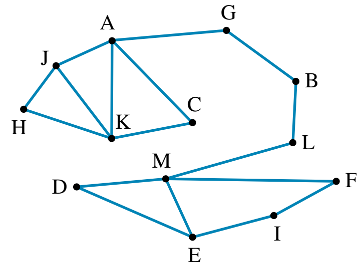
ii. AHGACBFCDE or similar

c. i.



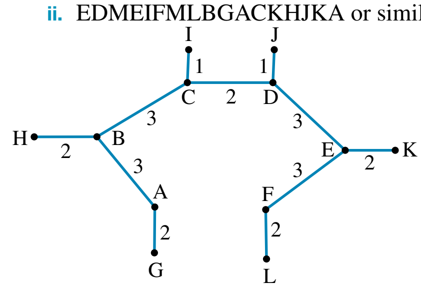
ii. KDEKHJKCFIAGBF or similar

d. i.

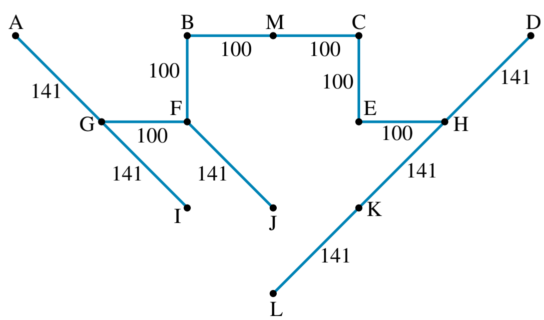


ii. EDMEIFMLBGACKHJKA or similar

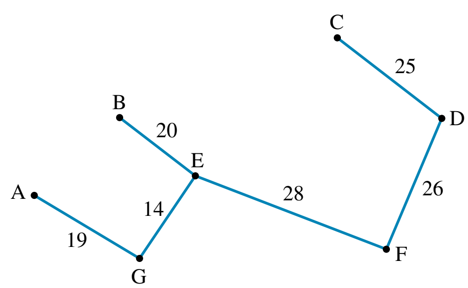
6.



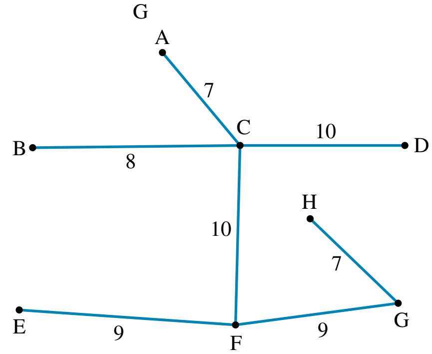
7.



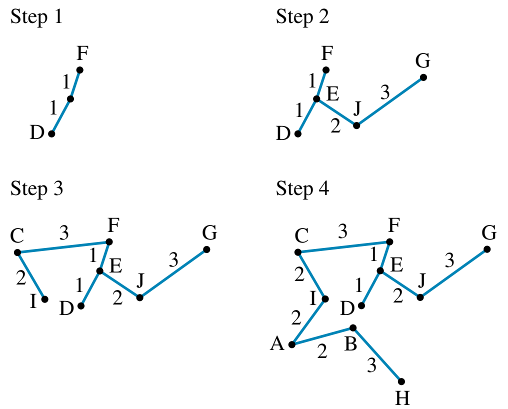
8. a.



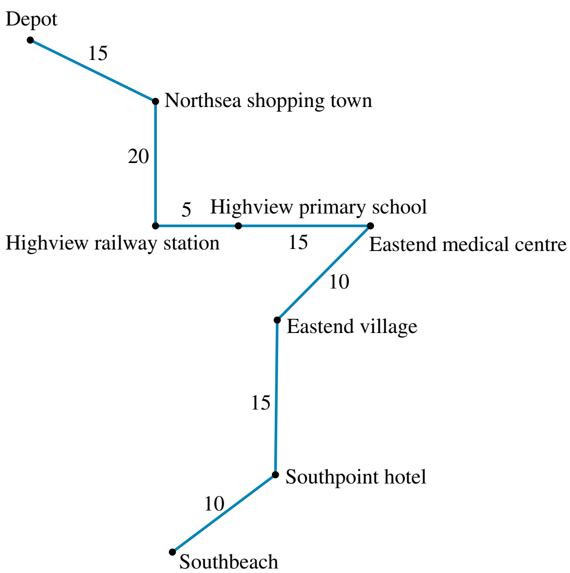
b.



9.

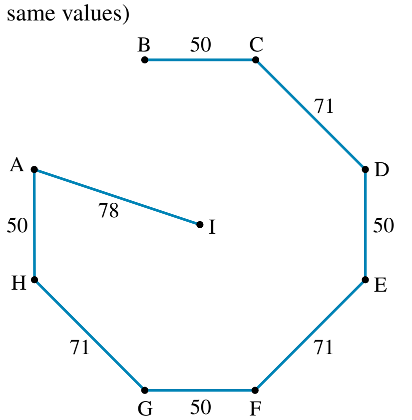


10.



11. a. Longest: IFEDCBAHG (or similar variation of the same values)
Shortest: IAHGFEDCB (or similar variation of the same values)

b.



12. a. FDCGBAE (other solutions exist)

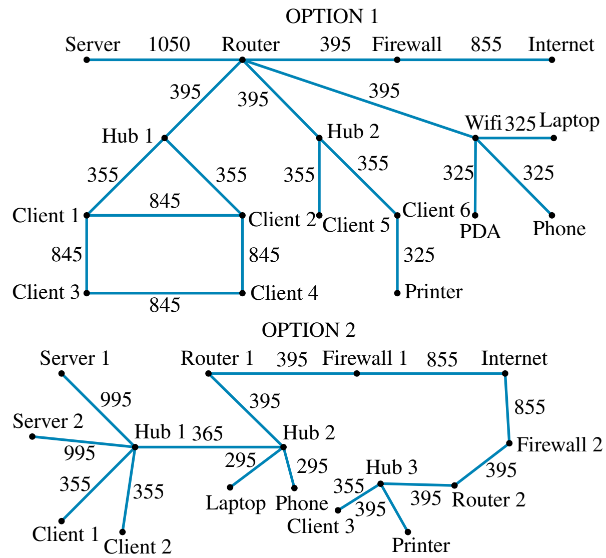
b. FDCBAEG (other solutions exist)

13. a. ADEG

b. BHG

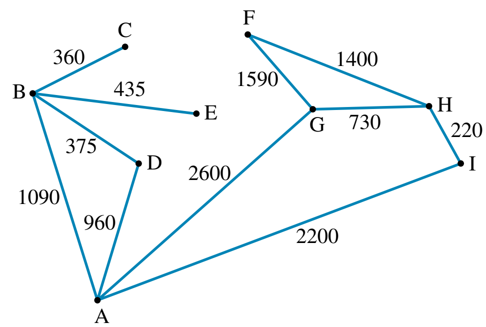
c. EGFCDABHE

14. a.



b. Option 2

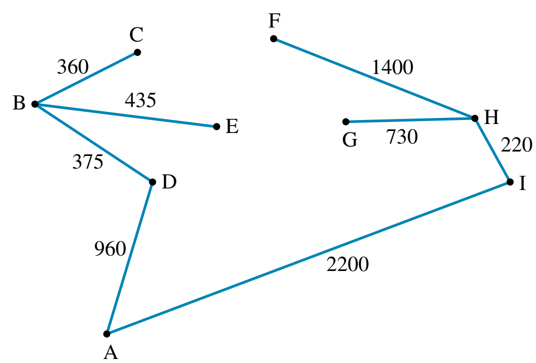
15. a.



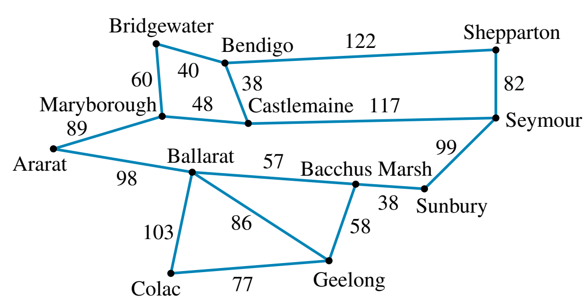
b. No; C and E are both only reachable from B.

c. **i.** 12 025 **ii.** 12 025

d.

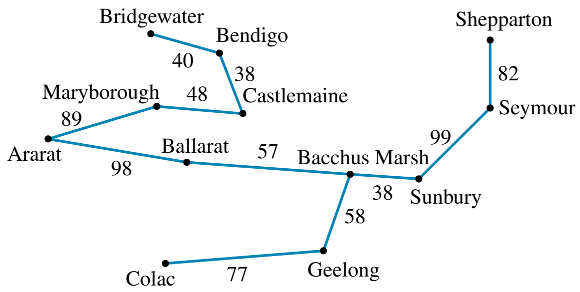


16. a.



b. 723 km

c.



d. 859 km

5.6 Review: exam practice

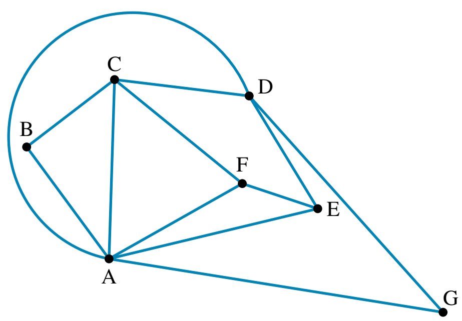
Multiple choice

1. C **2.** B **3.** C **4.** D **5.** E
6. A **7.** E **8.** A **9.** A **10.** E

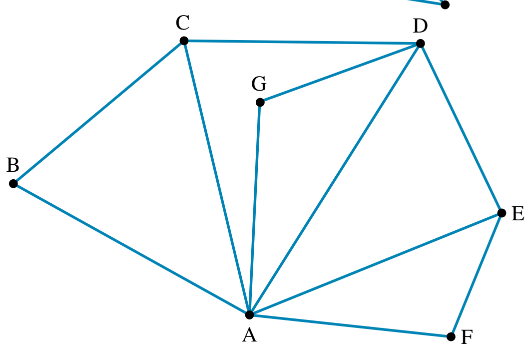
Short Answer

1. a. i. Planar **ii.** Planar
iii. Planar **iv.** Planar

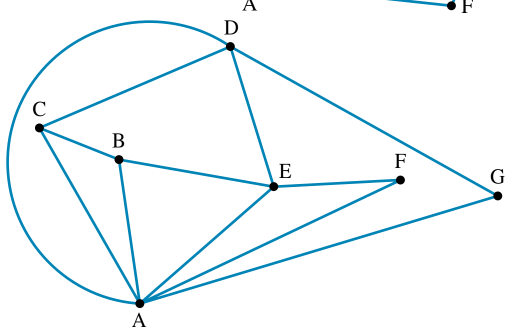
b. i.



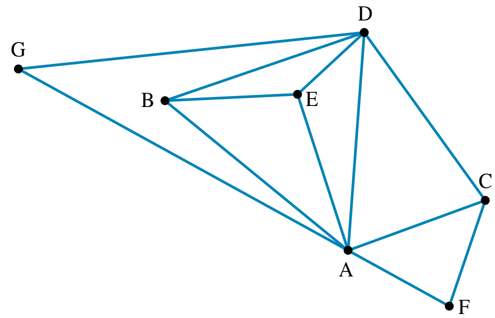
ii.



iii.



iv.



2. a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 2 & 2 & 0 & 2 & 3 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$

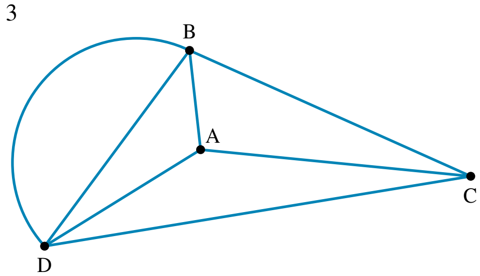
c. $\begin{bmatrix} 0 & 2 & 1 & 3 & 1 & 2 \\ 2 & 0 & 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 3 & 1 \\ 3 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 3 & 1 \\ 2 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 2 & 1 & 3 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 & 2 & 0 \end{bmatrix}$

3. a. Simple, planar **b.** Simple, planar
c. Simple, complete, planar **d.** Simple, planar
e. Simple, complete **f.** Simple, planar
g. Simple, planar **h.** Simple, planar

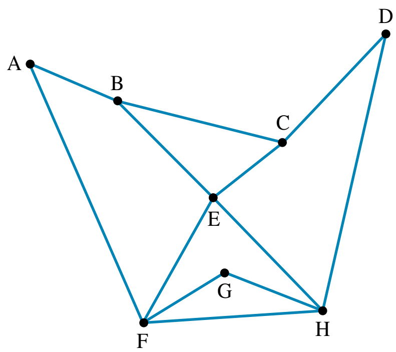
4. Graphs **a** and **d** are isomorphic.

5. a. i. 3



ii. ABDBCADC

b. i.

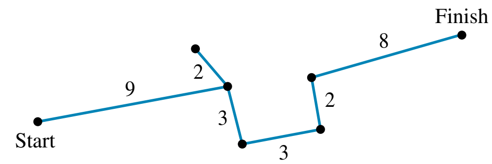


ii. BAFEHGFHDCEBC

6. a. 23

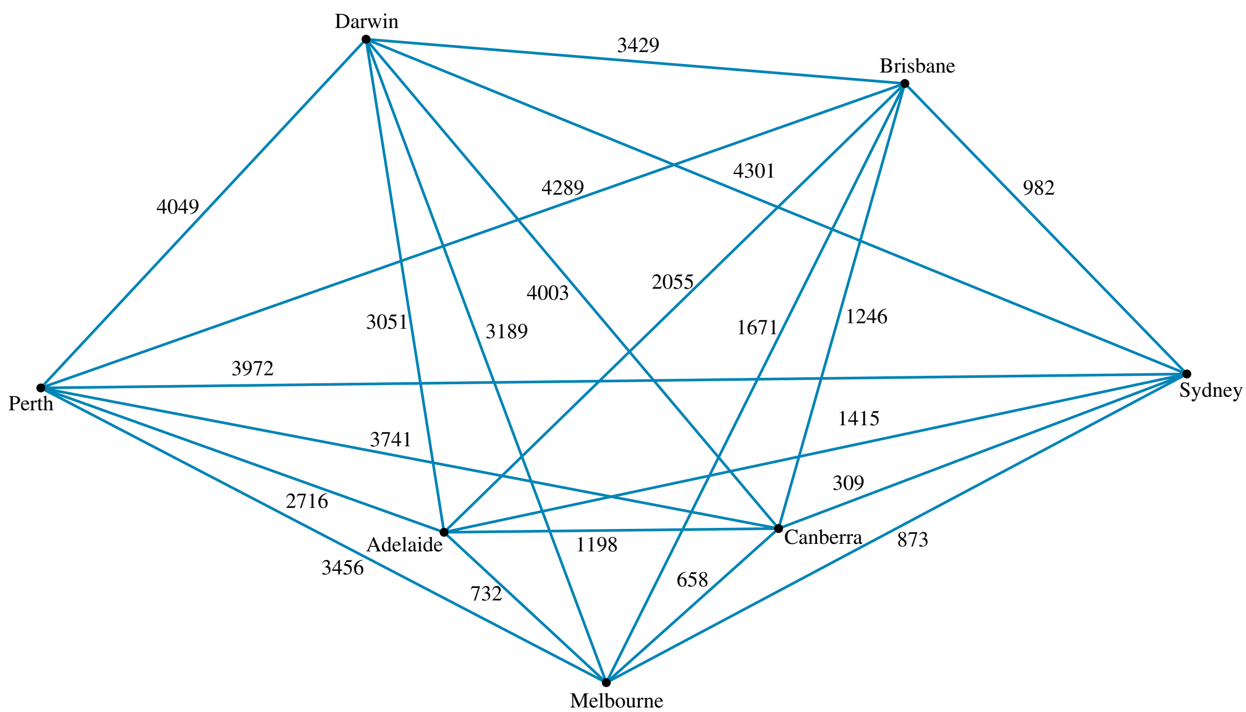
b. 34

c.



Extended response

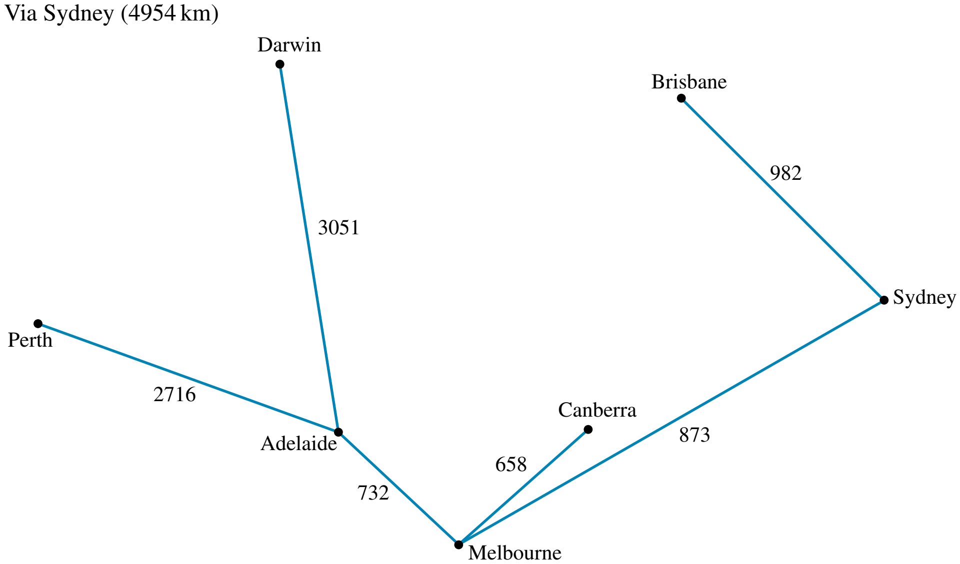
1. a.



b. Via Canberra (4661 km)

c. Via Sydney (4954 km)

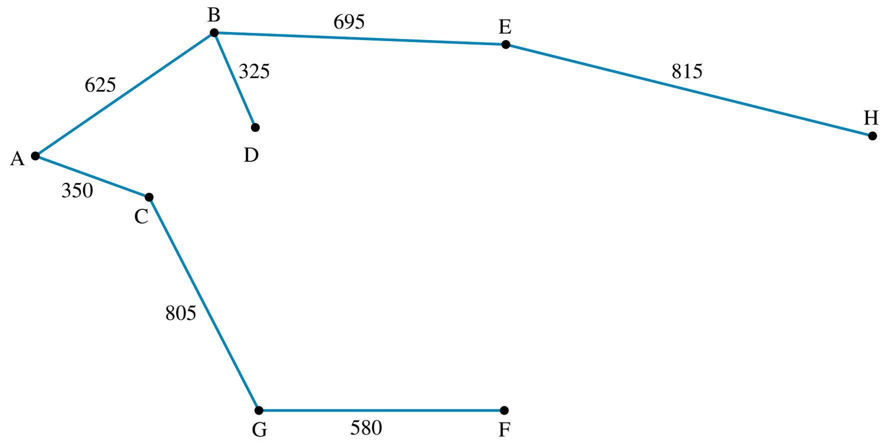
d.



The total distance is 9012 km.

2. a. i. 4195 m

ii.

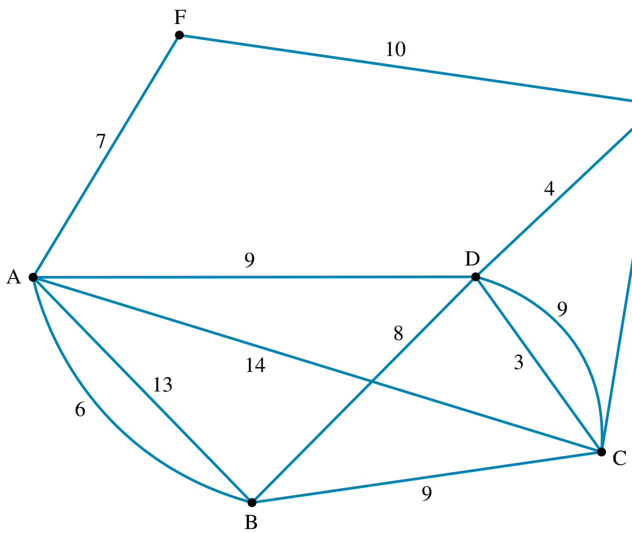


b. HEDA, 1860 m

c. 4905 m, DFGCABEH

d. DEHFGCABD, 5260 m

3. a.

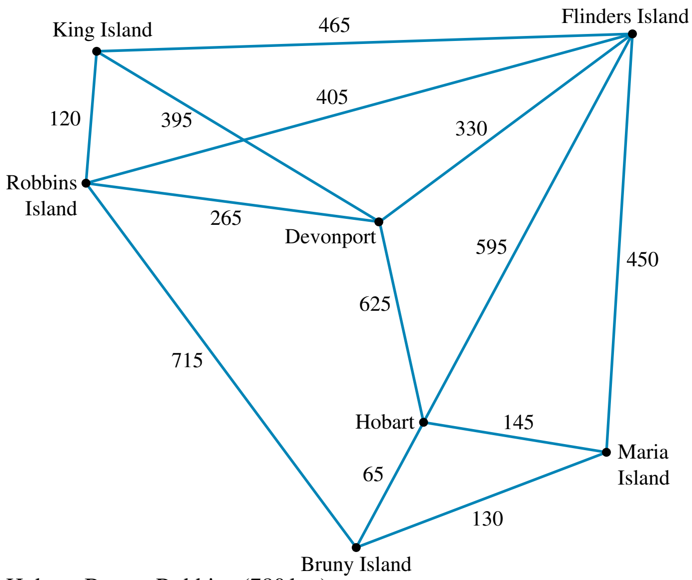


b. $\begin{bmatrix} 0 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

c. No, as there are more than two vertices of odd degree.

d. AFEDCBA(39)

4. a.



b. Hobart–Bruny–Robbins (780 km)

c. Hobart–Bruny–Robbins–King–Devonport–Flinders–Maria (2075 km)

d. King–Devonport–Flinders–Maria–Hobart–Bruny–Robbins–King (2220 km)

TOPIC 6
Sequences

6.1 Overview

6.1.1 Introduction

Leonardo of Pisa, more commonly known as Fibonacci, has been described as the greatest European mathematician of the Middle Ages. Fibonacci was born in 1175 in the Republic of Pisa and he travelled extensively around the Mediterranean coast. During his travels, he learned a lot about arithmetic from merchants. He became aware of the advantages of the Hindu-Arabic numeral system over all others. In 1202 he completed the *Liber Abaci*, also known as the *Book of Abacus* or *Book of Calculation*. This popularised Hindu-Arabic numerals in Europe.

The mathematical work Fibonacci was most famously linked with was that of sequences and series. These sequences are not only limited to the area of mathematics, but are also found in the fields of physics, chemistry and computer science. Probably the most famous of mathematical sequences is the Fibonacci series.

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Initially this just looks like a set of numbers; however, a closer look allows you to see the pattern is the addition of the two previous numbers to find the next number in the series

$1 + 1 = 2$
$1 + 2 = 3$
$2 + 3 = 5$
and so on.

This may not seem that remarkable, but this series of numbers occurs in nature, geometry, algebra, number theory, permutations and combinations and many other branches of mathematics. What's remarkable is that the series appears in nature in so many situations. Some examples are the number of spirals of bracts on a pine cone, or in arrangements of branches on various species of trees. Practically, the application of the Fibonacci sequence in nature appears to be boundless.

Leonardo da Vinci was well known for his usage of the Fibonacci sequence. The most famous of these is his painting of the Mona Lisa. Da Vinci used the golden spiral, which stems from the perfect rectangle. The perfect rectangle is formed by creating rectangles linked to descending Fibonacci numbers $(8, 5, 3, 2, 1, \ldots)$.

LEARNING SEQUENCE

6.1 Overview
6.2 Arithmetic sequences
6.3 Geometric sequences
6.4 Recurrence relations
6.5 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.

6.1.2 Kick off with CAS

Exploring the Fibonacci sequence with CAS

The Fibonacci sequence is a sequence of numbers that starts with 1 and 1, after which every subsequent number is found by adding the two previous numbers. Thus the sequence is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

This sequence is frequently found in nature. For example, the numbers of petals of many flowers fall within this sequence: a lily has 3 petals, a buttercup has 5 petals and a daisy has 34 petals, just to name a few. Within the head of a sunflower, seeds are produced at the centre and then migrate to the outside in spiral patterns, with the numbers of seeds in the spirals being numbers from the Fibonacci sequence.

1. Using CAS and a list and spreadsheet application, generate the first 30 terms of the Fibonacci sequence.
2. If the first 3 numbers in the Fibonacci sequence are called $t_1 = 1$ (term 1), $t_2 = 1$ (term 2) and $t_3 = 2$ (term 3), what is the value of t_{20}?
3. What is the smallest value of n for which $t_n > 1000$?
4. Calculate the ratios of consecutive terms for the first 12 terms; that is,

$$\frac{1}{1} = 1, \frac{2}{1} = 2, \frac{3}{2} = 1.5, \frac{5}{3} = ?, \frac{8}{5} = ?, \frac{13}{8} = ?, \frac{21}{13} = ?, \frac{34}{21} = ?, \frac{55}{34} = ?, \frac{89}{55} = ?, \frac{?}{?} = ?$$

5. What do you notice about the value of the ratios as the terms increase?

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

6.2 Arithmetic sequences

6.2.1 Defining mathematical sequences

A **sequence** is a related set of objects or events that follow each other in a particular order. Sequences can be found in everyday life, with some examples being:

- the opening share price of a particular stock each day
- the daily minimum temperature readings in a particular city
- the lowest petrol prices each day
- the population of humans counted each year.

When data is collected in the order that the events occur, patterns often emerge. Some patterns can be complicated, whereas others are easy to define.

In mathematics, sequences are always ordered, and the links between different terms of sequences can be identified and expressed using mathematical equations.

You may already be familiar with some mathematical sequences, such as the multiples of whole numbers or the square numbers.

Multiples of 3: 3, 6, 9, 12, …

Multiples of 5: 5, 10, 15, 20, …

Square numbers: 1, 4, 9, 16, …

For each of these patterns there is a link between the numbers in the sequence (known as **terms**) and their position in the sequence (known as the **term number**).

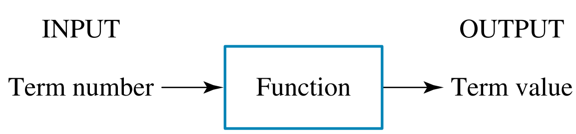
6.2.2 The language of mathematical sequences

> **In general, mathematical sequences can be displayed as:**
>
> $$t_1,\ t_2,\ t_3,\ t_4,\ t_5, \ldots t_n$$
>
> **where t_1 is the first term, t_2 is the second term, and so on.**

The first term of a mathematical sequence can also be referred to as a. The nth term is referred to as t_n (so $t_1 = a$), and n represents the ordered position of the term in the sequence, for example 1st, 2nd, 3rd, ...

6.2.3 Sequences expressed as functions

If we consider the term numbers in a sequence as the inputs of a **function**, then the term values of that sequence are the outputs of that function.



If we are able to define a sequence as a function, then we can input term numbers into that function to determine any term value in the sequence.

WORKED EXAMPLE 1

Determine the first five terms of the sequence $t_n = 2n + 3$.

THINK	WRITE
1. Substitute $n = 1$ into the function.	$t_1 = 2 \times 1 + 3$ $= 5$
2. Substitute $n = 2$ into the function.	$t_2 = 2 \times 2 + 3$ $= 7$
3. Substitute $n = 3$ into the function.	$t_3 = 2 \times 3 + 3$ $= 9$
4. Substitute $n = 4$ into the function.	$t_4 = 2 \times 4 + 3$ $= 11$
5. Substitute $n = 5$ into the function.	$t_5 = 2 \times 5 + 3$ $= 13$
6. State the answer.	The first five terms of the sequence are 5, 7, 9, 11 and 13.

Note: You can see that the terms of the sequence increase by the coefficient of n (i.e. the number n is multiplied by).

6.2.4 Arithmetic sequences

An **arithmetic sequence** is a sequence in which the difference between any two successive terms in the sequence is the same. In an arithmetic sequence, the next term in the sequence can be found by adding or subtracting a fixed value.

First consider the sequence 5, 9, 13, 17, 21. This is an arithmetic sequence, as each term is obtained by adding 4 (a fixed value) to the preceding term.

Now consider the sequence 1, 3, 6, 10, 15. This is not an arithmetic sequence, as each term does not increase by the same constant value.

6.2.5 The common difference

The difference between two consecutive terms in an arithmetic sequence is known as the common difference. If the common difference is positive, the sequence is increasing. If the common difference is negative, the sequence is decreasing.

In an arithmetic sequence, the first term is referred to as a and the common difference is referred to as d.

WORKED EXAMPLE 2

Determine which of the following sequences are arithmetic sequences, and for those sequences which are arithmetic, state the values of a and d.

a. $2, 5, 8, 11, 14, \ldots$

b. $4, -1, -6, -11, -16, \ldots$

c. $3, 5, 9, 17, 33, \ldots$

THINK	WRITE
a. 1. Calculate the difference between consecutive terms of the sequence.	**a.** $t_2 - t_1 = 5 - 2$ $= 3$ $t_3 - t_2 = 8 - 5$ $= 3$ $t_4 - t_3 = 11 - 8$ $= 3$ $t_5 - t_4 = 14 - 11$ $= 3$
2. If the differences between consecutive terms are constant, then the sequence is arithmetic. The first term of the sequence is *a* and the common difference is *d*.	The common differences are constant, so the sequence is arithmetic. $a = 2$ and $d = 3$
b. 1. Calculate the difference between consecutive terms of the sequence.	**b.** $t_2 - t_1 = -1 - 4$ $= -5$ $t_3 - t_2 = -6 - -1$ $= -6 + 1$ $= -5$ $t_4 - t_3 = -11 - -6$ $= -11 + 6$ $= -5$ $t_5 - t_4 = -16 - -11$ $= -16 + 11$ $= -5$
2. If the differences between consecutive terms are constant, then the sequence is arithmetic. The first term of the sequence is *a* and the common difference is *d*.	The common differences are constant, so the sequence is arithmetic. $a = 4$ and $d = -5$
c. 1. Calculate the difference between consecutive terms of the sequence.	**c.** $t_2 - t_1 = 5 - 3$ $= 2$ $t_3 - t_2 = 9 - 5$ $= 4$ $t_4 - t_3 = 17 - 9$ $= 8$ $t_5 - t_4 = 33 - 17$ $= 16$
2. If the differences between consecutive terms are constant, then the sequence is arithmetic.	The common differences are not constant, so the sequence is not arithmetic.

6.2.6 Equations representing arithmetic sequences

If we want to determine any term of an arithmetic sequence, we need to set up an equation to represent the sequence.

Any arithmetic sequence can be expressed by the equation $t_n = a + (n - 1)d$, where t_n is the *n*th term, *a* is the first term and *d* is the common difference.

Therefore, if we know or can determine the values of a and d, we can construct the equation for the sequence.

WORKED EXAMPLE 3

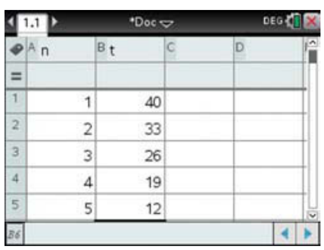
Determine the equations that represent the following arithmetic sequences.

a. **3, 6, 9, 12, 15, …** **b.** **40, 33, 26, 19, 12, …**

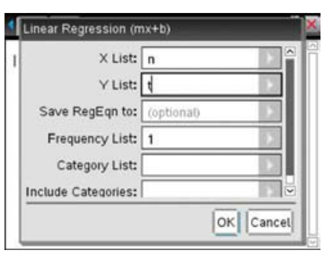
THINK	WRITE
a. 1. Determine the values of a and d.	**a.** $a = 3$ $d = t_2 - t_1$ $= 6 - 3$ $= 3$
2. Substitute the values for a and d into the formula for arithmetic sequences.	$t_n = a + (n - 1)d$ $= 3 + (n - 1) \times 3$ $= 3 + 3(n - 1)$ $= 3 + 3n - 3$ $= 3n$
b. 1. Determine the values of a and d.	**b.** $a = 40$ $d = t_2 - t_1$ $= 33 - 40$ $= -7$
2. Substitute the values for a and d into the formula for arithmetic sequences.	$t_n = a + (n - 1)d$ $= 40 + (n - 1) \times -7$ $= 40 - 7(n - 1)$ $= 40 - 7n + 7$ $= 47 - 7n$

TI | THINK **WRITE**

b. 1. In a Lists & Spreadsheets page, label the first column n and the second column t. As five terms in the sequence are given, place the numbers 1 to 5 in the first column. Enter the terms of the given sequence in the second column.

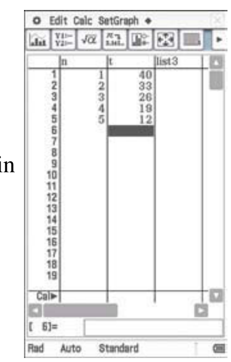


2. On the Calculator page, press MENU and select:
6: Statistics
1: Stat Calculations
3: Linear Regression $(mx + b)$ …
Select n as the X List and t as the Y List, then select OK.



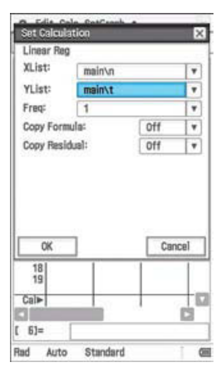
CASIO | THINK **WRITE**

b. 1. On a Statistics screen, change the label of list1 to n and list2 to t. As there are five terms given in this sequence, enter the numbers 1 to 5 in the first column. Enter the terms of the given sequence in the second column.

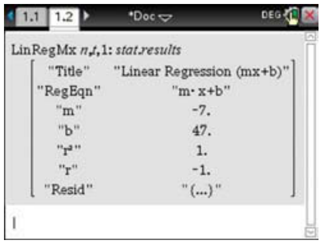


2. On the Statistics screen, select:
- Calc
- Regression
- Linear Reg

Select main\n as the XList and main\t as the YList, then select OK.



3. Interpret the output on the screen.

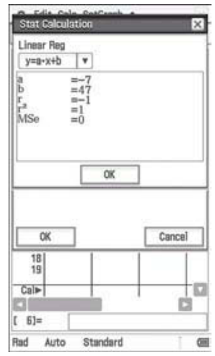


The equation is given in the form $y = mx + b$, where $y = t_n$, $m = -7$, $x = n$, and $b = 47$.

4. State the answer.

The equation is $t_n = -7n + 47$ or $t_n = 47 - 7n$.

3. Interpret the output on the screen.



The equation is given in the form $y = ax + b$, where $y = t_n$, $a = -7$, $x = n$, and $b = 47$.

4. State the answer.

The equation is $t_n = -7n + 47$ or $t_n = 47 - 7n$.

6.2.7 Determining future terms of an arithmetic sequence

After an equation has been set up to represent an arithmetic sequence, we can use this equation to determine any term in the sequence. Simply substitute the value of n into the equation to determine the value of that term.

6.2.8 Determining other values of an arithmetic sequence

We can obtain the values a, d and n for an arithmetic sequence by transposing the equation.

$$a = t_n - (n - 1)d$$

$$d = \frac{t_n - a}{n - 1}$$

$$n = \frac{t_n - a}{d} + 1$$

WORKED EXAMPLE 4

a. Find the 15th term of the sequence 2, 8, 14, 20, 26, …

b. Find the first term of the arithmetic sequence in which $t_{22} = 1008$ and $d = -8$.

c. Find the common difference of the arithmetic sequence which has a first term of 12 and an 11th term of 102.

d. An arithmetic sequence has a first term of 40 and a common difference of 12. Which term number has a value of 196?

THINK	WRITE
a. 1. As it has a common difference, this is an arithmetic sequence. State the known values.	a. $a = 2,\ d = 6,\ n = 15$

2. Substitute the known values into the equation for an arithmetic sequence and solve.

$$t_n = a + (n-1)d$$
$$\begin{aligned} t_{15} &= 2 + (15-1)6 \\ &= 2 + 14 \times 6 \\ &= 2 + 84 \\ &= 86 \end{aligned}$$

3. State the answer.

The 15th term of the sequence is 86.

b. 1. State the known values of the arithmetic sequence.

b. $d = -8$, $n = 22$, $t_{22} = 1008$

2. Substitute the known values into the equation to determine the first term and solve.

$$\begin{aligned} a &= t_n - (n-1)d \\ &= 1008 - (22-1)(-8) \\ &= 1008 - (21)(-8) \\ &= 1008 - -168 \\ &= 1008 + 168 \\ &= 1176 \end{aligned}$$

3. State the answer.

The first term of the sequence is 1176.

c. 1. State the known values of the arithmetic sequence.

c. $a = 12$, $n = 11$, $t_{11} = 102$

2. Substitute the known values into the equation to determine the common difference and solve.

$$\begin{aligned} d &= \frac{t_n - a}{n-1} \\ &= \frac{102-12}{11-1} \\ &= \frac{90}{10} \\ &= 9 \end{aligned}$$

3. State the answer.

The common difference is 9.

d. 1. State the known values of the arithmetic sequence.

d. $a = 40$, $d = 12$, $t_n = 196$

2. Substitute the known values into the equation to determine the term number and solve.

$$\begin{aligned} n &= \frac{t_n - a}{d} + 1 \\ &= \frac{196-40}{12} + 1 \\ &= 14 \end{aligned}$$

3. State the answer.

The 14th term in the sequence has a value of 196.

6.2.9 Graphical displays of sequences

Tables of values

When we draw a graph of a mathematical sequence, it helps to first draw a table of values for the sequence. The top row of the table displays the term number of the sequence, and the bottom of the table displays the term value.

Term number	1	2	3	...	n
Term value					

The data from the table of values can then be used to identify the points to plot in the graph of the sequence.

6.2.10 Drawing graphs of sequences

When we draw a graph of a numerical sequence, the term number is the independent variable, so it appears on the x-axis of the graph. The term value is the dependent value, so it appears on the y-axis of the graph.

Graphical displays of arithmetic sequences

Because there is a common difference between the terms of an arithmetic sequence, the relationship between the terms is a linear relationship. This means that when we graph the terms of an arithmetic sequence, we can join the points to form a straight line.

When we draw a graph of an arithmetic sequence, we can extend the straight line to determine values of terms in the sequence that haven't yet been determined.

WORKED EXAMPLE 5

An arithmetic sequence is given by the equation $t_n = 7 + 2(n - 1)$.

a. **Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.**

b. **Plot the graph of the sequence.**

c. **Use your graph of the sequence to determine the 12th term of the sequence.**

THINK

a. 1. Set up a table with the term number in the top row and the term value in the bottom row.

2. Substitute the first 5 values of n into the equation to determine the missing values.

WRITE/DRAW

a.

Term number	1	2	3	4	5
Term value					

$$\begin{aligned} t_1 &= 7 + 2(1 - 1) \\ &= 7 + 2 \times 0 \\ &= 7 + 0 \\ &= 7 \end{aligned}$$

$$\begin{aligned} t_2 &= 7 + 2(2 - 1) \\ &= 7 + 2 \times 1 \\ &= 7 + 2 \\ &= 9 \end{aligned}$$

$$\begin{aligned} t_3 &= 7 + 2(3 - 1) \\ &= 7 + 2 \times 2 \\ &= 7 + 4 \\ &= 11 \end{aligned}$$

$$\begin{aligned} t_4 &= 7 + 2(4 - 1) \\ &= 7 + 2 \times 3 \\ &= 7 + 6 \\ &= 13 \end{aligned}$$

$$
\begin{aligned}
t_5 &= 7 + 2(5-1)\\
&= 7 + 2 \times 4\\
&= 7 + 8\\
&= 15
\end{aligned}
$$

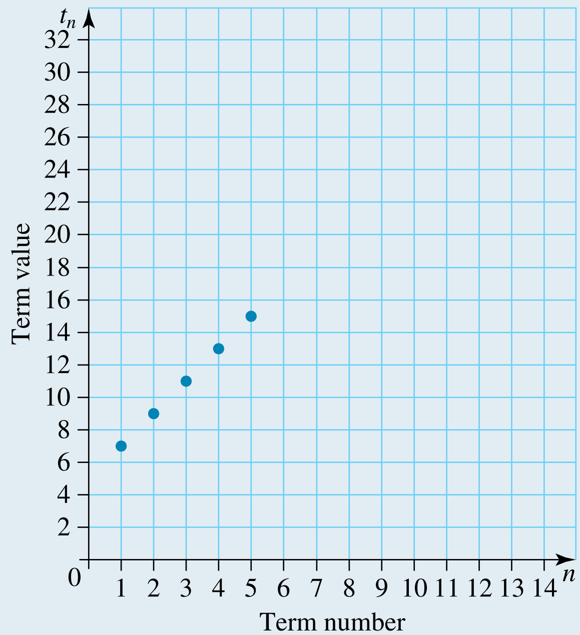
3. Complete the table with the calculated values.

Term number	1	2	3	4	5
Term value	7	9	11	13	15

b. 1. Use the table of values to identify the points to be plotted.

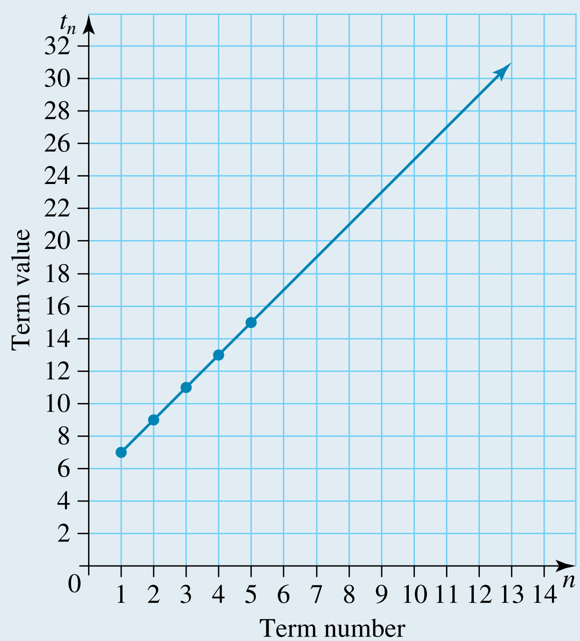
b. The points to be plotted are (1, 7), (2, 9), (3, 11), (4, 13) and (5, 15).

2. Plot the points on the graph.

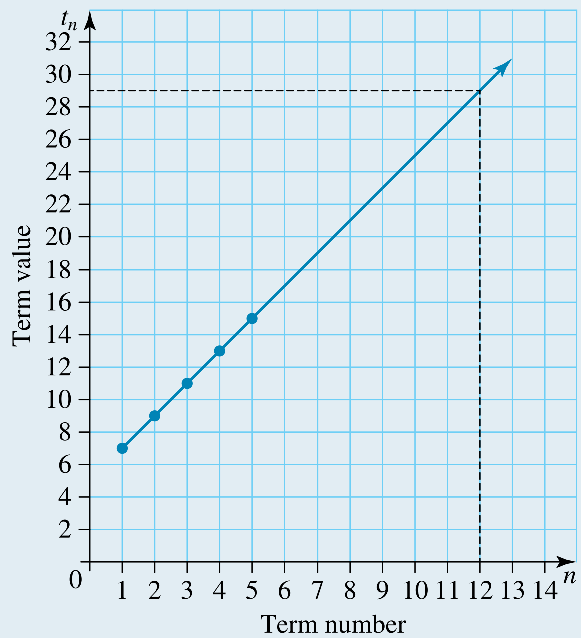


c. 1. Join the points with a straight line and extend the line to cover future values of the sequence.

c.



2. Read the required value from the graph (when $n = 12$).



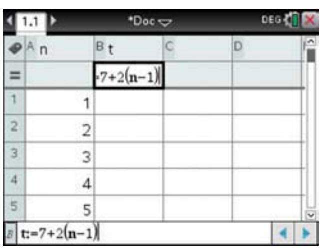
3. Write the answer.

The 12th term of the sequence is 29.

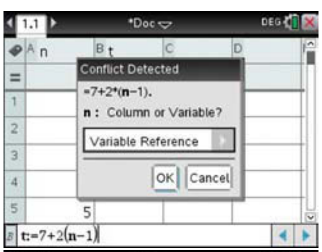
TI | THINK

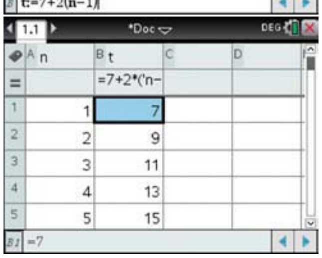
WRITE

a. 1. In a Lists & Spreadsheet page, label the first column n and the second column t. Enter the values 1 to 5 in the first column. In the formula cell underneath the label t, enter the rule for t_n starting with an = sign, then press ENTER.



2. When prompted, select Variable Reference for n and select OK.

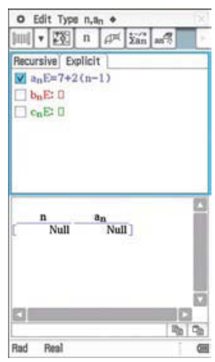




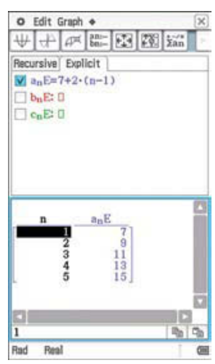
CASIO | THINK

WRITE

a. 1. In a Sequence screen, select the Explicit tab, and complete the entry line for a_n as: $a_nE = 7 + 2\,(n - 1)$ then click the tick box.



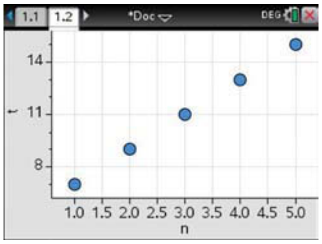
2. Click the # icon to view the terms in the sequence. Click the 8 icon to specify the number of terms to be displayed.



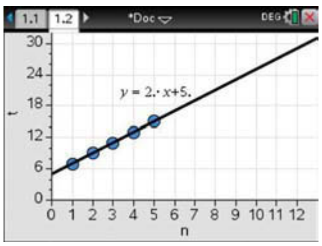
3. The table of values can be read from the screen.

Term number	1	2	3	4	5
Term value	7	9	11	13	15

b. 1. Open a Data & Statistics page. Click on the horizontal axis label and change it to *n*. Click on the vertical axis label and change it to *t*.



c. 1. Change the view so that the *x*-axis is visible up to $x = 12$. To do this, press MENU then select:
5: Window/Zoom
1: Window Settings …
To view the regression line, press MENU then select:
4: Analyze
6: Regression
1: Show Linear
$(mx + b)$



2. Estimate the value of t_{12} from the graph.

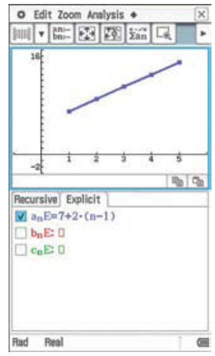
The 12th term is 29.

3. Check your answer using the table of values.
Enter the value 12 in the first column, then press ENTER. The value of the 12th term appears in the adjacent cell.

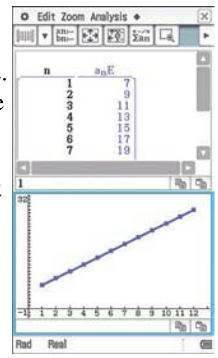
3. The table of values can be read from the screen.

Term number	1	2	3	4	5
Term value	7	9	11	13	15

b. 1. Click the $ icon to view the plot.



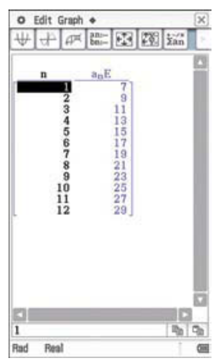
c. 1. Change the view so that the *x*-axis is visible up to $x = 12$.
To do this, press the 6 icon.
Press the 8 icon to generate the first 12 terms.
Press the $ icon to update the scatterplot.



2. Estimate the value of t_{12} from the graph.

The 12th term is 29.

3. Check your answer using the table of values.



6.2.11 Using arithmetic sequences to model practical situations

If we have a practical situation involving linear growth or decay in discrete steps, this situation can be modelled by an arithmetic sequence.

6.2.12 Simple interest

As covered in Topic 3, simple interest is calculated on the original amount of money invested. It is a fixed amount of interest paid at equal intervals, and as such it can be modelled by an arithmetic sequence.
Remember that simple interest is calculated by using the formula

$$I = \frac{PrT}{100},$$

where I is the amount of simple interest, P is the principal, r is the percentage rate and T is the amount of periods.

WORKED EXAMPLE 6

Jelena puts \$1000 into an investment that earns simple interest at a rate of 0.5% per month.

a. Set up an equation that represents Jelena's situation as an arithmetic sequence, where t_n is the amount in Jelena's account after n months.

b. Use your equation from part a to determine the amount in Jelena's account at the end of each of the first 6 months.

c. Calculate the amount in Jelena's account at the end of 18 months.

THINK	WRITE
a. 1. Use the simple interest formula to determine the amount of simple interest Jelena earns in one month.	**a.** $I = \frac{PrT}{100}$ $= \frac{1000 \times 0.5 \times 1}{100}$ $= \frac{500}{100}$ $= 5$
2. Calculate the amount in the account after the first month.	$a = 1000 + 5$ $= 1005$
3. State the known values in the arithmetic sequence equation.	$a = 1005,\ d = 5$
4. Substitute these values into the arithmetic sequence equation.	$t_n = 1005 + 5(n - 1)$

b. 1. Use the equation from part **a** to find the values of t_2, t_3, t_4, t_5 and t_6.

b. $t_2 = 1005 + 5(2 - 1)$
$= 1005 + 5 \times 1$
$= 1005 + 5$
$= 1010$
$t_3 = 1005 + 5(3 - 1)$
$= 1005 + 5 \times 2$
$= 1005 + 10$
$= 1015$
$t_4 = 1005 + 5(4 - 1)$
$= 1005 + 5 \times 3$
$= 1005 + 15$
$= 1020$
$t_5 = 1005 + 5(5 - 1)$
$= 1005 + 5 \times 4$
$= 1005 + 20$
$= 1025$
$t_6 = 1005 + 5(6 - 1)$
$= 1005 + 5 \times 5$
$= 1005 + 25$
$= 1030$

2. Write the answer.

The amounts in Jelena's account at the end of each of the first 6 months are \$1005, \$1010, \$1015, \$1020, \$1025 and \$1030.

c. 1. Use the equation from part **a** to find the values of t_{18}.

c. $t_{18} = 1005 + 5(18 - 1)$
$= 1005 + 5 \times 17$
$= 1005 + 85$
$= 1090$

2. Write the answer.

After 18 months Jelena has \$1090 in her account.

6.2.13 Depreciating assets

Many items, such as automobiles or electronic equipment, decrease in value over time as a result of wear and tear. At tax time individuals and companies use depreciation of their assets to offset expenses and to reduce the amount of tax they have to pay.

Unit cost depreciation

Unit cost depreciation is a way of depreciating an asset according to its use. For example, you can depreciate the value of a car based on how many kilometres it has driven. The unit cost is the amount of depreciation per unit of use, which would be 1 kilometre of use in the example of the car.

Future value and write-off value

When depreciating the values of assets, companies will often need to know the **future value** of an item. This is the value of that item at that specific time.

The **write-off value** or scrap value of an asset is the point at which the asset is effectively worthless (i.e. has a value of \$0) due to depreciation.

WORKED EXAMPLE 7

Loni purchases a new car for \$25 000 and decides to depreciate it at a rate of \$0.20 per km.

a. Set up an equation to determine the value of the car after *n* km of use.

b. Use your equation from part a to determine the future value of the car after it has 7500 km on its clock.

THINK	WRITE
a. 1. Calculate the value of the car after 1 km of use.	**a.** $a = 25\,000 - 0.2$ $= 24\,999.8$
2. State the known values in the arithmetic sequence equation.	$a = 24\,999.8,\ d = -0.2$
3. Substitute these values into the arithmetic sequence equation.	$t_n = a + (n-1)d$ $= 24\,999.8 + (n-1) \times -0.2$ $= 24\,999.8 - 0.2(n-1)$
b. 1. Substitute $n = 7500$ into the equation determined in part **a**.	**b.** $t_n = 24\,999.8 - 0.2(n-1)$ $t_{7500} = 24\,999.8 - 0.2(7500-1)$ $= 24\,999.8 - 0.2 \times 7499$ $= 24\,999.8 - 1499.8$ $= 23\,500$
2. Write the answer.	After 7500 km the car will be worth \$23 500.

on Resources

Interactivity: Terms of an arithmetic sequence (int-6261)

Interactivity: Arithmetic sequences (int-6258)

studyon

Units 1 & 2 > AOS 3 > Topic 3 > Concepts 1, 3, 4 & 7

Sequences Concept summary and practice questions

Arithmetic sequences Concept summary and practice questions

Modelling using arithmetic sequences Concept summary and practice questions

Depreciation Concept summary and practice questions

Exercise 6.2 Arithmetic sequences

1. WE1 Determine the first five terms of the sequence $t_n = 5n + 7$.
2. Determine the first five terms of the sequence $t_n = 3n - 5$.

3. WE2 Determine which of the following sequences are arithmetic sequences, and for those sequences which are arithmetic, state the values of a and d.

a. 23, 68, 113, 158, 203, … b. 3, 8, 23, 68, 203, … c. $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, ...$

4. Find the missing values in the following arithmetic sequences.

a. 13, −12, −37, f, −87, … b. 2.5, j, 8.9, 12.1, k, … c. $p, q, r, \frac{9}{2}, \frac{25}{4}, ...$

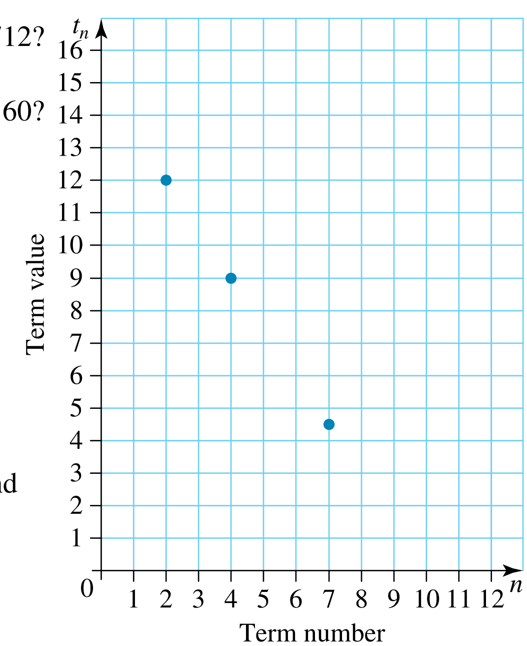
5. WE3 Determine the equations that represent the following arithmetic sequences.

a. −1, 3, 7, 11, 15, … b. 1.5, −2, −5.5, −8, −11.5 c. $\frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \frac{23}{2}, ...$

6. Determine the first five terms of the following arithmetic sequences.

a. $t_n = 5 + 3(n - 1)$ b. $t_n = -1 - 7(n - 1)$ c. $t_n = \frac{1}{3} + \frac{2}{3}(n - 1)$

7. WE4 **a.** Find the 20th term of the sequence 85, 72, 59, 46, 33, …
   b. Find the first value of the arithmetic sequence in which $t_{70} = 500$ and $d = -43$.
8. a. Find the common difference of the arithmetic sequence that has a first term of −32 and an 8th term of 304.
   b. An arithmetic sequence has a first term of 5 and a common difference of 40. Which term number has a value of 85?
   c. An arithmetic sequence has a first term of 40 and a common difference of 12. Which term number has a value of 196?
9. a. Find the 15th term of the arithmetic sequence 6, 13, 20, 27, 34, …
   b. Find the 20th term of the arithmetic sequence 9, 23, 37, 51, 65, …
   c. Find the 30th term of the arithmetic sequence 56, 48, 40, 32, 24, …
   d. Find the 55th term of the arithmetic sequence $\frac{72}{5}, \frac{551}{40}, \frac{263}{20}, \frac{501}{40}, \frac{119}{10}, ...$
10. a. Find the first value of the arithmetic sequence which has a common difference of 6 and a 31st term of 904.
   b. Find the first value of the arithmetic sequence which has a common difference of $\frac{2}{5}$ and a 40th term of −37.2.
   c. Find the common difference of an arithmetic sequence which has a first value of 564 and a 51st term of 54.
   d. Find the common difference of an arithmetic sequence which has a first value of −87 and a 61st term of 43.
11. a. An arithmetic sequence has a first value of 120 and a common difference of 16. Which term has a value of 712?
   b. An arithmetic sequence has a first value of 320 and a common difference of 4. Which term has a value of 1160?
12. The graph shows some points of an arithmetic sequence.
   a. What is the common difference between consecutive terms?
   b. What is the value of the first term of the sequence?
   c. What is the value of the 12th term of the sequence?



13. Three consecutive terms of an arithmetic sequence are $x - 5$, $x + 4$ and $2x - 7$. Find the value of x.
14. WE5 An arithmetic sequence is given by the equation $t_n = 5 + 10(n - 1)$.
   a. Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.
   b. Plot the graph of the sequence.
   c. Use your graph of the sequence to determine the 9th term of the sequence.

15. An arithmetic sequence is defined by the equation $t_n = 6.4 + 1.6(n - 1)$.
   a. Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.
   b. Plot the graph of the sequence.
   c. Use your graph of the sequence to determine the 13th term of the sequence.
16. Sketch the graph of $t_n = a + (n - 1)d$, where $a = 15$ and $d = 25$, for the first 10 terms.
17. **WE6** Grigor puts $1500 into an investment account that earns simple interest at a rate of 4.8% per year.
   a. Set up an equation that represents Grigor's situation as an arithmetic sequence, where t_n is the amount in Grigor's account after n months.
   b. Use your equation from part **a** to determine the amount in Grigor's account after each of the first 6 months.
   c. Calculate the amount in Grigor's account at the end of 18 months.
18. Justine sets up an equation to model the amount of her money in a simple interest investment account after n months. Her equation is $t_n = 8050 + 50(n - 1)$, where t_n is the amount in Justine's account after n months.
   a. How much did Justine invest in the account?
   b. What is the annual interest rate of the investment?
19. **WE7** Phillipe purchases a new car for $24 000 and decides to depreciate it at a rate of $0.25 per km.
   a. Set up an equation to determine the value of the car after n km of use.
   b. Use your equation from part **a** to determine the future value of the car after it has travelled 12 000 km.

20. Dougie is in charge of the equipment for his office. He decides to depreciate the value of a photocopier at the rate of x cents for every n copies made. Dougie's equation for the value of the photocopier after n copies is $t_n = 5399.999 - 0.001(n - 1)$.
   a. How much did the photocopier cost?
   b. What is the rate of depreciation per copy made?
21. An employee starts a new job with a $60 000 salary in the first year and the promise of a pay rise of $2500 a year.
   a. How much will her salary be in her 6th year?
   b. How long will it take for her salary to reach $85 000?

22. Nadia wants to invest her money and decided to place $90 000 into a credit union account earning simple interest at a rate of 6% per year.
   a. How much interest will Nadia receive after one year?
   b. What is the total amount Nadia has in the credit union after n years?
   c. For how long should Nadia keep her money invested if she wants a total of $154 800 returned?
23. Tom bought a car for $23 000, knowing it would depreciate in value by $210 per month.
   a. What is the value of the car after 18 months?
   b. By how much does the value of the car depreciate in 3 years?
   c. How many months will it take for the car to be valued at $6200?

24. A confectionary manufacturer introduces a new sweet and produces 50 000 packets of the sweets in the first week. The stores sell them quickly, and in the following week there is demand for 30% more. In each subsequent week the increase in production is 30% of the original production.
   a. How many packs are manufactured in the 20th week?
   b. In which week will the confectionary manufacturer produce 5 540 000 packs?

25. A canning machine was purchased for a total of $250 000 and is expected to produce 500 000 000 cans before it is written off.
   a. By how much does the canning machine depreciate with each can made?
   b. If the canning machine were to make 40 200 000 cans each year, when will the machine theoretically be written off?
   c. When will the machine have a book value of $89 200?
26. The local rugby club wants to increase its membership. In the first year they had 5000 members, and so far they have managed to increase their membership by 1200 members per year.
   a. If the increase in membership continues at the current rate, how many members will they have in 15 years' time?
   Tickets for membership in the first year were $200, and each year the price has risen by a constant amount, with memberships in the 6th year costing $320.
   b. How much would the tickets cost in 15 years' time?
   c. What is the total membership income in both the first and 15th years?

6.3 Geometric sequences

6.3.1 Geometric sequences

A **geometric sequence** is a pattern of numbers whose consecutive terms increase or decrease in the same ratio.

First consider the sequence 1, 3, 9, 27, 81, … This is a geometric sequence, as each term is obtained by multiplying the preceding term by 3.

Now consider the sequence 1, 3, 6, 10, 15, … This is not a geometric sequence, as the consecutive terms are not increasing in the same ratio.

6.3.2Common ratios

The ratio between two consecutive terms in a geometric sequence is known as the **common ratio**.

In a geometric sequence, the first term is referred to as *a* and the common ratio is referred to as *r*.

WORKED EXAMPLE 8

Determine which of the following sequences are geometric sequences, and for those sequences which are geometric, state the values of *a* and *r*.

a. **20, 40, 80, 160, 320, …**
b. $\mathbf{8, 4, 2, 1, \frac{1}{2}, ...}$
c. **3, −9, 27, −81, …**
d. **2, 4, 6, 8, 10, …**

THINK | **WRITE**

a. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

a. $\frac{t_2}{t_1} = \frac{40}{20}$
$= 2$

$\frac{t_3}{t_2} = \frac{80}{40}$
$= 2$

$\frac{t_4}{t_3} = \frac{160}{80}$
$= 2$

$\frac{t_5}{t_4} = \frac{320}{160}$
$= 2$

2. If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is a and the common difference is r.

The ratios between consecutive terms are all 2, so this is a geometric sequence.

$a = 20,\ r = 2$

b. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

b. $\frac{t_2}{t_1} = \frac{4}{8}$
$= \frac{1}{2}$

$\frac{t_3}{t_2} = \frac{2}{4}$
$= \frac{1}{2}$

$\frac{t_4}{t_3} = \frac{1}{2}$

$\frac{t_5}{t_4} = \frac{\left(\frac{1}{2}\right)}{1}$
$= \frac{1}{2}$

2. If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is a and the common difference is r.

The ratios between consecutive terms are all $\frac{1}{2}$ so this is a geometric sequence.

$a = 8,\ r = \frac{1}{2}$

c. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

c. $\frac{t_2}{t_1} = \frac{-9}{3}$
$= -3$

$\frac{t_3}{t_2} = \frac{27}{-9}$
$= -3$

$\frac{t_4}{t_3} = \frac{-81}{27}$
$= -3$

2. If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is a and the common difference is r.

The ratios between consecutive terms are all −3, so this is a geometric sequence.
$a = 3,\ r = -3$

d. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

d. $\frac{t_2}{t_1} = \frac{4}{2}$
$= 2$
$\frac{t_3}{t_2} = \frac{6}{4}$
$= \frac{3}{2}$
$\frac{t_4}{t_3} = \frac{8}{6}$
$= \frac{4}{3}$
$\frac{t_5}{t_4} = \frac{10}{8}$
$= \frac{5}{4}$

2. If the ratios between consecutive terms are constant, then the sequence is geometric.

All of the ratios between consecutive terms are different, so this is not a geometric sequence.

6.3.3 Equations representing geometric sequences

Any geometric sequence can be by the equation $t_n = ar^{n-1}$, where t_n is the nth term, a is the first term and r is the common ratio.

Therefore, if we know or can determine the values of a and r for a geometric sequence, we can construct the equation for the sequence.

WORKED EXAMPLE 9

Determine the equations that represent the following geometric sequences.

a. **7, 28, 112, 448, 1792, …**
b. **$8, -4, 2, -1, \frac{1}{2} \ldots$**

THINK

a. 1. Determine the values of a and r.

WRITE

a. $a = 7$

$r = \frac{t_2}{t_1}$
$= \frac{28}{7}$
$= 4$

2. Substitute the values for a and r into the formula for geometric sequences.

$$t_n = ar^{n-1}$$
$$= 7 \times 4^{n-1}$$

b. 1. Determine the values of a and r.

b. $a = 8$

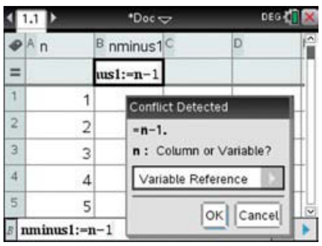
$$r = \frac{t_2}{t_1}$$
$$= \frac{-4}{8}$$
$$= -\frac{1}{2}$$

2. Substitute the values for a and r into the formula for geometric sequences.

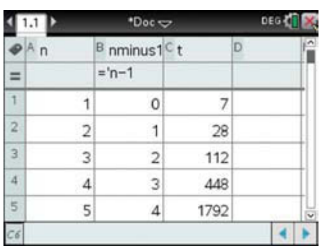
$$t_n = ar^{n-1}$$
$$= 8 \times \left(-\frac{1}{2}\right)^{n-1}$$

TI | THINK — **WRITE**

a. 1. In a Lists & Spreadsheets page, label the first column n. As there are five terms given in the sequence, enter the numbers 1 to 5 in the first column.
The rule will be of the form $t_n = ar^{n-1}$, so there needs to be a variable $n - 1$. Label the second column *nminus1* (n minus 1), then click on the formula cell below the label *nminus1* and complete the entry line as: $= n - 1$.
Select the variable reference for n, then select OK.

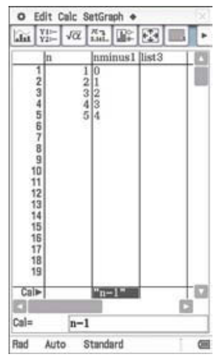


2. Label the third column t, and enter the terms of the given sequence underneath.

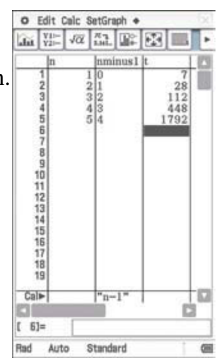


CASIO | THINK — **WRITE**

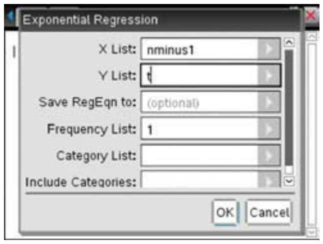
a. 1. On a Statistics screen, change the label of list1 to n. As there are five terms given in this sequence, enter the numbers 1 to 5 in the first column.The rule will be of the form $t_n = ar^{n-1}$, so there needs to be a variable $n - 1$.
Label the second column *nminus1* (n minus 1), then click on the cell at the bottom of the *nminus1* column and complete the entry line as: $n - 1$.
Press EXE.



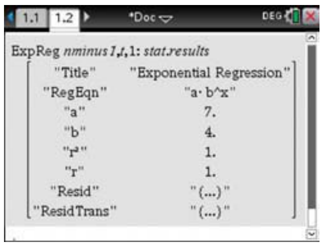
2. Label list3 as t, and enter the terms of the given sequence underneath.



3. On a Calculator page, press MENU then select:
6: Statistics
1: Stat Calculations
A: Exponential Regression …
Select *nminus1* as the X List and *t* as the Y List, then select OK.



4. Interpret the output on the screen.

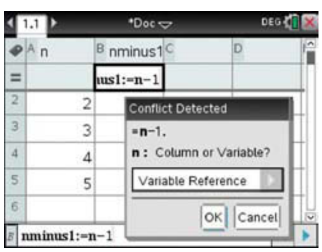


All terms in the sequence are positive, so the common ratio must be positive.
The output is given in the form $y = a \times b^x$, where $y = t_n$, $a = 7$ is the first term, $b = 4$ is the common ratio, and $x = n - 1$.

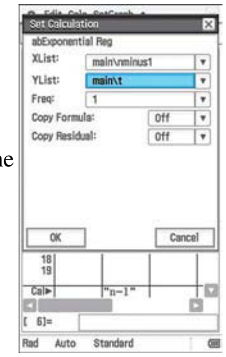
5. State the answer.

The equation is $t_n = 7 \times 4^{n-1}$.

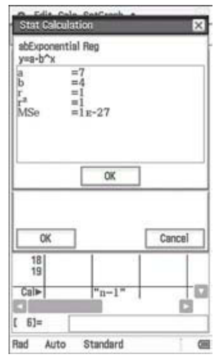
b. 1. In a Lists & Spreadsheets page, label the first column *n*. As there are five terms given in the sequence, enter the numbers 1 to 5 in the first column.
The rule will be of the form $t_n = ar^{n-1}$, so there needs to be a variable $n - 1$. Label the second column *nminus1* (*n* minus 1), then click on the formula cell below the label *nminus1* and complete the entry line as:
$= n - 1$.
Select the variable reference for *n*, then select OK.



3. On the Statistics screen, select:
- Calc
- Regression
- abExponential Reg.
Select main*nminus1* as the XList and main*t* as the YList, then select OK.



4. Interpret the output on the screen.

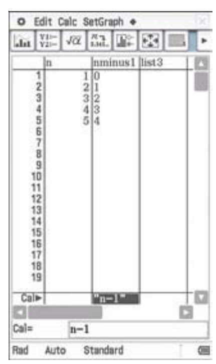


All terms in the sequence are positive, so the common ratio must be positive.
The output is given in the form $y = a \times b^x$, where $y = t_n$, $a = 7$ is the first term, $b = 4$ is the common ratio, and $x = n - 1$.

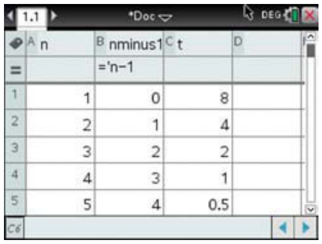
5. State the answer.

The equation is $t_n = 7 \times 4^{n-1}$.

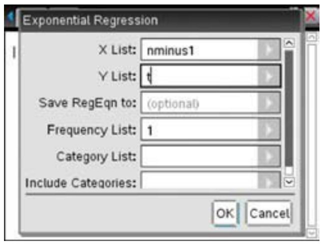
b. 1. On a Statistics screen, change the label of list1 to *n*.
As there are five terms given in this sequence, enter the numbers 1 to 5 in the first column.
The rule will be of the form $t_n = ar^{n-1}$, so there needs to be a variable $n - 1$. Label the second column *nminus1* (*n* minus 1), then click on the cell at the bottom of the *nminus1* column and complete the entry line as:
$n - 1$.
Press EXE.



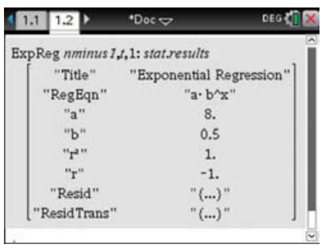
2. Label the third column t, and enter the absolute value (disregard the negative signs) of the terms in the given sequence underneath.



3. On a Calculator page, press MENU then select:
6: Statistics
1: Stat Calculations
A: Exponential Regression …
Select *nminus1* as the X List and t as the Y List, then select OK.



4. Interpret the output on the screen.

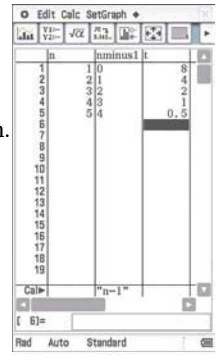


The terms in the given sequence alternate signs, so the common ratio must be negative.
The output is given in the form $y = a \times b^x$, where $y = t_n$, $a = 8$ is the first term, $b = 0.5$ so the common ratio is $-\frac{1}{2}$, and $x = n - 1$.

5. State the answer.

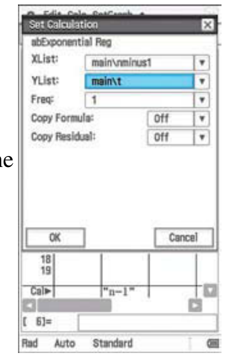
The equation is
$t_n = 8 \times \left(-\frac{1}{2}\right)^{n-1}$.

2. Label list3 as t, and enter the absolute value (disregard the negative signs) of the terms in the given sequence underneath.

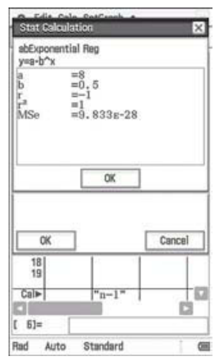


3. On the Statistics screen, select:
- Calc
- Regression
- abExponential Reg.

Select main*nminus1* as the XList and main\\t as the YList, then select OK.



4. Interpret the output on the screen.



The terms in the given sequence alternate signs, so the common ratio must be negative.
The output is given in the form $y = a \times b^x$, where $y = t_n$, $a = 8$ is the first term, $b = 0.5$ so the common ratio is $-\frac{1}{2}$, and $x = n - 1$.

5. State the answer.

The equation is
$t_n = 8 \times \left(-\frac{1}{2}\right)^{n-1}$.

6.3.4 Determining future terms of a geometric sequence

After an equation has been set up to represent a geometric sequence, we can use this equation to determine any term in the sequence. Simply substitute the value of n into the equation to determine the value of that term.

6.3.5 Determining other values of a geometric sequence

We can obtain the values a and r for a geometric sequence by transposing the equation.

$$a = \frac{t_n}{r^{n-1}}$$

$$r = \left(\frac{t_n}{a}\right)^{\frac{1}{n-1}}$$

Note: The value of n can also be determined, but this is beyond the scope of this course.

WORKED EXAMPLE 10

a. **Find the 20th term of the geometric sequence with $a = 5$ and $r = 2$.**

b. **A geometric sequence has a first term of 3 and a 20th term of 1 572 864. Find the common ratio between consecutive terms of the sequence.**

c. **Find the first term of a geometric series with a common ratio of 2.5 and a 5th term of 117.1875.**

THINK	WRITE
a. 1. Identify the known values in the question.	**a.** $a = 5$ $r = 2$ $n = 20$
2. Substitute these values into the geometric sequence formula and solve to find the missing value.	$t_n = ar^{n-1}$ $t_{20} = 5 \times 2^{20-1}$ $= 5 \times 2^{19}$ $= 2\,621\,440$
3. Write the answer.	The 20th term of the sequence is 2 621 440.
b. 1. Identify the known values in the question.	**b.** $t_{20} = 4\,194\,304$ $a = 4$ $n = 20$
2. Substitute these values into the formula to calculate the common ratio and solve to find the missing value.	$r = \left(\frac{t_n}{a}\right)^{\frac{1}{n-1}}$ $= \left(\frac{1\,572\,864}{3}\right)^{\frac{1}{20-1}}$ $= 524\,288^{\frac{1}{19}}$ $= 2$
3. Write the answer.	The common ratio between consecutive terms of the sequence is 2.

c. 1. Identify the known values in the question.

c. $t_5 = 117.1875$
$r = 2.5$
$n = 5$

2. Substitute these values into the formula to calculate the first term and solve to find the missing value.

$$a = \frac{t_n}{r^{n-1}}$$
$$= \frac{117.1875}{2.5^{5-1}}$$
$$= \frac{117.1875}{2.5^4}$$
$$= \frac{117.1875}{39.0625}$$
$$= 3$$

3. Write the answer.

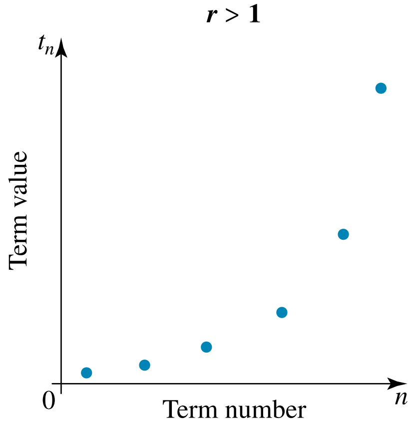
The first term of the sequence is 3.

6.3.6 Graphs of geometric sequences

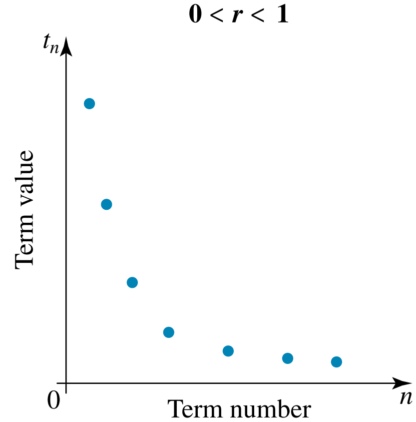
The shape of the graph of a geometric sequence depends on the value of r.

- When $r > 1$, the values of the terms increase or decrease at an exponential rate.
- When $0 < r < 1$, the values of the terms converge towards 0.
- When $-1 < r < 0$, the values of the terms oscillate on either side of 0 but converge towards 0.
- When $r < -1$, the values of the terms oscillate on either side of 0 and move away from the starting value at an exponential rate.

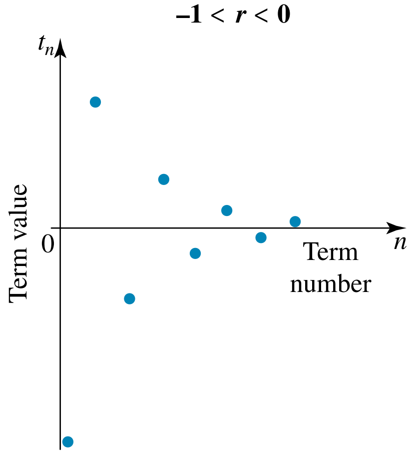
$r > 1$



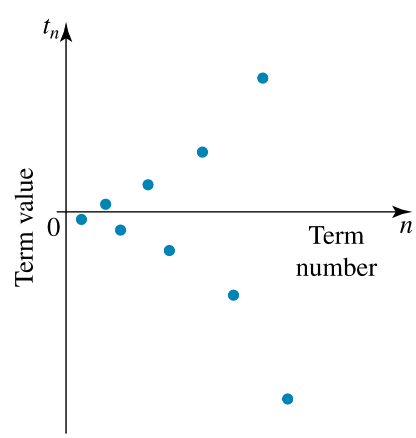
$0 < r < 1$



$-1 < r < 0$



$r < -1$



WORKED EXAMPLE 11

A geometric sequence is defined by the equation $t_n = 5 \times 2^{n-1}$.

a. **Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.**

b. **Plot the graph of the sequence.**

THINK

a. 1. Set up a table with the term number in the top row and the term value in the bottom row.

WRITE/DRAW

a.

Term number	1	2	3	4	5
Term value					

2. Substitute the first 5 values of n into the equation to determine the missing values.

$$\begin{aligned} t_1 &= 5 \times 2^{1-1} \\ &= 5 \times 2^0 \\ &= 5 \times 1 \\ &= 5 \end{aligned}$$

$$\begin{aligned} t_2 &= 5 \times 2^{2-1} \\ &= 5 \times 2^1 \\ &= 5 \times 2 \\ &= 10 \end{aligned}$$

$$\begin{aligned} t_3 &= 5 \times 2^{3-1} \\ &= 5 \times 2^2 \\ &= 5 \times 4 \\ &= 20 \end{aligned}$$

$$\begin{aligned} t_4 &= 5 \times 2^{4-1} \\ &= 5 \times 2^3 \\ &= 5 \times 8 \\ &= 40 \end{aligned}$$

$$\begin{aligned} t_1 &= 5 \times 2^{5-1} \\ &= 5 \times 2^4 \\ &= 5 \times 16 \\ &= 80 \end{aligned}$$

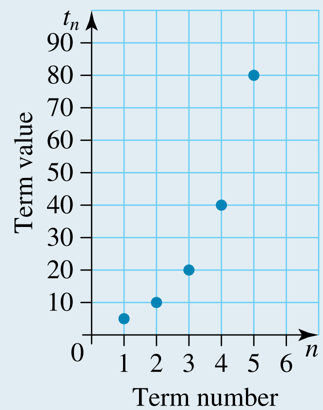
3. Complete the table with the calculated values.

Term number	1	2	3	4	5
Term value	5	10	20	40	80

b. 1. Use the table of values to identify the points to be plotted.

b. The points to be plotted are (1, 5), (2, 10), (3, 20), (4, 40) and (5, 80).

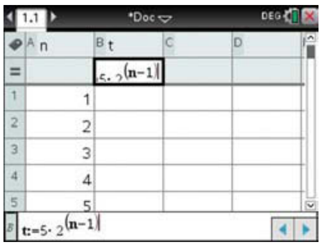
2. Plot the points on the graph.



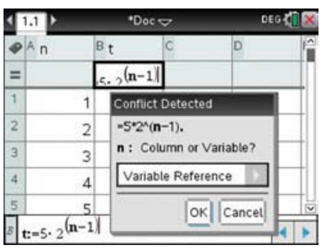
TI | THINK

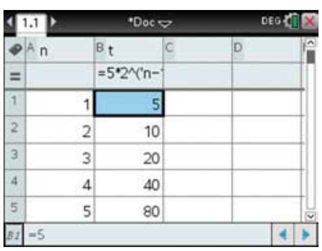
a. 1. In a Lists & Spreadsheet page, label the first column n and the second column t. Enter the values 1 to 5 in the first column. In the formula cell underneath the label t, enter the rule for t_n starting with an = sign, then press ENTER.

WRITE



2. When prompted, select Variable Reference for n and select OK.





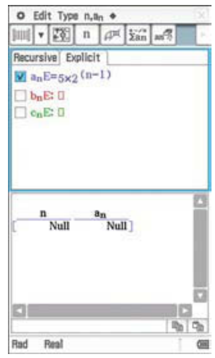
3. The table of values can be read from the screen.

Term number	1	2	3	4	5
Term value	5	10	20	40	80

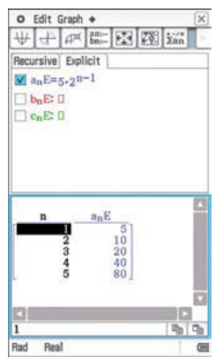
CASIO | THINK

a. 1. In a Sequence screen, select the Explicit tab, and complete the entry line for a_n as: $a_nE = 5 \times 2^{(n-1)}$ then click the tick box.

WRITE



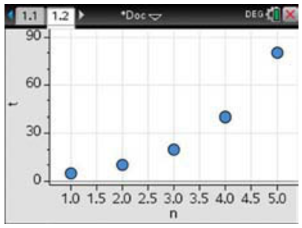
2. Click the # icon to view the terms in the sequence. Click the 8 icon to specify the number of terms to be displayed.



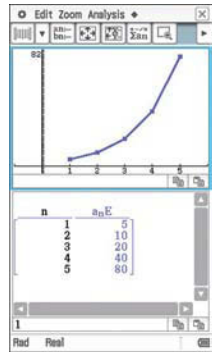
3. The table of values can be read from the screen.

Term number	1	2	3	4	5
Term value	5	10	20	40	80

b. 1. Open a Data & Statistics page. Click on the horizontal axis label and change it to n. Click on the vertical axis label and change it to t.



b. 1. Click the $ icon to view the plot.



6.3.7 Using geometric sequences to model practical situations

If we have a practical situation involving geometric growth or decay in discrete steps, this situation can be modelled by a geometric sequence.

6.3.8 Compound interest

As covered in Topic 3, compound interest is calculated on the sum of an investment at the start of each compounding period. The amount of interest accrued varies throughout the life of the investment and can be modelled by a geometric sequence.

Remember that simple interest is calculated by using the formula $A = P\left(1+\frac{r}{100}\right)^n$, where A is the total amount of the investment, P is the principal, r is the percentage rate and n is the number of compounding periods.

WORKED EXAMPLE 12

Alexis puts $2000 into an investment account that earns compound interest at a rate of 0.5% per month.

a. Set up an equation that represents Alexis's situation as a geometric sequence, where t_n is the amount in Alexis' account after n months.

b. Use your equation from part a to determine the amount in Alexis's account at the end of each of the first 6 months.

c. Calculate the amount in Alexis's account at the end of 15 months.

THINK	WRITE
a. 1. Determine the amounts in the account after each of the first two months.	**a.** $A = P\left(1+\frac{r}{100}\right)^n$ $= 2000\left(1+\frac{0.5}{100}\right)^1$ $= 2000 \times 1.005$ $= 2010$

$$A = P\left(1 + \frac{r}{100}\right)^n$$
$$= 2000\left(1 + \frac{0.5}{100}\right)^2$$
$$= 2000 \times 1.005^2$$
$$= 2020.05$$

2. Calculate the common ratio between consecutive terms.

$$r = \frac{t_2}{t_1}$$
$$= \frac{2020.05}{2010}$$
$$= 1.005$$

3. State the known values in the geometric sequence equation.

$a = 2010,\ r = 1.005$

4. Substitute these values into the geometric sequence equation.

$t_n = 2010 \times 1.005^{n-1}$

b. 1. Use the equation from part **a** to find the values of t_3, t_4, t_5 and t_6. Round all values correct to 2 decimal places.

b. $t_3 = 2010 \times 1.005^{n-1}$
$= 2010 \times 1.005^{3-1}$
$= 2010 \times 1.005^2$
$= 2030.150...$
≈ 2030.15

$t_4 = 2010 \times 1.005^{n-1}$
$= 2010 \times 1.005^{4-1}$
$= 2010 \times 1.005^3$
$= 2040.301...$
≈ 2040.30

$t_5 = 2010 \times 1.005^{n-1}$
$= 2010 \times 1.005^{5-1}$
$= 2010 \times 1.005^4$
$= 2050.502...$
≈ 2050.50

$t_6 = 2010 \times 1.005^{n-1}$
$= 2010 \times 1.005^{6-1}$
$= 2010 \times 1.005^5$
$= 2060.755...$
≈ 2060.76

2. Write the answer.

The amounts in Alexis' account at the end of each of the first 6 months are \$2010, \$2020.05, \$2030.15, \$2040.30, \$2050.50 and \$2060.76.

c. 1. Use the equation from part **a** to find the values of t_{15}, rounding your answer correct to 2 decimal place.

c. $t_{15} = 2010 \times 1.005^{n-1}$
$= 2010 \times 1.005^{15-1}$
$= 2010 \times 1.005^{14}$
$= 2155.365...$
≈ 2155.37

2. Write the answer.

After 15 months Alexis has $2155.37 in her account.

Note: The common ratio in the geometric sequence equation is equal to $1 + \frac{r}{100}$ (from the compound interest formula).

6.3.9 Reducing balance depreciation

Another method of depreciation is **reducing balance depreciation**. When an item is depreciated by this method, rather than the value of the item depreciating by a fixed amount each year, it depreciates by a percentage of the previous future value of the item.

Due to the nature of reducing balance depreciation, we can represent the sequence of the future values of an item that is being depreciated by this method as a geometric sequence.

WORKED EXAMPLE 13

A hot water system purchased for $1250 is depreciated by the reducing balance method at a rate of 8% p.a.

a. Set up an equation to determine the value of the hot water system after *n* years of use.

b. Use your equation from part a to determine the future value of the hot water system after 6 years of use (correct to the nearest cent).

THINK

a. 1. Calculate the common ratio by identifying the value of the item in any given year as a percentage of the value in the previous year. Convert the percentage to a ratio by dividing by 100.

WRITE

a. $100\% - 8\% = 92\%$

Each year the value of the item is 92% of the previous value.

$92\% = \frac{92}{100}$
$= 0.92$
$r = 0.92$

2. Calculate the value of the hot water system after 1 year of use.

$a = 1250 \times 0.92$
$= 1150$

3. Substitute the values of a and r into the geometric sequence equation.

$t_n = 1150 \times 0.92^{n-1}$

b. 1. Substitute $n = 6$ into the equation determined in part **a**. Give your answer correct to 2 decimal places.

b. $t_n = 1150 \times 0.92^{n-1}$

$t_6 = 1150 \times 0.92^{6-1}$

$= 1150 \times 0.92^5$

$= 757.943...$

≈ 757.94

2. Write the answer.

After 6 years the book value of the hot water system is \$757.94.

on Resources

Interactivity: Terms of a geometric sequence (int-6260)
Interactivity: Geometric sequences (int-6259)

study on

Units 1 & 2 > AOS 3 > Topic 3 > Concepts 5 & 6

Geometric sequences Concept summary and practice questions

Modelling using geometric sequences Concept summary and practice questions

Exercise 6.3 Geometric sequences

1. **WE8** Determine which of the following sequences are geometric sequences, and for those sequences which are geometric, state the values of a and r.
   a. 3, 6, 12, 24, 48, …
   b. $\frac{1}{2}, \frac{5}{4}, \frac{25}{8}, \frac{125}{16}, ...$
   c. 9, 6, 3, 0, −3, …
   d. $\frac{1}{2}, \frac{1}{5}, \frac{2}{25}, \frac{4}{125}, ...$
2. Find the missing values in the following geometric sequences.
   a. 1, 6, c, 216, 1296
   b. 3, g, h, −24, 48
   c. p, q, s, 300, 1500
3. Which of the following are geometric sequences? Where applicable state the first term and common ratio.
   a. 3, 15, 75, 375, 1875, …
   b. 7, 13, 25, 49, 97, …
   c. −8, 24, −72, 216, −648, …
   d. 128, 32, 8, 2, $\frac{1}{2}$, ...
   e. 2, 6, 12, 20, 30, …
   f. 3, $3\sqrt{3}$, 9, $9\sqrt{3}$, 27, ...
4. What is the value of x in the following geometric sequences?
   a. x, 14, 28, …
   b. $2x$, $4x$, $8 + 6x$, …
   c. $x + 1$, $3x + 3$, $10x + 5$, …
5. **WE9** Determine the equations that represent the following geometric sequences.
   a. −1, −5, −25, −125, −625, …
   b. 7, −3.5, 1.75, −0.875, 0.4375
   c. $\frac{5}{6}, \frac{5}{9}, \frac{10}{27}, \frac{20}{81}, \frac{40}{243}, ...$
6. Determine the first five terms of the following arithmetic sequences.
   a. $t_n = -2 \times 3^{n-1}$
   b. $t_n = 4 \times \left(\frac{1}{3}\right)^{n-1}$
   c. $t_n = \frac{1}{4} \times \left(-\frac{3}{2}\right)^{n-1}$

7. WE10 **a.** Find the 15th term of the geometric sequence with $a = 4$ and $r = 3$.
b. A geometric sequence has a first term of 2 and a 12th term of 97 656 250. Find the common ratio between consecutive terms of the sequence.
c. Find the first term of a geometric series with a common ratio of $-\frac{1}{2}$ and a 6th term of 13.125.

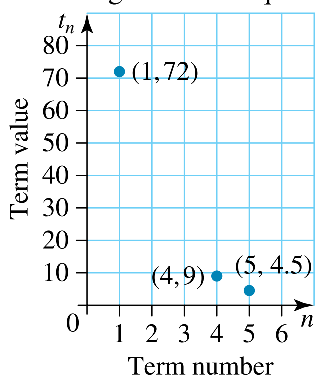
8. **a.** Find the 11th term of the geometric sequence with a first value of 1.2 and a common ratio of 4.
b. A geometric sequence has a first term of -1.5 and a 10th term of 768. Find the common ratio between consecutive terms of the sequence.
c. Find the first term of a geometric series with a common ratio of 0.4 and a 6th term of 6.5536.

9. **a.** Find the first four terms of the geometric sequence where the 6th term is 243 and the 8th term is 2187.
b. Find the first four terms of the geometric sequence where the 3rd term is 331 and the 5th term is 8275.

10. WE11 A geometric sequence is defined by the equation $t_n = 64 \times \left(\frac{1}{2}\right)^{n-1}$.
a. Draw a table of values showing the term number and term value for the first 5 terms of the sequence.
b. Plot the graph of the sequence.

11. A geometric sequence is defined by the equation $t_n = 1.5 \times 3^{n-1}$.
a. Draw a table of values showing the term number and term value for the first 5 terms of the sequence.
b. Plot the graph of the sequence.

12. Find the values of the 2nd and 3rd terms of the geometric sequence shown in the following graph.



13. WE12 Hussein puts $2500 into an investment that earns compound interest at a rate of 0.3% per month.
a. Set up an equation that represents Hussein's situation as a geometric sequence, where t_n is the amount in Hussein's account after n months.
b. Use your equation from part **a** to determine the amount in Hussein's account after each of the first 6 months.
c. Calculate the amount in Hussein's account at the end of 15 months.

14. Tim sets up an equation to model the amount of his money in a compound interest investment account after n months. His equation is $t_n = 4515.75 \times 1.0035^{n-1}$, where t_n is the amount in his account after n months.
a. How much did Tim invest in the account?
b. What is the annual interest rate of the investment?

15. Jonas starts a new job with a salary of $55 000 per year and the promise of a 3% pay rise for each subsequent year in the job.
a. Write an equation to determine Jonas' salary in his nth year in the job.
b. How much will Jonas earn in his 5th year in the job?

16. WE13 A refrigerator purchased for $1470 is depreciated by the reducing balance method at a rate of 7% p.a.
a. Set up an equation to determine the value of the refrigerator after n years of use.
b. Use your equation from part **a** to determine the future value of the refrigerator after 8 years of use.

17. Ivy buys a new oven and decides to depreciate the value of the oven by the reducing balance method. Ivy's equation for the value of the oven after n years is $t_n = 1665 \times 0.925^{n-1}$.
   a. How much did the oven cost?
   b. What is the annual rate of depreciation for the oven?

18. A geometric sequence has a 1st term of 200 and a 6th term of 2.048. Identify the values of the 2nd, 3rd, 4th and 5th terms.
19. The number of ants in a colony doubles every week. If there are 2944 ants in the colony at the end of 8 weeks, how many ants were in the colony at the end of the first week?
20. The 1st term of a geometric sequence is 13 and the 3rd term of the same sequence is 117.
   a. Explain why there are two possible values for the common ratio of the sequence.
   b. Calculate both possible values of the 6th term of the sequence.

21. Julio's parents invest $5000 into a college fund on his 5th birthday. The fund pays a compound interest rate of 5.5% p.a. How much will the fund be worth when Julio turns 18?
22. A meteoroid is burning up as it passes through the Earth's atmosphere. For every 5 km it travels, the mass of the meteoroid decreases by 5%. At the start of its descent into the Earth's atmosphere, at 100 km above ground level, the mass of the meteoroid is 675 g.
   a. Formulate an equation to determine the mass of the meteoroid after each 5-km increment of its descent.
   b. What is the mass of the meteoroid when it hits the Earth, correct to 2 decimal places?
23. The number of pieces of stone used to build a pyramid decreases in a ratio of $\frac{1}{3}$ for each layer of the pyramid. The pyramid has 9 layers. The top (9th) layer of the pyramid needed only 2 stones.
   a. How many stones were needed for the base layer of the pyramid?
   b. Write an equation to express how many stones were needed for the nth layer of the pyramid.
   c. How many stones were needed for the entire pyramid?

24. The populations of Melbourne and Sydney are projected to grow steadily over the next 20 years. A government agency predicts that the population of Melbourne will grow at a steady rate of 2.6% per year and the population of Sydney will grow at a steady rate of 1.7% per year.
   a. If the current population of Melbourne is 4.35 million, formulate an equation to estimate the population of Melbourne after n years.
   b. If the current population of Sydney is 4.65 million, formulate an equation to estimate the population of Sydney after n years.
   c. Using CAS, determine how long it will take for the population of Melbourne to exceed the population of Sydney. Give your answer correct to the nearest year.

6.4 Recurrence relations

6.4.1 Using first-order linear recurrence relations to generate number sequences

In a recurrence relation, the terms of a sequence are dependent on the previous terms of the sequence. A first-order linear recurrence relation is a relation whereby the terms of the sequence depend only on the previous term of the sequence, which means that we need only an initial value to be able to generate all remaining terms of the sequence.

In a recurrence relation, the *n*th term is represented by t_n, with the term directly after t_n being by t_{n+1} and the term directly before t_n being represented by t_{n-1}. The initial value of the sequence is represented by the term t_1.

If the initial value in a recurrence relation changes, then the whole sequence changes. If we are not given an initial value, we cannot determine any terms in the sequence.

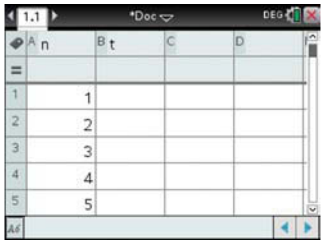
WORKED EXAMPLE 14

Determine the first five terms of the sequence represented by the recurrence relation $t_n = 2t_{n-1} + 5$, given that $t_1 = 8$.

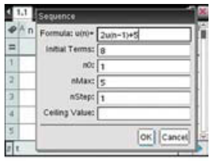
THINK	WRITE
1. Identify what the recurrence relation means.	$t_n = 2t_{n-1} + 5$ Each term of the sequence is given by multiplying the previous term of the sequence by 2 and adding 5 to the result.
2. Use the recurrence relation to determine the value of t_2.	$t_2 = 2t_1 + 5$ $= 2 \times 8 + 5$ $= 16 + 5$ $= 21$
3. Use the recurrence relation to determine the value of t_3.	$t_3 = 2t_2 + 5$ $= 2 \times 21 + 5$ $= 42 + 5$ $= 47$
4. Use the recurrence relation to determine the value of t_4.	$t_4 = 2t_3 + 5$ $= 2 \times 47 + 5$ $= 94 + 5$ $= 99$
5. Use the recurrence relation to determine the value of t_5.	$t_5 = 2t_4 + 5$ $= 2 \times 99 + 5$ $= 198 + 5$ $= 203$
6. Write the answer.	The first five terms of the sequence are 8, 21, 47, 99 and 203.

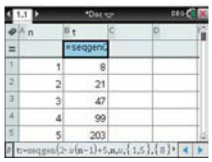
TI | THINK — **WRITE**

a. 1. In a Lists & Spreadsheet page, label the first column n and the second column t. Enter the values 1 to 5 in the first column.



2. Click on the formula cell underneath the label t, then press MENU and select:
3: Data
1: Generate Sequence
Complete the entry line for the Formula as
$u(n) = 2\,u(n-1) + 5$
and set the Initial Terms as 8, n0 as 1 and nMax as 5, then select OK.



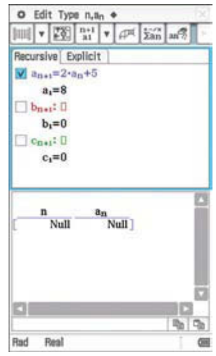


3. The first five terms can be read from the screen.

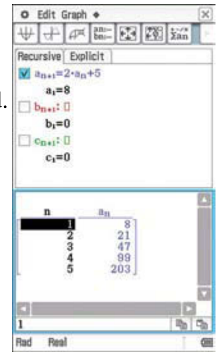
The first five terms are 8, 21, 47, 99 and 203.

CASIO | THINK — **WRITE**

a. 1. On the Sequence screen, select the Recursive tab then complete the entry line for a_{n+1} as:
$a_{n+1} = 2a_n + 5$
$a_1 = 8$
then click the tick box.
Note: To change the format so that the first term is labelled a_1 rather than a_0, select the & icon.



2. Click the # icon to view the terms in the sequence.
Click the 8 icon to specify the number of terms to be displayed.



The first five terms can be read from the screen.

The first five terms are 8, 21, 47, 99 and 203.

6.4.2 Using a recurrence relation to generate arithmetic sequences

If we know the values of a and d in an arithmetic sequence, we can set up a recurrence relation to generate the sequence.

> **A recurrence relation representing an arithmetic sequence will be of the form $t_{n+1} = t_n + d,\ t_1 = a$.**

WORKED EXAMPLE 15

Set up a recurrence relation to represent the arithmetic sequence $-9,\ -5,\ -1,\ 3,\ 7,\ \ldots$

THINK	WRITE
1. Determine the common difference by subtracting the first term from the second term.	$d = -5 - -9$ $= -5 + 9$ $= 4$
2. t_1 represents the first term of the sequence.	$t_1 = -9$
3. Set up the recurrence relation with the given information.	$t_{n+1} = t_n + 4,\ t_1 = -9$

6.4.3 Using a recurrence relation to generate geometric sequences

If we know the values of a and r in a geometric sequence, we can set up a recurrence relation to generate the sequence.

> **A recurrence relation representing a geometric sequence will be of the form $t_{n+1} = rt_n, t_1 = a$.**

WORKED EXAMPLE 16

Set up a recurrence relation to represent the sequence 10, 5, 2.5, 1.25, 0.625, …

THINK	WRITE
1. Determine the common ratio by dividing the second term by the first term.	$r = \frac{t_2}{t_1} = \frac{5}{10} = \frac{1}{2}$
2. t_1 represents the first term of the sequence.	$t_1 = 10$
3. Set up the recurrence relation with the given information.	$t_{n+1} = \frac{1}{2}t_n, t_1 = 10$

6.4.4 Using recurrence relations to model practical situations

When there is a situation that can be modelled by an arithmetic or geometric sequence, we can use a recurrence relation to model it. The first step we need to take is to decide whether the information suggests an arithmetic or geometric sequence. If there is a common difference between terms, we can use an arithmetic sequence, and if there is a common ratio between terms, we can use a geometric sequence.

6.4.5 Spotting arithmetic sequences

Arithmetic sequences are sequences that involve linear growth or decay. Examples include simple interest loans or investments, the revenue from the sale of a certain amount of items of the same price, and the number of flowers left in a field if the same amount is harvested each day.

6.4.6 Spotting geometric sequences

Geometric sequences are sequences that involve geometric growth or decay. Examples include compound interest loans or investments, the reducing height of a bouncing ball, and the number of bacteria in a culture after x periods of time.

If percentages are involved in generating a sequence of numbers, this can result in a geometric sequence.

When there is a percentage increase of x percent between terms, the value of the common ratio, r, will be $\left(1+\frac{x}{100}\right)$.
Similarly, when there is a percentage decrease of x percent between terms, the value of the common ratio, r, will be $\left(1-\frac{x}{100}\right)$.

WORKED EXAMPLE 17

to the International Federation of Tennis, a tennis ball must meet certain bounce regulations. The test involves the dropping of a ball vertically from a height of 254 cm and then measuring the rebound height. To meet the regulations, the ball must rebound 135 to 147 cm high, just over half the original distance.
Janine decided to test the ball bounce theory out. She dropped a ball from a height of 200 cm. She found that it bounced back up to 108 cm, with the second rebound reaching 58.32 cm and the third rebound reaching 31.49 cm.

a. **Set up a recurrence relation to model the bounce height of the ball.**
b. **Use your relation from part a to estimate the height of the 4th and 5th rebounds, giving your answers correct to 2 decimal places.**
c. **Sketch the graph of the number of bounces against the height of each bounce.**

THINK	WRITE/DRAW
a. 1. List the known information.	**a.** 1st bounce: 108 cm 2nd bounce: 58.32 cm 3rd bounce: 31.49 cm
2. Check if there is a common ratio between consecutive terms. If so, this situation can be modelled using a geometric sequence.	$\frac{t_2}{t_1}=\frac{58.32}{108}$ $=0.54$ $\frac{t_3}{t_2}=\frac{31.49}{58.32}$ $=0.541...$ ≈ 0.54 There is a common ratio between consecutive terms of 0.54.
3. Set up the equation to represent the geometric sequence.	$a=108$ $r=0.54$ $t_{n+1}=r\,t_n,\ t_1=a$ $t_{n+1}=0.54\,t_n,\ t_1=108$

b. 1. Use the formula from part **a** to find the height of the 4th rebound ($n = 4$).

b. $t_3 = 31.49$

$t_{n+1} = 0.54t_n$

$$\begin{aligned} t_4 &= 0.54t_3 \\ &= 0.54 \times 31.49 \\ &= 17.0046 \\ &= 17.00 \text{ (correct to 2 decimal places)} \end{aligned}$$

2. Use the formula from part **a** to find the height of the 5th rebound ($n = 5$).

$t_{n+1} = 0.54t_n$

$$\begin{aligned} t_5 &= 0.54t_4 \\ &= 0.54 \times 17.00 \\ &= 9.18 \end{aligned}$$

3. Write the answer.

The estimated height of the 4th rebound is 17.00 cm, and the estimated height of the 5th rebound is 9.18 cm.

c. 1. Draw up a table showing the bounce number against the rebound height.

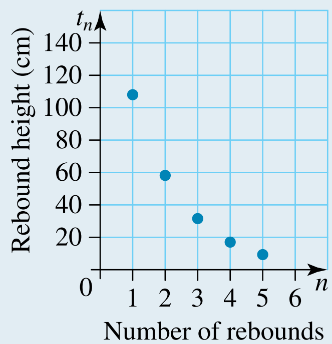
c.

Bounce number	1	2	3	4	5
Rebound height (cm)	108	58.32	31.49	17.00	9.18

2. Identify the points to be plotted on the graph.

The points to be plotted are (1, 108), (2, 58.32), (3, 31.49), (4, 17.01) and (5, 9.18).

3. Plot the points on the graph.



6.4.7 The Fibonacci sequence

In 1202, the Italian mathematician Leonardo Fibonacci introduced the Western world to a unique sequence of numbers which we now call the **Fibonacci sequence**.

The Fibonacci sequence begins with two 1s, and every subsequent term of the sequence is found by adding the two previous terms, giving the sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

6.4.8 Generating the Fibonacci sequence using a recurrence relation

Unlike the first-order recurrence relations that we have previously used to represent sequences, the Fibonacci sequence depends on two previous terms, and is therefore a second-order recurrence relation.

The Fibonacci sequence can be represented by the recurrence relation $F_{n+2} = F_n + F_{n+1}$, $F_n = 1$, $F_2 = 1$.

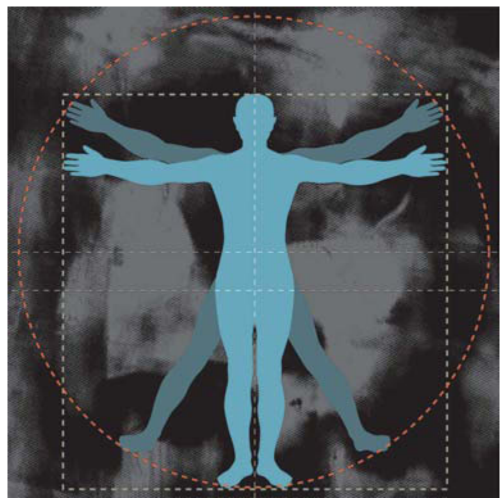
6.4.9 The Golden Ratio

The ratios between consecutive terms of the Fibonacci sequence is not a fixed ratio, as with the geometric sequences we've studied. However, the ratios do converge on a number that has special mathematical significance.

Ratio	$\frac{t_2}{t_1} = \frac{1}{1}$	$\frac{t_3}{t_2} = \frac{2}{1}$	$\frac{t_4}{t_3} = \frac{3}{2}$	$\frac{t_5}{t_4} = \frac{5}{3}$	$\frac{t_6}{t_5} = \frac{8}{5}$	$\frac{t_7}{t_6} = \frac{13}{8}$	$\frac{t_8}{t_7} = \frac{21}{13}$	$\frac{t_9}{t_8} = \frac{34}{21}$	$\frac{t_{10}}{t_9} = \frac{55}{34}$
Value	1	2	1.5	1.666 …	1.6	1.625	1.615 …	1.619 …	1.617 …

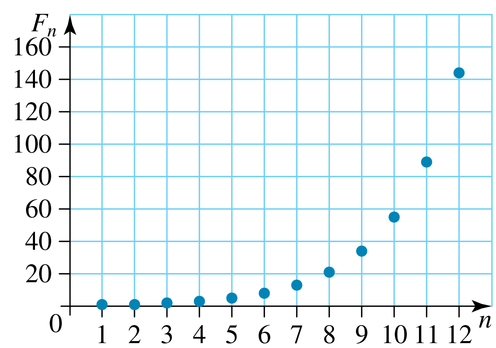
The number that the ratios converge to is called the **Golden Ratio**. It has an exact value of $\frac{1+\sqrt{5}}{2}$.

Throughout history many people have believed that the secrets of beauty lie in the Golden Ratio. Leonardo Da Vinci drew his picture of the Vitruvian Man using the Golden Ratio, and parts of the face of the Mona Lisa are in the proportions of the Golden Ratio.



6.4.10 Graphing the Fibonacci sequence

If you plot the graph of the term numbers of the Fibonacci sequence against the term values, the pattern forms a smooth curve, similar to the graphs of geometric sequences.



Variations of the Fibonacci sequence

The standard Fibonacci sequence begins with two 1s, which are used to formulate the rest of the sequence. If we change these two starting numbers, we get alternative versions of the Fibonacci sequence. For example, if the first two numbers are 2 and 5, the sequence becomes: 2, 5, 7, 12, 19, 31, 50, 81, 131, …

WORKED EXAMPLE 18

State the first 8 terms of the variation of the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_n + F_{n+1}$, $F_1 = -1$, $F_2 = 5$.

THINK	WRITE
1. State the known terms.	$F_1 = -1, F_2 = 5$
2. Use the recurrence relation to generate the remaining terms.	$F_{n+2} = F_n + F_{n+1}$ $F_3 = F_1 + F_2$ $= -1 + 5$ $= 4$ $F_4 = F_2 + F_3$ $= 5 + 4$ $= 9$ $F_5 = F_3 + F_4$ $= 4 + 9$ $= 13$ $F_6 = F_4 + F_5$ $= 9 + 13$ $= 22$ $F_7 = F_5 + F_6$ $= 13 + 22$ $= 35$ $F_8 = F_6 + F_7$ $= 22 + 35$ $= 57$
3. Write the answer.	The first 8 terms of the sequence are $-1, 5, 4, 9, 13, 22, 35, 57$.

on Resources

- **Interactivity:** Initial values and first-order recurrence relations (int-6262)
- **Interactivity:** First-order recurrence relations with a common ratio (int-6263)
- **Interactivity:** First-order recurrence relations with a common difference (int-6264)

studyon

Units 1 & 2 > AOS 3 > Topic 3 > Concepts 2 & 8

Linear recurrence relations Concept summary & practice questions

Generation and evaluation of the Fibonacci sequence Concept summary & practice questions

Exercise 6.4 Recurrence relations

1. **WE14** Determine the first five terms of the sequence represented by the recurrence relation $t_n = 0.5t_{n-1} + 8$, given that $t_1 = 24$.
2. Determine the first five terms of the sequence represented by the recurrence relation $t_n = 3t_{n-1} - 4$, given that $t_1 = 2$.
3. **WE15** Set up a recurrence relation to represent the arithmetic sequence 2, −3, −8, −13, −18.
4. An arithmetic sequence is represented by the recurrence relation. $t_{n+1} = t_n + 3.5,\ t_1 = -2.2$. Determine the first 5 terms of the sequence.
5. **WE16** Set up a recurrence relation to represent the geometric sequence 2.5, −7.5, 22.5, −67.5, 202.5, …
6. A geometric sequence is represented by the recurrence relation $t_{n+1} = -3.5t_n,\ t_1 = -4$. Determine the first five terms of the sequence.
7. The 3rd and 4th terms of an arithmetic sequence are −7 and −11.5. Set up a recurrence relation to define the sequence.
8. The 4th and 5th terms of a geometric sequence are −4 and −1. Set up a recurrence relation to define the sequence.
9. A variation of the Fibonacci sequence has a 3rd term of −2 and a 4th term of 6. Determine the recurrence relation for this sequence.
10. **WE17** Eric decided to test the rebound height of a tennis ball. He dropped a ball from a height of 300 cm and found that it bounced back up to 165 cm, with the second rebound reaching 90.75 cm, and the third rebound reaching 49.91 cm.
    a. Set up a recurrence relation to model the bounce height of the ball.
    b. Use your relation from part a to estimate the height of the 4th and 5th rebounds, giving your answers correct to 2 decimal places.
    c. Sketch the graph of the number of bounces against the height of the bounce.
11. Rosanna decided to test the ball rebound height of a basketball. She dropped the basketball from a height of 500 cm and noted that each successive rebound was two-fifths of the previous height.
    a. Set up a recurrence relation to model the bounce height of the ball.
    b. Use your relation to estimate the heights of the first 5 rebounds, correct to 2 decimal places.
    c. Sketch the graph of the first 5 bounces against the rebound height.
12. Brett invests $18 000 in an account paying simple interest. After 3 months he has $18 189 in his account.
    a. Set up a recurrence relation to determine the amount in Brett's account after n months.
    b. How much will Brett have in his account after 7 months?
13. **WE18** State the first 8 terms of the variation of the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_n + F_{n+1},\ F_1 = 3,\ F_2 = -5$.
14. The Lucas sequence is a special variation of the Fibonacci sequence that starts with the numbers 2 and 1. Determine the first 10 numbers of the Lucas sequence.
15. Graph the first 7 terms of the variation of the Fibonacci sequence that starts with the numbers −3 and 3.
16. Cassandra has $6615 in her bank account after 2 years and $6945.75 in her bank account after 3 years. Her account pays compound interest.
    a. Set up a recurrence relation to determine the amount in Cassandra's account after n years.
    b. How much does Cassandra have in her account after 5 years?
17. An ice shelf is shrinking at a rate of 1200 km^2 per year. When measurements of the ice shelf began, the area of the shelf was 37 000 km^2.
    a. Create a recurrence relation to express the area of the ice shelf after n years.
    b. Use your relation to determine the area of the ice shelf after each of the first 6 years.
    c. Plot a graph showing the area of the shrinking ice shelf over time.

18. The number of bacteria in a colony is increasing in line with a second-order recurrence relation of the form $t_{n+2} = 2t_n + t_{n+1}$, $t_1 = 3$, $t_2 = 9$, where n is the time in minutes. Determine the amount of bacteria in the colony after each of the first 10 minutes.

19. A bouncing ball rebounds to 70% of its previous height.

a. From how high would the ball have to be dropped for the 10th bounce to reach 50 cm in height? Give your answer correct to 1 decimal place.

b. Define a recurrence relation to determine the height of the ball after n bounces.

20. An abandoned island is slowly being overrun with rabbits. The population of the rabbits is approximately following a Fibonacci sequence. The estimated number of rabbits after 4 years of monitoring is 35 000, and the estimated number after 5 years of monitoring is 55 000.

a. Estimate the number of rabbits after the first year of monitoring.

b. Create a recurrence relation to determine the number of rabbits after n years.

c. Is it realistic to expect the population of rabbits to continue to increase at this rate? Explain your answer.

21. Luke and Lucinda are siblings who are given $3000 to invest by their parents. Luke invests his $3000 in a simple interest bond paying 4.8% p.a., and Lucinda invests her $3000 in a compound interest bond paying 4.3% p.a.

a. Write a recurrence relation to express the amount in Luke's account after n years.

b. Write a recurrence relation to express the amount in Lucinda's account after n years.

c. Determine the amount in each of their accounts for the first 7 years.

d. Draw a graph showing the amount in each account over the first 7 years.

22. 'Variations of the Fibonacci sequence will always tend towards plus or minus infinity.'
By altering the two starting numbers of the Fibonacci sequence, determine whether this statement is true or not.

6.5 Review: exam practice

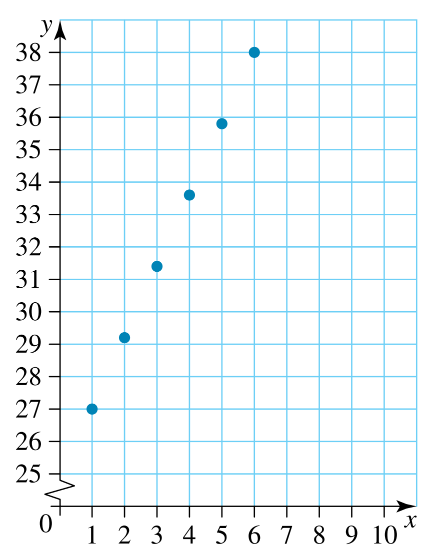
A summary of this topic is overall in the resources section of your eBookPLUS at www.jacplus.com.au

Multiple choice

1. **MC** Which of the following sequences is a geometric sequence?
   A. 12, 16, 20, 24, 28
   B. 12, 4.8, 1.92, 0.77, 0.31
   C. 12, 15, 21, 30, 42
   D. 12, 6.2, 3.3, 1.88, 0.42
   E. 12, 14.8, 17.6, 20.4, 23.2
2. **MC** For the sequence −16, −11.2, −6.4, −1.6, 3.2, the correct values for a and d are:
   A. $a = -16, d = -11.2$
   B. $a = 4.8, d = 3.2$
   C. $a = -16, d = 4.8$
   D. $a = -19.2, d = 3.2$
   E. $a = 16, d = -4.8$
3. **MC** If $t_1 = 65$, the third term in the recurrence relation $t_n = 2t_{n-1} - 10$ is:
   A. 120 **B.** 230 **C.** 119 **D.** 250 **E.** 35
4. **MC** The missing value in the arithmetic sequence of 65, x, 58, 54.5, 51 is:
   A. −3.5 **B.** 68.5 **C.** 60 **D.** 61.5 **E.** 62
5. **MC** The common ratio for the sequence 4.8, 14.4, 43.2, 129.6, 388.8 is:
   A. 3 **B.** 4.8 **C.** 9.6 **D.** 81 **E.** 4
6. **MC** If $t_3 = 18$, then according to the recurrence relation $t_n = 0.6t_{n-1} + 4.5$, t_4 will equal:
   A. 30 **B.** 22.5 **C.** 17.4 **D.** 10.8 **E.** 15.3
7. **MC** The first two numbers of a geometric sequence are 7 and 21. What is the fourth term?
   A. 35 **B.** 189 **C.** 63 **D.** 39 **E.** 1029
8. **MC** Which is the correct recurrence relation for the sequence 45, 11.25, 2.81, 0.70, 0.18?
   A. $t_{n+1} = t_n + 33.75,\ t_1 = 45$
   B. $t_{n+1} = t_n + 33.75,\ t_1 = 0.18$
   C. $t_{n+1} = 0.25t_n,\ t_1 = 33.75$
   D. $t_{n+1} = 0.25t_n,\ t_1 = 45$
   E. $t_{n+1} = 0.5t_n,\ t_1 = 45$
9. **MC** If $F_8 = 70$ and $F_{10} = 220$ in a Fibonacci sequence, then F_9 is equal to:
   A. 70 **B.** 150 **C.** 185 **D.** 220 **E.** 290
10. **MC** Which of the following equations represents a geometric sequence?
   A. $t_n = a + d(n-1)$
   B. $t_{n+1} = rt_n + d,\ t_1 = a$
   C. $t_{n+1} = t_n + d,\ t_1 = a$
   D. $t_{n+1} = t_n,\ t_1 = a$
   E. $t_n = ar^{n-1}$

Short answer

1. Determine the first five terms of each of the following sequences.
   a. Arithmetic sequence: $t_n = 2 + 5(n-1)$
   b. Geometric sequence: $t_n = 17 \times 2.2^{(n-1)}$
   c. Recurrence sequence: $t_{n+1} = t_n - 6,\ t_1 = 15$
2. Construct an equation to represent each of the following sequences.
   a. Arithmetic sequence: 3, 0.5, −2, −4.5, −7, −9.5
   b. Geometric sequence: −2, −8, −32, −128, −512
   c. Recurrence relation: 22, 143, 929.5, 6041.75, 39271.375
3. Find the 8th term of the following sequences, correct to 2 decimal places where appropriate.
   a. Arithmetic sequence: 14, 18.75, 23.5, 28.25
   b. Geometric sequence: $\frac{11}{25}, \frac{33}{50}, \frac{99}{100}, \frac{149}{100}, \frac{223}{100}$
   c. Recurrence relation: 45, 28, 11, −6, −23
4. The following graph shows some points of an arithmetic sequence.
   a. What is the common difference between consecutive terms?
   b. What is the value of the first term of the sequence?
   c. Determine the equation for this sequence.
   d. What is the value of the 9th term?



5. A geometric sequence is defined by the equation $t_n = 48 \times \left(\frac{1}{2}\right)^{(n-1)}$.
   a. Draw a table of values showing the first 5 terms.
   b. Plot the graph of the sequence.
6. Graph the first 6 terms of a variation of Fibonacci sequence that starts with the numbers −4 and 4.

Extended response

1. Chris is saving for his first car. He put $900 into a simple interest savings account that earns 8.2% per year.
   a. Set up an equation that represents Chris's situation as an arithmetic sequence, where t_n is the amount in the account after n months.
   b. Use your equation from part **a** to determine the amount in Chris's account after each of the first 5 months.
   c. Calculate the amount in the savings account at the end of 20 months.
   d. At this interest rate, how many months will it take Chris to save $1200?
2. Barry dropped a ball from a second story window, 6 m from the ground. The ball rebounded to two-thirds of the original height.
   a. Set up a recurrence relation to model the bounce of the height of the ball.
   b. Use your relation from part **a** to estimate the height of the 5th and 6th rebounds, giving your answer correct to 2 decimal places.
   c. Sketch the graph of the number of bounces against the height of the bounce.
3. Kane's salary in his first year at his job was $68 000. Each year his salary increases by 2.5%.
   a. Write an equation to reflect Kane's salary in his nth year in the job.
   b. How much will he earn in his 4th year in the job?
   Kane decided to put some of the money from his pay rise into a compound interest account. He established the equation $t_n = 1551 \times 1.034^{n-1}$ to model the amount of money in the account after n years.
   c. How much did Kane originally put in the account and at what interest rate?
   d. Using the equation, determine by how much the account will have increased after 5 years.
4. As part of an experiment, bacteria are grown in a laboratory. On day 7 of the experiment the bacteria count is 10 935. The number of bacteria has tripled each day.
   a. Calculate the amount of bacteria expected on the 9th day.
   b. Determine the original number of bacteria.

studyon

Units 1&2 — Sit topic test

Answers

Topic 6 Sequences

Exercise 6.2 Arithmetic sequences

1. 12, 17, 22, 27, 32

2. −2, 1, 4, 7, 10

3. **a.** Arithmetic; $a = 23$, $d = 45$
b. Not arithmetic
c. Arithmetic; $a = \frac{1}{2}$, $d = \frac{1}{4}$

4. **a.** $f = -62$
b. $j = 5.7$, $k = 15.3$
c. $p = -\frac{3}{4}$, $q = 1$, $r = \dfrac{11}{4}$

5. **a.** $t_n = -1 + 4(n - 1)$
b. $t_n = 1.5 - 3.5(n - 1)$
c. $t_n = \frac{7}{2} + 2(n - 1)$

6. **a.** 5, 8, 11, 14, 17
b. −1, −8, −15, −22, −29
c. $\frac{1}{3}$, 1, $\frac{5}{3}$, $\frac{7}{3}$, 3

7. **a.** −162 **b.** 3467

8. **a.** 48 **b.** The 3rd term
c. The 14th term

9. **a.** 104 **b.** 275 **c.** −176 **d.** $-\frac{387}{20}$

10. **a.** 724 **b.** −52.8 **c.** −10.2 **d.** $\frac{13}{6}$

11. **a.** The 38th term
b. The 211th term

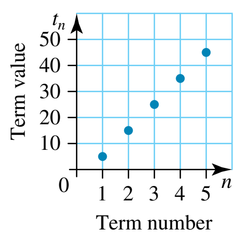
12. **a.** −1.5 **b.** 13.5 **c.** −3

13. $x = 20$

14. **a.**

Term number	1	2	3	4	5
Term value	15	15	25	35	45

b.

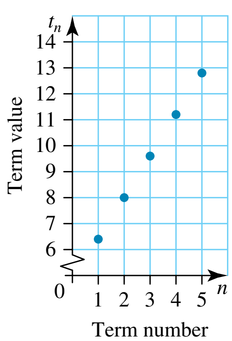


c. 85

15. **a.**

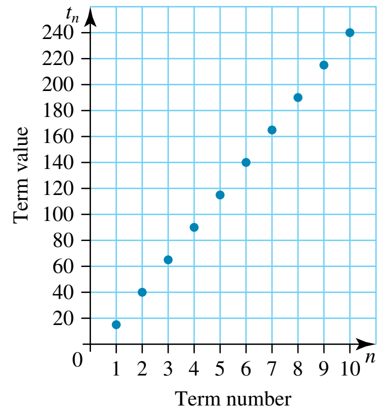
Term number	1	2	3	4	5
Term value	06.4	08.2	09.6	11.2	12.8

b.



c. 25.6

16.



17. **a.** $t_n = 1506 + 6(n - 1)$
b. \$1506, \$1512, \$1518, \$1524, \$1530, \$1536
c. \$1608

18. **a.** \$8000 **b.** 7.5%

19. **a.** $t_n = 23\,999.75 - 0.25(n - 1)$
b. \$21 000

20. **a.** \$5400 **b.** 0.1 cents

21. **a.** \$72 500 **b.** 11 years

22. **a.** \$5400
b. $t_n = 95\,400 + 5400(n - 1)$
c. 12 years

23. **a.** \$19 220 **b.** \$7560 **c.** 80 months

24. **a.** 335 000 **b.** The 367th week

25. **a.** 0.05 cents
b. In the 13th year
c. After 8 years

26. **a.** 21 800
b. \$536
c. Year 1: 1 000 000, Year 15: 11 684 800

Exercise 6.3 Geometric sequences

1. **a.** Geometric; $a = 3$, $r = 2$
b. Geometric; $a = \frac{1}{2}$, $r = 2\frac{1}{2}$
c. Not geometric
d. Geometric; $a = \frac{1}{2}$, $r = \frac{2}{5}$

2. **a.** $c = 36$
b. $g = -6$, $h = 12$
c. $p = 2.4$, $q = 12$, $s = 60$

3. **a.** Geometric; first term = 3, common ratio = 5
b. Not geometric
c. Geometric; first term = −8, common ratio = −3
d. Geometric; first term = 128, common ratio = $\frac{1}{4}$
e. Not geometric
f. Geometric; first term = 3, common ratio = $\sqrt{3}$

4. **a.** 7 **b.** 4 **c.** 4

5. **a.** $t_n = -1 \times 5^{n-1}$
b. $t_n = 7 \times (-0.5)^{n-1}$
c. $t_n = \frac{5}{6} \times \left(\frac{2}{3}\right)^{n-1}$

6. a. −2, −6, −18, −54, −162
b. 4, $\frac{4}{3}$, $\frac{4}{9}$, $\frac{4}{27}$, $\frac{4}{81}$
c. $\frac{1}{4}$, $-\frac{3}{8}$, $\frac{9}{16}$, $-\frac{27}{32}$, $\frac{81}{64}$

7. a. 19 131 876 b. 5 c. −420

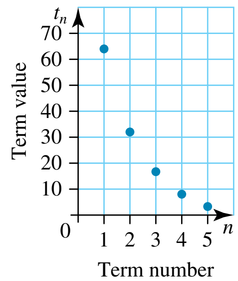
8. a. 1 258 291.2 b. −2 c. 640

9. a. 1, 3, 9, 27 b. 13.24, 66.2, 331, 1655

10. a.

Term number	1	2	3	4	5
Term value	64	32	16	18	14

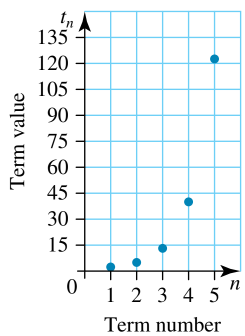
b.



11. a.

Term number	1	2	3	4	5
Term value	1.5	4.5	13.5	40.5	121.5

b.



12. 2nd term = 36, 3rd term = 18

13. a. $t_n = 2507.5 \times 1.003^{n-1}$
b. \$2507.50, \$2515.02, \$2522.57, \$2530.14, \$2537.73, \$2545.34
c. \$2614.89

14. a. \$4500 b. 4.2%

15. a. $t_n = 55\,000 \times 1.03^{n-1}$ b. \$61 902.98

16. a. $t_n = 1367.1 \times 0.93^{n-1}$ b. \$822.59

17. a. \$1800 b. 7.5%

18. 2nd term = 80, 3rd term = 32, 4th term = 12.8, 5th term = 5.12

19. 23

20. a. The second value could be either positive or negative.
b. 3159 and −3159

21. \$10 028.87

22. a. $t_n = 641.25 \times 0.95^{n-1}$, where n is the number of 5-km increments of the descent
b. 241.98 g

23. a. 13 122 b. $t_n = 13\,122 \times \left(\frac{1}{3}\right)^{n-1}$
c. 19 682

24. a. $t_n = 4\,463\,100 \times 1.026^{n-1}$
b. $t_n = 4\,729\,050 \times 1.017^{n-1}$
c. 8 years

Exercise 6.4 Recurrence relations

1. 24, 20, 18, 17, 16.5

2. 2, 2, 2, 2, 2

3. $t_{n+1} = t_n - 5,\ t_1 = 2$

4. −2.2, 1.3, 4.8, 8.3, 11.8

5. $t_{n+1} = -3t_n,\ t_1 = 2.5$

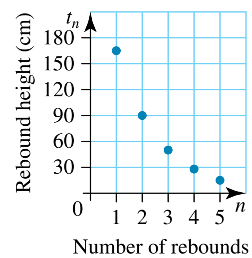
6. −4, 14, −49, 171.5, −600.25

7. $t_{n+1} = t_n - 4.5,\ t_1 = 2$

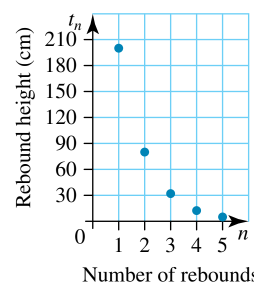
8. $t_{n+1} = \frac{1}{4}t_n,\ t_1 = -256$

9. $F_{n+2} = F_n + F_{n+1},\ F_1 = -10,\ F_2 = 8$

10. a. $t_{n+1} = 0.55t_n,\ t_1 = 165$
b. 4th rebound: 27.45 cm, 5th rebound: 15.10 cm
c.



11. a. $t_{n+1} = \frac{2}{5}t_n,\ t_1 = 200$
b. 1st rebound: 200 cm, 2nd rebound: 80 cm, 3rd rebound: 32 cm, 4th rebound: 12.8 cm, 5th rebound: 5.12 cm
c.

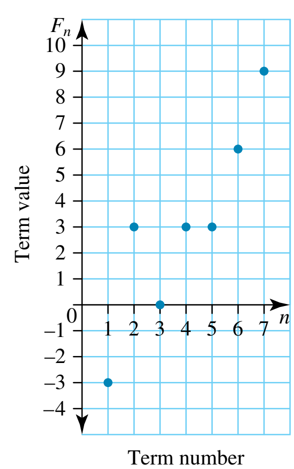


12. a. $t_{n+1} = t_n + 63,\ t_1 = 18\,063$ b. \$18 441

13. 3, −5, −2, −7, −9, −16, −25, −41

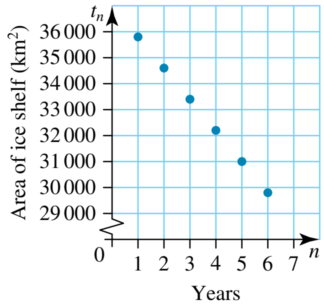
14. 2, 1, 3, 4, 7, 11, 18, 29, 47, 76

15.



16. a. $t_{n+1} = 1.05t_n$, $t_1 = 6300$
b. \$7657.69

17. a. $t_{n+1} = t_n - 1200$, $t_1 = 35\,800$
b. $35\,800\text{ km}^2$, $34\,600\text{ km}^2$, $33\,400\text{ km}^2$, $32\,200\text{ km}^2$, $31\,000\text{ km}^2$, $29\,800\text{ km}^2$

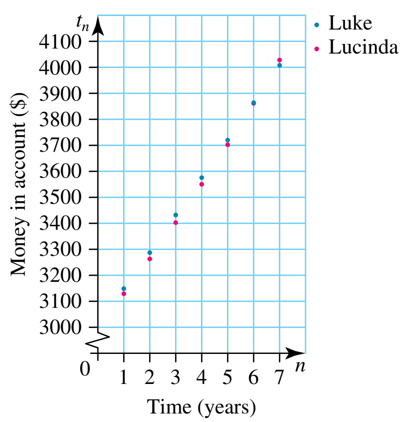
c. 

18. 3, 9, 15, 33, 63, 129, 255, 513, 1023, 2049

19. a. 17.7 metres
b. $t_{n+1} = 0.7t_n$, $t_1 = 12.39$

20. a. 5000 rabbits
b. $F_{n+2} = F_n + F_{n+1}$, $F_1 = 5000$, $F_2 = 15\,000$
c. No, there will be a natural limit to the population of the rabbits depending on resources such as food.

21. a. $t_{n+1} = t_n + 144$, $t_1 = 3144$
b. $t_{n+1} = 1.043t_n$, $t_1 = 3129$
c. Luke:
\$3144, \$3288, \$3432, \$3576, \$3720, \$3864, \$4008
Lucinda:
\$3129, \$3263.55, \$3403.88, \$3550.25, \$3702.91, \$3862.14, \$4028 21

d. 

22. Yes, this statement is true provided both of the starting numbers are not 0.

6.5 Review: exam practice

Multiple choice

1. B 2. C 3. B 4. D 5. A
6. E 7. B 8. D 9. B 10. E

Short answer

1. a. 2, 7, 12, 17, 22
b. 17, 37.4, 82.28, 181.02, 398.24
c. 15, 9, 3, −3, −9

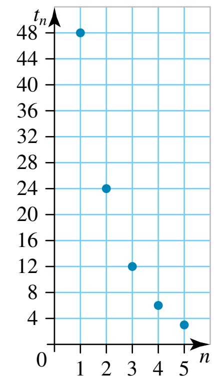
2. a. $t_n = 3 - 2.5(n - 1)$
b. $t_n = -2 \times 4^{(n-1)}$
c. $t_{n+1} = 6.5t_n$, $t_1 = 22$

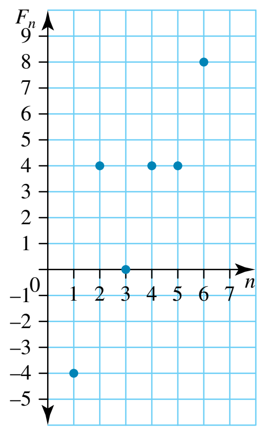
3. a. 47.25 b. 7.52 c. −74

4. a. 2.2 b. 2.7
c. $t_n = 27 + 2.2(n - 1)$ d. 44.6

5. a.

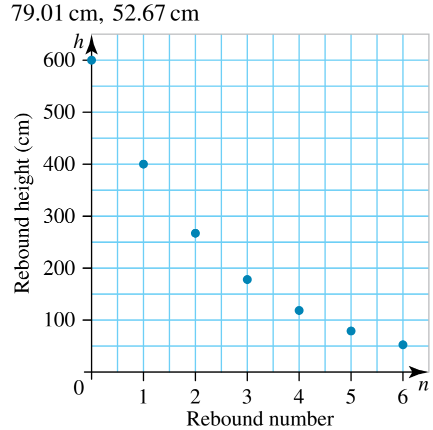
Term number	1	2	3	4	5
Term value	48	24	12	06	03

b. 

6. 

Extended response

1. a. $t_n = 906.15 + 6.15(n - 1)$
b. \$906.15, \$912.30, \$918.45, \$924.60, \$930.75
c. \$1023
d. 49 months

2. a. $t_{n+1} = \frac{2}{3}t_n$, $t_1 = 400$
b. 79.01 cm, 52.67 cm
c. 

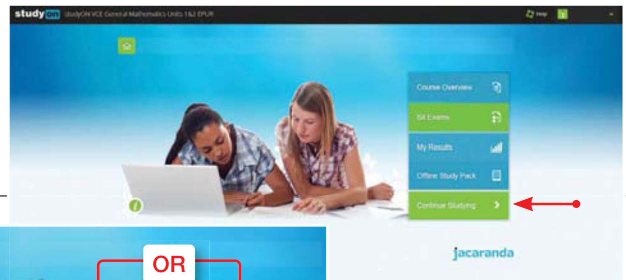
3. a. $t_n = 68\,000 \times 1.025^{n-1}$ b. \$73 228.56
c. \$1500, 3.4% d. \$272.94

4. a. 98 415 b. 15

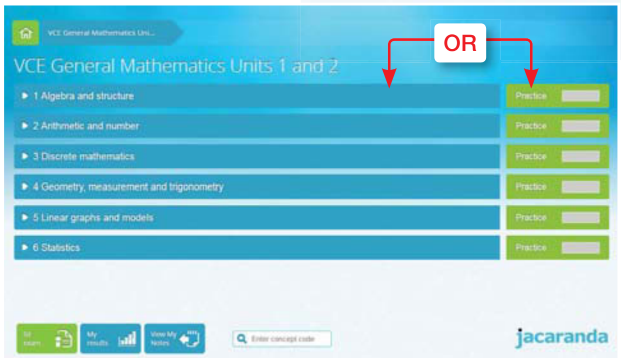
REVISION: AREA OF STUDY 3 Discrete mathematics

TOPICS 4 to 6

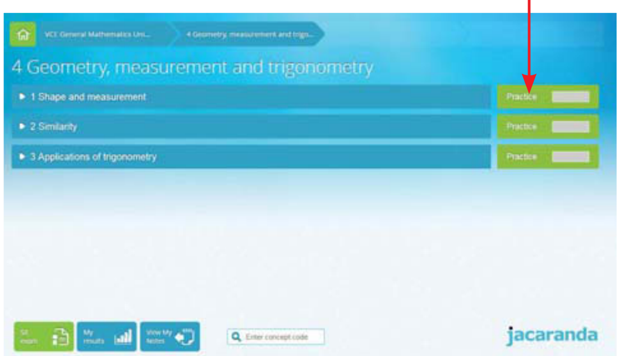
- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



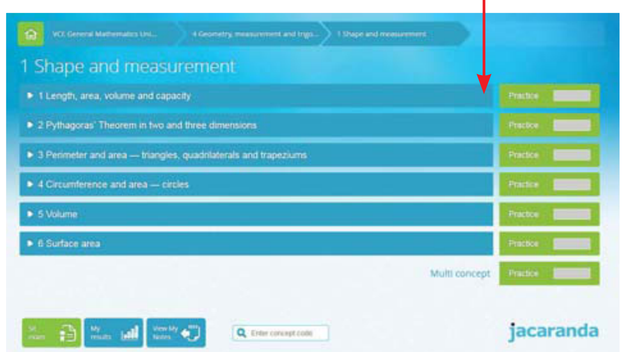
- Select your **course** *VCE General Mathematics Units 1 & 2* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.



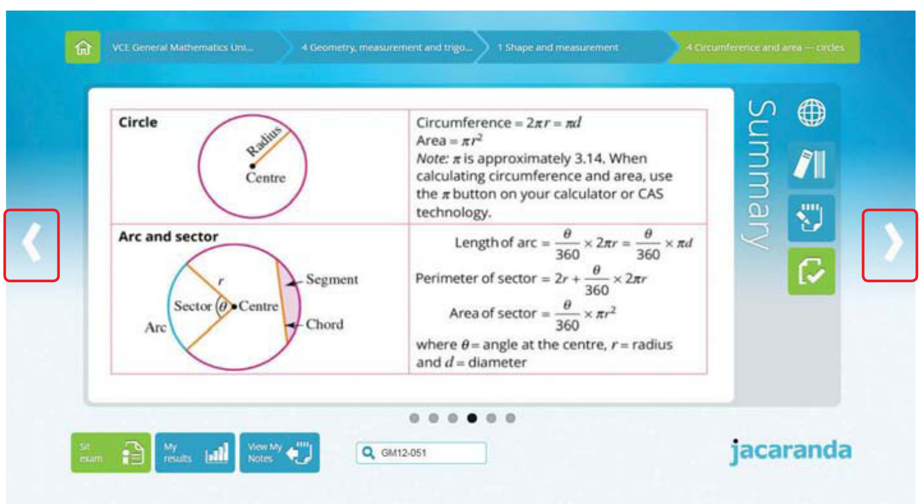
- Select **Practice** at the topic level to access all questions in the topic.



- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 7
Shape and measurement

7.1 Overview

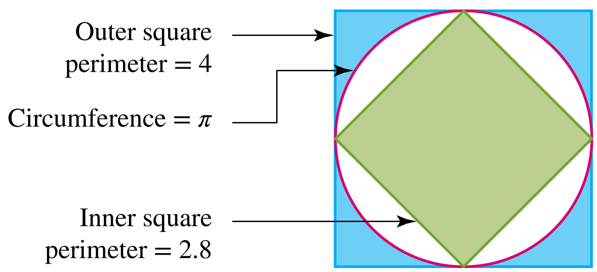
7.1.1 Introduction

When using formulas related to circles, we continually use π. We simply use 3.14 or push the button on our calculator, but what is π really and where did it come from?

π goes back to work done by Archimedes over 2000 years ago when calculators and computers didn't exist. He was able to find π to 99.9% accuracy, without using decimal places. The techniques he used to do this helped build the foundations of calculus.

π is the length of the circumference of a circle with a diameter of 1 unit. Archimedes did not know the circumference of a circle, so how did he work out its value? A possible method is to use a square of side length 1 and hence a circle of diameter 1as shown at right. Trigonometry shows that the value of π is between the outer square perimeter and the inner square perimeter. Thus $2.8 < \pi < 4$. If you take the centre of these two values, you calculate π to be equal to 3.4.

Archimedes didn't actually use squares; he started with hexagons (six-sided polygon) to find the range of values π is between. He then continued to 12, 24, 48 and 96 and stopped where he found $3\frac{10}{71} < \pi < 3\frac{1}{7}$. Decimals weren't invented until 250 BC, let alone any spreadsheet application that could easily perform these repeat calculations. So Archimedes had to spend a lot of time working through his formulas using fractions. The midpoint between $3\frac{10}{71} < \pi < 3\frac{1}{7}$ is 3.141 85, which is over 99.9% accurate. This is an amazing achievement considering is was done over 2000 years ago.



LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.

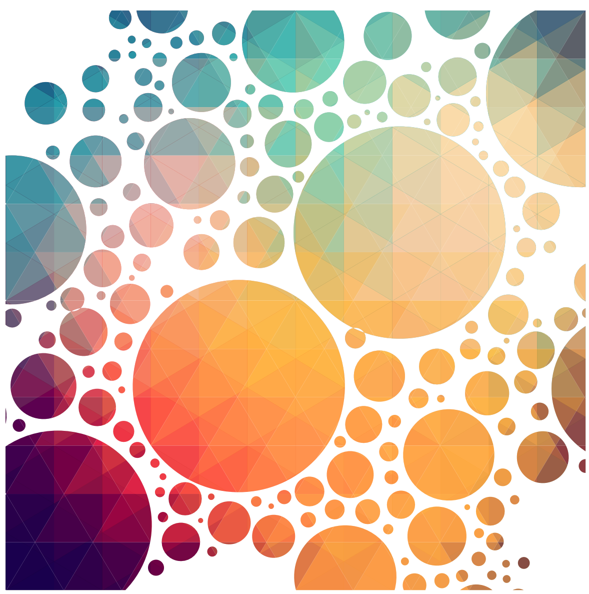
7.1.2 Kick off with CAS

Exploring area and volume with CAS

The area of a circle is defined by the formula $A = \pi r^2$, where r is the length of the radius of the circle. Areas of circles and other shapes will be studied in more detail in this topic.

1. Using CAS, define and save the formula for the area of a circle.
2. Use your formula to calculate the areas of circles with the following radii.
   a. $r = 3$
   b. $r = 7$
   c. $r = 12$
   d. $r = 15$
3. Using CAS, sketch a graph plotting the area of a circle (y-axis) against the radius of a circle (x-axis).
   The volume of a sphere is defined by the formula $V = \dfrac{4\pi r^3}{3}$.

4. Using CAS, calculate the volumes of spheres with the following radii.
   a. $r = 3$
   b. $r = 7$
   c. $r = 12$
   d. $r = 15$
5. Using CAS, sketch a graph plotting the volume of a sphere (y-axis) against the radius of a sphere (x-axis).
6. Comment on the differences and similarities between the two graphs you plotted in questions **3** and **5**.



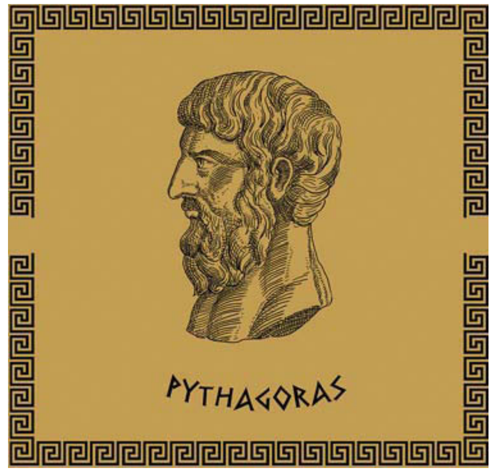
on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

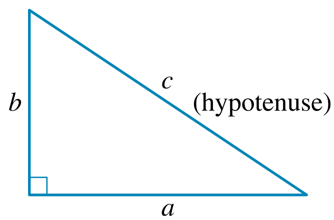
7.2 Pythagoras' theorem

7.2.1 Review of Pythagoras' theorem

Even though the theorem that describes the relationship between the side lengths of right-angled triangles bears the name of the famous Greek mathematician Pythagoras, who is thought to have lived around 550 BC, evidence exists in some of humanity's earliest relics that it was known and used much earlier than that.



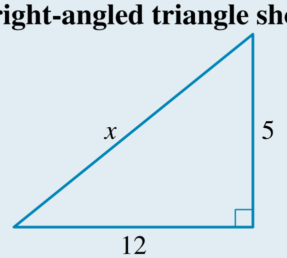
The side lengths of any right-angled triangle are related according to the rule $a^2 + b^2 = c^2$, where c represents the hypotenuse (the longest side), and a and b represent the other two side lengths.



The **hypotenuse** is always the side length that is opposite the right angle. **Pythagoras' theorem** can be used to find an unknown side length of a triangle when the other two side lengths are known.

WORKED EXAMPLE 1

Find the unknown side length in the right-angled triangle shown.

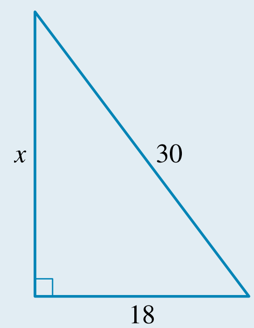


THINK	WRITE
1. Identify that the triangle is right-angled so Pythagoras' theorem can be applied.	$a^2 + b^2 = c^2$
2. Identify which of the side lengths is the hypotenuse.	x is opposite the right angle, so it is the hypotenuse; $a = 12$ and $b = 5$.
3. Substitute the known values into the theorem and simplify.	$12^2 + 5^2 = x^2$ $144 + 25 = x^2$ $169 = x^2$
4. Take the square root of both sides to obtain the value of x.	$\sqrt{169} = x$ $13 = x$

If the length of the hypotenuse and one other side length are known, the other side length can be found by subtracting the square of the known side length from the square of the hypotenuse.

WORKED EXAMPLE 2

Find the length of the unknown side in the right-angled triangle shown.



THINK	WRITE
1. Identify that the triangle is right-angled so Pythagoras' theorem can be applied.	$a^2 + b^2 = c^2$
2. Identify which of the side lengths is the hypotenuse.	30 is opposite the right angle, so it is the hypotenuse; $a = x$ and $b = 18$.
3. Substitute the known values into the theorem and simplify.	$x^2 + 18^2 = 30^2$ $x^2 = 30^2 - 18^2$ $= 576$
4. Take the square root of both sides to obtain the value of x.	$x = \sqrt{576}$ $= 24$

7.2.2 Pythagorean triads

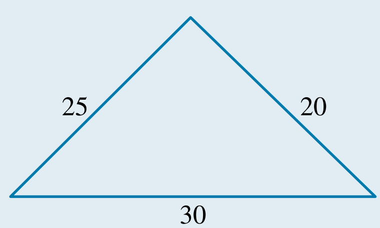
Any group of three numbers that satisfies Pythagoras' theorem is referred to as a **Pythagorean triad**. For example, because $6^2 + 8^2 = 10^2$, the numbers 6, 8 and 10 form a Pythagorean triad.

Demonstrating that three numbers form a Pythagorean triad is a way of showing that a triangle with these side lengths is right-angled.

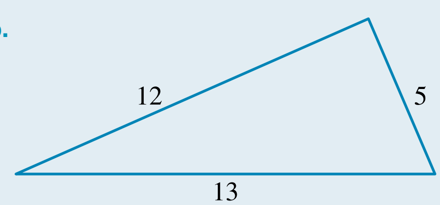
WORKED EXAMPLE 3

Demonstrate whether the given triangles are right-angled.

a.



b.



THINK	WRITE
a. 1. Write the rule.	a. For any right-angled triangle, $a^2 + b^2 = c^2$.
2. Substitute the two smaller numbers into the left-hand side of the rule and evaluate it.	$a^2 + b^2 = 20^2 + 25^2$ $= 1025$
3. Substitute the largest number into the right-hand side of the rule and evaluate it.	$c^2 = 30^2$ $= 900$
4. Identify whether $a^2 + b^2 = c^2$.	$a^2 + b^2 = 1025$ and $c^2 = 900$ Therefore, $a^2 + b^2 \neq c^2$.
5. State the final answer.	As $a^2 + b^2 \neq c^2$, this is not a right-angled triangle.
b. 1. Write the rule.	b. For any right-angled triangle, $a^2 + b^2 = c^2$.
2. Substitute the two smaller numbers into the left-hand side of the rule and evaluate it.	$a^2 + b^2 = 12^2 + 5^2$ $= 169$
3. Substitute the largest number into the right-hand side of the rule and evaluate it.	$c^2 = 13^2$ $= 169$
4. Identify whether $a^2 + b^2 = c^2$.	$a^2 + b^2 = 169$ and $c^2 = 169$ Therefore, $a^2 + b^2 = c^2$.
5. State the final answer.	As $a^2 + b^2 = c^2$, this is a right-angled triangle.

7.2.3 Pythagoras' theorem in three dimensions

Many three-dimensional objects contain right-angled triangles that can be modelled with two-dimensional drawings. Using this method we can calculate missing side lengths of three-dimensional objects.

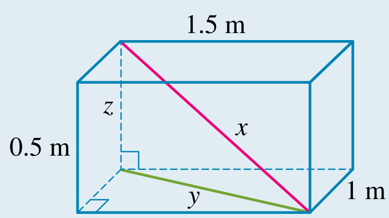
WORKED EXAMPLE 4

Calculate the maximum length of a metal rod that would fit into a rectangular crate with dimensions 1 m × 1.5 m × 0.5 m.

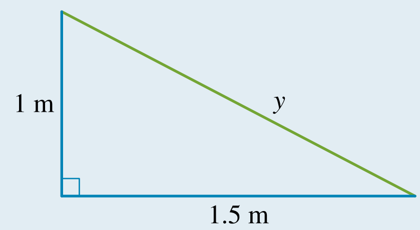
THINK

1. Draw a diagram of a rectangular box with a rod in it, labelling the dimensions.
2. Draw in a right-angled triangle that has the metal rod as one of the sides, as shown in pink.
The length of y in this right-angled triangle is not known.
Draw in another right-angled triangle, as shown in green, to calculate the length of y.

WRITE/DRAW

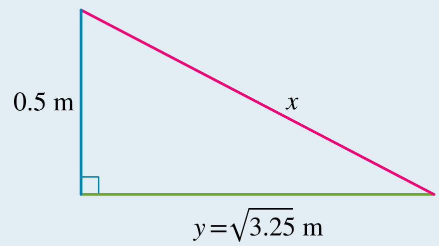


3. Calculate the length of y using Pythagoras' theorem. Calculate the exact value of y.



$$c^2 = a^2 + b^2$$
$$y^2 = 1.5^2 + 1^2$$
$$= 3.25$$
$$y = \sqrt{3.25}$$

4. Draw the right-angled triangle containing the rod and use Pythagoras' theorem to calculate the length of the rod (x).



$$c^2 = a^2 + b^2$$
$$x^2 = (\sqrt{3.25})^2 + 0.5^2$$
$$= 3.25 + 0.25$$
$$= 3.5$$
$$x = \sqrt{3.5}$$
$$\approx 1.87$$

5. Answer the question.

The maximum length of the metal rod is 1.87 m (correct to 2 decimal places).

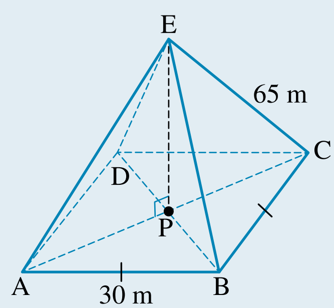
WORKED EXAMPLE 5

A square pyramid has a base length of 30 metres and a slant edge of 65 metres. Determine the height of the pyramid, giving your answer correct to 1 decimal place.

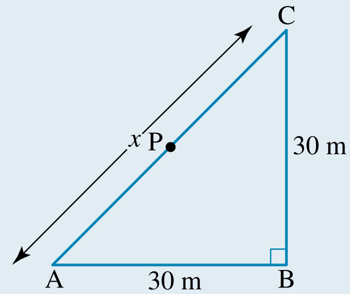
THINK

1. Draw a diagram to represent the situation. Add a point in the centre of the diagram below the apex of the pyramid.

WRITE/DRAW



2. Determine the diagonal distance across the base of the pyramid by using Pythagoras' theorem.

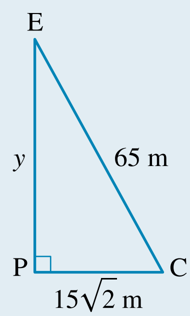


$$c^2 = a^2 + b^2$$
$$x^2 = 30^2 + 30^2$$
$$= 900 + 900$$
$$= 1800$$
$$x = \sqrt{1800}$$
$$= \sqrt{900 \times 2}$$
$$= 30\sqrt{2}$$

3. Calculate the distance from one of the corners on the base of the pyramid to the centre of the base of the pyramid.

$$AP = \frac{1}{2}AC$$
$$= \frac{1}{2} \times 30\sqrt{2}$$
$$= 15\sqrt{2}$$

4. Draw the triangle that contains the height of the pyramid and the distance from one of the corners on the base of the pyramid to the centre of the base of the pyramid.



5. Use Pythagoras' theorem to calculate the height of the pyramid, rounding your answer to 1 decimal place.

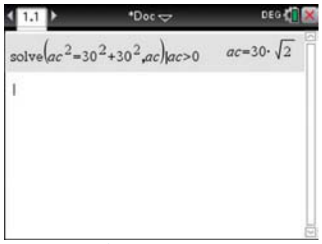
$$c^2 = a^2 + b^2$$
$$65^2 = y^2 + (15\sqrt{2})^2$$
$$4225 = y^2 + 450$$
$$y^2 = 4225 - 450$$
$$= 3775$$
$$y = \sqrt{3775}$$
$$\approx 61.4$$

6. State the answer.

The height of the pyramid is 61.4 metres (correct to 1 decimal place).

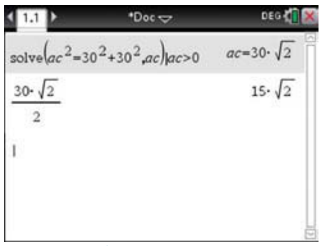
TI | THINK — **WRITE**

1. On a Calculator page, press MENU then select:
3: Algebra
1: Solve
Complete the entry line as:
solve$(ac^2 = 30^2 + 30^2, ac) \mid ac > 0$
then press ENTER.
The length of AC can be read from the screen.



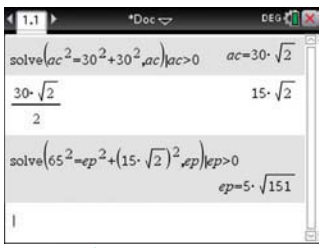
$AC = 30\sqrt{2}$

2. Complete the entry line as: $\dfrac{30\sqrt{2}}{2}$
then press ENTER.
The length of AP can be read from the screen.



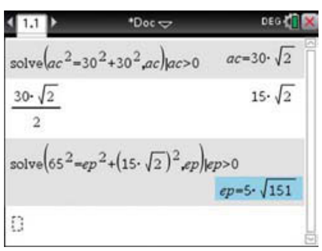
$AP = 15\sqrt{2}$

3. Complete the entry line as:
solve$(65^2 = ep^2 + (15\sqrt{2})^2, ep) \mid ep > 0$
then press ENTER.
The length of EP can be read from the screen.



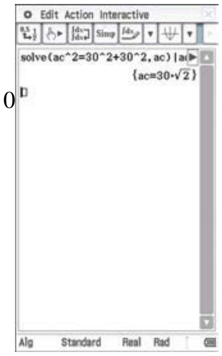
$EP = 5\sqrt{151}$

4. Use the up arrow to highlight the previous answer and then press ENTER to copy and paste it on a new entry line.



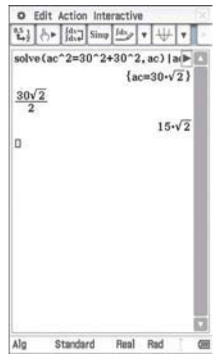
CASIO | THINK — **WRITE**

1. On the Main screen, complete the entry line as:
solve
$(ac^2 = 30^2 + 30^2, ac) \mid ac > 0$
then press EXE.
The length of AC can be read from the screen.



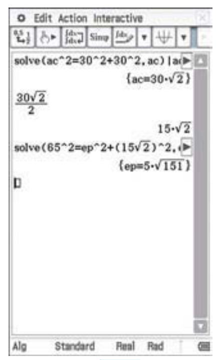
$AC = 30\sqrt{2}$

2. Complete the entry line as: $\dfrac{30\sqrt{2}}{2}$
then press EXE.
The length of AP can be read from the screen.



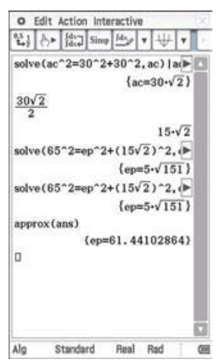
$AP = 15\sqrt{2}$

3. Complete the entry line as:
solve
$(65^2 = ep^2 + (15\sqrt{2})^2,$
$ep) \mid ep > 0$
then press EXE.
The length of EP can be read from the screen.

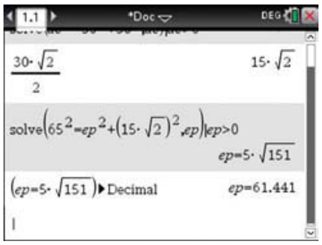


$EP = 5\sqrt{151}$

4. Select:
- Action
- Transformation
- approx.
then select ans from the Math1 tab on the Keyboard menu and press EXE.



5. Press MENU then select:
2: Number
1: Convert to Decimal
then press ENTER.



6. State the answer, rounded to 1 decimal place.

The height of the pyramid is 61.4 metres.

5. State the answer, rounded to 1 decimal place.

The height of the pyramid is 61.4 metres.

on Resources

Interactivity: Pythagoras' theorem (int-6473)

studyon

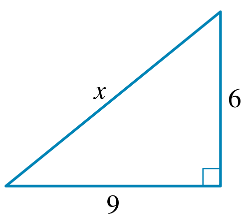
Units 1 & 2 > AOS 4 > Topic 1 > Concept 2

Pythagoras' theorem in two and three dimensions Concept summary and practice questions

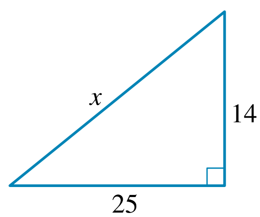
Exercise 7.2 Pythagoras' theorem

1. **WE1** Find the unknown side length in the right-angled triangles shown, giving your answers correct to 1 decimal place.

a.

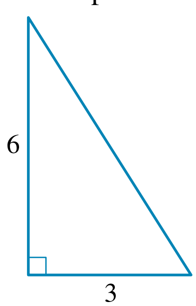


b.

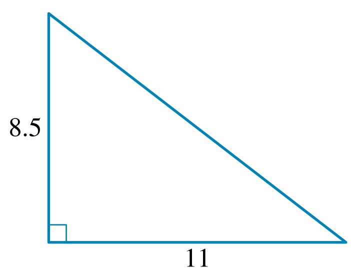


2. Show that a right-angled triangle with side lengths of 40 cm and 96 cm will have a hypotenuse of length 104 cm.
3. Evaluate the unknown side lengths in the right-angled triangles shown, giving your answers correct to 2 decimal places.

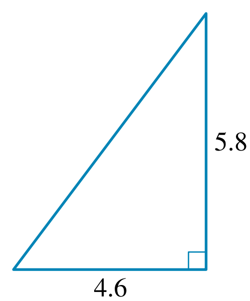
a.



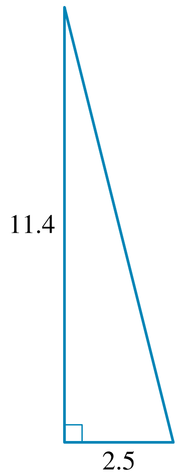
b.



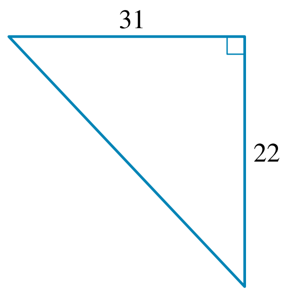
c.



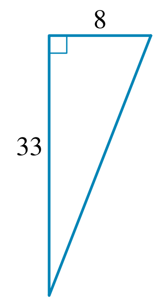
d.



e.

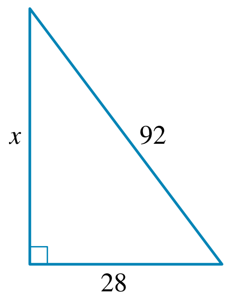


f.

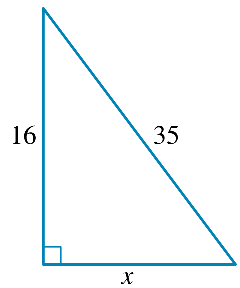


4. **WE2** Find the length of the unknown side in the right-angled triangles shown, giving your answers correct to 1 decimal place.

a.



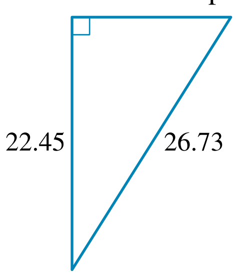
b.



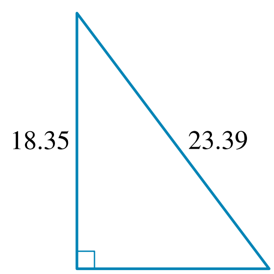
5. Show that if a right-angled triangle has a hypotenuse of 24 cm and a side length of 19.2 cm, the third side length will be 14.4 cm.

6. Evaluate the length of the unknown side in each of the right-angled triangles shown. Give your answers correct to 2 decimal places.

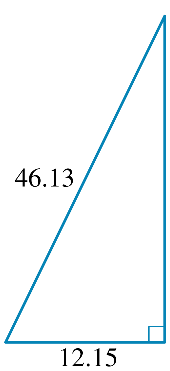
a.



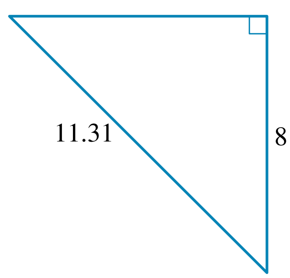
b.



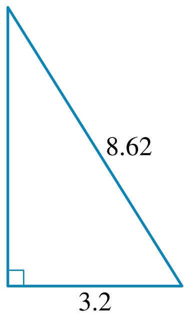
c.



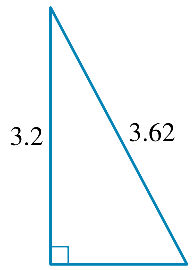
d.



e.

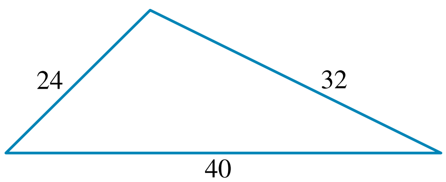


f.

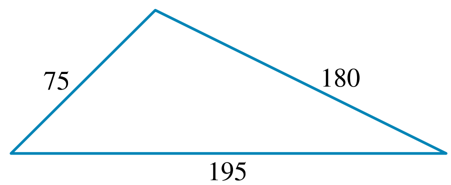


7. **WE3** Demonstrate whether the given triangles are right-angled.

a.



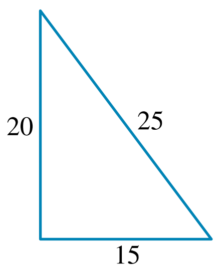
b.



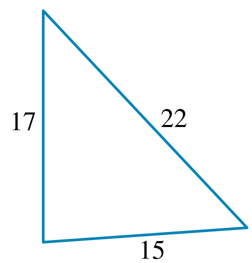
8. Demonstrate whether a triangle with side lengths of 1.5 cm, 2 cm and 2.5 cm is right-angled.

9. Demonstrate whether the given triangles are right-angled.

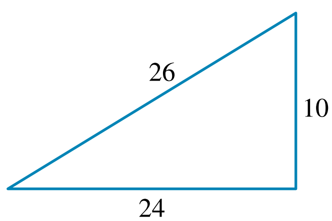
a.



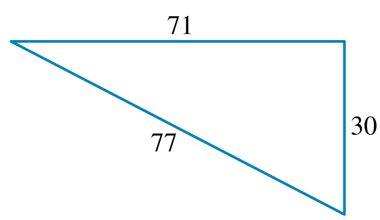
b.



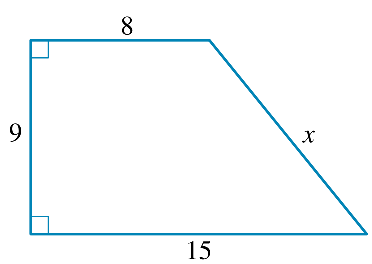
c.



d.



10. Calculate the length of the unknown side in the following diagram, giving your answer correct to 2 decimal places.



11. Find two possible values for x that would produce a Pythagorean triad with the two numbers listed. Where necessary, give your answers correct to 2 decimal places.

a. x, 5, 8 **b.** x, 64, 36 **c.** x, 15, 21

d. x, 33, 34 **e.** x, 6, 10 **f.** x, 15, 36

12. WE4 Calculate the maximum length of a metal rod that would fit into a rectangular crate with dimensions 1.2 m × 83 cm × 55 cm.

13. Determine whether a metal rod of length 2.8 metres would be able to fit into a rectangular crate with dimensions 2.3 m × 1.2 m × 0.8 m.

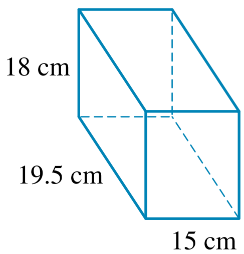
14. Calculate, correct to 2 decimal places, the lengths of the shorter sides of right-angled isosceles triangles that have a hypotenuse of length:

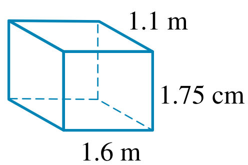
a. 20 cm
b. 48 cm
c. 5.5 cm
d. 166 cm.

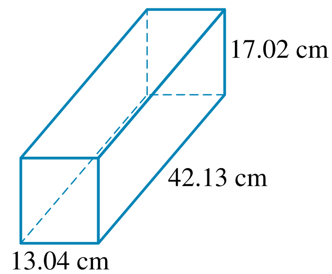
15. WE5 A square pyramid has a base length of 25 metres and a slant edge of 45 metres. Determine the height of the pyramid, giving your answer correct to 1 decimal place.

16. Determine which of the following square pyramids has the greatest height.
Pyramid 1: base length of 18 metres and slant edge of 30 metres
Pyramid 2: base length of 22 metres and slant edge of 28 metres

17. Calculate the length of the longest metal rod that can fit diagonally into the boxes shown below.

a.


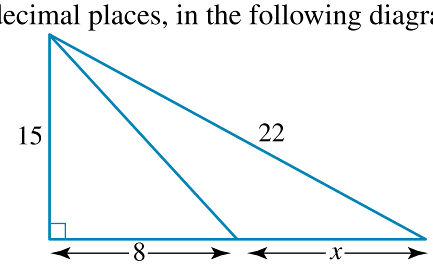
b.


c.


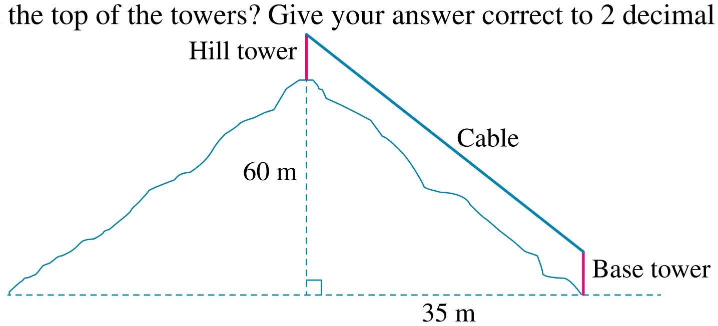
18. A friend wants to pack an umbrella into her suitcase.

a. If the suitcase measures 89 cm × 21 cm × 44 cm, will her 1-m umbrella fit in?
b. Give the length of the longest object that will fit in the suitcase.

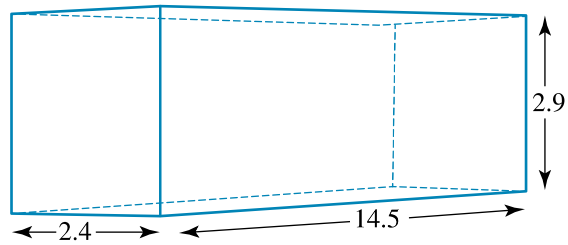
19. Find the value of x, correct to 2 decimal places, in the following diagram.



20. A cable joins the top of two vertical towers that are 12 metres high. One of the towers is at the bottom of a hill and the other is at the top. The horizontal distance between the towers is 35 metres and the vertical height of the base of the upper tower is 60 metres above ground level. What is the minimum length of cable required to join the top of the towers? Give your answer correct to 2 decimal places.



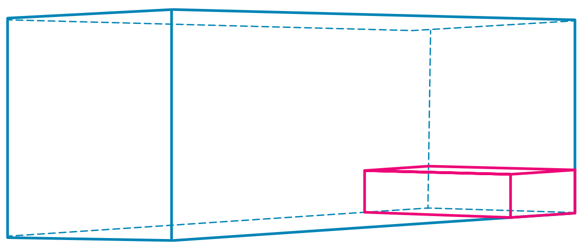
21. A semi-trailer carries a container that has the following internal dimensions: length 14.5 m, width 2.4 m and height 2.9 m. Give your answers to the following questions correct to 2 decimal places.



a. Calculate the length of the longest object that can be placed on the floor of the container.

b. Calculate the length of the longest object that can be placed in the container if only one end is placed on the floor.

c. If a rectangular box with length 2.4 m, width 1.2 m and height 0.8 m is placed on the floor at one end so that it fits across the width of the container, calculate the length of the longest object that can now be placed inside if it touches the floor adjacent to the box.



22. An ultralight aircraft is flying at an altitude of 1000 metres and a horizontal distance of 10 kilometres from its landing point.

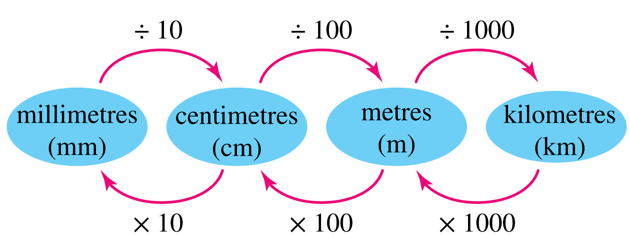
- **a.** If the aircraft travels in a straight line from its current position to its landing point, how far does it travel correct to the nearest metre? (Assume the ground is level.)
- **b.** If the aircraft maintained the same altitude for a further 4 kilometres, what would be the straight-line distance from the new position to the same landing point, correct to the nearest metre?
- **c.** From the original starting point the pilot mistakenly follows a direct line to a point on the ground that is 2.5 kilometres short of the correct landing point. He realises his mistake when he is at an altitude of 400 metres and a horizontal distance of 5.5 kilometres from the correct landing point. He then follows a straight-line path to the correct landing point. Calculate the total distance travelled by the aircraft from its starting point to the correct landing point, correct to the nearest metre.

7.3 Perimeter and area I

7.3.1 Units of length and area

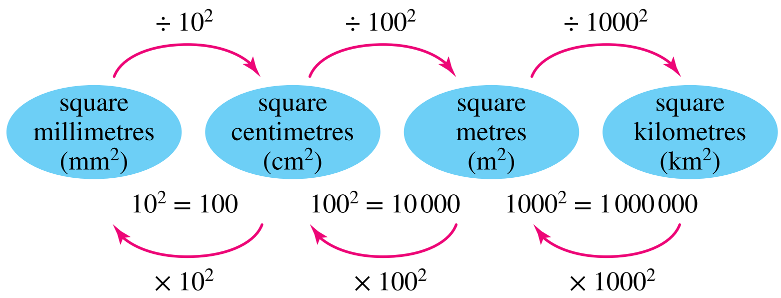
Units of length are used to describe the distance between any two points.

The standard unit of length in the metric system is the metre. The most commonly used units of length are the millimeter (mm), centimeter (cm), metre (m) and kilometer (km). These are related as shown in the following diagram.



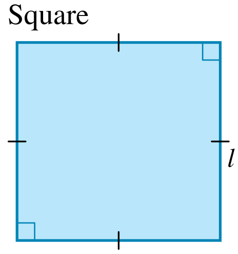
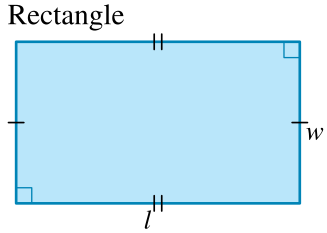
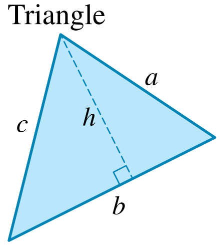
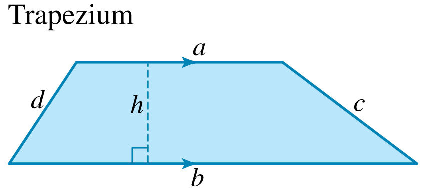
Units of **area** are named by the side length of the square that encloses that amount of space. For example, a square metre is the amount of space enclosed by a square with a side length of 1 metre.

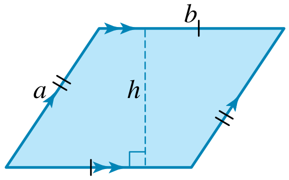
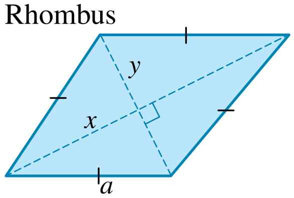
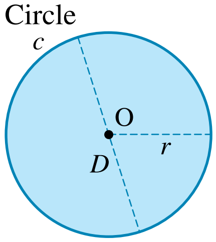
The most common units of area are related as shown in the following diagram:



7.3.2 Perimeter and area of standard shapes

You should now be familiar with the methods and units of measurement used for calculating the **perimeter** (distance around an object) and area (two-dimensional space taken up by an object) of standard **polygons** and other shapes. These are summarised in the following table.

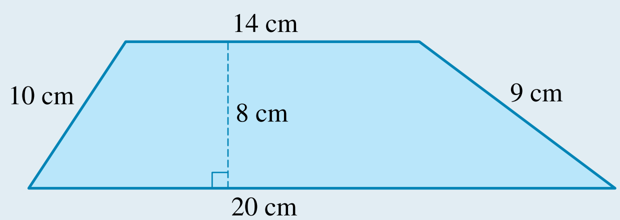
Shape	**Perimeter and area**
Square 	Perimeter: $P = 4l$ Area: $A = l^2$
Rectangle 	Perimeter: $P = 2l + 2w$ Area: $A = lw$
Triangle 	Perimeter: $P = a + b + c$ Area: $A = \frac{1}{2}bh$
Trapezium 	Perimeter: $P = a + b + c + d$ Area: $A = \frac{1}{2}(a + b)h$

Shape	**Perimeter and area**
Parallelogram 	Perimeter: $P = 2a + 2b$ Area: $A = bh$
Rhombus 	Perimeter: $P = 4a$ Area: $A = \frac{1}{2}xy$
Circle 	Circumference (perimeter): $C = 2\pi r$ $= \pi D$ Area: $A = \pi r^2$

Note: The approximate value of π is 3.14. However, when calculating **circumference** and area, always use the π button on your calculator and make rounding off to the required number of decimal places your final step.

WORKED EXAMPLE 6

Calculate the perimeter and area of the shape shown in the diagram.



THINK	WRITE
1. Identify the shape.	Trapezium
2. Identify the components for the perimeter formula and evaluate.	$P = 10 + 20 + 14 + 9$ $= 53$
3. State the perimeter including the units.	$P = 53\text{ cm}$
4. Identify the components for the area formula and evaluate.	$A = \frac{1}{2}(a + b)h$ $= \frac{1}{2}(20 + 14)8$ $= \frac{1}{2} \times 34 \times 8$ $= 136$
5. State the area and give the units.	$A = 136\text{ cm}^2$

7.3.3 Heron's formula

Heron's formula is a way of calculating the area of the triangle if you are given all three side lengths. It is named after Hero of Alexandria, who was a Greek engineer and mathematician.

Step 1: Calculate s, the value of half of the perimeter of the triangle:

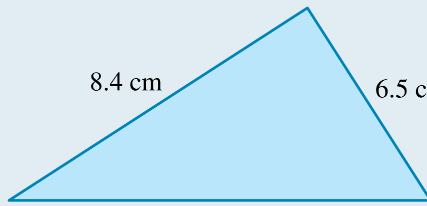
$$s = \frac{a+b+c}{2}$$

Step 2: Use the following formula to calculate the area of the triangle:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

WORKED EXAMPLE 7

Use Heron's formula to calculate the area of the following triangle. Give your answer correct to 1 decimal place.



THINK	WRITE
1. Calculate the value of s.	$s = \frac{a+b+c}{2}$ $= \frac{6.5+8.4+10.3}{2}$ $= \frac{25.2}{2}$ $= 12.6$
2. Use Heron's formula to calculate the area of the triangle correct to 1 decimal place.	$A = \sqrt{s(s-a)(s-b)(s-c)}$ $= \sqrt{12.6(12.6-6.5)(12.6-8.4)(12.6-10.3)}$ $= \sqrt{12.6 \times 6.1 \times 4.2 \times 2.3}$ $= \sqrt{742.4676}$ ≈ 27.2
3. State the area and give the units.	$A = 27.2\ \text{cm}^2$

on Resources

- **Interactivity:** Conversion of units of area (int-6269)
- **Interactivity:** Area and perimeter (int-6474)
- **Interactivity:** Using Heron's formula to find the area of a triangle (int-6475)

studyon

Units 1 & 2 | AOS 4 | Topic 1 | Concepts 1, 3 & 4

Length, area, volume and capacity Concept summary and practice questions

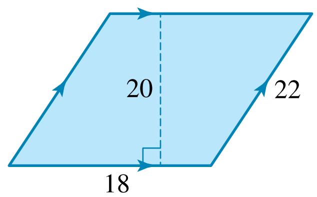
Perimeter and area — triangles, quadrilaterals and trapeziums Concept summary and practice questions

Circumference and area — circles Concept summary and practice questions

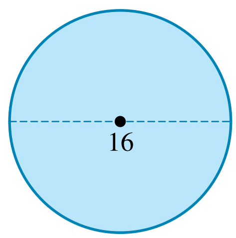
Exercise 7.3 Perimeter and area I

In the following questions, assume all measurements are in centimetres unless otherwise indicated.

1. **WE6** Calculate the perimeter and area of the shape shown in the diagram.

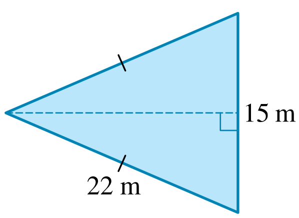


2. Calculate the circumference and area of the shape shown in the diagram, giving your final answers correct to 2 decimal places.

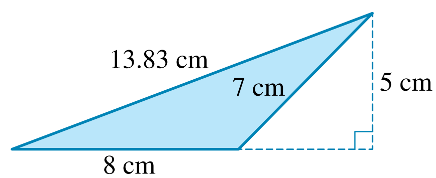


3. Calculate the perimeter and area of each of the following shapes, giving answers correct to 2 decimal places where appropriate.

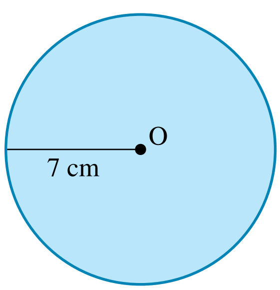
a.



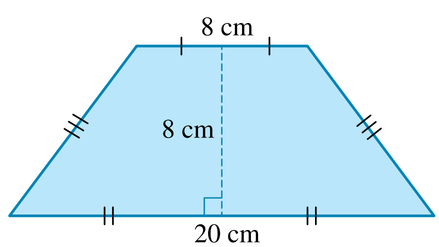
b.



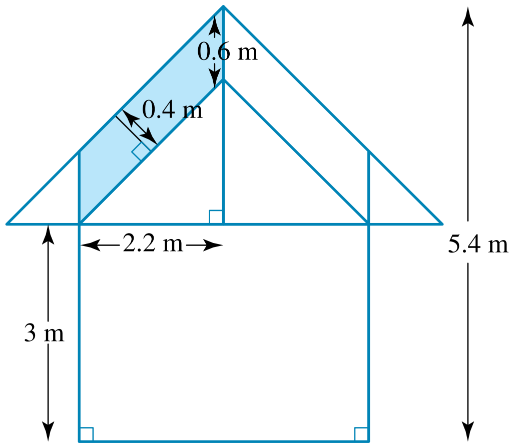
c.



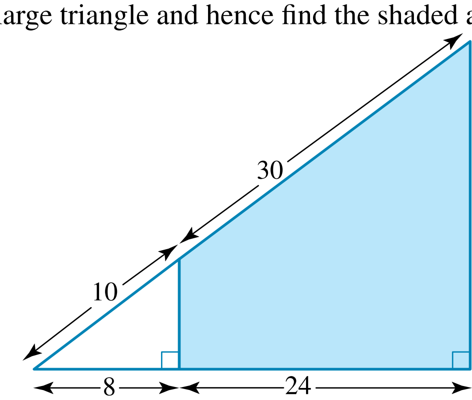
d.



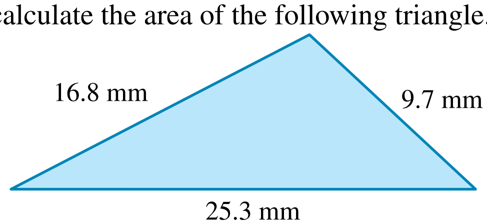
4. Correct to 2 decimal places, calculate the circumference and area of:
   a. a circle of radius 5 cm
   b. a circle of diameter 18 cm.
5. Calculate the perimeter and area of a parallelogram with side lengths of 12 cm and 22 cm, and a perpendicular distance of 16 cm between the short sides.
6. Calculate the area of a rhombus with diagonals of 11.63 cm and 5.81 cm.
7. Calculate the area of the shaded region shown in the diagram, giving your answer correct to 2 decimal places.



8. Calculate the perimeter of the large triangle and hence find the shaded area.



9. A circle has an area of 3140 cm^2. What is its radius correct to 2 decimal places?
10. **WE7** Use Heron's formula to calculate the area of the following triangle.



11. Use Heron's formula to determine which of the following triangles has the largest area.
    Triangle 1: side lengths of 10.6 cm, 13.5 cm and 16.2 cm
    Triangle 2: side lengths of 10.8 cm, 14.2 cm and 24.6 cm
    Triangle 3: side lengths of 12.1 cm, 12.6 cm and 12.7 cm

12. The Bayview Council wants to use the triangular park beside the beach to host a special Anzac Day barbecue. However, council rules stipulate that public areas can be used for such purposes only if the area chosen is over 350 m^2 in size. The sides of the triangular park measure 23 metres, 28 metres and 32 metres.

Calculate the area of the park using Heron's formula and determine whether it is of a suitable size to host the barbecue.

13. A rectangle has a side length that is twice as long as its width. If it has an area of 968 cm^2, find the length of its diagonal correct to 2 decimal places.

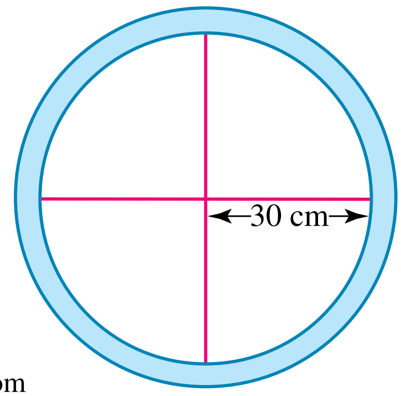
14. A window consists of a circular metal frame 2 cm wide and two straight pieces of metal that divide the inner region into four equal segments, as shown in the diagram.

a. If the window has an inner radius of 30 cm, calculate, correct to 2 decimal places:

i. the outer circumference of the window

ii. the total area of the circular metal frame.

b. If the area of the metal frame is increased by 10% by reducing the size of the inner radius, calculate the circumference of the new inner circle.

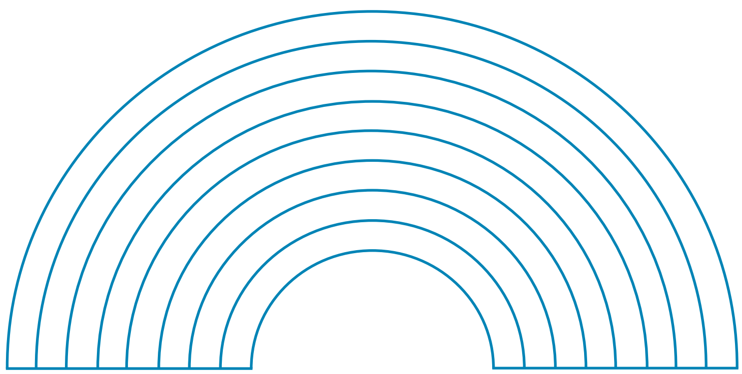


15. A semicircular section of a running track consists of eight lanes that are 1.2 m wide.

The innermost line of the first lane has a total length of 100 m.

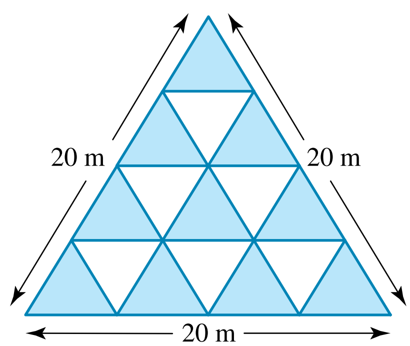
a. How much further will someone in lane 8 run around the curve from the start line to the finish line?

b. What is the total area of the curved section of the track?

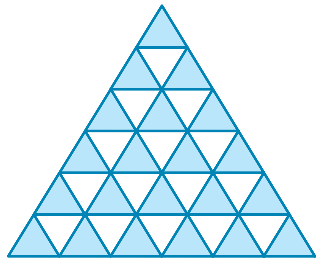


16. A paved area of a garden courtyard forms an equilateral triangle with a side length of 20 m. It is paved using a series of identically sized blue and white triangular pavers as shown in the diagram.

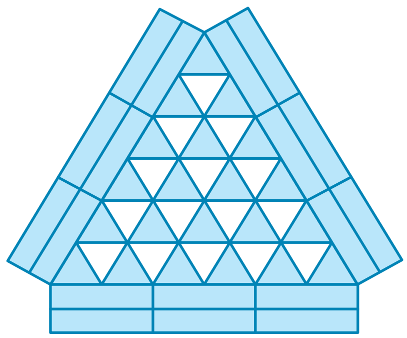
a. Calculate the total area of the paving correct to 2 decimal places.



b. If the pattern is continued by adding two more rows of pavers, calculate the new perimeter and area of the paving correct to 2 decimal places.



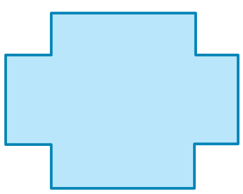
c. After the additional two rows are added, the architects decide to add two rows of rectangular pavers to each side. Each rectangular paver has a length that is twice the side length of a triangular paver, and a width that is half the side length of a triangular paver. If this was done on each side of the triangular paved area, calculate the perimeter and area of the paving.

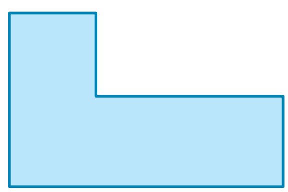


7.4 Perimeter and area II

7.4.1 Composite shapes

Many objects are not standard shapes but are combinations of them. For example:



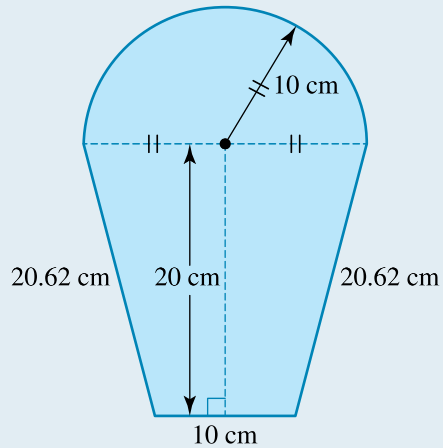


To find the areas of composite shapes, split them up into standard shapes, calculate the individual areas of these standard shapes and sum the answers together.

To find the perimeters of composite shapes, it is often easiest to calculate each individual side length and to then calculate the total, rather than applying any specific formula.

WORKED EXAMPLE 8

Calculate the area of the object shown correct to 2 decimal places.



THINK

WRITE

1. Identify the given information.

The shape is a combination of a trapezium and a semicircle.

2. Find the area of each component of the shape.

Area of trapezium: $A = \frac{1}{2}(a + b)h$

$= \frac{1}{2}(10 + 20)20$

$= 300 \text{ cm}^2$

Area of semicircle: $A = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi(10)^2$

$\approx 157.08 \text{ cm}^2$

3. Sum the areas of the components.

Total area: $300 + 157.08 = 457.08$

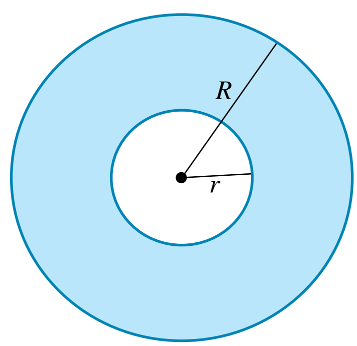
4. State the answer.

The area of the shape is 457.08 cm^2.

Some composite shapes do have specific formulas.

7.4.2 Annulus

The area between two circles with the same centre is known as an **annulus**. It is calculated by subtracting the area of the inner circle from the area of the outer circle.



$$\text{Area of annulus} = \text{area of outer circle} - \text{area of inner circle}$$

$$A = \pi R^2 - \pi r^2$$

$$= \pi\left(R^2 - r^2\right)$$

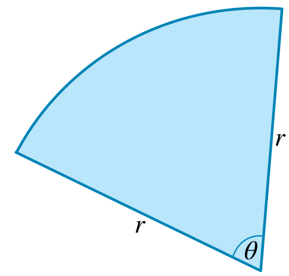
7.4.3 Sectors

Sectors are fractions of a circle. Because there are 360 degrees in a whole circle, the area of the sector can be found using $A = \frac{\theta}{360} \times \pi r^2$, where θ is the angle between the two radii that form the sector.

The perimeter of a sector is a fraction of the circumference of the related circle plus two radii:

$$P = \left(\frac{\theta}{360} \times 2\pi r\right) + 2r$$

$$= 2r\left(\frac{\theta}{360}\pi + 1\right)$$



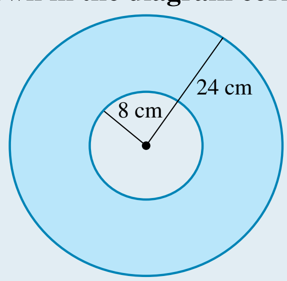
Area of an annulus $= \pi\left(R^2 - r^2\right)$

Area of a sector $= \frac{\theta}{360} \times \pi r^2$

Perimeter of a sector $= 2r\left(\frac{\theta}{360}\pi + 1\right)$

WORKED EXAMPLE 9

Calculate the area of the annulus shown in the diagram correct to 1 decimal place.



THINK	WRITE
1. Identify the given information.	The area shown is an annulus. The radius of the outer circle is 24 cm. The radius of the inner circle is 8 cm.
2. Substitute the information into the formula and simplify.	$A = \pi R^2 - \pi r^2$ $= \pi(R^2 - r^2)$ $= \pi(24^2 - 8^2)$ $= 512\pi$ ≈ 1608.5
3. State the answer.	The shaded area is 1608.5 cm^2.

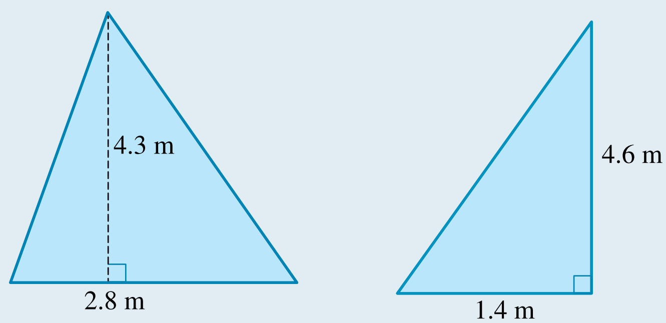
7.4.4 Applications

Calculations for perimeter and area have many and varied applications, including building and construction, painting and decorating, real estate, surveying and engineering.

When dealing with these problems it is often useful to draw diagrams to represent the given information.

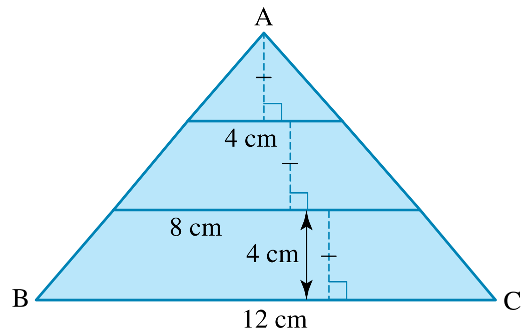
WORKED EXAMPLE 10

Calculate the total area of the sails on a yacht correct to 2 decimal places, if the apex of one sail is 4.3 m above its base length of 2.8 m, and the apex of the other sail is 4.6 m above its base of length of 1.4 m.

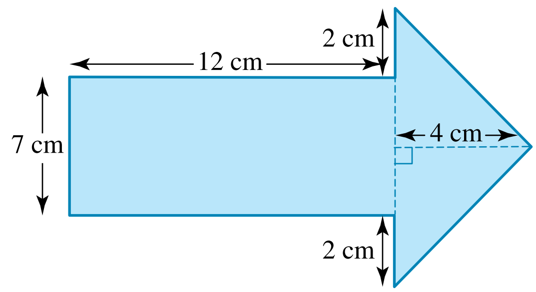
THINK	WRITE/DRAW
1. Draw a diagram of the given information.	
2. Identify the formulas required from the given information.	For each sail, use the formula for area of a triangle: $A = \frac{1}{2}bh$
3. Substitute the information into the required formulas for each area and simplify.	Sail 1: $A = \frac{1}{2}bh = \frac{1}{2} \times 2.8 \times 4.3 = 6.02$ Sail 2: $A = \frac{1}{2}bh = \frac{1}{2} \times 1.4 \times 4.6 = 3.22$
4. Add the areas of each of the required parts.	Area of sail $= 6.02 + 3.22 = 9.24$
5. State the answer.	The total area of the sails is 9.24 m^2.

Exercise 7.4 Perimeter and area II

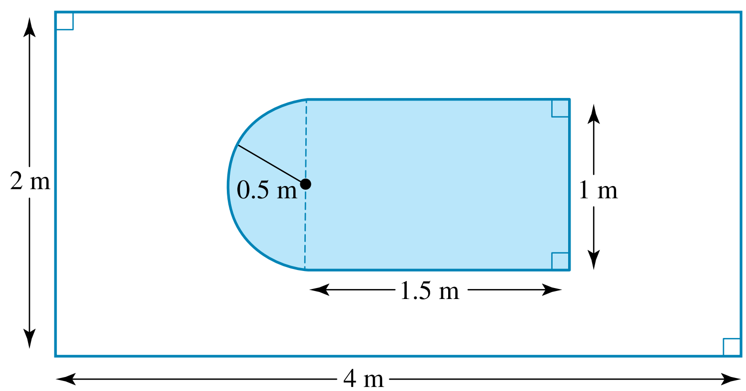
1. WE8 Calculate the area of the object shown.



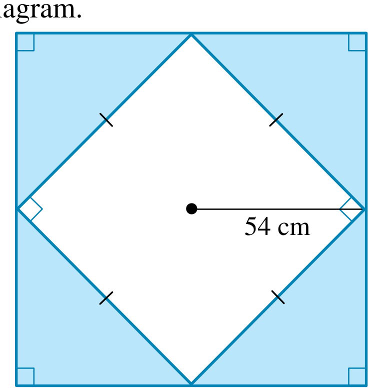
2. Calculate the perimeter and area of the object shown.



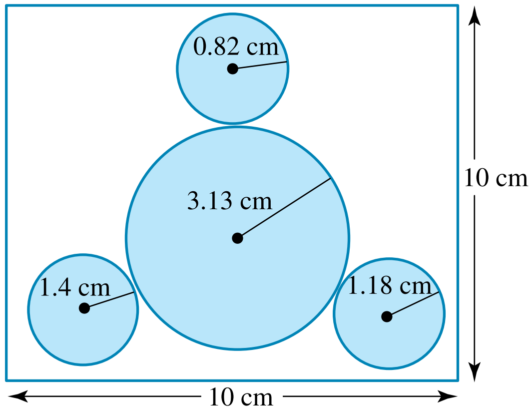
3. A circle of radius 8 cm is cut out from a square of side length 20 cm. How much of the area of the square remains? Give your answer correct to 2 decimal places.
4. a. Calculate the perimeter of the shaded area inside the rectangle shown in the diagram correct to 2 decimal places.



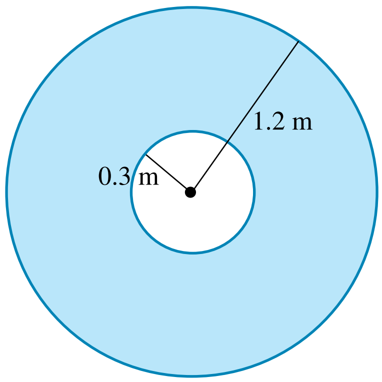
b. If the darker area inside the rectangle is removed, what area remains?
5. a. Calculate the shaded area in the diagram.



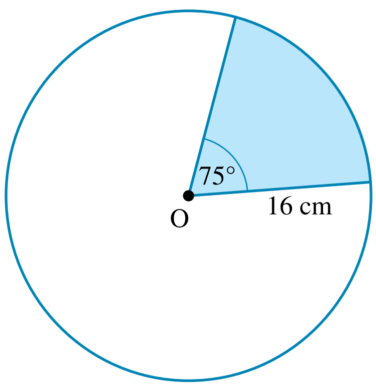
b. Calculate the unshaded area inside the square shown in the diagram, giving your answer correct to 2 decimal places.



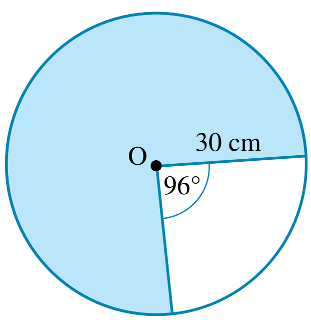
6. **WE9** Calculate the shaded area shown in the diagram correct to 2 decimal places.



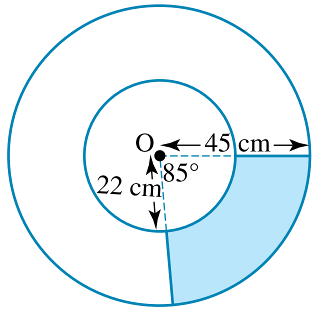
7. Calculate the area and perimeter of the shaded region shown in the diagram correct to 2 decimal places.



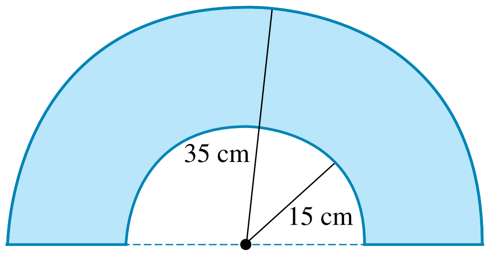
8. Calculate the area and perimeter of the shaded region shown in the diagram correct to 2 decimal places.



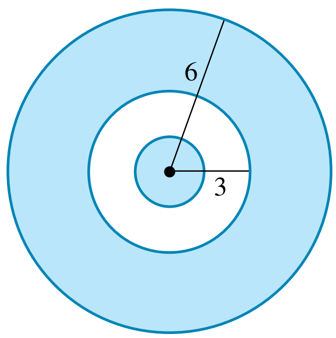
9. Calculate the area and perimeter of the shaded region shown in the diagram correct to 2 decimal places.



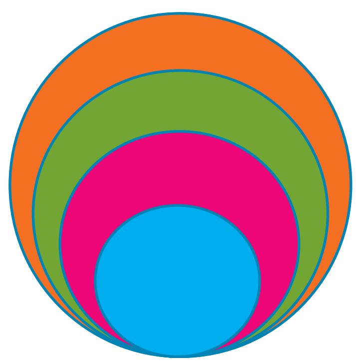
10. Calculate the perimeter and area of the shaded region in the half-annulus formed by 2 semicircles shown in the diagram. Give your answers correct to the nearest whole number.



11. The area of the inner circle in the diagram shown is $\frac{1}{9}$ that of the annulus formed by the two outer circles. Calculate the area of the inner circle correct to 2 decimal places given that the units are in centimetres.

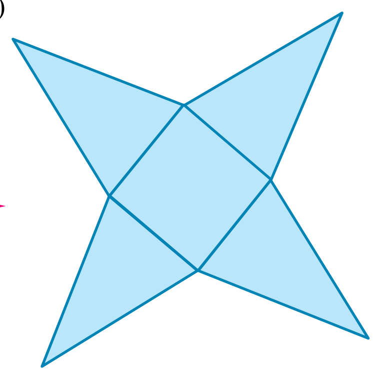


12. In the diagram the smallest circle has a diameter of 5 cm and the others have diameters that are progressively 2 cm longer than the one immediately before. Calculate the area that is shaded green, correct to 2 decimal places.

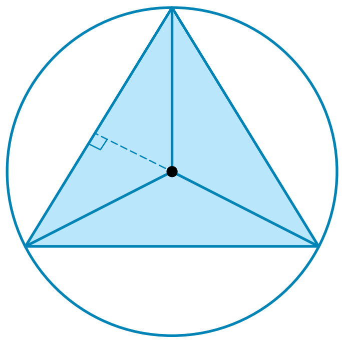


13. **WE10** Calculate the area of glass in a table that consists of three glass circles. The largest circle has a diameter of 68 cm. The diameters of the other two circles are 6 cm and 10 cm less than the diameter of the largest circle. Give your answer correct to 2 decimal places.

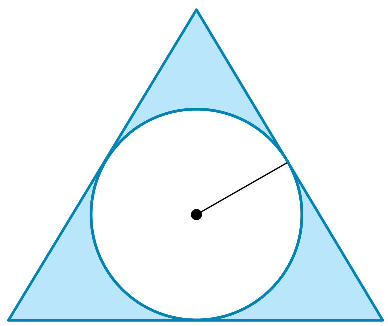
14. Part of the floor of an ancient Roman building was tiled in a pattern in which four identical triangles form a square with their bases. If the triangles have a base length of 12 cm and a height of 18 cm, calculate the perimeter and area they enclose, correct to 2 decimal places. (That is, calculate the perimeter and area of the shaded region shown on the right.)



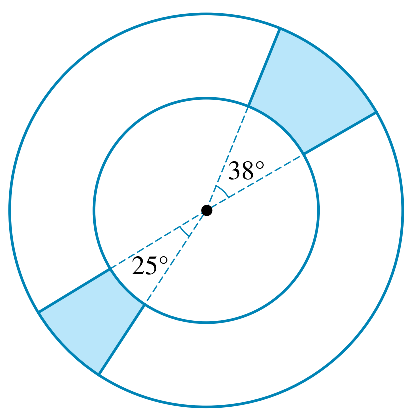
15. The vertices of an equilateral triangle of side length 2 metres touch the edge of a circle of radius 1.16 metres, as shown in the diagram. Calculate the area of the unshaded region correct to 2 decimal places.



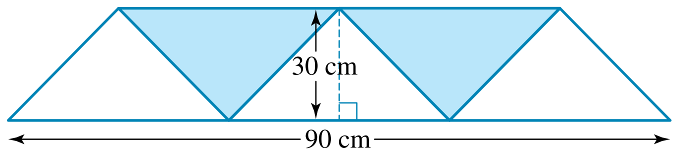
16. A circle of radius 0.58 metres sits inside an equilateral triangle of side length 2 metres so that it touches the edges of the triangle at three points. If the circle represents an area of the triangle to be removed, how much area would remain once this was done?



17. An annulus has an inner radius of 20 cm and an outer radius of 35 cm. Two sectors are to be removed. If one sector has an angle at the centre of 38° and the other has an angle of 25°, what area remains? Give your answer correct to 2 decimal places.



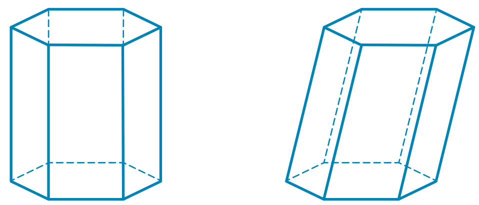
18. A trapezium is divided into five identical triangles of equal size with dimensions as shown in the diagram. Find the area and perimeter of the shaded region.



7.5 Volume

7.5.1 Volume

The amount of space that is taken up by any solid or three-dimensional object is known as its **volume**. Many standard objects have formulas that can be used to calculate their volume. If the centre point of the top of the solid is directly above the centre point of its base, the object is called a 'right solid'. If the centre point of the top is not directly above the centre point of the base, the object is an 'oblique solid'.



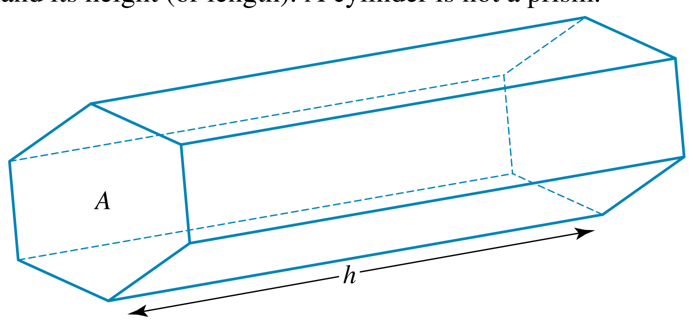
Note: For an oblique solid, the height, h, is the distance between the top and the base, not the length of one of the sides. (For a right solid, the distance between the top and the base equals the side length.)

Volume is expressed in cubic units of measurement, such as cubic metres $\left(\text{m}^3\right)$ or cubic centimetres $\left(\text{cm}^3\right)$. When calculations involve the amount of fluid that the object can contain, the units are commonly litres (L) or millilitres (mL).

To convert cubic centimetres to millilitres, use 1 cm^3 = 1 mL.

7.5.2 Prisms

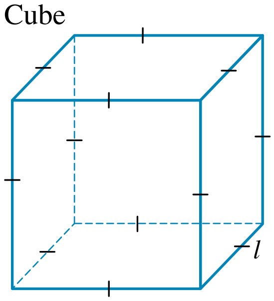
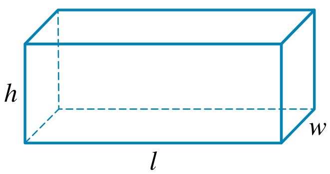
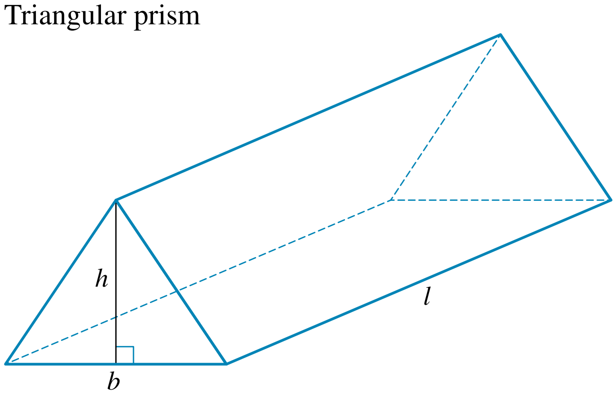
If a solid object has identical ends that are joined by flat surfaces, and the object's cross-section is a polygon and is the same along its length, the object is a **prism**. The volume of a prism is calculated by taking the product of the base area and its height (or length). A cylinder is not a prism.



$$V = A \times h \ (A = \text{base area})$$

7.5.3 Common prisms

The formulas for calculating the volume of some of the most common prisms are summarised in the following table.

Prism	Volume
Cube 	$V = A \times h$ $= (l \times l) \times l$ $= l^3$
Rectangular prism 	$V = (A \times h)$ $= (l \times w) \times h$ $= l \times w \times h$
Triangular prism 	$V = A \times h$ $= \left(\frac{1}{2}bh\right) \times l$ $= \frac{1}{2}bhl$

Note: These formulas apply to both right prisms and oblique prisms, as long as you remember that the height of an oblique prism is its perpendicular height (the distance between the top and the base).

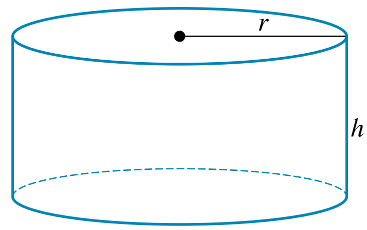
WORKED EXAMPLE 11

Calculate the volume of a triangular prism with length $l = 12$ cm, triangle base length $b = 6$ cm and triangle height $h = 4$ cm.

THINK	WRITE
1. Identify the given information.	Triangular prism, $l = 12\text{ cm},\ b = 6\text{ cm},\ h = 4\text{ cm}$

2. Substitute the information into the appropriate formula for the solid object and evaluate.	$V = \frac{1}{2}bhl$ $= \frac{1}{2} \times 6 \times 4 \times 12$ $= 144$
3. State the answer.	The volume is 144 cm^3.

7.5.4 Cylinders



A **cylinder** is a solid object with ends that are identical circles and a cross-section that is the same along its length (like a prism). As a result it has a curved surface along its length.

As for prisms, the volume of a cylinder is calculated by taking the product of the base area and the height:

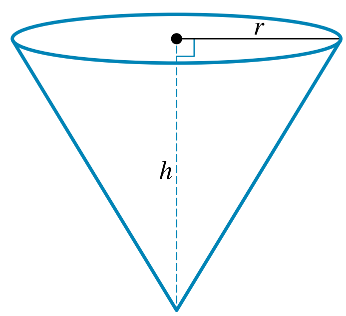
$$\begin{aligned} V &= \text{base area} \times \text{height} \\ &= \pi r^2 h \end{aligned}$$

WORKED EXAMPLE 12

Calculate the volume of a cylinder of radius 10 cm and height 15 cm correct to 2 decimal places.

THINK	WRITE
1. Identify the given information.	Cylinder, $r = 10$ cm, $h = 15$ cm
2. Substitute the information into the appropriate formula for the solid object and evaluate.	$V = \pi r^2 h$ $= \pi \times 10 \times 10 \times 15$ ≈ 4712.39
3. State the answer.	The volume is 4712.39 cm^3.

7.5.5 Cones



A **cone** is a solid object that is similar to a cylinder in that it has one end that is circular, but different in that at the other end it has a single vertex.

It can be shown that if you have a cone and a cylinder with identical circular bases and heights, the volume of the cylinder will be three times the volume of the cone. (The proof of this is beyond the scope of this course.) The volume of a cone can therefore be calculated by using the formula for a comparable cylinder and dividing by three.

$$\begin{aligned} V &= \frac{1}{3} \times \text{base area} \times \text{height} \\ &= \frac{1}{3}\pi r^2 h \end{aligned}$$

WORKED EXAMPLE 13

Calculate the volume of a cone of radius 20 cm and a height of 36 cm correct to 1 decimal place.

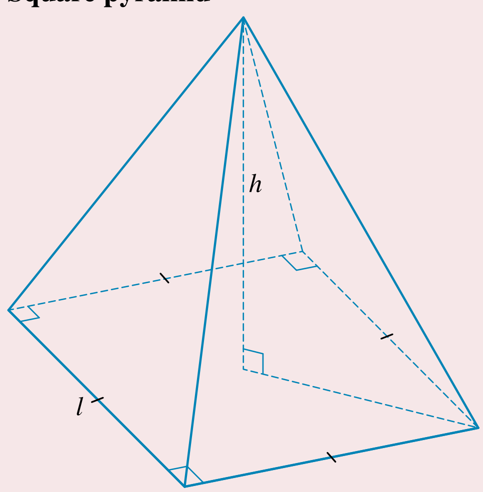
THINK	WRITE
1. Identify the given information.	Given: a cone with $r = 20$ cm and $h = 36$ cm
2. Substitute the information into the appropriate formula for the solid object and evaluate.	$V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3} \times \pi \times 20 \times 20 \times 36$ $\approx 15\,079.6$
3. State the answer.	The volume is $15\,079.6\ \text{cm}^3$.

7.5.6 Pyramids

A **pyramid** is a solid object whose base is a polygon and whose sides are triangles that meet at a single point. The most famous examples are the pyramids of Ancient Egypt, which were built as tombs for the pharaohs.

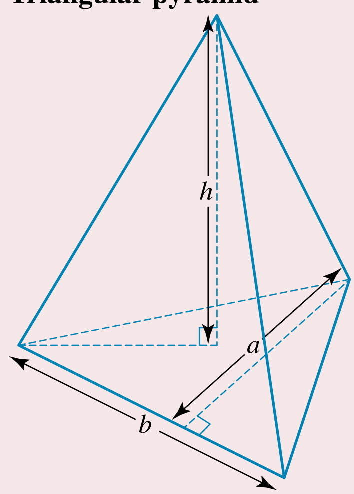
A pyramid is named after the shape of its base. For example, a hexagonal pyramid has a hexagon as its base polygon. The most common pyramids are square pyramids and triangular pyramids. As with cones, the volume of a pyramid can be calculated by using the formula of a comparable prism and dividing by three.

Square pyramid



$$V = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3} l^2 h$$

Triangular pyramid



$$V = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3}\left(\frac{1}{2}ab\right)h$$
$$= \frac{1}{6}abh$$

WORKED EXAMPLE 14

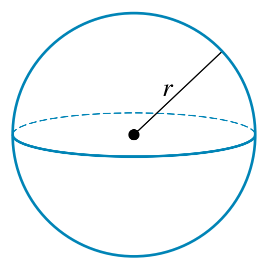
Calculate the volume of a pyramid that is 75 cm tall and has a rectangular base with dimensions 45 cm by 38 cm.

THINK	WRITE
1. Identify the given information.	Given: a pyramid with a rectangular base of 45×38 cm and a height of 75 cm
2. Substitute the information into the appropriate formula for the solid object and evaluate.	$V = \frac{1}{3}lwh$ $= \frac{1}{3} \times 45 \times 38 \times 75$ $= 42\,750$
3. State the answer.	The volume is $42\,750\ \text{cm}^3$.

7.5.7 Spheres

A **sphere** is a solid object that has a curved surface such that every point on the surface is the same distance (the radius of the sphere) from a central point.

The formula for calculating the volume of a sphere has been attributed to the ancient Greek mathematician Archimedes.



$$V = \frac{4}{3}\pi r^3$$

WORKED EXAMPLE 15

Calculate the volume of a sphere of radius 63 cm correct to 1 decimal place.

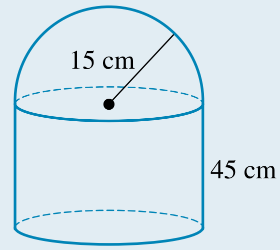
THINK	WRITE
1. Identify the given information.	Given: a sphere of $r = 63$ cm
2. Substitute the information into the appropriate formula for the solid object and evaluate.	$V = \frac{4}{3}\pi r^3$ $= \frac{4}{3} \times \pi \times 63 \times 63 \times 63$ $= 1\,047\,394.4$
3. State the answer.	The volume is $1\,047\,394.4\ \text{cm}^3$.

7.5.8 Volumes of composite solids

As with calculations for perimeter and area, when a solid object is composed of two or more standard shapes, we need to identify each part and add their volumes to evaluate the overall volume.

WORKED EXAMPLE 16

Calculate the volume of an object that is composed of a hemisphere (half a sphere) of radius 15 cm that sits on top of a cylinder of height 45 cm, correct to 1decimal place.



THINK	WRITE
1. Identify the given information.	Given: a hemisphere with $r = 15$ cm and a cylinder of $r = 15$ cm and $h = 45$ cm
2. Substitute the information into the appropriate formula for each component of the solid object and evaluate.	Hemisphere: $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$ $= \frac{1}{2} \times \left(\frac{4}{3} \times \pi \times 15 \times 15 \times 15\right)$ ≈ 7068.6 Cylinder: $V = \pi r^2 h$ $= \pi \times 15 \times 15 \times 45$ $\approx 31\,808.6$
3. Add the volume of each component.	Composite object: $V = 7068.6 + 31\,808.6$ $= 38\,877.2$
4. State the answer.	The volume is $38\,877.2\text{ cm}^3$.

on Resources

Interactivity: Volume (int-6476)

studyon

Units 1 & 2 > AOS 4 > Topic 1 > Concept 5 > **Volume** Concept summary and practice questions

Exercise 7.5 Volume

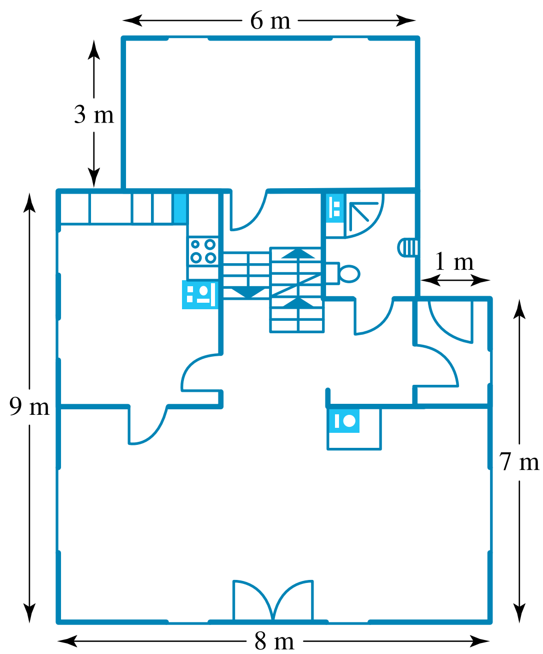
1. **WE11** Calculate the volume of a triangular prism with length $l = 2.5$ m, triangle base length $b = 0.6$ m and triangle height $h = 0.8$ m.
2. Giving answers correct to the nearest cubic centimetre, calculate the volume of a prism that has:
   a. a base area of 200 cm^2 and a height of 1.025 m
   b. a rectangular base 25.25 cm by 12.65 cm and a length of 0.42 m
   c. a right-angled triangular base with one side length of 48 cm, a hypotenuse of 73 cm and a length of 96 cm

d. a height of 1.05 m and a trapezium-shaped base with parallel sides that are 25 cm and 40 cm long and 15 cm apart.

3. The Gold Medal Pool Company sells three types of above-ground swimming pools, with base shapes that are cubic, rectangular or circular. Use the information in the table to list the volumes of each type in order from largest to smallest, giving your answers in litres.

Type	Depth	Base dimensions
Square pool	1.2 m	Length: 3 m
Rectangular pool	1.2 m	Length: 4.1 m Width: 2.25 m
Circular pool	1.2 m	Diameter: 3.3 m

4. A builder uses the floor plan of the house he is building to calculate the amount of concrete he needs to order for the foundations supporting the brick walls.



a. The foundation needs to go around the perimeter of the house with a width of 600 mm and a depth of 1050 mm. How many cubic metres of concrete are required?

b. The builder also wants to order the concrete required to pour a rectangular slab 3 m by 4 m to a depth of 600 mm. How many cubic metres of extra concrete should he order?

5. WE12 Giving answers correct to the nearest cubic centimetre, calculate the volume of a cylinder of radius 22.5 cm and a height of 35.4 cm.

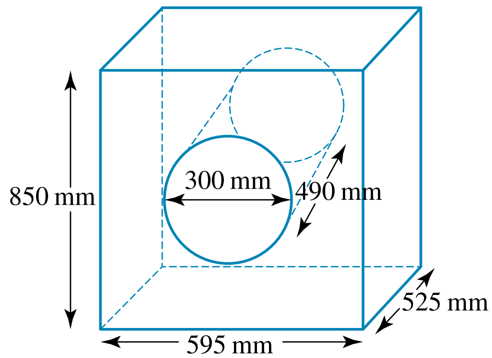
6. Calculate the volume of a cylinder that has:

a. a base circumference of 314 cm and a height of 0.625 m, giving your answer correct to the nearest cubic centimetre

b. a height of 425 cm and a radius that is three-quarters of its height, giving your answer correct to the nearest cubic metre.

7. A company manufactures skylights in the shape of a cylinder with a hemispherical lid.
When they are fitted onto a house, three-quarters of the length of the cylinder is below the roof. If the cylinder is 1.5 metres long and has a radius of 30 cm, calculate the volume of the skylight that is above the roof and the volume that is below it. Give your answers correct to the nearest cubic centimetre.

8. The outer shape of a washing machine is a rectangular prism with a height of 850 mm, a width of 595 mm and a depth of 525 mm. Inside the machine, clothes are washed in a cylindrical stainless steel drum that has a diameter of 300 mm and a length of 490 mm.

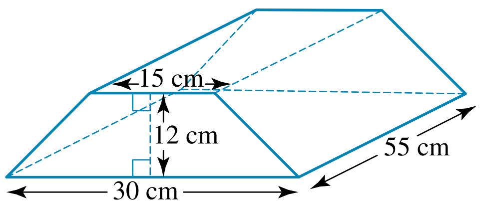


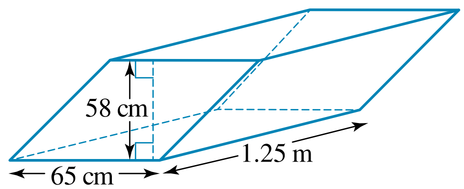
a. What is the maximum volume of water, in litres, that the stainless steel drum can hold?
b. Calculate the volume of the washing machine, in cubic metres, after subtracting the volume of the stainless steel drum.

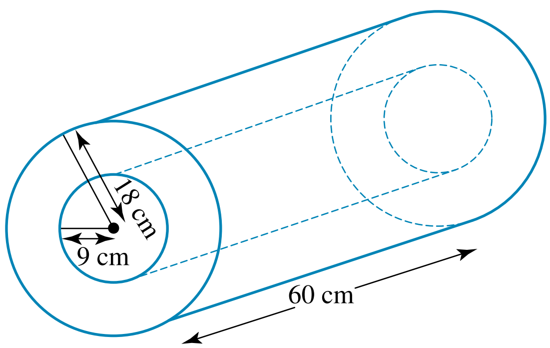
9. WE13 Calculate the volume of a cone of radius 30 cm and a height of 42 cm correct to 1 decimal place.

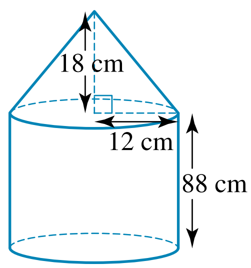
10. Calculate the volume of a cone that has:
a. a base circumference of 628 cm and a height of 0.72 m, correct to the nearest whole number
b. a height of 0.36 cm and a radius that is two-thirds of its height, correct to 3 decimal places.

11. Calculate the volumes of the solid objects shown in the following diagrams.

a.


b.


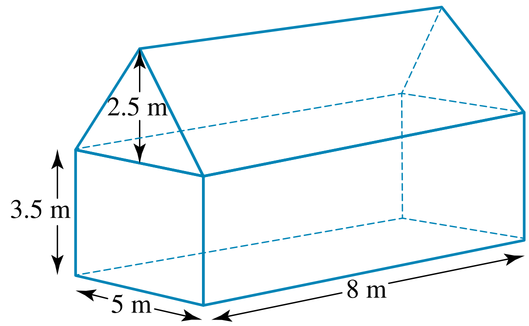
c.


d.


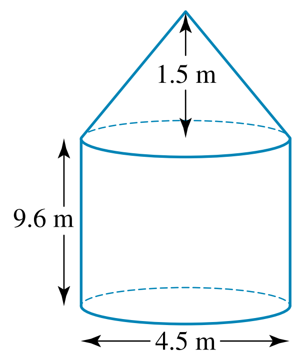
12. WE14 Calculate the volume of a pyramid that is 2.025 m tall and has a rectangular base with dimensions 1.05 m by 0.0745 m, correct to 4 decimal places.

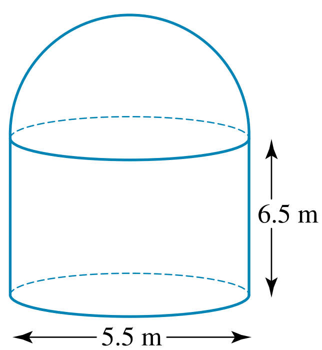
13. Calculate the volume of a pyramid that has:
a. a base area of 366 cm^2 and a height of 1.875 m
b. a rectangular base 18.45 cm by 26.55 cm and a length of 0.96 m
c. a height of 3.6 m, a triangular base with one side length of 1.2 m and a perpendicular height of 0.6 m.

14. The diagram shows the dimensions for a proposed house extension. Calculate the volume of insulation required in the roof if it takes up an eighth of the overall roof space.



15. **WE15** Calculate the volume of a sphere of radius 0.27 m correct to 4 decimal places.
16. Calculate the radius, correct to the nearest whole number, of a sphere that has:
    a. a volume of 248 398.88 cm^3
    b. a volume of 4.187 m^3.
17. **WE16** Calculate the volume of an object that is composed of a hemisphere (half a sphere) of radius 1.5 m that sits on top of a cylinder of height 2.1 m. Give your answer correct to 2 decimal places.
18. Calculate the volume of an object that is composed of:
    a. a square pyramid of height 48 cm that sits on top of a cube of side length 34 cm
    b. a cone of height 75 cm that sits on top of a 60-cm-tall cylinder of radius 16 cm.
19. A wheat farmer needs to purchase a new grain silo and has the choice of two sizes.
One is cylindrical with a conical top, and the other is cylindrical with a hemispherical top. Use the dimensions shown in the diagrams to determine which silo holds the greatest volume of wheat and by how much.





20. A tank holding liquid petroleum gas (LPG) is cylindrical in shape with hemispherical ends. If the tank is 8.7 metres from the top of the hemisphere at one end to the top of the hemisphere at the other, and the cylindrical part of the tank has a diameter of 1.76 metres, calculate the volume of the tank to the nearest litre.

21. The glass pyramid in the courtyard of the Louvre Museum in Paris has a height of 22 m and a square base with side lengths of 35 m.

a. What is the volume of the glass pyramid in cubic metres?
b. A second glass pyramid at the Louvre Museum is called the Inverted Pyramid as it hangs upside down from the ceiling. If its dimensions are one-third of those of the larger glass pyramid, what is its volume in cubic centimetres?

22. Tennis balls are spherical with a diameter of 6.7 cm. They are sold in packs of four in cylindrical canisters whose internal dimensions are 26.95 cm long with a diameter that is 5 mm greater than that of a ball. The canisters are packed vertically in rectangular boxes; each box is 27 cm high and will fit exactly eight canisters along its length and exactly four along its width.
a. Calculate the volume of free space that is in a canister containing four tennis balls.
b. Calculate the volume of free space that is in a rectangular box packed full of canisters.

23. Using CAS or otherwise, compare the volumes of cylinders that are 50 cm tall but have different radii.
a. Tabulate the results for cylinders with radii of 10, 20, 40, 80, 160 and 320 cm.
b. Graph your results.
c. Use your graph to estimate the volume of a cylinder that is 50 cm tall and has a radius of:
i. 100 cm
ii. 250 cm.

24. Using CAS or otherwise, compare the volumes of square pyramids that are 20 m tall but have different base lengths.
a. Tabulate the results for pyramids with base lengths of 5, 10, 15, 20, 25 and 30 m.
b. Graph your results.
c. Use your graph to estimate the volume of a square pyramid that is 20 m tall and a base length of:
i. 9 m
ii. 14 m.

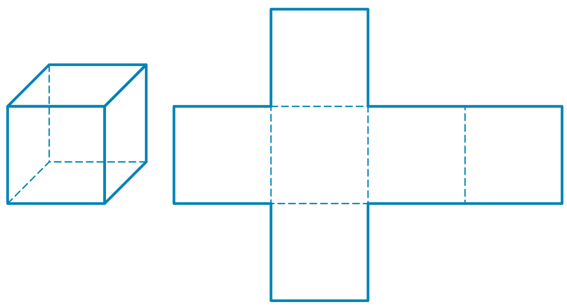
7.6 Surface area

7.6.1 Surface area

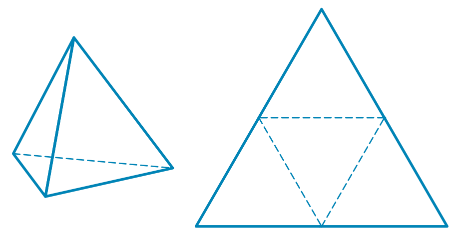
The **surface area** of a solid object is equal to the combined total of the areas of each individual surface that forms it. Some objects have specific formulas for the calculation of the total surface area, whereas others require the calculation of each individual surface in turn. Surface area is particularly important in design and construction when considering how much material is required to make a solid object. In manufacturing it could be important to make an object with the smallest amount of material that is capable of holding a particular volume. Surface area is also important in aerodynamics, as the greater the surface area, the greater the potential air resistance or drag.

7.6.2 Nets

The **net** of a solid object is like a pattern or plan for its construction. Each surface of the object is included in its net. Therefore, the net can be used to calculate the total surface area of the object. For example, the net of a cube will have six squares, whereas the net of a triangular pyramid (or tetrahedron) will have four triangles.



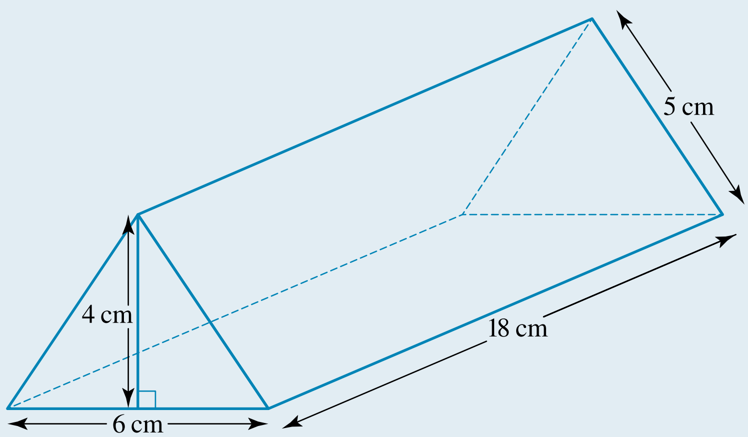
The net of a cube



The net of a tetrahedron

WORKED EXAMPLE 17

Calculate the surface area of the prism shown by first drawing its net.



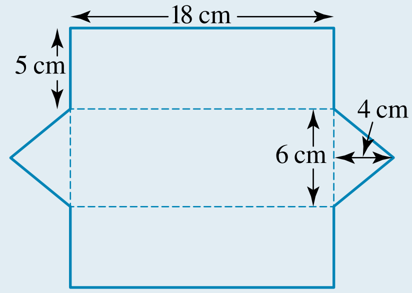
THINK

1. Identify the prism and each surface in it.

WRITE/DRAW

The triangular prism consists of two identical triangular ends, two identical rectangular sides and one rectangular base.

2. Redraw the given diagram as a net, making sure to check that each surface is present.



3. Calculate the area of each surface identified in the net.

Triangular ends:

$$A = 2 \times \left(\frac{1}{2}bh\right)$$
$$= 2 \times \left(\frac{1}{2} \times 6 \times 4\right)$$
$$= 24$$

Rectangular sides:

$$A = 2 \times (lw)$$
$$= 2 \times (18 \times 5)$$
$$= 180$$

Rectangular base:

$$A = lw$$
$$= 18 \times 6$$
$$= 108$$

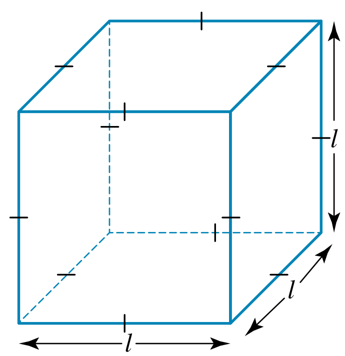
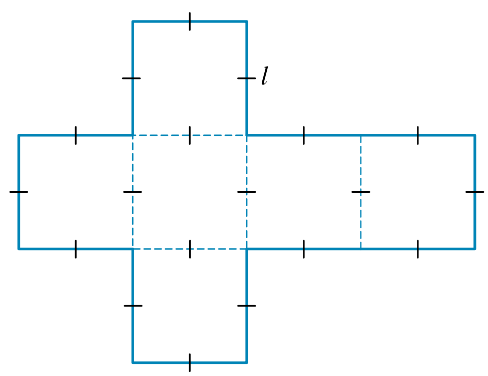
4. Add the component areas and state the answer.

Total surface area:

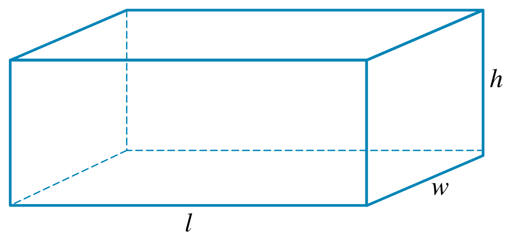
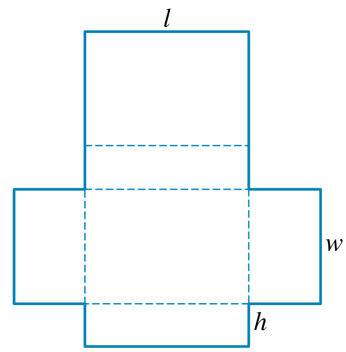
$$= 24 + 180 + 108$$
$$= 312\text{ cm}^2$$

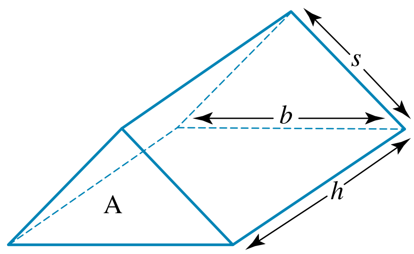
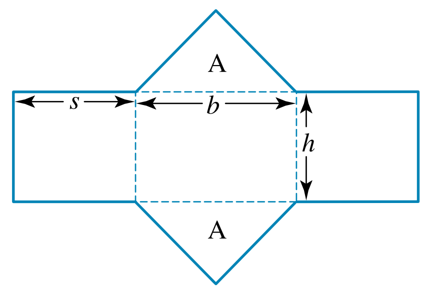
7.6.3 Surface area formulas

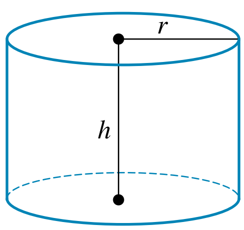
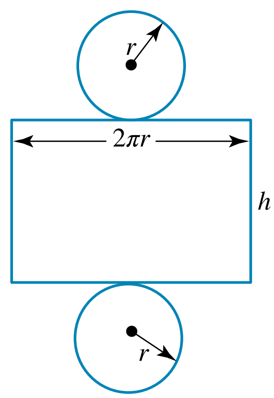
The surface area formulas for common solid objects are summarised in the following table.

Object	**Surface area**
Cube 	 SA $= 6l^2$

Continued

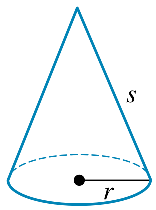
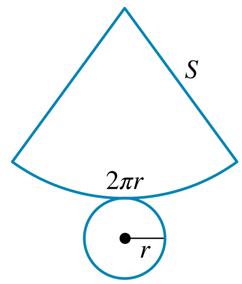
<table>
<tr><th>Object</th><th>Surface area</th></tr>
<tr><td>Rectangular prism

</td><td>

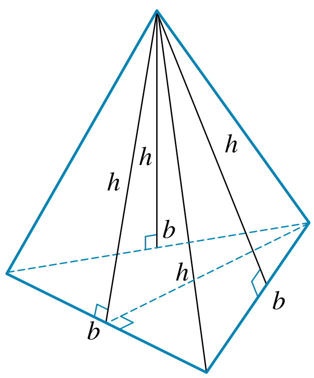
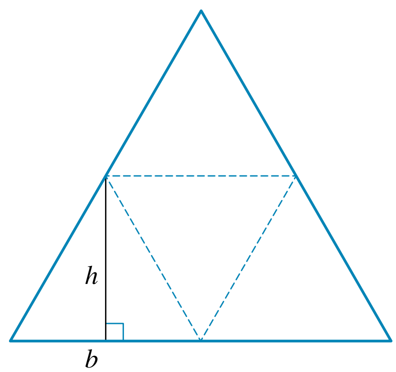
$\text{SA} = 2lw + 2lh + 2wh$</td></tr>
<tr><td>Triangular prism

</td><td>

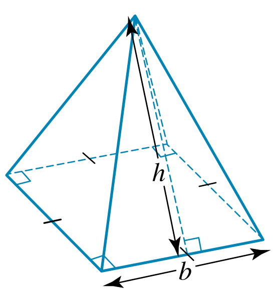
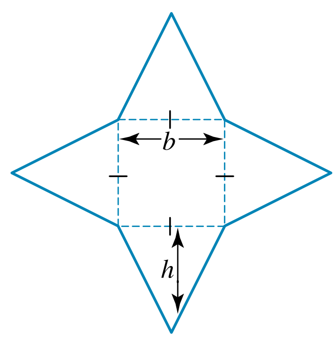
$\text{SA} = 2A + 2hs + bh$
(where A is the area of the triangular end)</td></tr>
<tr><td>Cylinder

</td><td>

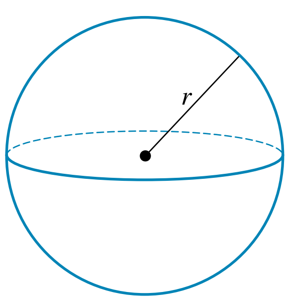
$\text{SA} = 2\pi r^2 + 2\pi rh$
$= 2\pi r(r + h)$</td></tr>
</table>

Continued

<table>
<tr><th>Object</th><th>Surface area</th></tr>
<tr><td>Cone

</td><td>

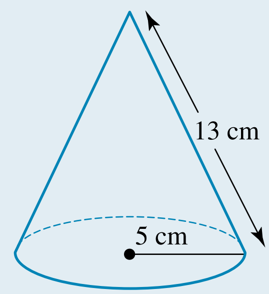
$\text{SA} = \pi rs + \pi r^2$
$= \pi r(s + r)$
(including the circular base)</td></tr>
<tr><td>Tetrahedron

</td><td>

$\text{SA} = 4 \times \left(\frac{1}{2}bh\right)$</td></tr>
<tr><td>Square pyramid

</td><td>

$\text{SA} = 4 \times \left(\frac{1}{2}bh\right) + b^2$</td></tr>
<tr><td>Sphere

</td><td>$\text{SA} = 4\pi r^2$</td></tr>
</table>

WORKED EXAMPLE 18

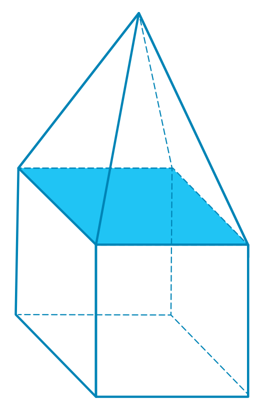
Calculate the surface area of the object shown by selecting an appropriate formula. Give your answer correct to 1 decimal place.



THINK	WRITE
1. Identify the object and the appropriate formula.	Given the object is a cone, the formula is $SA = \pi rs + \pi r^2$.
2. Substitute the given values into the formula and evaluate.	$SA = \pi rs + \pi r^2$ $= \pi r(s + r)$ $= \pi \times 5(5 + 13)$ $= \pi \times 90$ ≈ 282.7
3. State the final answer.	The surface area of the cone is 282.7 cm^2.

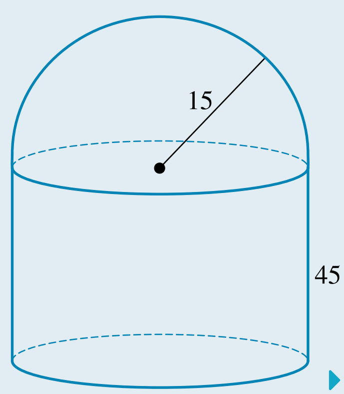
7.6.4 Surface areas of composite solids

For composite solids, be careful to include only those surfaces that form the outer part of the object. For example, if a solid consisted of a pyramid on top of a cube, the internal surface highlighted in blue would not be included.



WORKED EXAMPLE 19

Calculate the surface area of the object shown correct to 2 decimal places.



THINK	WRITE
1. Identify the components of the composite solid.	The object consists of a hemisphere that sits on top of a cylinder.
2. Substitute the given values into the formula for each surface of the object and evaluate.	Hemisphere: $SA = \frac{1}{2}(4\pi r^2)$ $= \frac{1}{2}(4 \times \pi \times 15^2)$ ≈ 1413.72 Cylinder (no top): $SA = \pi r^2 + 2\pi rh$ $= \pi \times 15^2 + 2 \times \pi \times 15 \times 45$ ≈ 4948.01
3. Add the area of each surface to obtain the total surface area.	Total surface area: $SA = 1413.72 + 4948.01$ $= 6361.73$
4. State the answer.	The total surface area of the object is 6361.73 cm^2.

on Resources

Interactivity: Surface area (int-6477)

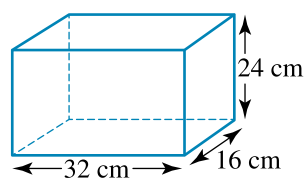
studyon

Units 1 & 2 > AOS 4 > Topic 1 > Concept 6

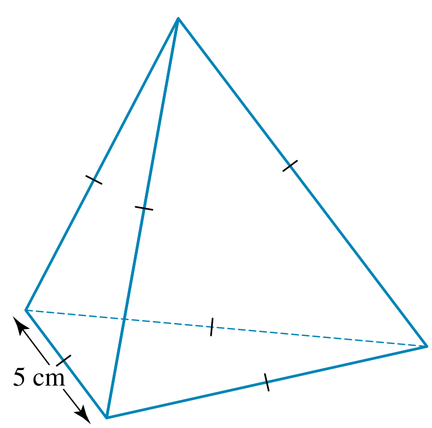
Surface area Concept summary and practice questions

Exercise 7.6 Surface area

1. **WE17** Calculate the surface area of the prism shown by first drawing its net.

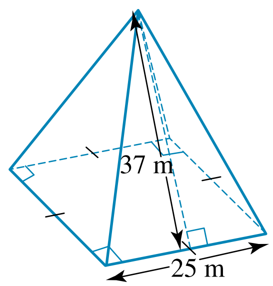


2. Calculate the surface area of the tetrahedron shown by first drawing its net.

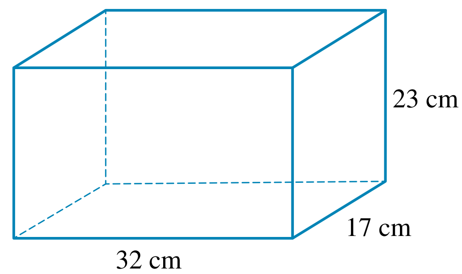


3. WE18 Calculate the surface areas of the objects shown by selecting appropriate formulas.

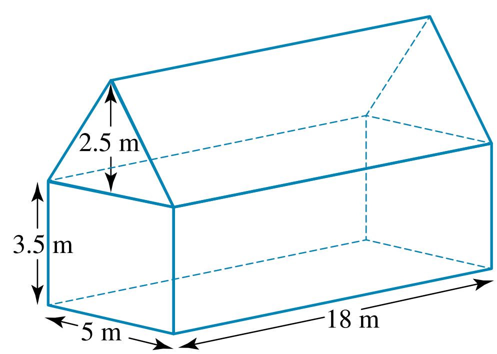
a.



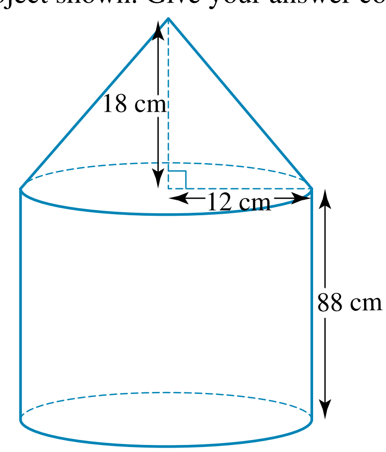
b.



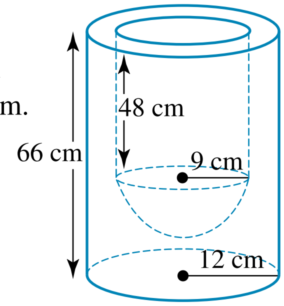
4. Calculate (correct to 2 decimal places where appropriate) the surface area of:
   a. a pyramid formed by four equilateral triangles with a side length of 12 cm
   b. a sphere with a radius of 98 cm
   c. a cylinder with a radius of 15 cm and a height of 22 cm
   d. a cone with a radius of 12.5 cm and a slant height of 27.2 cm.
5. Calculate (correct to 2 decimal places where appropriate) the total surface area of:
   a. a rectangular prism with dimensions 8 cm by 12 cm by 5 cm
   b. a cylinder with a base diameter of 18 cm and a height of 20 cm
   c. a square pyramid with a base length of 15 cm and a vertical height of 18 cm
   d. a sphere of radius 10 cm.
6. A prism is 25 cm high and has a trapezoidal base whose parallel sides are 8 cm and 12 cm long respectively, and are 10 cm apart.
   a. Draw the net of the prism.
   b. Calculate the total surface area of the prism.
7. WE19 Calculate the surface area of the object shown.



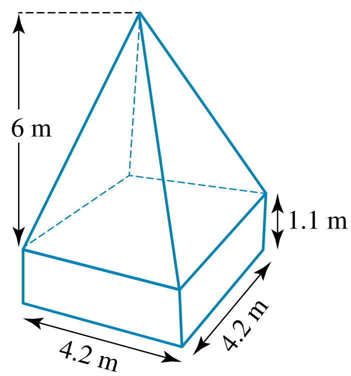
8. Calculate the surface area of the object shown. Give your answer correct to 2 decimal places.



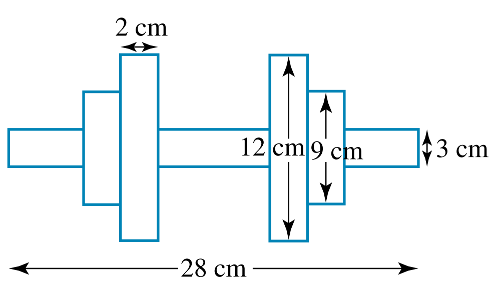
9. A hemispherical glass ornament sits on a circular base that has a radius of 5 cm.
   a. Calculate its total surface area to the nearest square centimetre.
   b. If an artist attaches it to an 8-cm-tall cylindrical stand with the same circumference, what is the new total surface area of the combined object that is created? Give your answer to the nearest square centimetre.
10. An ice-cream shop sells two types of cones. One is 6.5 cm tall with a radius of 2.2 cm. The other is 7.5 cm with a radius of 1.7 cm. By first calculating the slant height of each cone correct to 2 decimal places, determine which cone (not including any ice-cream) has the greater surface area and by how much.
11. A cylindrical plastic vase is 66 cm high and has a radius of 12 cm. The centre has been hollowed out so that there is a cylindrical space with a radius of 9 cm that goes to a depth of 48 cm and ends in a hemisphere, as shown in the diagram. Giving your answers to the nearest square centimetre:
   a. calculate the area of the external surfaces of the vase
   b. calculate the area of the internal surface of the vase.



12. The top of a church tower is in the shape of a square pyramid that sits on top of a rectangular prism base that is 1.1 m high. The pyramid is 6 m high with a base length of 4.2 m.
   Calculate the total external surface area of the top of the church tower if the base of the prism forms the ceiling of a balcony. Give your answer correct to 2 decimal places.

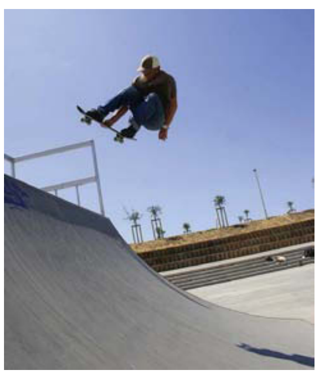


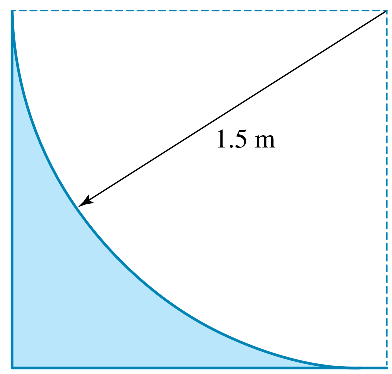
13. A dumbbell consists of a cylindrical tube that is 28 cm long with a diameter of 3 cm, and two pairs of cylindrical discs that are held in place by two locks. The larger discs have a diameter of 12 cm and a width of 2 cm, and the smaller discs are the same thickness with a diameter of 9 cm.
   Calculate the total area of the exposed surfaces of the discs when they are held in position as shown in the diagram. Give your answer to the nearest square centimetre.



14. A staircase has a section of red carpet down its centre strip. Each of the nine steps is 16 cm high, 25 cm deep and 120 cm wide. The red carpet is 80 cm wide and extends from the back of the uppermost step to a point 65 cm beyond the base of the lower step.
   a. What is the area of the red carpet?
   b. If all areas of the front and top of the stairs that are not covered by the carpet are to be painted white, what is the area to be painted?

15. A rectangular swimming pool is 12.5 m long, 4.3 m wide and 1.5 m deep. If all internal surfaces are to be tiled, calculate the total area of tiles required.
16. A quarter-pipe skateboard ramp has a curved surface that is one-quarter of a cylinder with a radius of 1.5 m. If the surface of the ramp is 2.4 m wide, calculate the total surface area of the front, back and sides.





17. Using CAS or otherwise, compare the surface areas of cones that have a slant height of 120 cm but different radii.
   a. Tabulate the results for cones with radii of 15, 30, 60, 120 and 240 cm.
   b. Graph your results.
   c. Use your graph to estimate the surface area of a cone that has a slant height of 120 cm and a radius of:
      i. 100 cm
      ii. 200 cm.
18. Using CAS or otherwise, compare the surface areas of cylinders that have a height that is twice the length of their radius.
   a. Tabulate the results for cylinders with radii of 5, 10, 15, 20, 25 and 30 cm.
   b. Graph your results.
   c. Use your graph to estimate the surface area of one of these cylinders that has a height twice the length of its radius and a radius of:
      i. 27 cm
      ii. 13 cm.

7.7 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. **MC** Which group of three numbers would be the side lengths of a right-angled triangle?

A. 6, 24, 25 **B.** 13, 14, 15 **C.** 7, 24, 25
D. 9, 12, 16 **E.** 6, 9, 12

2. **MC** If a right-angled isosceles triangle has a hypotenuse of length 32 units, the other sides will be closest to:

A. 21.54 **B.** 21.55 **C.** 22.62
D. 16 **E.** 22.63

3. **MC** An equilateral triangle with a side length of 4 units will have an altitude (height) closest to:

A. 3.46 **B.** 4.47 **C.** 4 **D.** 3.47 **E.** 4.48

4. **MC** A trapezium has a height of 8 cm and an area of 148 cm^2. Its parallel sides could be:

A. 11 cm and 27 cm **B.** 12 cm and 24 cm **C.** 12 cm and 26 cm
D. 12 cm and 25 cm **E.** 11 cm and 25 cm

5. **MC** A circle has a circumference of 75.4 cm. Its area is closest to:

A. 440 cm^2 **B.** 475 cm^2 **C.** 461 cm^2 **D.** 448 cm^2 **E.** 452 cm^2

6. **MC** A cylinder with a volume of 1570 cm^3 and a height of 20 cm will have a diameter that is closest to:

A. 5 cm **B.** 12 cm **C.** 15 cm **D.** 10 cm **E.** 18 cm

7. **MC** A cone with a surface area of 2713 cm^2 and a diameter of 24 cm will have a slant-height that is closest to:

A. 58 cm **B.** 48 cm **C.** 60 cm **D.** 46 cm **E.** 64 cm

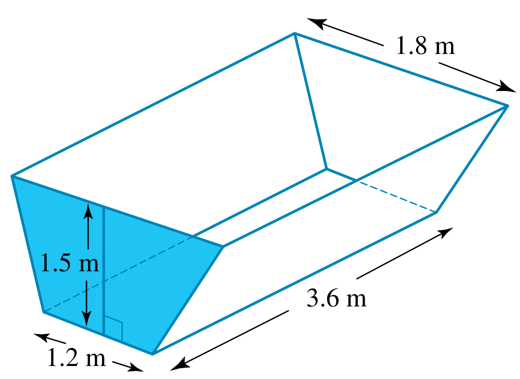
8. **MC** A hemisphere with a radius of 22.5 cm will have a volume and total surface area respectively that are closest to:

A. 47 689 cm^3 and 4764 cm^2 **B.** 47 689 cm^3 and 6358 cm^2
C. 23 845 cm^3 and 6359 cm^2 **D.** 23 856 cm^3 and 4771 cm^2
E. 47 688 cm^3 and 6359 cm^2

9. **MC** A square pyramid with a volume of 500 cm^3 and a vertical height of 15 cm will have a surface area that is closest to:

A. 416 cm^2 **B.** 492 cm^2 **C.** 359 cm^2 **D.** 316 cm^2 **E.** 635 cm^2

10. **MC** An open rubbish skip-container is in the shape of a trapezoidal prism with the dimensions indicated in the diagram.

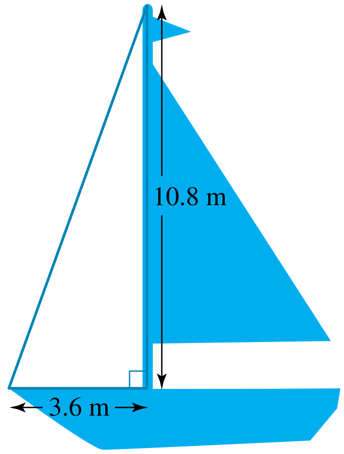


The surface area (m^2) and volume (m^3) respectively are:

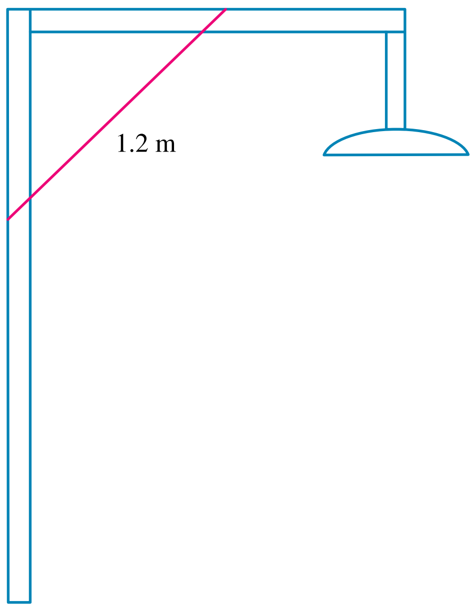
A. 8.1 and 19.8 **B.** 19.8 and 8.1 **C.** 17.75 and 8.1 **D.** 8.1 and 26.3 **E.** 26.3 and 8.1

Short answer

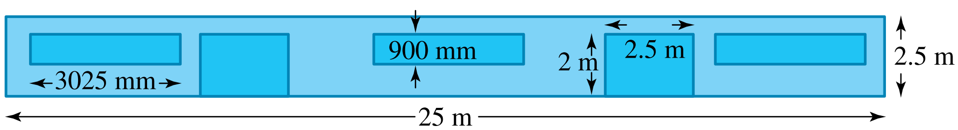
1. Calculate the length of wire required to support the mast of a yacht if the mast is 10.8 m long and the support wire is attached to the horizontal deck at a point 3.6 m from the base of the mast.



2. The supporting strut of a streetlight must be attached so that its ends are an equal distance from the top of the pole. If the strut is 1.2 m long, how far are the ends from the top of the pole?



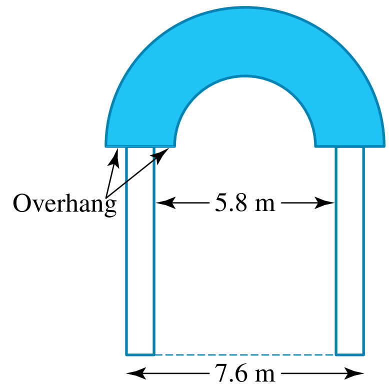
3. A surveyor is measuring a building site and wants to check that the guidelines for the foundations are square (i.e. at right angles). He places a marker 3600 mm from a corner along one line, and another marker 4800 mm from the corner along the other line. How far apart must the markers be for the lines to be square?
4. The side pieces of train carriages are made from rectangular sheets of pressed metal of length 25 metres and height 2.5 metres. Rectangular sections for the doors and windows are cut out. The dimensions of the spaces for the doors are 2 metres high by 2.5 metres wide. The window spaces are 3025 millimetres wide by 900 millimetres high. Each sheet must have spaces cut for two doors and three windows.



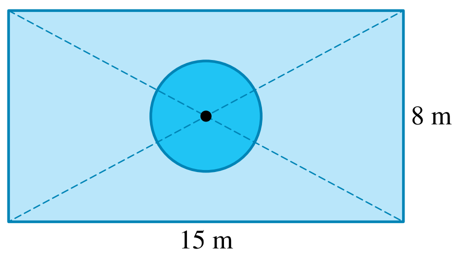
   a. Calculate the total area of pressed metal that remains once the sections for the doors and windows have been removed.

b. A thin edging strip is placed around each window and around the top and sides of the door opening. What is the total length of edging required?

5. A semicircular arch sits on two columns as shown in the diagram below. The outer edges of the columns are 7.6 metres apart and the inner edges are 5.8 metres apart. The width of each column is three-quarters the width of the arch, and the arch overhangs the columns by one-eighth of its width on each edge. The face of the arch (the shaded area) is to be tiled. What area will the tiles cover?



6. A circular pond is placed in the middle of a rectangular garden that is 15 m by 8 m.

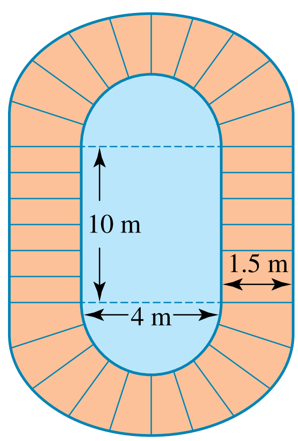


If the radius of the pond is a quarter of the distance from the centre to the corner of the garden, calculate:

a. the circumference of the pond
b. the area of the garden, not including the pond
c. the volume of water in the pond in litres if it is filled to a depth of 850 mm.

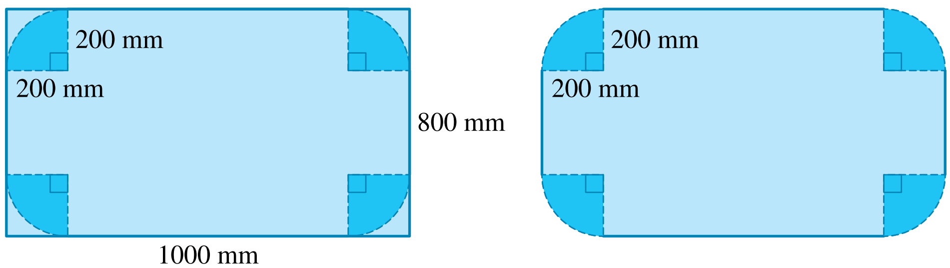
Extended response

1. When viewed from above, a swimming pool can be seen as a rectangle with a semicircle at each end, as shown in the diagram below. The area around the outside of the pool extending 1.5 metres from the edge is to be paved.

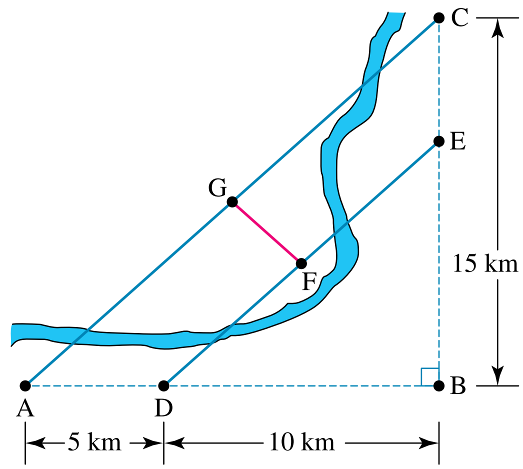


a. Calculate the paved area around the pool.
b. If the pool is to be filled to a depth of 900 mm in the semicircular sections and 1500 mm in the rectangular section, what is the total volume of water in the pool to the nearest litre?

2. A rectangular piece of glass with side lengths 1000 mm and 800 mm has its corners removed for safety, as shown in the diagrams below.
   a. Calculate the surface area of the glass after the corners have been removed.
   b. Calculate the perimeter of the glass after the corners have been removed.

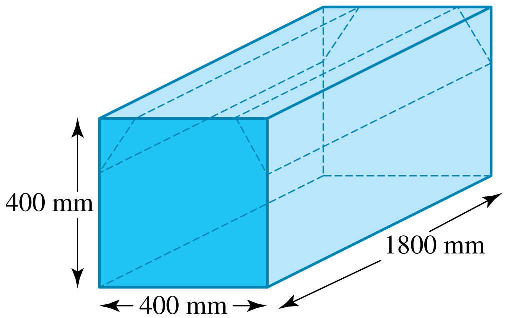


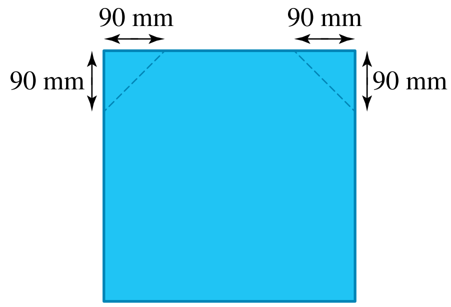
3. Two tunnels run under a bend in a river. One runs in a straight line from point A to point C, and the other runs in a straight line from point D to point E. Points A and D are 5 km apart, as are points C and E. Point B is 10 km from both D and E, and BD is perpendicular to BE. An access tunnel GF is to be constructed between the midpoints of AC and DE.



   a. Calculate the lengths of AC and DE correct to 2 decimal places.
   b. Calculate the length of GF correct to 2 decimal places.
   c. The tunnels running from A to C and from D to E are cylindrical in shape with an outer diameter of 40 metres. Calculate the volume of material that was removed to create the two tunnels.
   d. The trucks used to remove the excavated material during the construction of the tunnels can carry a maximum of 85 m^3. How many truck loads were required to make the two tunnels? Round your answer correct to the nearest whole number.
   e. The inner walls of the tunnels are formed of concrete that is 3.5 metres thick. Calculate the total volume of concrete used for the tunnels, correct to 1 decimal place.
   f. The inner surface of the concrete in the tunnels is sprayed with a sealant to prevent water seeping through. Calculate the total area that is sprayed with the sealant, correct to 1 decimal place.

4. A piece of timber has the dimensions 400 mm by 400 mm by 1800 mm. The top corners of the piece of timber are removed along its length. The cuts are made at an angle a distance of 900 mm from the corners, so the timber that is removed forms two triangular prisms.





a. Calculate the volume of the piece of timber in cubic metres after the corners are removed. Give your answer correct to 3 decimal places.
b. Calculate the surface area of the piece of timber after the corners are removed.
c. What is the total volume and surface area of the two smaller pieces of timber that are cut from the corners, assuming they each remain as one piece?

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Units 1&2 Sit topic test

7 Answers

Topic 7 Shape and measurement

Exercise 7.2 Pythagoras' theorem

1. a. 10.8 b. 28.7
2. $40^2 + 96^2 = 104^2$
3. a. 6.71 b. 13.90 c. 7.40
   d. 11.67 e. 38.01 f. 33.96
4. a. 87.6 b. 31.1
5. $24^2 - 19.2^2 = 14.4^2$
6. a. 14.51 b. 14.50 c. 44.50
   d. 7.99 e. 8.00 f. 1.69
7. a. $24^2 + 32^2 = 40^2$ b. $180^2 + 75^2 = 195^2$
8. $1.5^2 + 2^2 = 2.5^2$
9. a. The triangle is right-angled. b. The triangle is not right-angled.
   c. The triangle is right-angled. d. The triangle is not right-angled.
10. 11.40
11. a. 9.43, 6.24 b. 73.43, 52.92 c. 25.81, 14.70
    d. 47.38, 8.19 e. 11.66, 8.00 f. 39.00, 32.73
12. 1.56 m
13. No, the maximum length rod that could fit would be 2.71 m long.
14. a. 14.14 cm b. 33.94 cm
    c. 3.89 cm d. 117.38 cm
15. 41.4 m
16. Pyramid 1 has the greatest height.
17. a. 30.48 cm b. 2.61 cm c. 47.27 cm
18. a. Yes b. 1.015 m
19. 8.09
20. 69.46 m
21. a. 14.70 m b. 14.98 m c. 13.82 m
22. a. 10 050 m b. 6083 m c. 10 054 m

Exercise 7.3 Perimeter and area I

1. Perimeter = 80 cm, area = 360 cm^2
2. Circumference = 50.27 cm, area = 201.06 cm^2
3. a. Perimeter = 59 m, area = 155.12 m^2
   b. Perimeter = 28.83 cm, area = 20 cm^2
   c. Perimeter = 43.98 cm, area = 153.94 cm^2
   d. Perimeter = 48 cm, area = 112 cm^2
4. a. Circumference = 31.42 cm, area = 78.54 cm^2
   b. Circumference = 56.55 cm, area = 254.47 cm^2
5. Perimeter = 68 cm, area = 192 cm^2
6. 33.79 cm^2
7. Area = 1.14 m^2
8. Perimeter = 96 units, area = 360 $units^2$
9. 31.61 cm
10. 47.9 mm^2
11. Triangle 1 has the largest area.
12. The area of the park is 313.8 m^2, so it is not of a suitable size to host the barbecue.
13. 49.19 cm
14. a. i. 201.06 cm ii. 389.56 cm^2
    b. 187.19 cm
15. a. 26.39 m b. 1104.73 m^2
16. a. 173.21 m^2 b. Perimeter = 90 m, area = 389.71 m^2
    c. Perimeter = 120 m, area = 839.7 m^2

Exercise 7.4 Perimeter and area II

1. 72 cm^2
2. Perimeter = 48.6 cm, area = 106 cm^2
3. 198.94 cm^2
4. a. 5.57 m b. 6.11 m^2
5. a. 5831.62 cm^2 b. 56.58 cm^2
6. 4.24 cm^2
7. Perimeter = 52.94 cm, area = 167.55 cm^2
8. Perimeter = 198.23 cm, area = 2073.45 cm^2
9. Perimeter = 145.40 cm, area = 1143.06 cm^2
10. Perimeter = 197 cm, area = 1571 cm^2
11. 9.42 cm^2
12. 25.13 cm^2
13. 9292.83 cm^2
14. Perimeter = 151.76 cm, area = 576 cm^2
15. 2.50 m^2
16. 0.67 m^2
17. 2138.25 cm^2
18. Area = 900 cm^2, perimeter = 194.16 cm

Exercise 7.5 Volume

1. 0.6 m^3
2. a. 20 500 cm^3 b. 13 415 cm^3
   c. 126 720 cm^3 d. 51 188 cm^3
3. Rectangular pool: 11 070 litres
   Square pool: 10 800 litres
   Circular pool: 10 263.58 litres
4. a. 25.2 m^3 b. 7.2 m^3
5. 56 301 cm^3
6. a. 490 376 cm^3 b. 136 m^3
7. Volume above = 162 578 cm^3, volume below = 318 086 cm^3

8. a. 34.64 litres b. 0.231 m^3

9. 39 584.1 cm^3

10. a. 753 218 cm^3 b. 0.022 cm^3

11. a. 14 850 cm^3 b. 471 250 cm^3 c. 45 804.4 cm^3 d. 42 524.6 cm^3

12. 0.0528 m^3

13. a. 22 875 cm^3 b. 15 675.12 cm^3 c. 0.432 m^3

14. 6.25 m^3

15. 0.0824 m^3

16. a. 39 cm b. 1 m

17. 21.91 m^3

18. a. 57 800 cm^3 b. 68 361.05 cm^3

19. The hemispherical – topped silo holds 37.35 m^3 more.

20. 19 738.52 litres

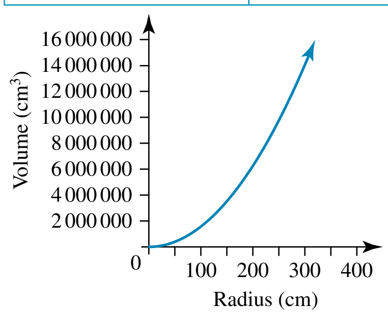
21. a. 8983.3 m^3 b. 332 716 049.4 cm^3

22. a. 467.35 cm^3 b. 9677.11 cm^3

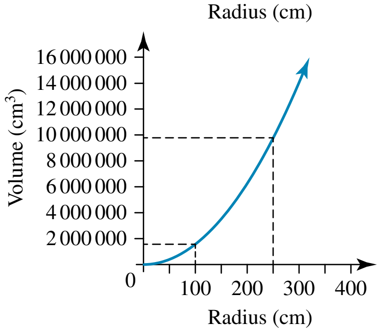
23. a.

Cylinder radius (cm)	Volume (cm^3)
10	15 708
20	62 832
40	251 327
80	1 005 310
160	4 021 239
320	16 084 954

b.



c.

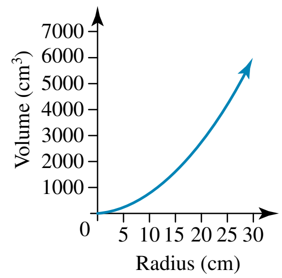


i. Approximately 1 600 000 cm^3 ii. Approximately 9 800 000 cm^3

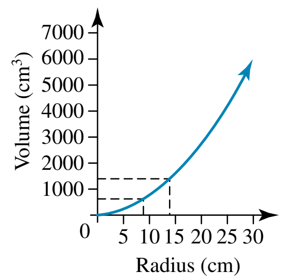
24. a.

Base length (cm)	Volume (cm^3)
5	166.7
10	666.7
15	1500.0
20	2666.7
25	4166.7
30	6000.0

b.



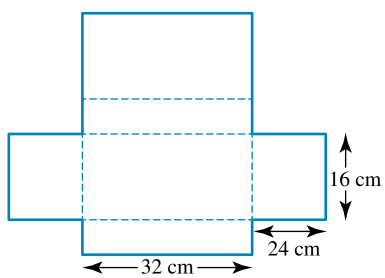
c.



i. Approximately 550 cm^3 ii. Approximately 1300 cm^3

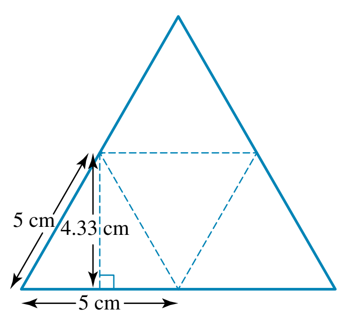
Exercise 7.6 Surface area

1.



3328 cm^2

2.



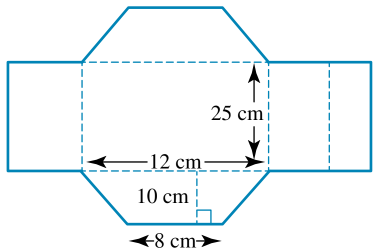
43.3 cm^2

3. a. $2475\ m^2$ b. $3342\ cm^2$

4. a. $249.42\ cm^2$ b. $120\,687.42\ cm^2$
c. $3487.17\ cm^2$ d. $1559.02\ cm^2$

5. a. $392\ cm^2$ b. $1639.91\ cm^2$
c. $810\ cm^2$ d. $1256.64\ cm^2$

6. a.



b. $1210\ cm^2$

7. $390.94\ m^2$

8. $7902.86\ cm^2$

9. a. $236\ cm^2$ b. $487\ cm^2$

10. The cone with height 6.5 cm and radius 2.2 cm has the greater surface area by $6.34\ cm^2$.

11. a. $5627\ cm^2$ b. $4241\ cm^2$

12. $71.90\ m^2$

13. $688\ cm^2$

14. a. $34\,320\ cm^2$ b. $14\,760\ cm^2$

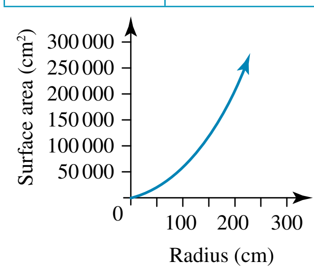
15. $104.15\ m^2$

16. $10.22\ m^2$

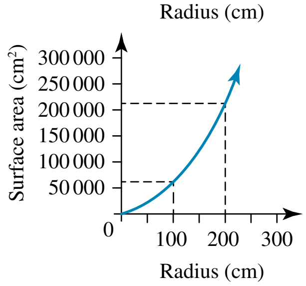
17. a.

Radius (cm)	Surface area (cm^2)
15	6361.7
30	14 1437.2
60	33 929.2
120	90 477.9
240	271 433.6

b.



c.

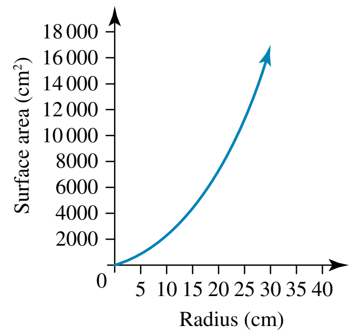


i. Approximately $60\,000\ cm^2$ ii. Approximately $210\,000\ cm^2$

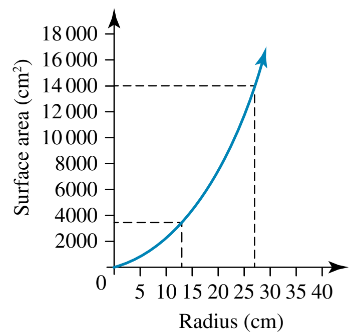
18. a.

Radius (cm)	Surface area (cm^2)
5	471.24
10	1884.96
15	4241.15
20	7539.82
25	11 780.97
30	16 964.60

b.



c.



i. Approximately $14\,000\ cm^2$ ii. Approximately $3200\ cm^2$

7.7 Review: exam practice

Multiple choice

1. C 2. E 3. A 4. D 5. E
6. D 7. C 8. D 9. A 10. B

Short answer

1. 11.38 m 2. 0.849 m 3. 6000 mm

4. a. $44.33\ m^2$ b. 36.55 m

5. $12.63\ m^2$

6. a. 13.352 m b. $105.8\ m^2$ c. 12 058.32 L

Extended response

1. a. $55.92\ m^2$ b. 71 309.7 L

2. a. $765\,663.7\ mm^2$ b. 3256.6 mm

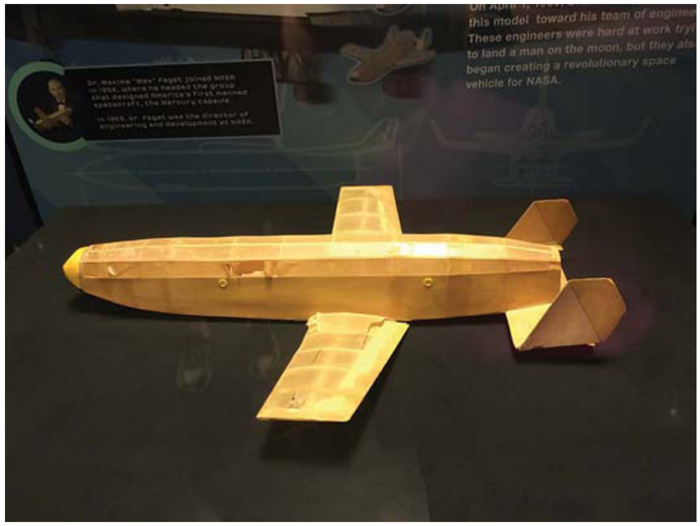
3. a. 21.21 km and 14.14 km b. 3.54 km
c. $44\,422\,120\ m^3$ d. 522 613
e. $14\,187\,314.6\ m^3$ f. $3\,664\,824.9\ m^2$

4. a. $0.273\ m^3$
b. $2\,994\,005\ mm^2$
c. Volume = $14\,580\,000\ mm^3$; surface area = $1\,122\,405\ mm^2$

TOPIC 8
Similarity

8.1 Overview

8.1.1 Introduction



The construction industry is constantly growing. You only need to look at the cranes on buildings in any major city in the world. Some of these buildings are used as homes. They are also hospitals, schools, offices and places of leisure.

To construct buildings and make sure they are functional, designers must apply the principles of mathematics. These principles involve scaled drawings in the form of plans. A plan is a drawing of the object to be built, reduced in size in a way that all the measurements correspond to the actual object.

Architects will often use a different set of scales than engineers, surveyors or furniture designers. This depends on the size of what is being designed, as well as the complexity of the design.

Scale is not just used for plans; it can also be used to create a scale model of the design. A scale model is generally a physical representation of an object that maintains accurate relationships between all important aspects of the model. The scale model allows you to see some behaviour of the original object without investigating the original object itself. Scale models are used in many fields including engineering, filmmaking, military, sales and hobby model building. To be considered a true scale model, all important aspects must be accurately modelled: not just the scale of the object, but also the material properties. An example could be an aerospace company wanting to test a new wing design. They could construct a scaled-down model and test it in a wind tunnel under simulated conditions. The scale model above is at the Kennedy Space Centre in Orlando, USA.

One famous model is that of a space shuttle by Dr Maxime Faget from NASA. He needed a model to demonstrate to his colleagues that a space shuttle would be able to glide back to Earth without power — a concept unknown until then.

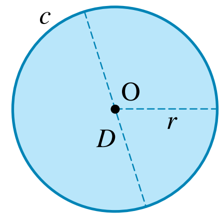
LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

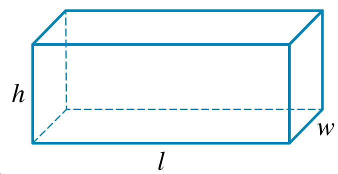
8.1.2 Kick off with CAS

Linear, area and volume scale factors

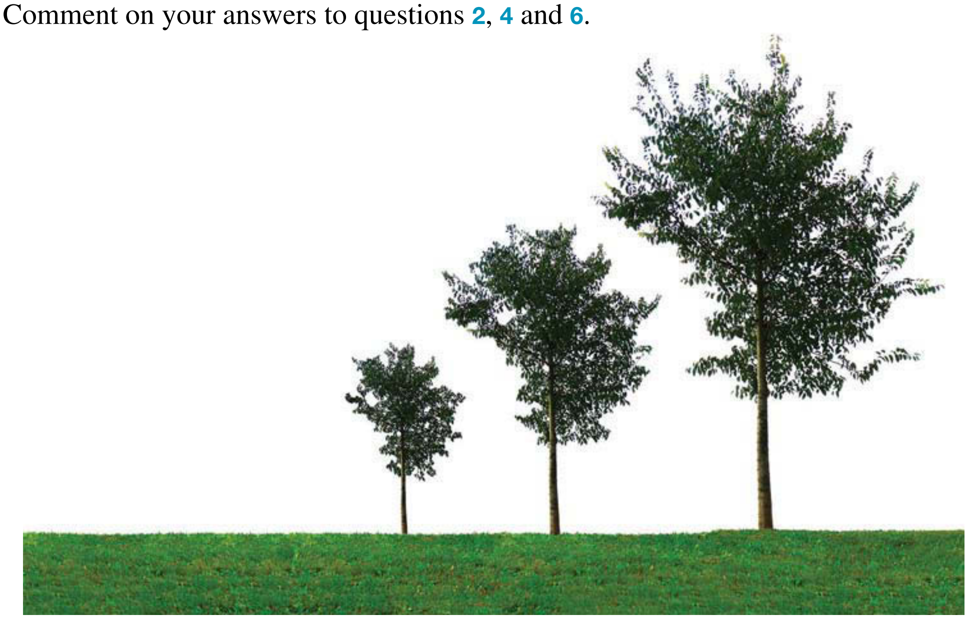
The area of a circle is given by the formula $A = \pi r^2$, where r is the length of the radius.



1. a. Using CAS, define and save the formula to calculate the area of a circle.
   b. Use the formula defined in part **a** to calculate the area of a circle with a radius of 7 cm.
2. If the radius in **1b** increased by a scale factor of 2 (i.e. was multiplied by 2), use your formula from **1a** to determine the scale factor the area of the circle has increased by.
   The area of a triangle is given by the formula $A = \frac{1}{2}bh$, where b is the length of the base and h is the perpendicular height.
3. a. Using CAS, define and save the formula to calculate the area of a triangle.
   b. Use the formula defined in part **a** to calculate the area of a triangle with a base length of 6 cm and perpendicular height of 9 cm.
4. If the base length and perpendicular height in **3b** both increased by a scale factor of 2 (i.e. were multiplied by 2), use your formula from **3a** to determine the scale factor the area of the triangle has increased by.
   The volume of a rectangular prism is given by the formula $V = lwh$, where l is the length, w is the width and h is the height.



5. a. Using CAS, define and save the formula to volume of a rectangular prism.
   b. Use the formula defined in part **a** to calculate the volume of a rectangular prism with a length of 3 cm, a width of 5 cm and a height of 7 cm.
6. If the length, width and height in **5b** all increased by a scale factor of 2 (i.e. were multiplied by 2), use your formula from **5a** to determine the scale factor the volume of the rectangular prism has increased by.
7. Comment on your answers to questions **2**, **4** and **6**.



on Resources

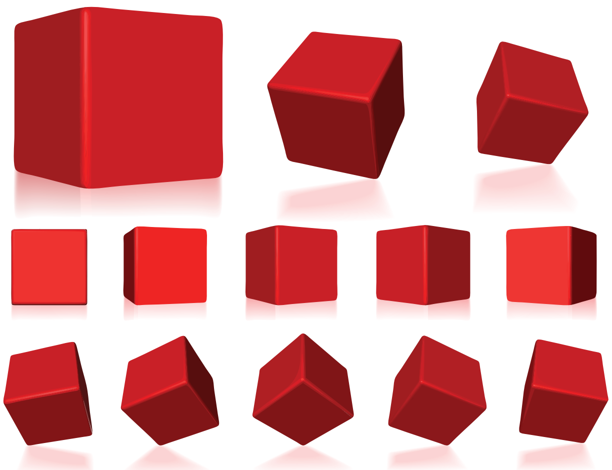
Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

8.2 Similar objects

8.2.1 Similar objects

Objects are called **similar** when they are exactly the same shape but have different sizes. Objects that are exactly the same size and shape are called **congruent**.

Similarity is an important mathematical concept that is often used for planning purposes in areas such as engineering, architecture and design. Scaled-down versions of much larger objects allow designs to be trialled and tested before their construction.

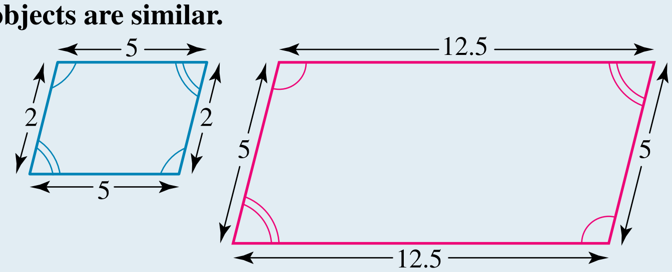


8.2.2 Conditions for similarity

Two-dimensional objects are similar when their internal angles are the same and their side lengths are **proportional**. This means that the ratios of corresponding side lengths are always equal for similar objects. We use the symbols ~ or ||| to indicate that objects are similar.

WORKED EXAMPLE 1

Show that these two objects are similar.



THINK	WRITE
1. Confirm that the internal angles for the objects are the same.	The diagrams indicate that all angles in both objects are equal.
2. Calculate the ratio of the corresponding side lengths and simplify.	Ratios of corresponding sides: $\frac{12.5}{5} = 2.5$ and $\frac{5}{2} = 2.5$
3. State the answer.	The two objects are similar as their angles are equal and the ratios of the corresponding side lengths are equal.

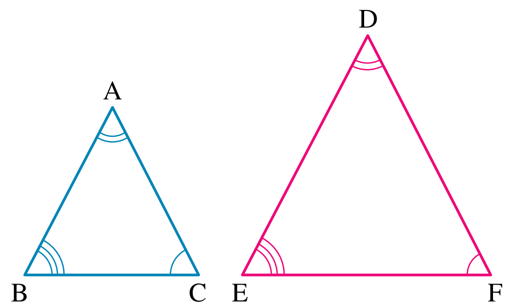
8.2.3 Similar triangles

The conditions for similarity apply to all objects, but not all of them need to be known in order to demonstrate similarity in triangles. If pairs of triangles have any of the following conditions in common, they are similar.

1. Angle–angle–angle (AAA)

If two different-sized triangles have all three angles identified as being equal, they will be similar.

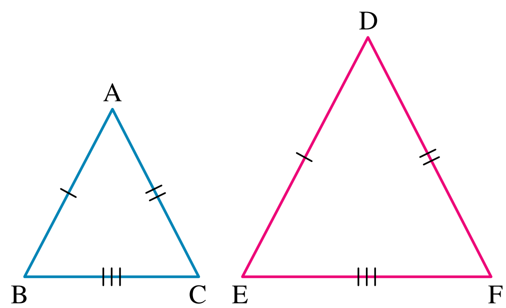
$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$$



2. Side–side–side (SSS)

If two different-sized triangles have all three sides identified as being in proportion, they will be similar.

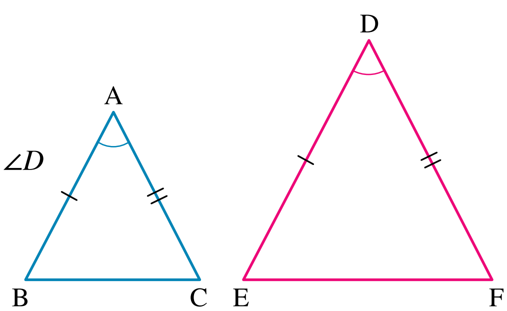
$$\frac{DE}{AB} = \frac{DF}{AC} = \frac{EF}{BC}$$



3. Side–angle–side (SAS)

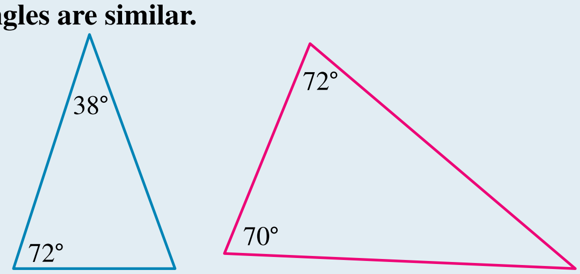
If two different-sized triangles have two pairs of sides identified as being in proportion and their included angles are equal, they will be similar.

$$\frac{DE}{AB} = \frac{DF}{AC} \text{ and } \angle A = \angle D$$



WORKED EXAMPLE 2

Show that these two triangles are similar.



THINK	WRITE
1. Identify all possible angles and side lengths.	The angles in the blue triangle are: 38°, 72° and $180 - (38 + 72) = 70°$. The angles in the red triangle are: 70°, 72° and $180 - (70 + 72) = 38°$
2. Use one of AAA, SSS or SAS to check for similarity.	All three angles in the two triangles are equal.
3. State the answer.	The two triangles are similar as they satisfy the condition AAA.

Resources

Interactivity: Similar triangles (int-6273)

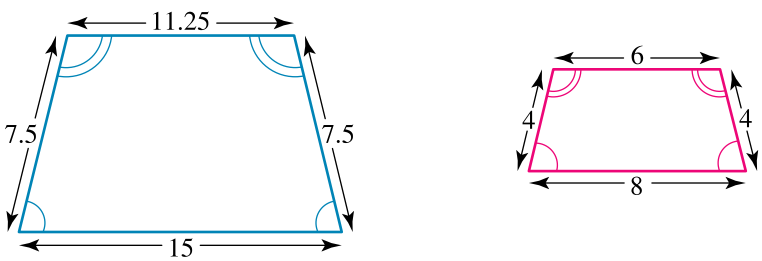
studyon

Units 1 & 2 > AOS 4 > Topic 2 > Concept 1 > **Similar figures** Concept summary and practice questions

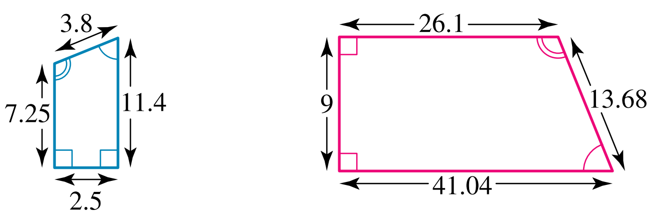
Exercise 8.2 Similar objects

1. **WE1** Show that the two objects in each of the following pairs are similar.

a.

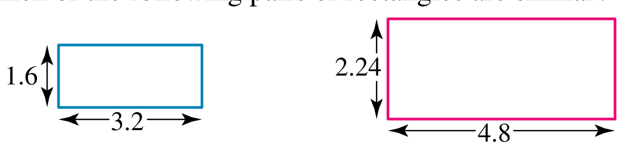


b.

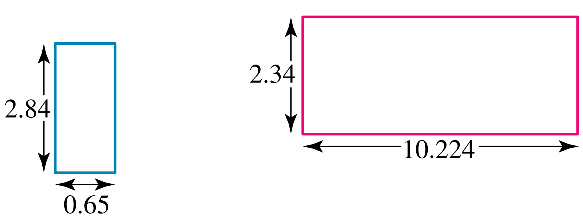


2. Show that a rectangle with side lengths of 4.25 cm and 18.35 cm will be similar to one with side lengths of 106.43 cm and 24.65 cm.
3. Which of the following pairs of rectangles are similar?

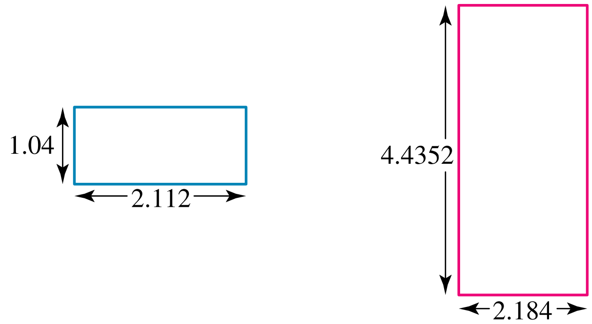
a.



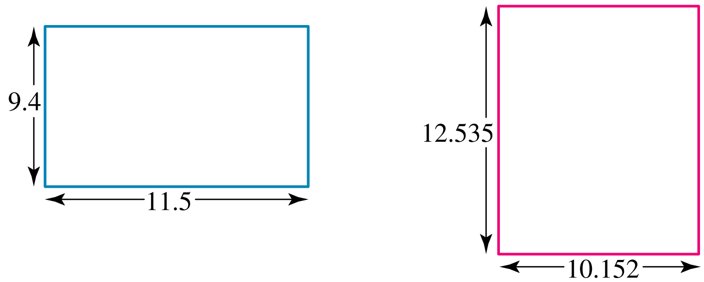
b.



c.

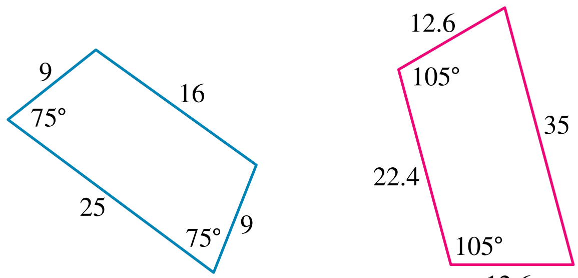


d.

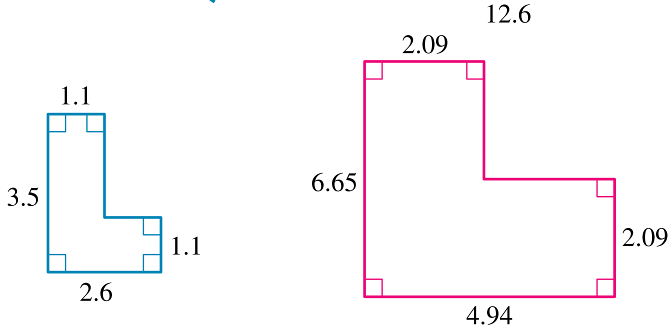


4. Which of the following pairs of polygons are similar?

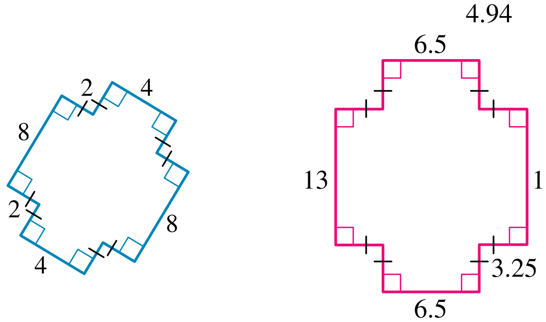
a.



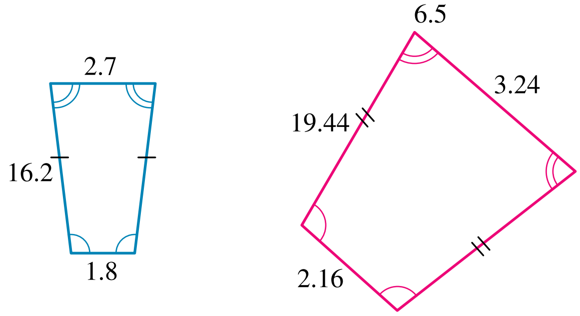
b.



c.

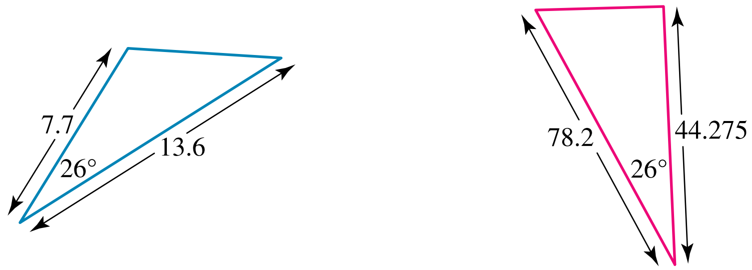


d.

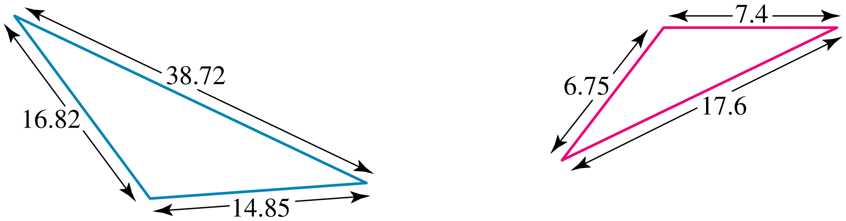


5. **WE2** Show that the two triangles in each of the following pairs are similar.

a.



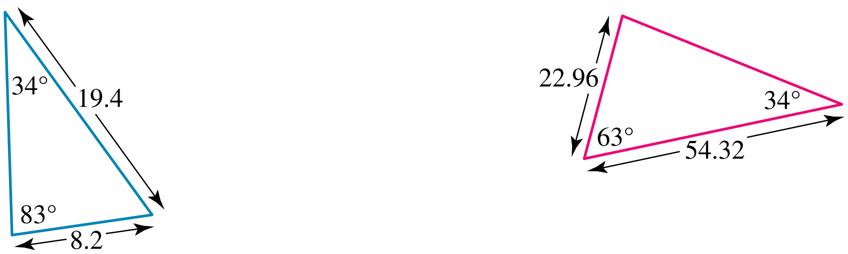
b.



6. Which of the following pairs of triangles are similar?

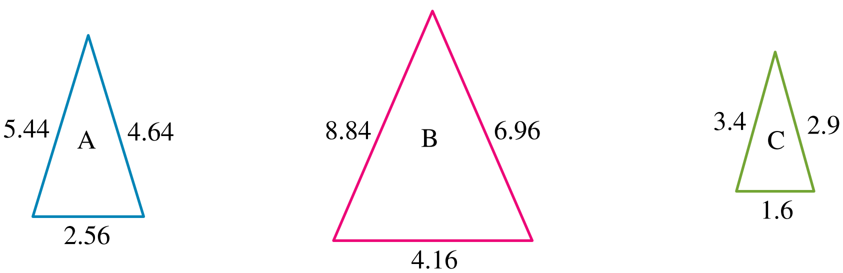
a.

b.

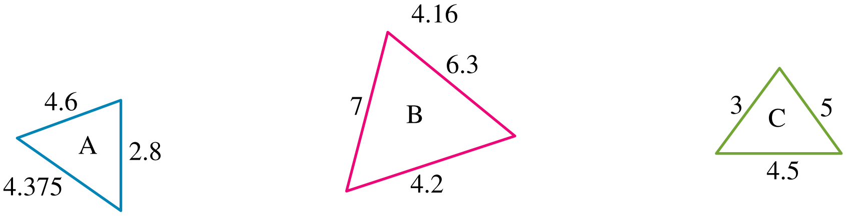


7. In each of the following groups, which two triangles are similar?

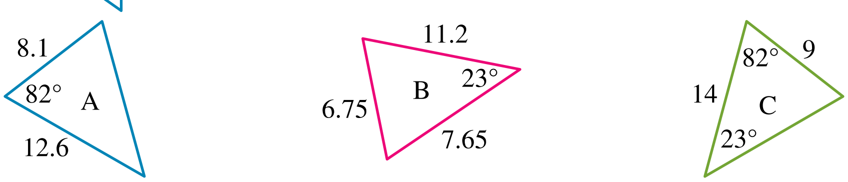
a.



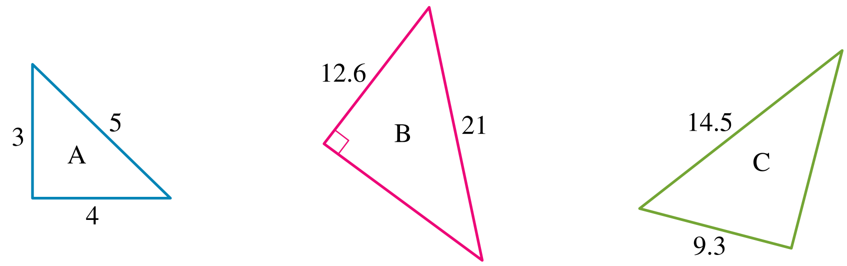
b.



c.



d.



8. Explain why each of the following pairs of objects must be similar.

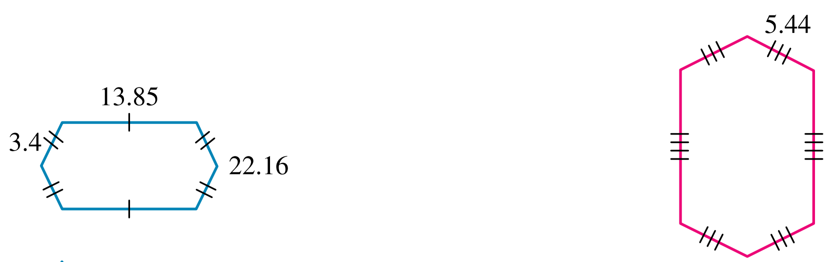
a.

b.

c.

9. Calculate the ratios of the corresponding sides for the following pairs of objects.

a.

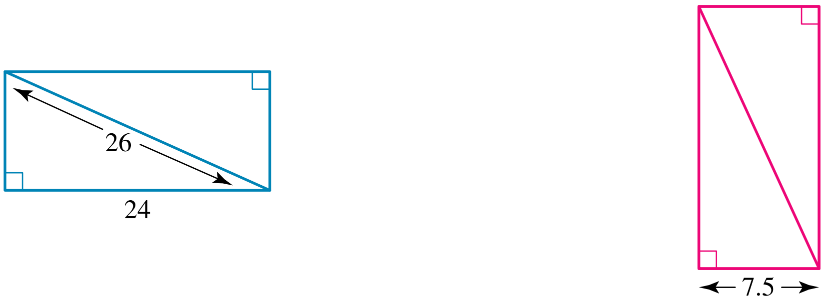


b.

c.

10. Evaluate the ratios of the corresponding side lengths in the following pairs of similar objects.

a.



b.

11. Evaluate the unknown side lengths in the following pairs of similar objects.

a.

b.

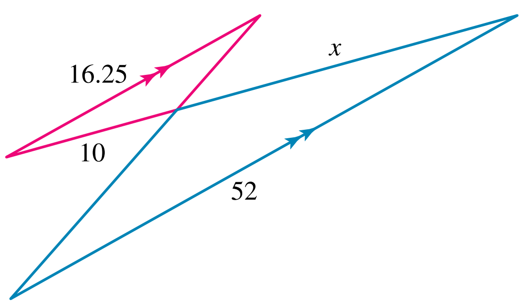
12. Verify that the following are similar.

a. A square of side length 8.2 cm and a square of side length 50.84 cm

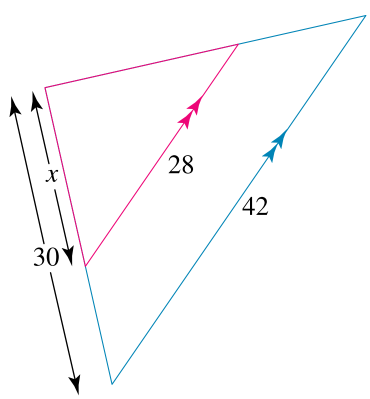
b. An equilateral triangle of side length 12.6 cm and an equilateral triangle of side length 14.34 cm

13. Calculate the value of x required to make the pairs of objects similar in each of the following diagrams.

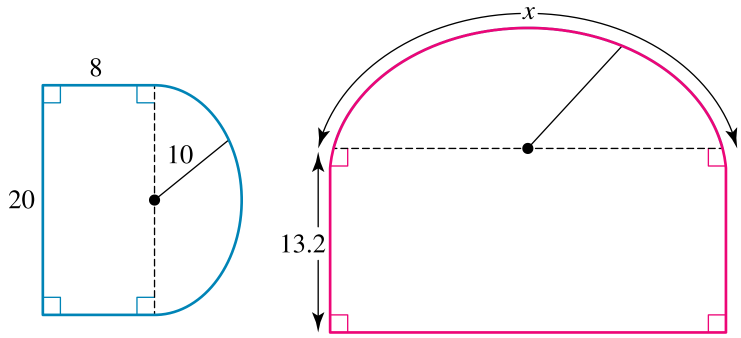
a.



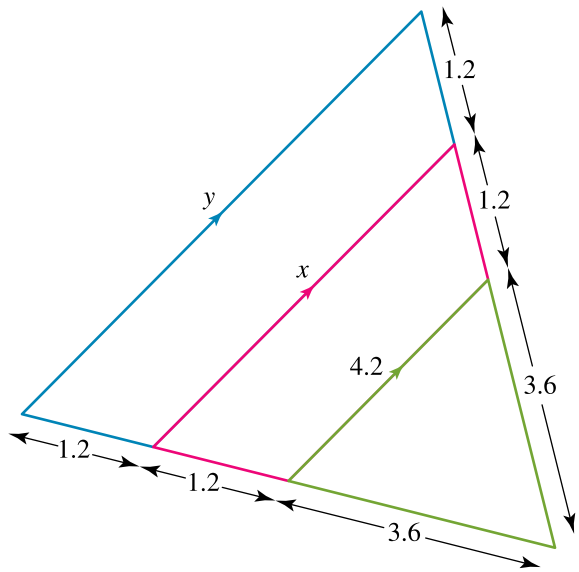
b.



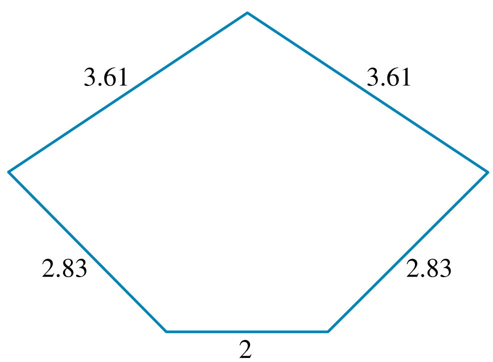
14. Calculate the value of x for the following similar shapes. Give your answer correct to 1 decimal place.



15. Calculate the values of x and y in the diagram.



16. For the polygon shown, draw and label a similar polygon where:



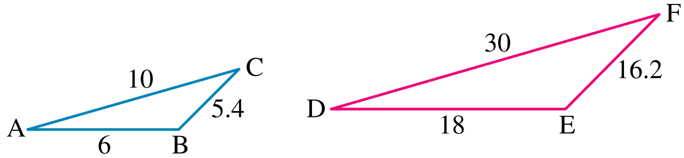
a. the corresponding sides are $\frac{4}{3}$ the size of those shown

b. the corresponding sides are $\frac{4}{5}$ the size of those shown.

8.3 Linear scale factors

8.3.1 Linear scale factors

Consider the pair of similar triangles shown in the diagram:



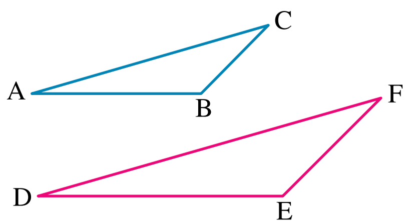
The ratios of the corresponding side lengths are:

$$\begin{aligned} DE : AB &= 18 : 6 \\ &= 3 : 1 \\ EF : BC &= 16.2 : 5.4 \\ &= 3 : 1 \\ DF : AC &= 30 : 10 \\ &= 3 : 1 \end{aligned}$$

Note: In this topic we will put the image first when calculating ratios of corresponding lengths.

In fact, the side lengths of triangle DEF are all three times the lengths of triangle ABC. In this case we would say that the **linear scale factor** is 3. The linear scale factor for similar objects can be evaluated using the ratio of the corresponding side lengths.

$\Delta ABC \sim \Delta DEF$ Linear scale factor: $\dfrac{DE}{AB} = \dfrac{EF}{BC} = \dfrac{DF}{AC} = k$

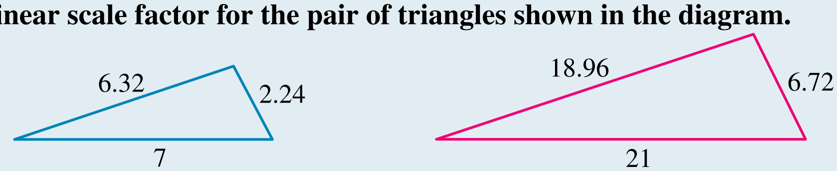


$$\textbf{Linear scale factor} = \frac{\textbf{length of image}}{\textbf{length of object}}$$

A linear scale factor greater than 1 indicates enlargement, and a linear scale factor less than 1 indicates reduction.

WORKED EXAMPLE 3

Calculate the linear scale factor for the pair of triangles shown in the diagram.



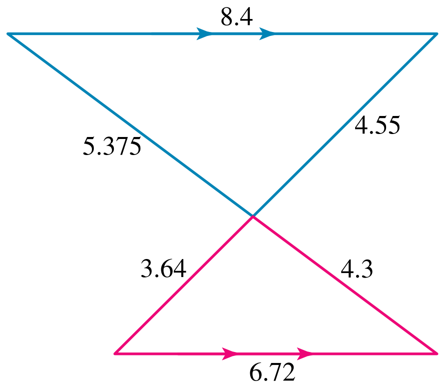
THINK	WRITE
1. Calculate the ratio of the corresponding side lengths and simplify.	$\dfrac{21}{7} = \dfrac{18.96}{6.32} = \dfrac{6.72}{2.24} = 3$
2. State the answer.	The linear scale factor is 3.

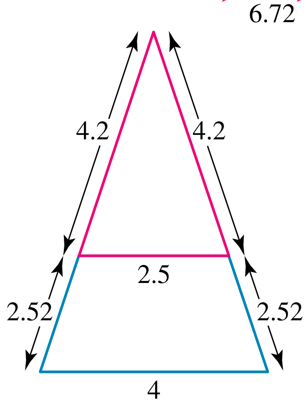
Exercise 8.3 Linear scale factors

1. **WE3** Calculate the linear scale factor for the pairs of triangles shown.

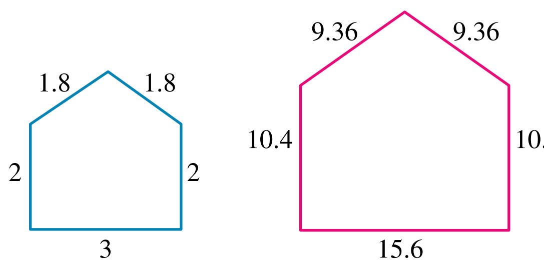
a. 

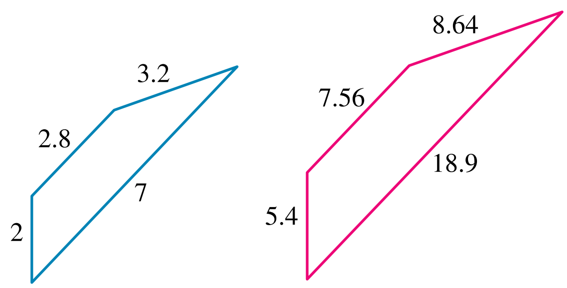
b. 

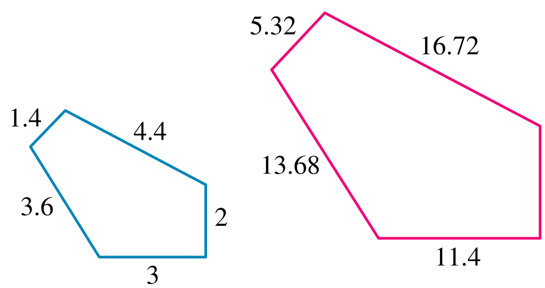
c. 

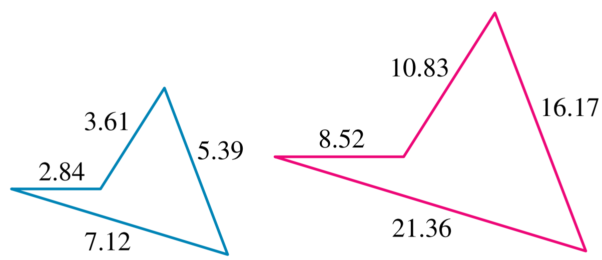
d. 

2. Calculate the linear scale factors for the pairs of similar objects shown.

a. 

b. 

c. 

d. 

3. Calculate the linear scale factors for the following ratios of corresponding side lengths.

a. 3 : 2 b. 12 : 5 c. 3 : 4 d. 85 : 68

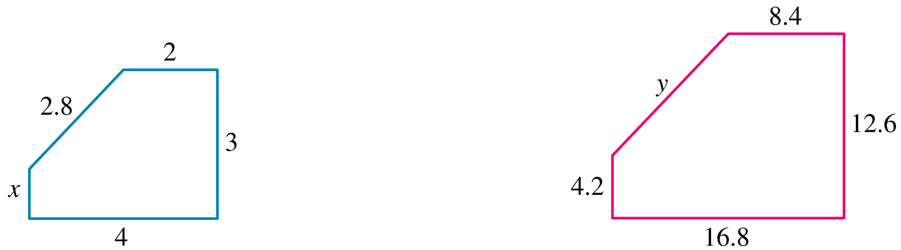
4. Calculate the missing values for the following.

a. $\frac{3}{\square} = \frac{\square}{12} = 6$ b. $\frac{5}{\square} = \frac{44}{11} = \square$ c. $\frac{\square}{7} = \frac{81}{9} = \square$

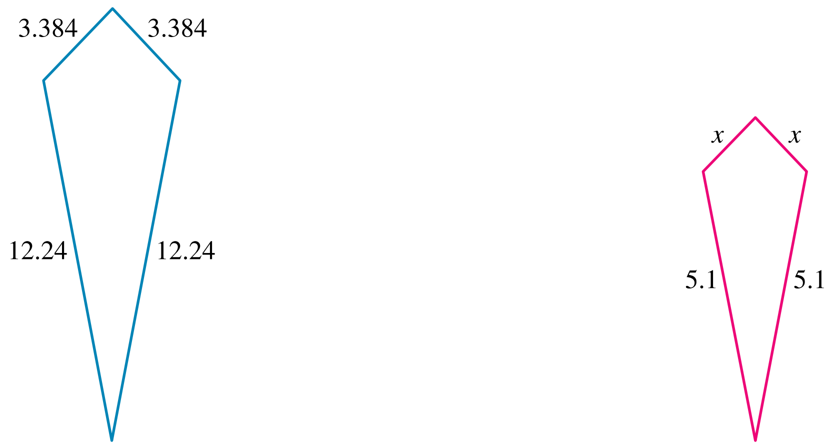
d. $\frac{2}{\square} = \frac{\square}{2} = 0.625$ e. $\frac{16.5}{\square} = \frac{\square}{34} = 5.5$

5. Calculate the unknown side lengths in the pairs of similar shapes shown.

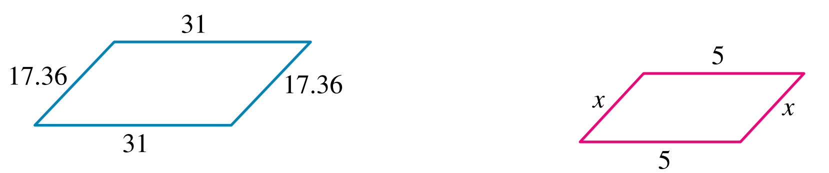
a.



b.



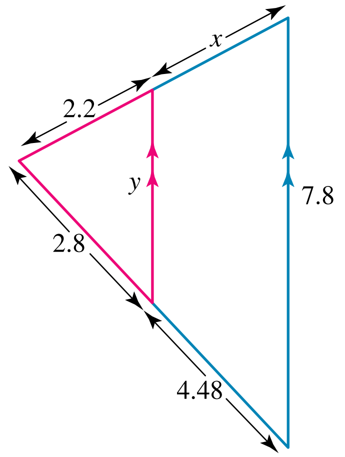
c.



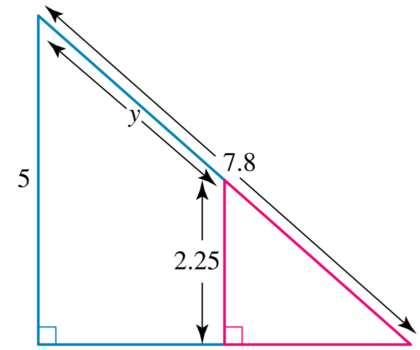
d.

6. Calculate the unknown side lengths in the diagrams shown.

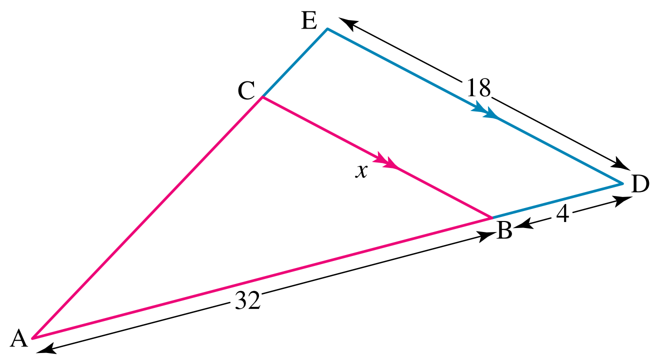
a.



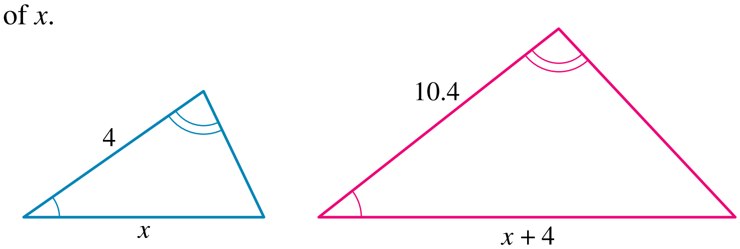
b.



7. Calculate the length of BC.

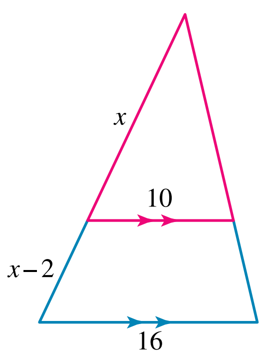


8. Calculate the value of x.

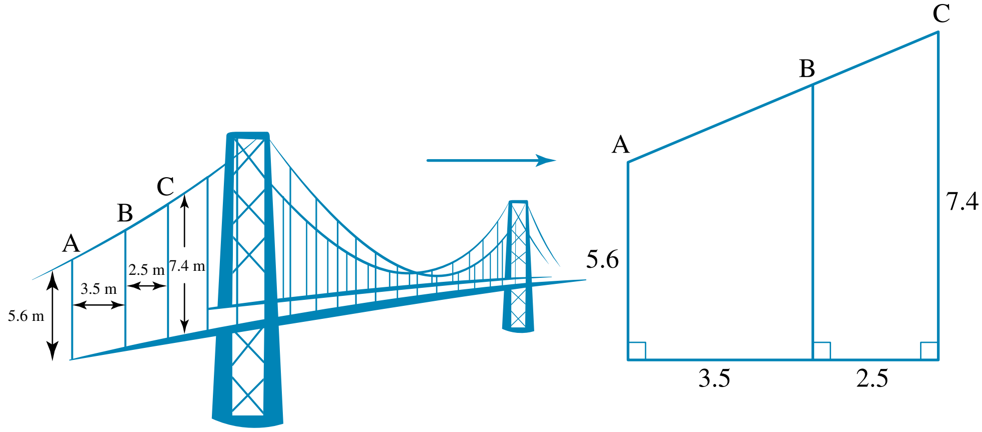


9. The side of a house casts a shadow that is 8.4 m long on horizontal ground.
   a. At the same time, an 800-mm vertical garden stake has a shadow that is 1.4 m long. What is the height of the house?
   b. When the house has a shadow that is 10 m long, how long is the garden stake's shadow?

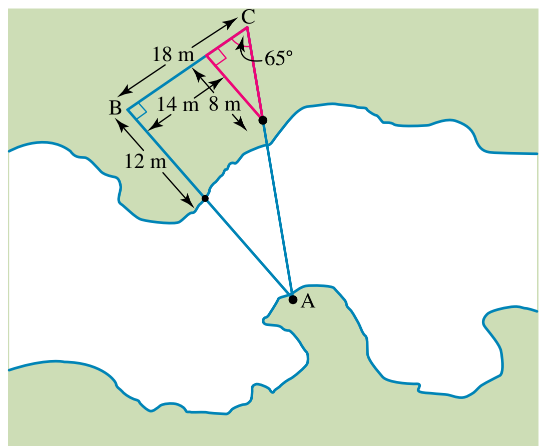
10. Calculate the value of x in the diagram.



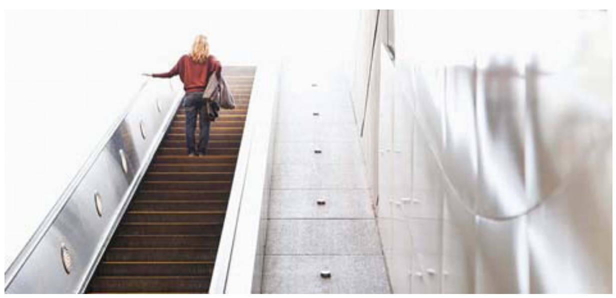
11. A section of a bridge is shown in the diagram. How high is point B above the roadway of the bridge?



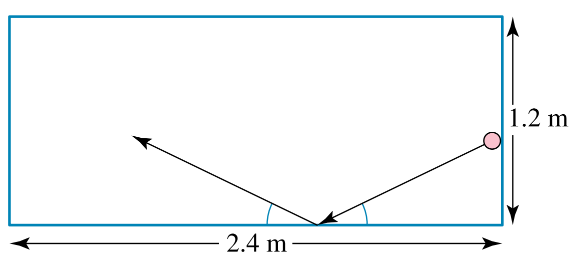
12. To calculate the distance across a ravine, a surveyor took a direct line of sight from the point B to a fixed point A on the other side and then measured out a perpendicular distance of 18 m. From that point the surveyor measured out a smaller similar triangle as shown in the diagram. Calculate the distance across the ravine along the line AB.



13. Over a horizontal distance of 6.5 m, an escalator rises 12.75 m. If you travel on the escalator for a horizontal distance of 4.25 m, what vertical distance have you risen?



14. In a game of billiards, a ball travels in a straight line from a point one-third of the distance from the bottom of the right side and rebounds from a point three-eighths of the distance along the bottom side. The angles between the bottom side and the ball's path before and after it rebounds are equal.



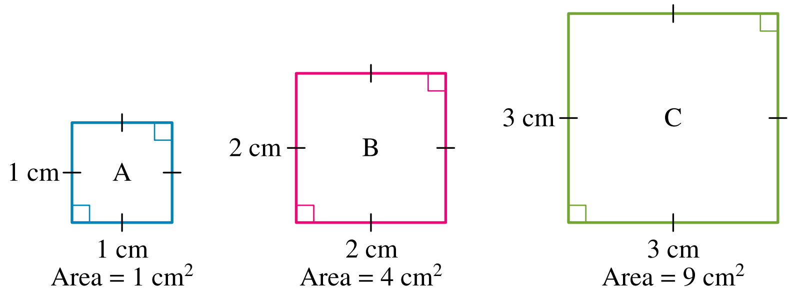
a. Calculate the perpendicular distance, correct to 2 decimal places, from the bottom side after the ball has travelled a distance of 0.8 m parallel with the bottom side after rebounding.

b. If the ball has been struck with sufficient force, at what point on an edge of the table will it next touch? Give your answer correct to 2 decimal places.

8.4 Area and volume scale factors

8.4.1 Area scale factor

Consider three squares with side lengths of 1, 2 and 3 cm. Their areas will be 1 cm^2, 4 cm^2 and 9 cm^2 respectively.



The linear scale factor between square A and square B will be 2, and the linear scale factor between square A and square C is 3. When we look at the ratio of the areas of the squares, we get 4 : 1 for squares A and B and 9 : 1 for squares A and C. In both cases, the **area scale factor** is equal to the linear scale factor raised to the power of two.

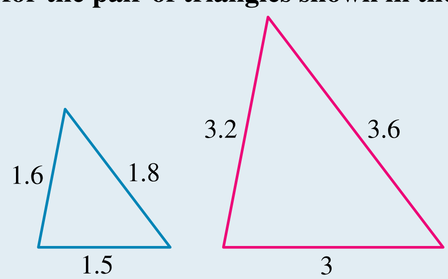
	Linear scale factor	Area scale factor
B : A	2	4
C : A	3	9

Comparing squares B and C, the ratio of the side lengths is 2 : 3, resulting in a linear scale factor of $\frac{3}{2}$ or 1.5. From the ratio of the areas we get 4 : 9, which once again indicates an area scale factor $\frac{9}{4} = 2.25$; that is, the linear scale factor to the power of two.

In general, if the linear scale factor for two similar objects is x, the area scale factor will be x^2.

WORKED EXAMPLE 4

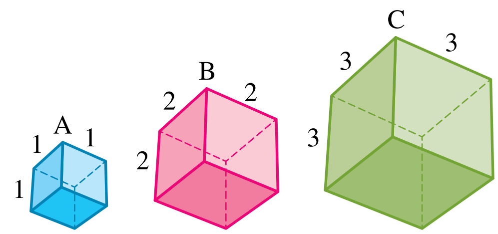
Calculate the area scale factor for the pair of triangles shown in the diagram.



THINK	WRITE
1. Calculate the ratio of the corresponding side lengths.	$\frac{3}{1.5} = \frac{3.6}{1.8} = \frac{3.2}{1.6} = 2$
2. Square the linear scale factor to obtain the area scale factor.	$2^2 = 4$
3. State the answer.	The area scale factor is 4.

8.4.2 Volume scale factor

Three-dimensional objects of the same shape are similar when the ratios of their corresponding dimensions are equal. When we compare the volumes of three similar cubes, we can see that if the linear scale factor is x, the **volume scale factor** will be x^3.

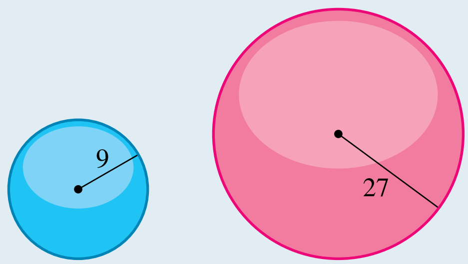


	Cube B : Cube A	Scale factor
Linear	2 : 1	2
Area	4 : 1	$2^2 = 4$
Volume	8 : 1	$2^3 = 8$

If the linear scale factor for two similar objects is x, the volume scale factor will be x^3.

WORKED EXAMPLE 5

Calculate the volume scale factor for the pair of spheres shown in the diagram.



THINK	WRITE
1 Calculate the ratio of the corresponding dimensions.	$\frac{27}{9} = 3$
2 Cube the result to obtain the volume scale factor.	$3^3 = 27$
3 State the answer.	The volume scale factor is 27.

on Resources

Interactivity: Area scale factor (int-6478)

Interactivity: Volume scale factor (int-6479)

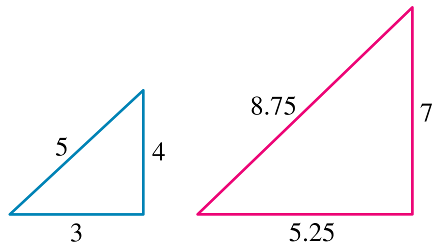
studyon

Units 1&2 > AOS 4 > Topic 2 > Concept 2 > **Similarity of solids** Concept summary and practice questions

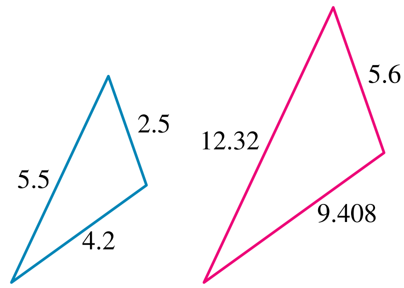
Exercise 8.4 Area and volume scale factors

1. **WE4** Calculate the area scale factor for each of the pairs of triangles shown

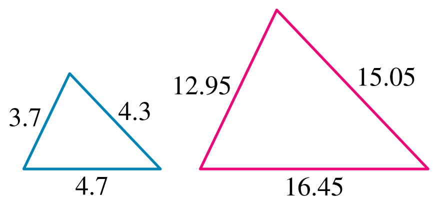
a.



b.



c.

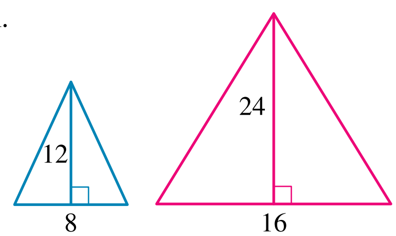


2. Calculate the area scale factor for each of the pairs of similar objects shown.

a.

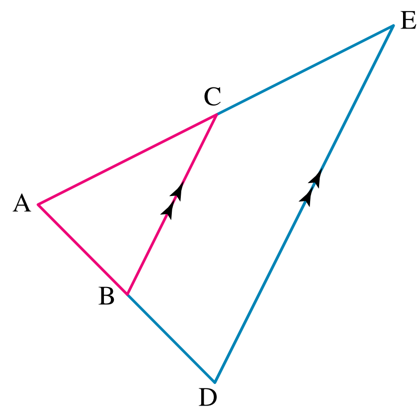
b.

3. a. Calculate the areas of the two similar triangles shown in the diagram.
   b. How many times larger in area is the biggest triangle?
   c. Calculate the linear scale factor.
   d. Calculate the area scale factor.



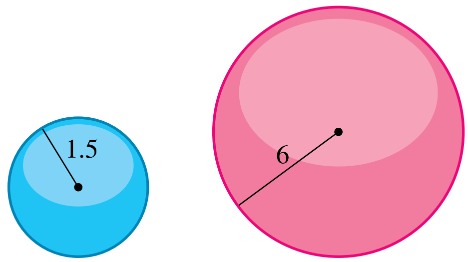
4. A hexagon is made up of six equilateral triangles of side length 2 cm. If a similar hexagon has an area of $24\sqrt{3}$ cm^2, calculate the linear scale factor.
5. A rectangular swimming pool is shown on the plans for a building development with a length of 6 cm and a width of 2.5 cm. If the scale on the plans is shown as 1 : 250:
   a. calculate the area scale factor
   b. calculate the surface area of the swimming pool.

6. The area of the triangle ADE in the diagram is 100 cm^2, and the ratio of $DE : BC$ is 2 : 1. Calculate the area of triangle ABC.



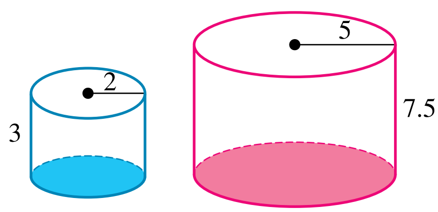
7. The floor of a square room has an area of 12 m^2. Calculate the area that the room takes up in a diagram with a scale of 1 : 250.

8. **WE5** Calculate the volume scale factor for the pair of spheres shown.

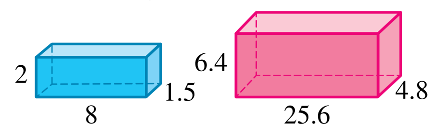


9. Calculate the volume scale factor for each of the pairs of similar objects shown.

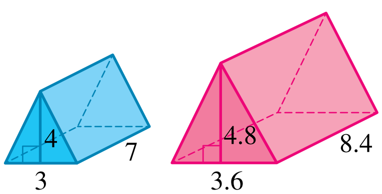
a.



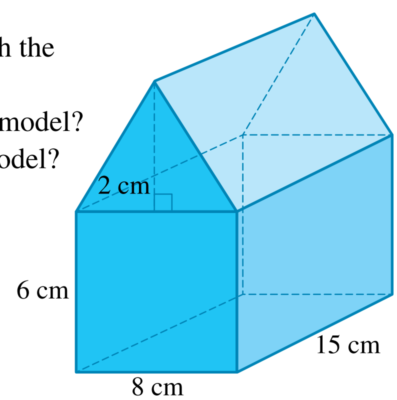
b.



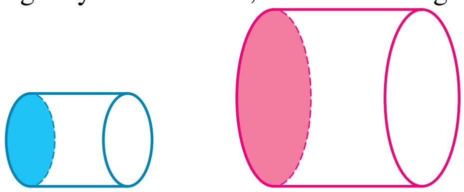
c.



10. An architect makes a small scale model of a house out of balsa wood with the dimensions shown in the diagram.
    a. If the actual length of the building is 26.25 m, what is the scale of the model?
    b. What is the ratio of the volume of the building to the volume of the model?



11. Two similar cylinders have volumes of $400\,cm^3$ and $50\,cm^3$ respectively.
  a. What is the linear scale factor?
  b. If the length of the larger cylinder is 8 cm, what is the length of the smaller one?



12. If a cube has a volume of $25\,cm^3$ and is then enlarged by a linear scale factor of 2.5, what will the new volume be?
13. Calculate the linear scale factor between two similar drink bottles if one has a volume of 600 mL and the other has a volume of 1.25 L.

14. If an area of $712\,m^2$ is represented on a scale drawing by an area of $44.5\,cm^2$, what is the actual length that a distance of 5.3 cm on the drawing represents?
15. A model car is an exact replica of the real thing reduced by a factor of 12.
  a. If the actual surface area of the car that is spraypainted is $4.32\,m^2$, what is the equivalent painted area on the model car?
  b. If the actual storage capacity of the car is $1\ 78\,m^3$, what is the equivalent volume for the model car?

16. A company sells canned fish in two sizes of similar cylindrical cans. For each size, the height is four-fifths of the diameter.

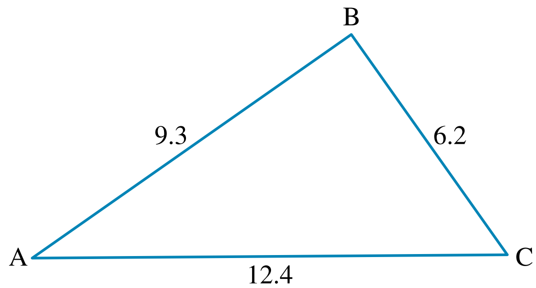
  a. Write an expression for calculating the volume of a can of fish in terms of its diameter.
  b. If the dimensions of the larger cans are 1.5 times those of the smaller cans, write an expression for calculating the volume of the larger cans of fish in terms of the diameter of the smaller cans.

8.5 Review: exam practice

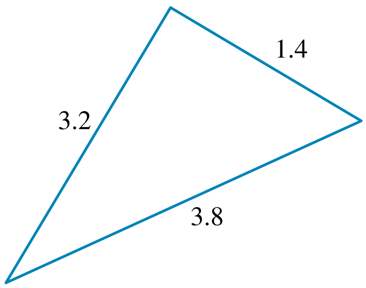
A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

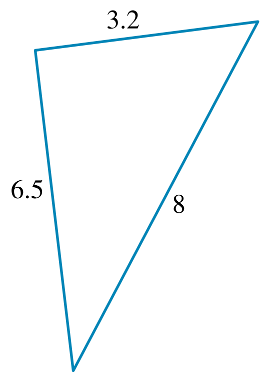
1. MC The triangle that is similar to ΔABC is:



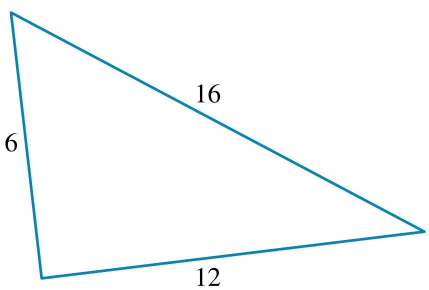
A.



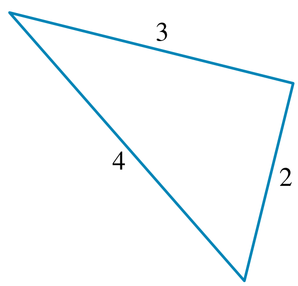
B.



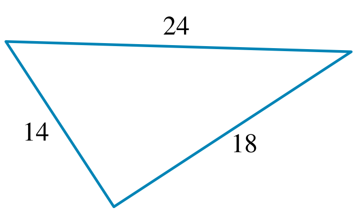
C.



D.



E.

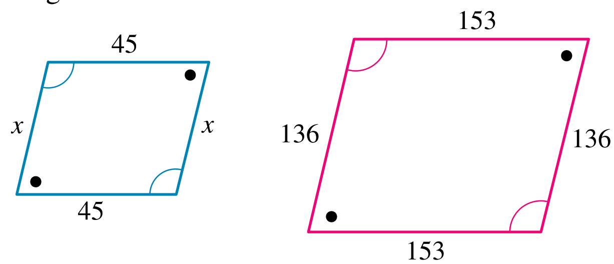


2. MC If a map has a scale factor of 1 : 50 000, an actual distance of 11 km would have a length on the map of:
   A. 21 cm B. 22 cm C. 21.5 cm D. 11 cm E. 5.5 cm
3. MC If the areas of two similar objects are 12 cm^2 and 192 cm^2 respectively, the linear scale factor will be:
   A. 3 B. 4 C. 16 D. 2 E. 9
4. MC If the volumes of two similar solids are 16 cm^3 and 128 cm^3 respectively, the area scale factor will be:
   A. 2 B. 3 C. 4 D. 8 E. 9

5. **MC** A tree casts a shadow that is 5.6 m long. At the same time a 5-m light pole casts a shadow that is 3.5 m long. The height of the tree is:

A. 4.4 m **B.** 7.5 m **C.** 8.0 m **D.** 6.0 m **E.** 5.6 m

6. **MC** The value of x in the diagram is:



A. 38 **B.** 41 **C.** 43 **D.** 36 **E.** 40

7. **MC** If the scale factor of the volumes of two similar cuboids is 64 and the volume of the larger one is 1728 cm^3, the surface area of the smaller cuboid will be:

A. 64 cm^2 **B.** 27 cm^2 **C.** 54 cm^2 **D.** 9 cm^2 **E.** 16 cm^2

8. **MC** An enlargement diagram of a very small object is drawn at a scale of 18 :1. If the diagram has an area of 162 cm^2, the equivalent area of the actual object will be closest to:

A. 0.5 cm^2 **B.** 0.4 cm^2 **C.** 9 cm^2 **D.** 0.6 cm^2 **E.** 18 cm^2

9. **MC** The plans of a house show the side of a building as 12.5 cm long. If the actual building is 15 m long, the scale of the plan will be:

A. 1 : 250 **B.** 1 : 150 **C.** 1 : 220 **D.** 1 : 120 **E.** 1 : 175

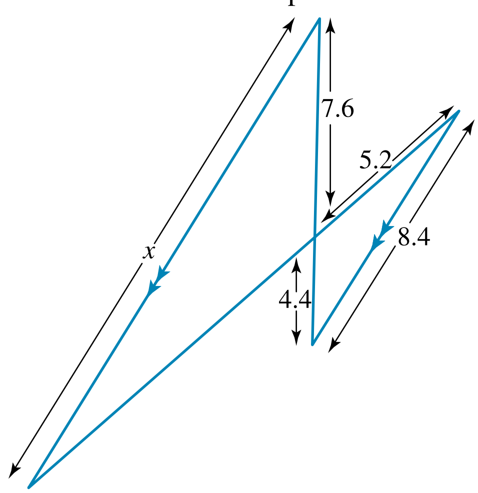
10. **MC** The plans for a building show a concrete slab covering an area of 12.5 cm × 8.4 cm to a depth of 0.25 cm. If the plans are drawn to a scale of 1 : 225, the actual volume of concrete will be closest to:

A. 26.25 m^3 **B.** 262.5 m^3 **C.** 5906.25 m^3 **D.** 299 m^3 **E.** 29.9 m^3

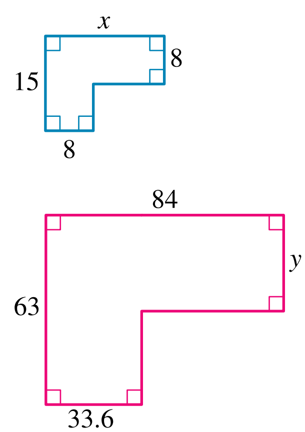
Short answer

1. Calculate the values of the pronumerals in the following diagrams.

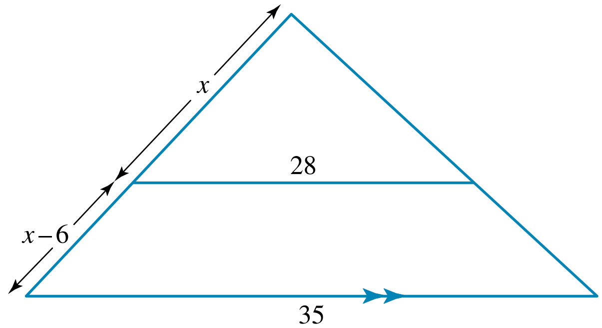
a.



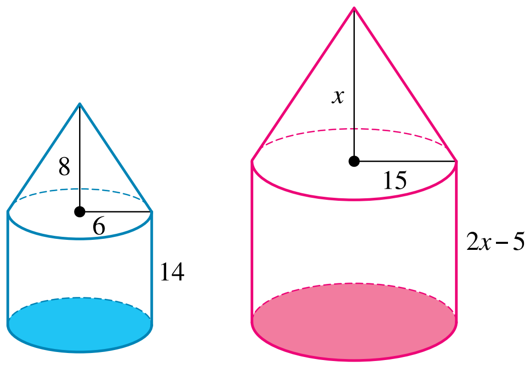
b.



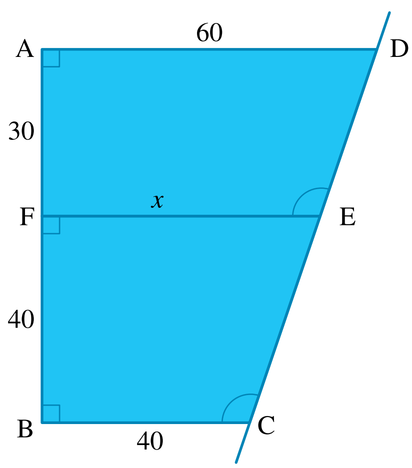
c.



2. Calculate the value of x in the diagram of two similar objects shown.

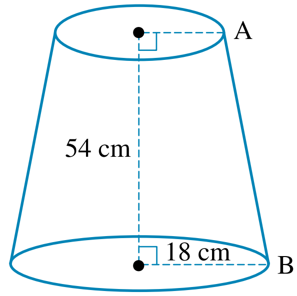


3. A triangle has side lengths of 32 mm, 45 mm and 58 mm.
   a. Calculate the side lengths of a larger similar triangle using a corresponding side ratio of 2 : 3.
   b. What would be the side lengths of the larger triangle in a drawing with a scale of 5 : 2?
4. The volume of a solid is 1600 cm^3. If the ratio of the corresponding dimensions between this solid and a similar solid is 4 : 5, calculate the volume of the similar solid.
5. A farmer divides paddock ABCD into two separate paddocks along the line FE as shown in the diagram. All distances are shown in metres.



   a. Express the ratio of the corresponding sides of the original paddock, ABCD, with paddock BCEF in its simplest form.
   b. Calculate the length of fencing required to separate the two paddocks along the line FE. Give your answer correct to the nearest centimetre.

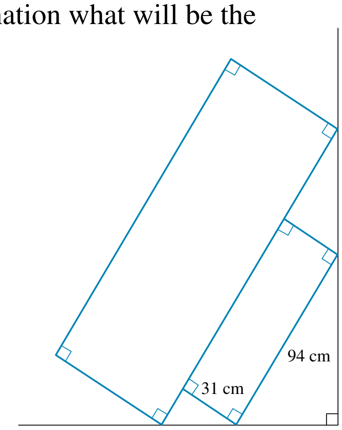
6. The top third of an inverted right cone is removed as shown in the diagram.



a. Calculate the height of the cone that has been removed.
b. Calculate the distance along the edge of the remaining part of the cone from A to B.

Extended response

1. On a map that is drawn to a scale of 1 : 225 000, the distance between two points is 88 cm.
   a. What is the actual distance between the two points?
   A boat sets out to travel from one point to the other but the navigator makes an error. After travelling 100 km, the crew realise they are directly south of a point that they should have reached after travelling 90 km in a direct line to their destination.
   b. If they continue in their current direction, how much further do they have to travel to be directly south of the intended destination?
   c. How far away from the intended destination will the boat be when it reaches the point on their course that is directly to the south?
   d. When the boat is at the point directly to the south of their intended destination what will be the distance on the map?
2. A rectangular box with dimensions 94 cm × 31 cm leans against a wall as shown in the diagram. A larger box leans against the first box.
   a. If the ratio of the corresponding side lengths between the two boxes is 4 : 7, what are the dimensions of the larger box?
   b. If the smaller box touches the floor at a point that is 52 cm from the base of the wall, how far up the wall does it reach?
   c. How far up the wall does the larger box reach?



3. A swimmer is observed in the water from the top of a vertical cliff that is 5 m tall. A line of sight is taken from a point 2 m back from the edge of the cliff.
   a. If the swimmer, the edge of the cliff and a point 1.8 m above the cliff are in line, how far is the swimmer from the base of the cliff?
   b. If the swimmer moves a further 2.45 m away from the cliff, from how far above the cliff should the new line of sight be taken?
4. A caterer sells takeaway coffee in three different-sized cups whose dimensions are in proportion.
   a. If the small cup has a capacity of 200 mL and the medium cup has a capacity of 300 mL, what is the linear scale factor correct to 2 decimal places?
   b. If the ratio of corresponding dimensions between the smaller and larger cups is 4 : 5, what is the capacity of the larger cup?
   c. If the caterer charges $3.00 for a small cup, $3.50 for a medium cup and $4.00 for a large cup, which is the better value for the customer?

studyon

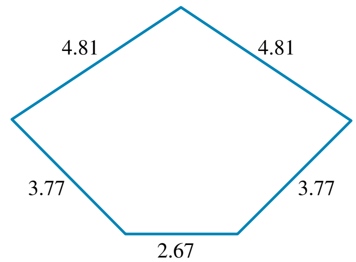
Units 1&2 Sit topic test

Answers

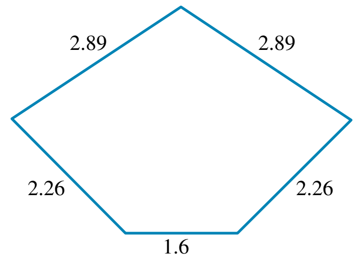
Topic 8 Similarity

Exercise 8.2 Similar objects

1. a. $\frac{15}{8} = \frac{11.25}{6} = \frac{7.5}{4} = 1.875$, and all angles are equal.
   b. $\frac{41.04}{11.4} = \frac{26.1}{7.25} = \frac{9}{2.5} = \frac{13.68}{3.8} = 3.6$, and all angles are equal.
2. $\frac{106.43}{18.35} = \frac{24.65}{4.25} = 5.8$, and all angles are equal.
3. a. $\frac{4.8}{3.2} = 1.5$, $\frac{2.24}{1.6} = 1.4$; not similar
   b. $\frac{10.224}{2.84} = 3.6$, $\frac{2.34}{0.65} = 3.6$; similar
   c. $\frac{4.4352}{2.112} = 2.1$, $\frac{2.184}{1.04} = 2.1$; similar
   d. $\frac{12.535}{11.5} = 1.09$, $\frac{10.152}{9.4} = 1.08$; not similar
4. a. $\frac{35}{25} = 1.4$, $\frac{22.4}{16} = 1.4$, $\frac{12.6}{9} = 1.4$ and all angles are equal; similar
   b. $\frac{4.94}{2.6} = 1.9$, $\frac{6.65}{3.5} = 1.9$, $\frac{2.09}{1.1} = 1.9$ and all angles are equal; similar
   c. $\frac{13}{8} = 1.625$, $\frac{6.5}{4} = 1.625$, $\frac{3.25}{2} = 1.625$ and all angles are equal; similar
   d. $\frac{3.24}{2.7} = 1.2$, $\frac{19.44}{16.2} = 1.2$, $\frac{2.16}{1.8} = 1.2$ and all angles are equal; similar
5. a. $\frac{44.275}{7.7} = \frac{78.2}{13.6} = 5.75$, SAS
   b. $\frac{38.72}{17.6} = \frac{16.28}{7.4} = \frac{14.85}{6.75} = 2.2$, SSS
6. a. $\frac{32.4}{18} = \frac{21.6}{12} = 1.8$, $\frac{23.8}{14} = 1.7$ not similar
   b. $\frac{54.32}{19.4} = \frac{22.96}{8.2} = 2.8$ and all angles are equal; similar
7. a. A and C   b. B and C
   c. A and C   d. A and B
8. a. All angles are equal and side lengths are in proportion.
   b. All measurements (radius and circumference) are in proportion.
   c. All angles are equal and side lengths are in proportion.
9. a. 1.6 : 1   b. 2.26 : 1
   c. 3.6 : 1
10. a. 4 : 3   b. 5.5 : 1
11. a. 4.8   b. 7.07
12. a. $\frac{50.84}{8.2} = 6.2$, all sides are in proportion and all angles equal.
    b. $\frac{14.34}{12.6} = 1.138$, all sides are in proportion and all angles are equal.
13. a. 32   b. 20
14. 51.8
15. $x = 5.6$, $y = 7$
16. a.



    b.



Exercise 8.3 Linear scale factors

1. a. 2.2   b. 3.4
   c. 1.25   d. 1.6
2. a. 5.2   b. 2.7
   c. 3.8   d. 3
3. a. 1.5   b. 2.4
   c. 0.75   d. 1.25
4. a. $\frac{3}{\boxed{0.5}} = \frac{\boxed{72}}{12} = 6$ b. $\frac{5}{\boxed{1.25}} = \frac{44}{11} = \boxed{4}$
   c. $\frac{\boxed{63}}{7} = \frac{81}{9} = \boxed{9}$ d. $\frac{2}{\boxed{3.2}} = \frac{\boxed{1.25}}{2} = 0.625$
   e. $\frac{16.5}{\boxed{3}} = \frac{\boxed{187}}{34} = 5.5$
5. a. $x = 1$, $y = 11.76$
   b. 1.41
   c. 2.8
   d. $x = 2$, $y = 1.6$, $z = 3.576$
6. a. $x = 3.52$, $y = 3$ b. $y = 4.29$
7. 16
8. 2.5
9. a. 4.8 m   b. 1.67 m
10. 5
11. 6.65 m
12. 24 m
13. 8.34 m
14. a. 0.36 m
    b. 0.67 m from the bottom of the left side

Exercise 8.4 Area and volume scale factors

1. a. $\frac{49}{16} = 3.0625$ b. $\frac{3136}{625} = 5.0176$
   c. $\frac{49}{4} = 12.25$
2. a. $\frac{64}{25} = 2.56$ b. $\frac{49}{25} = 1.96$

3. a. 48 and 192 square units
   b. 4
   c. 2
   d. 4
4. 2
5. a. 62 500 b. 93.75 m^2
6. 25 cm^2
7. 1.92 cm^2
8. 64
9. a. 15.625 b. 32.768 c. 1.728
10. a. 1 :175 b. 5359375 : 1
11. a. 2 b. 4 cm
12. 390.625 cm^3
13. 1.28
14. 21.2 m
15. a. 300 cm^2 b. 1030 cm^3
16. a. $V = \frac{\pi D^3}{5}$ b. $V_2 = \frac{27\pi D_1^3}{40}$

8.5 Review: exam practice

Multiple choice

1. D 2. B 3. B 4. C 5. C
6. E 7. C 8. A 9. D 10. D

Short answer

1. a. 14.51 b. $x = 20$, $y = 33.6$ c. 8
2. 20
3. a. 48 mm, 67.5 mm, 87 mm
   b. 120 mm, 168.75 mm, 217.5 mm
4. 3125 cm^3
5. a. 7 :4 b. 5143 cm
6. a. 27 cm b. 55.32 cm

Extended response

1. a. 198 km b. 120 km c. 95.90 km d. 42.62 cm
2. a. 164.5 cm × 54.25 cm
   b. 78.31 cm
   c. 137.04 cm
3. a. 5.6 m b. 1.24 m
4. a. 1.14 b. 390.6 mL c. Large cup

TOPIC 9
Applications of trigonometry

9.1 Overview

9.1.1 Introduction

Trigonometry is a branch of mathematics that describes the relationship between angles and lengths of triangles. Triangles were first studied in the second millennium BCE by Egyptians and Babylonians. The study of trigonometric functions began in Hellenistic (Greek) mathematics. This understanding enabled early explorers to plot the stars and navigate the seas.

Below are some areas in which trigonometry is widely used.

Architecture and engineering

Architecture and engineering rely on the formation of triangles for support structures. When an engineer wants to correctly lay out a curved wall, work out the slope of a roof or its correct height, trigonometry is used. The construction of the Golden Gate Bridge (pictured above) involved hundreds of thousands of trigonometric calculations.

Music theory

Music theory involves sound waves, which travel in a repeating wave pattern. This repeating pattern can be represented graphically by sine and cosine functions. A single note can be modelled on a sine curve and a chord can be modelled with multiple sine curves. This graphical representation of music allows computers to create sound, so sound engineers can adjust volume and pitch, and make different sound effects.

Electrical engineering

The electricity that is sent to our house requires an understanding of trigonometry. Power companies use alternating current (AC) to send electricity over long distances. The alternating current signal has a sinusoidal (sine wave) behaviour.

Video games

Consider the way the video game Mario uses trigonometry. When you see Mario smoothly glide over the road blocks, he doesn't really jump straight along the y-axis; it is a slightly curved path or a parabolic path. Trigonometry aids Mario to jump over these obstacles, demonstrating one of the many areas where trigonometry is used.

Trigonometry is also used in many other areas such as flight engineering, physics, archaeology and marine biology.

LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.

9.1.2 Kick off with CAS

Remembering the rules of rounding

Rounding to a certain number of decimal places is covered in an earlier topic. Use those rules and your CAS to calculate the following.

Note: When using trigonometric ratios to calculate angles or determine side lengths, ALWAYS ensure that your calculator has been set in degree mode.

1. Calculate the following.
   a. 4.23×6.890 (correct to 2 decimal places)
   b. 0.0352×1.33 (correct to 3 decimal places)
   c. 89.3×167.12 (correct to 1 decimal place)
   d. 9.314×13.1 (correct to 4 decimal places)
   e. $\dfrac{1561.45}{3.5}$ (correct to the nearest whole number)
   f. A 21.3-km stretch of road requires a post on the edge every 8 metres. If there is one post at the start, how many posts will be required?
   g. $0.0731 \div 1.24$ (correct to 3 decimal places)
   h. Convert 15 km/h into m/s (correct to the nearest whole number).
2. a. Using CAS, determine the value of $\sin(35°)$.
   b. Did you get -0.4282 or 0.5736? Only one of these is correct. Check the setting for angle mode on your calculator.
   c. Which is the correct answer?
3. Using your calculator, determine the value of the following, giving all answers correct to 4 decimal places.
   a. $\sin(24.6°)$
   b. $\cos(61.23°)$
   c. $\tan(18°)$
   d. $\sin(104.52°)$
   e. $\cos(133.8°)$
   f. $\tan(27.88°)$
   g. $\sin(45°)$
   h. $\cos(80°)$

on Resources

Please refer to the Resources tab in the Prelims section of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

9.2 Trigonometric ratios

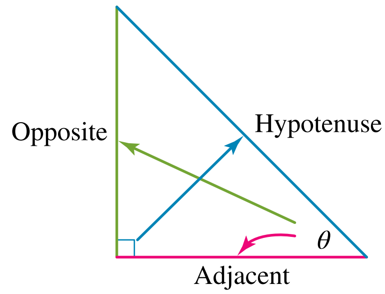
9.2.1 Trigonometric ratios

A ratio of the lengths of two sides of a right-angled triangle is called a **trigonometric ratio**. The three most common trigonometric ratios are **sine**, **cosine** and **tangent**. They are abbreviated as sin, cos and tan respectively. Trigonometric ratios are used to find the unknown length or acute angle size in right-angled triangles.

It is important to identify and label the features given in a right-angled triangle. The labelling convention of a right-angled triangle is as follows:

The longest side of a right-angled triangle is always called the hypotenuse and is opposite the right angle. The other two sides are named in relation to the reference angle, θ. The opposite side is opposite the reference angle, and the adjacent side is next to the reference angle.

When Egyptians first used a sundial around 1500 BC they were using trigonometry.

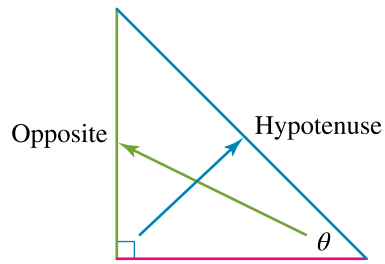


9.2.2 The sine ratio

The sine ratio is used when we want to find an unknown value given two out of the three following values: opposite, hypotenuse and reference angle.

The sine ratio of θ is written as sin (θ) and is defined as follows:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \quad or \quad \sin(\theta) = \frac{O}{H}$$

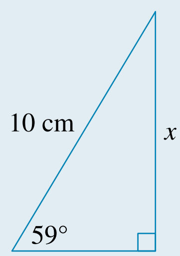


The inverse sine function is used to find the value of the unknown reference angle given the lengths of the hypotenuse and opposite side.

$$\theta = \sin^{-1}\left(\frac{O}{H}\right)$$

WORKED EXAMPLE 1

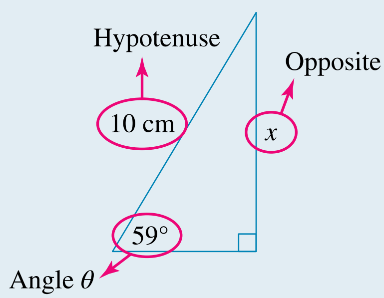
Calculate the length of x correct to 2 decimal places.



THINK

1. Label all the given information on the triangle.

WRITE/DRAW



2. Since we have been given the combination of opposite, hypotenuse and the reference angle θ, we need to use the sine ratio. Substitute the given values into the ratio equation.

$$\sin(\theta) = \frac{O}{H}$$
$$\sin(59°) = \frac{x}{10}$$

3. Rearrange the equation to make the unknown the subject and solve.
Make sure your calculator is in degree mode.

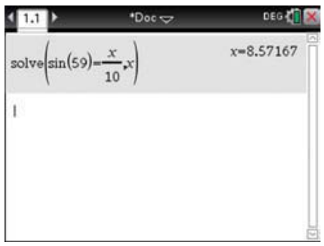
$$x = 10\sin(59°)$$
$$= 8.57$$

The opposite side length is 8.57 cm.

TI | THINK

1. Ensure your calculator is in DEGREE mode.
On a Calculator page, press MENU then select:
3: Algebra
1: Solve
Complete the entry line as:
$\text{solve}\left(\sin(59) = \frac{x}{10}, x\right)$
then press ENTER.

WRITE



2. The answer appears on the screen.

x = 8.57 cm (to 2 decimal places)

CASIO | THINK

1. Ensure your calculator is in DEGREE mode.
On the Main screen, complete the entry line as:
$\text{solve}\left(\sin(59) = \frac{x}{10}, x\right)$
then press EXE.

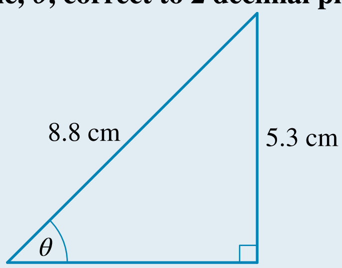
WRITE

2. The answer appears on the screen.

x = 8.57 cm (to 2 decimal places)

WORKED EXAMPLE 2

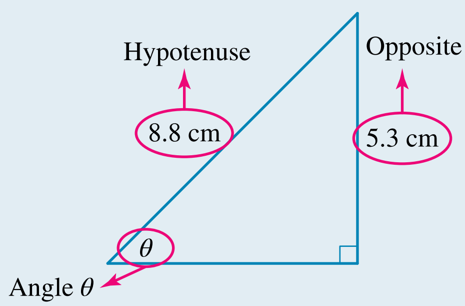
Find the value of the unknown angle, θ, correct to 2 decimal places.



THINK

1. Label all the given information on the triangle.

WRITE/DRAW



2. Since we have been given the combination of opposite, hypotenuse and the reference angle θ, we need to use the sine ratio. Substitute the given values into the ratio equation.

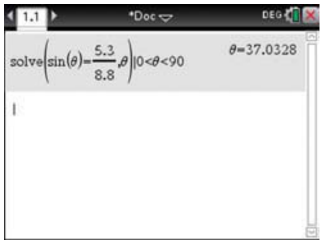
$$\sin(\theta) = \frac{\text{O}}{\text{H}} = \frac{5.3}{8.8}$$

3. To find the angle θ, we need to use the inverse sine function.
Make sure your calculator is in degree mode.

$$\theta = \sin^{-1}\left(\frac{5.3}{8.8}\right) = 37.03°$$

TI | THINK — **WRITE**

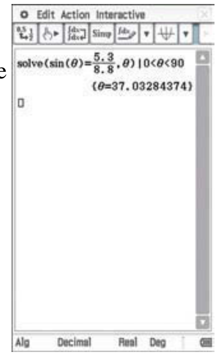
1. Ensure the calculator is in DEGREE mode.
On a Calculator page, press MENU then select:
3: Algebra
1: Solve
Complete the entry line as:
$\text{solve}\left(\sin(\theta) = \frac{5.3}{8.8}, \theta\right) | 0 < \theta < 90$
then press ENTER.
Note: The θ symbol can be found by pressing the ^{1} button. Alternatively, use a letter to represent the angle.



2. The answer appears on the screen. — $\theta = 37.03°$ (to 2 decimal places)

CASIO | THINK — **WRITE**

1. Ensure your calculator is in DEGREE mode.
On the Main screen, complete the entry line as:
$\text{solve}\left(\sin(\theta) = \frac{5.3}{8.8}, \theta\right)$
$| 0 < \theta < 90$
then press EXE.
Note: The θ symbol can be found in the Math2 tab in the Keyboard menu.
Alternatively, use a letter to represent the angle.



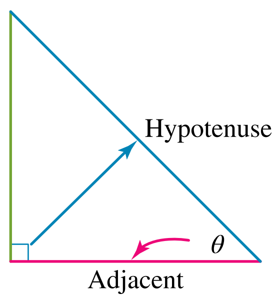
2. The answer appears on the screen. — $\theta = 37.03°$ (to 2 decimal places)

9.2.3 The cosine ratio

The cosine ratio is used when we want to find an unknown value given two out of the three following values: adjacent, hypotenuse and reference angle.

The cosine ratio of θ is written as $\cos(\theta)$ and is defined as follows:

$$\cos(\theta) = \frac{\textbf{adjacent}}{\textbf{hypotenuse}} \quad or \quad \cos(\theta) = \frac{\textbf{A}}{\textbf{H}}$$

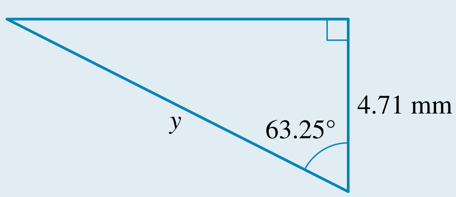


The inverse cosine function is used to find the value of the unknown reference angle when given lengths of the hypotenuse and adjacent side.

$$\theta = \cos^{-1}\left(\frac{\textbf{A}}{\textbf{H}}\right)$$

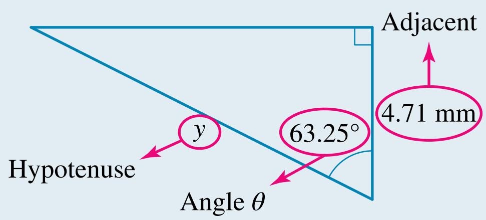
WORKED EXAMPLE 3

Calculate the length of y correct to 2 decimal places.



THINK | **WRITE/DRAW**

1. Label all the given information on the triangle.



2. Since we have been given the combination of adjacent, hypotenuse and the reference angle θ, we need to use the cosine ratio. Substitute the given values into the ratio equation.

$$\cos(\theta) = \frac{A}{H}$$

$$\cos(63.25°) = \frac{4.71}{y}$$

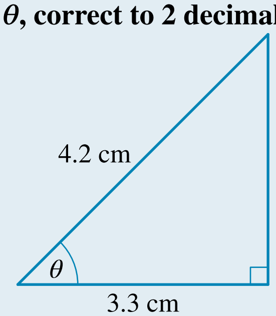
3. Rearrange the equation to make the unknown the subject and solve. Make sure your calculator is in degree mode.

$$y = \frac{4.71}{\cos(63.25°)}$$
$$= 10.46$$

The length of the hypotenuse is 10.46 mm.

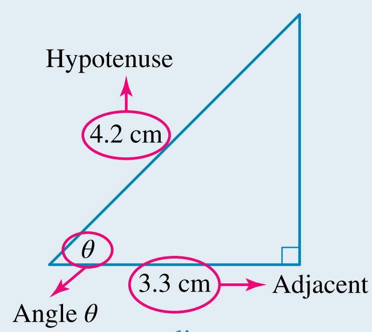
WORKED EXAMPLE 4

Find the value of the unknown angle, θ, correct to 2 decimal places.



THINK | **WRITE/DRAW**

1. Label all the given information on the triangle.



2. Since we have been given the combination of adjacent, hypotenuse and the reference angle θ, we need to use the cosine ratio. Substitute the given values into the ratio equation.

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H}$$

$$\cos(\theta) = \frac{3.3}{4.2}$$

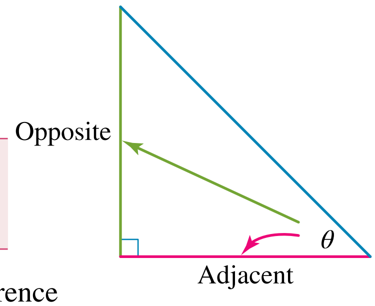
3. To find angle θ, we need to use the inverse cosine function. Make sure your calculator is in degree mode.

$$\theta = \cos^{-1}\left(\frac{3.3}{4.2}\right)$$
$$= 38.21°$$

9.2.4 The tangent ratio

The tangent ratio is used when we want to find an unknown value given two out of the three following values: opposite, adjacent and reference angle.

The tangent ratio of θ is written as tan (θ) and is defined as follows:



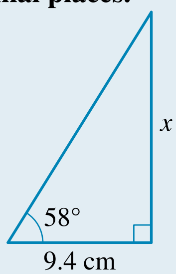
$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \quad \text{or} \quad \tan(\theta) = \frac{O}{A}$$

The inverse tangent function is used to find the value of the unknown reference angle given the lengths of the adjacent and opposite sides.

$$\theta = \tan^{-1}\left(\frac{O}{A}\right)$$

WORKED EXAMPLE 5

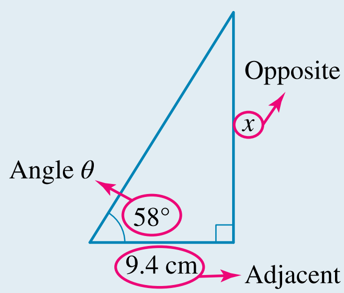
Calculate the length of x correct to 2 decimal places.



THINK

1. Label all the given information on the triangle.

WRITE/DRAW



2. Since we have been given the combination of opposite, adjacent and the reference angle θ, we need to use the tangent ratio. Substitute the given values into the ratio equation.

$$\tan(\theta) = \frac{O}{A}$$
$$\tan(58°) = \frac{x}{9.4}$$

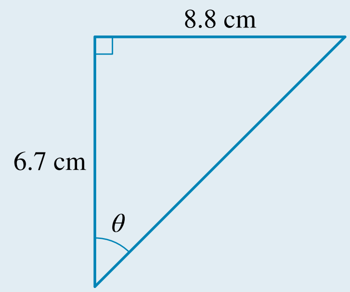
3. Rearrange the equation to make the unknown the subject and solve.
Make sure your calculator is in degree mode.

$$x = 9.4\tan(58°)$$
$$x = 15.04$$

The opposite side length is 15.04 cm.

WORKED EXAMPLE 6

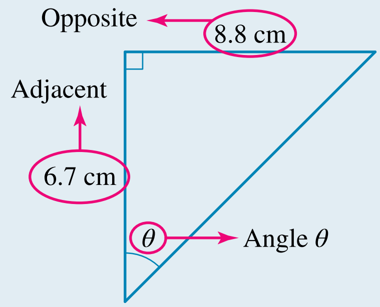
Find the value of the unknown angle, θ, correct to 2 decimal places.



THINK

1. Label all the given information on the triangle.

WRITE/DRAW



2. Since we have been given the combination of opposite, adjacent and the reference angle θ, we need to use the tangent ratio. Substitute the given values into the ratio equation.

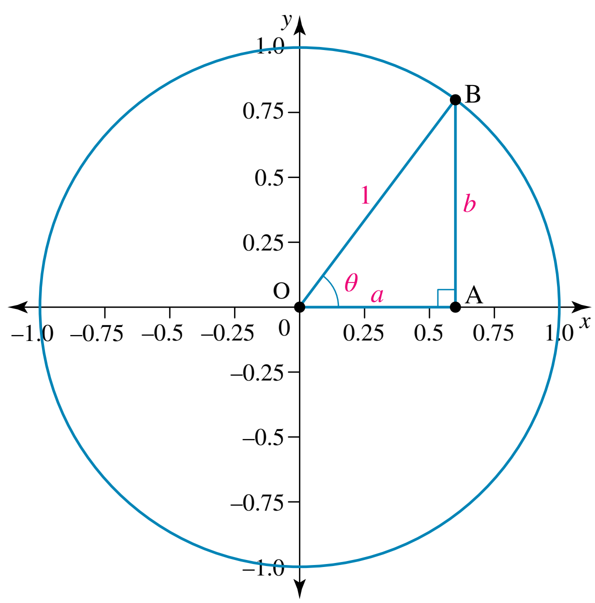
$$\tan(\theta) = \frac{O}{A} = \frac{8.8}{6.7}$$

3. To find the angle θ, we need to use the inverse tangent function. Make sure your calculator is in degree mode.

$$\theta = \tan^{-1}\left(\frac{8.8}{6.7}\right) = 52.72°$$

9.2.5 The unit circle

If we draw a circle of radius 1 in the Cartesian plane with its centre located at the origin, then we can locate the coordinates of any point on the circumference of the circle by using right-angled triangles.

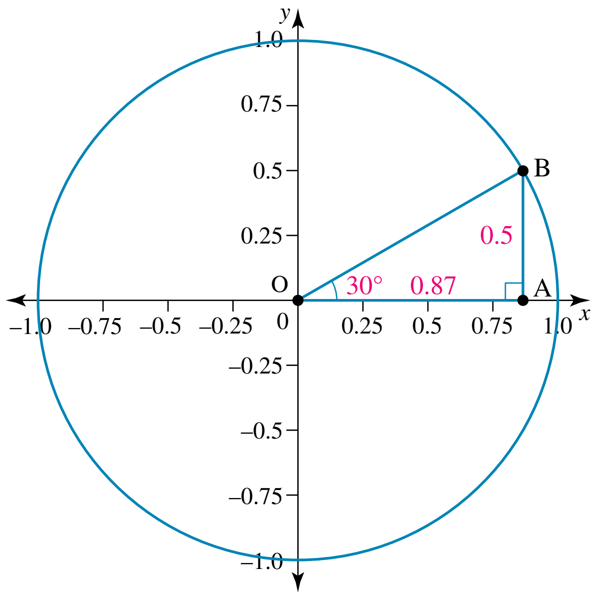


In this diagram, the length of the hypotenuse is 1 and the coordinates of B can be found using the trigonometric ratios.

$$\cos(\theta) = \frac{\text{A}}{\text{H}} \qquad \text{and} \qquad \sin(\theta) = \frac{\text{O}}{\text{H}}$$
$$= \frac{a}{1} \qquad\qquad\qquad = \frac{b}{1}$$
$$= a \qquad\qquad\qquad = b$$

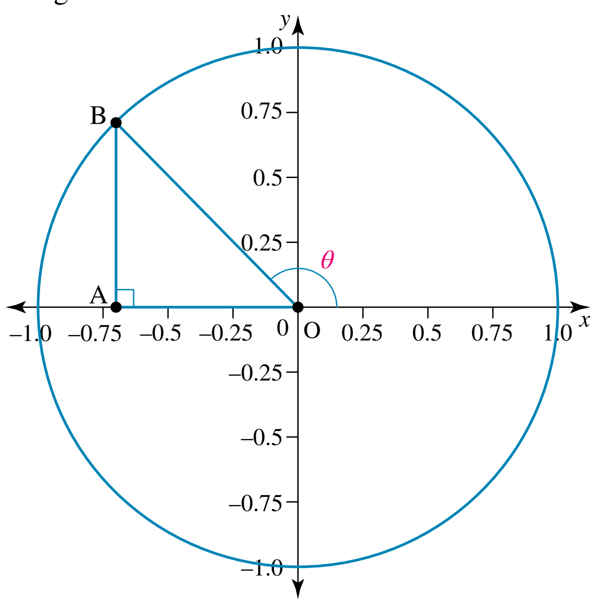
Therefore the base length of the triangle, a, is equal to $\cos(\theta)$, and the height of the triangle, b, is equal to $\sin(\theta)$. This gives the coordinates of B as $(\cos(\theta), \sin(\theta))$.

For example, if we have a right-angled triangle with a reference angle of 30° and a hypotenuse of length 1, then the base length of the triangle will be 0.87 and the height of the triangle will be 0.5, as shown in the following triangle.



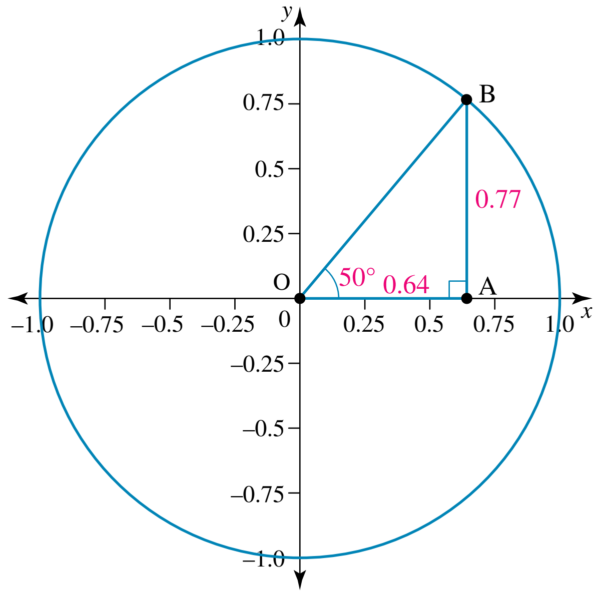
Similarly, if we calculate the value of cos(30°) and sin(30°), we get 0.87 and 0.5 respectively.

We can actually extend this definition to any point B on the unit circle as having the coordinates $(\cos(\theta), \sin(\theta))$, where θ is the angle measured in an anticlockwise direction.

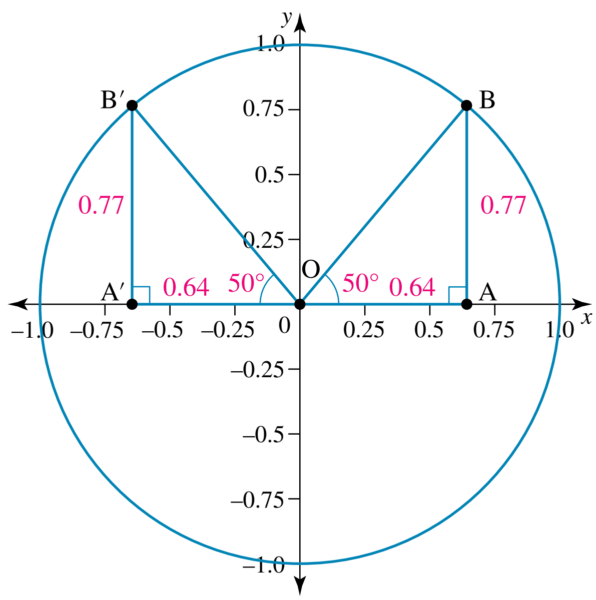


9.2.6 Extending sine and cosine to 180°

We can place any right-angled triangle with a hypotenuse of 1 in the unit circle so that one side of the triangle lies on the positive x-axis. The following diagram shows a triangle with base length 0.64, height 0.77 and reference angle 50°.



The coordinates of point B in this triangle are (0.64, 0.77) or (cos(50°), sin(50°)). Now reflect the triangle in the y-axis as shown in the following diagram.

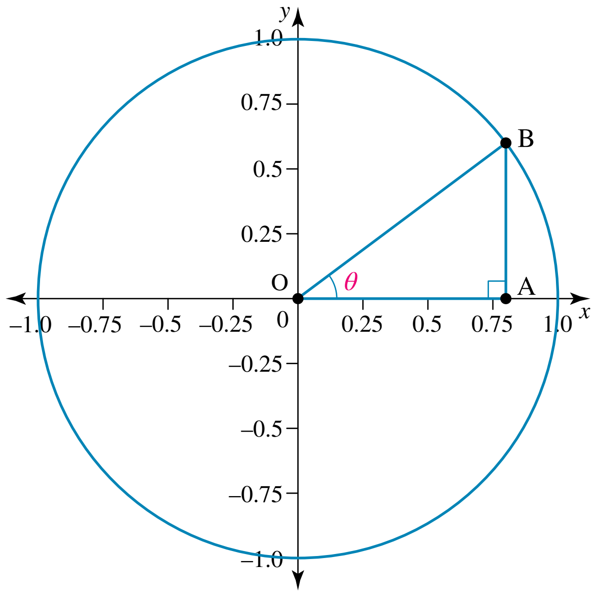


We can see that the coordinates of point B′ are (−0.64, 0.77) or (− cos(50°), sin(50°)).

We have previously determined the coordinates of any point B on the circumference of the unit circle as (cos(θ), sin(θ)), where θ is the angle measured in an anticlockwise direction. In this instance the value of $\theta = 180° - 50 = 130°$.

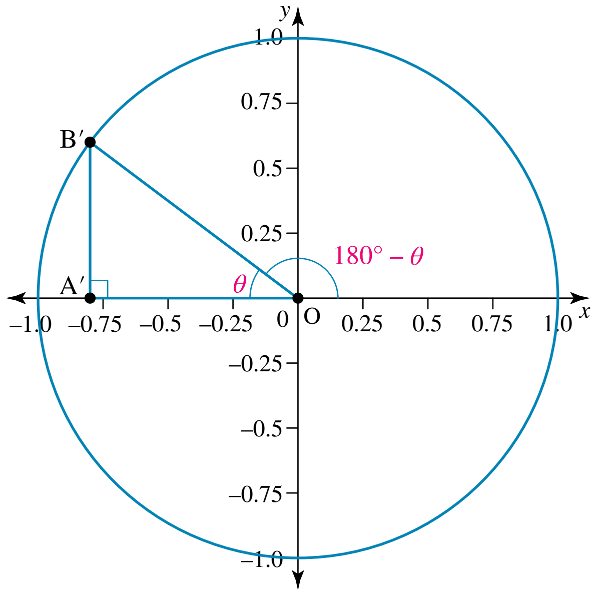
Therefore, the coordinates of point B′ are (cos(130°), sin(130°)).

This discovery can be extended when we place any right-angled triangle with a hypotenuse of length 1 inside the unit circle.



As previously determined, the point B has coordinates $(\cos(\theta),\ \sin(\theta))$.

Reflecting this triangle in the y-axis gives:



So the coordinates of point B′ are $(-\cos(\theta),\ \sin(\theta))$. We also know that the coordinates of B′ are $(\cos(180-\theta),\ \sin(180-\theta))$ from the general rule about the coordinates of any point on the unit circle.

Equating the two coordinates for B′ gives us the following equations:

$$-\cos(\theta) = \cos(180-\theta)$$
$$\sin(\theta) = \sin(180-\theta)$$

So, to calculate the values of the sine and cosine ratios for angles up to 180°, we can use:

$$\mathbf{\cos(\theta) = -\cos(180-\theta)}$$
$$\mathbf{\sin(\theta) = \sin(180-\theta)}$$

Remember that if two angles sum to 180°, then they are supplements of each other. So if we are calculating the sine or cosine of an angle between 90° and 180°, then start by finding the supplement of the given angle.

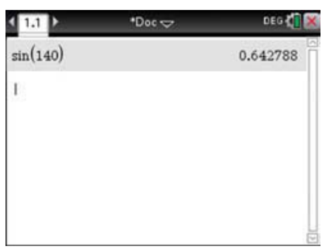
WORKED EXAMPLE 7

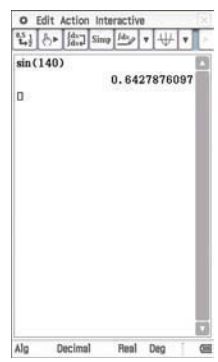
Find the values of:

a. $\sin(140°)$ **b.** $\cos(160°)$

giving your answers correct to 2 decimal places.

THINK	WRITE
a. 1. Calculate the supplement of the given angle.	**a.** $180° - 140° = 40°$
2. Calculate the sine of the supplement angle correct to 2 decimal places.	$\sin(40°) = 0.642\,787\ldots$ $= 0.64$ (to 2 decimal places)
3. The sine of an obtuse angle is equal to the sine of its supplement.	$\sin(140°) = \sin(40°)$ $= 0.64$ (to 2 decimal places)
b. 1. Calculate the supplement of the given angle.	**b.** $180° - 160° = 20°$
2. Calculate the cosine of the supplement angle correct to 2 decimal places.	$\cos(20°) = 0.939\,692\ldots$ $= 0.94$ (to 2 decimal places)
3. The cosine of an obtuse angle is equal to the negative cosine of its supplement.	$\cos(160°) = -\cos(20°)$ $= -0.94$ (to 2 decimal places)

TI | THINK	WRITE
1. Ensure your calculator is in DEGREE mode. On a Calculator page, complete the entry line as: sin(140) then press ENTER.	
2. The answer appears on the screen.	sin(140) = 0.64 (to 2 decimal places)

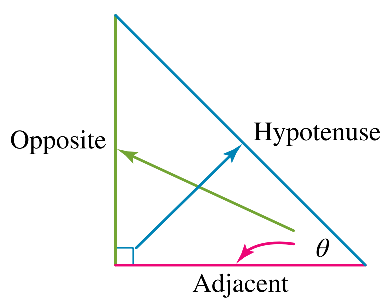
CASIO | THINK	WRITE
1. Ensure your calculator is in DEGREE mode. On the Main screen, complete the entry line as: sin(140) then press EXE.	
2. The answer appears on the screen.	sin(140) = 0.64 (to 2 decimal places)

9.2.7 SOH–CAH–TOA

Trigonometric ratios are relationships between the sides and angles of a right-angled triangle.

In solving trigonometric ratio problems for sine, cosine and tangent, we need to:

1. determine which ratio to use
2. write the relevant equation
3. substitute values from given information
4. make sure the calculator is in degree mode
5. solve the equation for the unknown lengths, or use the inverse trigonometric functions to find unknown angles.



To assist in remembering the trigonometric ratios, the mnemonic **SOH-CAH-TOA** has been developed.

SOH-CAH-TOA stands for:
- **S**ine is **O**pposite over **H**ypotenuse
- **C**osine is **A**djacent over **H**ypotenuse
- **T**angent is **O**pposite over **A**djacent.

on Resources

Interactivity: Trigonometric ratios (int-2577)

studyon

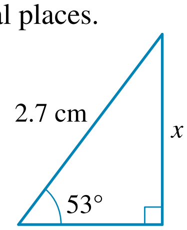
Units 1 & 2 > AOS 4 > Topic 3 > Concepts 1 & 2

Trigonometric ratios in rightangled triangles Concept summary and practice questions

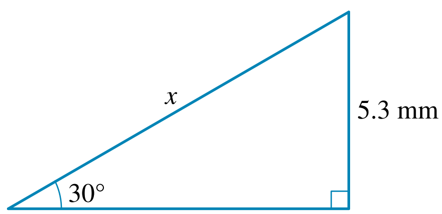
Trigonometric ratios of obtuse angles Concept summary and practice questions

Exercise 9.2 Trigonometric ratios

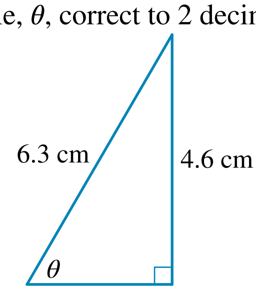
1. **WE1** Find the value of x correct to 2 decimal places.



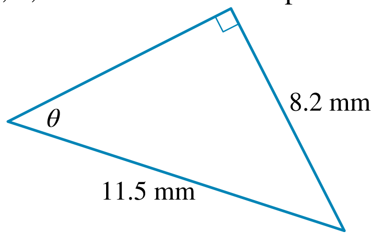
2. Find the value of x.



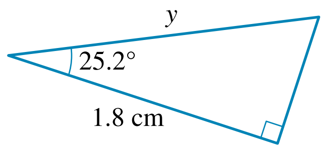
3. **WE2** Find the value of the unknown angle, θ, correct to 2 decimal places.



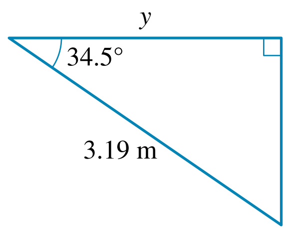
4. Find the value of the unknown angle, θ, correct to 2 decimal places.



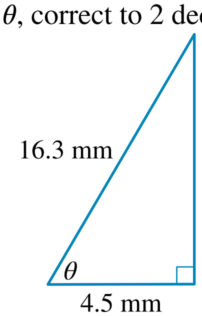
5. **WE3** Find the value of y correct to 2 decimal places.



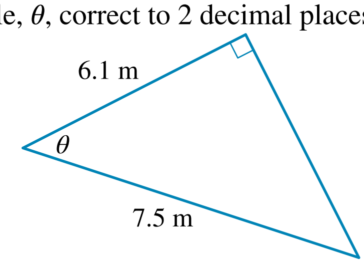
6. Find the value of y correct to 2 decimal places.



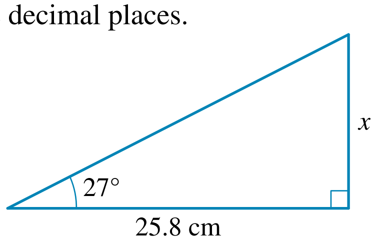
7. **WE4** Find the value of the unknown angle, θ, correct to 2 decimal places.



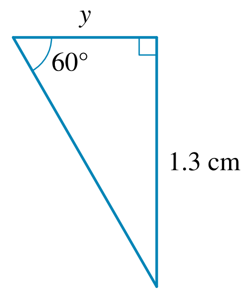
8. Find the value of the unknown angle, θ, correct to 2 decimal places.



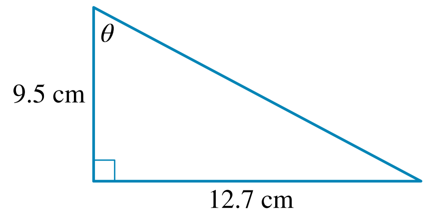
9. **WE5** Find the value of x correct to 2 decimal places.



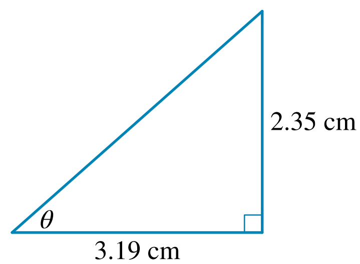
10. Find the value of y correct to 2 decimal places.



11. **WE6** Find the value of the unknown angle, θ, correct to 2 decimal places.



12. Find the value of the unknown angle, θ, correct to 2 decimal places.



13. **WE7** Find the values of:

a. $\sin(125°)$ **b.** $\cos(152°)$

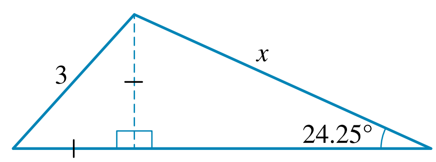
giving your answers correct to 2 decimal places.

14. Find the values of:

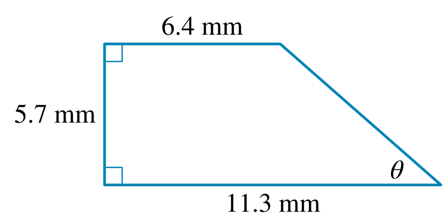
a. $\sin(99.2°)$ **b.** $\cos(146.7°)$

giving your answers correct to 2 decimal places.

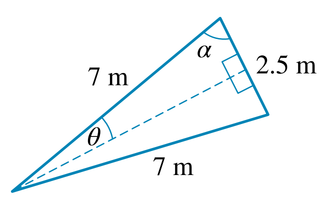
15. Find the value of x correct to 2 decimal places.



16. Find the value of the unknown angle, θ, correct to 2 decimal places.

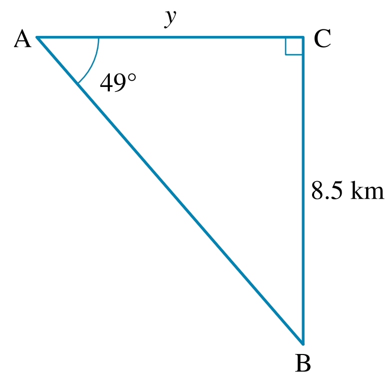


17. A kitesurfer has a kite of length 2.5 m and strings of length 7 m as shown. Find the values of the angles θ and α, correct to 2 decimal places.

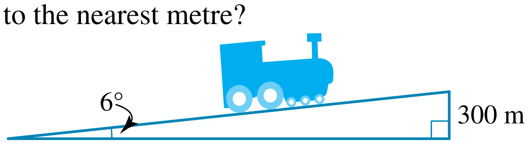


18. A daredevil is to be catapulted into the air in a capsule at an angle of 64.3° to the ground. Assuming the stuntman travels in a straight path, what horizontal distance will he have covered, correct to 2 decimal places, if he reached a height of 20 metres?

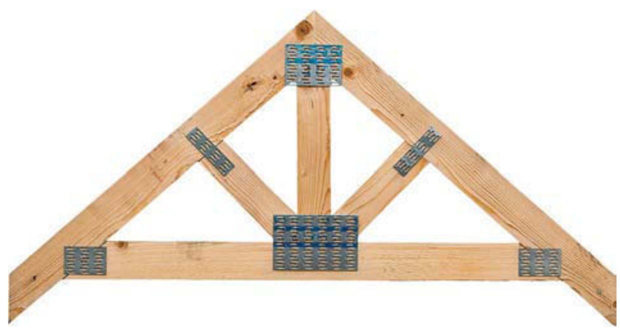
19. A yacht race follows a triangular course as shown below. Calculate, correct to 1 decimal place:
    a. the distance of the final leg, y
    b. the total distance of the course.



20. A railway line rises for 300 metres at a uniform slope of 6° with the horizontal. What is the distance travelled by the train, correct to the nearest metre?



21. A truss is used to build a section of a roof. If the vertical height of the truss is 1.5 metres and the span (horizontal distance between the walls) is 8 metres wide, calculate the pitch of the roof (its angle with the horizontal) correct to 1 decimal place.



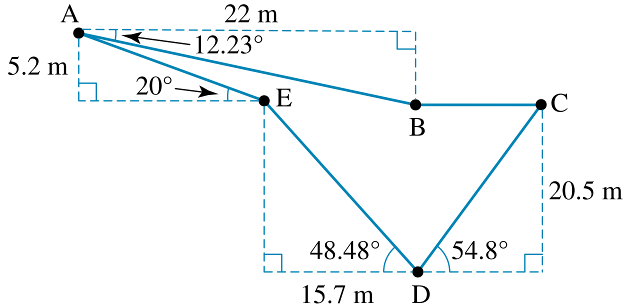
22. If 3.5 metres of Christmas lights are attached directly from the tip of the bottom branch to the top of a Christmas tree at an angle of 35.2° from the ground, how high is the Christmas tree, correct to 2 decimal places?

23. A 2.5 m ladder is placed against a wall. The base of the ladder is 1.7 m from the wall.
    a. Calculate the angle, correct to 2 decimal places, that the ladder makes with the ground.
    b. Find how far the ladder reaches up the wall, correct to 2 decimal places.

24. A school is building a wheelchair ramp of length 4.2 m to be inclined at an angle of 10.5°.
    a. Find the horizontal length of the ramp, correct to 1 decimal place.
    b. The ramp is too steep at 10.5°. Instead, the vertical height of the ramp needs to be 0.5 m and the ramp inclined at an angle of 5.7°. Calculate the new length of the wheelchair ramp, correct to 2 decimal places.

25. A play gym for monkeys is constructed at the zoo. A rope is tied from a tree branch 1.6 m above the ground to another tree branch 2.5 m above the ground. The monkey swings along the rope, which makes an angle of 11.87° to the horizontal. How far apart are the trees, correct to 2 decimal places?

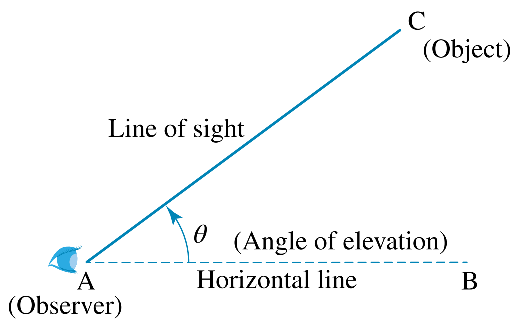
26. A dog training obstacle course ABCDEA is shown in the diagram below with point B vertically above point D. Find the total length of the obstacle course in metres, giving your answer correct to 2 decimal places.



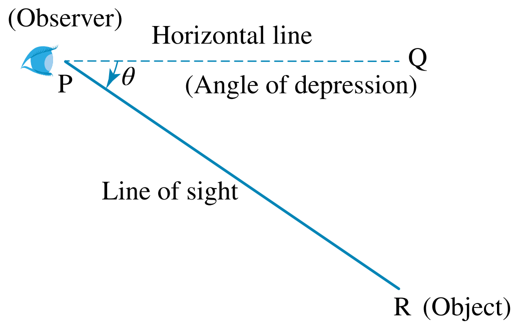
9.3 Applications of trigonometric ratios

9.3.1 Angles of elevation and depression

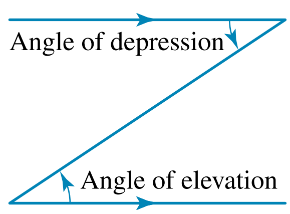
An **angle of elevation** is the angle between a horizontal line from the observer to an object that is above the horizontal line.



An **angle of depression** is the angle between a horizontal line from the observer to an object that is below the horizontal line.

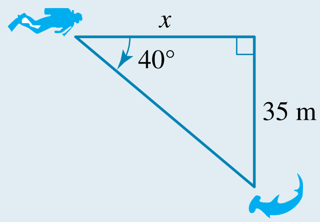
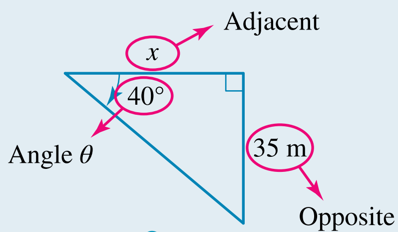


We use angles of elevation and depression to locate the positions of objects above or below the horizontal (reference) line. Angles of elevation and angles of depression are equal as they are alternate angles.



WORKED EXAMPLE 8

The angle of depression from a scuba diver at the water's surface to a hammerhead shark on the sea floor of the Great Barrier Reef is 40°. The depth of the water is 35 m. Calculate the horizontal distance from the scuba diver to the shark, correct to 2 decimal places.

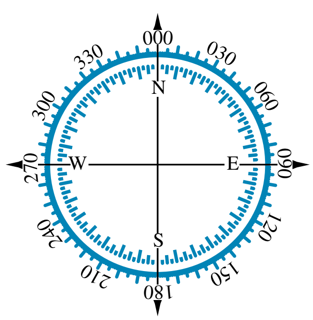
THINK	WRITE/DRAW
1. Draw a diagram to represent the information.	
2. Label all the given information on the triangle.	
3. Since we have been given the combination of opposite, adjacent and the reference angle θ, we need to use the tangent ratio. Substitute the given values into the ratio equation.	$\tan(\theta) = \frac{O}{A}$ $\tan(40°) = \frac{35}{x}$
4. Rearrange the equation to make the unknown the subject and solve. Make sure your calculator is in degree mode.	$x = \frac{35}{\tan(40°)}$ $= 41.71$ The horizontal distance from the scuba diver to the shark is 41.71 m.

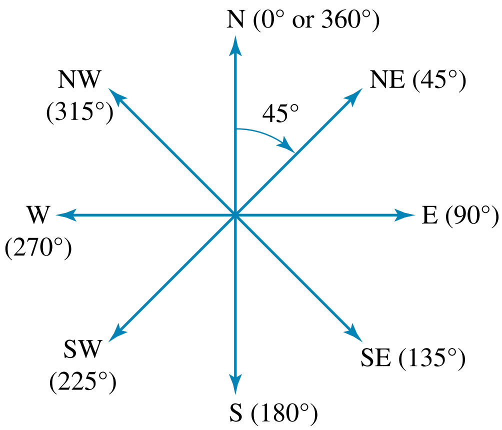
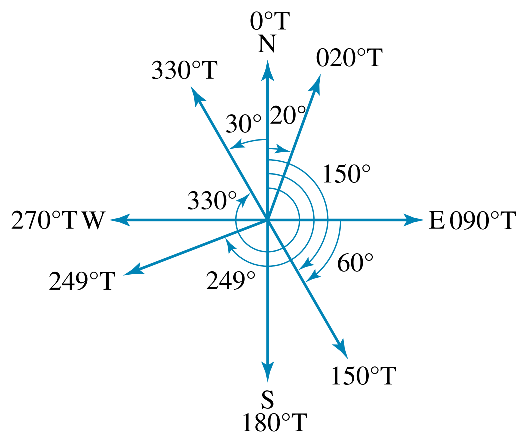
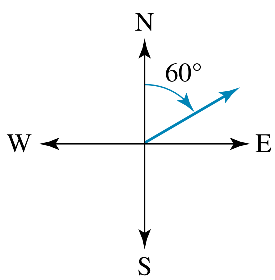
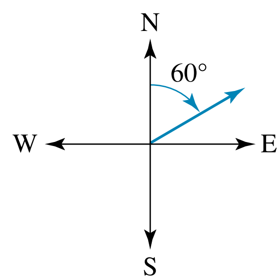
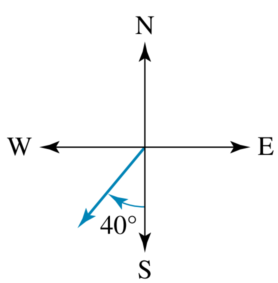
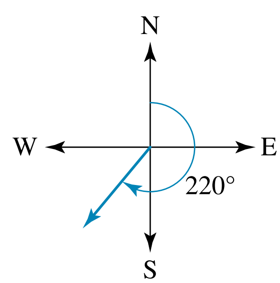
9.3.2 Bearings

Bearings are used to locate the positions of objects or the direction of a journey on a two-dimensional plane.

The four main directions or standard bearings of a directional compass are known as **cardinal points**. They are North (N), South (S), East (E) and West (W).

There are two types of bearings: **conventional (compass) bearings** and **true bearings**.



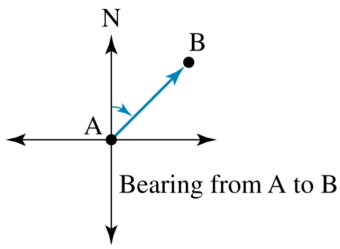
Conventional (compass) bearings	**True bearings**
Conventional bearings are written by first identifying whether the point is north or south, and then identifying whether the point is east or west.	True bearings are measured in a clockwise direction from the north–south line. They are written with all three digits of the angle stated.
The conventional bearing of a point is stated as the number of degrees east or west of the north–south line.	If the angle measured is less than 100°, place a zero in front of the angle. For example, if the angle measured is 20° from the north–south line the bearing is 020°T.
	
N60°E 	060°T 
S40°W 	220°T 

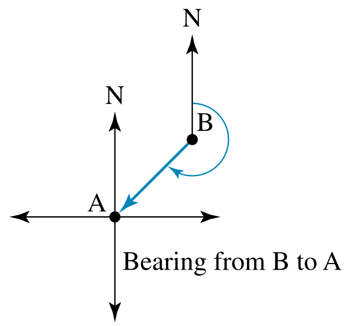
True bearings and conventional bearings are interchangeable, for example 030°T = N30°E.

For the remainder of this topic, we will only use true bearings.

9.3.3 Bearings from A to B

The bearing from A to B is **not** the same as the bearing from B to A.





When determining a bearing from a point to another point, it is important to follow the instructions and draw a diagram. Always draw the centre of the compass at the starting point of the direction requested.

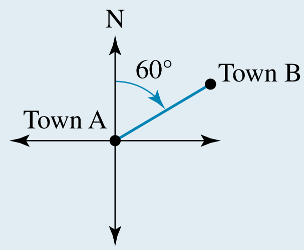
When a problem asks to find the bearing of B from A, mark in north and join a directional line to B to work out the bearing. To return to where you came from is a change in bearing of 180°.

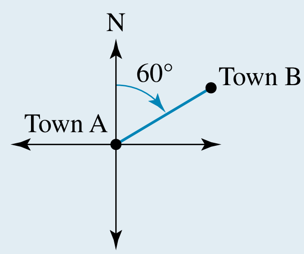
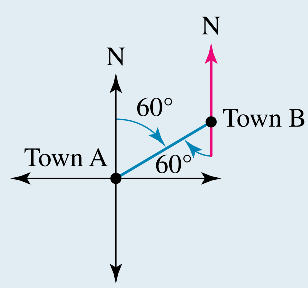
WORKED EXAMPLE 9

Find the true bearing from:

a. Town A to Town B

b. Town B to Town A.



THINK	WRITE/ DRAW
a. 1. To find the bearing from Town A to Town B, make sure the centre of the compass is marked at town A. The angle is measured clockwise from north to the bearing line at Town B.	**a.**  The angle measure from north is 60°.
2. A true bearing is written with all three digits of the angle followed by the letter T.	The true bearing from Town A to Town B is 060°T.
b. 1. To find the bearing from Town B to Town A, make sure the centre of the compass is marked at Town B. The angle is measured clockwise from north to the bearing line at Town A.	**b.**  The angle measure from north is 60° + 180° = 240°.
2. A true bearing is written with all three digits of the angle followed by the letter T.	The true bearing from Town B to Town A is 240°T.

9.3.4 Using trigonometry in bearings problems

As the four cardinal points (N, E, S, W) are at right angles to each other, we can use trigonometry to solve problems involving bearings.

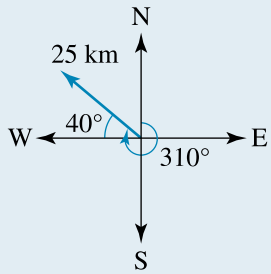
When solving a bearings problem with trigonometry, always start by drawing a diagram to represent the problem. This will help you to identify what information you already have, and determine which trigonometric ratio to use.

WORKED EXAMPLE 10

A boat travels for 25 km in a direction of 310°T.

a. How far north does the boat travel, correct to 2 decimal places?

b. How far west does the boat travel, correct to 2 decimal places?

THINK	WRITE/ DRAW
a. 1. Draw a diagram of the situation, remembering to label the compass points as well as all of the given information.	**a.** 
2. Identify the information you have in respect to the reference angle, as well as the information you need.	Reference angle = 40° Hypotenuse = 25 Opposite = ?
3. Determine which of the trigonometric ratios to use.	$\sin(\theta) = \frac{O}{H}$
4. Substitute the given values into the trigonometric ratio and solve for the unknown.	$\sin(40°) = \frac{O}{25}$ $25\ \sin(40°) = O$ $O = 16.069\ldots$ $= 16.07$ (to 2 decimal places)
5. Write the answer.	The ship travels 16.07 km north.
b. 1. Use your diagram from part **a** and identify the information you have in respect to the reference angle, as well as the information you need.	**b.** Reference angle = 40° Hypotenuse = 25 Adjacent = ?
2. Determine which of the trigonometric ratios to use.	$\cos(\theta) = \frac{A}{H}$
3. Substitute the given values into the trigonometric ratio and solve for the unknown.	$\cos(40°) = \frac{A}{25}$ $25\ \cos(40°) = A$ $A = 19.151\ldots$ $= 19.15$ (to 2 decimal places)
4. Write the answer.	The boat travels 19.15 km west.

Resources

Interactivity: Bearings (int-6481)

studyon

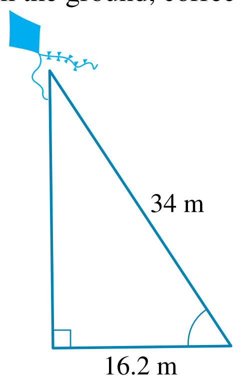
Units 1 & 2 | AOS 4 | Topic 3 | Concepts 3 & 4

Angles of elevation and depression Concept summary and practice questions

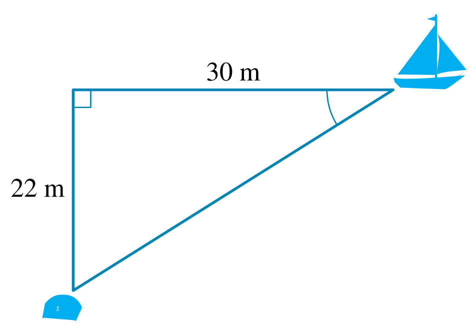
Bearings Concept summary and practice questions

Exercise 9.3 Applications of trigonometric ratios

1. **WE8** The angle of depression from a scuba diver at the water's surface to a hammerhead shark on the sea floor of the Great Barrier Reef is 41°. The depth of the water is 32 m. Calculate the horizontal distance from the scuba diver to the shark.
2. The angle of elevation from a hammerhead shark on the sea floor of the Great Barrier Reef to a scuba diver at the water's surface is 35°. The depth of the water is 33 m. Calculate the horizontal distance from the shark to the scuba diver.
3. Find the angle of elevation of the kite from the ground, correct to 2 decimal places.



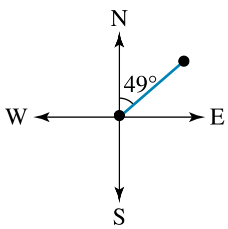
4. Find the angle of depression from the boat to the treasure at the bottom of the sea, correct to 2 decimal places.

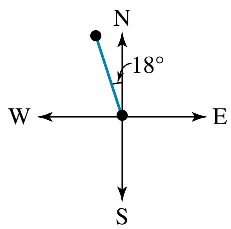


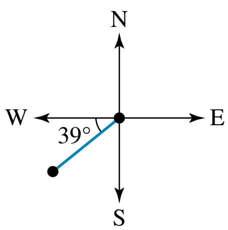
5. A crocodile is fed on a 'jumping crocodile tour' on the Adelaide River. The tour guide dangles a piece of meat on a stick at an angle of elevation of 60° from the boat, horizontal to the water. If the stick is 2 m long and held 1 m above the water, find the vertical distance the crocodile has to jump out of the water to get the meat, correct to 2 decimal places.

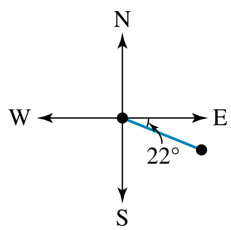
6. A ski chair lift operates from the Mt Buller village and has an angle of elevation of 45° to the top of the Federation ski run. If the vertical height is 707 m, calculate the ski chair lift length, correct to 2 decimal places.

7. State each of the following as a true bearing.

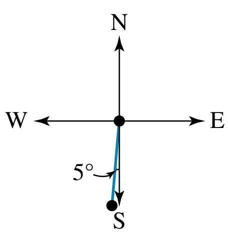
a. 

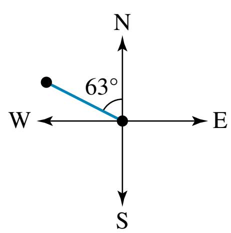
b. 

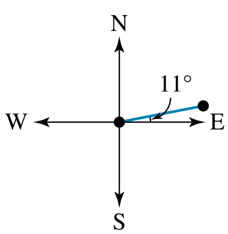
c. 

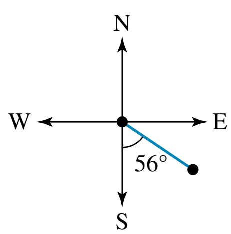
d. 

8. State each of the following as a true bearing.

a. 

b. 

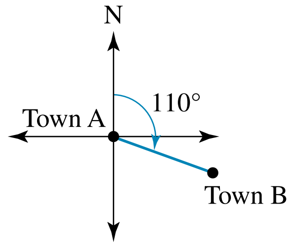
c. 

d. 

9. **WE9** In the figure, find the true bearing from:

a. Town A to Town B

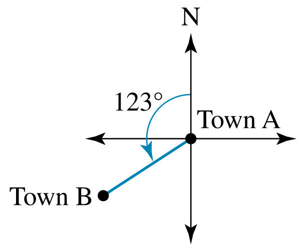
b. Town B to Town A.



10. In the figure, find the true bearing from:

a. Town A to Town B

b. Town B to Town A.

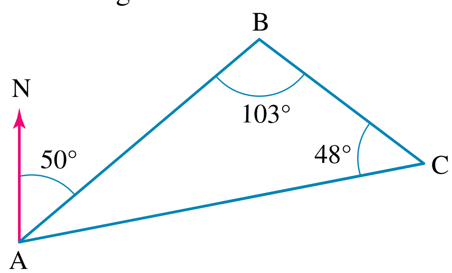


11. From the figure below, find the true bearing of

a. B from A

b. C from B

c. A from C.



12. **WE10** A boat travels for 36 km in a direction of 155°T.

a. How far south does the boat travel, correct to 2 decimal places?

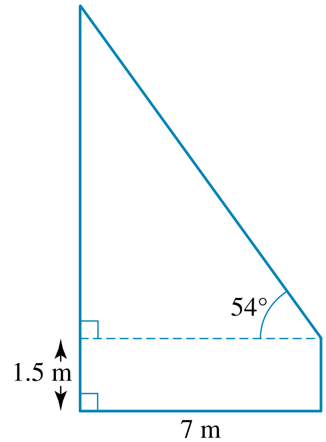
b. How far east does the boat travel, correct to 2 decimal places?

13. A boat travels north for 6 km, west for 3 km, then south for 2 km. What is the boat's true bearing from its starting point? Give your answer in decimal form to 1 decimal place.

14. A student uses an inclinometer to measure an angle of elevation of 50° from the ground to the top of Uluru (Ayers Rock). If the student is standing 724 m from the base of Uluru, determine the height of Uluru correct to 2 decimal places.

15. A tourist looks down from the Eureka Tower's Edge on the Skydeck to see people below on the footpath. If the angle of depression is 88° and the people are 11 m from the base of the tower, how high up is the tourist standing in the glass cube?

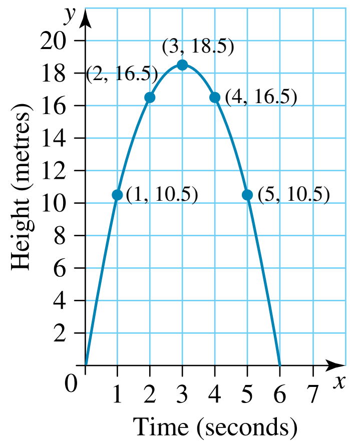
16. A student uses an inclinometer to measure the height of his house. The angle of elevation is 54°. He is 1.5 m tall and stands 7 m from the base of the house. Calculate the height of the house correct to 2 decimal places.



17. A tourist 1.72 m tall is standing 50 m away from the base of the Sydney Opera House. The Opera House is 65 m tall. Calculate the angle of elevation, to the nearest degree, from the tourist to the top of the Opera House.

18. A parachutist falls from a height of 5000 m to the ground while travelling over a horizontal distance of 150 m. What was the angle of depression of the descent, correct to 2 decimal places?
19. Air traffic controllers in two control towers, which are both 87 m high, spot a plane at an altitude of 500 metres. The angle of elevation from tower A to the plane is 5° and from tower B to the plane is 7°. Find the distance between the two control towers correct to the nearest metre.

20. A footballer takes a set shot at goal, with the graph showing the path that the ball took as it travelled towards the goal.



If the footballer's eye level is at 1.6 metres, calculate the angle of elevation from his eyesight to the ball, correct to 2 decimal places, after:

a. 1 second **b.** 2 seconds **c.** 3 seconds **d.** 4 seconds **e.** 5 seconds.

9.4 The sine rule

9.4.1 The sine rule

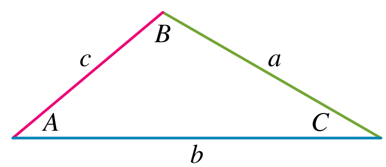
The **sine rule** can be used to find the side length or angle in non-right-angled triangles.

To help us solve non-right-angled triangle problems, the labelling convention of a non-right-angled triangle, ABC, is as follows:

Angle A is opposite side length a.

Angle B is opposite side length b.

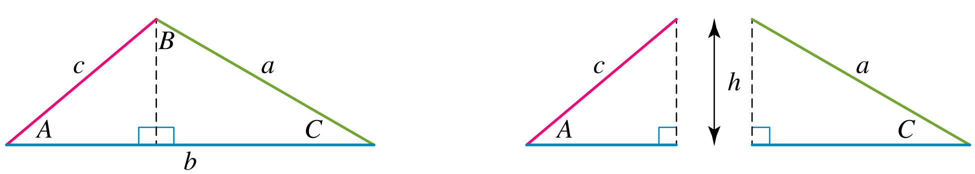
Angle C is opposite side length c.



The largest angle will always be opposite the longest side length, and the smallest angle will always be opposite to the smallest side length.

9.4.2 Formulating the sine rule

We can divide an acute non-right-angled triangle into two right-angled triangles as shown in the following diagrams.



If we apply trigonometric ratios to the two right-angled triangles we get:

$$\frac{h}{c} = \sin(A) \quad \text{and} \quad \frac{h}{a} = \sin(C)$$

$$h = c\sin(A) \qquad h = a\sin(C)$$

Equating the two expressions for h gives:

$$c \sin(A) = a \sin(C)$$
$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

In a similar way, we can split the triangle into two using side a as the base, giving us:

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

This gives us the sine rule:

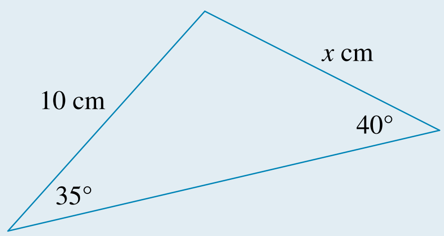
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

We can apply the sine rule to determine all of the angles and side lengths of a triangle if we are given either:

- 2 side lengths and 1 corresponding angle
- 1 side length and 2 angles.

WORKED EXAMPLE 11

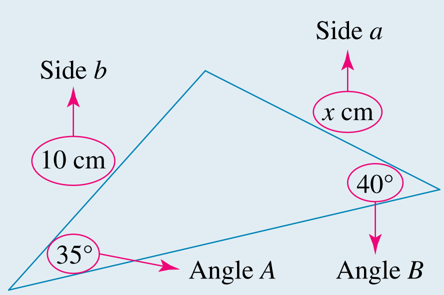
Find the value of the unknown length x, correct to 2 decimal places.



THINK

1. Label the triangle with the given information, using the conventions for labelling.
   Angle A is opposite to side a.
   Angle B is opposite to side b.

WRITE/DRAW



2. Substitute the known values into the sine rule.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$
$$\frac{x}{\sin(35°)} = \frac{10}{\sin(40°)}$$

3. Rearrange the equation to make x the subject and solve.
   Make sure your calculator is in degree mode.

$$\frac{x}{\sin(35°)} = \frac{10}{\sin(40°)}$$
$$x = \frac{10 \sin(35°)}{\sin(40°)}$$
$$x = 8.92$$

4. Write the answer.

The unknown side length x is 8.92 cm.

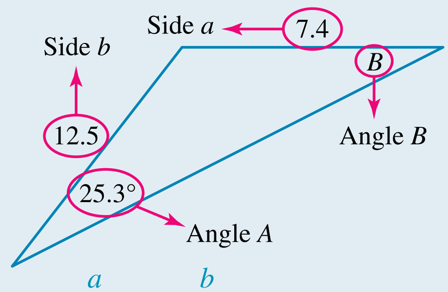
WORKED EXAMPLE 12

A non-right-angled triangle has values of side b = 12.5, angle A = 25.3° and side a = 7.4. Calculate the value of angle B, correct to 2 decimal places.

THINK

1. Draw a non-right-angled triangle, labelling with the given information.
   Angle A is opposite to side a.
   Angle B is opposite to side b.

WRITE/DRAW



2. Substitute the known values into the sine rule.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$
$$\frac{7.4}{\sin(25.3°)} = \frac{12.5}{\sin(B)}$$

3. Rearrange the equation to make $\sin(B)$ the subject and solve.
   Make sure your calculator is in degree mode.

$$\frac{7.4}{\sin(25.3°)} = \frac{12.5}{\sin(B)}$$
$$7.4\ \sin(B) = 12.5\ \sin(25.3°)$$
$$\sin(B) = \frac{12.5\ \sin(25.3°)}{7.4}$$
$$B = \sin^{-1}\left(\frac{12.5\ \sin(25.3°)}{7.4}\right)$$
$$= 46.21°$$

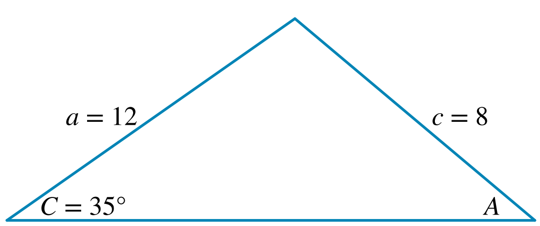
4. Write the answer.

Angle B is 46.21°.

9.4.3 The ambiguous case of the sine rule

When we are given two sides lengths of a triangle and an acute angle opposite one of these side lengths, there are two different triangles we can draw. So far we have only dealt with triangles that have all acute angles; however, it is also possible to draw triangles with an obtuse angle. This is known as the ambiguous case of the sine rule.

For example, take the triangle ABC, where $a = 12$, $c = 8$ and $C = 35°$.



When we solve this for angle A we get an acute angle as shown:

$$\frac{8}{\sin(35°)} = \frac{12}{\sin(A)}$$

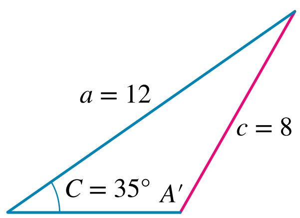
$$8\sin(A) = 12\sin(35°)$$

$$\sin(A) = \frac{12\sin(35°)}{8}$$

$$A = \sin^{-1}\left(\frac{12\sin(35°)}{8}\right)$$

$$= 59.36°$$

However, there is also an obtuse-angled triangle that can be drawn from this given information.

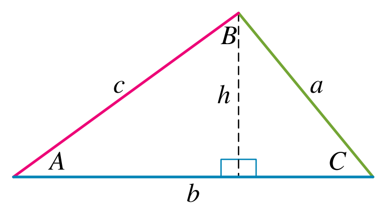


In this case, the size of the obtuse angle is the supplement of the acute angle calculated previously.

$$\begin{aligned} A' &= 180° - A \\ &= 180° - 59.36° \\ &= 120.64° \end{aligned}$$

9.4.4 Determining when we can use the ambiguous case

The ambiguous case of the sine rule does not work for every example. This is due to the way the ratios are set in the development of the sine rule, since side length a must be longer than h, where h is the length of the altitude from angle B to the base line b.



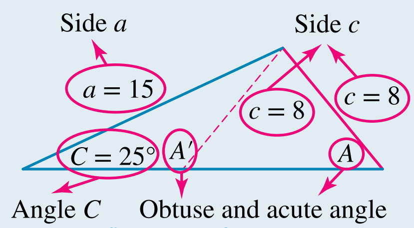
For the ambiguous case to be applicable, the following conditions must be met:

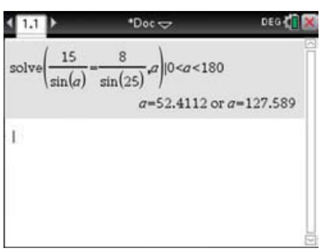
- The given angle must be acute.
- The adjacent side must be bigger than the opposite side.
- The opposite side must be bigger than the adjacent side multiplied by the sine of the given angle.

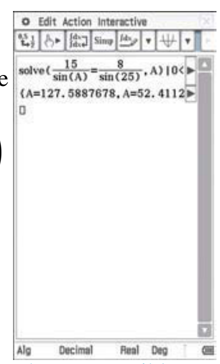
When using the sine rule to calculate a missing angle, it is useful to first identify whether the ambiguous case is applicable to the problem or not.

WORKED EXAMPLE 13

Find the two possible values of angle A for triangle ABC, given $a = 15$, $c = 8$ and $C = 25°$. Give your answers correct to 2 decimal places.

THINK	WRITE/DRAW
1. Draw a non-right-angled triangle, labelling with the given information. Angle A is opposite to side a. Angle C is opposite to side c. Note that two triangles can be drawn, with angle A being either acute or obtuse.	
2. Substitute the known values into the sine rule.	$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$ $\frac{15}{\sin(A)} = \frac{8}{\sin(25°)}$
3. Rearrange the equation to make $\sin(A)$ the subject and solve. Make sure your calculator is in degree mode. The calculator will only give the acute angle value.	$\frac{15}{\sin(A)} = \frac{8}{\sin(25°)}$ $15\sin(25°) = 8\sin(A)$ $\sin(A) = \frac{15\sin(25°)}{8}$ $A = \sin -1\left(\frac{15\sin(25°)}{8}\right)$ $= 52.41°$
4. Solve for the obtuse angle A'.	$A' = 180° - A$ $= 180° - 52.41°$ $= 127.59°$
5. Write the answer.	The two possible values for A are $52.41°$ and $127.59°$.

TI | THINK	WRITE
1. Ensure your calculator is in DEGREE mode. On a Calculator page, press MENU then select: 3: Algebra 1: Solve Complete the entry line as: $\text{solve}\left(\frac{15}{\sin(a)} = \frac{8}{\sin(25)}, a\right) \mid 0 < a < 180$ then press ENTER.	
2. The answer appears on the screen.	The two possible values for A are 52.41° and 127.59°.

CASIO | THINK	WRITE
1. Ensure your calculator is in DEGREE mode. On the Main screen, complete the entry line as: $\text{solve}\left(\frac{15}{\sin(A)} = \frac{8}{\sin(A)}, A\right)$ $\mid 0 < A < 180$ then press EXE.	
2. The answer appears on the screen.	The two possible values for A are 52.41° and 127.59°.

on Resources

Interactivity: The sine rule (int-6275)

Interactivity: Solving non-right-angled triangles (int-6482)

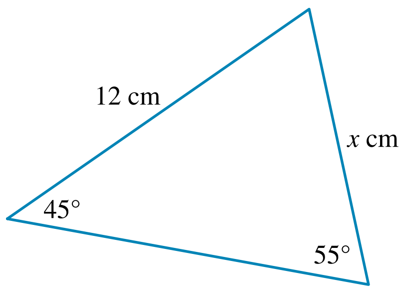
studyon

Units 1 & 2 > AOS 4 > Topic 3 > Concept 5

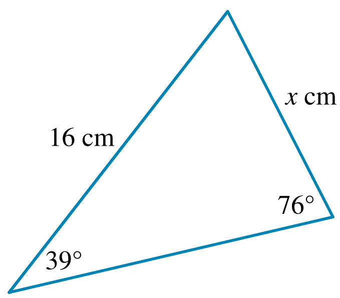
The sine rule Concept summary and practice questions

Exercise 9.4 The sine rule

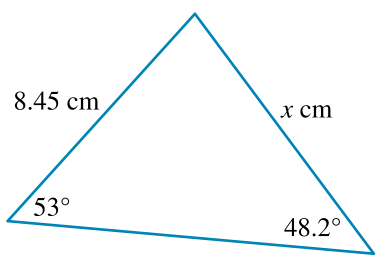
1. **WE11** Find the value of the unknown length x correct to 2 decimal places.



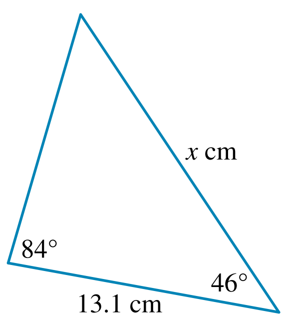
2. Find the value of the unknown length x correct to 2 decimal places.



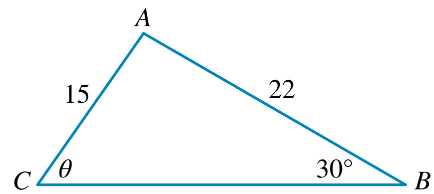
3. Find the value of the unknown length x correct to 2 decimal places.



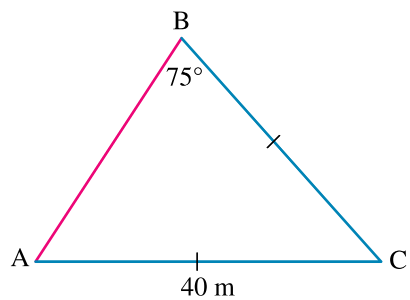
4. Find the value of the unknown length x correct to the nearest cm.



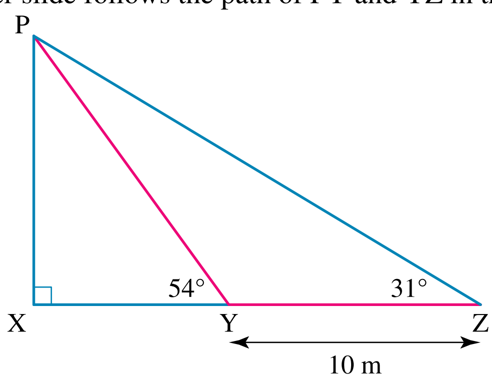
5. **WE12** A non-right-angled triangle has values of side $b = 10.5$, angle $A = 22.3°$ and side $a = 8.4$. Calculate the value of angle B correct to 1 decimal place.
6. A non-right-angled triangle has values of side $b = 7.63$, angle $A = 15.8°$ and side $a = 4.56$. Calculate the value of angle B correct to 1 decimal place.
7. For triangle ABC shown below, find the acute value of θ correct to 1 decimal place.



8. **WE13** Find the two possible values of angle A for triangle ABC, given $a = 8$, $c = 6$ and $C = 43°$. Give your answers correct to 2 decimal places.
9. Find the two possible values of angle A for triangle ABC, given $a = 7.5$, $c = 5$ and $C = 32°$. Give your answers correct to 2 decimal places.
10. If triangle ABC has values $b = 19.5$, $A = 25.3°$ and $a = 11.4$, find both possible angle values of B correct to 2 decimal places.
11. Find all the side lengths, correct to 2 decimal places, for the triangle ABC, given $a = 10.5$, $B = 60°$ and $C = 72°$.
12. Find the acute and obtuse angles that have a sine value of approximately 0.573 58. Give your answer to the nearest degree.
13. Part of a roller-coaster track is in the shape of an isosceles triangle, ABC, as shown in the following triangle. Calculate the track length AB correct to 2 decimal places.



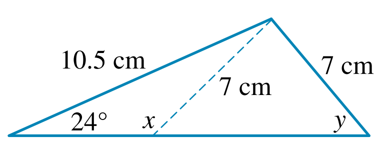
14. The shape and length of a water slide follows the path of PY and YZ in the following diagram.



Calculate, correct to 2 decimal places:

a. the total length of the water slide
b. the height of the water slide, PX.

15. Find the two unknown angles shown in the diagram below, correct to 1 decimal place.



16. In the triangle ABC, $a = 11.5$ m, $c = 6.5$ m and $C = 25°$.
   a. Draw the two possible triangles with this information.
   b. Find the two possible values of angle A, and hence the two possible values of angle B. Give all values correct to 2 decimal places.
17. At a theme park, the pirate ship swings back and forth on a pendulum. The centre of the pirate ship is secured by a large metal rod that is 5.6 metres in length. If one of the swings covers an angle of 122°, determine the distance between the point where the rod meets the ship at both extremes of the swing. Give your answers correct to 2 decimal places.

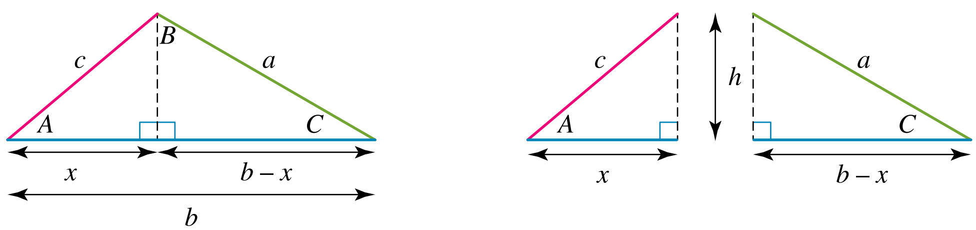
18. Andariel went for a ride on her dune buggy in the desert. She rode east for 6 km, then turned 125° to the left for the second stage of her ride. After 5 minutes riding in the same direction, she turned to the left again, and from there travelled the 5.5 km straight back to her starting position.
   How far did Andariel travel in the second section of her ride, correct to 2 decimal places?

9.5 The cosine rule

9.5.1 Formulating the cosine rule

The **cosine rule**, like the sine rule, is used to find the length or angle in a non-right-angled triangle. We use the same labelling conventions for non-right-angled triangles as when using the sine rule.

As with the sine rule, the cosine rule is derived from a non-right-angled triangle being divided into two right-angled triangles, where the base side lengths are equal to $(b - x)$ and x.



Using Pythagoras' theorem we get:

$$c^2 = x^2 + h^2 \qquad a^2 = (b-x)^2 + h^2$$
$$h^2 = c^2 - x^2 \quad \text{and} \quad h^2 = a^2 - (b-x)^2$$

Equating the two expressions for h^2 gives:

$$c^2 - x^2 = a^2 - (b-x)^2$$
$$a^2 = (b-x)^2 + c^2 - x^2$$
$$a^2 = b^2 - 2bx + c^2$$

Substituting the trigonometric ratio $x = c\ \cos(A)$ from the right-angled triangle into the expression, we get:

$$\begin{aligned} a^2 &= b^2 - 2b\,(c\cos(A)) + c^2 \\ &= b^2 + c^2 - 2bc\cos(A) \end{aligned}$$

This is known as the cosine rule, and we can interchange the pronumerals to get:

$$\mathbf{a^2 = b^2 + c^2 - 2bc\ \cos(A)}$$
$$\mathbf{b^2 = a^2 + c^2 - 2ac\ \cos(B)}$$
$$\mathbf{c^2 = a^2 + b^2 - 2ab\ \cos(C)}$$

We can apply the cosine rule to determine all of the angles and side lengths of a triangle if we are given either:

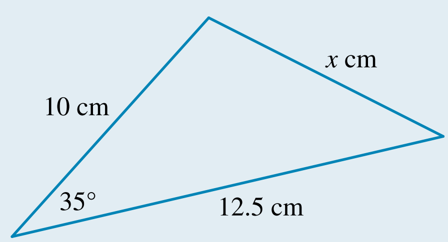
- 3 side lengths or
- 2 side lengths and the included angle.

The cosine rule can also be transposed to give:

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

WORKED EXAMPLE 14

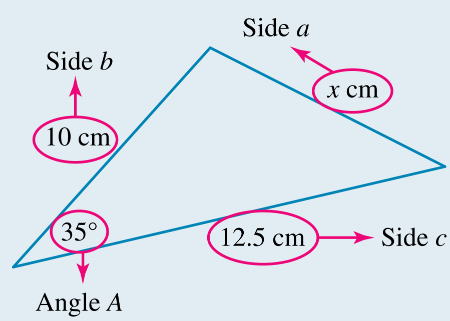
Find the value of the unknown length x correct to 2 decimal places.



THINK

1. Draw the non-right-angled triangle, labelling with the given information.
   Angle A is opposite to side a.
   If three side lengths and one angle are given, always label the angle as A and the opposite side as a.

WRITE/DRAW



2. Substitute the known values into the cosine rule.

$a^2 = b^2 + c^2 - 2bc\cos(A)$

$x^2 = 10^2 + 12.5^2 - 2 \times 10 \times 12.5\cos(35°)$

3. Solve for x.
Make sure your calculator is in degree mode.

$x^2 = 51.462$

$x = \sqrt{51.462}$

≈ 7.17

4. Write the answer.

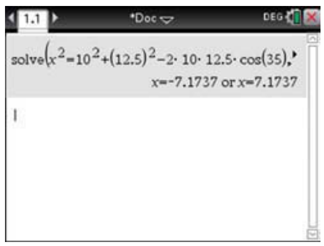
The unknown length x is 7.17 cm.

TI | THINK

1. Ensure your calculator is in DEGREE mode.
On a Calculator page, press MENU then select:
3: Algebra
1: Solve
Complete the entry line as:
solve($x^2 = 10^2 + 12.5^2 - 2 \times 10 \times 12.5\cos(35), x$)
then press ENTER.

2. Reject the negative solution as lengths can only be positive.

WRITE



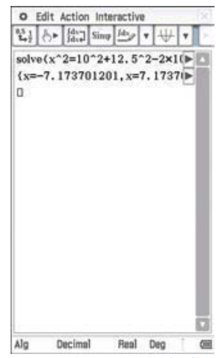
$x = 7.17$ cm (to 2 decimal places)

CASIO | THINK

1. Ensure your calculator is in DEGREE mode.
On the Main screen, complete the entry line as:
solve($x^2 = 10^2 + 12.5^2 - 2 \times 10 \times 12.5\cos(35), x$)
then press EXE.

2. Reject the negative solution as lengths can only be positive.

WRITE



$x = 7.17$ cm (to 2 decimal places)

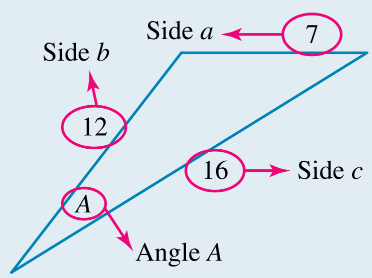
WORKED EXAMPLE 15

A non-right-angled triangle ABC has values $a = 7$, $b = 12$ and $c = 16$. Find the magnitude of angle A correct to 2 decimal places.

THINK

WRITE/DRAW

1. Draw the non-right-angled triangle, labelling with the given information.



2. Substitute the known values into the cosine rule.

$a^2 = b^2 + c^2 - 2bc\cos(A)$

$7^2 = 12^2 + 16^2 - 2 \times 12 \times 16\cos(A)$

3. Rearrange the equation to make cos (A) the subject and solve.
Make sure your calculator is in degree mode.

$$\cos(A) = \frac{12^2 + 16^2 - 7^2}{2 \times 12 \times 16}$$

$$A = \cos^{-1}\left(\frac{12^2 + 16^2 - 7^2}{2 \times 12 \times 16}\right)$$

$$\approx 23.93°$$

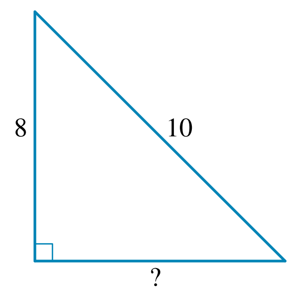
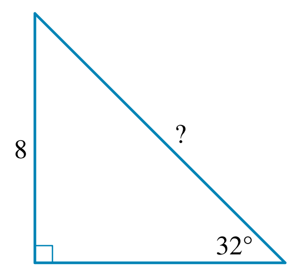
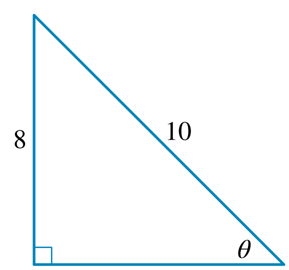
4. Write the answer.

The magnitude of angle A is 23.93°.

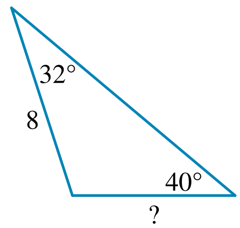
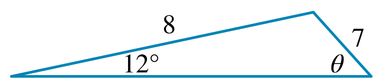
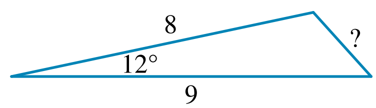
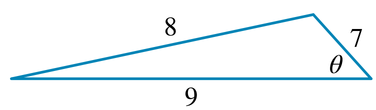
Note: In the example above, it would have been quicker to substitute the known values directly into the transposed cosine rule for cos (A).

9.5.2 Sets of sufficient information to determine a triangle

Knowing which rule to use for different problems will save time and help to reduce the chance for errors to appear in your working. The following table should help you determine which rule to use.

Type of triangle	What you want	What you know	What to use	Rule	Example
Right-angled	Side length	Two other sides	Pythagoras' theorem	$a^2 + b^2 = c^2$	
	Side length	A side length and an angle	Trigonometric ratios	$\sin(\theta) = \frac{\text{O}}{\text{H}}$ $\cos(\theta) = \frac{\text{A}}{\text{H}}$ $\tan(\theta) = \frac{\text{O}}{\text{A}}$	
	Angle	Two side lengths	Trigonometric ratios	$\sin(\theta) = \frac{\text{O}}{\text{H}}$ $\cos(\theta) = \frac{\text{A}}{\text{H}}$ $\tan(\theta) = \frac{\text{O}}{\text{A}}$	

Continued

Type of triangle	What you want	What you know	What to use	Rule	Example
Non-right-angled	Side length	Angle opposite unknown side and another side/angle pair	Sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$	
	Angle	Side length opposite unknown side and another side/angle pair	Sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$	
	Side length	Two sides and the angle between them	Cosine rule	$a^2 = b^2 + c^2 - 2bc\cos(A)$	
	Angle	Three sides	Cosine rule	$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$	

on Resources

Interactivity: The cosine rule (int-6276)

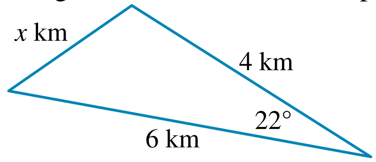
studyon

Units 1 & 2 > AOS 4 > Topic 3 > Concept 6

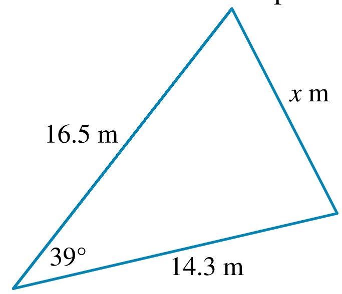
The cosine rule Concept summary and practice questions

Exercise 9.5 The cosine rule

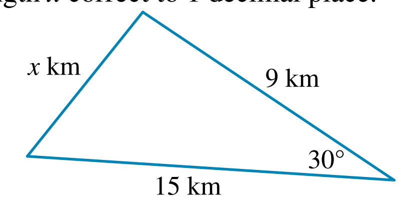
1. **WE14** Find the value of the unknown length x correct to 2 decimal places.



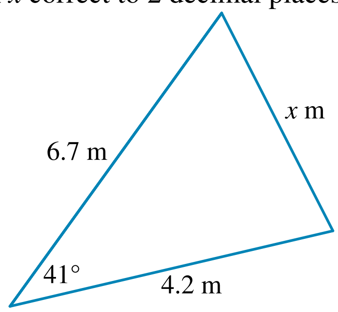
2. Find the value of the unknown length x correct to 2 decimal places.



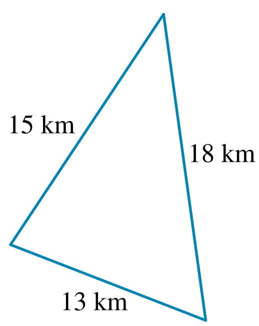
3. Find the value of the unknown length x correct to 1 decimal place.



4. Find the value of the unknown length x correct to 2 decimal places.

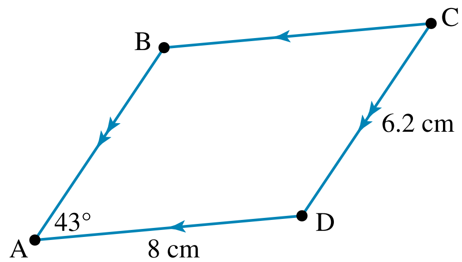


5. **WE15** A non-right-angled triangle ABC has values $a = 8$, $b = 13$ and $c = 17$. Find the magnitude of angle A correct to 2 decimal places.
6. A non-right-angled triangle ABC has values $a = 11$, $b = 9$ and $c = 5$. Find the magnitude of angle A correct to 2 decimal places.
7. For triangle ABC, find the magnitude of angle A correct to 2 decimal places, given $a = 5$, $b = 7$ and $c = 4$.
8. For triangle ABC with $a = 12$, $B = 57°$ and $c = 8$, find the side length b correct to 2 decimal places.
9. Find the largest angle, correct to 2 decimal places, between any two legs of the following sailing course.

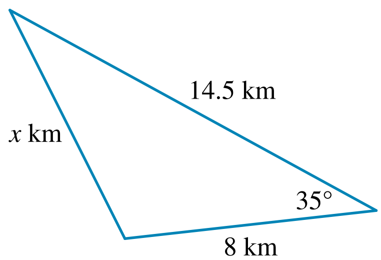


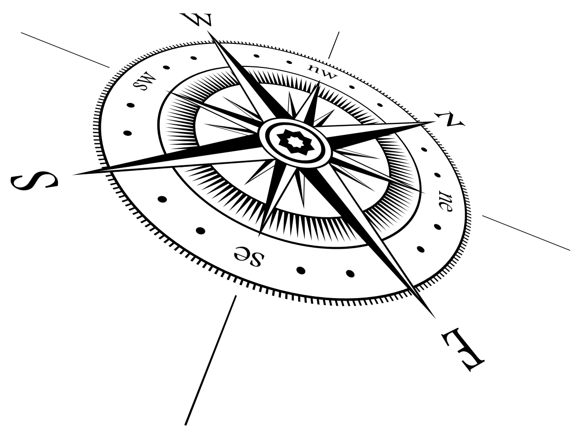
10. A triangular paddock has sides of length 40 m, 50 m and 60 m. Find the magnitude of the largest angle between the sides, correct to 2 decimal places.
11. A triangle has side lengths of 5 cm, 7 cm and 9 cm. Find the size of the smallest angle correct to 2 decimal places.

12. ABCD is a parallelogram. Find the length of the diagonal AC correct to 2 decimal places.

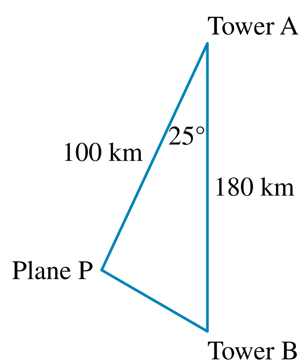


13. An orienteering course is shown in the following diagram. Find the total distance of the course correct to 2 decimal places.





14. Two air traffic control towers are 180 km apart. At the same time, they both detect a plane, P. The plane is at a distance 100 km from Tower A at the bearing shown in the diagram below. Find the distance of the plane from Tower B correct to 2 decimal places.



15. Britney is mapping out a new running path around her local park. She is going to run west for 2.1 km, before turning 105° to the right and running another 3.3 km. From there, she will run in a straight line back to her starting position.
How far will Britney run in total? Give your answer correct to the nearest metre.

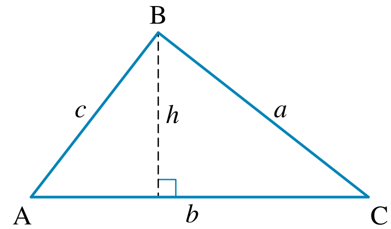
16. A cruise boat is travelling to two destinations. To get to the first destination it travels for 4.5 hours at a speed of 48 km/h. From there, it takes a 98° turn to the left and travels for 6 hours at a speed of 54 km/h to reach the second destination.
The boat then travels directly back to the start of its journey. How long will this leg of the journey take if the boat is travelling at 50 km/h? Give your answer correct to the nearest minute.

9.6 Area of triangles

9.6.1 Area of triangles

You should be familiar with calculating the area of a triangle using the rule: area $= \frac{1}{2}bh$ where b is the base length and h is the perpendicular height of the triangle. However, for many triangles we are not given the perpendicular height, so this rule cannot be directly used.

Take the triangle ABC as shown below.



If h is the perpendicular height of this triangle, then we can calculate the value of h by using the sine ratio:

$$\sin(A) = \frac{h}{c}$$

Transposing this equation gives $h = c\sin(A)$, which we can substitute into the rule for the area of the triangle to give:

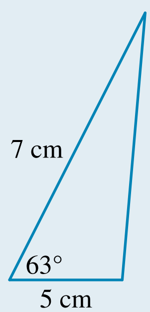
$$\textbf{Area} = \frac{1}{2}bc\sin(A)$$

Note: We can label any sides of the triangle a, b and c, and this formula can be used as long as we have the length of two sides of a triangle and know the value of the included angle.

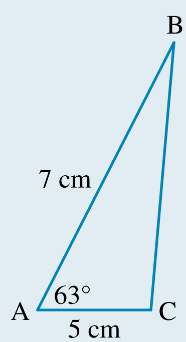
WORKED EXAMPLE 16

Find the area of the following triangles. Give both answers correct to 2 decimal places.

a.



b. **A triangle with sides of length 8 cm and 7 cm, and an included angle of 55°.**

THINK	WRITE/DRAW
a. 1. Label the vertices of the triangle.	a. 

2. Write down the known information. | $b = 5\text{ cm}$

$c = 7\text{ cm}$

$A = 63°$

3. Substitute the known values into the formula to calculate the area of the triangle.

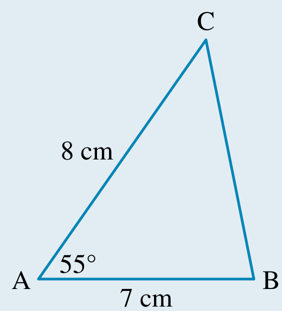
$$\begin{aligned}\text{Area} &= \frac{1}{2}bc\sin(A)\\ &= \frac{1}{2}\times 5\times 7\times \sin(63°)\\ &= 15.592\ldots\\ &= 15.59 \text{ (to 2 decimal places)}\end{aligned}$$

4. Write the answer, remembering to include the units.

The area of the triangle is 15.59 cm^2 correct to 2 decimal places.

b. 1. Draw a diagram to represent the triangle.

b.



2. Write down the known information.

$b = 8\text{ cm}$
$c = 7\text{ cm}$
$A = 55°$

3. Substitute the known values into the formula to calculate the area of the triangle.

$$\begin{aligned}\text{Area} &= \frac{1}{2}bc\sin(A)\\ &= \frac{1}{2}\times 8\times 7\times \sin(55°)\\ &= 22.936\ldots\\ &= 22.94 \text{ (to 2 decimal places)}\end{aligned}$$

4. Write the answer, remembering to include the units.

The area of the triangle is 22.94 cm^2 correct to 2 decimal places.

9.6.2 Heron's formula

As shown in Topic 7, we can also use Heron's formula to calculate the area of a triangle.

To use Heron's formula, we need to know the length of all three sides of the triangle.

$$\textbf{Area} = \sqrt{s(s-a)(s-b)(s-c)},\quad \textbf{where}\ s = \frac{a+b+c}{2}$$

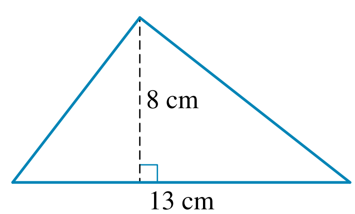
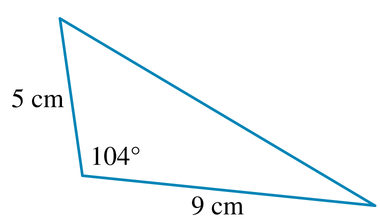
WORKED EXAMPLE 17

Find the area of a triangle with sides of 4 cm, 7 cm and 9 cm, giving your answer correct to 2 decimal places.

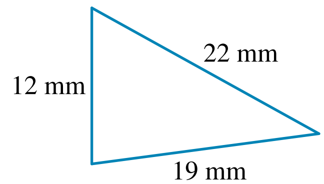
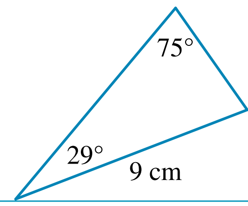
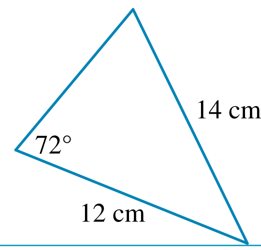
THINK	WRITE
1. Write down the known information.	$a = 4$ cm $b = 7$ cm $c = 9$ cm
2. Calculate the value of s (the semi-perimeter).	$s = \frac{a+b+c}{2}$ $= \frac{4+7+9}{2}$ $= \frac{20}{2}$ $= 10$
3. Substitute the values into Heron's formula to calculate the area.	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$ $= \sqrt{10(10-4)(10-7)(10-9)}$ $= \sqrt{10 \times 6 \times 3 \times 1}$ $= \sqrt{180}$ $= 13.416\ldots$ ≈ 13.42
4. Write the answer, remembering to include the units.	The area of the triangle is 13.42 cm^2 correct to 2 decimal places.

9.6.3 Determining which formula to use

In some situations you may have to perform some calculations to determine either a side length or angle size before calculating the area. This may involve using the sine or cosine rule. The following table should help if you are unsure what to do.

Given	What to do	Example
The base length and perpendicular height	Use area $= \frac{1}{2}bh$.	
Two side lengths and the included angle	Use area $= \frac{1}{2}bc\sin(A)$.	

Continued

Given	What to do	Example
Three side lengths	Use Heron's formula: Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$	
Two angles and one side length	Use the sine rule to determine a second side length, and then use area $= \frac{1}{2}bc\sin(A)$. *Note:* The third angle may have to be calculated.	
Two side lengths and an angle opposite one of these lengths	Use the sine rule to calculate the other angle opposite one of these lengths, then determine the final angle before using area $= \frac{1}{2}bc\sin(A)$. *Note:* Check if the ambiguous case is applicable.	

on Resources

Interactivity: Area of triangles (int-6483)

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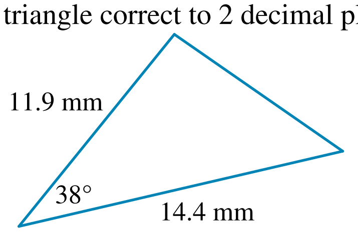
Units 1 & 2 > AOS 4 > Topic 3 > Concept 7

Area of a triangle Concept summary and practice questions

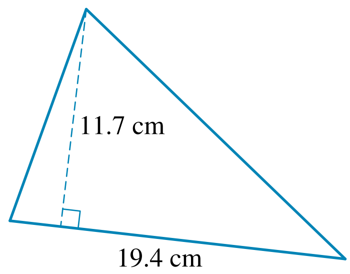
Exercise 9.6 Area of triangles

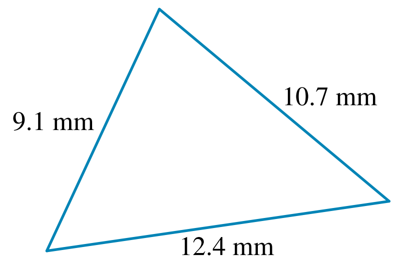
1. **WE16** Find the area of the following triangle correct to 2 decimal places.

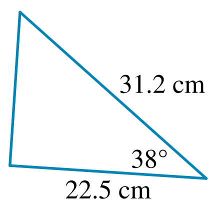


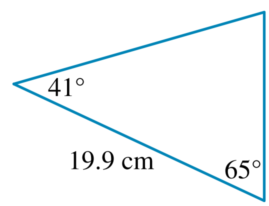
2. Find the area of a triangle with sides of length 14.3 mm and 6.5 mm, and an inclusive angle of 32°. Give your answer correct to 2 decimal places.
3. A triangle has one side length of 8 cm and an adjacent angle of 45.5°. If the area of the triangle is 18.54 cm^2, calculate the length of the other side that encloses the 45.5° angle, correct to 2 decimal places.

4. The smallest two sides of a triangle are 10.2 cm and 16.2 cm respectively, and the largest angle of the same triangle is 104.5°. Calculate the area of the triangle correct to 2 decimal places.
5. **WE17** Find the area of a triangle with sides of 11 cm, 12 cm and 13 cm, giving your answer correct to 2 decimal places.
6. Find the area of a triangle with sides of 22.2 mm, 13.5 mm and 10.1 mm, giving your answer correct to 2 decimal places.
7. Find the area of the following triangles, correct to 2 decimal places where appropriate.

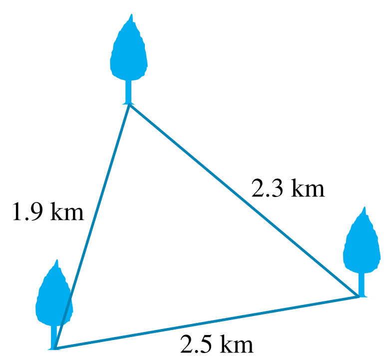
a. 

b. 

c. 

d. 

8. Find the area of the following triangles, correct to 2 decimal places where appropriate.
   a. Triangle ABC, given $a = 12$ cm, $b = 15$ cm, $c = 20$ cm
   b. Triangle ABC, given $a = 10.5$ mm, $b = 11.2$ mm and $C = 40°$
   c. Triangle DEF, given $d = 19.8$ cm, $e = 25.6$ cm and $D = 33°$
   d. Triangle PQR, given $p = 45.9$ cm, $Q = 45.5°$ and $R = 67.2°$
9. A triangular field is defined by three trees, each of which sits in one of the corners of the field, as shown in the following diagram.

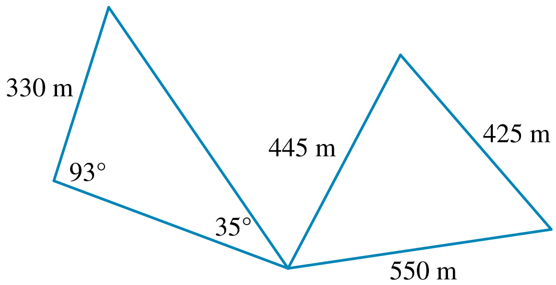


Calculate the area of the field in km^2 correct to 3 decimal places.
10. A triangle ABC has values $a = 11$ cm, $b = 14$ cm and $A = 31.3°$. Answer the following correct to 2 decimal places.
    a. Calculate the size of the other two angles of the triangle.
    b. Calculate the other side length of the triangle.
    c. Calculate the area of the triangle.
11. A triangle has side lengths of $3x$, $4x$ and $5x$. If the area of the triangle is 121.5 cm^2, use any appropriate method to determine the value of x.

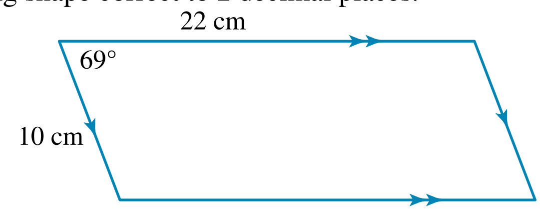
12. A triangular-shaped piece of jewellery has two side lengths of 8 cm and an area of 31.98 cm^2. Use trial and error to find the length of the third side correct to 1 decimal place.

13. A triangle has two sides of length 9.5 cm and 13.5 cm, and one angle of 40.2°. Calculate all three possible areas of the triangle correct to 2 decimal places.

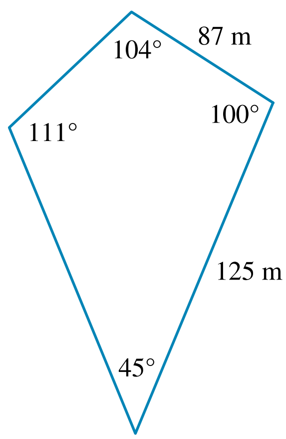
14. A BMX racing track encloses two triangular sections, as shown in the following diagram. Calculate the total area that the race track encloses to the nearest m^2.



15. Find the area of the following shape correct to 2 decimal places.



16. A dry field is in the shape of a quadrilateral, as shown in the following diagram.



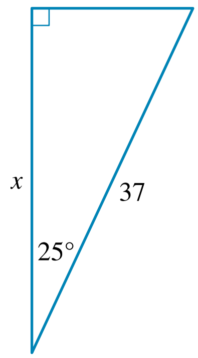
How much grass seed is needed to cover the field in 1 mm of grass seed? Give your answer correct to 2 decimal places.

9.7 Review: exam practice

A summary of this topic is avaiblable in the resources section of your eBook PLUS at www.jacplus.com.au.

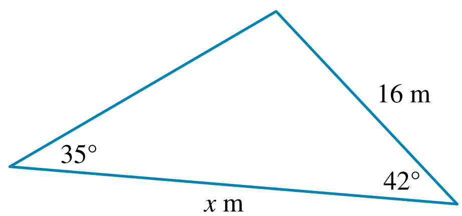
Multiple choice

1. **MC** The length x in the triangle shown can be calculated by using:



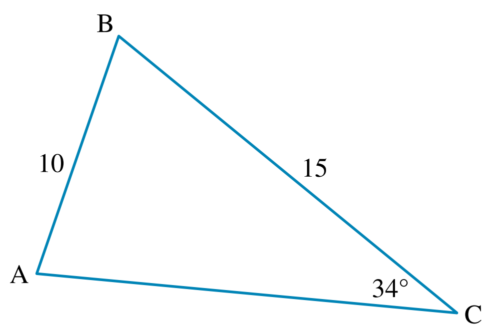
A. $37\cos(25°)$ **B.** $37\sin(25°)$ **C.** $\dfrac{37}{\cos(25°)}$ **D.** $\dfrac{37}{\sin(25°)}$ **E.** $\dfrac{\cos(25°)}{37}$

2. **MC** The length x in the triangle shown, correct to the nearest metre, is:



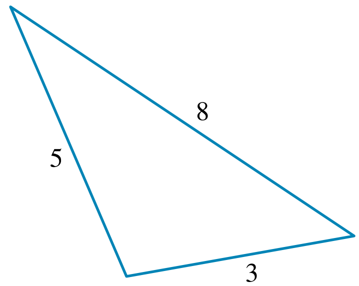
A. 9 **B.** 14 **C.** 19 **D.** 23 **E.** 27

3. **MC** The magnitude of angle A in the triangle shown, correct to the nearest degree, is:



A. 22° **B.** 30° **C.** 34° **D.** 56° **E.** 57°

4. **MC** The largest angle in the triangle shown can be calculated by using:



A. $\cos^{-1}\left(\dfrac{8^2+3^2-5^2}{2\times3\times8}\right)$ **B.** $\cos^{-1}\left(\dfrac{8^2+5^2-3^2}{2\times5\times8}\right)$ **C.** $\cos^{-1}\left(\dfrac{5^2+3^2-8^2}{2\times3\times5}\right)$

D. $\cos^{-1}\left(\dfrac{5^2+3^2-8^2}{2\times3\times8}\right)$ **E.** $\cos^{-1}\left(\dfrac{8^2-3^2-5^2}{2\times3\times5}\right)$

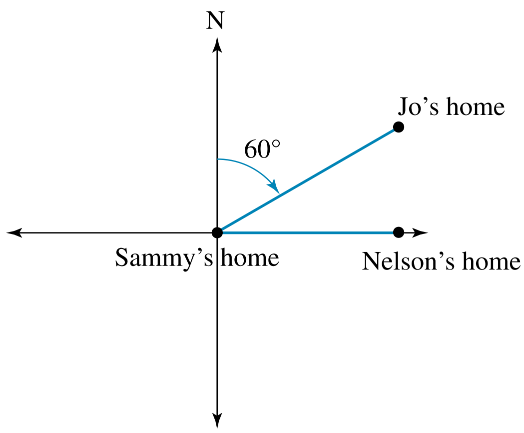
5. **MC** The acute and obtuse angles that have a sine of approximately 0.52992, correct to the nearest degree, are respectively:

A. 31° and 149° **B.** 32° and 58° **C.** 31° and 59° **D.** 32° and 148° **E.** 58° and 148°

6. **MC** Using Heron's formula, the area of a triangle with sides 4.2 cm, 5.1 cm and 9 cm is:

A. 5.2 cm^2 **B.** 9.2 cm^2 **C.** 13.7 cm^2 **D.** 18.3 cm^2 **E.** 28.2 cm^2

7. **MC** The locations of Jo's, Nelson's and Sammy's homes are shown in the diagram. Jo's home is due north of Nelson's home. The bearings of Jo's and Nelson's homes from Sammy's home are respectively:



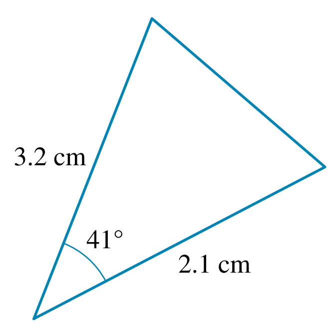
A. 030°T and 090°T **B.** 060°T and 090°T **C.** 030°T and 180°T

D. 060°T and 180°T **E.** 210°T and 270°T

8. **MC** A boy is standing 150 m away from the base of a building. His eye level is 1.65 m above the ground. He observes a hot-air balloon hovering above the building at an angle of elevation of 30°. If the building is 20 m high, the distance the hot air balloon is above the top of the building is closest to:

A. 64 m **B.** 65 m **C.** 66 m **D.** 67 m **E.** 68 m

9. **MC** The area of the triangle shown, correct to 2 decimal places, is:



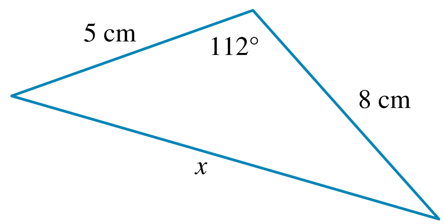
A. 2.20 cm^2 **B.** 2.54 cm^2 **C.** 3.36 cm^2 **D.** 4.41 cm^2 **E.** 5.07 cm^2

10. **MC** A unit of cadets walked from their camp for 7.5 km on a bearing of 064°T. They then travelled on a bearing of 148°T until they came to a signpost that indicated they were 14 km in a straight line from their camp. The bearing from the signpost back to their camp is closest to:

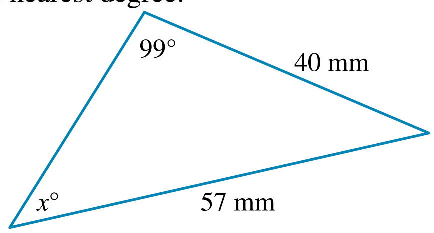
A. 032°T **B.** 064°T **C.** 096°T **D.** 296°T **E.** 328°T

Short answer

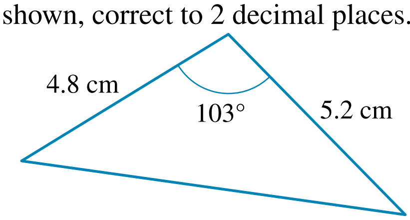
1. a. Find the value of the unknown length x correct to 2 decimal places.



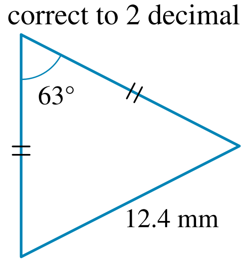
b. Find the value of x correct to the nearest degree.



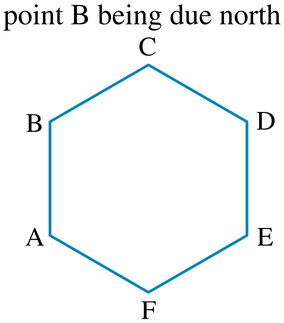
2. In the triangle ABC, $a = 18.5$ m, $c = 12.6$ m and $C = 31.35°$.
   a. Draw the two possible triangles with this information.
   b. Find the two possible values of A and hence the two possible values of B.
3. a. Calculate the area of the triangle shown, correct to 2 decimal places.



b. Calculate the area of the triangle shown, correct to 2 decimal places.

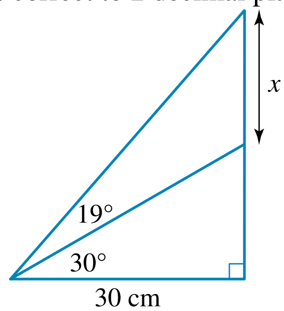


4. ABCDEF is a regular hexagon with the point B being due north of A. Find the true bearing of the point:

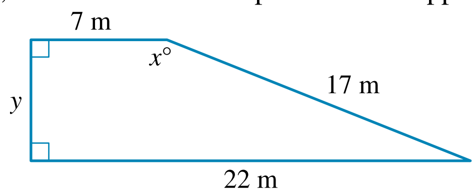


a. C from B b. D from C c. F from E d. E from B.

5. Three immunity idols are hidden on an island for a TV reality game. Idol B is 450 m on a bearing of 072°T from Idol A. Immunity idol C is 885 m on a bearing of S75°E from Idol A.
   a. Calculate the distance between immunity idols B and C.
   b. Calculate the triangular area between immunity idols A, B and C that contestants need to search to find all three idols.
6. a. Find the value of the unknown length x correct to 2 decimal places.



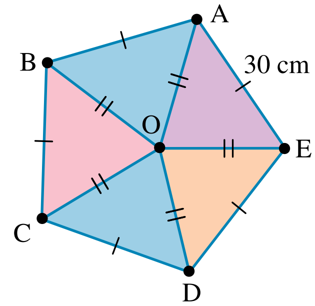
   b. Calculate the values of x and y, correct to 2 decimal places where appropriate.



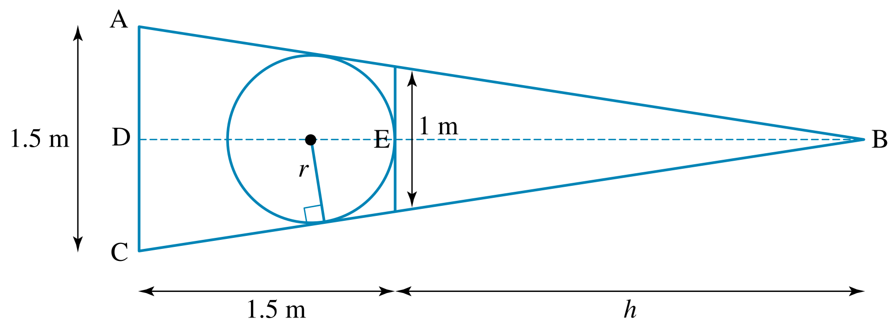
Extended response

1. Three treasure chests are buried on an island. Treasure chest B is 412 m on a bearing of 073°T from treasure chest A. Treasure chest C is 805 m on a bearing of 108°T from treasure chest A.
   a. Draw a diagram to represent the information.
   b. Show that angle A is 35°.
   c. Calculate the distance between treasure chests B and C correct to 1 decimal place.
   d. Calculate the triangular area between treasure chests A, B and C correct to 1 decimal place.
   e. A treasure hunter misreads the information as 'Treasure chest B is 412 m on a bearing of 078°T from treasure chest A' rather than 'Treasure chest B is 412 m on a bearing of 073°T from treasure chest A.'
      i. Draw a diagram to represent the misread information.
      ii. Find how far the treasure hunter has travelled from the actual position of treasure chest B to his incorrect location of treasure chest B, correct to 1 decimal place.
      iii. Calculate the true bearing from his incorrect location of treasure chest B to the actual location of treasure chest B.
2. A dog kennel is placed in the corner of a triangular garden at point C. The dog kennel is positioned 30.5 m at an angle of 32.8° from one corner of the backyard fence (A) and 20.8 m from the other corner of the backyard fence (B).
   a. Draw a diagram to represent the information.
   b. Find, correct to 1 decimal place:
      i. the shortest distance between the dog kennel and the backyard fence
      ii. the length of the backyard fence between points A and B.
   c. Using Heron's formula, find the triangular area between the dog kennel and the two corners of the backyard fence, correct to 1 decimal place.
3. A stained glass window frame consisting of five triangular sections is to be made in the shape of a regular pentagon with a side length of 30 cm.
   a. Show that ∠AOB = 72°.
   b. Calculate ∠ABO.
   c. Use the sine rule to find the length of OB, correct to 1 decimal place.

d. Find the total length of frame required to construct the window, correct to 1 decimal place.
e. Three of the triangular panels must have coloured glass. Calculate the total area of coloured glass required correct to the nearest cm^2.



4. A triangular flag ABC has a printed design with a circle touching the sides of a flag and a 1-metre vertical line as shown in the diagram.



a. Calculate h, the horizontal distance from the vertex point of the flag B to the vertical line at point E, correct to 1 decimal place.
b. Calculate ∠ACB correct to 2 decimal places.
c. Calculate r, the radius of the circle, correct to 2 decimal places.
d. The circle printed design in the flag is to be coloured yellow. Calculate the area of the circle correct to 2 decimal places.
e. Calculate the total area of the flag correct to 2 decimal places.

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Answers

Topic 9 Applications of trigonometry

Exercise 9.2 Trigonometric ratios

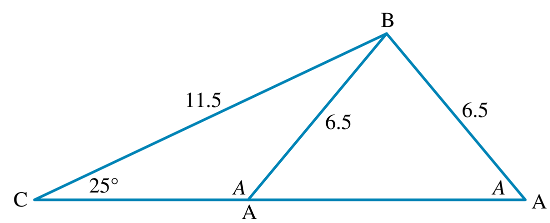
1. $x = 2.16$ cm
2. $x = 10.6$ mm
3. $\theta = 46.90°$
4. $\theta = 45.48°$
5. $y = 1.99$ cm
6. $y = 2.63$ cm
7. $\theta = 73.97°$
8. $\theta = 35.58°$
9. $x = 13.15$ cm
10. $y = 0.75$ cm
11. $\theta = 53.20°$
12. $\theta = 36.38°$
13. a. 0.82 b. −0.88
14. a. 0.99 b. −0.84
15. $x = 5.16$
16. $\theta = 49.32°$
17. $\theta = 10.29°$, $\alpha = 79.71°$
18. 9.63 m
19. a. 7.4 km b. 27.2 km
20. 2870 m
21. 20.56°
22. 2.02 m
23. a. 47.16° b. 1.83 m
24. a. 4.1 m b. 5.03 m
25. 4.28 m
26. 100.94 m

Exercise 9.3 Applications of trigonometric ratios

1. 36.81 m
2. 47.13 m
3. 61.54°
4. 36.25°
5. 2.73 m
6. 999.85 m
7. a. 49° b. 342° c. 231° d. 112°
8. a. 185° b. 297° c. 79° d. 124°
9. a. 110°T b. 290°T
10. a. 237°T b. 057°T
11. a. 050°T b. 127°T c. 259°T
12. a. 32.63 km b. 15.21 km
13. 323.1°T
14. 862.83 m
15. 315 m
16. 11.13 m
17. 52°
18. 88.28°
19. 1357 m
20. a. 83.59° b. 82.35° c. 79.93° d. 74.97° e. 60.67°

Exercise 9.4 The sine rule

1. $x = 10.36$ cm
2. $x = 10.38$ cm
3. $x = 9.05$ cm
4. $x = 17$ cm
5. $B = 28.3°$
6. $B = 27.1°$
7. $\theta = 47.2°$
8. $A = 65.41°$ or $114.59°$
9. $A = 52.64°$ or $127.36°$
10. $B = 46.97°$ or $133.03°$
11. $b = 12.24$, $c = 13.43$
12. 35° and 145°
13. 20.71 m
14. a. 23.18 m b. 10.66 m
15. $x = 142.4°$, $y = 37.6°$
16. a.



   b. $A = 48.39°$ or $131.61°$; $B = 106.61°$ or $23.39°$
17. 9.80 m
18. 5.91 km

Exercise 9.5 The cosine rule

1. $x = 2.74$ km
2. $x = 10.49$ m
3. $x = 8.5$ km
4. $x = 4.48$ m
5. $A = 26.95°$
6. $A = 99.59°$
7. $A = 44.42°$
8. $b = 10.17$
9. 79.66°
10. 82.82°
11. 33.56°
12. 13.23 cm
13. 31.68 km

14. 98.86 km

15. 8822 m

16. 7 hours, 16 minutes

Exercise 9.6 Area of triangles

1. 52.75 mm^2

2. 24.63 mm^2

3. 6.50 cm

4. 79.99 cm^2

5. 61.48 cm^2

6. 43.92 mm^2

7. a. 113.49 cm^2 b. 47.45 mm^2
   c. 216.10 cm^2 d. 122.48 cm^2

8. a. 89.67 cm^2 b. 37.80 mm^2
   c. 247.68 cm^2 d. 750.79 cm^2

9. 2.082 km^2

10. a. $B = 41.39°$, $C = 107.31°$
    b. $c = 20.21$ cm
    c. 73.51 cm^2

11. $x = 4.5$ cm

12. 11.1 cm

13. 41.39 cm^2, 61.41 cm^2 and 59.12 cm^2

14. 167 330 m^2

15. 205.39 cm^2

16. 8.14 m^3

9.7 Review: exam practice

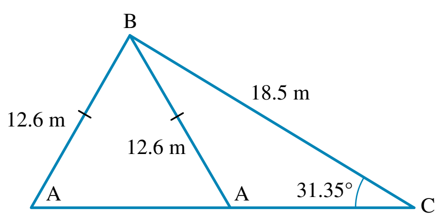
Multiple choice

1. A	2. E	3. E	4. C	5. D
6. A	7. B	8. E	9. A	10. D

Short answer

1. a. 10.91 cm b. 44°

2. a.



b. $A = 49.81°$, $B = 98.84°$ or $A = 130.19°$, $B = 18.46°$

3. a. 12.16 cm^2 b. 62.73 mm^2

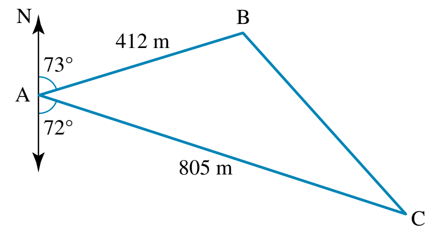
4. a. 060°T b. 120°T
   c. 240°T d. 120°T

5. a. 563.67 m b. 108 451 m^2

6. a. 17.19 cm
   b. $x = 151.93°$, $y = 8$ m

Extended response

1. a.

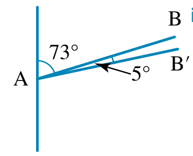


b. $A = 180 - (73 + 72)$
$= 35°$

c. 523.8 m

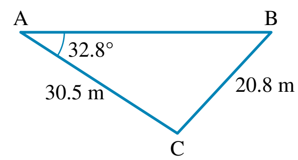
d. 95 116.2 m^2

e. i.



ii. 35.9 m iii. 345.6°T

2. a.



b. i. 16.5 m ii. 38.3 m

c. 316.1 m^2

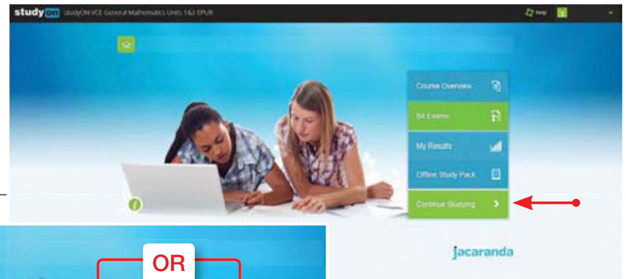
3. a. $\frac{360}{5} = 72°$ b. 54° c. 25.5 cm
   d. 277.5 cm e. 928 cm^3

4. a. 3 m b. 80.54° c. 0.59 m
   d. 1.09 m^2 e. 3.38 m^2

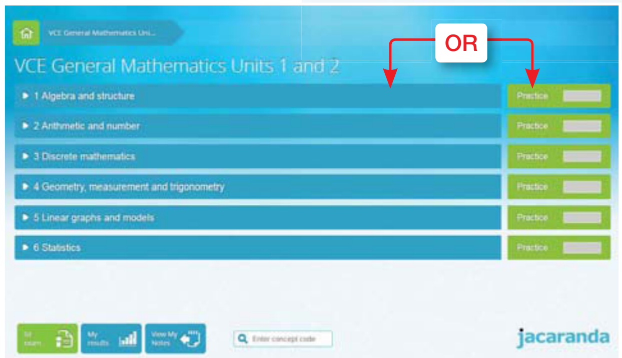
REVISION: AREA OF STUDY 4 Geometry, Measurement and Trigonometry

TOPICS 7 to 9

- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

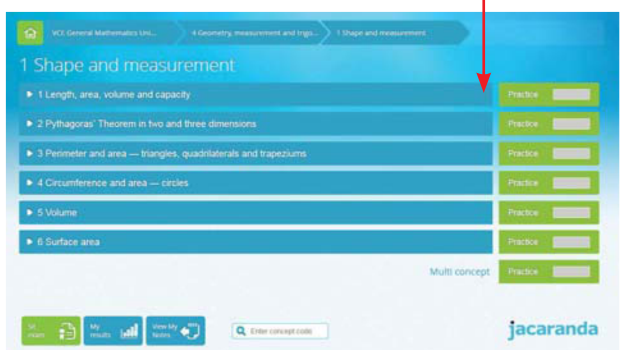


- Select your **course** *VCE General Mathematics Units 1 & 2* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.

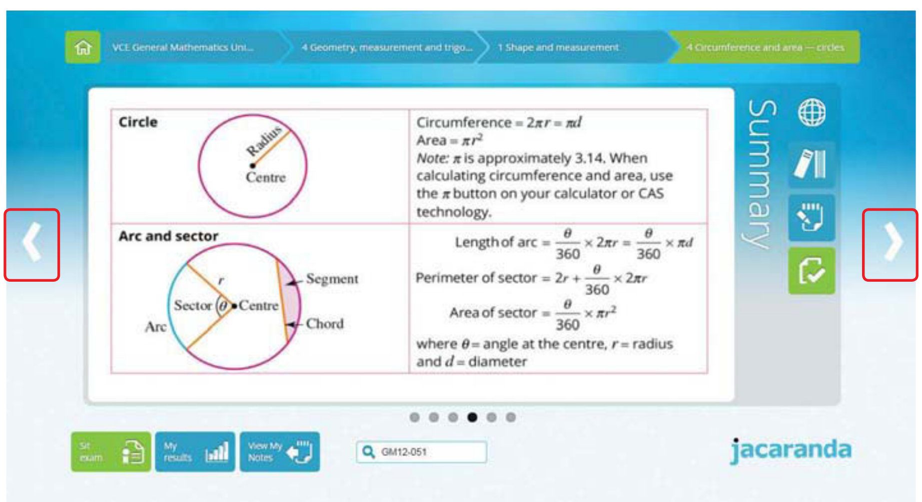


- Select **Practice** at the topic level to access all questions in the topic.

- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



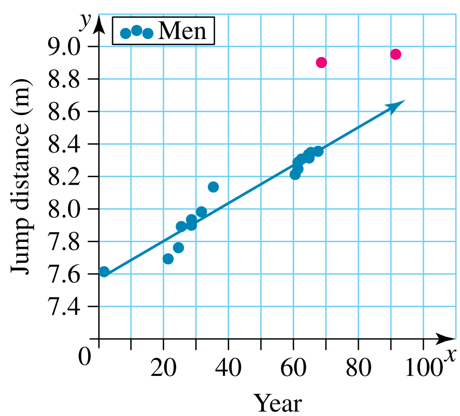
TOPIC 10
Linear graphs and models

10.1 Overview

10.1.1 Introduction

The chapter 1 introduction mentioned that linear equations have been around for over 4000 years, but that it wasn't until the 17th century that real progress was made with linear algebra. Linear equations use one or more variable where one variable is dependent on the other. Almost any situation where there is an unknown quantity can be represented by a linear equation, such as predicting profit or calculating the cost of a taxi trip. A useful way to apply linear equations is to make predictions about what will happen in the future. For example, if a linear profit equation is modelled, then this model could be used to predict future profits. One way of modelling a linear equation is to plot the relevant data and then draw a linear line of best fit. The graph that the data is plotted on is known as a scatter plot. This enables people to determine the relationship between the two variables.

A more precise method for determining the line of best fit is known as the least squares method. This is commonly used by calculators and computers to model data. To determine how well one variable explains the behaviour of the other, a value called the coefficient of determination is used.



It is always interesting to note that a lot of world records follow a linear trend over time. One event that challenges this is the men's long jump world record. In 1968 Bob Beamon smashed the record by an amazing 55 cm with a jump of 8.90 m at the Olympics. This jump certainly went against the linear trend. This record stood until 1991 when Mike Powell jumped 8.95 m at the World Championships in Athletics. If you plot the previous world records and draw a line of best fit, it clearly shows this went against the linear trend over the previous 60-odd years.

LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

10.1.2 Kick off with CAS

Gradient–intercept form

Linear equations produce straight-line graphs. All linear equations can be put into gradient–intercept form, from which we can easily identify the gradient and y-intercept of the equation.

1. Use CAS to draw graphs of the following linear equations.
   a. $2y = 6x + 5$
   b. $y - 3x + 4 = 0$
   c. $3y - 4x = 9$

The y-intercept is the point on a graph where the line crosses the y-axis (the vertical axis).

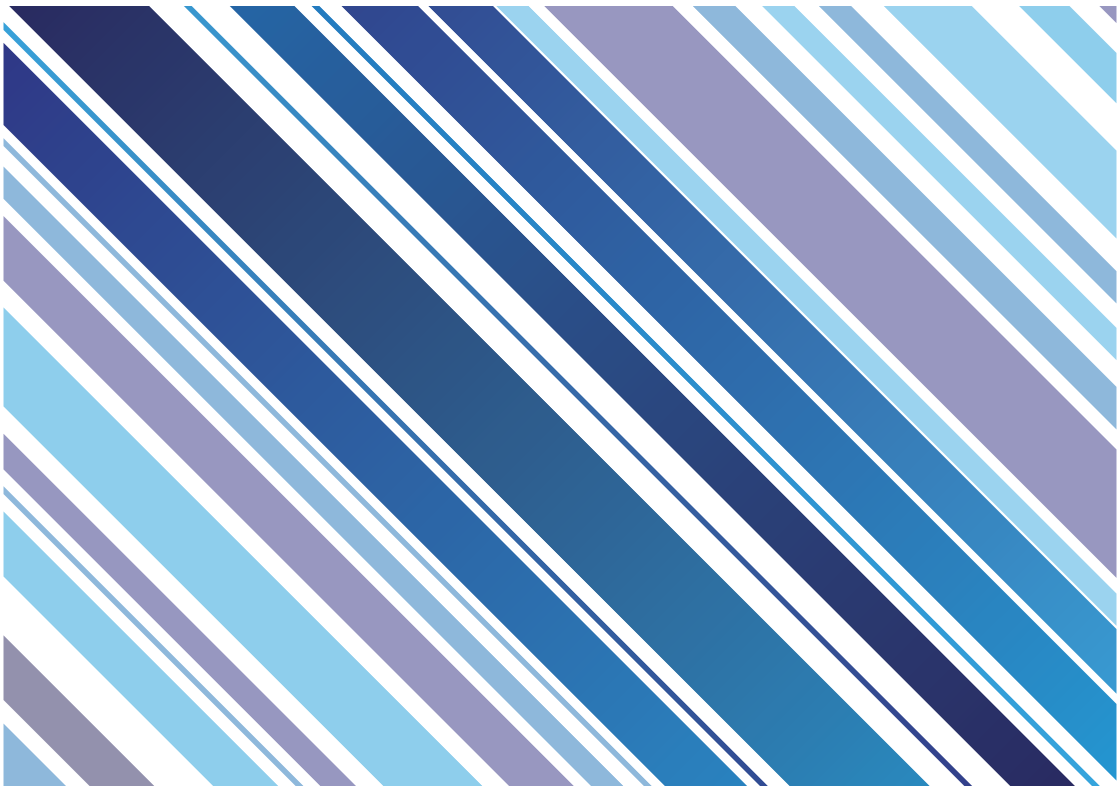
2. Use your graphs from question 1 to identify the y-intercept for each of the equations.

The gradient of a linear equation is the amount by which the y-value increases or decreases for each increase of 1 in the x-value.

3. Use your graphs from question 1 to determine the gradient for each of the equations.

When a linear equation is written in gradient–intercept form, it appears as $y = ax + b$, where a is the value of the gradient and b is the value of the y-intercept.

4. Use your answers from questions 2 and 3 to write the three equations in gradient–intercept form.
5. Confirm your answers to question 4 by transposing the equations in question 1.



on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

10.2 Linear functions and graphs

10.2.1 Linear functions

A function is a relationship between a set of inputs and outputs, such that each input is related to exactly one output. Each input and output of a function can be expressed as an ordered pair, with the first element of the pair being the input and the second element of the pair being the output.

A function of x is denoted as $f(x)$. For example, if we have the function $f(x) = x + 3$, then each output will be exactly 3 greater than each input.

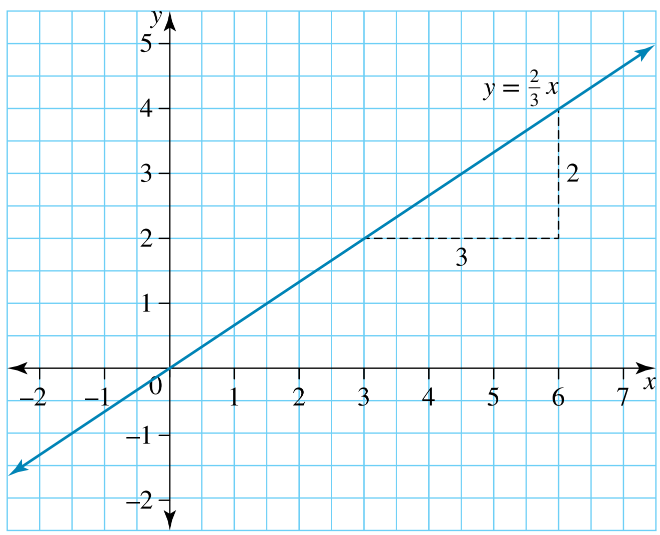
A linear function is a set of ordered pairs that form a straight line when graphed.

10.2.2 The gradient of a linear function

The **gradient** of a straight-line function, also known as the slope, determines the change in the y-value for each change in x-value. The gradient can be found by analysing the equation, by examining the graph or by finding the change in values if two points are given. The gradient is typically represented with the pronumeral m.

A positive gradient means that the y-value is increasing as the x-value increases, and a negative gradient means that the y-value is decreasing as the x-value increases.

A gradient of $\frac{a}{b}$ means that there for every increase of b in the x-value, there is an increase of a in the y-value. For example, a gradient of $\frac{2}{3}$ means that for every increase of 3 in the x-value, the y-value increases by 2.

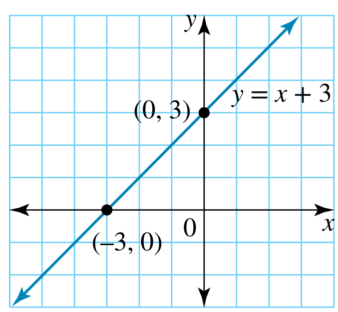


10.2.3 x- and y-intercepts

The ***x*-intercept** of a linear function is the point where the graph of the equation crosses the x-axis. This occurs when $y = 0$.

The ***y*-intercept** of a linear function is the point where the graph of the equation crosses the y-axis. This occurs when $x = 0$.

In the graph of $y = x + 3$, we can see that the x-intercept is at $(-3, 0)$ and the y-intercept is at $(0, 3)$. These points can also be determined algebraically by putting $y = 0$ and $x = 0$ into the equation.



10.2.4 Gradient–intercept form

All linear equations relating the variables x and y can be rearranged into the form $y = mx + c$, where m is the gradient. This is known as the **gradient–intercept form** of the equation.

If a linear equation is in gradient–intercept form, the number and sign in front of the x-value gives the value of the gradient of the equation. For example, in $y = 4x + 5$, the gradient is 4.

The value of c in linear equations written in gradient–intercept form is the y-intercept of the equation. This is because the y-intercept occurs when $x = 0$, and when $x = 0$ the equation simplifies to $y = c$. The value of c in $y = 4x + 5$ is 5.

WORKED EXAMPLE 1

State the gradients and y-intercepts of the following linear equations.

a. $y = 5x + 2$ b. $y = \frac{x}{2} - 3$ c. $y = -2x + 4$

d. $2y = 4x + 3$ e. $3y - 4x = 12$

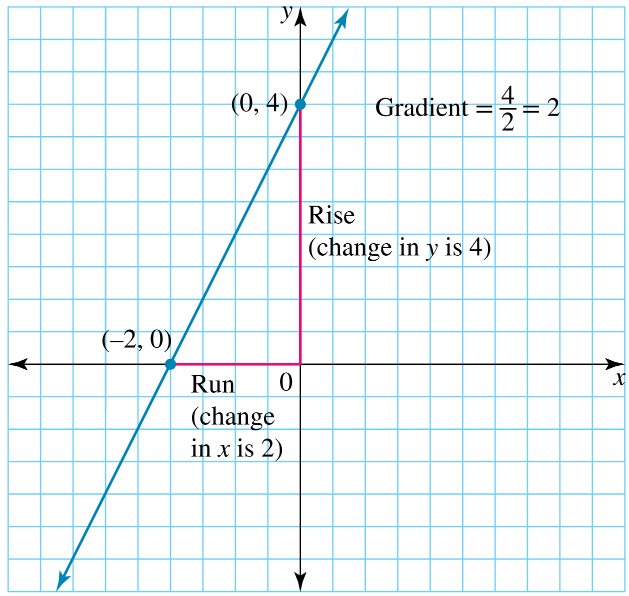
THINK	WRITE
a. 1. Write the equation. It is in the form $y = mx + c$.	a. $y = 5x + 2$
2. Identify the coefficient of x.	The coefficient of x is 5.
3. Identify the value of c.	The value of c is 2.
4. Answer the question.	The gradient is 5 and the y-intercept is 2.
b. 1. Write the equation. It is in the form $y = mx + c$.	b. $y = \frac{x}{2} - 3$
2. Identify the coefficient of x.	x has been multiplied by $\frac{1}{2}$, so the coefficient is $\frac{1}{2}$.
3. Identify the value of c.	The value of c is -3.
4. Answer the question.	The gradient is $\frac{1}{2}$ and the y-intercept is -3.
c. 1. Write the equation. It is in the form $y = mx + c$.	c. $y = -2x + 4$
2. Identify the coefficient of x.	The coefficient of x is -2 (the coefficient includes the sign).
3. Identify the value of c.	The value of c is 4.
4. Answer the question.	The gradient is -2 and the y-intercept is 4.
d. 1. Write the equation. Rearrange the equation so that it is in the form $y = mx + c$.	d. $2y = 4x + 3$ $\frac{2y}{2} = \frac{4x}{2} + \frac{3}{2}$ $y = 2x + \frac{3}{2}$
2. Identify the coefficient of x.	The coefficient of x is 2.
3. Identify the value of c.	The value of c is $\frac{3}{2}$.
4. Answer the question.	The gradient is 2 and the y-intercept is $\frac{3}{2}$.
e. 1. Write the equation. Rearrange the equation so that it is in the form $y = mx + c$.	e. $3y - 4x = 12$ $3y - 4x + 4x = 12 + 4x$ $3y = 4x + 12$ $\frac{3y}{3} = \frac{4}{2}x + \frac{12}{3}$ $y = \frac{4}{3}x + 4$
2. Identify the coefficient of x.	The coefficient of x is $\frac{4}{3}$.
3. Identify the value of c.	The value of c is 4.
4. Answer the question.	The gradient is $\frac{4}{3}$ and the y-intercept is 4.

10.2.5 Determining the gradient from a graph

The value of the gradient can be found from a graph of a linear function. The gradient can be found by selecting two points on the line, then finding the change in the y-values and dividing by the change in the x-values.

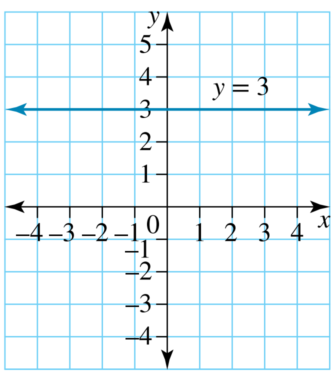
In other words, the general rule to find the value of a gradient that passes through the points (x_1, y_1) and (x_2, y_2) is:

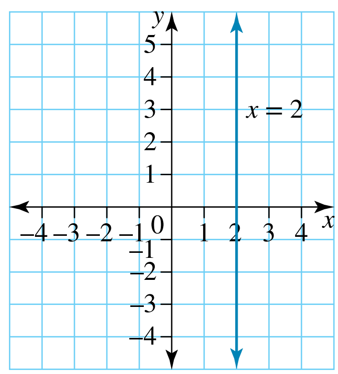
$$\textbf{Gradient, } m = \frac{\textbf{rise}}{\textbf{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$



For all horizontal lines the y-values will be equal to each other, so the numerator of $\frac{y_2 - y_1}{x_2 - x_1}$ will be 0. Therefore, the gradient of horizontal lines is 0.

For all vertical lines the x-values will be equal to each other, so the denominator of $\frac{y_2 - y_1}{x_2 - x_1}$ will be 0. Dividing a value by 0 is undefined; therefore, the gradient of vertical lines is undefined.

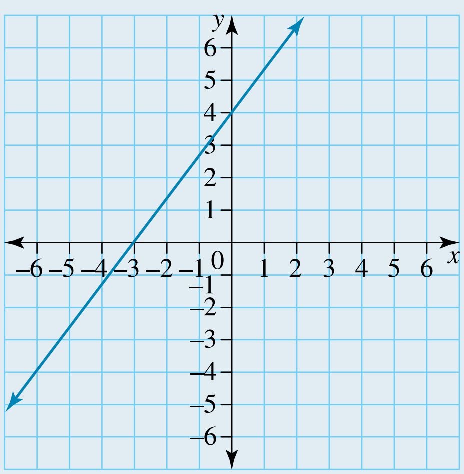




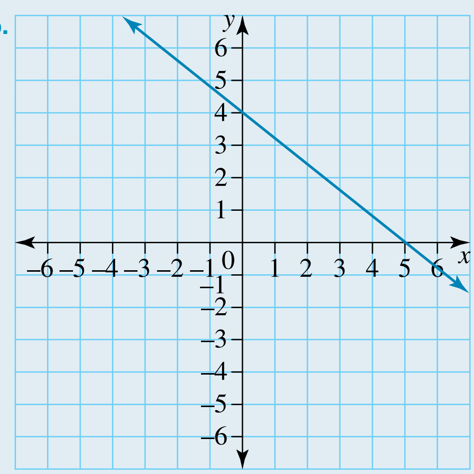
WORKED EXAMPLE 2

Find the values of the gradients of the following graphs.

a.



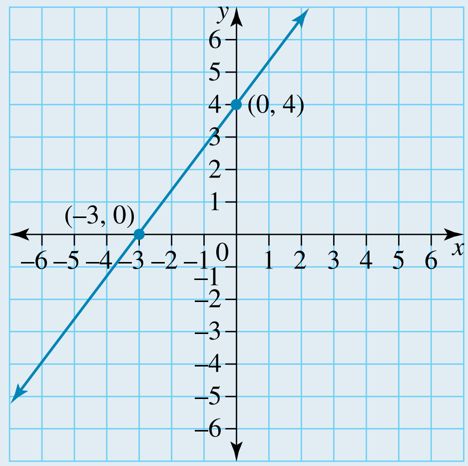
b.



THINK | **WRITE/DRAW**

a. 1. Find two points on the graph. (Select the x- and y-intercepts.)

a.



(−3, 0) and (0, 4)

2. Determine the rise in the graph (change in y-values).

$4 - 0 = 4$

3. Determine the run in the graph (change in x-values).

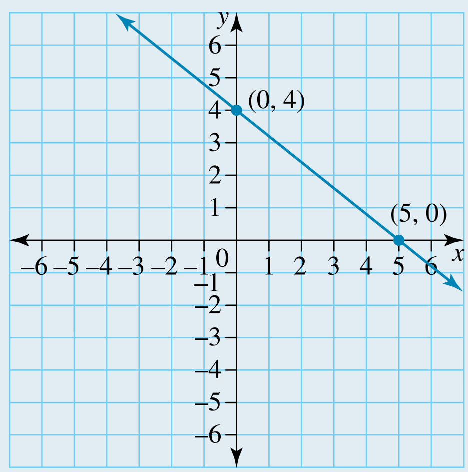
$0 - -3 = 3$

4. Substitute the values into the formula for the gradient.

$$\text{Gradient} = \frac{\text{rise}}{\text{run}} = \frac{4}{3}$$

b. 1. Find two points on the graph. (Select the x- and y-intercepts.)

b.



(0, 4) and (5, 0)

2. Determine the rise in the graph (change in y-values).

$0 - 4 = -4$

3. Determine the change in the x-values.

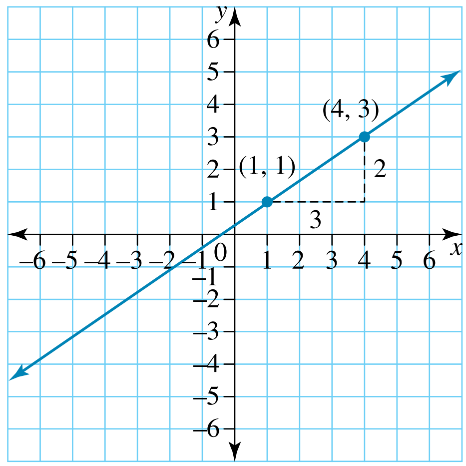
$5 - 0 = 5$

4. Substitute the values into the formula for the gradient.

$$\text{Gradient} = \frac{\text{rise}}{\text{run}} = -\frac{4}{5}$$

10.2.6 Finding the gradient given two points

If a graph is not provided, we can still find the gradient if we are given two points that the line passes through. The same formula is used to find the gradient by finding the difference in the two y-coordinates and the difference in the two x-coordinates:



For example, the gradient of the line that passes through the points (1, 1) and (4, 3) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{4-1} = \frac{2}{3}$$

WORKED EXAMPLE 3

Find the value of the gradients of the linear graphs that pass through the following points.

a. (4, 6) and (5, 9)

b. (2, −1) and (0, 5)

c. (0.5, 1.5) and (−0.2, 1.8)

THINK	WRITE
a. 1. Number the points.	**a.** Let $(4, 6) = (x_1, y_1)$ and $(5, 9) = (x_2, y_2)$.
2. Write the formula for the gradient and substitute the values.	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-6}{5-4} = \frac{3}{1}$
3. Simplify the fraction and answer the question.	The gradient is 3 or $m = 3$.
b. 1. Number the points.	**b.** Let $(2, -1) = (x_1, y_1)$ and $(0, 5) = (x_2, y_2)$.
2. Write the formula for the gradient and substitute the values.	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - -1}{0-2} = \frac{6}{-2}$
3. Simplify the fraction and answer the question.	The gradient is −3 or $m = -3$.

c. 1. Number the points.

c. Let $(0.5, 1.5) = (x_1, y_1)$ and $(-0.2, 1.8) = (x_2, y_2)$.

2. Write the formula for the gradient and substitute the values.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1.8 - 1.5}{-0.2 - 0.5} = \frac{0.3}{-0.7}$$

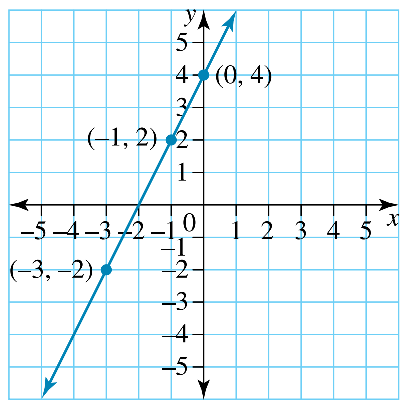
3. Simplify the fraction and answer the question.

The gradient is $-\frac{3}{7}$ or $m = -\frac{3}{7}$.

10.2.7 Plotting linear graphs

Linear graphs can be constructed by plotting the points and then ruling a line between the points as shown in the diagram.

If the points or a table of values are not given, then the points can be found by substituting x-values into the rule and finding the corresponding y-values. If a table of values is provided, then the graph can be constructed by plotting the points given and joining them.



WORKED EXAMPLE 4

Construct a linear graph that passes through the points (−1, 2), (0, 4), (1, 6) and (3, 10):

a. without technology

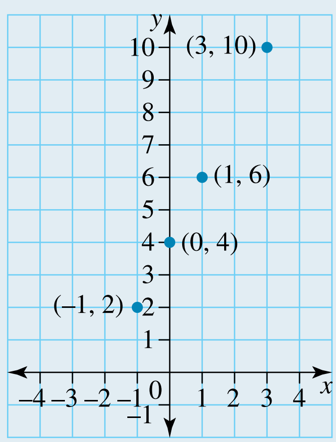
b. using CAS or otherwise.

THINK

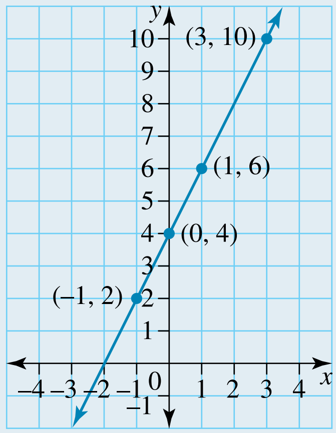
a. 1. Using grid paper, rule up the Cartesian plane (set of axes) and plot the points.

DRAW/DISPLAY

a.



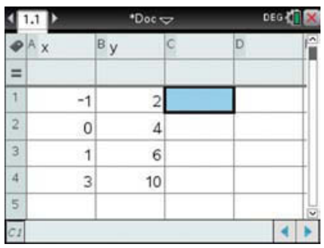
2. Using a ruler, rule a line through the points.



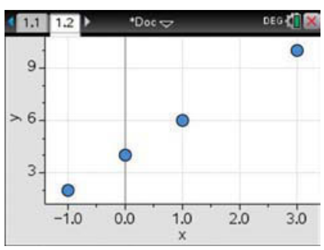
TI | THINK

WRITE

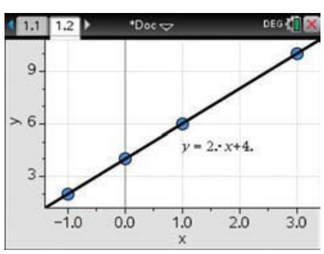
b1. On a Lists & Spreadsheets page, label the first column x and the second column y. Enter the x-coordinates of the given points in the first column, and y-coordinates in the second column.



2. On a Data & Statistics page, click on the horizontal axis label and select x, then click on the vertical axis label and select y.



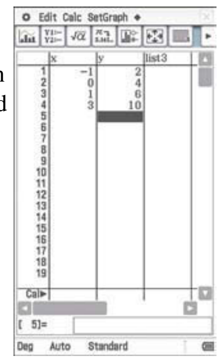
3. Press MENU and select:
-4: Analyze
-6: Regression
-1: Show Linear (mx + b)



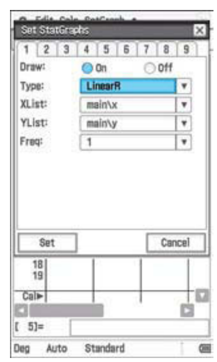
CASIO | THINK

WRITE

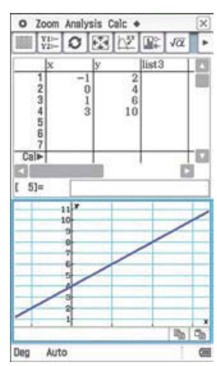
b1. On a Statistics screen, label list1 as x and list2 as y. Enter the x-coordinates of the given points in the first column, and y-coordinates in the second column.



2. Click the G icon and complete the fields as:
Draw: On
Type: LinearR
XList: main\x
YList: main\y
Freq: 1 then select Set.

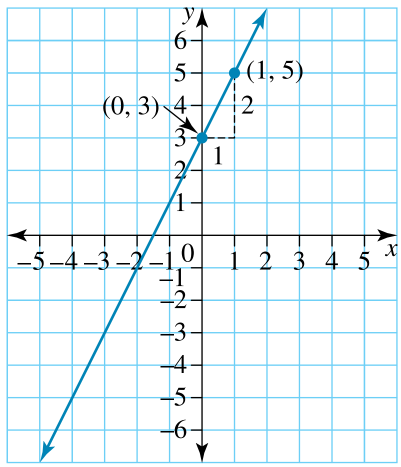


3. Click the y icon.



10.2.8 Sketching graphs using the gradient and y-intercept method

A linear graph can be constructed by using the gradient and y-intercept. The y-intercept is marked on the y-axis, and then another point is found by using the gradient. For example, a gradient of 2 means that for an increase of 1 in the x-value, the y-value increases by 2. If the y-intercept is (0, 3), then add 1 to the x-value $(0 + 1)$ and 2 to the y-value $(3 + 2)$ to find another point that the line passes through, (1, 5).



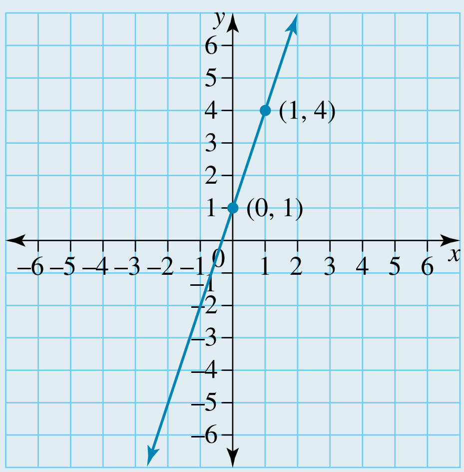
WORKED EXAMPLE 5

Using the gradient and the y-intercept, sketch the graph of each of the following.

a. **A linear graph with a gradient of 3 and a y-intercept of 1**

b. $y = -2x + 4$

c. $y = \frac{3}{4}x - 2$

THINK	WRITE/DRAW
a. 1. Interpret the gradient.	**a.** A gradient of 3 means that for an increase of 1 in the x-value, there is an increase of 3 in the y-value.
2. Write the coordinates of the y-intercept.	y-intercept: (0, 1)
3. Find the x- and y-values of another point using the gradient.	New x-value $= 0 + 1 = 1$ New y-value $= 1 + 3 = 4$ Another point on the graph is (1, 4).
4. Construct a set of axes and plot the two points. Using a ruler, rule a line through the points.	
b. 1. Identify the value of the gradient and y-intercept.	**b.** $y = -2x + 4$ has a gradient of -2 and a y-intercept of 4.
2. Interpret the gradient.	A gradient of -2 means that for an increase of 1 in the x-value, there is a decrease of 2 in the y-value

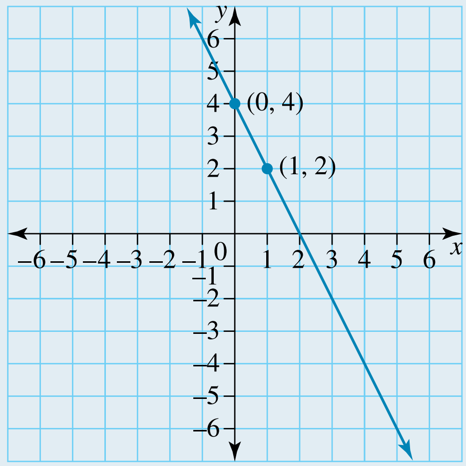
3. Write the coordinates of the y-intercept.

y-intercept: (0, 4)

4. Find the x- and y-values of another point using the gradient.

New x-value = 0 + 1 = 1
New y-value = 4 − 2 = 2
Another point on the graph is (1, 2).

5. Construct a set of axes and plot the two points. Using a ruler, rule a line through the points.



c. 1. Identify the value of the gradient and y-intercept.

c. $y = \frac{3}{4}x - 2$ has a gradient of $\frac{3}{4}$ and a y-intercept of −2.

2. Interpret the gradient.

A gradient of $\frac{3}{4}$ means that for an increase of 4 in the x-value, there is an increase of 3 in the y-value

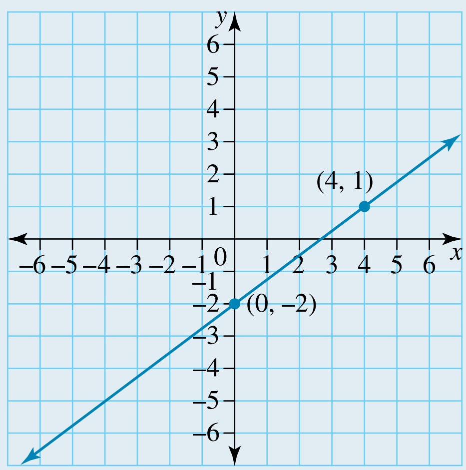
3. Write the coordinates of the y-intercept.

y-intercept: (0, −2)

4. Find the x- and y-values of another point using the gradient.

New x-value = 0 + 4 = 4
New y-value = −2 + 3 = 1
Another point on the graph is (4, 1).

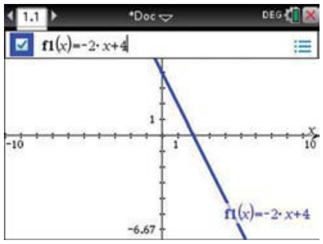
5. Construct a set of axes and plot the two points. Using a ruler, rule a line through the points.



TI | **THINK**

b1. On a Graphs page, complete the entry line for function 1 as:
$f1(x) = -2x + 4$ then press ENTER.

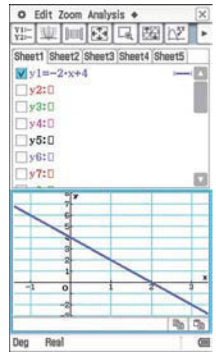
WRITE



CASIO | **THINK**

b1. On a Graph & Table screen, complete the entry line for equation 1 as:
$y1 = -2x + 4$
then click the tick box.
Click the $ icon to view the graph.

WRITE

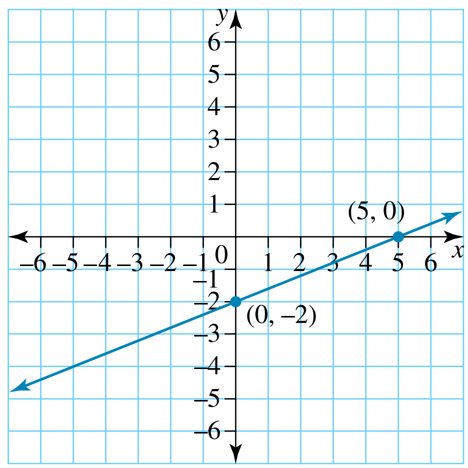


10.2.9 Sketching graphs using the *x*- and *y*-intercepts

If the points of a linear graph where the line crosses the x- and y-axes (the x- and y-intercepts) are known, then the graph can be constructed by marking these points and ruling a line through them.

To find the x-intercept, substitute $y = 0$ into the equation and then solve the equation for x.

To find the y-intercept, substitute $x = 0$ into the equation and then solve the equation for y.



WORKED EXAMPLE 6

Find the value of the x- and y-intercepts for the following linear equations, and hence sketch their graphs.

a. $3x + 4y = 12$ **b.** $y = 5x$ **c.** $3y = 2x + 1$

THINK

a. 1. To find the x-intercept, substitute $y = 0$ and solve for x.

WRITE/DRAW

a. x-intercept: $y = 0$

$$\begin{aligned} 3x + 4y &= 12 \\ 3x + 4 \times 0 &= 12 \\ 3x &= 12 \\ \frac{3x}{3} &= \frac{12}{3} \\ x &= 4 \end{aligned}$$

x-intercept: $(4, 0)$

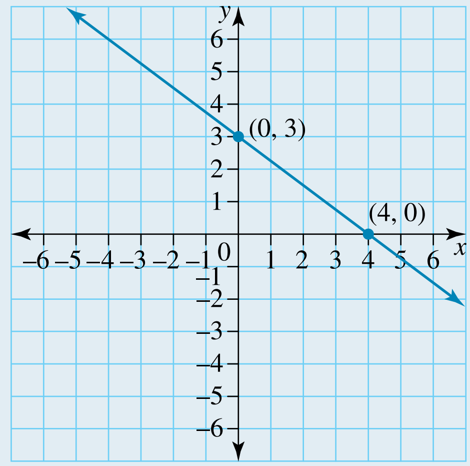
2. To find the y-intercept, substitute $x = 0$ into the equation and solve for y.

y-intercept: $x = 0$

$$\begin{aligned} 3x + 4y &= 12 \\ 3 \times 0 + 4y &= 12 \\ \frac{4y}{4} &= \frac{12}{4} \\ y &= 3 \end{aligned}$$

y-intercept: $(0, 3)$

3. Draw a set of axes and plot the x- and y-intercepts. Draw a line through the two points.



b. 1. To find the x-intercept, substitute $y = 0$ into the equation and solve for x.

b. x-intercept: $y = 0$

$y = 5x$

$0 = 5x$

$x = 0$

x-intercept: $(0, 0)$

2. To find the y-intercept, substitute $x = 0$ into the equation and solve for y.

y-intercept: $x = 0$

$y = 5x$

$= 5 \times 0$

$= 0$

y-intercept: $(0, 0)$

3. As the x- and y-intercepts are the same, we need to find another point on the graph. Substitute $x = 1$ into the equation.

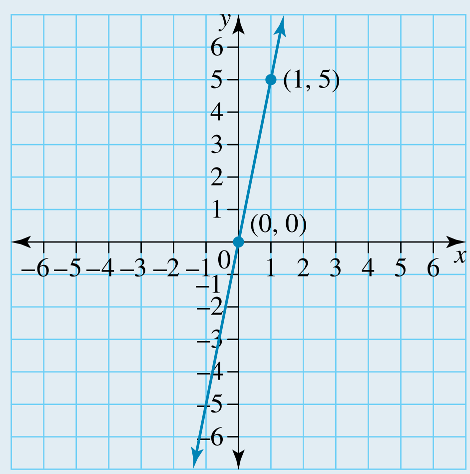
$y = 5x$

$= 5 \times 1$

$= 5$

Another point on the graph is $(1, 5)$.

4. Draw a set of axes. Plot the intercept and the second point. Draw a line through the intercepts.



c. 1. To find the x-intercept, substitute $y = 0$ into the equation.

c x-intercept: $y = 0$

$3y = 2x + 1$

$3 \times 0 = 2x + 1$

$0 = 2x + 1$

2. Solve the equation for x.

$$0 - 1 = 2x + 1 - 1$$
$$-1 = 2x$$
$$\frac{-1}{2} = \frac{2x}{2}$$
$$x = \frac{-1}{2}$$

x-intercept: $\left(-\frac{1}{2}, 0\right)$

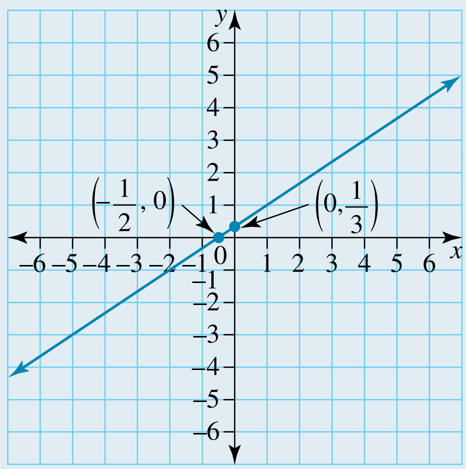
3. To find the y-intercept, substitute $x = 0$ into the equation and solve for y.

y-intercept: $x = 0$

$$3y = 2 \times 0 + 1$$
$$3y = 1$$
$$\frac{3y}{3} = \frac{1}{3}$$
$$y = \frac{1}{3}$$

y-intercept: $\left(0, \frac{1}{3}\right)$

4. Draw a set of axes and mark the x- and y-intercepts. Draw a line through the intercepts.



on Resources

Interactivity: Linear graphs (int-6484)

Interactivity: Equations of straight lines (int-6485)

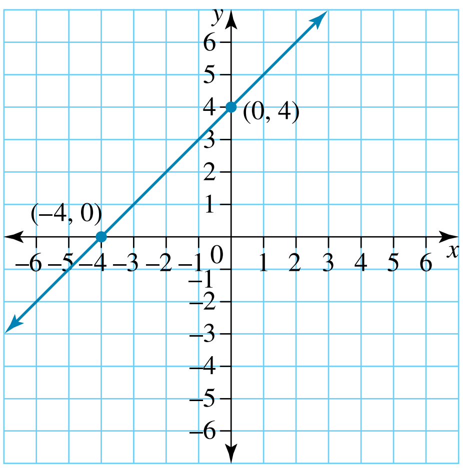
studyon

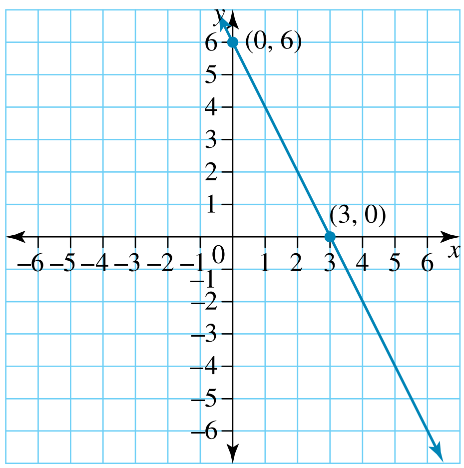
Units 1 & 2 > AOS 5 > Topic 1 > Concept 1

Linear functions and graphs Concept summary and practice questions

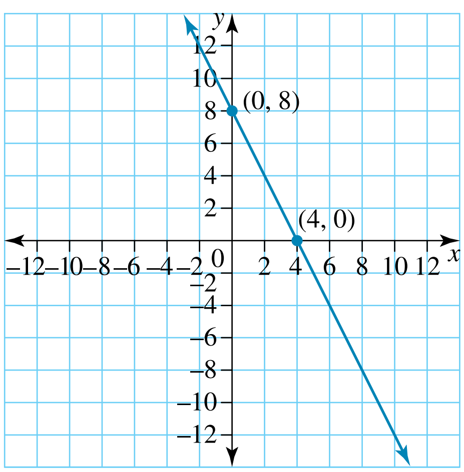
Exercise 10.2 Linear functions and graphs

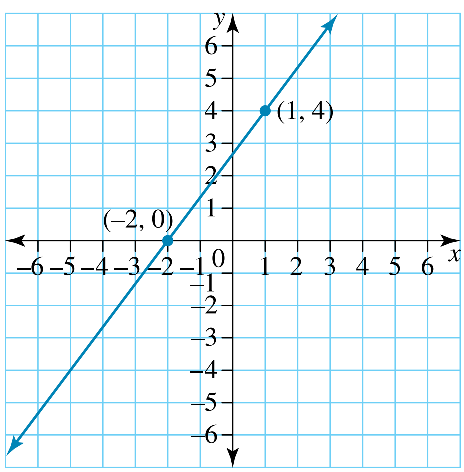
1. **WE1** State the gradients and y-intercepts of the following linear equations.
   a. $y = 2x + 1$
   b. $y = -x + 3$
   c. $y = \frac{1}{2}x + 4$
   d. $4y = 4x + 1$
   e. $2y + 3x = 6$
2. Find the gradients and y-intercepts of the following linear equations.
   a. $y = \dfrac{3x - 1}{5}$
   b. $y = 5(2x - 1)$
   c. $y = \dfrac{3 - x}{2}$
3. **WE2** Find the value of the gradient of each of the following graphs.

a.


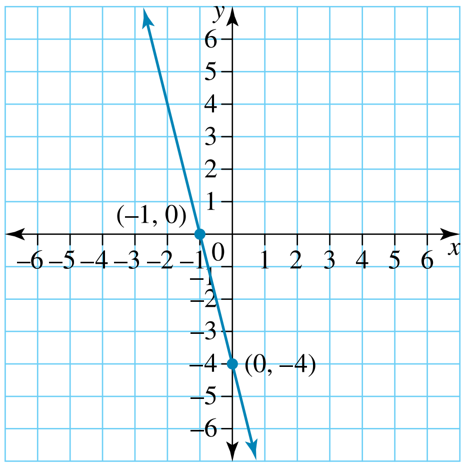
b.


4. **MC** Which of the following graphs has a gradient of $-\frac{1}{4}$?

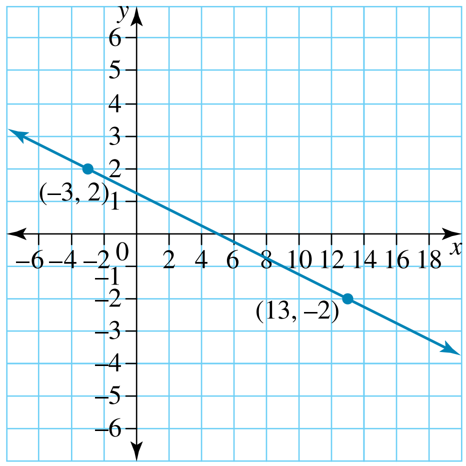
A.


B.


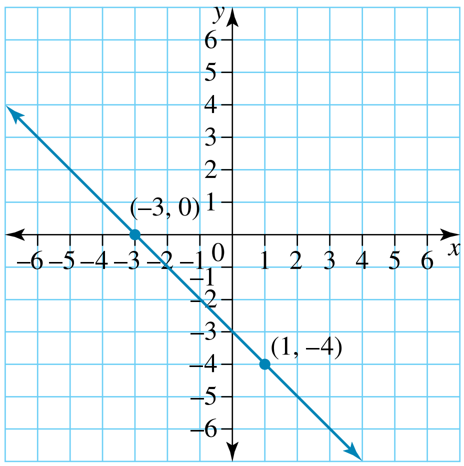
C.



D.



E.



5. Using the gradient, find another point in addition to the y-intercept that lies on each of the following straight lines. Hence, sketch the graph of each straight line.

a. Gradient = 4, y-intercept = 3

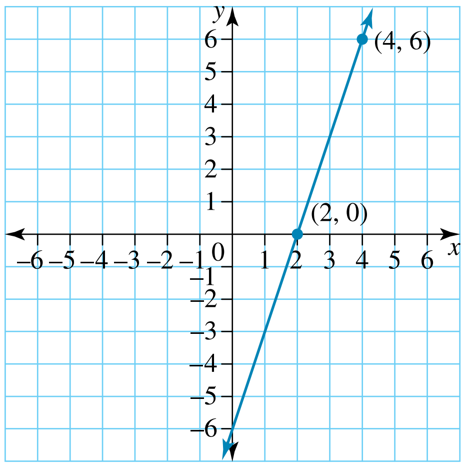
b. Gradient = −3, y-intercept = 1

c. Gradient = $\frac{1}{4}$, y-intercept = 4

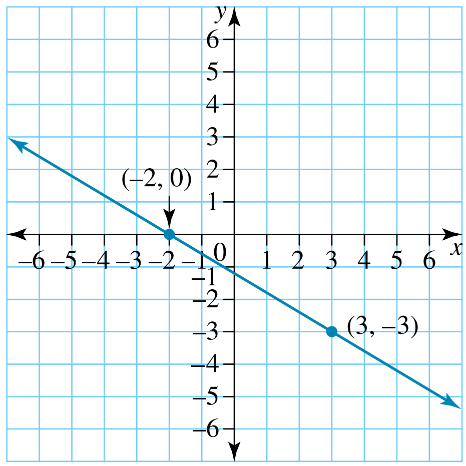
d. Gradient = $-\frac{2}{5}$, y-intercept = −2

6. Find the value of the gradient and y-intercept of each of the following graphs.

a.



b.



7. **WE3** Find the value of the gradients of the straight-line graphs that pass through the following points.
   a. $(2, 3)$ and $(5, 12)$
   b. $(-1, 3)$ and $(2, 7)$
   c. $(-0.2, 0.7)$ and $(0.5, 0.9)$
8. A line has a gradient of -2 and passes through the points $(1, 4)$ and $(a, 8)$. Find the value of a.
9. Find the values of the gradients of the straight-line graphs that pass through the following points.
   a. $(3, 6)$ and $(2, 9)$
   b. $(-4, 5)$ and $(1, 8)$
   c. $(-0.9, 0.5)$ and $(0.2, -0.7)$
   d. $(1.4, 7.8)$ and $(3.2, 9.5)$
   e. $\left(\frac{4}{5}, \frac{2}{5}\right)$ and $\left(\frac{1}{5}, -\frac{6}{5}\right)$
   f. $\left(\frac{2}{3}, \frac{1}{4}\right)$ and $\left(\frac{3}{4}, -\frac{2}{3}\right)$
10. **WE4** Construct a straight-line graph that passes through the points $(2, 5)$, $(4, 9)$ and $(0, 1)$:
   a. without technology
   b. using CAS, a spreadsheet or otherwise.
11. A straight line passes through the following points: $(3, 7)$, $(0, a)$, $(2, 5)$ and $(-1, -1)$. Construct a graph and hence find the value of the unknown, a.
12. A straight line passes through the points $(2, 5)$, $(0, 9)$, $(-1, 11)$ and $(4, a)$. Construct a graph of the straight line and hence find the value of the unknown, a.
13. A line has a gradient of 5. If it passes through the points $(-2, b)$ and $(-1, 7)$, find the value of b.
14. **WE5** Using the gradient and the y-intercept, sketch the following linear graphs.
   a. Gradient = 2, y-intercept = 5
   b. Gradient = -3, y-intercept = 0
   c. Gradient = $\frac{1}{2}$, y-intercept = 3
15. Using an appropriate method, find the gradients of the lines that pass through the following points.
   a. $(0, 5)$ and $(1, 8)$
   b. $(0, 2)$ and $(1, -2)$
   c. $(0, -3)$ and $(1, -5)$
   d. $(0, -1)$ and $(2, -3)$
16. **WE6** Find the values of the x- and y-intercepts for the following linear equations, and hence sketch their graphs.
   a. $2x + 5y = 20$
   b. $y = 2x + 4$
   c. $4y = 3x + 5$
17. Find the values of the x- and y-intercepts for the following linear equations, and hence sketch their graphs.
   a. $2x + y = 6$
   b. $y = 3x + 9$
   c. $2y = 3x + 4$
   d. $3y - 4 = 5x$
18. Mohammed was asked to find the x- and y-intercepts of the following graphs. His responses are shown in the table.

Graph		**Mohammed's response**
i	$3x + 4y = 12$	x-intercept = 4, y-intercept = 3
ii	$2x - y = 6$	x-intercept = 3, y-intercept = 6
iii	$5x - 4y = 20$	x-intercept = 4, y-intercept = 5
iv	$2x + 8y = 30$	x-intercept = 15, y-intercept = 3.75

   a. Some of Mohammed's answers are incorrect. By finding the x- and y-intercepts of the graphs shown, determine which answers are incorrect.
   b. Write an equation that Mohammed would get correct using his method.
   c. What advice would you give Mohammed so that he answers these types of questions correctly?
19. Otis was asked to find the gradient of the line that passes through the points $(3, 5)$ and $(4, 2)$. His response was $\frac{5-2}{4-3} = 3$.
   a. Explain the error in Otis's working out. Hence find the correct gradient.
   b. What advice would you give Otis so that he can accurately find the gradients of straight lines given two points?

20. a. Explain why the gradient of a horizontal line is zero. Support your answer with calculations.
    b. The gradient of a vertical line is undefined. Explain using values why this is the case.
21. A straight-line graph passes through the points (2, 0), (0, a) and (1, 3).
    a. Explain why the value of a must be greater than 3.
    b. Which two points can be used to determine the gradient of the line? Justify your answer by finding the value of the gradient.
    c. Using your answer from part **b**, find the value of a.
22. The table below shows the value of x- and y-intercepts for the linear equations shown.

Equation	x-intercept	y-intercept
$y = 2x + 7$	$-\frac{7}{2}$	7
$y = 3x + 5$	$-\frac{5}{3}$	5
$y = 4x - 1$	$\frac{1}{4}$	−1
$y = 2x - 4$	$\frac{4}{2} = 2$	−4
$y = x + 2$	−2	2
$y = \frac{1}{2}x + 1$	$\frac{-1}{\frac{1}{2}} = -2$	1
$y = \frac{x}{3} + 2$	$\frac{-2}{\frac{1}{3}} = -6$	2

    a. Explain how you can find the x- and y-intercepts for equations of the form shown. Does this method work for all linear equations?
    b. Using your explanation from part a, write the x- and y-intercept for the equation $y = mx + c$.
    c. A straight line has x-intercept $= -\frac{4}{5}$ and y-intercept $= 4$. Write its rule.
23. Using CAS, a spreadsheet or otherwise, find the values of the x- and y-intercepts for the following straight line graphs.
    a. $\frac{4(5 - x)}{3} + 2y = 0$
    b. $\frac{x + 6}{7} = 2(y - 1)$
    c. $\frac{3x}{5} + \frac{2}{7} = \frac{5y}{3}$
24. Using CAS, a spreadsheet or otherwise, sketch the following straight line graphs. Label all key features with their coordinates.
    a. $\frac{2 - 3x}{5} = y + \frac{1}{2}$
    b. $\frac{y + 4}{6} = \frac{2x - 1}{5}$
    c. $\frac{5 - 2x}{3} = \frac{y}{2}$

10.3 Linear modelling

10.3.1 Linear models

Practical problems in which there is a constant change over time can be modelled by linear equations. The constant change, such as the rate at which water is leaking or the hourly rate charged by a tradesperson, can be represented by the gradient of the equation. Usually the y-value is the changing quantity and the x-value is time.

The starting point or initial point of the problem is represented by the y-intercept, when the x-value is 0. This represents the initial or starting value. In situations where there is a negative gradient the x-intercept represents when there is nothing left, such as the time taken for a leaking water tank to empty.

10.3.2 Forming linear equations

Identifying the constant change and the starting point can help to construct a linear equation to represent a practical problem. Once this equation has been established we can use it to calculate specific values or to make predictions as required.

WORKED EXAMPLE 7

Elle is an occupational therapist who charges an hourly rate of $35 on top of an initial charge of $50. Construct a linear equation to represent Elle's charge, *C*, for a period of *t* hours.

THINK	WRITE
1. Find the constant change and the starting point.	Constant change = 35 Starting point = 50
2. Construct the equation in terms of C by writing the value of the constant change as the coefficient of the pronumeral (t) that affects the change, and writing the starting point as the y-intercept.	$C = 35t + 50$

10.3.3 Solving practical problems

Once an equation is found to represent the practical problem, solutions to the problem can be found by sketching the graph and reading off important information such as the value of the x- and y-intercepts and the gradient. Knowing the equation can also help to find other values related to the problem.

When we have determined important values in practical problems, such as the value of the intercepts and gradient, it is important to be able to relate these back to the problem and to interpret their meaning.

For example, if we are given the equation $d = -60t + 300$ to represent the distance a car is in kilometres from a major city after t hours, the value of the gradient (-60) would represent the speed of the car in km/h(60 km/h), the y-intercept (300) would represent the distance the car is from the city at the start of the problem (300 km), and the x-intercept ($t = 5$) would represent the time it takes for the car to reach the city (5 hours).

Note that in the above example, the value of the gradient is negative because the car is heading towards the city, as opposed to away from the city, and we are measuring the distance the car is from the city.

WORKED EXAMPLE 8

A bike tyre has 500 cm^3 of air in it before being punctured by a nail. After the puncture, the air in the tyre is leaking at a rate of 5 cm^3/minute.

a. Construct an equation to represent the amount of air, A, in the tyre t minutes after the puncture occurred.

b. Interpret what the value of the gradient in the equation means.

c. Determine the amount of air in the tyre after 12 minutes.

d. By solving your equation from part a, determine how long, in minutes, it will take before the tyre is completely flat (i.e. there is no air left).

THINK	WRITE
a. 1. Find the constant change and the starting point.	**a.** Constant change = −5 Starting point = 500
2. Construct the equation in terms of A by writing the value of the constant change as the coefficient of the pronumeral that affects the change, and writing the starting point as the y-intercept.	$A = -5t + 500$
b. 1. Identify the value of the gradient in the equation.	**b.** $A = -5t + 500$ The value of the gradient is −5.
2. Identify what this value means in terms of the problem.	The value of the gradient represents the rate at which the air is leaking from the tyre. In this case it means that for every minute, the tyre loses 5 cm^3 of air.
c. 1. Using the equation found in part **a**, substitute $t = 12$ and evaluate.	**c.** $A = -5t + 500$ $= -5 \times 12 + 500$ $= 440$
2. Answer the question using words.	There are 440 cm^3 of air in the tyre after 12 minutes.
d. 1. When the tyre is completely flat, $A = 0$.	**d.** $0 = -5t + 500$
2. Solve the equation for t.	$0 - 500 = -5t + 500 - 500$ $-500 = -5t$ $\frac{-500}{-5} = \frac{-5t}{-5}$ $100 = t$
3. Answer the question using words.	After 100 minutes the tyre will be flat.

10.3.4 The domain of a linear model

When creating a linear model, it is important to interpret the given information to determine the **domain** of the model, that is, the values for which the model is applicable. The domain of a linear model relates to the values of the independent variable in the model (x in the equation $y = mx + c$). For example, in the previous example about air leaking from a tyre at a constant rate, the model will stop being valid after there is no air left in the tyre, so the domain only includes x-values for when there is air in the tyre.

The domains of linear models are usually expressed using the less than or equal to sign ($\leq$) and the greater than or equal to sign ($\geq$). If we are modelling a car that is travelling at a constant rate for 50 minutes before it arrives at its destination, the domain would be $0 \leq t \leq 50$, with t representing the time in minutes.

WORKED EXAMPLE 9

Express the following situations as linear models and give the domains of the models.

a. **A truck drives across the country for 6 hours at a constant speed of 80 km/h before reaching its destination.**

b. **The temperature in an ice storage room starts at −20 °C and falls at a constant rate of 0.8 °C per minute for the next 22 minutes.**

THINK	WRITE
a. 1. Use pronumerals to represent the information given in the question.	a. Let d = the distance travelled by the truck in km. Let t = the time of the journey in hours.
2. Represent the given information as a linear model.	$d = 80t$
3. Determine the domain for which this model is valid.	The model is valid from 0 to 6 hours.
4. Express the domain with the model in algebraic form.	$d = 80t,\ 0 \leq t \leq 6$
b. 1. Use pronumerals to represent the information given in the question.	b. Let i = the temperature of the ice room. Let t = the time in minutes.
2. Represent the given information as a linear model.	$i = -20 - 0.8t$
3. Determine the domain for which this model is valid.	The model is valid from 0 to 22 minutes.
4. Express the domain with the model in algebraic form.	$i = -20 - 0.8t,\ 0 \leq t \leq 22$

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Units 1 & 2 > AOS 5 > Topic 1 > Concept 2

Constructing a linear model Concept summary and practice questions

Exercise 10.3 Linear modelling

1. **WE7** An electrician charges a call out fee of \$90 plus an hourly rate of \$65 per hour. Construct an equation that determines the electrician's charge, C, for a period of t hours.

2. An oil tanker is leaking oil at a rate of 250 litres per hour. Initially there was 125 000 litres of oil in the tanker. Construct an equation that represents the amount of oil, A, in litres in the oil tanker t hours after the oil started leaking.
3. A children's swimming pool is being filled with water. The amount of water in the pool at any time can be found using the equation $A = 20t + 5$, where A is the amount of water in litres and t is the time in minutes.
   a. Explain why this equation can be represented by a straight line.
   b. State the value of the y-intercept and what it represents.
   c. Sketch the graph of $A = 20t + 5$ on a set of axes.
   d. The pool holds 500 litres. By solving an equation, determine how long it will take to fill the swimming pool. Write your answer correct to the nearest minute.

4. **WE8** A yoga ball is being pumped full of air at a rate of 40 cm^3/second. Initially there is 100 cm^3 of air in the ball.
   a. Construct an equation that represents the amount of air, A, in the ball after t seconds.
   b. Interpret what the value of the y-intercept in the equation means.
   c. How much air, in cm^3, is in the ball after 2 minutes?
   d. When fully inflated the ball holds 100 000 cm^3 of air. Determine how long, in minutes, it takes to fully inflate the ball. Write your answer to the nearest minute.

5. Kirsten is a long-distance runner who can run at rate of 12 km/h. The distance, d, in km she travels from the starting point of a race can be represented by the equation $d = at - 0.5$.
   a. Write the value of a.
   b. Write the y-intercept. In the context of this problem, explain what this value means.
   c. How far is Kirsten from the starting point after 30 minutes?
   d. The finish point of this race is 21 km from the starting point. Determine how long, in hours and minutes, it takes Kirsten to run the 21 km. Give your answer correct to the nearest minute.

6. Petrol is being pumped into an empty tank at a rate of 15 litres per minute.

   a. Construct an equation to represent the amount of petrol in litres, P, in the tank after t minutes.
   b. What does the value of the gradient in the equation represent?
   c. If the tank holds 75 litres of petrol, determine the time taken, in minutes, to fill the tank.
   d. The tank had 15 litres of petrol in it before being filled. Write another equation to represent the amount of petrol, P, in the tank after t minutes.
   e. State the domain of the equation formulated in part **c**.

7. Gert rides to and from work on his bike. The distance and time taken for him to ride home can be modelled using the equation $d = 37 - 22t$, where d is the distance from home in km and t is the time in hours.

   a. Determine the distance, in km, between Gert's work and home.
   b. Explain why the gradient of the line in the graph of the equation is negative.
   c. By solving an equation determine the time, in hours and minutes, taken for Gert to ride home. Write your answer correct to the nearest minute.
   d. State the domain of the equation.
   e. Sketch the graph of the

8. A large fish tank is being filled with water. After 1 minute the height of the water is 2 cm and after 4 minutes the height of the water is 6 cm. The height of the water in cm, h, after t minutes can be modelled by a linear equation.

   a. Determine the gradient of the graph of this equation.
   b. In the context of this problem, what does the gradient represent?
   c. Using the gradient found in part **a**, determine the value of the y-intercept. Write your answer correct to 2 decimal places.
   d. Was the fish tank empty of water before being filled? Justify your answer using calculations.

9. Fred deposits \$40 in his bank account each week. At the start of the year he had \$120 in his account. The amount in dollars, A, that Fred has in his account after t weeks can be found using the equation $A = at + b$.
   a. State the values of a and b.
   b. In the context of this problem, what does the y-intercept represent?
   c. How many weeks will it take Fred to save \$3000?

10. Michaela is a real estate agent. She receives a commission of 1.5% on house sales, plus a payment of \$800 each month. Michaela's monthly wage can be modelled by the equation $W = ax + b$, where W represents Michaela's total monthly wage and x represents her house sales in dollars.
   a. State the values of a and b.
   b. Is there an upper limit to the domain of the model? Explain your answer.
   c. In March Michaela's total house sales were \$452 000. Determine her monthly wage for March.
   d. In September Michaela earned \$10 582.10. Determine the amount of house sales she made in September.

11. An electrician charges a call-out fee of \$175 on top of an hourly rate of \$60.
 a. Construct an equation to represent the electrician's fee in terms of his total charge, C, and hourly rate, h.
 b. Claire is a customer and is charged \$385 to install a hot water system. Determine how many hours she was charged for.
 The electrician changes his fee structure. The new fee structure is summarised in the following table.

Time	Call-out fee	Quarter-hourly rate
Up to two hours	\$100	\$20
2–4 hours	\$110	\$25
Over 4 hours	\$115	\$50

 c. Using the new fee structure, how much, in dollars, would Claire be charged for the same job?
 d. Construct an equation that models the new fee structure for between 2 and 4 hours.

12. WE9 Express the following situations as linear models and give the domains of the models.
 a. Julie works at a department store and is paid \$19.20 per hour. She has to work for a minimum of 10 hours per week, but due to her study commitments she can work for no more than 20 hours per week.
 b. The results in a driving test are marked out of 100, with 4 marks taken off for every error made on the course. The lowest possible result is 40 marks.

13. Monique is setting up a new business selling T-shirts through an online auction site. Her supplier in China agrees to a deal whereby they will supply each T-shirt for \$3.50 providing she buys a minimum of 100 T-shirts. The deal is valid for up to 1000 T-shirts.
 a. Set up a linear model (including the domain) to represent this situation.
 b. Explain what the domain represents in this model.
 c. Why is there an upper limit to the domain?

14. The Dunn family departs from home for a caravan trip. They travel at a rate of 80 km/h. The distance they travel from home, in km, can be modelled by a linear equation.
 a. Write the value of the gradient of the graph of the linear equation.
 b. Write the value of the y-intercept. Explain what this value means in the context of this problem.
 c. Using your values from parts **a** and **b**, write an equation to represent the distance the Dunn family are from home at any given point in time.
 d. How long, in hours, have the family travelled when the nearest minute.
 e. The Dunns travel for 2.5 hours before stopping. Determine the distance they are from home.

15. The table shows the amount of money in Kim's savings account at different dates. Kim withdraws the same amount of money every five days.

Date	26/11	1/12	6/12
Amount	\$1250	\$1150	\$1050

a. The amount of money at any time, t days, in Kim's account can be modelled by a linear equation. Explain why.
b. Using a calculator, spreadsheet or otherwise, construct a straight line graph to represent the amount of money Kim has in her account from 26 November.
c. Determine the gradient of the line of the graph, and explain the meaning of the gradient in the context of this problem.
d. Find the linear equation that models this situation.
e. In the context of this problem, explain the meaning of the x-intercept.
f. Is there a limit to the domain of this problem? If so, do we know the limit?
g. Kim will need at least \$800 to go on a beach holiday over the Christmas break (starting on 23 December). Show that Kim will not have enough money for her holiday.

16. There are two advertising packages for Get2Msg.com. Package A charges per cm^2 and package B charges per letter. The costs for both packages increase at a constant rate. The table shows the costs for package A for areas from 4 to 10 cm^2.
a. Determine the cost per cm^2 for package A. Write your answer correct to 2 decimal places.
b. Construct an equation that determines the cost per cm^2.
c. Using your equation from part b, determine the cost for an advertisement of 7.5 cm^2.
d. Package B costs 58 cents per standard letter plus an administration cost of \$55. Construct an equation to represent the costs for package B.
e. Betty and Boris of B'n'B Bedding want to place this advertisement on Get2Msg.com. Which package would be the better option for them? Justify your answer by finding the costs they would pay for both packages.

Area, incm^2	Cost (excluding administration charge of \$25)
4	30
6	45
8	60
10	75

17. A basic mobile phone plan designed for school students charges a flat fee of \$15 plus 13 cents per minute of a call. Text messaging is free.
a. Construct an equation that determines the cost, in dollars, for any time spent on the phone, in minutes.
b. In the context of this problem what do the gradient and y-intercept of the graph of the equation represent?
c. Using a spreadsheet, CAS or otherwise, complete the following table to determine the cost at any time, in minutes.

Time (min)	Cost (\$)	Time (min)	Cost (\$)
5		35	
10		40	
15		45	
20		50	
25		55	
30		60	

d. Bill received a monthly phone bill for \$66.50. Determine the number of minutes he spent using his mobile phone. Give your answer to the nearest minute.

18. Carly determines that the number of minutes she spends studying for History tests affects her performances on these tests. She finds that if she does not study, then her test performance in History is 15%. She records the number of minutes she spends studying and her test scores, with her results shown in the following table.

Time in minutes studying, t	15	55	35
Test scores, y (%)	25	a	38

Carly decides to construct an equation that determines her test scores based on the time she spends studying. She uses a linear equation because she finds that there is a constant increase between her number of minutes of study and her test results.

a. Using CAS, determine the value of a. Give your answer correct to 2 decimal places.
b. Using CAS, a spreadsheet or otherwise, represent Carly's results from the table on a graph.
c. Explain why the y-intercept is 15.
d. Construct an equation that determines Carly's test score, in %, given her studying time in minutes, t. State the domain of the equation.
e. Carly scored 65% on her final test. Using your equation from part **d**, determine the number of minutes she spent studying. Give your answer correct to the nearest second.

10.4 Linear equations and predictions

10.4.1 Finding the equation of straight lines given the gradient and y-intercept

When we are given the gradient and y-intercept of a straight line, we can enter these values into the equation $y = mx + c$ to determine the equation of the straight line. Remember that m is equal to the value of the gradient and c is equal to the value of the y-intercept.

For example, if we are given a gradient of 3 and a y-intercept of 6, then the equation of the straight line would be $y = 3x + 6$.

10.4.2 Finding the equation of straight lines given the gradient and one point

When we are given the gradient and one point of a straight line, we need to establish the value of the y-intercept to find the equation of the straight line. This can be done by substituting the coordinates of the given point into the equation $y = mx + c$ and then solving for c. Remember that m is equal to the value of the gradient, so this can also be substituted into the equation.

10.4.3 Finding the equation of straight lines given two points

When we are given two points of a straight line, we can find the value of the gradient of a straight line between these points as discussed in Section 10.2 (by using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$). Once the gradient has been found, we can find the y-intercept by substituting one of the points into the equation $y = mx + c$ and then solving for c.

WORKED EXAMPLE 10

Find the equations of the following straight lines.

a. A straight line with a gradient of 2 passing through the point (3, 7)

b. A straight line passing through the points (1, 6) and (3, 0)

c. A straight line passing through the points (2, 5) and (5, 5)

THINK	WRITE
a. 1. Write the gradient–intercept form of a straight line.	**a.** $y = mx + c$
2. Substitute the value of the gradient into the equation (in place of m).	Gradient $= m = 2$ $y = 2x + c$
3. Substitute the values of the given point into the equation and solve for c.	(3, 7) $7 = 2(3) + c$ $7 = 6 + c$ $c = 1$
4. Substitute the value of c back into the equation and write the answer.	The equation of the straight line is $y = 2x + 1$.
b. 1. Write the formula to find the gradient given two points.	**b.** $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
2. Let one of the given points be (x_1, y_1) and let the other point be (x_2, y_2).	Let $(1, 6) = (x_1, y_1)$. Let $(3, 0) = (x_2, y_2)$.
3. Substitute the values into the equation to determine the value of m.	$m = \dfrac{0 - 6}{3 - 1}$ $= \dfrac{-6}{2}$ $= -3$
4. Substitute the value of m into the equation $y = mx + c$.	$y = mx + c$ $y = -3x + c$
5. Substitute the values of one of the points into the equation and solve for c. *Note:* The point (1, 6) could also be used.	(3, 0) $0 = -3(3) + c$ $0 = -9 + c$ $c = 9$
6. Substitute the value of c back into the equation and write the answer.	The equation of the straight line is $y = -3x + 9$.
c. 1. Write the equation to find the gradient given two points.	**c.** $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
2. Let one of the given points be (x_1, y_1) and let the other point be (x_2, y_2).	Let $(2, 5) = (x_1, y_1)$. Let $(5, 5) = (x_2, y_2)$.
3. Substitute the values into the equation to determine the value of m.	$m = \dfrac{5 - 5}{5 - 2}$ $= \dfrac{0}{3}$ $= 0$
4. A gradient of 0 indicates that the straight line is horizontal and the equation of the line is of the form $y = c$.	$y = c$

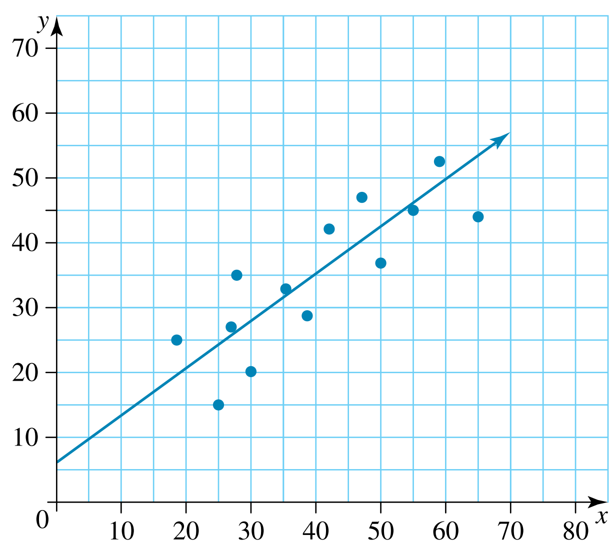
5. Substitute the values of one of the points into the equation and solve for c.	(2, 5) $5 = c$ $c = 5$
6. Substitute the value of c back into the equation and write the answer.	The equation of the straight line is $y = 5$.

For part **b** of Worked example 10, try substituting the other point into the equation at stage 5. You will find that the calculated value of c is the same, giving you the same equation as an end result.

10.4.4 Lines of best fit by eye

Sometimes the data for a practical problem may not be in the form of a perfect linear relationship, but the data can still be modelled by an approximate linear relationship.

When we are given a scatterplot representing data that appears to be approximately represented by a linear relationship, we can draw a **line of best fit** by eye so that approximately half of the data points are on either side of the line of best fit.



After drawing a line of best fit, the equation of the line can be determined by picking two points on the line and determining the equation, as demonstrated in the previous section.

10.4.5 Creating a line of best fit when given only two points

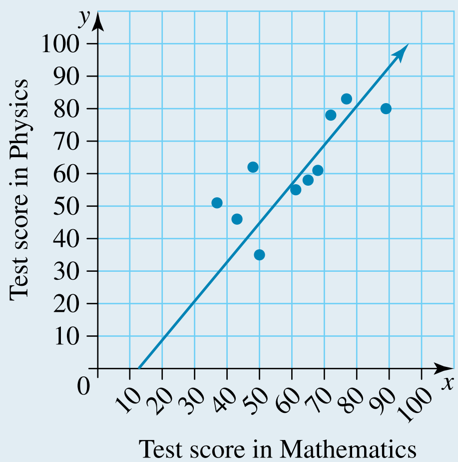
In some instances we may be given only two points of data in a data set. For example, we may know how far someone had travelled after 3 and 5 hours of their journey without being given other details about their journey. In these instances we can make a line of best fit using these two values to estimate other possible values that might fit into the data set.

Although this method can be useful, it is much less reliable than drawing a line of best fit by eye, as we do not know how typical these two points are of the data set. Also, when we draw a line of best fit through two points of data that are close together in value, we are much more likely to have an inaccurate line for the rest of the data set.

WORKED EXAMPLE 11

The following table and scatterplot represent the relationship between the test scores in Mathematics and Physics for ten Year 11 students. A line of best fit by eye has been drawn on the scatterplot.

Test score in Mathematics	65	43	72	77	50	37	68	89	61	48
Test scores in Physics	58	46	78	83	35	51	61	80	55	62



Choose two appropriate points that lie on the line of best fit and determine the equation for the line.

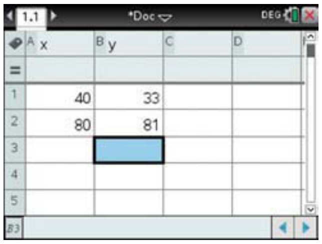
THINK	WRITE
1. Look at the scattergraph and pick two points that lie on the line of best fit.	Two points that lie on the line of best fit are (40, 33) and (80, 81).
2. Calculate the value of the gradient between the two points.	Let $(40, 33) = (x_1, y_1)$. Let $(80, 81) = (x_2, y_2)$. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $= \dfrac{81 - 33}{80 - 40}$ $= 1.2$
3. Substitute the value of m into the equation $y = mx + c$.	$y = 1.2x + c$
4. Substitute the values of one of the points into the equation and solve for c.	(40, 33) $33 = 1.2 \times 40 + c$ $33 = 48 + c$ $33 - 48 = c$ $c = -15$
5. Substitute the value of c back into the equation and write the answer.	The line of best fit for the data is $y = 1.2x - 15$.

TI | THINK — **WRITE**

1. Select two points that lie on the line of best fit.

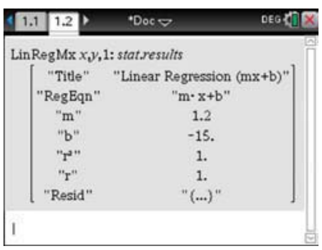
Two points are (40, 33) and (80, 81).

2. On a Lists & Spreadsheets page, label the first column x and the second column y. Enter the x-coordinates of the chosen points in the first column, and y-coordinates in the second column.



3. Press MENU then select:
6: Statistics
1: Stat Calculations
3: Linear Regression (mx + b) …
Complete the fields as:
X List: x
Y List: y
then select OK.

4. Interpret the output shown on the screen.



The equation is given in the form $y = mx + b$, where $m = 1.2$ and $b = -15$.

5. State the answer.

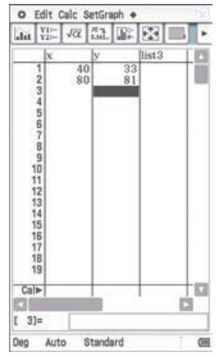
The line of best fit is $y = 1.2x - 15$.

CASIO | THINK — **WRITE**

1. Select two points that lie on the line of best fit.

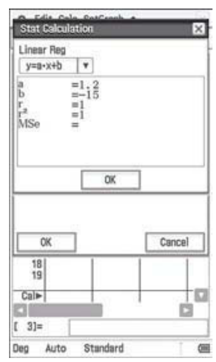
Two points are (40, 33) and (80, 81).

2. On a Statistics screen, label list1 as x and list2 as y. Enter the x-coordinates of the chosen points in the first column, and y-coordinates in the second column.



3. Select:
- Calc
- Regression
- Linear Reg
Complete the fields as:
XList: main\x
YList: main\y
then select OK.

4. Interpret the output shown on the screen.



The equation is given in the form $y = ax + b$, where $a = 1.2$ and $b = -15$.

5. State the answer.

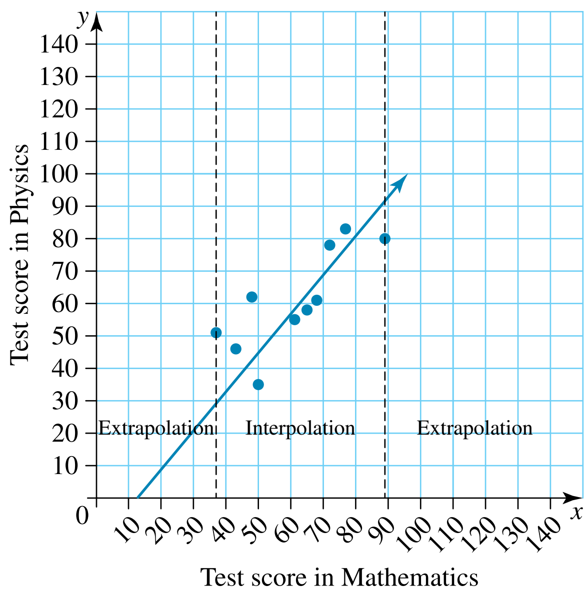
The line of best fit is $y = 1.2x - 15$.

Note: If you use CAS, there are shortcuts you can take to find the equation of a straight line given two points. Refer to the CAS instructions available on the eBookPLUS.

10.4.6 Making predictions: Interpolation

When we use **interpolation**, we are making a prediction from a line of best fit that appears within the parameters of the original data set.

If we plot our line of best fit on the scatterplot of the given data, then interpolation will occur between the first and last points of the scatterplot.



10.4.7 Making predictions: Extrapolation

When we use **extrapolation**, we are making a prediction from a line of best fit that appears outside the parameters of the original data set.

If we plot our line of best fit on the scatterplot of the given data, then extrapolation will occur before the first point or after the last point of the scatterplot.

10.4.8 Reliability of predictions

The more pieces of data there are in a set, the better the line of best fit you will be able to draw. More data points allow more reliable predictions.

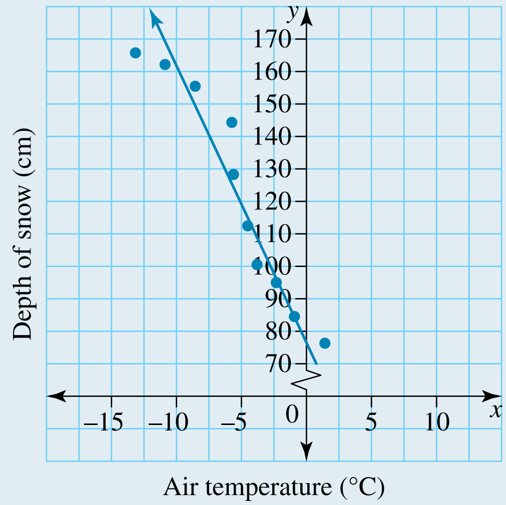
In general, interpolation is a far more reliable method of making predictions than extrapolation. However, there are other factors that should also be considered. Interpolation closer to the centre of the data set will be more reliable that interpolation closer to the edge of the data set. Extrapolation that appears closer to the data set will be much more reliable than extrapolation that appears further away from the data set.

A strong correlation between the points of data will give a more reliable line of best fit to be used. This is shown when all of the points appear close to the line of best fit. The more points there are that appear further away from the line of best fit, the less reliable other predictions will be.

When making predictions, always be careful to think about the data that you are making predictions about. Be sure to think about whether the prediction you are making is realistic or even possible!

WORKED EXAMPLE 12

The following data represent the air temperature (°C) and depth of snow (cm) at a popular ski resort.



Air temperature(°C)	−4.5	−2.3	−8.9	−11.0	−13.3	−6.2	−0.4	1.5	−3.7	−5.4
Depth of snow (cm)	111.3	95.8	155.6	162.3	166.0	144.7	84.0	77.2	100.5	129.3

The line of best fit for this data set has been calculated as $y = -7.2x + 84$.

a. Use the line of best fit to estimate the depth of snow if the air temperature is −6.5 °C.

b. Use the line of best fit to estimate the depth of snow if the air temperature is 25.2 °C.

c. Comment on the reliability of your estimations in parts a and b.

THINK

a. 1. Enter the value of x into the equation for the line of best fit.

2. Evaluate the value of x.

3. Write the answer.

b. 1. Enter the value of x into the equation for the line of best fit.

2. Evaluate the value of x.

3. Write the answer.

c. Relate the answers back to the original data to check their reliability.

WRITE

a $x = -6.5$

$y = -7.2x + 84$

$= -7.2 \times -6.5 + 84$

$= 130.8$

The depth of snow if the air temperature is −6.5 °C will be approximately 130.8 cm.

b. $x = 25.2$

$y = -7.2x + 84$

$= -7.2 \times 25.2 + 84$

$= -97.4$ (1 decimal point).

The depth of snow if the air temperature is 25.2 °C will be approximately −97.4 cm.

c. The estimate in part **a** was made using interpolation, with the point being comfortably located within the parameters of the original data. The estimate appears to be consistent with the given data and as such is reliable.

The estimate in part **b** was made using extrapolation, with the point being located well outside the parameters of the original data. This estimate is clearly unreliable, as we cannot have a negative depth of snow.

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Units 1 & 2 > AOS 5 > Topic 1 > Concepts 3 & 4

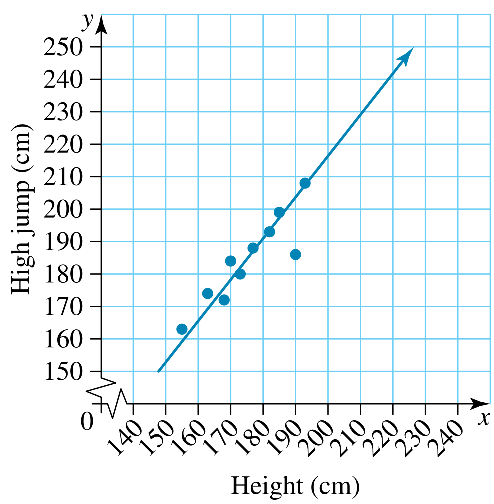
Interpolations and extrapolations Concept summary and practice questions
Line of best fit Concept summary and practice questions

Exercise 10.4 Linear equations and predictions

1. WE10 Find the equations of the following straight lines.
   a. A straight line with a gradient of 5 passing through the point $(-2, -5)$
   b. A straight line passing through the points $(-3, 4)$ and $(1, 6)$
   c. A straight line passing through the points $(-3, 7)$ and $(0, 7)$
2. MC Which of the following equations represents the line that passes through the points $(3, 8)$ and $(12, 35)$?

   A. $y = 3x + 1$ B. $y = -3x + 1$ C. $y = 3x - 1$ D. $y = \frac{1}{3}x + 1$

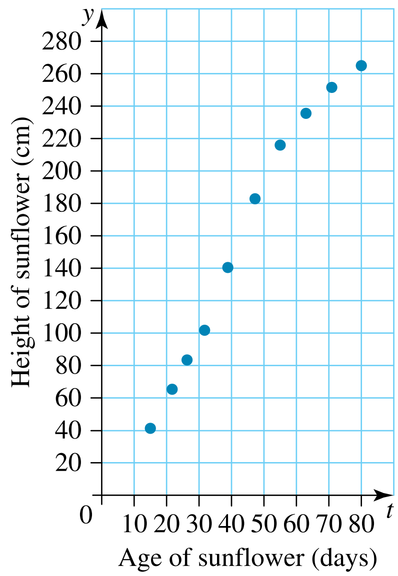
   E. $y = \frac{1}{3}x - 1$
3. Steve is looking at data comparing the size of different music venues across the country and the average ticket price at these venues. After plotting his data in a scatterplot, he calculates a line of best fit for his data as $y = 0.04x + 15$, where y is the average ticket price in dollars and x is the capacity of the venue.
   a. What does the value of the gradient (m) represent in Steve's equation?
   b. What does the value of the y-intercept represent in Steve's equation?
   c. Is the y-intercept a realistic value for this data?
4. WE11 A sports scientist is looking at data comparing the heights of athletes and their performance in the high jump. The following table and scatterplot represent the data they have collected. A line of best fit by eye has been drawn on the scatterplot. Choose two appropriate points that lie on the line of best fit and determine the equation for the line.



Height (cm)	168	173	155	182	170	193	177	185	163	190
High jump (cm)	172	180	163	193	184	208	188	199	174	186

5. Nidya is analysing the data from question 4, but a clerical error means that she only has access to two points of data: (170, 184) and (177, 188).
   a. Determine Nidya's equation for the line of best fit, rounding all decimal numbers to 2 places.
   b. Add Nidya's line of best fit to the scatterplot of the data.
   c. Comment on the similarities and differences between the two lines of best fit.
6. The following table and scatterplot shows the age and height of a field of sunflowers planted at different times throughout summer.

Age of sunflower (days)	63	71	15	33	80	22	55	47	26	39
Height of sunflower (cm)	237	253	41	101	264	65	218	182	82	140

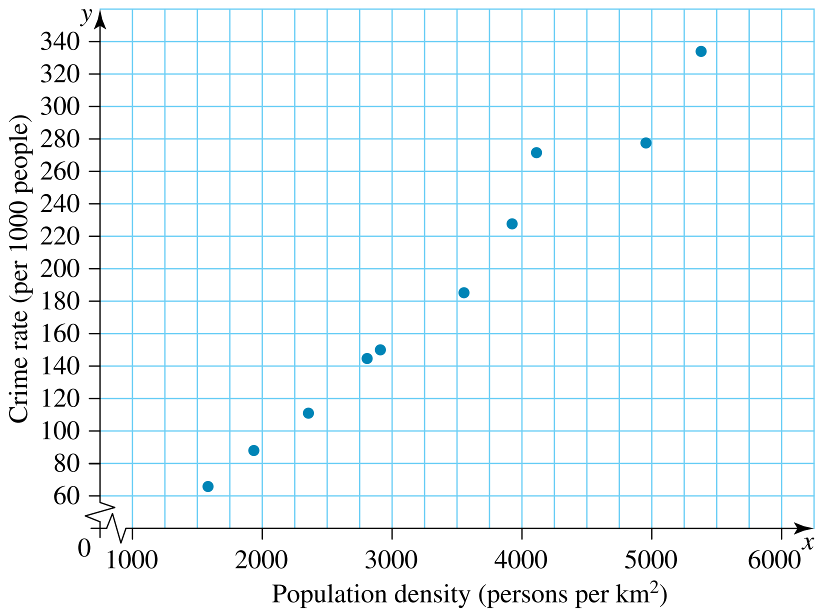


   a. Xavier draws a line of best fit by eye that goes through the points (10, 16) and (70, 280). Draw his line of best fit on the scatterplot and comment on his choice of line.
   b. Calculate the equation of the line of best fit using the two points that Xavier selected.
   c. Patricia draws a line of best fit by eye that goes through the points (10, 18) and (70, 258). Draw her line of best fit on the scatterplot and comment on her choice of line.
   d. Calculate the equation of the line of best fit using the two points that Patricia selected.
   e. Why is the value of the y-intercept not 0 in either equation?
7. Olivia is analysing historical figures for the prices of silver and gold. The price of silver (per ounce) at any given time (x) is compared with the price of gold (per gram) at that time (y). She asks her assistant to note down the points she gives him and to create a line of best fit from the data.

   On reviewing her assistant's notes, she has trouble reading his handwriting. The only complete pieces of information she can make out are one of the points of data (16, 41.5) and the gradient of the line of best fit (2.5).
   a. Use the gradient and data point to determine an equation of the line of best fit.
   b. Use the equation from part **a** to answer the following questions.
      i. What is the price of a gram of gold if the price of silver is \$25 per ounce?
      ii. What is the price of an ounce of silver if the price of gold is \$65 per gram?
      iii. What is the price of a gram of gold if the price of silver is \$11 per ounce?
      iv. What is the price of an ounce of silver if the price of gold is \$28 per gram?
8. A government department is analysing the population density and crime rate of different suburbs to see if there is a connection. The following table and scatterplot display the data that has been collected so far.

a. Draw a line of best fit on the scatterplot of the data.
b. Choose two points from the line of best fit and find the equation of the line.
c. What does the value of the x-intercept mean in terms of this problem?
d. Is the x-intercept value realistic? Explain your answer.

Population density (persons per km^2)	3525	2767	4931	3910	1572	2330	2894	4146	1968	5337
Crime rate (per 1000 people)	185	144	279	227	65	112	150	273	87	335



9. Kari is calculating the equation of a straight line passing through the points (−2, 5) and (3, 1). Her working is shown below.

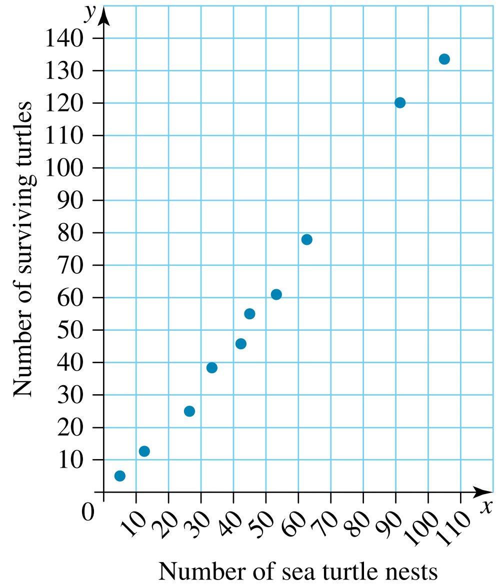
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-5}{-2-3} = \frac{-4}{-5} = \frac{4}{5}$$

$$y = \frac{4}{5}x + c$$
$$1 = \frac{4}{5} \times 3 + c$$
$$c = 1 - \frac{12}{5} = \frac{-7}{5}$$
$$y = \frac{4}{5}x - \frac{7}{5}$$

a. Identify the error in Kari's working.
b. Calculate the correct equation of the straight line passing through these two points.

10. Horace is a marine biologist studying the lives of sea turtles. He collects the following data comparing the number of sea turtle egg nests and the number of survivors from those nests. The following table and scatterplot display the data he has collected.

Number of sea turtle nests	45	62	12	91	27	5	53	33	41	105
Number of surviving turtles	55	78	13	120	25	5	61	39	46	133



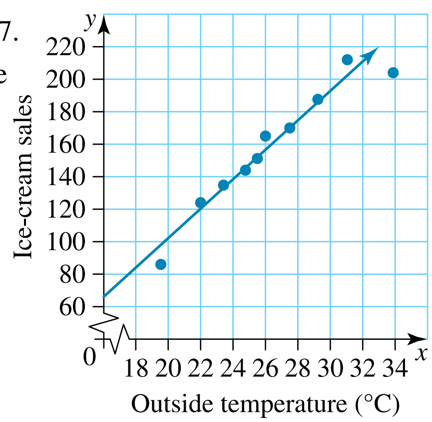
a. Horace draws a line of best fit for the data that goes through the points $(0, -5)$ and $(100, 127)$. Determine the equation for Horace's line of best fit.
b. What does the gradient of the line represent in terms of the problem?
c. What does the y-intercept represent in terms of the problem? Is this value realistic?
d. Use the equation to answer the following questions.
   i. Estimate how many turtles you would expect to survive from 135 nests.
   ii. Estimate how many eggs would need to be laid to have 12 surviving turtles.
e. Comment on the reliability of your answers to part **d**.

11. A straight line passes through the points $(-2, 2)$ and $(-2, 6)$.
   a. What is the gradient of the line?
   b. Determine the equation of this line.

12. WE12 An owner of an ice-cream parlour has collected data relating the outside temperature to ice-cream sales.

Outside temperature (°C)	23.4	27.5	26.0	31.1	33.8	22.0	19.7	24.6	25.5	29.3
Ice-cream sales	135	170	165	212	204	124	86	144	151	188

A line of best fit for this data has been calculated as $y = 9x - 77$.



a. Use the line of best fit to estimate ice-cream sales if the outside temperature is 27.9 °C.
b. Use the line of best fit to estimate ice-cream sales if the air temperature is 15.2 °C.
c. Comment on the reliability of your answers to parts **a** and **b**.

13. Georgio is comparing the cost and distance of various long-distance flights, and after drawing a scatterplot he creates an equation for a line of best fit to represent his data. Georgio's line of best fit is $y = 0.08x + 55$, where y is the cost of the flight and x is the distance of the flight in kilometres.
   a. Estimate the cost of a flight between Melbourne and Sydney (713 km) using Georgio's equation.
   b. Estimate the cost of a flight between Melbourne and Broome (3121 km) using Georgio's equation.
   c. All of Georgio's data came from flights of distances between 400 km and 2000 km. Comment on the suitability of using Georgio's equation for shorter and longer flights than those he analysed. What other factors might affect the cost of these flights?

14. Mariana is a scientist and is collecting data measuring lung capacity (in L) and time taken to swim 25 metres (in seconds). Unfortunately a spillage in her lab causes all of her data to be erased apart from the records of a person with a lung capacity of 3.5 L completing the 25 metres in 55.8 seconds and a person with a lung capacity of 4.8 L completing the 25 metres in 33.3 seconds.

a. Use the remaining data to construct an equation for the line of best fit relating lung capacity (x) to the time taken to swim 25 metres (y). Give any numerical values correct to 2 decimal places.

b. What does the value of the gradient (m) represent in the equation?

c. Use the equation to estimate the time it takes people with the following lung capacities to swim 25 metres.

i. 3.2 litres **ii.** 4.4 litres **iii.** 5.3 litres

d. Comment on the reliability of creating the equation from Mariana's two remaining data points.

15. Mitch is analysing data comparing the kicking efficiency (x) with the handball efficiency (y) of different AFL players. His data is shown in the table at the top of the following page.

a. A line of best fit for the data goes through the points (66, 79) and (84, 90). Determine the equation for the line of best fit for this data. Give any numerical values correct to 2 decimal places.

b. Use the equation from part **a** and the figures for kicking efficiency to create a table for the predicted handball efficiency of the same group of players.

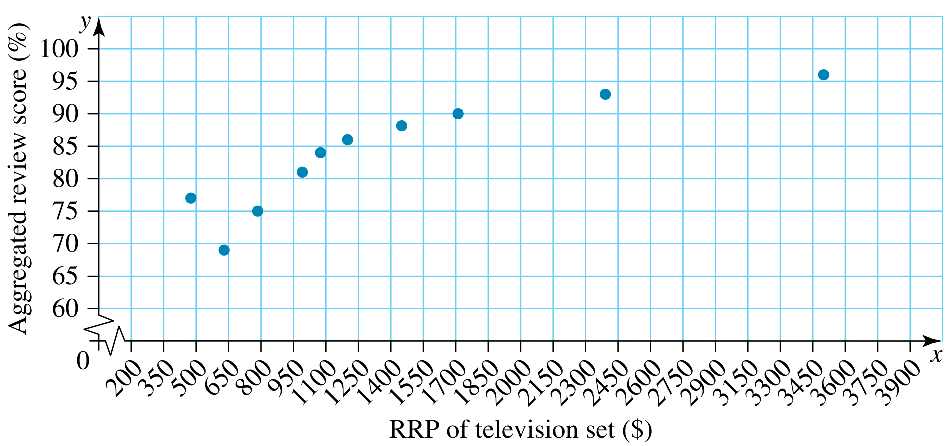
Kicking efficiency (%)	75.3	65.6	83.1	73.9	79.0	84.7	64.4	72.4	68.7	80.2
Handball efficiency (%)	84.6	79.8	88.5	85.2	87.1	86.7	78.0	81.3	82.4	90.3

Kicking efficiency (%)	75.3	65.6	83.1	73.9	79.0	84.7	64.4	72.4	68.7	80.2
Predicted handball efficiency (%)										

c. Comment on the differences between the predicted kicking efficiency and the actual kicking efficiency.

16. Chenille is comparing the price of new television sets versus their aggregated review scores (out of 100). The following table and scatterplot display the data she has collected.

RRP of television set ($)	799	1150	2399	480	640	999	1450	1710	3500	1075
Aggregated review score (%)	75	86	93	77	69	81	88	90	96	84



a. Using CAS, a spreadsheet or otherwise, calculate the equations of the straight lines that pass through each of the following pairs of points.

i. (400, 800) and (3400, 97)
ii. (300, 78) and (3200, 95)
iii. (400, 75) and (2400, 95)
iv. (430, 67) and (1850, 95)

b. Draw these lines on the scatterplot of the data.

c. Which line do you think is the most appropriate line of best fit for the data? Give reasons for your answer.

17. Karyn is investigating whether the salary of the leading actors/actresses in movies has any impact on the box office receipts of the movie.

The following table and scatterplot display the data that Karyn has collected.

Actor/actress salary ($m)	0.2	3.5	8.0	2.5	10.0	6.5	1.6	14.0	4.7	7.5
Box office receipts ($m)	135	72	259	36	383	154	25	330	98	232

a. Explain why a line of best fit for the data would never go through the point of data (0.2, 135).

b. Draw a line of best fit on the scatterplot.

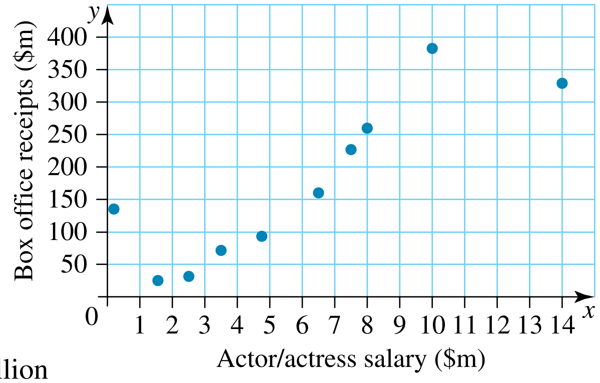
c. Use two points from your line of best fit to determine the equation for this line.

d. Explain what the value of the gradient (m) means in the context of this problem.

e. Calculate the expected box office receipts for films where the leading actor/actress is paid the following amounts.

i. $1.2 million
ii. $11 million
iii. $50 000
iv. $20 million

f. Comment on the reliability of your answers to part **e**.



18. Giant sequoias are the world's largest trees, growing up to 100 or more metres in height and 10 or more metres in diameter. Throughout their lifetime they continue to grow in size, with the largest of them among the fastest growing organisms that we know of. Sheila is examining the estimated age and diameter of giant sequoias. The following table and scatterplot show the data she has collected.

Estimated age (years)	450	1120	330	1750	200	1400	630	800	980	2050
Diameter (cm)	345	485	305	560	240	525	390	430	465	590

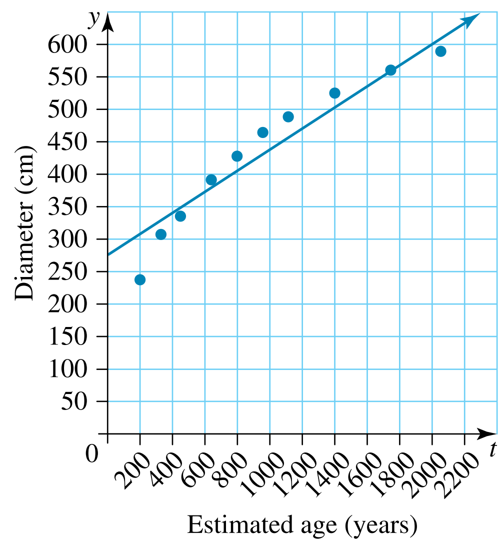
a. The line of best fit shown on the scatterplot passes through the points (0, 267) and (2000, 627). Determine the equation for the line of best fit.

b. Using a spreadsheet, CAS or otherwise, calculate the average (mean) age of the trees in Sheila's data set.

c. Using a spreadsheet, CAS or otherwise, calculate the average (mean) height of the trees in Sheila's data set.

d. Subtract the y-intercept from your equation in part **a** from the average age calculated in part **b**, and divide this total by the average height calculated in part c.

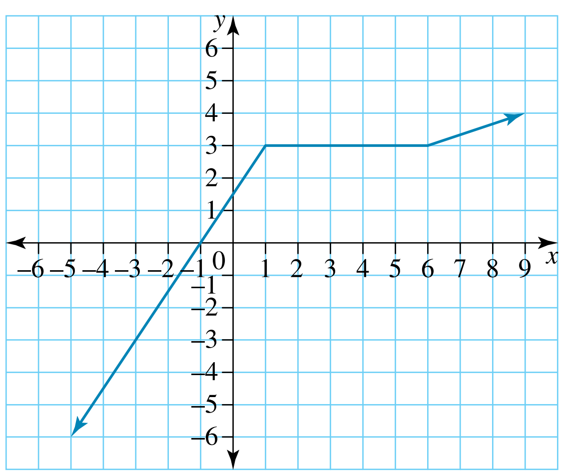
e. How does the answer in part **d** compare to the gradient of the equation calculated in part **a**?



10.5 Further linear applications

10.5.1 Piecewise linear and step graphs

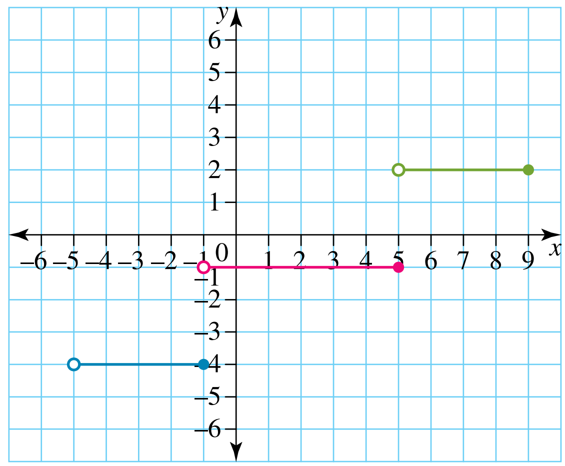
Piecewise graphs are formed by two or more linear graphs that are joined at points of intersection. A piecewise graph is continuous, which means there are no breaks or gaps in the graph, as shown in the first diagram.



Step graphs are formed by two or more linear graphs that have zero gradients. Step graphs have breaks, as shown in the second diagram.

The end points of each line depend on whether the point is included in the interval. For example, the interval $-1 < x \leq 5$ will have an open end point at $x = -1$, because x does not equal -1 in this case. The same interval will have a closed end point at $x = 5$, because x is less than or equal to 5.

A closed end point means that the x-value is also 'equal to' the value. An open end point means that the x-value is not equal to the value; that is, it is less than or greater than only.



WORKED EXAMPLE 13

A piecewise linear graph is constructed from the following linear graphs.

$$y = 2x + 1,\ x \leq a$$
$$y = 4x - 1,\ x > a$$

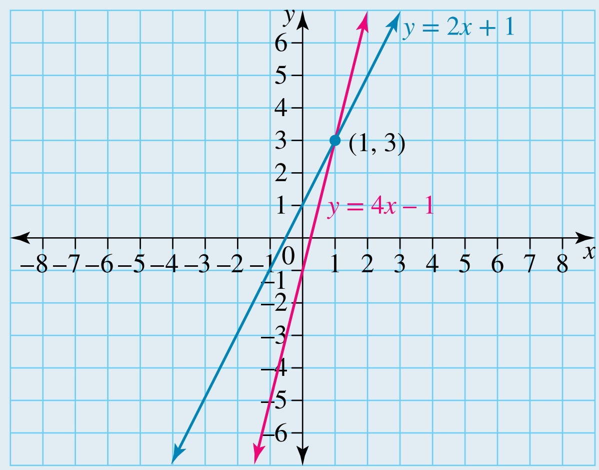
a. **By solving the equations simultaneously, find the point of intersection and hence state the value of a.**

b. **Sketch the piecewise linear graph.**

THINK	WRITE/DRAW
a. 1. Find the intersection point of the two graphs by solving the equations simultaneously.	a. $y = 2x + 1$ $y = 4x - 1$ Solve by substitution: $2x + 1 = 4x - 1$ $2x - 2x + 1 = 4x - 2x - 1$ $1 = 2x - 1$ $1 + 1 = 2x - 1 + 1$ $2 = 2x$ $x = 1$ Substitute $x = 1$ to find y: $y = 2(1) + 1$ $= 3$ The point of intersection is (1, 3). $x = 1$ and $y = 3$
2. The x-value of the point of intersection determines the x-intervals for where the linear graphs meet.	$x = 1$, therefore $a = 1$.

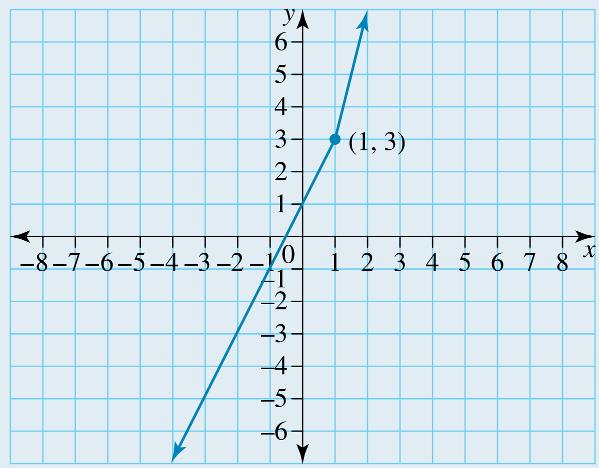
b. 1. Using CAS, a spreadsheet or otherwise, sketch the two graphs without taking into account the intervals.

b.



2. Identify which graph exists within the stated x-intervals to sketch the piecewise linear graph.

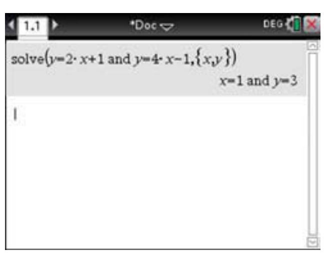
$y = 2x + 1$ exists for $x \leq 1$.
$y = 4x - 1$ exists for $x > 1$.
Remove the sections of each graph that do not exist for these values of x.



TI | THINK

a1. On a Calculator page, press MENU then select:
3: Algebra
1: Solve
Complete the entry line as:solve $(y = 2x + 1 \text{ and } y = 4x - 1, \{x, y\})$
then press ENTER.

WRITE



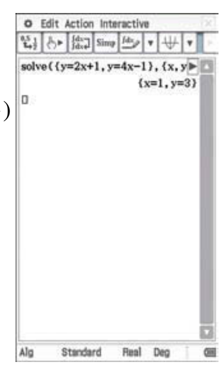
2. The answer is shown on the screen.

$x = 1$, hence $a = 1$.

CASIO | THINK

a1. On the Main screen, complete the entry line as:
solve
$(\{y = 2x + 1, y = 4x - 1\}, \{x, y\})$
then press EXE.

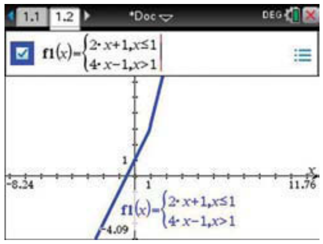
WRITE



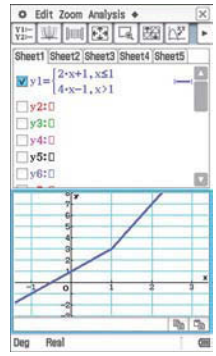
2. The answer is shown on the screen.

$x = 1$, hence $a = 1$.

b1. On a Graphs page, complete the entry line for function 1 as:
$f1(x) = \begin{cases} 2x + 1, & x \le 1 \\ 4x - 1, & x > 1 \end{cases}$
then press ENTER.
Note: The piecewise template can be found by pressing the t button.



b1. On a Graph & Table screen, complete the entry line for equation 1 as:
$y1 = \begin{cases} 2x + 1, & x \le 1 \\ 4x - 1, & x > 1 \end{cases}$
Click the tick box and then click the $ icon.
Note: The piecewise template can be found in the Math3 tab in the Keyboard menu.



WORKED EXAMPLE 14

Construct a step graph from the following equations, making sure to take note of the relevant end points.

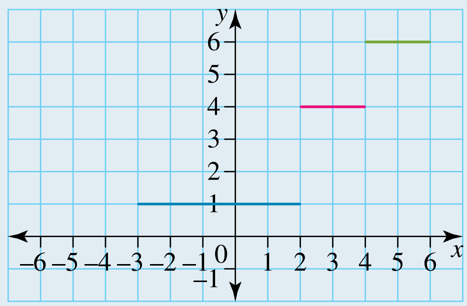
$$y = 1,\ -3 < x \le 2$$
$$y = 4,\ 2 < x \le 4$$
$$y = 6,\ 4 < x \le 6$$

THINK

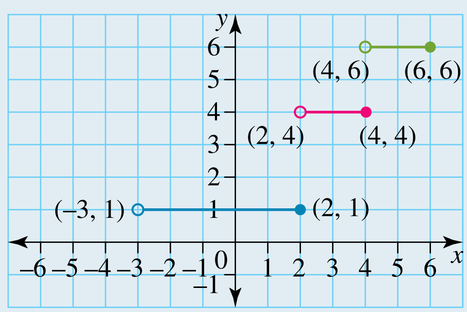
1. Construct a set of axes and draw each line within the stated x-intervals.

2. Draw in the end points.

WRITE/DRAW



For the line $y = 1$:
$-3 < x \le 2$
$x > -3$ is an open circle.
$x \le 2$ is a closed circle.
For the line $y = 4$:
$2 < x \le 4$
$x > 2$ is an open circle.
$x \le 4$ is a closed circle.
For the line $y = 6$:
$4 < x \le 6$
$x > 4$ is an open circle.
$x \le 6$ is a closed circle.



10.5.2 Modelling with piecewise linear and step graphs

Consider the real-life situation of a leaking water tank. For the first 3 hours it leaks at a constant rate of 12 litres per minute; after 3 hours the rate of leakage slows down (decreases) to 9 litres per minute. The water leaks at a constant rate in both situations and can therefore be represented as a linear graph. However, after 3 hours the slope of the line changes because the rate at which the water is leaking changes.

WORKED EXAMPLE 15

The following two equations represent the distance travelled by a group of students over 5 hours. Equation 1 represents the first section of the hike, when the students are walking at a pace of 4 km/h. Equation 2 represents the second section of the hike, when the students change their walking pace.

Equation 1: $d = 4t,\ 0 \le t \le 2$

Equation 2: $d = 2t + 4,\ 2 \le t \le 5$

The variable d is the distance in km from the campsite, and t is the time in hours.

a. **Determine the time, in hours, for which the group travelled in the first section of the hike.**

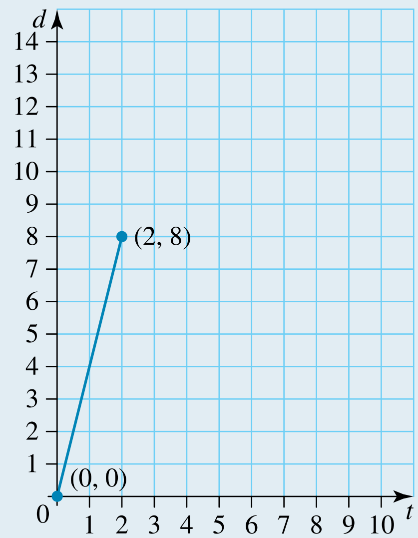
b. i. **What was their walking pace in the second section of their hike?**

ii. **For how long, in hours, did they walk at this pace?**

c. **Sketch a piecewise linear graph to represent the distance travelled by the group of students over the five hour hike.**

THINK		WRITE/DRAW
a.	1. Determine which equation the question applies to.	a. This question applies to Equation 1.
	2. Look at the time interval for this equation.	$0 \le t \le 2$
	3. Interpret the information.	The group travelled for 2 hours.
b. i.	1. Determine which equation the question applies to.	b. i. This question applies to Equation 2.
	2. Interpret the equation. The walking pace is found by the coefficient of t, as this represents the gradient.	$d = 2t + 4,\ 2 \le t \le 5$ The coefficient of t is 2.
	3. Answer the question.	The walking pace is 2 km/h.
ii.	1. Look at the time interval shown.	ii. $2 \le t \le 5$
	2. Interpret the information and answer the question.	They walked at this pace for 3 hours.
c.	1. Find the distance travelled before the change of pace.	c. Change after $t = 2$ hours: $d = 4t$ $d = 4 \times 2$ $d = 8$ km

2. Using a calculator, spreadsheet or otherwise sketch the graph $d = 4t$ between $t = 0$ and $t = 2$.



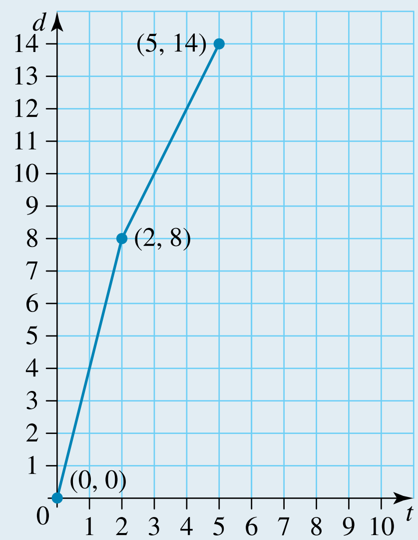
3. Solve the simultaneous equations to find the point of intersection.

$$4t = 2t + 4$$
$$4t - 2t = 2t - 2t + 4$$
$$2t = 4$$

Substitute $t = 2$ into $d = 4t$:

$$d = 4 \times 2$$
$$= 8$$

4. Using CAS, a spreadsheet or otherwise, sketch the graph of $d = 2t + 4$ between $t = 2$ and $t = 5$.



WORKED EXAMPLE 16

The following sign shows the car parking fees in a shopping carpark.

CARPARK FEES

0 – 2 hours $ 1.00

2 – 4 hours $ 2.50

4 – 6 hours $ 5.00

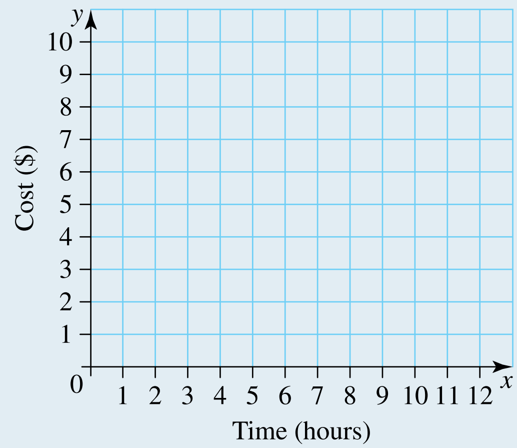
6+ hours $ 6.00

Construct a step graph to represent this information.

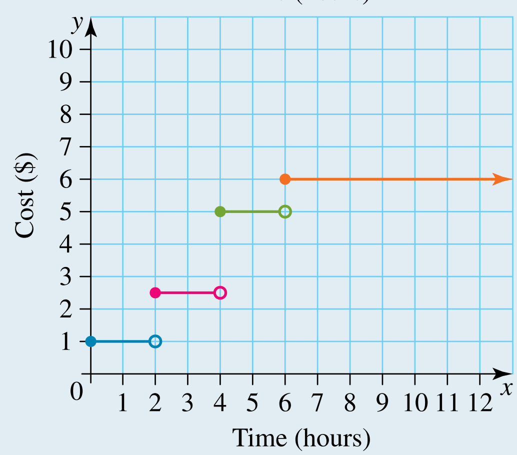
THINK

WRITE/DRAW

1. Draw up a set of axes, labelling the axes in terms of the context of the problem; that is, the time and cost. There is no change in cost during the time intervals, so there is no rate (i.e. the gradient is zero). This means we draw horizontal line segments during the corresponding time intervals.



2. Draw segments to represent the different time intervals.



on Resources

Interactivity: Piecewise linear graphs (int-6486)

Interactivity: Step functions (int-6281)

studyon

Units 1 & 2 > AOS 5 > Topic 1 > Concept 5 > **Step graphs** Concept summary and practice questions

Exercise 10.5 Further linear applications

1. WE13 A piecewise linear graph is constructed from the following linear graphs.

$$y = -3x - 3,\ x \le a$$
$$y = x + 1,\ x \ge a$$

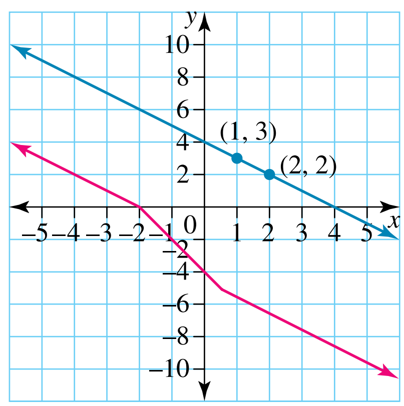
   a. By solving the equations simultaneously, find the point of intersection and hence state the value of a.
   b. Sketch the piecewise linear graph.

2. Consider the following linear graphs that make up a piecewise linear graph.

$$y = 2x - 3,\ x \le a$$
$$y = 3x - 4,\ a \le x \le b$$
$$y = 5x - 12,\ x \ge b$$

   a. Using CAS, a spreadsheet or otherwise, sketch the three linear graphs.
   b. Determine the two point of intersection.
   c. Using the points of intersection, find the values of a and b.
   d. Sketch the piecewise linear graph.

3. MC The diagram shows a piecewise linear graph. Which one of the following options represents the linear graphs that make up the piecewise graph?



A. $y = -2x - 4,\ x \le -2$
$y = -x - 2,\ -2 \le x \le 0.5$
$y = -x - 4.5,\ x \ge 0.5$

B. $y = -x - 2,\ x \le -2$
$y = -2x - 4,\ -2 \le x \le 0.5$
$y = -x - 4.5,\ x \ge 0.5$

C. $y = -2x - 4,\ x \le 0$
$y = -x - 2,\ 0 \le x \le -5$
$y = -x - 4.5,\ x \ge -5$

D. $y = -x - 2,\ x \le 0$
$y = -2x - 4,\ 0 \le x \le -5$
$y = -x - 4.5,\ x \ge -5$

E. $y = -x - 4.5,\ x \le -2$
$y = -x - 2,\ -2 \le x \le 0.5$
$y = -2x - 4,\ x \ge 0.5$

4. The growth of a small tree was recorded over 6 months. It was found that the tree's growth could be represented by three linear equations, where h is the height in centimetres and t is the time in months.

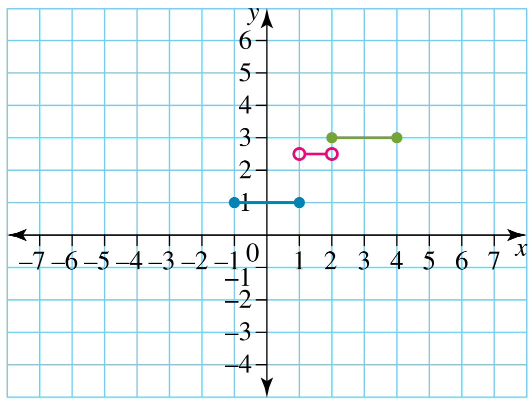
Equation 1: $h = 2t + 20, 0 \le t \le a$
Equation 2: $h = t + 22, a \le t \le b$
Equation 3: $h = 3t + 12, b \le t \le c$

   a. i. By solving equations 1 and 2 simultaneously, determine the value of a.
      ii. By solving equations 2 and 3 simultaneously, determine the value of b.
   b. Explain why $c = 6$.
   c. During which time interval did the tree grow the most?
   d. Sketch the piecewise linear graph that shows the height of the tree over the 6-month period.

5. WE14 Construct a step graph from the following equations, making sure to take note of the relevant end points.

$$y = 3,\ 1 < x \le 4$$
$$y = 1.5,\ 4 < x \le 6$$
$$y = -2,\ 6 < x \le 8$$

6. A step graph is shown below. Write the equations that make up the graph.

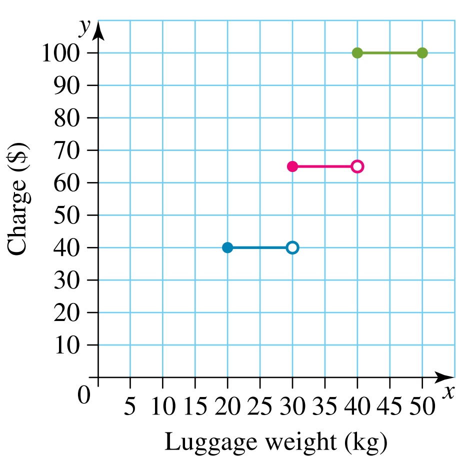


7. The following table shows the costs to hire a plumber.

Time (minutes)	Cost ($)
0–15	45
15–30	60
30–45	80
45–60	110

a. Represent this information on a step graph.
b. Anton hired the plumber for a job that took 23 minutes. How much will Anton be expected to be charged for this job?

8. Airline passengers are charged an excess for any luggage that weighs 20 kg or over. The following graph shows these charges for luggage weighing over 20 kg.



a. How much excess would a passenger be charged for luggage that weighs 31 kg?
b. Nerada checks in her luggage and is charged \$40. What is the maximum excess luggage she could have without having to pay any more?
c. Hilda and Hanz have two pieces of luggage between them. One piece weighs 32 kg and the other piece weighs 25 kg. Explain how they could minimise their excess luggage charges.

9. WE15 The following two equations represent water being added to a water tank over 15 hours, where w is the water in litres and t is the time in hours.

Equation 1: $w = 25t,\ 0 \le t \le 5$
Equation 2: $w = 30t - 25,\ 5 \le t \le 15$

a. Determine how many litres of water are in the tank after 5 hours.
b. i. At what rate is the water being added to the tank after 5 hours?
ii. For how long is the water added to the tank at this rate?
c. Sketch a piecewise graph to represent the water in the tank at any time, t, over the 15-hour period.

10. A car hire company charges a flat rate of \$50 plus 75 cents per kilometre up to and including 150 kilometres. An equation to represent this cost, C, in dollars is given as $C = 50 + ak,\ 0 \le k \le b$, where k is the distance travelled in kilometres.
a. Write the values of a and b.
b. Using CAS, a spreadsheet or otherwise, sketch this equation on a set of axes, using appropriate values.
The cost charged for distances over 150 kilometres is given by the equation $C = 87.50 + 0.5k$.
c. Determine the charge in cents per kilometre for distances over 150 kilometres.
d. By solving the two equations simultaneously, find the point of intersection and hence show that the graph will be continuous.
e. Sketch the equation $C = 87.50 + 0.5k$ for $150 \le k \le 300$ on the same set of axes as part **b**.

11. The temperature of a wood-fired oven, T °C, steadily increases until it reaches a 200 °C. Initially the oven has a temperature of 18 °C and it reaches the temperature of 200 °C in 10 minutes.

a. Construct an equation that finds the temperature of the oven during the first 10 minutes. Include the time interval, t, in your answer.
Once the oven has heated up for 10 minutes, a loaf of bread is placed in the oven to cook for 20 minutes. An equation that represents the temperature of the oven during the cooking of the bread is $T = 200,\ a \le t \le b$.
b. i. Write the values of a and b.
ii. In the context of this problem, what do a and b represent?
After the 20 minutes of cooking, the oven's temperature is lowered. The temperature decreases steadily, and after 30 minutes the oven's temperature reaches 60°C. An equation that determines the temperature of the oven during the last 30 minutes is $T = mt + 340,\ d \le t \le e$.
c. Find the values of m, d and e.
d. What does m represent in this equation?
e. Using your values from the previous parts, sketch the graph that shows the changing temperature of the wood fired oven during the 60-minute interval.

12. **WE16** The costs to hire a paddle boat are listed in the following table. Construct a step graph to represent the cost of hiring a paddle boat for up to 40 minutes.

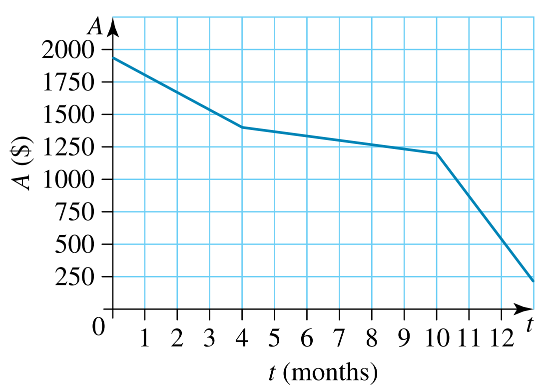
Time (minutes)	Hire cost ($)
0–20	15
20–30	20
30–40	25

13. The postage costs to send parcels from the Northern Territory to Sydney are shown in the following table:

Weight of parcel (kg)	Cost ($)
0–0.5	6.60
0.5–1	16.15
1–2	21.35
2–3	26.55
3–4	31.75
4–5	36.95

a. Represent this information in a step graph.
b. Pammie has two parcels to post to Sydney from the Northern Territory. One parcel weighs 450 g and the other weighs 525 g. Is it cheaper to send the parcels individually or together? Justify your answer using calculations.

14. The amount of money in a savings account over 12 months is shown in the following piecewise graph, where A is the amount of money in dollars and t is the time in months.



One of the linear graphs that make up the piecewise linear graph is $A = 2000 - 150t$, $0 \le t \le a$.

a. Determine the value of a.

b. The equation that intersects with $A = 2000 - 150t$ is given by $A = b - 50t$. If the two equations intersect at the point (4, 1400), show that $b = 1600$.

c. The third equation is given by the rule $A = 4100 - 300t$. By solving a pair of simultaneous equations, find the time interval for this equation.

d. Using an appropriate equation, determine the amount of money in the account at the end of the 12 months.

15. The following linear equations represent the distance sailed by a yacht from the yacht club during a race, where d is the distance in kilometres from the yacht club and t is the time in hours from the start of the race.

Equation 1: $d = 20t$, $0 \le t \le 0.75$

Equation 2: $d = 15t + 3.75$, $0.75 \le t \le 1.25$

Equation 3: $d = -12t + 37.5$, $1.25 \le t \le b$

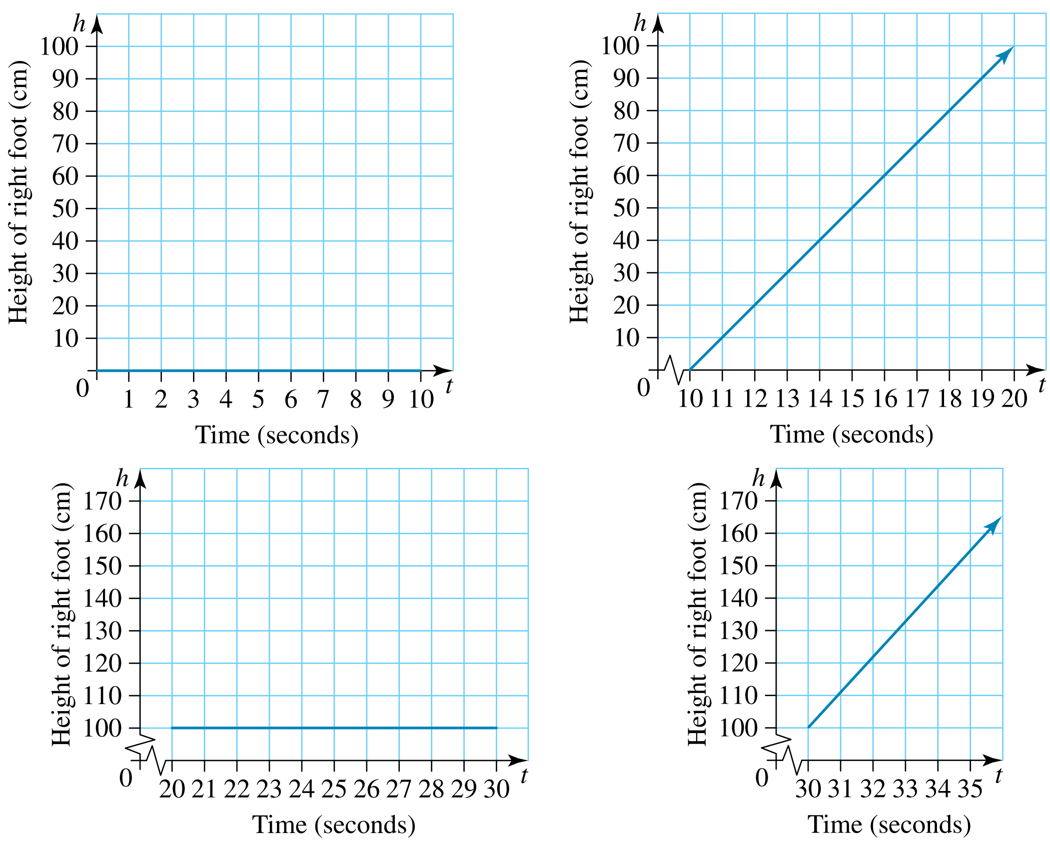
a. Using CAS, a spreadsheet or otherwise, find the points of intersection.

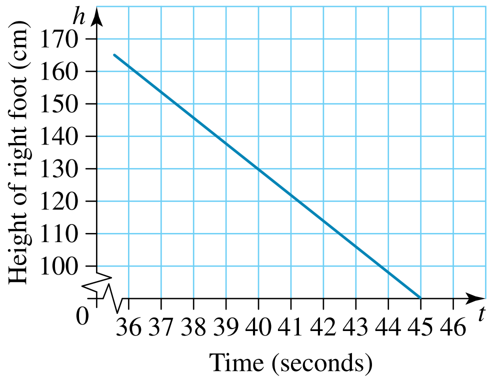
b. In the context of this problem, explain why equation 3 has a negative gradient.

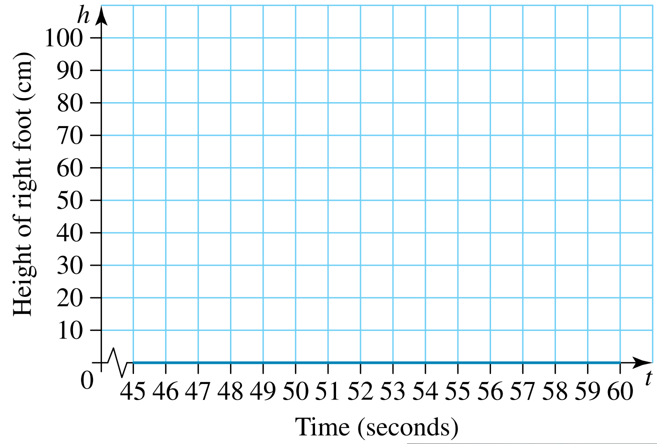
c. How far is the yacht from the starting point before it turns and heads back to the yacht club?

d. Determine the duration, to the nearest minute, of the yacht's sailing time for this race. Hence, find the value for b. Write your answer correct to 2 decimal places.

16. The distance of a dancer's right foot from the floor during a dance recital can be found using the following linear graphs, where h is the height in centimetres from the floor and t is the duration of the recital in seconds.







a. During which time interval(s) was the dancer's right foot on the floor? Explain your answer.

b. What was the maximum height the dancer's right foot was from the floor?

c. How long, in seconds, was the recital?

d. Sketch the graph that shows the distance of the dancer's right foot from the floor at any time during the recital. Clearly label all key features.

17. Stamp duty is a government charge on the purchase of items such as cars and houses. The table shows the range of stamp duty charges for purchasing a car in South Australia.

Car price ($\$P$)	Stamp duty (S)
0–1000	1%
1000–2000	$10 + 2% ($P$–1000)
2000–3000	$30 + 3% ($P$–2000)
3000+	$60 + 4% ($P$–3000)

a. Explain why the stamp duty costs for cars can be modelled by a piecewise linear graph.

The stamp duty charge for a car purchased for $1000 or less can be expressed by the equation $S = 0.01P$, where S is the stamp duty charge and P is the purchase price of the car for $0 \leq P \leq 1000$. Similar equations can be used to express the charges for cars with higher prices.

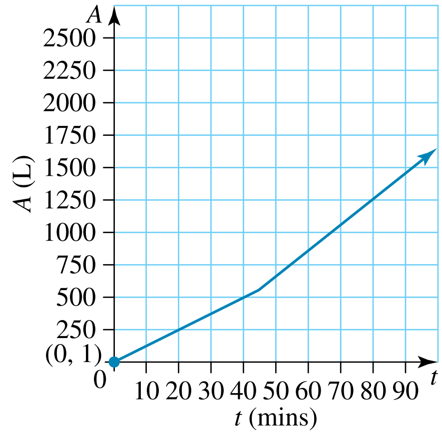
Equation 1: $S = 0.01P,\ 0 \leq P \leq 1000$
Equation 2: $S = 0.02P - 10,\ a < P \leq b$
Equation 3: $S = 0.03P - c,\ 2000 < P \leq d$
Equation 4: $S = fP - e,\ P > 3000$

b. For equations 2, 3 and 4, determine the values of a, b, c, d, e and f.

c. Using CAS, a spreadsheet or otherwise, find the points of intersections for the equations in part **b**.

d. Suki and Boris purchase a car and pay $45 in stamp duty. What price did they pay for their car?

18. A small inflatable swimming pool that holds 1500 litres of water is being filled using a hose. The amount of water, A, in litres in the pool after t minutes is shown in the following graph.



a. Estimate the amount of water, in litres, in the pool after 45 minutes.
b. Determine the amount of water being added to the pool each minute during the first 45 minutes.
   After 45 minutes the children become impatient and turn the hose up. The equation $A = 20t - 359$ determines the amount of water, A, in the pool t minutes after 45 minutes.
c. Using this equation, determine the time taken, in minutes, to fill the pool. Give your answer to the nearest whole minute.

19. a. Using CAS, a spreadsheet or otherwise, find the points of intersection for the following four linear graphs.
   i. $y = x + 4,\ x \le a$
   ii. $y = 2x + 3,\ a \le x \le b$
   iii. $y = x + 6,\ b \le x \le c$
   iv. $y = 3x + 1,\ x \ge c$
b. Using your values from part **a**, complete the x-intervals for the linear graphs by finding the values of a, b and c.
c. What problem do you encounter when trying to sketch a piecewise linear graph formed by these four linear graphs?

20. The Slippery Slide ride is a new addition to a famous theme park. The slide has a horizontal distance of 20 metres and is comprised of four sections. The first section is described by the equation $h = -3x + 12,\ 0 \le x \le a$, where h is the height in metres from the ground and x is the horizontal distance in metres from the start. In the first section, the slide drops 3 metres over a horizontal distance of 1 metre before meeting the second section.

a. What is the maximum height of the slide above ground?
b. State the value of a.
   The remaining sections of the slides are modelled by the following equations.

$$\text{Section 2}: h = -\frac{2x}{3} + \frac{29}{3},\ a \le x \le b$$

$$\text{Section 3}: h = -2x + 13,\ b \le x \le c$$

$$\text{Section 4}: h = -\frac{5x}{16} + \frac{25}{4},\ c \le x \le d$$

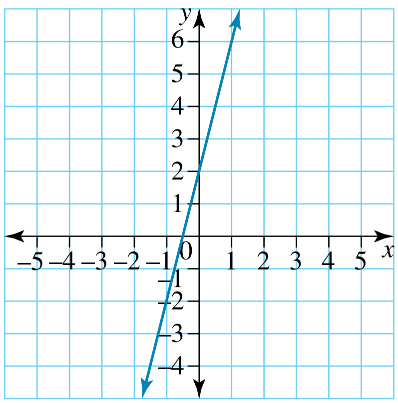
c. Using CAS, a spreadsheet or otherwise, find the points of intersection between each section of the slide and hence find the values of b and c.
d. Explain why $d = 20$.
e. Sketch the graph that shows the height at any horizontal distance from the start of the slide.

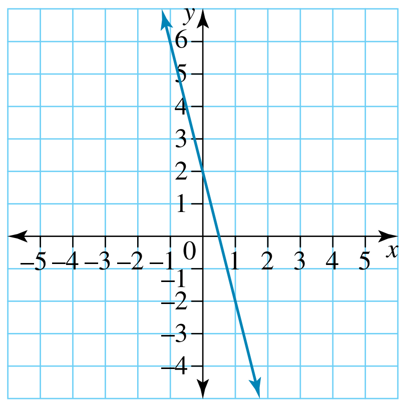
10.6 Review: exam practice

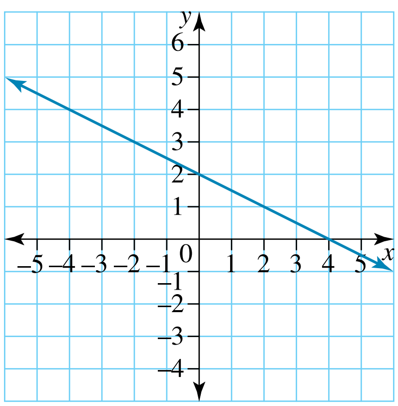
A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

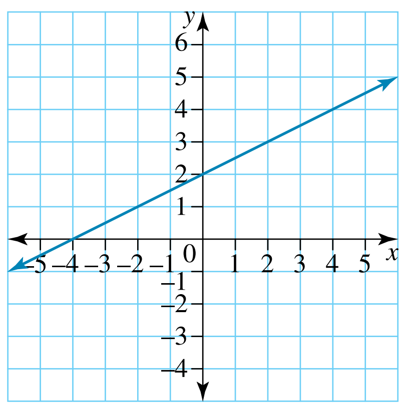
Multiple choice

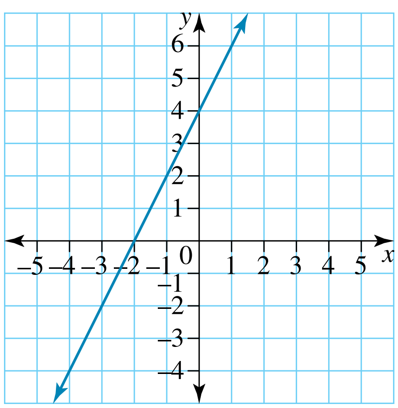
1. **MC** The gradient of the line passing through the points (4, 6) and (−2, −6) is:
   A. −2 **B.** −0.5 **C.** 0 **D.** 0.5 **E.** 2
2. **MC** The x- and y-intercepts of the linear graph with equation $3x - y = 6$ are:
   A. (2, 6) **B.** (0, 2) and (−6, 0)
   C. (0, 2) and (6, 0) **D.** (2, 0) and (0, −6)
   E. (2, 0) and (0, 6)
3. **MC** Which of the following is a sketch of the graph with equation $y = 4x + 2$?

A.


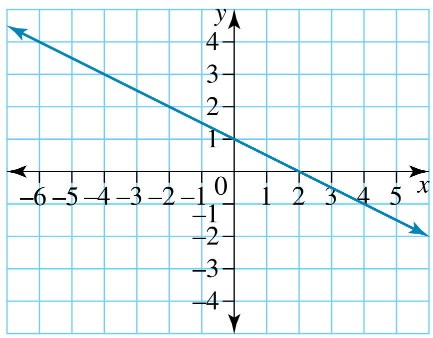
B.


C.


D.


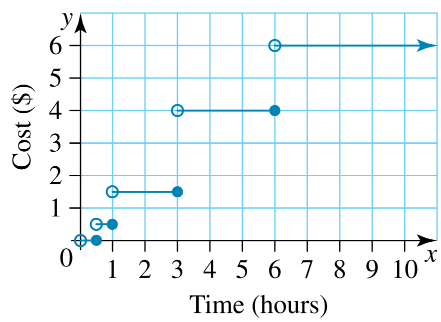
E.


4. MC The gradient of the graph shown in the following diagram is:



A. -2 **B.** -1 **C.** $-\frac{1}{2}$ **D.** 1 **E.** $\frac{1}{2}$

5. MC This step graph shows the parking fees for a multilevel carpark in a major city.



Roberta parked in the car park and was charged $4.00. If she arrived in the carpark at 9.30 am, the time that she most likely drove out of the carpark was:

A. 10.30 am **B.** 11.00 pm **C.** 12.30 pm **D.** 1.30 pm **E.** 3.00 pm

6. MC A piecewise linear graph is constructed from the following linear equations:

$$y = 3x - 5, \quad x \leq a$$
$$y = 4x - 4, \quad a \leq x \geq b$$
$$y = 6x - 5, \quad x \geq b$$

The values of a and b are respectively:

A. -1 and $\frac{1}{2}$ **B.** $\frac{1}{2}$ and -1 **C.** $-\frac{1}{2}$ and 1 **D.** 0 and 1 **E.** -4 and -5

The following information relates to Questions 7 and 8.

An inflated party balloon has a small hole and is slowly deflating. The initial volume of the balloon is 1000 cm^3 and the balloon loses 5 cm^3 of air every minute.

7. MC If V represents the volume of the balloon in cm^3 and t represents the time in minutes, an equation to represent the volume of the balloon after t minutes is:

A. $V = \frac{1000}{5t}$ **B.** $V = \frac{1000 - 5t}{60}$ **C.** $V = 1000 + 5t$ **D.** $V = 1000 - 5t$ **E.** $V = 200 - t$

8. MC The time taken for the balloon to have lost 650 cm^3 of air is:

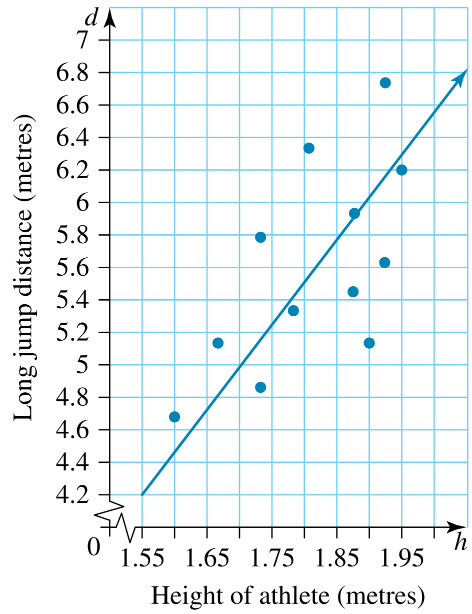
A. 70 minutes **B.** 130 minutes **C.** 270 minutes **D.** 330 minutes **E.** 930 minutes

9. MC The equation of a straight line passing through the points $(-2, 3)$ and $(5, 1)$ is:

A. $y = -\frac{2}{7}x + 2\frac{3}{7}$ **B.** $y = 2\frac{3}{7}x - \frac{2}{7}$ **C.** $y = \frac{2}{7}x + 2\frac{3}{7}$

D. $y = \frac{2}{7}x - 2\frac{3}{7}$ **E.** $y = 2\frac{3}{7} + \frac{2}{7}x$

10. MC The following scatterplot represents the relationship between the height of an athlete and the distance that they can long jump in metres.



Using the line of best fit, the estimated distance that a 1.8-metre-tall athlete could long jump would be:

A. 5 m **B.** 6.02 m **C.** 5.46 m **D.** 5.26 **E.** 5.52 m

Short answer

1. Sketch the following graphs by finding the x- and y-intercepts. Hence, state the gradient of each graph.
   a. $2x + y = 5$
   b. $y - 4x = 8$
   c. $4(x + 3y) = 16$
   d. $3x + 4y - 10 = 0$
2. Find the gradients of the lines passing through the following pairs of points.
   a. $(3, -2)$ and $(0, 4)$
   b. $(5, 11)$ and $(-2, 18)$
   c. $(0.3, 4.1)$ and $(1.2, 5.3)$
   d. $\left(\frac{2}{5}, \frac{1}{4}\right)$ and $\left(-\frac{1}{4}, \frac{3}{5}\right)$
3. A line has a gradient of $-\frac{3}{4}$. If the line passes through the points $(-a, 3)$ and $(-2, 6)$, find the value of a.
4. Complete the following table.

	Equation	Gradient	y-intercept	x-intercept
a	$y = 5x - 3$	5		
b	$y = 3x + 1$			
c	$3y = 6x - 9$			
d	$2y + 4x = 8$			
e			5	−5
f		2		2

5. A step graph is formed by the following equations:

$$y = -3, 0 < x \le 3$$
$$y = 2, 3 < x < 5$$
$$y = 5, x \ge 5$$

Construct a step graph to represent this information.

6. The following two linear equations make a piecewise linear graph.

$$y = -2x + 1, x \le a$$
$$y = -3x + 2, x \ge a$$

a. Solve the equations simultaneously, and hence find the value of a.
b. Sketch the piecewise linear graph.

Extended response

1. The recommended maximum heart rate during exercise is given by the equation

$$H = 0.85\,(220 - A)$$

where H is the person's heart rate in beats per minute and A is their age in years.
a. Explain why the maximum heart rate is given by a linear equation.
b. Determine the recommended maximum heart rate for a 25-year-old person. Write your answer correct to the nearest whole number.
c. Determine the gradient and y-intercept of the linear equation.
d. Using your answers from part **c**, sketch the graph that shows the recommended maximum heart rate for persons aged 20 to 70 years.
e. Charlie is working out at the recommended maximum heart rate. His measured heart rate is 162 beats per minute. By solving a linear equation, find Charlie's age.
f. In the context of this problem, explain why finding the x-intercept would be meaningless.

2. Jerri and Samantha have both entered a 10-km fun run for charity. The distance travelled by Jerri can be modelled by the linear equation

$$d = 6t - 0.1$$

where d is the distance in km from the starting point and t is time in hours.
a. Determine the time taken for Jerri to run the 10 kilometres. Give your answer correct to the nearest minute.
b. In the context of this problem, explain the meaning of the d-intercept (y-intercept).
The distance Samantha is from the starting point at any time, t hours, can be modelled by the piecewise linear graph

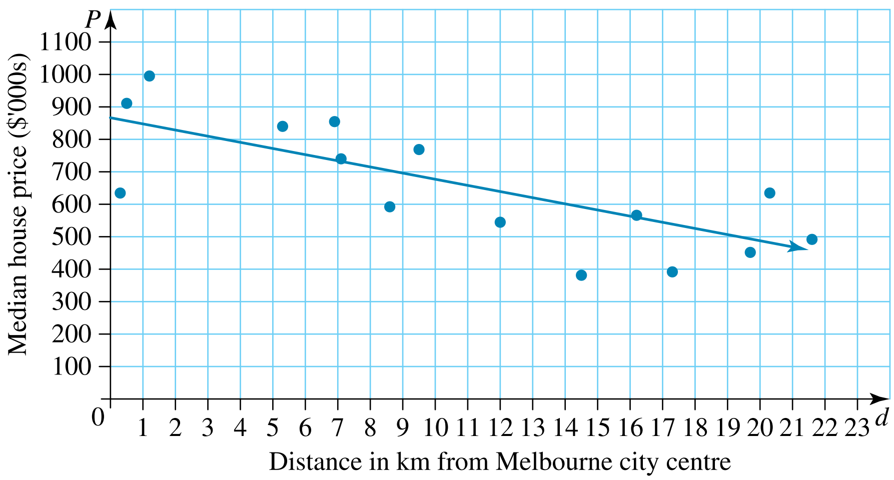
$$d = 4t,\ 0 \le t \le \frac{1}{2}$$
$$d = 8t - 2,\ \frac{1}{2} \le t \le b$$

c. How far, in kilometres, did Samantha travel in the first 30 minutes?
d. Determine the speed at which Samantha was travelling in the first 30 minutes.
e. Explain how Samantha's run changed after 30 minutes.
f. i. Determine the value of b.
ii. Hence, show that Samantha crossed the finishing line ahead of Jerri by 11 minutes.
g. By solving a pair of simultaneous equations, determine:
i. the time when Samantha passed Jerri on the run
ii. the distance from the starting point at which Samantha passed Jerri.
h. Construct two graphs on the same set of axes to represent the distances travelled by Jerri and Samantha for the 10-km race.

3. Trudy is unaware that there is a small hole in the petrol tank of her car. Petrol is leaking out of the tank at a constant rate of 5 mL/min. Trudy has parked her car in a long-term carpark at the airport and gone on a holiday. Initially there is 45 litres of petrol in the tank.

a. In terms of linear graphs, what does the leaking rate and the initial amount of petrol determine?
b. How many litres of petrol leak out of the tank each hour? What is the assumption that is being made about the rate of petrol leaking each hour?
c. How many litres of petrol are lost after four hours?
d. After how many hours will there be 39.75 litres of petrol in the tank?
e. An equation is used to represent the amount of petrol left in the tank, l, after t hours.
  i. Explain why the amount of petrol in the tank would be best modelled by a linear equation.
  ii. Explain why the linear equation will have a negative gradient
  iii. Write an equation to determine the amount of petrol left in the tank, l, after t hours.
f. Using CAS, a spreadsheet or otherwise, sketch the graph for the equation found in part **eiii**. Clearly label the x- and y-intercepts.
g. Determine how many hours it will take for the petrol tank to become empty.

4. The median house prices ($′000) from fifteen western suburbs and the distance in kilometres from the centre of Melbourne were collected. The graph shows these results.



To determine an equation that could be used to determine the median house price, p, at a distance, d km west from the city centre, the following points were used: (3, 808) and (20, 468).

a. Using these two points, determine the equation of the line of best fit relating the median house price, p, to the distance, d, from the city centre. Write your answers correct to the nearest whole number.
b. In the context of this problem, explain the meaning of the gradient and y-intercept.
c. Using your equation from part **a**, determine the median house price in dollars for the suburbs at the following distances from the city centre.
  i. 15.5 km **ii.** 4.8 km **iii.** 18.7 km
d. Xena and Hertz purchase a house for $650 800 in the western suburbs. Using your equation from part **a**, determine how far in kilometres their house is likely to be located from the city centre. Write your answer correct to 1 decimal place.
e. Determine the value of the x-intercept and explain the meaning of this in the context of the problem.
f. Freda and Benny are planning on purchasing a holiday house in the Grampians located 229 km west of the city centre. Explain why the equation would not be reliable in determining their expected house price.

studyon

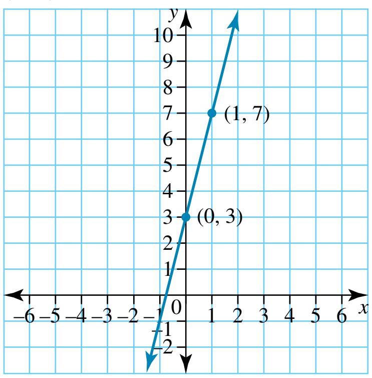
Units 1 & 2 Sit topic test

Answers

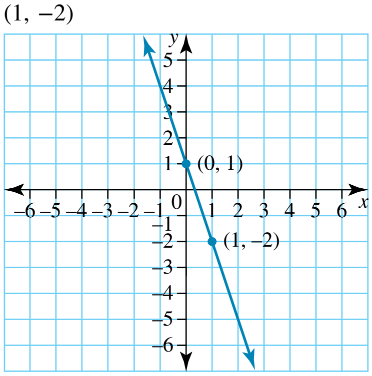
Topic 10 Linear graphs and models

Exercise 10.2 Linear functions and graphs

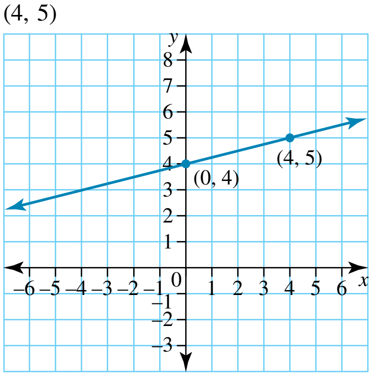
1. a. Gradient = 2, y-intercept = 1
   b. Gradient = −1, y-intercept = 3
   c. Gradient = $\frac{1}{2}$, y-intercept = 4
   d. Gradient = 1, y-intercept = $\frac{1}{4}$
   e. Gradient = $-\frac{3}{2}$, y-intercept = 3
2. a. Gradient = $\frac{3}{5}$, y-intercept = $-\frac{1}{5}$
   b. Gradient = 10, y-intercept = −5
   c. Gradient = $-\frac{1}{2}$, y-intercept = $\frac{3}{2}$
3. a. 1 b. −2
4. D
5. a. (1, 7)



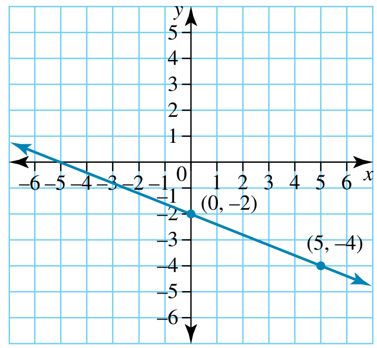
   b. (1, −2)



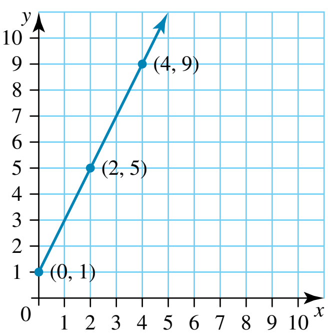
   c. (4, 5)



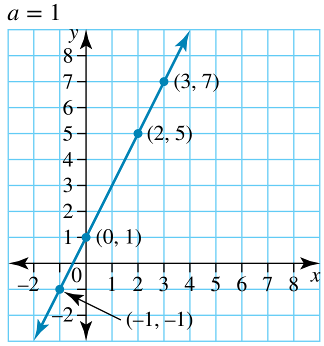
   d. (5, −4)



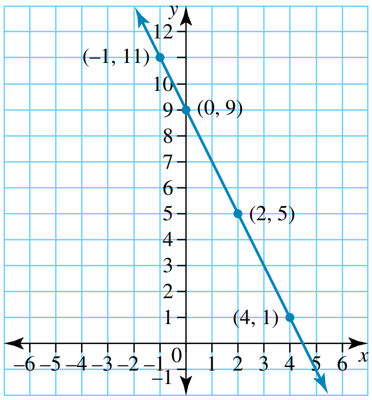
6. a. Gradient = 3, y-intercept = −6
   b. Gradient = $-\frac{3}{5}$, y-intercept = $-\frac{6}{5}$
7. a. 3 b. $\frac{4}{3}$ c. $\frac{2}{7}$
8. −1
9. a. −3 b. $\frac{3}{5}$ c. $-\frac{12}{11}$
   d. $\frac{17}{18}$ e. $\frac{8}{3}$ f. −11
10. a, b



11. $a = 1$



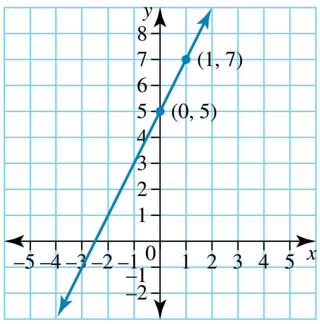
12.



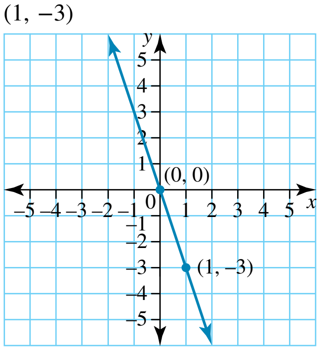
$a = 1$

13. $b = 2$

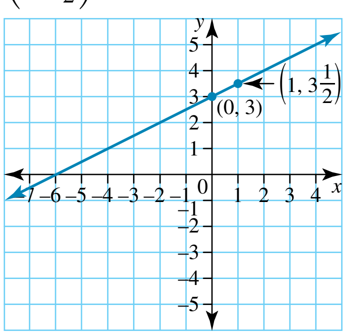
14. **a.** (1, 7)



b. (1, −3)



c. $\left(1, 3\frac{1}{2}\right)$



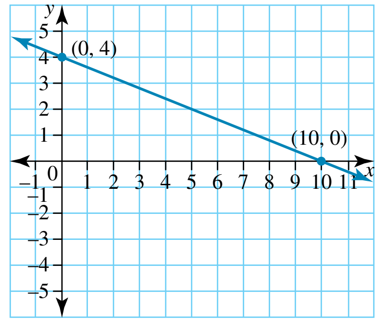
15. **a.** 3

b. −4

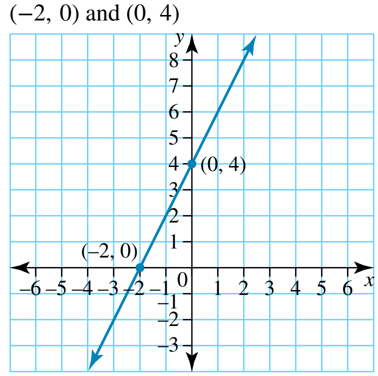
c. −2

d. −1

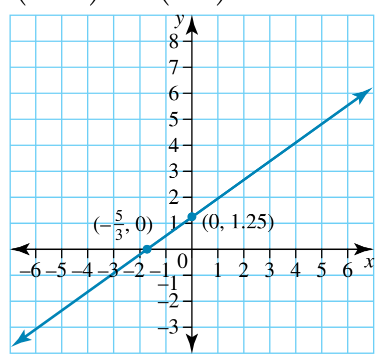
16. **a.** (10, 0) and (0, 4)



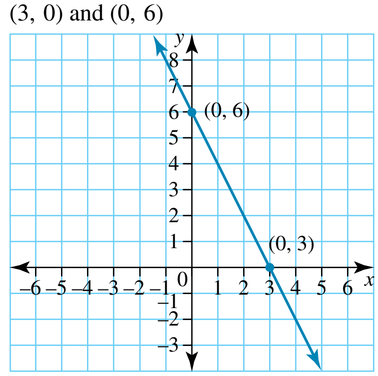
b. (−2, 0) and (0, 4)



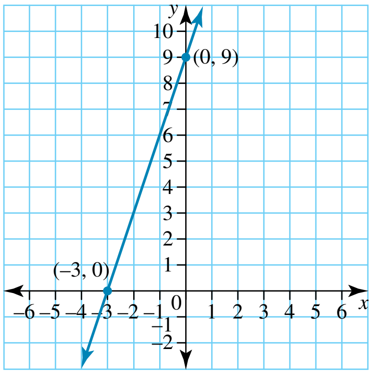
c. $\left(-\frac{5}{3}, 0\right)$ and $\left(0, \frac{5}{4}\right)$



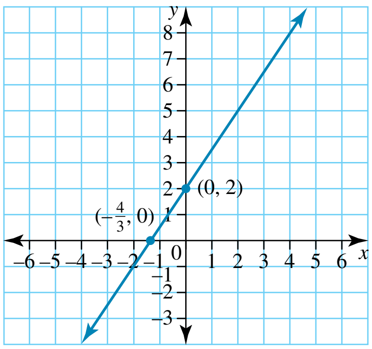
17. **a.** (3, 0) and (0, 6)



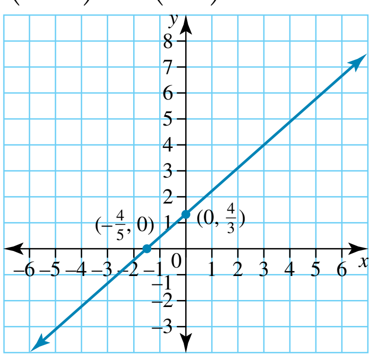
b. (−3, 0) and (0, 9)



c. $\left(-\frac{4}{3}, 0\right)$ and (0, 2)



d. $\left(-\frac{4}{5}, 0\right)$ and $\left(0, \frac{4}{3}\right)$



18. a. i. x-intercept = 4, y-intercept = 3; correct

ii. x-intercept = 3, y-intercept = −6; incorrect

iii. x-intercept = 4, y-intercept = −5; incorrect

iv. x-intercept = 15, y-intercept = 3.75; correct

b. Any equation that has a positive y-coefficient instead of a negative y-coefficient, for example $3x + 7y = 21$

c. Don't ignore positive or negative signs when calculating the intercepts.

19. a. Otis swapped the x- and y-values, calculating $\frac{y_2 - y_1}{x_1 - x_2}$.

The correct gradient is −3.

b. Label each x and y pair before substituting them into the formula.

20. a. For all horizontal lines, the y-values of any two points will be the same. Therefore, when calculating the gradient, $y_2 - y_1$ will be 0, and the gradient will be 0.

b. For all vertical lines, the x-values of any two points will be the same. Therefore, when calculating the gradient, $x_2 - x_1$ will be 0. Dividing any number by 0 gives an undefined result, so the gradient is also undefined.

21. a. The points (1, 3) and (2, 0) tell us that the graph has a negative gradient, so the y-intercept must have a greater value than 3.

b. (2, 0) and (1, 3); gradient = −3

c. $a = 6$

22. a. The y-intercept is the number separate from the x (the constant), and the x-intercept is equal to $\frac{-y\text{-intercept}}{\text{gradient}}$.

This method only works when the equation is in the form $y = mx + c$.

b. y-intercept = c, x-intercept = $\frac{-c}{m}$

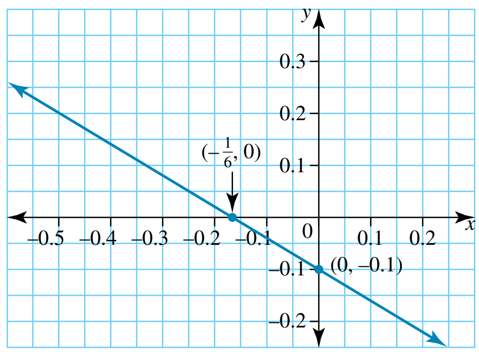
c. $y = 5x + 4$

23. a. x-intercept = 5, y-intercept = $-\frac{10}{3}$

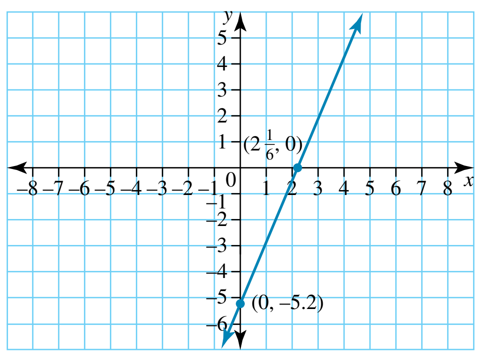
b. x-intercept = −20, y-intercept = $\frac{10}{7}$

c. x-intercept = $-\frac{10}{21}$, y-intercept = $\frac{6}{35}$

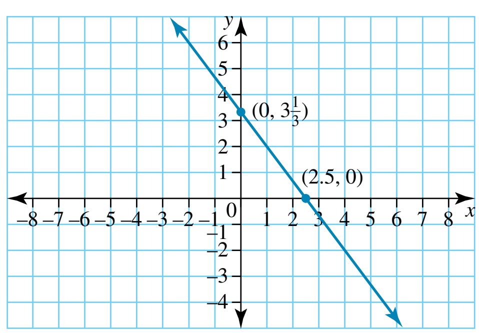
24. a.



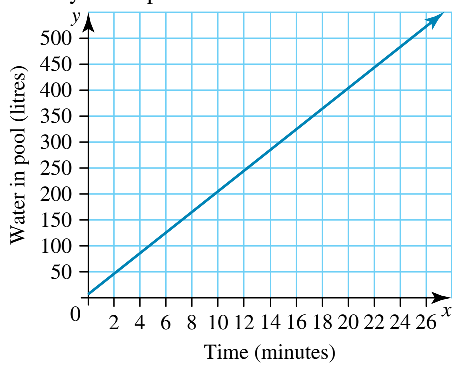
b.

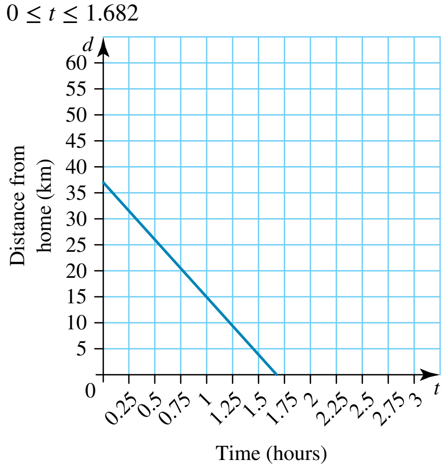


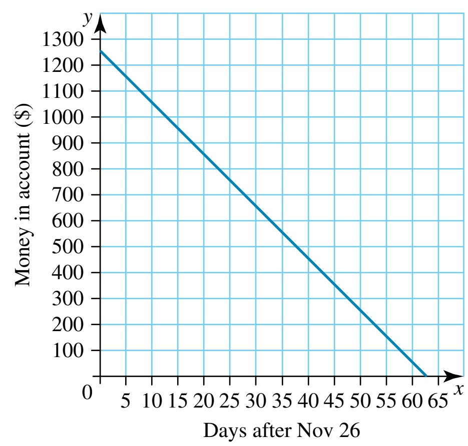
c.



Exercise 10.3 Linear modelling

1. $C = 65t + 90$
2. $a = -250t + 125\,000$
3. a. Both variables in the equation have a power of 1.
   b. y-intercept = 5. This represents the amount of water initially in the pool.
   c. 

   d. 25 minutes
4. a. $A = 40t + 100$
   b. How much air was initially in the ball
   c. $4900\,\text{cm}^3$
   d. 41 minutes 38 seconds
5. a. 12
   b. y-intercept = -0.5. This means that Kirsten starts 0.5 km before the starting point of the race.
   c. 5.5 km
   d. 1 hour, 48 minutes
6. a. $P = 15t$
   b. The additional amount of petrol in the tank each minute
   c. 5 minutes
   d. $P = 15t + 15$
   e. $0 \le t \le 5$
7. a. 37 km
   b. The distance to Gert's home is reducing as time passes.
   c. 1 hour, 41 minutes
   d. $0 \le t \le 1.682$
   e. 

8. a. $\frac{4}{3}$
   b. The increase in the height of the water each minute.
   c. 0.67 or $\frac{2}{3}$
   d. No, the y-intercept calculated in part **c** is not 0, so there was water in the tank to start with.
9. a. $a = 40,\ b = 120$
   b. The amount of money in Fred's account at the start of the year
   c. 72 weeks
10. a. $a = 0.015,\ b = 800$
    b. No, there is no limit to how much Michaela can earn in a month.
    c. \$7580
    d. \$652 140
11. a. $C = 60h + 175$
    b. 3.5 hours
    c. \$460
    d. $C = 110 + 100h,\ 2 \le h \le 4$
12. a. $P = 19.2t,\ 10 \le t \le 20$
    b. $R = 100 - 4e,\ 0 \le e \le 15$
13. a. $C = 3.5t,\ 100 \le t \le 1000$
    b. The domain represents the number of T-shirts Monique can buy.
    c. There is an upper limit as the deal is valid only up to 1000 T-shirts.
14. a. 80
    b. y-intercept = 0. This means that they start from home.
    c. $d = 80t$
    d. 2 hours, 11 minutes
    e. 200 km
15. a. Kim withdraws the same amount each 5 days, so there is a constant decrease.
    b. 

    c. Gradient = -20. This means that Kim withdraws an average of \$20 each day.
    d. $M = 1250 - 20t$
    e. The x-intercept represents when there will be no money left in Kim's account.
    f. There will be a limit to the domain, possibly $0 \le t \le 62.5$ days, but we do not know this limit as it depends on how much Kim's account can be overdrawn.
    g. After 25 days (on December 21) Kim will have \$750 in her account, so she will not have enough for her holiday.

16. a. $7.50/cm^2

b. $C = 7.5A + 25$

c. $81.25

d. $C = 0.58l + 55$

e. Package A = $126.25; Package B = $113.00. Package B is the better option.

17. a. $C = 0.13t + 15$

b. The gradient represents the call cost per minute and the y-intercept represent the flat fee.

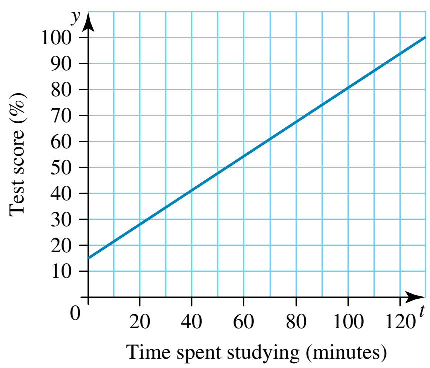
c.

Time	Cost ($)
5	15.65
10	16.30
15	16.95
20	17.60
25	18.25
30	18.90
35	19.55
40	20.20
45	20.85
50	21.50
55	22.15
60	22.80

d. 396 minutes

18. a. $a = 51$

b.



c. If Carly doesn't study, she will score 15%.

d. $y = 0.65t + 15,\ 0 \le t \le 130.77$

e. 76 minutes, 55 seconds

Exercise 10.4 Linear equations and predictions

1. a. $y = 5x + 5$

b. $y = 0.5x + 5.5$

c. $y = 7$

2. C

3. a. The increase in price for every additional person the venue holds

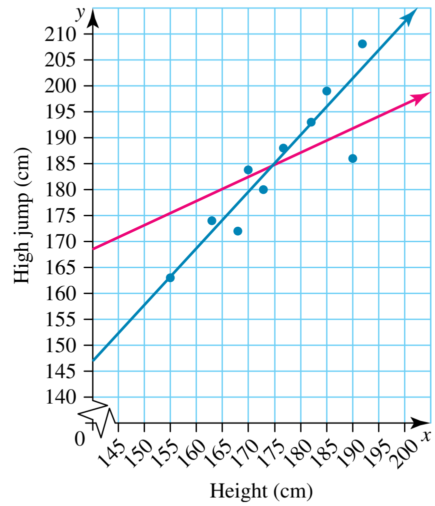
b. The price of a ticket if a venue has no capacity

c. No, as the smallest venues would still have some capacity

4. $y = 1.25x - 33.75$

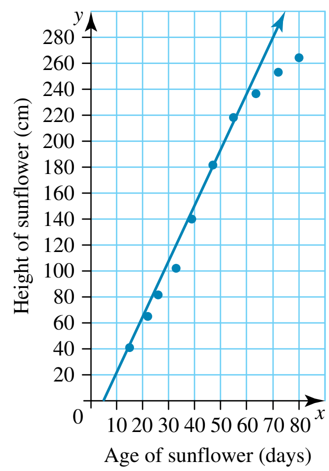
5. a. $y = 0.57x + 86.86$

b.



c. Nidya's line of best fit is not a good representation of the data. In this instance having only two points on data to create the line of best fit was not sufficient.

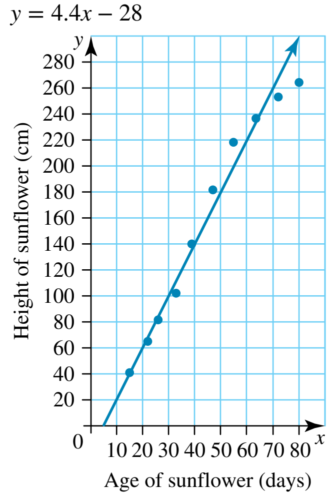
6. a.



Xavier's line is closer to the values above the line than those below it, and there are more values below the line than above it, so this is not a great line of best fit.

b. $y = 4.4x - 28$

c.



Patricia's line is more appropriate as the data points lie on either side of the line and the total distance of the points from the line appears to be minimal.

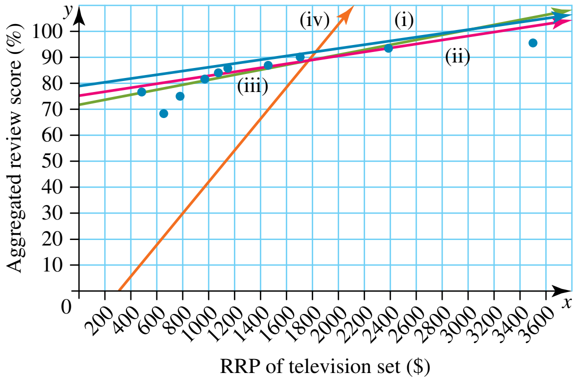
d. $y = 4x - 22$

e. The line of best fit does not approximate the height for values that appear outside the parameters of the data set, and the y-intercept lies well outside these parameters.

7. a. $y = 2.5x + 1.5$
   b. i. \$64.00   ii. \$25.40   iii. \$29.00   iv. \$10.60
8. a. Lines of best fit will vary but should split the data points on either side of the line and minimise the total distance from the points to the line.
   b. Answers will vary.
   c. The amount of crime in a suburb with 0 people
   d. No; if there are 0 people in a suburb there should be no crime.
9. a. Kari did not assign the x- and y-values for each point before calculating the gradient, and she mixed up the values.
   b. $y = -\frac{4}{5}x + \frac{17}{5}$
10. a. $y = 1.32x - 5$
   b. The number of surviving turtles from each nest
   c. The y-intercept represents the number of surviving turtles from 0 nests. This value is not realistic as you cannot have a negative amount of turtles.
   d. i. 173   ii. 13
   e. The answer to **di** was made using extrapolation, so it is not as reliable as the answer to part **dii**, which was made using interpolation. However, due to the nature of the data in question, we would expect this relationship to continue and for both answers to be quite reliable.
11. a. The gradient is undefined.
   b. $x = -2$
12. a. 174 ice-creams
   b. 60 ice-creams
   c. The estimate in part **a** is reliable as it was made using interpolation, it is located within the parameters of the original data set, and it appears consistent with the given data.
   The estimate in part **b** is unreliable as it was made using extrapolation and is located well outside the parameters of the original data set.
13. a. \$112
   b. \$305
   c. All estimates outside the parameters of Georgio's original data set (400 km to 2000 km) will be unreliable, with estimates further away from the data set being more unreliable than those closer to the data set.
   Other factors that might affect the cost of flights include air taxes, fluctuating exchange rates and the choice of airlines for various flight paths.
14. a. $y = -17.31x + 116.37$
   b. For each increase of 1 L of lung capacity, the swimmer will take less time to swim 25 metres.
   c. i. 61.0 seconds
   ii. 40.2 seconds
   iii. 24.6 seconds
   d. As Mariana has only two data points and we have no idea of how typical these are of the data set, the equation for the line of best fit and the estimates established from it are all very unreliable.
15. a. $y = 0.61x + 38.67$
   b.

Kicking efficiency (%)	75.3	65.6	83.1	73.9	79.0	84.7	64.4	72.4	68.7	80.2
Predicted handball efficiency (%)	84.6	78.7	89.4	83.7	86.9	90.3	78.0	82.8	80.6	87.6

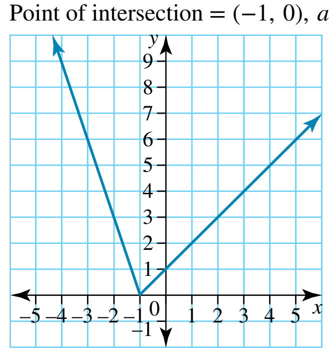
   c. The predicted and actual kicking efficiencies are very similar in values. A couple of the results are identical, and only a couple of the results are significantly different.
16. a. i. $y = 0.0057x + 77.7333$
   ii. $y = 0.0059x + 76.242$
   iii. $y = 0.01x + 71$
   iv. $y = 0.0197x + 58.5211$
   b.



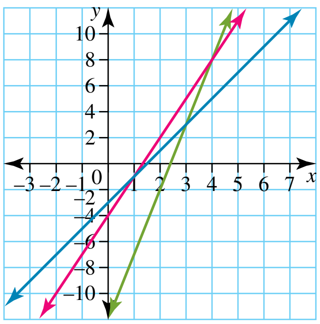
   c. Line **iii** is the most appropriate line of best fit for this data
17. a. This point of data is clearly an outlier in terms of the data set
   b. Lines of best fit will vary but should split the data points on either side of the line and minimise the total distance from the points to the line.
   c. Answers will vary.
   d. The increase in box office taking per \$1 m increase in the leading actor/actress salary
   e. Answers will vary.
   f. The answers to parts **iii** and **iv** are considerably less reliable than the answers to parts **i** and **ii**, as they are created using extrapolation instead of interpolation.
18. a. $y = 0.18x + 267$
   b. 971 cm
   c. 433.5years
   d. 0.446
   e. The answers are very similar

Exercise 10.5 Further linear applications

1. a. Point of intersection = $(-1, 0)$, $a = -1$
   b.



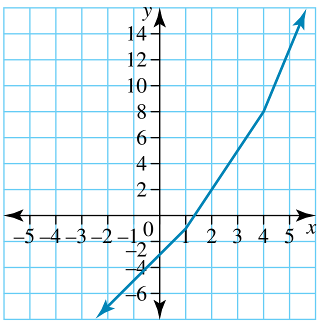
2. a.



b. (1, −1) and (4, 8)

c. $a = 1$ and $b = 4$

d.



3. B

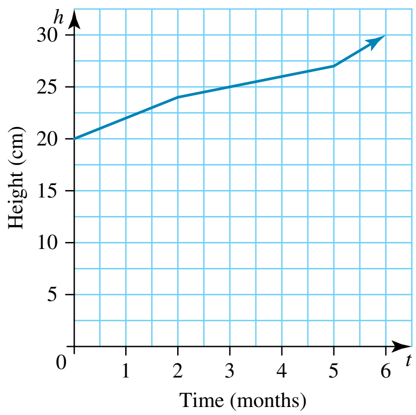
4. a. i. $a = 2$

ii. $b = 5$

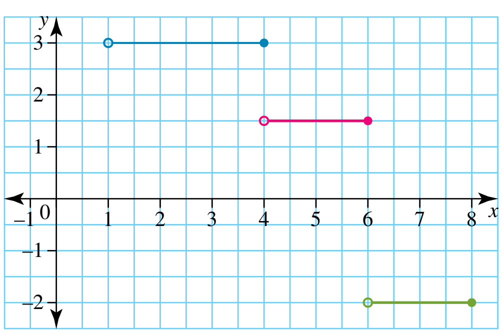
b. The data is only recorded over 6 months.

c. $5 \leq t \leq 6$ (between 5 and 6 months)

d.

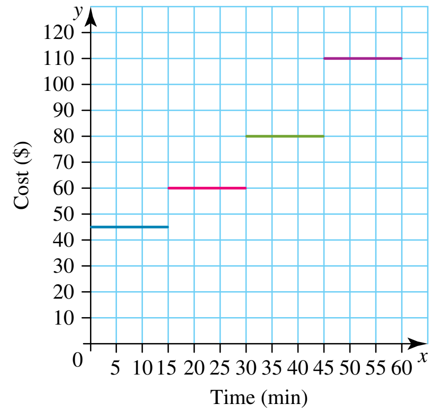


5.



6. $y = 1,\ 1 \leq x \leq 1$; $y = 2.5,\ 1 < x < 2$; $y = 3,\ 2 \leq x \leq 4$

7. a.



b. $60

8. a. $65

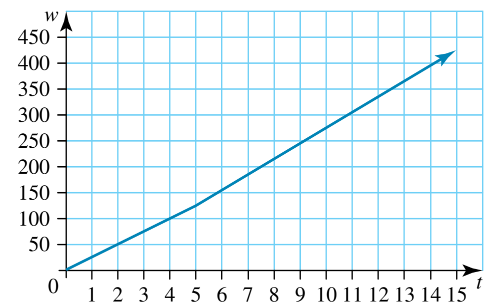
b. 10 kg

c. Place 2–3-kg from the 32-kg bag into the 25-kg bag and pay $80 rather than $105.

9. a. 125 L

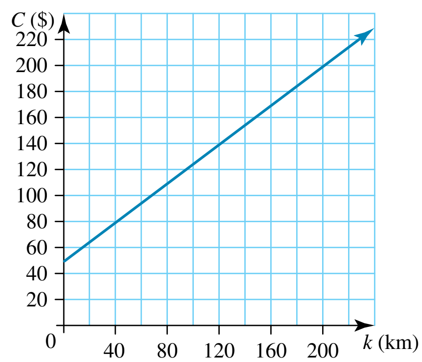
b. i. 30 L/h **ii.** 10 h

c.



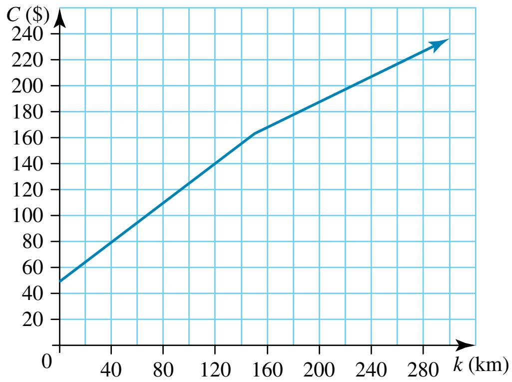
10. a. $a = 0.75,\ b = 150$

b.



c. 50 cents/km

d. $k = 150,\ c = 162.50$. This means that the point of intersection (150, 162.5) is the point where the charges change. At this point both equations will have the same value, so the graph will be continuous.

e. 

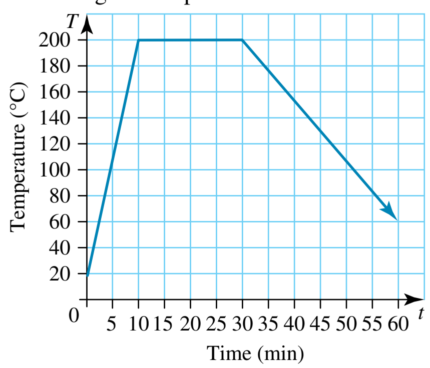
11. a. $T = 18 + 18.2t, 0 \leq t \leq 10$

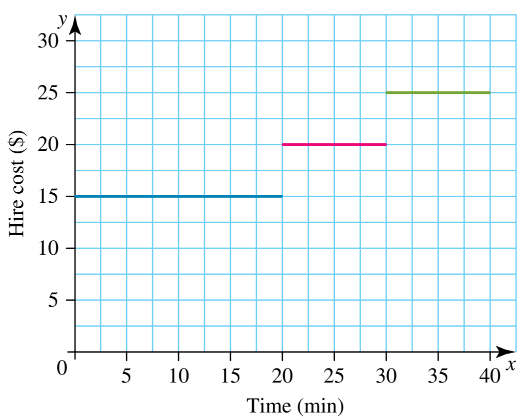
b. i. $a = 10, b = 30$

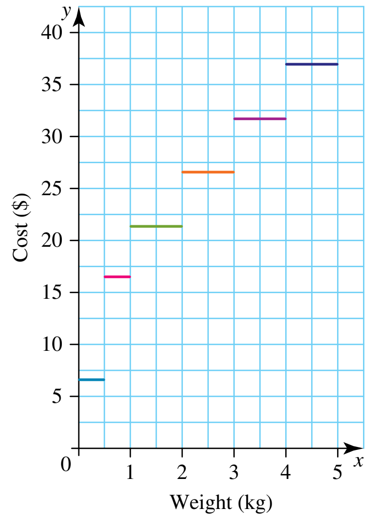
ii. a is the time the oven first reaches 200 °C and b is the time at which the bread stops being cooked.

c. $m = \frac{-14}{3}, d = 30, e = 60$

d. The change in temperature for each minute in the oven

e. 

12. 

13. a. 

b. It is cheaper to post them together ($16.15 together versus $22.75 individually).

14. a. $a = 4$

b. $b - 50(4) = 1400$

$b = 1600$

c. $10 \leq t \leq 12$

d. $500

15. a. (0.75, 15) and (1.25, 22.5)

b. The yacht is returning to the yacht club during this time period.

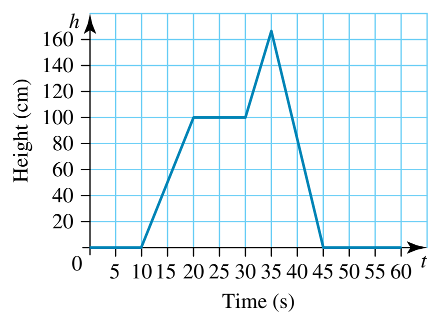
c. 22.5 km

d. 3 hours, 8 minutes; $b = 3.13$

16. a. $0 \leq t \leq 10$ and $45 \leq t \leq 60$; these are the intervals when $y = 0$.

b. 165 cm

c. 60 seconds

d. 

17. a. There is a change in the rate for different x-values (i.e. different car prices).

b. $a = 1000, b = 2000, c = 30, d = 3000, e = 60, f = 0.04$

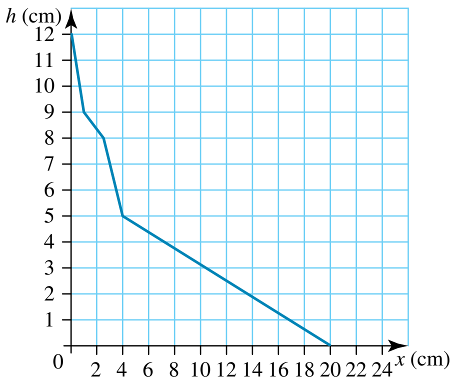
c. (1000, 10), (2000, 30) and (3000, 60)

d. $2500

18. a. 540 L
b. 12 L/min
c. 93 min

19. a. (1, 5), (3, 9) and (2.5, 8.5)
b. $a = 1$, $b = 3$, $c = 2.5$
c. $b > c$, which means that graph **iii** is not valid and the piecewise linear graph cannot be sketched.

20. a. 12 m
b. $a = 1$
c. (1, 9), (2.5, 8) and (4, 5); $b = 2.5$, $c = 4$
d. The horizontal distance of the slide is 20 m.
e.



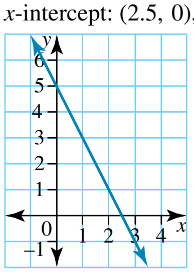
10.6 Review: exam practice

Multiple choice

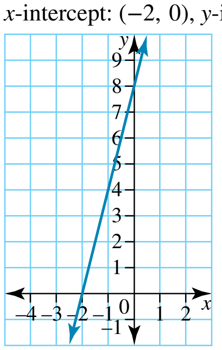
1. E 2. D 3. A 4. C 5. D
6. A 7. D 8. B 9. A 10. E

Short answer

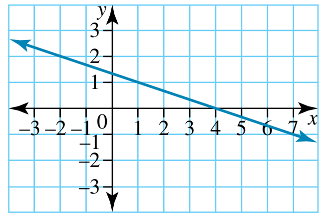
1. a. x-intercept: (2.5, 0), y-intercept: (0, 5)



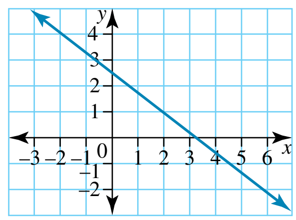
b. x-intercept: (−2, 0), y-intercept: (0, 8)



c. x-intercept: (4, 0), y-intercept: $\left(0, \frac{4}{3}\right)$



d. x-intercept: $\left(\frac{10}{3}, 3\right)$, y-intercept: (0, 2.5)



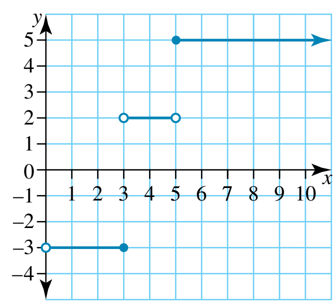
2. a. −2
b. −1
c. $\frac{4}{3}$
d. $-\frac{7}{13}$

3. −2

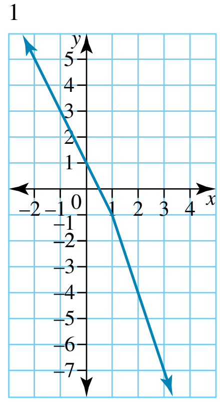
4.

	Equation	Gradient	y-intercept	x-intercept
a	$y = 5x - 3$	5	−3	0.6
b	$y = 3x + 1$	3	1	$-\frac{1}{3}$
c	$3y = 6x - 9$	2	−3	1.5
d	$2y + 4x = 8$	−2	4	2
e	$y = x + 5$	1	5	−5
f	$y = 2x - 4$	2	−4	2

5.

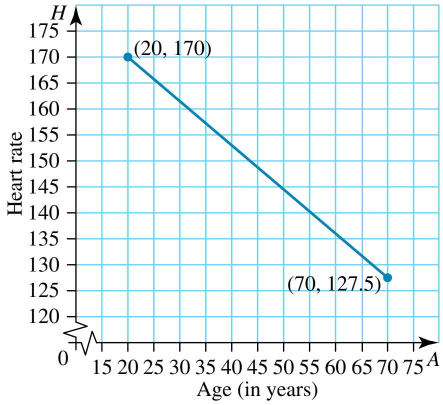


6. a. 1
b.



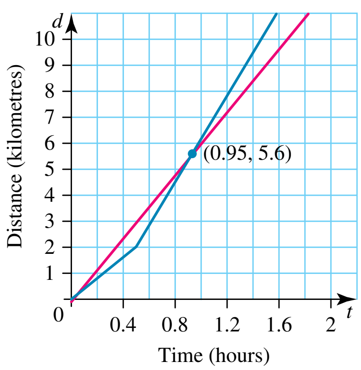
Extended response

1. **a.** The power of both variables in the equation (H and A) is 1.
   b. 166
   c. Gradient = −0.85, y-intercept = 187
   d.

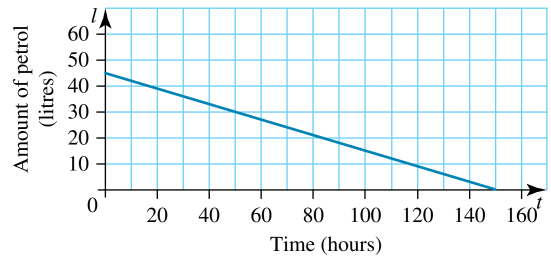


   e. 29
   f. At the x-intercept, heart rate = 0; therefore, the person would no longer be alive.

2. **a.** 1 hour, 41 minutes
   b. Jerri started 0.1 km (100 metres) behind the starting line.
   c. 2 km
   d. The gradient of the equation equals the speed; therefore, Samantha was travelling at 4 km/h.
   e. After 30 minutes, Samantha increased her speed from 4 km/h to 8 km/h.
   f. **i.** 1.5
   ii. Samantha took 1 hour, 30 minutes hours to run 10 km; Jerri took 1 hour 41 minutes. Difference: 41 − 30 minutes = 11 minutes
   g. **i.** 0.95 hours (57 minutes)
   ii. 5.6 km
   h.



3. **a.** Leaking rate = gradient, initial petrol = y-intercept
   b. 0.3 L/h. It is assumed that the petrol is leaking at a constant rate.
   c. 1.2 L
   d. 17.5 hours
   e. **i.** Petrol is leaking at a constant rate (the gradient).
   ii. Petrol is leaking from the tank; therefore, the amount of petrol is decreasing.
   iii. $l = 45 - 0.3t$
   f.



   g. 150 hours

4. **a.** $p = -20d + 868$
   b. Gradient = −20 means that for every 1 km from the city centre, the median house price decreases by \$20 000. y-intercept = 868 means that the median house price in the city centre is \$868 000.
   c. **i.** \$558 000
   ii. \$772 000
   iii. \$494 000
   d. 10.9 km
   e. x-intercept = 43.4. The x-intercept is where the value of properties would equal 0, which is not possible in the context of this problem.
   f. A distance of 229 km from the city centre is well outside the data set. Therefore, the equation would not be reliable.

TOPIC 11
Inequalities and linear programming

11.1 Overview

11.1.1 Introduction

With mathematics, we can write an equation that would take many words to explain. Using mathematical symbols consolidates space and words. Furthermore, mathematical symbols are universal, thus allowing individuals to share information that they would find difficult to share worldwide in different languages.

To be able to write an equation we first need to understand what an equals sign is. Robert Recorde (1510–1558), a Welsh physician and mathematician, was the first to use the symbol represented by two parallel lines.

It wasn't until 1631 that the book *Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas* first used greater than (>) and less than (<) signs. The book was written by British mathematician Thomas Harriot. Interestingly, Harriot also used parallel lines to denote equality. However, Harriot's equal sign was vertical (||) rather than horizontal (=). The symbols less than, greater than and equal to were not used until 1734 by French mathematician Pierre Bouguer.

The understanding of inequalities allows us to solve linear programming problems. These problems require optimization of resources. Linear programming can be applied to manufacturing, to calculate how to assign labour and machinery to minimise the cost of operation. It could also be applied to business operations, to decide which products to sell and how much to order to maximise profit.

The airline industry is a good example of an industry that uses linear programming. They use it to optimise profits and minimise expenses to their business. Initially, airlines charged the same price for any seat on the plane. To try and make more money, they decided to charge different fares for different seats and promote different prices depending on how early people bought their ticket. Airlines needed to have an understanding of how many people would be willing to pay a higher price for a ticket if they booked at the last minute, or how many people would only purchase low-price tickets, without an inflight meal. With the use of linear programming, they were able to find the optimal breakdown of how many tickets to sell and at what price. Airlines also use linear programming for plane routes, pilot schedules, direct and indirect flights and layovers.

LEARNING SEQUENCE

Fully worked solutions for this topic are available in the resources section of your eBookPLUS at www.jacplus.com.au.

11.1.2 Kick off with CAS

Linear inequalities and shaded regions

A linear inequality is a linear function with an inequality sign. This type of function divides the Cartesian plane into two regions. One region satisfies the inequality, and the other region does not satisfy the inequality. The sign used in the inequality will determine which region satisfies the inequality.

The > symbol means greater than, and the < symbol means less than.

1. Use CAS to graph the following linear inequalities.
   a. $x > 6$ b. $y < 3$ c. $x - y > 2$
2. How are the regions that satisfy the inequalities in question 1 identified on your CAS?

Linear inequalities may also use the symbols ≥ or ≤, which mean greater than or equal to, and less than or equal to.

3. Use CAS to graph the following linear inequalities.
   a. $x \geq -2$ b. $y \leq 5$ c. $2x - y \geq 4$
4. How do the lines of the graphs in question 3 differ from the lines of the graphs in question 1?

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology

11.2 Graphs of linear inequalities

11.2.1 Linear inequalities

When a linear graph is drawn on the **Cartesian plane**, the plane is divided into two distinct regions or sections.

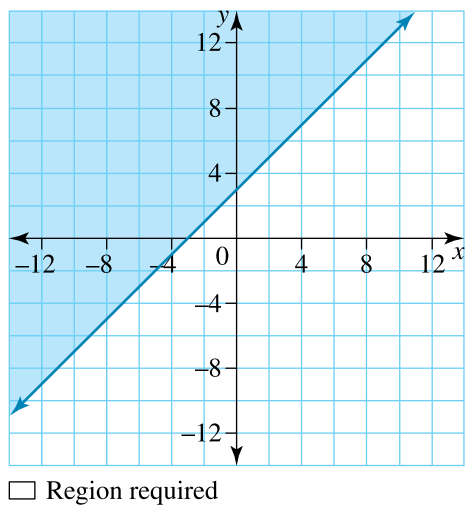
A linear inequation is a linear equation with the equals sign replaced with an inequality sign. This sign determines which one of the two regions drawn is the solution to the inequality. The line which divides the plane into two regions may or may not be included in the inequality, depending on the sign used.

Inequality sign	Meaning
$>$	Greater than
$<$	Less than
$\geq$	Greater than or equal to
$\leq$	Less than or equal to

11.2.2 Graphing linear inequalities

When graphing a **linear inequality**, the first step is to graph the equivalent linear equation. Once this is done, we need to determine which of the two regions represents our linear inequality. To do this we can take any point on either side of the line (known as a **test point**) and substitute the x- and y-values of this point into the inequality to determine whether it satisfies the inequality.

Once we've determined which region of the graph satisfies our inequality, we need to represent the **required region**. In this text we leave the required region unshaded; however, you could chose to shade the required region. Be sure to include a legend with your diagram, as shown, to indicate the required region.



Finally we need to determine whether or not the line belongs in our inequation.

Style of line	
A **solid line** represents: $\geq$ Greater than or equal to $\leq$ Less than or equal to $=$ Equal to	A **dashed line** represents: $>$ Greater than $<$ Less than
A solid line means that the values on the line **are** included in the region.	A dashed line means that the values on the line **are not** included in the region.

Now we have all of the necessary information needed to graph linear inequalities.

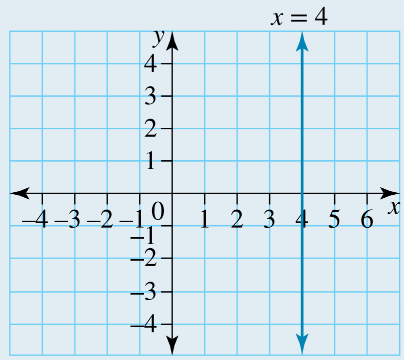
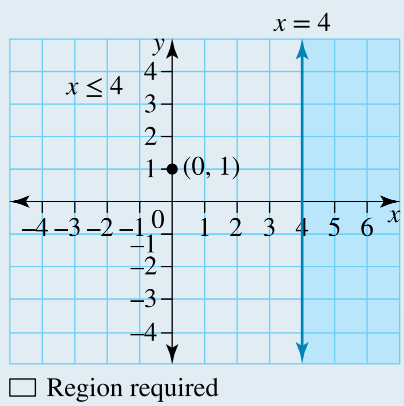
11.2.3 Linear inequalities in one variable

When graphing a linear inequality in one variable, the result will either be a vertical or horizontal line. Linear inequalities in one variable may also be displayed on number lines, but in this text we display them solely on the Cartesian plane.

WORKED EXAMPLE 1

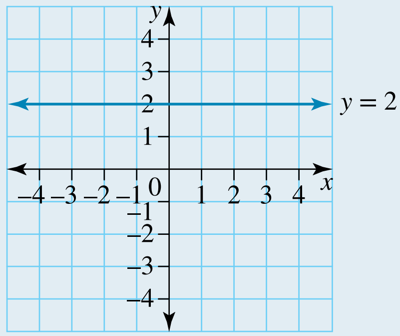
Sketch the following linear inequalities on separate Cartesian planes, leaving the required regions unshaded.

a. $x \le 4$ **b.** $y > 2$

THINK	WRITE/DRAW
a. 1. For sketching purposes, replace the inequality sign with an equals sign. Sketch the line $x = 4$.	**a.** 
2. Determine whether the line should be dashed or solid by looking at the inequality sign.	As the inequality sign is $\le$, the line will be solid (meaning values on the line are included).
3. Select any point not on the line to be the test point. Substitute the x- and y-values of this point into the inequality.	Test point: (0, 1) $x \le 4$ $0 \le 4$
4. Test if the statement is true or false, and shade the region that is not required.	$0 \le 4$ is true and the test point (0, 1) lies in the required region. So we shade the opposing region, in other words the region opposite to where the test point lies.
5. Add a legend indicating the required region.	

b. 1. For sketching purposes, replace the inequality sign with an equals sign. Sketch the line $y = 2$.

b.



2. Determine whether the line should be dashed or solid by looking at the inequality sign.

As the inequality sign is >, the line will be dashed (meaning values on the line are not included).

3. Select any point not on the line to be the test point. Substitute the x- and y-values of this point into the inequality.

Test point: (2, 5)

$y > 2$

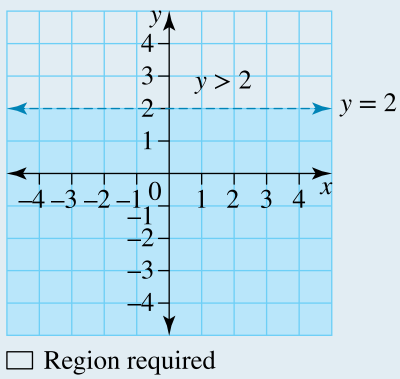
$5 > 2$

4. Test if the statement is true or false, and shade the region that is not required.

$5 > 2$ is true, so the test point (2, 5) lies in the required region.

5. Add a legend indicating the required region.

So we shade the opposing region, in other words the region opposite to where the test point lies.



11.2.4 Transposing linear inequalities

In some situations we need to transpose (rearrange) the inequality to make x or y the subject before we can sketch it.

When dividing both sides of an inequality by a negative number, the direction of the sign of the inequality changes to its opposite direction.

For example, if we are dividing both sides of the linear inequality $-x < 7$ by -1, then the result is $x > -7$. Note that the 'less than' sign has become a 'greater than' sign.

WORKED EXAMPLE 2

Sketch the linear inequality $-x + 6 < 4$ on a Cartesian plane, leaving the required region unshaded.

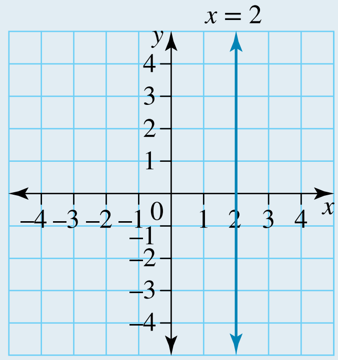
THINK

1. Transpose (rearrange) the inequality so x is the subject.

WRITE/DRAW

$$-x + 6 < 4$$
$$-x + 6 - 6 < 4 - 6$$
$$-x < -2$$
$$x > 2$$

2. For sketching purposes, replace the inequality sign with an equals sign. Sketch the line $x = 2$.



3. Determine whether the line should be solid or dashed by looking at the inequality sign.

As the inequality sign is >, the line will be dashed (meaning the values on the line are not included).

4. Select any point not on the line to be the test point. Substitute the x- and y-values of this point into the inequality.

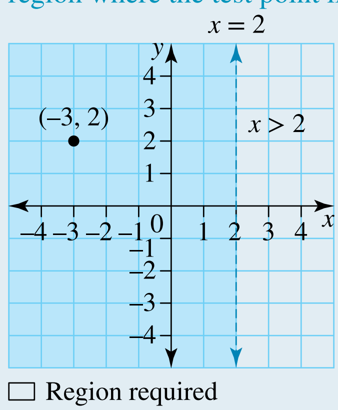
Test point: $(-3, 2)$
$x > 2$
$-3 > 2$

5. Check whether the statement is true or false, and shade the region which is not required.

$-3 > 2$ is false, so the test point $(-3, 2)$ does not lie in the required region.

6. Add a legend indicating the required region.

So shade this region, in other words the region where the test point lies.

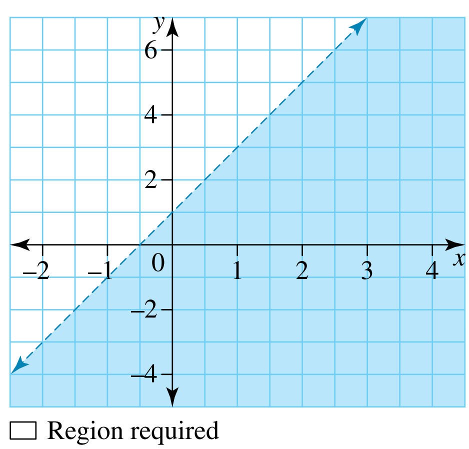


11.2.5 Linear inequalities in two variables

Linear inequalities in two variables work in much the same way as linear inequalities in one variable; however, they must be shown on the Cartesian plane.

For example $y > 2x + 1$ is shown in the diagram.

Notice that the line is dashed to indicate that it does not appear in the required region.



WORKED EXAMPLE 3

Sketch the following linear inequalities on separate Cartesian planes, leaving the required regions unshaded.

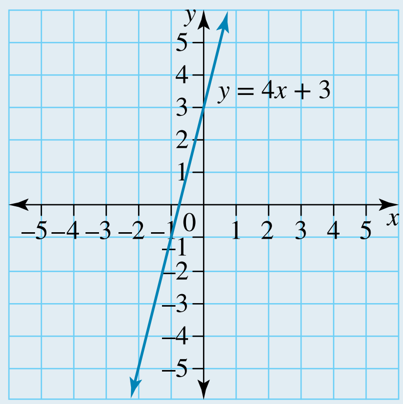
a. $y < 4x + 3$ **b.** $-5x - y > -10$

THINK

WRITE/DRAW

a. 1. For sketching purposes replace the inequality sign with an equals sign. Sketch the line $y = 4x + 3$ using the y-intercept and gradient method (y-intercept = 3, gradient = 4).

a.



2. Determine whether the line should be solid or dashed by looking at the inequality sign.

As the inequality sign is <, the line will be dashed (meaning the line is not included).

3. Select any point not on the line as your test point. Substitute the x- and y-values of this point into the inequality.

Test point: (2, 1)

$y < 4x + 3$

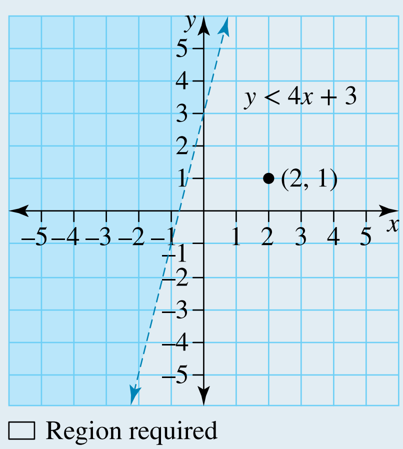
$1 < 4 \times 2 + 3$

$1 < 11$

4. Check if the statement is true or false, and shade the region that is not required.

$1 < 11$ is true, so the test point (2, 1) lies in the required region.
So we shade the opposing region, in other words the region opposite to where the test point lies.

5. Add a legend indicating the required region.



b. 1. For sketching purposes replace the inequality sign with an equals sign. Sketch the line $-5x - y = -10$ using the x-intercept and y-intercept method.

b. To find the x-intercept, $y = 0$:

$$\begin{aligned} -5x - y &= -10 \\ -5x - 0 &= -10 \\ -5x &= -10 \\ x &= 2 \end{aligned}$$

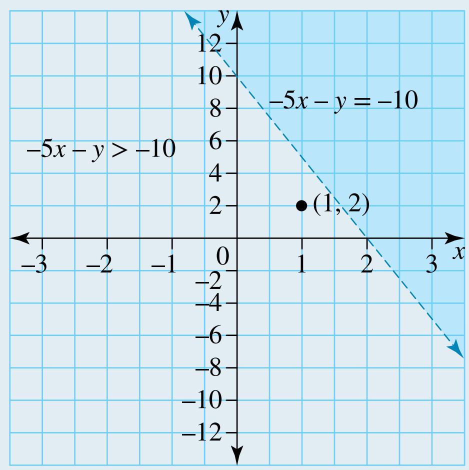
The x-intercept is at $(2, 0)$.

To find the y-intercept, $x = 0$:

$$\begin{aligned} -5x - y &= -10 \\ -5 \times 0 - y &= -10 \\ -y &= -10 \\ y &= 10 \end{aligned}$$

The y-intercept is at $(0, 10)$.

2. Determine whether the line should be solid or dashed by looking at the inequality sign.



☐ Region required

As the inequality sign is <, the line will be dashed (meaning the line is not included).

3. Select any point not on the line to be the test point. Substitute the x- and y-values of this point into the inequality.

Test point: $(1, 2)$

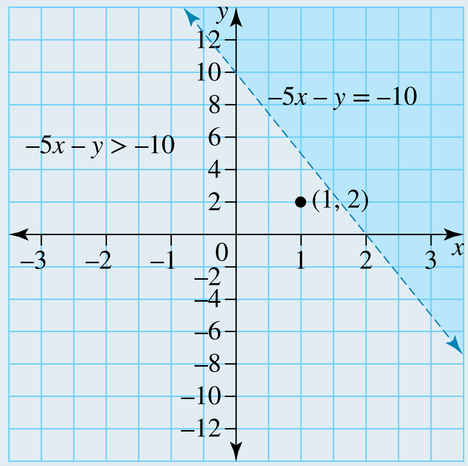
$$\begin{aligned} -5x - y &> -10 \\ -5 \times 1 - 2 &> -10 \\ -5 - 2 &> -10 \\ -7 &> -10 \end{aligned}$$

4. Check if the statement is true or false, and shade the region that is not required.

$-7 > -10$ is true, so the test point $(1, 2)$ lies in the required region.

So we shade the opposing region, in other words the region opposite to where the test point lies.

5. Add a legend indicating the required region.

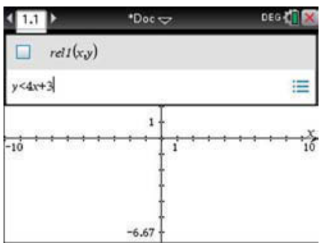


☐ Region required

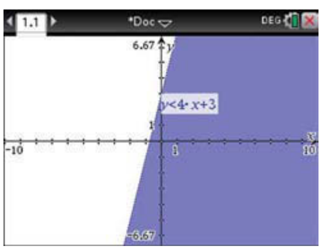
TI | THINK

WRITE

a. 1. On a Graphs page, press MENU then select:
3: Graph Entry/Edit
2: Relation
Complete the entry line as
$y < 4x + 3$
then press ENTER.



2. The inequality is represented by the shaded region on the screen.

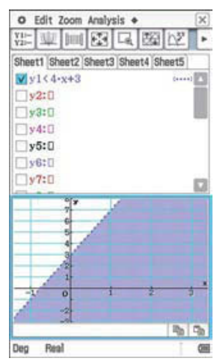


CASIO | THINK

WRITE

a. 1. On a Graph & Table screen, select:
- Type
- Inequality
-Y < Type
Complete the entry line for y1 as
$y1 < 4x + 3$
then click the tick box.

2. Click the $ icon. The inequality is represented by the shaded region on the screen.



Resources

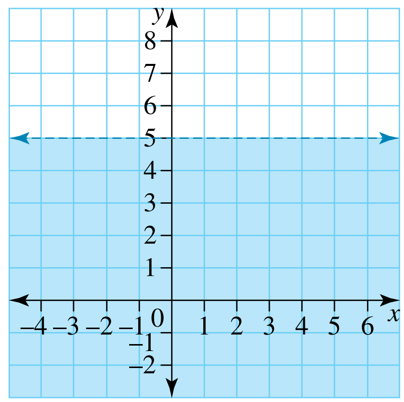
Interactivity: Linear inequalities in one variable (int-6487)
Interactivity: Linear inequalities in two variables (int-6488)

studyon

Units 1 & 2 > AOS 5 > Topic 2 > Concept 1 > **Linear inequalities** Concept summary and practice questions

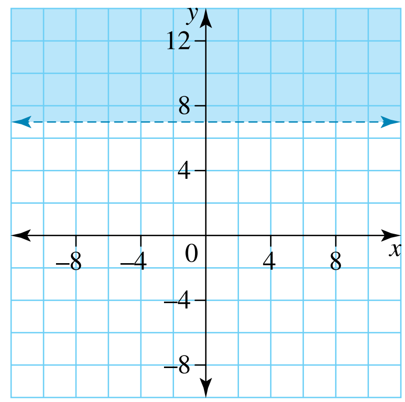
Exercise 11.2 Graphs of linear inequalities

1. **WE1** Sketch the following linear inequalities on separate Cartesian planes, leaving the required regions unshaded.
   a. $x \leq 5$
   b. $y > 7$
2. Write the linear inequality for each of the following graphs.
   a.



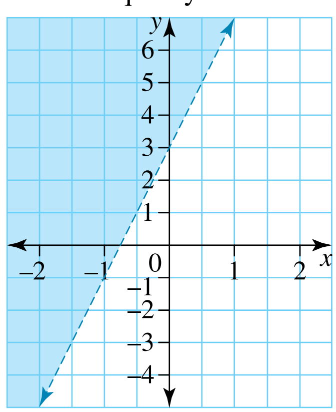
☐ Region required

   b.



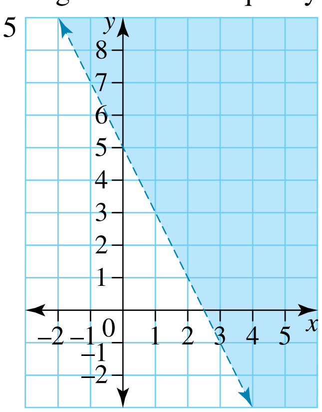
☐ Region required

3. Sketch the following linear inequalities on separate Cartesian planes, leaving the required regions unshaded.
   a. $x > 5$
   b. $y < -3$
   c. $x \leq 4$
   d. $y \geq 12$
4. **WE2** Sketch the linear inequality $-x+8 < 10$ on a Cartesian plane, leaving the required region unshaded.
5. Sketch the linear inequality $-2y + 4 < 7$ on a Cartesian plane, leaving the required region unshaded.
6. **WE3** Sketch the following linear inequalities on separate Cartesian planes, leaving the required regions unshaded.
   a. $y < 5x + 3$
   b. $-5x - y > -20$
7. Complete the linear inequality for each of the following graphs by placing the correct inequality in the box.
   a. $y\square 4x + 3$



☐ Region required

   b. $y\square - 2x + 5$



☐ Region required

8. Sketch the following linear inequalities on separate Cartesian planes, leaving the required regions unshaded.
   a. $2x + 4y > 10$
   b. $10 < -3x + 9$
   c. $3y + 2x + 6 \geq 0$
   d. $0 \leq 6x + 4y + 24$
9. **MC** When sketching linear inequalities, which of the following is represented by a dashed line?
   A. $-6 \geq 3x + 8y$
   B. $y \leq 2x + 4$
   C. $x \leq 7 + 2y$
   D. $5 > x + y$
   E. $x + 2y \geq 3$
10. **MC** When sketching linear inequalities, which of the following is represented by a solid line?
   A. $2x + 4y > 7$
   B. $y < 3x + 5$
   C. $5x - 7y \geq 9$
   D. $3x + 4y < 12$
   E. $10 > x + 3y$

11. MC Which of the following is not a suitable test point for the linear inequality $y < 2x + 4$?

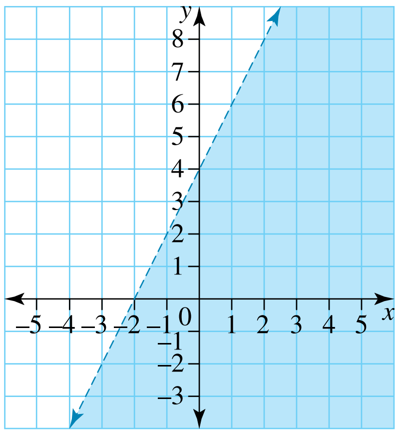
A. $(-3, 2)$ **B.** $(-1, 2)$ **C.** $(0, 2)$ **D.** $(3, 5)$ **E.** $(8, 2)$

12. MC Which of the following is a suitable test point for the linear inequality $y > -3x + 6$?

A. $(0, 6)$ **B.** $(2, 0)$ **C.** $(-1, 9)$ **D.** $(1, 3)$ **E.** $(1, 4)$

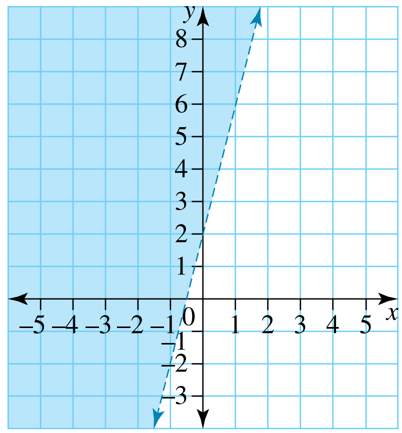
13. MC Which of the following linear inequalities has been incorrectly sketched?

A. $y > 2x + 4$



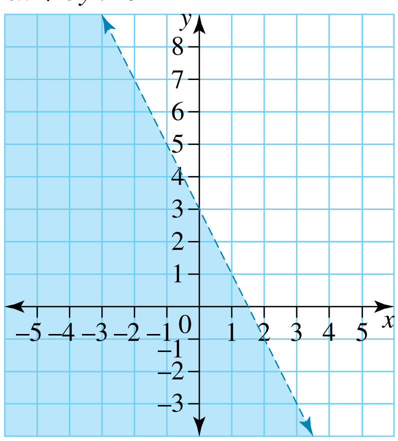
☐ Region required

B. $y > 4x + 2$



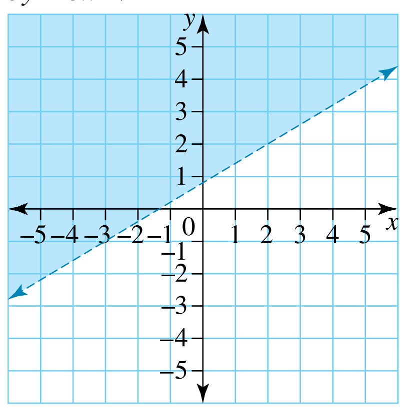
☐ Region required

C. $6x + 3y > 9$



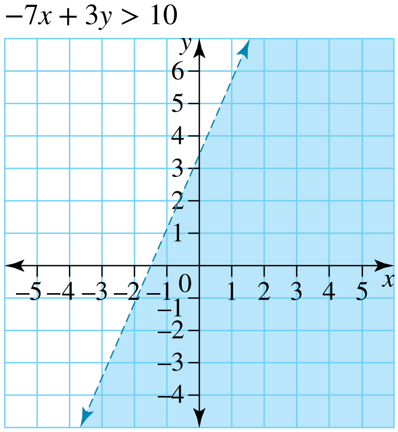
☐ Region required

D. $5y - 3x < 4$



☐ Region required

E. $-7x + 3y > 10$



☐ Region required

14. Sketch the following linear inequalities.

a. $y < \frac{-x}{2} + 4$

b. $\frac{y}{3} \geq 2x + 1$

15. **MC** Which linear inequality represents the graph?

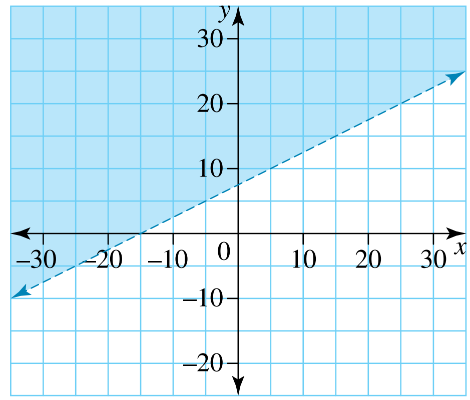
A. $0 > 15x + 7.5y$

B. $0 < 15x + 7.5y$

C. $2x - 4y + 30 > 0$

D. $2x - 4y + 30 < 0$

E. $x - 2y > 30$



☐ Region required

16. **MC** Which linear inequality best represents the graph?

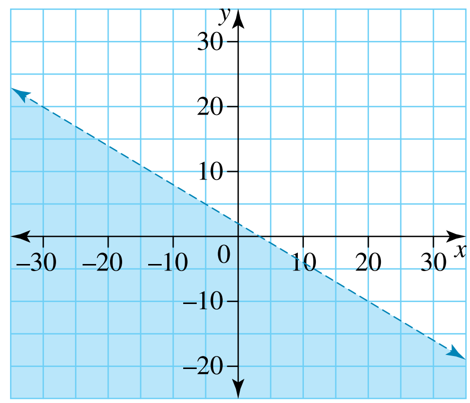
A. $5y + 3x > 10$

B. $5y - 3x > 10$

C. $5y + 3x < 10$

D. $3x - 2y > 0$

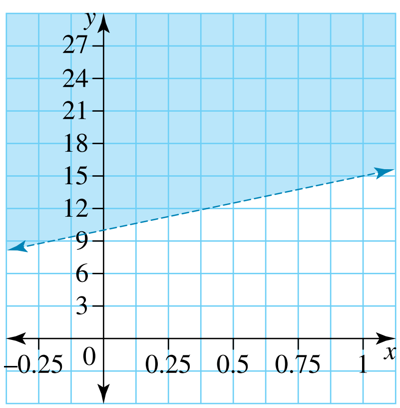
E. $3x - 2y < 0$



☐ Region required

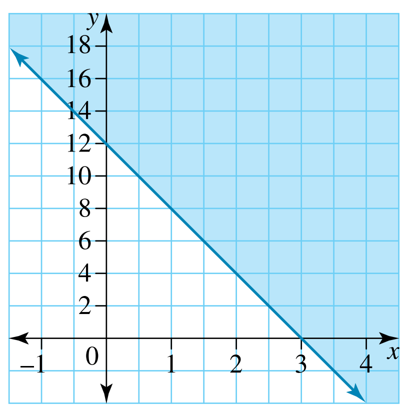
17. State the inequality that defines the following graphs.

a.



☐ Region required

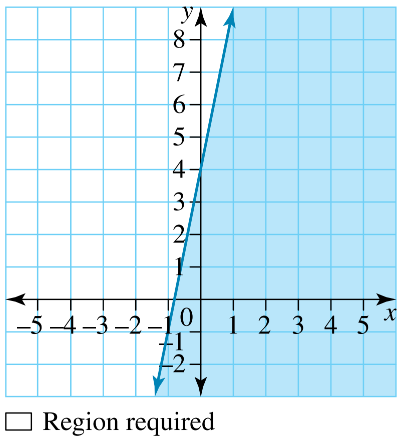
b.



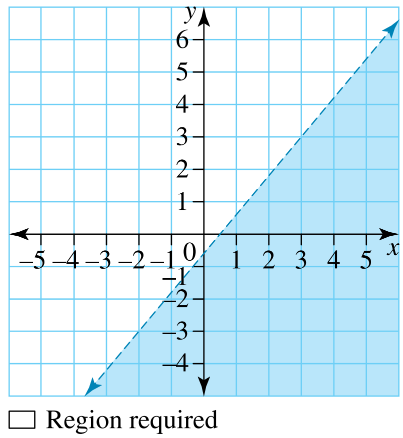
☐ Region required

18. State the inequality that defines the following graphs.

a.



b.



11.3 Linear programming

11.3.1 Simultaneous linear inequalities

If we want to solve more than one linear inequality simultaneously, we can do so by graphing the solutions (or **feasible region**) for all of the linear inequalities on the same Cartesian plane and finding the intersection of the required regions.

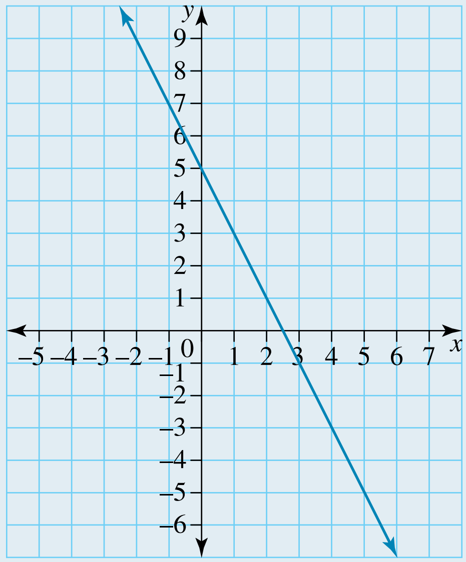
Keeping the required region unshaded allows us to easily identify which region we require, as we can simply shade all of the regions that don't fit into the solution. However, you may find that CAS shades the required regions, so use test points to ensure that you have the correct region. Always remember to include a legend with your graphs.

WORKED EXAMPLE 4

Find the solution to the following simultaneous linear inequalities, leaving the required region unshaded.

$$\mathbf{4x + 2y \le 10}$$
$$\mathbf{3x + y \le 9}$$

THINK	WRITE/DRAW
1. Sketch the linear inequalities individually, starting with the first inequality ($4x + 2y \le 10$). For sketching purposes, remember to replace the inequality sign with an equals sign. Sketch the graph of $4x + 2y = 10$ using the x-intercept and y-intercept method.	To find the x-intercept, $y = 0$: $4x + 2y = 10$, $4x + 2 \times 0 = 10$, $4x = 10$, $x = 2.5$. The x-intercept is at (2.5, 0). To find the y-intercept, $x = 0$: $4x + 2y = 10$, $4 \times 0 + 2y = 10$, $2y = 10$, $y = 5$. The y-intercept is at (0, 5).



2. Determine whether the line should be solid or dashed by looking at the inequality sign.

As the inequality sign is $\leq$, the line will be solid (meaning the line is included).

3. Select any point not on the line as the test point. Substitute the x- and y-values of this point into the inequality.

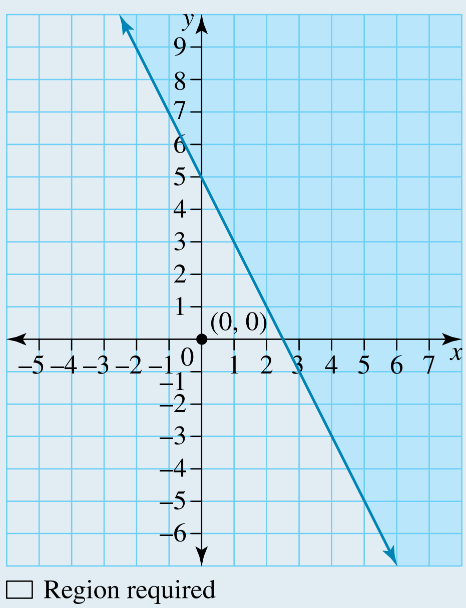
Test point: (0, 0)

$$4x + 2y \leq 10$$
$$4 \times 0 + 2 \times 0 \leq 10$$
$$0 \leq 10$$

4. Check if the statement is true or false, and shade the region that is not required.

$0 \leq 10$ is true, so the test point (0, 0) lies in the required region.
So we shade the opposing region, in other words the region opposite to where the test point lies.

5. Add a legend indicating the required region.



6. Repeat this process with the second inequality ($3x + y \leq 9$). Sketch $3x + y = 9$ using the x-intercept and y-intercept method.

When $y = 0$:

$3x + y = 9$

$3x + 0 = 9$

$x = 3$

x-intercept $= (3, 0)$

When $x = 0$:

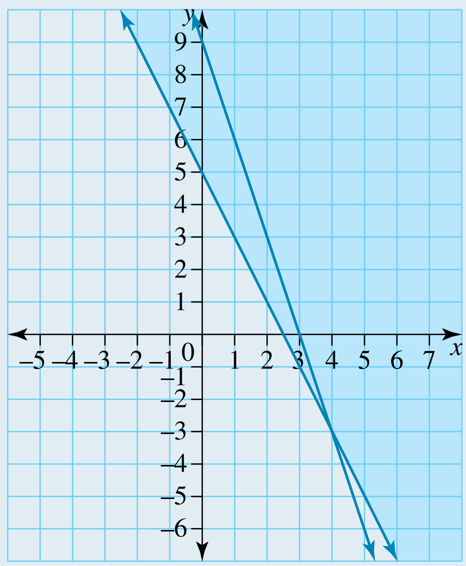
$3x + y = 9$

$3 \times 0 + y = 9$

$y = 9$

y-intercept $= (0, 9)$

As the inequality sign is $\leq$, the line will be solid (meaning the line is included).



Region required

7. Select a test point to find the required region. All required regions have now been found, so the remaining unshaded region is the solution (or feasible) region.

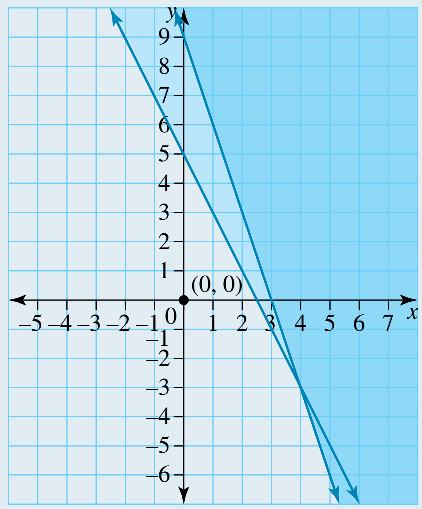
Test point: (0, 0)

$3x + y \leq 9$

$3 \times 0 + 0 \leq 9$

$0 \leq 9$

In this case, the statement is true and the test point (0, 0) lies in the required region, so we shade the opposing region.

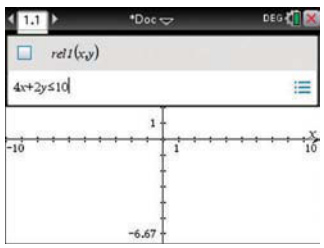


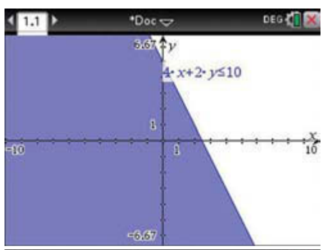
Region required

TI | THINK

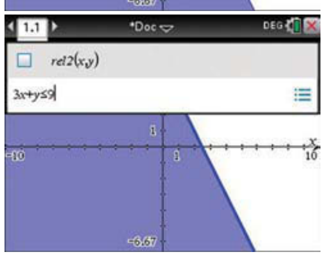
1. On a Graphs page, press MENU then select:
3: Graph Entry/Edit
2: Relation
Complete the entry line as
$4x + 2y \le 10$
then press ENTER.

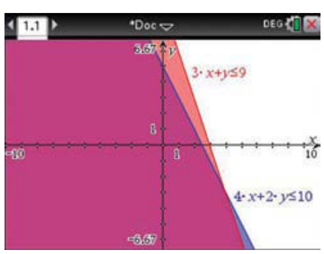
WRITE





2. Press MENU then select3:
Graph Entry/Edit
2: Relation
Complete the entry line as
$3x + y \le 9$
then press ENTER.





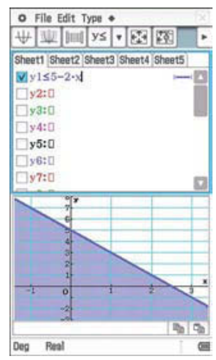
3. The answer is shown on the screen.

The required region is the *darker* region.

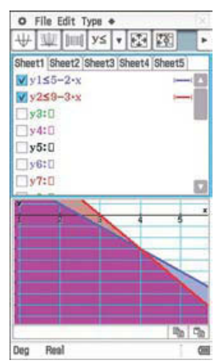
CASIO | THINK

1. Rearrange the first equation to make y the subject:
$y \le 5 - 2x$.
On a Graph & Table screen, select:
- Type
- Inequality
-y ≤ Type
Complete the entry line for y1 as
$y1 \le 5 - 2x$
then click the tick box.
Click the $ icon.

WRITE



2. Rearrange the second equation to make y the subject:
$y \le 9 - 3x$.
Complete the entry line for y2 as
$y2 \le 9 - 3x$
then click the tick box.
Click the $ icon.



3. The answer is shown on the screen.

The required region is the *darker* region.

Note: The method to find the solution to three or more simultaneous linear inequalities is exactly the same as the method used to find the solution to two simultaneous linear inequalities.

11.3.2 Linear programming

Linear programming is a method used to achieve the best outcome in a given situation. It is widely used in many industries, but in particular is used in the business, economics and engineering sectors. In these industries, companies try to maximise profits while minimising costs, which is where linear programming is useful.

The **constraints** in a linear programming problem are the set of linear inequalities that define the problem. In this topic, the constraints are displayed in the form of inequalities that you are already familiar with. Real-life linear programming problems may have hundreds of constraints; however, the problems we will deal with only have a small number of constraints.

When the constraints of a linear programming problem are all sketched on the same grid, the feasible region is acquired. The feasible region is every required point that is a possible solution for the problem.

WORKED EXAMPLE 5

Sketch the feasible region for a linear programming problem with the following constraints.

$$\begin{aligned} 5x + 4y &\le 10 \\ -2x + 3y &\le 3 \\ x &> 0 \\ y &> 0 \end{aligned}$$

THINK

1. Sketch the linear inequalities individually, starting with the first inequality ($5x + 4y \le 10$). Remember, for sketching purposes, to replace the inequality sign with an equals sign. Sketch the graph of $5x + 4y = 10$ using the x-intercept and y-intercept method.

WRITE/DRAW

To find the x-intercept, $y = 0$:

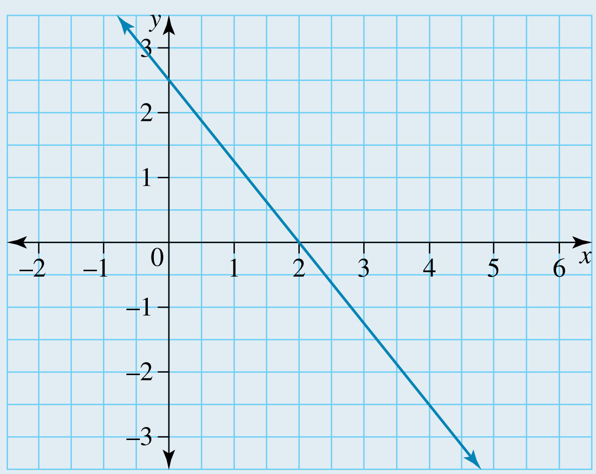
$$\begin{aligned} 5x + 4y &= 10 \\ 5x + 4 \times 0 &= 10 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

The x-intercept is at (2, 0).

To find the y-intercept, $x = 0$:

$$\begin{aligned} 5x + 4y &= 10 \\ 5 \times 0 + 4y &= 10 \\ 4y &= 10 \\ y &= \frac{10}{4} \\ &= 2.5 \end{aligned}$$

The y-intercept is at (0, 2.5).



2. Determine whether the line should be solid or dashed by looking at the inequality sign.

As the inequality sign is $\leq$, the line will be solid (meaning the line is included).

3. Select any point not on the line to be the test point. Substitute the x- and y-values of this point into the inequality.

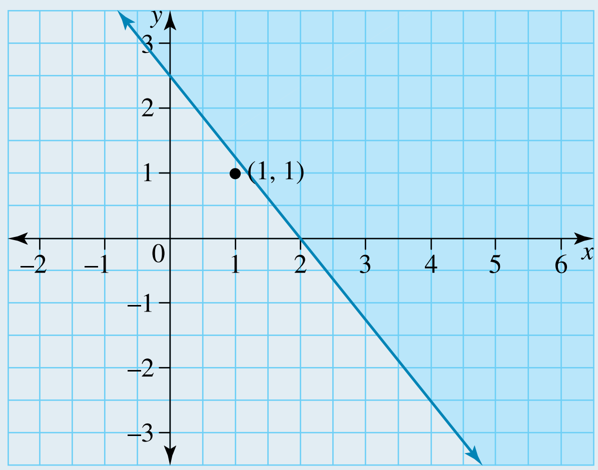
Test point: (1, 1)

$$\begin{aligned} 5x + 4y &\leq 10 \\ 5 \times 1 + 4 \times 1 &\leq 10 \\ 9 &\leq 10 \end{aligned}$$

4. Check if the statement is true or false, and shade the region that is not required.

$9 \leq 10$ is true, so the test point (1, 1) lies in the required region.
So we shade the opposing region, in other words the region opposite to where the test point lies.

5. Add a legend indicating the required region.



6. Repeat this process with the second inequality ($-2x + 3y \leq 3$).
Sketch $-2x + 3y = 3$ using the x-intercept and y-intercept method.

When $y = 0$:

$$\begin{aligned} -2x + 3y &= 3 \\ -2x + 3 \times 0 &= 3 \\ -2x &= 3 \\ x &= -1.5 \end{aligned}$$

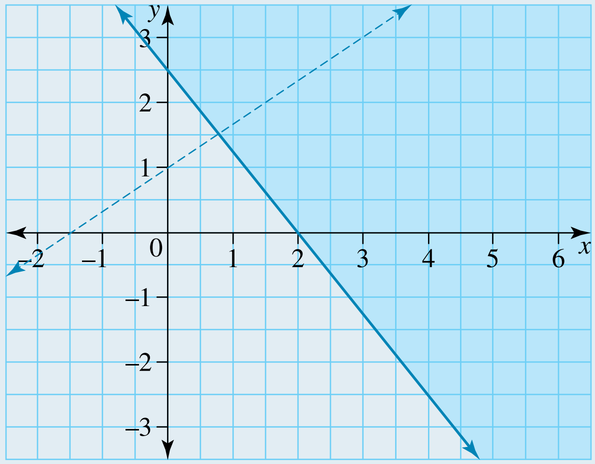
x-intercept = $(-1.5, 0)$

When, $x = 0$:

$$\begin{aligned} -2x + 3y &= 3 \\ -2 \times 0 + 3y &= 3 \\ 3y &= 3 \\ y &= 1 \end{aligned}$$

y-intercept = $(0, 1)$

As the inequality sign is <, the line will be a dashed line (meaning the line is not included).



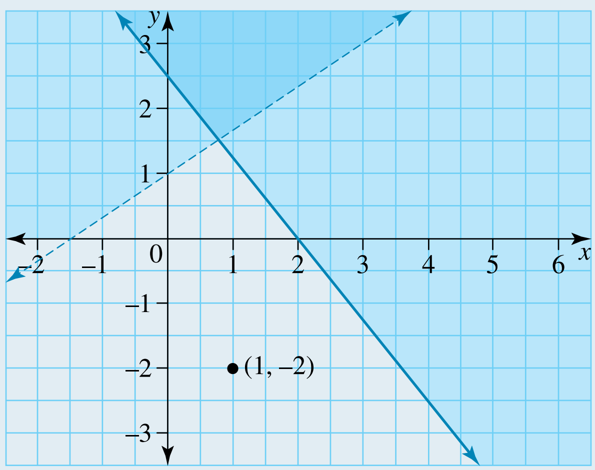
☐ Region required

7. Select a test point to find the required region.

Test point: (1, –2)

$$x - 4y > 3$$
$$1 - 4 \times -2 > 3$$
$$1 + 8 > 3$$
$$9 > 3$$

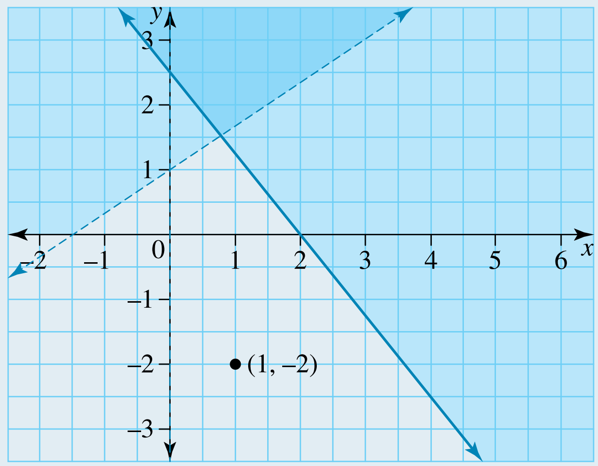
9 > 2 is true, so the test point (1, −2) lies in the required region, so we shade the opposing region.



☐ Region required

8. Repeat this process with the third inequality ($x > 0$).
Sketch $x = 0$.

Place $x > 0$ on the Cartesian plane (use a dashed line due to the $>$ sign).



Region required

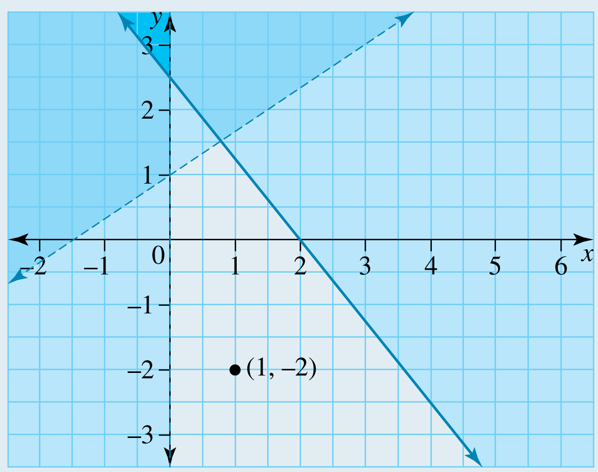
9. Select a test point to find the required region.

Test point: $(1, -2)$

$x > 0$

$1 > 0$

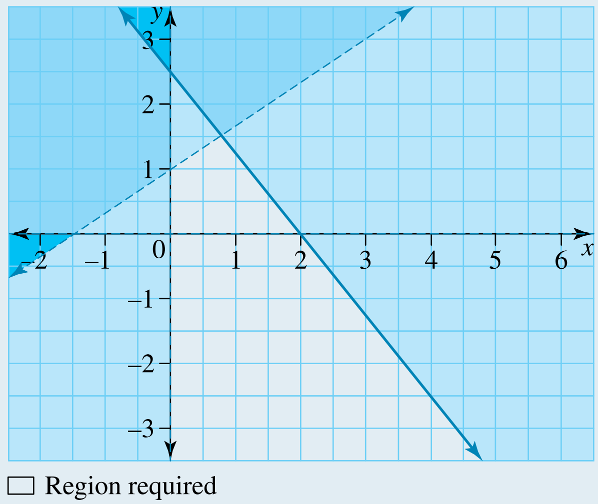
The test point $(1, -2)$ lies in the required region, so we shade the opposing region.



Region required

10. Repeat this process with the fourth inequality ($y > 0$).
Sketch $y = 0$.

Place $y > 0$ on the Cartesian plane. Use a dashed line due to the > sign.



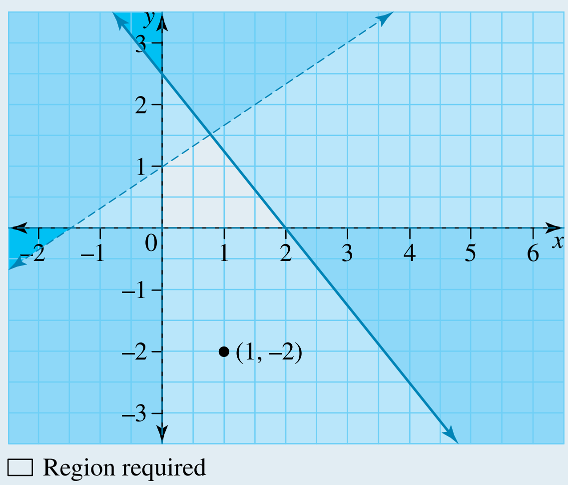
11. Select a test point to find the required region. All required regions have now been found, so the remaining unshaded region is the feasible region.

Test point: $(1, -2)$

$y > 0$

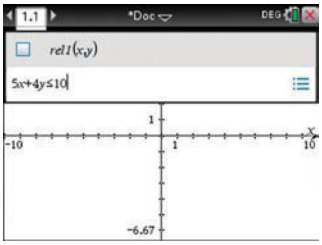
$-2 > 0$

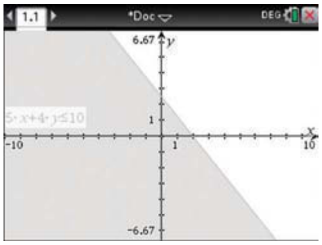
The test point $(1, -2)$ does not lie in the required region, so we shade this region.



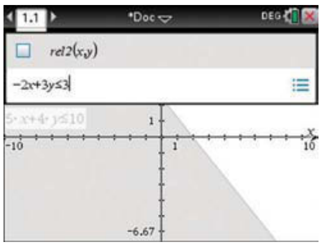
TI | THINK **WRITE**

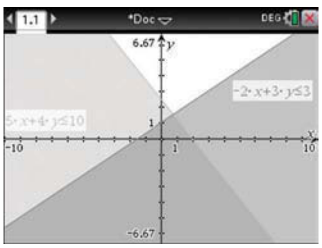
1. On a Graphs page, press MENU then select:
3: Graph Entry/Edit
2: Relation
Complete the entry line as
$5x + 4y \le 10$
then press ENTER.



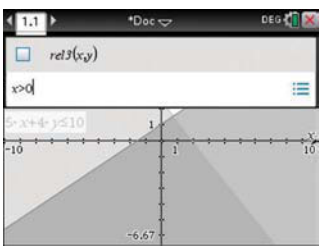


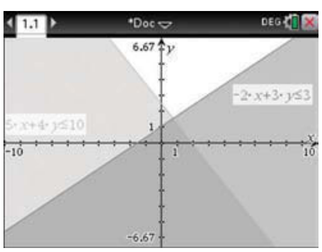
2. Press the e button, then complete the entry line as
$-2x + 3y \le 3$
and press ENTER.





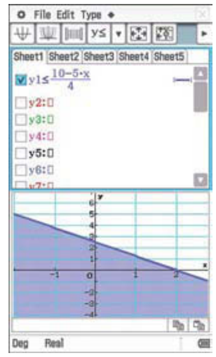
3. Press the e button, then complete the entry line as
$x > 0$
and press ENTER.



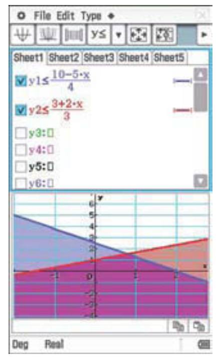


CASIO | THINK **WRITE**

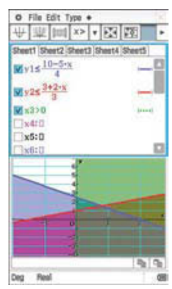
1. Rearrange the first equation to make y the subject:
$y \le \dfrac{10 - 5x}{4}$.
On a Graph & Table screen, select:
- Type
- Inequality
- y ≤ Type
Complete the entry line for y1 as
$y1 \le \dfrac{10 - 5x}{4}$
then click the tick box.
Click the $ icon.



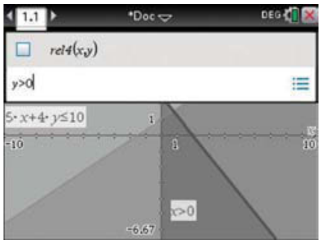
2. Rearrange the second equation to make y the subject:
$y \le \dfrac{3 + 2x}{3}$.
Complete the entry line for y2 as
$y2 \le \dfrac{3 + 2x}{3}$
then click the tick box.
Click the $ icon.

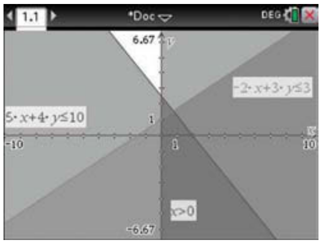


3. Select:
- Type
- Inequality
- x > Type
Complete the entry line for x3 as
x3 > 0
then click the tick box.
Click the $ icon.

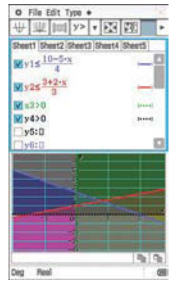


4. Press the e button, then complete the entry line as
$y > 0$
and press ENTER. The feasible region is the *darkest* region on the screen.

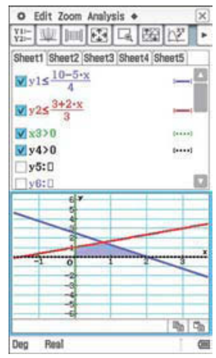




4. Select:
- Type
- Inequality
- y > Type
Complete the entry line for y4 as
$y4 > 0$
then click the tick box.
Click the $ icon.



5. Click the settings icon and select Graph Format. Underneath Inequality Plot, select Intersection, then select Set. The feasible region is shaded.



11.3.3 Identifying the constraints in a linear programming problem

For many linear programming problems, you won't be given the constraints as linear inequalities, and instead will need to identify them from the text of the problem.

To solve these types of problems, first define the variables using appropriate pronumerals, and then identify the key bits of information from the question necessary to write the constraints as linear inequalities.

WORKED EXAMPLE 6

The Cake Company makes two different types of cakes: a lemon sponge cake and a Black Forest cake. In order to meet demand, the Cake Company makes at least 40 batches of lemon sponge cakes a week and at least 16 batches of Black Forest cakes a week.

Every batch consists of 50 cakes, with a batch of lemon sponge cakes taking $1\frac{1}{2}$ hours to be made and a batch of Black Forest cakes taking 2 hours.

The equipment used to make the cakes can be used for a maximum of 144 hours a week.

a. Write the constraints of the problem as linear inequalities.
b. Sketch the solution to the problem (the feasible region).

THINK | **WRITE/DRAW**

a. 1. Define the variables.

a. Let s = number of sponge cake batches
Let b = number of Black Forest cake batches

2 Write the number of sponge cake batches as a constraint.

$s \geq 40$

3. Write the number of Black Forest cake batches as a constraint.

$b \geq 16$

4. Write the number of sponge and Black Forest cake batches that can be made in the given time as a constraint.

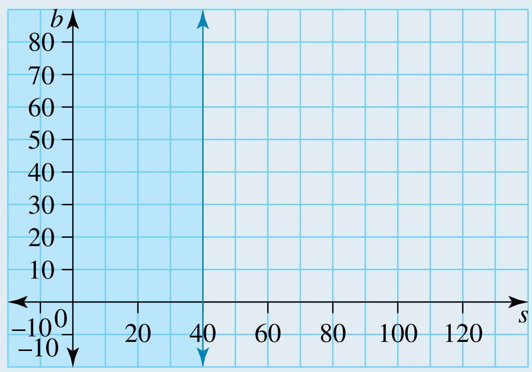
$1.5s + 2b \leq 144$

b. 1. Sketch the first constraint ($s \geq 40$) on a Cartesian plane. Add a legend for the required region.

b. Test point: (0, 0)

$s \geq 40$
$0 \geq 40$

The test point does not lie in the required region.



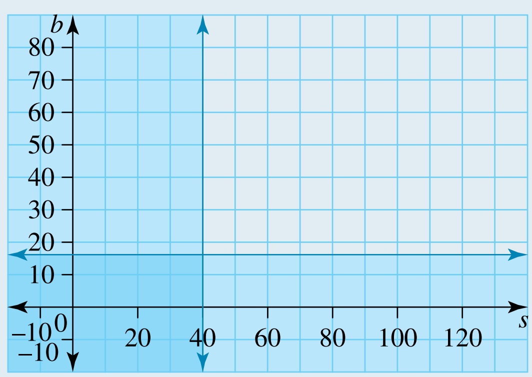
☐ Region required

2. Sketch the second constraint ($b \geq 16$) on the same Cartesian plane.

Test point: (0, 0)

$b \geq 40$
$0 \geq 16$

The test point does not lie in the required region.



☐ Region required

3. Sketch the third constraint ($1.5s + 2b \leq 144$) on the same Cartesian plane. The unshaded region is the solution to the problem.

When $s = 0$:

$$\begin{aligned} 1.5s + 2b &= 144 \\ 1.5 \times 0 + 2b &= 144 \\ 2b &= 144 \\ b &= 72 \end{aligned}$$

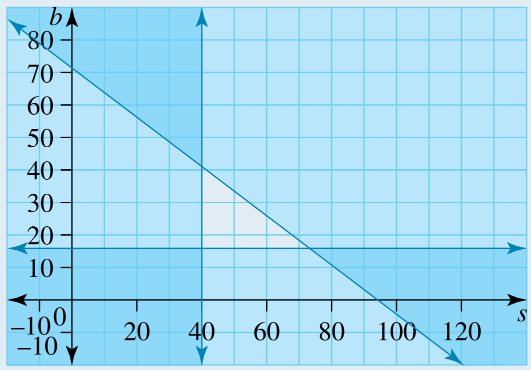
Intercept is at (0, 72).
When $b = 0$:

$$\begin{aligned} 1.5s + 2b &= 144 \\ 1.5s + 2 \times 0 &= 144 \\ 1.5s &= 144 \\ s &= 96 \end{aligned}$$

Intercept is at (0, 96).
Test point: (0, 0)

$$\begin{aligned} 1.5s + 2b &\leq 144 \\ 1.5 \times 0 + 2 \times 0 &\leq 144 \\ 0 &\leq 96 \end{aligned}$$

The test point lies in the required region.



☐ Region required

4. Interpret the graph.

The unshaded region in the graph above shows the range of batches that the Cake Company could make each week.

11.3.4 The objective function

The **objective function** is a function of the variables in a linear programming problem (e.g. cost and time). If we can find the maximum or minimum value of the function within the required region, that is, within the possible solutions to the problem, then we have found the **optimal solution** to the problem.

WORKED EXAMPLE 7

a. Emma owns a hobby store, and she makes a profit of \$3 every model car and \$6 for every model plane she sells. Write an equation to find her maximum profit (the objective function).

b. Domenic owns a fast food outlet with his two best-selling products being chips and onion rings. He buys 2 kg bags of chips for \$2 and 1 kg bags of onion rings for \$1.50. Write an equation to find his minimum cost (the objective function)

c. A stationery manufacturer makes two types of products: rulers and erasers. It costs the manufacturer \$0.10 to make the rulers and \$0.05 to make the erasers. The manufacturer sells its products to the distributors who buy the rulers for \$0.12 and the erasers for \$0.08. Write an equation to find the manufacturer's minimum cost and maximum profit (two objective functions).

THINK	WRITE
a. 1. Define the variables.	**a.** Let c = the number of model cars sold. Let p = the number of model planes sold.
2. Determine what is to be maximised or minimised.	Our objective is to maximise the profit.
3. Write the objective function.	The objective function is: Profit $= 3c + 6p$
b. 1. Define the variables.	**b.** Let c = the number of 2 kg bags of chips purchased Let r = the number of 1 kg bags of onion rings purchased
2. Determine what is to be maximised or minimised.	Our objective is to minimise the cost.
3. Write the objective function.	The objective function is: Cost $= 2c + 1.5r$
c. 1. Define the variables.	**c.** Let r = the number of rulers manufactured. Let e = the number of erasers manufactured.
2. Determine what is to be maximised or minimised.	Our two objectives are to minimise the cost and maximise the profit.
3. Write the objective functions. To find the profit we need to subtract the costs from the selling price.	Profit on ruler $= 0.12 - 0.10$ $= 0.02$ Profit on eraser $= 0.08 - 0.05$ $= 0.03$ The objective functions are: Cost $= 0.1r + 0.05e$ Profit $= 0.02r + 0.03e$

on Resources

Interactivity: Graphing simultaneous linear inequalities (int-6283)

studyon

Units 1 & 2 > AOS 5 > Topic 2 > Concepts 2 & 3

Linear programming Concept summary and practice questions

Objective function Concept summary and practice questions

Exercise 11.3 Linear programming

1. **WE4** Find the solution to the following simultaneous linear inequalities, leaving the required region unshaded.

$$5x + 3y \geq 15$$
$$x + 4y > 8$$

2. Find the solution to the following simultaneous linear inequalities, leaving the required region unshaded.

$$x + 3y < 9$$
$$2y - x > 6$$

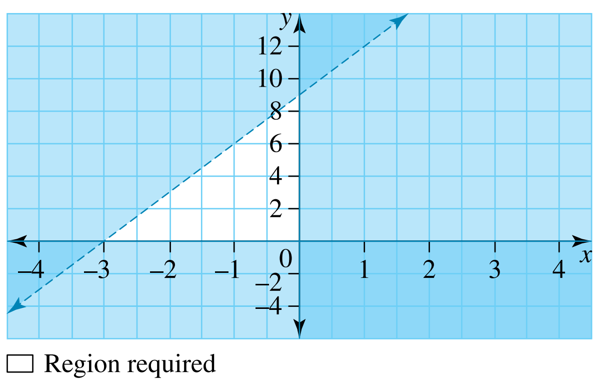
3. **MC** (8, 3) is a feasible solution for which of the following linear inequalities?
   - **A.** $4x + 2y < 12$
   - **B.** $3x - 2y < 16$
   - **C.** $3x - 7y > 10$
   - **D.** $3x + 5y > 1$
   - **E.** $2x + 2y > 28$
4. **MC** (−2, 10) is not a feasible solution for which of the following linear inequalities?
   - **A.** $x \leq 0$
   - **B.** $4x + 3y > 14$
   - **C.** $2y + 5x > 20$
   - **D.** $-3y + 6x < 4$
   - **E.** $y - 4x \geq 12$
5. **WE5** Sketch the feasible region for a linear programming problem with the following constraints.

$$6x + 2y \leq 120$$
$$x - 3y > 15$$
$$x > 0$$
$$y > 0$$

6. Sketch the feasible region for a linear programming problem with the following constraints.

$$36x - 10y \leq 540$$
$$2x + 3y < 900$$
$$x > 0$$
$$y > 0$$

7. **MC** Which of the following groups of constraints represent the feasible region shown?



A. $x \geq 0$, $y \leq 0$, $y > 3x + 9$

B. $x \geq 0$, $y \leq 0$, $y < 3x + 9$

C. $x \leq 0$, $y \geq 0$, $y < 3x + 9$

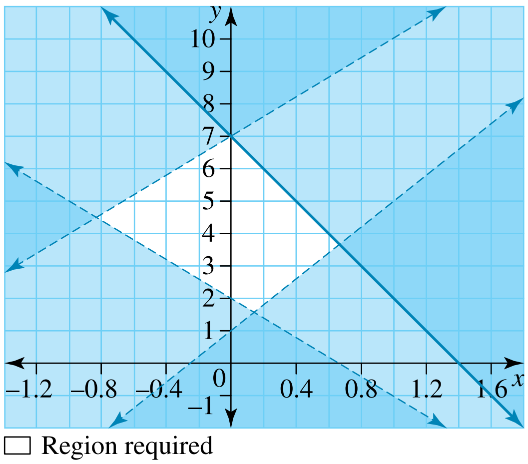
D. $x \leq 0$, $y \geq 0$, $y > 3x + 9$

E. $x \geq 0$, $y \geq 0$, $y > 3x + 9$

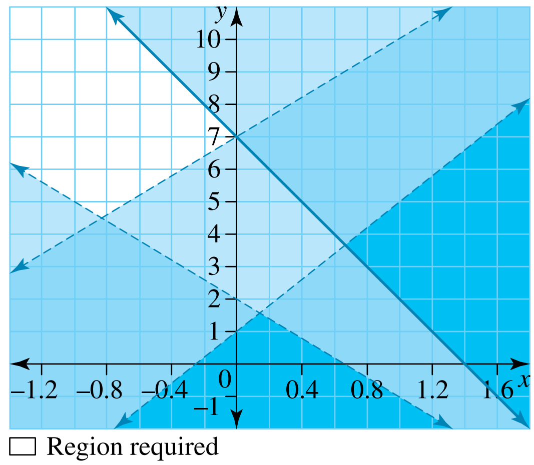
8. **MC** Which of the following graphs represents the feasible region for the listed constraints?

$$y > -3x + 2$$
$$y \leq -5x + 7$$
$$y > 4x + 1$$
$$y > 3x + 7$$

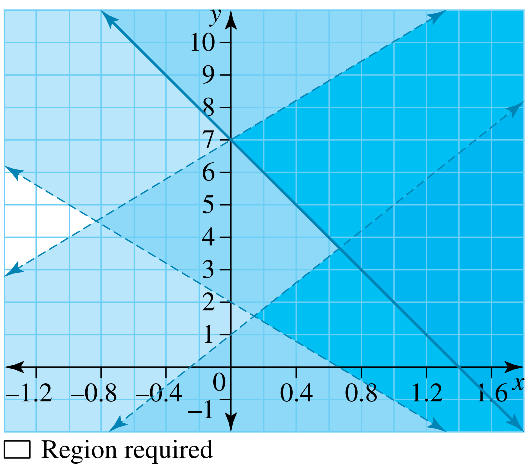
A.



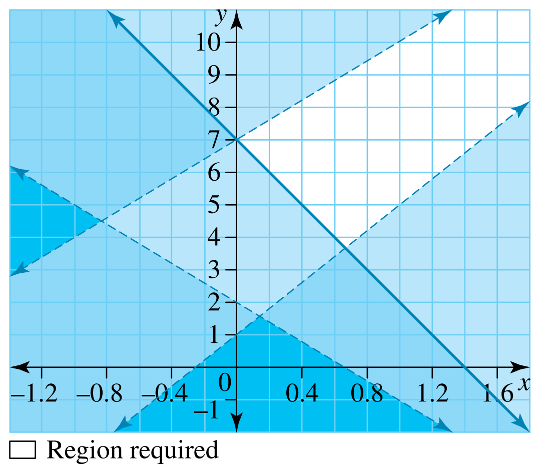
B.



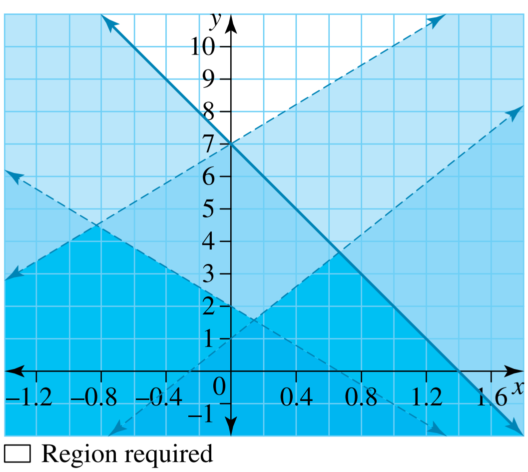
C.



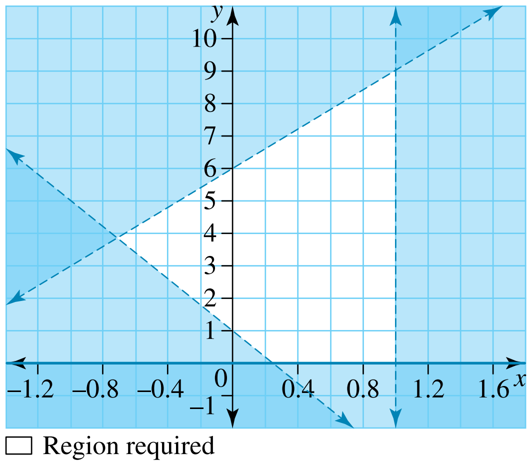
D.



E.



9. Identify the constraints that represent the following feasible region.



10. Sketch the feasible regions of the following sets of constraints.

a. $y > 5x + 4$
$x + y \le 50$
$x \le 0$
$y \ge 0$

b. $y > 3x + 4$
$y \le -4x + 10$
$x \ge 0$
$y \ge 0$

11. **WE6** The Biscuit Company makes two different types of biscuits: chocolate cookies and plain biscuits. In order to meet demand, the Biscuit Company makes at least 30 batches of chocolate cookies a week and at least 60 batches of plain biscuits a week.

Each batch consists of 1000 individual biscuits. One batch of chocolate cookies takes 1 hour to be made, and one batch of plain biscuits takes $\frac{1}{2}$ an hour. The equipment used to make the biscuits can be used for a maximum of 144 hours a week.

a. Write the constraints of the problem as linear inequalities.
b. Sketch the solution to the problem (the feasible region).

12. The Trinket Company makes two different types of trinkets: necklaces and bracelets. In order to meet demand, each week the Trinket Company makes at least 20 boxes of necklaces and at least 30 boxes of bracelets. A box of necklaces takes 2 hours to be made, while a box of bracelets takes 1.5 hours to be made. Each box contains 100 items.

The equipment used to make the trinkets can be used for a maximum of 100 hours a week.

a. Write the constraints of the problem as linear inequalities.
b. Sketch the solution to the problem (the feasible region).

13. Rocco is a vet who specialises in cats and dogs only. On any given day, Rocco can have a maximum of 45 appointments. He is booked for appointments to see at least 15 cats and at least 10 dogs each day.

a. Determine the constraints in the situation.
b. Sketch the feasible region for this problem.

14. **WE7** Samantha decides to sell items at the country fair. She makes a profit of $10 for every pair of

shoes she sells and \$6 for every hat she sells. Write an equation to find her maximum profit (the objective function).

15. Morris creates tables and chairs. It costs Morris \$20.50 to make a chair and \$50.25 to make a table. He sells these items to distributors, who buy the tables for \$70.00 and chairs for \$30.00. Write an equation to help find Morris's minimum cost and maximum profit (two objective functions).
16. Write the objective function for the following situations.
   a. Terri sells items of clothing and shoes. She makes a profit of \$12 for every piece of clothing she sells and \$15 for every pair of shoes she sells.
   b. Emily buys boxes of oranges at a cost of \$6.00 and boxes of avocados at a cost of \$15.00 for her fruit shop.
   c. A manufacturing company makes light globes. Small light globes sell for \$3.00 but cost \$0.30 to make; large light globes sell for \$5.00 but cost \$0.45 to make. (two objective functions)
17. A service station sells regular petrol and ethanol blended petrol. Each day the service station sells at least 9500 litres of regular petrol and at least 4500 litres of ethanol blended petrol. In total, a maximum of 30 000 litres of petrol is sold on any given day.

   Let R = the number of litres of regular petrol sold and E = the number of litres of ethanol blended petrol sold.
   a. Identify all of the constraints related to this problem.
   b. Sketch the feasible region for this problem.

18. Dan is a doctor who specialises in knee surgery. On any given day, Dan can perform arthroscopies or knee reconstructions. He can perform a maximum of 40 surgeries a week. He is booked weekly to perform at least 5 arthroscopies and at least 7 knee reconstructions.

   Let A = the number of arthroscopies Dan performs and R = the number of knee reconstructions Dan performs.
   a. Identify all of the constraints related to this problem.
   b. Sketch the feasible region for this problem.
19. Helen buys and sells second-hand fridges and televisions. She buys fridges at a cost of \$40 and then sells them for \$120. She buys televisions at a cost of \$20 and then sells them for \$55.

   Let F = the number of fridges sold and T = the number of televisions sold.
   a. Write the objective function for the maximum profit Helen makes.
   b. Helen buys at least 15 fridges and 30 televisions in a year, with a maximum of 120 items bought in total. Sketch the feasible region for this problem.
20. Anna manufactures cutlery, specialising in dessert and coffee spoons. It costs Anna \$2.50 to make a dessert spoon and \$1.25 to make a coffee spoon. She sells these items to distributors, who buy the dessert spoons for \$3.50 and the coffee spoons for \$2.00.

   Let D = the number of dessert spoons made and C = the number of coffee spoons made.
   a. Write the objective function for the maximum profit Anna makes.
   b. Anna manufactures at least twice as many coffee spoons as dessert spoons, and makes less than 1000 spoons each week. Sketch the feasible region for this problem.

11.4 Applications of linear programming

11.4.1 The corner point principle

After we have found the feasible region for a linear programming problem, all of the points within the feasible region satisfy the objective function.

The **corner point principle** states that the maximum or minimum value of the objective function must lie at one of the corners (vertices) of the feasible region. So if we place all of the corner coordinates into the objective function, we can determine the solution to the problem.

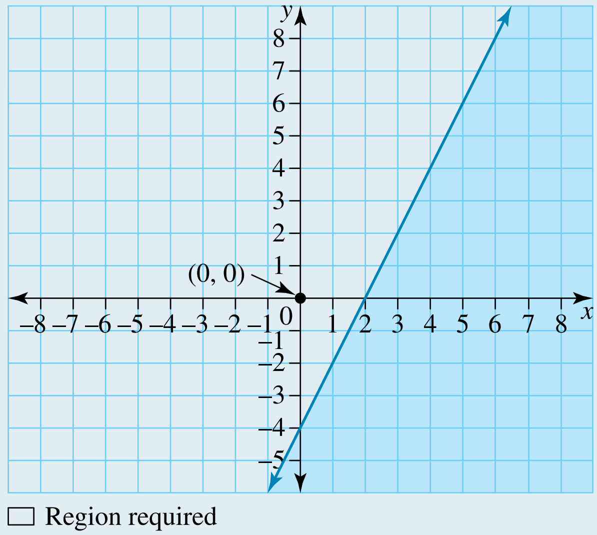
WORKED EXAMPLE 8

a. **Sketch the feasible region of a linear programming problem with the following constraints.**

$$\begin{aligned} y - 2x &\geq -4 \\ 6x + y &\geq 6 \\ y + x &\leq 7 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

b. **Use the corner point principle to determine the maximum and minimum solution of the objective function $G = 4x + 2y$.**

THINK	WRITE/DRAW
a. 1. Sketch the inequalities individually, starting with $y - 2x \geq -4$. For sketching purposes, replace the inequality sign with an equals sign ($y - 2x = -4$). Use the x-intercept and y-intercept method to sketch the graph, and use a test point to determine the required region.	a. When $y = 0$: $y - 2x = -4$ $0 - 2x = -4$ $-2x = -4$ $x = 2$ The x-intercept is (2, 0). When $x = 0$: $y - 2x = -4$ $y - 2 \times 0 = -4$ $y = -4$ The y-intercept is (0, −4). As the inequality sign is $\geq$, the line will be solid (meaning the line is included). Test point: (0, 0) $y - 2x \geq -4$ $0 - 2 \times 0 \geq -4$ $0 \geq -4$ $0 \geq -4$ is true, so the test point is in the required region and we shade this region.



2. Sketch $6x + y \geq 6$ on the same Cartesian plane.

When $y = 0$:

$$6x + y = 6$$
$$6x + 0 = 6$$
$$x = 1$$

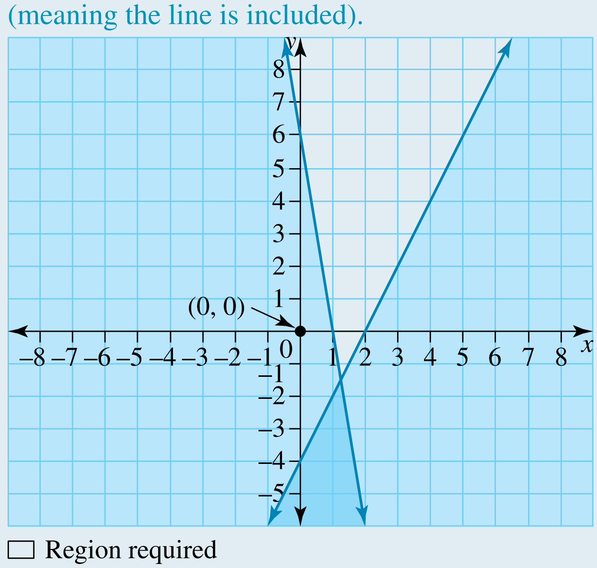
The x-intercept is (1, 0).

When $x = 0$:

$$6x + y = 6$$
$$6 \times 0 + y = 6$$
$$y = 6$$

The y-intercept is (0, 6).

As the inequality sign is $\geq$, the line will be solid (meaning the line is included).



Test point: (0, 0)

$$6x + y \geq 6$$
$$6 \times 0 + 0 \geq 6$$
$$0 \geq 6$$

$0 \geq 6$ is false, so the test point is not in the required region and we shade this region.

3. Sketch $y + x \leq 7$ on the same Cartesian plane.

When $y = 0$:

$$y + x = 7$$
$$0 + x = 7$$
$$x = 7$$

The x-intercept is $(7, 0)$.

When $x = 0$:

$$y + x = 7$$
$$y + 0 = 7$$
$$y = 7$$

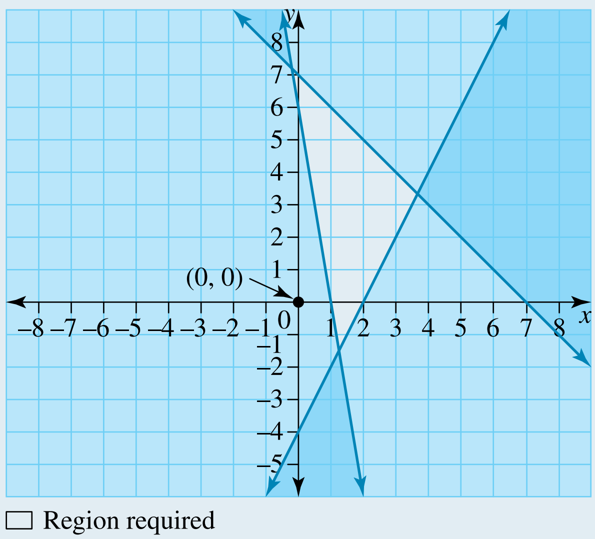
The y-intercept is $(0, 7)$.

As the inequality sign is $\leq$, the line will be solid (meaning the line is included).

Test point: $(0, 0)$

$$y + x \leq 7$$
$$0 + 0 \leq 7$$
$$0 \leq 7$$

$0 \leq 7$ is true, so the test point is in the required region and we shade the other region.

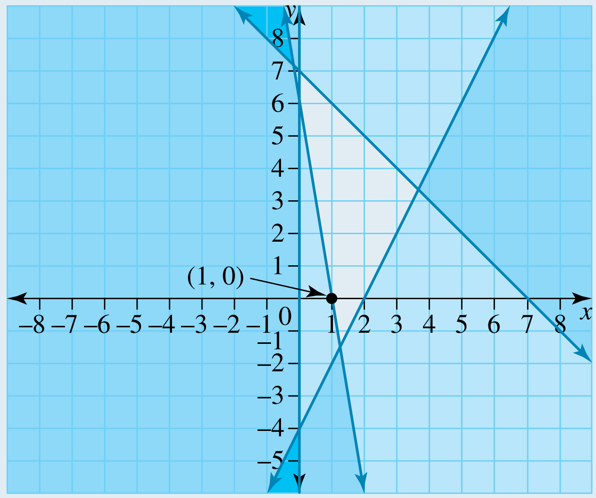


4. Sketch $x \geq 0$ on the same Cartesian plane.

As the inequality sign is $\geq$, the line will be solid (meaning the line is included).

Test point: $(1, 0)$

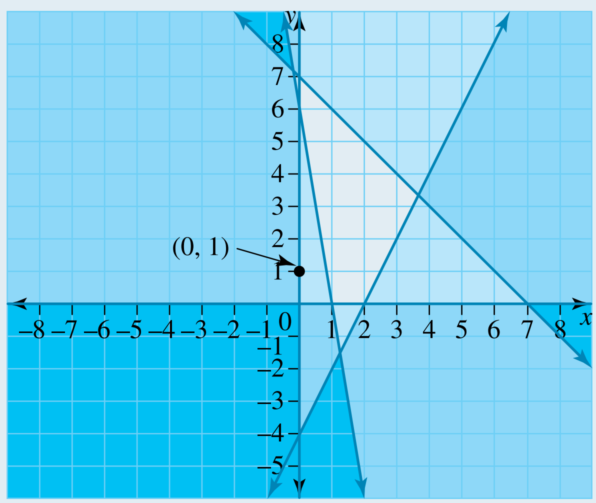
$1 \geq 0$ is true, so the test point is in the required region and we shade the other region.



☐ Region required

5. Sketch $y \geq 0$ on the same Cartesian plane.

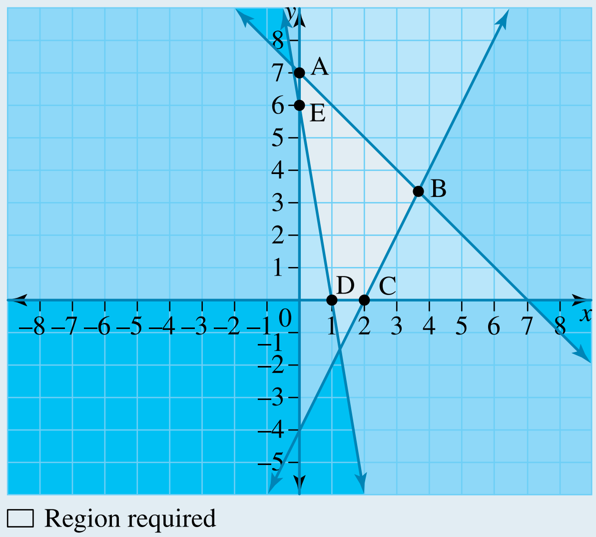
As the inequality sign is $\geq$, the line will be solid (meaning the line is included).
Test point: $(0, 1)$
$1 \geq 0$ is true, so the test point is in the required region and we shade the other region.



☐ Region required

b. 1. Label the vertices of the required region. List the values of the vertices that are easily identifiable.

b.



A = (0, 7) B = (?, ?) C = (2, 0) D = (1, 0) E = (0, 6)

2. Calculate the coordinates of the remaining points. This is done by solving the simultaneous equations where these points meet.

We need to solve for B:

$y - 2x = -4[1]$

$y + x = 7[2]$

[1] − [2]:

$-3x = -11$

$x = \frac{11}{3}$

Substitute $x = \frac{11}{3}$ into [2]:

$y + \frac{11}{3} = 7$

$$\begin{aligned} y &= 7 - \frac{11}{3} \\ &= \frac{21}{3} - \frac{11}{3} \\ &= \frac{10}{3} \end{aligned}$$

$B = \left(\frac{11}{3}, \frac{10}{3}\right)$

3. Calculate the value of the objective function $G = 4x + 2y$ at each of the corners.

At A (0, 7):

$$\begin{aligned} G &= 4x + 2y \\ &= 4 \times 0 + 2 \times 7 \\ &= 14 \end{aligned}$$

At B $\left(\frac{11}{3}, \frac{10}{3}\right)$:

$$\begin{aligned} G &= 4x + 2y \\ &= 4 \times \frac{11}{3} + 2 \times \frac{10}{3} \\ &= 21\frac{1}{3} \end{aligned}$$

At C (2, 0):

$$\begin{aligned} G &= 4x + 2y \\ &= 4 \times 2 + 2 \times 0 \\ &= 8 \end{aligned}$$

At D (1, 0):

$$\begin{aligned} G &= 4x + 2y \\ &= 4 \times 1 + 2 \times 0 \\ &= 4 \end{aligned}$$

At E (0, 6):

$G = 4x + 2y$

$= 4 \times 0 + 2 \times 6$

$= 12$

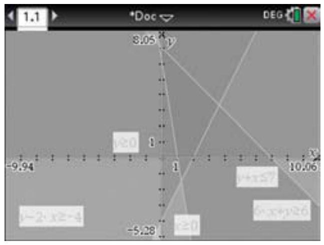
4. State the answer.

The maximum value of G is $21\frac{1}{3}$ at B $\left(\frac{11}{3}, \frac{10}{3}\right)$.

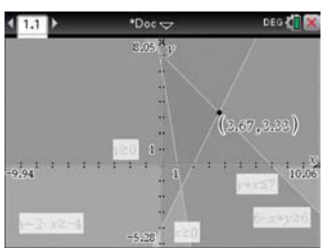
The minimum value of G is 4 at D $(1, 0)$.

TI | THINK — **WRITE**

a. 1. On a Graphs page, press MENU then select:
3: Graph Entry/Edit
2: Relation
Complete the entry lines for relations 1 to 5 as
$y - 2x \geq -4$
$6x + y \geq 6$
$y + x \leq 7$
$x \geq 0$
$y \geq 0$
then press ENTER.
The feasible region is the *darkest* region on the screen.

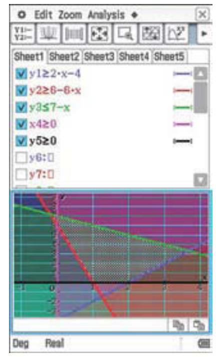


b. 1. To find the intersection of the boundaries of regions 1 and 3, press MENU then select:
6: Analyze Graph
4: Intersection
Click on the boundary of region 1 and the boundary of region 3. When prompted for the lower bound, click to the left of the point of intersection and when prompted for the upper bound click to the right of the point of intersection. Press ENTER to mark the point of intersection between boundaries 1 and 3 on the graph.

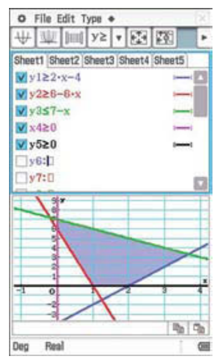


CASIO | THINK — **WRITE**

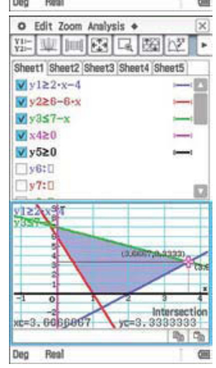
a. 1. Rearrange all inequations to make either x or y the subject.
On a Graph & Table screen, select:
- Type
- Inequality
and select the appropriate option.
Complete the entry lines for relations 1 to 5 as
$y1 \geq 2x - 4$
$y2 \geq 6 - 6x$
$y3 \leq 7 - x$
$x4 \geq 0$
$y5 \geq 0$
then click the tick boxes for these relations.
Click the $ icon.



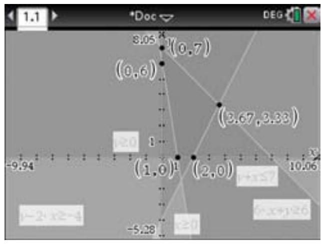
2. Click the settings icon and select Graph Format. Underneath Inequality Plot, select Intersection, then select Set. The feasible region is shaded.



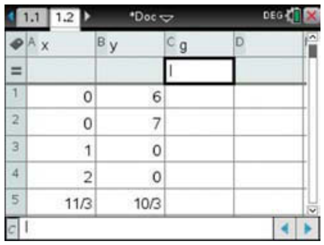
b. 1. To find the intersection of the boundaries of regions 1 and 3, select:
- Analysis
- G-Solve
- Intersection
Select the boundary of region 1 and the boundary of region 3, then press EXE.



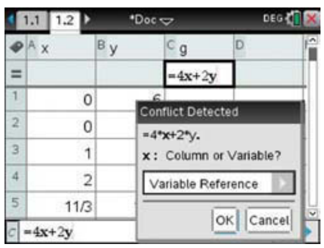
2. Repeat this process to find the points of intersection at each corner of the feasible region. *Note:* For intersections with the boundary of region 4, click above and to the left of the point of intersection when prompted for the 1st corner, then below and to the right of the point of intersection when prompted for the 2nd corner.



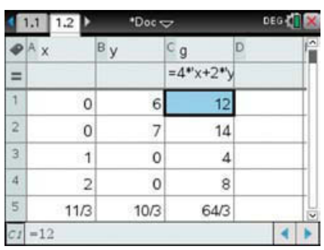
3. In a Lists & Spreadsheet page, label the first column as x and the second column as y. Enter the x-coordinates of the corner points in the first column and the y-coordinates in the second column.



4. In the function cell underneath the label g, complete the entry line as:
$= 4x + 2y$
then select the variable reference for x and y when prompted.

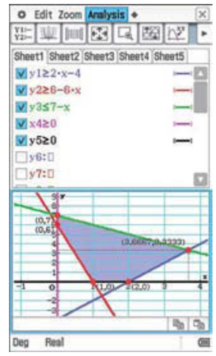


5. Read the maximum and minimum values of G from the table.

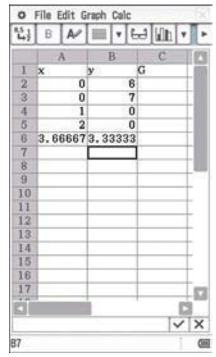


The maximum value of G is $21\frac{1}{3}$ at the point $\left(\frac{11}{3}, \frac{10}{3}\right)$.
The minimum value of G is 4 at the point $(1, 0)$.

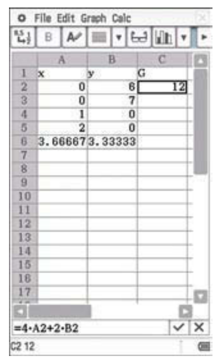
2. Repeat this process to find the point of intersection of boundaries 1 and 5 and the point of intersection of boundaries 2 and 5.
Select:
- Analysis
- Trace
to find the points of intersection with the boundary of region 4.



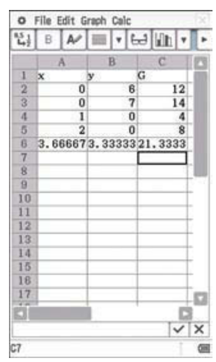
3. On a Spreadsheet screen, enter the label x in cell A1, y in B1 and G in C1.
Enter the x-coordinates of the corner points in cells A2 to A6 and the y-coordinates in cells B2 to B6.



4. In cell C2, complete the entry line as
$= 4A2 + 2B2$
then press EXE.



5. Copy cell C2 and paste into cells C3 to C6.
Read the maximum and minimum values of G from the table.

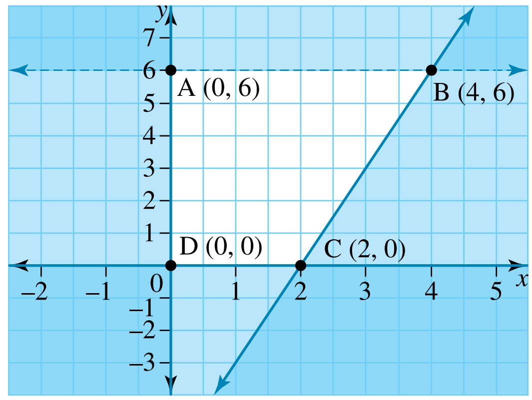


The maximum value of G is $21\frac{1}{3}$ at the point $\left(\frac{11}{3}, \frac{10}{3}\right)$.
The minimum value of G is 4 at the point $(1, 0)$.

11.4.2 The sliding-line method

After we have found the corner points of the feasible region, we can also use the **sliding-line method** to find the optimal solution(s) to our linear programming problem.

To use the sliding-line method you need to graph the objective function that you want to maximise (or minimise). As the objective function will not be in the form $y = mx + c$, we first need to transpose it into that form. For example, if the objective function was $F = 2x + y$, we would transpose this into the form $y = 2x - F$. This line has a fixed gradient, but not a fixed y-intercept.



Region required

If we slide this line up (by adjusting the value of F) to meet the last point the line touches in the feasible region, then this point is the maximum value of the function.

Similarly if we slide this line down (by adjusting the value of F) to meet the last point the line touches in the feasible region, then this point is the minimum value of the function.

The following graph shows a feasible region with corners A (0, 6), B (4, 6), C (2, 0) and D (0, 0).

If the objective function was $T = -x + y$, we would transpose this equation to make y the subject:

$y = x - T$.

This equation has a gradient of 1 and a y-intercept of $-T$.

If we plot this graph for different values of T, we get a series of parallel graphs (same gradient).

To find the maximum value using the objective function, we slide the line $y = x - T$ to the highest vertex, in this case A.

To find the minimum value using the objective function, we slide the line $y = x - T$ to the lowest vertex, in this case D.

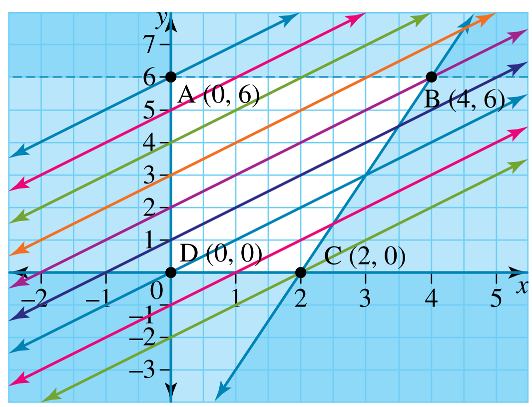
We can check that these vertices give the maximum and minimum values by calculating $T = -x + y$ at each of the vertices:

At A (0, 6):
$T = -0 + 6$
$= 6$

At B (4, 6):
$T = -4 + 6$
$= 2$

At C (0, 0):
$T = 0 + 0$
$= 0$

At D (2, 0):
$T = -2 + 0$
$= -2$



Region required

WORKED EXAMPLE 9

a. Sketch the feasible region of a linear programming problem with the following constraints.

$$\begin{aligned} 2y + x &\geq 14 \\ y - 3x &\geq 9 \\ y + x &\leq 15 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

b. Use the sliding-rule method to determine the maximum and minimum solution of the objective function $P = 10y - 4x$.

THINK

a. 1. Sketch the inequalities individually, starting with $2y + x \geq 14$. For sketching purposes, replace the inequality sign with an equals sign ($2y + x = 14$). Use the x-intercept and y-intercept method to sketch the graph, and use a test point to determine the required region.

WRITE/DRAW

a. When $y = 0$:

$$2y + x = 14$$
$$2 \times 0 + x = 14$$
$$x = 14$$

The x-intercept is (14, 0).

When $x = 0$:

$$2y + x = 14$$
$$2y + 0 = 14$$
$$2y = 14$$
$$y = 7$$

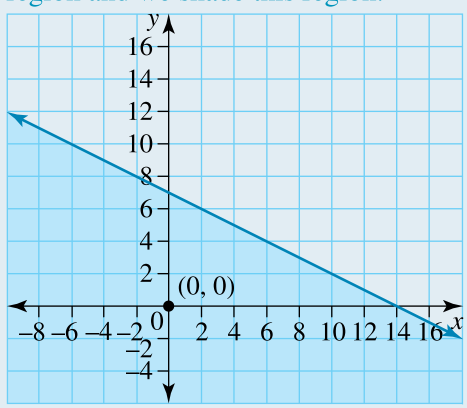
The y-intercept is (0, 7).

As the inequality sign is $\geq$, the line will be solid (meaning the line is included).

Test point: (0, 0)

$$2y + x \geq 14$$
$$2 \times 0 + 0 \geq 14$$
$$0 \geq 14$$

$0 \geq 14$ is false, so the test point is not in the required region and we shade this region.



☐ Region required

2. Sketch $y - 3x \geq 9$ on the same Cartesian plane.

When $y = 0$:

$$y - 3x = 9$$
$$0 - 3x = 9$$
$$x = -3$$

The x-intercept is $(-3, 0)$.

When $x = 0$:

$$y - 3x = 9$$
$$y - 3 \times 0 = 9$$
$$y = 9$$

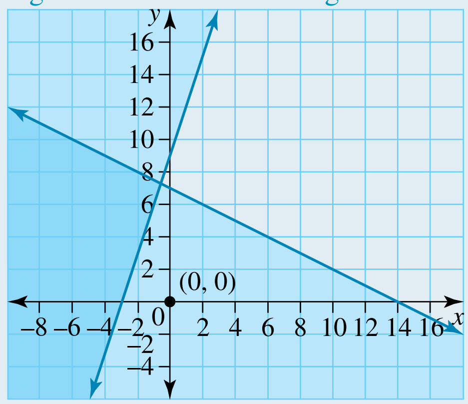
The y-intercept is (0, 9).

As the inequality sign is $\geq$, the line will be solid (meaning the line is included).

Test point: (0, 0)

$$y - 3x \geq 9$$
$$0 - 3 \times 0 \geq 9$$
$$0 \geq 9$$

$0 \geq 9$ is false, so the test point is not in the required region and we shade this region.



☐ Region required

3. Sketch $y + x \leq 15$ on the same Cartesian plane.

When $y = 0$:

$$y + x = 15$$
$$0 + x = 15$$
$$x = 15$$

The x-intercept is (15, 0).

When $x = 0$:

$$y + x = 15$$
$$y + 0 = 15$$
$$y = 15$$

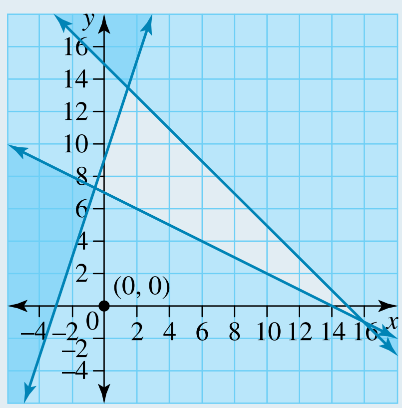
The y-intercept is (0, 15).

As the inequality sign is $\leq$, the line will be solid (meaning the line is included).

Test point: (0, 0)

$$y + x \leq 15$$
$$0 + 0 \leq 15$$
$$0 \leq 15$$

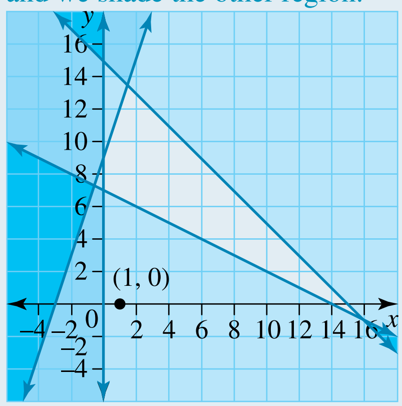
$0 \leq 15$ is true, so the test point is in the required region and we shade the other region.



☐ Region required

4. Sketch $x \geq 0$ on the same Cartesian plane.

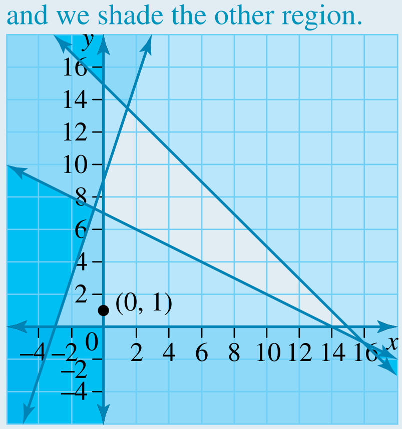
As the inequality sign is $\geq$, the line will be solid (meaning the line is included).
Test point: (1, 0)
$1 \geq 0$ is true, so the test point is in the required region and we shade the other region.



☐ Region required

5. Sketch $y \geq 0$ on the same Cartesian plane.

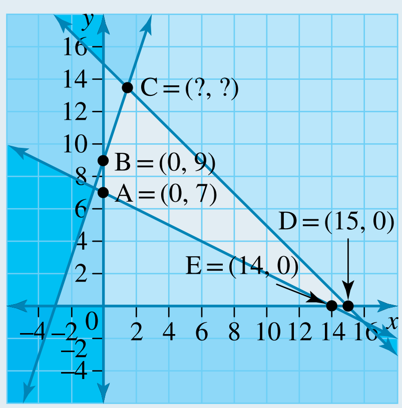
As the inequality sign is $\geq$, the line will be solid (meaning the line is included).
Test point: (0, 1)
$1 \geq 0$ is true, so the test point is in the required region and we shade the other region.



☐ Region required

b. 1. Label the vertices of the required region. List the values of the vertices that are easily identifiable.

b.



☐ Region required

A = (0, 7) B = (9, 9) C = (?, ?) D = (15, 0) E = (14, 0)

2. Calculate the coordinates of the remaining points. This is done by solving the simultaneous equations where these points meet.

We need to solve for C:
$y - 3x = 9$ [1]
$x + y = 15$ [2]
[2] − [1]:
$4x = 6$
$x = 1.5$
Substitute $x = 1.5$ into [2]:
$1.5 + y = 15$
$y = 15 - 1.5$
$= 13.5$
$\text{C} = (1.5,\ 13.5)$

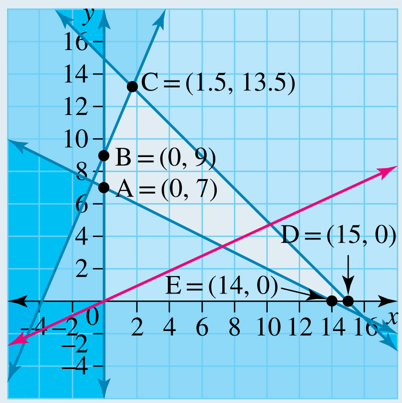
3. Transpose the objective function to make y the subject.

$P = 10y - 4x$
$P + 4x = 10y$
$y = 0.1P + 0.4x$

4. Choose a value for P and calculate the equation of the line.

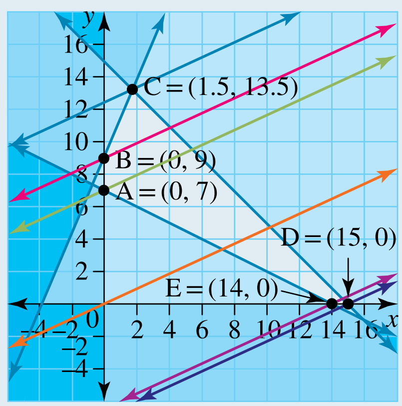
When $P = 0$:
$y = 0.1P + 0.4x$
$= 0.1 \times 0 + 0.4x$
$= 0.4x$

5. Draw this line on the graph of the feasible region of the problem.



☐ Region required

6. Slide this line up and down by drawing parallel lines that meet the vertices of the feasible region.



☐ Region required

7. The maximum value of the objective function is at the vertex that meets the highest parallel line. The minimum value of the objective function is at the vertex that meets the lowest parallel line.

The maximum solution is at C (1.5, 13.5).
The minimum solution is at D (15, 0).

8. Calculate the maximum and minimum values of the objective function.

At C (1.5, 13.5):
$P = 10y - 4x$
$= 10 \times 13.5 - 4 \times 1.5$
$= 135 - 6$
$= 129$
At D (15, 0):
$P = 10y - 4x$
$= 10 \times 0 - 4 \times 15$
$= 0 - 60$
$= -60$

9. State the answer.

The maximum value of P is 129 at C (1.5, 13.5).
The minimum value of P is −60 at D (15, 0).

11.4.3 Solving linear programming problems

There are seven steps we need to take to formulate and solve a linear programming problem.

1. Define the variables.
2. Find the constraints.
3. Find the objective function.
4. Sketch the constraints.
5. Find the coordinates of the vertices of the feasible region.
6. Use the corner point principle or sliding-line method.
7. Find the optimal solution.

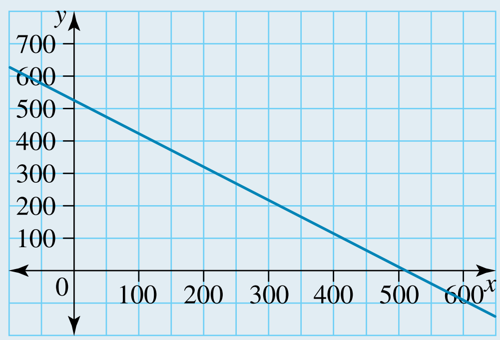
WORKED EXAMPLE 10

Jennifer and Michael's company sells shirts and jeans to suppliers. From previous experience, the company can sell a maximum of 530 items per day. They have a minimum order of 50 shirts and 100 jeans per day. It costs the company $20 to buy a shirt and $30 for a pair of jeans, while they sell each shirt for $35 and each pair of jeans for $55. How many of each item should be sold to make the greatest profit?

THINK	WRITE
1. Define the variables.	x = the number of shirts y = the number of jeans
2. Determine the constraints.	$x + y \leq 530$ $x \geq 50$ $y \geq 100$
3. Determine the objective function.	$P = 15x + 25y$

4. Sketch the constraints to find the feasible region.

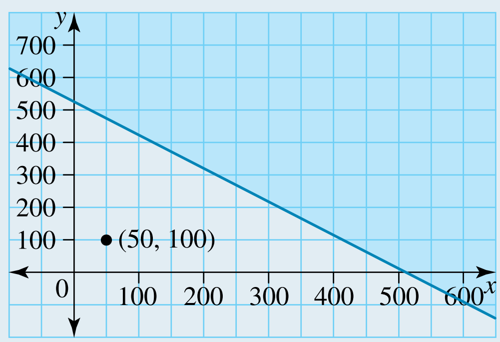
$x + y \leq 530$



Test point: (50, 100)

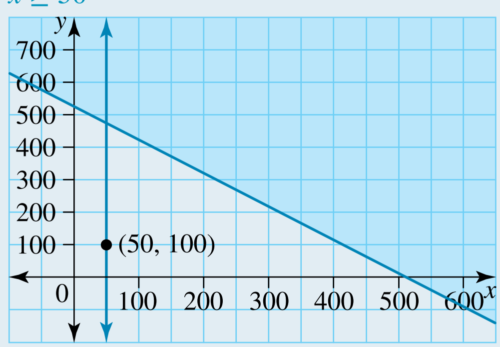
$$\begin{aligned} x + y &\leq 530 \\ 50 + 100 &\leq 530 \\ 150 &\leq 530 \end{aligned}$$

$150 \leq 530$ is true, so the test point is in the required region and we shade the other region.



Region required

$x \geq 50$

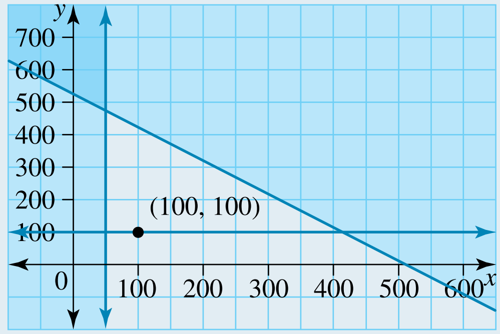


Region required

Test point: (100, 100)

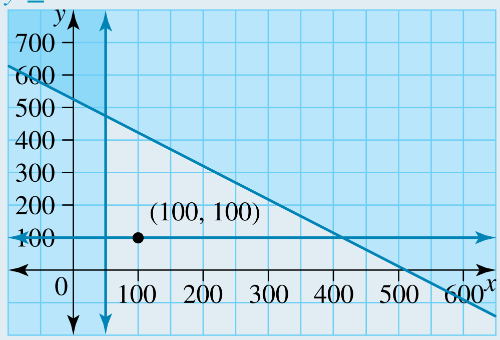
$$\begin{aligned} x &\geq 50 \\ 100 &\geq 50 \end{aligned}$$

$100 \geq 50$ is true, so the test point is in the required region and we shade the other region.



☐ Region required

$y \geq 100$



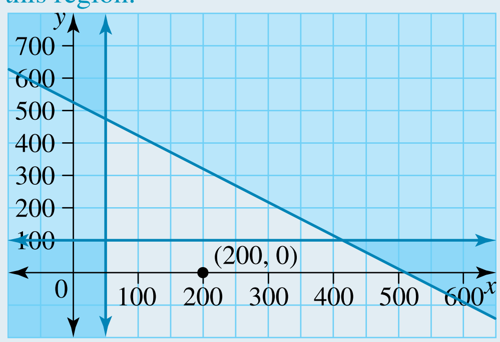
☐ Region required

Test point: (200, 0)

$y \geq 100$

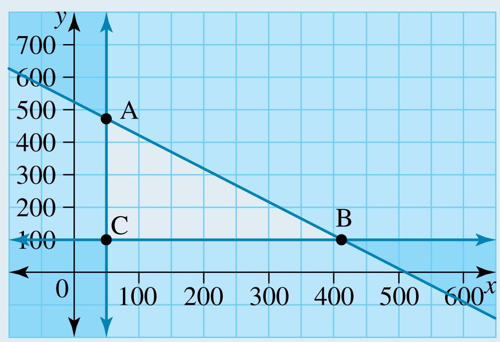
$0 \geq 100$

$0 \geq 100$ is false, so the test point is not in the required region and we shade this region.



☐ Region required

5. Label the vertices of the feasible region and find the coordinates of these vertices.
Start with finding the coordinates of A.



☐ Region required

At A, the following two lines meet:

$x + y = 530$

$x = 50$

So $x = 50$

Substitute $x = 50$ into the first equation:

$x + y = 530$

$50 + y = 530$

$y = 480$

A = (50, 480)

6. Find the coordinates of B.

At B, the following two lines meet:

$x + y = 530$

$y = 100$

So $y = 100$

Substitute $y = 100$ into the first equation:

$x + y = 530$

$x + 100 = 530$

$x = 430$

B = (430, 100)

7. Find the coordinates of C.

At C, the following two lines meet:

$x = 50$

$y = 100$

So C = (50, 100)

8. Use the corner point principle.

	$P = 15x + 25y$
A(50, 480)	$P = 15 \times 50 + 25 \times 480$ $= \$12\ 750$
B(430, 100)	$P = 15 \times 430 + 25 \times 100$ $= \$8950$
C(50, 100)	$P = 15 \times 50 + 25 \times 100$ $= \$3250$

9. Find the optimal solution and write the answer.

The optimal solution occurs at point A(50, 480). Therefore 50 shirts and 480 jeans should be sold to make the maximum profit.

on Resources

Interactivity: Linear programming: corner point method (int-6282)

Exercise 11.4 Applications of linear programming

1. WE8 **a.** Sketch the feasible region of a linear programming problem with the following constraints.

$$\begin{aligned} 2y + 6x &\le 60 \\ 3y - 7x &\le 42 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

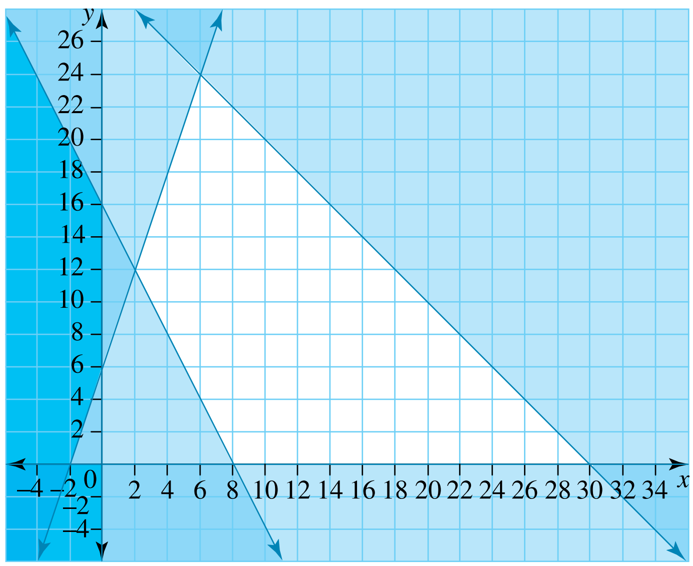
b. Use the corner point principle to determine the maximum and minimum solution of $P = 3x + 2y$.

2. **a.** Sketch the feasible region of a linear programming problem with the following constraints.

$$\begin{aligned} x + 2y &\ge -4 \\ 4x - y &\ge -2 \\ x &\le 6 \\ y &\ge 10 \end{aligned}$$

b. Use the corner point principle to determine the maximum and minimum solution of $M = 30x - 7y$.

3. The following graph shows the feasible region for a linear programming problem. Determine the minimum and maximum values for the objective function $S = 6x + 5y$.



☐ Region required

4. WE9 **a.** Sketch the feasible region of a linear programming problem with the following constraints.

$$\begin{aligned} y - x &\le 20 \\ x + y &\le 60 \\ x &\ge 10 \\ y &\ge 10 \end{aligned}$$

b. Use the sliding-rule method to determine the maximum and minimum solution of $L = 8y + 12x$.

5. **a.** Sketch the feasible region of a linear programming problem with the following constraints.

$$\begin{aligned} 2x + y &\le 12 \\ -2x + y &\le 10 \\ y + x &\ge 5 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

b. Use the sliding-rule method to determine the maximum and minimum solution of $S = 6y - 2x$.

6. WE10 Gabe and Kim work on their hobby farm. They can have a maximum of 300 animals on their farm due to council regulations. They buy calves at $450 and lambs at $80, which they then sell when they have matured for $1200 per cow and $120 per sheep. They have a minimum order of 35 cows and 50 sheep from their local animal market seller per month.
How many of each should be sold to make the greatest profit?

7. Brian's company sells mobile phones and laptops to suppliers. From previous experience, the company can sell a maximum of 350 items a day. They have a minimum order of 20 mobile phones and 40 laptops per day. It costs the company $30 to buy a mobile phone and $50 to buy a laptop, and the company sells each mobile phone for $50 and each laptop for $80. How many of each item should be sold to make the greatest profit?

8. The Fresh Food Grocery is trying to determine how to maximise the profit they can make from selling apples and pears. They have a limited amount of space, so they can only have 120 pieces of fruit in total, and the supplier's contract states that they must always have 15 pears and 20 apples in stock and no more than 80 pieces of either type of fruit. They make a profit of $0.25 for each apple and $0.20 profit for each pear.
   a. Determine how many apples and pears they should have to maximise their profit.
   b. Determine the maximum profit.

9. Beach Side Resorts are expanding their operations and have bought a new site on which to build chalets and apartments. They want each of their sites to have a minimum of 2 chalets and 5 apartments, and no more than 18 accommodation options in total.
Each chalet takes up 150 m^2 in ground space; each apartment takes up 100 m^2; and their new site has 2300 m^2 of ground space in total.
If they can rent the chalets for $335 a night and the apartments for $205 a night, determine the maximum weekly profit they can make from their new site.

10. A hairdresser offers both quick haircuts and stylised haircuts. The quick haircuts take an average of 15 minutes each and the stylised haircuts take an average of 36 minutes each. The hairdresser likes to do at least 4 quick haircuts and 3 stylised haircuts in any given day, and no more than 21 haircuts in a day. If the hairdresser makes $21 on each quick haircut and $48 on each stylised haircut, and works for 7 hours in a day:
   a. draw the feasible region that represents this problem
   b. determine how many of each type of haircut the hairdresser should do to maximise his daily income
   c. determine his maximum daily income.

11. A new airline company is trying to determine the layout of the cabins on their new planes. They have two different types of tickets: economy and business class. Each economy seat requires $0.8\,\text{m}^2$ of cabin space and each business class seat requires $1.2\,\text{m}^2$ of cabin space. The airline can make a profit of \$55 on each economy ticket and \$185 on each business class seat, although regulations state that they must have at least 6 times more economy seats than business class seats, as well as at least 8 business class seats.
If there is a total of $240\,\text{m}^2$ of cabin space for seating:
a. draw the feasible region that represents this problem
b. determine how many of each type of seat the airline should install to maximise their profit
c. determine the maximum profit for each flight.

12. Rubio runs an app development company that creates both simple and complex apps for other companies. His company can make a maximum of 25 simple apps and 15 complex apps in one week, but cannot make more than 30 apps in total. It takes 4 hours to make each simple app and 6 hours to make each complex app, and the company can put a maximum of 138 hours per week towards app development. If the company makes \$120 profit for each simple app and \$200 profit for each complex app, how many of each should they aim to make to maximise their weekly profit?

13. A school is planning an excursion for all of their students to see the penguins on Phillip Island. They find a company who can provide two different types of coaches for the trip: one that holds 34 passengers and one that holds 51 passengers. The coach company has 8 of the smaller coaches and 4 larger coaches available. The hire cost \$500 for a smaller coach and \$700 for a larger coach. If there are 374 students and teachers going on the trip, determine the minimum total cost for the coach hire.

14. The feasible region for a linear programming problem is defined by the following constraints:
$x + y \leq 850$
$x \geq 25$
$y \geq 35$
There are two objective functions for the problem:
Objective $A = 5.5x + 7y$
Objective $B = 8x + 5y$
a. Which objective function has the greater maximum value?
b. Which objective function has the smaller minimum value?

15. Swish Phone Cases make two different covers for the latest iPhone. It takes them 24 minutes to manufacture the parts for case A and 36 minutes to manufacture the parts for case B. There is a total of 10 hours available for manufacturing the cases each week.
It takes a further 24 minutes to assemble case A and 48 minutes to assemble case B. There is a total of 12 hours available for assembling each week.
Case A retails for \$59 and case B retails for \$89.
a. What are the coordinates of the vertices of the feasible region in this problem?
b. How many of case A and case B should be made each week to maximise profit?
c. If the prices of case A and case B were swapped, would this affect your answer to part **b**?

16. Superb Desserts makes two different types of chocolate cake, as shown in the following table:

Cake	Sugar (g)	Chocolate (g)	Butter (g)
Chocolate ripple	300	120	80
Death by chocolate	200	360	100

The total amount of sugar available to make the cakes is 3.6 kg; the total amount of chocolate available is 3.96 kg; and the total amount of butter is 1.24 kg.
Each chocolate ripple cake retails for $24 and each death by chocolate cake retails for $28.
Let x represent the number of chocolate ripple cakes made and y represent the number of death by chocolate cakes made.

a. Determine the constraints for x and y.
b. What is the objective function for the maximum revenue?
c. What is the maximum revenue?

17. A jewellery company makes two special pieces of jewellery for Mother's Day. Each piece requires two specialists to work on it; one to shape and set the stones, and one to finish the piece.
The first piece of jewellery takes the setter 1.2 hours to complete and the finisher 0.9 hours, while the second piece of jewellery takes the setter 0.8 hours to complete and the finisher 1.35 hours.
Each week the setter can work for up to 24 hours and the finisher can work for up to 27 hours.
A profit of $215 is made on each of the first pieces of jewellery sold, and a profit of $230 is made on each of the second pieces of jewellery sold.

a. Determine the constraints for the problem.
b. Draw the feasible region and identify the values of the vertices.
c. Write the objective function to maximise profit.
d. Determine how many pieces of each type of jewellery should be made each week to maximise profit.
e. Determine the maximum weekly profit.

18. A pharmacy is making two new drugs that are made from the same two compounds. Drug A requires 5 mg of compound P and 10 mg of compound R, while drug B requires 9 mg of compound P and 6 mg of compound R.
The pharmacy can make a profit of 25 cents for each unit of drug A it sells and 22 cents for each unit of drug B it sells.
The company has 1.2 kg of compound P and 1.5 kg of compound R to make the drugs, and wants to make at least 60 000 units of each drug A and 50 000 units of drug B.

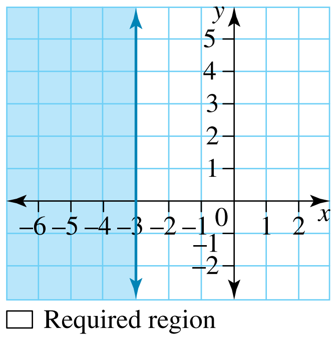
a. Determine the constraints for the problem.
b. Draw the feasible region and identify the values of the vertices.
c. Determine how much of each drug the company should produce to maximise profit.
d. Determine the maximum profit.
e. How much of each compound will remain after making the drugs?

11.5 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC The following graph represents which linear inequality?

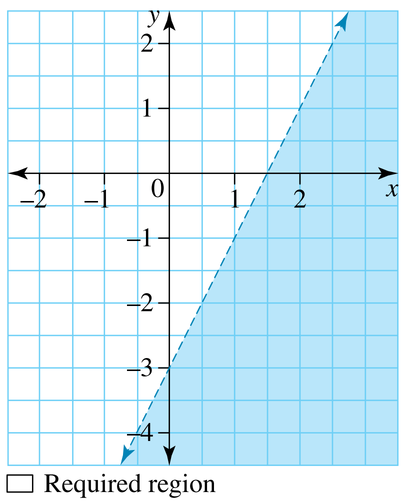


A. $x < -3$ **B.** $x \leq -3$ **C.** $x \geq -3$ **D.** $x > -3$ **E.** $x = -3$

2. MC When sketching a linear inequality, which of the following is represented by a solid line?

A. $4y + 2x > 5$ **B.** $4x < -3y - 6$ **C.** $2x \geq -3y - 3$
D. $5 + 3y < 2x$ **E.** $3x + 2y < 0$

3. MC The correct linear inequality for the following graph is:



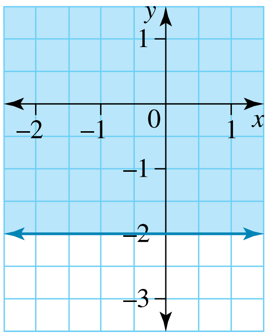
A. $y > 2x - 3$ **B.** $y \geq 3x + 1.5$ **C.** $y \geq 1.5x - 3$
D. $y > 2x + 3$ **E.** $y > 0.5x - 3$

4. MC Which of the following is a suitable test point for the linear inequality $y > 2x + 7$?

A. (0, 7) **B.** (4, −2) **C.** (−1, 5) **D.** (−3, 1) **E.** (2, 11)

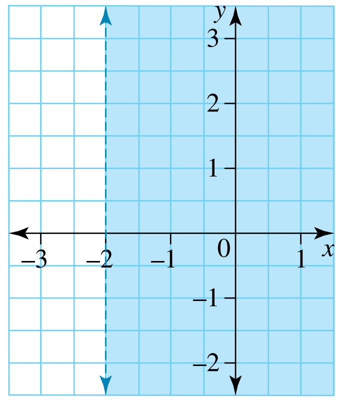
5. **MC** Which linear inequality has been incorrectly sketched?

A. $y \leq -2$



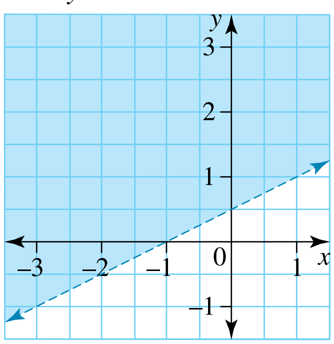
☐ Required region

B. $x \leq -2$



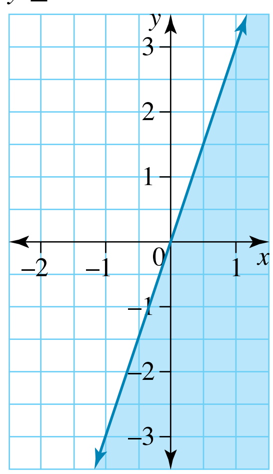
☐ Required region

C. $x > 2y - 1$



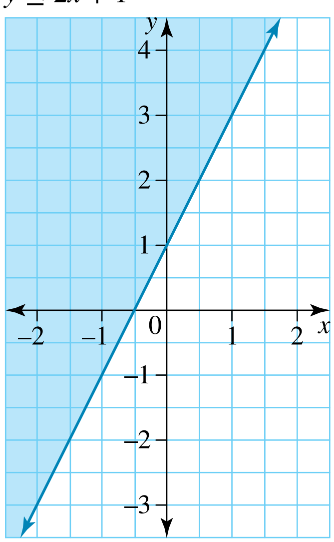
☐ Required region

D. $y \geq 3x$



☐ Required region

E. $y \leq 2x + 1$



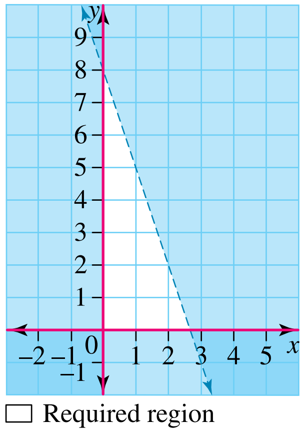
☐ Required region

6. **MC** The point $(5, -10)$ is not a feasible solution for which of the following linear inequalities?

A. $5x + 2y < 30$ **B.** $3y + 5x > 14$ **C.** $2y - 3x < 17$

D. $4x - 7y > 10$ **E.** $y + 2x > 5$

7. **MC** Which of following groups of constraints represents the feasible region shown?



A. $x \leq 0$
$y \geq 0$
$y > -3x + 8$

B. $x \geq 0$
$y \geq 0$
$y < -3x + 8$

C. $x \leq 0$
$y \leq 0$
$y < -3x + 8$

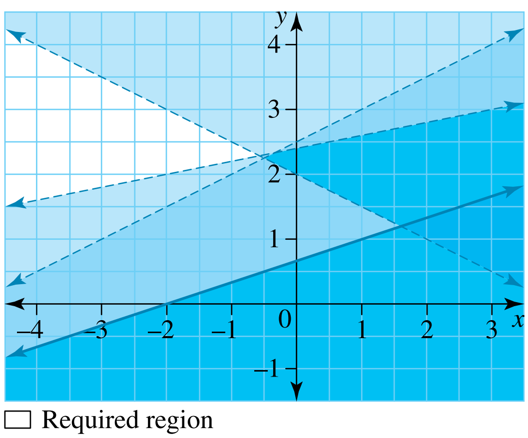
D. $x \geq 0$
$y \leq 0$
$y > -3x + 8$

E. $x \geq 0$
$y \geq 0$
$y > -3x + 8$

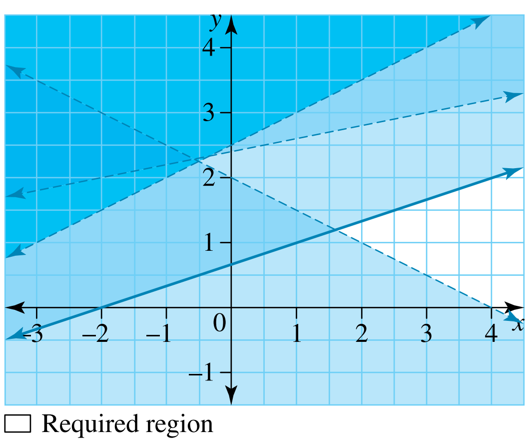
8. **MC** Which of the following graphs represents the feasible region for the listed constraints?

$$x \leq 3y - 2$$
$$x > 2y - 5$$
$$x > -2y + 4$$
$$x > 5y - 12$$

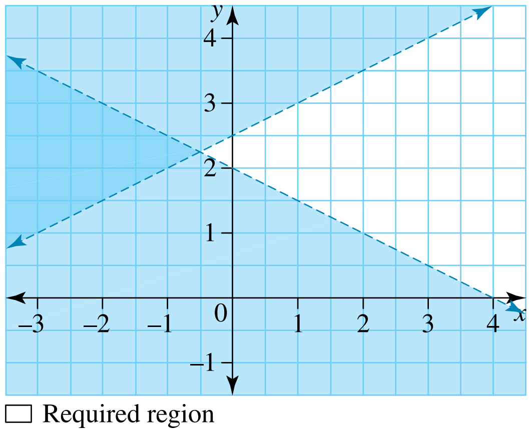
A.



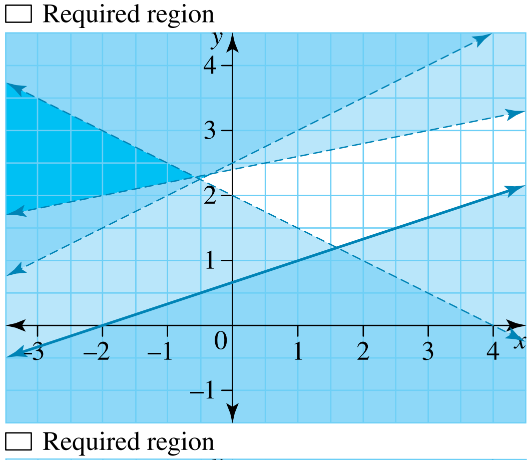
B.



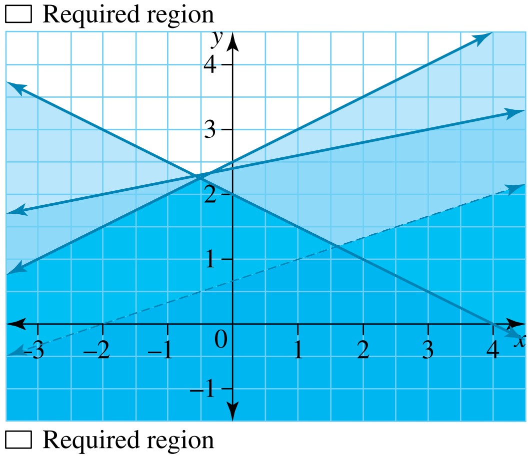
C.



D.



E.



9. MC The equation of a linear inequality generates the corner points A(2, 3), B(0, 4), C(−2, 0) and D(4, 2). The points used for the maximum and minimum values of the function $S = 2x + 3y$ respectively would be:

A. C and D **B.** D and B **C.** D and C
D. B and A **E.** C and B

10. MC A linear inequality with the following constraints is drawn.

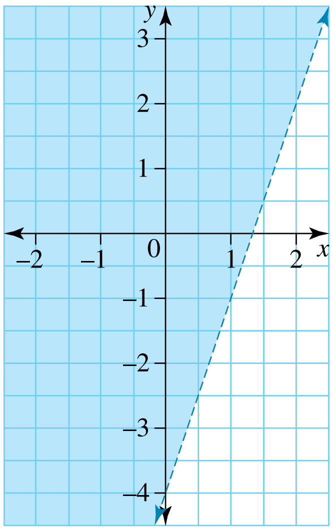
$$3x - 3y < 6$$
$$3x + 2y \leq 10$$
$$x \geq 0$$
$$y \geq 0$$

Three of the four coordinates of the feasible region are known: (0, 0), (2, 0) and (0, 5). The unknown set of coordinates is:

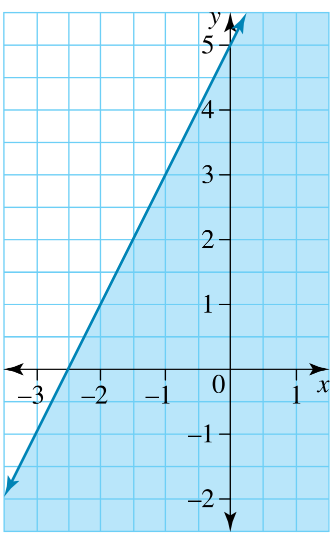
A. (2.8, 0.8) **B.** (2.75, 0.75) **C.** (4, 6)
D. (1, 3) **E.** (5, 2)

Short answer

1. Sketch the following linear inequalities on separate Cartesian planes, leaving the required regions unshaded.
   - a. $x \leq -4$
   - b. $y > 6$
   - c. $-x + 6 > y$
   - d. $3y + 2 \leq x$
   - e. $4y + 2x - 7 < 5$
   - f. $3 \leq 9x - 6y$
2. Determine the linear inequalities that define the following graphs.

a.


☐ Required region

b.


☐ Required region

3. Find the solution to each of the following sets of simultaneous linear inequalities, leaving the required region unshaded.
   a. $4x + 2y \geq 16$
      $x + 3y > 6$
   b. $90x - 36y \leq 360$
      $108x - 18y > -54$
      $y < 0$
      $x > 0$
4. A dairy company produces both full-cream milk and low-fat milk. Each day the company produces and dispatches at most 28 000 L of milk, with a minimum of 10 000 L of full-cream milk and 8500 L of low-fat milk.
   Let F = the number of litres of full-cream milk produced and L = the number of litres of low-fat milk sold.
   a. Identify all of the constraints for this situation.
   b. Sketch the feasible region for this problem.
5. a. Sketch the feasible region of a linear programming problem with the following constraints:

$$\begin{aligned} 2y + 5x &> 10 \\ -2y - 5x &\geq -40 \\ y &\geq 0 \\ x &\geq 0 \end{aligned}$$

   b. Use the corner-point principle to determine the maximum and minimum solution for $R = 14x - 12y$.
6. a. Sketch the feasible region of a linear programming problem with the following constraints:

$$\begin{aligned} -6x + 4y &< 20 \\ 2x + 4y &< 24 \\ x + 4y &> 16 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

   b. Use the sliding-line method to determine the maximum and minimum solution for $M = -2x + 6y$.

Extended response

1. Alice is a landscape photographer who sells her images through local stores. She can edit and print a small photograph in 25 minutes and a large photograph in 1 hour. She spends a maximum of 12 hours a week editing and printing her work. Each week she completes no more than 19 images, with a minimum of 3 small and 2 large prints.
   a. Determine the constraints for this situation.
   b. Sketch the feasible region for this situation.
   The print company that Alice uses charges \$46.50 to print the small images and \$75.80 for the larger images. She is able to sell her prints to local stores at a price of \$120.00 for small images and \$180.00 for large images.
   c. Construct an equation to find Alice's maximum profit.
   d. Determine how many images of each image Alice should produce to maximise her income.
   e. If she was to sell all the images she could print in a week, calculate her maximum weekly profit.

2. A school holiday sports program organises round-robin competitions for soccer, baseball and tennis. Staff are required to set up and pack up the equipment for each sport.

Sport	Soccer	Baseball	Tennis
Set-up time (minutes)	10	20	20
Pack-up time (minutes)	15	15	8

Throughout the day, staff members are allocated 70 minutes for set-up and pack-up for baseball, and 1 hour for soccer and tennis. Due to the number of staff required, set-up time costs the school \$22 and pack-up time costs \$18.

Let x represent the time allocated to set up equipment, and let y represent the time allocated to pack up.

a. Determine the constraints for x and y.
b. Draw the feasible region and identify the values of the vertices.
c. What is the objective function for the maximum cost?
d. What is the maximum cost?

3. Cameron's Concrete Company is a family business that provides two services: standard concreting and heavy-duty concreting. In addition to the cement base required for each product, the mixture for standard concreting uses 2 buckets of sand and 4 buckets of gravel. The mixture for heavy-duty concreting uses 2 buckets of sand and 3 buckets of gravel.

Cameron's Concrete Company has 40 buckets of sand and 65 buckets of gravel in stock. They are able to make a profit of \$75.00 on a mixture of standard concrete and \$130.00 on a mixture of heavy-duty concrete.

a. Determine all of the constraints for this situation.
b. Draw the feasible region that represents this situation.
c. Identify the objective function and calculate the maximum possible profit.

4. Frank's florist shop sells bunches of flowers and boxed arrangements of flowers. Frank allows 4 hours in a day to cut and prepare flowers before they are bunched or boxed. Each bunch of flowers takes Frank 6 minutes to cut and prepare, whereas a boxed arrangement takes him only 4 minutes. Once the flowers are prepared, Frank takes a further 3 minutes to arrange a bunch of flowers and 6 minutes to arrange a box of flowers. Throughout the day Frank spends at most 3 hours arranging flowers. He sells each bunch of flowers for \$48 and each box of flowers for \$65.

a. What are the coordinates of the vertices of the feasible region in this problem?
b. How many bunches and boxes of flowers should Frank produce each week to maximise his profit?
c. If the prices of a bunch of flowers and a box of flowers were reversed, would this change your answer to part **b**?

studyon

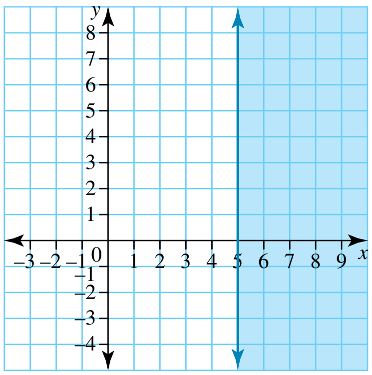
Units 1 & 2 **Sit topic test**

Answers

Topic 11 Inequalities and linear programming

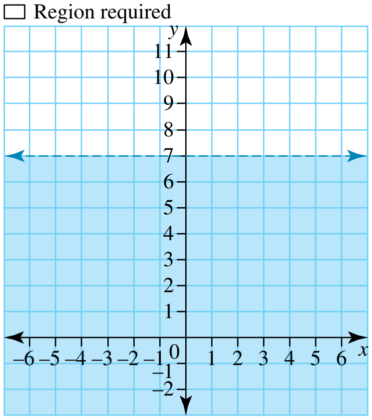
Exercise 11.2 Graphs of linear inequalities

1. a.



☐ Region required

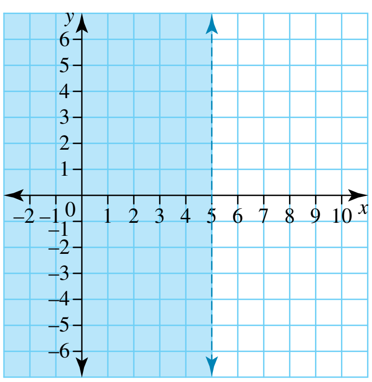
b.



☐ Region required

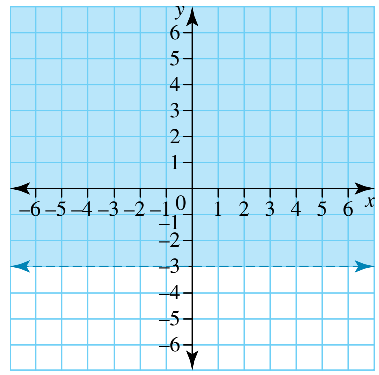
2. a. $y > 5$ **b.** $y < 7$

3. a.



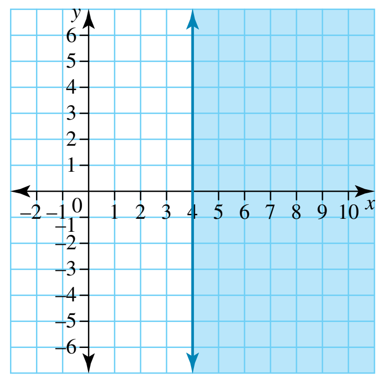
☐ Region required

b.



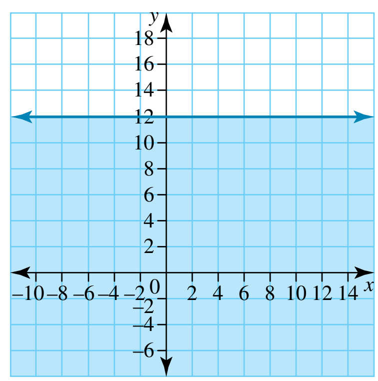
☐ Region required

c.



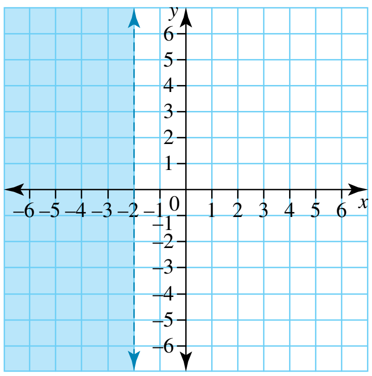
☐ Region required

d.



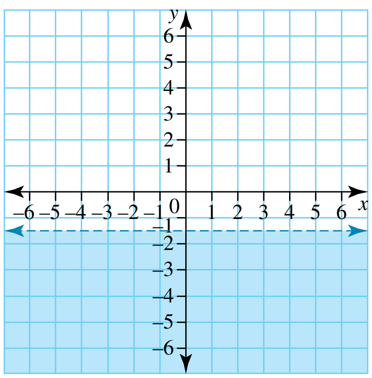
☐ Region required

4.



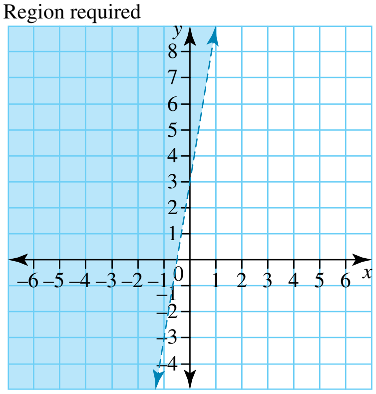
☐ Region required

5.



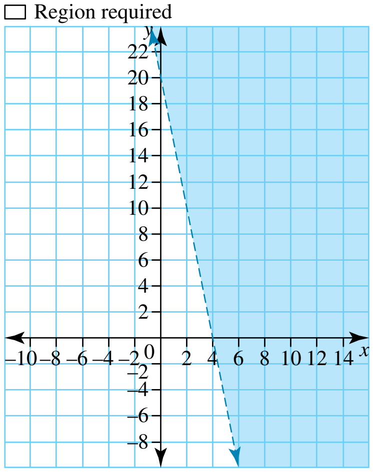
☐ Region required

6. a.



☐ Region required

b.

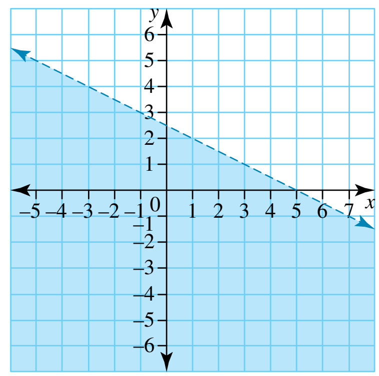


☐ Region required

7. a. $y < 4x + 3$

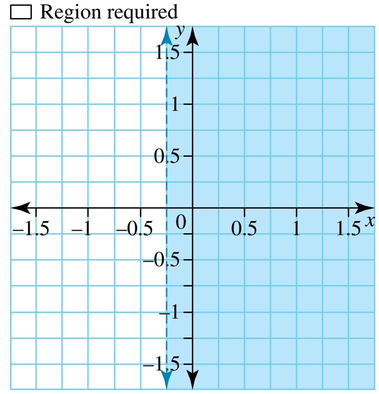
b. $y \le -2x + 5$

8. a.



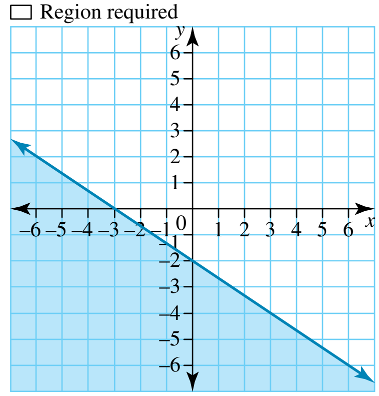
☐ Region required

b.



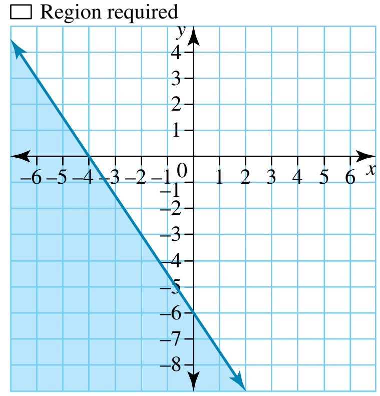
☐ Region required

c.



☐ Region required

d.



☐ Region required

9. D

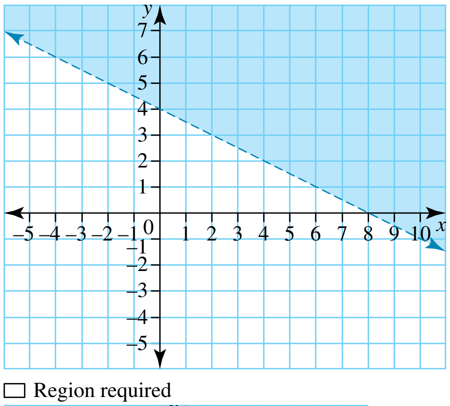
10. C

11. B

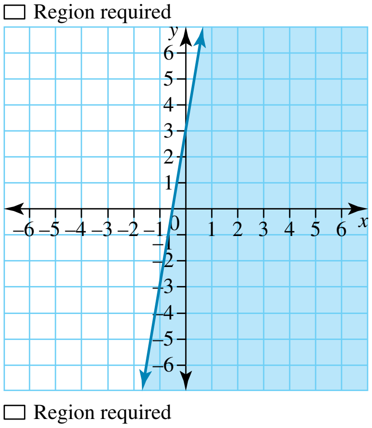
12. E

13. B

14. a.



b.



15. C

16. A

17. a. $y < 5x + 10$

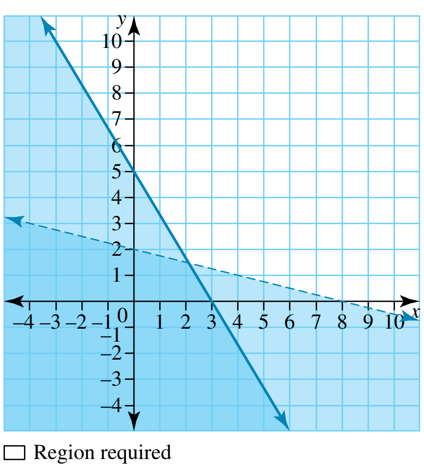
b. $y \leq -4x + 12$

18. a. $y \geq 5x + 4$

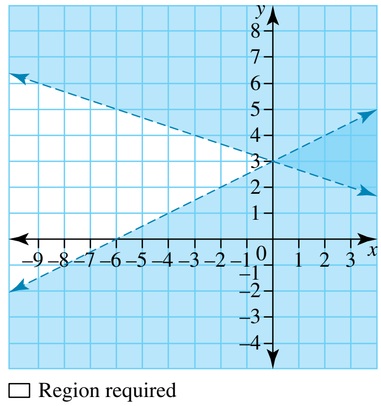
b. $5y > 6x - 3$

Exercise 11.3 Linear programming

1.



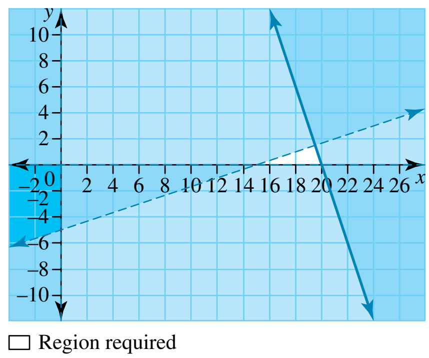
2.



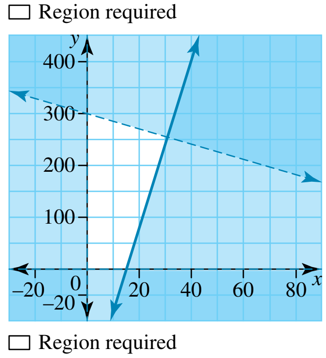
3. D

4. C

5.



6.



7. C

8. B

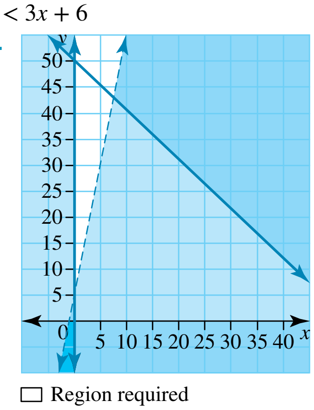
9. $x < 1$

$y \geq 0$

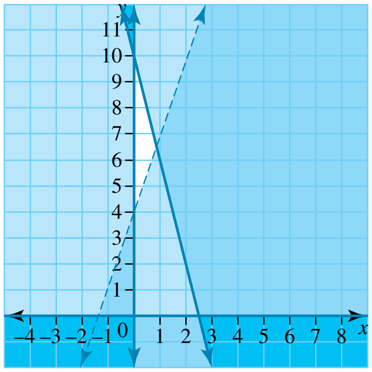
$y > -4x + 1$

$y < 3x + 6$

10. a.



b.



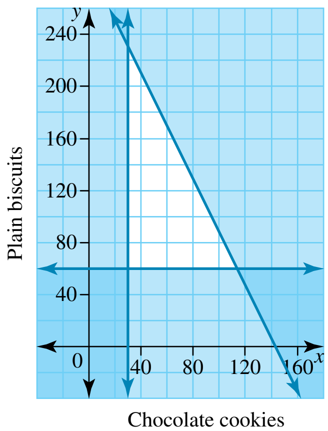
☐ Region required

11. a. $c \geq 30$

$p \geq 60$

$c + 0.5p \leq 144$

b.



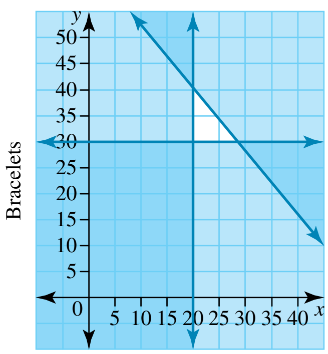
☐ Region required

12. a. $n \geq 20$

$b \geq 30$

$2n + 1.5b \leq 100$

b.



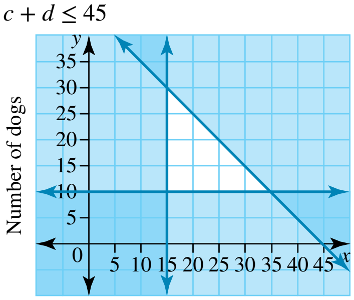
☐ Region required Necklaces

13. a. $c \geq 15$

$d \geq 10$

$c + d \leq 45$

b.



☐ Region required Number of cats

14. Profit $= 10s + 6h$

15. Cost $= 20.5c + 50.25t$

Profit $= 9.5c + 19.75t$

16. a. Profit $= 12c + 15s$

b. Cost $= 60 + 15a$

c. Cost $= 0.3s + 0.45l$

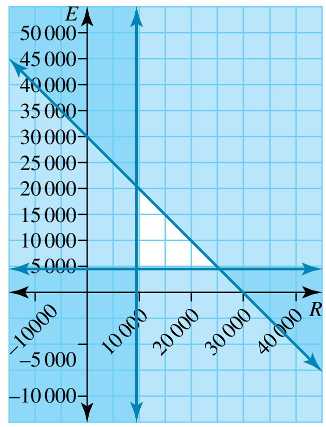
Profit $= 2.7s + 4.55l$

17. a. $R \geq 9500$

$E \geq 4500$

$R + E \leq 30000$

b.



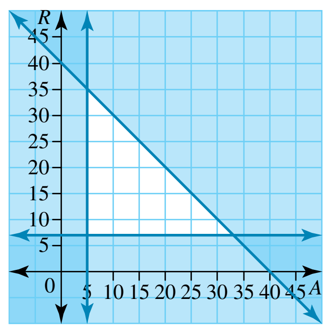
☐ Region required

18. a. $A \geq 5$

$R \geq 7$

$A + R \leq 40$

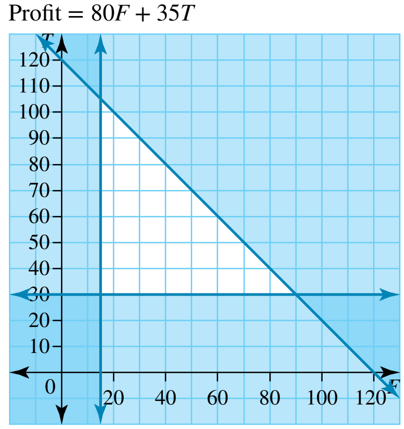
b.



☐ Region required

19. a. Profit $= 80F + 35T$

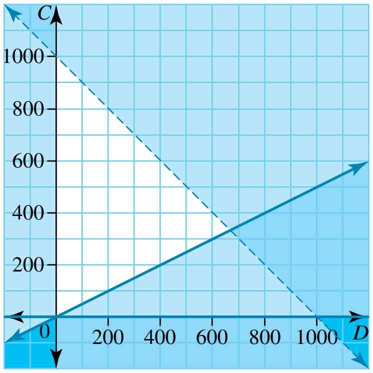
b.



☐ Region required

20. a. Profit $= D + 0.75C$

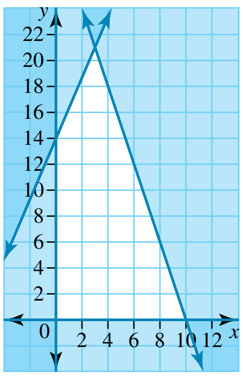
b.



☐ Region required

Exercise 11.4 Applications of linear programming

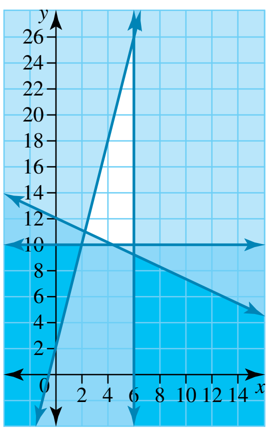
1. a.



☐ Region required

b. Maximum = 51, minimum = 0

2. a.

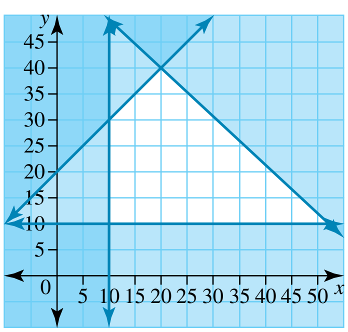


☐ Region required

b. Maximum = 110, minimum = $-9\frac{5}{9}$

3. Maximum = 180, minimum = 48

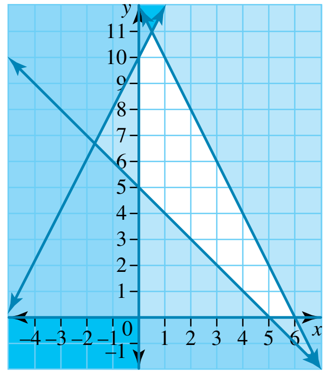
4. a.



☐ Region required

b. Maximum = 680, minimum = 200

5. a.



☐ Region required

b. Maximum= 65, minimum = −12

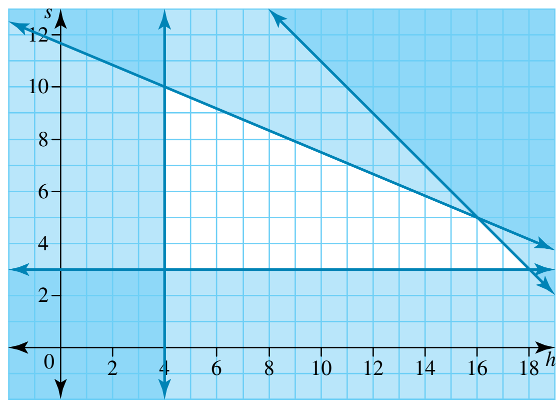
6. 250 cows and 50 sheep ($189 500)

7. 20 mobile phones and 330 laptops ($10 300)

8. a. 80 apples and 40 pears **b.** $28

9. $5045

10. a.

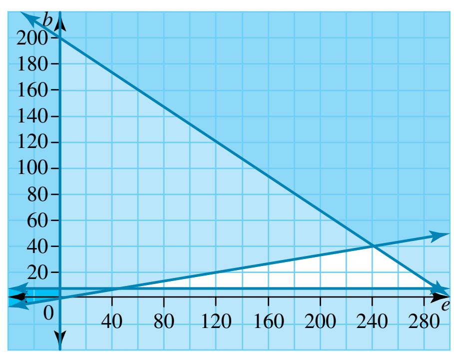


☐ Region required

b. 16 quick haircuts and 5 stylised haircuts

c. $576

11. a.



☐ Region required

b. 240 economy class seats and 40 business class seats

c. $20 600

12. 15 complex apps and 12 simple apps ($4440)

13. $5300

14. a. Objective *B* (6695)

b. Objective *B* (375)

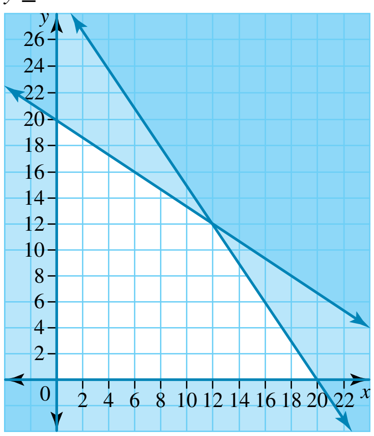
15. a. (0, 0), (0, 15), (10, 10) and (25, 0)

b. 10 of each case ($1480)

c. Yes, it would then be best to make 25 of case A and 0 of case B ($2225).

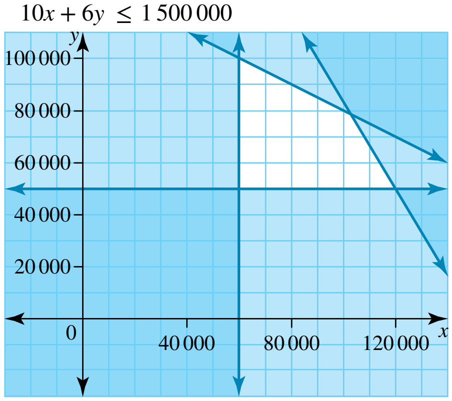
16. a. $x \geq 0$
$y \geq 0$
$300x + 200y \leq 3600$
$120x + 360y \leq 3960$
$80x + 100y \leq 1240$
b. Profit $= 24x + 28y$
c. $360

17. a. $1.2x + 0.8y \leq 24$
$0.9x + 1.35y \leq 27$
$x \geq 0$
$y \geq 0$
b.



☐ Region required
(0, 0), (0, 20), (12, 12) and (20, 0)
c. Profit $= 215x + 230y$
d. 12 of each type of jewellery
e. $5340

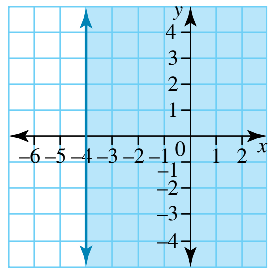
18. a. $x \geq 60\,000$
$y \geq 50\,000$
$5x + 9y \leq 1\,200\,000$
$10x + 6y \leq 1\,500\,000$
b.



☐ Region required
(60 000, 50 000), (60 000, 100 000), (105 000, 75 000) and (120 000, 50 000)
c. 105 000 units of Drug A and 75 000 units of Drug B
d. $42 750
e. There will be nothing left of either compound.

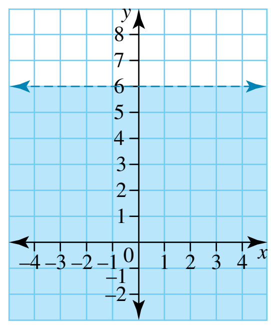
Short answer

1. a.



☐ Required region

b.



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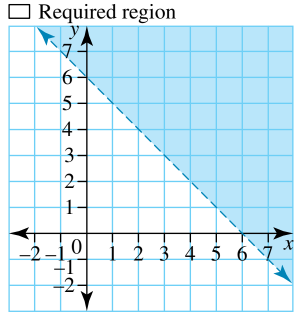
c.



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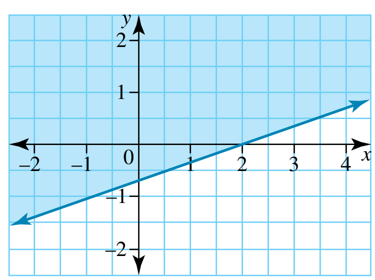
d.



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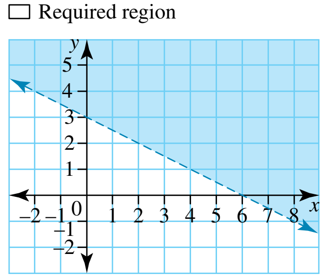
e.



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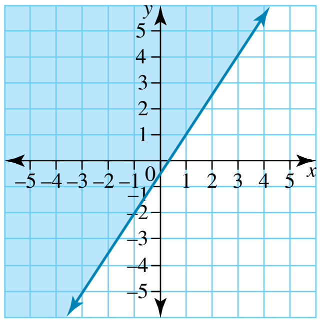
11.5 Review: exam practice

Multiple choice

1. C	2. C	3. A	4. B	5. B
6. B	7. B	8. D	9. C	10. A

f.

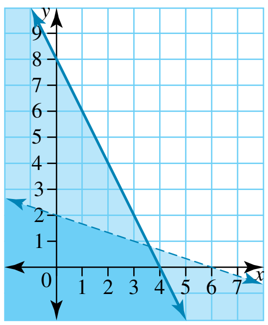


☐ Required region

2. a. $y < 3x - 4$

b. $y \geq 2x + 5$

3. a.



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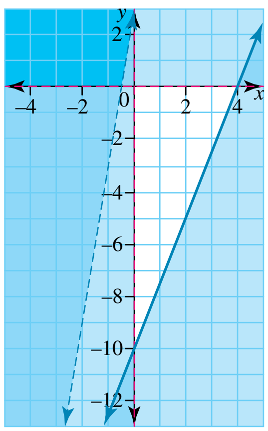
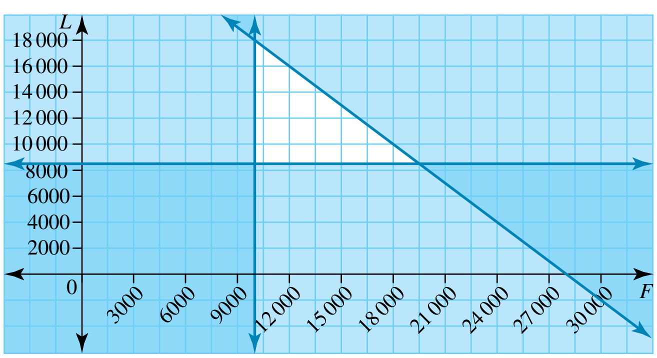
b.



☐ Required region

4. a.
$$\begin{aligned} F + L &\leq 28\,000 \\ F &\geq 10\,000 \\ L &\geq 8500 \end{aligned}$$

b.



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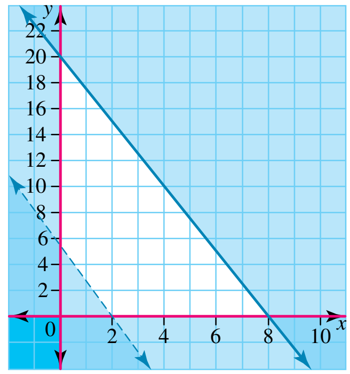
5. a.



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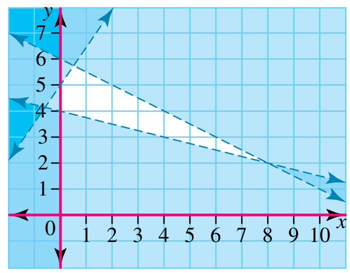
b. Maximum = 112, minimum = −240

6. a.



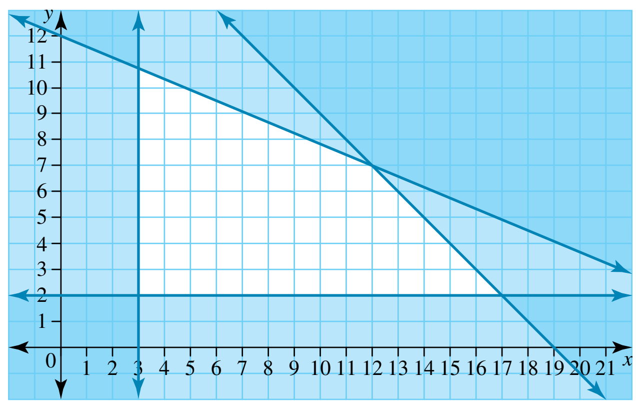
☐ Required region

b. Maximum = 33.5, minimum = −4

Extended response

1. a.
$$\begin{aligned} 25x + 60y &\leq 720 \\ x + y &\leq 19 \\ x &\geq 3 \\ y &\geq 2 \end{aligned}$$

b.



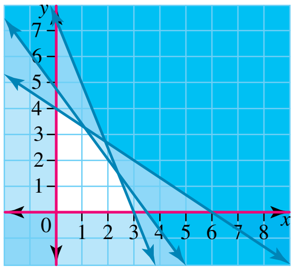
☐ Required region

c. $P = 73.5x + 104.2y$

d. 12 small images and 7 large images

e. \$1611.40

2. a. $10x + 15y \leq 60$
$20x + 15y \leq 70$
$20x + 8y \leq 60$
$x \geq 0$
$y \geq 0$

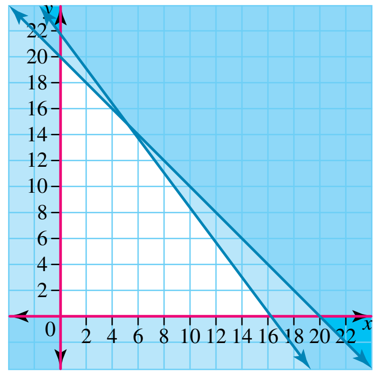
b. 

☐ Required region

c. $P = 22x + 18y$

d. \$82

3. a. $2x + 2y \leq 40$
$4x + 3y \leq 65$
$x \geq 0$
$y \geq 0$

b. 

☐ Required region

c. \$2600

4. a. (0, 0), (0, 30), (30, 15), (40, 0)

b. 30 bunches and 15 boxes of flowers

c. No

TOPIC 12
Variation

12.1 Overview

12.1.1 Introduction

So far in mathematics, you've only seen linear relationships, or relationships with two that were directly proportional. Even though a lot of variables are linearly related, this is certainly not always the case. Depending on what the scatterplot of the data shows, the data can also be modelled by relationships that are not linear. These could be reciprocal $\left(\frac{1}{x}\right)$, square (x^2) and logarithmic transformation ($\log_{10} x$), to name a few.

Let's look at logarithmic transformation and a couple of commonly used logarithmic transformations. The Richter scale assigns a magnitude number to quantify the size of an earthquake. The Richter scale was developed in the 1930s and it is a base 10 logarithmic scale. The logarithmic scale allows for the data to cover a huge range of values for different earthquakes. It defines magnitude as the logarithm of the ratio of the amplitude of the seismic waves to an arbitrary, minor amplitude, as recorded on a standardised seismograph at a standard distance. As a measure with a seismometer, an earthquake that registers 5.0 on the Richter scale has a shaking amplitude 10 times greater than an earthquake that registered 4.0 at the same distance.

Another well-known use of logarithmic transformation is the pH scale. The pH scale tells us the acidity or basicity of an aqueous solution. It is the negative of the base 10 logarithm of the activity of the hydrogen ion. Solutions with a pH less than 7 are classified as acidic and solutions with a pH greater than 7 are basic. The pH values are very important for industries such as medicine, biology, chemistry, agriculture, food science nutrition, engineering and oceanography.

LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.

12.1.2 Kick off with CAS

Exploring variation with CAS

In mathematics, graphs come in many different shapes. Understanding what causes the changing of shapes in different graphs will help to increase your understanding of the underlying mathematics.

1. Use CAS to draw the following graphs.
   a. $y = 2x$
   b. $y = 4x$
   c. $y = \frac{1}{2}x$
2. How do the graphs in question 1 change as the coefficient in front of x changes?
3. Use CAS to draw the following graphs.
   a. $y = x^2$
   b. $y = 3x^2$
   c. $y = \frac{1}{2}x^2$
4. How do the graphs in question 3 change as the coefficient in front of x^2 changes?
5. Use CAS to draw the following graphs.
   a. $y = \dfrac{1}{x}$
   b. $y = \dfrac{3}{x}$
   c. $y = \dfrac{1}{2x}$
6. How do the graphs in question 5 change as the coefficient in the fraction changes?

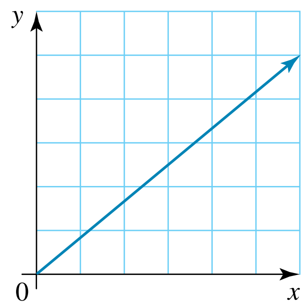
on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology

12.2 Direct, inverse and joint variation

12.2.1 Direct variation

Frequently in mathematics we deal with investigating how the changes that occur in one quantity cause changes in another quantity. Understanding how these quantities vary in relation to each other allows the development of equations that provide mathematical models for determining all possible values in the relationship.



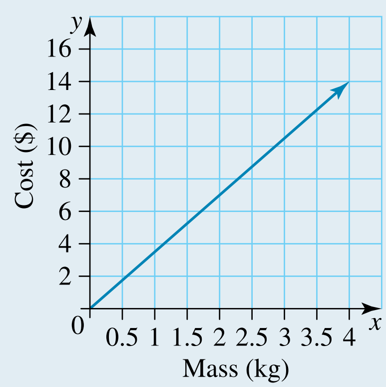
Direct variation involves quantities that are proportional to each other. If two quantities vary directly, then doubling one doubles the other. In direct variation, as one value increases, so does the other; likewise, as one decreases, so does the other. This produces a linear graph that passes through the origin.

If two quantities are directly proportional, we say that they 'vary directly' with each other. This can be written as $y \propto x$.

The proportion sign, $\propto$, is equivalent to '$= k$', where k is called the **constant of proportionality** or the constant of variation. The constant of variation, k, is equal to the ratio of y to x for any data pair. Another way to put this is that k is the rate at which y varies with x, otherwise known as the gradient. This means that $y \propto x$ can be written as $y = kx$.

WORKED EXAMPLE 1

The cost of apples purchased at the supermarket varies directly with their mass, as shown in the following graph. Calculate the constant of proportionality and use it to write a rule connecting cost, *C*, and mass, *m*.

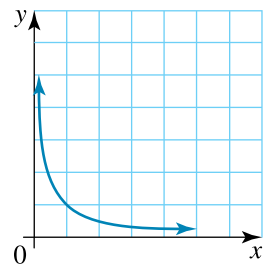


THINK	WRITE
1. Use the gradient formula to find k.	$k = \frac{\text{rise}}{\text{run}}$ $= \frac{14}{4}$ $= 3.5$
2. Substitute the value of k into the equation that relates the two variables (that is, in the form $y = kx$).	Here the variables are cost ($) and mass (kg), so: $C = km$ $= 3.5m$

12.2.2 Inverse variation

If two quantities vary inversely, then increasing one variable decreases the other.

Inverse variation produces the graph of a hyperbola.



The statement 'x varies inversely with y' can be written as $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$.

As in direct variation, k is called the proportionality constant or the constant of variation.

WORKED EXAMPLE 2

The time taken to complete a task is inversely proportional to the number of workers as shown in the following table.

Write the rule that relates the time to complete the task with the number of workers.

Number of workers (n)	1	2	3	4	6
Hours to complete (T)	12	6	4	3	2

THINK	WRITE
1. Write the statement for inverse proportion using the variables from the question.	$T \propto \frac{1}{n} \rightarrow T = \frac{k}{n}$
2. Select a pair of coordinates from the table and substitute them into the equation to solve for k.	$(n, T) \rightarrow T = \frac{k}{n}$ $(1, 12) \rightarrow 12 = \frac{k}{1}$ $\therefore k = 12$
3. Write the equation.	$T = \frac{12}{n}$

12.2.3 Joint variation

In some situations there may be multiples of more than one independent variable. When this happens, it is known as **joint variation**. Some examples are displayed in the following table.

Joint variation	**Rule**	**Description**
$A \propto BC^2$	$A = kBC^2$	A varies jointly as B and the square of C.
$A \propto \frac{B}{\sqrt{C}}$	$A = \frac{kB}{\sqrt{C}}$	A varies jointly as B and inversely as the square root of C.
$A \propto \frac{B^3}{C}$	$A = \frac{kB^3}{C}$	A varies jointly as the cube of B and inversely as C.

WORKED EXAMPLE 3

Use the following information to write a variation statement and a rule connecting the quantities. The gravitational force, F, between two objects is proportional to each of the masses, m_1 and m_2, in kg, and to the inverse square of the distance between them, d, in metres.

THINK	WRITE
1. Write a variation statement as a product and quotient of the variables indicated.	$F \propto \frac{m_1 \times m_2}{d^2}$
2. Replace '∝' with '= k' and write the rule.	$F = \frac{km_1m_2}{d^2}$

on Resources

Interactivity: Direct, inverse and joint variation (int-6490)

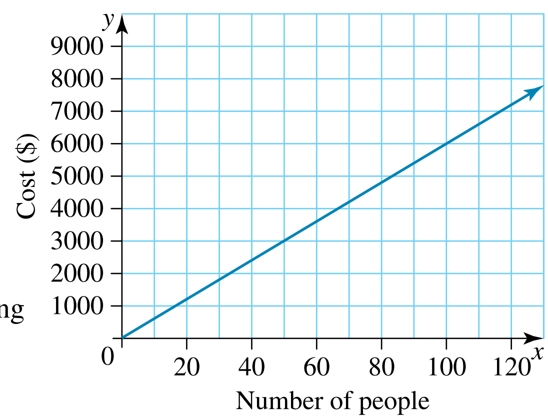
studyon

Units 1 & 2 > AOS 5 > Topic 3 > Concept 1

Direct, inverse and joint variation Concept summary and practice questions

Exercise 12.2 Direct, inverse and joint variation

1. WE1 The cost of a wedding reception varies directly with the number of people attending as shown in the graph.
   Calculate the constant of proportionality and use it to write a rule connecting cost, C, and the number of people attending, n.



2. The distance travelled by a vehicle, d (in kilometres) is directly proportional to the time, t (in hours). Use the information in the following table to write a rule connecting distance and time.

Time (h)	1	2	3	4
Distance (km)	90	180	270	360

3. Identify whether y is directly proportional to x in each of the following tables:

a.

x	0.7	1.2	4	4.1
y	2.8	4.8	12	16.4

b.

x	0.4	1.5	2.2	3.4
y	0.84	3.15	4.62	7.14

4. The amount of interest earned by an investment is proportional to the amount of money invested.
   a. Use the pronumerals I for the amount of interest and A for the amount of money invested, together with the proportionality sign ($\propto$), to write this in mathematical shorthand.
   b. Write your answer to part **a** using an equals sign and a constant, k.
   c. If an investment of \$30 000 earns \$15 000 interest, find k.
   d. Write an equation connecting I and A.
   e. Use the equation to find the interest earned by an investment of \$55 000.
   f. Use the equation to calculate the investment needed to earn \$75 000 in interest.
5. If $K \propto m$ and $K = 27.9$ when $m = 6.2$:
   a. calculate the constant of proportionality
   b. find the rule connecting K and m
   c. calculate the value of K when $m = 72$
   d. calculate the value of m when $K = 450$.

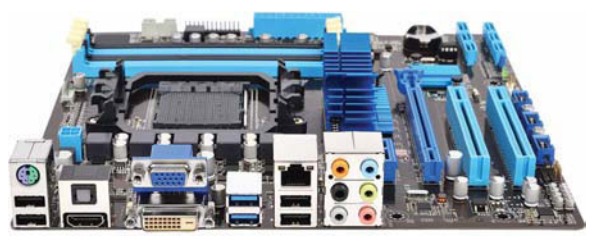
6. **WE2** The time taken to travel 100 km is inversely proportional to the speed, as shown in the following table. Write the rule that relates the time taken to travel 100 km (t) with the speed (s).

Speed (km/h)	100	50	25	10	5	1
Time (h)	1	2	4	10	20	100

7. A box of chocolates contains 20 pieces which are divided equally among family members.
   a. Copy and complete the table to show the number of sweets that each receives for various numbers of relatives.

Family members	20	10		4	
Number of chocolates	1		4		20

   b. Show this information in a graph.
   c. Write a rule connecting the number of family members (n) with the number of chocolates each receives (C).
8. A company making electronic parts finds that the number sold depends on the price. If the price is higher, fewer parts are sold. The market research gives the following expected sales results:



Price (\$)	1	5	20	50	100	200	400
Number sold (thousands)	400	80	20	8	4	2	1

   a. Draw a graph of the number sold versus the price.
   b. Use the information to write an equation that connects the price, P, with the number sold, n.
   c. Use the equation to predict the number sold if the price is \$25.
   d. Use the equation to calculate the price if 250 000 are sold.

9. The percentage of harmful bacteria, P, found to be present in a sample of cooked food at certain temperatures, t, is shown in the following table.

Food temperature, t(°C)	10	20	40	50	80	100
Harmful bacteria percentage, P	80	40	20	16	10	8

a. Draw a graph of this information.
b. Find an equation connecting the quantities.
c. Use the equation to predict the percentage of harmful bacteria present when the food temperature is 25 °C.
d. Use the equation to calculate the temperature required to ensure that the food contains no more than 3.2% of harmful bacteria.

10. WE3 Use the following information to write variation statements and rules connecting the quantities.
a. y varies jointly as the square of x and the cube of z.
b. A varies jointly as the product of B, C and D.
c. V is jointly proportional to the square root of r and to h.
d. U varies jointly as the square of p and inversely as the square root of q.

11. X varies jointly as the square of Q and inversely as the square root of p.
a. Write a variation statement connecting the quantities.
b. Write a rule connecting the quantities, using a constant, k.
c. If $X = 150$ when $Q = 3$ and $p = 9$, find the constant of variation, k.
d. Calculate the value of X when $Q = 7$ and $p = 16$.

12. R varies jointly as the square of q and inversely as the square root of s.
a. Write a variation statement connecting the quantities.
b. Write a rule connecting the quantities, using a constant, k.
c. If $R = 150$ when $q = 3$ and $s = 9$, find the constant of variation, k.
d. Calculate the value of R when $q = 7$ and $s = 16$.
e. Calculate the value of R when $q = 5$ and $s = 4$.

13. P is directly proportional to the cube of Q and inversely proportional to R squared. If $P = 33.75$ when $Q = 3$ and $R = 2$:
a. calculate the constant of variation.
b. calculate P when $Q = 5$ and $R = 7$, correct to 2 decimal places.

14. The acceleration, a, of an object varies directly with the force, F, acting on it and inversely with its mass, m.
a. Write a statement to describe the proportions in this relationship.
b. If a mass of 2 kilograms is acted on by a force of 10 newtons and has an acceleration of 5 m/s^2, calculate the constant of variation and state a rule connecting the quantities.
c. Calculate the acceleration of an object of mass 200 kilograms acted on by a force of 500 newtons.

15. The velocity, v km/h, of a communications satellite in its orbit around the earth varies directly with the radius, r km, of the orbit and inversely with the orbital time period, T hours.

a. Write a proportion statement to describe this relationship.
b. A satellite in an orbit of radius 10 000 km has a time period of 8 hours and a speed of 1875 km/h. Calculate the constant of variation.
c. Calculate the speed of a satellite whose orbital radius is 15 600 km and whose orbital time period is 12 hours.

16. The distance travelled, d m, by an object that starts from rest varies directly with its acceleration, a m/s^2, and the square of the time, t s.
   a. Write a proportion statement to describe this.
   b. An object starting from rest and moving with an acceleration of 5 m/s^2 for 4 seconds travels 32 m. Calculate the constant of variation.
   c. Calculate the distance travelled by an object whose acceleration is 8 m/s^2 for 6 seconds.
17. The sound level of a public address system, L, varies directly with the power output, P watts, of the speakers and inversely with the square of the distance from them, d metres. A speaker of power 60 watts produces a sound level of 0.6 watts/m^2 at a distance of 5 m.
   a. Find the rule that connects sound level, power output and distance.
   b. Use the rule to calculate the sound level 10 m from a speaker that produces 80 watts of power.

18. The masses of stars and planets can be calculated by observing the orbit of objects around them. A planet's mass, M, varies directly with the cube of the radius, r, of an object's orbit around it, and inversely with the square of the orbital period, t.
   a. Write a variation statement for this information.
   b. The mass of the Earth is estimated to be 5.972×10^{24} kg. If the Moon orbits the Earth with a period of 2.42×10^6 s at a distance of 3.844×10^8 m, what is the constant of variation?
   c. Write a rule for calculating the mass of planetary bodies.
   d. Neptune's main moon, Triton, orbits it at a distance of 3.55×10^8 m with a period of 5.08×10^5 s. What is the mass of Neptune?

12.3 Data transformations

12.3.1 Linearising data

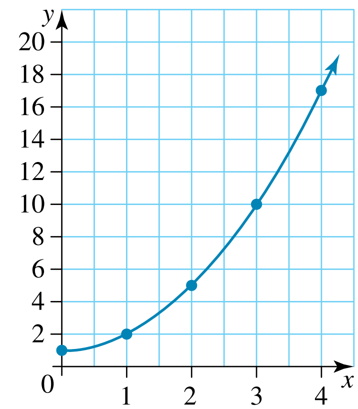
We have seen that when quantities have a relationship that is directly proportional, their graphs are straight lines. If we are using the linear rule obtained from investigating the direct relationship, it becomes much easier to identify any values that don't match the graph.

If a data set displays non-linear behaviour, we can often apply a mathematical approach to **linearise** it. From a graphical point of view this is achieved by adjusting the scale of one of the axes. This is known as a data transformation, and it can often be achieved through a process of squaring one of the coordinates or taking their reciprocal.

12.3.2 Transforming data with x^2

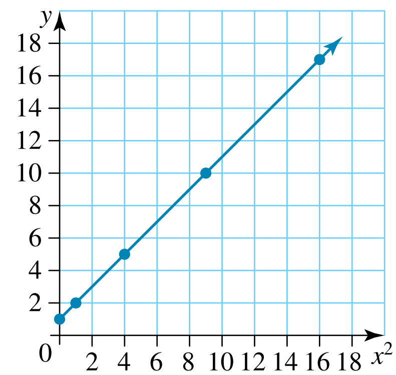
When the relationship between two variables appears to be quadratic or parabolic in shape, we can often transform the non-linear relation to a linear relation by plotting the y-values against x^2-values instead of x-values.

For example, when we graph the points from the table below, we obtain a typical parabolic shape.



x	0	1	2	3	4
y	1	2	5	10	17

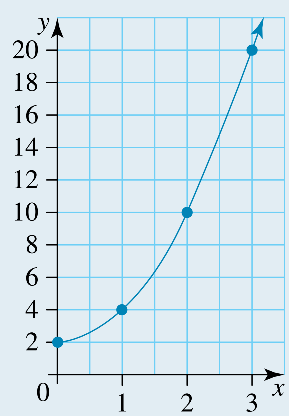
If instead the y-values are plotted against the square of the x-values, the graph becomes linear.



x	0	1	2	3	4
x^2	0	1	4	9	16
y	1	2	5	10	17

WORKED EXAMPLE 4

Redraw the following graph by plotting y-values against x^2-values.



THINK

1. Construct a table of values for the points shown in the graph.

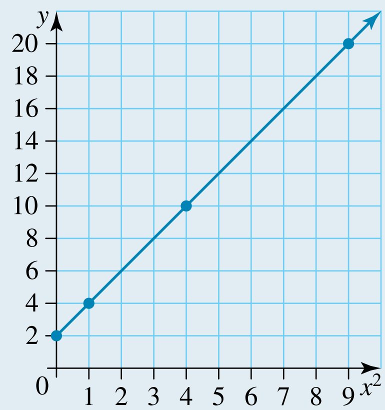
WRITE/DRAW

x	0	1	2	3
y	2	4	10	20

2. Add a row for calculating the values of x^2.

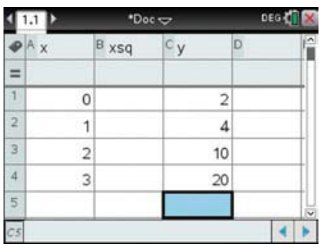
x	0	1	2	3
x^2	0	1	4	9
y	2	4	10	20

3. Plot the points using the x^2- and y-values.

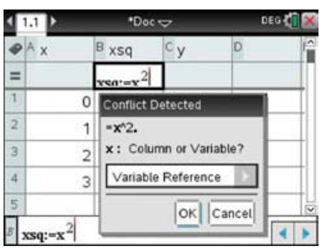


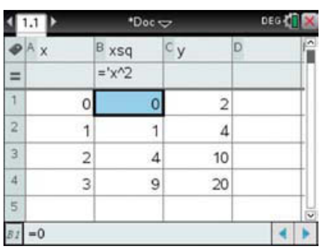
TI | THINK / **WRITE**

1. In a Lists & Spreadsheet page, label the first column as x, the second column as *xsq* (x-square) and the third column as y. Enter the x-values into the first column and the y-values into the third column.

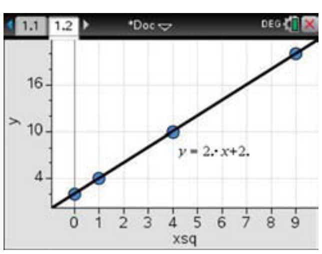


2. In the function cell below *xsq*, complete the entry line as: $= x^2$ then press ENTER. Select the variable reference for x when prompted, then select OK.



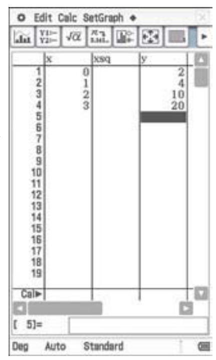


3. In a Data & Statistics page, click on the label of the horizontal axis and select *xsq*. Click on the label of the vertical axis and select y. Press MENU then select:
4: Analyze
6: Regression
1: Show Linear ($mx + b$)
The graph is displayed on the screen.

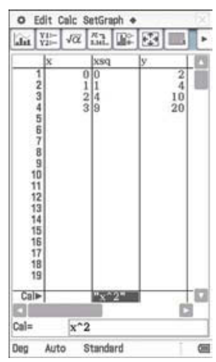


CASIO | THINK / **WRITE**

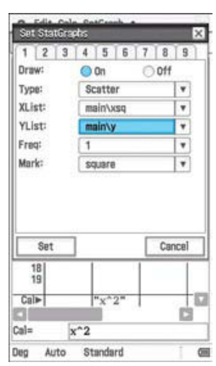
1. In a Statistics screen, label list1 as x, list2 as *xsq* (x-square), and list3 as y. Enter the x-values into the first column and the y-values into the third column.



In the function cell below the *xsq* column, complete the entry line as x^2 then press EXE.



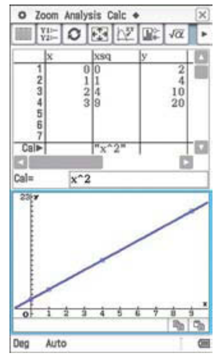
3. Select:
- SetGraph
- Setting …
Complete the fields as:
Draw: On
Type: Scatter
XList: main\xsq
YList: main\y
Freq: 1
then select Set.



4. Click the y icon then select:
- Calc
- Regression
- Linear Reg

Complete the fields as:
XList: main\xsq
YList: main\y
Select OK then select OK again.

5. The graph is displayed on the screen.

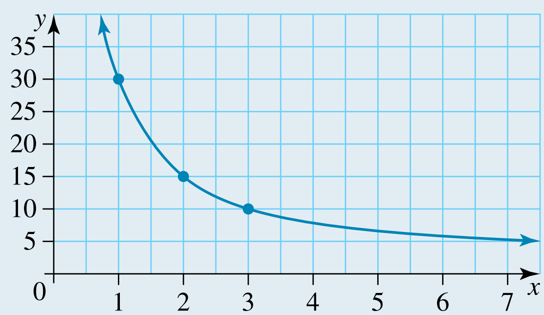


12.3.3 Transforming data with $\frac{1}{x}$

For data that is not linearised by an x^2 transformation, other adjustments to the scale of the x-axis can be made. Another common transformation is to use $\frac{1}{x}$, called the reciprocal of x.

WORKED EXAMPLE 5

Redraw the following graph by plotting y-values against $\frac{1}{x}$-values.



THINK

1. Construct a table of values for the points shown in the graph.

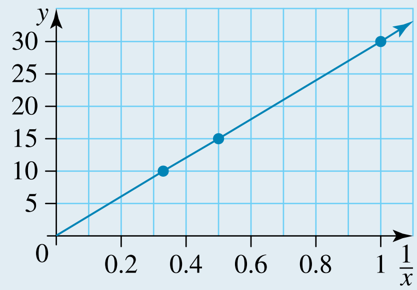
WRITE/DRAW

x	1	2	3
y	30	15	10

2. Add a row for calculating the values of $\frac{1}{x}$.

x	1	2	3
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$
y	30	15	10

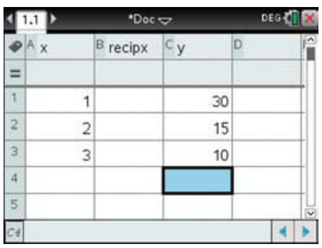
3. Plot the points using the $\frac{1}{x}$- and y-values.



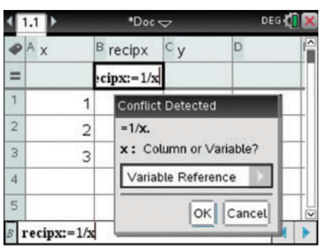
TI | THINK

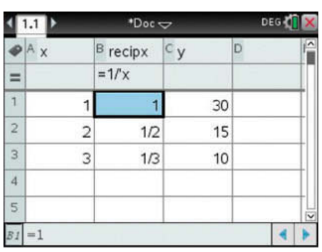
1. In a Lists & Spreadsheet page, label the first column as x, the second column as *recipx* (reciprocal x) and the third column as y.
Enter the x-values into the first column and the y-values into the third column.

WRITE

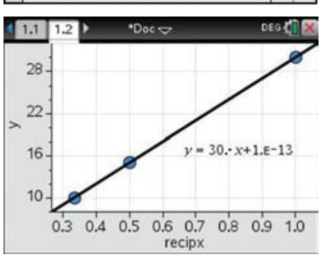


2. In the function cell below *recipx*, complete the entry line as:
$= \frac{1}{x}$
then press ENTER.
Select the variable reference for x when prompted, then select OK.





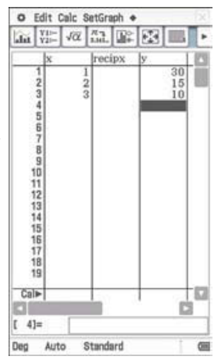
3. In a Data & Statistics page, click on the label of the horizontal axis and select *recipx*. Click on the label of the vertical axis and select y.
Press MENU then select:
4: Analyze
6: Regression
1: Show Linear ($mx + b$)
The graph is displayed on the screen.



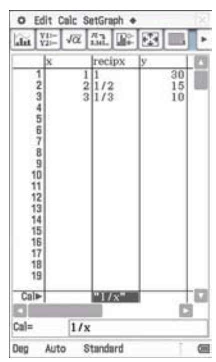
CASIO | THINK

1. In a Statistics screen, label list1 as x, list2 as *recipx* (reciprocal x) and list3 as y.
Enter the x-values into the first column and the y-values into the third column.

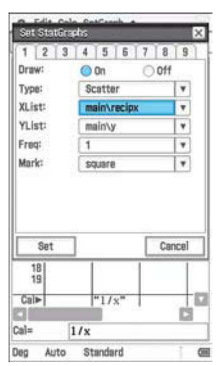
WRITE



2. In the function cell below the *recipx* column, complete the entry line as $\frac{1}{x}$ then press EXE.

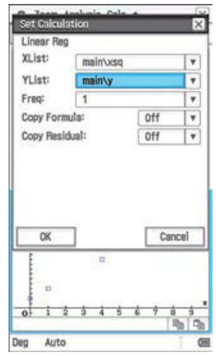


3. Select:
- SetGraph
- Setting …
Complete the fields as:
Draw: On
Type: Scatter
XList: main\recipx
YList: main\y
Freq: 1
then select Set.

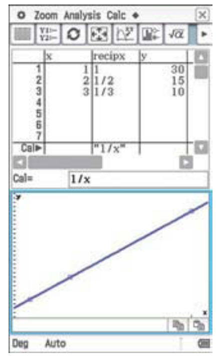


4. Click the y icon then select:
- Calc
- Regression
- Linear Reg

Complete the fields as:
XList: main\recipx
YList: main\y
Select OK then select OK again.



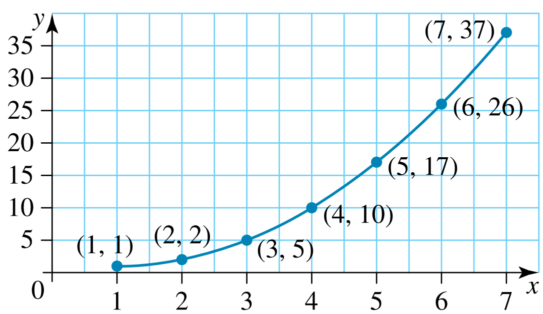
5. The graph is displayed on the screen.



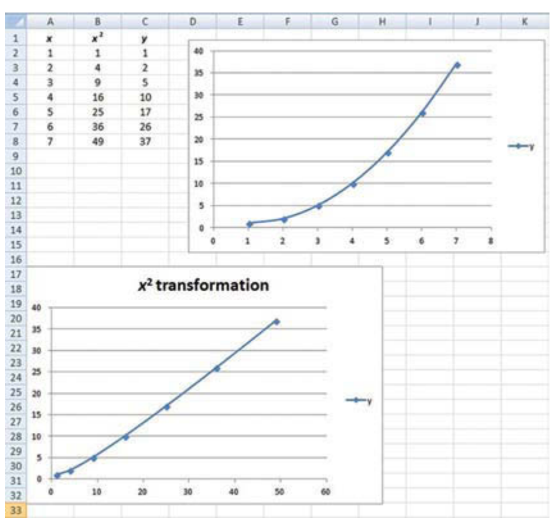
12.3.4 Transforming data with technology

In practice, technology such as CAS and spreadsheets can be used to quickly and efficiently transform data and identify equations for the relationships. Consider the following table of values with its corresponding graph.

x	1	2	3	4	5	6	7
y	1	2	5	10	17	26	37



Columns in a spreadsheet can be set up to calculate transformed values and graphs.



on Resources

Interactivity: Linearising data (int-6491)
Interactivity: Transforming to linearity (int-6253)

studyon

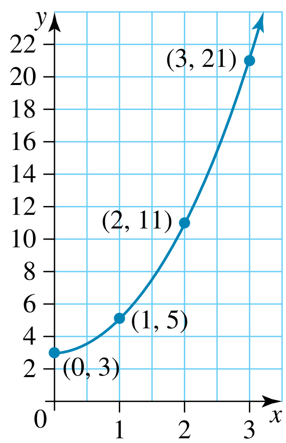
Units 1 & 2 › AOS 5 › Topic 3 › Concept 3

Finding relationships using data transformations Concept summary and practice questions

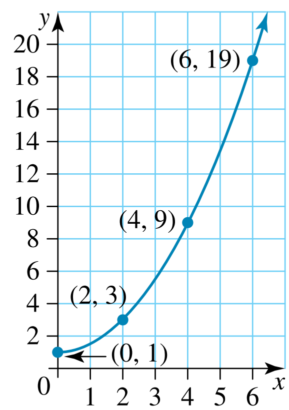
Exercise 12.3 Data transformations

1. **WE4** Redraw the following graphs by plotting y-values against x^2-values.

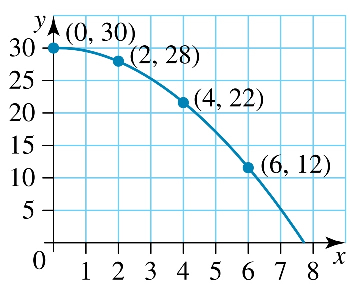
a.



b.

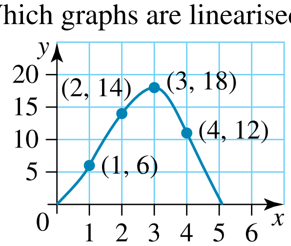


c.

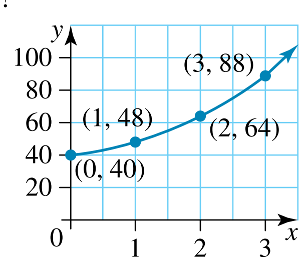


2. **i.** Redraw each of these graphs by plotting the y-values against x^2-values.
   ii. Which graphs are linearised by the x^2 transformation?

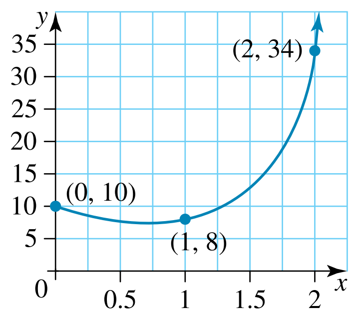
a.



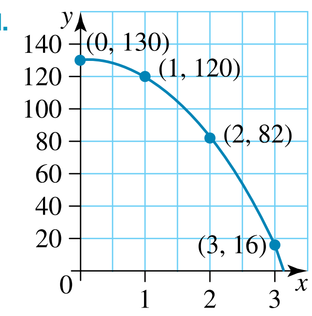
b.



c.



d.



3. For each of the table of values shown:
   i. draw a graph of the original values
   ii. construct a table of values to show x^2-values against the original y-values
   iii. draw a graph of the transformed values
   iv. write a comment that compares the transformed graph with the original one.

a.

x	0	1	2	3
y	20	18	12	2

b.

x	1	3	5	7	9	11	15
y	4	22	56	106	172	254	466

c.

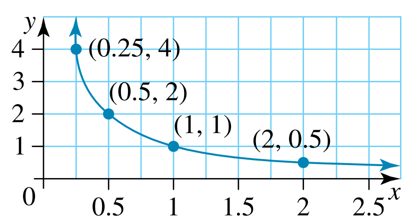
x	9	10	11	12	13	14	15
y	203	275	315	329	323	303	275

d.

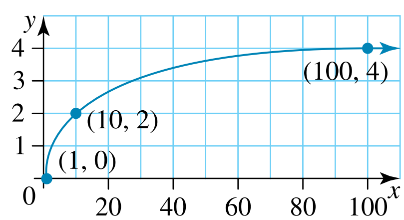
x	0	1	4	9	16	25	36	49	64
y	16	17	18	19	20	21	22	23	24

4. **WE5** Redraw the following graph by plotting y-values against $\frac{1}{x}$-values.

a.

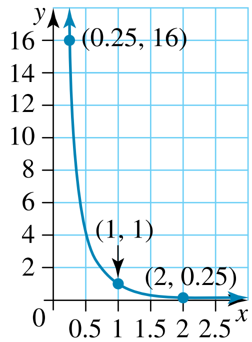


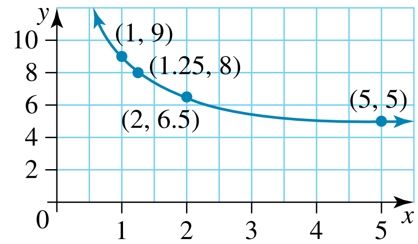
b.



5. i. Redraw each of these graphs by plotting the y-values against $\frac{1}{x}$-values.

ii. Which of the graphs are linearised by the $\frac{1}{x}$ transformation?

a. 

b. 

6. For each of the table of values shown:

i. draw a graph of the original values

ii. construct a table of values to show $\frac{1}{x}$-values against the original y-values

iii. draw a graph of the transformed values

iv. write a comment that compares the transformed graph with the original one.

a.

x	2	4	6	8	10
y	40	25	18	17	16

b.

x	1	2	5	10	20	25
y	40	27.5	20	17.5	16.25	16

c.

x	0	1	2	3	4	5	6
y	60	349	596	771	844	785	564

d.

x	0	1	4	6	8	9	10	11
y	72	66	48	36	24	18	12	6

7. a. Use the rule $y = x^3 + x^2 + x + 1$ to complete the following table of values.

x	0	1	2	3	4	5	6	7	8	9	10
y											

b. Draw a graph of the values shown in the table.

c. Complete an x^2 transformation for the data and graph the results.

d. Comment on the effectiveness of this transformation for linearising this data.

e. Suggest a transformation that might be more effective in this case.

8. a. Use the rule $y = \frac{-x^3}{5} + 3x^2 - 3x + 5$ to complete the following table of values, giving your answers correct to 1 decimal place.

x	1	2	3	4	5	6	7	8	9
x^2									
x^3									
y									

b. Which transformation produces the best linear relationship for this data?

9. a. Use the rule $y = \frac{1}{x^2}$ to complete the following table of values, giving your answers correct to 1 decimal place.

x	0.25	0.5	1	1.25	2	2.5	4	5
$\frac{1}{x}$								
y								

b. Draw graphs of the original data and the transformed data.
c. Comment on the effectiveness of this transformation for linearising this data.

10. A science experiment measured the distance an object travels when dropped from a height of 5 m. The results of are shown in the following table.

Time (s)	0.2	0.4	0.6	0.8	1.0
Distance (cm)	20	78	174	314	490

a. Draw a graph to represent the data.
b. Select an appropriate data transformation to linearise the data and show the resultant table of values and graph.
c. Draw a graph to show the actual height of the object for the time period shown in the table.
d. Would the same data transformation linearise the data this time? Explain your answer.

11. Data for the population of Australia is shown in the table.
a. Draw a graph to represent the data. (Use $x = 0$ for 1880, $x = 40$ for 1920 etc.)
b. Select an appropriate data transformation to linearise the data and show the resultant table of values and graph.

c. Use CAS to complete the following table, then draw the resultant graph.

Year	Population (millions)
1880	2.2
1920	5.4
1960	10.4
1980	14.8
1990	17.2
2000	19.1
2010	22.2

x	0.2	0.4	0.6	0.8	1.0
log (population) (correct to 2 d.p.)					

d. Is the transformed data in the table from part **c** a better linearisation than the one you chose in part **b**? Explain your answer.

12. a. Use CAS to complete the details of the table correct to 2 decimal places.

x	10	20	30	40	50	60	70	80
y	4	5.2	5.9	6.4	6.8	7.1	7.4	7.6
$\frac{1}{x}$								
$\log(x)$								

b. Use CAS to draw the two data transformations.

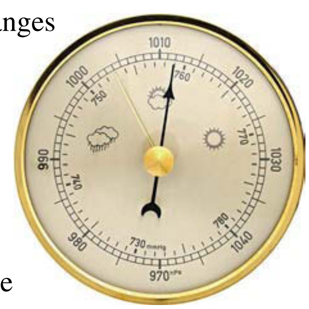
c. Which transformation gives the better linearisation of the data?

13. Measurements of air pressure are important for making predictions about changes in the weather. Because air pressure also changes with altitude, any predictions made must also take into account the height at which any measurements were taken.

The table gives a series of air pressure measurements taken at various altitudes.

a. Draw a graph to represent the data.

b. Select an appropriate data transformation to linearise the data and show the resultant table of values and graph.



c. Does the transformation result in a relationship that puts all of the points in a straight line? Explain what this would mean.

Altitude (m)	Air pressure (kPa)
0	100
300	90
1500	75
2500	65
3000	60
3500	55
4000	50
5000	45
5500	40
6000	35
8000	25
9000	18
12 000	12
15 000	10

14. Data comparing consumption of electricity with the maximum daily temperature in a country is shown in the following table.

Max. temp. (°C)	21.2	22.6	25.7	30.1	35.1	38.7	37.1	32.7	27.9	23.9	21.7	39.6
Consumption (GWh)	3800	3850	3950	4800	5900	6600	6200	5200	4300	3900	3825	7200

a. Which variable should be used for the x-axis? Explain your answer.
b. Use CAS or other technology to:
   i. draw the graph of the original data.
   ii. redraw the graph using both an x^2 and a $\frac{1}{x}$ transformation
   iii. comment on the effect of both transformations.

15. Use CAS or other technology to answer the following questions.
a. Use the rule $y = -0.25x^2 + 4x + 5$ to complete the details for the following table.

x	1	2	3	4	5	6	7	8
x^2								
y								

b. Has the x^2 transformation linearised the data for this group of x-values?

c. Repeat part **a** for whole number x-values between 8 and 17. Does this transformation linearise this section of the data?

d. Compare the effectiveness of the two data transformations for linearising the data.

16. Use CAS or other technology to answer the following questions.

a. Use the rule $y = -2x^2 + 25$ to complete the details for the following table.

x	0.5	1	1.5	2	2.5	3	3.5
x^2							
y							

b. Has the x^2 transformation linearised the data for this group of x-values?

c. Complete the details for the following table and draw the transformed graph.

x	2	3	4	5	6	7
y						
y^2						

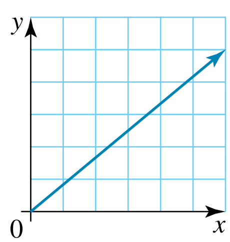
d. Compare the effectiveness of the two data transformations for linearising the data.

12.4 Data modelling

12.4.1 Modelling non-linear data

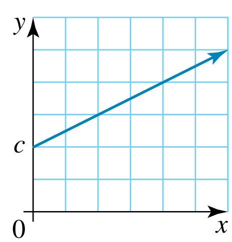
In the previous section we saw how data can be linearised through data transformations. Non-linear data relationships can thus be represented with straight line graphs and variation statements can be used to establish the rules that connect them.

The **constant of variation**, k, will be the gradient of the straight line.



$y \propto x \Rightarrow y = kx$

Straight line through origin



$y = kx + c$

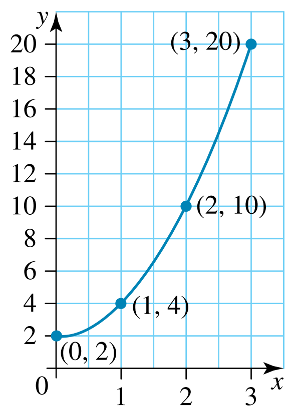
Straight line with y-intercept at c

WORKED EXAMPLE 6

Find the rule for the transformed data shown in the table:

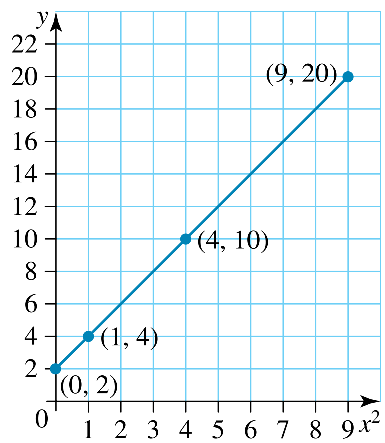
Original data

x	0	1	2	3
y	2	4	10	20



Transformed data

x	0	1	2	3
x^2	0	1	4	9
y	2	4	10	20



THINK

1. Identify the variation statement from the labels of the axes of the straight line graph.

WRITE

The straight line graph indicates that y varies directly with x^2: $y \propto x^2$, which indicates that the rule will be of the form $y = kx^2$.

2. Find k by calculating the gradient of the straight line.

$k = \frac{\text{rise}}{\text{run}}$

Using (9, 20) and (0, 2):

$$\begin{aligned} k &= \frac{20 - 2}{9 - 0} \\ &= \frac{18}{9} \\ &= 2 \end{aligned}$$

3. Identify the value of the y-intercept, c.

The straight line graph intercepts the y-axis at $y = 2$.

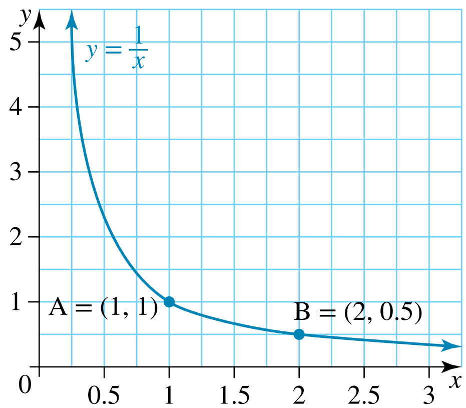
4. Substitute the values of k and c in $y = kx^2 + c$ to state the answer.

$$\begin{aligned} y &= kx^2 + c \\ &= 2x^2 + 2 \end{aligned}$$

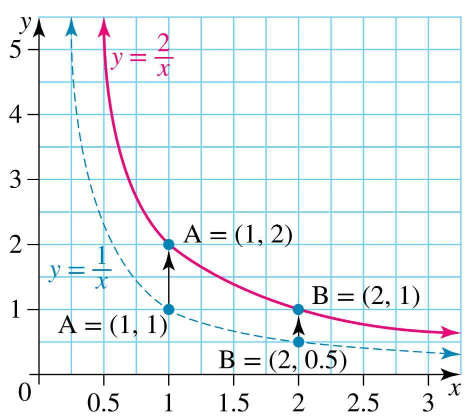
12.4.2 Modelling with $y = \frac{k}{x} + c$

We know that when x and y vary inversely, the rules $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$ apply, and the graph will be hyperbolic in shape.

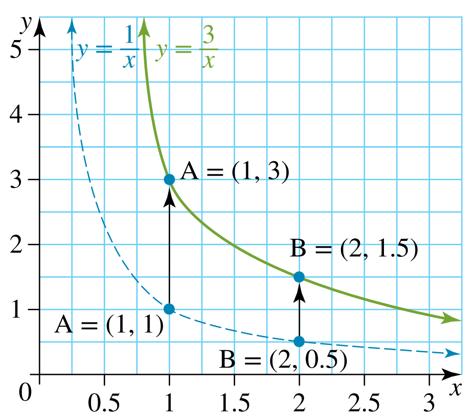
As the constant of variation, k, increases, the graph stretches out away from the horizontal axis and parallel to the vertical axis.



$k = 1 \Rightarrow y = \frac{1}{x}$

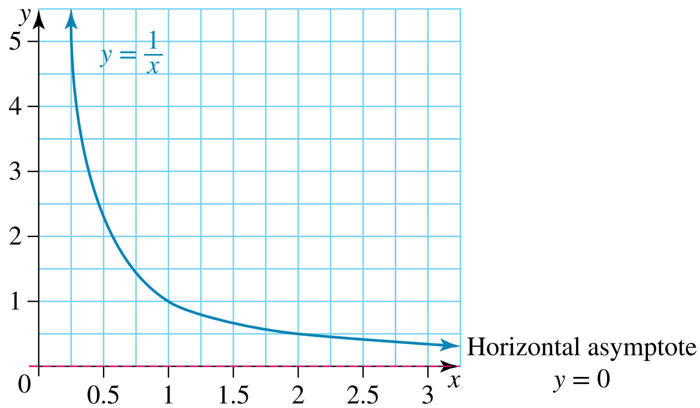


$k = 2 \Rightarrow y = \frac{2}{x}$

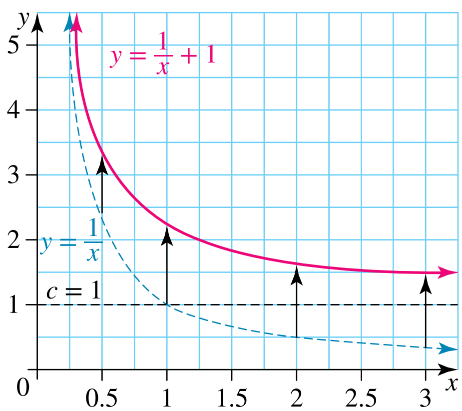


$k = 3 \Rightarrow y = \frac{3}{x}$

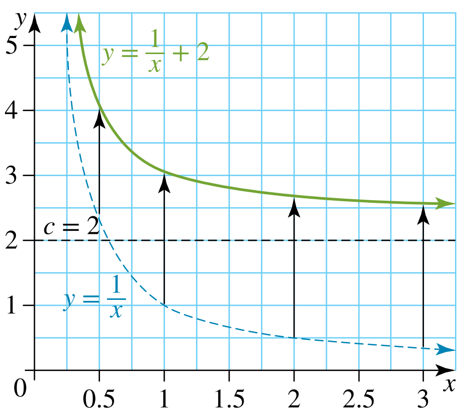
For inverse relationships of the type $y = \frac{k}{x} + c$, as the value of c increases, every point on the graph translates (moves) parallel to the horizontal axis by a distance of c. In addition, no y-coordinate on the graph will ever actually equal the value of c, because $\frac{1}{x}$ will never equal zero. The line $y = c$ is known as the **horizontal asymptote**.



$c = 0 \Rightarrow y = \frac{1}{x}$



$c = 1 \Rightarrow y = \frac{1}{x} + 1$



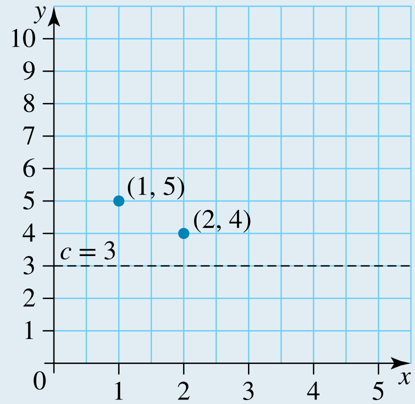
$c = 2 \Rightarrow y = \frac{1}{x} + 2$

WORKED EXAMPLE 7

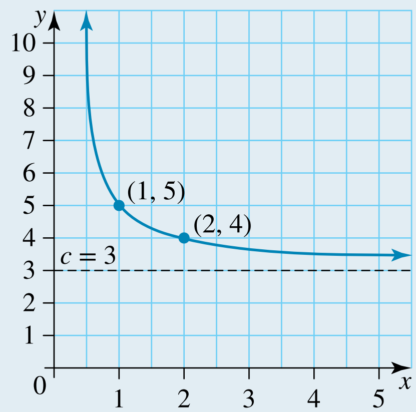
Draw the graph of $y = \frac{2}{x} + 3$, indicating the position of the horizontal asymptote and the coordinates for when $x = 1$ and $x = 2$.

THINK	WRITE/DRAW
1. Identify the value of the horizontal asymptote from the general form of the equation, $y = \frac{k}{x} + c$.	$y = \frac{2}{x} + 3$, so the horizontal asymptote is $c = 3$.
2. Substitute the values into the rule for the coordinates required.	When $x = 1, y = \frac{2}{1} + 3 = 5$, and when $x = 2, y = \frac{2}{2} + 3 = 4$. The required coordinates are (1, 5) and (2, 4).

3. Mark the asymptote and the required coordinates on the graph.



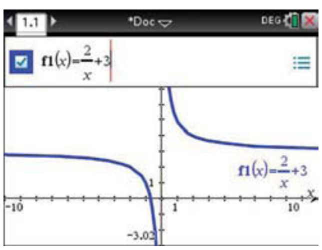
4. Complete the graph by drawing the hyperbola.



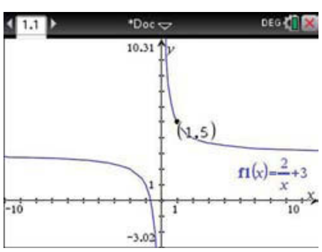
TI | THINK

1. On a Graphs page, complete the entry line for function 1 as:
$f1(x) = \frac{2}{x} + 3$
then press ENTER.

WRITE



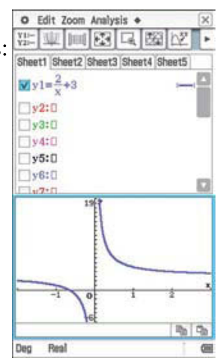
2. Press MENU then select:
5: Trace
1: Graph Trace
Type '1' then press ENTER twice to mark the point (1, 5) on the graph.



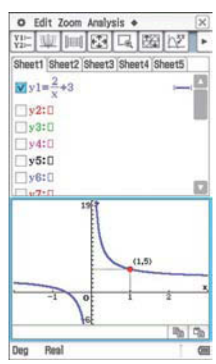
CASIO | THINK

1. In a Graph & Table screen, complete the entry line for $y1$ as:
$y1 = \frac{2}{x} + 3$
Select the tick box then press the $ icon.

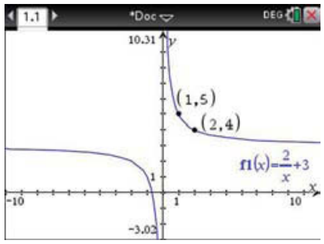
WRITE



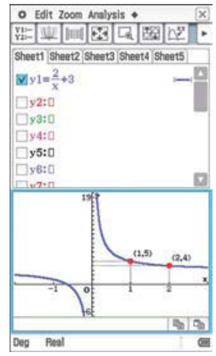
2. Select:
- Analysis
- Trace
Type '1' then select OK. Press EXE to mark the point (1, 5) on the graph.



3. Press MENU then select:
5: Trace
1: Graph Trace
Type ‘2’ then press ENTER twice to mark the point (2, 4) on the graph.



3. Select:
- Analysis
- Trace
Type ‘2’ then select OK. Press EXE to mark the point (2, 4) on the graph.



12.4.3 Logarithmic functions

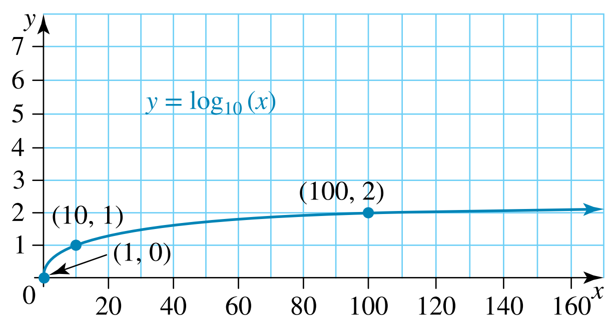
Logarithms (‘logs’) can be very useful when dealing with various calculations in mathematics, as they are the inverse of exponentials. Log transformations can also be very useful for linearising some data types.

12.4.4 Functions of the type $y = a\log_{10}(x) + c$

Base 10 logarithms, written ‘$\log_{10}$’, are usually an inbuilt function in calculators and are obtained using the ‘log’ button. We can work between exponential and logarithmic functions by using the relationship $y = A^x \Leftrightarrow \log_A(y) = x$. The following tables show some values for the functions $y = 10^x$ and $y = \log_{10}(x)$.

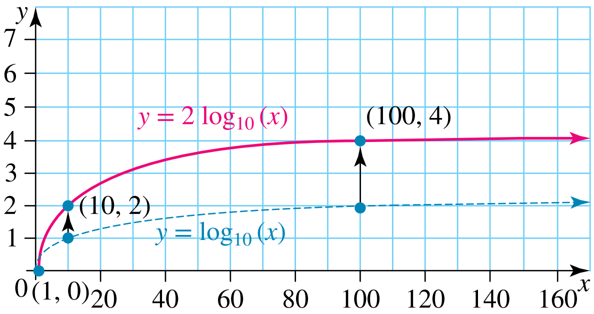
x	0	1	2	3
$y = 10^x$	$10^0 = 1$	$10^1 = 10$	$10^2 = 100$	$10^3 = 1000$

x	1	10	100	1000
$y = \log_{10}(x)$	$\log_{10}(1) = 0$	$\log_{10}(10) = 1$	$\log_{10}(100) = 2$	$\log_{10}(1000) = 3$

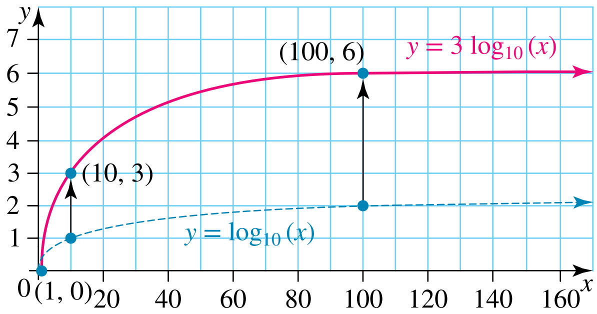


$y = \log_{10}(x)$

For functions of the type $y = a\log_{10}(x) + c$, as the value of a increases, the graph stretches out away from the horizontal axis and parallel to the vertical axis.

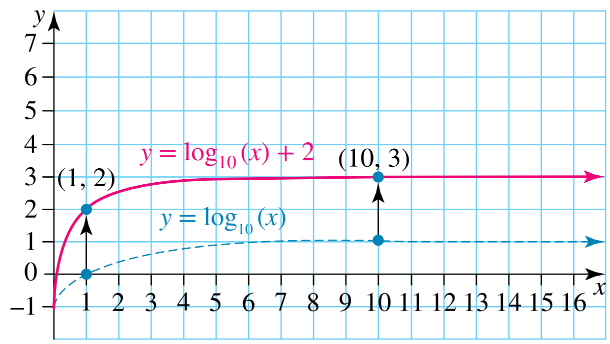


$a = 2$



$a = 3$

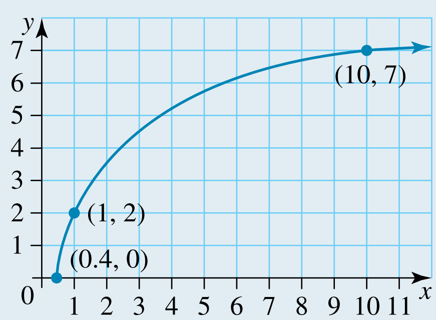
As the value of c increases, every point on the graph translates (moves) parallel to the horizontal axis by a distance of c. This will also change the x-axis intercept.



$c = 2$

WORKED EXAMPLE 8

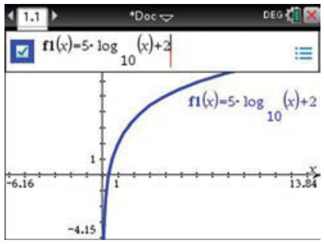
Draw the graph of $y = 5\log_{10}(x) + 2$ indicating the coordinates for when $x = 1$ and $x = 10$. Use CAS to find the x-intercept correct to 1 decimal place.

THINK	WRITE/DRAW
1. Substitute the values into the rule for the coordinates required.	When $x = 1$, $y = 5\log_{10}(1) + 2$ $= 2$ and when $x = 10$, $y = 5\log_{10}(10) + 2$ $= 7$ The required coordinates are (1, 2) and (10, 7).
2. Use CAS to find the x-axis intercept by solving the equation equal to zero.	Solve: $5\log_{10}(x) + 2 = 0$ $\Rightarrow x = 10^{\frac{-2}{5}} \approx 0.4$
3. Draw the graph with the indicated coordinates.	

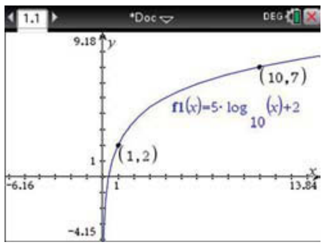
TI | THINK

WRITE

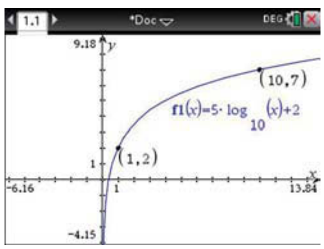
1. On a Graphs page, complete the entry line for function 1 as:
$f1(x) = 5\log_{10}(x) + 2$
then press ENTER.



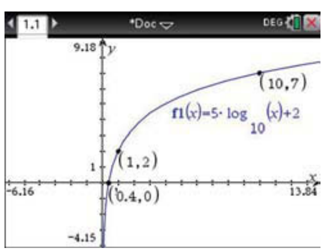
2. Press MENU then select:
5: Trace
1: Graph Trace
Type '1' then press ENTER twice to mark the point (1, 2) on the graph.



3. Press MENU then select:
5: Trace
1: Graph Trace
Type '10' then press ENTER twice to mark the point (10, 7) on the graph.



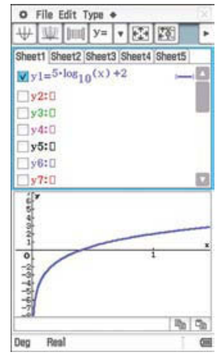
4. Press MENU then select:
6: Analyze Graph
1: Zero
When prompted for the lower bound, move the cursor to the left of the x-intercept and click.
When prompted for the upper bound, move the cursor to the right of the x-intercept and click.



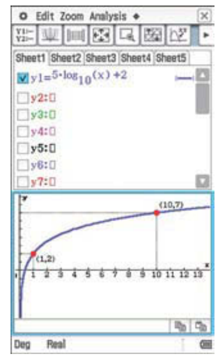
CASIO | THINK

WRITE

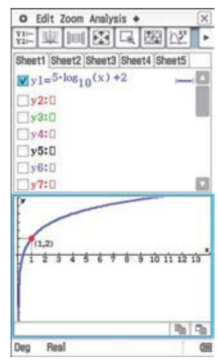
1. In a Graph & Table screen, complete the entry line for $y1$ as:
$y1 = 5\log_{10}(x) + 2$
Select the tick box then press the $ icon.



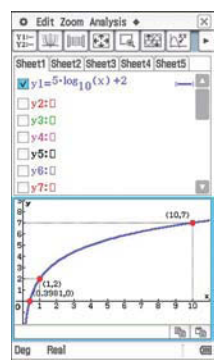
2. Select:
- Analysis
- Trace
Type '1' then select OK.
Press EXE to mark the point (1, 2) on the graph.



3. Select:
- Analysis
- Trace
Type '10' then select OK.
Press EXE to mark the point (10, 7) on the graph.



4. Select:
- Analysis
- G-Solve
- Root
then press EXE to mark the x-intercept on the graph.



on Resources

Interactivity: Modelling non-linear data (int-6492)

studyon

Units 1 & 2 > AOS 5 > Topic 3 > Concepts 2 & 4

Modelling given non-linear data Concept summary and practice questions

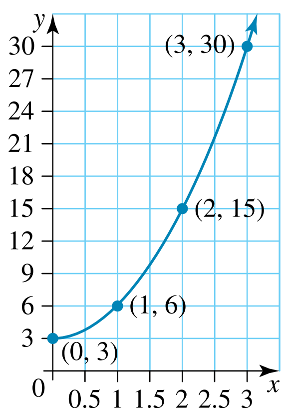
Modelling with the logarithmic function Concept summary and practice questions

Exercise 12.4 Data modelling

1. **WE6** Find the rule for the transformed data shown in the table.

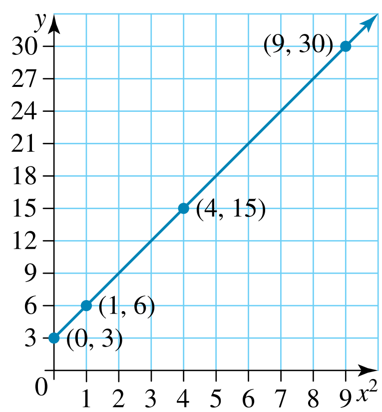
Original data

x	0	1	2	3
y	3	6	15	30



Transformed data

x	0	1	2	3
x^2	0	1	4	9
y	3	6	15	30



2. Find the rule in the form $y = kx^2 + c$ that relates the variables in the following tables.

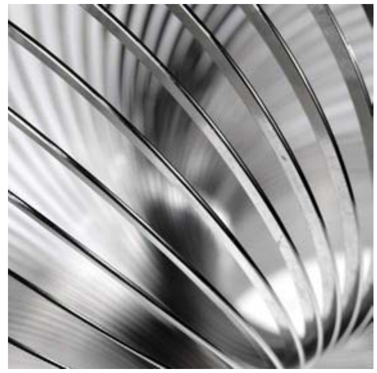
a.

x	0	1	2	3
y	25	23	17	7

b.

x	0	1	2	3	4	5	6	7
y	10	10.4	11.6	13.6	16.4	20	24.4	29.6

3. For each of the following applications:
   i. write a formula to represent the relationship
   ii. calculate the required values.
      a. According to Hooke's law, the distance, d, that a spring is stretched by a hanging object varies directly with the object's mass, m. If a 6-kg mass stretches a spring by 40 cm, what will the distance be when the mass is 15 kg?
      b. The length of a radio wave, L, is inversely proportional to its frequency, f. If a radio wave has a length of 150 metres when the frequency is 600 kHz, what is the length of a radio wave that has a frequency of 1600 kHz?
      c. The stopping distance of a car, d, after the brakes have been applied varies directly as the square of the speed, s. If a car travelling 100 km/h can stop in 40 m, how fast can a car go and still stop in 25 m?



4. a. Use the rule $y = 2x^2 + 5$ to complete the following table of values.

x	0	1	2	3	4	5
y						

   b. Draw a graph of the table.
   c. Use an x^2 transformation and redraw the graph using the transformed data.
   d. What is the gradient of the transformed data?
5. **WE7** Draw the graph of $y = \frac{4}{x} + 5$, indicating the position of the horizontal asymptote and the coordinates for when $x = 0.5$ and $x = 8$.
6. Find the rule in the form $y = \frac{k}{x} + c$ that relates the variables in the following tables.

   a.

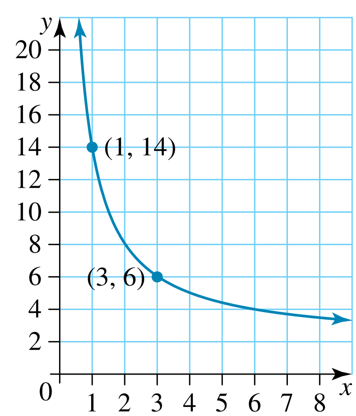
x	1	2	3	4	6	12
y	18	12	10	9	8	7

   b.

x	1	2	4	5
y	16	6	1	0

7. Consider the inverse relation $y \propto \frac{k}{x} + c$ with the graph shown below.
   a. Use the coordinates (1, 14) and (3, 6) to find the values of k and c.
   b. Complete the following table.

x						
y	50	26	18	10	8	3
$\frac{1}{x}$						



   c. Draw a graph of y against $\frac{1}{x}$ for the points in the table.
   d. What is the gradient of the transformed data?

8. Consider the following table.

x	0	1	2	3	4	5
y	6	6.75	9	12.75	18	24.75

Given that y varies directly with x^2:

a. draw a straight-line representation of the relationship

b. find the rule for the straight line graph.

9. WE8 Draw the graph of $y = 4\log_{10}(x) + 4$, indicating the coordinates for when $x = 1$ and $x = 10$. Use CAS to find the x-axis intercept correct to 2 decimal places.

10. Find the rule in the form $y = a\log_{10}(x) + c$ that relates the variables in the following tables.

a.

x	1	10	100
y	1	3	5

b.

x	10	100	1000
y	1	7	13

11. A study of a marsupial mouse population on an isolated island finds that it changes according to the rule $P = 10\log_{10}(t) + 600$, for $t \geq 1$, where P is the total population and t is the time in days for the study.

a. Complete the table.

t	1	10	100
P			

b. Draw a graph of the changing population over this time period.

c. What will be the population on the 1000th day of the study?

12. The relationship between the speed of a car, s km/h, and its exhaust emissions, E g/1000 km, is shown in the following table.

s	50	55	60	65	70	75	80	85
E	1260	1522.5	1810	2122.5	2460	2822.5	3210	3622.5

a. Draw a straight-line representation of the relationship.

b. Find the rule for the straight line graph.

13. The maximum acceleration possible for various objects of differing mass is shown in the following table.

Mass (kg)	0.5	3	10	15	20	30	75
Acceleration (m/s^2)	30	3	1.5	1	0.75	0.5	0.2

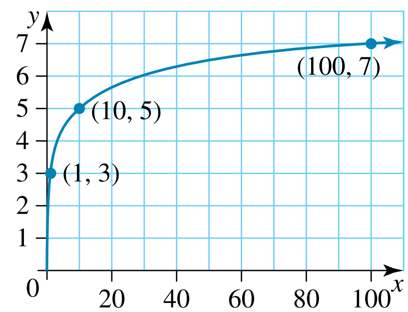
a. Describe the type of variation that is present for this data.
b. Draw a graph of the relationship.
c. Find the rule for the graph.

14. The frequency, f, of the vibrations of the strings of a musical instrument will vary inversely with their length, l.
a. A string in Katya's cello is 62 cm long and vibrates 5.25 times per second. Complete the following table, giving your answers correct to 1 decimal place.

String length (cm)	70	65	60	55	50	45	40
Vibrations per second							

b. Draw a graph of the table of values.
c. Use CAS to draw a graph of a $\frac{1}{x}$ transformation.
d. Comment on the effect of the transformation.

15. The following graph is of the form $y = a\log_{10}(x) + c$.
a. Use CAS to find the values of a and c.
b. Redraw the graph with a and c increased by adding two to their value. Indicate the coordinates on the graph that correspond to the x-values of 1, 10 and 100.



16. The total number of people, N, infected by a virus after t days can be found using the rule $N = 100 \times \log_{10}(t) + 20$.
a. Giving your answers to the nearest whole number, what is the total number of people infected after:
i. 1 day?
ii. 2 days?
iii. 10 days?
b. Draw a graph of $N = 1000 \times \log_{10}(t) + 20$.
c. Use CAS to find how many days it takes to reach double the number of infected people compared to when $t = 10$.

17. Use CAS to investigate graphs of the form $y = \log_{10}(x - b)$.
a. Draw graphs for:
i. $y = \log_{10}(x - 1)$
ii. $y = \log_{10}(x - 2)$
iii. $y = \log_{10}(x - 3)$.
b. What happens to the graphs when the value of b is changed?

18. Use CAS to investigate relationships of the form $y = 10\log_{10}(x+1) + 2$.

a. Complete the table for the given values.

x	9	99	999
y			

b. Draw a graph for the values in the table.

c. Complete the table of values for $a\log_{10}(x)$ transformation. Give answers correct to 4 decimal places.

d.

x	9	99	999
$\log_{10}(x)$			
y			

e. Draw a graph of the transformed data.

f. Comment on the transformed graph.

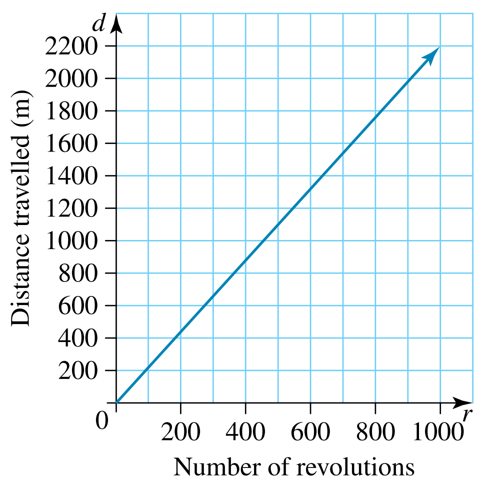
12.5 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. MC The graph shows the connection between the number of revolutions of a bicycle wheel and the distance travelled.
Which of the following is true?

Graph of distance travelled versus number of revolutions



A. $k = 2.2$

B. $\frac{r}{d} = 2.2$ for any point on the graph

C. The units for k are revolutions per metre.

D. $r = 2.2d$

E. $k = 22$

2. MC If b is directly proportional to c^2, what is the constant of variation if $b = 72$ when $c = 12$?

A. 0.5

B. 2

C. 6

D. 36

E. 144

3. MC If $y \propto \frac{1}{x}$ and $y = 12$ when $x = 50$, then when $x = 25$:

A. $y = 6$ B. $y = 18$ C. $y = 12.5$ D. $y = 24$ E. $y = 11.5$

4. MC When an amount of gas is enclosed in a container, the pressure, P, is inversely proportional to the volume, V. Which of the following is *not* true?

A. P varies inversely with V.

B. As V increases, P decreases.

C. If V is halved, P is doubled.

D. $P \propto \frac{1}{V}$

E. $\frac{P}{V} = k$

5. **MC** If y varies with the square of x and $y = 100$ when $x = 2$, then the constant of variation is:

A. 400 **B.** 200 **C.** 100 **D.** 50 **E.** 25

6. **MC** If R varies directly with the square root of p and inversely with the cube of s, and $R = 2$ when $p = 16$ and $s = 2$, then:

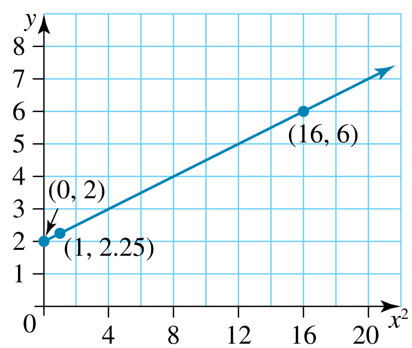
A. $R = \frac{4\sqrt{s}}{p^3}$ **B.** $s = \frac{4\sqrt{R}}{p^3}$ **C.** $R = \frac{4\sqrt{p}}{s^3}$ **D.** $R = \frac{6\sqrt{s}}{p^3}$ **E.** $R = \frac{4\sqrt{s}}{p}$

7. **MC** The table of values shown follows the rule $y = kx^n$, where k and n are respectively equal to:

x	2	3	4	5
y	11.2	25.2	44.8	70

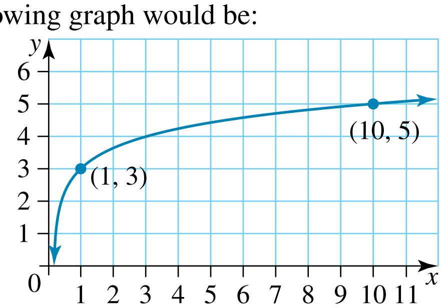
A. 2 and 2.6 **B.** 2 and 2.8 **C.** 2.8 and 2 **D.** 1 and 2.8 **E.** 1 and 2.6

8. **MC** The graph shown follows the general rule $y = ax^2 + b$. A possible rule for the graph would be:



A. $y = 4x^2 + 2$ **B.** $y = \frac{3x^2}{4} + 2$ **C.** $y = 0.2x^2 + 2$ **D.** $y = \frac{x^2}{4} + 2$ **E.** $y = 0.5x^2 + 2$

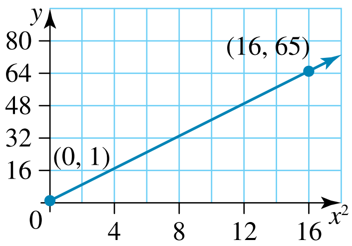
9. **MC** A possible rule for the following graph would be:



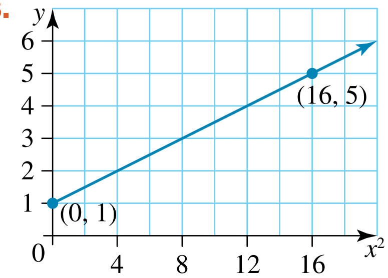
A. $y = \log_{10}(x)$ **B.** $y = 2\log_{10}(x) + 2$ **C.** $y = 3\log_{10}(x) + 2$
D. $y = 2\log_{10}(x) + 3$ **E.** $y = 2\log_{10}(x)$

10. MC The graph with the rule $y = \frac{x^2}{4} + 1$ is:

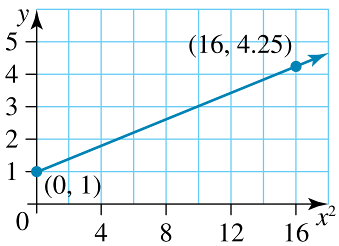
A.



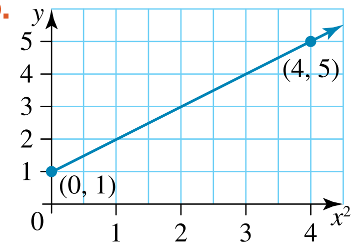
B.



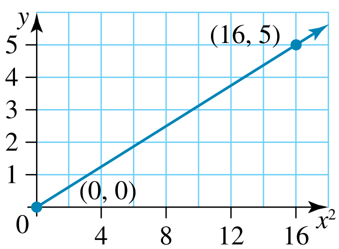
C.



D.



E.



Short answer

1. The distance, d metres, that an object has fallen from rest after a time, t seconds, is given in the table.

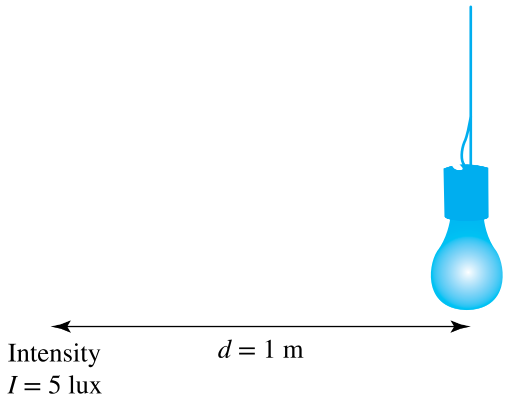
Time, *t* (seconds)	0	1	2	3	4	5
Distance, *d* (metres)	0	4.9	19.6	44.1	78.4	122.5

a. What type of variation is this?
b. What is the rule that relates the distance, d, and the time, t?
c. Use the rule to find the distance fallen in 12 seconds.
d. Use the rule to calculate the time taken to fall a distance of 25 metres, correct to 2 decimal places.

2. The length, L metres, of a light wave varies directly with its speed, v (m/s), and inversely with its frequency, f (Hz). For a light wave of frequency 660 Hz travelling at 330 m/s, the wavelength is 0.5 m.
a. Find a rule that connects the wavelength, the frequency and the speed.
b. Calculate the wavelength of a light wave of speed 3.0×10^8 m/s and frequency 5.0×10^{14} Hz.

3. The time, t hours, to complete a 100-kilometre journey varies inversely with the traveller's speed, v (m/s).
a. Write a proportionality statement connecting t and v.
b. Write a rule connecting t and v using a constant of variation, k.
c. Calculate the time to complete the 100 km at 50 km/h.
d. Calculate the constant of variation.
e. State the rule connecting t and v.

4. The intensity, I (lux), of the light from a light bulb is inversely proportional to the square of the distance, d (m), from the bulb. At a distance of 1.0 m, the intensity is 5.0 lux, as shown in the diagram.



a. Write a variation statement using a '$\propto$' sign for the relationship between I and d.
b. Calculate the constant of variation and hence write a rule for the relationship.
c. Use the rule to calculate the intensity of the light at a distance of 3.5 m from the bulb. Give your answer correct to 2 decimal places.
d. Calculate the distance at which the intensity is 0.05 lux.

5. A call-centre employee is paid by the hour. The table shows the number of hours worked in a 5-day week and the wages earned each day. The wages earned vary directly with the number of hours worked.

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Hours worked (h)	7.0	6.5	9.0	7.0	8.5
Wages (w)	\$119.00	\$110.50	\$153.00	\$119.00	\$144.50

a. Write an equation for the relationship between the variables.
b. Calculate the constant of variation, k.
c. Calculate the value of $\frac{\text{wages}}{\text{hours}}$ for each day.
d. Comment on what your answer to part **c** indicates.

6. Consider the data in the following table.

x	3.5	6.1	9.7	11.2
y	7.35	12.81	21.34	23.52

a. Explain why y is not directly proportional to x.
b. One y-value can be changed so that y is directly proportional to x. State the required change.

Extended response

1. Johann Kepler formulated three laws of planetary motion in the early seventeenth century. The second of these stated that for any planetary system, $R^3 \propto T^2$, where R is the radius of orbit and T is the time taken for the orbit (the period).

a. Copy and complete the following table to test Kepler's Second Law.

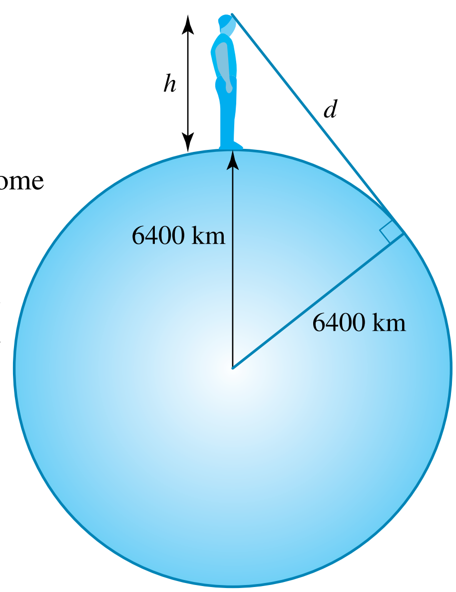
Planet	**Orbital radius, *R* (m)**	R^3	**Orbital period, *T* (s)**	T^2	$\frac{R^3}{T^2}$
Mercury	5.79×10^{10}		7.60×10^{6}		
Venus	1.08×10^{11}		1.94×10^{7}		
Earth	1.49×10^{11}		3.16×10^{7}		
Mars	2.28×10^{11}		5.94×10^{7}		
Jupiter	7.78×10^{11}		3.74×10^{8}		
Saturn	1.43×10^{12}		9.30×10^{8}		
Uranus	2.87×10^{12}		2.66×10^{9}		
Neptune	4.50×10^{12}		5.20×10^{9}		

b. Explain how the data supports Kepler's Second Law.

2. A rectangle has a length of 8 cm and an area of 24 cm^2.
   a. Write a variation statement relating the width, w, and length, l, for a rectangle whose area is 24 cm^2.
   b. What type of variation exists in this situation?

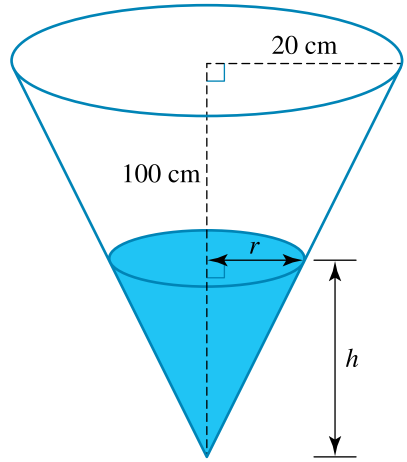
3. The Earth is roughly a sphere of radius 6400 km. We can use this, together with Pythagoras' theorem and some geometry, to make some predictions about how the distance to the horizon changes with height.

   The diagram shows a person of height h standing on the Earth's surface and looking to the horizon, which is a distance d km away. The line of sight is a tangent to the Earth and meets the radius at a right angle.



   a. Use Pythagoras' theorem to show that $d = \sqrt{h^2 + 12\,800 \times h}$.
   b. Use $d = \sqrt{h^2 + 12\,800 \times h}$ to calculate the distance, d km, to the horizon when a person of height $h = 1.8$ m (0.0018 km) to eye level stands on the sea shore.
   c. Similarly, calculate the distance to the horizon when the person is standing on higher ground and is looking out to sea with their eye level at a height of 100 m. Give your answer correct to 1 decimal place.
   d. As long as the height, h, is reasonably small compared to the Earth's radius, the h^2 term under the square root sign contributes very little to the result, and the distance effectively becomes $d = \sqrt{12\,800 \times h}$ or $d \propto \sqrt{h}$.
   Calculate the constant of variation in this relationship.

4. Water is being poured into a cone of height 100 cm and radius 20 cm. When the water has reached a height, h, in the vessel, the radius of the surface is r.



a. Determine the rule that relates the height and the radius of the vessel, and use this rule to calculate the height when the radius is 4 cm.
What is the height, h, when $r = 4$ cm?

b. Complete the table by calculating the volume of water in the cone at various radii.

***r* (cm)**	4	6	8	10	15	20
Volume, $V = \dfrac{\pi r^2 h}{3}$ **(cm³)**						

c. Using CAS, a spreadsheet or otherwise, draw a graph of V against r.

d. Does V vary directly as r?

e. Complete the table and draw a graph for an r^3 transformation.

***r* (cm)**	4	6	8	10	15	20
r^3 **(cm³)**						
Volume, $V = \dfrac{\pi r^2 h}{3}$ **(cm³)**						

f. Identify the rule that connects V and r.

studyon

Units 1 & 2 Sit topic test

Answers

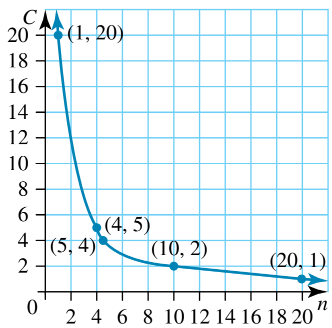
Topic 12 Variation

Exercise 12.2 Direct, inverse and joint variation

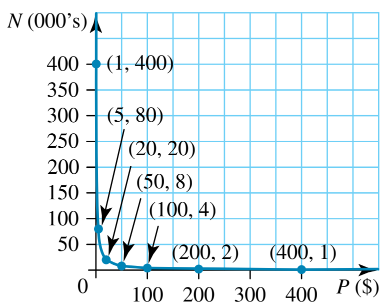
1. $k = 60$, $C = 60N$
2. $d = 90t$
3. a. No, $y \neq kx$. b. Yes, $y = 2.1x$.
4. a. $I \propto A$ b. $I = kA$ c. $k = \frac{1}{2}$
   d. $I = \frac{1}{2}A$ e. \$27 500 f. \$150 000
5. a. $k = 4.5$ b. $K = 4.5$ m
   c. 324 d. 100
6. $T = \dfrac{100}{s}$
7. a.

Family members	20	10	5	4	1
Number of chocolates	1	2	4	5	20

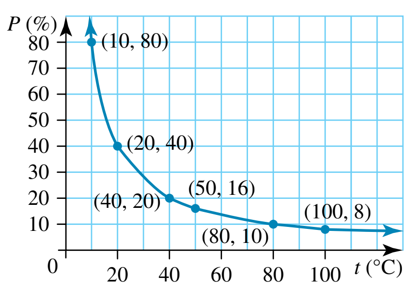
   b.



   c. $C = \dfrac{20}{n}$
8. a.



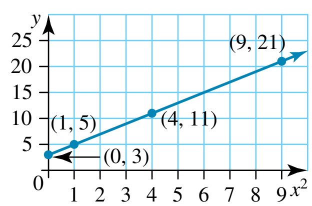
   b. $n = \dfrac{400}{P}$
   c. 16
   d. \$0.0016
9. a.



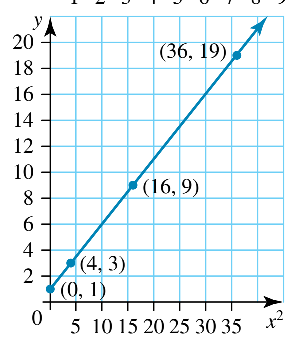
   b. $P = \dfrac{800}{t}$
   c. 32%
   d. 250 °C
10. a. $y \propto x^2z^3$; $y = kx^2z^3$ b. $A \propto BCD$; $A = kBCD$
    c. $V \propto h\sqrt{r}$; $V = kh\sqrt{r}$ d. $U \propto \dfrac{p^2}{\sqrt{q}}$; $U = \dfrac{kp^2}{\sqrt{q}}$
11. a. $X \propto \dfrac{Q^2}{\sqrt{p}}$ b. $X = \dfrac{kQ^2}{\sqrt{p}}$
    c. $k = 50$ d. $X = 612.5$
12. a. $R \propto \dfrac{q^2}{\sqrt{s}}$ b. $R = \dfrac{kq^2}{\sqrt{s}}$
    c. $k = 50$ d. 612.5
    e. 625
13. a. $k = 5$ b. 12.76
14. a. $a \propto \dfrac{F}{m}$ b. $k = 1$ c. 2.5 m/s^2
15. a. $v \propto \dfrac{r}{T}$ b. $k = 1.5$ c. 1950 km/h
16. a. $d \propto at^2$ b. $k = 0.4$ c. 115.2 m
17. a. $L = \dfrac{0.25P}{d^2}$ b. 0.2 W/m^2
18. a. $M \propto \dfrac{r^3}{t^2}$ b. $k = 6.157 \times 10^{11}$
    c. $M = \dfrac{(6.157 \times 10^{11})r^3}{t^2}$ d. 1.067×10^{26} kg

Exercise 12.3 Data transformations

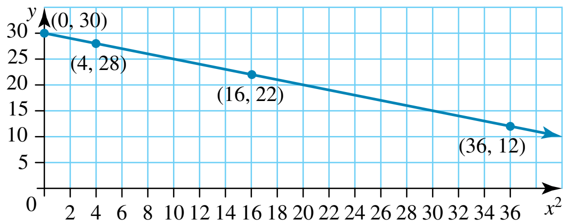
1. a.



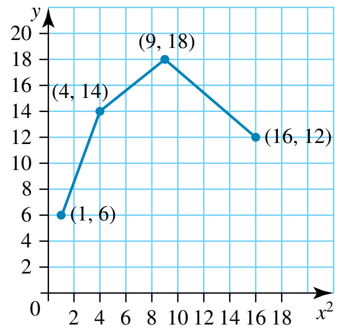
   b.



c.

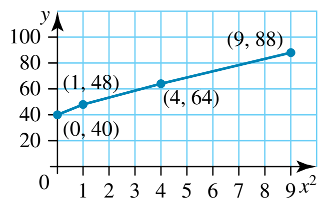


2. a. i.



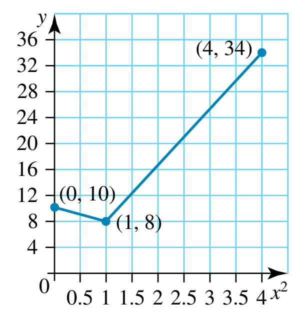
ii. Not linearised

b. i.



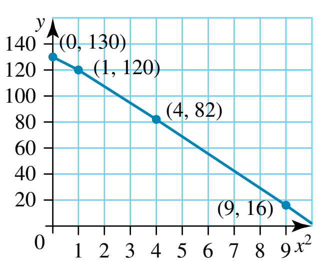
ii. Linearised (not perfectly)

c. i.



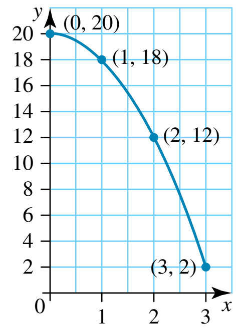
ii. Not linearised

d. i.



ii. Linearised (not perfectly)

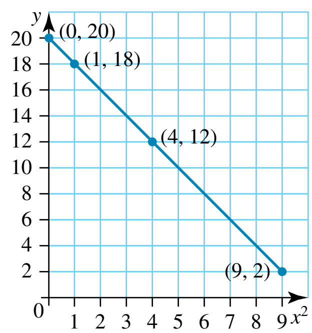
3. a. i.



ii.

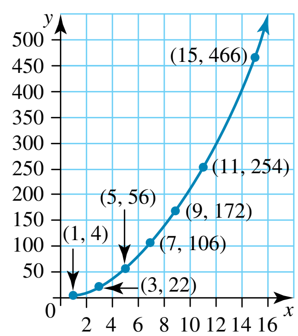
x^2	0	1	4	9
y	20	18	12	2

iii.



iv. The transformed data has been linearised.

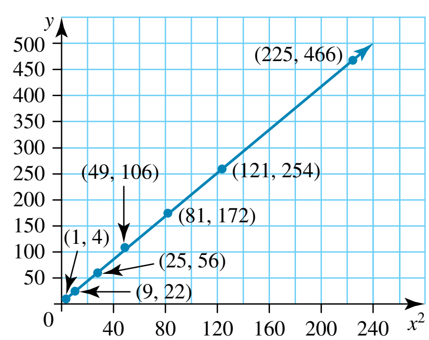
b. i.



ii.

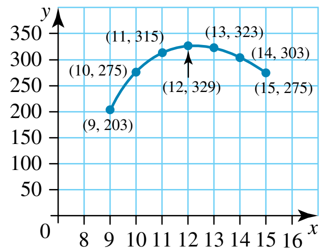
x^2	1	9	25	49	81	121	225
y	4	22	56	106	172	254	466

iii.



iv. The transformed data has been linearised.

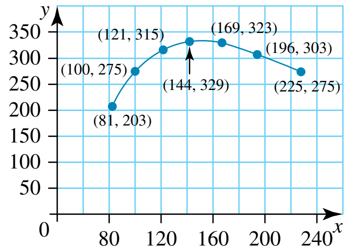
c. **i.**



ii.

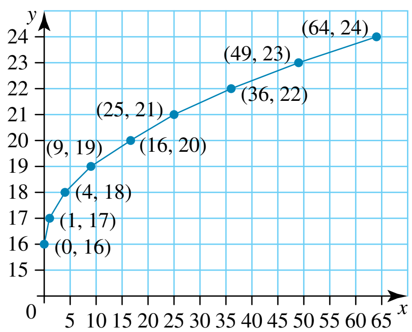
x^2	81	100	121	144	169	196	225
y	203	275	315	329	323	303	275

iii.



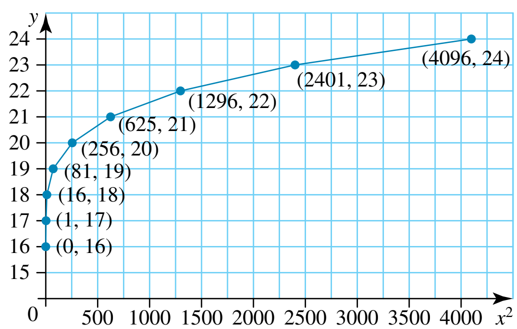
iv. The transformed data has not been linearised.

d. **i.**



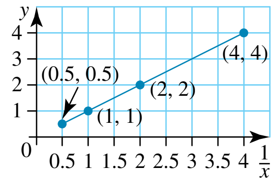
ii. *

iii.



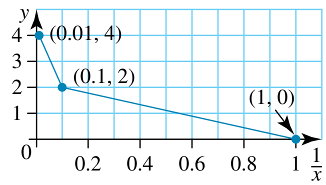
iv. The transformed data has not been linearised.

4. **a.** **i.**



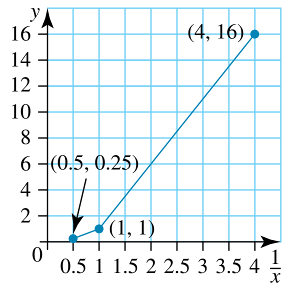
ii. Linearised

b. **i.**



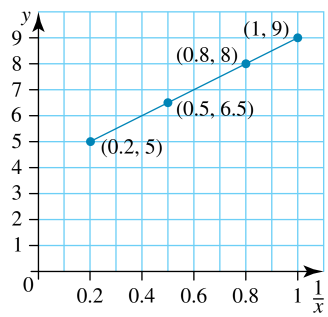
ii. Not linearised

5. **a.** **i.**



ii. Not linearised (but more linear than the original)

b. **i.**

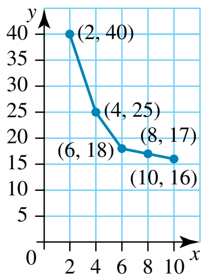


ii. Linearised

3. **d.** **i.** *

x^2	0	1	16	81	256	625	1296	2401	4096
y	16	17	18	19	20	21	22	23	24

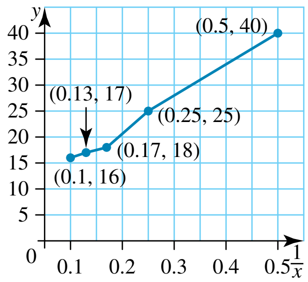
6. a. i.



ii.

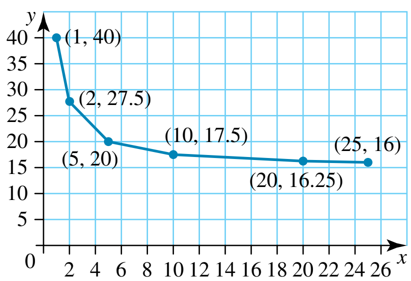
$\frac{1}{x}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$
y	40	25	18	17	16

iii.



iv. The transformed data has been made more linear.

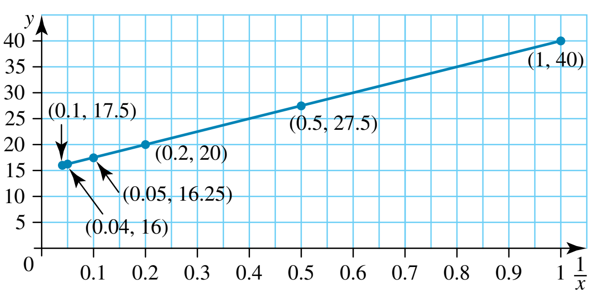
b. i.



ii.

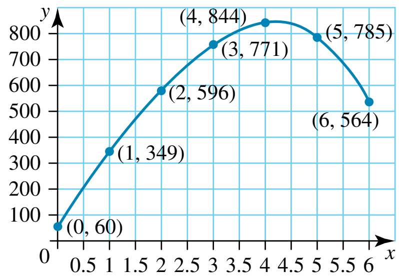
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{25}$
y	40	27.5	20	17.5	16.25	16

iii.



iv. The transformed data has been linearised.

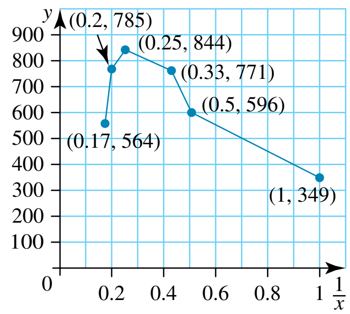
c. i.



ii.

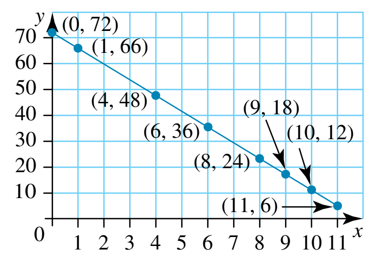
$\frac{1}{x}$	Undefined	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
y	60	349	596	771	844	785	564

iii.



iv. The transformed data has not been linearised.

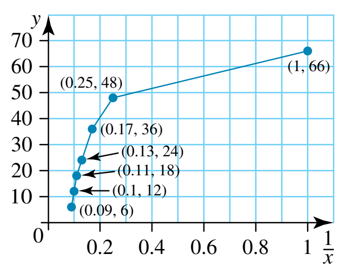
d. i.



ii.

$\frac{1}{x}$	Undefined	1	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$
y	72	66	48	36	24	18	12	6

iii.

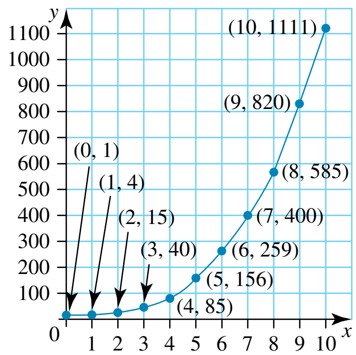


iv. The original data was linear but the transformed data is not.

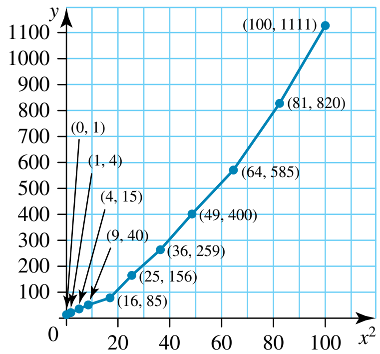
7. a.

x	0	1	2	3	4	5	6	7	8	9	10
y	1	4	15	40	85	156	259	400	585	820	1111

b.



c.



d. The transformed data appears a little more linear.

e. A $\frac{1}{x}$ transformation might work better but would need to be checked.

8. a.

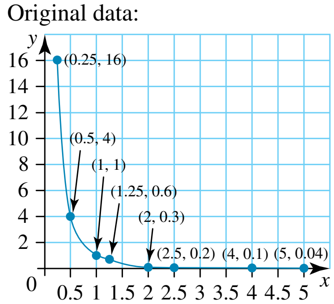
x	1	2	3	4	5	6	7	8	9
x^2	1	4	9	16	25	36	49	64	81
x^3	1	8	27	64	125	216	343	512	729
y	4.8	9.4	17.6	28.2	40	51.8	62.4	70.6	75.2

b. The x^2 transformation gives the best linear transformation for this data.

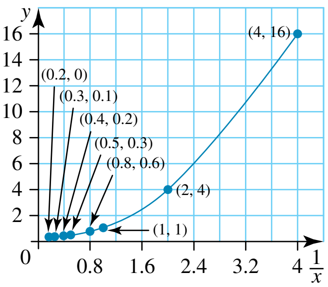
9. a.

x	0.25	0.5	1	1.25	2	2.5	4	5
$\frac{1}{x}$	4	2	1	0.8	0.5	0.4	0.25	0.2
y	16	4	1	0.64	0.25	0.16	0.0625	0.04

b. Original data:

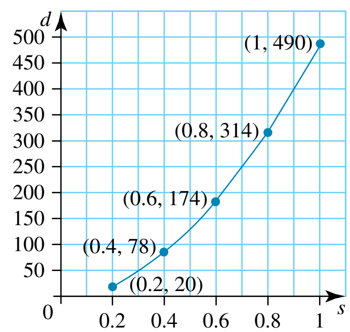


Transformed data:



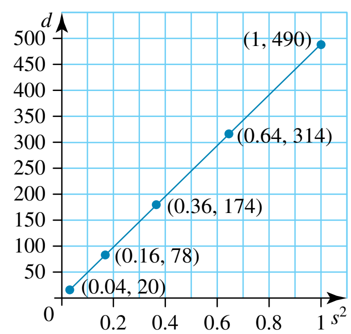
c. This transformation is not very effective at linearising this data.

10. a.

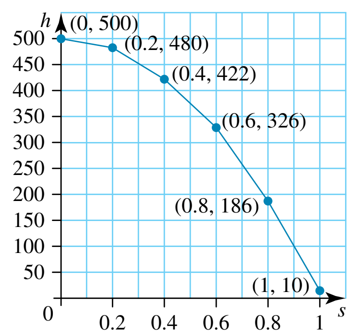


b.

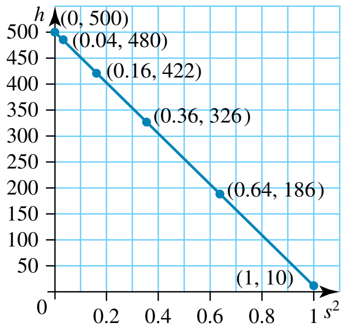
Time^2 (s^2)	0.04	0.16	0.36	0.64	1.0
Distance (cm)	20	78	174	314	490



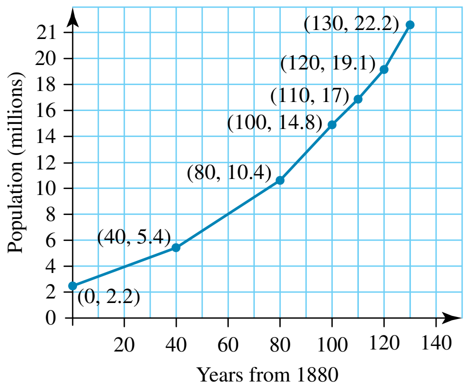
c.



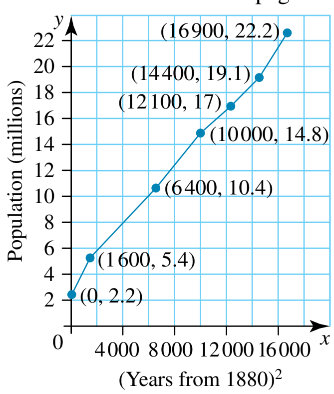
d. The same data transformation will work in this case as the original data is parabolic in its shape, so the transformed data will look like this.



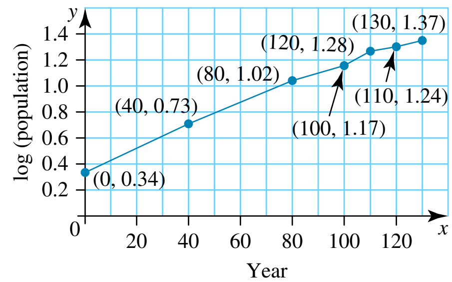
11. a.



b. See table at the foot of the page*



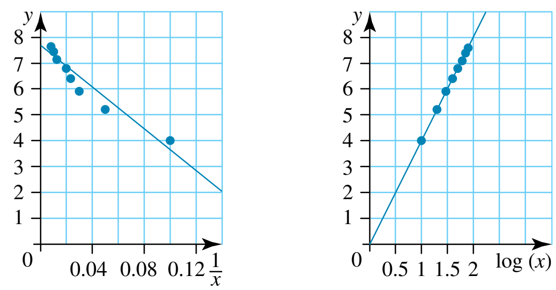
c. See table at the foot of the page*



d. Both transformations show a similar level of linearisation of the data.

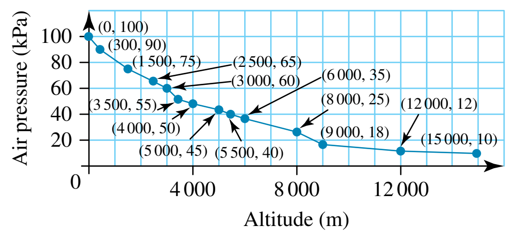
12. a. See table at the foot of the page*

b.



c. When using the calculator, the log (x) transformation gives the best linearisation of the data.

13. a.



11. b. *

Years from 1880	0	40	80	100	110	120	130
(Years from 1880)^2	0	1600	6400	10 000	12 100	14 400	16 900
Population (millions)	2.2	5.4	10.4	14.8	17.2	19.1	22.2

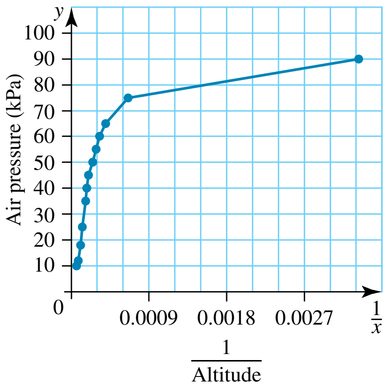
c. *

Years from 1880	0	40	80	100	110	120	130
log (population) (correct to 2 d.p.)	0.34	0.73	1.02	1.17	1.24	1.28	1.37

12. a. *

x	10	20	30	40	50	60	70	80
y	4	5.2	5.9	6.4	6.8	7.1	7.4	7.6
$\frac{1}{x}$	0.10	0.05	0.033	0.025	0.02	0.017	0.014	0.013
log (x)	1.00	1.30	1.48	1.60	1.70	1.78	1.85	1.90

b.

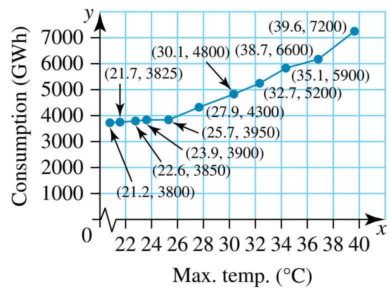


Altitude (m)	Air pressure (kPa)	$\frac{1}{\text{Altitude (m)}}$
0	100	Undefined
300	90	0.003 33
1500	75	0.000 66
2500	65	0.000 40
3000	60	0.000 33
3500	55	0.000 29
4000	50	0.00025
5000	45	0.000 20
5500	40	0.000 18
6000	35	0.000 17
8000	25	0.000 13
9000	18	0.000 11
12 000	12	0.000 08
15 000	10	0.000 06

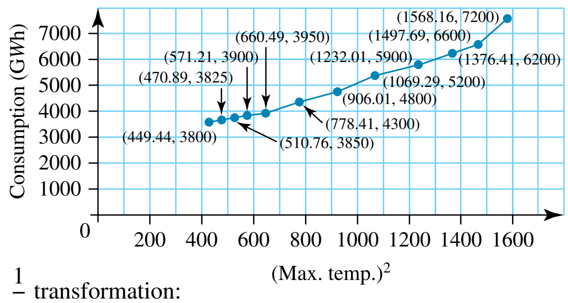
c. The $\frac{1}{x}$ transformation doesn't put all the data in a straight line. Most of the values seem to be more linear, but the lower altitude values curve sharply. This would seem to indicate that an alternative transformation (other than $\frac{1}{x}$ or x^2) is needed to linearise the data.

14. a. As it is likely that the consumption of electricity would be influenced by the maximum temperature, the temperature should be used for the x-axis.

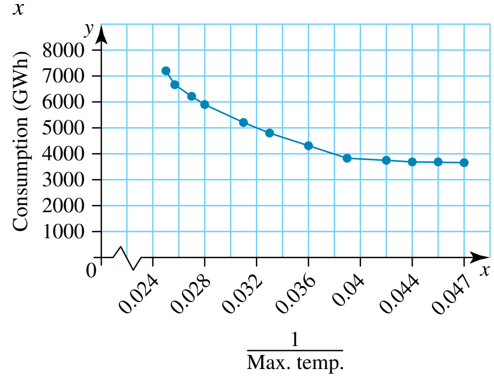
b. i.



ii. x^2 transformation:



$\frac{1}{x}$ transformation:



iii. Both transformations have a linearising effect, but neither appears to be substantially better than the original data.

15. a.

x	1	2	3	4	5	6	7	8
x^2	1	4	9	16	25	36	49	64
y	8.75	12	14.75	17	18.75	20	20.75	21

b. See table at the foot of the page*

c. This time the x^2 transformation has made the data look a little more linear.

d. The x^2 transformation was more effective in linearising the second group of data than the first, but neither were perfect.

16. a.

x	0.5	1	1.5	2	2.5	3	3.5
x^2	0.25	1	2.25	4	6.25	9	12.25
y	24.5	23	20.5	17	12.5	7	0.5

15. b. * The x^2 transformation has not linearised this data.

x	8	9	10	11	12	13	14	15	16	17
x^2	64	81	100	121	144	169	196	225	256	289
y	21	20.75	20	18.75	17	14.75	12	8.75	5	0.75

b. The x^2 transformation has linearised this data.
c. See table at the foot of the page*
d. Both transformations make the data more linear, but the x^2 transformation is better in this case.

Exercise 12.4 Data modelling

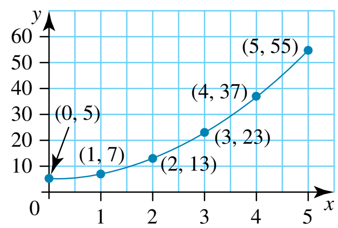
1. $y = 3x^2 + 3$

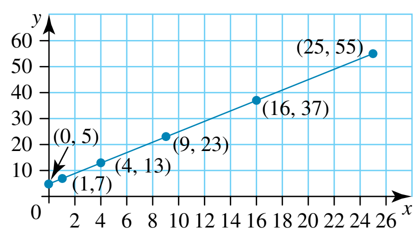
2. a. $y = 22x^2 + 25$
b. $y = 0.4x^2 + 10$

3. a. i. $d = 6.7$ m
ii. 100.5 cm
b. i. $L = \dfrac{90\,000}{f}$
ii. 56.25 m
c. i. $d = 0.004\ s^2$
ii. 79 km/h

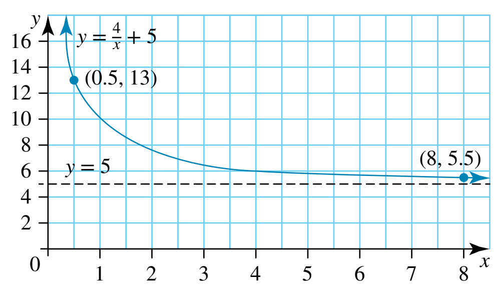
4. a.

x	0	1	2	3	4	5
y	5	7	13	23	37	55

b.


c.


d. 2

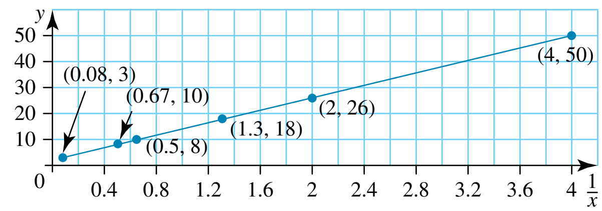
5.


6. a. $y = \dfrac{12}{x} + 6$ b. $y = \dfrac{20}{x} - 4$

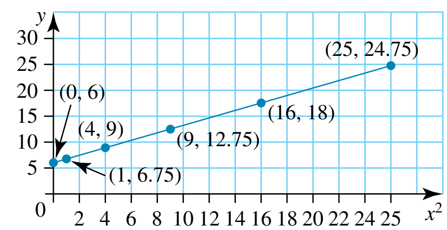
7. a. $k = 12$ and $c = 2$

b.

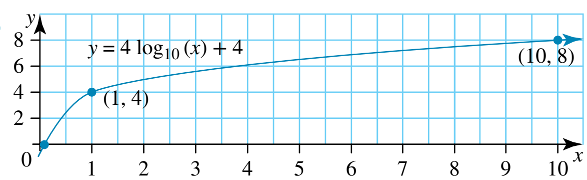
x	0.25	0.5	0.75	1.5	2	12
y	50	26	18	10	8	3
$\frac{1}{x}$	4	2	1.33	0.67	0.5	0.08

c.


d. 12

8. a.


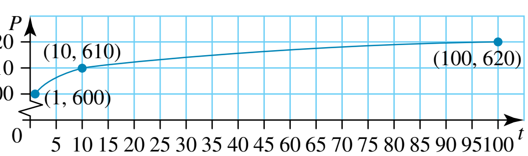
b. $y = 0.75x^2 + 6$

9.


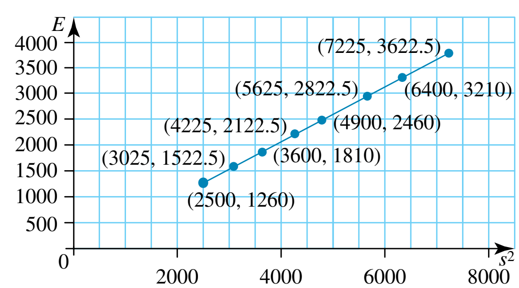
10. a. $y = 2\log_{10}(x) + 1$
b. $y = 6\log_{10}(x) - 5$

11. a.

t	1	10	100
P	600	610	620

b.


c. 630

12. a.


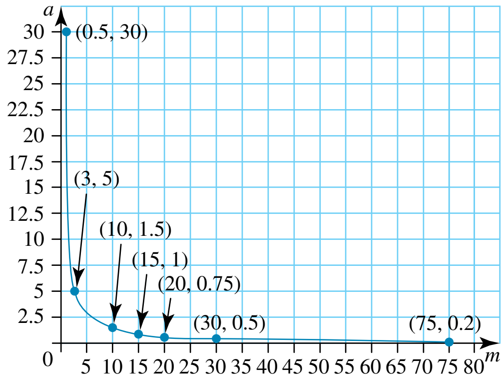
b. $E = 0.5s^2 + 10$

16. c. *

x	0.5	1	1.5	2	2.5	3	3.5
x^2	24.5	23	20.5	17	12.5	7	0.5
y	600.25	529	420.25	289	156.25	49	0.25

13. a. Inverse variation

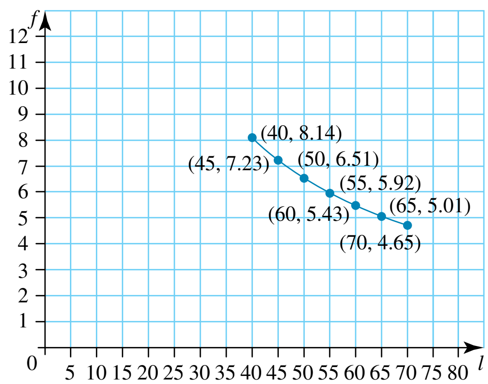
b.



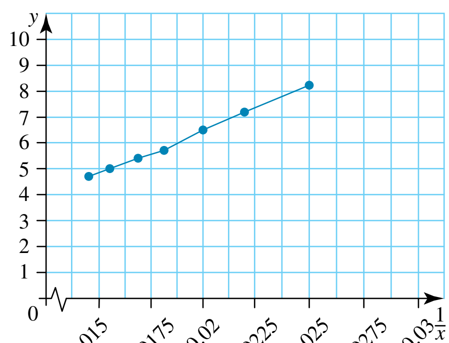
c. $a = \dfrac{15}{m}$

14. a. See table at the foot of the page*

b.



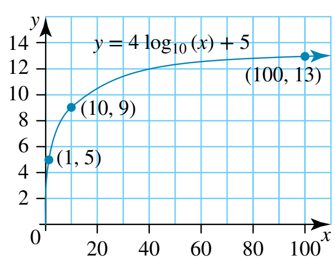
c.



d. The transformed data is linearised.

15. a. $a = 2$ and $c = 3$

b.

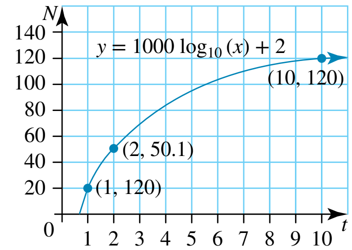


16. a. i. 20

ii. 50

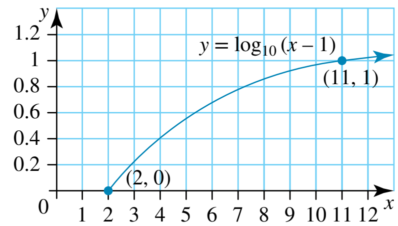
iii. 120

b.

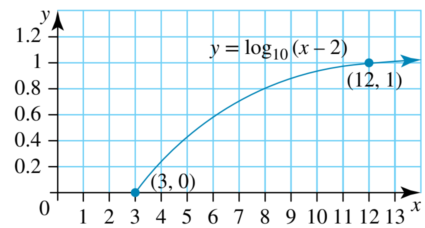


c. 158 days

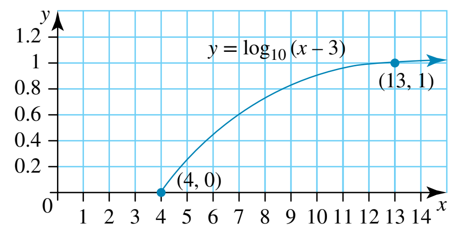
17. a. i.



ii.



iii.

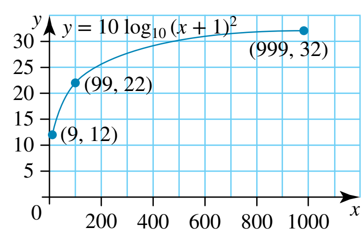


b. As the value of b increases, the graph moves an equivalent distance from the y-axis. The x-axis intercept is one more than the value of b.

18. a.

x	9	99	999
y	12	22	32

b.



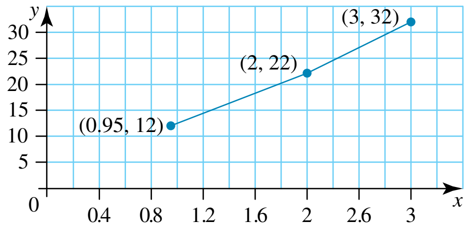
c.

x	9	99	999
$\log_{10}(x)$	0.9542	1.9956	2.9995
y	12	22	32

14. a. *

String length (cm)	70	65	60	55	50	45	40
Vibrations per second	4.7	5.0	5.4	5.9	6.5	7.2	8.1

d.



e. The transformed data has been linearised.

Review: exam practice

Multiple choice

1. A **2.** A
3. D **4.** E
5. E **6.** C
7. C **8.** D
9. D **10.** B

Short answer

1. a. Direct variation **b.** $d = 4.9t^2$
c. 705.6 m **d.** 2.26 s

2. a. $L = \frac{v}{f}$ **b.** 6×10^{-7}

3. a. $t \propto \frac{1}{v}$ **b.** $t = \frac{k}{v}$ **c.** 2 hours
d. $k = 100$ **e.** $t = \frac{100}{v}$

4. a. $I \propto \frac{1}{d^2}$ **b.** $k = 5, I = \frac{5}{d^2}$ **c.** 0.41 lux **d.** 10 m

5. a. $w = kh$ **b.** $k = 17$ **c.** See table at the foot of the page*

6. a. $k \neq \frac{y}{x}$ for all values in the table.
b. Change the y-value from 21.34 to 20.37.

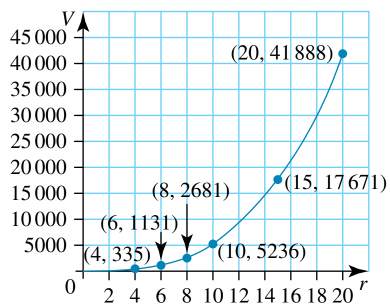
Extended response

1. a. See table at the foot of the page*
b. $\frac{R^3}{T^2}$ is approximately the same value for all the planets in the table.

2. a. $w \propto \frac{1}{l}$ **b.** Inverse variation

3. a. See worked solutions for a complete answer. **b.** 4.8 km
c. 35.8 km **d.** 113.1

4. a. $h = 5r$, so **b.** $r = 4 \Rightarrow h = 20$ cm
c. See table at the foot of the page*
d.



e. No **f.** See table at the foot of the page* **g.** $V = 5.24r^3$

5. c. *

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Wages/hour (w/h)	\$17	\$17	\$17	\$17	\$17

1. a. *

Planet	**Orbital radius, R (m)**	R^3	**Orbital period, T (s)**	T^2	$\frac{R^3}{T^2}$
Mercury	5.79×10^{10}	1.94×10^{32}	7.60×10^{6}	5.78×10^{13}	3.36×10^{18}
Venus	1.08×10^{11}	1.26×10^{33}	1.94×10^{7}	3.76×10^{14}	3.35×10^{18}
Earth	1.49×10^{11}	3.31×10^{33}	3.16×10^{7}	9.99×10^{14}	3.31×10^{18}
Mars	2.28×10^{11}	1.19×10^{34}	5.94×10^{7}	3.53×10^{15}	3.37×10^{18}
Jupiter	7.78×10^{11}	4.71×10^{35}	3.74×10^{8}	1.40×10^{17}	3.36×10^{18}
Saturn	1.43×10^{12}	2.92×10^{36}	9.30×10^{8}	8.65×10^{17}	3.38×10^{18}
Uranus	2.87×10^{12}	2.36×10^{37}	2.66×10^{9}	7.08×10^{18}	3.33×10^{18}
Neptune	4.50×10^{12}	9.11×10^{37}	5.20×10^{9}	2.70×10^{19}	3.37×10^{18}

4. c. *

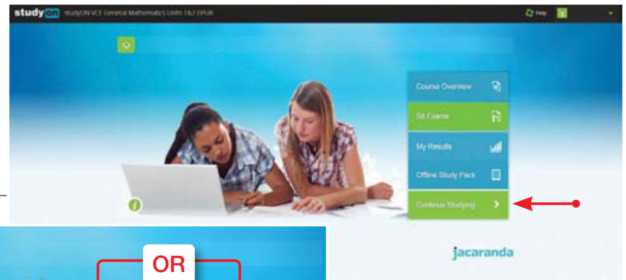
r (cm)	4	6	8	10	15	20
Volume, $V = \frac{\pi r^2 h}{3}$ (cm^3)	335	1131	2681	5236	17 671	41 888

4. f. *

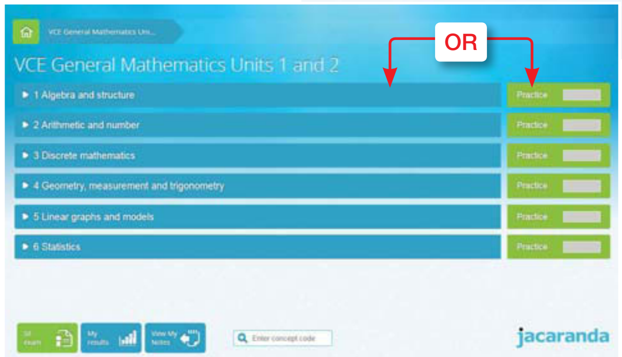
r (cm)	4	6	8	10	15	20
r^3 (cm^3)	64	216	512	1000	3375	8000
Volume, $V = \frac{\pi r^2 h}{3}$ (cm^3)	335	1131	2681	5236	17 671	41 888

TOPICS 10 to 12

- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

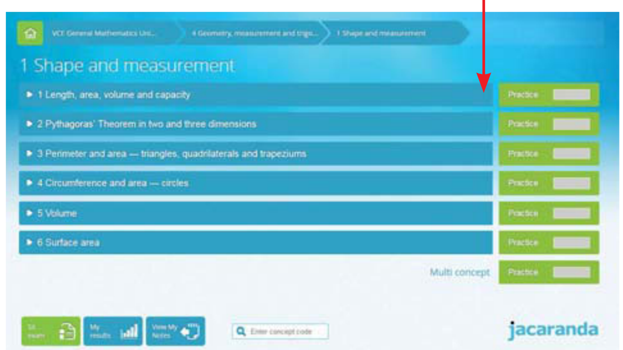


- Select your **course** *VCE General Mathematics Units 1 & 2* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.

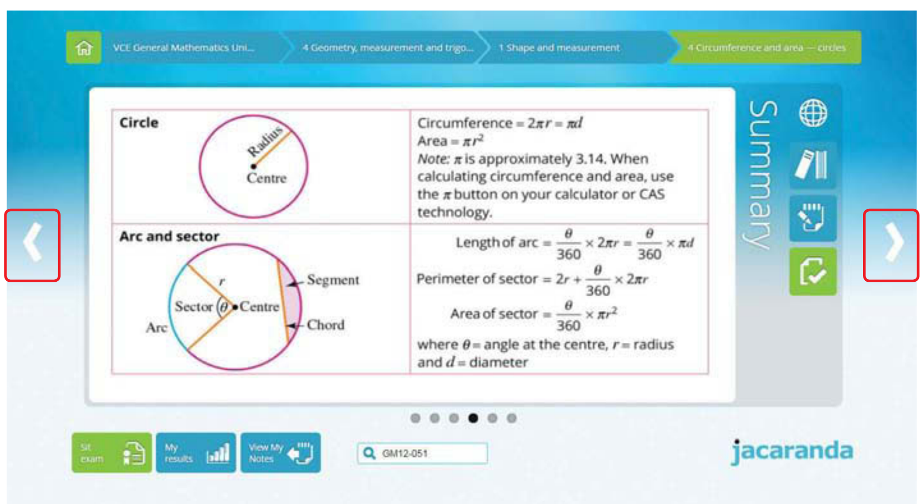


- Select **Practice** at the topic level to access all questions in the topic.

- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 13 Investigating and comparing data distributions

13.1 Overview

13.1.1 Introduction

A boxplot is a way of graphically representing groups of numerical data through their quartiles. The spaces between the different sections of the box indicate the degree of spread and skewness in the data. The boxplot gives a snapshot of a number of values, such as: interquartile range, maximum value, minimum value, range and medium of the data. It is also used to determine and show outliers.

The boxplot was introduced by the mathematician John Tukey. He is regarded as one of the most influential statisticians of the past 50 years. Some of his work in modern statistics led to concepts that have played a central role to the creation of today's telecommunication technology. He is credited for the invention of the computer term 'bit'.

John Tukey was born in New Bedford, Massachusetts (USA) in 1915. He obtained a Bachelor of Arts and a Master of Science in Chemistry from Brown University in 1937, before moving to Princeton University where he completed his PhD in mathematics. He became a professor at 35 and founded the Princeton statistics department in 1965. He was awarded the IEEE Medal of Honor in 1982 for his contributions to the spectral analysis of random processes and the fast Fourier transform (FFT) algorithm. It was in his 1977 book *Exploratory Data Analysis* that he introduced the boxplot.

LEARNING SEQUENCE

- **13.1** Overview
- **13.2** Data types and displays
- **13.3** Numerical data distributions
- **13.4** Measures of centre
- **13.5** Measures of spread
- **13.6** Comparing numerical distributions
- **13.7** Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

13.1.2 Kick off with CAS

Sample statistics with CAS

Sample statistics allow us to analyse and compare different sets of data.

1. Calculate the mean (the numerical average) of each of the following data sets by hand.
   a. 11, 14, 17, 12, 9, 13, 12, 16, 13
   b. 22, 27, 30, 21, 29, 25, 18, 25, 33, 24, 21
   c. 41, 45, 42, 44, 48, 51, 40, 45, 49
2. Calculate the median (the middle value) of each of the data sets given in question 1.
3. Use CAS to find the summary statistics for each of the data sets given in question 1.
4. Which symbols represent the mean and median statistics on your CAS?
5. From the list of summary statistics on your CAS, how is the minimum value of each data set represented on your CAS?
6. How is the maximum value of each data set represented on your CAS?
7. Use the minimum and maximum values from each data set to calculate the range of each data set in question 1 (range = maximum value − minimum value).

Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology

13.2 Data types and displays

13.2.1 Data types

When analysing data it is important to know what type of data you are dealing with. This can help to determine the best way to both display and analyse the data.

Data can be split into two major groups: **categorical data** and **numerical data**. Both of these can be further divided into two subgroups.

13.2.2 Categorical data

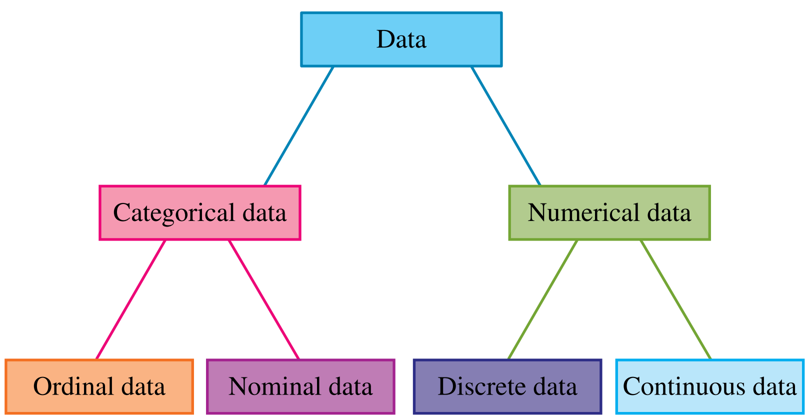
Data that can be organised into groups or categories is known as categorical data. Categorical data is often an ‘object’, ‘thing’ or ‘idea’, with examples including brand names, colours, general sizes and opinions. Categorical data can be classified as either ordinal or nominal. **Ordinal data** is placed into a natural order or ranking, whereas **nominal data** is split into subgroups with no particular order or ranking.

For example, if you were collecting data on income in terms of whether it was ‘High’, ‘Medium’ or ‘Low’, the assumed order would be to place the ‘Medium’ category between the other two, so this is ordinal data. On the other hand, if you were investigating preferred car colours the order doesn’t really matter, so this is nominal data.

13.2.3 Numerical data

Data that can be counted or measured is known as numerical data. Numerical data can be classified as either discrete or continuous. **Discrete data** is counted in exact values, with the values often being whole numbers, whereas **continuous data** can have an infinite number of values, with an additional value always possible between any two given values.

For example, the housing industry might consider the number of bedrooms in residences offered for sale. In this case, the data can only be a restricted group of numbers (1, 2, 3, etc.), so this is discrete data. Now consider meteorological data, such as the maximum daily temperatures over a particular time period. Temperature data could have an infinite number of decimal places (23 °C, 25.6 °C, 18.21 °C, etc.), so this is continuous data.



WORKED EXAMPLE 1

Data on the different types of cars on display in a car yard is collected.
Verify that the collected data is categorical, and determine whether it is ordinal or nominal.

THINK	WRITE
1. Identify the type of data.	The data collected is the brand or model of cars, so this is categorical data.
2. Identify whether the order of the data is relevant.	When assessing the types of different cars, the order is not relevant, so this is nominal data.
3. State the answer.	The data collected is nominal data.

WORKED EXAMPLE 2

Data on the number of people attending matches at sporting venues is collected.
Verify that the collected data is numerical, and determine whether it is discrete or continuous.

THINK	WRITE
1. Identify the type of data.	The data collected is the number of people in sporting venues, so this is numerical data.
2. Does the data have a restricted or infinite set of possible values?	The data involves counting people, so only whole number values are possible.
3. State the answer.	The data collected is discrete data.

13.2.4 Displaying categorical data

Once raw data has been collected, it is helpful to summarise the information into a table or display. Categorical data is usually displayed in either **frequency tables** or **bar charts**. Both of these display the frequency (number of times) that a piece of data occurs in the collected data.

13.2.5 Frequency tables

Frequency tables split the collected data into defined categories and register the frequency of each category in a separate column. A tally column is often included to help count the frequency.

For example, if we have collected the following data about people's favourite colours, we could display it in a frequency table.

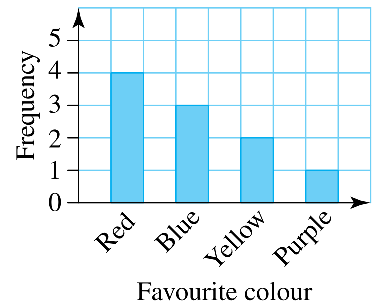
Red, Blue, Yellow, Red, Purple, Blue, Red, Yellow, Blue, Red

Favourite colour	Tally	Frequency
Red	||||	4
Blue	|||	3
Yellow	||	2
Purple	|	1

13.2.6 Bar charts

Bar charts display the categories of data on the horizontal axis and the frequency of the data on the vertical axis. As the categories are distinct, there should be a space between all of the bars in the chart.

The bar chart on the right displays the previous data about favourite colours.



WORKED EXAMPLE 3

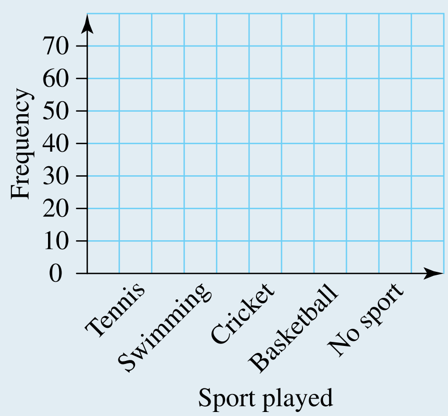
The number of students from a particular school who participate in organised sport on weekends is shown in the frequency table.
Display the data in a bar chart.

Sport	Frequency
Tennis	40
Swimming	30
Cricket	60
Basketball	50
No sport	70

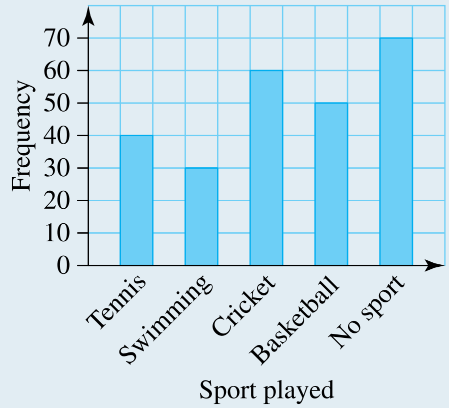
THINK

1. Choose an appropriate scale for the bar chart. As the frequencies go up to 70 and all of the values are multiples of 10, we will mark our intervals in 10s. Display the different categories along the horizontal axis.

WRITE/DRAW



2. Draw bars to represent the frequency of each category, making sure there are spaces between the bars.



13.2.7 The mode

For categorical data, the **mode** is the category that has the highest frequency. When displaying categorical data in a bar chart, the modal category is the highest bar.

Identifying the mode allows us to know which category is the most common or most popular, which can be particularly useful when analysing data.

In some instances there may be either no modal category or more than one modal category. If the data has no modal category then there is no mode, if it has 2 modal categories then it is bimodal, and if it has 3 modal categories it is trimodal.

WORKED EXAMPLE 4

Thirty students were asked to pick their favourite time of the day between the following categories: Morning (M), Early afternoon (A), Late afternoon (L), Evening (E)
The following data was collected:
A, E, L, E, M, L, E, A, E, M, E, L, E, A, L, M, E, E, L, M, E, A, E, M, L, L, E, E, A, E

a. **Represent the data in a frequency table.**
b. **Draw a bar chart to represent the data.**
c. **Which time of day is the most popular?**

THINK

a. 1. Create a frequency table to capture the data.

WRITE/DRAW

a.

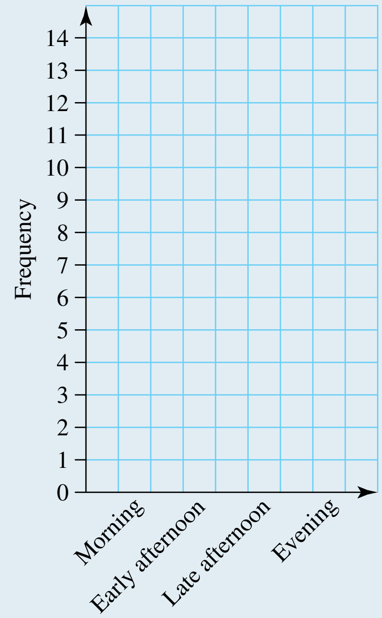
Time of day	Tally	Frequency
Morning		
Early afternoon		
Late afternoon		
Evening		

2. Go through the data, filling in the tally column as you progress. Sum the tally columns to complete the frequency column.

Time of day	Tally	Frequency
Morning	~~||||~~	5
Early afternoon	~~||||~~	5
Late afternoon	~~||||~~ ||	7
Evening	~~||||~~ ~~||||~~ |||	13

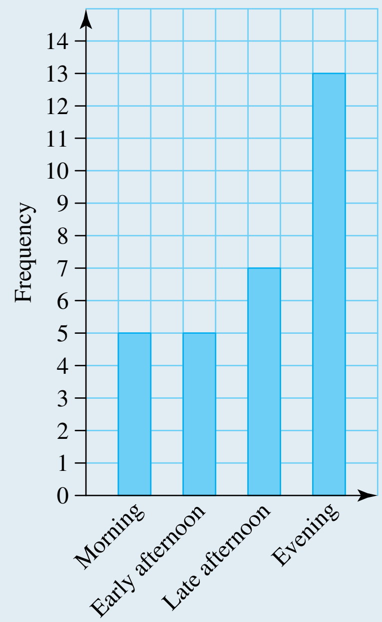
b. 1. Choose an appropriate scale for the bar chart. As the frequencies only go up to 13, we will mark our intervals in single digits. Display the different categories along the horizontal axis.

b.



Favourite time of day

2. Draw bars to represent the frequency of each category, making sure there are spaces between the bars.



Favourite time of day

c. 1. The highest bar is the modal category. This is the most popular category. Write the answer.

c. Evening is the most popular time of day among the students.

on Resources

Interactivity: Create a bar chart (int-6493)

study on

Units 1 & 2 > AOS 6 > Topic 1 > Concepts 1 & 2

Classification of data Concept summary and practice questions
Categorical data Concept summary and practice questions

Exercise 13.2 Data types and displays

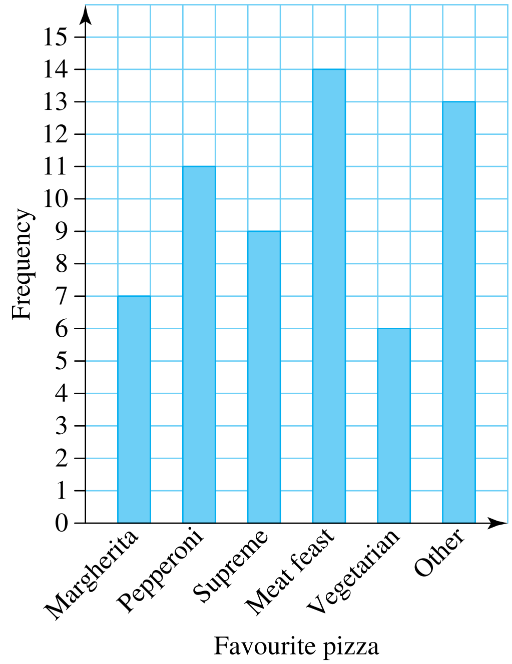
1. **WE1** Data on the different types of cereal on supermarket shelves is collected.
Verify that the collected data is categorical, and determine whether it is ordinal or nominal.

2. Data on the rating of hotels from 'one star' to 'five star' is collected.
Verify that the collected data is categorical, and determine whether it is ordinal or nominal.
3. **WE2** Identify whether the following numerical data is discrete or continuous.
   a. The amount of daily rainfall in Geelong
   b. The heights of players in the National Basketball League
   c. The number of children in families
4. Identify whether the following numerical data is nominal, ordinal, discrete or continuous.
   a. The times taken for the place getters in the Olympic 100 m sprinting final
   b. The number of gold medals won by countries competing at the Olympic Games
   c. The type of medals won by a country at the Olympic Games
5. **WE3** The preferred movie genre of 100 students is shown in the following frequency table.

Favourite movie genre	Frequency
Action	32
Comedy	19
Romance	13
Drama	15
Horror	7
Musical	4
Animation	10

Display the data in a bar chart.

6. The favourite pizza type of 60 students is shown in the following bar chart.



Display the data in a frequency table.

7. A group of students at a university were surveyed about their usual method of travel, with the results shown in the following table.

Student	Transport method	Student	Transport method
A	Bus	N	Car
B	Walk	O	Bus
C	Train	P	Car
D	Bus	Q	Bus
E	Car	R	Bicycle
F	Bus	S	Car
G	Walk	T	Train
H	Bicycle	U	Bus
I	Bus	V	Walk
J	Car	W	Car
K	Car	X	Train
L	Train	Y	Bus
M	Bicycle	Z	Bus

a. What type of data is being collected?
b. Organise the data into a frequency table.
c. Display the data as a bar chart.

8. In a telephone survey people were asked the question, ‘Do you agree that convicted criminals should be required to serve their full sentence and not receive early parole?’ They were required to respond with either ‘Yes’, ‘No’ or ‘Don’t care’ and the results are as follows.

Person	Opinion	Person	Opinion
A	Yes	N	Yes
B	Yes	O	No
C	Yes	P	No
D	Yes	Q	Yes
E	Don't care	R	Yes
F	No	S	Yes
G	Don't care	T	Yes
H	Yes	U	No
I	No	V	Yes
J	No	W	Yes
K	Yes	X	Don't care
L	No	Y	Yes
M	Yes	Z	Yes

a. Organise the data into an appropriate table.
b. Display the data as a bar graph.
c. Identify the data as either nominal or ordinal. Explain your answer.

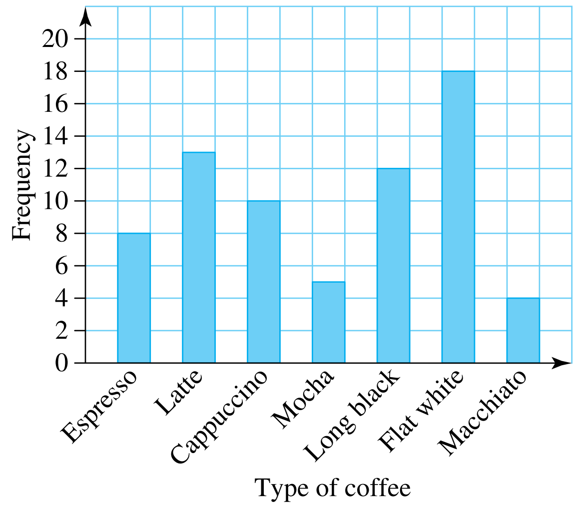
9. Complete the following table by indicating the type of data.

Data	Type	
Example: The types of meat displayed in a butcher shop.	Categorical	Nominal
a. Wines rated as high, medium or low quality		
b. The number of downloads from a website		
c. Electricity usage over a three-month period		
d. The volume of petrol sold by a petrol station per day		

10. **WE4** Twenty-five students were asked to pick their favourite type of animal to keep as a pet. The following data was collected.
Dog, Cat, Cat, Rabbit, Dog, Guinea pig, Dog, Cat, Cat, Rat, Rabbit, Ferret, Dog, Guinea pig, Cat, Rabbit, Rat, Dog, Dog, Rabbit, Cat, Cat, Guinea pig, Cat, Dog
a. Represent the data in a frequency table.
b. Draw a bar chart to represent the data.
c. Which animal is the most popular?

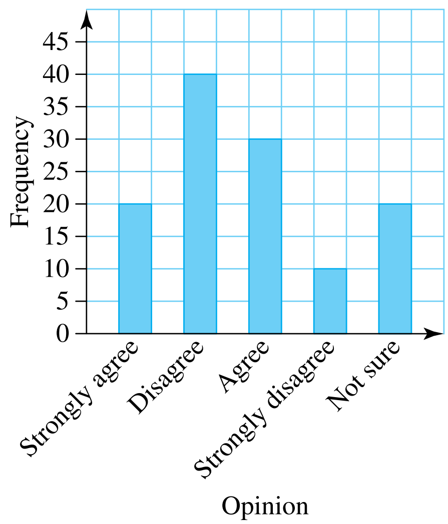
11. Thirty students were asked to pick their favourite type of music from the following categories: Pop (P), Rock (R), Classical (C), Folk (F), Electronic (E).
The following data was collected:
E, R, R, P, P, E, F, E, E, P, R, C, E, P, E, P, C, R, P, F, E, P, P, E, R, R, E, F, P, R
a. Represent the data in a frequency table.
b. Draw a bar chart to represent the data.
c. Which type of music is the most popular?

12. The different types of coffee sold at a café in one hour are displayed in the following bar chart.



a. What is the modal category of the coffees sold?
b. How many coffees were sold in that hour?

13. The result of an opinion survey are displayed in the following bar chart.



a. What type of data is being displayed?
b. Explain what is wrong with the current data display.
c. Redraw the bar chart displaying the data correctly.

14. Exam results for a group of students are shown in the following table.

Student	Result	Student	Result	Student	Result	Student	Result
1	A	6	C	11	B	16	C
2	B	7	C	12	C	17	A
3	D	8	C	13	C	18	C
4	E	9	E	14	C	19	D
5	A	10	D	15	D	20	E

a. Display the exam result data in a frequency table.
b. Display the data in a bar chart.
c. What is the type of data collected?

15. The number of properties sold in the capital cities of Australia for a particular time period is shown in the following table.

	Number of bedrooms			
City	**2**	**3**	**4**	**5**
Adelaide	8	12	5	4
Brisbane	15	11	8	6
Canberra	8	12	9	2
Hobart	3	9	5	1
Melbourne	16	18	12	11
Sydney	23	19	15	9
Perth	7	9	12	3

Use the given information to create a bar graph that represents the number of bedrooms of properties sold in the capital cities during this time period.

16. The maximum daily temperatures (°C) in Adelaide during a 15-day period in February are listed in the following table.

Day	1	2	3	4	5	6	7	8
Temp (°C)	31	32	40	42	32	34	41	29

Day	9	10	11	12	13	14	15
Temp (°C)	25	33	34	24	22	24	30

Temperatures greater than or equal to 39 °C are considered above average and those less than 25 °C are considered below average.

a. Organise the data into three categories and display the results in a frequency table.
b. Display the organised data in a bar graph.
c. What is the type of data displayed in your bar graph?

17. The following frequency table displays the different categories of purchases in a shopping basket.

Category	**Frequency**
Fruit	6
Vegetables	8
Frozen goods	5
Packaged goods	11
Toiletries	3
Other	7

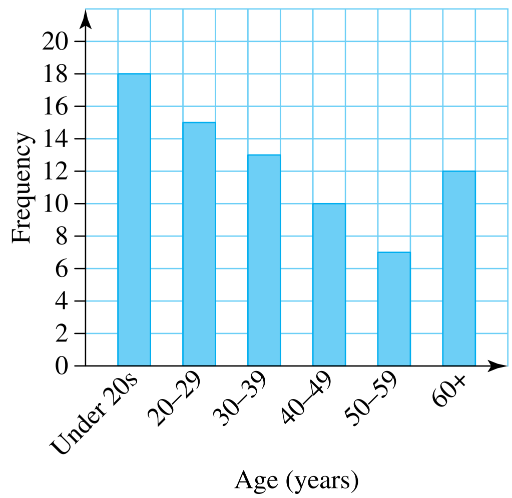
a. How many items were purchased in total?
b. What percentage of the total purchases were fruit?

18. The birthplaces of 200 Australian citizens were recorded and are shown in the following frequency table.

Birthplace	Frequency
Australia	128
United Kingdom	14
India	10
China	9
Ireland	6
Other	33

a. What type of data is being collected?
b. Represent this information in a bar chart.
c. What percentage of the respondents were born in Australia?

19. The following bar chart represents the ages of attendees at a local sporting event.



a. Represent the data in a frequency table.
b. Which is the modal category?
c. The age groups are changed to 'Under 20', '20–39', '40–59' and '60+'. Redraw the bar chart with these new categories.
d. Does this change the modal category?

20. Data for the main area of education and study for a selected group of people aged 15 to 64 during a particular year in Australia is shown in the following table.

Number of people (thousands)					
Main area of education and study	**15–19**	**20–24**	**25–34**	**35–44**	**45–64**
Agriculture	10	9	14	5	5
Creative arts	36	51	20	10	9
Engineering	59	75	50	13	6
Health	44	76	64	32	32
Management and commerce	71	155	135	86	65

a. Create separate bar charts for each area of education and study to represent the data.
b. Create separate bar charts for each age group to represent the data.

13.3 Numerical data distributions

13.3.1 Grouped data

Numerical data may be represented as either **grouped data** or **ungrouped data**. When assessing ungrouped data, the analysis we do is exact; however, if we have a large data set, the data can be difficult to work with. Grouping data allows us to gain a clearer picture of the data's distribution, and the resultant data is usually easier to work with.

When grouping data, we try to pick class sizes so that between 5 and 10 classes are formed. Ensure that all of the classes are distinct and that there are no overlaps between classes.

When creating a frequency table to represent grouped continuous data, we will represent our class intervals in the form 12−<14. This interval covers all values from 12 up to 14 but does not include 14.

WORKED EXAMPLE 5

The following data represents the time (in seconds) it takes for each individual in a group of 20 students to run 100 m.

18.2, 20.1, 15.6, 13.5, 16.7, 15.9, 19.3, 22.5, 18.4, 15.9, 12.4, 14.1, 17.7, 19.4, 21.0, 20.4, 18.2, 15.8, 16.1, 14.6

Group and display the data in a frequency table.

THINK

1. Identify the smallest and largest values in the data set. This will help you to choose your class size and decide what the first class should be.

WRITE

Smallest value = 12.4
Largest value = 22.5
We will have class intervals of 2, starting with 12−<14.

THINK

2. Draw a frequency table to represent the data. Complete the tally column in your table, and use this to fill in the frequency column.

WRITE

Time (seconds)	Tally	Frequency
12−<14	||	2
14−<16	|||| ||	6
16−<18	|||	3
18−<20	||||	5
20−<22	|||	3
22−<24	|	1

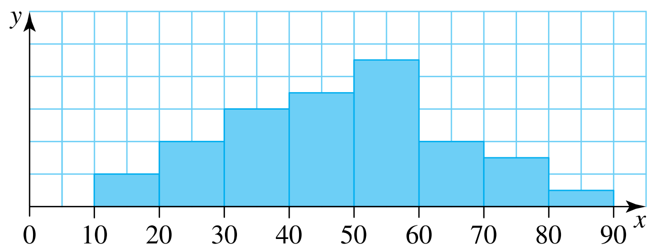
13.3.2 Displaying numerical distributions

The types of display we chose to represent numerical data depend on whether that data is discrete or continuous. Representations of discrete data should imply that irrelevant values are impossible, so we usually insert gaps between the data values. On the other hand, continuous data displays often have no gaps between whole numbers, as all possible values between the listed values are possible.

13.3.3 Histograms

We can represent continuous numerical data using a **histogram**, which is very similar to a bar chart with a few essential differences.

In a histogram, the width of each column represents a range of data values, while the height represents their frequencies. For example, in the following histogram the first column represents the frequency of data values that are greater than or equal to 10 but less than 20 $(10 \le x < 20)$.



WORKED EXAMPLE 6

The following frequency table represent the heights of players in a basketball squad.

Height (cm)	175–<180	180–<185	185–<190	190–<195	195–<200	200–<205
Frequency	1	3	6	3	1	1

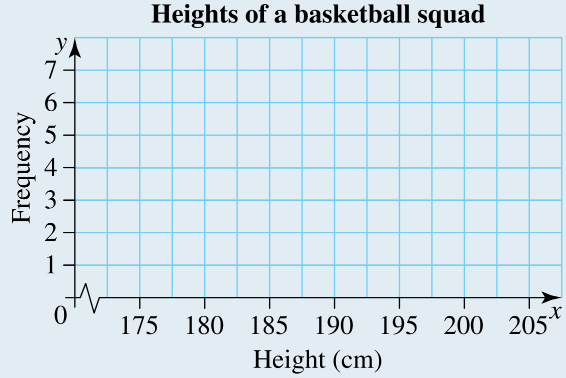
Draw a histogram to represent this data.

THINK

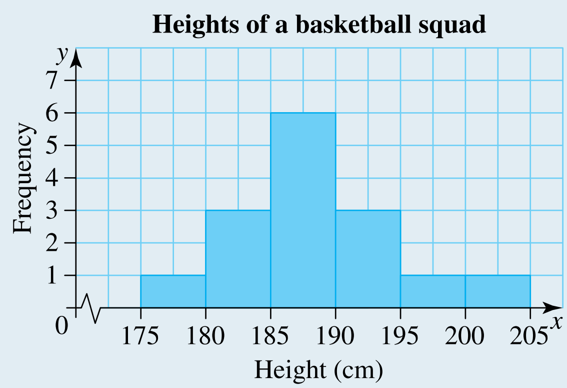
1. Look at the data range and use the leading values from each interval in the table for the scale of the horizontal axis.

WRITE/DRAW

The height data in the table has intervals starting from 175 cm and increasing by 5 cm.

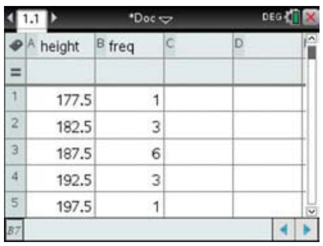


2. Draw rectangles for each interval to the height of the frequency indicated by the data in the table.

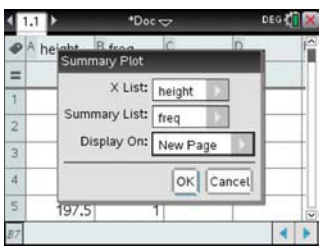


TI | THINK **WRITE**

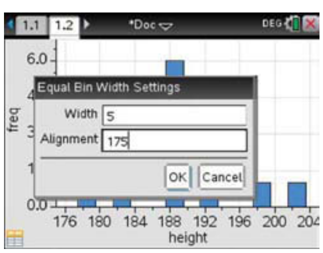
1. In a Lists & Spreadsheet page, label the first column as *height* and the second column as *freq*. Enter the midpoint of each height interval in the first column and the frequencies in the second column.



2. Press MENU then select:
3: Data
8: Summary Plot
Complete the fields as:
X List: height
Summary List: freq
Display On: New Page
then select OK.

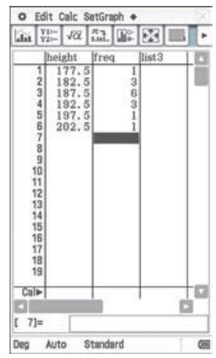


3. On the Summary Plot page, press MENU then select:
2: Plot Properties
2: Histogram Properties
2: Bin Settings
1: Equal Bin Width
Complete the fields as:
Width: 5
Alignment: 175
then select OK.

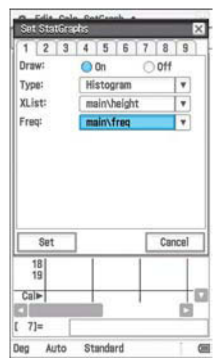


CASIO | THINK **WRITE**

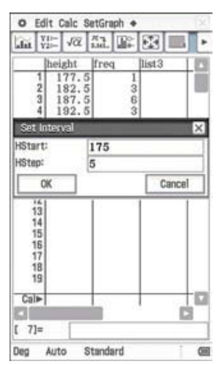
1. On a Statistics screen, relabel list1 as *height* and list2 as *freq*.
Enter the midpoint of each height interval in the first column and the frequencies in the second column.



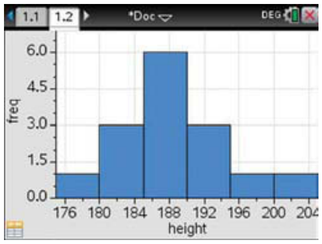
2. Select:
- SetGraph
- Setting …
Complete the fields as:
Draw: On
Type: Histogram
XList: main\height
Freq: main\freq
then select Set.



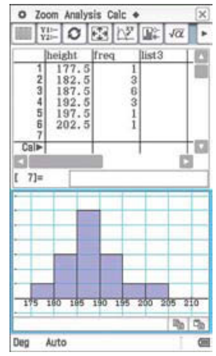
3. Press the y icon. Complete the fields as:
HStart: 175
HStep: 5
then select OK.



4. The histogram is displayed on the screen.



4. The histogram is displayed on the screen.



13.3.4 Stem plots

Stem plots (or stem-and-leaf plots) can be used to display both discrete and continuous numerical data. The data is grouped according to its numerical place value (the 'stem') and then displayed horizontally as a single digit (the 'leaf'). In an unordered stem plot, the data values have been placed into categories but do not appear in order. In an ordered stem plot, the values are placed in numerical order with the smallest values closest to the stem. When answering questions relating to stem plots, present your final answer as an ordered stem plot.

If there are 4 or fewer different place values in your data, it may be preferable to make the stems of the plot represent a class set of 5 instead of a class set of 10. This can be done by inserting an asterisk (∗) after the second of the stems with the same number, as shown in the following example.

Note that the data has been presented in neat vertical columns, making it easy to read.

Always remember to include a key with your stem plot to indicate what the stem and the leaf represents when put together.

Key: 1|4 = 14

Stem	Leaf
1	4
1*	7 7 9
2	2 3 4 4
2*	6 8
3	0 1 3
3*	5 6 9
4	0

13.3.5 Back-to-back stem plots

As we will see later in this topic, **back-to-back stem plots** can be used to compare two different sets of data. Back-to-back stem plots share the same stem, with one data set appearing on the left of the stem and the other data set appearing on the right.

Key: 1|4 =14

Leaf	Stem	Leaf
3 2	1	4
9 8 6 6	1*	7 7 9
4 2 1 0	2	2 3 4 4
7 6	2*	6 8
4 2 1	3	0 1 3
6 5	3*	5 6 9
	4	0

WORKED EXAMPLE 7

The following data set (of 31 values) shows the maximum daily temperature during the month of January in a particular area.

26, 22, 24, 26, 28, 28, 27, 42, 25, 25, 29, 31, 23, 33, 34, 27, 39, 44, 35, 34, 27, 30, 36, 30, 30, 28, 33, 23, 24, 34, 37

Draw a stem plot to represent the data.

THINK	WRITE/DRAW
1. Identify the place values for the data. If there are 4 or less different place values, split each into two.	The temperature data has values in the 20s, 30s and 40s.
2. Write the units for each stem place value in numerical order, with the smallest values closest to the stem. Make sure to keep consecutive numbers level as they move away from the stem. Remember to add a key to your plot.	see stem plots below

Stem	Leaf
2	
2*	
3	
3*	
4	

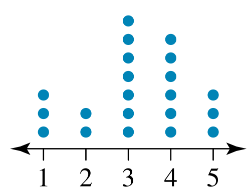
Key: 2|2 = 22° C

Stem	Leaf
2	2 3 3 4 4
2*	5 5 6 6 7 7 7 8 8 8 9
3	0 0 0 1 3 3 4 4 4
3*	5 6 7 9
4	2 4

13.3.6 Dot plots

Discrete numerical data can also be displayed as a **dot plot**. In these plots every data value is represented by a dot. The most common values can then be clearly identified. You can also use dot plots to represent categorical data.

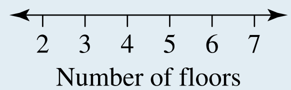
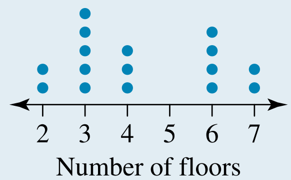
When drawing a dot plot, be careful to make sure that the dots are evenly and consistently spaced.



WORKED EXAMPLE 8

The frequency table shows the number of floors in apartment buildings in a particular area. Draw a dot plot to represent the data.

Number of floors	Frequency
2	2
3	5
4	3
5	0
6	4
7	2

THINK	WRITE/DRAW
1. Draw a horizontal scale using the discrete data values shown.	The discrete data values are given by the number of floors. 
2. Place one dot directly above the number on the scale for each discrete data value present, making sure to keep corresponding dots at the same level.	

13.3.7 Choosing which plot to use

Grouped data should be represented by a histogram, boxplot or dot plot. On the other hand, we should usually represent ungrouped data by using stem plots.

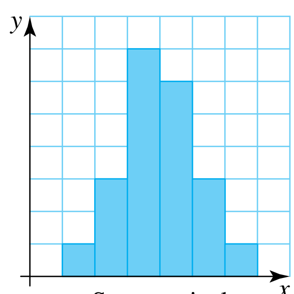
We should not use a stem plot to represent our data if the range of values in the data set is large, or if the data values have a high number of units in them (ignoring decimal places), as these stem plots can become unwieldy and difficult to use.

13.3.8 Describing distributions

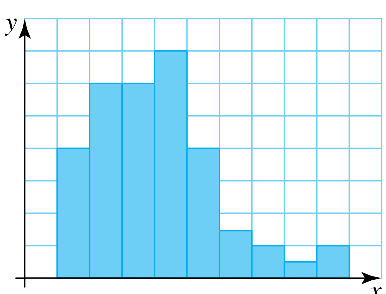
The distribution of a set of data can be described in terms of a number of key features, including shape, modality, spread and outliers.

13.3.9 Shape

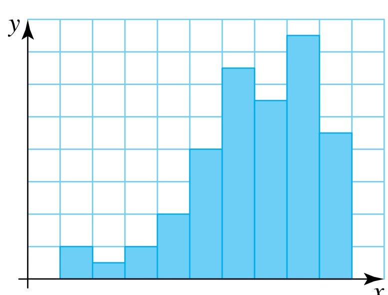
The shape of a numerical distribution is an important indicator of some of the key measures for further analysis and is one of the most important reasons for displaying the data in a graphical form. Shape will generally be described in terms of symmetry or skew. Symmetrical data distributions have higher frequencies around their centres with a relatively evenly balanced spread to either side, while skewed distributions have the majority of their values towards one end. Distributions with higher frequencies on the left side of the graph are **positively skewed**, while those with higher frequencies on the right side are **negatively** skewed.



Symmetrical



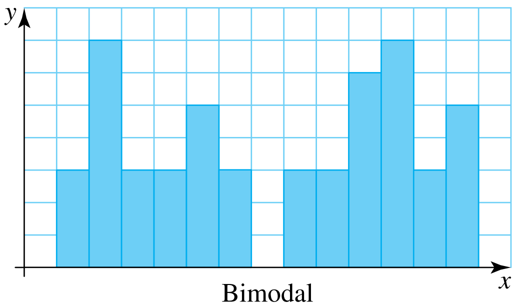
Positively skewed



Negatively skewed

13.3.10 Modality

The mode of a distribution is the data value or class interval that has the highest frequency. This will be the column or row on the display that is the longest. When there is more than one mode, the data distribution is multimodal. This can indicate that there may be subgroups within the distribution that may require further investigation. Bimodal distributions can occur when there are two distinct groups present, such as in data values that typically have clear differences between male and female measurements.



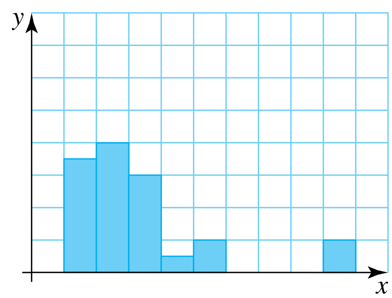
Bimodal

13.3.11 Spread

An awareness of how widely spread the data is can be an important consideration when conducting any further analysis. Common indicators of spread include the measures of **range** and the **standard deviation**. The graph will again point to which measures might be most appropriate to use.

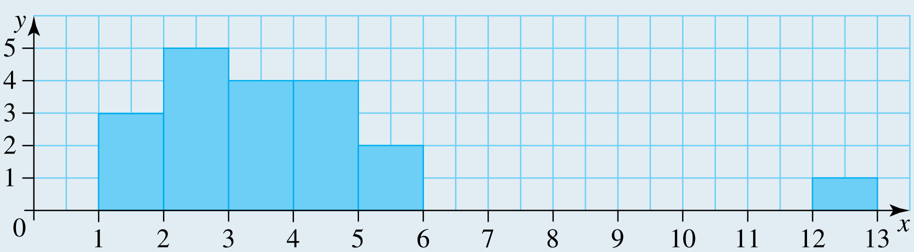
13.3.12 Outliers

An **outlier** is a data value that is an anomaly when compared to the majority of the sample. Sometimes outliers are just unusual readings or measurements, but they can also be the result of errors when recording the data. Outliers can have a significant effect on some of the measures that are used for further data analysis, and they are sometimes removed from the sample for those calculations. The graphical display of the data can alert us to the presence of potential outliers.



WORKED EXAMPLE 9

Describe the distribution of the data shown in the following histogram.



THINK	WRITE
1. Look for the mode and comment on its value.	The distribution has one mode with data values most frequently in the $2 \leq x < 3$ interval.
2. Identify the presence of any potential outliers.	There is one potential outlier in the interval between 12 and 13.
3. Describe the shape in terms of symmetry or skewness.	If we include the outlier, the data set can be described as positively skewed as it is clustered to the left. If we don't include the outlier, the distribution can be considered to be approximately symmetrical.

on Resources

- **Interactivity:** Create a histogram (int-6494)
- **Interactivity:** Stem plots (int-6242)
- **Interactivity:** Create stem plots (int-6495)
- **Interactivity:** Dot plots, frequency tables and histograms, and bar charts (int-6243)

studyon

Units 1 & 2 > AOS 6 > Topic 1 > Concepts 3 & 4

Display of numerical distributions Concept summary and practice questions
The shape of numerical distributions Concept summary and practice questions

Exercise 13.3 Numerical data distributions

1. **WE5** The following data represents the time (in seconds) it takes for each individual in a group of 20 students to swim 50 m.
   48.5, 54.1, 63.0, 39.7, 51.3, 57.7, 68.4, 59.4, 37.5, 41.8,
   72.3, 56.3, 45.4, 39.2, 60.3, 56.6, 48.1, 42.9, 53.3, 64.1
   Group and display the data in a frequency table.
2. The following data set indicates the time, in seconds, it takes for a tram to travel between two stops on 20 weekday mornings.
   95, 112, 99, 91, 105, 110, 97, 122, 108, 101, 95,
   89, 100, 115, 124, 98, 87, 111, 115, 106
   a. Group and display the data in a frequency table with intervals of width 10 seconds.
   b. Group and display the data in a frequency table with intervals of width 5 seconds.
3. **WE6** Draw a histogram to represent the following data.
   a. The cholesterol levels measured for a group of people

Cholesterol level (mmol/L)	1–<2.5	2.5–<4.0	4.0–<5.5	5.5–<7.0	7.0–<8.5
Frequency	2	8	12	14	10

   b. The distances travelled to school by a group of students

Distance travelled (km)	0–<22	2–<4	4–<6	6–<8	8–<10
Frequency	18	26	14	8	2

4. Organise each of the following data sets into a frequency table using intervals of five, commencing from the lowest value. Then draw a histogram to represent the data.
   a. 5, 7, 14, 17, 13, 24, 22, 15, 12, 26, 17, 15, 14, 13, 15, 7, 8, 13, 17, 24,
   22, 7, 13, 20, 12, 15, 23, 20, 17, 15, 17, 16, 20, 23, 15, 16, 18, 17, 14, 15
   b. 34, 28, 45, 46, 13, 24, 11, 33, 41, 35, 16, 15, 35, 13, 14, 28, 27, 22, 36, 31,
   11, 18, 24, 20, 12, 15, 41, 50, 27, 13, 14, 16, 20, 23, 31, 26, 25, 27, 34, 35

5. A group of 40 workers were surveyed on their average hours worked per week. The results were:
36, 40, 42, 40, 34, 33, 38, 36, 43, 39, 35, 36, 22, 39, 37,
37, 40, 38, 25, 27, 41, 34, 33, 28, 36, 25, 37, 39, 35,
36, 36, 37, 28, 42, 39, 40, 33, 35, 37, 38
   a. Display the data in a frequency table using intervals of 5 commencing from 20 hours.
   b. Use the frequency table to draw a histogram of the data.
   c. Describe the distribution of the data.
6. WE7 Draw a stem plot for each of the following data sets.
   a. The dollars spent per day on lunch by a group of 15 people:
   22, 21, 22, 24, 19, 22, 24, 21, 22, 23, 25, 26, 22, 23, 22
   b. The number of hours spent per week playing computer games by a group of 20 students at a particular school:
   14, 21, 25, 7, 25, 20, 21, 14, 21, 20, 6, 23, 26, 23, 17, 13, 9, 24, 17, 24
7. Draw a stem plot for the following data sets.

   a. The number of passengers per day transported by a taxi driver (40 values):
   33, 27, 44, 47, 23, 24, 22, 35, 42, 36 17, 25, 34, 13, 15, 27, 28, 23, 37, 34,
   22, 27, 23, 20, 12, 15, 43, 30, 27, 15 27, 36, 20, 23, 35, 36, 28, 17, 14, 15
   b. The number of patients per day treated by a doctor (40 values):
   44, 38, 55, 56, 23, 34, 31, 43, 51, 45, 26, 25, 45, 23, 24, 38, 37, 32, 46, 41,
   21, 28, 34, 30, 22, 25, 51, 60, 17, 23, 24, 26, 30, 33, 41, 26, 35, 17, 24, 25
8. The following set of data indicates the number of people who attend early morning fitness classes run by a business for its workers:
14, 17, 13, 8, 16, 21, 25, 16, 19, 17, 21, 8, 13
   a. Display the data as a stem plot.
   b. Describe the distribution.

9. The total number of games played by the players from two basketball squads is shown in the following stem plots.

Key: 0|1 = 1 game played

Stem	Leaf
0	1
1	4 7
2	4 4 8
3	3 3 5 6
4	1 2 3
5	1 1
6	5
7	
8	
9	1

Key: 2|4 = 24 games played

Stem	Leaf
2	4
3	1 2 6
4	3 4 5
5	2
6	
7	
8	2 5 7
9	3

a. Describe the shape of each distribution.
b. Draw a stem plot that combines the data for the two teams.

10. **WE8** Draw a dot plot to represent each of the following collections of data.

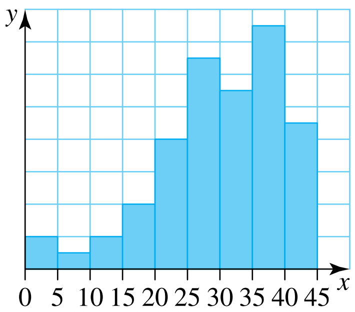
a. The number of wickets per game taken by a bowler in a cricket season

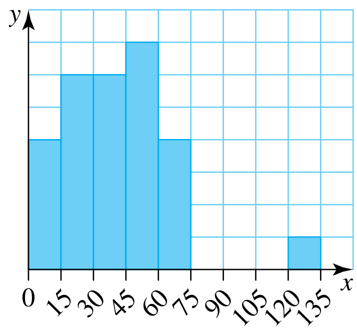
Number of wickets	Frequency
0	4
1	6
2	4
3	2
4	1
5	1

b. The number of hours per week spent checking emails by a group of workers at a particular company

Hours checking emails	Frequency
1	1
2	1
3	2
4	4
5	8
6	4

11. Draw dot plots to represent the following collections of data.
   a. The scores per round of a golfer over a particular time period (40 values):
      73, 77, 74, 77, 73, 74, 72, 75, 72, 76, 77, 75, 74, 73, 75, 77, 78, 73, 77, 74,
      72, 77, 73, 70, 72, 75, 73, 70, 77, 75, 77, 76, 70, 73, 75, 76, 78, 77, 74, 75
   b. The scores out of 10 in a multiple choice test for a group of students (30 values):
      6, 7, 4, 7, 3, 7, 7, 5, 7, 6, 7, 5, 1, 3, 5, 7, 8, 3, 7, 4, 9, 5, 4, 6, 7, 9, 10, 5, 7, 4
12. A group of 26 students received the following marks on a test:
    6, 4, 3, 8, 6, 9, 5, 6, 9, 7, 7, 8, 5, 7, 4, 3, 8, 6, 5, 7, 9, 5, 6, 6, 7, 8
   a. Display the data as a dot plot.
   b. Describe the distribution.
13. WE9 Describe the numerical distributions shown by the following histograms.

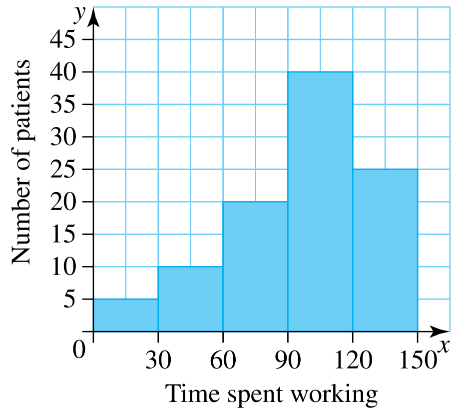
a. 

b. 

14. Describe the distribution of the following data sets after drawing histograms with intervals of 10 commencing with the smallest values.
   a. 105, 70, 140, 127, 132, 124, 122, 125, 123, 126, 107, 105, 104, 113, 125, 70, 88,
      103, 107, 124, 122, 76, 103, 120, 112, 115, 123, 120, 117, 115, 107, 106, 120, 123,
      115, 74, 128, 119
   b. 4, 18, 35, 26, 12, 25, 21, 34, 43, 37, 6, 25, 25, 23, 34, 38, 37, 22, 36, 31,
      21, 28, 34, 30, 32, 25, 31, 40, 37, 33, 24, 26, 10, 13, 21, 36, 35, 37, 24, 25
15. A group of people were surveyed about the number of pets they owned.
   a. Complete the following table.

Number of pets	Frequency	Percentage
0–1		
2–3		30%
4–5	8	
More than 5	2	5%
Total		100%

   b. What type of data was collected in the survey?
   c. How many people were surveyed?
   d. Display the data in an appropriate graph.
   e. Describe the distribution.
16. The waiting time for patients to receive treatment in a hospital emergency department is shown in the following histogram.



a. According to the histogram, what percentage of patients had to wait more than 90 minutes?
b. What percentage of patients received treatment in less than one hour?
c. Describe the distribution.

17. The systolic blood pressure readings for a group of 38 adults are listed below.
118, 125, 130, 122, 123, 128, 135, 128, 117, 121, 123, 126, 129,
142, 144, 148, 146, 122, 123, 118, 148, 126, 126, 144, 139, 147,
144, 142, 118, 124, 122, 145, 144, 143, 124, 125, 140, 119

a. Use the data to draw a histogram with intervals starting at 115 and increasing by 10.
b. Describe the distribution shown in the histogram.
c. Now use the data to draw a histogram with intervals starting at 115 and increasing by 3.
d. Describe how the second histogram is different from the first.
e. What might the second histogram be demonstrating about this data?

18. Consider the set of data in the stem plot shown.

Key: 0|1 = 1

Stem	Leaf
0	1
1	1 1 1 4 4 6 6 7 8
2	3 3 4 4 7 7 9

a. Instead of grouping the data in 10s, the stems could be split in half to use groups of 5. For example, the following stem plot places the data values from 10 to 14 in the column labelled '1', while data values from 15 to 19 are put in the column labelled '1*'.
Use the data from the original stem plot to complete the split stem plot.
b. Comment on the effect of splitting the stem for the data in this question.

19. The prices (in $000s) of 42 properties sold in a particular area are as shown:
289, 345, 456, 340, 678, 856, 446, 432, 468, 495, 521, 534, 499, 653, 453, 783,
921, 486, 965, 875, 656, 567, 874, 459, 521, 632, 621, 340, 560, 567, 535, 532,
428, 296, 346, 789, 632, 450, 555, 468, 696, 867

a. Use the data to draw a histogram with intervals starting at $289 000 and increasing by amounts of $50 000.
b. Now use the data to draw a histogram with intervals starting at $289 000 and increasing by amounts of $100 000.
c. Describe each display and comment on which may be more useful to analyse the data.
d. If a property selling for $1 450 000 was included with the data, redraw the histogram from part **b**.
e. Describe the histogram drawn in part **d**.

20. The average price of a litre of petrol on 20 days during a three-month period is as shown.
$1.67, $1.77, $1.61, $1.78, $1.73, $1.56, $1.66, $1.63, $1.82, $1.72,
$1.75, $1.56, $1.63, $1.71, $1.70, $1.45, $1.40, $1.78, $1.68, $1.72

a. Use the data to draw a histogram with intervals starting at $1.40 and increasing by amounts of 5 cents.
b. Now use the data to draw a histogram with intervals starting at $1.40 and increasing by 10 cents.
c. Describe each display and comment on the differences.

21. The heights in centimetres of a sample of AFL footballers are shown below.
183, 182, 196, 175, 198, 186, 195, 184, 181, 193, 174, 181, 177,
194, 202, 196, 200, 176, 178, 188, 199, 204, 192, 193, 191, 183,
174, 187, 184, 176, 194, 195, 188, 180, 189, 191, 196, 189, 181,
181, 183, 185, 184, 185, 208

a. Use CAS to display the data as a histogram using:
   i. intervals of 5 commencing with the smallest data value
   ii. intervals of 10 commencing with the smallest data value
   iii. intervals of 15 commencing with the smallest data value.
b. Describe each display and comment on the effect of changing the size of the intervals.

22. The average maximum temperature (in °C) in Victoria for two 20-year time periods is shown in the following tables.

Year	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Temp.	22.3	22.6	21.8	22.1	22.4	22.7	22.1	22.7	23.1	23.1
Year	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
Temp.	22.7	23.4	23.4	23.1	22.7	22.1	22.9	22.6	22.6.	22.7

Year	1893	1894	1895	1896	1897	1898	1899	1900	1901	1902
Temp.	20.5	20.9	21.0	20.9	21.5	21.3	21.0	20.9	21.0	21.0
Year	1903	1904	1905	1906	1907	1908	2009	2010	2011	1912
Temp.	20.7	20.9	20.9	21.5	21.4	21.3	21.1	21.4	21.3	21.4

Use CAS to display the data:
a. for the period 1993–2012 as a histogram using intervals of 0.4 °C commencing with the data value 20 °C
b. for the period 1893–1912 as a histogram using intervals of 0.4 °C commencing with the data value 20 °C.
c. Describe each display and comment on the differences between the two data sets.

13.4 Measures of centre

13.4.1 Representative measurements

In many practical settings it is common to use a single measurement to represent an entire set of data. For example, discussions about fuel costs will often focus on the average price of petrol, while in real estate the median house price is considered an important measurement. These representative values are known as **measures of centre** as they are located in the central region of the data. The **mean**, **median** and **mode** are all measures of centre, and the most appropriate one to use depends on various characteristics of the data set.

13.4.2 The mean (or arithmetic mean)

The mean of a data set is what we commonly refer to as the average. It is calculated by dividing the sum of the data values by the number of data values. If the data set is a sample of the population, the symbol used for the mean is $\bar{x}$ (pronounced 'x-bar'), whereas if the data set is the whole population, we use the The variance is calculated by findingGreek letter μ (pronounced 'mu').

$$\bar{x} = \frac{\text{sum of data values}}{\text{number of data values}}$$
$$= \frac{\sum x}{n}$$
$$\left(\text{or } \mu = \frac{\sum x}{n}\right)$$

The Greek letter Σ (sigma) indicates calculating the sum of these values.

WORKED EXAMPLE 10

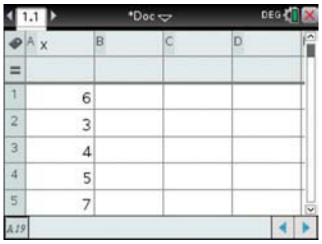
Calculate the mean of the following data set, correct to 2 decimal places.

6, 3, 4, 5, 7, 7, 4, 8, 5, 10, 6, 10, 9, 8, 3, 6, 5, 4

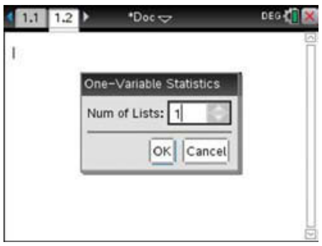
THINK	WRITE
1. Calculate the sum of the data values.	$6+3+4+5+7+7+4+8+5+10+6$ $+10+9+8+3+6+5+4=110$
2. Divide the sum by the number of data values.	$\bar{x}=\frac{110}{18}$ $=6.111...$
3. State the answer.	The mean of the data set is 6.11.

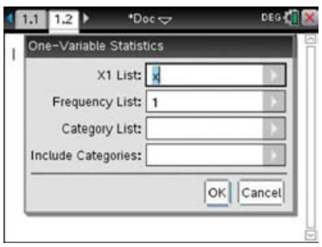
TI | THINK — **WRITE**

1. In a Lists & Spreadsheet page, label the first column as x. Enter the given data values in the first column.

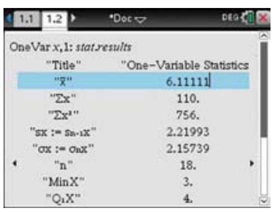


2. In a Calculator page, press MENU and select:
6: Statistics
1: Stat Calculations
1: One-Variable Statistics …
When prompted, enter the number of lists as 1.
Complete the fields as:
X1 List: x
Frequency List: 1
then select OK.





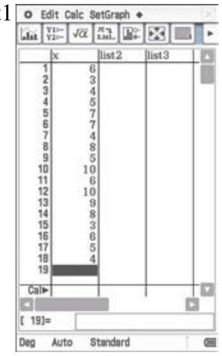
3. The mean is displayed on the screen as $\bar{x}$.



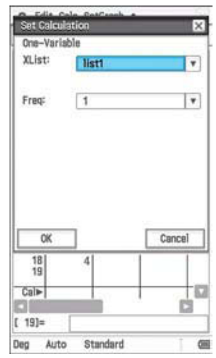
The mean is 6.11 (2 decimal places).

CASIO | THINK — **WRITE**

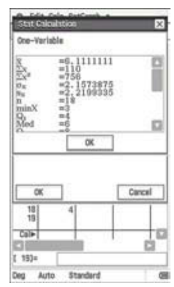
1. On a Statistics screen, relabel list1 as x.
Enter the given data values in the first column.



2. Select:
- Calc
- One-Variable
Complete the fields as:
XList: main\x
Freq: 1
then select OK.



3. The mean is displayed on the screen as $\bar{x}$.



The mean is 6.11 (2 decimal places).

13.4.3 Calculating the mean for grouped data

To calculate the mean from a table of data that has been organised into groups, we first need to calculate the midpoints of the intervals. We then multiply the values of the midpoints by the corresponding frequencies, and find the sum of these values. Finally, we divide this sum by the total of the frequencies.

If f = the values of the frequencies and x = the values of the midpoints, then $\bar{x} = \dfrac{\sum xf}{\sum f}$.

WORKED EXAMPLE 11

Calculate the mean of the data set displayed in the following frequency table.

Intervals	Frequency
0−<5	3
5−<10	12
10−<15	3
15−<20	2

THINK

WRITE

1. Add a column to the table and enter the midpoints for the corresponding intervals.

Intervals	Frequency (f)	Midpoint (x)
0−<5	3	2.5
5−<10	12	7.5
10−<15	3	12.5
15−<20	2	17.5

2. Add a column to the table and enter the product of the frequencies and midpoints (xf) for the corresponding intervals.

Intervals	Frequency (f)	Midpoint (x)	xf
0−<5	3	2.5	7.5
5−<10	12	7.5	90
10−<15	3	12.5	37.5
15−<20	2	17.5	35

3. Calculate the totals of the f and xf columns.

Intervals	Frequency (f)	Midpoint (x)	xf
0−<5	3	2.5	7.5
5−<10	12	7.5	90
10−<15	3	12.5	37.5
15−<20	2	17.5	35
	$\sum f = 20$		$\sum fx = 170$

4. Calculate the mean using the formula $\bar{x} = \dfrac{\sum xf}{\sum f}$.

$$\bar{x} = \frac{\sum xf}{\sum f} = \frac{170}{20} = 8.5$$

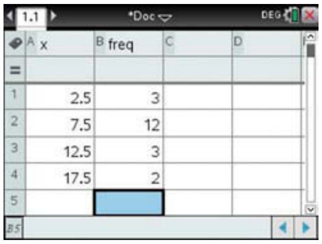
5. State the answer.

The mean of the distribution is 8.5.

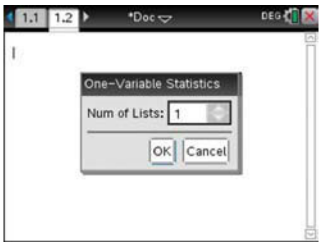
TI | THINK

WRITE

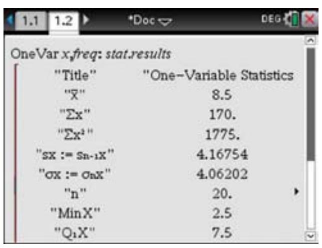
1. In a Lists & Spreadsheet page, label the first column as x and the second column as *freq*. Enter the midpoint of each interval in the first column and the frequencies in the second column.



2. In a Calculator page, press MENU and select:
6: Statistics
1: Stat Calculations
1: One-Variable Statistics …
When prompted, enter the number of lists as 1.
Complete the fields as:
X1 List: x
Frequency List: freq
then select OK.



3. The mean is displayed on the screen as $\bar{x}$.

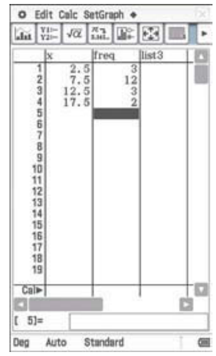


The mean is 8.5.

CASIO | THINK

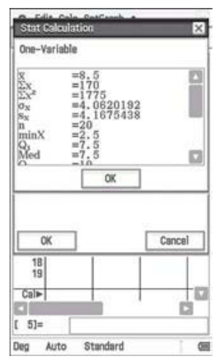
WRITE

1. On a Statistics screen, relabel list1 as x and list2 as *freq*. Enter the midpoint of each interval in the first column and the frequencies in the second column.



2. Select:
- Calc
- One-Variable
Complete the fields as:
XList: main\x
Freq: main\freq
then select OK.

3. The mean is displayed on the screen as $\bar{x}$.



The mean is 8.5.

13.4.4 The median

When considering a value that truly indicates the centre of a distribution, it would make sense to look at the number that is actually in the middle of the data set. The median of a distribution is the middle value of the ordered data set if there are an odd number of values. If there are an even number of values, the median is halfway between the two middle values. It can be found using the rule:

$$\textbf{Median} = \left(\frac{n+1}{2}\right)\textbf{th position}$$

WORKED EXAMPLE 12

Calculate the median of the following data sets.

a. **5, 3, 4, 5, 7, 7, 4, 8, 5, 10, 6, 10, 9, 8, 3, 6, 5, 4**

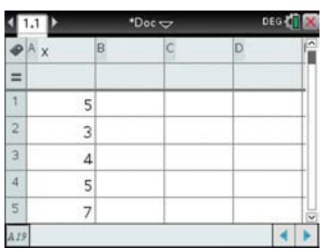
b. **16, 3, 4, 5, 17, 27, 14, 18, 15, 10, 6, 10, 9, 8, 23, 26, 35**

THINK	WRITE
a. 1. Put the data set in order from lowest to highest.	**a.** 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 7, 7, 8, 8, 9, 10, 10
2. Identify the data value in the $\left(\frac{n+1}{2}\right)$th position.	There are 18 data values, so the median will be in position $\left(\frac{18+1}{2}\right) = 9.5$, or halfway between position 9 and position 10. 3, 3, 4, 4, 4, 5, 5, 5, [5, 6,] 6, 7, 7, 8, 8, 9, 10, 10 median = 5.5
3. State the answer.	The median of the data set is 5.5.
b. 1. Put the data set in order from lowest to highest.	**b.** 3, 4, 5, 6, 8, 9, 10, 10, 14, 15, 16, 17, 18, 23, 26, 27, 35
2. Identify the data value in the $\left(\frac{n+1}{2}\right)$th position.	There are 17 data values, so the median will be in position $\left(\frac{17+1}{2}\right) = 9$. 3, 4, 5, 6, 8, 9, 10, 10, [14,] 15, 16, 17, 18, 23, 26, 27, 35 median = 14
3. State the answer.	The median of the data set is 14.

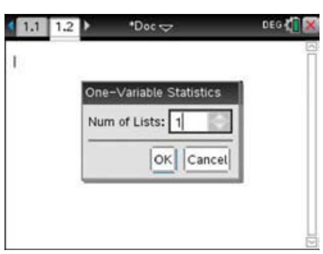
TI | THINK

a. 1. In a Lists & Spreadsheet page, label the first column as x. Enter the given data values in the first column.

WRITE



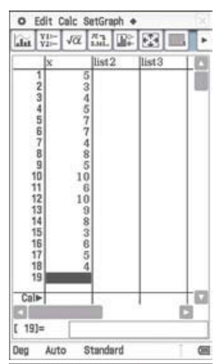
2. In a Calculator page, press MENU and select:
6: Statistics
1: Stat Calculations
1: One-Variable Statistics …
When prompted, enter the number of lists as 1.
Complete the fields as:
X1 List: x
Frequency List: 1
then select OK.



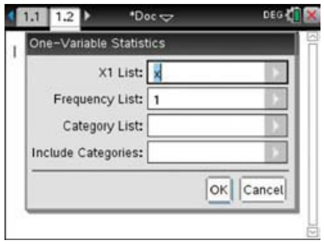
CASIO | THINK

a. 1. On a Statistics screen, relabel list1 as x. Enter the given data values in the first column.

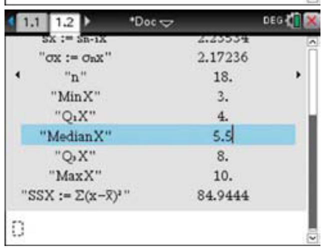
WRITE



2. Select:
- Calc
- One-Variable
Complete the fields as:
XList: main\x
Freq: 1
then select OK.

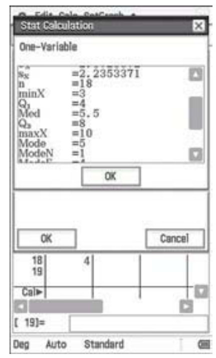


3. The median is displayed on the screen as MedianX.



The median is 5.5.

The median is displayed on the screen as Med.



The median is 5.5.

13.4.5 Limitations of the mean as a measures of centre

An important property of the mean is that it includes all the data in its calculation. As such, it has genuine credibility as a representative value for the distribution. On the other hand, this property also makes it susceptible to being adversely affected by the presence of extreme values when compared to the majority of the distribution.

Consider the data set: 3, 4, 5, 6, 7, 8, 9, which would have a mean of

$$\frac{3+4+5+6+7+8+9}{7} = 6.$$

Compare this to a data set with the same values with the exception of the largest one: 3, 4, 5, 6, 7, 8, 90, which has a mean of

$$\frac{3+4+5+6+7+8+90}{7} = 17.6.$$

As we can see, the mean has been significantly influenced by the one extreme value. When the data is skewed or contains extreme values, the mean becomes less reliable as a measure of centre.

13.4.6 Limitations of the median as a measure of centre

In the previous example we saw that the mean is affected by an extreme value in the data set; however, the same cannot be said for the median. In both data sets the median will be the value in the fourth position.

3, 4, 5, 6, 7, 8, 9   3, 4, 5, 6, 7, 8, 90

median

The median is therefore more reliable than the mean when the data is skewed or contains extreme values. Another potential advantage of median is that it is often one of the data values, whereas the mean often isn't. However, the median can be considered unrepresentative as it is not calculated by taking into account the actual values in the data set.

13.4.7 Choosing between measures of centre

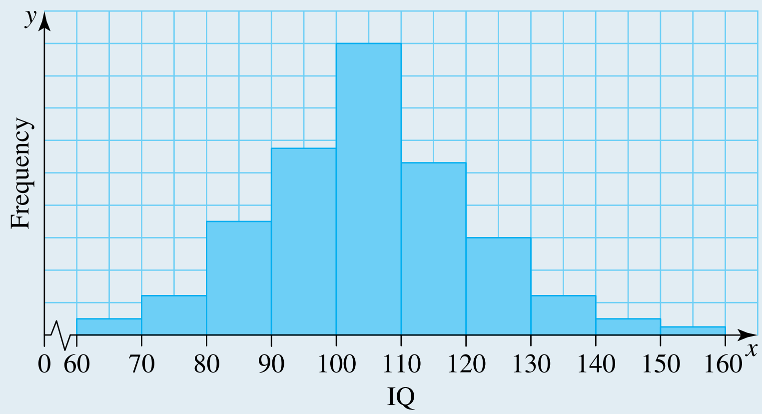
In most situations it is preferable to give both the median and the mean as a measure of centre, as between them they portray a more accurate picture of the data set. However, sometimes it is only possible to give one of these values to represent our data set.

When choosing which measure of centre to use to represent a data set, take into account the distribution of the data. If the data has no outliers and is approximately symmetrical, then the mean is probably the best measure of centre to represent the data. If there are outliers, the median will be significantly less affected by these and would be a better choice to represent the data. The median is also a good choice to represent skewed data.

Also consider what each measure of centre tells you about the data. The values of the mean and median can vary significantly, so choosing which one to represent the data set can be important, and you will need to justify your choice.

WORKED EXAMPLE 13

The following histogram represents the IQ test results for a group of people.



Determine which measure of centre is best to represent the data set.

THINK	WRITE
1. Look at the distribution of the data set.	The data set is approximately symmetrical and has no outliers.
2. If the data set is approximately symmetrical with no outliers, the mean is probably the best measure of centre to represent the data set. If there are outliers or the data is skewed, the median is probably the best measure of centre to use. State the answer.	The mean is the best measure of centre to represent this data set.

on Resources

Interactivity: Mean, median, mode and quartiles (int-6496)

studyon

Units 1 & 2 > AOS 6 > Topic 1 > Concept 5

Measures of centre Concept summary and practice questions

Exercise 13.4 Measures of centre

1. **WE10** Calculate the mean of the following data set.
108, 135, 120, 132, 113, 138, 125, 138, 107, 131, 113, 136, 119, 152, 134, 158, 136, 132, 113, 128
2. a. Calculate the mean of the following data set correct to 2 decimal places.
25, 23, 24, 25, 27, 26, 23, 28, 24, 20, 25, 20, 29, 28, 23, 27, 24
b. Replace the highest value in the data set from part **a** with the number 79, and then calculate the mean again, correct to 2 decimal places.
c. How did changing the highest value in the data set affect the mean?
3. **WE11** Calculate the means of the data sets displayed in the following tables, giving your answer correct to 2 decimal places.

a.

Intervals	Frequency
0–<5	12
5–<10	10
10–<15	1
15–<20	4

b.

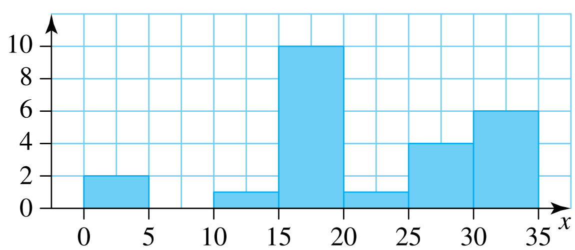
Intervals	Frequency
20–<35	12
35–<50	6
50–<65	13
65–<80	4

4. For each of the following sets of data, estimate the mean by creating a table using intervals that commence with the lowest data value and increase by an amount that is equal to the difference between the highest and lowest data value divided by 5. Give your answers correct to 2 decimal places.
a. 205, 203, 204, 205, 207, 216, 213, 218, 214, 220, 225, 220, 229, 228, 233, 238, 234
b. 5, 13, 24, 5, 27, 16, 13, 18, 24, 10, 5, 20, 30, 18, 13, 7, 14
5. Calculate the means of the following data sets correct to 2 decimal places.

a. Key: 1|2 = 12

Stem	Leaf
0	1 1 5 7
1	2 6
2	3 4 4 5
3	1 3
4	0 0 3
5	5
6	5

b.



6. a. Calculate the mean (correct to 2 decimal places) and median for the following data set.

Average annual rainfall in selected Australian cities	
City	**Rainfall (mm)**
Sydney	1276
Melbourne	654
Brisbane	1194
Adelaide	563
Perth	745
Hobart	576
Darwin	1847
Canberra	630
Alice Springs	326

b. Which would be the most appropriate measure of centre to represent this data?

7. WE12 Calculate the medians of the following data sets.

a. 15, 3, 54, 53, 27, 72, 41, 85, 15, 11, 62, 16, 49, 81, 53, 56, 75, 42

b. 126, 301, 422, 567, 179, 267, 149, 198, 165, 170, 602, 180, 109, 85, 223, 206, 335

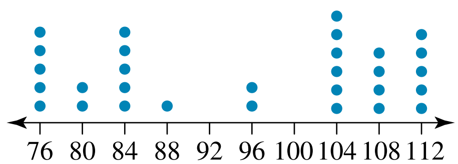
8. a. Calculate the median of the following data set.
21, 22, 23, 24, 27, 26, 22, 27, 23, 21, 24, 20, 31, 25, 24, 28, 23

b. Replace the highest value in the data set from part **a** with the number 96 and then calculate the median again.

c. How does changing the highest value in the data set affect the median?

9. Find the median of the following data sets.

a.



b. 1.02, 2.01, 3.21, 4.63, 1.49, 3.45, 1.17, 1.38, 1.47, 1.70, 5.02, 1.38, 1.91, 8.54

10. WE13 The following stem plot represents the lifespan of different animals at an animal sanctuary. Determine which measure of centre is best to represent the data set.

Key: 1|2 = 12

Stem	Leaf
0	3 5 9
1	2 4 6 8
2	0 1 4 5 5 7 9
3	0 2 6
4	
5	
6	0 3

11. The following data set represents the salaries (in $000s) of workers at a small business.
45, 50, 55, 55, 55, 60, 65, 65, 70, 70, 75, 80, 220

a. Calculate the mean of the salaries correct to 2 decimal places.

b. Calculate the median of the salaries.

c. When it comes to negotiating salaries, the workers want to use the mean to represent the data and the management want to use the median. Explain why this might be the case.

12. On a particular weekend, properties sold at auction for the following 30 prices:
$4 700 000, $3 160 000, $2 725 000, $2 616 000, $2 560 000, $241 000,
$265 000, $266 000, $310 000, $320 000, $3 010 000, $2 580 000,
$2 450 000, $2 300 000, $2 275 000, $286 000, $325 000, $330 000,
$435 500, $456 000, $1 350 000, $1 020 000, $900 000, $735 000,
$733 000, $305 000, $330 000, $347 000, $357 000, $408 000
   a. Calculate the mean and median for the data.
   b. Draw a histogram of the data using intervals commencing at the lowest value and increasing by amounts of $250 000.
   c. Mark in the location of the mean and median on the histogram.
   d. Which would be the more appropriate measure of centre to represent this data?
13. The waiting times in minutes and seconds for a group of passengers at a railway station are:
19:28, 17:35, 14:21, 16:22, 12:18, 11:09, 13:15, 16:21, 11:45, 12:26, 14:16, 17:12,
13:42, 14:51, 15:26, 15:13, 18:02, 11:22, 12:26, 13:10, 13:18, 13:41, 13:23, 14:06
   a. Calculate the mean of the data correct to the nearest second.
   b. Use intervals of 2 minutes starting with 10−<12 to display the data in a frequency table.
   c. Use the frequency table to calculate the mean correct to the nearest second.
   d. Comment on any differences between the two means.
14. The heights in metres of fruit trees in an orchard were measured with the following results:
1.83, 1.94, 1.98, 1.91, 1.88, 1.76, 2.12, 2.05, 2.11, 2.01, 2.04, 2.08,
2.07, 2.06, 2.05, 2.03, 1.94, 1.96, 2.12, 2.14, 2.04, 2.01, 2.03, 2.06,
2.02, 1.94, 1.98, 2.25, 2.04, 2.06
   a. Use intervals of 0.05 m starting with 1.75−<1.80 to display the data in a frequency table.
   b. Display the data in a histogram and use it to comment on the appropriateness of using either the mean or the median to represent the data.
   c. Use the frequency table to calculate the mean.

15. The winning margins in the NRL over a particular period of time were as follows.

Winning margin	Frequency
2	4
4	12
6	8
8	5
10	4
12	4
16	1
20	1
34	1

   a. Calculate the mean and the median.
   b. Which is the most appropriate measure of centre for this data set and why?

16. The value of the Australian dollar in US cents over a particular period of time was as follows:
93, 91, 88, 94, 86, 90, 93, 95, 84, 81, 91, 96, 99, 101, 106, 104, 104, 99, 99, 96, 94, 95, 91, 90, 89, 88, 89, 86, 88, 87, 83, 88, 84, 85, 86, 86, 87, 88, 87, 84
    a. Calculate the mean and median of the raw data.
    b. Display the data in a histogram using intervals commencing at 80−<85.
    c. Mark in the positions of the mean and median on the histogram.
    d. Comment on the positions of the mean and median.
17. The annual earnings of a group of professional tennis players are as follows:
$5 700 000, $1 125 000, $620 000, $4 950 000, $275 000, $220 000, $242 000, $350 000, $375 000, $300 000, $422 000, $2 150 000, $270 000, $420 000, $300 000, $245 000, $385 000, $284 000, $320 000, $444 000, $185 000, $200 500, $264 000, $290 000

    a. Calculate the mean and median of the raw data. Give your answers correct to the nearest dollar.
    b. Display the data in a histogram using intervals commencing with $180 000−<$380 000.
    c. Mark in the positions of the mean and median on the histogram.
    d. Which is the most appropriate measure of centre for this data and why?
18. The total cost of the weekly phone calls made by each of the employees of a business for a particular week is listed below.

Employee	Phone call costs	Employee	Phone call costs	Employee	Phone call costs
A	$64.50	G	$33.45	M	$58.35
B	$96.65	H	$87.90	N	$71.35
C	$110.25	I	$48.45	O	$63.50
D	$74.25	J	$66.50	P	$52.45
E	$53.55	K	$86.65		
F	$104.75	L	$82.96		

    a. Calculate the mean and median for the raw data. Give your answers correct to the nearest cent.
    b. How many employees are above the mean weekly phone costs?
    c. Assume all employees whose phone calls were above the mean calculated in part a reduced their costs by 15%. Calculate the new mean value, correct to the nearest cent, if all the values below the mean remain the same.
    d. How does the change from part c affect the median?

19. Use CAS to answer the following questions.
The body mass index (BMI) is an accepted measure of obesity with a value of 30 or more being the obese category. The BMI results for a group of people are shown in the table.

22.5	31.4	28.4	18.5	33.2	26.3	27.1	28.6	31.2	21.2	19.8	20.4	20.7	26.4
29.4	27.1	31.6	21.4	34.1	32.1	26.3	21.4	27.3	23.2	28.3	21.4	26.1	26.3
28.4	29.1	22.8	23.7	20.4	28.1	30.4	22.4	18.1	22.5	24.3	25.2	24.7	30.2

a. Display the data in two frequency tables and draw the corresponding histograms.
 i. In the first frequency table, use intervals commencing at the lowest value and increasing by an amount that is calculated by dividing the difference between the lowest and highest data value by 5.
 ii. In the second frequency table, use intervals commencing at the lowest value and increasing by an amount that is calculated by dividing the difference between the lowest and highest data value by 10.

b. Describe the two histograms.

c. Calculate the mean for each frequency table and compare them to the mean of the raw data. Give your answers correct to 2 decimal places.

d. Which measure of centre is the best representation of this data?

20. Use CAS to answer the following questions.
A health centre measured the heights (in cm) of a group of young children aged from two to three years of age, with the results shown in the table.

82.5	71.4	88.4	88.5	93.2	96.3
97.1	88.6	61.2	101.2	99.8	100.4
100.7	96.4	89.4	87.1	81.6	81.4
94.1	92.1	96.3	91.4	97.3	83.2
88.3	91.4	76.1	96.3	98.4	89.1
92.8	93.7	100.4	98.1	100.4	102.4
88.1	92.5	94.3	85.2	94.7	90.2

a. Calculate the median of the raw data.

b. Use the data values that are below the median to draw a histogram starting with the lowest value and increasing by intervals that are calculated by dividing the difference between the lowest and highest data value (of the lower half of the data set) by 5.

c. Repeat part **b** for the data values that are above the median.

d. Calculate the mean for each of the two groups correct to 2 decimal places.

e. Describe the two histograms.

13.5 Measures of spread

13.5.1 Range and quartiles

While measures of centre such as the mean or median give valuable information about a set of data, taken in isolation they can be quite misleading. Take for example the data sets {36, 43, 44, 59, 68} and {1, 2, 44, 80, 123}. Both groups have a mean of 50 and a median of 44, but the values in the second set are much further apart from each other. Measures of centre tell us nothing about how variable the data values in a set might be; for this we need to consider the **measures of spread** of the data.

13.5.2 Range

In simplest terms the spread of a data set can be determined by looking at the difference between the smallest and largest values. This is called the range of the distribution. While the range is a useful calculation, it can also be limited. Any extreme values (outliers) will result in the range giving a false indication of the spread of the data.

Range = largest value − smallest value

13.5.3 Quartiles

A clearer picture of the spread of data can be obtained by looking at smaller sections. A common way to do this is to divide the data into quarters, known as quartiles.

The **lower quartile** ($\mathbf{Q_1}$) is the value that indicates the median of the lower half of the data.

The second quartile ($\mathbf{Q_2}$) is the median.

The **upper quartile** ($\mathbf{Q_3}$) is the value that indicates the median of the upper half of the data.

When calculating the values of the lower and upper quartiles, the median should not be included. If the median is between values, then these values should be considered in your calculations.

13.5.4 The interquartile range

The **interquartile range** is found by calculating the difference between the third and first quartiles ($Q_3 - Q_1$), which gives an indication of the spread of the middle 50% of the data.

WORKED EXAMPLE 14

Calculate the interquartile range of the following set of data.
23, 34, 67, 17, 34, 56, 19, 22, 24, 56, 56, 34, 23, 78, 22, 16, 15, 35, 45

THINK	WRITE
1. Put the data in order.	15, 16, 17, 19, 22, 22, 23, 23, 24, 34, 34, 34, 35, 45, 56, 56, 56, 67, 78
2. Identify the median.	There are 19 data values, so the median will be in position $\left(\frac{19+1}{2}\right) = 10$. median ↓ 15, 16, 17, 19, 22, 22, 23, 23, 24, (34) The median is 34.

3. Identify Q_1 by finding the median of the lower half of the data.

There are 9 values in the lower half of the data, so Q_1 will be the 5th of these values.

Q_1 ↓

15, 16, 17, 19, (22), 22, 23, 23, 24

$Q_1 = 22$

4. Identify Q_3 by finding the median of the upper half of the data.

There are 9 values in the upper half of the data, so Q_3 will be the 5th of these values.

Q_3 ↓

34, 34, 35, 45, (56), 56, 56, 67, 78

$Q_3 = 56$

5. Calculate the interquartile range using $\text{IQR} = (Q_3 - Q_1)$.

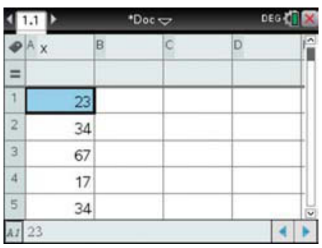
$$\begin{aligned}\text{IQR} &= (Q_3 - Q_1)\\ &= 56 - 22\\ &= 34\end{aligned}$$

6. State the answer.

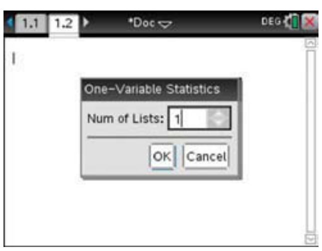
The interquartile range is 34.

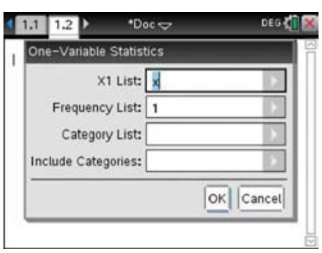
TI | THINK — **WRITE**

1. In a Lists & Spreadsheet page, label the first column as x. Enter the given data values in the first column.



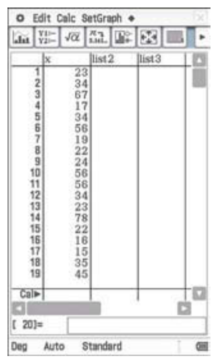
2. In a Calculator page, press MENU and select:
6: Statistics
1: Stat Calculations
1: One-Variable Statistics …
When prompted, enter the number of lists as 1.
Complete the fields as:
X1 List: x
Frequency List: 1
then select OK.



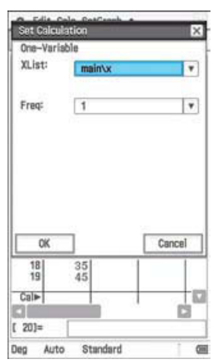


CASIO | THINK — **WRITE**

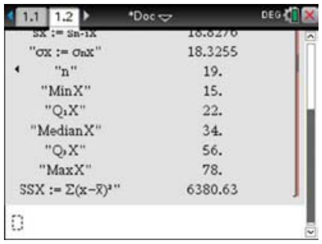
1. On a Statistics screen, relabel list1 as x.
Enter the given data values in the first column.



2. Select:
- Calc
- One-Variable
Complete the fields as:
XList: main\x
Freq: 1
then select OK.



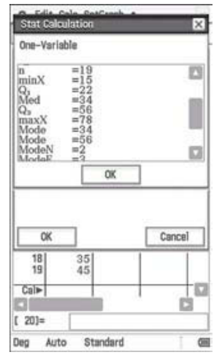
3. The first quartile is displayed as Q_1X and the second quartile is displayed as Q_3X.



4. Calculate the interquartile range.

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 56 - 22\\ &= 34\end{aligned}$$

3. The first quartile is displayed as Q_1 and the second quartile is displayed as Q_3.



Calculate the interquartile range.

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 56 - 22\\ &= 34\end{aligned}$$

13.5.5 Spread around the mean

When the mean is used as a representative value for data, it makes sense to take note of how much the data varies in comparison to the mean. Two indicators of the spread of data around the mean are the **variance** and the **standard deviation**. These measures generally only apply to continuous numerical data. The larger the variance and standard deviation are, the more spread out the data is away from the mean.

13.5.6 Variance

The variance is calculated by finding the difference between each data value and the mean. To adjust for the fact that values below the mean will result in a negative number, the results are then squared. These values are then averaged to give a single number. The variance is calculated using the following formula:

$$\textbf{Sample variance: } s^2 = \frac{\sum f(x-\bar{x})^2}{\left(\sum f\right)-1}$$

13.5.7 Standard deviation

The standard deviation is calculated by taking the square root of the variance.

$$\textbf{Sample standard deviation: } s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\left(\sum f\right)-1}}$$

This reverses the previous mathematical process of squaring the differences between the data values and the mean, so that the standard deviation reverts to a comparative unit of measurement for the original data.

The following example shows that the variance and standard deviation can become very messy to calculate once you have large groups of data. Spreadsheets, calculators and similar technologies are a more practical and reliable option for these computations.

The table shows a grouped distribution of a sample of data with a mean of 6.5.

Intervals	Frequency (f)	Midpoint (x)	xf
0−<5	2	2.5	5
5−<10	8	7.5	60
			$\Sigma xf = 65$

To calculate the variance and standard deviation, two columns are added. The first column shows the square of the difference between the midpoint and the mean, and the second shows this value multiplied by the frequency for the interval.

Intervals	Frequency (f)	Midpoint (x)	xf	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
0−<5	2	2.5	5	$(2.5 - 6.5)^2 = 16$	32
5−<10	8	7.5	60	$(7.5 - 6.5)^2 = 1$	8
	$\sum f = 10$		$\sum xf = 65$		$\sum f(x - \bar{x})^2 = 40$

The sum of the final column can then be used in the formulas to calculate the variance and standard deviation of the sample.

Sample variance: $$s^2 = \frac{\sum f(x - \bar{x})^2}{(\sum f) - 1} = \frac{40}{9} \approx 4.44$$

Sample standard deviation: $$s = \sqrt{4.44} \approx 2.11$$

WORKED EXAMPLE 15

Calculate the variance and standard deviation for the sample from the information shown in the table. Give your answers correct to 2 decimal places.

Intervals	Frequency (f)	Midpoint (x)	xf	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
10−<5	8	12.5	100	$(12.5 - 15.5)^2 = 9$	72
5−<10	12	17.5	210	$(17.5 - 15.5)^2 = 4$	48
	$\sum f = 20$		$\sum xf = 310$		

THINK	WRITE
1. Sum the $f(x - \overline{x})^2$ column.	$\sum f(x - \overline{x})^2 = 72 + 48$ $= 120$
2. Substitute the values into the formulas for variance and standard deviation.	$s^2 = \frac{120}{19}$ ≈ 6.32 $s = \sqrt{6.32}$ ≈ 2.51
3. State the answer.	The variance of the sample is 6.32 and the standard deviation of the sample is 2.51.

13.5.8 Preferred measures of spread

The standard deviation is generally considered the preferred measure of the spread of a distribution when there are no outliers and no skew, as all of the data contributes to its calculation. When there are outliers or the data is skewed, the interquartile range is a better option as it is not adversely influenced by extreme values.

As the interquartile range is calculated on the basis of just two numbers that may or may not be actual values from the data set, it could be considered to be unrepresentative of the data set.

on Resources

- **Interactivity:** The median, the interquartile range, the range and the mode (int-6244)
- **Interactivity:** The mean and the standard deviation (int-6246)

studyon

Units 1 & 2 > AOS 6 > Topic 1 > Concept 6

Measures of spread Concept summary and practice questions

Exercise 13.5 Measures of spread

1. **WE14** Calculate the interquartile range of the following set of data.
   421, 331, 127, 105, 309, 512, 129, 232, 124, 154, 246, 124, 313, 218, 112, 136, 155, 305, 415
2. Calculate the interquartile range of the following set of data.
   3.11, 3.16, 1.13, 1.56, 3.19, 4.43, 1.98, 4.89, 2.12, 4.78, 3.21, 8.88, 1.21, 5.67, 2.22, 3.34
3. The results for a multiple choice test for 20 students in two different classes are as follows.
   Class A: 7, 13, 14, 13, 14, 14, 12, 8, 18, 13, 14, 12, 16, 14, 12, 11, 13, 14, 13, 15
   Class B: 18, 19, 12, 12, 11, 17, 9, 18, 17, 14, 13, 11, 17, 13, 17, 14, 14, 15, 13, 12
   a. Compare the spread of the marks for each class by using the range.
   b. Compare the spread of the marks for each class by using the interquartile range.

4. The competition ladder of the Australian and New Zealand netball championship is as follows.

Position	Team	Win	Loss	Goals for	Goals against
1	Adelaide Thunderbirds	12	1	688	620
2	Melbourne Vixens	9	4	692	589
3	Waikato BOP Magic	9	4	749	650
4	Queensland Firebirds	9	4	793	691
5	Central Pulse	8	5	736	706
6	Southern Steel	6	7	812	790
7	West Coast Fever	5	8	715	757
8	NSW Swifts	4	9	652	672
9	Canterbury Tactix	2	11	700	882
10	Northern Mystics	1	12	699	879

a. Calculate the spread for the 'Goals for' column by using the range.
b. Calculate the spread for the 'Goals for' column by using the interquartile range.
c. Compare the spread of the 'Goals for' column with the spread of the 'Goals against' column.

5. WE15 Calculate the variance and standard deviation for the sample from the information shown in the table. Give your answers correct to 2 decimal places.

Intervals	Frequency (f)	Midpoint (x)	xf	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
0–<10	14	5	70	$(5 - 8.3)^2 = 10.89$	152.46
10–<20	7	15	105	$(15 - 8.3)^2 = 44.89$	314.23
	$\sum f = 21$		$\sum xf = 175$		

6. Complete the table and calculate the variance and standard deviation for the following sample correct to 3 decimal places.

Intervals	Frequency (f)	Midpoint (x)	xf	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
0–<50	35				
50–<100	125				
	$\sum f =$		$\sum xf =$		$\Sigma f(x - \bar{x})^2 =$

7. a. Complete the details of the following table, which shows the results of a survey of the ages of a sample of workers in the hospitality industry.

Age group (years)	Frequency (f)	Midpoint (x)	xf
15–<20	14		
20–<25	18		
25–<30	11		
30–<35	7		
35–<40	5		
	$\sum f =$		$\sum xf =$

8. A survey of the number of motor vehicles that pass a school between 8.30 am and 9.30 am on 10 days during a term are as follows.

72, 89, 94, 78, 83, 84, 88, 97, 82, 88

a. Use CAS to calculate the standard deviation of the sample correct to 2 decimal places.

b. Calculate the interquartile range of the sample.

c. The lowest number is reduced by 10 and the highest value increased by 10. Recalculate the values of the standard deviation and interquartile range.

d. How is each of the measures affected by the change in the values?

9. A survey of a large sample of people from particular areas of employment found the following average Australian salary ranges.

Employment area	Average minimum	Average maximum
Mining	\$65 795	\$262 733
Management	\$66 701	\$240 000
Engineering	\$56 572	\$233 451
Legal	\$53 794	\$193 235
Building and construction	\$46 795	\$186 412
Telecommunications	\$47 354	\$193 735
Science	\$47 978	\$211 823
Medical	\$42 868	\$228 806
Sales	\$42 917	\$176 783

a. Calculate the interquartile range for the average minimum salaries.

b. Calculate the interquartile range for the average maximum salaries.

c. Comment on the two interquartile ranges.

10. A sample of crime statistics over a two-year period are shown in the following table.

Crime	Year 1	Year 2
Theft from motor vehicle	46 700	42 900
Theft (steal from shop)	19 800	20 600
Theft of motor vehicle	15 650	14 670
Theft of bicycle	4200	4660
Theft (other)	50 965	50 650

a. Calculate the interquartile range and standard deviation (correct to 1 decimal place) for both years.
b. Recalculate the interquartile range and standard deviation for both years after removing the smallest category.
c. Comment on the effect of removing the smallest category on the interquartile ranges and standard deviations.

11. The table shows the number of registered passenger vehicles in two particular years for the states and territories of Australia.

Number of passenger vehicles		
State	**Year 1**	**Year 2**
New South Wales	3 395 905	3 877 515
Victoria	2 997 856	3 446 548
Queensland	2 138 364	2 556 581
South Australia	915 059	1 016 590
Western Australia	1 205 266	1476 743
Tasmania	271 365	305 913
Northern Territory	73 302	91 071
Australian Capital Territory	191 763	229 060

a. Calculate the interquartile range and standard deviation (correct to 1 decimal place) for both years.
b. Recalculate the interquartile range and standard deviation for both years after removing the three smallest values.
c. Comment on the effect of the removal of the three smallest values on the interquartile ranges and standard deviations.

12. Data collected on the number of daylight hours in Alice Springs is as shown.

10.3, 9.8, 9.6, 9.5, 8.5, 8.4, 9.1, 9.8, 10.0, 10.0, 10.1, 10.0, 10.1, 10.1, 10.6, 8.7, 8.8, 9.0, 8.0, 8.5, 10.6, 10.8, 10.5, 10.9, 8.5, 9.5, 9.3, 9.0, 9.4, 10.6, 8.3, 9.3, 9.0, 10.3, 8.4, 8.9

a. Calculate the range of the data.
b. Calculate the interquartile range of the data.
c. Comment on the difference between the two measures and what this indicates.

13. The volume of wine ('000 litres) available for consumption in Australia for a random selection of months over a 10-year time period is shown in the following table.

38 595	41 301	44 212	39 362	38 914	38 273	39 456	38 823
41 123	42 981	44 567	41 675	41 365	42 845	43 987	41 583
39 347	42 673	44 835	39 773	38 586	38 833	39 756	39 095
42 946	46 382	44 892	41 038	41 402	42 587	43 689	41 209

a. Use CAS to calculate the mean and standard deviation of the data correct to 2 decimal places.

b. Calculate the median and interquartile range of the data.

c. What percentage, correct to 2 decimal places, of the actual data values from the sample are within one standard deviation of the mean (i.e. between the number obtained by subtracting the standard deviation from the mean and the number obtained by adding the standard deviation to the mean)?

d. What percentage of the actual data values from the sample are between the first and third quartiles?

e. Comment on the differences between your answers for parts **c** and **d**.

14. A random sample of the monthly consumer price indices in various cities of Australia is shown in the following table. Answer the following questions, giving answers correct to 2 decimal places where appropriate.

Sydney	Melbourne	Brisbane	Adelaide	Perth	Hobart	Darwin	Canberra
0.4	0.8	0.9	0.7	0.6	0.3	1.2	0.8
0.9	1.0	1.1	1.1	0.8	0.8	0.3	1.0
1.4	1.3	1.3	1.5	1.4	1.3	0.9	1.4
1.5	1.2	1.7	1.3	1.6	1.0	1.5	1.2
1.1	1.2	1.4	1.3	1.0	1.1	1.8	1.5
0.1	0.3	0.2	0.2	0.1	0.2	0.1	0.3
0.4	0.3	0.5	0.5	0.9	0.5	1.1	0.6
1.1	0.5	1.4	1.1	0.8	1.2	1.9	0.9
0.5	0.6	0.3	0.4	0.5	0.6	0.1	0.4
0.8	1.3	0.7	0.5	1.2	0.7	0.5	0.6

a. Use CAS to calculate the standard deviation and interquartile range of the entire data set.
b. Use CAS to calculate the standard deviation and interquartile range for each city.
c. Which city bears the closest similarity to the entire data set?
d. Which city bears the least similarity to the entire data set?

15. Use CAS to answer the questions on the data in the following table. Where appropriate, give answers correct to 2 decimal places.

Carbon dioxide emissions (million metric tons of carbon dioxide)						
Country	**2012**	**2014**	**2015**	**2016**	**2017**	**2018**
Australia	374.05	382.65	380.68	391.03	416.89	417.06
Canada	553.55	573.25	602.46	614.69	632.01	614.33
China	3107.99	3440.60	4061.64	4847.33	5429.30	6017.69
Germany	877.71	857.35	874.04	871.88	852.57	857.60
India	1035.42	1033.52	1048.11	1151.33	1194.01	1293.17
Indonesia	300.18	314.88	305.44	323.29	323.51	280.36
Japan	1197.15	1203.33	1253.29	1257.89	1249.62	1246.76
Russia	1571.14	1571.77	1626.86	1663.44	1698.56	1704.36
United Kingdom	575.19	563.89	575.17	582.29	584.65	585.71
United States	5762.33	5823.80	5877.73	5969.28	5994.29	5902.75

a. Calculate the interquartile range and standard deviation for the Australian data.
b. Compare the measures of spread for the Australian data with those for India, China, the United Kingdom and the United States.
c. For this data, which measure of spread is more appropriate?

16. Use CAS to answer the questions on the data in the following table. Where appropriate, give your answers correct to 2 decimal places.

Alcohol consumption per adult (litres)	
Country	**Consumption per adult (litres)**
Australia	10.21
Canada	10.01
France	12.48
Germany	12.14
Greece	11.01
Indonesia	0.56
Ireland	14.92
New Zealand	9.99
Russia	16.23
South Africa	10.16
Spain	11.83
Sri Lanka	0.81
United Kingdom	13.24
United States	9.7
Yemen	0.2

a. Calculate the interquartile range and variance for the data set correct to 2 decimal places.
b. Calculate the interquartile range and variance after removing the three lowest values, correct to 2 decimal places.
c. Compare the results from parts **a** and **b**.

13.6 Comparing numerical distributions

13.6.1 The five-number summary

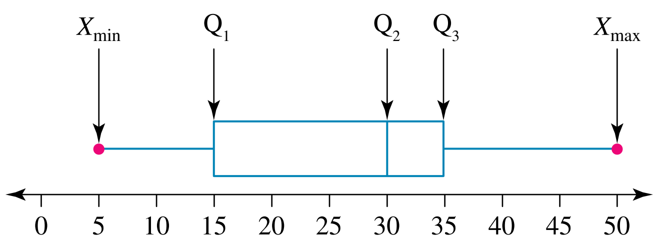
The five-number summary gives five key values that provide information about the spread of a data set. These values are:

1. the lowest score (X_{min})
2. the lower quartile (Q_1)
3. the median (Q_2)
4. the upper quartile (Q_3)
5. the highest score (X_{max}).

13.6.2 Boxplots

We use a **boxplot** to represent the five-number summary in a graphical form. The boxplot is often displayed either above or below a number line, which allows easy identification of the key values.

Boxplots usually consist of both a central box and 'whiskers' on either side of the box. The box represents the interquartile range of the data set, with the distance between the start of the first whisker and the end of the second whisker representing the range of the data. If either the lowest score is equal to the lower quartile or the highest score is equal to the upper quartile, then there will be no whisker on that side of the boxplot.

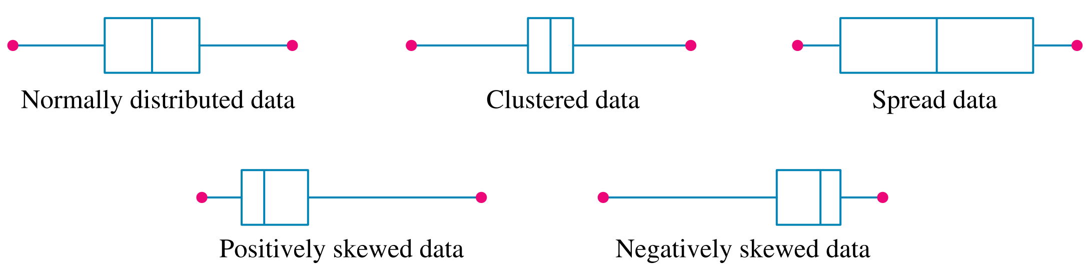


13.6.3 The shape of boxplots

The shape of a boxplot will mirror the distribution of the data set. For example, a boxplot with a small central box and large whiskers will indicate that the majority of the data is clustered around the median, whereas a boxplot with a large central box and small whiskers will indicate that the data is spread more evenly across the range.

Positively skewed data will have the central box on the left-hand side of the boxplot with a large whisker to the right, while negatively skewed data will have the central box on the right-hand side of the boxplot with a large whisker to the left.

Learning to interpret the shape of boxplots will help you to better understand the data that the boxplot represents.



WORKED EXAMPLE 16

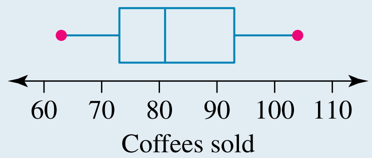
Draw a boxplot for the data contained in the following stem plot, which shows the number of coffees sold by a café each day over a 21-day period.

Key: 6|3 = 63 coffees

Stem	Leaf
6	3 5 8
7	0 2 4 5 7 9
8	1 1 3 6 8
9	0 1 5 6 7
10	1 4

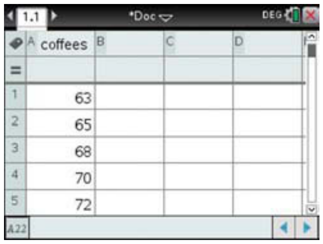
THINK	WRITE/DRAW
1. Determine the median of the data.	There are 21 values, so the median is in the $\left(\frac{21+1}{2}\right) = 11\text{th position}$. median ↓ 63, 65, 68, 70, 72, 74, 75, 77, 79, 81, (81) $Q_2 = 81$
2. Determine the value of the lower quartile.	$\left(\frac{10+1}{2}\right) = 5.5$ There are 10 values in the lower half of the data, so Q_1 will be between the 5th and 6th values. Q_1 ↓ 63, 65, 68, 70, 72, 74, 75, 77, 79, 81 $Q_1 = \frac{72+74}{2}$ $= 73$
3. Determine the value of the upper quartile.	$\left(\frac{10+1}{2}\right) = 5.5$ There are 10 values in the upper half of the data, so Q_3 will be between the 5th and 6th values. Q_3 ↓ 83, 86, 88, 90, 91, 95, 96, 97, 101, 104 $Q_3 = \frac{91+95}{2}$ $= 93$
4. Write the five-number summary.	$X_{min} = 63$ $Q_1 = 73$ $Q_2 = 81$ $Q_3 = 93$ $X_{max} = 104$

5. Rule a suitable scale for your boxplot which covers the full range of values. Draw the central box first (from Q_1 to Q_3, with a line at Q_2) and then draw in the whiskers from the edge of the box to the minimum and maximum values.

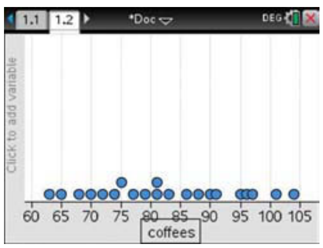


TI | THINK **WRITE**

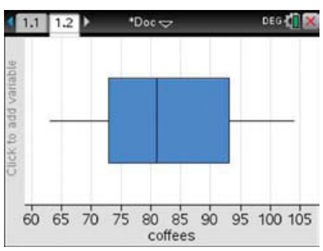
1. In a Lists & Spreadsheet page, label the first column as *coffees*. Enter the given data values in the first column.



2. In a Data & Statistics page, click on the label of the horizontal axis and select *coffees*.

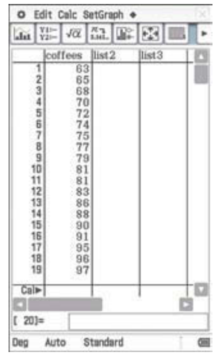


3. Press MENU then select:
   1: Plot Type
   2: Box Plot

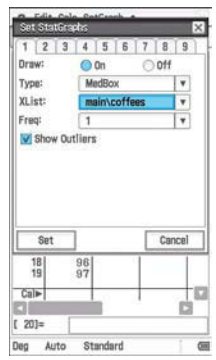


CASIO | THINK **WRITE**

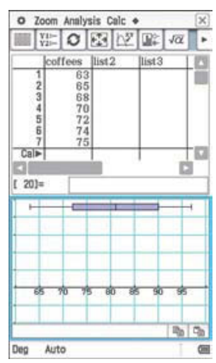
1. On a Statistics screen, relabel list1 as *coffees*. Enter the given data values in the first column.



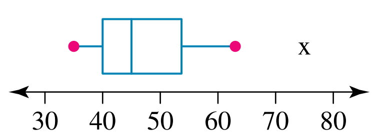
2. Select:
   - SetGraph
   - Setting …
   Complete the fields as:
   Draw: On
   Type: MedBox
   XList: main\coffees
   Freq: 1
   Tick the Show Outliers box then select Set.



Click the y icon.



13.6.4 Identifying possible outliers



If there is an outlier (an extreme value) in the data set, then rather than extending the whiskers to reach this value, we extend the whiskers to the next smallest or largest value and indicate the outlier value with an ‘x’, as shown in the diagram.

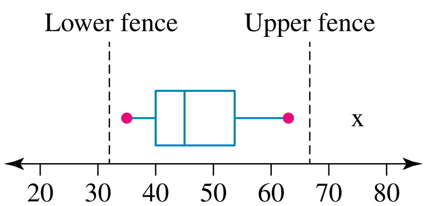
We can identify whether a value in our data set is an outlier or not by calculating the lower and upper fences of our data set.

13.6.5 Lower and upper fences

To calculate the **lower fence** and **upper fence** of the data set, we first need to calculate the interquartile range (IQR). Once this has been calculated, the lower and upper fences are given by the following rules:

Lower fence = $Q_1 - 1.5 \times IQR$
Upper fence = $Q_3 - 1.5 \times IQR$

If a data value lies outside the lower or upper fence then it can be considered an outlier.



WORKED EXAMPLE 17

The following stem plot represents the time taken (in minutes) for 25 students to finish a maths puzzle.

Key: 1|5 = 1.5 minutes

Stem	Leaf
1	5
2	8
3	4 6 6 9
4	0 0 2 3 5 5 7 8 8
5	0 2 4 4 5 8 9
6	0 4
7	
8	5

a. **Calculate the values of the lower and upper fences.**
b. **Identify any outliers in the data set.**
c. **Draw a boxplot to represent the data.**

THINK	WRITE/ DRAW
a. 1. Determine the median of the data.	a. There are 25 values, so the median is in the $\left(\frac{25+1}{2}\right) = 13\text{th position.}$ median ↓ 1.5, 2.8, 3.4, 3.6, 3.6, 3.9, 4.0, 4.0, 4.2, 4.3, 4.5, 4.5, (4.7) $Q_2 = 4.7$
2. Determine the value of the lower quartile.	$\left(\frac{12+1}{2}\right) = 6.5$ There are 12 values in the lower half of the data so Q_1 will be between the 6th and 7th values.

Q1 ↓

1.5, 2.8, 3.4, 3.6, 3.6, 3.9, 4.0, 4.0, 4.2, 4.3, 4.5, 4.5

$$Q_1 = \frac{3.9 + 4.0}{2}$$
$$= 3.95$$

3. Determine the value of the upper quartile.

$$\left(\frac{12+1}{2}\right) = 6.5$$

There are 12 values in the upper half of the data, so Q_3 will be between the 6th and 7th values.

Q3 ↓

4.8, 4.8, 5.0, 5.2, 5.4, 5.4, 5.5, 5.8, 5.9, 6.0, 6.4, 8.5

$$Q_3 = \frac{5.4 + 5.5}{2}$$
$$= 5.45$$

4. Calculate the IQR.

$$\begin{aligned} IQR &= Q_3 - Q_1 \\ &= 5.45 - 3.95 \\ &= 1.5 \end{aligned}$$

5. Calculate the values of the lower and upper fences.

$$\begin{aligned} \text{Lower fence} &= Q_1 - 1.5 \times IQR \\ &= 3.95 - 1.5 \times 1.5 \\ &= 3.95 - 2.25 \\ &= 1.7 \end{aligned}$$

$$\begin{aligned} \text{Upper fence} &= Q_3 + 1.5 \times IQR \\ &= 5.45 + 1.5 \times 1.5 \\ &= 5.45 + 2.25 \\ &= 7.7 \end{aligned}$$

b. 1. Identify whether any values lie below the lower fence or above the upper fence.

b. Values below the lower fence (1.7): 1.5
Values above the upper fence (7.7): 8.5

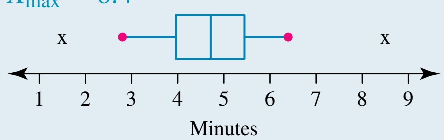
2. State the answer.

There are two outliers: 1.5 and 8.5.

c. 1. Write the five-number summary, giving the minimum and maximum values as those that lie within the lower and upper fences.

c.
$X_{min} = 2.8$
$Q_1 = 3.95$
$Q_2 = 4.7$
$Q_3 = 5.45$
$X_{max} = 6.4$

2. Rule a suitable scale for your boxplot to cover the full range of values. Draw the central box first (from Q_1 to Q_3, with a line at Q_2) and then draw in the whiskers from the edge of the box to the minimum and maximum values. Mark the outliers with an 'x'.



13.6.6 Comparing data sets

In some situations you will be required to compare two or more data sets. We can use two different graphical representations to easily compare and contrast data sets.

13.6.7 Back-to-back stem plots

As mentioned earlier in this topic, back-to-back stem plots are plotted with the same stem, with one of the plots displayed to the left of the stem and the other plot to the right.

Key: 7|5 = 7.5

Leaf	Stem	Leaf
7 3	6	
9 6 2	7	5 7
5 4 2	8	0 1 4 8
8 6 6 0 0	9	2 6 6 9 9
9 5 3	10	3 5 7
	11	1 4
	12	2

Remember to start the numbering on both sides with the smallest values closest to the stem and increasing in value as you move away from the stem. Also include a key with your back-to-back stem plot

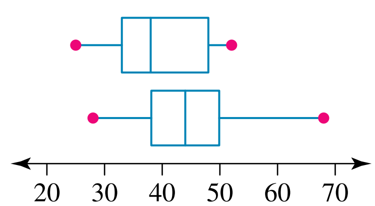
After drawing back-to-back stem plots you can easily identify key points such as:

- which data set has the lowest and/or highest values
- which data set has the largest range
- the spread of both data sets.

13.6.8 Parallel boxplots

Parallel boxplots are plotted with one of the boxplots above the other. Both boxplots share the same scale.

Parallel boxplots allow us to easily make comparisons between data sets, as we can see the key features of the boxplots in the same picture. The position and size of the interquartile ranges of the data sets can be seen, as well as their range. However, while parallel boxplots do display information about the general distribution of the data sets they cover, they lack the detail about this distribution that a histogram or stem plot gives.



WORKED EXAMPLE 18

The following back-to-back stem plot shows the size (in kg) of two different breeds of dog.

a. Draw parallel boxplots of the two sets of data.

b. Compare and contrast the two sets of data.

Key: 2|6 = 26 kg

Breed X Leaf	**Stem**	**Breed Y Leaf**
9 8 7 7 6 4 4	**1**	
8 7 5 4 3 3 1 0	**2**	**6 9**
	3	**3 5 5 7 8**
	4	**0 2 4 5 6 9**
	5	**1 3**

THINK | **WRITE**

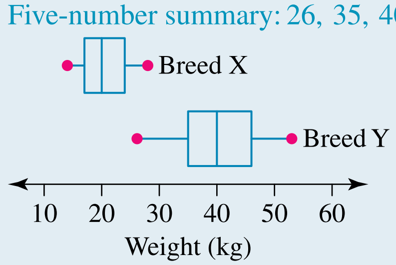
a. 1. Calculate the five-number summary for the first data set (Breed X).

a. $X_{min} = 14$
$X_{max} = 28$
There are 15 pieces of data, so the median is the $\left(\dfrac{15+1}{2}\right) = 8$th piece of data.
$Q_2 = 20$
There are 7 pieces of data in the lower half, so Q_1 is the 4th value.
$Q_1 = 17$
There are 7 pieces of data in the upper half, so Q_3 is the 4th value.
$Q_3 = 24$
Five-number summary: 14, 17, 20, 24, 28

2. Calculate the five-number summary for the second data set (Breed Y).

$X_{min} = 26$
$X_{max} = 53$
There are 15 pieces of data, so the median is the $\left(\dfrac{15+1}{2}\right) = 8$th piece of data.
$Q_2 = 40$
There are 7 pieces of data in the lower half, so Q_1 is the 4th value.
$Q_1 = 35$
There are 7 pieces of data in the upper half, so Q_3 is the 4th value.
$Q_3 = 46$
Five-number summary: 26, 35, 40, 46, 53

3. Use the five-number summaries to plots the parallel boxplots. Use a suitable scale that will cover the full range of values for both data sets.



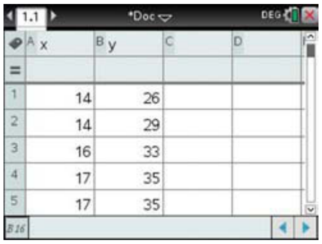
b. Compare and contrast the data sets, looking at where the key points of each data set lie. Comment on any noticeable differences in the centre and spread of the scores, as well as the shape of the distributions.

b. On the whole, Breed X is considerably lighter than Breed Y, with only a small overlap in the data sets. Breed X has a smaller interquartile range than Breed Y, although both spreads are balanced with no noticeable skew.

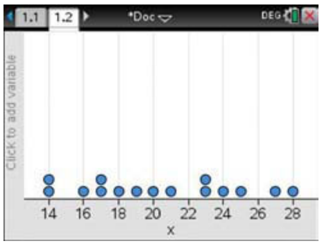
TI | THINK

1. In a Lists & Spreadsheet page, label the first column as *x* and the second column as *y*. Enter the values for breed X in the first column and the values for breed Y in the second column.

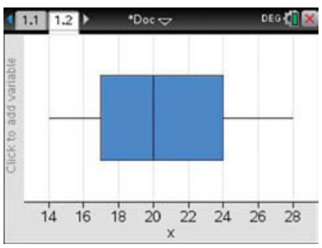
WRITE



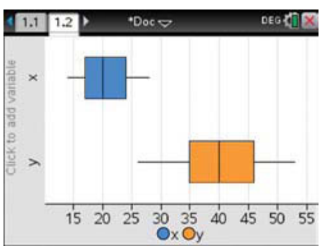
2. In a Data & Statistics page, click on the label of the horizontal axis and select *x*.



3. Press MENU then select:
   1: Plot Type
   2: Box Plot



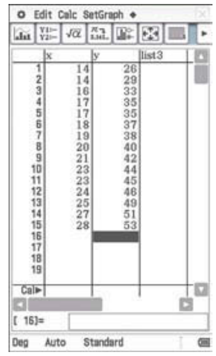
4. Press MENU then select:
   2: Plot Properties
   5: Add X Variable
   then select *y*.



CASIO | THINK

1. On a Statistics screen, relabel list1 as *x* and list2 as *y*. Enter the values for breed X in the first column and the values for breed Y in the second column.

WRITE

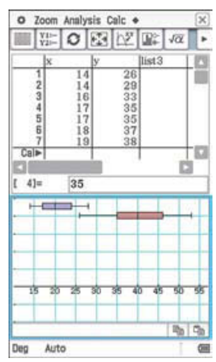


2. Select:
   - SetGraph
   - Setting …

   Select tab 1 then complete the fields as:
   Draw: On
   Type: MedBox
   XList: main\x
   Freq: 1
   Tick the Show Outliers box.

3. Select tab 2 then complete the fields as:
   Draw: On
   Type: MedBox
   XList: main\y
   Freq: 1
   Tick the Show Outliers box then select Set.

4. Click the y icon.



on Resources

- **Interactivity:** Boxplots (int-6245)
- **Interactivity:** Back-to-back stem plots (int-6252)
- **Interactivity:** Parallel boxplots (int-6248)

study on

Units 1 & 2 > AOS 6 > Topic 1 > Concepts 7, 8 & 9

Boxplots Concept summary and practice questions
Outliers and fences Concept summary and practice questions
Comparisons between two data sets Concept summary and practice questions

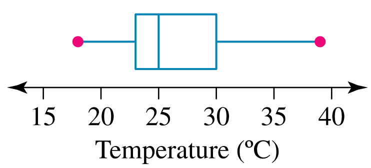
Exercise 13.6 Comparing numerical distributions

1. **WE16** Draw a boxplot for the data contained in the following stem plot, which shows the number of sandwiches sold by a café per day over a 21-day period.

Key: 2|9 = 29 sandwiches

Stem	Leaf
2	9
3	1 3 6 8 9
4	2 4 5 5 6 7 7 8
5	0 0 3 5 8
6	1 2

2. The boxplot shows the temperatures in Melbourne over a 23-day period.



a. What is the median temperature?
b. What is the range of the temperatures?
c. What is the interquartile range of temperatures?

3. The following stem plot shows the ages of 25 people when they had their first child.

Key: 1*|7 = 17 years old

Stem	Leaf
1*	7 8 8
2	0 2 3 3 4
2*	5 6 6 7 8 9
3	0 0 1 2 2 4
3*	6 8 9
4	1 3

a. Prepare a five-number summary of the data.
b. Draw a boxplot of the data.
c. Comment on the distribution of the data.

4. **MC** The five-number summary for a data set is 45, 56, 70, 83, 92. Which of the following statements is definitely not true?
A. There are no outliers in the data set.
B. Half of the scores are between 56 and 70.
C. The range is 47.
D. The value of the lower fence is 15.5.
E. The data has no noticeable skew.

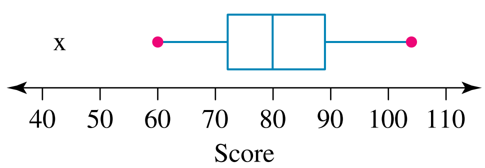
5. **WE17** The following stem plot represents the time taken (in minutes) for 25 students to finish a logic problem.

Key: 4|4 = 4.4 minutes

Stem	Leaf
4	4
5	
6	2 6 9
7	0 4 7 7 8
8	0 3 3 5 6 8 9
9	1 2 4 6 7
10	2 4 4
11	5
12	

a. Calculate the values of the lower and upper fences.
b. Identify any outliers in the data set.
c. Draw a boxplot to represent the data.

6. The boxplot represents the scores made by an Australian football team over a season.



a. What was the highest amount of points the team scored in the season?
b. What was the lowest amount of points the team scored in the season?
c. What was the range of points scored?
d. What was the interquartile range of points scored?

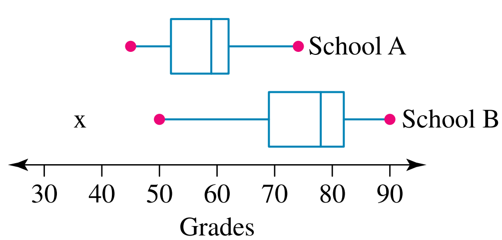
7. **WE18** The following back-to-back stem plot shows the amount of sales (in \$000s) for two different high street stores.

Key: 3|4 = \$3400

Store 1 Leaf	Stem	Store 2 Leaf
	3	4 7 9
7 4	4	2 4 6 8
6 2 1	5	1 2 5 5 9
8 8 5 5	6	3 5 7
5 3 2	7	
6 1	8	
0	9	

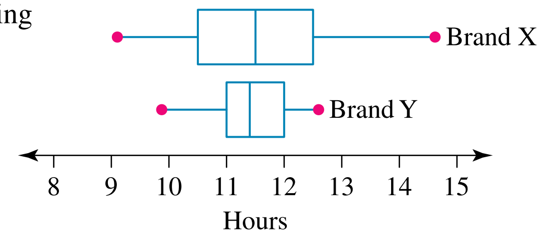
a. Draw a parallel boxplot of the two sets of data.
b. Compare and contrast the two sets of data.

8. The parallel boxplot shows the difference in grades (out of 100) between students at two schools.



a. Which school had the highest overall grade?
b. Which school had the lowest overall grade?
c. Calculate the difference between the interquartile ranges of the grades of the two schools.

9. The parallel boxplot shows the performance of two leading brands of battery in a test of longevity.



a. Which brand had the better median performance?
b. Which brand gave the most consistent performance?
c. Which brand had the worst performing battery?
d. Which brand had the best performing battery?

10. The prices of main meals at two restaurants which appear in the Good Food Guide are shown in the following back-to-back stem plot.

Key: 1|8 = $18

Restaurant A Leaf	Stem	Restaurant B Leaf
	1	8 9
9 9 8 5 5 4	2	2 5 5 7
8 6 5 5 2	3	0 0 2 5 5 8
2 0 0	4	0 3 6
	5	
9	6	

a. Identify any outliers in either set of data.
b. Prepare five-number summaries for the price of the meals at each restaurant.
c. Draw a parallel boxplot to compare the two data sets.
d. Compare and contrast the cost of the main meals at each restaurant.

11. The following table displays the number of votes that two political parties received in 15 different constituencies in the local elections.

Party A	425	630	813	370	515	662	838	769
Party B	632	924	514	335	748	290	801	956

Party A	541	484	745	833	497	746	651
Party B	677	255	430	789	545	971	318

a. Prepare five-number summaries for both parties' votes.
b. Display the data sets on a parallel boxplot.
c. Comment on the distributions of both data sets.

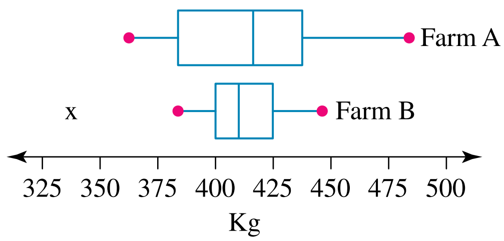
12. The following back-to-back stem plot shows the share prices (in $) of two companies from 18 random months out of a 10-year period.

Key: 1*|7 = $17

Company A Leaf	Stem	Company B Leaf
4 2	1	
9 7 5	1*	7 9
4 4 1	2	0 3 4
9 8 6 6	2*	7 8 8
3 3 2 0	3	1 2 4
8 6	3*	6 9
	4	0 1 2
	4*	5 6

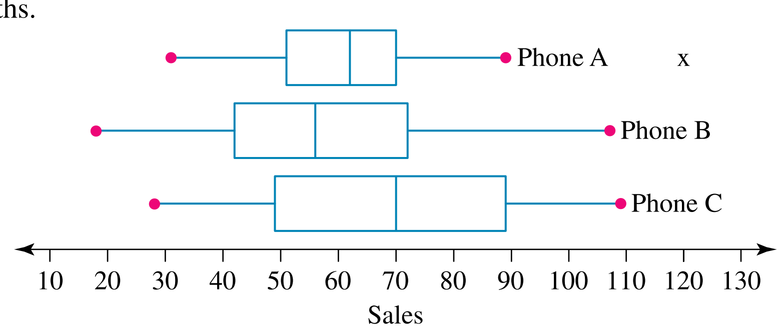
a. Display the data in two frequency tables with intervals of 5.
b. Display the data on a parallel boxplot.
c. Comment on the distributions of both data sets.

13. The following parallel boxplot details the amount of strawberries harvested in kg at two different farms for the month of March over a 15-year period.



Decide whether the following statements are true or false.
a. Farm A produced a larger harvest of strawberries in March than Farm B more often than not.
b. The strawberry harvest at Farm B in March is much more reliable than the strawberry harvest at farm A.
c. Farm A had the highest producing month for strawberries on record.
d. Farm A had the lowest producing month for strawberries on record.

14. The following parallel boxplot shows the weekly sales figures of three different mobile phones across a period of six months.



a. Which phone had the highest weekly sales overall?
b. Which phone had the most consistent sales?
c. Which phone had the largest range in sales?
d. Which phone had the largest interquartile range in sales?
e. Which phone had the highest median sales figure?

15. Determine whether the following statements are true or false.
a. You can always determine the median from a boxplot.
b. A stem plot contains every piece of data from a data set.
c. Boxplots show the complete distribution of scores within a data set.

16. The five-number summaries for the amount collected by three different charities in collection tins over a series of weeks are as follows.
Charity 1: 225, 310, 394, 465, 580
Charity 2: 168, 259, 420, 493, 667
Charity 3: 262, 312, 349, 388, 445
a. Draw a parallel boxplot to compare the collections for the three charities.
b. Compare and contrast the amount collected by the three charities.

17. The following data sets show the daily sales figures for three new drinks across a 21-day period.
Drink 1: 35, 51, 47, 56, 53, 64, 44, 39, 50, 47, 62, 66, 58, 41, 39, 55, 52, 59, 47, 42, 60
Drink 2: 48, 53, 66, 51, 37, 44, 70, 59, 41, 68, 73, 62, 56, 40, 65, 77, 74, 63, 54, 49, 61
Drink 3: 57, 49, 51, 49, 52, 60, 46, 48, 53, 56, 52, 49, 47, 54, 61, 50, 33, 48, 54, 57, 50
   a. Prepare a five-number summary for each drink, excluding any outliers.
   b. Plot a parallel boxplot to compare the sales of the three drinks.
   c. Compare and contrast the sales of the three drinks.

18. The following back-to-back stem plot displays the rental price (in $) of one-bedroom apartments in two different suburbs.

Key: 25|0 = $250

Suburb A Leaf	Stem	Suburb B Leaf
	25	0
	26	5 9
5 5	27	0 0 5
9 9 5 0	28	5 9 9
5 5 0	29	0
5 5 0 0 0 0	30	0 0 0
5 5 0 0	31	0 5 5
	32	9 9
	33	
	34	0 0
0	35	

   a. Prepare a five-number summary for each suburb, excluding any outliers.
   b. Plot a parallel boxplot to compare the data sets.
   c. Compare and contrast the rental price in the two suburbs.
   d. The rental prices in a third suburb, Suburb C, were also analysed, with the data having a five-number summary of 280, 310, 325, 340, 375. Add the third data set to your parallel boxplot.
   e. Compare the rent in the third suburb with the other two suburbs.

13.7 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Multiple choice

1. **MC** The interquartile range of the data distribution shown in the stem plot is:

Key: 2 | 6 = 26

Stem	Leaf
0	2
1	1 5
2	6 6 7 8
3	8 8 9
4	3 4
5	2

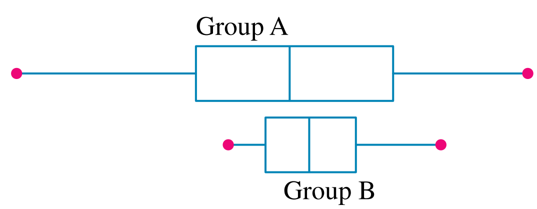
A. 41 **B.** 50 **C.** 28 **D.** 20.5 **E.** 26

2. **MC** The mean of the data distribution shown in the table is:

Interval	Frequency (f)
0–<15	5
15–<30	7
30–<45	6
45–<60	2

A. 22.4 **B.** 26.25 **C.** 24.35 **D.** 25.65 **E.** 27.45

3. **MC** For the following parallel boxplots, which statement is correct?



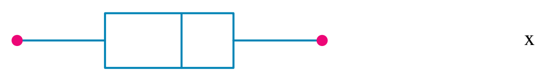
A. Group A has a smaller IQR than Group B.
B. Group B has a greater range than Group A.
C. Group A has a higher median than Group B.
D. Group A has a smaller highest value than Group B.
E. 25% of Group B is greater than Group A’s Q_1 and less than its median.

4. **MC** Data gathered on the number of home runs in a baseball season would be classified as:

A. discrete **B.** nominal
C. continuous **D.** ordinal
E. categorical

5. **MC** For the data set 789, 211, 167, 321, 432, 222, 234, 456, 456, 234, the five-figure summary in order from the smallest value to the largest is:
A. 167, 222, 321, 456, 789
B. 167, 222, 277.5, 456, 789
C. 167, 234, 432, 456, 789
D. 167, 234, 432, 456, 789
E. 167, 222, 287.5, 444, 789

6. **MC** The following boxplot would best be described as:



A. positively skewed
B. symmetrical with an outlier
C. positively skewed with an outlier
D. negatively skewed with an outlier
E. negatively skewed

7. **MC** For the sample data set 2, 3, 5, 2, 3, 6, 3, 8, 9, 2, 8, 9, 2, 6, 7, the mean and standard deviation respectively would be closest to:
A. 5 and 6 **B.** 5 and 2.6 **C.** 2.6 and 5 **D.** 5 and 2.7 **E.** 2.7 and 5

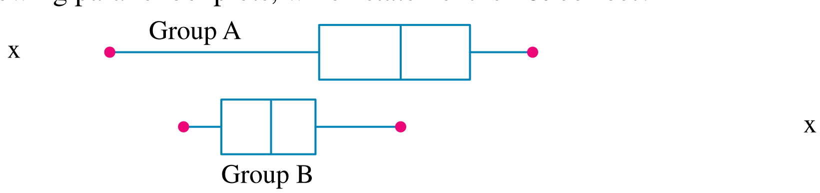
8. **MC** For the following stem plot, the median and range respectively are:

Key: 5 | 1 = 51

Stem	Leaf
5	1 2
6	2 3 4
7	3 4 4 5
8	6 6
9	2

A. 73 and 41 **B.** 73.5 and 41 **C.** 71 and 39 **D.** 71 and 41 **E.** 73 and 39

9. **MC** For the following parallel boxplots, which statement is **not** correct?



A. 75% of Group A is larger than 75% of group B.
B. 25% of Group A has a larger spread than 75% of group B.
C. The median of Group A is equal to the highest value of Group B
D. Group A is negatively skewed and Group B is positively skewed.
E. The middle 50% of both groups are approximately symmetrical.

10. **MC** For the data set 21, 56, 110, 15, 111, 45, 250, 124, 78, 24, the number of outliers and the value of $1.5 \times$ IQR will respectively be:
A. 0 and 87 **B.** 1 and 87 **C.** 1 and 111 **D.** 1 and 130.5 **E.** 0 and 130.5

Short answer

1. Use the data on the incidence of communicable diseases in Australia to answer the following questions.

Incidence of communicable diseases in Australia over two consecutive years

Disease	Year 1	Year 2
Hepatitis C	11 089	7286
Typhoid fever	116	96
Legionellosis	302	298
Meningococcal disease	259	230
Tuberculosis	1324	1327
Influenza (laboratory confirmed)	59 090	13 419
Measles	104	70
Mumps	165	95
Chickenpox	1753	1743
Shingles	2716	2978
Dengue virus infection	1406	1201
Malaria	508	399
Ross River virus infection	4796	5147

a. Calculate the mean (correct to 1 decimal place) and median number of cases of communicable diseases of the sample for each year.

b. Comment on the differences between the mean and median values calculated in part **a**.

2. The price of a barrel of oil in US dollars over a particular 18-month time period is shown in the following table.

Month	Price (US$)
Jan	102.96
Feb	97.63
Mar	108.76
Apr	105.25
May	106.17
Jun	83.17
Jul	83.72
Aug	88.99
Sep	95.34

Month	Price (US$)
Oct	92.44
Nov	87.05
Dec	88.69
Jan	93.14
Feb	97.46
Mar	90.71
Apr	97.1
May	90.74
Jun	93.41

a. Calculate the mean and median for this data set. Give your answers correct to 1 decimal place.
b. Calculate the interquartile range and the standard deviation for this data set. Give your answers correct to 2 decimal places.
c. Which would be the best measures of centre and spread for this data set?

3. The number of passengers arriving from overseas during a particular time period at various airports in Australia is shown in the following table.

Airport	Number of passengers
Adelaide	5743
Brisbane	480 625
Cairns	5110
Coolangatta	7 655
Darwin	5318
Melbourne	594 286
Perth	318 493

a. Calculate the mean and standard deviation for the sample. Give your answers correct to 1 decimal place.
b. Compare the mean and standard deviation to the median and interquartile range for the sample.

4. The following table shows data on the Top 10 tourist destinations in Europe in 2009.

Country	Nights in country (× 1000)
Spain	200 552
Italy	158 527
France	98 700
United Kingdom	80 454
Austria	72 225
Germany	54 097
Greece	46 677
Portugal	25 025
Netherlands	25 014
Czech Republic	17 747

a. Display the data as a boxplot.
b. Describe the distribution of the data using the five-figure summary and identify any outliers.

5. The biggest winning margins in AFL Grand Finals up to the year 2013 are shown in the following table.

Biggest winning margins

Winning margin	Year	Winning team	Winning score	Losing team	Losing score
119	2007	Geelong	163	Port Adelaide	44
96	1988	Hawthorn	152	Melbourne	56
83	1983	Hawthorn	140	Essendon	57
81	1980	Richmond	159	Collingwood	78
80	1994	West Coast	143	Geelong	63
78	1985	Essendon	170	Hawthorn	92
73	1949	Essendon	125	Carlton	52
73	1956	Melbourne	121	Collingwood	48
63	1946	Essendon	150	Melbourne	87
61	1995	Carlton	141	Geelong	80
61	1957	Melbourne	116	Essendon	55
60	2000	Essendon	135	Melbourne	75

a. i. Display the winning margin data as a boxplot.
   ii. Display the winning score data as a boxplot.
   iii. Display the losing score data as a boxplot.

b. Describe each boxplot from part a.

6. The following table shows data on military expenditure by region (US $billion) for the years 1999–2012.

Year	Africa	The Americas	Asia and Oceania	Europe	Middle East
1999	22	441	198	348	69
2000	19	458	202	360	80
2001	20	466	213	362	85
2002	21	515	224	374	80
2003	21	571	234	380	84
2004	23	620	247	383	89
2005	24	652	260	387	98
2006	25	665	275	397	105
2007	26	685	296	408	110
2008	30	737	312	419	106
2009	32	793	348	428	109
2010	34	817	355	419	115
2011	38	808	369	411	117
2012	38	768	382	419	128

a. Display the data for the Americas, Asia and Oceania, and Europe as parallel boxplots.

b. Use the boxplots to compare military spending between the three regions.

Extended response

1. Use the data shown to answer the following questions.
Women who gave birth and Indigenous status by states and territories, 2009

Status	NSW	Vic	Qld	WA	SA	Tas	ACT	NT	Aust
Indigenous	2904	838	3332	1738	607	284	107	1474	11 284
Non-Indigenous	91 958	70 328	57 665	29 022	18 994	5996	5601	2369	281 933

a. Display the data in an appropriate display.
b. Calculate the mean births per state/territory of Australia in 2009 for both Indigenous and Non-Indigenous groups. Give your answers correct to 1 decimal place.
c. Calculate the median births per state/territory of Australia in 2009 for both Indigenous and Non-Indigenous groups.
d. Calculate the standard deviation (correct to 1 decimal place) and IQR for the data on births per state/territory of Australia in 2009 for both Indigenous and Non-Indigenous groups.
e. Comment on the measures of centre and spread you have calculated for this data.

2. Use the data on Tokyo's average maximum temperatures to answer the questions.
Tokyo average maximum temperature, 1980–2007

Year	Temp. (°C)	Year	Temp. (°C)	Year	Temp. (°C)	Year	Temp. (°C)
1980	19.3	1985	19.4	2003	19.6	2008	20.1
1981	19.0	1986	18.8	2004	21.2	2009	20.3
1982	19.6	1987	20.0	2005	20.4	2010	20.6
1983	19.6	1988	19.0	2006	19.9	2011	20.2
1984	18.8	1989	19.9	2007	20.6	2012	20.0

a. Calculate the mean and standard deviation of the temperature data for the two 10-year periods of 1980–89 and 2003–12. Give your answers correct to 2 decimal places.
b. What do the means and standard deviations calculated indicate about the two 10-year periods?
c. Calculate the mean and standard deviation of the total 20 years of the sample data. Give your answers correct to 2 decimal places.
d. How do these measurements compare to the calculations you made in part a?

3. The following table shows the AFL Grand Final statistics for a sample of players who have kicked a total of 5 or more goals from the clubs Carlton and Collingwood.

Player	Team	Kicks	Marks	Handballs	Disposals	Goals	Behinds
Alex Jesaulenko	Carlton	23	11	9	32	11	0
John Nicholls	Carlton	29	3	1	30	13	1
Wayne Johnston	Carlton	78	19	17	95	5	7
Robert Walls	Carlton	19	9	5	24	11	1

Continued

Player	Team	Kicks	Marks	Handballs	Disposals	Goals	Behinds
Craig Bradley	Carlton	61	11	37	98	6	2
Mark MacLure	Carlton	34	16	14	48	5	4
Stephen Kernahan	Carlton	44	26	8	52	17	5
Ken Sheldon	Carlton	36	5	12	48	5	2
Syd Jackson	Carlton	13	3	1	14	5	1
Rodney Ashman	Carlton	25	4	10	35	5	2
Greg Williams	Carlton	30	6	29	59	6	4
Alan Didak	Collingwood	46	17	24	70	6	2
Peter Moore	Collingwood	42	22	13	55	11	7
Ricky Barham	Collingwood	42	15	16	58	5	5
Travis Cloke	Collingwood	26	16	9	35	5	4
Ross Dunne	Collingwood	17	6	6	23	5	2
Craig Davis	Collingwood	27	8	8	35	6	3

a. Use the data for goals to compare the two clubs using parallel boxplots.
b. Comment on what the parallel boxplots indicate about the data for goals.
c. Compare the data for kicks and handballs using parallel boxplots.
d. Comment on what the parallel boxplots indicate about the data for kicks and handballs.

4. The following table shows some key nutritional information about a sample of fruits and vegetables.

Food	Calcium (mg)	Serve weight (g)	Water (%)	Energy (kcal)	Protein (g)	Carbohydrate (g)
Avocado	19	173	73	305	4.0	12.0
Blackberries	46	144	86	74	1.0	18.4
Broccoli	205	180	90	53	5.3	10
Cantaloupe	29	267	90	94	2.4	22.3
Carrots	19	72	88	31	0.7	7.3
Cauliflower	17	62	92	15	1.2	2.9
Celery	14	40	95	6	0.3	1.4
Corn	2	77	70	83	2.6	19.4

Continued

Food	Calcium (mg)	Serve weight (g)	Water (%)	Energy (kcal)	Protein (g)	Carbohydrate (g)
Cucumber	4	28	96	4	0.2	0.8
Eggplant	10	160	92	45	1.3	10.6
Lettuce	52	163	96	21	2.1	3.8
Mango	21	207	82	135	1.1	35.2
Mushrooms	2	35	92	9	0.7	1.6
Nectarines	6	136	86	67	1.3	16.0
Peaches	4	87	88	37	0.6	9.6
Pears	19	166	84	98	0.7	25.1
Pineapple	11	155	86	76	0.6	19.2
Plums	10	95	84	55	0.5	14.4
Spinach	55	56	92	12	1.6	2.0
Strawberries	28	255	73	245	1.4	66.1

a. Use CAS, a spreadsheet or otherwise to convert the water data into its equivalent weight in grams.
b. Compare the data for serve weight with your data for the weight of the water content using parallel boxplots.
c. Comment on the parallel boxplots from part **b**.
d. Use CAS, a spreadsheet or otherwise to compare the data for protein and carbohydrate using parallel boxplots.
e. Comment on the parallel boxplots from part **d**.

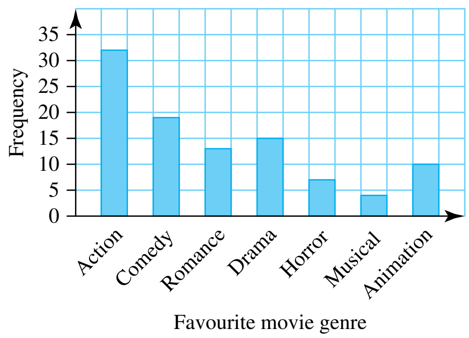
studyon

Units 1 & 2 — **Sit topic test**

Answers

Topic 13 Investigating and comparing data distributions

Exercise 13.2 Data types and displays

1. Nominal
2. Ordinal
3. a. Continuous b. Continuous c. Discrete
4. a. Continuous b. Discrete c. Ordinal
5. 

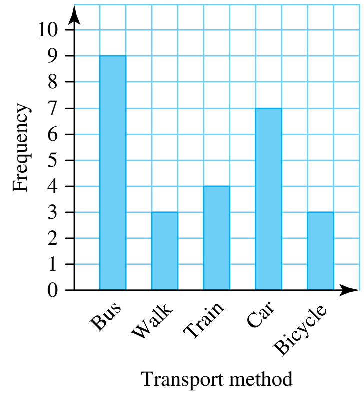
6.

Favourite pizza	Frequency
Margherita	7
Pepperoni	11
Supreme	9
Meat feast	14
Vegetarian	6
Other	13

7. a. Nominal categorical

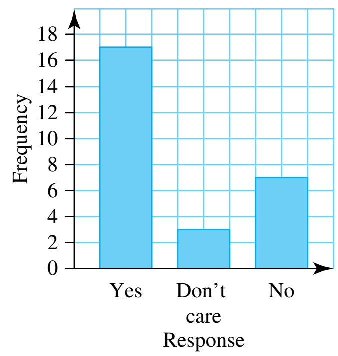
b.

Transport method	Frequency
Bus	9
Walk	3
Train	4
Car	7
Bicycle	3

c. 

8. a.

Response	Frequency
Yes	16
Don't care	3
No	7

b. 

c. Ordinal, as it makes sense to arrange the data in order from 'Yes' to 'No', with 'Don't care' between them.

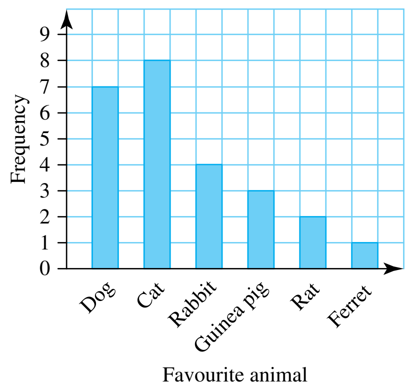
9.

Data	Type	
a Wines rated as high, medium or low quality	Categorical	Ordinal
b The number of downloads from a website	Numerical	Discrete
c The electricity usage over a three-month period	Numerical	Continuous
d A volume of petrol sold by a petrol station per day	Numerical	Continuous

10. a.

Favourite animal	Frequency
Dog	7
Cat	8
Rabbit	4
Guinea pig	3
Rat	2
Ferret	1

b.

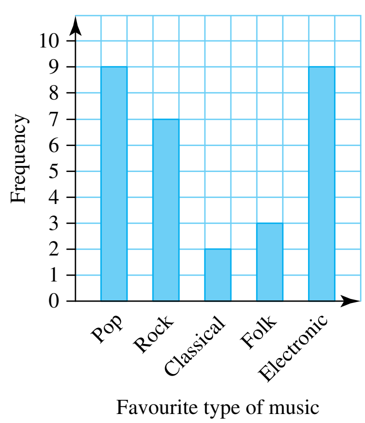


c. Cat

11. a.

Favourite type of music	Frequency
Pop	9
Rock	7
Classical	2
Folk	3
Electronic	9

b.



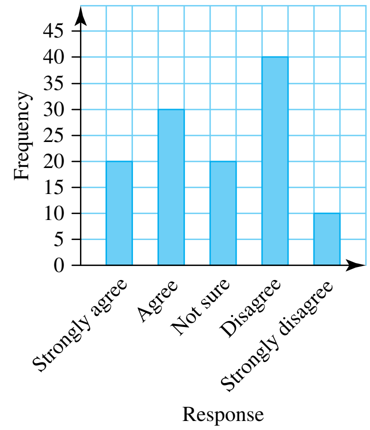
c. Pop and electronic

12. a. Flat white **b.** 70

13. a. Ordinal categorical

b. The data should be in order from 'Strongly agree' through to 'Strongly disagree'.

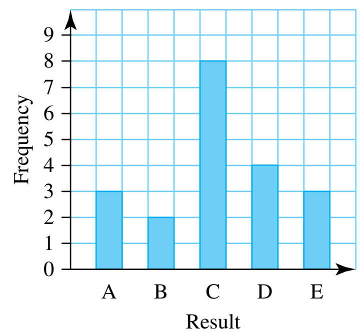
c.



14. a.

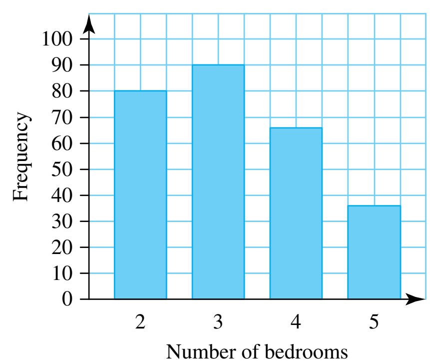
Result	Frequency
A	3
B	2
C	8
D	4
E	3

b.



c. Ordinal categorical

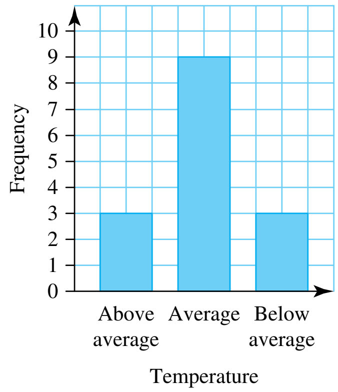
15.



16. a.

Temperature	Frequency
Above average	3
Average	9
Below average	3

b.

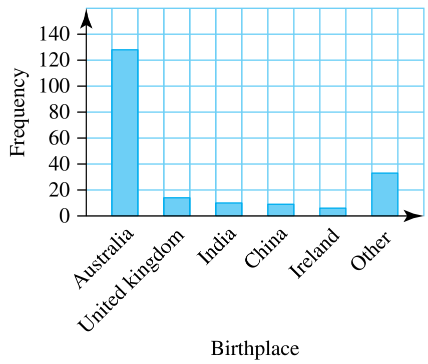


c. Ordinal categorical

17. a. 40 **b.** 15%

18. a. Nominal categorical

b.



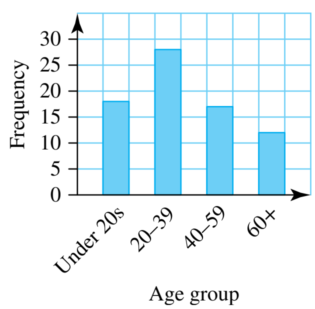
c. 64%

19. a.

Age group	Frequency
Under 20s	18
20–29	15
30–39	13
40–49	10
50–59	7
60+	12

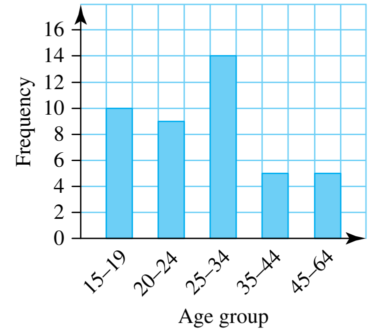
b. Under 20

c.

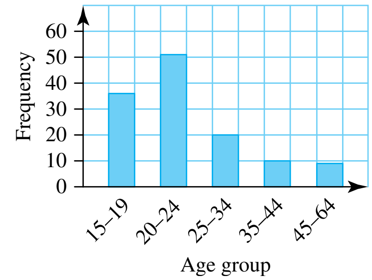


d. Yes, the modal category is now 20–39.

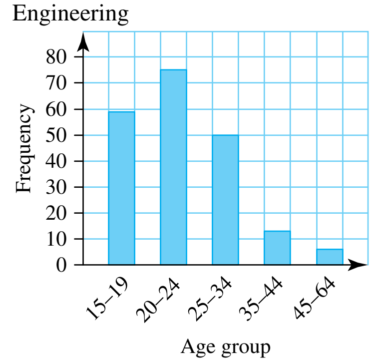
20. a. Agriculture



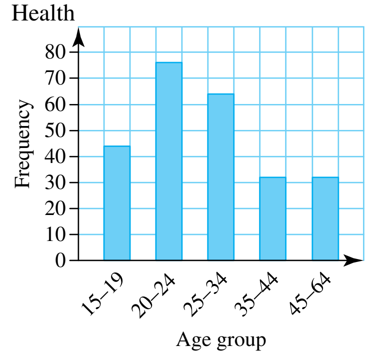
Creative arts



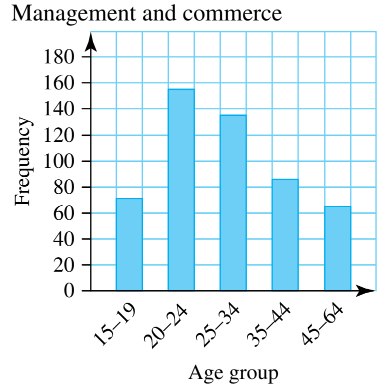
Engineering



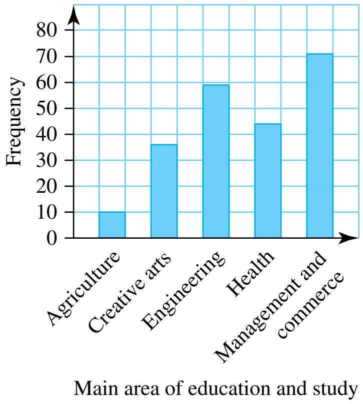
Health



Management and commerce

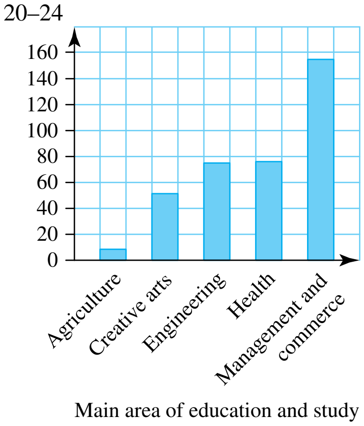


b. 15–19

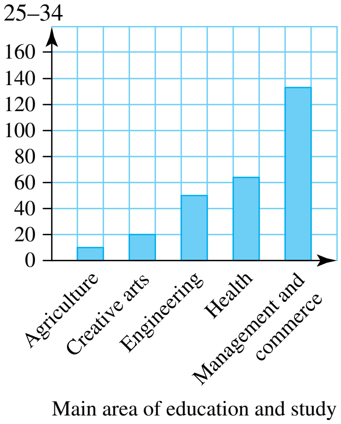


21.

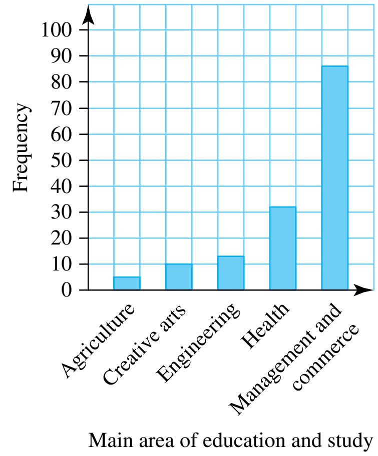
20–24



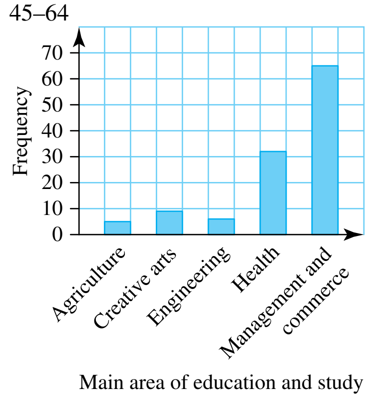
25–34



35–44



45–64



Exercise 13.3 Numerical data distributions

1.

Time (seconds)	Frequency
30–<40	3
40–<50	5
50–<60	7
60–<70	4
70–<80	1

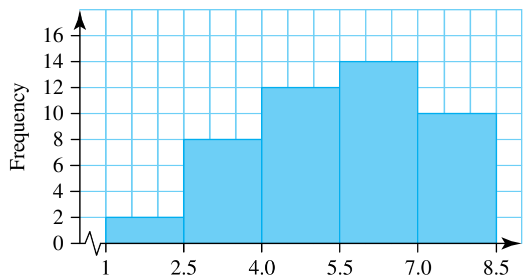
2. a.

Time (seconds)	Frequency
80–<90	2
90–<100	6
100–<110	5
110–<120	5
120–<130	2

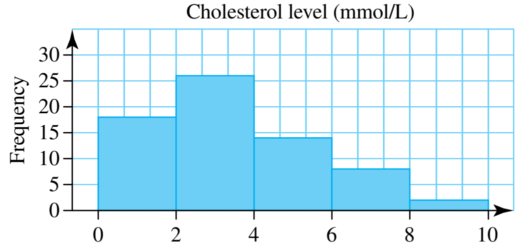
b.

Time (seconds)	Frequency
85–<90	2
90–<95	1
95–<100	5
100–<105	2
105–<110	3
110–<115	3
115–<120	2
120–<125	2

3. a.

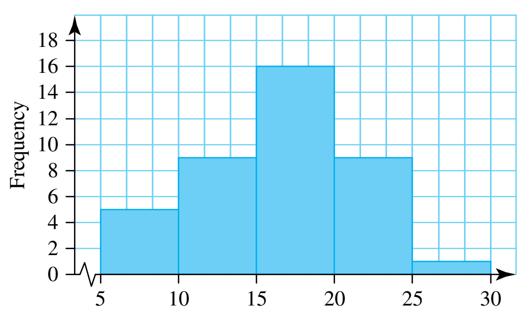


b.



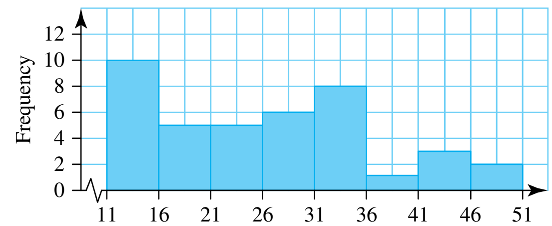
4. a.

Class interval	Frequency
5–<10	5
10–<15	9
15–<20	16
20–<25	9
25–<30	1



b.

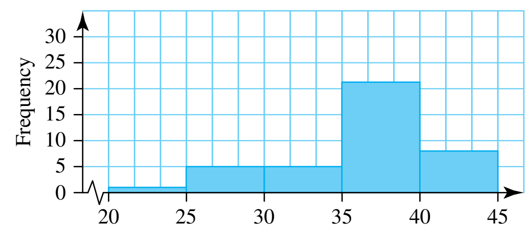
Class interval	Frequency
11–<16	10
16–<21	5
21–<26	5
26–<31	6
31–<36	8
36–<41	1
41–<46	3
46–<51	2



5. a.

Average hours	Frequency
20–<25	1
25–<30	5
30–<35	5
35–<40	21
40–<45	8

b.



c. The distribution has one mode with data values that are most frequent in the 35–<40 interval. There are no obvious outliers, and the distribution has a negative skew.

6. a. Key: 1*|9 = $19

Stem	Leaf
1*	9
2	1 1 2 2 2 2 2 2 3 3 4 4
2*	5 6

b. Key: 0*|6 = 6 hours

Stem	Leaf
0*	6 7 9
1	3 4 4
1*	7 7
2	0 0 1 1 1 3 3 4 4
2*	5 5 6

7. a. Key: 1|2 = 12 passengers

Stem	Leaf
1	2 3 4
1*	5 5 5 5 7 7
2	0 0 2 2 3 3 3 3 4
2*	5 7 7 7 7 7 8 8
3	0 3 4 4
3*	5 5 6 6 6 7
4	2 3 4
4*	7

b. Key: 1|7 = 17 patients

Stem	Leaf
1	7 7
2	1 2 3 3 3 4 4 4 5 5 5 6 6 6 8
3	0 0 1 2 3 4 4 5 7 8 8
4	1 1 3 4 5 5 6
5	1 1 5 6
6	0

8. a. Key: 0*|8 = 8 people

Stem	Leaf
0*	8 8
1	3 4 4
1*	6 6 7 7 9
2	1 1
2*	5

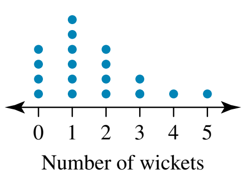
b. The distribution has one mode with data values that are most frequent in the 16−<20 interval. There are no obvious outliers, and the distribution appears to be symmetrical.

9. a. The first stem plot has one mode with data values that are most frequent in the 30−<40 interval. There is a possible outlier at 91, and the distribution appears to be symmetrical.
The second stem plot has 3 modes and two distinct groups of data. There are no obvious outliers, and there is a slight positive skew to the distribution.

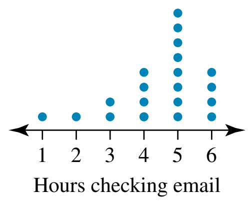
b. Key: 0|1 = 1 game played

Stem	Leaf
0	1
1	4 7
2	4 4 4 8
3	1 2 3 3 5 6 6
4	1 2 3 3 4 5
5	1 1 2
6	5
7	
8	2 5 7
9	1 3

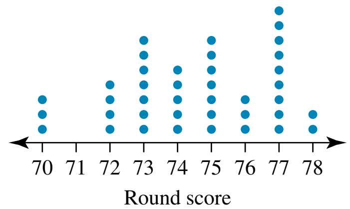
10. a.



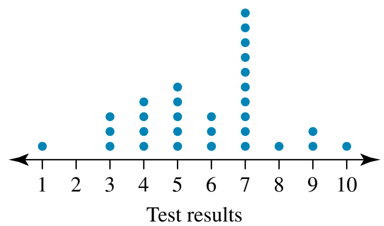
b.



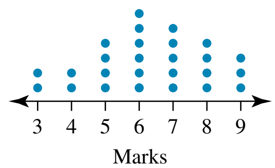
11. a.



b.



12. a.

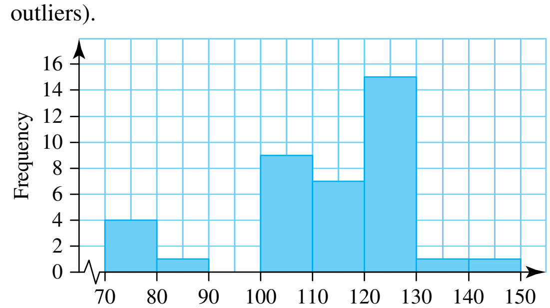


b. The distribution has one mode with a value of 6. There are no obvious outliers and there is a slight negative skew to the distribution.

13. a. The distribution has one mode with data values that are most frequent in the 35−<40 interval. There are no obvious outliers, and there is a negative skew to the distribution.

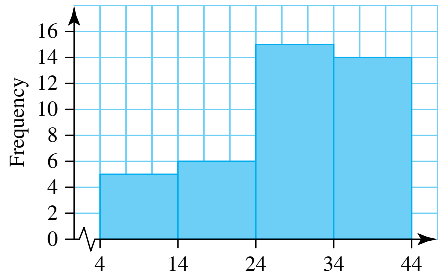
b. The distribution has one mode with data values that are most frequent in the 45−<60 interval. There are two potential outliers in the 120−<135 interval, and the distribution is either symmetrical (excluding the outliers) or has a slight negative skew (including the outliers).

14. a.



The distribution has one mode with data values that are most frequent in the 120−<130 interval. There are potential outliers in the 70−<80 interval, and there is a negative skew to the distribution.

b.



The distribution has one mode with data values that are most frequent in the 24−<34 interval. There are no obvious outliers, and there is a negative skew to the distribution.

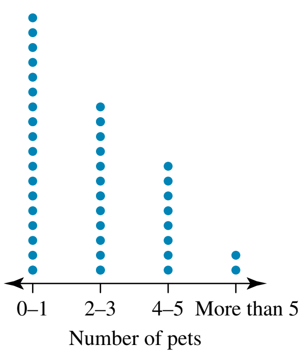
15. a.

Number of pets	Frequency	Percentage
0–1	18	45%
2–3	12	30%
4–5	8	20%
More than 5	2	5%
Total	40	100%

b. Discrete numerical

c. 40

d.



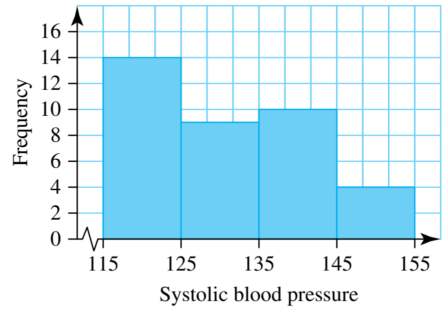
e. The distribution has one mode with data values that are most frequent in the 0–1 interval. There are no obvious outliers, and the distribution has a positive skew.

16. a. 65%

b. 15%

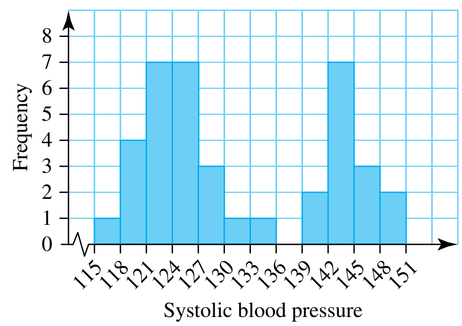
c. The distribution has one mode with data values that are most frequent in the 90−<120 interval. There are no obvious outliers, and the distribution has a negative skew.

17. a.



b. The distribution has one mode with data values that are most frequent in the 115−<125 interval. There are no obvious outliers, and the distribution has a positive skew.

c.



d. The second histogram is split into two distinct groups with three modes. The lower group is symmetrical around the interval 121−<128. The upper group has a slight positive skew.

e. The second histogram might be demonstrating that there are two distinct groups present in the data, for example a younger age group and an older age group.

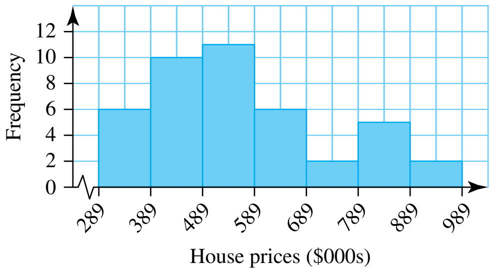
18. a. Key: 0|1 = 1

Stem	Leaf
0	1
0*	
1	1 1 1 4 4
1*	6 6 7 8
2	3 3 4 4
2*	7 7 9

b. Splitting the stem for this data gives a clearer picture of the spread and shape of the distribution of the data set.

19. a.

b.

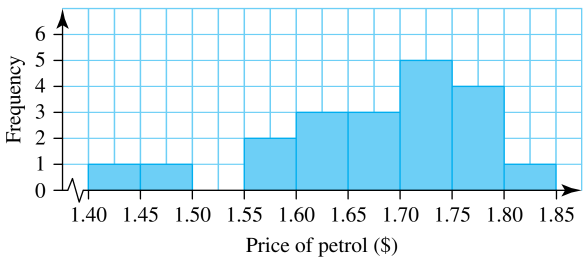


c. The first histogram has the data spread out across the range, making it difficult to see aspects of the distribution as clearly as the second histogram.

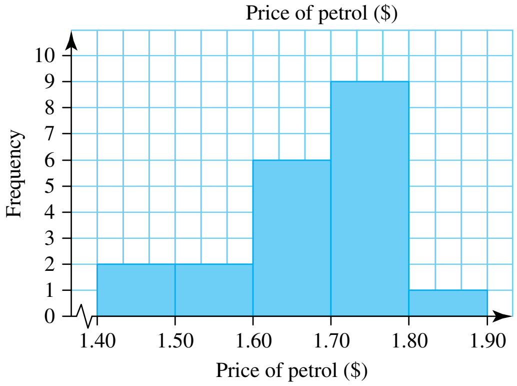
d. See table at the foot of the page*

e. This distribution has one mode, with 1 potential outlier and a positive skew.

20. a.

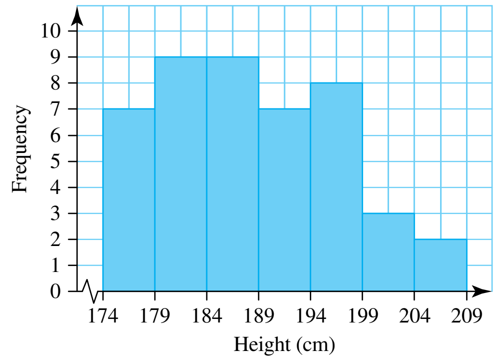


b.

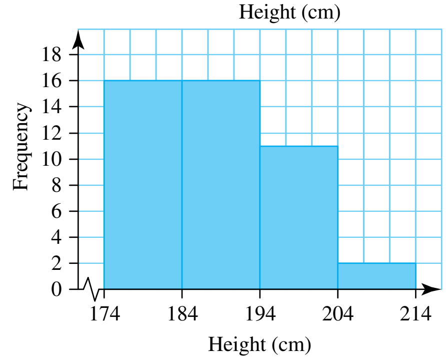


c. Both histograms have one mode with a negative skew to the distribution. The first histogram gives the impression that there may be outliers at the start of the data set, but this is not as evident in the second histogram.

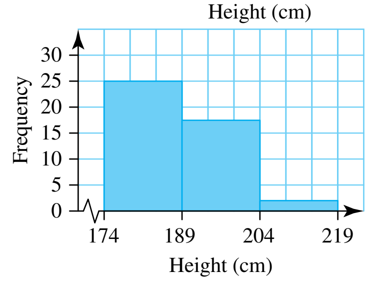
21. a. i.



ii.



iii.



b. The histogram with more intervals allows you to see the shape and details of the distribution better, although the overall shape is fairly similar in all three histograms.

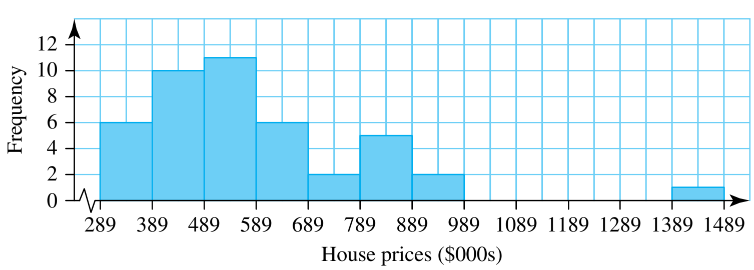
i. This distribution has two modes, with a negatives skew and no obvious outliers.

ii. This distribution also has two modes, with a negatives skew and no obvious outliers.

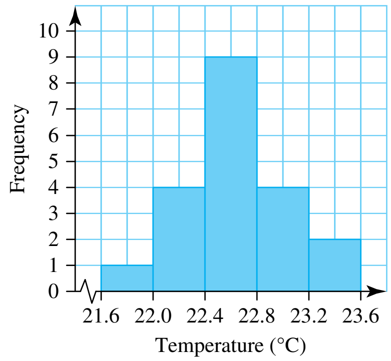
iii. This distribution has one mode, with a negative skew and no obvious outliers.

---

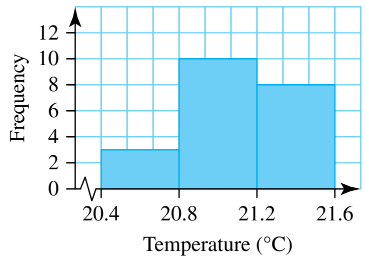
19. d. *



22. a.



b.



c. The distribution for the interval 1993–2012 is approximately symmetrical, with a slight negative skew. The distribution for the interval 1893–1912 is more symmetrical with a smaller overall range. Both distributions have one mode.

Exercise 13.4 Measures of centre

1. 128.4

2. a. 24.76
   b. 27.71
   c. The mean increased significantly.

3. a. 6.94 b. 46.36

4. a. See table at the foot of the page*
   b. See table at the foot of the page*

5. a. 26.18 b. 21.67

6. a. Mean = 867.89, median = 654
   b. The median, as it is not affected by the extreme values present in the data set.

7. a. 51 b. 198

8. a. 24 b. 24 c. The median is unchanged.

9. a. 104 b. 1.805

10. The median, as the data set has two clear outliers

11. a. \$74 231
    b. \$65 000
    c. It would be in the workers' interest to use a higher figure when negotiating salaries, whereas it would be in the management's interest to use a lower figure.

---

4. a. *

Interval	Frequency (f)	Midpoint (x)	xf
203–<210	5	206.5	1032.5
210–<217	3	213.5	640.5
217–<224	3	220.5	661.5
224–<231	3	227.5	682.5
231–<238	2	234.5	469
238–<245	1	241.5	241.5
	$\Sigma f = 17$		$\Sigma fx = 3727.5$

$\bar{x} = 219.26$

b. *

Interval	Frequency (f)	Midpoint (x)	xf
5–<10	4	7.5	30
10–<15	5	12.5	62.5
15–<20	3	17.5	52.5
20–<25	3	22.5	67.5
25–<30	1	27.5	27.5
30–<35	1	32.5	32.5
	$\Sigma f = 17$		$\Sigma fx = 272.5$

$\bar{x} = 16.03$

12. a. Mean = \$1 269 850, median = \$594 500

b. and **c.** See table at the foot of the page*

d. The median, as the mean is affected by a few very high values.

13. a. Mean = 14 : 23

b.

Interval	Frequency
10−<12	3
12−<14	9
14−<16	6
16−<18	4
18−<20	2

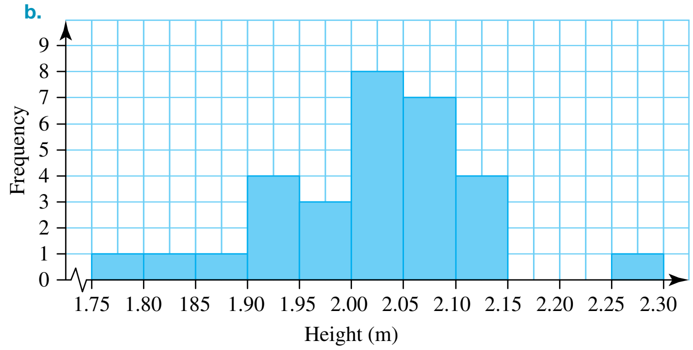
c. Mean = 14 : 25

d. The two means are very similar in value.

14. a.

Interval	Frequency
1.75−<1.80	1
1.80−<1.85	1
1.85−<1.90	1
1.90−<1.95	4
1.95−<2.00	3
2.00−<2.05	8
2.05−<2.10	7
2.10−<2.15	4
2.15−<2.20	0
2.20−<2.25	0
2.25−<2.30	1

b.



The median would be the preferred choice due to the extreme values in the data set.

c. 202 cm

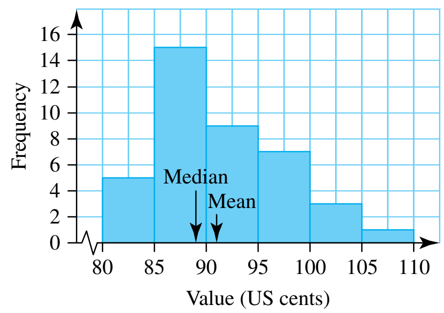
15. a. Mean = 7.55, Median = 6

b. The median would be the preferred choice due to the extreme value of 34.

16. a. Mean = 91.125, Median = 89.5

b. and

c.



d. The mean is higher than the median as it has been more influenced by the values at higher end of the distribution.

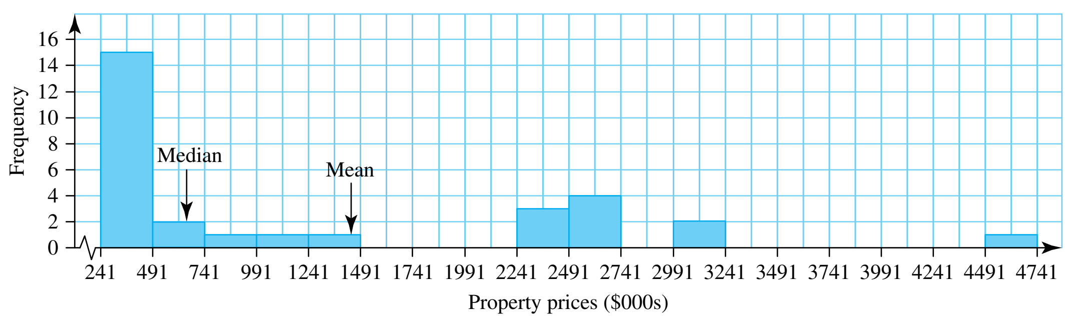
17. a. Mean = \$847 354, median = \$310 000

b. and **c.** See table at the foot of the page*

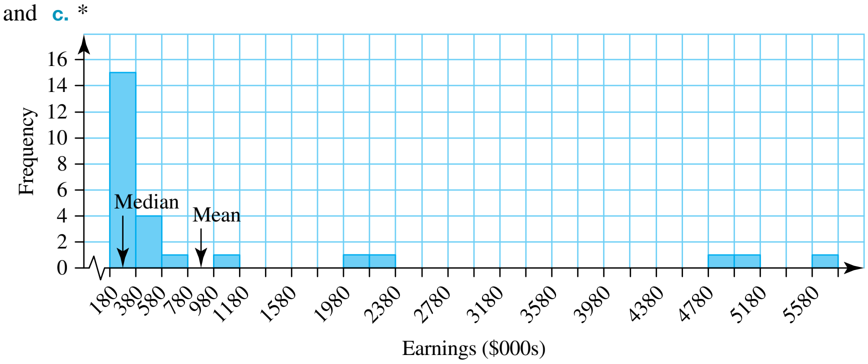
d. The median is the best measure as the mean is affected by the extreme values.

---

12. b. and **c.** *



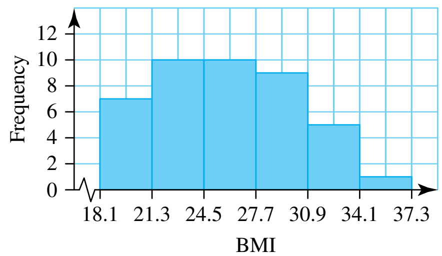
17. b. and **c.** *



18. a. Mean = \$72.22, median = \$68.93
b. 7
c. \$66.19
d. The median is reduced to \$65.50.

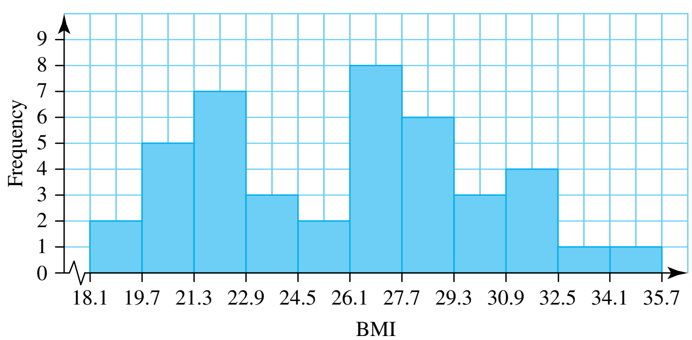
19. a. i.

Interval	Frequency
18.1−<21.3	7
21.3−<24.5	10
24.5−<27.7	10
27.7−<30.9	9
30.9−<34.1	5
34.1−<37.3	1



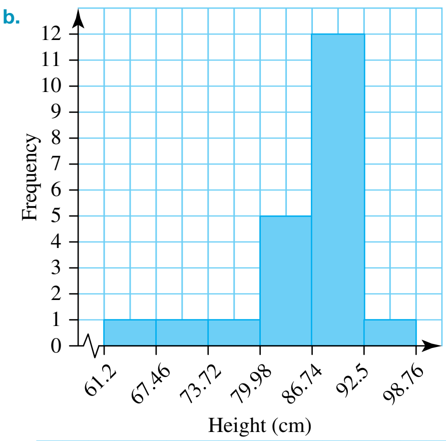
ii.

Interval	Frequency
18.1−<19.7	2
19.7−<21.3	5
21.3−<22.9	7
22.9−<24.5	3
24.5−<26.1	2
26.1−<27.7	8
27.7−<29.3	6
29.3−<30.9	3
30.9−<32.5	4
32.5−<34.1	1
34.1−<35.7	1

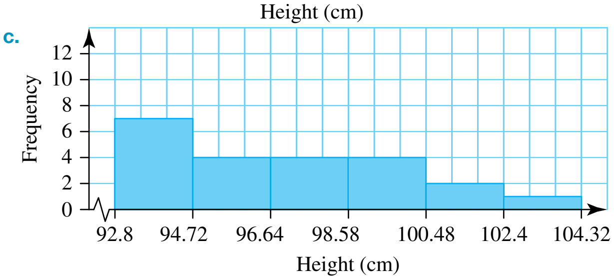


b. The first histogram has two modes and is near symmetrical, with a slight positive skew. The second histogram shows two distinct groups, with a symmetrical lower group and a positively skewed upper group.
c. Table 1: 25.95, table 2: 25.91, raw data: 25.76
Both of the tables give a higher value for the mean than the raw data, although the differences are small.
d. The total data set is generally symmetrical with no obvious outliers, so the mean is the best measure of centre.

20. a. 92.65
b.



c.



d. Group 1: 85.13, Group 2: 97.35
e. The lower half of the data has one mode, with a strong negative skew.
The upper half of the data also has one mode, with a more even spread and a positive skew.

Exercise 13.5 Measures of spread

1. 186
2. 2.555
3. a. Class A = 11, Class B = 10
b. Class A = 2, Class B = 5
4. a. 160
b. 57
c. Goals against: range = 293, interquartile range = 140
The 'Goals against' column is significantly more spread out than the 'Goals for' column.
5. Variance = 23.33, standard deviation = 4.83
6. See table at the foot of the page*

6. *

Intervals	Frequency (f)	Midpoint (x)	xf	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
0−<50	35	25	875	1525.879	53 405.762
50−<100	125	75	9375	119.629	14 953.613
	$\sum f = 160$		$\sum xf = 10\,250$		$\Sigma f(x-\bar{x})^2 = 68\,359.375$

Variance 427 246, standard deviation 20 670

7. See table at the foot of the page*

8. a. 7.37
   b. 7
   c. Standard deviation = 11.49, interquartile range = 7
   d. The standard deviation increased by 4.12, while the interquartile range was unchanged.

9. a. $16 327.50
   b. $46 902
   c. There is a much larger spread in the maximum salaries than the minimum salaries.

10. a. Year 1: standard deviation = 20 382.8, interquartile range = 38 907.5
    Year 2: standard deviation = 19 389.0, interquartile range = 37 110
    b. Year 1: standard deviation = 18 123.5, interquartile range = 31107.5
    Year 2: standard deviation = 17 289.2, interquartile range = 29140
    c. Both values are reduced by a similar amount.

11. a. Year 1: standard deviation = 1 301 033.5, interquartile range = 2 336 546
    Year 2: standard deviation = 1 497 303.5, interquartile range = 2 734 078
    b. Year 1: standard deviation = 1 082 470.9, interquartile range = 2 136 718
    Year 2: standard deviation = 1 228 931.0, interquartile range = 2 415 365
    c. Both values are reduced, but there is a bigger impact on the interquartile range than the standard deviation.

12. a. 2.9
    b. 1.25
    c. The range is less than double the value of the interquartile range. This indicates that the data is quite tightly bunched with no outliers.

13. a. Mean = 41 440.78, standard deviation = 2248.92
    b. Median = 41 333, interquartile range = 3609
    c. 59.38%
    d. 50%
    e. There is a greater percentage of the sample within one standard deviation of the mean than between the first and third quartiles.

14. a. Standard deviation = 0.46, interquartile range = 0.7
    b. See table at the foot of the page*
    c. Perth
    d. Darwin

15. a. Standard deviation = 18.81, interquartile range = 36.21
    b. India: standard deviation = 105.87, interquartile range = 158.59
    China: standard deviation = 1143.53, interquartile range = 1988.7
    United Kingdom: standard deviation = 8.21, interquartile range = 9.48
    USA: standard deviation = 87.34, interquartile range = 145.48
    c. The standard deviation is appropriate, as there appear to be no obvious outliers in the data for any country.

16. a. Interquartile range = 2.78, variance = 25.40
    b. Interquartile range = 2.78, variance = 4.43
    c. The interquartile range has stayed the same value, while the variance has reduced significantly.

---

7. *

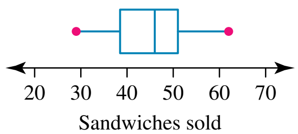
Age group (years)	Frequency (f)	Midpoint (x)	xf
15−<20	14	17.5	245
20−<25	18	22.5	405
25−<30	11	27.5	302.5
30−<35	7	32.5	227.5
35−<40	5	37.5	187.5
	$\sum f = 55$		$\sum xf = 1367.5$

14. b. *

	Sydney	Melbourne	Brisbane	Adelaide	Perth	Hobart	Darwin	Canberra
Std dev.	0.46	0.40	0.51	0.45	0.44	0.38	0.67	0.41
IQR	0.7	0.7	0.9	0.8	0.6	0.6	1.2	0.6

Exercise 13.6 Comparing numerical distributions

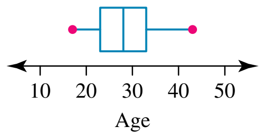
1.



2. **a.** 25 °C **b.** 21 °C **c.** 7 °C

3. **a.** 17, 23, 28, 33, 43

b.



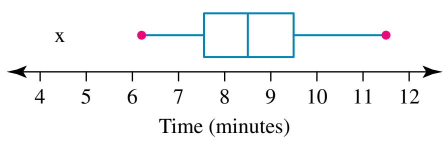
c. The data is fairly symmetrical with no obvious outliers.

4. B

5. **a.** Lower fence = 4.625, higher fence = 12.425

b. 4.4 is an outlier.

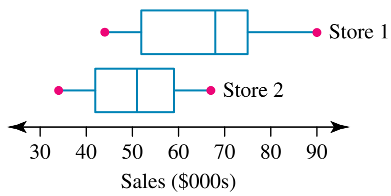
c.



6.

a. 104 **b.** 43 **c.** 61 **d.** 17

7. **a.**



b. On the whole, store 2 has less sales than store 1; however, the sales of store 2 are much more consistent than store 1's sales.
The sales of store 1 have a negative skew, while the sales of store 2 are symmetrical. There are no obvious outliers in either data set.

8. **a.** School B

b. School B

c. 3 (School B has a bigger interquartile range.)

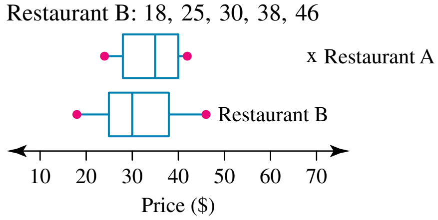
9. **a.** Brand X **b.** Brand Y

c. Brand X **d.** Brand X

10. **a.** $69 in Restaurant A is an outlier

b. Restaurant A: 24, 28, 35, 40, 42
Restaurant B: 18, 25, 30, 38, 46

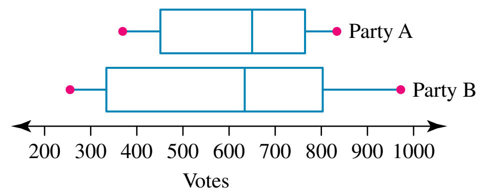
c.



d. The meals in Restaurant A are more consistently priced, but are also in general higher priced. The distribution of prices at Restaurant A has a positive skew, while the distribution of prices at Restaurant B is nearly symmetrical.

11. **a.** Party A: 370, 497, 651, 769, 838
Party B: 255, 335, 632, 801, 971

b.



c. The spread of votes for Party B is far larger than it is for Party A. Party A polled more consistently and had a higher median number of votes. Party A had a nearly symmetrical distribution of votes, while Party B's votes had a slight negative skew.

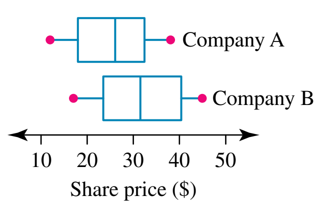
12. **a.** Company A

Share price ($)	Frequency
10–<15	2
15–<20	3
20–<25	3
25–<30	4
30–<35	4
35–<40	2

Company B

Share price ($)	Frequency
15–<20	2
20–<25	3
25–<30	3
30–<35	3
35–<40	2
40–<45	3
45–<50	2

b.



c. On the whole, the share price of Company B is greater than the share price of Company A. However, the share price of Company A is more consistent than the share price of Company B. The share price of Company A has a negative skew, while the share price of Company B has a nearly symmetrical distribution.

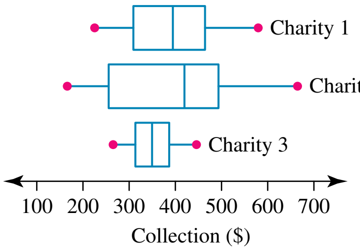
13. **a.** True **b.** True **c.** True **d.** False

14. **a.** Phone A **b.** Phone A **c.** Phone B

d. Phone C **e.** Phone C

15. **a.** True **b.** True **c.** False

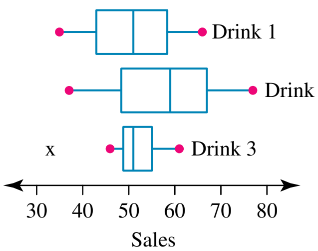
16. a.



b. The collections for Charity 3 were the most consistent of the three charities. Charity 2 collected more money on average than the other charities, but also had the poorest performing week in total. There are no outliers in any of the data sets.

17. a. Drink 1: 35, 43, 51, 58.5, 66
Drink 2: 37, 48.5, 59, 67, 77
Drink 3: 46, 48.5, 51, 55, 61

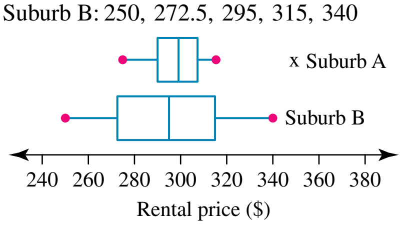
b.



c. The sales of Drink 3 are by far the most consistent, although overall Drink 2 has the highest sales. Drink 2's sales are also the most inconsistent of all the drinks. There is one outlier in the data sets (33 in Drink 3).

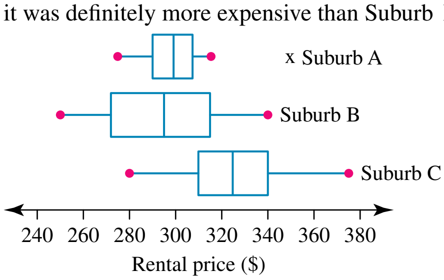
18. a. Suburb A: 275, 289, 300, 307.5, 315
Suburb B: 250, 272.5, 295, 315, 340

b.



c. The rental prices in Suburb A are far more consistent than the rental prices in Suburb B. There is one outlier in the data sets ($350 in Suburb A). Although Suburb A has a higher median rental price, you could not say that it was definitely more expensive than Suburb B.

d.



e. Suburb C has a higher average rental price than either Suburb A or B. The spread of the prices in Suburb C is more similar to those in Suburb B than Suburb A.

13.7 Review: exam practice

Multiple choice

1. D	**2.** B	**3.** E	**4.** A	**5.** B
6. B	**7.** D	**8.** B	**9.** C	**10.** D

Short answer

1. a. Year 1: mean = 6432.9, median = 1324
Year 2: mean = 2637.6, median = 1201

b. The mean values are significantly different but the medians are very similar. This would seem to indicate the presence of extreme values in the data.

2. a. Mean = 94.6, median = 93.3

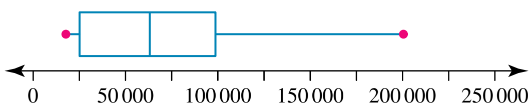
b. IQR = 8.64, standard deviation = 7.49

c. For this sample, the mean and the standard deviation would be preferred as there are no clear outliers and no apparent skew.

3. a. Mean = 202 461.4, standard deviation = 257 819.6

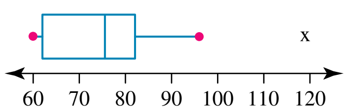
b. Median = 7655, IQR = 475 307
The median and interquartile range are not influenced by the presence of the much smaller values for Darwin and Adelaide.

4. a.

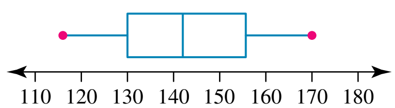


b. Five-figure summary:
17 747, 25 025, 63 161, 98 700, 200 552
The data shows a positive skew with the upper 25% having a much greater spread than the lower 25%. The middle 50% is approximately symmetrical. There are no outliers.

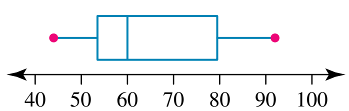
5. a. i. $w = kh$



ii.



iii.



b. i. Negative skew with an upper outlier
ii. Symmetrical with no outliers
iii. Positive skew with no outliers

6. a. See table at the foot of the page*

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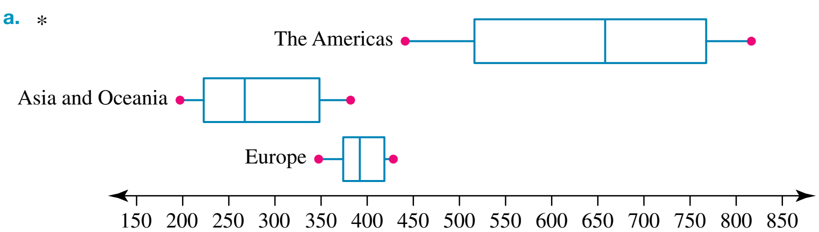
6. a. *



b. Military spending is greater but more variable in the Americas as indicated by the larger range and IQR. Europe has the least variable spending. Asia and Oceania spend the least but are more variable than Europe.

Extended response

1. a. See table at the foot of the page*

b. Indigenous mean = 1410.5, Non-Indigenous mean = 35 241.6

c. Indigenous median = 1156, Non-Indigenous median = 24 008

d. Indigenous: standard deviation = 1193.8, IQR = 1875.5
Non-Indigenous: standard deviation = 33 949.03, IQR = 58 198

e. The median and IQR are probably more appropriate due to the presence of potential extreme values in the data

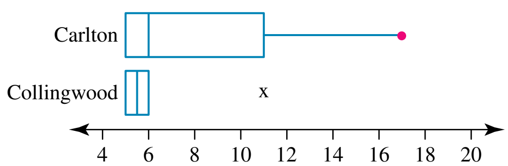
2. a. 1980–89: mean = 19.34, standard deviation = 0.44
2003–12: mean = 20.28, standard deviation = 0.42

b. The mean temperature is about one degree higher in the period 2003–12, but the standard deviations indicate that the data have similar spreads.

c. Total data: mean = 19.82, standard deviation = 0.65

d. The mean of the total data is halfway between the two separate time periods. The standard deviation indicates a much greater variation from the mean for the total data.

3. a.



b. Goals scored in Grand Finals for this sample of players is greater but more variable among the Carlton players, as indicated by the larger range and IQR. Collingwood players are concentrated at 5 or 6 with the exception of the one upper outlier of 11.

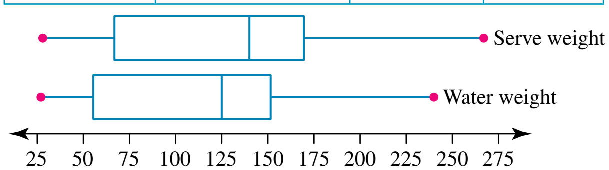
c. See table at the foot of the page*

d. Kicks for this sample of players are greater but more variable than the handballs, as indicated by the larger range and IQR. Both are positively skewed with one upper outlier.

4. a.

Food	Serve weight (g)	Water (%)	Water weight (g)
Avocado	173	73	126
Blackberries	144	86	124
Broccoli	180	90	162
Cantaloupe	267	90	240
Carrots	72	88	63
Cauliflower	62	92	57
Celery	40	95	38
Corn	77	70	54
Cucumber	28	96	27
Eggplant	160	92	147
Lettuce	163	96	156
Mango	207	82	170
Mushrooms	35	92	32
Nectarines	136	86	117
Peaches	87	88	77
Pears	166	84	139
Pineapple	155	86	133
Plums	95	84	80
Spinach	56	92	52
Strawberries	255	73	186

b.



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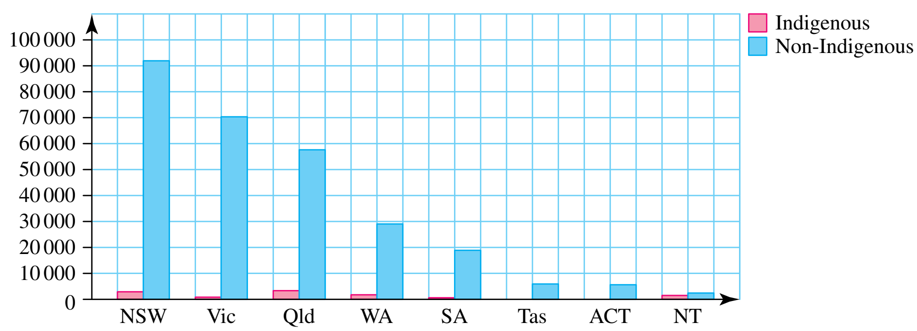
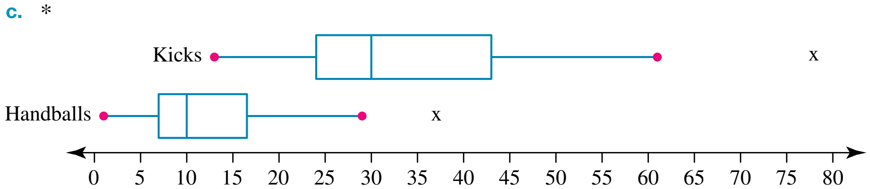
1. a. *

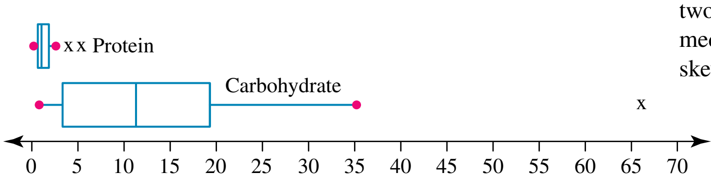


3. c. *



c. The boxplots appear to indicate that there are only slight differences between the serve weights and water weights of the samples. The distributions are very similar in shape, with the water weights being slightly less overall.

d.

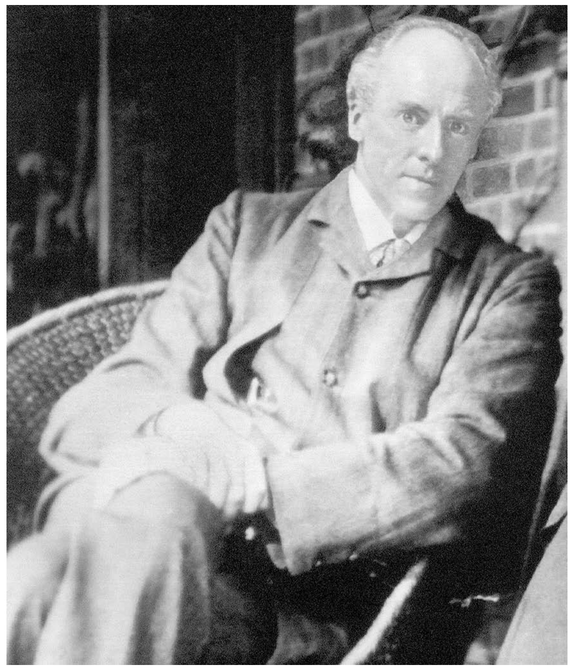


e. Carbohydrate for this sample of foods is much greater but more variable than protein, as indicated by the larger range and IQR. The protein amounts are all less than the Q_1 for the carbohydrate amounts, with the exception of two upper outliers for protein that lie between the Q_1 and median for carbohydrate. Carbohydrate is positively skewed with one upper outlier.

TOPIC 14 Relationships between two numerical variables

14.1 Overview

14.1.1 Introduction



Karl Pearson, born in London in 1857, was an influential mathematician and biostatistician. He was educated privately at University College School in London before moving to King's College, Cambridge, in 1876 to study mathematics. Pearson then travelled to Germany to study physics at the University of Heidelberg under GH Quincke and Kuno Fischer. In 1884 Pearson was appointed professor of applied mathematics and mechanics at University College, London. His work there sparked his interest in statistics. In 1892 he published *The Grammar of Science*, in which he argued that scientific method is essentially descriptive rather than explanatory. He then made similar arguments about statistics, highlighting the importance of quantification for biology, medicine and social sciences.

Through hishis mathematical work and institution building, Pearson played a leading role in the creation of modern statistics. The basis for hishis statistical mathematics came from the long tradition of work on the method of least square approximation in order to estimate quantities from repeated astronomical and geodetic measures using probability theory. As a statistician, Pearson emphasised measuring correlations and fitting curves to data (Pearson's correlation coefficient), and later went on to develop the chi-squared distribution (test of goodness of fit of observed data).

A 23-year-old Albert Einstein started the Olympia Academy study group in 1902 with his friends Maurice Solovine and Conrad HabichtHabicht. The first reading suggestion made was Pearson's *The Grammar of Science*. This book covered a number of themes that were later to become part of Einstein's and other scientists' theories.

LEARNING SEQUENCE

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

14.1.2 Kick off with CAS

What information can be obtained from a scatterplot?

1. Using CAS, plot each of the following four scatterplots on a different set of axes. Remember to label the axes with the appropriate variable.

a.

Week number	1	2	3	4	6	8	10	13	14	17	20
Length (cm)	20	21	23	24	25	30	32	35	36	37	39

b.

Price ($)	14	18	20	21	24	25	28	30	32	35
Number sold	21	22	18	19	17	17	15	16	14	11

c.

Number of guests	30	40	50	60	70	80	90	100	110	120
Total cost (×$1000)	10	12	8	5.5	8	6	7	10	13	14

d.

Number	2	4	7	10	5	2	6	3	9	4	8	3
Score	22	39	69	100	56	18	60	36	87	45	84	32

For each of the scatterplots, answer the following questions.

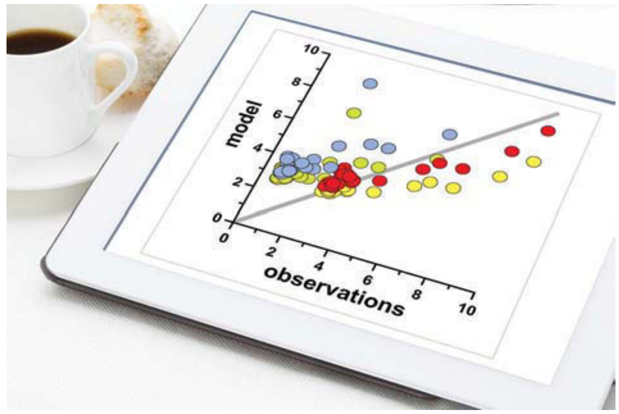
2. As the variable on the horizontal axis increases, what happens to the variable on the vertical axis?
3. Can you identify or describe a pattern for the scatterplot?
4. Would it be appropriate to join the points to make a curve or a line?
5. Does the scatterplot provide any other information about the data?

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

14.2 Scatterplots and basic correlation

14.2.1 Bivariate data



Often when we look at a situation we are trying to assess how much one variable has caused or influenced another to create the end result. **Bivariate data** is the term used for information relating to two different variables.

When exploring bivariate data, it is necessary to identify which of the two variables is the **explanatory variable** (represented on the x-axis) and which is the **response variable** (represented on the y-axis). For example, we are given the length of a pair of pants (variable 1) and the age of a person (variable 2). In this example, the age of a person is not going to depend on the length of their pants, while the length of the pants will generally be explained by the age of the person. Therefore, the explanatory variable is the age, while the length of the pants is the response variable.

WORKED EXAMPLE 1

Identify the explanatory and response variables in each of the following scenarios.

a. Distance walked in an hour and the age of a person

b. The cost of bananas and the average daily temperature in Queensland

THINK	WRITE
a. Consider which variable does not respond to the other. The age of a person will not be changed due to the distance they walk; however, their age could explain the distance they have covered.	**a.** Explanatory variable = age Response variable = distance walked in an hour
b. The cost of bananas is influenced by supply and demand. If the growing season has been affected by higher than expected daily temperatures, the number of bananas produced will be less, therefore increasing the price.	**b.** Explanatory variable = average daily temperature in Queensland Response variable = cost of bananas

14.2.2 Scatterplots

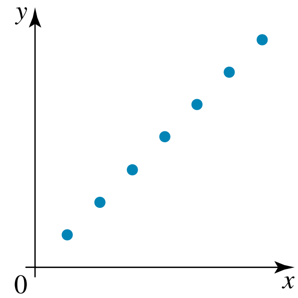
A common way to interpret bivariate data is through the use of a **scatterplot**. Scatterplots provide a visual display of the data and can be used to draw **correlations** and **causations** between two variables.

When constructing a scatterplot, it is important to place the explanatory variable along the x-axis and the response variable along the y-axis.

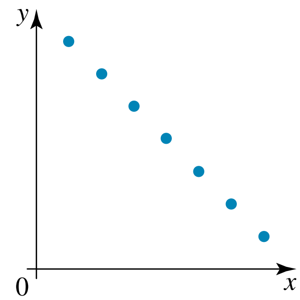
14.2.3 Correlation

When interpreting a scatterplot the correlation provides an insight into the relationship between the two variables. The correlation is a measure of the strength of the linear relationship between the two variables. There are three classifications for the correlation of data:

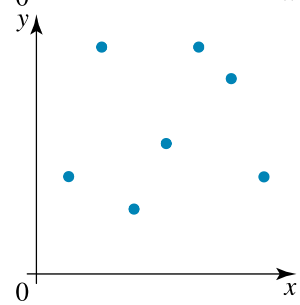
1. Positive correlation: as the explanatory variables (x-axis) increase, the response variables (y-axis) also increase, forming an upwards trend.

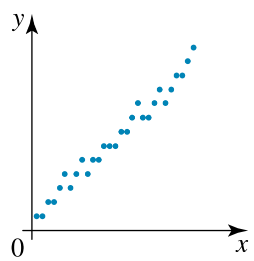
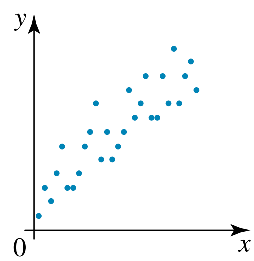
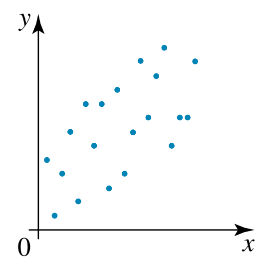
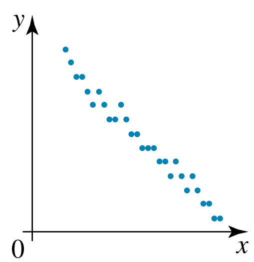
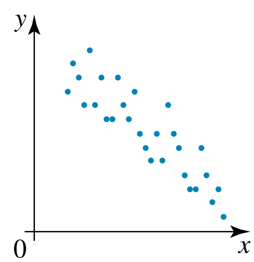
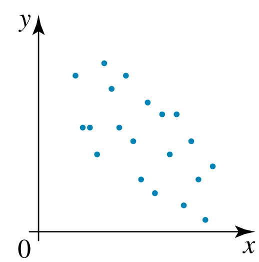


2. Negative correlation: as the explanatory variables (x-axis) increase, the response variables (y-axis) decrease, forming a downwards trend.



3. No correlation: no visible pattern formed by the data points, which appear to be randomly placed.



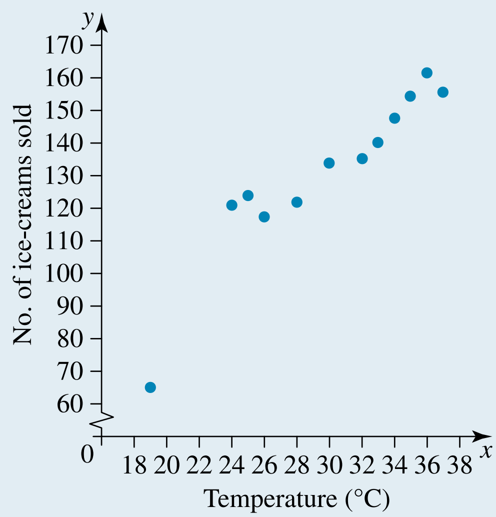
Strong positive correlation	Moderate positive correlation	Weak positive correlation
		
Strong negative correlation	**Moderate negative correlation**	**Weak negative correlation**
		

WORKED EXAMPLE 2

A local cafe recorded the number of ice-creams sold per day as well as the daily maximum temperature for 12 days.

Temp (°C)	36	32	28	26	30	24	19	25	33	35	37	34
No. of ice-creams sold	162	136	122	118	134	121	65	124	140	154	156	148

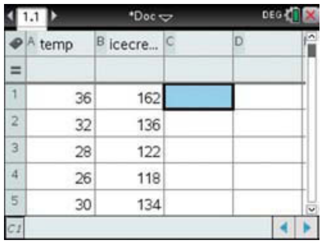
a. **Identify the response and explanatory variables.**
b. **Represent the data in a scatterplot.**
c. **Discuss the strength of the relationship between the variables.**

THINK	WRITE/DRAW
a. Consider which variable does not rely on the other. This will be the explanatory variable.	a. Explanatory variable = temperature Response variable = number of ice-creams sold
b. Select a reasonable scale for each variable that covers the full range of the data set. Plot the given points, remembering that the explanatory variable should be represented on the x-axis and the response variable should be represented on the y-axis.	b. 
c. Look at the pattern of the data points. Do they form a linear pattern? Are they progressing in a similar direction, either positive or negative? How strong is the correlation between the variables?	c. There is a linear relationship between the two variables. As the temperature increases so does the number of ice-creams sold. The correlation between the variables is strong. Therefore, this graph could be described as having a strong positive correlation.

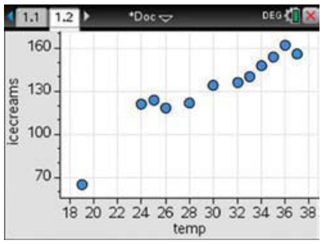
Note: The lowest point on the scatterplot could be considered a potential outlier.

TI | THINK — **WRITE**

b. 1. On a Lists & Spreadsheet page, label the first column as *temp* and the second column as *iceceams*.
Enter the temperature values in the first column and the corresponding ice-cream values in the second column.

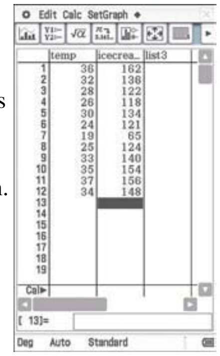


2. On a Data & Statistics page, click on the horizontal axis label and select *temp*. Click on the vertical axis label and select *icecreams*.

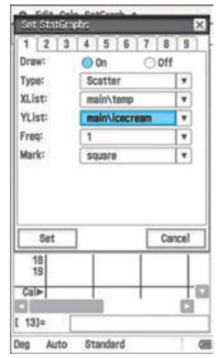


CASIO | THINK — **WRITE**

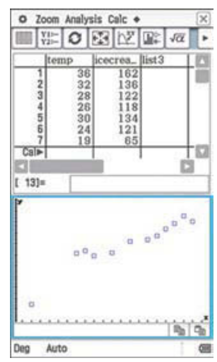
b. 1. On a Statistics screen, relabel list1 as *temp* and list2 as *icecream*.
Enter the temperature values in the first column and the corresponding ice-cream values in the second column.



2. Click the G icon, and complete the fields as:
Draw: On
Type: Scatter
XList: main\temp
YList: main\icecream
Press Set.



3. Click the *y* icon.



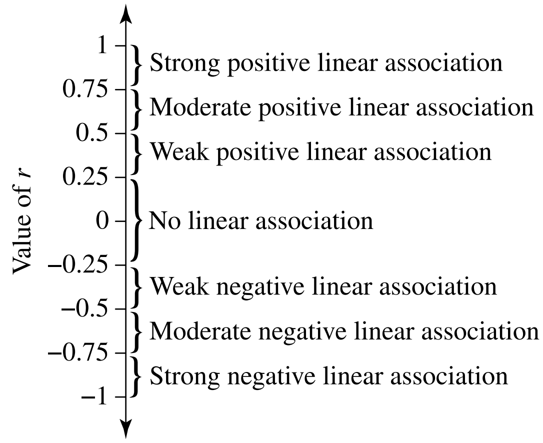
14.2.4 Pearson's product-moment correlation coefficient

The strength of the linear relationship can be observed from a scatterplot of the data. However, to determine exactly how strong this relationship is we can use **Pearson's product-moment correlation coefficient**, r, which measures the strength of a linear trend and associates it with a numerical value between -1 and $+1$. A value of either -1 or $+1$ indicates a perfect linear correlation, while a result closer to zero indicates no correlation between the variables. The following scale is a guide when using r to describe the strength of a linear relationship.

The formula to calculate Pearson's product-moment coefficient is:

$$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$, **where:**

n is the numbers of pieces of data in the data set
x_i is an x-value (explanatory variable)
y_i is a y-value (response variable)
s_x is the standard deviation of the x-values
s_y is the standard deviation of the y-values
$\bar{x}$ is the mean of the x-values
$\bar{y}$ is the mean of the y-values.



As this formula can be difficult to work with, once the raw data has been gathered we will use CAS to generate the Pearson's product-moment correlation coefficient value.

14.2.5 Causation and coefficient of determination

The measure of how much the change in one variable is caused by the other is referred to as causation. It is important to note that a strong linear relationship between two variables does not necessarily mean that a change in one variable will cause a change in the other. There are often other factors that need to be considered.

When there is a clear explanatory and response variable, the **coefficient of determination**, r^2, can be calculated to explore the impact a change in one variable may have on the other. For example, if a data set generated an r-value of 0.9, indicating a very strong linear relationship, the r^2-value would be 0.81. This indicates that 81% of the variation in the y-variable is explained by the variation in the x-variable, and 19% can be explained by other factors.

WORKED EXAMPLE 3

Use the data from Worked example 2 to answer the following questions.

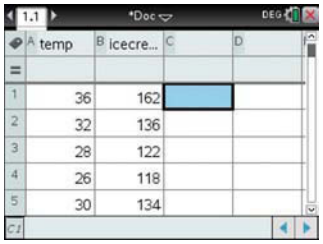
a. Use CAS to calculate Pearson's product-moment correlation coefficient correct to 4 decimal places.

b. Determine the coefficient of determination for this situation correct to 4 decimal places. What causations could be drawn from this information?

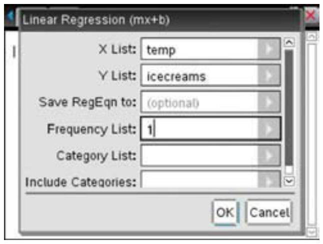
THINK	WRITE
a. 1. Enter the data values into a CAS to generate the r-value.	**a.** $r = 0.9355$
b. 1. Square the value of r to determine the coefficient of determination (r^2).	**b.** $r^2 = 0.8752$
2. Consider the value to be a percentage. What does this value tell you about the relationship between the two variables?	$r^2 = 0.8752$ indicates that 87.52% of sales of ice-cream are related to the temperature.

TI | THINK — **WRITE**

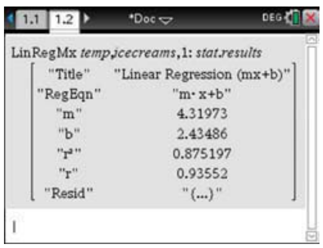
a. 1. On a Lists & Spreadsheet page, label the first column as *temp* and the second column as *iceceams*.
Enter the temperature values in the first column and the corresponding ice-cream values in the second column.



2. On a Calculator page, press MENU then select:
6: Statistics
1: Stat Calculations
3: Linear Regression (mx+b)
Complete the fields as:
X List: temp
Y List: icecreams then select OK.

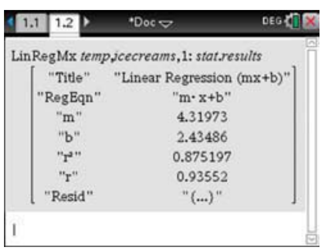


3. The Pearson's product-moment correlation coefficient can be read from the screen.



$r = 0.9355$

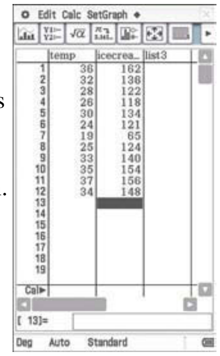
b. 1. The coefficient of determination can be read from the screen.



$r^2 = 0.8752$, meaning that 87.52% of the variation in ice-cream sales can be explained by the variation in temperature.

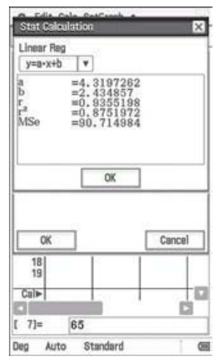
CASIO | THINK — **WRITE**

a. 1. On a Statistics screen, relabel list1 as *temp* and list2 as *icecream*.
Enter the temperature values in the first column and the corresponding ice-cream values in the second column.



2. Select:
- Calc
- Regression
- Linear Reg
Complete the fields as:
XList: main\temp
YList: main\icecream then select OK.

3. The Pearson's product-moment correlation coefficient can be read from the screen.



$r = 0.9355$

b. 1. The coefficient of determination can be read from the screen.

$r^2 = 0.8752$, meaning that 87.52% of the variation in ice-cream sales can be explained by the variation in temperature.

on Resources

- **Interactivity:** Create scatterplots (int-6497)
- **Interactivity:** Scatterplots (int-6250)
- **Interactivity:** Pearson's product-moment correlation coefficient and the coefficient of determination (int-6251)

studyon

Units 1 & 2 > AOS 6 > Topic 2 > Concepts 1 & 2

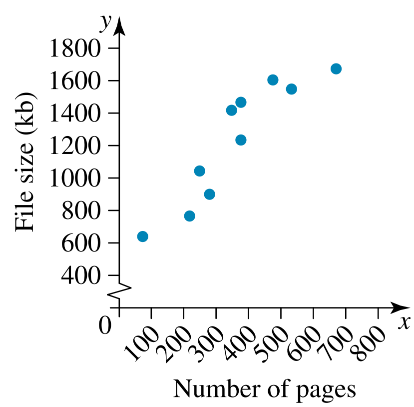
Scatterplots and correlation Concept summary and practice questions

Pearson's correlation coefficient Concept summary and practice questions

Exercise 14.2 Scatterplots and basic correlation

1. **WE1** A survey was conducted to record how long it takes to eat a pizza and the time of day. Identify the explanatory and response variables.
2. A study recorded the amount of data needed on a phone plan and. the time spent using phone apps. Identify the explanatory and response variables.

3. For each of the following scenarios, identify the explanatory and response variable.
   a. The age of people (in years) and the number of star jumps they can complete in one minute
   b. The cost of purchasing various quantities of chocolate
   c. The number of songs stored on a media player and the memory capacity used
   d. The growth rate of bacteria in a laboratory and the quantity of food supplied
4. **WE2** The scatterplot on the right has been established.
   a. Which variable is the response variable?
   b. How would you describe the relationship between these variables?



5. With reference to the data in the following table:
   a. identify the response and explanatory variables
   b. represent the data in a scatterplot
   c. identify the type of correlation, if any, that is evident from the scatterplot of these two variables.

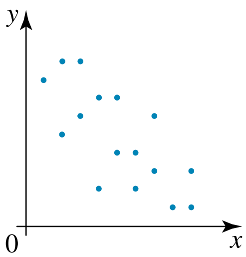
Time (minutes)	2	4	6	8	10	12	14	16	18	20
Weight that can be held (kg)	55	52	46	33	28	25	19	20	17	12

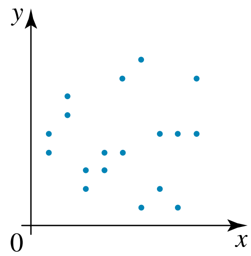
6. The weights and heights of a random sample of people were collected, with the following table displaying the collected data.

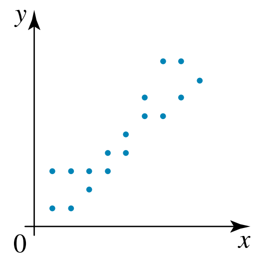
Height (cm)	140	145	150	155	160	165	170	175	180	185
Weight (kg)	58	62	66	70	75	77	78	80	88	90

a. Identify the explanatory and response variables.
b. Using a reasonable scale, plot the data.

7. Comment on the type and strength of the correlation displayed in each of the following scatterplots:

a. 

b. 

c. 

8. Suggest a combination of explanatory variable and response variable that may produce each of the following correlation trends:
a. negative correlation
b. no correlation.

9. **WE3** The following table outlines the cost of an annual magazine subscription along with the number of magazine issues per year.

No. of issues per year	7	9	10	6	8	4	4	5	11	9	10	5	11	3	7	12	7	6	12
Subscription cost ($)	34	40	52	38	50	25	28	40	55	55	45	28	65	24	38	55	50	33	59

a. Use CAS to determine Pearson's product-moment correlation coefficient for this data correct to 4 decimal places. What does this tell you about the strength of the linear relationship between the variables?
b. Calculate the coefficient of determination correct to 4 decimal places. What causations could be drawn from this information? What other factors might contribute to this result?

10. After assessing a series of bivariate data, a coefficient of determination value of 0.52 was calculated.
a. What does this value tell you about the strength of relationship between the two variables?
b. Referring to the variables as x and y, what causation could be suggested?
c. Why can't we use the coefficient of determination to draw exact conclusions?

11. Use your understanding of Pearson's product-moment correlation coefficient to explain what the following results indicate.
a. $r = 0.68$
b. $r = -0.97$
c. $r = -0.1$
d. $r = 0.30$

12. Find the value of the coefficient of determination in the following scenarios and interpret the meaning behind the result. Give your answers correct to 4 decimal places.
a. A survey found that the correlation between a child's diet and their health is $r = 0.8923$.
b. The correlation between global warming and the amount of water in the ocean was found to be $r = 0.9997$.

13. Using the data table from question 6, calculate Pearson's product-moment correlation coefficient and the coefficient of determination. Give your answers correct to 3 decimal places.

14. The coefficient of determination for a data set is found to be 0.5781. What is the percentage of variation that can be explained by other factors?

15. A series of data looked at the amount of time rugby teams spent warming up before a match and the number of wins. The coefficient of determination is 0.86. What conclusions could be drawn from this?

16. A survey asked random people for their house number and the combined age of the household members. The following data was collected:

House no.	Total age of household
14	157
65	23
73	77
58	165
130	135
95	110
54	94
122	25
36	68

House no.	Total age of household
101	53
57	64
34	120
120	180
159	32
148	48
22	84
9	69

a. Using the house number as the explanatory variable, plot this data.
b. Comment on the resulting scatterplot.
c. Determine Pearson's product-moment correlation coefficient and the coefficient of determination. Give your answers correct to 4 decimal places.
d. What conclusions can you draw from these values?
e. What percentage of variation could be contributed by other factors?

17. A class of Year 11 students were asked to record the amount of time in hours that they spent on a History assignment and the mark out of 100 that they received for the assignment.

Time spent (hours)	Mark (%)	Time spent (hours)	Mark (%)
2	72	0.75	58
0.5	52	1.5	69
1.5	76	1	62
2.5	82	2	78
0.25	36	3	90
2	73	3.5	94
2.5	84	1	70
2.5	80	3	92
2	74	2.5	88
0.5	48	3	97

a. Identify the explanatory and response variables.
b. Draw a scatterplot to represent this data.
c. Comment on the direction and correlation of the data points.
d. Explain why the data is not perfectly linear.
e. Using the data table, calculate Pearson's product-moment correlation coefficient and the coefficient of determination. Give your answers correct to 3 decimal places.
f. What do these values suggest about the relationship between a student's assignment mark and the time spent on it?

18. Use CAS to design a data set that meets the following criteria:
- contains 10–15 data points
- produces a negative trend
- has a coefficient of determination between 0.25 and 0.75.

14.3 Further correlation coefficients

14.3.1 Line of best fit

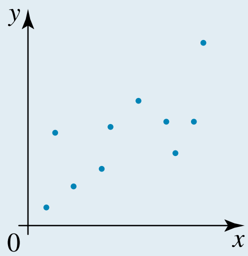
After raw data has been plotted as a scatterplot, the scatterplot can be used to determine findings and predictions can be made. A line of best fit can be used to generate information about the data and form an equation for the individual scatterplot. A line of best fit is the straight line that is most representative of the data, with the average distance between the data points and the line being minimised. There are numerous ways to draw a line of best fit, some more accurate than others.

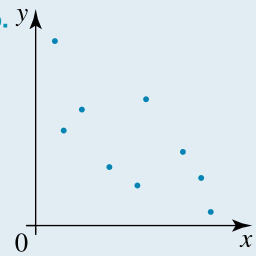
14.3.2 Line of best fit by eye

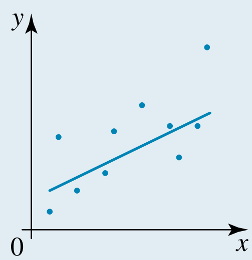
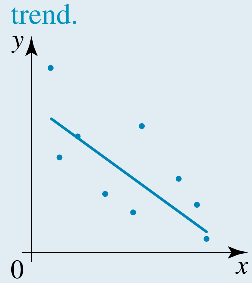
A quick way to draw a line of best fit is by eye. This method aims to draw a straight line with approximately the same number of data points above and below the line. The line should follow the direction of the general trend of the data. This method, while quick, leaves significant room for error.

WORKED EXAMPLE 4

For each of the scatterplots below, use a ruler to draw a line of best fit by eye.

a. 

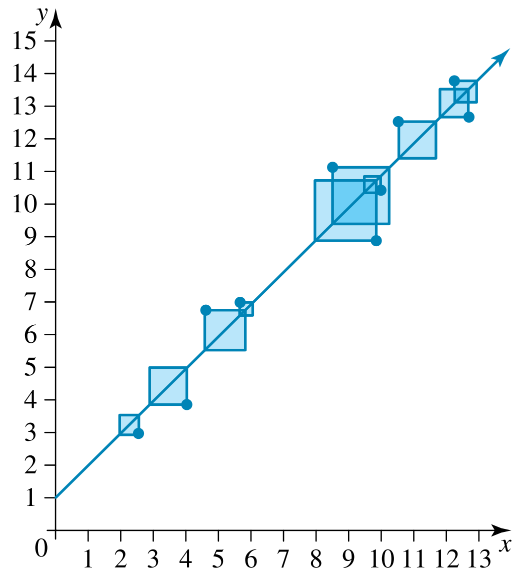
b. 

THINK	WRITE/DRAW
a. 1. Count the number of data points.	a. In this example there are 10 data points; therefore, the line of best fit should have 5 points on each side.
2. Consider the direction of the data points.	In this example, the trend is positive. The line of best fit will follow this trend.
3. Draw a straight line through the data points using a pencil. Review the line to confirm an even distribution of data points.	
b. 1. Count the number of data points.	b. In this example there are 9 data points. As there is an odd number of points, either a data point sits on the line or there will be a slight imbalance of points above and below the line.
2. Consider the direction of the data points.	In this example, the trend is negative. The line of best fit will follow this trend.
3. Draw a straight line through the data points using a pencil. Review the line to confirm an even distribution of data points.	

14.3.3 Least squares regression

Sometimes a line of best fit can be drawn by eye; however, in other situations it is necessary to be more accurate. When there are no outliers in a scatterplot, we can generate an equation using the **least squares regression line.**

This line minimises the vertical distances between the data points and the line of best fit. It is called the least squares regression line because if we took the squares of these vertical distances, this line would represent the smallest possible sum of all of these squares.



The equation for the least squares regression line takes the form $y = a + bx$, where y is the response variable, x is the explanatory variable, b is the gradient or slope of the line and a is the y-intercept.

You can use CAS to generate these values.

14.3.4 Manually calculating the equation for the least squares regression line

To find the equation for the least squares regression line, the following elements need to be collected.

$\bar{x}$: the mean of the explanatory variable
$\bar{y}$: the mean of the response variable
s_x: the standard deviation of the explanatory variable
s_y: the standard deviation of the response variable
r: Pearson's product-moment correlation coefficient

These values can then be substituted to find a and b, using the following formulas:

To find the slope: $b = r\dfrac{s_y}{s_x}$

To find the y-intercept: $a = \bar{y} - b\bar{x}$

Once the values for b and a have been found, substitute the results into $y = a + bx$ to calculate the least squares regression equation.

WORKED EXAMPLE 5

A school recorded the age and height of a group of male students, with the following data being collected:
Average age (years) = 15.59
Average height (cm) = 160.8
Standard deviation of age = 1.60
Standard deviation of height = 7.50
Pearson's product-moment correlation coefficient = 0.81

a. **Identify the explanatory variable.**
b. **Calculate the value of b correct to 4 significant figures.**
c. **Calculate the value of a correct to 4 significant figures.**
d. **Determine the equation of the least squares regression line.**

THINK	WRITE
a. Consider which variable is more likely to influence the other.	**a.** Explanatory variable = age
b. 1. Identify the necessary variables.	**b.** $s_x = 1.60$ $s_y = 7.50$ $r = 0.81$
2. Substitute these values into the equation for b.	$b = r\dfrac{s_y}{s_x}$ $= 0.81 \times \dfrac{7.5}{1.6}$ $= 3.797$
c. 1. Identify the necessary variables.	**c.** $\bar{x} = 15.59$ $\bar{y} = 160.8$ $m = 3.797$
2. Substitute these values into the equation for a.	$a = \bar{y} - b\bar{x}$ $= 160.8 - 3.797 \times 15.59$ $= 101.6$
d. Substitute the values for b and a into $y = a + bx$.	**d.** $y = a + bx$ $= 101.6 + 3.797x$

on Resources

Interactivity: Fitting a straight line using least-squares regression (int-6254)

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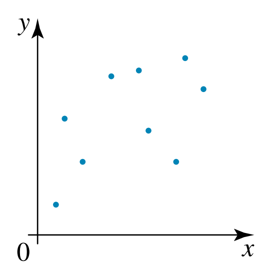
Units 1 & 2 > AOS 6 > Topic 2 > Concept 3

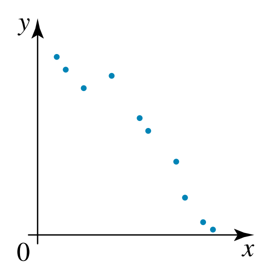
Least squares regression line Concept summary and practice questions

Exercise 14.3 Further correlation coefficients

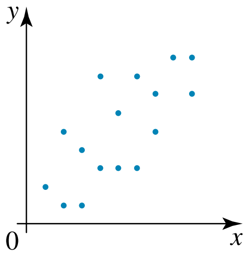
1. **WE4** For the scatterplot below, use a ruler to draw a line of best fit by eye.

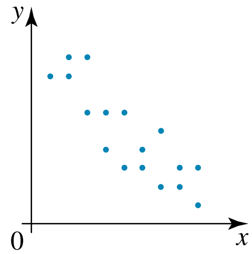


2. For the scatterplot below, use a ruler to draw a line of best fit by eye.



3. Draw a line of best fit by eye for the following scatterplots.

a. 

b. 

4. For each of scatterplots in question **3**, comment on the type and strength of the correlation displayed.
5. **WE5** Each student in a class recorded the amount of time they spent on a computer each day and the number of siblings they had. The following data was collected.
Average number of siblings = 2.75
Average time spent on a computer per day (min) = 39.5
Standard deviation of number of siblings = 1.39
Standard deviation of time spent on a computer = 20.61
Pearson's product-moment correlation coefficient = −0.73
   a. Identify the explanatory variable.
   b. Calculate the value of b correct to 4 significant figures.
   c. Calculate the value of a correct to 4 significant figures.
   d. Determine the equation of the least squares regression line.
6. People were asked to record the distance they travel to work and the time it takes. Where appropriate, give answers to the following questions correct to 4 significant figures.

Distance travelled (km)	33	50	10	19	65	24	42	46	52
Time taken (mins)	60	38	21	30	58	30	41	47	48

   a. Use CAS to plot this data on a scatterplot.
   b. Determine the values of the following variables.
      i. $\bar{x}$ ii. $\bar{y}$ iii. s_x iv. s_y v. r
   c. Substitute the values into $b = r\frac{s_y}{s_x}$ to find the value of the slope.
   d. Solve $a = \bar{y} - b\bar{x}$ to find the y-intercept.
   e. Identify the least squares regression equation for this data.
7. Use the following information to calculate the values for b and a in the least squares regression line. Give your answers correct to 3 significant figures.

$$\bar{x} = 164.11,\ \bar{y} = 7.6,\ s_x = 16.37,\ s_y = 1.67,\ r = -0.98$$

8. Use the following data set to answer the questions below. Where necessary, give your answers correct to 3 significant figures.

x	2	4	6	8	10	12	14	16	18	20
y	55	52	46	33	28	25	19	20	17	12

   a. Determine these values:
      i. $\bar{x}$ ii. $\bar{y}$ iii. s_x iv. s_y v. r
   b. Substitute the values from your answers to part **a** into $b = r\frac{s_y}{s_x}$ to find the value of the slope.
   c. Solve $a = \bar{y} - b\bar{x}$ to find the value of the y-intercept.
   d. Identify the least squares regression equation for this data.

9. The point (1, 70) was added to the data set from question 8. How would this data point affect the equation of the line of best fit?

10. Use your understanding of the least squares regression line equation to sketch the equation $y = 14 - 3x$ on a blank Cartesian plane.

11. A researcher investigating the proposition that 'tall mothers have tall sons' measures the height of 12 mothers and the height of their adult sons. The results are shown below.

Height of mother (cm)	Height of son (cm)	Height of mother (cm)	Height of son (cm)
185	188	158	159
155	157	156	150
171	172	168	172
169	173	169	175
170	174	179	180
175	180	173	190

a. Which variable is the response variable?

b. Draw a scatterplot and a line of best fit.

c. Determine the equation of the line of best fit, expressing the equation in terms of height of mother (M) and height of son (S). Give values correct to 4 significant figures.

12. An equation for a regression line is $y = 3.2 - 1.56x$. What conclusions about the trend of the regression line can be determined from the equation?

13. Data on the daily sales of gumboots and the maximum daily temperature were collected.

Temp (°C)	Daily sales (no. of pairs)	Temp (°C)	Daily sales (no. of pairs)
17	2	23	1
16	3	19	2
12	8	17	3
10	16	15	3
14	7	12	12
17	3	15	9
18	2	20	1
22	1		

a. Draw a scatterplot of this data.

b. Find the equation of the line of best fit, expressed in terms of temperature (T) and daily sales (D). Give values correct to 4 significant figures.

c. Find Pearson's product-moment correlation coefficient and the coefficient of determination. Give your answers correct to 4 significant figures.

d. Interpret these values in the context of the data.

14. A data set produced a positive trend and for each incremental increase in the explanatory variable, the response variable increased by 2.5. If $y = 4$ when $x = 0$, determine the equation for the regression line.

15. a. Use CAS to plot the regression line $y = -1.6 + 2.5x$.
    b. Would a data point of (3, 4) be found above or below the regression line?

16. Use CAS to design a data set that meets the following criteria:
    - contains 10–15 data points
    - produces a positive trend
    - has a b value between 2 and 5 in its regression equation.

14.4 Making predictions

14.4.1 Interpreting the regression line equation

Often data is collected in order to make informed decisions or predictions about a situation. The regression line equation from a scatterplot can be used for this purpose.

Remember that the equation for the regression line is in the form $y = a + bx$, where b is the gradient or slope, a is the y-intercept, and x and y refer to the two variables. Two important pieces of information can be attained from this equation.

1. **When the explanatory variable is equal to 0, the value of the response variable is indicated by the y-intercept, a.**
2. **For each increment of 1 unit of change in the explanatory variable, the change in the response variable is indicated by the value of the slope, b.**

WORKED EXAMPLE 6

The least squares regression equation for a line is $y = 62 - 8x$.

a. **Identify the y-intercept.**

b. **For each unit of change in the explanatory variable, by how much does the response variable change?**

c. **What does your answer to part b tell you about the direction of the line?**

THINK	WRITE
a. Consider the equation in the form $y = a + bx$. Identify the value that represents a.	y-intercept = 62
b. The change in the response variable due to the explanatory variable is reflected in the slope. Identify the b value in the equation.	$b = -8$
c. A positive b value indicates a positive trending line, while a negative b value indicates a negative trending line.	As the b value is negative, the trend of the line is negative.

14.4.2 Interpolation and extrapolation

The regression line can be used to explore data points both inside and outside of the scatterplot range. When investigating data inside the variable range, the data is being interpolated. Data points that lie above or below the scatterplot range can also be used to make predictions; this is referred to as the data being extrapolated.

14.4.3 Using the regression line equation to make predictions

The regression equation can be used to make predictions from the data by substituting in a value for either the explanatory variable (x) or the response variable (y) in order to find the value of the other variable.

WORKED EXAMPLE 7

Flowers with a diameter of 5–17 cm were measured and the number of petals for each flower was documented. A regression equation of $N = 0.41 + 1.88d$, where N is the number of petals and d is the diameter of the flower (in cm) was established.

a. **Identify the explanatory variable.**

b. **Determine the number of petals that would be expected on a flower with a diameter of 15 cm. Round to the nearest whole number.**

c. **Is the value found in part b an example of interpolated or extrapolated data?**

d. **A flower with 35 petals is found. Use the equation to predict the diameter of the flower correct to 1 decimal place.**

e. **Is part d an example of interpolated or extrapolated data?**

THINK	WRITE
a. Consider the format of the equation. The variable on the right-hand side will be the explanatory variable.	**a.** Explanatory variable = flower diameter
b. 1. Using the equation, substitute 15 in place of d.	**b.** $N = 0.41 + 1.88d$ $= 0.41 + 1.88 \times 15$ $= 28.61$
2. Round to the nearest whole value.	29 petals
c. Consider the data range given in the opening statement.	**c.** 15 cm is inside the data range, so this is interpolation, not extrapolation.
d. 1. Using the equation, substitute 30 in place of N.	**d.** $35 = 0.41 + 1.88d$
2. Transpose the equation to solve for d.	$d = \dfrac{35 - 0.41}{1.88}$ $= 18.40$
3. Round to 1 decimal place.	$= 18.4$ (correct to 1 decimal place)
e. Consider the data range given in the opening statement.	**e.** 18.4 cm is outside the data range, so this is an example of extrapolated data.

14.4.4 Limitations of regression line predictions

When reviewing predictions drawn from a scatterplot, it is necessary to question the reliability of the results. As with any conclusion or prediction, the results rely heavily on the initial data. If the data was collected from a small sample, then the limited information could contain biases or a lack of diversity that would not be present in a larger sample. The more data that can be provided at the start, the more accurate the result that will be produced.

The strength of the correlation between the variables also provides an indication of the reliability of the data. Data that produces no correlation or a low correlation would suggest that any conclusions drawn from the data will be unreliable.

When extrapolating data it is assumed that additional data will follow the same pattern as the data already in use. This assumption means extrapolated data is not as reliable as interpolated data.

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Units 1 & 2 > AOS 6 > Topic 2 > Concepts 4 & 5

Interpretation of slope and intercept Concept summary and practice questions

Predictions and extrapolation Concept summary and practice questions

Exercise 14.4 Making predictions

1. **WE6** The least squares regression equation for a line is $y = -1.837 + 1.701x$.
   a. Identify the y-intercept.
   b. For each unit of change in the explanatory variable, by how much does the response variable change?
   c. What does your answer to part **b** tell you about the direction of the line?
2. The least squares regression equation for a line is $y = 105.90 - 1.476x$.
   a. Identify the y-intercept.
   b. For each unit of change in the explanatory variable, by how much does the response variable change?
   c. What does your answer to part **b** tell you about the direction of the line?
3. Answer the following questions for the equation $y = 60 - 5x$.
   a. Identify the y-intercept.
   b. For each unit of change in the explanatory variable, by how much does the response variable change?
   c. Is the trend of the data positive or negative?
   d. Calculate the value of y when $x = 40$.
4. Lucy was given the equation $y = -12.9 + 7.32x$ and asked to find the value of x when $y = 15.68$. Her working steps are below:

$$\begin{aligned} y &= -12.9 + 7.32x \\ 15.68 &= -12.9 + 7.32x \\ x &= 12.9 + \frac{15.68}{7.32} \\ &= 15.04 \end{aligned}$$

   Her teacher indicates her answer is wrong.
   a. Calculate the correct value of x. Give your answer correct to 2 decimal places.
   b. Identify and explain Lucy's error.
5. Answer the following questions for the equation $y = -12 + 25x$.
   a. Identify the y-intercept.
   b. For each unit of change in the explanatory variable, by how much does the response variable change?

c. Is the trend of the data positive or negative?
d. Calculate the value of y when $x = 3.5$.

6. WE7 A brand of medication for babies bases the dosage on the age (in months) of the child. The regression equation for this situation is $M = 0.157 + 0.312A$, where M is the amount of medication in mL and A is the age in months.
   a. Identify the explanatory variable.
   b. Calculate the amount of medication required for a child aged 6 months.
   c. Determine the age of a child who requires 2.5 mL of the medication. Give your answer correct to 1 decimal place.
7. A survey of the nightly room rate for Sydney hotels and their proximity to the Sydney Harbour Bridge produced the regression equation $C = 281.92 - 50.471d$, where C is the cost of a room per night in dollars and d is the distance to the bridge in kilometres.
   a. Identify the response variable.
   b. Based on this equation, calculate the cost of a hotel room 2.5 km from the bridge. Give your answer correct to the nearest cent.
   c. Determine the distance of a hotel room from the bridge if the cost of the room was \$115. Give your answer correct to 2 decimal places.
8. Answer the following questions for the equation $I = 0.43 + 1.1s$, where I is the number of insects caught and s is the area of a spider's web in cm^2.
   a. Identify the response variable.
   b. For each unit of change in the explanatory variable, by how much does the response variable change?
   c. Is the trend of the data positive or negative?
   d. Determine how many insects are likely to be caught if the area of the spider's web is $60\ cm^2$. Give your answer correct to the nearest whole number.

9. a. Use the data given to draw a scatterplot and a line of best fit by eye.

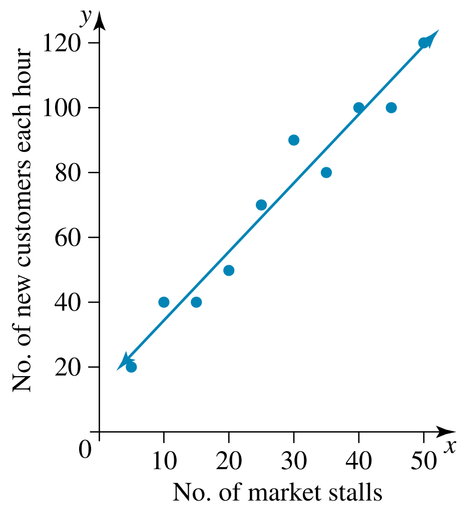
x	1	2	3	4	5	6	7	8	9	10
y	35.3	35.9	35.7	36.2	37.3	38.6	38.4	39.1	40.0	41.1

   b. Find the equation of the line of best fit and use it to predict the value of y when $x = 15$. Give your answers correct to 4 significant figures.
10. Use the data given below to complete the following questions.

x	1	2	3	4	5	6	7	8	9	10
y	4	1	2	3	5	5	3	6	8	7

   a. Draw a scatterplot and a line of best fit by eye.
   b. Determine the equation of the line of best fit. Give values correct to 2 significant figures.
   c. Predict the value of y when $x = 20$.
   d. Predict the value of x when $y = 9$. Give your answers correct to 2 decimal places.

11. The following scatterplot has the equation of $c = 13.33 + 2.097m$, where c is the number of new customers each hour and m is the number of market stalls.



a. Using the line of best fit, interpolate the data to find the number of new customers expected if there are 30 market stalls.
b. Use the formula to extrapolate the number of market stalls required in order to expect 150 new customers.
c. Explain why part a is an example of interpolating data, while part b demonstrates extrapolation.

12. Use the data given below to complete the following questions.

x	10	11	12	13	14	15	16	17	18	19
y	22	18	20	15	17	11	11	7	9	8

a. Draw a scatterplot and a line of best fit by eye.
b. Determine the equation of the line of best fit. Give values correct to 4 significant figures.
c. Extrapolate the data to predict the value of y when $x = 23$.
d. What assumptions are made when extrapolating data?

13. While camping a mathematician estimated that:
number of mosquitos around fire $= 10.2 + 0.5 \times$ temperature of the fire (°C)
a. Determine the number of mosquitoes that would be expected if the temperature of the fire was 240°C. Give your answer correct to the nearest whole number.
b. What would be the temperature of the fire if there were only 12 mosquitoes in the area?
c. Identify some factors that could affect the reliability of this equation.

14. Data on people's average monthly income and the amount of money they spend at restaurants was collected.

Average monthly income (\$000s)	Money spent at restaurants per month (\$)
2.8	150
2.5	130
3.0	220
3.1	245
2.2	100
4.0	400
3.7	380
3.8	200

Average monthly income \$(000s)	Money spent at restaurants per month (\$)
4.1	600
3.5	360
2.9	175
3.6	350
2.7	185
4.2	620
3.6	395

a. Draw a scatterplot of this data on your calculator.
b. Find the equation of the line of best fit in terms of average monthly income in thousands of dollars (I) and money spent at restaurants in dollars (R). Give values correct to 4 significant figures.
c. Extrapolate the data to predict how much a person who earns $5000 a month might spend at restaurants each month.
d. Explain why part **c** is an example of extrapolation.

e. A person spent $265 eating out last month. Estimate their monthly income, giving your answer to the nearest $10. Is this an example of interpolation or extrapolation?

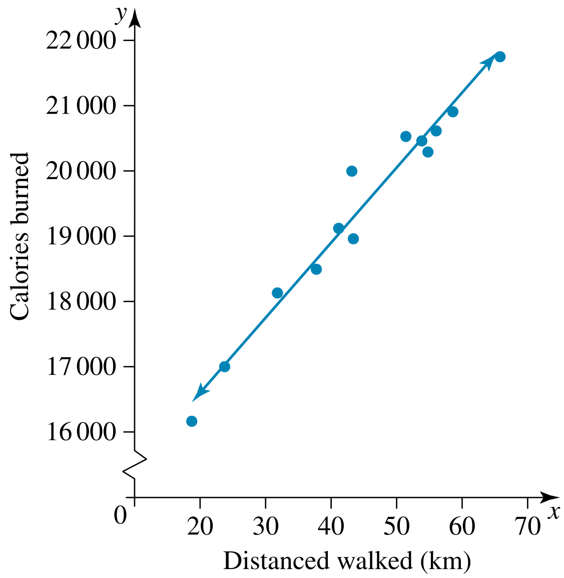
15. Data on students' marks in Geography and Music were collected.

Geography	Music
65	91
80	57
72	77
61	89
99	51
54	76
39	62
66	87

Geography	Music
78	88
89	64
84	90
73	45
68	60
57	79
60	69

a. Is there an obvious explanatory variable in this situation?
b. Draw a scatterplot of this data on your calculator, using the marks in Geography as the explanatory variable.
c. Find the equation of the line of best fit. Give values correct to 4 significant figures.
d. Based on your equation, if a student received a mark of 85 for Geography, what mark (to the nearest whole number) would you predict they would receive for Music?
e. How confident do you feel about making predictions for this data? Explain your response.
f. Calculate Pearson's product-moment correlation coefficient for this data. How can you use this value to evaluate the reliability of your data?

16. For three months, Cameron has been wearing an exercise-tracking wristband that records the distance he walks and the number of calories he burns. The graph shows his weekly totals. The regression line equation for this data is $y = 14\,301 + 115.02x$.



a. Identify the response variable in this situation.
b. Rewrite the equation in terms of the explanatory and response variables.
c. Using the equation for the regression line, determine the number of calories burned if a person walked 50 km in a week. Is this an example of interpolation or extrapolation? Explain your response.
d. Due to an injury, in one week Cameron only walked 10 km. Use the data to determine the number of calories this distance would burn. Is this an example of interpolation or extrapolation? Explain your response.
e. Pearson's product-moment correlation coefficient for this data is 0.9678. How can you use this value to evaluate the reliability of the data?
f. List at least two other factors that could influence this data set.

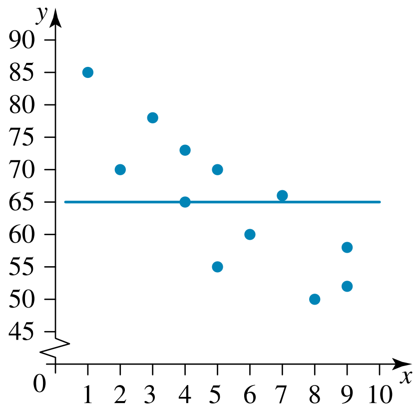
14.5 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

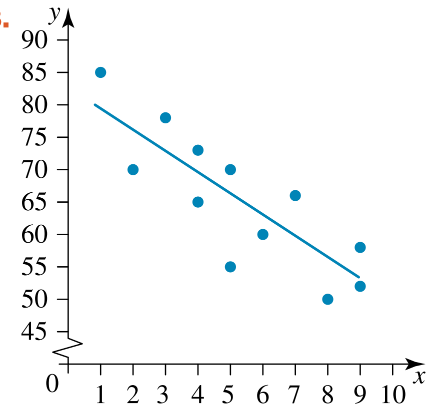
Multiple choice

1. MC Which of the following scatterplots best demonstrates a line of best fit?

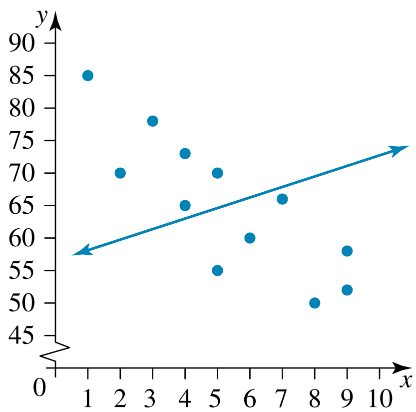
A.



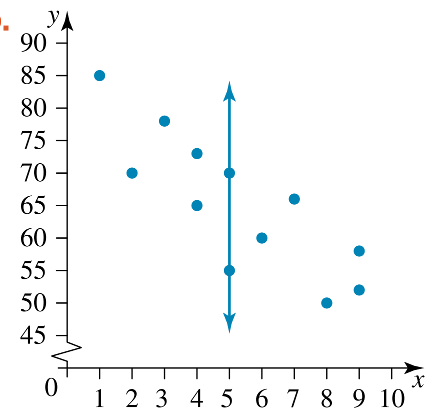
B.



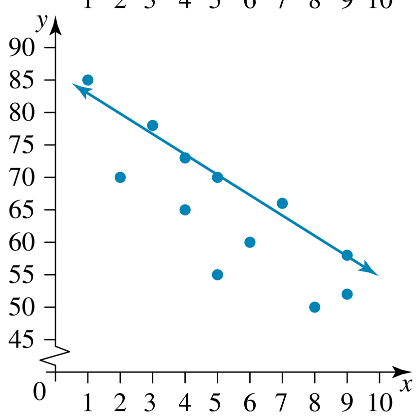
C.



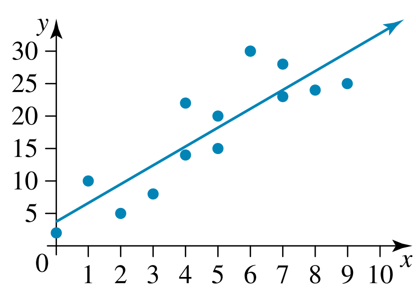
D.



E.



2. MC The regression line equation for the following graph is closest to:



A. $y = 3.8 + 2.9x$
B. $y = -3.8 - 2.9x$
C. $y = -3.8 + 2.9x$
D. $y = 3.8 - 2.9x$
E. $y = 2.9 + 3.8x$

3. MC The type of correlation shown in the graph for question 2 would best be described as:

A. weak, positive correlation
B. moderate, positive correlation
C. strong, positive correlation
D. no correlation
E. moderate, negative correlation

4. MC What type of correlation does an r-value of 0.64 indicate?

A. Strong, positive correlation
B. Strong, negative correlation
C. Moderate, positive correlation
D. Moderate, negative correlation
E. Weak, positive correlation

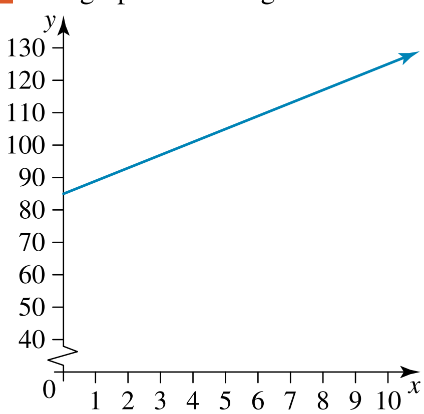
5. MC A gardener tracks a correlation coefficient of 0.79 between the growth rate of his trees and the amount of fertiliser used. What can the gardener conclude from this result?

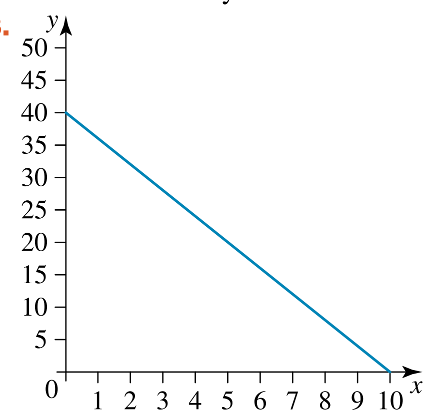
A. An increase in tree growth increases the use of fertiliser.
B. An increase in the use of fertiliser increases the health of the tress.
C. The growth rate of the trees is influenced by the amount of fertiliser used.
D. The growth rate of the trees influences the quality of the fertiliser used.
E. There is no correlation between the growth rate of the tress and the amount of fertiliser used.

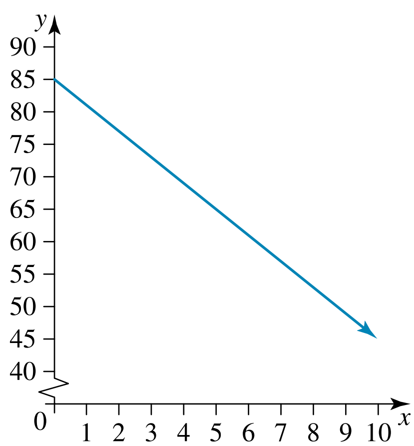
6. MC When $y = 0.54 + 15.87x$, the value of y when $x = 2.5$ is:

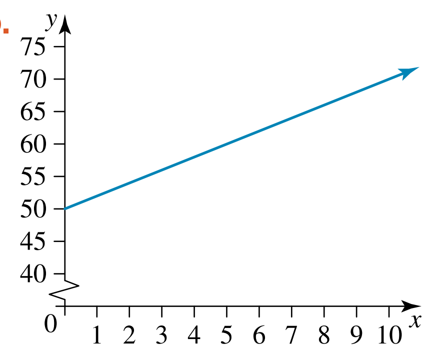
A. 18.91
B. 40.215
C. 39.135
D. 6.888
E. 34.019

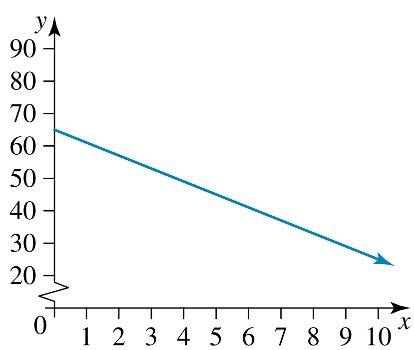
7. MC The graph for the regression line equation $y = 85 - 4x$ is most likely to be:

A.


B.


C.


D.


E.


8. MC A series of data points recorded a coefficient of determination value of 0.82. Calculate the Pearson's product-moment coefficient.

A. 82% B. 0.18 C. 0.67 D. 0.91 E. 18%

9. MC For the following sample data set, which of the following is an example of interpolating data?

x	1	5	15	25
y	10	16	18	22

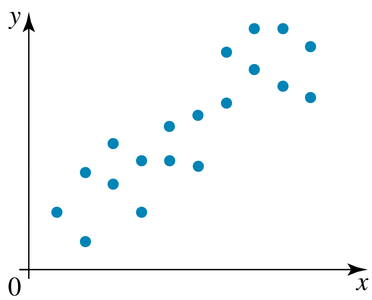
A. Finding the value of x when $y = -7$
B. Finding the value of y when $x = 17$
C. Finding the value of x when $y = 27$
D. Finding the value of y when $x = 37$
E. Finding the value of x when $y = 5$

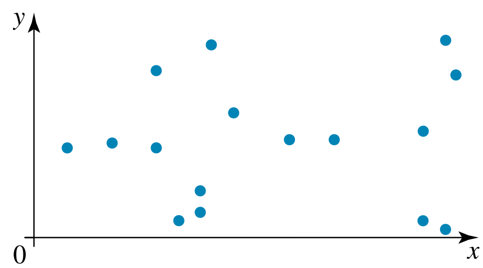
10. MC For the data set from question 9, the regression line equation is:

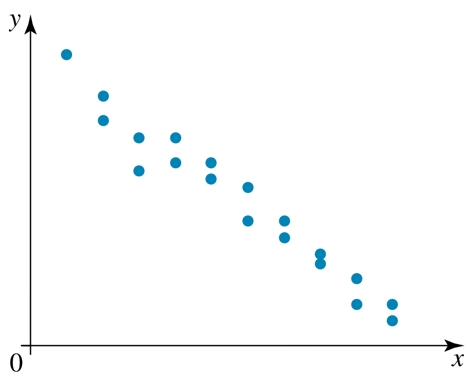
A. $y = 10 + x$
B. $y = 0.435 + 11.456x$
C. $y = 0.876 + 0.936x$
D. $y = 11.456 + 0.435x$
E. $y = 0.936 + 0.876x$

Short answer

1. For each of the following graphs, describe the strength of correlation between the explanatory and response variables.

a. 

b. 

c. 

2. For each of the graphs in question 1, draw a line of best fit where possible.
3. Identify the explanatory and response variable for each of the following scenarios:
   a. In a junior Science class, students plot the time taken to boil various quantities of water.
   b. Extra buses are ordered to transport a number of students to the school athletics carnival.
4. Use the following data to complete this question.

x	10	9	8	7	6	5	4	3	2	1
y	6	10	4	11	13	18	15	19	21	26

   a. Plot the data on a scatterplot.
   b. Comment on the direction and strength of the data.

c. Find Pearson's product-moment correlation coefficient and the coefficient of determination for the data.
d. Use you answer from part **c** to further discuss the relationship between the variables.

5. Pearson's product moment correlation coefficient for a scatterplot was found to be −0.7564.
   a. Find the value of the coefficient of determination.
   b. What would these values indicate to you about the strength of relationship between the two variables?
6. During an interview investigating the link between the sales of healthy snack foods (functional foods) and the increasing consumer demand for these products, an advertising expert made the following comment: 'There is a correlation but it's not causation … our increasing need for healthy food and our laziness has resulted in mass innovation of functional foods.'
   Explain why he might have stated there is no causative link between the sales of healthy foods and laziness.

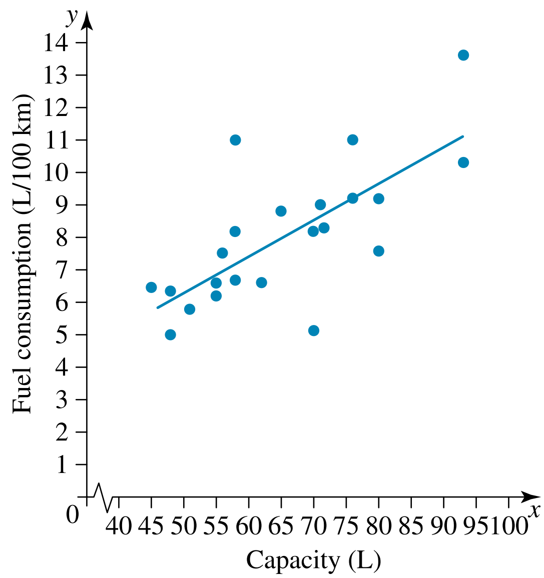
Extended response

1. Data on 15 people's shoe size and the length of their hair was collected.

Shoe size	Length of hair (cm)
6	9
8	14
7	12
8	1
9	7
6	8
7	5
12	22
8	15
9	8
10	18
12	4
7	5
9	9
11	3

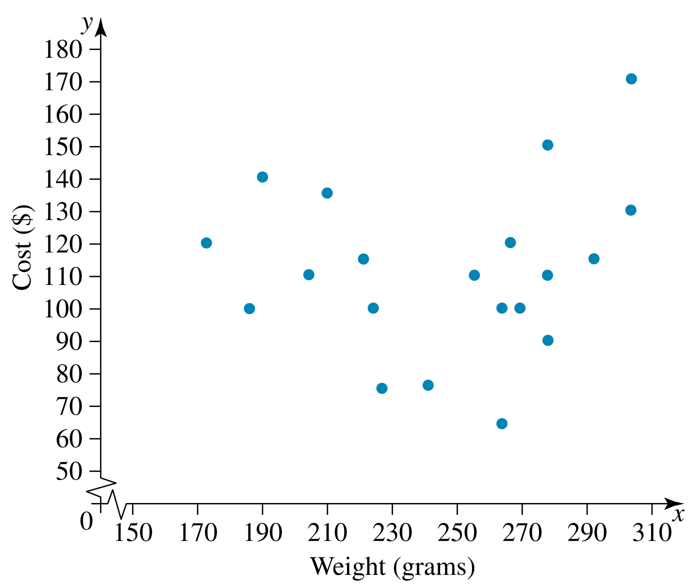
   a. Draw a scatterplot of this data.
   b. Find the equation of the line of best fit.
   c. Find Pearson's product-moment correlation coefficient and the coefficient of determination for the data.
   d. What conclusions could you draw from this data?

2. An independent agency test-drove a random sample of current model vehicles and measured their fuel tank capacity against the average fuel consumption. Along with the following scatterplot, a regression equation of $y = 0.1119x + 0.6968$ was established.



a. Identify the response variable in this situation.
b. Rewrite the equation in terms of the explanatory and response variables.
c. It is often said that smaller vehicles are more economical. Determine correct to 2 decimal places the fuel consumption of a vehicle that had a 40-litre fuel tank.
d. Is your answer to part **c** an example of interpolation or extrapolation? Explain your response.
e. Calculate, correct to the nearest whole number, the tank size of a vehicle that had a fuel consumption rate of 10.2 L per 100 km.
f. Pearson's product-moment correlation coefficient for this data is 0.516. How can you use this value to evaluate the reliability of your data?
g. List at least two other factors that could influence the data.

3. The weight of top brand runners was tracked against the recommended retail price, and the results were recorded in the following scatterplot.



a. Identify the explanatory variable for this situation.
b. How would you describe the relationship between these two variables?
c. The coefficient of determination for this data is $r^2 = 0.018\,72$. What conclusions can be established from this result?
d. Identify two external factors that could explain the distribution of the data points.

4. The Bureau of Meteorology records data such as maximum temperatures and solar exposure on a daily and monthly basis. The following data table, for the Botanical Gardens in Melbourne, shows the monthly average amount of solar energy that fall on a horizontal surface and the monthly average maximum temperature. (*Note:* The data values have been rounded to the nearest whole number.)

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Average solar exposure (MJ)	25	21	17	11	8	6	7	10	13	18	21	24
Average max daily temp. (°C)	43	41	34	33	24	19	24	24	28	32	25	40

a. Identify the explanatory and response variables for this situation.
b. Using CAS, plot the data on a scatterplot.
c. Describe the trend of the data.
d. Calculate Pearson's product-moment coefficient and coefficient of determination for this data. What do these values tell you about the reliability of the data?
e. Plot the regression line for this data and write the equation in terms of the variables.
f. Using your equation, calculate the amount of solar exposure for a monthly maximum temperature of 37°C.
g. Extrapolate the data to find the average maximum temperature expected for a month that recorded an average solar exposure of 3 MJ.
h. Explain why part **g** is an example of extrapolation.

study on

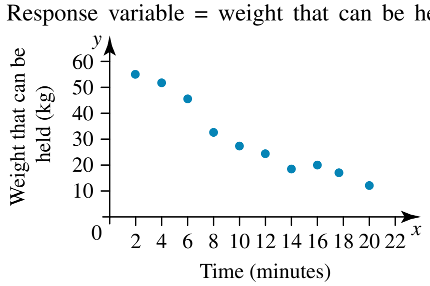
Units 1&2 **Sit topic test**

Answers

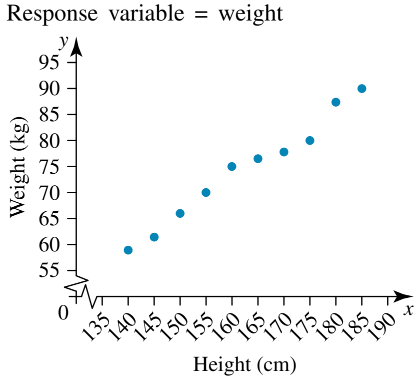
Topic 14 Relationships between two numerical variables

Exercise 14.2 Scatterplots and basic correlation

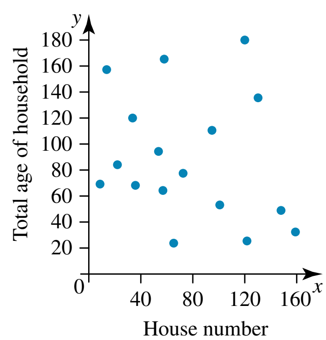
1. Explanatory variable = time of day
   Response variable = time taken to eat a pizza
2. Explanatory variable = time spend using phone apps
   Response variable = amount of data required
3. a. Explanatory variable = age
   Response variable = number of star jumps
   b. Explantory variable = quantity of chocolate
   Response variable = cost
   c. Explantory variable = number of songs
   Response variable = memory used
   d. Explantory variable = food supplied
   Response variable = growth rate
4. a. File size
   b. Strong positive correlation
5. a. Explanatory variable = time
   Response variable = weight that can be held (kg)
   b.



   c. Strong negative correlation
6. a. Explanatory variable = height
   Response variable = weight
   b.

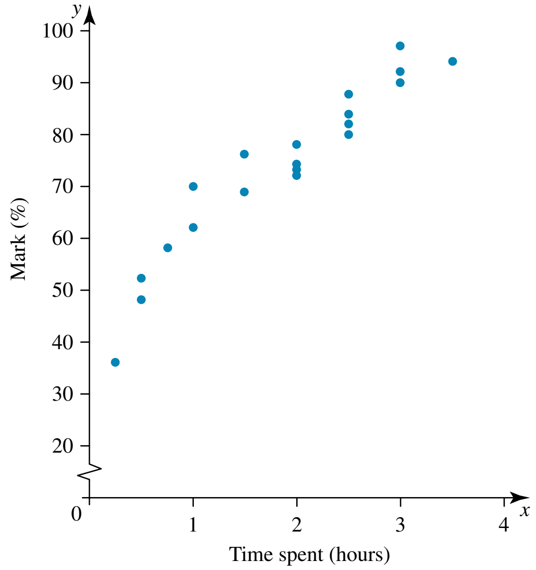


7. a. Weak negative correlation
   b. No correlation
   c. Moderate positive correlation
8. Various possible answers, for example:
   a. Loss of money over time
   b. Temperature and number of shoes owned
9. a. $r = 0.8947$, which indicates a strong positive linear association.
   b. $r^2 = 0.8005$, which suggests that just over 80% of the subscription cost is due to the number of issues per year. Other factors might include the number of coloured pages, weight of postage, amount of advertising in each issue.
10. a. It indicates a moderate positive linear association between the x and y variables.
    b. 52% of the variation in y is a result of the variation in x.
    c. The coefficient of determination provides information about the strength of the data rather than the causation.
11. a. A moderate positive linear association
    b. A strong negative linear association
    c. No linear association
    d. A weak positive linear association
12. a. $r^2 = 0.7962$. Diet and health have a strong positive association.
    b. $r^2 = 0.9994$. Global warming and the amount of water in the ocean have a very strong positive association.
13. $r = 0.989$
    $r^2 = 0.978$
14. 42.19%
15. A strong positive relationship between the amount of time spent warming up and the number of matches won
16. a.



    b. The data points appear random, indicating no correlation.
    c. $r = -0.2135$
    $r^2 = 0.0456$
    d. There is no relationship between the house number and the age of the household.
    e. 95.44%

17. a. Explanatory variable = time spent
Response variable = mark

b.


c. Strong positive linear correlation

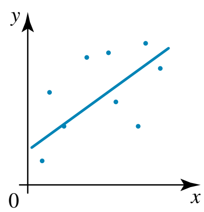
d. Each person's understanding of the topic is different and their study habits are unique. Therefore 1 hour spent on the assignment does not guarantee a consistent result. Individual factors will also influence the assignment mark.

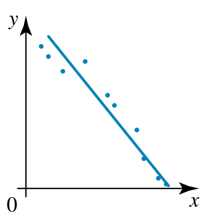
e. $r = 0.952$
$r^2 = 0.906$

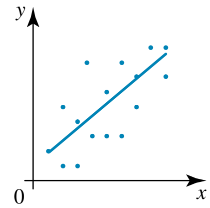
f. There is a strong relationship between the time spent on an assignment and the resulting grade. As the time spent increased, so did the mark.

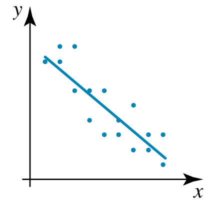
18. Various answers are possible. An example data set would be:*

Exercise 14.3 Further correlation coefficients

1.


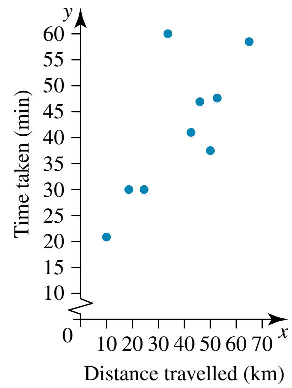
2.


3. a.


b.


4. a. Weak positive correlation
b. Moderate negative correlation

5. a. Explanatory variable = number of siblings
b. −10.82
c. 69.26
d. $y = 69.26 - 10.82x$

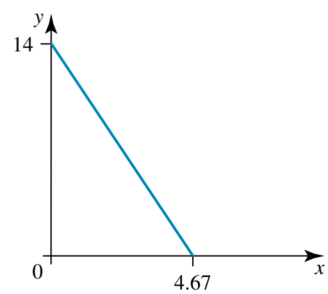
6. a.


b. i. 37.89 ii. 41.44 iii. 17.73 iv. 13.15 v. 0.749
c. 0.5557
d. 20.40
e. $y = 20.40 + 0.5557x$

7. $a = -8.795$, $b = -0.0999$

8. a. i. 11 ii. 30.7 iii. 6.06 iv. 15.3 v. −0.967
b. −2.44
c. 57.5
d. $y = 57.5 - 2.44x$

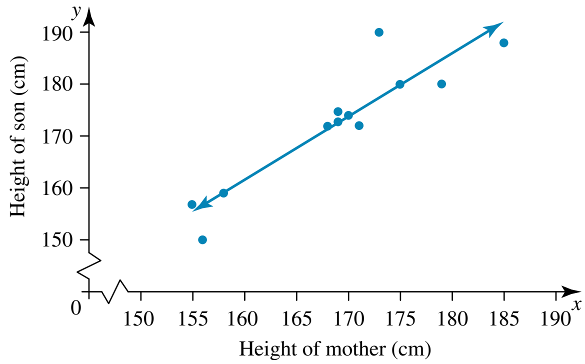
9. The equation for the graph becomes $y = 62.4 - 2.78x$. The y-intercept is higher and the slope is slightly steeper.

10.


18. *

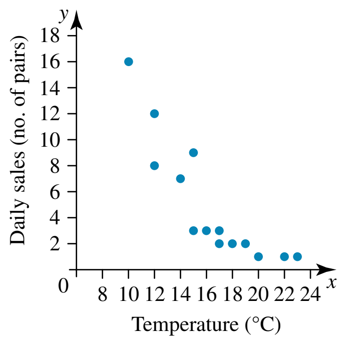
x	1	2	2	3	4	5	5	6	7	7	8	9	10	12	14
y	30	25	18	11	18	17	10	11	16	8	6	4	12	10	7

11. a. Response variable = height of son

b. 

c. $S = -33.49 + 1.219M$

12. As the b value is negative, the trend is negative. The y-intercept is 3.2, therefore when $x = 0$, $y = 3.2$.

13. a. 

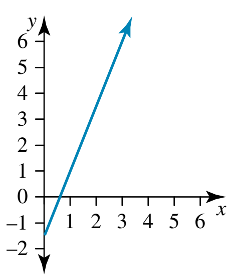
b. $D = 22.50 - 1.071T$

c. $r = -0.8621$

$r^2 = 0.7432$

d. There is a moderate negative relationship between the number of gumboots sold and the temperature. The data indicates that 74% of the sales are due to the temperature; therefore 26% of sales are due to other factors.

14. $y = 4 + 2.5x$

15. a. 

b. Below the regression line

16. Various answers are possible.
An example data set would be:

x	1	2	2	3	4	5	5	6	7	7	8	9	10	12	14
y	2	5	6	8	9	7	12	15	16	22	25	29	32	33	35

Exercise 14.4 Making predictions

1. a. −1.837 b. 1.701 c. Positive trend

2. a. 105.9 b. −1.476 c. Negative trend

3. a. 60 b. −5 c. Negative
d. −140

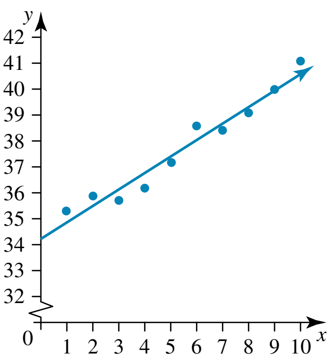
4. a. 3.90
b. Lucy incorrectly transposed the 12.9. She should have moved this first before dividing by 7.32.

5. a. −12 b. 25 c. Positive d. 75 5

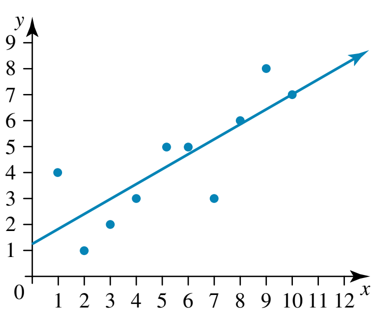
6. a. Amount of medication b. 2.029 mL c. 7.5 months old

7. a. Cost per night b. $155.74 c. 3.31 km

8. a. Number of insects caught b. 1.1 c. Positive
d. 66

9. a. 

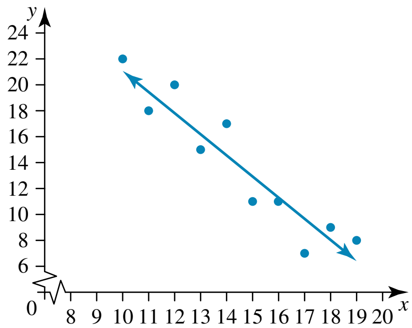
b. $y = 34.23 + 0.641x$. When $x = 15$, $y = 43.85$.

10. a. 

b. $y = 1.2 + 0.58x$
c. 12.8
d. 13.45

11. a. 76
b. 65
c. Part **a** looks at data within the original data set range, while part **b** predicts data outside of the original data set range of 0–125 new customers each hour.

12. a.



b. $y = 37.70 - 1.648x$

c. -0.204

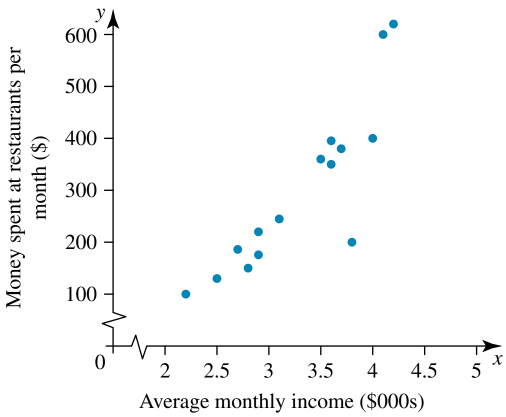
d. It is assumed the data will continue to behave in the same manner as the data originally supplied.

13. a. 130

b. 3.6 °C

c. The location of the fire, air temperature, proximity to water, etc.

14. a.



b. $R = -459.8 + 229.5I$

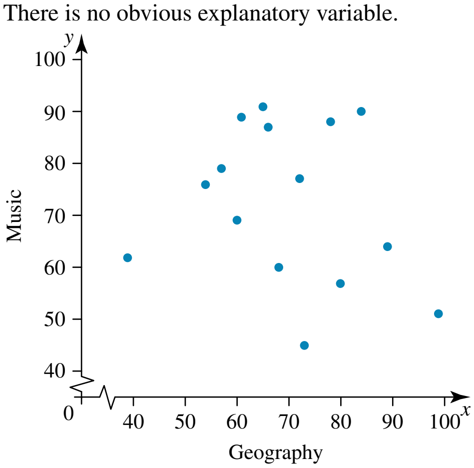
c. $687.70

d. Part **c** asks you to predict data outside of the original data set range.

e. $3160, interpolation

15. a. There is no obvious explanatory variable.

b.



c. $G = 87.63 - 0.2195M$

d. 12

e. Not very confident. The graph does not indicate a strong correlation between the two variables.

f. $r = -0.2172$. This indicates very weak correlation between the data, which supports the view that conclusions cannot be drawn from this data.

16. a. Calories burned

b. Calories burned = 14 301 + 115.02 × distance walked

c. 20 052. Interpolation, as this data is inside the original range.

d. 15 451.2. Extrapolation, as the explanatory variable provided is outside the original data range.

e. An r value of 0.9678 indicates a very strong positive linear relationship, showing that the relationship between the two variables is very strong and can be used to draw conclusions.

f. Examples: speed of walking, difficulty of walking surface, foods eaten.

14.5 Review: exam practice

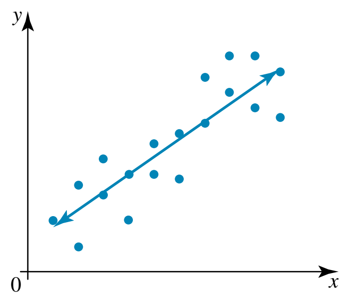
Multiple choice

1. B	**2.** A	**3.** B	**4.** C	**5.** C
6. B	**7.** C	**8.** D	**9.** B	**10.** D

Short answer

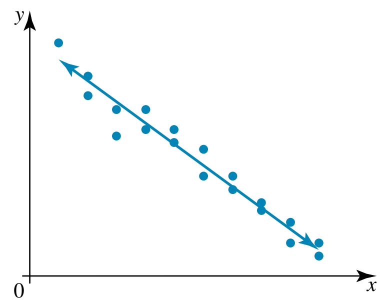
1. a. Moderate positive correlation **b.** No correlation **c.** Strong negative correlation

2. a.



b. No line of best fit possible

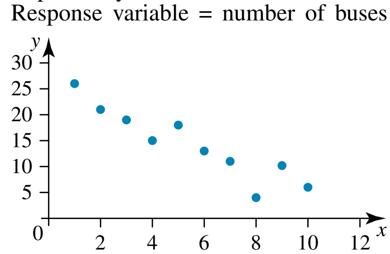
c.



3. a. Explanatory variable = quantity of water
Response variable = time

b. Explanatory variable = number of students
Response variable = number of buses required

4. a.



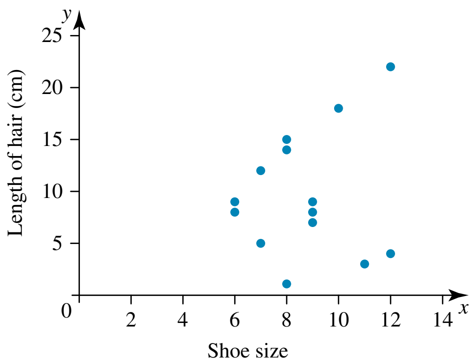
b. Strong negative correlation

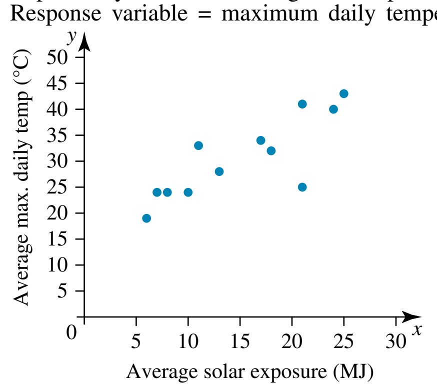
c. $r = 0.9329$; $r^2 = 0.8703$

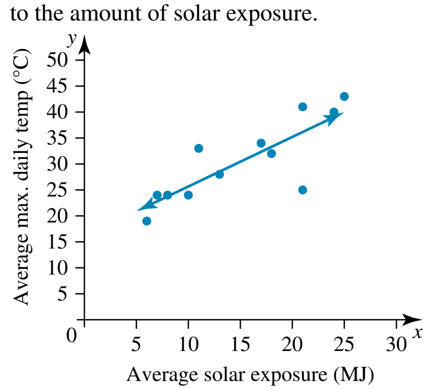
d. Pearson's product-moment correlation coefficient and the coefficient of determination confirm a strong relationship between the two variables.

5. a. 0.5721
   b. There is a moderate negative relationship between the variables. The coefficient of determination suggests that 57% of the variation in the y-variable is due to changes in the x-variable, and 43% is due to other factors.
6. Although there appears to be a link between the laziness of people and the increase in sales of healthy foods, there are also many other possible factors besides laziness. Based on this observation alone, the cause of an increase in sales of healthy foods cannot be concluded to be due to laziness.

Extended response

1. a. 

   b. $y = 4.0386 + 0.6157x$
   c. $r = 0.6497$; $r^2 = 0.4221$
   d. There is a weak link between the two variables. Therefore, no solid conclusions can be made to suggest a change in a person's shoe size will affect the length of their hair.
2. a. Average fuel consumption
   b. Average fuel consumption = 0.6968 + 0.1119 × fuel tank capacity
   c. 5.17
   d. Extrapolation, as the x-value is outside of the original data range.
   e. 85 L
   f. This value indicates a moderate relationship between the variables. Therefore, the data can be used, but other factors should also be considered.
   g. Various answers are possible, e.g. the manner in which a person drives the vehicle, weather conditions and road conditions.
3. a. Weight (grams)
   b. No correlation
   c. This supports the view that there is no correlation between the variables. Based on this value, no conclusions can be made from the data.
   d. Various answers are possible, e.g. popularity of the shoe or desired profits.
4. a. Explanatory variable = average solar exposure
   Response variable = maximum daily temperature
   b. 

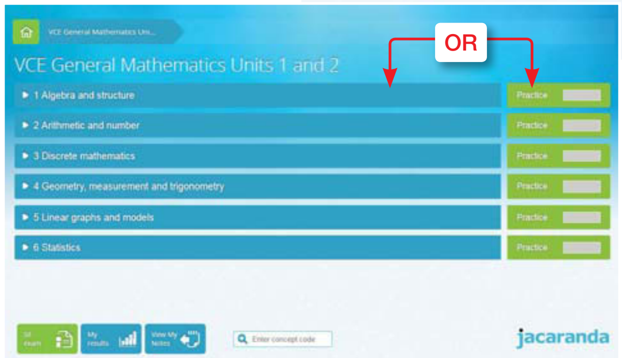
   c. Strong positive correlation
   d. $r = 0.8242$; $r^2 = 0.6793$
   These values indicate a strong relationship between the two variables. The coefficient of determination suggests that nearly 70% of the maximum daily temperature is due to the amount of solar exposure.
   e. 

   Maximum daily temperature = 16.232 + 0.9515 × average solar exposure
   f. 22 MJ
   g. 19 °C
   h. An x-value of 3 MJ is outside the original data set.

TOPICS 13 and 14

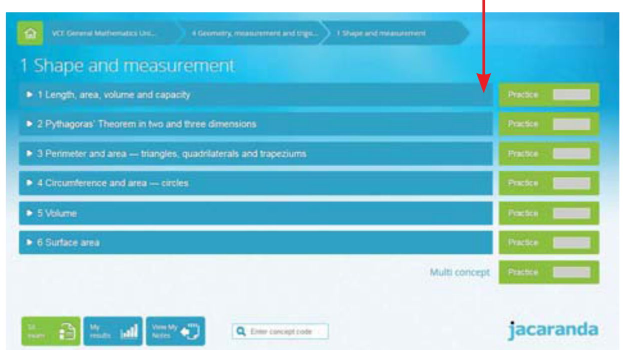
- For revision of this entire area of study, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *VCE General Mathematics Units 1 & 2* to see the entire course divided into areas of study.
- Select the **area of study** you are studying to navigate into the topic level **OR** select **Practice** to answer all practice questions available for each area of study.

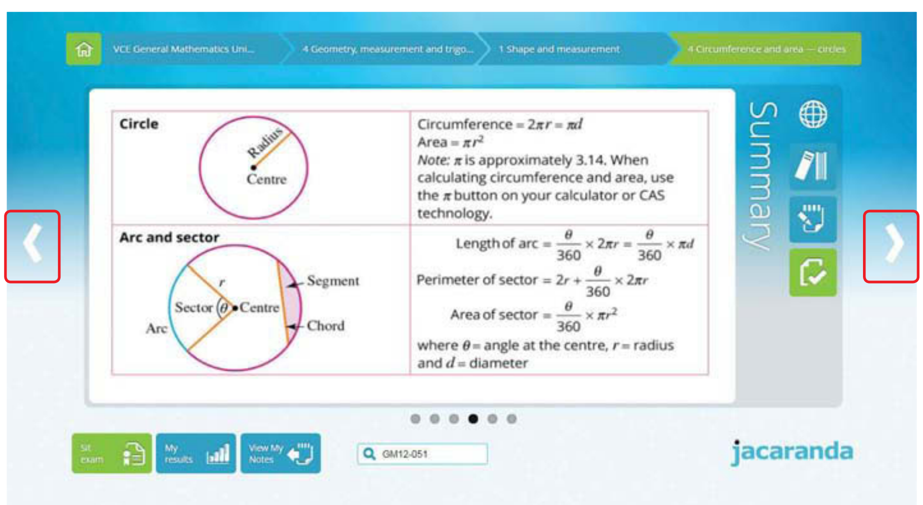


- Select **Practice** at the topic level to access all questions in the topic.

- At **topic level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts.
  - Select the **next arrow** to revise all concepts in the topic.
  - Select this icon to practise a more granular set of questions at the concept level.



TOPIC 15
Financial extension

15.1 Overview

Have you ever thought about how you could afford to buy a new car or a home? Have you ever wondered what superannuation is and how you might fund your retirement? How much will you need to have invested to live comfortably when you turn 70? This topic may help you understand some of these concepts. When we invest money with a financial institution, the institution is using our money to lend to others, so they pay us interest. However, when we borrow money from an institution, we are using their money so they charge us interest. The rates of interest charged, or given, may vary and can change at any time.

In this topic, we will be considering mainly reducing balance loans. Loans are given for specific purposes, such as to buy a home or a car. The financial institution loans the money and the borrower agrees to repay the money, plus interest, over a given period of time. In reducing balance loans, interest is charged, usually every month, on the outstanding balance and repayments are made by the borrower on a regular basis. To reduce the balance, these repayments need to be more than the interest charged for the same period of time. The outstanding balance will continually decrease with each repayment until it is paid out. Consequently, the interest charged each month will also decrease throughout the period of the loan.

The theory behind reducing balance loans can also be applied to other situations, such as scholarships and superannuation payments. These payments are known as annuities, when the amount of the investment is decreasing with each payment. Superannuation is probably the most common annuity known. In Australia, all employers must pay a superannuation contribution to a fund for each employee, set by the Australian Government at a minimum of 9.5% per annum. Employees may also contribute to these funds to increase their balance and hence their payments at retirement.

Not all superannuation or investment funds reduce their balance with payments. If only the interest is distributed or paid out, then the principal (the amount invested) will stay the same. These types of investments are called perpetuities, as the balance remains the same indefinitely. Certain long-term interest bonds are perpetuities. Trusts are often used to distribute the interest, with the capital held in perpetuity.

15.2 Reducing balance loans modelled using recurrence relations

15.2.1 Revisiting recurrence relations

Recurrence relations were introduced in Topic 6. We learned the following:

- A recurrence relation generates a sequence of numbers.
- A term of the sequence is dependent on the previous term of the sequence.
- The first (initial) term of the recurrence relation needs to be known to generate the sequence.
- A recurrence relation representing an arithmetic sequence will be of the form $t_{n+1} = t_n + d, t_1 = a$.
- A recurrence relation representing a geometric sequence will be of the form $t_{n+1} = rt_n$, $t_1 = a$.
- Practical situations can be modelled by recurrence relations. These include simple and compound interest, depreciation and growth and decay.
- The sequence may be a combination of an arithmetic or geometric sequence of the form $t_{n+1} = rt_n + d, t_1 = a$.

WORKED EXAMPLE 1

Determine the first six terms of the sequence represented by the recurrence relation $t_{n+1} = 3t_n + 2$, given that $t_1 = 4$.

THINK	WRITE
1. Identify what the recurrence relation means.	Each term is given by multiplying the previous term by 3 and adding 2 to the result.
2. State t_1.	$t_1 = 4$
3. Use the recurrence relation to determine t_2.	$t_2 = 3t_1 + 2$ $= 3 \times 4 + 2$ $= 14$
4. Use the recurrence relation to determine t_3.	$t_3 = 3t_2 + 2$ $= 3 \times 14 + 2$ $= 44$
5. Use the recurrence relation to determine t_4.	$t_4 = 3t_3 + 2$ $= 3 \times 44 + 2$ $= 134$
6. Use the recurrence relation to determine t_5.	$t_5 = 3t_4 + 2$ $= 3 \times 134 + 2$ $= 404$
7. Use the recurrence relation to determine t_6.	$t_6 = 3t_5 + 2$ $= 3 \times 404 + 2$ $= 1214$
8. Write the answer.	The first six terms of the sequence are 4, 14, 44, 134, 404 and 1214.

Using a calculator

1. On a calculator page, type 4.
2. Press ENTER/EXE.
3. Type $\times 3 + 2$.
4. Press ENTER/EXE 5 times (for the next 5 terms of the sequence).

4	4
$4 \times 3 + 2$	14
$14 \times 3 + 2$	44
$44 \times 3 + 2$	134
$134 \times 3 + 2$	404
$404 \times 3 + 2$	1214

5. Write the answer.

The first six terms of the sequence are 4, 14, 44, 134, 404 and 1214.

WORKED EXAMPLE 2

A sequence is defined by the recurrence relation $u_{n+1} = 1.02u_n - 4$, where $u_1 = 10$. Show that the fourth term is negative.

THINK	WRITE
1. Identify what the recurrence relation means.	Each term is given by multiplying the previous term by 1.02 and subtracting 4 from the result.
2. State u_1.	$u_1 = 10$
3. Use the recurrence relation to determine u_2.	$u_2 = 1.02u_1 - 4$ $= 1.02 \times 10 - 4$ $= 6.2$
4. Use the recurrence relation to determine u_3.	$u_3 = 1.02u_2 - 4$ $= 1.02 \times 6.2 - 4$ $= 2.324$
5. Use the recurrence relation to determine u_4.	$u_4 = 1.02u_3 - 4$ $= 1.02 \times 2.324 - 4$ $= -1.629\,52$
6. Write the answer.	The fourth term is negative.

Using a calculator

1. On a calculator page, type 10.
2. press ENTER/EXE.
3. Type $\times 1.02 - 4$.
4. Press ENTER/EXE until the result is negative.

10	10
$10 \times 1.02 - 4$	6.2
$6.2 \times 1.02 - 4$	2.324
$2.324 \times 1.02 - 4$	−1.62952

5. Write the answer.

The fourth term is negative.

WORKED EXAMPLE 3

A company charges $150 for the first hour and then $90 for every hour, or part thereof, for general landscaping.

a. Write a recurrence relation to describe this situation.

b. Determine the cost of landscaping if it took:

i. 3 hours ii. 4 hours 20 minutes.

THINK		WRITE
a.	1. State the values of a and d.	$a = 150 \quad d = 90$
	2. Write the recurrence relation.	$t_{n+1} = t_n + 90, \quad t_1 = 150$
b. i	3 hours:	$t_1 = 150$
	1. For 2 hours, find t_2.	$t_2 = t_1 + 90$ $= 150 + 90$ $= 240$
	2. For 3 hours, find t_3.	$t_3 = t_2 + 90$ $= 240 + 90$ $= 330$
	3. Write the answer.	The cost of 3 hours of landscaping is $330.
b. ii	4 hours and 20 minutes:	
	1. For 4 hours, find t_4.	$t_4 = t_3 + 90$ $= 330 + 90$ $= 420$
	2. For 5 hours, find t_5.	$t_5 = t_4 + 90$ $= 420 + 90$ $= 510$
	3. Write the answer.	The cost for 4 hours 20 minutes will be charged as 5 hours, a total of $510.

An easier method, using a calculator

b. ii 4 hours and 20 minutes: charge 5 hours

1. On a calculator page, type 150.
2. Press ENTER/EXE.
3. Type + 90.
4. Press ENTER/EXE 4 times (for the next 4 hours after the initial hour).
5. Write the answer.

150	150
150 + 90	240
240 + 90	330
330 + 90	420
420 + 90	510

The cost for 4 hours 20 minutes will be charged as 5 hours, a total of \$510.

WORKED EXAMPLE 4

The population in a small regional town is growing at a constant rate of 2.5% per year. The population at the beginning of 2019 is 24 350 people.

a. Write a recurrence relation to describe this situation.

b. Determine the expected population of the town at the end of 2022.

THINK	WRITE
a. 1. State the values of a and r.	$a = 24\,350$ $r = (100 + 2.5)\,\% = 1.025$
2. Write the recurrence relation.	$t_{n+1} = 1.025\ t_n,\ t_1 = 24\,350$
b. 1. Determine the term for the end of 2022.	End of 2022 would be beginning of 2023, t_5.
2. Beginning of 2020:	$t_2 = 1.025\ t_1$ $= 1.025 \times 24\,350 = 24\,958.8$
3. Beginning of 2021:	$t_3 = 1.025\ t_2$ $= 1.025 \times 24\,958.8 = 25\,582.7$
4. Beginning of 2022:	$t_4 = 1.025\ t_3$ $= 1.025 \times 25\,582.7 = 26\,222.3$
5. Beginning of 2023: (end of 2022)	$t_5 = 1.025\ t_4$ $= 1.025 \times 26\,222.3 = 26\,877.8$
6. Write the answer.	The population at the end of 2022 is 26 878 people (to the nearest whole number).

An easier method, using a calculator

b. 1. On a calculator page, type 24 350.

2. Press ENTER/EXE.

3. Type × 1.025.

4. Press ENTER/EXE 4 times (for the 4 years).

24 350	24 350
24 350 × 1.025	24958.8
24958.8 × 1.025	25582.7
25582.7 × 1.025	26222.3
26222.3 × 1.025	26877.8

5. Write the answer.

The population at the end of 2022 is 26 878 people (to the nearest whole number).

15.2.2 Modelling reducing balance loans with a recurrence relation

When we borrow money from a financial institution, such as a bank, we are using the institution's money, so it charges us interest on the outstanding balance. We need to repay both the amount borrowed, the principal, as well as the interest charged by the institution. This will be achieved by regular repayments over the period of the loan.

In reducing balance loans:

- interest is charged by the institution per compounding period, normally monthly
- repayments are made by the borrower on a regular basis
- repayments need to be more than the interest for the same period to reduce the amount still owing
- interest is charged on the outstanding balance before the repayment is made
- a recurrence relation can be used to model a reducing balance loan.

$$V_{n+1} = R\,V_n - d, \quad V_0 = a$$

where:
V_0 = the initial amount borrowed (the principal)
V_n = the balance of the loan after n payments
$R = 1 + \frac{r}{100}$, where r is the interest rate per compounding period
d = the payment made per compounding period

WORKED EXAMPLE 5

Heidi takes out a loan for \$3500 to pay for her business course. The interest rate charged by the financial institution is 9.6% per annum, compounded monthly. She has agreed to make monthly payments of \$325.

a. Write a recurrence relation to describe this situation.
b. Calculate the outstanding balance on the loan after Heidi has made:
i. 3 payments ii. 6 payments.
c. Use your calculator to determine the number of monthly repayments needed to pay off the loan.

THINK		WRITE
a	1. State the values of V_0 and d.	$V_0 = 3500 \quad d = 325$
	2. Calculate the value of r and R.	$r = \frac{9.6}{12} = 0.8$ $R = 1 + \frac{0.8}{100} = 1.008$
	3. Write the general form of the recurrence relation and substitute.	$V_{n+1} = RV_n - d, \quad V_0 = a$ $V_{n+1} = 1.008V_n - 325, \quad V_0 = 3500$
b i	1. After 1 payment: V_1	$V_1 = 1.008 \times 3500 - 325$ $= 3203$
	2. After 2 payments: V_2	$V_2 = 1.008 \times 3203 - 225$ $= 2903.62$
	3. After 3 payments: V_3	$V_3 = 1.008 \times 2903.62 - 225$ $= 2601.85$

4. Write the answer.

The outstanding balance after 3 payments is \$2601.85.

b ii ***Using a calculator***

1. On a calculator page, type 3500.
2. Press ENTER/EXE.
3. Type $\times$ 1.008 − 325.
4. Press ENTER/EXE 6 times (for 6 payments).

3500	3500
3500 × 1.008 − 325	3203
3203 × 1.008 − 325	2901.62
2901.62 × 1.008 − 325	2601.85
2601.85 × 1.008 − 325	2297.67
2297.67 × 1.008 − 325	1991.05
1991.05 × 1.008 − 325	1681.98

5. Write the answer.

The outstanding balance after 6 payments is \$1681.98.

c ***Using a calculator***

1. On a calculator page, type 3500.
2. Press ENTER/EXE.
3. Type × 1.008 − 325.
4. Press ENTER/EXE until the answer is less than zero (the loan has been repaid and there is no outstanding balance).
5. Count the number of times you pressed ENTER/EXE (or the number of repayments).

3500	3500
3500 × 1.008 − 325	3203
3203 × 1.008 − 325	2901.62
2901.62 × 1.008 − 325	2601.85
2601.85 × 1.008 − 325	2297.67
2297.67 × 1.008 − 325	1991.05
1991.05 × 1.008 − 325	1681.98
1681.98 × 1.008 − 325	1370.43
1370.43 × 1.008 − 325	1056.40
1056.40 × 1.008 − 325	739.85
739.85 × 1.008 − 325	420.77
420.77 × 1.008 − 325	99.13
99.13 × 1.008 − 325	−225 07

6. Write the answer.

After 12 months the account balance is −225.07, so it would take 12 months to pay off the loan.
Note: The last payment would only be \$99.93 \$ (325 − 225.07) or \$99.13 × 1.008.

WORKED EXAMPLE 6

Henry purchased a car for \$10 000. He had saved \$7600 and needed to borrow the remainder. His financial institution would give him a loan at 8.4% p.a., compounded monthly. Henry has agreed to repay the loan with monthly instalments of \$410.

a. **How much did Henry borrow?**
b. **Write a recurrence relation to describe this situation.**
c. **Use your calculator to determine the number of monthly repayments needed to pay off the loan.**

THINK / **WRITE**

a **1.** Subtract savings from the purchase price.
$10 000 − $7600 = $2400

2. Write the answer.
Henry borrowed $2400.

b **1.** State the values of V_0 and d.
$V_0 = 2400 \quad d = 410$

2. Calculate the value of r and R.
$r = \frac{8.4}{12} = 0.7$

$R = 1 + \frac{0.7}{100} = 1.007$

3. Write the general form of the recurrence relation and substitute.
$V_{n+1} = RV_n - d, V_0 = a$
$V_{n+1} = 1.007V_n - 410, V_0 = 2400$

c ***Using a calculator***

1. On a calculator page, type 2400.

2. Press ENTER/EXE.

3. Type × 1.007 − 410.

4. Press ENTER/EXE until the answer is less than zero (the loan has been repaid and there is no outstanding balance).

5. Count the number of times you pressed ENTER/EXE (or the number of repayments).

2400	2400
2400 × 1.007 − 410	2006.80
2006.80 × 1.007 − 410	1610.85
1610.85 × 1.007 − 410	1212.12
1212.12 × 1.007 − 410	810.61
810.61 × 1.007 − 410	406.28
406.28 × 1.007 − 410	−0.87

6. Write the answer.
The loan will be repaid in 6 months.

Note: The final payment will be adjusted for the overpayment of 87 cents. The above calculations have been written correct to 2 decimal places, but kept accurate on the calculator.

15.2.3 Amortisation of a reducing balance loan using a table

- The process of paying off a loan by regular payments over a period of time is known as amortisation.
- The amortisation of a loan can be tracked on a step-by-step basis by following the payments made, the interest added and the reduction in the principal (the initial value of the loan).
- Remember, the repayments need to cover both the interest and part of the principal so that the principal is reduced in each period.
- A recurrence relation gives the last column of the amortisation table without showing any of the intermediate steps.

In an amortisation table for reducing balance loans:

- payments remain fixed
- interest is charged on the previous period's balance (normally months)
- principal reduction = payment − interest for the month
- balance of the loan = previous month′s balance − principal reduction.

Consider an amortisation table for **Worked example 6**.

- Payments: $410 per month
- Interest: 8.4% p.a. = 0.7% per month
- Principal: $2400

Payment number, n	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0				2400
1	410	0.007 × 2400 = 16.80	410 − 16.80 = 393.20	2400 − 393.20 = 2006.80
2	410	0.007 × 2006.80 = 14.05	410 − 14.05 = 395.95	2006.80 − 395.95 = 1610.85
3	410	0.007 × 1610.85 = 11.28	410 − 11.28 = 398.72	1610.85 − 398.72 = 1212.13
4	410	0.007 × 1212.13 = 8.48	410 − 8.48 = 401.52	1212.13 − 401.52 = 810.62
5	410	0.007 × 810.62 = 5.67	410 − 5.67 = 404.33	810.62 − 404.33 = 406.29
6	410	0.007 × 406.29 = 2.84	410 − 2.84 = 407.12	406.29 − 407.12 = −0.87

Note: Calculations may vary by a few cents with rounding to 2 decimal places.

WORKED EXAMPLE 7

Drew and Elise are renovating their kitchen. To complete the renovation, they borrow $4800 from a financial institution at 10.4% p.a., compounded quarterly, with quarterly payments of $1280. The incomplete amortisation table for this loan is below.

a. What is the interest rate per payment period?

b. Calculate the interest charged for the first quarter.

c. Determine the balance of the loan after two payments have been made.

d. What is the principal reduction after three payments have been made?

e. **Determine the amount of the final payment to ensure a balance of zero at the end of the loan period.**

f. **Calculate the total interest paid on this loan.**

Payment number, n	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0				4800.00
1	1280.00		1155.20	3644.80
2	1280.00	94.76	1185.24	
3	1280.00	63.95		1243.51
4	1280.00	32.33	1247.67	−4.16

THINK / **WRITE**

a. 1. State the annual interest rate.
10.4% per year

2. Calculate the quarterly interest rate.
$= \frac{10.4}{4}\%$ per quarter

$= 2.6\%$ per quarter

3. Write the answer.
The interest rate is 2.6% per quarter.

b. 1. Calculate 2.6% of \$4800.
$2.6\% \times 4800$

$= \frac{2.6}{100} \times 4800$

$= 124.80$

2. Write the answer.
Interest charged in the first quarter is \$124.80.

c. 1. Calculate the balance after two payments.
Previous balance − principal reduction

$= 3644.80 - 1185.24$

$= 2459.56$

2. Write the answer.
Balance owing after 2 payments is \$2459.56.

d. 1. Calculate principal reduction after three payments.
Payment − interest charged

$= 1280 - 63.95$

$= 1216.05$

2. Write the answer.
The principal is reduced by \$1216.05 after 3 payments.

e. 1. Last payment is an overpayment of \$4.16.
Adjusted last payment

$= 1280 - 4.16$

$= 1275.84$

2. Write the answer.
The last payment should be \$1275.84.

f. 1. Total interest paid is sum of the interest column.
$124.80 + 94.76 + 63.95 + 32.33 = 315.84$

2. Write the answer.
Total interest paid is \$315.84.

Exercise 15.2 Reducing balance loans modelled using recurrence relations

1. **WE1** Determine the first six terms of the sequence represented by the recurrence relation $t_{n+1} = 2t_n + 5$, given that $t_1 = 3$.
2. Determine the first five terms of the sequence defined by the recurrence relation $t_{n+1} = 4t_n + 3$, given that $t_1 = 2$.
3. **WE2** A sequence is defined by the recurrence relation $u_{n+1} = 1.2u_n - 3$, where $u_1 = 5$. Show that the fourth term is negative.
4. A sequence is defined by the recurrence relation $u_{n+1} = 3u_n - 10$, where $u_1 = 6$. Show that the sixth term is greater than 200.
5. Calculate the first five terms of the sequences defined by the following recurrence relations.
   a. $t_{n+1} = t_n + 5$, given that $t_1 = 5$
   b. $t_{n+1} = 5\,t_n$, given that $t_1 = 6$
   c. $t_{n+1} = 3\,t_n - 6$, given that $t_1 = 4$
   d. $u_{n+1} = u_n - 15$, where $u_1 = 50$
   e. $u_{n+1} = 2.5\,u_n$, where $u_1 = 8$
   f. $u_{n+1} = 1.5\,u_n - 25$, where $u_1 = 100$
6. **WE3** A company charges \$180 for the first hour and then \$120 for every hour, or part thereof, for general plumbing repairs.
   a. Write a recurrence relation to describe this situation.
   b. Determine the cost of plumbing repairs if it took:
      i. 2 hours
      ii. 6 hours 15 minutes.
7. An outdoor adventure company hires canoes. They charge \$50 for the first hour and then \$40 for every hour, or part thereof.
   a. Write a recurrence relation to describe this situation.
   b. Determine the cost of hiring a canoe for:
      i. 3 hours
      ii. 9 hours 45 minutes.

8. **WE4** The population in a small regional town is growing at a constant rate of 3.4% per year. The population at the beginning of 2019 is 34 680 people.
   a. Write a recurrence relation to describe this situation.
   b. Determine the expected population of the town at the end of 2025.
9. Scientists are monitoring a culture of bacteria in a laboratory. Initially, there were 20 bacteria present. Scientists estimate that the culture is growing at a constant rate of 15% per hour.
   a. Write a recurrence relation to describe this situation.
   b. Determine the expected number of bacteria in the culture *after* 3 hours. Give your answer to the nearest whole number.
   c. Find how long it would take for the culture of bacteria to triple in size.
10. **WE5** Stephen takes out a loan of \$4200. The interest rate charged by the financial institution is 10.5% per annum, compounded monthly. He has agreed to make monthly payments of \$380.
    a. Write a recurrence relation to describe this situation.
    b. Calculate the outstanding balance on the loan after Stephen has made:
       i. 2 payments
       ii. 5 payments.
    c. Use your calculator to determine the number of monthly repayments Stephen needed to make to pay off the loan.

11. Sallyanne took out a personal loan of $2600 for a new TV, with the loan having an interest rate of 15% p.a., compounded monthly. She has agreed to make regular monthly payments of $345.

a. Write a recurrence relation to describe this situation.
b. Calculate the outstanding balance on the loan after Sallyanne has made:
   i. 3 payments
   ii. 7 payments.
c. Use your calculator to determine the number of monthly repayments Sallyanne needed to make to completely repay the loan.

12. WE6 Hazel purchased her first car for $12 500. She had saved $7200 and needed to borrow the remainder. Her financial institution would give her a loan at 12.6% p.a., compounded monthly. Hazel has agreed to repay the loan with monthly instalments of $575.
a. How much did Hazel borrow?
b. Write a recurrence relation to describe this situation.
c. Use your calculator to determine the number of monthly repayments needed to pay off the loan.

13. Leo purchased his first car for $7500. His grandparents gave him $5000 for his birthday and he borrowed the remainder from a bank at an interest rate of 8.4% p.a., compounded monthly. He agreed to make monthly repayments of $300.
a. How much did Leo borrow?
b. Write a recurrence relation to describe this situation.
c. Use your calculator to determine the number of monthly repayments needed to pay off the loan.

14. WE7 Paul and Penny are moving into their first home. To pay for some furniture, they need to borrow $4500 from a financial institution at 11.8% p.a., compounded quarterly. They agree to make quarterly payments of $1200.
The incomplete amortisation table for this loan is below.
a. What is the interest rate per payment period?
b. Calculate the interest charged for the first quarter.
c. Determine the balance of the loan after two payments have been made.
d. What is the principal reduction after three payments have been made?
e. Determine the amount of the final payment to ensure a balance of zero at the end of the loan period.
f. Calculate the total interest paid on this loan.

Payment number, *n*	**Payment ($)**	**Interest ($)**	**Principal reduction ($)**	**Balance of loan ($)**
0				4500.00
1	1200.00		1067.25	3432.75
2	1200.00	101.27	1098.73	
3	1200.00	68.85		1202.87
4	1200.00	35.48	1164.52	38.35

15. Michael wants to purchase a new bike. He has taken out a $7600 loan at 9.6% p.a., compounded quarterly, with quarterly payments of $1375. The incomplete amortisation table for this loan is below.

a. What is the interest rate per payment period?
b. Calculate the interest charged for the first quarter.
c. Determine the balance of the loan after two payments have been made.
d. What is the principal reduction after three payments have been made?
e. Complete the last two rows of the table.
f. Michael's final payment needs to be adjusted to ensure a zero balance on the loan. How much is Michael's final payment?

Payment number, n	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0				7600.00
1	1375.00		1192.60	6407.40
2	1375.00	153.78	1121.22	
3	1375.00	124.47		3935.65
4	1375.00	94.46	1280.54	2655.11
5	1375.00			
6	1375.00			

16. Adam has taken out a loan of $25 000 for a small business venture. There is an interest rate of 15% per annum, compounded quarterly, with equal quarterly payments of $3250.
a. What is the interest rate per quarter?
b. Write a recurrence relation to describe this situation.
c. Calculate the outstanding balance on the loan after Adam has made 3 payments.
d. Calculate the outstanding balance after 18 months.

17. You are Adam's accountant (from question 16). Adam has asked you to prepare a table to show him how his quarterly payments are reducing the balance of his loan. Complete the amortisation table below to illustrate this for the first year of the loan.

Payment number, n	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0				25 000.00
1	3250.00			
2				
3				
4				

18. Jamie took out a personal loan of $3750 to replace her washing machine and dryer. The terms of the loan were 10 monthly payments of $390 with an interest rate of 9% p.a., compounded monthly.
  a. What is the interest rate per month?
  b. Write a recurrence relation to describe this situation.
  c. Calculate the outstanding balance on the loan after Jamie has made 2 payments.
  d. What would Jamie's final payment be to ensure a zero balance on the loan?

15.3 Reducing balance loan problems using the Finance Solver

In the previous subtopic, recurrence relations were used to solve reducing balance loan problems. However, it is very time consuming, particularly if the loan is to be repaid over a long period of time. For example, if a home loan is to be repaid with monthly payments over a 30-year period, this would mean 360 iterations of the recurrence relation. Instead, the Finance Solver application on your CAS calculator can be used to solve reducing balance loan problems.

The Finance Solver on your CAS calculator has a particular set of fields, similar to those in the following table.

N	The total **n**umber of repayment periods
I%	The **i**nterest rate per annum
PV	The **p**resent **v**alue of the loan – this is positive as the money is coming to you.
PMT (Pmt)	The **pay**me**nt** per period – this is negative if you are repaying a loan.
FV	The **f**uture **v**alue of the loan – positive, negative or zero
P/Y (PpY)	The number of **p**ayments **p**er **y**ear
C/Y (CpY)	The number of **c**ompounding periods **p**er **y**ear (nearly always the same as P/Y)

Remember:

- When entering values into the Finance Solver, pay particular attention to the sign of the value.
- Cash flowing towards you is considered to be positive.
  - If you take out a loan, the money is coming to you, so PV is positive.

- Cash flowing away from you is considered to be negative.
  - If you make a payment, the money is leaving you, so PMT is negative.
- If the FV is:
  - negative, then there is still money to be paid on the loan
  - zero, then the loan has been completely paid out
  - positive, then the borrower has overpaid the loan.
- Fill in all of the fields except the unknown field. Place your cursor in that field and press ENTER/EXE.
- To move between fields, use the TAB key.
- Total interest paid equals the total of the payments minus the principal repaid.

15.3.1 Finding the future value

The balance of a loan can be found at any point in time by completing all of the fields except FV, the future value of the loan. Remember the future value of the loan will be negative if money is still owed (i.e. the loan has not been repaid in full) and positive if the loan has been overpaid.

WORKED EXAMPLE 8

Noah has taken out a $45 000 loan to purchase equipment for his business. The conditions of the loan were that it was to be repaid over 10 years with interest charged at 7.5% p.a., compounded monthly. Noah has agreed to make monthly payments of $535. Find the amount still owing after 6 years. Give your answer to the nearest cent.

THINK

1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
   N = 72 payments (6 × 12 months)
   I% = 7.5 (annual interest rate)
   PV = 45 000 (amount borrowed)
   PMT = −535 (monthly repayment)
   FV = **??? unknown**
   P/Y = 12 (payments per year – monthly)
   C/Y = 12 (compounding periods per year)
2. TAB to place cursor in the FV field; press ENTER.
3. Write the answer.

WRITE

N	72
I%	7.5
PV	45 000
PMT	−535
FV	
P/Y	12
C/Y	12

FV = −22 015.63

The amount of the loan still owing after 6 years is $22 015.63.

15.3.2 Finding the payments

The payments required to repay a loan can be found at any point in time by completing all of the fields except PMT. When a loan is repaid, the future value will be zero. Remember, money is leaving so payments will be negative.

WORKED EXAMPLE 9

Olivia is considering taking out a 30-year loan of $350 000 to purchase a unit. The bank will lend her the money at 4.5% p.a., with interest compounded monthly.

a. What would Olivia need to pay each month if she is to repay the loan in the 30 years?

b. How much would Olivia pay in total over the 30 years?

c. How much interest will Olivia pay during this time?

THINK

a 1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
N = 360 payments (30 × 12 months)
I% = 4.5 (annual interest rate)
PV = 350 000 (amount borrowed)
PMT = **??? unknown**
FV = 0 (loan is repaid in full)
P/Y = 12 (payments per year – monthly)
C/Y = 12 (compounding periods per year)

WRITE

N	360
I%	4.5
PV	350 000
PMT	
FV	0
P/Y	12
C/Y	12

2. TAB to place cursor in the PMT field, then press ENTER.

PMT = −1773.398

3. Write the answer.

The amount of each monthly payment would be $1773.40.

b. 1. Total = payments × number of payments

$1773.40 × 360 = $638 424

2. Write the answer.

Olivia would have paid $638 424 over the 30 years of the loan.

c. 1. Total interest = total payments – principal

$638 424 – $350 000 = $288 424

2. Write the answer.

Olivia would have paid $288 424 in interest.

15.3.3 Finding the present value

The present value of a loan is the amount of money that can be borrowed under certain conditions. This is found by completing all of the fields except PV. Remember, the present value, money coming into the account, will be positive, but payments, money leaving the account, will be negative. The future value will be zero since the loan will be fully repaid over the given period of time.

WORKED EXAMPLE 10

William knows he is earning enough to repay a loan with payments of $1000 per month. The current interest rate for a car loan is 8.2% p.a., compounded monthly. He wishes to repay the loan fully in 5 years. How much can William borrow for a new car? Give your answer to the nearest dollar.

THINK

1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
N = 60 payments (5 × 12 months)
I% = 8.2 (annual interest rate)
PV = **??? unknown**
PMT = −1000 (monthly repayment)
FV = 0 (loan is repaid in full)
P/Y = 12 (payments per year)
C/Y = 12 (compounding periods per year)

WRITE

N	60
I%	8.2
PV	
PMT	−1000
FV	0
P/Y	12
C/Y	12

2. TAB to place cursor in the PV field, then press ENTER.

PV = 49 086.388 45

3. Write the answer.

The amount William could borrow is \$49 086 (to the nearest dollar).

15.3.4 Finding time

The time of the loan depends on the value of N. This is the number of payment periods. The periods may be in months, quarters or years. To convert the number of payment periods to years, divide by 12 if the periods are in months; divide by 4 if the periods are in quarters. Remember the present value will be positive as money is coming into the account and the payments negative as money is leaving the account.

WORKED EXAMPLE 11

A loan of \$24 000 is being repaid by quarterly instalments of \$2050. The interest rate charged is 9.4%.

a. **How long would it take to repay the loan in full?**

b. **What would the last payment need to be to ensure a zero balance at the end of this time? Give your answer to the nearest cent.**

THINK

a 1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
N = **??? unknown**
I% = 9.4 (annual interest rate)
PV = 24 000 (amount borrowed)
PMT = −2050 (monthly repayment)
FV = 0 (loan is repaid in full)
P/Y = 4 (payments per year – quarters)
C/Y = 4 (compounding periods per year – quarters)

WRITE

N	
I%	9.4
PV	24 000
PMT	−2050
FV	0
P/Y	4
C/Y	4

2. TAB to place cursor in the N field, then press ENTER.

N = 13.851 82

3. Write the answer.

It would take 14 quarters (3.5 years) to repay the loan.

b. 1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
N = 14 (number of payments)
I% = 9.4 (annual interest rate)
PV = 24 000 (amount borrowed)
PMT = −2050 (monthly repayment)
FV = **??? unknown**
P/Y = 4 (payments per year – quarters)
C/Y = 4 (compounding periods per year – quarters)

N	14
I%	9.4
PV	24 000
PMT	−2050
FV	
P/Y	4
C/Y	4

2. TAB to place cursor in the FV field, then press ENTER.

FV = 300.769 80
This is an overpayment of $300.77, since the value is positive.

3. Subtract overpayment from PMT to adjust the last payment.

Adjusted last payment
= $2050 − 300.77
= $1749.23

4. Write the answer.

The final payment is $1749.23.

15.3.5 Finding the interest rate

Interest rates are set by individual financial institutions and may vary according to the type of loan being considered. The interest rate is found by completing all of the fields except I. The value of I is the interest rate as a percent per annum, or per year.

WORKED EXAMPLE 12

Charlie and Chloe approached their banks to take out a personal loan of $15 000, to be repaid over a 4-year period. The conditions from Charlie's bank were payments of $370 per month, with interest charged monthly. Chloe's bank asked for payments of $1090 per quarter, with interest adjusted quarterly. Which person has the lower interest rate? State the difference.

THINK

Charlie:

1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
N 48 (payments: 4 × 12 months)
I% = **??? unknown**
PV = 15 000 (amount borrowed)
PMT = −370 (monthly repayment)
FV = 0 (loan is repaid in full)
P/Y = 12 (payments per year – months)
C/Y = 12 (compounding periods per year – months)

WRITE

N	48
I%	
PV	15 000
PMT	−370
FV	0
P/Y	12
C/Y	12

2 TAB to place cursor in the N field, then press ENTER.

I = 8.538 886 65

Chloe:

3. Using Finance Solver on your CAS calculator, fill the appropriate fields.
N 16 (payments: 4 × 4 quarters)
I% = **??? unknown**
PV = 15 000 (amount borrowed)
PMT = −1090 (quarterly repayment)
FV = 0 (loan is repaid in full)
P/Y = 4 (payments per year – quarters)
C/Y = 4 (compounding periods per year – quarters)

N	16
I%	
PV	15 000
PMT	−1090
FV	0
P/Y	14
C/Y	14

4. TAB to place cursor in the N field, then press ENTER.	I = 7.323 216
5. State the interest rates, correct to 2 decimal places.	Charlie: 8.54% p.a. Chloe: 7.32% p.a.
6. Write the answer.	Chloe has the lower interest rate by 1.22%.

Exercise 15.3 Reducing balance loan problems using the Finance Solver

1. **WE8** Zoe has taken out a $28 500 loan to purchase equipment for her business. The loan is to be repaid over 6 years with interest charged at 8.25% p.a., compounded monthly. Zoe has agreed to make monthly payments of $503. Find the amount still owing after 4 years. Give your answer to the nearest cent.
2. Jack wants to borrow $3750 to purchase a new sound system. He has arranged for a loan at 12.45% p.a., with interest adjusted monthly, to be repaid with monthly payments of $126 over a three-year period.

   Find the amount outstanding after 18 months. Give your answer to the nearest cent.

3. **WE9** Charlotte is considering taking out a 30-year loan of $475 000 to purchase a unit. The bank will lend her the money at 4.28% p.a., with interest compounded monthly.
   a. What would Charlotte need to pay each month if she is to repay the loan in 30 years?
   b. How much would Charlotte pay in total over the 30 years?
   c. How much interest will Charlotte pay during this time?
4. Thomas borrowed $75 800 to purchase a work vehicle. He agreed to repay the loan with monthly instalments for four years, with interest being charged at 9.8% p.a., compounding monthly.
   a. Calculate Thomas's monthly payments. Give your answer to the nearest cent.
   b. How much would Thomas pay in total over the 4 years?
   c. How much interest has Thomas paid during this time?

5. **WE10** Henry knows he is earning enough to repay a loan with payments of $4580 per month. Knowing interest rates may change, he allows for an interest rate of 6% p.a., compounded monthly. He wishes to repay the loan fully in 25 years. How much can Henry borrow? Give your answer to the nearest dollar.
6. Calculate the amount that can be borrowed if it is to be fully repaid in 18 months, with payments of $350 per month and an interest rate of 5.2% per annum, compounded monthly.
   Give your answer to the nearest hundred dollars.

7. **WE11** A loan of $32 500 is being repaid by quarterly instalments of $1910. The interest rate charged is 6.35% p.a., compounding quarterly.
   a. How long would it take to repay the loan in full?
   b. What would the last payment need to be to ensure a zero balance at the end of this time? Give your answer to the nearest cent.
8. Joshua agrees to repay a loan of $17 500, with quarterly instalments of $1535.75. The interest rate, compounding quarterly, is 7.75% per annum.
   a. How long would it take Joshua to repay the loan in full?
   b. What would his last payment need to be to ensure a zero balance at the end of this time? Give your answer to the nearest cent.
9. **WE12** Liam and Amelia approached their banks for a personal loan of $12 500, to be repaid over a 3-year period. The conditions from Liam's bank were payments of $388.55 per month, with interest charged monthly. Amelia's bank asked for payments of $1177.50 per quarter, with interest adjusted quarterly. Which person has the lower annual interest rate? State the difference.
10. You are comparing two deals for a personal loan of $34 000 for a new car. The reducing balance loan from A-bank requires monthly payments of $850 with the loan repaid in 4 years, with interest adjusted monthly. B-bank is advertising quarterly payments of $2551 for the same time, with interest adjusted quarterly. Which bank has the lower annual interest rate? State the difference.
11. Grace wanted to borrow $315 000 for a home loan and was offered a reducing balance loan over a period of 25 years at 4.9% p.a., with interest adjusted monthly.

   a. Calculate her monthly repayments. Give your answer to the nearest cent.
   b. What was the total Grace would repay for this loan?
   c. Find the total interest paid on the loan.
12. James borrowed $35 650 to purchase some new equipment for his business. He agreed to repay the loan over 8 years, with quarterly payments and interest charged quarterly at 9.8% p.a.
   a. Find the amount James needs to pay quarterly to fully repay the loan in 8 years. Give your answer to the nearest cent.
   b. After 5 years, James decides to pay out the loan. How much did James need to pay? Give your answer to the nearest cent.
13. Charlie and Matilda took out a reducing balance loan of $475 000 to purchase their first home. The loan is to be repaid with monthly instalments over a period of 30 years. The interest rate, compounded monthly, is 4.25% p.a.
   a. Find their monthly instalments, correct to the nearest cent.
   b. How much is still owing after 15 years?
   c. They now wish to repay the loan in full in the next 5 years. Find their new monthly instalments, correct to the nearest cent.
14. Aria has taken out a reducing balance loan of $38 500 to pay for a new kitchen in her unit. The loan is to be repaid over a period of 5 years with an interest rate, compounded per period of time, of 6.2% p.a.

   a. Find the payments, to the nearest cent, to fully repay the loan if the payments were made:
      i. quarterly ii. monthly
      iii. fortnightly (assume 26 fortnights per year).
   b. Which would be the better deal?

15. Stuart and Kate have taken out a home loan of \$395 000 to purchase their first home. They agreed to repay the loan in full in 25 years, with monthly payments. The interest rate initially was 3.95% p.a., compounded monthly.
   a. What are their monthly payments? Give your answer to the nearest cent.
   b. After 10 years, the interest rate was increased to 5.2% p.a. Find their new monthly payments if they still want to repay the loan in full in 25 years.

15.4 Annuities

An annuity is an investment account that at some future date will provide a regular stream of income for a set period of time. An annuity is similar to a reducing balance loan, but with the payment coming to you from an investment account you have already in place. Annuities are often scholarships, bursaries or retirement funds.

15.4.1 Modelling an annuity using a recurrence relation

A recurrence relation can be used to model an annuity. Since it is using the same concepts as those in the reducing balance loan problems, the recurrence relation is written in the same form.

$$V_{n+1} = RV_n - d, \quad V_0 = a$$

where:
V_0 = the initial amount invested (the principal)
V_n = the balance of the investment after n payments
$R = 1 + \frac{r}{100}$ (r is the interest rate per compounding period)
d = the payment made per compounding period

WORKED EXAMPLE 13

To study overseas for 3 months, Mia was awarded a scholarship of \$10 250. The scholarship was invested in an annuity that pays 3.96% p.a., compounded monthly. Mia is paid \$3440 per month from the annuity.
a. Write a recurrence relation to describe this situation.
b. Calculate how much money is left in the annuity at the end of 2 months.
c. What adjustment would need to be made to Mia's last payment to ensure zero balance in the investment account?

THINK	WRITE
a 1. State the values of V_0 and d.	$V_0 = 10\,250 \quad d = 3440$
2. Calculate the value of r and R.	$r = \frac{3.96}{12} = 0.33$ $R = 1 + \frac{0.33}{100} = 1.0033$

3. Write the general form of the recurrence relation and substitute.

$V_{n+1} = RV_n - d, \quad V_0 = a$
$V_{n+1} = 1.0033V_n - 3440, \quad V_0 = 10\,250$

b 1. After 1 payment: V_1

$V_1 = 1.0033 \times 10\,250 - 3440$
$= 6843.83$

2. After 2 payments: V_2

$V_2 = 1.0033 \times 6843.83 - 3440$
$= 3426.41$

3. Write the answer.

The balance of the annuity investment account after 2 payments is \$3426.41.

c 1. After 3 payments: V_3

$V_3 = 1.0033 \times 3426.41 - 3440$
$= -2.283\,23$

2. Interpret the result to give a zero balance.

Overdrawn by \$2.28

3. Write the answer.

Last payment will be \$3437.72. (\$3440 – \$2.28)

WORKED EXAMPLE 14

Peter was granted a sporting scholarship for a 6-month season overseas. The scholarship of \$18 500 has been invested in an annuity that pays 4.2% p.a. compounded monthly. Peter receives a monthly payment of \$3120 from the annuity.

a. **Write a recurrence relation to describe this situation.**
b. **Use a calculator to find the balance of the annuity account after:**
   i. **3 months** ii. **6 months.**
c. **What adjustment would need to be made to Peter's last payment to ensure a zero balance in the investment account?**

THINK / **WRITE**

a 1. State the values of V_0 and d.

$V_0 = 18\,500 \quad d = 3120$

2. Calculate the value of r and R.

$r = \frac{4.2}{12} = 0.35$

$R = 1 + \frac{0.35}{100} = 1.0035$

3. Write the general form of the recurrence relation and substitute.

$V_{n+1} = RV_n - d, \qquad V_0 = a$
$V_{n+1} = 1.0035V_n - 3120, \quad V_0 = 18\,500$

b ***Using a calculator***

1. On a calculator page, type 18 500.
2. Press ENTER/EXE.
3. Type × 1.0035 – 3120.
4. Press ENTER/EXE 6 times (for 6 payments).

18500	18500
18500 × 1.0035 – 3120	15444.75
15444.75 × 1.0035 – 3120	12378.81
12378.81 × 1.0035 – 3120	9302.13
9302.13 × 1.0035 – 3120	6214.69
6214.69 × 1.0035 – 3120	3116.44
3116.44 × 1.0035 – 3120	7.35

5. Write the answers: i. after 3 months ii. after 6 months.	i. The balance of the annuity after 3 months is \$9302.13. ii. The balance of the annuity after 6 months is \$7.35.
c 1. Interpret the balance after 6 months.	Balance = \$7.35 (remaining in account)
2. Adjust the last payment by adding balance.	Payment = \$3120 + \$7.35 = \$3127.35
3. Write the answer.	Peter's last payment will be \$3127.35.

15.4.2 Annuity problems and the Finance Solver

The Finance Solver application on your CAS calculator can be used to solve annuity problems. The fields are the same as used in the reducing balance loans but are interpreted differently.

N	The total **n**umber of repayment periods
I%	The **i**nterest rate per annum
PV	The **p**resent **v**alue of the investment – this is negative as money went into account.
PMT (Pmt)	The **pay**m**ent** per period – this is positive as money is coming to you.
FV	The **f**uture **v**alue of the investment – positive, negative or zero
P/Y (PpY)	The number of **p**ayments **p**er **y**ear
C/Y (CpY)	The number of **c**ompounding periods **p**er **y**ear (nearly always the same as P/Y)

Remember:

- When entering values into the Finance Solver, pay particular attention to the sign of the value.
- Cash flowing towards you is considered to be positive.
  - In an annuity, the money is coming to you as a payment, so PMT is positive.
- Cash flowing away from you is considered to be negative.
  - In an annuity, the money left you to go into an investment account, so PV is negative.
- If the FV is:
  - negative, the annuity has been overdrawn
  - zero, the annuity has been completely paid out
  - positive, money is still in the annuity account.
- Fill in all of the fields except the unknown field, place your cursor in that field and press ENTER/EXE.
- To move between fields, use the TAB key.

WORKED EXAMPLE 15

Leo has \$450 000 in his superannuation fund. He is about to retire and invests his funds in an annuity paying 5.6% p.a. compounded monthly.

a. How long will Leo's funds last if the monthly payment is \$4800? Give your answer in years and months, to the nearest month.

b. **If Leo wishes the annuity to last 20 years, how much should he receive each month? Give your answer to the nearest cent.**

THINK

a **1.** Using Finance Solver on your CAS calculator, fill the appropriate fields.
N = **??? unknown**
I% = 5.6 (annual interest rate)
PV = −450 000 (amount investment in annuity)
PMT = 4800 (monthly repayment)
FV = 0 (annuity balance of zero)
P/Y = 12 (payments per year – months)
C/Y = 12 (compounding periods per year – months)

WRITE

N	
I%	5.6
PV	−450 000
PMT	4800
FV	0
P/Y	12
C/Y	12

2. TAB to place cursor in the N field, then press ENTER.

N = 123.579

3. Write the answer.

The annuity would last for 124 months, or 10 years and 4 months (to the nearest month).

b **1.** Using Finance Solver on your CAS calculator, fill the appropriate fields.
N = 240 (20 years × 12 for months)
I% = 5.6 (annual interest rate)
PV = −450 000 (amount investment in annuity)
PMT= **??? unknown**
FV = 0 (annuity balance of zero)
P/Y = 12 (payments per year – months)
C/Y = 12 (compounding periods per year – months)

N	240
I%	5.6
PV	−450 000
PMT	
FV	0
P/Y	12
C/Y	12

2. TAB to place cursor in the PMT field, then press ENTER.

PMT = 3120.9638

3. Write the answer.

For the annuity to last 20 years, the monthly payment would be $3120.96 (to the nearest cent).

Exercise 15.4 Annuities

1. WE13 To study overseas for 4 months, Thomas was awarded a scholarship of $8500. The scholarship was invested in an annuity that pays 4.8% p.a., compounded monthly. Thomas is paid $2145 per month from the annuity.

a. Write a recurrence relation to describe this situation.
b. Calculate how much money is left in the annuity at the end of 3 months.
c. What adjustment would need to be made to Thomas's last payment to ensure zero balance in the annuity investment account?

2. Lisa is travelling overseas for 3 months after winning a bursary of $6500. She invested the bursary in an annuity that pays 3.24% p.a., compounded monthly. Lisa will be paid $2180 per month from the annuity.

a. Write a recurrence relation to describe this situation.
b. Calculate how much money is left in the annuity at the end of 2 months.
c. What adjustment would need to be made to Lisa's last payment to ensure zero balance in the investment account?
d. Complete the amortisation table below for this annuity.

Payment number, *n*	Payment ($)	Interest ($)	Annuity reduction ($)	Balance of annuity ($)
0				6500.00
1	2180.00			
2				
3				

3. WE14 Terry has saved for an eight-month overseas trip. To pay for this, Terry has invested his savings of $12 500 in an annuity that pays 3.6% p.a. compounded monthly, from which he will receive a monthly payment of $1580.
a. Write a recurrence relation to describe this situation.
b. Use a calculator to find the balance of the annuity account after:
i. 4 months
ii. 8 months.
c. What adjustment would need to be made to Terry's last payment to ensure a zero balance in the investment account?

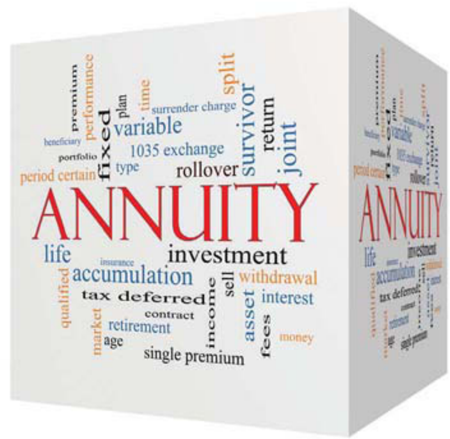
4. Melanie was granted a scholarship to study abroad for 9 months. The scholarship is for $15 250 and has been invested in an annuity that pays 4.5% p.a. compounded monthly. Melanie receives a monthly payment of $1725 from the annuity.

a. Write a recurrence relation to describe this situation.
b. Use a calculator to find the balance of the annuity account after 6 months.
c. What adjustment would need to be made to Melanie's last payment to ensure a zero balance in the investment account?

5. Anita has been granted a scholarship of $16 000 to assist with the expenses of attending a sporting institute for one year. The money has been invested in an annuity that pays 6.4% p.a., compounded quarterly. The scholarship pays Anita $4150 per quarter from the annuity.
a. Write a recurrence relation to describe this situation.
b. Use your calculator to determine how much money is left in the annuity at the end of the year.
c. What adjustment would need to be made to Anita's last payment to ensure a zero balance in the annuity?
d. Complete the amortisation table below for this annuity.

Payment number, *n*	Payment ($)	Interest ($)	Annuity reduction ($)	Balance of annuity ($)
0				16 000.00
1	4150.00			
2				
3				
4				

6. WE15 Sam has \$725 000 in his superannuation fund. He is about to retire and invests his funds in an annuity paying 5.25% p.a. compounded monthly.
   a. How long will Sam’s funds last if the monthly payment is \$5500? Give your answer in years and months, to the nearest month.
   b. If Sam wishes the annuity to last for 25 years, how much should he receive each month? Give your answer to the nearest cent.
7. Georgia has \$850 000 in her superannuation fund. She is about to retire and invests her funds in an annuity paying 5.4% p.a. compounded monthly.
   a. Georgia wishes the annuity to last for 30 years. How much should she receive each month? Give your answer to the nearest cent.
   b. How long will Georgia’s funds last if her monthly payment is \$5500? Give your answer in years and months, to the nearest month.



8. Patricia has invested her inheritance of \$236 000 in an annuity paying 7.3% p.a., compounded quarterly. Determine her quarterly payments if Patricia wishes the annuity to last for 15 years.
9. Stephen is investigating superannuation funds. He hopes to have enough in a fund that will give him a monthly payment of \$6200 for 25 years. Currently, his superannuation fund is paying an annuity at the rate of 6.2% p.a., compounded monthly. How much would Stephen need to have invested in his superannuation fund to achieve his goals? Give your answer to the nearest dollar.

10. A superannuation fund is advertising an annuity paying 7.32% p.a., compounded quarterly. Marion wished to have a quarterly payment of \$13 200 for 20 years. Determine how much Marion would need to have invested in this annuity to achieve her retirement goals. Give your answer to the nearest dollar.
11. James has invested his superannuation of \$560 000 in an annuity paying 8.45% p.a., compounded monthly. He wishes to receive a monthly payment of \$5000.
   a. How long would this annuity last? Give your answer to the nearest month.
   b. Determine the balance of the annuity after 5 years. Give your answer to the nearest dollar.
   c. After 5 years, James wishes to increase his monthly payments by \$1500 to cover some unexpected expenses. How many more years will the annuity now last? Give your answer to the nearest month.
12. Samantha has received a scholarship of \$120 000 to study for another two years. The scholarship has been invested in an annuity, paying 5.6% p.a. for the two years, with Samantha receiving regular payments.
   a. Calculate her payments, correct to the nearest cent, if she receives:
      i. quarterly payments, with interest compounded quarterly
      ii. monthly payments, with interest compounded monthly
      iii. weekly payments, with interest compounded weekly (assume 52 weeks/year).
   b. What payment period would give Samantha the best value from her scholarship?

13. Billy has just retired and invested his superannuation of \$468 000 into an annuity paying 6.48% p.a., compounded monthly.
   a. How much should Billy receive each month if he wishes the annuity to last 20 years? Give your answer to the nearest cent.
   b. After 6 years, Billy received an inheritance of \$75 000 and decided to add this to his annuity. How much would Billy now receive each month if he still wishes the annuity to last the same time?
14. Matilda has invested her life savings of \$256 000 in an annuity paying 6.73% p.a., compounded monthly. For the first 5 years, Matilda received a monthly payment of \$4000.
   Due to the economic climate, the annuity's interest rate was then changed to 5.95% p.a., compounded monthly. If Matilda continued to receive the same monthly payment, how long, in total, would this annuity last? Give your answer to the nearest month.
15. Paul and Mary have contributed to their superannuation fund all of their working lives. They have both now retired and their superannuation fund has a balance of \$1 645 250. They have invested these funds into an annuity paying 7.24% p.a., compounded monthly. For the first10 years, they wish to receive a monthly payment of \$10 500. After 10 years, the interest rate is increased by 1.25% p.a. If they had planned for this annuity to last 25 years in total, determine the new monthly payment, correct to the nearest cent.

15.5 Perpetuities

A perpetuity is a permanently invested sum of money that provides regular payments that continue forever.

- The funds will last indefinitely, provided the amount paid out is no more than the interest earned on the funds.
- If using the perpetuity formula, the period of the regular payment must be the same as the compounding period of the given interest rate.
- The balance of the amount invested does not change, so it will remain the same indefinitely.
- The type of investment that is used to earn the interest is often called a bond.
- Many scholarships or grants offered by universities or colleges are provided by funds invested as perpetuities.
- Perpetuities are often said to have long-term interest rates.

15.5.1 The perpetuity formula

The perpetuity formula is:

$$d = \frac{V_0 r}{100}$$

where:
V_0 = the principal, the amount invested in the perpetuity (\$)
r = the interest rate earned per period (%)
d = the amount of the regular payment per period (\$)

The perpetuity formula can be transposed to make V_0 or r the subject.

$$V_0 = \frac{100d}{r} \text{ and } r = \frac{100d}{V_0}$$

WORKED EXAMPLE 16

The charity has invested \$150 000 to set up a perpetuity to give grants to a local community centre. The funds have been invested in a bond that offers a long-term interest rate of 4.5% p.a. Calculate the amount of the annual grant to the local community centre if the interest is calculated yearly.

THINK	WRITE
1. Write the perpetuity formula.	$d = \dfrac{V_0 r}{100}$
2. State the values of V_0 and r.	$V_0 = 150\,000$ and $r = 4.5$
3. Substitute and calculate the value of d.	$d = \dfrac{150\,000 \times 4.5}{100}$ $= 6750$
4. Write the answer.	The annual grant to the community centre will be \$6750.

WORKED EXAMPLE 17

A family trust wishes to provide a scholarship of \$20 000 per year, in perpetuity, for a talented music student to attend a college. The trust has secured a bond at an interest rate of 8% p.a. How much needs to be invested if the payments and the interest were paid:

a. annually

b. quarterly?

THINK	WRITE
a 1. Write the perpetuity formula to find V_0.	$V_0 = \dfrac{100d}{r}$
2. State the values of d and r.	$d = 20\,000 \quad r = 8$
3. Substitute and calculate the value of V_0.	$V_0 = \dfrac{100 \times 20\,000}{8}$ $= 250\,000$
4. Write the answer.	The amount invested in the perpetuity would be \$250 000.
b 1. Write the perpetuity formula to find V_0.	$V_0 = \dfrac{100\,d}{r}$
2. State the values of d and r. *Remember they need to be same time period to use the formula.*	$d = 5000 \quad r = \dfrac{8}{4} = 2$
3. Substitute and calculate the value of V_0.	$V_0 = \dfrac{100 \times 5000}{2}$ $= 250\,000$
4. Write the answer.	The amount invested in the perpetuity would be \$250 000.

15.5.2 Perpetuity problems and the Finance Solver

The Finance Solver application on your CAS calculator can be used to solve perpetuity problems.

The fields are the same as used in annuities. However, the principal that is invested, PV, needs to be known for the Finance Solver to be used.

N	1 – it can take any value as the balance never changes.
I%	The **i**nterest rate per annum
PV	The **p**resent **v**alue of the investment – this is negative as money went into account.
PMT (Pmt)	The **pay**m**ent** per period – this is positive as money is coming to you.
FV	The **f**uture **v**alue of the investment – positive and equal to PV
P/Y (PpY)	The number of **p**ayments **p**er **y**ear
C/Y (CpY)	The number of **c**ompounding periods **p**er **y**ear

Remember:

- When entering values into the Finance Solver, pay particular attention to the sign of the value.
- The number of payments, N, can take any value since the balance never changes.
  - For simplicity, let N = 1.
- Cash flowing towards you is considered to be positive.
  - In a perpetuity, the money is coming to you as a payment, so PMT is positive.
- Cash flowing away from you is considered to be negative.
  - In a perpetuity, the money left you to goes into an investment account, so PV is negative.
- The amount invested never changes, so FV equals PV but with a positive sign.
- In perpetuities, the number of payments per year, P/Y, and the compounding periods per year, C/Y, are not necessarily the same.
- Fill in all of the fields except the unknown field, place your cursor in that field and press ENTER/EXE.
- To move between fields, use the TAB key.

WORKED EXAMPLE 18

A charity has invested $175 000 to set up a perpetuity to give grants to a local sporting team. The funds have been invested in a bond that offers a long-term interest rate of 5.25% p.a. Calculate the amount of the annual grant to the local sporting team if the interest is calculated:

i yearly **ii monthly** **iii daily (assuming 365 days).**

THINK

i. Interest compounded yearly

1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
   N = 1
   I% = 5.25 (annual interest rate)
   PV = −175 000 (amount investment in perpetuity)
   PMT = **??? unknown**
   FV = 175 000 (balance always the same)
   P/Y = 1 (payments per year)
   C/Y = 1 (compounding periods)

WRITE

N	1
I%	5.25
PV	−175 000
PMT	
FV	175 000
P/Y	1
C/Y	1

2. TAB to place cursor in the PMT field, then press ENTER.

PMT = 9187.50

3. Write the answer.

With interest compounded yearly, the annual payment would be $9187.50 in perpetuity.

ii. Interest compounded monthly

1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
N = 1
I% = 5.25 (annual interest rate)
PV = −175 000 (amount investment in perpetuity)
PMT = **??? unknown**
FV = 175 000 (balance always the same)
P/Y = 1 (payments per year)
C/Y = 12 (compounding periods – months)

N	1
I%	5.25
PV	−175 000
PMT	
FV	175 000
P/Y	1
C/Y	12

2. TAB to place cursor in the PMT field, then press ENTER.

PMT = 9411.8302

3. Write the answer.

With interest compounded monthly, the annual payment would be $9411.83 in perpetuity.

iii Interest compounded daily

1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
N = 1
I% = 5.25 (annual interest rate)
PV = −175 000 (amount investment in perpetuity)
PMT = **??? unknown**
FV = 175 000 (balance always the same)
P/Y = 1 (payments per year)
C/Y = 365 (compounding periods – days)

N	1
I%	5.25
PV	−175 000
PMT	
FV	175 000
P/Y	1
C/Y	365

2. TAB to place cursor in the PMT field, then press ENTER.

PMT = 9432.252 07

3. Write the answer.

With interest compounded daily, the annual payment would be $9432.25 in perpetuity.

WORKED EXAMPLE 19

The owner of a company wishes to set up an annual grant of $25 000 for a research centre. He has $450 000 to invest in a perpetuity bond. What interest rate would he need to find, with interest calculated daily, for this grant to be possible? Give your answer correct to 1 decimal place.

THINK

Interest compounded daily

1. Using Finance Solver on your CAS calculator, fill the appropriate fields.
   N = 1
   I% = **??? unknown**
   PV = −450 000 (amount investment in perpetuity)
   PMT = 25 000 (annual payment)
   FV = 450 000 (balance always the same)
   P/Y = 1 (payments per year)
   C/Y = 365 (compounding periods – days)
2. TAB to place cursor in the I field, then press ENTER.
3. Write the answer.

WRITE

N	1
I%	
PV	−475 000
PMT	25 000
FV	450 000
P/Y	1
C/Y	365

I = 5.407 122

The interest rate would need to be 5.4% p.a. in perpetuity.

Exercise 15.5 Perpetuities

1. **WE16** A charity has invested $75 000 to set up a perpetuity to give grants to a local language centre. The funds have been invested in a bond that offers a long-term interest rate of 6.25% p.a. Calculate the amount of the annual grant to the local language centre if the interest is calculated yearly.
2. A football club has invested $200 000 to set up a perpetuity to give grants to a local sporting club. The funds have been invested in a bond that offers a long-term interest rate of 5.82% p.a.

   a. Calculate the amount of the annual grant to the local sporting club if the interest is calculated yearly.
   b. How much would the sporting club receive in total in 10 years?
3. **WE17** A family trust wishes to provide a scholarship of $15 000 per year for a talented art student to attend a college. The trust has secured a bond with a long-term interest rate of 5% p.a. How much needs to be invested if the payments and the interest were paid:
   a. annually
   b. quarterly?
4. A benefactor wishes to provide a scholarship for an outstanding Year 11 student to attend a particular college. He wishes to provide $23 400 annually in four equal payments. The benefactor has secured a bond at an interest rate of 6% p.a., with interest compounded quarterly.
   a. How much is each payment?
   b. What is the principal that needs to be invested for the benefactor to meet his commitments?
5. **WE18** A university has invested $1 500 000 to set up a perpetuity to give grants to students in need. The funds have been invested in a bond that offers a long-term interest rate of 4.95% p.a. Calculate the amount of the annual grants available to students if the interest is calculated:
   i. yearly
   ii. monthly
   iii. daily (assuming 365 days).

6. Mr and Mrs Smith have set up an investment of \$350 000 in perpetuity to assist with their grandchildren's education. The funds have been invested in a bond that offers a long-term interest rate of 7.85% p.a. Calculate the amount, to the nearest dollar, that can be provided each year for their grandchildren if the interest is calculated:
   i. quarterly
   ii. monthly
   iii. daily (assuming 365 days).

7. **WE19** The Penny Company wishes to set up a biannual grant of \$12 500 for research. The company has \$625 000 to invest in a perpetuity bond. What interest rate would they need to find, with interest calculated daily, for this grant to be possible? Give your answer correct to 2 decimal places.
8. A benefactor has \$85 000 to set up a perpetuity as an annual grant to help refugees. The money has been invested in bonds that return 5.8% p.a. compounded monthly.
   a. Find the amount of the annual grant. Give your answer correct to the nearest cent.
   b. What interest rate would he need to find, with interest calculated daily, for the perpetuity to provide an annual grant of \$6750? Give your answer correct to 1 decimal place.
9. An Old Scholar wishes to set up a perpetuity as a scholarship fund for his old school, to be paid annually. He intends to invest \$185 000 in bonds that return 6.75% p.a.
   a. Find the amount of the scholarship fund, to the nearest dollar, if the interest is compounded monthly.
   b. What interest rate would be needed, with interest calculated daily, if he wished to increase the scholarship by \$1000 the next year? Give your answer correct to 2 decimal places.
10. The Farthing Company has set up a perpetuity for research and development. On average, they anticipate a return of 9.6% p.a., compounded quarterly, from their investment.
   a. How much does the Farthing Company need to invest in the perpetuity bond if they wish to receive \$48 000 annually, paid in equal quarterly payments?
   b. The Farthing Company has found a different perpetuity bond paying 9.6% p.a., but with interest compounded daily. Calculate the new quarterly payments.

11. Gemma wishes to give an annual grant of \$4600 to a deserving student each year to assist with the cost of their education. To achieve this, she has set up a fund in perpetuity, paying 6.25% p.a., compounded annually.
   a. How much did Gemma invest in the fund?
   b. Another fund is offering 6.22% p.a., compounded daily. Would Gemma be able to increase her annual grant if she changed to this fund?
12. A rugby club has been given half a million dollars to invest in perpetuity to encourage participation at the local level. The club has found a bond that offers a long-term interest rate of 5.8% p.a., compounded daily.
   a. How much can the rugby club give annually to the local clubs? Give your answer to the nearest dollar.
   b. If the rugby club continues to support the local clubs in this way for the next 15 years, how much would they have contributed to the local clubs?

13. A club is considering investing \$950 000 in a perpetuity to give grants to needy athletes. They have found two options, both paying 6.2% p.a. Plan A will give quarterly payments with interest calculated monthly. Plan B gives monthly payments with interest calculated daily. Over a 2-year period, which plan gives the club more funds to give as grants, and by how much?

14. An organisation wishes to support a charity with monthly donations of \$2750. They were able to secure a bond offering a long-term interest rate of 6.6% p.a., compounded monthly.
   a. What is the amount the organisation needs to invest to support the charity in this way?
   b. By how much could the organisation increase its monthly donations if the bond, for the same amount, offered an interest rate of 6.6% p.a., compounded daily?

15. Penelope, the accountant of a large institution, is in charge of various trusts that have been set up in perpetuity for specific projects. The details of three perpetuities are:
   Trust A: \$800 000 providing \$12 500 biannually, interest compounded monthly
   Trust B: \$800 000 providing \$2100 monthly, interest compounded monthly
   Trust C: \$800 000 providing \$980 per fortnight, interest compounded daily.
   Which trust is paying the better annual interest rate?

15.6 Review: exam practice

1. Calculate the first four terms of the sequence defined by the following recurrence relations.
   a. $t_1 = 8,\ t_{n+1} = 3t_n - 4$
   b. $t_1 = 4,\ t_{n+1} = 0.5t_n + 6$
2. Determine the fifth term of the sequence defined by the following recurrence relations.
   a. $u_1 = 6,\ u_{n+1} = 2u_n - 7$
   b. $u_1 = 40,\ u_{n+1} = 1.5u_n - 10$
3. From his part-time job, Tristan has saved for his holiday away. He had \$850 on his first day and spent \$56 on each of the following days.
   a. Write a recurrence relation to describe this situation.
   b. How much did Tristan have left after the seven days of his holiday?
4. Tiffany has just started in a new position on an annual salary of \$65 000. Her employer has promised her a 6% salary increase per year.
   a. Write a recurrence relation to describe this situation.
   b. What will Tiffany's annual salary be in her fourth year in this position? Give your answer to the nearest dollar.
5. Claire has borrowed \$16 000 to purchase a new car. The terms of the loan were monthly payments of \$340, and an interest rate of 9.6% p.a., charged monthly.
   a. Write a recurrence relation to describe this situation.
   b. Calculate the outstanding balance on the loan after Claire has made 3 payments.

6. Tom borrowed \$6500 to purchase more equipment for his online business. The financial institution charges interest, compounded monthly, at a rate of 12.24% p.a. Tom agrees to repay the loan with monthly payments of \$750.
   a. Write a recurrence relation to show the outstanding balance on the loan at any time.
   b. Calculate the outstanding balance after Tom has made four payments.
   c. Use your calculator to determine the number of monthly payments needed to reduce the loan to under \$100.
   d. Calculate the last payment needed to ensure a zero balance.

7. a. Complete the amortisation table below for a personal loan of $5000 from a financial institution charging 9.2% p.a., compounded quarterly, with quarterly payments of $1325.

Payment number, *n*	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0				5000.00
1	1325.00			
2	1325.00			
3	1325.00			
4	1325.00			

b. What would the last payment be to ensure a zero balance on the loan?

8. Zoe needed to purchase equipment to expand her business. She has taken out a loan of $76 000 from her bank, whose current interest rate is 7.35% p.a., compounded monthly.

a. Calculate Zoe's monthly payments if she wishes to repay the loan in 5 years. Give your answer to the nearest cent.

b. How much interest did Zoe pay during the 5 years?

9. Penelope earned a grant, worth $85 000, to continue her research for the next year. She has invested the grant in an annuity that pays 10.08% p.a., compounded monthly. Penelope will receive monthly payments of $7475.

a. Write a recurrence relation to illustrate the outstanding monthly balance of Penelope's annuity.

b. Calculate the balance of the annuity after 3 months. Give your answer correct to the nearest dollar.

10. A benefactor has set up a scholarship fund for his old college. The money, $550 000, has been placed in a long-term bond. Find the annual payment available to give the college if the bond paid 3.52% p.a., compounded annually.

11. A company wishes to establish a fund that will give a monthly grant of $8250 for a research and development project. They have found a fund that gives a long-term return of 10% p.a.
How much would the company need to invest in this fund for it to continue indefinitely?

12. A local bank is advertising home loans for 25 years at 3.95% p.a., compounded monthly. Jack and Jill are looking for their first home together and wish to take out a home loan of $480 000. What would Jack and Jill's monthly payment be from this local bank? Give your answer correct to the nearest cent.

13. Anita knows she can afford to repay $3500 per month on a home loan. If the expected interest rate is 4.2% p.a., compounded monthly, and she would like to repay the loan fully in 20 years, how much can Anita borrow? Give your answer to the nearest dollar.

14. Paul took out a reducing balance loan of $125 000 for an extension on his home. The loan is to be repaid with monthly instalments over a period of 15 years. The interest rate, compounded monthly, is 4.65% p.a.

a. Find Paul's monthly instalment, correct to the nearest cent.

b. How much is owing after 10 years?

c. If Paul wishes to repay the outstanding balance of the loan in the next 2 years, determine, to the nearest cent, his new monthly instalments.

15. Mabel has $657 500 in her superannuation fund. She will convert these funds to an annuity when she retires. The annuity from her fund pays 5.8% p.a. compounded monthly.
   a. How long will Mabel's funds last if she receives a monthly payment of $5250? Give your answer in years and months, to the nearest month.
   b. How much should her monthly payments be if Mabel wishes the annuity to last 20 years? Give your answer to the nearest dollar.
16. You approached your bank about taking out a personal loan of $23 400, repaid over 4 years, to purchase a car.

   a. Your bank asks for fortnightly payments of $275, with interest charged fortnightly. Calculate the bank's interest rate per annum, correct to 3 decimal places.
   b. Another bank offered the same rate of interest, but charged monthly. Calculate their monthly repayments.
   c. Considering the total repaid over the 4 years, which would be the better loan?
17. A sporting club has invested $250 000 in a perpetuity fund to give grants to promising athletes. The funds have been invested in a bond offering a long-term interest rate of 9.25% p.a. Calculate the amount, to the nearest dollar, the club can give as an annual grant if the interest is compounded:
   i. monthly ii. daily.
18. You are comparing the amount payable from various superannuation funds. You have $750 000 to invest. Plan A returns 7.8% p.a. compounded quarterly and Plan B returns 7.8% p.a. compounded daily. You would like both of these to still have a balance of $750 000 at any time. Which plan gives the better monthly payment and by how much?
19. Jacinta has $525 000 to invest.
   a. Calculate her monthly payment, to the nearest cent:
      i. if she invests in an annuity paying 5.32% p.a., compounded monthly, lasting 25 years
      ii. if she invests in a perpetuity paying 5.32% p.a., compounded daily.
   b. Determine the balance of the two different investments after 15 years.
20. Sally has a reducing balance home loan of $540 000. Interest is charged at 3.85% p.a. monthly for 30 years.

   a. Calculate Sally's monthly payment, correct to the nearest cent.
   b. After 5 years, the interest rate was increased by 0.5% p.a. Sally still wishes to pay off the loan within 30 years. Calculate her new monthly instalment, correct to the nearest cent.
   c. After another 10 years, Sally has received an inheritance of $150 000 which she wants to use to reduce the balance of her loan. She also wants to repay the loan in 20 years instead of the original 30 years. Calculate her new loan repayments, correct to the nearest dollar, after she received her inheritance.

Answers

Financial Extension

Exercise 15.2 – Reducing balance loans modelled using recurrence relations

1. 3, 11, 27, 59, 123 and 251
2. 2, 11, 47, 191 and 767
3. −2.28
4. 248
5. a. 5, 10, 15, 20 and 25.
   b. 6, 30, 150, 750 and 3750.
   c. 4, 6, 12, 30 and 84.
   d. 50, 35, 20, 5 and −10.
   e. 8, 20, 50, 125 and 312.5
   f. 100, 125, 162.5, 218.75 and 303.125
6. a. $t_{n+1} = t_n + 120,\ t_1 = 180$
   b. i. $300 ii. $900
7. a. $t_{n+1} = t_n + 40,\ t_1 = 50$
   b. i. $130 ii. $410
8. a. $t_{n+1} = 1.034t_n,\ t_1 = 34\,680$
   b. 43 825 people
9. a. $t_{n+1} = 1.15t_n,\ t_1 = 20$
   b. 30 bacteria
   c. 8 hours
10. a. $V_{n+1} = 1.00875\,V_n - 380,\ V_0 = 4200$
    b. i. $3510.50 ii. $2453.45
    c. 12 payments
    (Note: the last payment would be reduced by $123.12 to give a zero balance)
11. a. $V_{n+1} = 1.0125\,V_n - 345,\ V_0 = 2600$
    b. i. $1650.73 ii. $328.74
    c. 8 payments
    (Note: the last payment would be reduced by $12.15 to give a zero balance)
12. a. $5300
    b. $V_{n+1} = 1.0105\,V_n - 575,\ V_0 = 5300$
    c. 10 payments
    (Note: the last payment would be reduced by $145.89 to give a zero balance)
13. a. $2500
    b. $V_{n+1} = 1.007\,V_n - 300,\ V_0 = 2500$
    c. 9 payments
    (Note: the last payment would be reduced by $114.87 to give a zero balance)
14. a. 2.95% per quarter
    b. $132.75
    c. $2334.02
    d. $1131.15
    e. $1238.35
    f. $338.35
15. a. 2.4% per quarter
    b. $182.40
    c. $5186.18
    d. $1250.53
    e.

5	1375.00	63.72	1311.28	1343.83
6	1375.00	32.25	1342.75	1.08

    f. $1376.08
16. a. 3.75% per quarter
    b. $V_{n+1} = 1.0375\,V_n - 3250,\ V_0 = 25000$
    c. $17799.10
    d. $9757.32
17. See table at the foot of the page*
18. a. 0.75% per month
    b. $V_{n+1} = 1.0075\,V_n - 390,\ V_0 = 3750$
    c. $3 023.54
    d. $396.64

Exercise 15.3 – Reducing balance loan problems using the Finance Solver

1. $11 108.05
2. $2036.11
3. a. $2345.06
   b. $844 221.60
   c. $369 221.60
4. a. $1915.21
   b. $91 930.08
   c. $16 130.08
5. $710 847
6. $6000
7. a. The loan would be repaid in 20 quarterly payments, a total of 5 years.
   b. $1894.95
8. a. The loan would be repaid in 13 quarterly payments, a total of 3 years and 3 months.
   b. $1535.39
9. Liam has the lower interest rate of 7.45%, lower by 0.3%.
10. B-bank has the lower interest rate of 8.94%, lower by 0.3%.
11. a. $1823.15
    b. $546 945
    c. $231 945

17*

Paym. num, n	Paym. ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0				25 000.00
1	3250	$0.0375 \times 25\,000 = 937.50$	$3250 - 937.50 = 2312.5$	$25\,000 - 2312.5 = 22\,687.50$
2	3250	$0.0375 \times 22\,687.5 = 850.78$	$3250 - 850.78 = 2399.22$	$22\,687.5 - 2399.22 = 20\,288.28$
3	3250	$0.0375 \times 20\,288.28 = 760.81$	$3250 - 760.81 = 2489.19$	$20\,288.28 - 2489.19 = 17\,799.09$
4	3250	$0.0375 \times 17\,700.09 = 667.47$	$3250 - 667.47 = 2582.53$	$17\,799.09 - 2582.53 = 15\,216.56$

12. a. $1620.19 per quarter
b. $16 669.86

13. a. $2336.71
b. $310 619.44
c. $5755.64

14. a. i. $2253.53 ii. $747.90 iii. $344.78
b. The better deal is paying fortnightly (by a minimum of $52.60)

15. a. $2074.07 b. $2254.31

Exercise 15.4 – Annuities

1. a. $V_{n+1} = 1.004\,V_n - 2145,\ V_0 = 8500$
b. $2141.63
c. The annuity still has a balance of $5.20, so last payment will have to be increased by $5.20 to $2150.20.

2. a. $V_{n+1} = 1.0027\,V_n - 2180,\ V_0 = 6500$
b. $2169.26
c. The annuity has been overdrawn by $4.88, so last payment will have to be decreased by $4.88 to $2175.12
d. See table at the foot of the page*

3. a. $V_{n+1} = 1.003\,V_n - 1580,\ V_0 = 12\,500$
b. i. $6302.18 ii. $29.65
c. The annuity still has a balance of $29.65, so Terry's last payment will have to be increased by $29.65 to $1 609.65.

4. a. $V_{n+1} = 1.00375\,V_n - 1725,\ V_0 = 15250$
b. $5148.84
c. The annuity still has a balance of $12.55, so Melanie's last payment will have to be increased by $12.55 to $1737.55.

5. a. $V_{n+1} = 1.016\,V_n - 4150,\ V_0 = 16000.$
b. $46.17
c. The annuity still has a balance of $46.17, so the last payment will have to be increased by $46.17 to $4196.17.
d. See table at the foot of the page*

6. a. 16 years and 5 months
b. $4344.55

7. a. $4773.01
b. 22 years and 1 month

8. $6 504.65 per quarter

9. $944 284

10. $552 243

11. a. 18 years and 6 months
b. $481 442
c. 8 years and 9

12. a. $15 960.32
$5296.87
$1220.29
b. Quarterly

13. a. $3483.77 b. $4164.04

14. 6 years and 7 months

15. $15 201.59

Exercise 15.5 – Perpetuities

1. $4687.50

2. a. $11 640 b. $116 400

3. a. $300 000 b. $300 000

4. a. $5 850 b. $390 000

5. i. $74 250
ii. $75 957.93
iii. $76 113.10

6. i. $28 294 ii. $28 485 iii. $28 579

7. 3.96% p.a.

8. a. $5063.19 b. 7.6% p.a.

9. a. $12 881 b. 7.24% p.a.

10. a. $500 000
b. $12 143.54

11. a. $73 600
b. $4722.88, an increase of $122.88 from the original fund.

12. a. $29 855
b. $447 825

13. Plan A would give more funds for grants by $314.80.

14. a. $500 000
b. $2757.33, an increase of $7.33 per month.

15. Trust C

Exercise 15.6 – Review: exam practice

1. a. 8, 20, 56 and 164
b. 4, 8, 10 and 11

2. a. −9 b. 121.25

3. a. $t_{n+1} = t_n - 56,\quad t_1 = 850$
b. $514

2. d*

Payment number, n	Payment ($)	Interest ($)	Annuity reduction ($)	Balance of annuity ($)
0				6500.00
1	2180.00	$0.0027 \times 6500 = 17.55$	$2180 - 17.55 = 2162.45$	$6500 - 2162.45 = 4337.55$
2	2180.00	$0.0027 \times 4337.55 = 11.71$	$2180 - 11.71 = 2168.29$	$4337.55 - 2168.29 = 2169.26$
3	2180.00	$0.0027 \times 2169.26 = 5.86$	$2180 - 5.86 = 2174.14$	$2169.26 - 2174.14 = -4.88$

5. d*

Payment number, n	Payment ($)	Interest ($)	Annuity reduction ($)	Balance of annuity ($)
0				16 000.00
1	4150	$0.016 \times 16\,000 = 256$	$4150 - 256 = 3894$	$16\,000 - 3894 = 12\,106$
2	4150	$0.016 \times 12\,106 = 193.70$	$4150 - 193.70 = 3956.30$	$12106 - 3956.3 = 8149.70$
3	4150	$0.016 \times 8149.7 = 130.39$	$4150 - 130.39 = 4019.61$	$8149.70 - 4019.61 = 4130.09$
4	4150	$0.016 \times 4130.09 = 66.08$	$4150 - 66.08 = 4083.92$	$4130.09 - 4083.02 = 46.17$

4. a. $t_{n+1} = 1.06t_n, \ \ t_1 = 65\,000$
   b. $77 416
5. a. $V_{n+1} = 1.008\, V_n - 340, \ \ V_0 = 16\,000$
   b. $15 358.90
6. a. $V_{n+1} = 1.0102\, V_n - 750, \ \ V_0 = 6500$
   b. $3723.07
   c. Nine
   d. $839.58
7. a. See table at the foot of the page*
   b. $1315.45
8. a. $1517.47
   b. $15 048.20
9. a. $V_{n+1} = 1.0084\, V_n - 7475, \ \ V_0 = 85\,000$
   b. $64 546
10. $19 360
11. $990 000
12. $2520.38
13. $567 656
14. a. $965.85
   b. $51 618.96
   c. $2256.51
15. a. 16 years and 1
   b. $4635
16. a. 10.307% p.a.
   b. $596.94
   c. The fortnightly loan would be the better choice as you would pay $54.12 less over the 4 years.
17. i. $24 131
   ii. $24 225
18. Plan B is the better plan by an extra $46.70 per month.
19. a. i. $3167.77   ii. $2332.50
   b. $545 000
20. a. $2531.56
   b. $2666.83
   c. $3755

7. a*

Payment number, n	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0				5000.00
1	1325.00	$0.023 \times 5000 = 115$	$1325 - 115 = 1210$	$5000 - 1210 = 3790$
2	1325.00	$0.023 \times 3790 = 87.17$	$1325 - 87.17 = 1237.83$	$3790 - 1237.83 = 2552.17$
3	1325.00	$0.023 \times 2552.17 = 58.70$	$1325 - 58.70 = 1266.30$	$2552.17 - 1266.30 = 1285.87$
4	1325.00	$0.023 \times 1285.87 = 29.58$	$1325 - 29.58 = 1295.42$	$1285.87 - 1295.42 = -9.55$

GLOSSARY

adjacency matrix: a matrix that represents the number of connections between objects in a network or the number of edges that connect the vertices of a graph
angle of depression: the angle measured down from the horizontal line (through the observation point) to the line of vision
angle of elevation: the angle measured up from the horizontal line (through the observation point) to the line of vision
annulus: the area between two circles with the same centre. The formula is $A = \pi(R^2 - r^2)$, where R is the radius of the outer circle and r is the radius of the inner circle.
area: the two-dimensional space taken up by an object
area scale factor: the ratio of the corresponding areas of similar objects. It is equal to the linear scale factor raised to the power of two.
arithmetic sequence: a sequence in which the difference between any two successive terms is the same
back-to-back stem plot: a stem plot used to compare two different sets of data. Back-to-back stem plots share the same stem, with one data set appearing on the left of the stem and the other data set appearing on the right.
bar chart: a display with the categories of data on one axis (usually the horizontal axis) and the frequency of the data on the other (usually the vertical axis)
bearing: the direction of a fixed point, or the path of an object, from the point of observation
bivariate data: a comparison between two data variables to assess how much one may have influenced the other
boxplot: a graphical representation of the five-number summary
break-even point: the point at which revenue begins to exceed the cost of production
cardinal point: one of the four main directions or standard bearings: north (N), south (S), east (E) and west (W)
Cartesian plane: a region in which any point can be defined by specifying its x- and y-values as an ordered pair: (x, y)
categorical data: data that can be organised into groups or categories and is often an 'object', 'thing' or 'idea'. Examples include brand names, colours, general sizes and opinions.
causation: the measure of how much the change in one variable is caused by the other
circuit: a trail beginning and ending at the same vertex
circumference: the perimeter of a curved figure
coefficient of determination: the square of the product-moment correlation coefficient (r^2). This is a measure of how much a change in the response variable is influenced by a change in the explanatory variable.
column matrix: a matrix that has only one column
common difference: the difference between each term in an arithmetic sequence: $d = t_{n+1} - t_n$
common ratio: the ratio between two consecutive terms in a geometric sequence: $r = \frac{t_{n+1}}{t_n}$
complete graph: a graph in which there is an edge connecting each vertex to all other vertices in the graph
compound interest: interest calculated on the changing value throughout the time period of a loan or investment. Interest is added to the balance before the next interest calculation is made. The amount of the loan or investment can be calculated using $A = P\left(1 + \frac{r}{100}\right)^n$.
cone: a solid object in which one end is circular and the other end is a single vertex. Its cross-section is a series of circles that gradually get smaller as they approach the vertex.
congruent: objects that are exactly the same size and shape

connected graph: a graph with a path between each pair of vertices that makes it possible to reach every vertex of the graph by moving along the edges

constant of proportionality: or **constant of variation** denoted as k, a value that is equal to the ratio of y to x for any data pair. It is the rate at which y varies with x, or the gradient.

constraint: the set of linear inequalities that define a problem

consumer price index (CPI): a measure of changes, over time, in retail prices of a constant basket of goods and services representative of consumption expenditure by resident households in Australian metropolitan areas

continuous data: numerical data that can take any value that lies within an interval. Continuous data values are subject to the accuracy of the measuring device being used.

conventional (compass) bearing: a bearing that is measured first in terms of north and south, then in terms of east and west

corner point principle: the principle that the maximum or minimum value of an objective function must lie at one of the corners (vertices) of the feasible region

correlation: a measure of the strength of the linear relationship between two variables

cosine: the ratio of the side length adjacent to an internal angle of a right-angled triangle with the hypotenuse: $\cos(\theta) = \frac{A}{H}$

cosine rule: a rule that can enable unknown side lengths and angles of triangles to be determined if either two side lengths and an included angle, or all three side lengths are known: $a^2 = b^2 + c^2 = 2bc \times \cos(A)$ or $\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$

credit card: a method of purchasing whereby a financial institution loans an amount of money to an individual up to a pre-approved limit

cycle: a path beginning and ending at the same vertex

cylinder: a solid object with ends that are identical circles and a cross-section that is the same along its length

debit card: a method of purchasing in which the money is debited directly from a bank account or a pre-loaded amount

degree (of a vertex): the number of edges that are directly connected to a vertex

determinant: In the 2×2 matrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, the determinant ($\det A$ or $|A|$) is the difference between the product of the diagonal elements: $ad - bc$. If $\det A = 0$, the inverse matrix will not exist.

directed graph: a graph in which it is only possible to move along the edges of the graph in one direction

directed number: a number that has a designated direction (positive or negative) and a size

direct variation: when quantities are proportional to each other; for example, doubling one doubles the other

disconnected graph: a graph in which it is not possible to reach every vertex by moving along the edges

discrete data: numerical data that is counted in exact values, with the values often being whole numbers

dividend: portion of earnings distributed to shareholders

domain: the values for which a mathematical model is applicable. For an equation where y is expressed in terms of x, the domain is the set of all x-values of the ordered pairs (x, y) that make up the relation

dot plot: a plot in which every data value is represented by a dot, used to identify the most common values

edge: a line of a graph

effective rate of interest: the rate of interest that takes into account the reducing balance owing after each payment has been made: $R_{\text{ef}} = \frac{2400I}{P(m+1)}$

element: an entry in a matrix

elimination: the process of simplifying a mathematical expression by removing a variable; common when solving simultaneous equations

Euler circuit: a circuit in which every edge is used once

Euler trail: a trail in which every edge is used once

explanatory variable: the variable represented on the x-axis, used in a bivariate analysis to explain or predict a difference in the response variable

extrapolation: making a prediction from a line of best fit that appears outside the parameters of the original data set

face: distinct area created by the non-intersecting edges and vertices of planar graphs

feasible region: every required point that is a possible solution for the problem

Fibonacci sequence: a sequence in which all terms other than the first two are defined by the recurrence relation $F_{n+2} = F_n + F_{n+1}$, $F_1 = F_2 = 1$. After the first two terms, each subsequent term is the sum of the previous two. The Fibonacci sequence begins with two 1s, and every subsequent term of the sequence is found by adding the two previous terms, giving the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, . . .

frequency table: a display that tabulates data according to the frequencies of predetermined groupings

function: a relation in which ordered pairs of unique numbers can be identified by a given rule or equation

future value: the value of an asset at a time in the future, based on the original cost less depreciation

geometric sequence: a pattern of numbers whose consecutive terms increase or decrease in the same ratio. Each term after the first is found by multiplying the previous one by a fixed non-zero number called the common ratio.

Golden Ratio: the ratio $\phi = \dfrac{a+b}{a} = \dfrac{a}{b}$ obtained from a line segment $a + b$ where $a > b$. The value of ϕ is $\frac{1+\sqrt{5}}{2}$.

goods and services tax (GST): a tax that is charged on most purchases

gradient: also known as the slope; determines the change in the y-value for each change in x-value. This measures the steepness of a line as the ratio $m = \dfrac{\text{rise}}{\text{run}}$. If (x_1, y_1) and (x_2, y_2) are two points on the line, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

gradient–intercept form: the form of a linear equation expressed as $y = mx + c$, where m is the gradient and c is the y-intercept

graph: a visual representation of the connections and/or relationships that exist between things. In graph theory, a graph is a series of points and lines that are used to represent the connections that exist in various settings.

grouped data: numerical data that is arranged in groups to allow a clearer picture of the distribution and make it easier to work with

Hamiltonian cycle: a cycle that reaches all vertices of a network

Hamiltonian path: a path that reaches all vertices of a network

Heron's formula: a method of calculating the area of the triangle if you are given all three side lengths: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the side lengths and s is half the perimeter

histogram: a display of continuous numerical data similar to a bar chart, in which the width of each column represents a range of data values and the height of the column represents that range's frequency

horizontal asymptote: Asymptotes are lines a graph approaches but never reaches. A horizontal asymptote shows the long-term behaviour as, for example, $x \to \infty$.

hypotenuse: the longest side of a right-angled triangle. The hypotenuse is opposite the right angle.

identity matrix: a square matrix in which all of the elements on the diagonal line from the top left to bottom right are 1s and all of the other elements are 0s

inflation: a general increase in prices over time that effectively decreases the purchasing power of a currency. This is an application of compound interest.

integer: one of the set of positive and negative whole numbers and zero: $Z = \{\ldots\ -2, -1, 0, 1, 2, \ \ldots\}$

interpolation: making a prediction from a line of best fit that appears within the parameters of the original data set

interquartile range: the difference between the upper and lower quartiles of a data set

inverse matrix: The inverse of a square matrix is a matrix such that the resultant product matrix will be the identity matrix: $A \times A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

inverse variation: when increasing one variable decreases another

isomorphic graph: a graph that has the same number of vertices and edges, with corresponding vertices having identical degrees and connections

joint variation: variation involving multiples of more than one independent variable

least squares regression line: an accurate linear trend line that is calculated by minimising the vertical distances between the data points and the line of best fit. When taking the squares of these vertical distances, this line would represent the smallest possible sum of all of these squares.

linear equation: an equation in which the highest power of any variable is one

linear inequality: a linear function with an inequality sign

linearise: to create a more linear (straight line) form

linear programming: a method used to achieve the best outcome in a given situation in a mathematical model whose requirements are represented by linear relationships

linear relation: a relation between up to two variables of degree 1 that produces a straight line

linear scale factor: the ratio of the corresponding side lengths of similar objects

line of best fit: a straight line that best represents the general trend of the data in a scatterplot

literal equation: an equation that contains pronumerals rather than numerals as terms or coefficients

logarithm: the power to which a fixed number (the base) must be raised to produce a given value. The relationship between exponential and logarithmic functions is $y = A^x \Leftrightarrow \log_A(y) = x$.

logarithmic scale: a scale that uses the exponent (power) of a base number. In base 10 numbers, each unit increase or decrease represents a tenfold increase or decrease in the quantity being measured.

loop: an edge in a graph that joins a vertex to itself

lower fence: the lower boundary beyond which a data value is considered to be an outlier: $Q_1 - 1.5 \times \text{IQR}$

lower quartile (Q^1): the median of the lower half of an ordered data set

matrix: a rectangular array of rows and columns that is used to store and display information

mean: commonly referred to as the average; a measure of the centre of a set of data. The mean is calculated by dividing the sum of the data values by the number of data values.

measure of centre: a representative value for a data distribution located in the central region of the data: mean, median and mode

measure of spread: a measure that indicates how variable the data values are: range, interquartile range, variance and standard deviation

median: the middle value of the ordered data set if there are an odd number of values, or halfway between the two middle values if there are an even number of values

mode: the category or data value(s) with the highest frequency. It is the most frequently occurring value in a data set.

negatively skewed: distributions with higher frequencies on the right-hand side of the graph

net: a 2-dimensional plan of the surfaces that make up a 3-dimensional object

nominal data : categorical data that has no natural order or ranking

numerical data: data that can be counted or measured

objective function: a function of the variables in a linear programming problem

optimal solution: the maximum or minimum value of an objective function within the required feasible region

order (of a matrix): the number of rows, m, and columns, n, of a matrix expressed as $m \times n$

order of magnitude: an exponential change in a value. In base 10 numbers, changing the order of magnitude is the same as multiplying by 10.

order of operations: the correct sequence for performing the mathematical operations in an expression; often characterised by a mnemonic such as BODMAS (Brackets, Order, Division or Multiplication from left to right, Addition or Subtraction from left to right)

ordinal data: categorical data that can be placed into a natural order or ranking

outlier: an extreme value or unusual reading in the data set, generally considered to be any value beyond the lower or upper fences

parallel boxplot: a display in which two or more boxplots share the same scale to enable comparisons between data sets

path: a walk in which no vertices are repeated, except possibly the start and finish

Pearson's product-moment correlation coefficient: a measure of the strength of a linear trend that is designated with the letter r and associated with a numerical value between -1 and $+1$. Values close to $+1$ or -1 indicate strong linear trends; values close to zero indicate weak or no linear trends.

percentage: a rate per hundred

percentage dividend: the percentage of earnings paid as a dividend

perimeter: the distance around an object

personal loan: a loan made by a lending institution to an individual. A personal loan will usually have a fixed interest rate attached to it, with the interest paid by the customer calculated on a reduced balance.

piecewise graph: a continuous graph formed by two or more linear graphs that are joined at points of intersection

planar graph: a graph that can be redrawn to represent information so that it has no intersecting edges

polygon: a 2-dimensional shape consisting of at least three straight sides

positively skewed: distributions with higher frequencies on the left side of the graph

price-to-earnings ratio (P/E ratio): the ratio of the current price of a share and the dividend per share, giving an indication of how much shares cost per dollar of profit

Prim's algorithm: a set of logical steps that can be used to identify the minimum spanning tree for a weighted connected graph

principal: the amount that is borrowed or invested

prism: a solid object that has identical ends that are joined by flat surfaces, and a cross-section that is the same along its length

product matrix: the result of multiplying two matrices where the number of columns in the first matrix equals the number of rows in the second matrix. For matrices of order $m \times n$ and $n \times r$, the product matrix will have order $m \times r$.

pronumeral: a letter or symbol representing a number (usually a variable) in a mathematical expression or equation

proportional: when two quantities have the same ratio; therefore, they always have the same size in relation to each other

pyramid: a solid object whose base is a polygon and whose sides are triangles that meet at a single point

Pythagoras' theorem: The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.

Pythagorean triad: any group of three numbers that satisfies Pythagoras' theorem

range: a measure of the spread of a numerical data set determined by calculating the difference between the smallest and largest values

rate: a measure of how one quantity is changing compared to another

ratio: the relationship between two or more values commonly expressed as a quotient: $\frac{a}{b} \Leftrightarrow a: b$

recurrence relation: an equation that recursively defines a sequence; that is, once one or more initial terms are given, each further term of the sequence is defined as a function of the preceding terms

reducing balance depreciation: a method of depreciation, sometimes called 'diminishing value' depreciation, in which the value of an asset is reduced by a fixed percentage of its previous book value. This is an application of compound interest. The depreciated future value can be calculated using $FV = P\left(1 - \frac{r}{100}\right)^n$.

required region: the region representing the required inequality

response variable: the variable represented on the y-axis that is explained or predicted by the explanatory variable in a bivariate analysis

row matrix: a matrix that has only one row

scalar multiplication: Each element of the matrix is multiplied by the same number, called a 'scalar'. A scalar quantity can be any real number, such as negative or positive numbers, fractions or decimal numbers.

scatterplot: a visual display of bivariate data

scientific notation (or standard form): When a number is written as $a \times 10^b$, the product of a number a where $1 \leq a < 10$ and a power of 10, it is expressed in scientific notation.

sector: fraction of a circle. The area of a sector can be calculated using $A = \frac{\theta}{360}\pi r^2$.

sequence: a related set of objects or events that follow each other in a particular order

share: unit of ownership in a company

significant figures: the number of digits that would occur in a if the number was expressed in scientific notation as $a \times 10^b$ or as $a \times 10^{-b}$

similar: objects that are exactly the same shape but have different sizes

simple graph: a graph in which pairs of vertices are connected by one edge at most, so there are no loops or multiple edges

simple interest: interest calculation based on the original amount borrowed or invested; also known as, 'flat rate' as it is a constant amount: $I = \frac{PrT}{100}$

simultaneous equation: an equation belong to a system of equations in which the solutions for the values of the unknowns must satisfy each equation

sine: the ratio of the side length opposite an internal angle of a right-angled triangle with the hypotenuse: $\sin(\theta) = \frac{O}{H}$

sine rule: The ratios of a side length with the sine of the angle opposite it are equal throughout a triangle: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$. This can be used to find the unknown side length or angle in non-right-angled triangles.

sliding-line method: a method to find the optimal solution(s) to a linear programming problem by changing the y-intercept of the objective function. Sliding the line up to meet the last point it touches in the feasible region will identify the maximum value of the function. Sliding the line down to meet the last point it touches in the feasible region will identify the minimum value of the function.

SOH-CAH-TOA: a mnemonic to assist in remembering the trigonometric ratios:

- **S**ine is **O**pposite over **H**ypotenuse
- **C**osine is **A**djacent over **H**ypotenuse
- **T**angent is **O**pposite over **A**djacent.

spanning tree: a tree that is a sub-graph which includes all of the vertices of the original graph

spending power: also known as 'purchasing power'; the number of goods or services that can be purchased with a unit of currency
sphere: a solid object that has a curved surface such that every point on the surface is the same distance (the radius of the sphere) from a central point
square matrix: a matrix that has the same number of rows and columns
standard deviation: the most common measure of the spread of data around the mean; found by taking the square root of the variance
stem plot: an arrangement used for numerical data in which data points are grouped according to their numerical place values (the 'stem') and then displayed horizontally as single digits (the 'leaf')
step graph: a discontinuous graph formed by two or more linear graphs that have zero gradients
substitute: replace with an equivalent value or expression
surface area: the combined total of the areas of each individual surface that forms a solid object
tables of values: tables of x- and y-values that are true for a line, used for solving problems and plotting graphs
tangent: the ratio of the side length opposite an internal angle of a right-angled triangle with the side length adjacent to it: $\tan(\theta) = \frac{O}{A}$
term number: the position of a term in a sequence
test point: a point used to determine the correct region that represents a linear inequality by substituting its x- and y-values into the inequality
time payment: also known as 'hire purchase'; a method of purchasing in which the customer receives an item by paying a small amount as a deposit followed by weekly or monthly instalments
trail: a walk in which no edges are repeated
transpose: to rearrange an expression or formula
tree: a simple connected graph with no circuits
trigonometric ratio: a ratio of the lengths of two sides of a right-angled triangle
true bearing: a bearing that is measured in a clockwise direction from the north–south line. A true bearing is written as a three-digit expression of the angle.
undirected graph: a graph in which it is possible to move along the edges of the graph in any direction
ungrouped data: numerical data that is not arranged in groups to enable exact analysis
unit cost: the cost of a single item
unit cost depreciation: a method of depreciating an asset according to its use — the more it is used, the faster it will depreciate
upper fence: the upper boundary beyond which a data value is considered to be an outlier: $Q_3 + 1.5 \times \text{IQR}$
upper quartile (Q_3): the median of the upper half of an ordered data set
variable: a quantity that can take on a range of values depending on its relationship to other values; typically represented by pronumerals
variance: a measure of the spread of a data set from the mean: $s^2 = \frac{\sum f(x - \bar{x})^2}{\left(\sum f\right) - 1}$
vertex: the point or 'node' of a graph
volume: the amount of space that is taken up by any solid or 3-dimensional object
volume scale factor: the ratio of the corresponding volumes of similar objects. This is equal to the linear scale factor raised to the power of 3.
walk: any route taken through a network, including those that repeat edges and vertices
weighted graph: a graph with values attached to the edges
write-off value: the value at which an asset is removed from the books of a company as it is considered effectively worthless; also called the 'scrap value'
***x*-intercept:** the point where the graph of an equation crosses the x-axis. This occurs when $y = 0$
***y*-intercept:** the point where the graph of an equation crosses the y-axis. This occurs when $x = 0$.
zero matrix: a square matrix that consists entirely of '0' elements

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