1 Investigating data distributions

Topic	1	Investigating data distributions
Subtopic	1.2	Types of data

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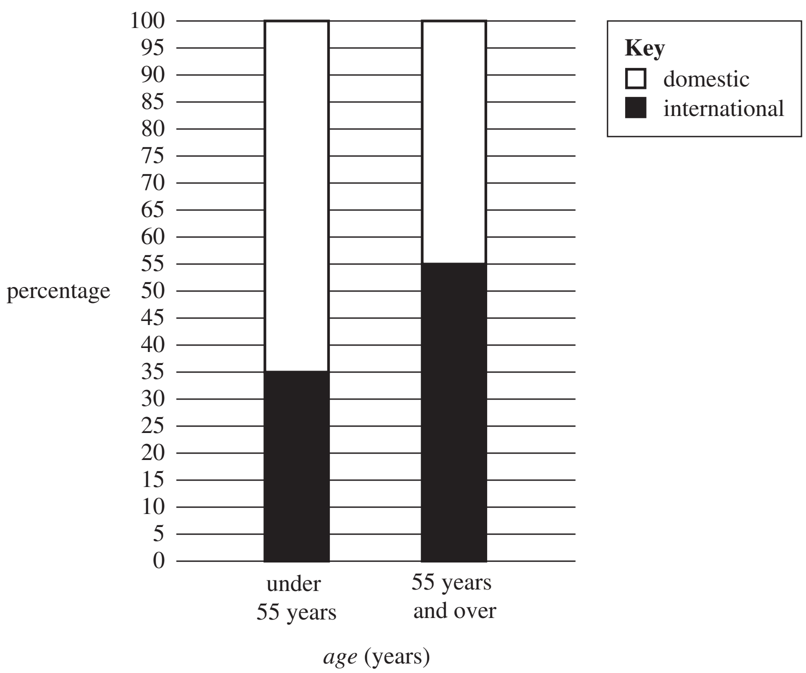
Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.1; © VCAA*

Question 1 (1 mark)

The percentaged segmented bar chart below shows the age (under 55 years, 55 years and over) of visitors at a travel convention, segmented by *preferred travel destination* (domestic, international).



The variables *age* (under 55 years, 55 years and over) and *preferred travel destination* (domestic, international) are

A. both categorical variables.
B. both numerical variables.
C. a numerical variable and a categorical variable respectively.
D. a categorical variable and a numerical variable respectively.
E. a discrete variable and a continuous variable respectively.

Topic 1 | Subtopic 1.2 Types of data

Source: *VCE 2020, Further Mathematics 1, Section A, Q.7; © VCAA*

Question 2 (1 mark)

Data relating to the following five variables was collected from insects that were caught overnight in a trap:

- *colour*
- *name of species*
- *number of wings*
- *body length (in millimetres)*
- *body weight (in milligrams)*

The number of these variables that are discrete numerical variables is

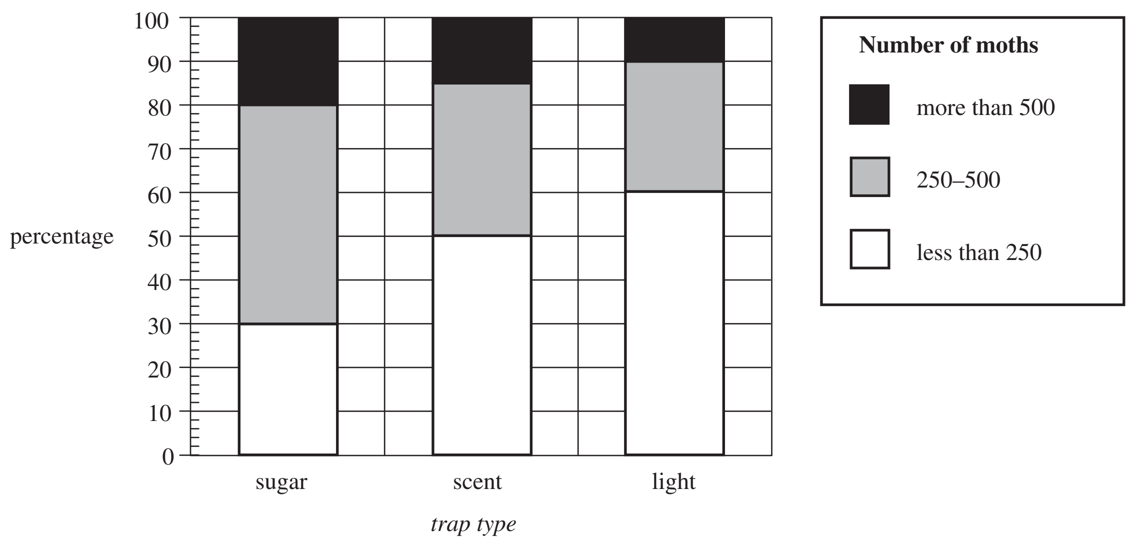
A. 1
B. 2
C. 3
D. 4
E. 5

Topic 1 Subtopic 1.2 Types of data

Source: *VCE 2017, Further Mathematics 1, Section A, Core, Q.7; © VCAA*

Question 3 (1 mark)

A study was conducted to investigate the association between the *number of moths* caught in a moth trap (less than 250, 250–500, more than 500) and the *trap type* (sugar, scent, light). The results are summarised in the percentaged segmented bar chart below.



The variables *number of moths* (less than 250, 250– 500, more than 500) and *trap type* (sugar, scent, light) are

A. both nominal variables.
B. both ordinal variables.
C. a numerical variable and a categorical variable respectively.
D. a nominal variable and an ordinal variable respectively.
E. an ordinal variable and a nominal variable respectively.

Topic 1 Subtopic 1.2 Types of data

Source: *VCE 2016, Further Mathematics 1, Section A, Q.2; © VCAA*

Question 4 (1 mark)

The *blood pressure* (low, normal, high) and the *age* (under 50 years, 50 years or over) of 110 adults were recorded. The results are displayed in the two-way frequency table below.

Blood pressure	Age	
	Under 50 years	**50 years or over**
low	15	5
normal	32	24
high	11	23
Total	58	52

The variables *blood pressure* (low, normal, high) and *age* (under 50 years, 50 years or over) are

A. both nominal variables.
B. both ordinal variables.
C. a nominal variable and an ordinal variable respectively.
D. an ordinal variable and a nominal variable respectively.
E. a continuous variable and an ordinal variable respectively.

Source: *VCE 2014, Further Mathematics 1, Section A, Q.4; © VCAA*

Question 5 (1 mark)

The following table shows the data collected from a sample of seven drivers who entered a supermarket car park. The variables in the table are:

- *distance* – the distance that each driver travelled to the supermarket from their home
- *sex* – the sex of the driver (female, male)
- *number of children* – the number of children in the car
- *type of car* – the type of car (sedan, wagon, other)
- *postcode* – the postcode of the driver's home.

Distance **(km)**	***Sex*** **(F = female, M = male)**	***Number of children*** **(1 = sedan, 2 = wagon, 3 = other)**	***Type of car***	**Postcode**
4.2	F	2	1	8148
0.8	M	3	2	8147
3.9	F	3	2	8146
5.6	F	1	3	8245
0.9	M	1	3	8148
1.7	F	2	2	8147
2.5	M	2	2	8145

The number of categorical variables in this data set is

A. 0
B. 1
C. 2
D. 3
E. 4

Source: *VCE 2014, Further Mathematics 1, Section A, Q.5; © VCAA*

Question 6 (1 mark)

The following table shows the data collected from a sample of seven drivers who entered a supermarket car park. The variables in the table are:

- *distance* – the distance that each driver travelled to the supermarket from their home
- *sex* – the sex of the driver (female, male)
- *number of children* – the number of children in the car
- *type of car* – the type of car (sedan, wagon, other)
- *postcode* – the postcode of the driver's home.

Distance (km)	Sex (F = female, M = male)	Number of children (1 = sedan, 2 = wagon, 3 = other)	Type of car	Postcode
4.2	F	2	1	8148
0.8	M	3	2	8147
3.9	F	3	2	8146
5.6	F	1	3	8245
0.9	M	1	3	8148
1.7	F	2	2	8147
2.5	M	2	2	8145

The number of female drivers with three children in the car is

A. 0
B. 1
C. 2
D. 3
E. 4

Question 7 (1 mark)

The variables **mass** (g) and **gender** (male and female) are

A. both continuous variables.
B. both nominal variables.
C. continuous and nominal variables respectively.
D. nominal and continuous variables respectively.
E. neither continuous nor nominal variables.

Question 8 (1 mark)

The level of participation in sport (never, sometimes, often) for 150 people of varying ages is indicated in the table below.

Participation in sport	Age group		
	0 - 10	11 - 20	20 - 30
Never	8	3	1
Sometimes	16	20	46
Often	25	16	15

The variables **Participation in sport** and **Age group**, as recorded in this table, are respectively:

A. categorical and numerical
B. both categorical
C. both numerical
D. numerical and categorical
E. discrete and numerical

Question 9 (1 mark)

The two variables of Type of phone ($1 = $ Samsung, $2 = $ iPhone, $3 = $ other) and Postcode ($5986, 4251, ..$) are:

A. categorical and numerical
B. both categorical
C. both numerical
D. numerical and categorical
E. discrete and numerical

Question 10 (1 mark)

Which of the following statements is the best example of a numerical variable?

A. year level of students
B. position in a queue at the canteen
C. the total attendance at the AFL Grand Final
D. favourite foods
E. attitudes towards Australia becoming a republic

Question 11 (1 mark)
The weight of each cargo item loaded onto a plane at Melbourne Airport was recorded.
This data is best described as
A. discrete data.
B. continuous and numerical data.
C. numerical and discrete data.
D. categorical and discrete data.
E. continuous and categorical data.

Question 12 (1 mark)
Forty Year 12 students were asked to list their top three favourite movies.
Their responses are best classified as
A. numerical data.
B. nominal and categorical data.
C. ordinal and numerical data.
D. ordinal and categorical data.
E. ordinal and discrete data.

Question 13 (1 mark)
Which of the following statements is the best example of a categorical variable?
A. The number of goals scored in a basketball game.
B. The distance walked each day in the month of June.
C. The names of the attendees at a local concert.
D. The height a ball is thrown into the air.
E. The minutes spent watching television.

Question 14 (1 mark)

A group of shoppers recorded the name of the fruits they had in their shopping trolley when they exited their local supermarket. The fruits were then listed in alphabetical order.

The information collected is best described as:

A. numerical and discrete
B. numerical and continuous
C. categorical and discrete
D. categorical and nominal
E. categorical and ordinal

Topic	1	Investigating data distributions
Subtopic	1.3	Stem plots

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Source: *VCE 2016, Further Mathematics 1, Section A, Q.3; © VCAA*

Question 1 (1 mark)

The stem plot below displays 30 temperatures recorded at a weather station.

temperature

Stem	Leaf
2	2 2 4 4
2	8
3	5 7 8 8 8 8 8 8 9 9 9 9
3	1 2 3 3 4 4 4
4	5 6 7 7 7 7
	1

Key : $2|2 = 2.2°\text{C}$

The modal temperature is

A. 2.8°C
B. 2.9°C
C. 3.7°C
D. 8.0°C
E. 9.0°C

Topic 1 > Subtopic 1.3 Stem plots

Source: *VCE 2013, Further Mathematics 1, Section A, Core, Q.1; © VCAA*

Question 2 (1 mark)

The following ordered stem plot shows the percentage of homes connected to broadband internet for 24 countries in 2007.

Stem	Leaf
1	
1	6 7
2	0 1 1 3 4 4
2	5 7 8 9
3	0 0 1 1 1 2 2 3
3	5 7 8 8
4	

Key : 1|6 = 16%

The number of these countries with more than 22% of homes connected to broadband internet in 2007 is

A. 4
B. 5
C. 19
D. 20
E. 22

Question 3 (1 mark)

The marks students received in a recent test are displayed in the stem plot. How many outliers does the data contain?

Stem	*Leaf*
0	7 8
1	8
2	7 8 8 9
3	2 4 5 6 8 9 9
4	0 2 8 9 9
5	0

Key : 2|7 = 27

A. 0
B. 1
C. 2
D. 3
E. 4

Question 4 (1 mark)

The number of cars passing through an intersection each hour is displayed as an ordered stem plot.

Stem	*Leaf*
1	2 5
2	3 3 4 5
3	6 9
4	0
5	3 5 5 6 9
6	1 2 3
7	0 2 6

Key : 7|0 = 70

The number of hours over which the data was collected is:

A. 12
B. 20
C. 7
D. 70
E. 76

Topic	1	Investigating data distributions
Subtopic	1.4	Dot plots, frequency tables, histograms, bar charts and logarithmic scales

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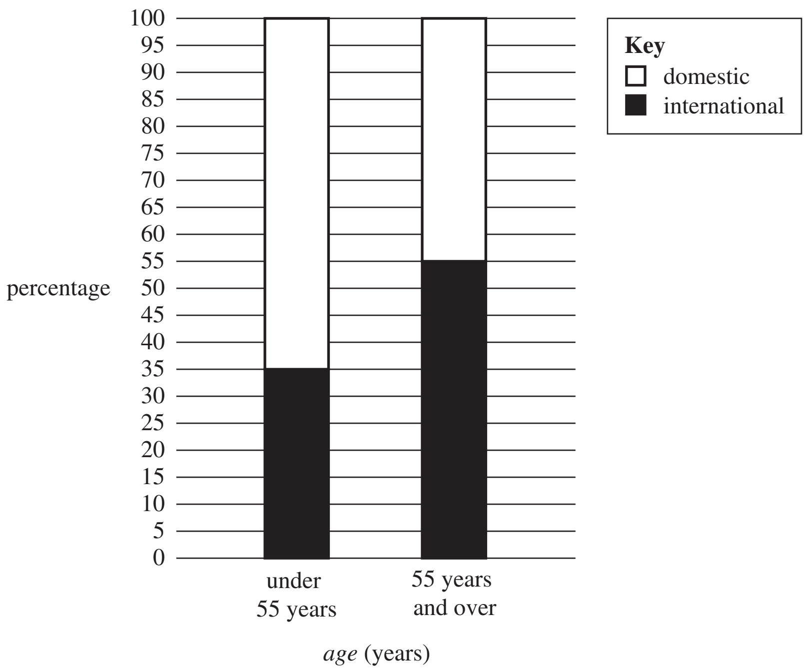
Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.2; © VCAA*

Question 1 (1 mark)

The percentaged segmented bar chart below shows the *age* (under 55 years, 55 years and over) of visitors at a travel convention, segmented by *preferred travel destination* (domestic, international).



The data displayed in the percentaged segmented bar chart supports the contention that there is an association between *preferred travel destination* and *age* because

A. more visitors favour international travel.
B. 35% of visitors under 55 years favour international travel.
C. 45% of visitors 55 years and over favour domestic travel.
D. 65% of visitors under 55 years favour domestic travel while 45% of visitors 55 years and over favour domestic travel.
E. the percentage of visitors who prefer domestic travel is greater than the percentage of visitors who prefer international travel.

Source: *VCE 2020, Further Mathematics 1, Section A, Q.6; © VCAA*

Question 2 (1 mark)

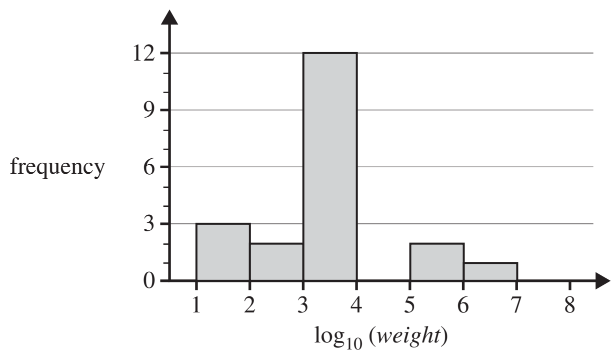
A percentaged segmented bar chart would be an appropriate graphical tool to display the association between *month of the year* (January, February, March, etc.) and the

A. *monthly average rainfall* (in millimetres).
B. *monthly mean temperature* (in degrees Celsius).
C. *annual median wind speed* (in kilometres per hour).
D. *monthly average rainfall* (below average, average, above average).
E. *annual average temperature* (in degrees Celsius).

Source: *VCE 2020, Further Mathematics 1, Section A, Q.5; © VCAA*

Question 3 (1 mark)

The histogram below shows the distribution of *weight*, in grams, for a sample of 20 animal species. The histogram has been plotted on a $\log_{10}$ scale.



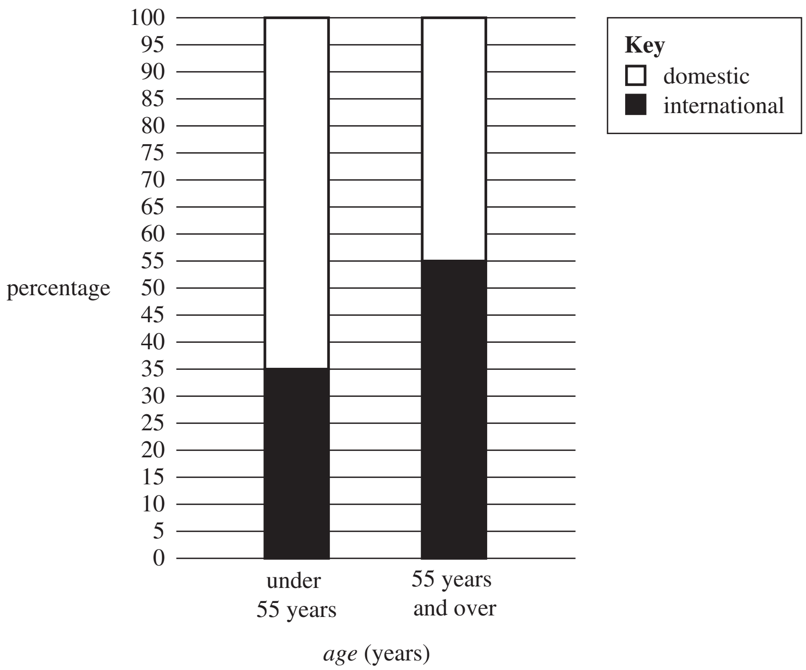
The percentage of these animal species with a weight of less than 10 000 g is

A. 17%
B. 70%
C. 75%
D. 80%
E. 85%

Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.3; © VCAA*

Question 4 (1 mark)

The percentaged segmented bar chart below shows the age (under 55 years, 55 years and over) of visitors at a travel convention, segmented by preferred travel destination (domestic, international).



The results could also be summarised in a two-way frequency table.

Which one of the following frequency tables could match the percentaged segmented bar chart?

A.

Preferred travel destination	**Age**	
	Under 55 years	**55 years and over**
domestic	91	90
international	49	110
Total	140	200

B.

Preferred travel destination	**Age**	
	Under 55 years	**55 years and over**
domestic	65	35
international	45	55
Total	110	90

C.

Preferred travel destination	**Age**	
	Under 55 years	**55 years and over**
domestic	35	55
international	65	45
Total	100	100

D.

Preferred travel destination	Age	
	Under 55 years	**55 years and over**
domestic	50	70
international	100	50
Total	150	120

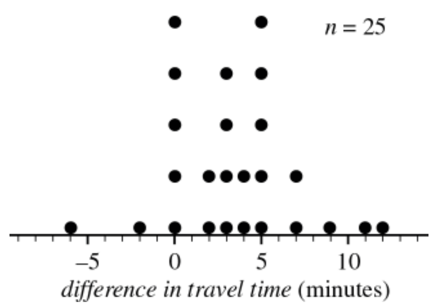
E.

Preferred travel destination	Age	
	Under 55 years	**55 years and over**
domestic	71	39
international	29	61
Total	100	100

Source: *VCE 2018, Further Mathematics 1, Section A, Q.1; © VCAA*

Question 5 (1 mark)

The dot plot below displays the *difference in travel time* between the morning peak and the evening peak travel times for the same journey on 25 days.



The percentage of days when there was five minutes *difference in travel time* between the morning peak and the evening peak travel times is

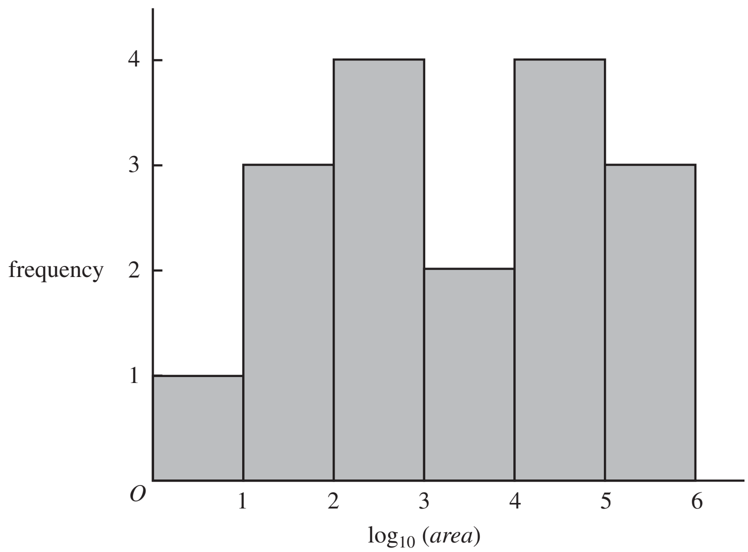
A. 0%
B. 5%
C. 20%
D. 25%
E. 28%

Topic 1 Subtopic 1.4 Dot plots, frequency tables, histograms, bar charts and logarithmic scales

Source: *VCE 2017, Further Mathematics 1, Section A, Core, Q.4; © VCAA*

Question 6 (1 mark)

The histogram below shows the distribution of the $\log_{10}(area)$, with area in square kilometres, of 17 islands.



The median area of these islands, in square kilometres, is between

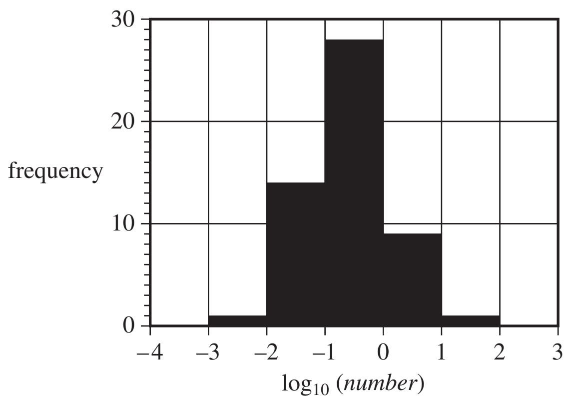
A. 2 and 3
B. 3 and 4
C. 10 and 100
D. 1000 and 10 000
E. 10 000 and 100 000

Topic 1 | Subtopic 1.4 Dot plots, frequency tables, histograms, bar charts and logarithmic scales

Source: *Adapted from VCE 2016, Further Mathematics 1, Section A, Q.7; © VCAA*

Question 7 (1 mark)

The histogram below shows the distribution of the *number* of billionaires per million people for 53 countries, plotted on a $\log_{10}$ scale.



Data: Gapminder

Based on this histogram, the number of countries with one or more billionaires per million people is

A. 1

B. 3

C. 8

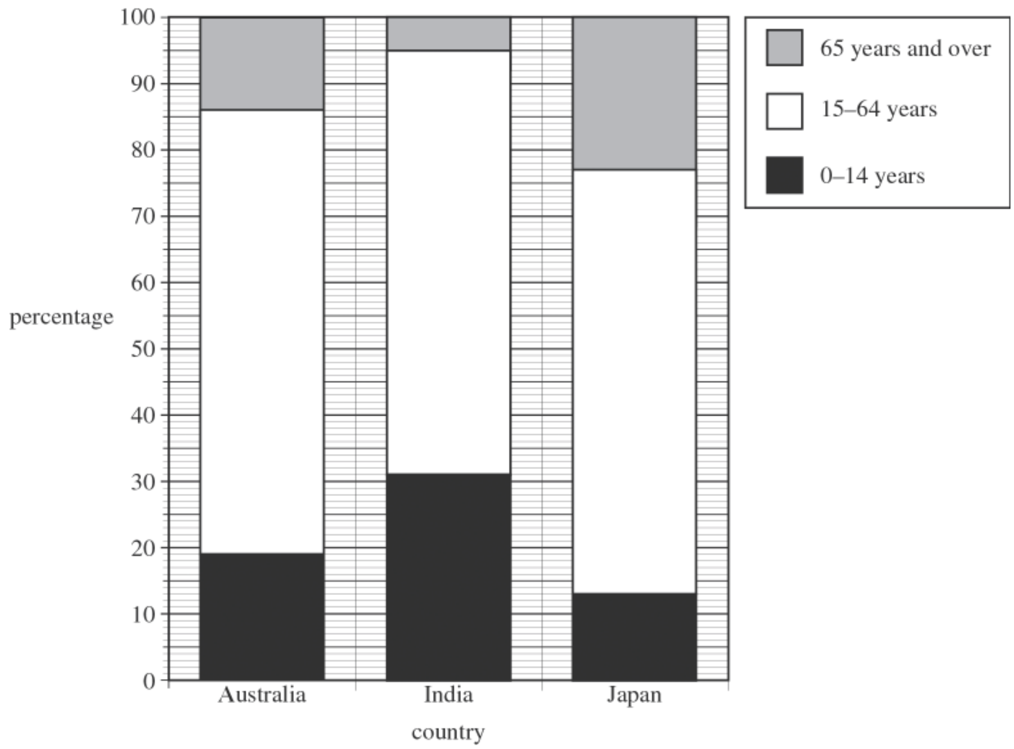
D. 9

E. 10

Source: *VCE 2014, Further Mathematics 2, Core, Q.1; © VCAA*

Question 8 (3 marks)

The segmented bar chart below shows the age distribution of people in three countries, Australia, India and Japan, for the year 2010.



Source: Australian Bureau of Statistics, *3201.0 – Population by Age and Sex, Australian States and Territories*, June 2010

a. Write down the percentage of people in Australia who were aged $0-14$ years in 2010. Write your answer, correct to the nearest percentage. **(1 mark)**

b. In 2010, the population of Japan was 128 000 000. How many people in Japan were aged 65 years and over in 2010? **(1 mark)**

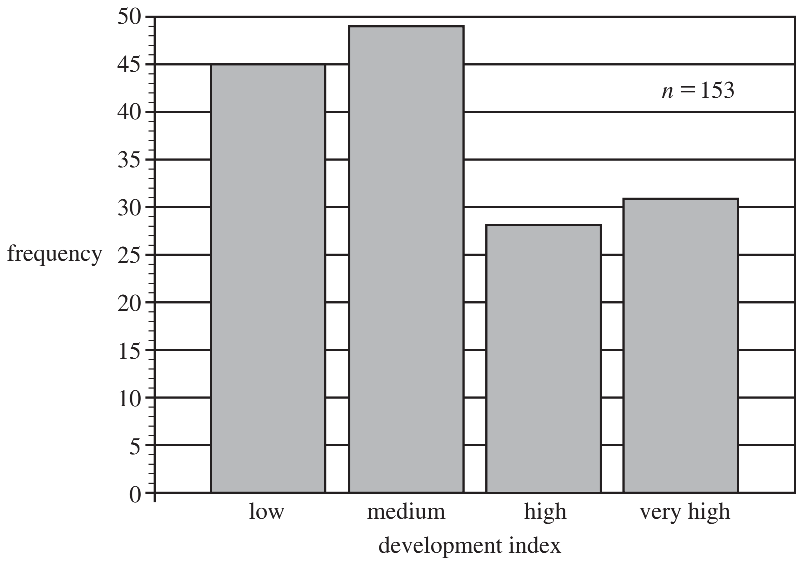
c. From the graph above, it appears that there is no association between the percentage of people in the $15-64$ age group and the country in which they live. Explain why, quoting appropriate percentages to support your explanation. **(1 mark)**

Source: *VCE 2013, Further Mathematics 2, Core, Q.1; © VCAA*

Question 9 (2 marks)

A development index is used as a measure of the standard of living in a country.

The bar chart below displays the development index for 153 countries in four categories: low, medium, high and very high.

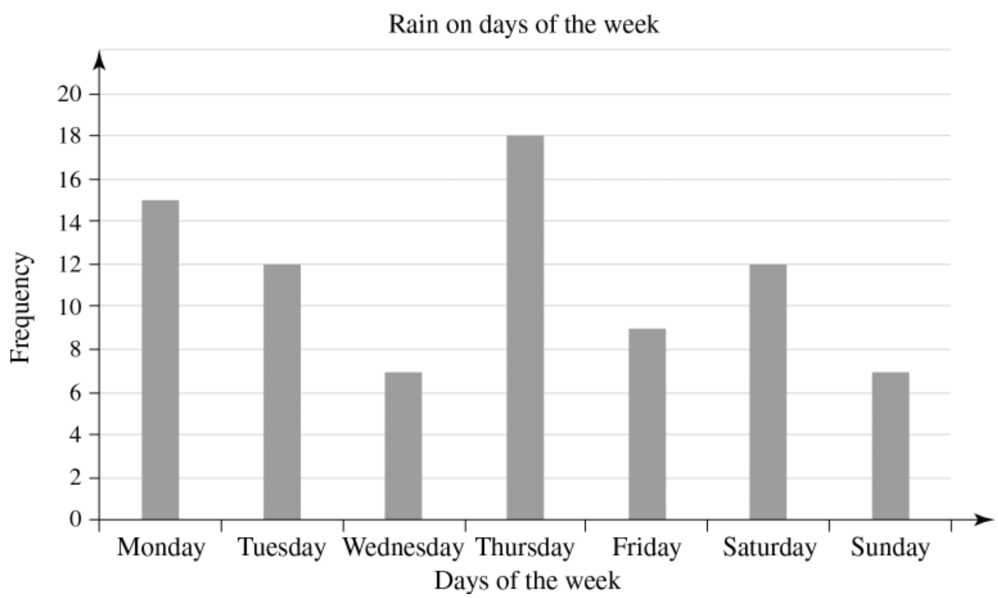


a. How many of these countries have a very high development index? **(1 mark)**

b. What percentage of the 153 countries has either a low or medium development index?
Write your answer, correct to the nearest percentage. **(1 mark)**

Question 10 (1 mark)

The following bar chart displays the frequency of days on which rain was recorded one year.

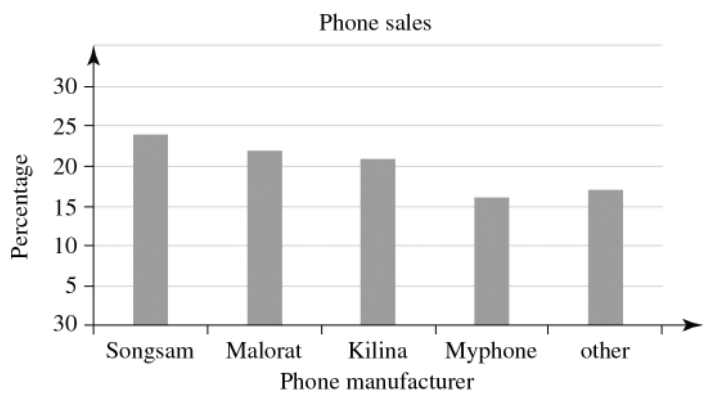


The percentage of weekend days of rain is closest to

A. 12%
B. 19%
C. 24%
D. 56%
E. 80%

Question 11 (1 mark)

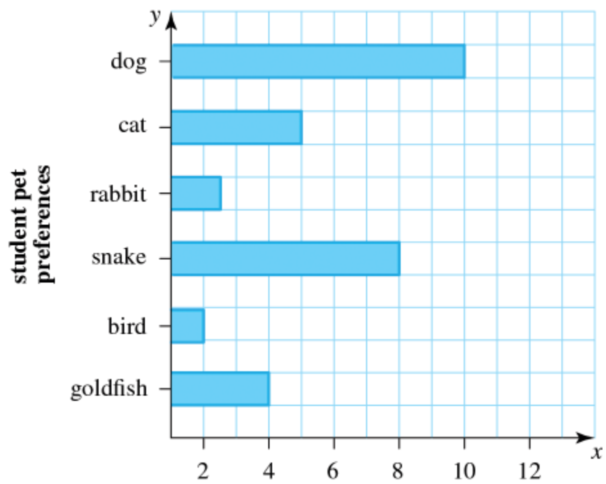
The distribution of smart phone purchases from a small phone retailer is shown in the bar chart below.



According to the bar chart the most frequently purchased phone was

A. Songsam
B. Malorat
C. Kilina
D. Myphone
E. Other

Question 12 (1 mark)



For the bar chart shown, the variable is

A. the number of students.

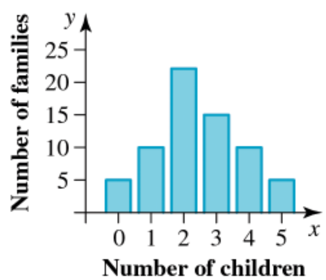
B. dog, cat, rabbit, snake, bird or goldfish.

C. preferences.

D. students.

E. student pet preferences.

Question 13 (1 mark)



For the bar chart shown, which of the following statements is false?

A. There are five families with zero children.

B. There are zero families with five children.

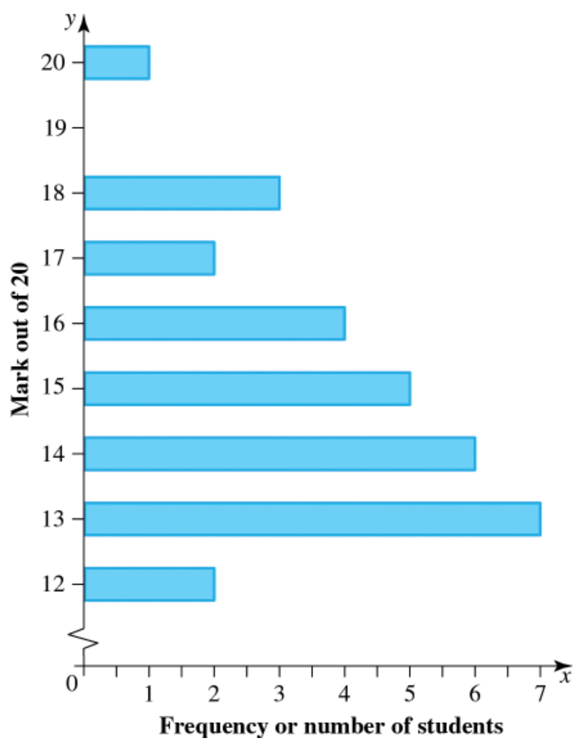
C. There are more families with three children than there are families with four children.

D. The modal number of children is two.

E. There are ten families with four children.

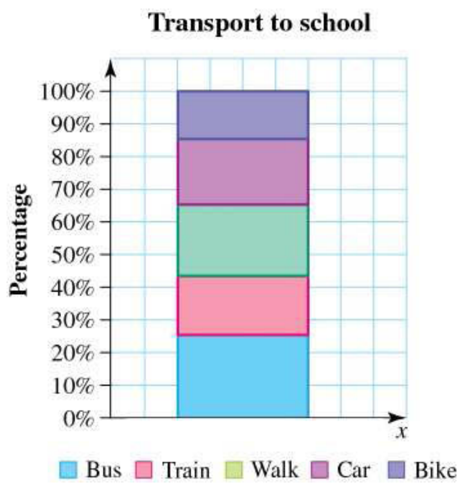
Question 14 (1 mark)

For the bar chart shown below, which of the following statements is true?



A. No student scored 95%.
B. Two students scored 100%.
C. Two students scored 12%.
D. Seven students scored 13%.
E. Six students scored 15 marks out of 20.

Question 15 (1 mark)
In a small survey students were asked to state the mode of transport they used to get to school. The following data was collected.



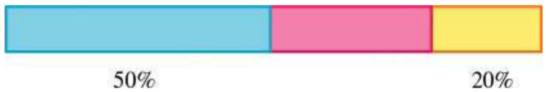
The percentage of students who travelled to school by car is closest to
A. 10%
B. 14%
C. 20%
D. 66%
E. 86%

Question 16 (1 mark)
The exercise preferences of a group of people were recorded in the following table:

Exercise	Frequency
Gymnasium	15
Cycling	22
Running	13
Aerobics	17
Yoga	25

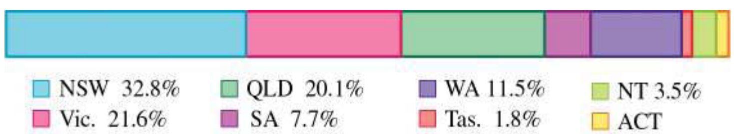
In a segmented bar chart, the percentage of the bar that would be allocated to 'Running' would be closest to
A. 16%
B. 27%
C. 18%
D. 24%
E. 14%

Question 17 (1 mark)
What percentage does the pink (middle) section of the graph represent?



A. 20%
B. 50%
C. 80%
D. 70%
E. 30%

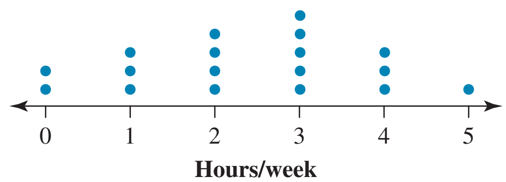
Question 18 (1 mark)
What is the missing data, for the ACT, in the segmented bar graph shown?



A. 3.5%
B. 1.8%
C. 1.1%
D. 7.7%
E. 1.9%

Question 19 (1 mark)
Which of the following statements about a segmented bar chart is false?
A. A segmented bar chart is a single bar that is used to represent all the data being studied.
B. In a segmented bar chart, each segment represents a particular group being studied.
C. The total bar length in a segmented bar chart is equal to 100%.
D. Percentages are always used to represent the segments on a segmented bar chart.
E. Segmented bar charts require a key.

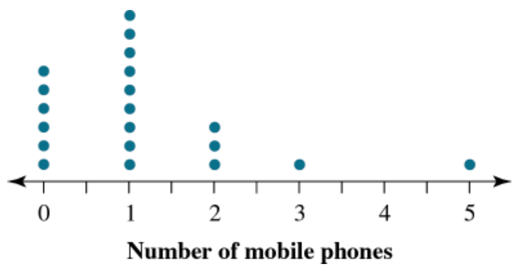
Question 20 (1 mark)
The dot plot shown represents the amount of time spent studying by a group of students during a week.



Which of the following statements is false?
A. 18 students were surveyed.
B. The modal number of hours spent studying was 3 per day.
C. The most number of hours spent studying by any student was 5 per week.
D. Two students did no study.
E. Half of the students spent two or less hours per week studying.

Question 21 (1 mark)
Which of the following statements about dot plots is false?
A. A dot plot contains a scaled axis.
B. Each dot represents a single data value.
C. Dot plots are suitable for small sets of data.
D. Dot plots display the distribution of the data.
E. Dot plots are best used for large sets of data.

Question 22 (1 mark)
The number of mobile phones in each of the 20 households in a sample is shown in the dot plot below.



In this sample, what is the percentage of households with more than one mobile phone?
A. 45%
B. 25%
C. 70%
D. 30%
E. 55%

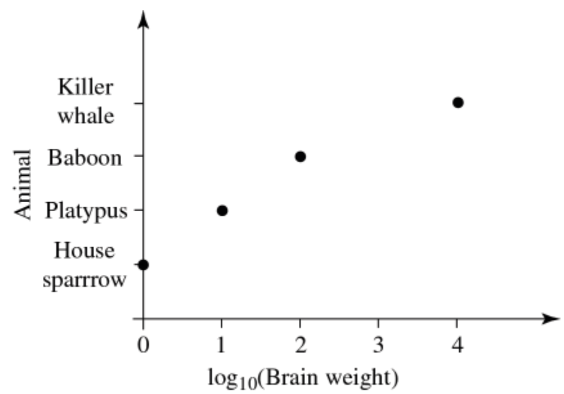
Question 23 (1 mark)
The flow of water, in mL/day, into the Verywet Dam for four consecutive seasons is shown in the table below:

Season	**Flow (mL/day)**
Spring	91 711
Summer	5411
Autumn	16 602
Winter	196 923

The correct value, written to 2 decimal places, to be plotted for the summer flow rate using a base 10 log scale would be
A. 12.40
B. 3.73
C. 4.22
D. 8.60
E. 4.96

Question 24 (1 mark)

The graph shown indicates a selection of animals and their brain weights plotted using a base 10 log scale.



If a polar bear with a brain weight of 498 grams was added to the dot plot, its weight would be

A. less than the house sparrow
B. between the house sparrow and platypus
C. between the platypus and baboon
D. between the baboon and killer whale
E. more than the killer whale

Topic	1	Investigating data distributions
Subtopic	1.5	Describing stem plots and histograms

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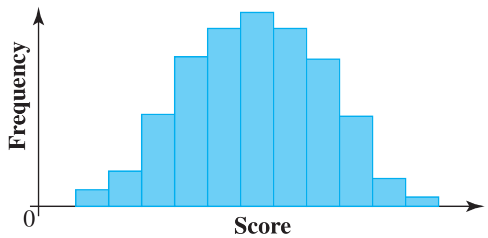
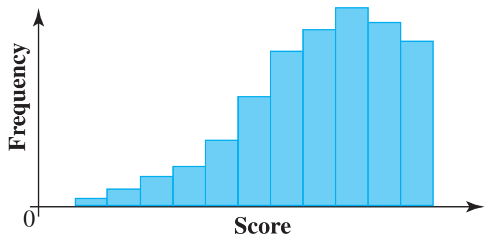
Question 1 (1 marks)

The histogram that represents a set of negatively skewed data is

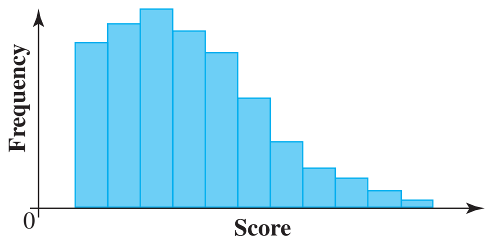
A.



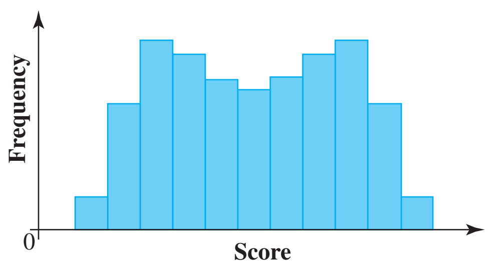
B.



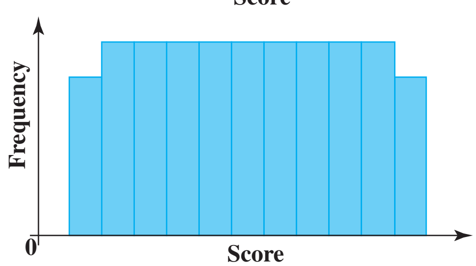
C.



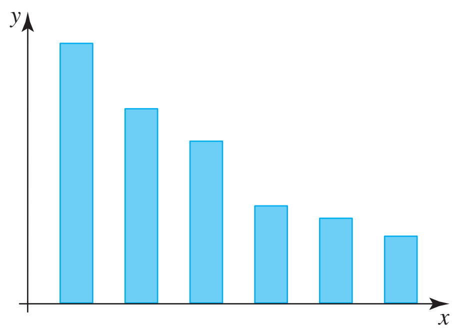
D.



E.



Question 2 (1 mark)
How is the shape of the distribution of the data in the graph best described?



A. symmetrical
B. positively skewed
C. negatively skewed
D. bimodal
E. positively skewed with an outlier

Question 3 (1 mark)
How is the distribution of the data in the following stem plot best described?

stem	leaf
0	1
0	2
0	4 4 5
0	5 6 5 7
0	8 8 8 8 9 9
1	0 0 0 1 1 1 1
1	2 2 2 3 3 3
1	4 4 5 5
1	6 7 7
1	89

key: 1|8 = 18

A. negatively skewed
B. negatively skewed and symmetric
C. positively skewed
D. positively skewed and symmetric
E. symmetric

Topic	**1**	**Investigating data distributions**
Subtopic	**1.6**	**Summary statistics**

online only

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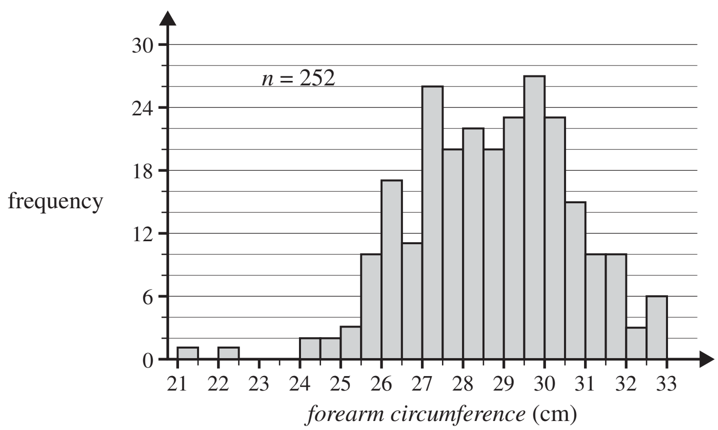
Source: *VCE 2020, Further Mathematics 1, Section A, Q.4; © VCAA*

Question 1 (1 mark)

The histogram below shows the distribution of the *forearm circumference*, in centimetres, of 252 men.
Assume that the *forearm circumference* values were all rounded to one decimal place.



The third quartile (Q_3) for this distribution could be

A. 29.3
B. 29.8
C. 30.3
D. 30.8
E. 31.3

Source: *VCE 2019, Further Mathematics 1, Section A, Q.4; © VCAA*

Question 2 (1 mark)

The stem plot below shows the distribution of mathematics *test scores* for a class of 23 students.

key: 4|2 = 42 $n = 23$

4	0	1	4	4		
5	2	7	9	9	9	
6	5	6	8	8	9	9
7	0	0	5	6	7	8
8	5	9				

For this class, the range of *test scores* is

A. 22
B. 40
C. 45
D. 49
E. 89

Source: *VCE 2019, Further Mathematics 1, Section A, Q.5; © VCAA*

Question 3 (1 mark)

The stem plot below shows the distribution of mathematics *test scores* for a class of 23 students.

key: 4|2 = 42 $n = 23$

4	0	1	4	4		
5	2	7	9	9	9	
6	5	6	8	8	9	9
7	0	0	5	6	7	8
8	5	9				

For this class, the interquartile range (IQR) of *test scores* is

A. 14.5
B. 17.5
C. 18
D. 24
E. 49

Source: *VCE 2020, Further Mathematics 1, Section A, Q.1; © VCAA*

Question 4 (1 mark)

The times between successive nerve impulses (time), in milliseconds, were recorded.

Table 1 shows the mean and the five-number summary calculated using 800 recorded data values.

Table 1

	Time (milliseconds)
Mean	220
Minimum value	10
First quartile (Q_1)	70
Median	150
Third quartile (Q_3)	300
Maximum value	1380

Data: Adapted from P Fatt and B Katz, 'Spontaneous subthreshold activity at motor nerve endings', *The Journal of Physiology*, 117, 1952, pp. 109 – 128

The difference, in milliseconds, between the mean time and the median time is

A. 10
B. 70
C. 150
D. 220
E. 230

Source: *VCE 2015, Further Mathematics 1, Section A, Q.2; © VCAA*

Question 5 (1 mark)

For an ordered set of data containing an odd number of values, the middle value is always

A. the mean.
B. the median.
C. the mode.
D. the mean and the median.
E. the mean, the median and the mode.

Topic 1 | Subtopic 1.6 Summary statistics

Source: *VCE 2014, Further Mathematics 1, Section A, Q.1; © VCAA*

Question 6 (1 mark)

The following ordered stem plot shows the areas, in square kilometres, of 27 suburbs of a large city.

1	5 6 7 8
2	1 2 4 5 6 8 9 9
3	0 1 1 2 2 8 9
4	0 4 7
5	0 1
6	1 9
7	
8	4

Key: 1|6 = 1.6 km^2

The median area of these suburbs, in square kilometres, is

A. 3.0
B. 3.1
C. 3.5
D. 30.0
E. 30.5

Question 7 (1 mark)

The number of cars passing through an intersection each hour is displayed as an ordered stem plot.

Stem	Leaf
1	2 5
2	3 3 4 5
3	6 9
4	0
5	3 5 5 6 9
6	1 2 3
7	0 2 6

Key: 7|0 = 70

The interquartile range for the number of cars passing through the intersection is closest to

A. 61.5
B. 37
C. 24.5
D. 20
E. 12

Question 8 (1 mark)
The number of communication devices in each of 25 households is recorded in the frequency table below.

Number of communication devices	Frequency
1	4
3	3
4	2
5	3
6	7
8	6
	Total: 25

For this sample of 25 households, the mean number of communicating devices per household is

A. 1.00
B. 1.08
C. 2.08
D. 4.50
E. 5.04

Question 9 (1 mark)
A golfer has scored an average (mean) of 72.6 shots with his last 15 rounds of golf. He would like to reduce his mean number of shots per round to 71. The total number of shots the golfer can have over the next two rounds to make sure this happens is

A. 71
B. 72.6
C. 118
D. 142
E. 144

Question 10 (1 mark)

The following table shows the distribution of car sales by a dealership in one particular month.

$\log_{10}$(price)	**Frequency**
0–	0
1–	0
2–	5
3–	28
4–	64
5	3

The mean sale price of cars during this month is closest to

A. \$4470
B. \$9680
C. \$53 270
D. \$96 850
E. \$106 250

Question 11 (1 mark)

The marks students received in a recent test are displayed in the stem plot below:

Stem	**Leaf**
0	7 8
1	8
2	7 8 8 9
3	2 4 5 6 8 9 9
4	0 2 8 9 9
5	0

Key: 2|7 = 27

The mean test mark is

A. 31.7
B. 33.8
C. 35.5
D. 37.5
E. 39.2

Topic	**1**	**Investigating data distributions**
Subtopic	**1.7**	**The five-number summary and boxplots**

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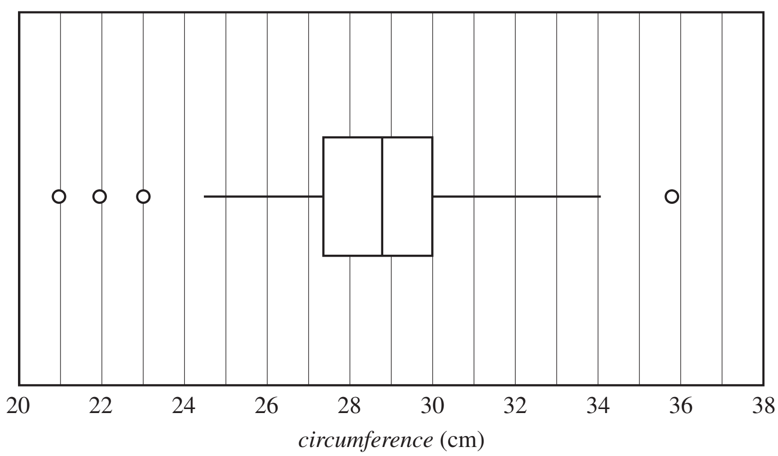
Source: *VCE 2017, Further Mathematics 1, Section A, Core, Q.1; © VCAA*

Question 1 (1 mark)

The boxplot below shows the distribution of the forearm *circumference*, in centimetres, of 252 people.



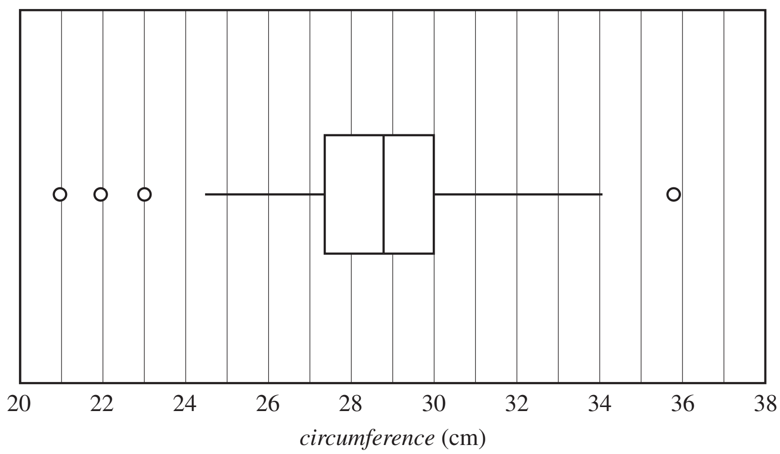
The percentage of these 252 people with a forearm *circumference* of less than 30 cm is closest to

A. 15%
B. 25%
C. 50%
D. 75%
E. 100%

Source: *VCE 2017, Further Mathematics 1, Section A, Core, Q.2; © VCAA*

Question 2 (1 mark)

The boxplot below shows the distribution of the forearm *circumference*, in centimetres, of 252 people.



The five-number summary for the forearm *circumference* of these 252 people is closest to

A. 21, 27.4, 28.7, 30, 34
B. 21, 27.4, 28.7, 30, 35.9
C. 24.5, 27.4, 28.7, 30, 34
D. 24.5, 27.4, 28.7, 30, 35.9
E. 24.5, 27.4, 28.7, 30, 36

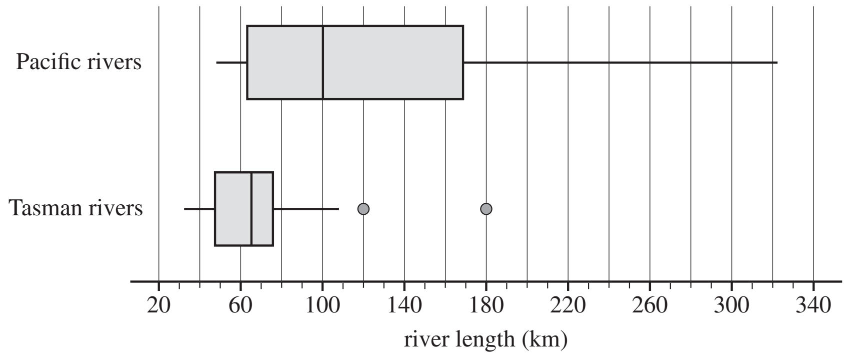
Topic 1 Subtopic 1.7 The five-number summary and boxplots

Source: *VCE 2015, Further Mathematics 1, Section A, Q.6; © VCAA*

Question 3 (1 mark)

In New Zealand, rivers flow into either the Pacific Ocean (the Pacific rivers) or the Tasman Sea (the Tasman rivers).

The boxplots below can be used to compare the distribution of the lengths of the Pacific rivers and the Tasman rivers.



Source: *The New Zealand Yearbook,* 1982

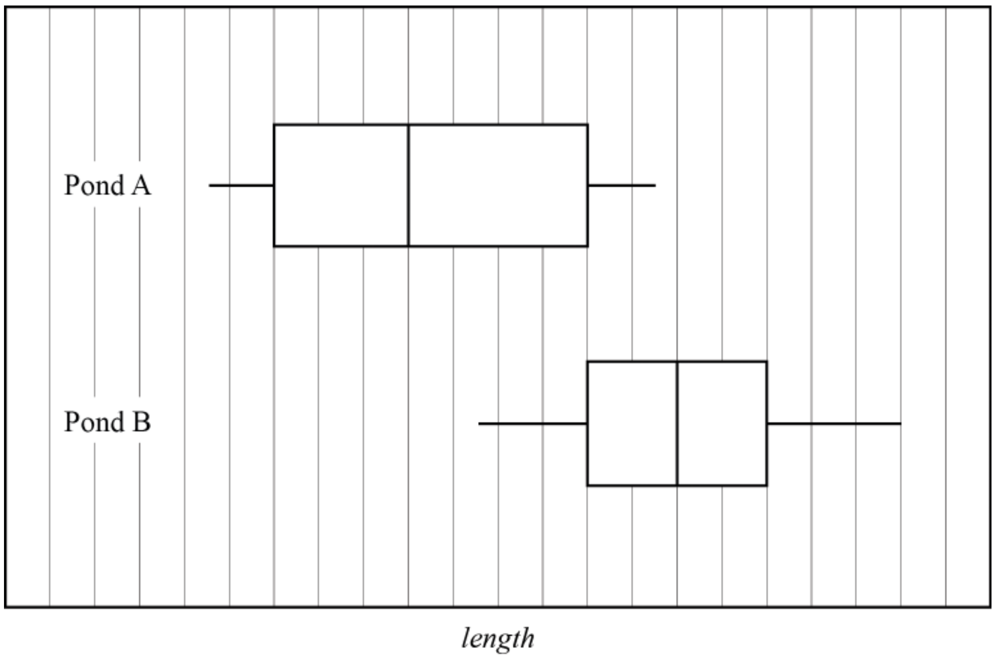
The five-number summary for the lengths of the Tasman rivers is closest to

A. 32, 48, 64, 76, 108
B. 32, 48, 64, 76, 180
C. 32, 48, 64, 76, 322
D. 48, 64, 97, 169, 180
E. 48, 64, 97, 169, 322

Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.4; © VCAA*

Question 4 (1 mark)

The boxplots below show the distribution of the length of fish caught in two different ponds, Pond A and Pond B.



Based on the boxplots above, it can be said that

A. 50% of the fish caught in Pond A are the same length as the fish caught in Pond B.
B. 50% of the fish caught in Pond B are longer than all of the fish caught in Pond A.
C. 50% of the fish caught in Pond B are shorter than all of the fish caught in Pond A.
D. 75% of the fish caught in Pond A are shorter than all of the fish caught in Pond B.
E. 75% of the fish caught in Pond B are longer than all of the fish caught in Pond A.

Topic 1 Subtopic 1.7 The five-number summary and boxplots

Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.5; © VCAA*

Question 5 (1 mark)

The stem plot below shows the *height*, in centimetres, of 20 players in a junior football team.

14	2 2 4 7 8 8 9
15	0 0 1 2 5 5 6 8
16	0 1 1 2
17	9

Key: 14|2 = 142 cm $n = 20$

A player with a height of 179 cm is considered an outlier because 179 cm is greater than

A. 162 cm
B. 169 cm
C. 172.5 cm
D. 173 cm
E. 175.5 cm

Source: *VCE 2020, Further Mathematics 1, Section A, Q.3; © VCAA*

Question 6 (1 mark)

The times between successive nerve impulses (time), in milliseconds, were recorded.

Table 1 shows the mean and the five-number summary calculated using 800 recorded data values.

Table 1

	Time (milliseconds)
Mean	220
Minimum value	10
First quartile (Q1)	70
Median	150
Third quartile (Q3)	300
Maximum value	1380

Data: Adapted from P Fatt and B Katz, 'Spontaneous subthreshold activity at motor nerve endings', *The Journal of Physiology*, 117, 1952, pp. 109–128

The shape of the distribution of these 800 times is best described as

A. approximately symmetric.
B. positively skewed.
C. positively skewed with one or more outliers.
D. negatively skewed.
E. negatively skewed with one or more outliers.

Question 7 (1 mark)
The total rainfall values (in millimetres) measured over many months for a mountainous region of Australia are:
15, 24, 7, 12, 15, 24, 3, 9, 3, 7, 15, 7, 12, 9, 3, 15, 5, 9, 24, 12.
The range and interquartile range respectively for this data are
A. 15, 24
B. 8, 24
C. 15, 21
D. 8, 21
E. 21, 8

Question 8 (1 mark)
The marks students received in a recent test are displayed in the stem plot below:

Stem	Leaf
0	7 8
1	8
2	7 8 8 9
3	2 4 5 6 8 9 9
4	0 2 8 9 9
5	0

Key: 2|7 = 27
The difference between the range and interquartile range is
A. 13
B. 28
C. 30
D. 41
E. 43

Topic	1	Investigating data distributions
Subtopic	1.8	The mean of a sample

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Source: *VCE 2012, Further Mathematics 1, Section A, Q.3; © VCAA*

Question 1 (1 mark)

The total weight of nine oranges is 1.53 kg.

Using this information, the mean weight of an orange would be calculated to be closest to

A. 115 g
B. 138 g
C. 153 g
D. 162 g
E. 170 g

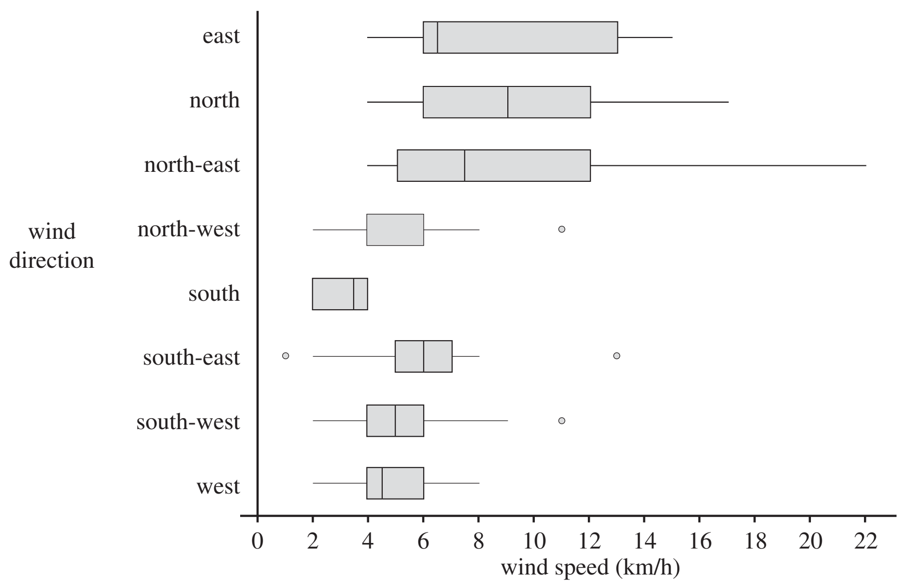
Source: *VCE 2012, Further Mathematics, Exam 2, Core, Q.3; © VCAA*

Question 2 (2 marks)

A weather station records the wind speed and the wind direction each day at 9.00 am.

The wind speed is recorded, correct to the nearest whole number.

The parallel boxplots below have been constructed from data that was collected on the 214 days from June to December in 2011.



a. Complete the following statements.

The wind direction with the lowest recorded wind speed was ________________________.

The wind direction with the largest range of recorded wind speeds was __________________. **(1 mark)**

b. The wind blew from the south on eight days.
Reading from the parallel boxplots above we know that, for these eight wind speeds, the
first quartile $Q1 = 2$ km/h
median $M = 3.5$ km/h
third quartile $Q3 = 4$ km/h
Given that the eight wind speeds were recorded to the nearest whole number, write down the eight wind speeds. **(1 mark)**

Question 3 (1 mark)
A golfer has scored an average (mean) of 72.6 shots with his last 15 rounds of golf. He would like to reduce his mean number of shots per round to 71. The total number of shots the golfer can have over the next two rounds to make sure this happens is
A. 71
B. 72.6
C. 118
D. 142
E. 144

Topic	1	Investigating data distributions
Subtopic	1.9	Standard deviation of a sample

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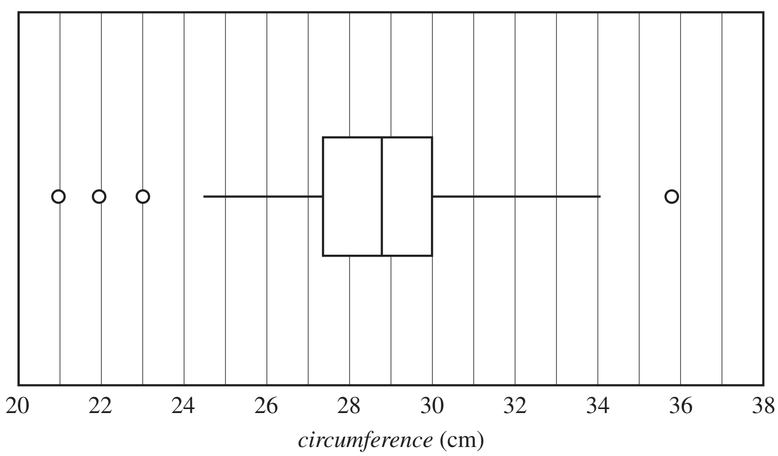
Source: *VCE 2017, Further Mathematics 1, Section A, Core, Q.3; © VCAA*

Question 1 (1 mark)

The boxplot below shows the distribution of the forearm *circumference*, in centimetres, of 252 people.



The table below shows the forearm *circumference*, in centimetres, of a sample of 10 people selected from this group of 252 people.

Circumference	26.0	27.8	28.4	25.9	28.3	31.5	28.2	25.9	27.9	27.8

The mean, $\bar{x}$, and the standard deviation, s_x, of the forearm *circumference* for this sample of people are closest to

A. $\bar{x} = 1.58 \quad s_x = 27.8$

B. $\bar{x} = 1.66 \quad s_x = 27.8$

C. $\bar{x} = 27.8 \quad s_x = 1.58$

D. $\bar{x} = 27.8 \quad s_x = 1.66$

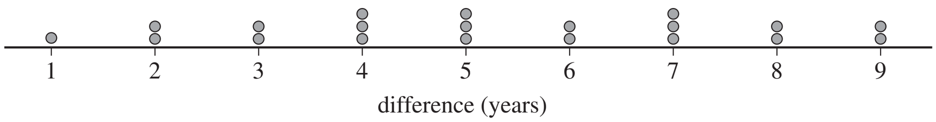
E. $\bar{x} = 27.8 \quad s_x = 2.30$

Topic 1 Subtopic 1.9 Standard deviation of a sample

Source: *VCE 2015, Further Mathematics 1, Section A, Q.3; © VCAA*

Question 2 (1 marks)

The dot plot below displays the difference between female and male life expectancy, in years, for a sample of 20 countries.



The mean $(\bar{x})$ and standard deviation (s) for this data are

A. mean = 2.32 standard deviation = 5.25

B. mean = 2.38 standard deviation = 5.25

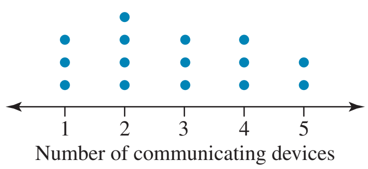
C. mean = 5.0 standard deviation = 2.0

D. mean = 5.25 standard deviation = 2.32

E. mean = 5.25 standard deviation = 2.38

Question 3 (1 mark)

The dot plot below shows the distribution of the number of communicating devices in 15 households.



In this distribution, the standard deviation is:

A. 1.33

B. 1.37

C. 1.41

D. 1.58

E. 2.03

Question 4 (1 mark)

The number of text messages sent during a week by a group of 8 students has a mean of 25 messages sent and a standard deviation of 2.

If, during the next week, a ninth student is included in the sample, and they send 25 text messages while all others send the same number of text messages they had the previous week, we can be certain that

A. the mean and standard deviation will remain at 25 and 2 respectively
B. the mean will remain at 25 and the standard deviation will increase above 2
C. the mean will remain at 25 and the standard deviation will decrease below 2
D. the mean will increase above 25 and the standard deviation remains at 2
E. the mean will decrease below and the standard deviation remains at 2

Question 5 (1 mark)

The number of communication devices in each of 25 households is recorded in the frequency table below.

Number of communication device	Frequency
1	4
3	3
4	2
5	3
6	7
8	6
	Total: 25

For this sample of 25 households, the standard deviation of communicating devices is

A. 6.00
B. 5.04
C. 3.21
D. 2.41
E. 2.36

Topic 1 | Subtopic 1.9 Standard deviation of a sample

Question 6 (1 mark)

The total rainfall values (in millimetres) measured over many months for a mountainous region of Australia are:

15, 24, 7, 12, 15, 24, 3, 9, 3, 7, 15, 7, 12, 9, 3, 15, 5, 9, 24, 12.

The standard deviation for this set of rainfall values, correct to 2 decimal places, is

A. 6.55
B. 6.72
C. 8
D. 10.5
E. 11.5

Topic	1	Investigating data distributions
Subtopic	1.10	The 68–95–99% rule and z-scores

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Source: *VCE 2021, Further Mathematics Exam 1, Section A, Core, Q.7; © VCAA*

Question 1 (1 mark)

800 participants auditioned for a stage musical. Each participant was required to complete a series of ability tests for which they received an overall score.

The overall scores were approximately normally distributed with a mean score of 69.5 points and a standard deviation of 6.5 points.

Only the participants who scored at least 76.0 points in the audition were considered successful.

Using the 68−95−99.7% rule, how many of the participants were considered unsuccessful?

A. 127
B. 128
C. 272
D. 672
E. 673

Source: *VCE 2021, Further Mathematics Exam 1, Section A, Core, Q.8; © VCAA*

Question 2 (1 mark)

800 participants auditioned for a stage musical. Each participant was required to complete a series of ability tests for which they received an overall score.

The overall scores were approximately normally distributed with a mean score of 69.5 points and a standard deviation of 6.5 points.

To be offered a leading role in the stage musical, a participant must achieve a standardised score of at least 1.80

Three participants' names and their overall scores are given in the table below.

Participant	Overall score
Amy	81.5
Brian	80.5
Cherie	82.0

Which one of the following statements is true?

A. Only Amy was offered a leading role.
B. Only Cherie was offered a leading role.
C. Only Brian was not offered a leading role.
D. Both Brian and Cherie were offered leading roles.
E. All three participants were offered leading roles.

Source: *VCE 2021, Further Mathematics Exam 1, Section A, Core, Q.9; © VCAA*

Question 3 (1 mark)

The heights of females living in a small country town are normally distributed:

- 16% of the females are more than 160 cm tall.
- 2.5% of the females are less than 115 cm tall.

The mean and the standard deviation of this female population, in centimetres, are closest to

A. mean = 135 standard deviation = 15
B. mean = 135 standard deviation = 25
C. mean = 145 standard deviation = 15
D. mean = 145 standard deviation = 20
E. mean = 150 standard deviation = 10

Source: *VCE 2020, Further Mathematics 1, Section A, Q.8; © VCAA*

Question 4 (1 mark)

The *wing length* of a species of bird is approximately normally distributed with a mean of 61 mm and a standard deviation of 2 mm.

Using the 68–95–99.7% rule, for a random sample of 10 000 of these birds, the number of these birds with a *wing length* of less than 57 mm is closest to

A. 50
B. 160
C. 230
D. 250
E. 500

Source: *VCE 2020, Further Mathematics 1, Section A, Q.9; © VCAA*

Question 5 (1 mark)

The lifetime of a certain brand of light globe, in hours, is approximately normally distributed.

It is known that 16% of the light globes have a lifetime of less than 655 hours and 50% of the light globes have a lifetime that is greater than 670 hours.

The mean and the standard deviation of this normal distribution are closest to

A. mean = 655 hours standard deviation = 10 hours
B. mean = 655 hours standard deviation = 15 hours
C. mean = 670 hours standard deviation = 10 hours
D. mean = 670 hours standard deviation = 15 hours
E. mean = 670 hours standard deviation = 20 hours

Source: *VCE 2019, Further Mathematics 1, Section A, Q.6; © VCAA*

Question 6 (1 mark)

The time taken to *travel* between two regional cities is approximately normally distributed with a mean of 70 minutes and a standard deviation of 2 minutes.

The percentage of *travel* times that are between 66 minutes and 72 minutes is closest to

A. 2.5%
B. 34%
C. 68%
D. 81.5%
E. 95%

Topic 1 Subtopic 1.10 The 68–95–99% rule and z-scores

Source: *VCE 2019, Further Mathematics 1, Section A, Q.7; © VCAA*

Question 7 (1 mark)

The volume of a cup of soup served by a machine is normally distributed with a mean of 240 mL and a standard deviation of 5 mL.

A fast-food store used this machine to serve 160 cups of soup.

The number of these cups of soup that are expected to contain less than 230 mL of soup is closest to

A. 3
B. 4
C. 8
D. 26
E. 156

Source: *VCE 2016, Further Mathematics 1, Section A, Q;4; © VCAA*

Question 8 (1 mark)

The weights of male players in a basketball competition are approximately normally distributed with a mean of 78.6 kg and a standard deviation of 9.3 kg.

There are 456 male players in the competition.

The expected number of male players in the competition with weights above 60 kg is closest to

A. 3
B. 11
C. 23
D. 433
E. 445

Source: *VCE 2016, Further Mathematics 1, Section A, Q.5; © VCAA*

Question 9 (1 mark)

The weights of male players in a basketball competition are approximately normally distributed with a mean of 78.6 kg and a standard deviation of 9.3 kg.

Brett and Sanjeeva both play in the basketball competition.

When the weights of all players in the competition are considered, Brett has a standardised weight of $z = -0.96$ and Sanjeeva has a standardised weight of $z = -0.26$

Which one of the following statements is **not** true?

A. Brett and Sanjeeva are both below the mean weight for players in the basketball competition.
B. Sanjeeva weighs more than Brett.
C. If Sanjeeva increases his weight by 2 kg, he would be above the mean weight for players in the basketball competition.
D. Brett weighs more than 68 kg.
E. More than 50% of the players in the basketball competition weigh more than Sanjeeva.

Source: *VCE 2015, Further Mathematics 1, Section A, Q.4; © VCAA*

Question 10 (1 mark)

The foot lengths of a sample of 2400 women were approximately normally distributed with a mean of 23.8 cm and a standard deviation of 1.2 cm.

The expected number of these women with foot lengths less than 21.4 cm is closest to

A. 60
B. 120
C. 384
D. 2280
E. 2340

Topic 1 Subtopic 1.10 The 68–95–99% rule and z-scores

Source: *VCE 2015, Further Mathematics 1, Section A, Q.5; © VCAA*

Question 11 (1 mark)

The foot lengths of a sample of 2400 women were approximately normally distributed with a mean of 23.8 cm and a standard deviation of 1.2 cm.

The standardised foot length of one of these women is $z = -1.3$

Her actual foot length, in centimetres, is closest to

A. 22.2
B. 22.7
C. 25.3
D. 25.6
E. 31.2

Source: *VCE 2014, Further Mathematics 1, Section A, Q.2; VCAA*

Question 12 (1 mark)

The time spent by shoppers at a hardware store on a Saturday is approximately normally distributed with a mean of 31 minutes and a standard deviation of 6 minutes.

If 2850 shoppers are expected to visit the store on a Saturday, the number of shoppers who are expected to spend between 25 and 37 minutes in the store is closest to

A. 16
B. 68
C. 460
D. 1900
E. 2400

Topic	1	Investigating data distributions
Subtopic	1.11	Review

online only

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Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.5; © VCAA*

Question 1 (1 mark)

The stem plot below shows the *height*, in centimetres, of 20 players in a junior football team.

14	2 2 4 7 8 8 9
15	0 0 1 2 5 5 6 8
16	0 1 1 2
17	9

Key: 14|2 = 142 cm $\quad n = 20$

A player with a height of 179 cm is considered an outlier because 179 cm is greater than

A. 162 cm
B. 169 cm
C. 172.5 cm
D. 173 cm
E. 175.5 cm

Source: *VCE 2021, Further Mathematics 2, Section A, Core, Q.1; © VCAA*

Question 2 (7 marks)

In the sport of heptathlon, athletes compete in seven events.

These events are the 100 m hurdles, high jump, shot-put, javelin, 200 m run, 800 m run and long jump.

Fifteen female athletes competed to qualify for the heptathlon at the Olympic Games.

Their results for three of the heptathlon events – high jump, shot-put and javelin – are shown in Table 1.

Table 1

Athlete number	High jump (metres)	Shot-put (metres)	Javelin (metres)
1	1.76	15.34	41.22
2	1.79	16.96	42.41
3	1.83	13.87	46.53
4	1.82	14.23	40.62
5	1.87	13.78	45.64
6	1.73	14.50	42.33
7	1.68	15.08	40.88
8	1.82	13.13	39.22
9	1.83	14.22	42.51
10	1.87	13.62	42.75
11	1.87	12.01	38.12
12	1.80	12.88	42.65
13	1.83	12.68	45.68
14	1.87	12.45	41.32
15	1.78	11.31	42.88

a. Write down the number of numerical variables in Table 1. **(1 mark)**

b. Complete Table 2 below by calculating the mean height jumped for the high jump, in metres, by the 15 athletes. Write your answer in the space provided in the table. **(1 mark)**

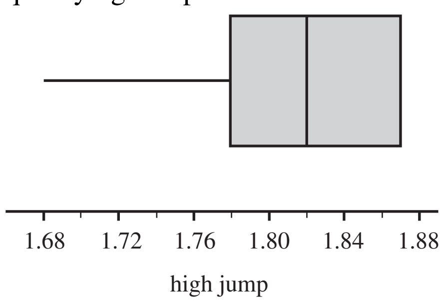
Table 2

Statistic	High jump (metres)	Shot-put (metres)
mean		13.74
standard deviation	0.06	1.43

c. In shot-put, athletes throw a heavy spherical ball (a shot) as far as they can.
Athlete number six, Jamilia, threw the shot 14.50 m.
Calculate Jamilia's standardised score (z).
Round your answer to one decimal place. **(1 mark)**

d. In the qualifying competition, the heights jumped in the high jump are expected to be approximately normally distributed.
Chara's jump in this competition would give her a standardised score of $z = -1.0$
Use the $68 - 95 - 99.7\%$ rule to calculate the percentage of athletes who would be expected to jump higher than Chara in the qualifying competition. **(1 mark)**

e. The boxplot below was constructed to show the distribution of high jump heights for all 15 athletes in the qualifying competition. **(1 mark)**



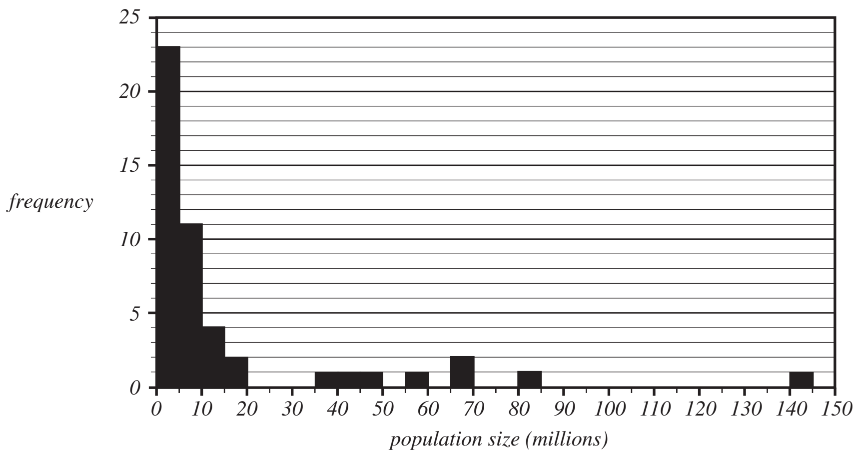
Explain why the boxplot has no whisker at its upper end.

f. For the javelin qualifying competition (refer to Table 1), another boxplot is used to display the distribution of athletes' results.
An athlete whose result is displayed as an outlier at the upper end of the plot is considered to be a potential medal winner in the event.
What is the minimum distance that an athlete needs to throw the javelin to be considered a potential medal winner? **(2 marks)**

Source: *VCE 2019, Further Mathematics 1, Section A, Q.2; © VCAA*

Question 3 (1 mark)

The histogram below shows the distribution of the *population size* of 48 countries in 2018.



Data: Worldometers, <www.worldometers.info/>

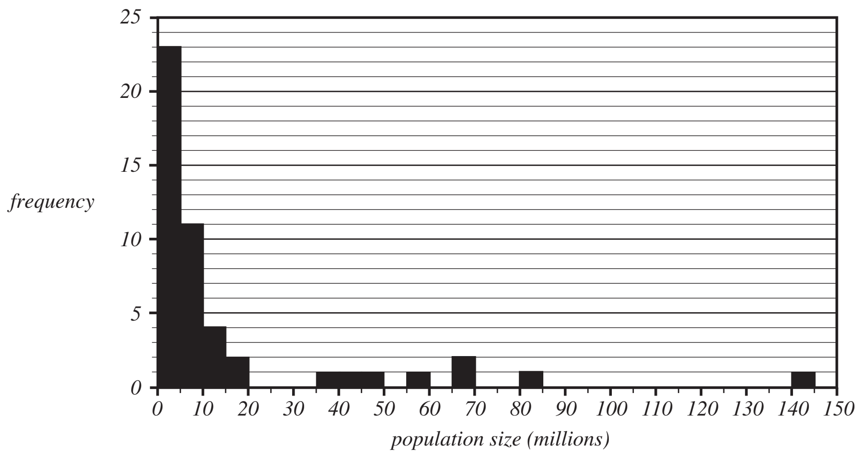
The shape of this histogram is best described as

A. positively skewed with no outliers.
B. positively skewed with outliers.
C. approximately symmetric.
D. negatively skewed with no outliers.
E. negatively skewed with outliers.

Source: *VCE 2019, Further Mathematics 1, Section A, Q.3; © VCAA*

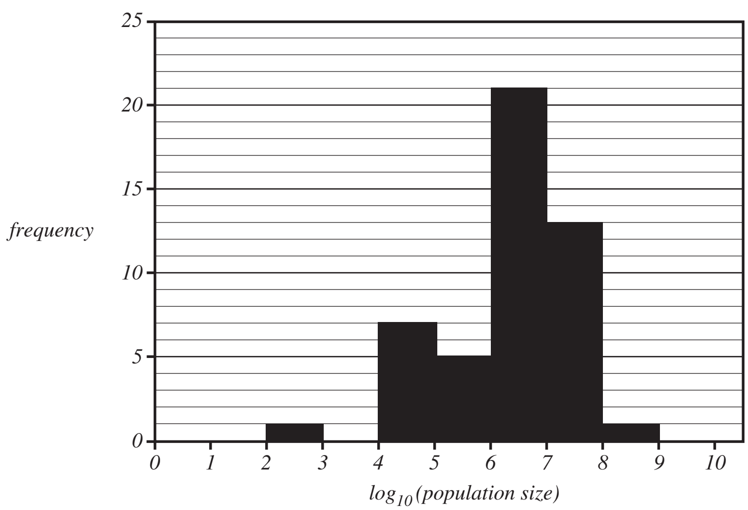
Question 4 (1 mark)

The histogram below shows the distribution of the *population size* of 48 countries in 2018.



Data: Worldometers, <www.worldometers.info/>

The histogram below shows the *population size* for these 48 countries plotted on a $\log_{10}$ scale.



Data: Worldometers, <www.worldometers.info/>

Based on this histogram, the number of countries with a *population size* that is less than 100 000 people is

A. 1
B. 5
C. 7
D. 8
E. 48

Source: *VCE 2018, Further Mathematics 2, Section A, Q.1; © VCAA*

Question 5 (8 marks)

Table 1

City	*Congestion level*	*Size*	*Increase in travel time* (minutes per day)
Belfast	high	small	52
Edinburgh	high	small	43
London	high	large	40
Manchester	high	large	44
Brighton and Hove	high	small	35
Bournemouth	high	small	36
Sheffield	medium	small	36
Hull	medium	medium	40
Bristol	medium	small	39
Newcastle-Sunderland	medium	large	34
Leicester	medium	small	36
Liverpool	medium	large	29
Swansea	low	small	30
Glasgow	low	large	34
Cardiff	low	small	31
Nottingham	low	small	31
Birmingham-Wolverhampton	low	large	29
Leeds-Bradford	low	large	31
Portsmouth	low	small	27
Southampton	low	small	30
Reading	low	small	31
Coventry	low	small	30
Stoke-on-Trent	low	small	29

The data in Table 1 relates to the impact of traffic congestion in 2016 on travel times in 23 cities in the United Kingdom (UK).

The four variables in this data set are:

- *city* – name of city
- *congestion level* – traffic congestion level (high, medium, low)
- *size* – size of city (large, small)
- *increase in travel time* – increase in travel time due to traffic congestion (minutes per day).

a. How many variables in this data set are categorical variables? **(1 mark)**

b. How many variables in this data set are ordinal variables? **(1 mark)**

c. Name the large UK cities with a medium level of traffic congestion. **(1 mark)**

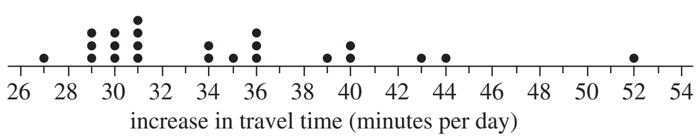
d. Use the data in Table 1 to complete the following two-way frequency table, Table 2. **(2 marks)**

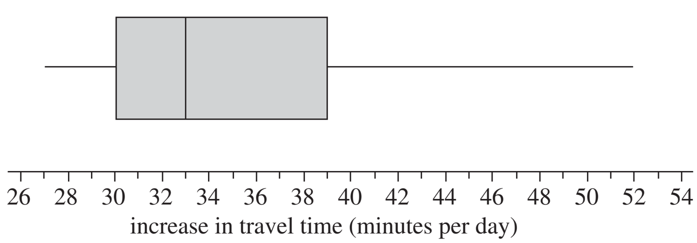
Table 2

	City size	
Congestion level	**Small**	**Large**
high	4	
medium		
low		
Total	16	

e. What percentage of the small cities have a high level of traffic congestion? **(1 mark)**

f. Traffic congestion can lead to an increase in travel times in cities. The dot plot and boxplot below both show the increase in travel time due to traffic congestion, in minutes per day, for the 23 UK cities.





Describe the shape of the distribution of the increase in travel time for the 23 cities. **(1 mark)**

g. The data value 52 is below the upper fence and is not an outlier.
Determine the value of the upper fence. **(1mark)**

Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.6; © VCAA*

Question 6 (1 mark)

The relationship between *resting pulse rate*, in beats per minute, and *age group* (15–20 years, 21–30 years, 31–50 years, over 50 years) is best displayed using

A. a histogram.
B. a scatterplot.
C. parallel boxplots.
D. a time series plot.
E. a back-to-back stem plot.

Source: *VCE 2020, Further Mathematics 2, Section A, Q.1; © VCAA*

Question 7 (3 marks)

Body mass index (*BMI*), in kilograms per square metre, was recorded for a sample of 32 men and displayed in the ordered stem plot below.

21	6 9 9
22	1 2 5 6
23	0 1 4 6 6 7 8
24	4 5 6 7 7 9
25	6 8
26	1 7 9
27	3 7
28	2
29	1 8
30	4
31	1

Key: 14|2 = 142 cm $\quad n = 32$

a. Describe the shape of the distribution. **(1 mark)**

b. Determine the median *BMI* for this group of men. **(1 mark)**

c. People with a *BMI* of 25 or over are considered to be overweight.
What percentage of these men would be considered to be overweight? **(1 mark)**

Source: *VCE 2020, Further Mathematics 1, Section A, Q.10; © VCAA*

Question 8 (1 mark)

The data in Table 2 was collected in a study of the association between the variables *frequency of nightmares* (low, high) and snores (no, yes).

Table 2

Frequency of nightmares	***Total***		**Snores**
	No	**Yes**	
low	80	58	138
high	11	12	23
Total	91	70	161

Data: Adapted from RA Hicks and J Bautista, 'Snoring and nightmares', *Perceptual and Motor Skills*, 1 October 1993, <https://doi.org/10.2466/pms.1993.77.2.433>

The variables in this study, *frequency of nightmares* (low, high) and *snores* (no, yes), are

A. ordinal and nominal respectively.
B. nominal and ordinal respectively.
C. both numerical.
D. both ordinal.
E. both nominal.

Source: *VCE 2020, Further Mathematics 2, Section A, Q.1; © VCAA*

Question 9 (3 marks)

Body mass index (*BMI*), in kilograms per square metre, was recorded for a sample of 32 men and displayed in the ordered stem plot below.

21	6 9 9
22	1 2 5 6
23	0 1 4 6 6 7 8
24	4 5 6 7 7 9
25	6 8
26	1 7 9
27	3 7
28	2
29	1 8
30	4
31	1

Key: 14|2 = 142 cm $\quad n = 32$

a. Describe the shape of the distribution. **(1 mark)**

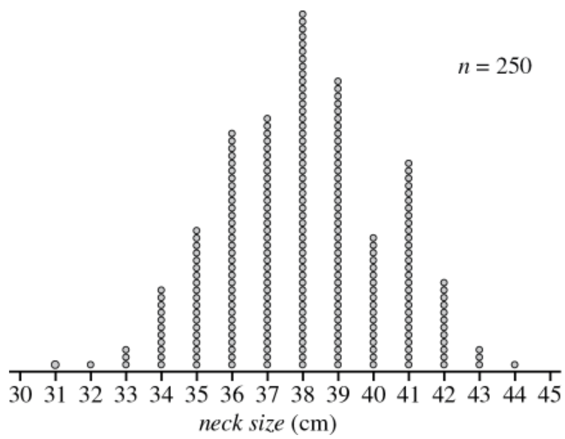
b. Determine the median *BMI* for this group of men. **(1 mark)**

c. People with a *BMI* of 25 or over are considered to be overweight.
What percentage of these men would be considered to be overweight? **(1 mark)**

Source: *VCE 2020, Further Mathematics 2, Section A, Q.2; © VCAA*

Question 10 (5 marks)

The *neck* size, in centimetres, of 250 men was recorded and displayed in the dot plot below.



Data: RW Johnson, 'Fitting percentage of body fat to simple body measurements', *Journal of Statistics Education*, 4:1, 1996, <https://doi.org/10.1080/10691898.1996.11910505>

a. Write down the modal *neck size*, in centimetres, for these 250 men. **(1 mark)**

b. Assume that this sample of 250 men has been drawn at random from a population of men whose neck size is normally distributed with a mean of 38 cm and a standard deviation of 2.3 cm.

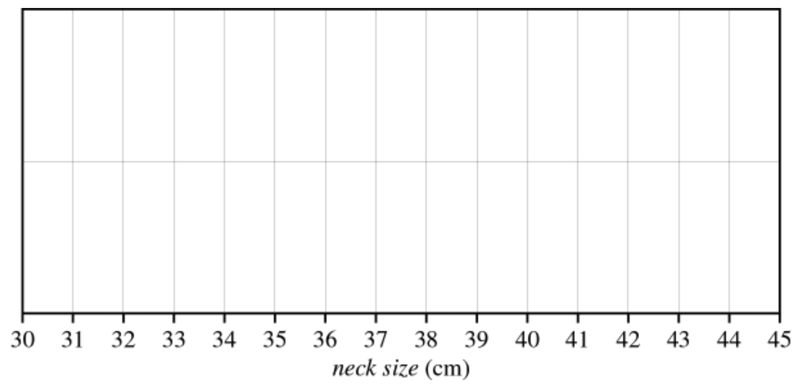
i. How many of these 250 men are **expected** to have a *neck size* that is more than three standard deviations above or below the mean?
Round your answer to the nearest whole number. **(1 mark)**

ii. How many of these 250 men **actually** have a neck size that is more than three standard deviations above or below the mean? **(1 mark)**

c. The five-number summary for this sample of neck sizes, in centimetres, is given below.

Minimum	First quartile (Q_1)	Median	Third quartile (Q_3)	Maximum
31	36	38	39	44

Use the five-number summary to construct a boxplot, showing any outliers if appropriate, on the grid below. **(2 marks)**



Source: *VCE 2020, Further Mathematics 2, Section A, Q.3; © VCAA*

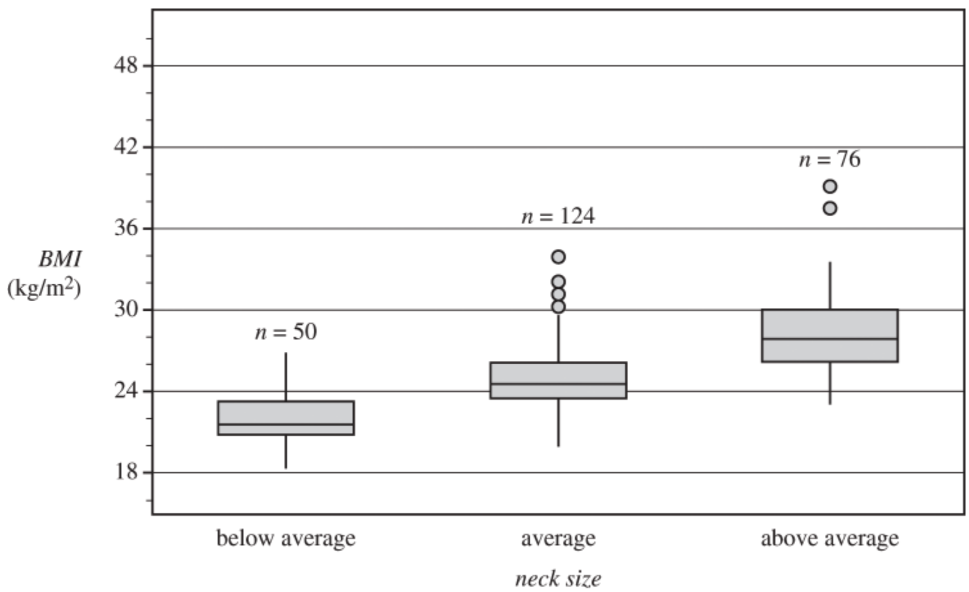
Question 11 (5 marks)

In a study of the association between *BMI* and *neck size*, 250 men were grouped by *neck size* (below average, average and above average) and their *BMI* recorded.

Five-number summaries describing the distribution of *BMI* for each group are displayed in the table below along with the group size.

The associated boxplots are shown below the table.

		BMI(kg/m^2)				
Neck size	**Group size**	**Min.**	**(Q_1)**	**Median**	**(Q_3)**	**Max.**
below average	50	18.1	20.6	21.6	23.2	26.8
average	124	19.8	23.4	24.6	26.0	33.9
above average	76	23.1	26.25	28.1	29.95	39.1



Data: RW Johnson, 'Fitting percentage of body fat of simple body measurement', *Journal of Statistics Education*, 4:1, 1996, <http://doi.org/10.1080/1069 1898.1996.11910505>

a. What percentage of these 250 men are classified as having a below average neck size? **(1 mark)**

b. What is the interquartile range (IQR) of *BMI* for the men with an average neck size? **(1 mark)**

c. People with a *BMI* of 30 or more are classified as being obese.
Using this criterion, how many of these 250 men would be classified as obese? Assume that the *BMI* values were all rounded to one decimal place. **(1 mark)**

d. Do the boxplots support the contention that *BMI* is associated with *neck size*? Refer to the values of an appropriate statistic in your response. **(2 marks)**

Source: *VCE 2020, Further Mathematics 1, Section A, Q.2; © VCAA*

Question 12 (1 mark)

The times between successive nerve impulses (time), in milliseconds, were recorded.
Table 1 shows the mean and the five-number summary calculated using 800 recorded data values.

	Time **(milliseconds)**
Mean	220
Minimum value	10
First quartile (Q_1)	70
Median	150
Third quartile (Q_3)	300
Maximum value	1380

Data: Adapted from P Fatt and B Katz, 'Spontaneous subthreshold activity at motor nerve endings', *The Journal of Physiology*, 117, 1952, pp. 109 – 128

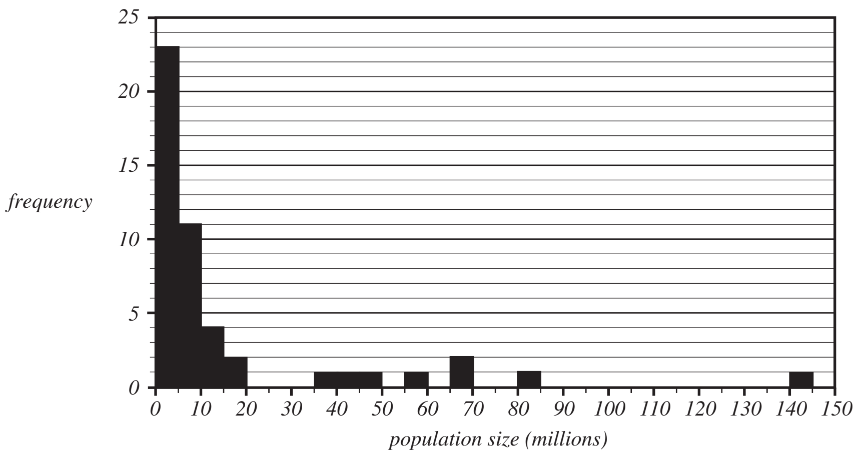
Of these 800 times, the number of times that are longer than 300 milliseconds is closest to

A. 20
B. 25
C. 75
D. 200
E. 400

Source: *VCE 2019, Further Mathematics 1, Section A, Q.1; © VCAA*

Question 13 (1 mark)

The histogram below shows the distribution of the *population size* of 48 countries in 2018.



Data: Worldometers, <www.worldometers.info/>

The number of these countries with a *population size* between 5 million and 20 million people is

A. 11
B. 17
C. 23
D. 34
E. 35

Source: *VCE 2019, Further Mathematics 2, Section A, Q.1; © VCAA*

Question 14 (4 marks)

Table 1 shows the *day number* and the *minimum temperature*, in degrees Celsius, for 15 consecutive days in May 2017.

Table 1

Day number	***Minimum temperature*** (°C)
1	12.7
2	11.8
3	10.7
4	9.0
5	6.0
6	7.0
7	4.1
8	4.8
9	9.2
10	6.7
11	7.5
12	8.0
13	8.6
14	9.8
15	7.7

Data: Australian Government, Bureau of Meteorology, <www.bom.gov.au/>

a. Which of the two variables in this data set is an ordinal variable? **(1 mark)**

b. The incomplete ordered stem plot below has been constructed using the data values for days 1 to 10.

key: $4|1 = 4.1$ $\quad n = 15$

minimum temperature (°C)

4	1	8
5		
6	0	7
7	0	
8		
9	0	2
10	7	
11	8	
12	7	

Complete the **stem plot above** by adding the data values for days 11 to 15.

(*Answer on the stem plot above*.) **(1 mark)**

c. The ordered stem plot below shows the *maximum temperature*, in degrees Celsius, for the same 15 days.

key: 9|2 = 9.2 $n = 15$

minimum temperature (°C) **(2 marks)**

9	2
10	
11	5 6
12	2 5
13	5 5 7
14	9 9
15	0 2 5 6
16	0

Data: Australian Government, Bureau of Meteorology, <www.bom.gov.au/>

Use this stem plot to determine

i. the value of the first quartile (Q_1) **(1 mark)**

ii. the percentage of days with a maximum temperature higher than 15.3 °C. **(1 mark)**

Source: *VCE 2019, Further Mathematics 2, Section A, Q.1; © VCAA*

Question 15 (4 marks)

Table 1 shows the *day number* and the *minimum temperature*, in degrees Celsius, for 15 consecutive days in May 2017.

Table 1

Day number	*Minimum temperature*(°*C*)
1	12.7
2	11.8
3	10.7
4	9.0
5	6.0
6	7.0
7	4.1
8	4.8
9	9.2
10	6.7
11	7.5
12	8.0
13	8.6
14	9.8
15	7.7

Data: Australian Government, Bureau of Meteorology, <www.bom.gov.au/>

a. Which of the two variables in this data set is an ordinal variable? **(1 mark)**

b. The incomplete ordered stem plot below has been constructed using the data values for days 1 to 10.

key: $4|1 = 4.1$ $n = 15$

minimum temperature (°C)

4	1 8
5	
6	0 7
7	0
8	
9	0 2
10	7
11	8
12	7

Complete the **stem plot above** by adding the data values for days 11 to 15.

(*Answer on the stem plot above.*) **(1 mark)**

c. The ordered stem plot below shows the *maximum temperature*, in degrees Celsius, for the same 15 days.

key: 9|2 = 9.2 $n = 15$

minimum temperature (°C)

9	2
10	
11	5 6
12	2 5
13	5 5 7
14	9 9
15	0 2 5 6
16	0

Data: Australian Government, Bureau of Meteorology, <www.bom.gov.au/>

Use this stem plot to determine **(2 marks)**

i. the value of the first quartile (Q_1) **(1 mark)**

ii. the percentage of days with a maximum temperature higher than 15.3 °C. **(1 mark)**

Source: *VCE 2018, Further Mathematics 1, Section A, Q.4; © VCAA*

Question 16 (1 mark)

The pulse rates of a population of Year 12 students are approximately normally distributed with a mean of 69 beats per minute and a standard deviation of 4 beats per minute.

Another student selected at random from this population has a standardised pulse rate of $z = -1$.

The percentage of students in this population with a pulse rate greater than this student is closest to

A. 2.5%
B. 5%
C. 16%
D. 68%
E. 84%

Source: *VCE 2018, Further Mathematics 1, Section A, Q.5; © VCAA*

Question 17 (1 mark)

The pulse rates of a population of Year 12 students are approximately normally distributed with a mean of 69 beats per minute and a standard deviation of 4 beats per minute.

A sample of 200 students was selected at random from this population.

The number of these students with a pulse rate of less than 61 beats per minute or greater than 73 beats per minute is closest to

A. 19
B. 37
C. 64
D. 95
E. 190

Source: *VCE 2017, Further Mathematics 2, Section A, Core, Q.1; © VCAA*

Question 18 (5 marks)

The number of eggs counted in a sample of 12 clusters of moth eggs is recorded in the table below.

Number of eggs	172	192	159	125	197	135	140	140	138	166	136	131

a. From the information given, determine **(2 marks)**

i. the range **(1 mark)**

ii. the percentage of clusters in this sample that contain more than 170 eggs. **(1 mark)**

b. In a large population of moths, the number of eggs per cluster is approximately normally distributed with a mean of 165 eggs and a standard deviation of 25 eggs.

Using the 68 – 95 – 99.7% rule, determine **(2 marks)**

i. the percentage of clusters expected to contain more than 140 eggs. **(1 mark)**

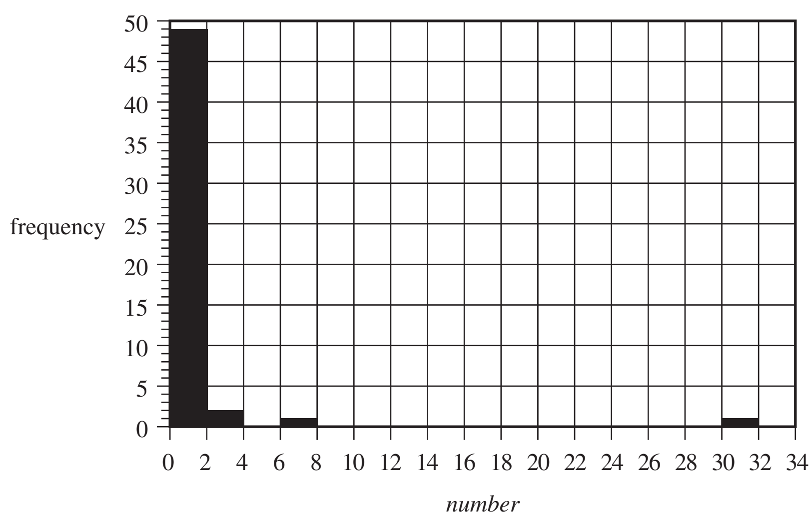
ii. the number of clusters expected to have less than 215 eggs in a sample of 1000 clusters. **(1 mark)**

c. The standardised number of eggs in one cluster is given by $z = -2.4$
Determine the actual number of eggs in this cluster. **(1 mark)**

Source: *VCE 2016, Further Mathematics 1, Section A, Q.6; © VCAA*

Question 19 (1 mark)

The histogram below shows the distribution of the *number* of billionaires per million people for 53 countries.



Data: Gapminder

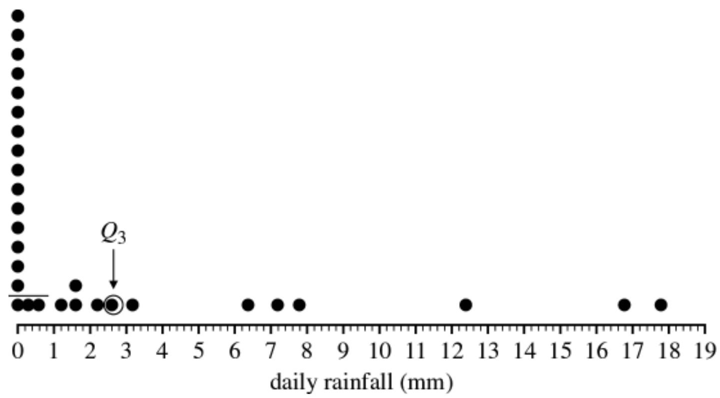
Using this histogram, the percentage of these 53 countries with less than two billionaires per million people is closest to

A. 49%
B. 53%
C. 89%
D. 92%
E. 98%

Source: *VCE 2016, Further Mathematics 2, Core, Q.1; © VCAA*

Question 20 (6 marks)

The dot plot below shows the distribution of daily rainfall, in millimetres, at a weather station for 30 days in September.



a. Write down the **(2 marks)**

i. range **(1 mark)**

ii. median. **(1 mark)**

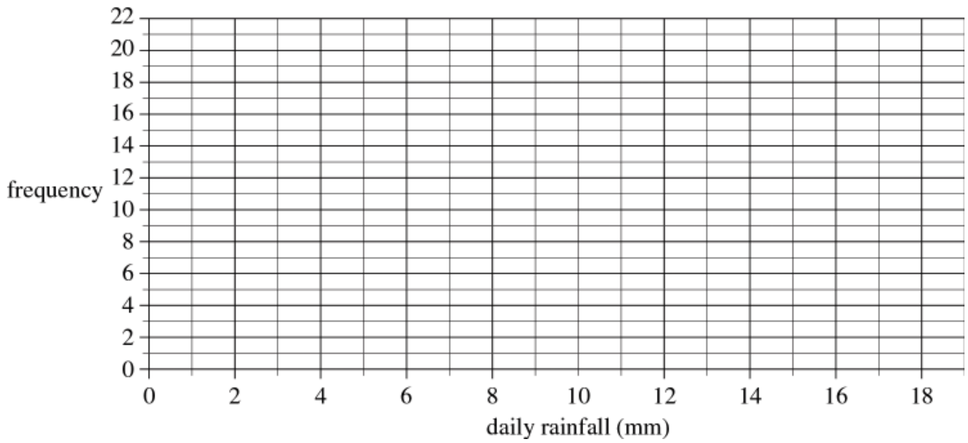
b. Circle the data point on the **dot plot above** that corresponds to the third quartile (Q_3). **(1 mark)**

c. Write down **(2 marks)**

i. the number of days on which no rainfall was recorded **(1 mark)**

ii. the percentage of days on which the daily rainfall exceeded 12 mm. **(1 mark)**

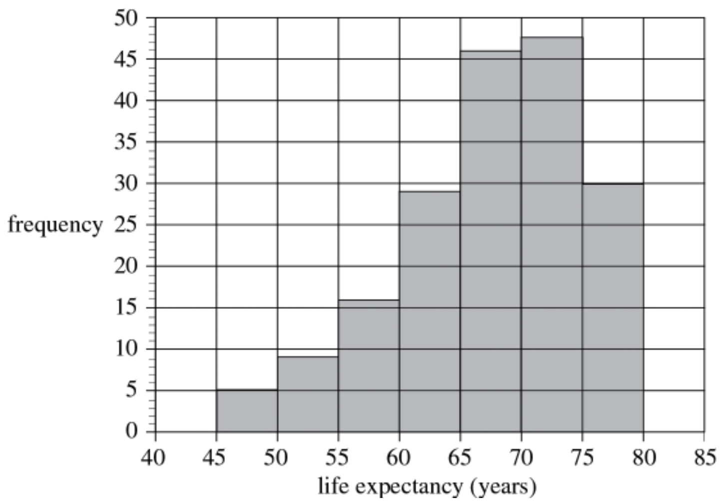
d. Use the grid below to construct a histogram that displays the distribution of daily rainfall for the month of September. Use interval widths of two with the first interval starting at 0. **(1 mark)**



Source: *VCE 2015, Further Mathematics 2, Core, Q.1; © VCAA*

Question 21 (3 marks)

The histogram below shows the distribution of life expectancy of people for 183 countries.



a. For this distribution, the modal interval is ________. **(1 mark)**

b. In how many of these countries is life expectancy less than 55 years? **(1 mark)**

c. In what percentage of these 183 countries is life expectancy between 75 and 80 years? Write your answer correct to one decimal place. **(1 mark)**

Source: *VCE 2015, Further Mathematics 1, Section A, Q.1; © VCAA*

Question 22 (1 mark)

The stem plot below displays the average number of decayed teeth in 12-year-old children from 31 countries.

key: 0|2 = 0.2

0	2
0	5 6 7 7 8 9
1	0 0 0 0 1 4 4 4
1	5 6 7
2	3 3 4
2	7 7 8 9
3	0 4
3	5 6
4	1
4	7

Data: Gapminder

Based on this stem plot, the distribution of the average number of decayed teeth for these countries is best described as

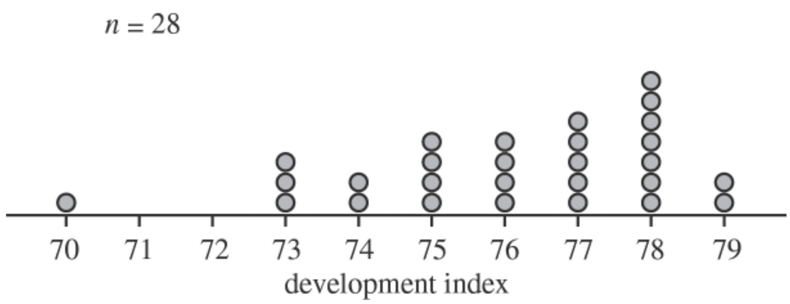
A. negatively skewed with a median of 15 decayed teeth and a range of 45
B. positively skewed with a median of 15 decayed teeth and a range of 45
C. approximately symmetric with a median of 1.5 decayed teeth and a range of 4.5
D. negatively skewed with a median of 1.5 decayed teeth and a range of 4.5
E. positively skewed with a median of 1.5 decayed teeth and a range of 4.5

Source: *VCE 2013, Further Mathematics 2, Core, Q.2; © VCAA*

Question 23 (2 marks)

The development index for each country is a whole number between 0 and 100.

The dot plot below displays the values of the development index for each of the 28 countries that has a high development index.



a. Using the information in the dot plot, determine each of the following.

- the mode
- the range **(1 mark)**

b. Write down an appropriate calculation and use it to explain why the country with a development index of 70 is an outlier for this group of countries. **(1 mark)**

Source: *VCE 2013, Further Mathematics 1, Section A, Core, Q.5; © VCAA*

Question 24 (1 mark)

The time, in hours, that each student spent sleeping on a school night was recorded for 1550 secondary-school students. The distribution of these times was found to be approximately normal with a mean of 7.4 hours and a standard deviation of 0.7 hours.

The time that 95% of these students spent sleeping on a school night could be

A. less than 6.0 hours.
B. between 6.0 and 8.8 hours.
C. between 6.7 and 8.8 hours.
D. less than 6.0 hours or greater than 8.8 hours.
E. less than 6.7 hours or greater than 9.5 hours.

Topic 1 Subtopic 1.11 Review

Source: *VCE 2013, Further Mathematics 1, Section A, Core, Q.6; © VCAA*

Question 25 (1 mark)

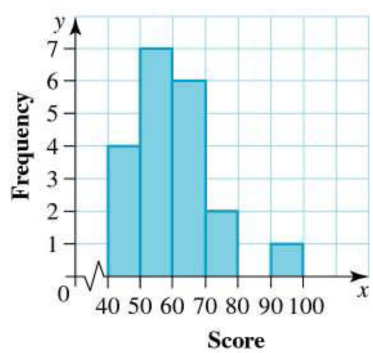
The time, in hours, that each student spent sleeping on a school night was recorded for 1550 secondary-school students. The distribution of these times was found to be approximately normal with a mean of 7.4 hours and a standard deviation of 0.7 hours.

The number of these students who spent more than 8.1 hours sleeping on a school night was closest to

A. 16
B. 248
C. 1302
D. 1510
E. 1545

Question 26 (1 mark)

The histogram shows the test scores of a group of students studying General Mathematics 12.



What is the total number of students in the class?

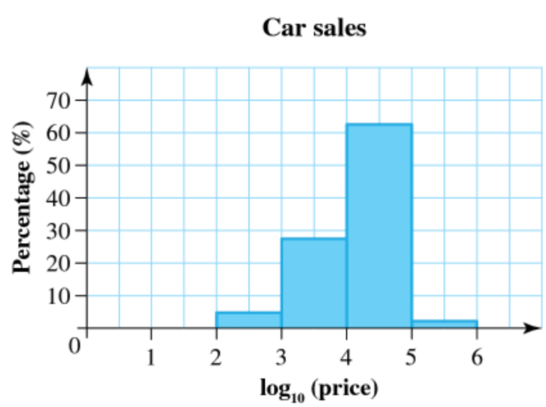
A. 5
B. 20
C. 7
D. 18
E. 22

Question 27 (1 mark)
Numerical data showing the estimated average number of visitors in one month to a selection of websites was recorded using a base 10 log scale. The table below shows the results.

Website	$\log_{10}$ (number of visitors)
Yourtube	6.301
Yippee	9
Goodday	8.699
Televise	7.301

Based on these figures the number of actual visitors to Yippee compared to Televise was approximately:
A. 1.7 times more
B. 5 times more
C. 10 times more
D. 20 times more
E. 50 times more

Question 28 (1 mark)
The graph below shows the distribution of car prices during a particular month of the year.



The percentage of sales with car prices below $100 000 was:
A. 33%
B. 64%
C. 67%
D. 90%
E. 97%

Question 29 (1 mark)
The graph below displays the distribution of real estate sales made by a local real estate agency.

The most frequent prices of real estate sales were between:
A. 5 – 6
B. 4 – 5
C. \$1000 – \$1000
D. \$10 000 – \$100 000
E. \$100 000 – \$1 000 000

Question 30 (1 mark)
The marks students received in a recent test are displayed in the stem plot below:

Stem	Leaf
0	7 8
1	8
2	7 8 8 9
3	2 4 5 6 8 9 9
4	0 2 8 9 9
5	0

key: 2|7 = 27
The median test mark is:
A. 25
B. 27
C. 33.5
D. 35.5
E. 50

Question 31 (1 mark)

The mean weekly pay for shop assistants is \$480.

If Liam earns \$460 and this represents a z-score of -2, what is the standard deviation?

A. 2
B. 5
C. 10
D. 15
E. 20

Question 32 (1 mark)

The mean number of hours per week spent studying by VCE students is 14.25 hours, with a standard deviation of 4.32. Michelle studies for 15 hours per week.

This can be represented as a z-score of

A. 0.1736.
B. −0.1736.
C. 0.7495.
D. −0.7295.
E. 0.6620.

Question 33 (1 mark)

The mean score in an IQ test completed by a group of young footballers is 105. The standard deviation is 12.

If Imran's z-score is 2, what is his score on the IQ test?

A. 107
B. 93
C. 81
D. 117
E. 129

Question 34 (1 mark)

Consider the following exam results for a Year 7 class that have been summarised according to the mean and standard deviation scores for the class.

English: $\bar{x} = 60;\ s = 11.5$

Maths: $\bar{x} = 72;\ s = 6$

Humanities: $\bar{x} = 56;\ s = 7$

Art: $\bar{x} = 82;\ s = 4$

Science: $\bar{x} = 46;\ s = 18$

John obtained 64 marks for the English exam, 61 for Humanities, 80 for Maths, 83 for Art and 60 for Science.

John's worst result, when compared to the class, was in

A. English

B. Maths

C. Humanities

D. Art

E. Science

Answers and marking guide

1.2 Types of data

Question 1

The variables *age* and **preferred travel destination** are both categorical variables. Categories have been given for both.

Question 2

A discrete variable means whole countable numbers. This is only true for *number of wings*, so only 1 of these variables is a discrete numerical variable.

Question 3

Number of moths is a categorical variable that can be ordered, so it is classified as ordinal.
Trap type is a categorical variable that is not ordered, so it is classified as nominal.
***VCAA Examination Report* note:**
This question required students to identify the type of variable for each of those given in the question. While almost half of the students did so correctly, many incorrectly chose option C, evidently because they incorrectly assumed that the variable number of moths was numerical because it involved numbers. The use of numbers in a variable definition does not automatically make the variable numerical, and students should be careful to analyse whether the numbers are referring to categories for that variable, as was the case in this question.

Question 4

Both *blood pressure* and *age*, in this instance, are ordinal variables, as they have an *order*.
***VCAA Assessment Report* note:**
Many students incorrectly identified the variable *age* (under 50 years, 50 years or over) as nominal. The variable *age* (under 50 years, 50 years or over) is ordinal because the process of allocating each of the people to one of these two categories 'under 50 years' or ' 50 years or over' orders the group of people by age.

Question 5

There are three categorical variables: sex, type of car and postcode. These are categorical variables because there are a limited number of options to choose as a response
***VCAA Assessment Report* note:**
In Question 4 students were asked to identify the number of categorical variables in a data set. There were three: *sex, type of car and postcode* (option D); however, the majority of students did not choose this option. Most students decided that there were only two categorical variables (option C), possibly rejecting *postcode* because its data values were numbers. Postcodes are numbers, as are phone numbers. However, in both cases, these numbers only serve as identifiers. They have no other numerical properties. If students are in doubt about classifying a variable as categorical or numerical, they should ask, 'Does it make sense to calculate the mean of this variable?' If the answer is 'No', the variable is categorical. For postcodes, the answer is 'No'.

Question 6

Reading down the 'Number of children' column, there are two drivers who have three children in the car, but only one of them is female.

Question 7

Mass is numerical and continuous. Gender is categorical and nominal.

Question 8
Participation in sport is categorised as Never, Sometimes or Often. **Age group** is grouped into numerical class intervals.

Question 9
Type of phone is categorised as Samsung, iPhone and other. Postcode is a categorical variable as well since each postcode represents a region and the numerical order of the codes is meaningless.

Question 10
Numerical data involves counting on a measurable scale.

Question 11
Weight is continuous numerical data as it can be measured.

Question 12
The group, favourite movies, is an example of categorical data that has some order.

Question 13
The only option that is not counted or measured is 'The names of the attendees at a local concert.' This would require the names of the attendees to be recorded. This is the definition of a categorical variable.

Question 14
The name of the fruit is categorical and, as there is no specific order associated with the names, it is also classed as nominal. The listing of fruit in alphabetical order is a chosen order but not the only possible option. It is not an order that ranks the fruit in any way.

1.3 Stem plots

Question 1
Mode means the most frequent score.
$\therefore 2.8°\text{C}$ is the modal temperature.

Question 2
There are 19 scores that are above 22.

Question 3
Given there are 20 pieces of data, the median is halfway between the 10^{th} and 11^{th} data pieces.
Q_1 is then halfway between the 5^{th} and 6^{th} pieces of data. That is, $Q_1 = 28$
Q_3 is then halfway between the 15^{th} and 16^{th} pieces of data. That is, $Q_3 = 41$
$\text{IQR} = Q_3 - Q_1 = 41 - 28 = 13$
Lower bound $= 28 - 1.5 \times 13 = 8.5$
Upper bound $= 41 + 1.5 \times 13 = 60.5$
Any number less than 8.5 or greater than 60.5 is considered an outlier.
Therefore, there are two outliers: 7 and 8.

Question 4
Count the number of leaves: 20

1.4 Dot plots, frequency tables, histograms, bar charts and logarithmic scales

Question 1
The contention: *there is an association between preferred travel destination and age*.
The answer will need to reference the different age groups and preferred travel destination.
Option D fits the contention.

Question 2
Segmented bar charts show bars stacked on top of one another to give a single bar with several parts. The lengths are determined by frequencies.
The only option that is broken into parts is D: below average, average, above average rainfall.

Question 3
$\log_{10} 10\,000 = 4$
Therefore, the number of species with a weight of less than 4 is 17.
$\frac{17}{20} \times 100 = 85\%$

Question 4
Check that the numbers match the percentages from the segmented bar chart. Option A:

Preferred travel destination	**Age**	
	Under 55 years	**55 years and over**
domestic	91	90
international	49	110
Total	140	200

Question 5
Out of the 25 dots, 5 are at 5 minutes.
$\frac{5}{25} = \frac{1}{5} = 20\%$ **[1 mark]**

Question 6
Of 17 islands, the median value is the $\left(\frac{17+1}{2}\right)$th or 9th value.
The 9$^{\text{th}}$ value is located between $\log_{10}(\text{area}) = 3$ and $\log_{10}(\text{area}) = 4$.
So, area is between 10^3 and 10^4 , that is, between 1000 and 10 000.
***VCAA Examination Report* note:**
The use of a $\log_{10}$ scale on the horizontal axis for the histogram caused difficulty for some students. Many students identified the location of the median correctly (between 3 and 4 on the horizontal axis) but some did not convert these $\log_{10}$ values to actual area values (1000 and 10 000), leading to the choice of option B, which was incorrect.

Question 7
For $x \geq 1$, $\log_{10} x \geq 0$
Frequency $= 9 + 1 = 10$
***VCAA Assessment Report* note:**
A common error was to ignore the log scale to arrive at the answer 1 (option A).

Question 8
a. Reading off Australia's column, the black section ends at 19. Therefore 19% of people were aged $0 - 14$ years old in 2010. **[1 mark]**
b. Reading off Japan's column, the grey section starts at 77, so the percentage of people in Japan 65 years and over is $100 - 77 = 23\%$.
$\frac{23}{100} \times 128\,000\,000 = 29\,440\,000$ **[1 mark]**
***VCAA Assessment Report* note:**
The answer is expected to be written in full and not, for example using technology syntax such as 2.944E7, a technology representation of scientific form. Many students gave the percentage as the answer rather than the required number. Technology syntax is not to be used in providing answers; standard mathematical notation is to be used.

c. 15 − 64 age group:
Australia's population percentage = 86 − 19 = 67%
India's population percentage = 95 − 31 = 64%
Japan's population percentage = 77 − 13 = 64%
As the percentage of people aged 15 − 64 is almost the same for all three countries, there is no association between these percentages and the country in which they live.
[1 mark for appropriate calculations and correct reference to them in explanation]
***VCAA Assessment Report* note:**
Most students were able to explain that, because the percentages were all close to each other in the 15 − 64 age group, there was no association between the percentage of people in this age group and the country in which they lived. However, some students contradicted the given statement and claimed that 'there **was** an association because…'

Question 9

a. From the graph, the frequency for the last column, representing category 'very high', is 31. Therefore, 31 countries have a very high development index. **[1 mark]**

b. There are 45 countries with low development indices and 49 countries with medium development indices. The percentage of countries that have a low or medium development index is:
$\frac{45 + 49}{153} \times 100\% = 61\%$ (correct to the nearest percentage). **[1 mark]**

Question 10

Total days of rain: 15 + 12 + 7 + 18 + 9 + 12 + 7 = 80
Days of rain on the weekend: 12 + 7 = 19
Percentage of weekend days of rain:

$$\begin{aligned}\frac{\text{Days of rain on the weekend}}{\text{Total days of rain}} \times 100\% &= \frac{19}{80} \times 100\% \\ &= 24\%\end{aligned}$$

Question 11

Most frequent is the phone with the tallest column. That is the Songsam.

Question 12

The variable is linked to the labels used to describe each row.

Question 13

There are 5 families with 5 children, therefore B is false.

Question 14

95% is 19 out of 20 marks and no student scored 19 marks.

Question 15

The car section begins at 66% and stretches to 86%. This represents 20%.

Question 16

The number of people surveyed is 15 + 22 + 13 + 17 + 25 = 92
The percentage of the bar chart allocated to 'Running' is $\frac{13}{92} \times 100\% = 14\%$.

Question 17

The total % on the bar is 100% thus pink represents 30%.

Question 18

The missing ACT data is 1.1% as the total of the graph is 100%.

Question 19
Percentages are not always used to represent the segments on a segmented bar chart.

Question 20
The modal number of hours spent studying was 3 per WEEK not per day.

Question 21
Dot plots should not be used for large sets of data.

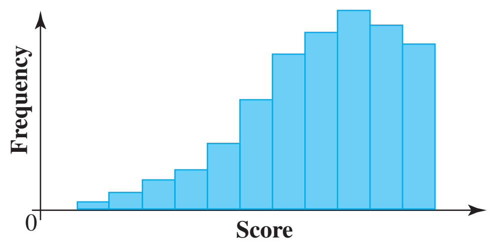
Question 22
Only 5 households have more than one mobile phone.
$\frac{5}{20} \times \frac{100}{1} = 25\%$

Question 23
Use a calculator to find $\log_{10}(5411)$
(5411 = the flow rate for summer).
The answer, correct to 2 decimal places, is 3.73.

Question 24
Use a calculator to find $\log_{10}(498)$ (498 = the brain weight of the polar bear). The calculated value will be 2.697, which places it between the baboon and killer whale.

1.5 Describing stem plots and histograms

Question 1
This histogram represents negatively distributed data.



The correct option is **B**.

Question 2
The data trails off on the positive end so the distribution is positively skewed.

Question 3
Turning the image 90°and viewing the shape of the distribution confirms symmetrical.

1.6 Summary statistics

Question 1
Divide the data in half to find the median. It will be in the 126th position.
Divide the upper half in half to find the third quartile. It will be in the 189th position.
Count up the frequencies until you get a bar that contains the 189th data point.

Question 2
Range = highest value − lowest value
$= 89 - 40$
$= 49$

Question 3

$$\begin{aligned} IQR &= Q3 - Q1 \\ &= 75 - 57 \\ &= 18 \end{aligned}$$

Question 4

$$\begin{aligned} \text{mean} - \text{median} &= 220 - 150 \\ &= 70 \end{aligned}$$

Question 5

The middle value of any data set is always the median.
The mean is dependent on extreme values, and we do not have the information to know if there are any, so we cannot assume that the middle value will also be the mean.

Question 6

There are 27 numbers, so the middle term $= \dfrac{27+1}{2}$th term. That is, the 14$^{\text{th}}$ term.
Counting along, the 14$^{\text{th}}$ term is 3.1.

Question 7

Given there are 20 pieces of data, the median is halfway between the 10th and 11th data pieces.
Q_1 is then halfway between the 5th and 6th pieces of data. That is, $Q_1 = 24.5$
Q_3 is then halfway between the 15th and 16th pieces of data. That is, $Q_3 = 61.5$
$IQR = Q_3 - Q_1 = 61.5 - 24.5 = 37$

Question 8

$$\text{Mean} = \frac{1 \times 4 + 3 \times 3 + 4 \times 2 + 5 \times 3 + 6 \times 7 + 8 \times 6}{25} = 5.04$$

Question 9

Total shots for 15 rounds $= 15 \times 72.6 = 1089$
Total shots needed over 17 rounds for a mean score of $71 = 17 \times 71 = 1207$
The total number of shots needed over the next two rounds will be $1207 - 1089 = 118$

Question 10

Finding the midpoints of each interval and calculating midpoints $\times$ frequency gives:

$\log_{10}$(price)	midpoint (m) of the actual price	frequency (f)	$f \times m$
0–	5	0	0
1–	55	0	0
2–	550	5	2750
3–	5500	28	154 000
4–	55 000	64	3 520 000
5	550 000	3	1 650 000
	Total	100	5 326 750

$$\text{Mean car price} = \frac{5326750}{100} = \$53267.50$$

Question 11

Adding all the values and dividing by 20 finds 33.8 as the mean value.

1.7 The five-number summary and boxplots

Question 1

A circumference of 30 cm coincides with Q_3, so 75% of the 252 people lie below this measurement.

Question 2
The five-number summary consists of the smallest measurement of 21 cm, lower quartile, $Q_1 = 27.4$ cm, median of 28.7 cm, upper quartile, $Q_3 = 30$ cm and the largest measurement of 35.9 cm.
***VCAA Examination Report* note:**
Students were required to interpret the provided boxplot and choose the correct five-number summary for the data. While many students did this correctly, a large number of students ignored the outlier points when determining the maximum and minimum values for the five-number summary, leading to the choice of option C, which was incorrect. Even though the points are identified as outliers, they are still valid data points within the data set and must be used as maximum and minimum values if appropriate.

Question 3
Tasman rivers are represented on the second box plot. The five-number summary is Min, Q_1, Med, Q_3, Max.
Looking at the minimum and maximum first, the minimum is between 30 and 40 km and the maximum is at 180 km – even though it is an outlier, it is still the maximum! And that is enough to work out that the answer is B, as that is the only option that has the correct minimum and maximum.
***VCAA Assessment Report* note:**
Students were asked to construct a five-number summary from a boxplot with outliers. Many students incorrectly selected option A because they failed to recognise that, when a boxplot displays outliers, these values cannot be ignored when determining the minimum and maximum values in the distribution.

Question 4
The median of Pond B is above the maximum value of Pond A. Therefore, 50% of the fish caught in Pond B are longer than all of the fish caught in Pond A.

Question 5
$Q_1 = 148$, $Q_3 = 159$, therefore $IQR = 159 - 148 = 11$
$UF = 159 + 1.5 \times 11 = 175.5$

Question 6
The value of the mean (220) being greater than the median (150) suggests a positively skewed data set. To determine if there are outliers, we need to calculate the fences:
$LF = Q_1 - 1.5 \times IQR = 10 - 1.5 \times 230 = -335$ $\quad \therefore$no outliers
$UF = Q_3 + 1.5 \times IQR = 300 + 1.5 \times 230 = 750$
This suggests that *at least* the maximum value (1380) is an outlier, but there may be more.

Question 7
Writing in ascending order:
3, 3, 3, 5, 7, 7, 7, 9, 9, 9, 12, 12, 12, 15, 15, 15, 15, 24, 24, 24
Range = 24 − 3 = 21 (difference between highest and lowest values)
$Q_1 = 7$ (between the 5^{th} and 6th data values).
$Q_3 = 15$ (between the 15th and 16th data values).
IQR $= Q_3 - Q_1 = 15 - 7 = 8$

Question 8
20 pieces of data defines Q_3 to be between the15th and 16th data pieces. $Q_3 = 41$
Q_1 is between the 5^{th} and 6th pieces of data. $Q_1 = 28$
IQR $= Q_3 - Q_1 = 41 - 28 = 13$
Range = 50 − 7 = 43
Difference between range and IQR is 43 − 13 = 30

1.8 The mean of a sample

Question 1

Mean weight $= \dfrac{1.53}{9} = 0.17$ kg $= 170$ g

Question 2

a. The wind direction with the lowest recorded wind speed was south-east.
The wind direction with the largest range of recorded wind speeds was north-east. **[1 mark]**

b. The minimum speed (from the boxplot) was 2, so the first score is 2. The first quartile is also 2 and all scores are whole numbers, therefore the 2nd and the 3rd scores are both 2 as well. The median is 3.5; therefore, the 4th score is 3 and the 5th score is 4. The third quartile and the maximum score are both 4; therefore, the 6th, 7th and 8th scores are all 4. Hence the scores are 2, 2, 2, 3, 4, 4, 4 and 4. **[1 mark]**
***VCAA Assessment Report* note:**
Few students answered this question correctly.
Often, incorrect answers included 3.5 as a value, while others included numbers less than 2 or greater than 4.

Question 3

Total shots for 15 rounds $= 15 \times 72.6 = 1089$
Total shots needed over 17 rounds for a mean score of 71 $= 17 \times 71 = 1207$
The total number of shots needed over the next two rounds will be $1207 - 1089 = 118$

1.9 Standard deviation of a sample

Question 1

Calculating the *one-variable statistics* with the CAS calculator,
$\bar{x} = 27.77$ and $s_x = 1.66403....$

Question 2

Enter the data into CAS.
$\bar{x} = 5.25$
$s = 2.38$

Question 3

Using a calculator, the standard deviation is $s = 1.37$ (don't confuse with the population standard deviation of 1.33).

Question 4

An additional 25 messages from the new student (which is the mean value) will not change the mean of 25 during the next week.

Mean $= \dfrac{8 \times 25 + 25}{9} = 25$

Even though there is no further deviation from the number of text messages sent per person and the overall mean, the standard deviation will decrease slightly. The number of students has increased to nine, which spreads the same total deviation among one more student.

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}} \qquad \frac{\leftarrow \textit{no change}}{\leftarrow \textit{increased by one}}$$

Question 5

Entering the data into a calculator and using the one-variable statistics calculation finds the standard deviation value to be 2.41 (don't confuse with the population standard deviation of 2.36).

Question 6
Entering the data into a calculator and using the one-variable statistics calculation finds the standard deviation value to be 6.72. (don't confuse with the population standard deviation of 6.55)

1.10 The 68–95–99% rule and z-scores

Question 1
76.0 is a score 1 standard deviation above the mean. Therefore, there would be 84% of participants below this value.
$84\% \times 800 = 672$ unsuccessful participants.

Question 2
Amy z-score: $\dfrac{81.5 - 69.5}{6.5} = 1.846...$
Brian z-score: $\dfrac{80.5 - 69.5}{6.5} = 1.692...$
Cherie z-score: $\dfrac{82 - 69.5}{6.5} = 1.923...$
Amy and Cherie scored over 1.80, so will be offered a leading role, only Brian will not.

Question 3
Greater than 16% is at 1 standard deviation above the mean: $\bar{x} + \sigma = 160$
Less than 2.5% is at 2 standard deviations below the mean: $\bar{x} - 2\sigma = 115$
Solve on your CAS: $\bar{x} = 145$ and $\sigma = 15$

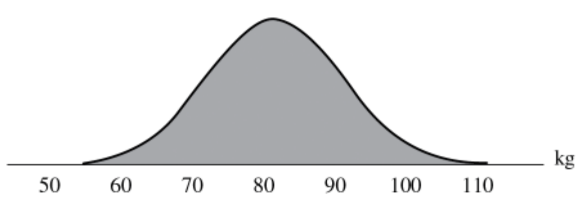
Question 4
57 mm is 2 standard deviations below the mean. Hence, *less than* 57 *mm* 2.5% of the data.
$2.5 \times 10\,000 = 250$

Question 5
If 50% of the light globes have a lifetime greater than 670 hours, this means that the **mean** is 670 hours.
If 16% have a lifetime less than 655 hours, this is 1 standard deviation below the mean, so the standard deviation must be 15 hours.

Question 6
66 minutes is two standard deviations below the mean.
72 minutes is one standard deviation above the mean.
$47.5\% + 34\% = 81.5\%$

Question 7
230 mL is two standard deviations below the mean.
Less than 230 mL is 2.5% of all serves.
$2.5\% \times 160 = 4$

Question 8
The data can be represented on a graph.



60 kg is 2 standard deviations below the mean of 78.6 kg.
So weights above 60 kg represent 97.5% of the data.
Therefore, the expected number of male players with weights above 60kg is
$456 \times .975 = 444.6 \simeq 445$

Question 9

Brett:

$$z = \frac{x - \mu}{\sigma}$$
$$-0.96 = \frac{x - 78.6}{9.3}$$
$$x = 69.672$$

Sanjeeva:

$$z = \frac{x - \mu}{\sigma}$$
$$-0.26 = \frac{x - 78.6}{9.3}$$
$$x = 69.672$$

If Sanjeeva increases his weight by 2 kg, he will weigh 78.182 kg, which is still less than the mean weight of 78.6 kg.

Question 10

21.4 cm is two standard deviations below the mean, so the probability of a foot length below 21.4 cm is 2.5%.
Therefore, the expected number of women with foot lengths less than 21.4 cm is $0.025 \times 2400 = 60$.

Question 11

$$Z = \frac{x - \overline{x}}{s}$$
$$-1.3 = \frac{x - 2.8}{1.2}$$
$$x = 22.24$$

Question 12

$$\mu \pm 1\sigma = 31 \pm 6$$
$$= [25, \quad 37]$$

Therefore, approximately 68% of shoppers spend between 25 and 37 minutes in the store.

$$\text{Number} = \frac{68}{100} \times 2850$$
$$= 1938$$
$$\approx 1900$$

1.11 Review

Question 1

$Q_1 = 148$, $Q_3 = 159$, therefore $IQR = 159 - 148 = 11$
$UF = 159 + 1.5 \times 11 = 175.5$

Question 2

a. There are 3 numerical variables **[1 mark]**
b. 1.81 (use your CAS) **[1 mark]**
c. $z = \frac{14.50 - 13.74}{1.43} \simeq 0.5$ [**1 mark** – note that rounding applies here]
d. A z-score of -1 means 1 standard deviation below the mean. Therefore, the percentage of athletes who would be expected to jump higher than Chara is $34\% + 50\% = 84\%$ **[1 mark]**
e. Q_3 and the maximum are the same value of 1.87 **[1 mark]**
f. Calculate the upper fence: $UF = 42.88 + 1.5 \times (42.88 - 40.88) = 45.88$ [**1 mark** for calculating 45.88]

Question 3

The data is bunched down the lower end of the histogram with a tail going to the right, therefore it is positively skewed.
Given the large range of the data, it is likely that the data at the upper end is an outlier.

Question 4

$\log_{10}(100\,000) = \log_{10}(10^5) = 5$
Therefore, less than 5 on the horizontal scale is $7 + 1 = 8$ countries.

Question 5

a. There are three categorical variables: City, Congestion level and Size. **[1 mark]**

b. There are two categories that are ordinal: Congestion level and Size. **[1 mark]**
VCAA Examination Report **note:**
Most students were able to answer this question correctly, but 1 and 3 were also common answers.

c. The large cities with medium traffic congestion levels are Newcastle-Sunderland and Liverpool. **[1 mark]**
VCAA Examination Report **note:**
It was evident that some students did not read the question carefully as they gave the full list of all cities with a medium level of congestion instead of giving only the large cities with a medium level of congestion as required by the question.

d.

	City size	
Congestion level	**Small**	**Large**
high	4	2
medium	4	2
low	8	3
Total	16	7

[**1 mark** per 3 correct responses, for a total of **2 marks**]

e. The percentage of small cities with high traffic congestion is $\frac{4}{16} \times 100 = 25\%$. **[1 mark]**
VCAA Examination Report **note:**
This question was answered well, although some students gave 17% by taking the percentage of all 23 cities.

f. The distribution of the increase in travel time is positively skewed. **[1 mark]**
VCAA Examination Report **note:**
This question was answered well; however, some students went on to comment on the centre and spread, which was not required and made it difficult to determine if the student understood which had been asked for. The mark could not be awarded in this case.

g. Q_1 is at 30 minutes and Q_3 is at 39 minutes. The IQR is $Q_3 - Q_1 = 39 - 30 = 9$.
The upper fence is $1.5 \times \text{IQR} + Q_3 = 1.5 \times 9 + 39 = 52.5$. **[1 mark]**
VCAA Examination Report **note:**
This question was answered well; however, some students added the relationship between the data value and the upper fence, which was not part of this question.

Question 6

There is a numerical variable (*resting pulse rate*) with three categories (*age group*), so **parallel boxplots** are the best way to display the data.

Question 7

a. Positively skewed **[1 mark]**

b. The median will be in the $\left(\dfrac{32+1}{2}\right) = 16.5$th position. Therefore, the median is 24.55. **[1 mark]**

c. $\dfrac{12}{32} \times 100 = 37.5\%$. **[1 mark]**

Question 8

Frequency of nightmares is an ordinal variable, as the options are ordered — low, high.
Snores is a nominal variable, as the options have no order.

Question 9

a. Positively skewed **[1 mark]**

b. The median will be in the $\left(\dfrac{32+1}{2}\right) = 16.5$th position. Therefore, the median is 24.55. **[1 mark]**

c. $\dfrac{12}{32} \times 100 = 37.5\%$ **[1 mark]**

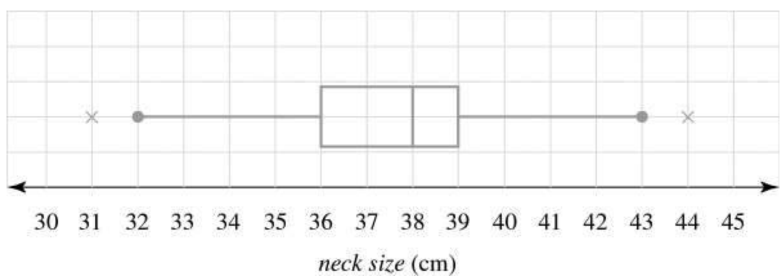
Question 10

a. The mode is the most frequent value, which is 38 cm. **[1 mark]**

b. i. 2 standard deviations above and below the mean is 0.3% of the data.
$0.3\% \times 250 = 0.75 \simeq 1$ **[1 mark]**

ii. Below: $\bar{x} - 3s = 38 - 3(2.3) = 31.1$
Above: $\bar{x} + 3s = 38 + 3(2.3) = 44.9$
There is ONE man with a neck size more than 3 standard deviations away from the mean (neck size of 31). **[1 mark]**

c. First, check for outliers. $LF = 36 - 1.5 \times 3 = 31.5$ (one outlier at the lower end),
$UF = 39 + 1.5 \times 3 = 43.5$ (one outlier)



Award **1 mark** for outliers.
Award **1 mark** for the boxplot.

Question 11

a. $\dfrac{50}{250} \times 100 = 20\%$ **[1 mark]**

b. IQR for average neck size $= 26.0 - 23.4 = 2.6$ **[1 mark]**

c. There could be one or more men for each outlier. So for average neck size, there are at least 4 men with a BMI over 30. For above average neck size, there are $25\% \times 76 = 19$ men with a BMI over 30. This means that there *at least* 23 men with a BMI over 30. i.e. total ≥ 23. **[1 mark]**

d. The boxplots do show that BMI is associated with neck size. **[1 mark]**
The larger the neck size, the higher the median BMI. **[1 mark]**

Question 12

300 milliseconds is the value at the third quartile (Q_3); therefore, *longer than* 300 *milliseconds* will be 25% of the times:
$0.25 \times 800 = 200$

Question 13

Use the vertical (frequency) scale to count the number of countries between five and 20 million people.

$11 + 4 + 2 = 17$

Question 14

a. Day number **[1 mark]**

***VCAA Examination Report* note:**

Most responses given to this question were correct. A small number of students answered 'neither'.

b. key: $4|1 = 4.1$ $n = 15$

minimum temperature (°C)

4	1	8	
5			
6	0	7	
7	0	***5***	***7***
8	***0***	***6***	
9	0	2	***8***
10	7		
11	8		
12	7		

Award 1 mark for all five numbers in correct positions.

***VCAA Examination Report* note:**

Some students entered only the values for day 11 and day 15 rather than the five days from day 11 to 15.

c. i. $Q_1 = 12.2$ **[1 mark]**

ii. There are three days with a maximum value higher than 15.3 °C.

$\frac{3}{15} \times 100 = 20\%$ **[1 mark]**

Question 15

a. Day number **[1 mark]**

***VCAA Examination Report* note:**

Most responses given to this question were correct. A small number of students answered 'neither'.

b. key: $4|1 = 4.1$ $n = 15$

minimum temperature (°C)

4	1	8	
5			
6	0	7	
7	0	***5***	***7***
8	***0***	***6***	
9	0	2	***8***
10	7		
11	8		
12	7		

Award 1 mark for all five numbers in correct positions.

***VCAA Examination Report* note:**

Some students entered only the values for day 11 and day 15 rather than the five days from day 11 to 15.

c. i. $Q_1 = 12.2$ **[1 mark]**

ii. There are three days with a maximum value higher than 15.3 °C.

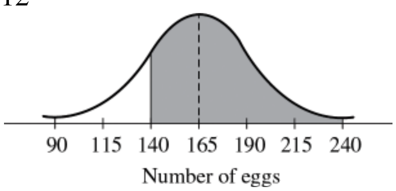
$\frac{3}{15} \times 100 = 20\%$ **[1 mark]**

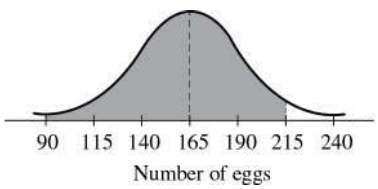
Question 16

A z-score of -1 is a score that is 1 standard deviation below the mean.

The percentage of scores within 1 standard deviation of the mean is 68%. By halving this, 34% of scores fall between $z = -1$ and the mean.
Since 50% of scores are above the mean, and 34% of scores are between $z = -1$ and the mean, the percentage of students with a pulse rate greater than this student is 84%.

Question 17
Using the $68 - 95 - 99.7$ rule:
The percentage of students with a pulse rate less than 61 beats per minute is $\left(\frac{100-95}{2}\right)\% = 2.5\%$.
The percentage of students with a pulse rate greater than 73 beats per minute is $\left(\frac{100-68}{2}\right)\% = 16\%$.
The total number of students outside the range of $61 - 73$ beats per minute is $(2.5 + 16)\% \times 200 = 37$.

Question 18
a. i. The range = largest value − smallest value
$= 197 - 125$
$= 72$ **[1 mark]**
***VCAA Examination Report* note:**
This question was generally answered well. A small number of students wrote the answer as an interval, which was not accepted.
ii. Three of the 12 values are greater than 170
$\frac{3}{12} \times 100 = 25\%$ **[1 mark]**
b. i.


The percentage expected to contain more than 140 eggs is $50 + 34 = 84\%$ **[1 mark]**
ii.


The percentage of clusters containing fewer than 215 eggs is $50 + 47.5 = 97.5\%$
$97.5\% \text{ of } 1000 = \frac{97.5}{100} \times 1000$
$= 975$ **[1 mark]**
***VCAA Examination Report* note:**
Many students overlooked the requirement to convert from percentage to quantity and gave an answer of 97.5%.
c. $z = \frac{x - \bar{x}}{s_x}$ $\quad z = -2.4,\ \bar{x} = 165 \text{ and } s_x = 25$
$-2.4 = \frac{x - 165}{25}$
Solving for x, $x = 105$
The actual number of eggs is 105. **[1 mark]**

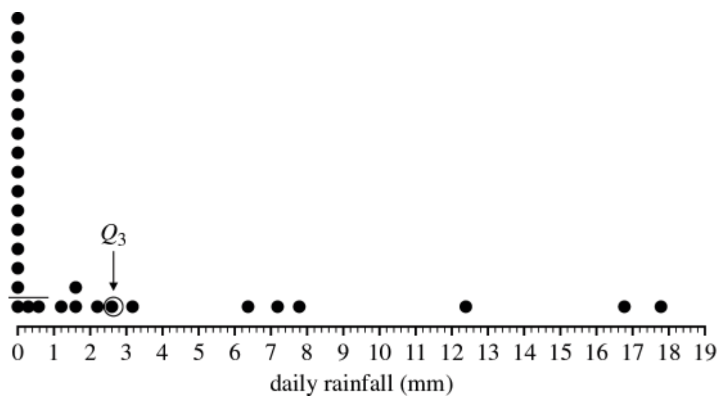
Question 19
$\frac{49}{53} \times 100 = 92.45\% = 92\%$

Question 20

a. i. Range = max − min = 17.8 **[1 mark]**

ii. The median is the $\left(\frac{n+1}{2}\right)$th = $\left(\frac{30+1}{2}\right)$th = 15.5th number = 0 **[1 mark]**

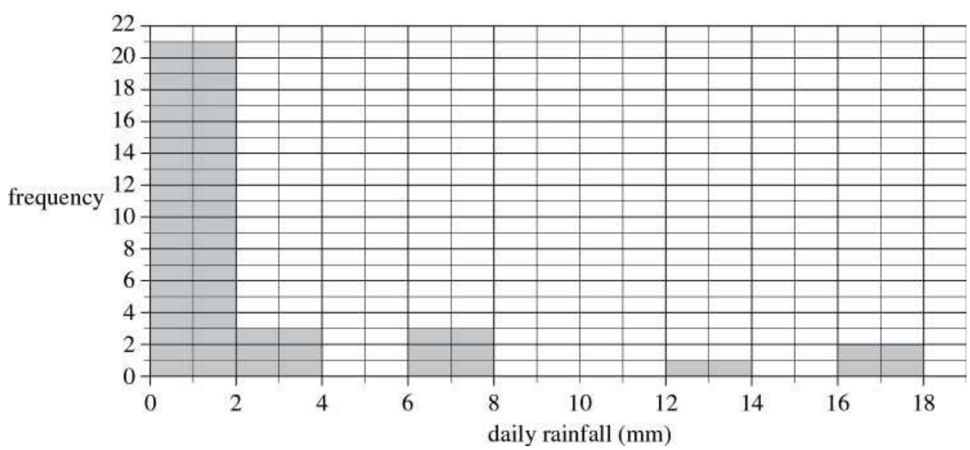
b. The dot at 2.6 is Q_3. **[1 mark]**



c. i. 16 days **[1 mark]**

ii. 3 days exceed 12 mm of rainfall. $\frac{3}{30} \times 100 = 10\%$ **[1 mark]**

d.



Award 1 mark for the correct placement of columns and 1 mark for the overall graph.

***VCAA Assessment Report* note:**

Many students inappropriately drew columns with interval widths of only one.

Question 21

a. The mode is the interval with the highest frequency. Therefore, the modal interval is 70 − 75. **[1 mark]**

***VCAA Assessment Report* note:**

This question was not well answered. Many students simply wrote '5', while others wrote 'negatively skewed'.

b. Count the frequencies for less than 55 years, so 5 + 9 = 14. **[1 mark]**

c. $\frac{30}{183} \times 100 = 16.4\%$. **[1 mark]**

VCAA Assessment Report **note:**
An answer rounded to one decimal place was expected. A number of students simply wrote 16% or an unrounded figure. This answer was not accepted. It could not be considered as a rounding error as there was no evidence that rounding had occurred.

Question 22
There are 31 countries, so the median is the $\frac{31+1}{2}$ = 16 th term. The 1th term is 1.5.
The range is the maximum – minimum, so 4.7 − 0.2 = 4.5.
Finally, the stem plot is positively skewed.

Question 23
a. The mode is the score with the highest frequency; therefore, the mode is 78.
The range = highest score − lowest score = 79 − 70 = 9; therefore, the range is 9. **[1 mark]**
b. $Q_1 = 75; Q_3 = 78; \text{IQR} = 78 - 75 = 3$

$$\begin{aligned} 1.5 \times \text{IQR} &= 1.5 \times 3 \\ &= 4.5 \end{aligned}$$

$$\begin{aligned} \text{Lower boundary} &= Q_1 - 1.5\,\text{IQR} \\ &= 75 - 4.5 \\ &= 70.5 \quad [1 \text{ mark}] \end{aligned}$$

Because 70 < 70.5, the development index of 70 is below the lower boundary and hence is an outlier for this group of countries. **[1 mark]**
VCAA Assessment Report **note:**
This question asked for an explanation of why 70 was an outlier for this group of countries. Many students calculated a value of 70.5 and then wrote that ‘it is therefore an outlier’. Further explanation, including a direct comparison between 70 and 70.5, was expected.

Question 24
95% of scores are within 2 standard deviations either side of the mean.

$$\begin{aligned} 7.4 - 2 \times 0.7 \le x &\le 7.4 + 2 \times 0.7 \\ 6 \le x &\le 8.8 \end{aligned}$$

Question 25
8.1 = 7.4 + 0.7
Therefore, we need the number of students whose sleeping time is more than 1 standard deviation above the mean.
Because 16% of scores are more than 1 standard deviation above the mean, the number of students who spent more than 8.1 hours
sleeping $= \frac{16}{100} \times 1550 = 248$

Question 26
Add the frequencies.
4 + 7 + 6 + 2 + 1 = 20 students.

Question 27
For the actual number of visitors, evaluate 10^9 = 1 000 000 000 for Yippee $10^{7.301} \approx$ 199 986 18 and for Televise.
Dividing these values gives approximately 50. Therefore, there are 50 times as many visitors to Yippee than Televise in the recorded month.

Question 28
Interpreting the horizontal scale shows $10^5 = 100\,000$. The percentage of car prices below 100 000 is 5 + 28 + 64 = 97%

Question 29
Given the horizontal axis is written as a base 10 log scale, the most frequent prices are between $10^5 = \$100\,000$ and $10^6 = \$1\,000\,000$.

Question 30
There are 20 pieces of data ranked in ascending order. The middle value is the $\frac{20+1}{2} = 10.5$th piece of data.
The median is between 35 and 36. The median is 35.5.

Question 31

$$\begin{aligned} z &= \frac{x-\bar{x}}{s} \\ -2 &= \frac{460-480}{s} \\ -2 &= \frac{-20}{s} \\ s &= \frac{-20}{-2} = 10 \end{aligned}$$

Question 32

$$\begin{aligned} z &= \frac{x-\bar{x}}{s} \\ z &= \frac{15-14.25}{4.32} \\ z &= 0.1736 \end{aligned}$$

Question 33
Imran scored 2 standard deviations above the mean ($z = 2$).
Therefore, he scored $105 + 2 \times 12 = 129$ on the IQ test.

Question 34
To compare performances use

$$z = \frac{x-\bar{x}}{s}$$

$$Z_{\text{eng}} = \frac{64-60}{11.5} = 0.348$$

$$Z_{\text{hum}} = \frac{61-56}{7} = 0.714$$

$$Z_{\text{math}} = \frac{80-72}{6} = 1.33$$

$$Z_{\text{art}} = \frac{83-82}{4} = 0.25$$

$$Z_{\text{sci}} = \frac{60-46}{18} = 0.778$$

Therefore, John's worst result was in Art.

2 Investigating associations between two variables

Topic	2	Investigating associations between two variables
Subtopic	2.2	Response and explanatory variables

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Source: *VCE 2021, Further Mathematics Exam 1, Section A, Core, Q.1; © VCAA*

Question 1 (1 mark)

A survey of 103 students who recently joined the workforce was taken. The focus of the survey was to determine the area of employment the students had joined. The results are in the table below.

		Age	
		20 – 25	26 – 30
Area of employment	Emergency services	12	7
	Health services	13	19
	Hospitality	24	20
	Clerical	7	34
	Retail	25	22

The two variables of age and employment area would be defined as

A. both numerical.
B. both categorical.
C. numerical and categorical respectively.
D. categorical and numerical respectively.
E. categorical and ordinal respectively.

Topic 2 Subtopic 2.2 Response and explanatory variables

Question 2 (1 mark)

The relationship between pulse rate (beats per minute) and exercise regime (walking, jogging, running) is best displayed using

A. a histogram.
B. a scatterplot.
C. a back-to-back stem plot.
D. a time series plot.
E. parallel boxplots.

Question 3 (1 mark)

Paul owns a gym specialising in weight loss. He has collected data on each client in order to determine whether there is a relationship between the number of weekly training sessions and the weekly weight loss. Paul wants to graph the data so that he can more easily interpret the results.

Determine which of the following statements describes how Paul should graph the data.

A. When graphing, the number of weekly training sessions should be on the vertical axis as it is the response variable.
B. When graphing, the weekly weight loss should be on the vertical axis because it is the explanatory variable.
C. When graphing, the weekly weight loss should be on the horizontal axis because it is the explanatory variable.
D. When graphing, the weekly training sessions should be on the horizontal axis because it is the explanatory variable.
E. When graphing, a scatterplot should be used because there is neither an independent nor response variable.

Question 4 (1 mark)

In an experimental study of rabbits, the response variable was the time (in weeks) it took the male rabbits to recover from Illness Y after being administered Drug X.

The explanatory variable would most likely have been

A. the number of scientists conducting the experiment.

B. the time taken to administer the daily dosage of experimental Drug X to the rabbits.

C. the daily dosage of experimental Drug X administered to the rabbits.

D. the number of rabbits.

E. the sex of the rabbits.

Question 5 (1 mark)

A study was conducted over a number of weeks into the volume of traffic and the air pollution level on a busy Melbourne freeway.

Which of the following variables would be the response variable?

A. time

B. number of weeks

C. volume of traffic

D. air pollution level

E. volume of air pollution

Topic	2	Investigating associations between two variables
Subtopic	2.3	Contingency (two-way) frequency tables and segmented bar charts

online only

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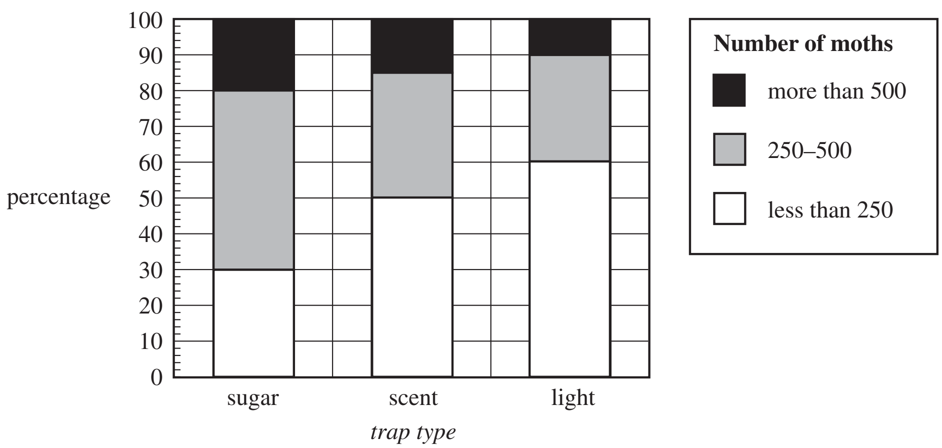
Source: *VCE 2017, Further Mathematics Exam 1, Section A, Core, Q.6; © VCAA*

Question 1 (1 mark)

A study was conducted to investigate the association between the *number of moths* caught in a moth trap (less than 250, 250 – 500, more than 500) and the *trap type* (sugar, scent, light). The results are summarised in the percentaged segmented bar chart below.



The data displayed in the percentaged segmented bar chart supports the contention that there is an association between the *number of moths* caught in a moth trap and the trap type because

A. most of the light traps contained less than 250 moths.
B. 15% of the scent traps contained 500 or more moths.
C. the percentage of sugar traps containing more than 500 moths is greater than the percentage of scent traps containing less than 500 moths.
D. 20% of sugar traps contained more than 500 moths while 50% of light traps contained less than 250 moths.
E. 20% of sugar traps contained more than 500 moths while 10% of light traps contained more than 500 moths.

Topic 2 | Subtopic 2.3 Contingency (two-way) frequency tables and segmented bar charts

Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.1; © VCAA*

Question 2 (1 mark)

The *blood pressure* (low, normal, high) and the *age* (under 50 years, 50 years or over) of 110 adults were recorded. The results are displayed in the two-way frequency table below.

	Age	
Blood pressure	**Under 50 years**	**50 years or over**
low	15	5
normal	32	24
high	11	23
Total	58	52

The percentage of adults under 50 years of age who have high blood pressure is closest to

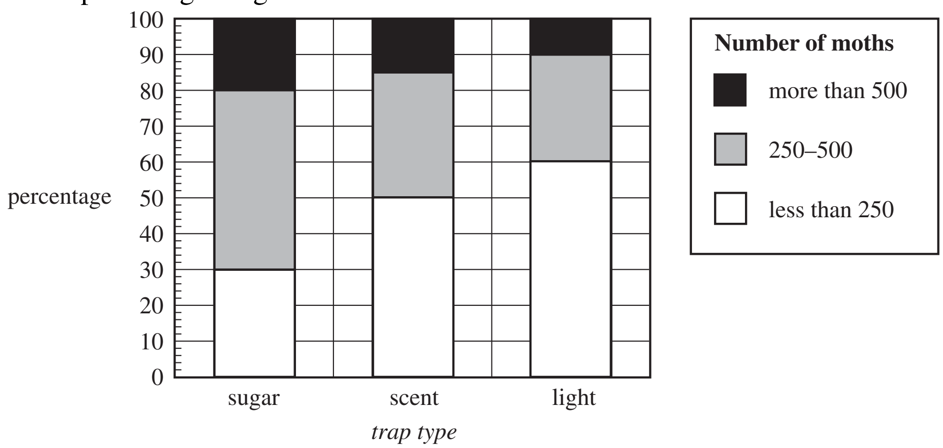
A. 11%
B. 19%
C. 26%
D. 44%
E. 58%

Topic 2 | Subtopic 2.3 Contingency (two-way) frequency tables and segmented bar charts

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Core, Q.5; © VCAA*

Question 3 (1 mark)

A study was conducted to investigate the association between the *number of moths* caught in a moth trap (less than 250, 250 – 500, more than 500) and the *trap type* (sugar, scent, light). The results are summarised in the percentaged segmented bar chart below.



There were 300 sugar traps.

The number of sugar traps that caught less than 250 moths is closest to

A. 30
B. 90
C. 250
D. 300
E. 500

Source: *VCE 2020, Further Mathematics 1, Section A, Q.11; © VCAA*

Question 4 (1 mark)

The data in Table 2 was collected in a study of the association between the variables *frequency of nightmares* (low, high) and snores (no, yes).

	Total		
Frequency of nightmares	**No**	**Yes**	**Snores**
low	80	58	138
high	11	12	23
Total	91	70	161

Data: Adapted from RA Hicks and J Bautista, 'Snoring and nightmares', *Perceptual and Motor Skills*, 1 October 1993, <https://doi.org/10.2466/pms.1993.77.2.433>

The percentage of participants in the study who did not snore is closest to

A. 42.0%
B. 43.5%
C. 49.7%
D. 52.2%
E. 56.5%

Source: *VCE 2020, Further Mathematics 1, Section A, Q.12; © VCAA*

Question 5 (1 mark)

The data in Table 2 was collected in a study of the association between the variables *frequency of nightmares* (low, high) and snores (no, yes).

Table 2

	Total		
Frequency of nightmares	**No**	**Yes**	**Snores**
low	80	58	138
high	11	12	23
Total	91	70	161

Data: Adapted from RA Hicks and J Bautista, 'Snoring and nightmares', *Perceptual and Motor Skills*, 1 October 1993, <https://doi.org/10.2466/pms.1993.77.2.433>

Of those people in the study who did snore, the percentage who have a high frequency of nightmares is closest to

A. 7.5%
B. 17.1%
C. 47.8%
D. 52.2%
E. 58.0%

Source: *VCE 2019, Further Mathematics 1, Section A, Q.8; © VCAA*

Question 6 (1 mark)

Percy conducted a survey of people in his workplace. He constructed a two-way frequency table involving two variables.

One of the variables was attitude towards shorter working days (for, against). The other variable could have been

A. *age* (in years).
B. *sex* (male, female).
C. *height* (to the nearest centimetre).
D. *income* (to the nearest thousand dollars).
E. *time* spent travelling to work (in minutes).

Topic	2	Investigating associations between two variables
Subtopic	2.4	Back-to-back stem plots

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Source: *VCE 2014, Further Mathematics Exam 1, Section A, Q.8; © VCAA*

Question 1 (1 mark)

A single back-to-back stem plot would be an appropriate graphical tool to investigate the association between a car's speed, in kilometres per hour, and the

A. driver's age, in years.
B. car's colour (white, red, grey, other).
C. car's fuel consumption, in kilometres per litre.
D. average distance travelled, in kilometres.
E. driver's sex (female, male).

Question 2 (1 mark)

To ensure the IQR value for both the males and females is equal, the missing value from the female data must be

Male	Stem	Female
3 3 2	2	
9 8 6 5	3	
4 3	4	
7	5	1 2 5
	6	5 5 7
	7	4 6
	8	2

Key: 5|2 = 52

A. 43
B. 45
C. 74
D. 75
E. 84

Question 3 (17 marks)

The following stem-and-leaf plot details the age of 26 offenders who were caught drink driving during a Friday night blitz.

Male offenders (Leaf)	Stem	Female offenders (Leaf)
7 7 7 7 8 8 8 8 8 9	1	8 8 8
1 1 1 1 1 1	2	1 1 2 4
7	3	
	4	2 5
Key:	1|8 =	18 years old

Note: Give answers to 1 decimal place where appropriate.

a. Calculate the mean, median, mode and standard deviation for the male offenders. **(4 marks)**

b. Calculate the five-figure summary, and hence the range and IQR, for the male offenders. **(3 marks)**

c. Calculate the mean, median, mode and standard deviation for the female offenders. **(4 marks)**

d. Calculate the five-figure summary, and hence the range and IQR, for the female offenders. **(3 marks)**

e. Using statistics, write a paragraph about the results of the Friday night drink driving blitz. **(3 marks)**

Source: *VCE 2017, Further Mathematics 2, Section A, Q.2;*© VCAA

Question 4 (8 marks)

The back-to-back stem plot below displays the wingspan, in millimetres, of 32 moths and their place of capture (forest or grassland).

Key 1|8 = 18 ***wingspan*** **(mm)**

forest (n=13)		**grassland** (n=19)
6	1	8
2 1 1 0 0 0 0	2	2 2 4 4
7	2	5 5 9
4 0	3	0 0 1 2 3 4
5	3	6 8
	4	0 3
	4	5
2	5	

a. Which variable, *wingspan* or *place of capture*, is a categorical variable? **(1 mark)**

b. Write down the modal wingspan, in millimetres, of the moths captured in the forest. **(1 mark)**

c. Use the information in the back-to-back stem plot to complete the table below. **(2 marks)**

	Wingspan **(mm)**				
Place of capture	**minimum**	Q_1	**median (M)**	Q_3	**maximum**
forest		20	21	32	52
grassland	18	24	30		45

d. Show that the moth captured in the forest that had a wingspan of 52 mm is an outlier. **(2 marks)**

e. The back-to-back stem plot suggests that *wingspan* is associated with *place of capture* **(2 marks)**
Explain why, quoting the values of an appropriate statistic.

Question 5 (4 marks)

A group of 28 students had their heights recorded. See the data set below.

123	125	126	124	111	135	147	125	128	123	126	128	124	124
120	122	125	123	110	130	140	127	128	123	120	149	125	127

a. Construct a frequency table using a suitable class interval. **(1 mark)**

b. Construct a histogram to represent these results. **(1 mark)**

c. Comment on the distribution. **(2 marks)**

Question 6 (1 mark)
The back-to-back stem plot displays the heights of a sample of students in Year 7 and Year 12:

Year 7	Stem	Year 12
5 3	12	
5 4 2 2 0	13	
6 5 3 2	14	8
8 5	15	3 3 5
2	16	5 5 8 9 9
1	17	2 2 7 7
	18	55 6 7
	19	4 6
	20	1

Key: 16|2 = 162 cm

Which of the following is incorrect?
A. More than 50% of the Year 7 heights are less that the minimum Year 12 height
B. More than 50% of the Year 12 heights are greater than the maximum Year 12 height
C. The Year 12 median height is 20 cm greater than the Year 7 median height
D. The IQR for the Year 12 heights is less than the IQR for the Year 7 heightsy
E. The variation in the Year 12 heights is greater than the variation in the Year 7 heights.

Question 7 (1 mark)
Which of the following pairs of variables could be displayed on a back-to-back stem plot?
A. religion; number of holidays
B. weight; height
C. number of Christmas presents received; age in years
D. time spent shopping; annual salary
E. scores on a board game for Year 12 girls; scores on a board game for Year 12 boys

Topic	2	Investigating associations between two variables
Subtopic	2.5	Parallel boxplots and dot plots

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Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.8; © VCAA*

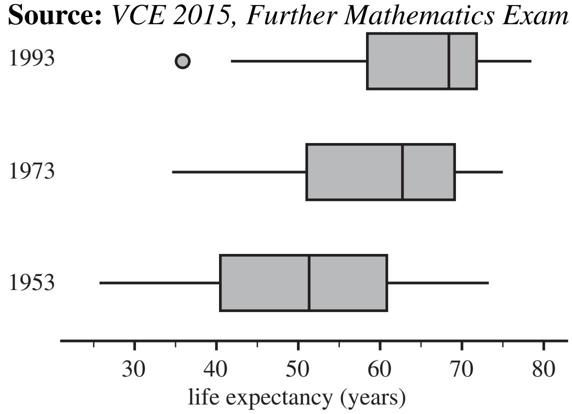
Question 1 (1 mark)

Parallel boxplots would be an appropriate graphical tool to investigate the association between the monthly median rainfall, in millimetres, and the

A. monthly median wind speed, in kilometres per hour.
B. monthly median temperature, in degrees Celsius.
C. month of the year (January, February, March, etc.).
D. monthly sunshine time, in hours.
E. annual rainfall, in millimetres.

Question 2 (2 marks)

Source: *VCE 2015, Further Mathematics Exam 2, Core, Q.2; © VCAA*



The parallel boxplots below compare the distribution of life expectancy for 183 countries for the years 1953,1973 and 1993.

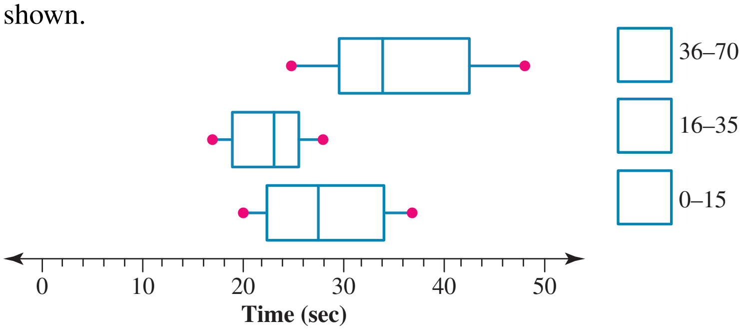
a. Describe the shape of the distribution of life expectancy for 1973. **(1 mark)**

b. Explain why life expectancy for these countries is associated with the year. Refer to specific statistical values in your answer. **(1 mark)**

Topic 2 Subtopic 2.5 Parallel boxplots and dot plots

Question 3 (1 mark)

The time taken, measured in seconds, for a sample of people to type a 250 character SMS was recorded. The results were grouped into the age brackets of $0-15,\ 16-35,\ 36-70$ and displayed in the boxplots shown.



In comparing the information given in the boxplots, which of the following is inaccurate?

A. At least 75% of the times recorded by the $0-15$ age bracket were less than the median of the $36-70$ age bracket.

B. At least 75% of the times recorded by the $36-70$ age bracket were less than the median of the $0-15$ age bracket.

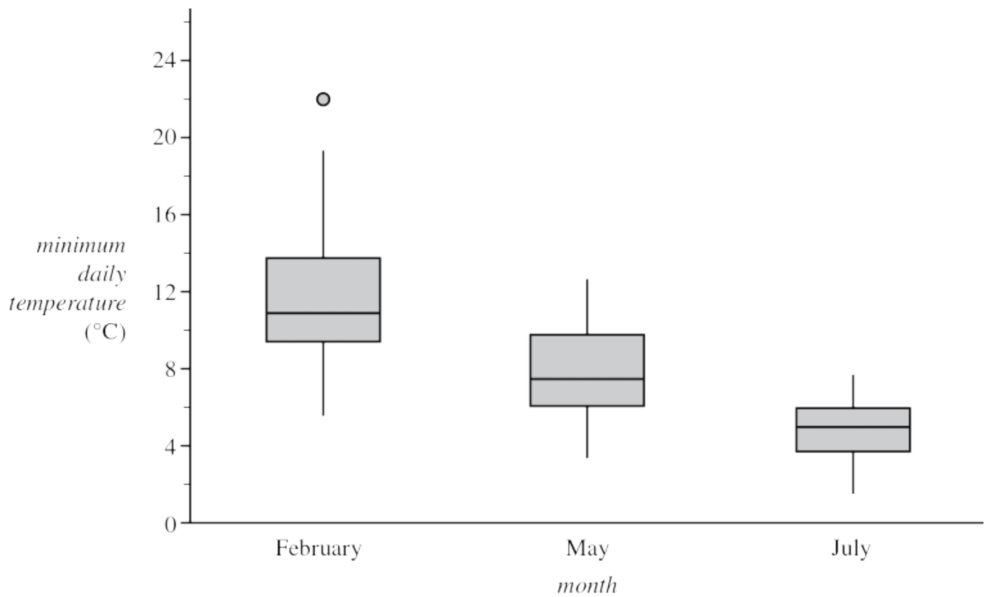
C. At least 25% of the times recorded by the $36-70$ age bracket were less than the median of the $0-15$ age bracket.

D. 100% of the times recorded by the $36-35$ age bracket were less than the median of the $0-15$ age bracket.

E. At least 25% of the times recorded by the $16-35$ age bracket were less than the minimum recorded time of the $0-15$ age bracket.

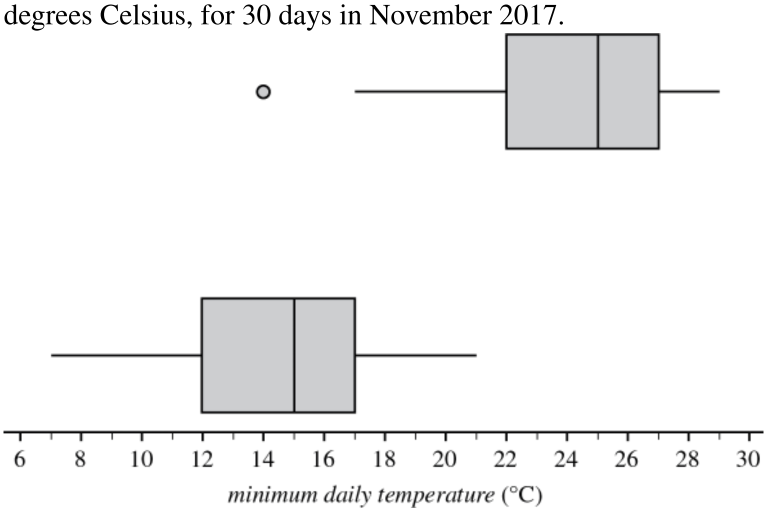
Question 4 (1 mark)
Source: *VCE 2017, Further Mathematics 2, Section A, Q.3; © VCAA*
The five-number summary for the distribution of *minimum daily temperature* for the months of February, May and July in 2017 is shown in Table 2.
The associated boxplots are shown below the table.
Table 2. Five-number summary for *minimum daily temperature*

Month	Minimum	Q_2	Median	Q_3	Maximum
February	5.9	9.5	10.9	13.9	22.2
May	3.3	6.0	7.5	9.8	12.7
July	1.6	3.7	5.0	5.9	7.7



Data: Australian Government, Bureau of Meteorology, <www.bom.gov.au/>
Explain why the information given above supports the contention that *minimum daily temperature* is associated with the *month*. Refer to the values of an appropriate statistic in your response.

Question 5 (4 marks)
Source: *VCE 2019, Further Mathematics Exam 2, Section A, Q.2; © VCAA*
The parallel boxplots below show the *maximum daily temperature* and *minimum daily temperature*, in degrees Celsius, for 30 days in November 2017.



Data: Australian Government, Bureau of Meteorology, <www.bom.gov.au/>

a. Use the information in the boxplots to complete the following sentences.
For November 2017

i. the interquartile range for the *minimum daily temperature was* _______ °C **(1 mark)**

ii. the median value for *maximum daily temperature* was _______ °C higher than the median value for *minimum daily temperature* **(1 mark)**

iii. the number of days on which the *maximum daily temperature* was less than the median value for *minimum daily temperature* was _______. **(1 mark)**

b. The *temperature difference* between the *minimum daily temperature* and the *maximum daily temperature* in November 2017 at this location is approximately normally distributed with a mean of 9.4°C and a standard deviation of 3.2°C.
Determine the number of days in November 2017 for which this temperature difference is expected to be greater than 9.4°C. **(1 mark)**

Topic	2	Investigating associations between two variables
Subtopic	2.6	Scatterplots

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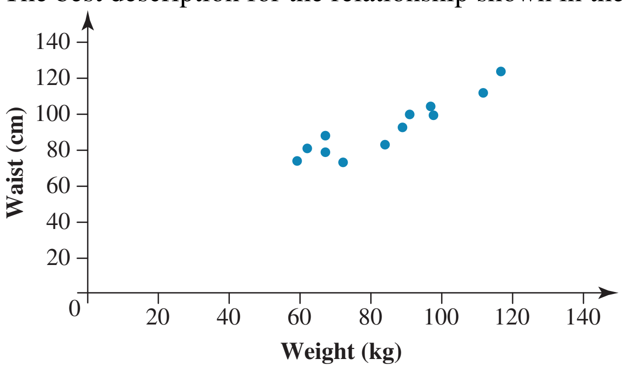
Source: *VCE 2012, Further Mathematics Exam 1, Section A, Q.1; © VCAA*

Question 1 (1 mark)

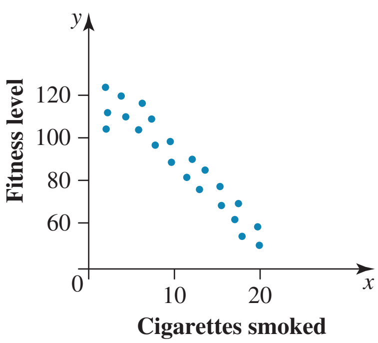
The best description for the relationship shown in the scatterplot would be



A. moderate, positive and linear.
B. strong, negative and linear.
C. moderate, negative and non-linear.
D. strong, positive and linear.
E. weak, positive and linear.

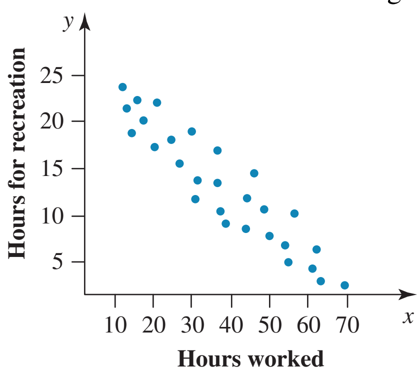
Question 2 (1 mark)

Determine which of the following conclusions can be drawn from the scatterplot shown.



A. As the number of cigarettes smoked increases, fitness level increases.
B. As the number of cigarettes smoked increases, fitness level decreases.
C. As the number of cigarettes smoked decreases, fitness level remains unchanged.
D. As fitness level decreases, there is little change in the number of cigarettes smoked.
E. There is no relationship between the number of cigarettes smoked and fitness level.

Question 3 (1 mark)
Determine which of the following conclusions can be drawn from the scatterplot shown.

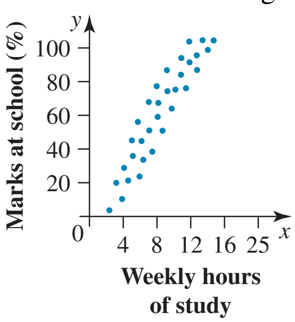


A. There is a strong, linear, positive relationship between the number of hours worked and the number of hours for recreation.
B. There is a moderate, linear, positive relationship between the number of hours worked and the number of hours for recreation.
C. There is a strong, linear, negative relationship between the number of hours worked and the number of hours for recreation.
D. There is a moderate, linear, negative relationship between the number of hours worked and the number of hours for recreation.
E. There is no relationship between the number of hours worked and the number of hours for recreation.

Question 4 (1 mark)
A scatterplot cannot be used to display the relationship between which of the following pairs of variables?
A. The marks obtained by a group of students and their gender.
B. The masses (kg) of newborn male and female giraffes born in captivity
C. The number of couriers delivering packages to two nearby factories.
D. The weights and ages of females attending a gym.
E. The scores obtained by basketball teams from two neighbouring schools over the course of a season.

Question 5 (1 mark)
What variables are typically displayed on a scatterplot?

Question 6 (1 mark)
Which of the following conclusions can be drawn from the scatterplot shown below?



A. As the number of hours of study increases, the marks achieved at school decrease.
B. As the number of hours of study decreases, the marks achieved at school increase.
C. As the number of hours of study increases, the marks achieved at school increase.
D. As the number of hours of study decreases, there is no change in the marks achieved at school.
E. There is no relationship between weekly hours of study and marks achieved at school.

Topic	2	Investigating associations between two variables
Subtopic	2.7	Estimating and interpreting Pearson's product-moment correlation coefficient

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Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.12;* © VCAA

Question 1 (1 mark)

There is a strong positive association between a country's Human Development Index and its carbon dioxide emissions.

From this information, it can be concluded that

A. increasing a country's carbon dioxide emissions will increase the Human Development Index of the country.

B. decreasing a country's carbon dioxide emissions will increase the Human Development Index of the country.

C. this association must be a chance occurrence and can be safely ignored.

D. countries that have higher human development indices tend to have higher levels of carbon dioxide emissions.

E. countries that have higher human development indices tend to have lower levels of carbon dioxide emissions.

Question 2 (1 mark)

A scatterplot cannot be used to display the relationship between which of the following pairs of variables?

A. The marks obtained by a group of students and their gender.

B. The masses kg of newborn male and female giraffes born in captivity.

C. The number of couriers delivering packages to two nearby factories.

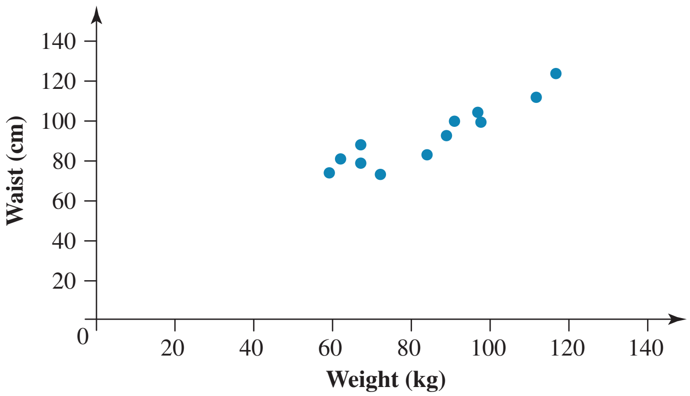
D. The weights and ages of females attending a gym.

E. The scores obtained by basketball teams from two neighbouring schools over the course of a season.

Topic 2 Subtopic 2.7 Estimating and interpreting Pearson's product-moment correlation coefficient

Question 3 (1 mark)

An estimate of the correlation coefficient for the data shown is:



A. 0.93
B. −0.93
C. 0.67
D. −0.67
E. 0.28

Question 4 (1 mark)

A set of data comparing age and stress levels was found to have a Pearson's product moment correlation coefficient of $r = 0.8553$.

Which of the following conclusions can be drawn from this data?

A. Old age causes stress.
B. Young people are more stressed than older people.
C. Stress is caused by aging.
D. Age and stress levels are unrelated.
E. The older a person is, the higher their stress level.

Question 5 (1 mark)

The relationship between the amount of savings in the bank accounts of a group of people and the amount they spent each week was found to have a Pearson's product moment correlation coefficient of $r = -0.87$.

Which of the following conclusions can be drawn from this data?

A. Increased spending per week caused people to be unable to save money.
B. Saving money caused people to have less money to spend each week.
C. The more money people spent each week, the more they could save.
D. The more money people spent each week, the less they saved.
E. The more money people had, the more they spent.

Topic	2	Investigating associations between two variables
Subtopic	2.8	Calculating r and the coefficient of determination, r^2

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Source: *VCE 2013, Further Mathematics Exam 1, Section A, Core, Q.8; © VCAA*

Question 1(1 mark)

The table below shows the hourly rate of pay earned by 10 employees in a company in 1990 and in 2010.

Employee	Hourly rate of pay($)	
	1990	2010
Ben	9.53	17.02
Lani	9.15	16.71
Freya	8.88	15.10
Jill	8.60	15.93
David	7.67	14.40
Hong	7.96	13.32
Stuart	6.42	15.40
Mei Lien	11.86	19.79
Tim	14.64	23.38
Simon	15.31	25.11

The value of the correlation coefficient, r, for this set of data is closest to

A.0.74
B.0.86
C.0.92
D.0.93
E.0.96

Question 2 (1 mark)
Source: *VCE 2013, Further Mathematics Exam 1, Section A, Core, Q.7; © VCAA*
For a city, the correlation coefficient between
- population density and distance from the centre of the city is $r = -0.563$
- house size and distance from the centre of the city is $r = 0.357$.

Given this information, determine which of the following statements is true.
A. Around 31.7% of the variation observed in house size in the city can be explained by the variation in distance from the centre of the city.
B. Population density tends to increase as the distance from the centre of the city increases.
C. House sizes tend to be larger as the distance from the centre of the city decreases.
D. The slope of a least-squares regression line relating population density to distance from the centre of the city is positive.
E. Population density is more strongly associated with distance from the centre of the city than is house size.

Question 3 (1 mark)
Source: *VCE 2011, Further Mathematics Exam 1, Section A, Q.11*; *© VCAA*
For a group of 15-year-old students who regularly played computer games, the correlation between the time spent playing computer games and fitness level was found to be $r = -0.56$.
On the basis of this information it can be concluded that
A. 56% of these students were not very fit.
B. these students would become fitter if they spent less time playing computer games.
C. these students would become fitter if they spent more time playing computer games.
D. the students in the group who spent a short amount of time playing computer games tended to be fitter.
E. the students in the group who spent a large amount of time playing computer games tended to be fitter.

Question 4 (1 mark)

The following data was recorded from a survey of 10 people:

Age (years)	Armspan (cm)
12	134
13	132
15	140
15	144
18	156
19	163
22	175
25	189
26	157
28	168

Correct to 4 decimal places, the value of the Pearson's product-moment correlation coefficient for this data is

A. 0.8198
B. 0.6721
C. −0.8198
D. −0.6721
E. 0.2515

Question 5 (1 mark)

The regression equation for a set of data is: volume = 2300 − 7.61× time and the coefficient of determination is 0.8192. The value of Pearson's product-moment correlation for this data, correct to 3 decimal places, is

A. 0.905
B. −0.90
C. 0.671
D. −0.671
E. 0.410

Topic	**2**	**Investigating associations between two variables**
Subtopic	**2.9**	**Cause and effect**

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Source: *VCE 2017, Further Mathematics Exam 1, Section A, Core, Q.12*; © VCAA

Question 1 (1 mark)

Data collected over a period of 10 years indicated a strong, positive association between the number of stray cats and the number of stray dogs reported each year ($r = 0.87$) in a large, regional city.

A positive association was also found between the population of the city and both the number of stray cats ($r = 0.61$) and the number of stray dogs ($r = 0.72$).

During the time that the data was collected, the population of the city grew from 34 564 to 51 055.

From this information, we can conclude that

A. if cat owners paid more attention to keeping dogs off their property, the number of stray cats reported would decrease.

B. the association between the number of stray cats and stray dogs reported cannot be causal because only a correlation of $+1$ or -1 shows causal relationships.

C. there is no logical explanation for the association between the number of stray cats and stray dogs reported in the city, so it must be a chance occurrence.

D. because larger populations tend to have both a larger number of stray cats and stray dogs, the association between these two numbers can be explained by a common response to a third variable, which is the increasing population size of the city.

E. more stray cats were reported because people are no longer as careful about keeping their cats properly contained on their property as they were in the past.

Question 2 (1 mark)

Following an investigation into the relationship between the two variables of tissue sales and hot chocolate sales, a correlation value of $r = 0.95$ was found. Which of the following conclusions can be drawn from this result?

A. strong tissue sales cause an increase in sales of hot chocolate

B. 90.25% of the variation in hot chocolate sales can be explained by the variation in tissue sales

C. Tere is no causal relationship between the two variable

D. 90.25% of the variation in tissue sales can be explained by the variation in hot chocolate sales

E. Higher tissue sales are the cause of greater hot chocolate sales.

Question 3 (1 mark)
A recent study over a summer period found a strong, positive and linear association between ice-cream sales and house burglaries. Which of the following statements is false?
A. House burglaries tended to increase as ice-cream sales increased
B. Higher ice-cream sales caused more burglaries.
C. A confounding factor is evident.
D. Non-causal reasons are evident in this study.
E. More homes are vacant in hot weather as more people go out to enjoy warmer weather.

Question 4 (1 mark)
Determine which one of the following comparisons would have a definite linked or causal relationship.
A. High ATAR scores and high university score
B. High red meat intake and higher incidence of prostate cancer
C. High mortality rate and low level of personal happiness
E. High quantity of food consumed in the last hour and low level of hunger.

Question 5 (1 mark)
Which of the following descriptors associating two variables together are classified as non-causal?

I. Pearson's correlation
II. Confounded
III. Coincidence
IV. Categorical
V. Common response

A. I, II and III
B. II, III and V
C. I and IV
D. III, IV and V
E. V only

Question 6 (1 mark)
The correlation between the variables of lung cancer and smoking is due, in part, to the fact that lung cancer is responding to changes in some unobserved third variable.
This can be best explained by

A. the cause and effect between smoking and lung cancer.
B. the effect of a non-causal explanation.
C. extrapolation.
D. interpolation.
E. statistical inference.

Topic	2	Investigating associations between two variables
Subtopic	2.10	Review

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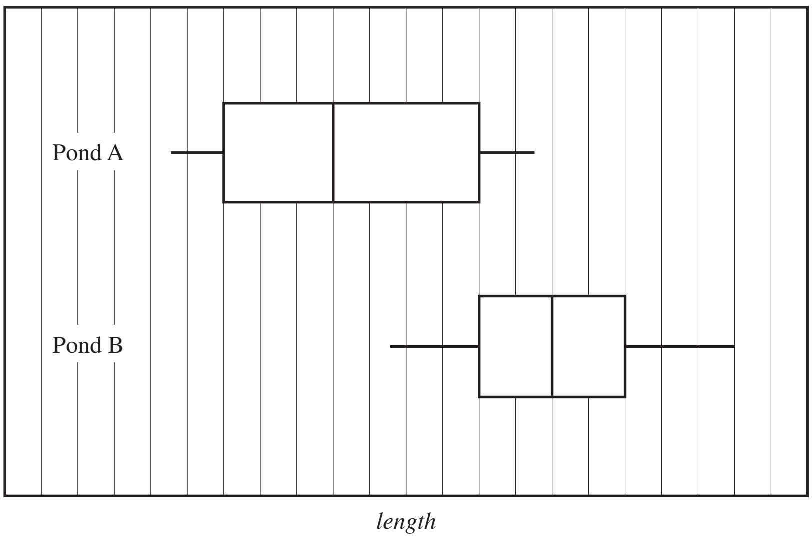
Source: *VCE 2021, Further Mathematics Exam 1, Section A, Core, Q.4; © VCAA*

Question 1 (1 mark)

The boxplots below show the distribution of the length of fish caught in two different ponds, Pond A and Pond B.



Based on the boxplots above, it can be said that

A. 50% of the fish caught in Pond A are the same length as the fish caught in Pond B.
B. 50% of the fish caught in Pond B are longer than all of the fish caught in Pond A.
C. 50% of the fish caught in Pond B are shorter than all of the fish caught in Pond A.
D. 75% of the fish caught in Pond A are shorter than all of the fish caught in Pond B.
E. 75% of the fish caught in Pond B are longer than all of the fish caught in Pond A.

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.11;* © *VCAA*

Question 2 (1 mark)

A study was conducted to investigate the effect of drinking *coffee* on sleep.

In this study, the amount of *sleep*, in hours, and the amount of *coffee* drunk, in cups, on a given day were recorded for a group of adults.

The following summary statistics were generated.

	***Sleep* (hours)**	***Coffee* (cups)**
Mean	7.08	2.42
Standard deviation	1.12	1.56
Correlation coefficient (*r*)	−0.770	

On average, for each additional cup of *coffee* drunk, the amount of *sleep*

A. decreased by 0.55 hours.
B. decreased by 0.77 hours.
C. decreased by 1.1 hours.
D. increased by 1.1 hours.
E. increased by 2.3 hours.

Source: *VCE 2017, Further Mathematics Exam 2, Section A, Core, Q.2; © VCAA*

Question 3 (1 mark)

The back-to-back stem plot below displays the *wingspan*, in millimetres, of 32 moths and their *place of capture* (forest or grassland).

Key 1|8 = 18 ***wingspan*** **(mm)**

forest (n = 13)		**grassland** (n = 19)
6	1	8
2 1 1 0 0 0 0	2	2 2 4 4
7	2	5 5 9
4 0	3	0 0 1 2 3 4
5	3	86 8
	4	30 3
	4	5
2	5	

a. Which variable, *wingspan* or *place of capture*, is a categorical variable? **(1 mark)**

b. Write down the modal wingspan, in millimetres, of the moths captured in the forest. **(1 mark)**

c. Use the information in the back-to-back stem plot to complete the table below. **(2 marks)**

	***Wingspan* (mm)**				
Place of capture	**minimum**	**Q_1**	**median (M)**	**Q_3**	maximum
forest		20	21	32	52
grassland	18	24	30		45

d. Show that the moth captured in the forest that had a *wingspan* of 52 mm is an outlier. **(2 marks)**

e. The back-to-back stem plot suggests that *wingspan* is associated with *place of capture*. Explain why, quoting the values of an appropriate statistic. **(2 marks)**

Source: VCE 2016, Further Mathematics Exam 1, Section A, Q.12; © VCAA

Question 4 (1 mark)

There is a strong positive association between a country's Human Development Index and its carbon dioxide emissions.

From this information, it can be concluded that

A. increasing a country's carbon dioxide emissions will increase the Human Development Index of the country.

B. decreasing a country's carbon dioxide emissions will increase the Human Development Index of the country.

C. this association must be a chance occurrence and can be safely ignored.

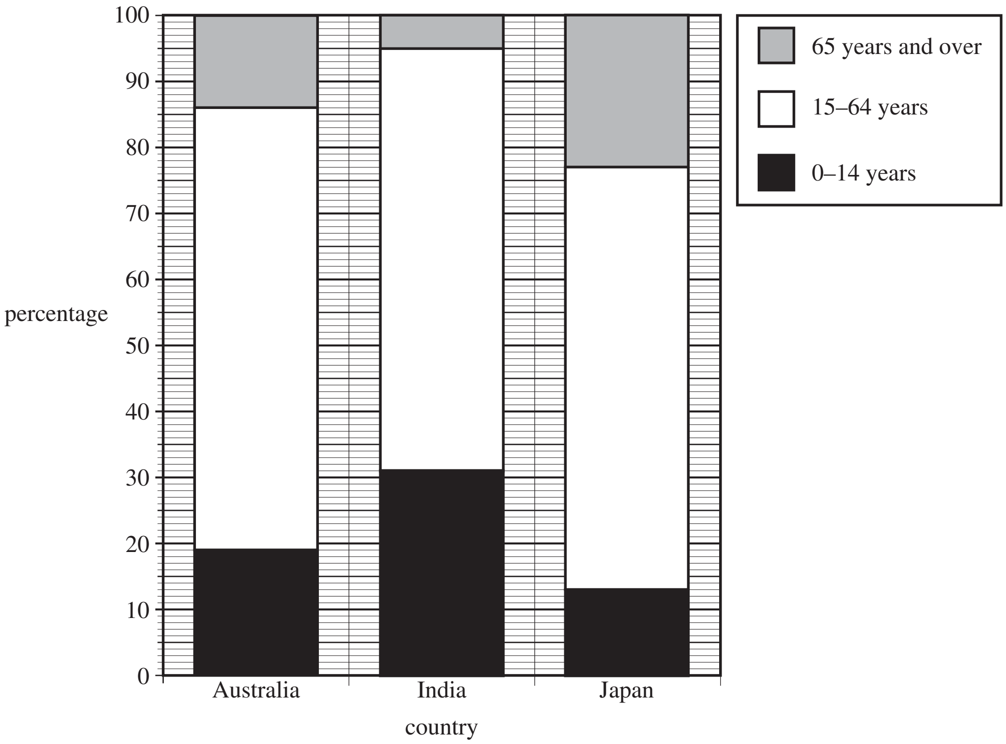
D. countries that have higher human development indices tend to have higher levels of carbon dioxide emissions.

E. countries that have higher human development indices tend to have lower levels of carbon dioxide emissions.

***Source:** VCE 2014, Further Mathematics Exam 2, Core, Q.1; © VCAA*

Question 5 (1 mark)

The segmented bar chart below shows the age distribution of people in three countries, Australia, India and Japan, for the year 2010.



Source: Australian Bureau of Statistics, 3201.0 – Population by Age and Sex, Australian States and Territories, June 2010

a. Write down the percentage of people in Australia who were aged 0 – 14 years in 2010.
Write your answer, correct to the nearest percentage. **(1 mark)**

b. In 2010, the population of Japan was 128 000 000.
How many people in Japan were aged 65 years and over in 2010? **(1 mark)**

c. From the graph above, it appears that there is no association between the percentage of people in the 15 – 64 age group and the country in which they live.
Explain why, quoting appropriate percentages to support your explanation. **(1 mark)**

Source: VCE 2018, Further Mathematics Exam 1, Section A, Q.6; © VCAA

Question 6 (1 mark)

Data was collected to investigate the association between the following two variables:

- *age* (29 and under, 30 – 59, 60 and over)
- *uses public transport* (yes, no)

Which one of the following is appropriate to use in the statistical analysis of this association?

A. a scatterplot
B. parallel box plots
C. a least squares line
D. a segmented bar chart
E. a segmented bar chart *r*

Source: VCE 2016, Further Mathematics Exam 1, Section A, Q.1; © VCAA

Question 7 (1 mark)

The *blood pressure* (low, normal, high) and the age (under 50 years, 50 years or over) of 110 adults were recorded. The results are displayed in the two-way frequency table below.

Blood pressure	**Age**	
	Under 50 years	**50 years or over**
low	15	5
normal	32	24
high	11	23
Total	58	52

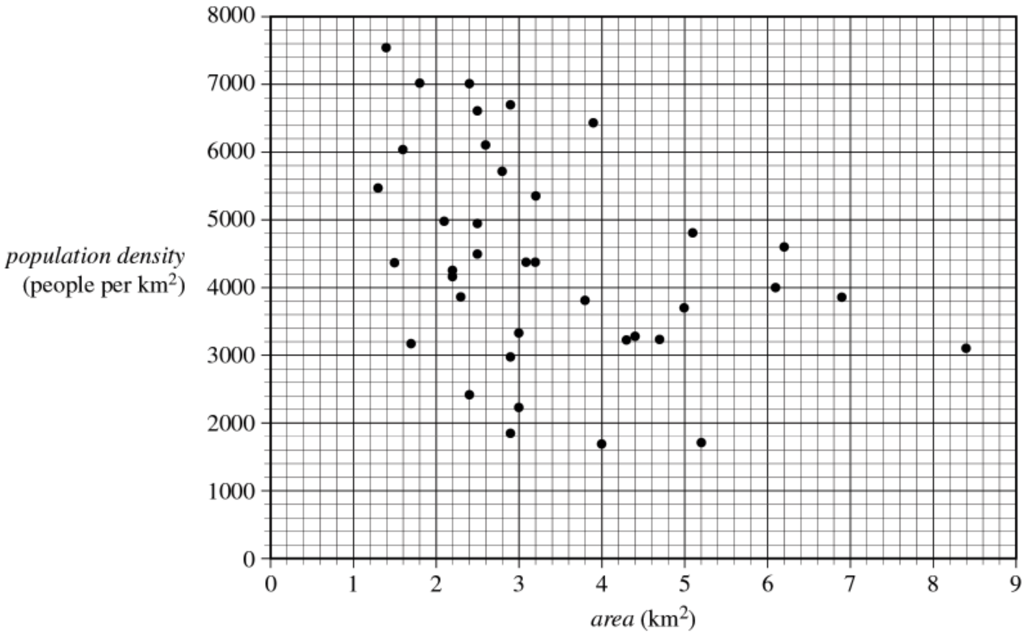
The **percentage** of adults under 50 years of age who have high blood pressure is closest to

A. 11%
B. 19%
C. 26%
D. 44%
E. 58%

Source: *VCE 2014, Further Mathematics Exam 2, Core, Q.4; © VCAA*

Question 8

The scatterplot below shows the *population density*, in people per square kilometre, and the area, in square kilometres, of 38 inner suburbs of the same city.



For this scatterplot, $r^2 = 0.141$

a. Describe the association between the variables *population density* and *area* for these suburbs in terms of strength, direction and form. **(1 mark)**

b. The mean and standard deviation of the variables *population density* and area for these 38 inner suburbs are shown in the table below.

	Population density (people per km^2**)**	***Area*** **(**km^2**)**
Mean	4370	3.4
Standard deviation	1560	1.6

i. One of these suburbs has a population density of 3082 people per square kilometre.
Determine the standard z-score of this suburb's population density.
Write your answer, correct to one decimal place. **(1 mark)**

ii. Assume the areas of these inner suburbs are approximately normally distributed.
How many of these 38 suburbs are **expected** to have an area that is two standard deviations or more above the mean?
Write your answer, correct to the nearest whole number. **(1 mark)**

iii. How many of these 38 inner suburbs **actually** have an area that is two standard deviations or more above the mean? **(1 mark)**

Source: *VCE 2013, Further Mathematics Exam 1, Section A, Core, Q.3; © VCAA*

Question 9 (1 mark)

The heights of 82 mothers and their eldest daughters are classified as 'short', 'medium' or 'tall'. The results are displayed in the frequency table below.

		Mother		
		Short	**Medium**	**Tall**
Eldest daughter	**Short**	16	10	3
	Medium	8	14	11
	Tall	5	7	8

The number of mothers whose height is classified as 'medium' is

A. 7
B. 10
C. 14
C. 31
D. 33

Source: *VCE 2013, Further Mathematics Exam 1, Section A, Core, Q.4; © VCAA*

Question 10 (1 mark)

The heights of 82 mothers and their eldest daughters are classified as 'short', 'medium' or 'tall'. The results are displayed in the frequency table below.

		Mother		
		Short	**Medium**	**Tall**
Eldest daughter	**Short**	16	10	3
	Medium	8	14	11
	Tall	5	7	8

Of the mothers whose height is classified as 'tall', the percentage who have eldest daughters whose height is classified as 'short' is closest to

A. 3%
B. 4%
C. 14%
D. 17%
E. 27%

Question 11 (1 mark)

The level of participation in sport (never, sometimes, often) for 150 people of varying ages is indicated in the table below. Some of the values are missing.

Participation in sport	Age group			
	0 – 10	11 – 20	21 – 30	Total
Never	8	3	1	
Sometimes	6			82
Often	25		15	
Total			62	150

For this sample, the total number of 11 – 20 year olds who sometimes participated in sport is

A. 12
B. 16
C. 20
D. 39
E. 46

Question 12 (1 mark)

A number of people were surveyed and had their hair and eye colour recorded.

Eye colour	Hair colour			
	Brown	Black	Red	Total
Blue	5	8	2	15
Green	4	7	4	15
Hazel	5	3	7	15
Total	14	18	13	45

Of the people who had their hair colour recorded as red, the percentage who also had an eye colour of green is closest to

A. 9%
B. 27%
C. 29%
D. 31%
E. 100%

Question 13 (1 mark)

A back-to-back stem plot cannot be used to display the relationship between

A. the marks obtained by a group of students in their end-of-semester Chemistry and French exams.
B. the masses (kg) of newborn male and female giraffes born in captivity.
C. the number of couriers delivering packages to two nearby factories.
D. the weights and ages of females attending a gym.
E. the scores obtained by basketball teams from two neighbouring schools over the course of a season.

Question 14 (1 mark)

The back-to-back stem plot below shows the scores obtained by a class of students studying Physics.

leaf females	Stem	leaf males
6 2	3	0 1
8 5 3	4	3 4 5
1	5	5 7 9
5	6	
2 1	7	4 5 3 1
5 4	8	6 2
7 4 0	9	4

Key: 3|0 = 30%

Which of the following statements is true?

A. The median for females is lower than the median for males.
B. The range for females is lower than the range for males.
C. There are equal numbers of males and females studying Physics.
D. The *IQR* for females is lower than the *IQR* for males.
E. The mean for females is higher than the mean for males.

Question 15

A survey of **103** car drivers recorded the following information.

Gender	Car type			
	SUV	**4WD**	**Sedan**	**Total**
Male	15	23	7	45
Female	7	9	42	58
Total	22	32	49	103

The percentage of male SUV owners in this surveyed group is closest to

A. 15%

B. 22%

C. 33%

D. 68%

E. 72%

Answers and marking guide

2.2 Response and explanatory variables

Question 1
Both variables are defined as categorical variables.

Question 2
The data represents a numerical and a categorical variable. Parallel boxplots graph this data.
A histogram requires two numerical variables.
A scatterplot requires two sets of numerical data.
A time series plot requires two sets of numerical data.
A back-to-back stem plot requires categorical sets (only two options) and numerical data.

Question 3
Training sessions would generate or create data linked to weight loss. Therefore training sessions are the explanatory (horizontal) variable and weight loss is the response (vertical) variable.

Question 4
Drug X was administered to generate or create a response in the recovery time of the rabbits. Therefore, drug X is the explanatory variable.

Question 5
The volume of traffic would create or generate the air pollution. Therefore, air pollution is the response variable.

2.3 Contingency (two-way) frequency tables and segmented bar charts

Question 1
Select the option that shows an *association* between two variables, not just a fact about the table (eliminate A and B).
The supportive statement containing correct values is option E.
20% of sugar traps contained more than 500 moths while 10% of light traps contained more than 500 moths.

Question 2
There are 11 adults with high blood pressure who are also under 50 years of age. There are a total of 58 adults who are under 50 years of age.

$$\begin{aligned}\frac{11}{58}\times 100 &= 18.97\% \\ &= 19\%\end{aligned}$$

Question 3
From the graph, 30% of 300 sugar traps caught less than 250 moths.

$$\frac{30}{100}\times 300 = 90$$

Question 4

$$\frac{91}{161}\times 100 = 56.5217\%$$

Question 5

$$\frac{12}{70}\times 100 = 17.14\%$$

Question 6
The second variable also needs to have a small number of categories. The only possibility is *sex* (male, female).
VCAA Examination Report **note:**
Students needed to recognise that for a two way-frequency table to be used, both variables had to be categorical variables.

2.4 Back-to-back stem plots

Question 1
Back-to-back stem plots are used when comparing categorical data with numerical data. The car's speed is the numerical data. Options B and E both contain categorical data, but for a back-to-back stem plot, the categorical data can only contain two categories, i.e. M or F.
VCAA Assessment Report **note:**
An understanding of types of variables was also required to answer this question. The key was to recognise that the type of variable plays a role in choosing an appropriate statistical plot when displaying data. Of the plots listed in this question, only a back-to-back stem plot was suitable for displaying the association between a car's speed (a numerical variable) and the sex of the driver (a categorical variable with two categories). A parallel box plot would also have been appropriate, but was not given as an option.

Question 2
The male IQR = 43 − 23 = 21.
Adding 43 or 45 to the female data would find $Q_1 = 52$ and $Q_3 = 74$. Does not give IQR = 20
Adding 74 to the female data would find $Q_1 = 55$ and $Q_3 = 74$. Does not give IQR = 20
Adding 75 to the female data would find $Q_1 = 55$ and $Q_3 = 75$. Gives IQR = 20
Adding 84 to the female data would find $Q_1 = 55$ and $Q_3 = 76$. Does not give the IQR = 20

Question 3
a. Using CAS: mean $\left(\bar{x}\right) = 20$, median = 18, mode = 21,
standard deviation = 4.7
Award 1 mark for each correct value = 4 marks

b. Min $X = 17$
$Q_1 = 17.5$
Med = 18
$Q_3 = 21$
Max $X = 37$ **[1 mark]**
Range = 37 − 17
= 20 **[1 mark]**
IQR = 21 − 17.5
= 3.5 **[1 mark]**

c. Using CAS: mean $\left(\bar{x}\right) = 20$, median = 21, mode = 18, standard deviation = 10.5
Award 1 mark for each correct value = 4 marks

d. Min $X = 18$
$Q_1 = 18$
Med = 21
$Q_3 = 33$
Max $X = 45$ **[1 mark]**

Range = 45 − 18
= 27 **[1 mark]**
IQR = 33 − 18
= 15 **[1 mark]**

e. Answers will vary. Students should include at least three of the following.
- There are many more male offenders than female offenders. **[1 mark]**
- The ages of the male offenders include many younger males, even below legal driving age. **[1 mark]**
- The mean age of male offenders is 20, while the mean age for female offenders is 25.4 (much higher). **[1 mark]**
- There is a greater deviation in the female results as compared to the male results, as seen by the different standard deviation values. **[1 mark]**
- Most of the offenders were below 25 (only 3 out of 26 were older). **[1 mark]**

Question 4

a. Place of capture **[1 mark]**

b. The most frequently occurring value in the forest section of the stem plot is 20, so the modal wingspan is 20 mm. **[1 mark]**

c. The minimum wingspan in the forest is 16 mm. **[1 mark]**
The upper quartile (Q_3) in the grassland is 36 mm. **[1 mark]**

d. $IQR = Q_3 - Q_1$
$= 32 - 20$
$= 12$
The wingspan of 52 mm is at the upper end of the forest values. Students need to show that this value is greater than the upper fence, that is, greater than $Q_3 + 1.5 \times IQR$.
$Q_3 + 1.5 \times IQR$
$= 32 + 1.5 \times 12$
$= 50$
As 52 mm is greater than this upper-fence value of 50 mm, it is an outlier. **[1 mark]**

e. Possible solution:
The wingspan is associated with the place of capture. Those captured in the grassland had a median wingspan of 30 mm, which is greater than the median wingspan of 21 mm of the moths captured in the forest.
[Award **1 mark** for stating that the median wingspan of those captured in grassland is greater than the median wingspan of those captured in the forest. Award **1 mark** for stating the values of the two medians, 30 mm and 21 mm.]
VCAA Examination Report **note:**
The mean is not part of a stemplot five-number summary and was not appropriate to use because of the outlier of 52 in the forest data.
Students who initially gave the required median comparisons and then went further by quoting comparisons of other irrelevant statistics were not awarded full marks.

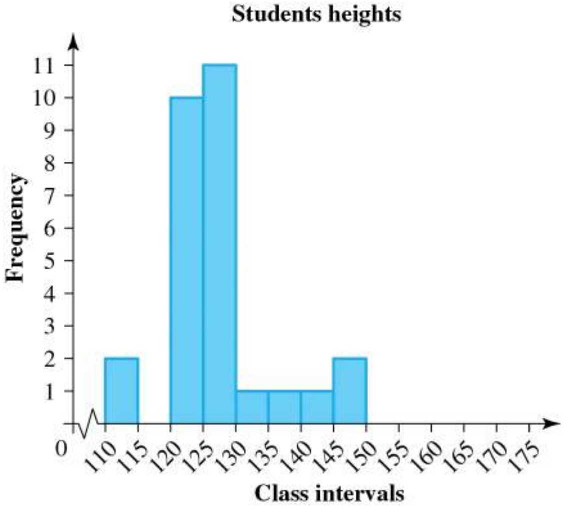
Question 5

a.

Class interval (5 cm)	**Frequency**
110 – 114	2
115 – 199	0
120 – 124	10
125 – 129	11
130 – 134	1
135 – 139	1
140 – 144	1
145 – 149	2

[1 mark]

b. **[1 mark]**



c. The data is positively skewed. **[1 mark]**

The data is clustered between 120 cm and 130 cm. **[1 mark]**

Question 6

The five figure summary for the Year 7 heights is:

123, 132, 142, 155, 171

The five figure summary for the Year 12 heights is:

148, 165, 172, 186, 201

The Year 12 median height is 30 cm greater than the Year 7 median height, not 20 cm as indicated in Option C. Incorrect.

Question 7

The variables of year 12 girls and year 12 boys are categorical and they both generate numerical data.

2.5 Parallel boxplots and dot plots

Question 1

Parallel boxplots are used to investigate the association between a numerical variable and a categorical variable.

In this question, the variable - *monthly rainfall* (in mm) is numerical, so the unknown second variable must be categorical. Therefore, *Month of the year*, option C is the only categorical variable in the given options.

***VCAA Assessment Report* note:**

In this question, the variable *monthly rainfall* (in mm) was numerical, so the unknown second variable must be categorical. *Month of the year* (January, February, March, etc.), option C, was the only categorical variable in the list of options provided.

Question 2

a Negatively skewed **[1 mark]**

b The median life expectancy increases as the years increase (in 1953 the median is 51; in 1973 the median is 63; in 1993 the median is 69). There is a positive correlation.
Award **1 mark** for discussion about median, and **1 mark** for explaining the association.
***VCAA Assessment Report* note:**
Some students referred to mean values, which are not discernible from a box plot unless it is perfectly symmetrical.

Question 3

Noting the positions of each of the respective five figure summary statistics shows that less than 100% of the 16–35 age bracket were less than the median of the 0–15 age bracket. (The entire boxplot is not below the median (middle) value of the 0–15 age bracket.) This indicates Option D is inaccurate.

Question 4

The *month* is the explanatory variable and the *minimum daily temperature* is the response variable.
The *median* values decrease with the month, which is expected as the year moves from summer into winter months.
Award **1 mark** for identifying an appropriate statistic – e.g. median.
Award **1 mark** for the explanation.
***VCAA Examination Report* note:**
A statement that a **decrease** or **change** in median (or IQR) signals an association was required for the first mark to be awarded. Median (or IQR) values for all three months needed to be quoted correctly for the second mark to be awarded.
Successful responses focused on one statistic only (usually the median) and quoted the values from the table rather than estimating from the boxplots.
Incorrect answers included using the word 'averages' or 'means' rather than medians and quoting only two medians rather than all three.
Some students went on to comment on the minimums and maximums; this additional information compromised an otherwise correct answer. Comments about the shape of the boxplots were also not appropriate.

Question 5

a. **i.** $IQR = 17° - 12° = 5°\text{C}$ **[1 mark]**

ii. Median for maximum temperature is 25°C and median for minimum temperature is 15°C, so the median for maximum temperature was 10°C higher. **[1 mark]**

iii. One – the outlier **[1 mark]**

b. If the median is 9.4°C, then it is expected that 50% will be above this. There are 30 days in November, so $50\% \times 30 = 15$ days. **[1 mark]**
***VCAA Examination Report* note:**
Some gave the percentage of 50% rather than the number of days.

2.6 Scatterplots

Question 1

Points are tightly grouped in a good linear format. Best description is strong association between the points in a linear form with a positive gradient.

Question 2
As the number of cigarettes smoked increases, fitness level decreases.

Question 3
There is a moderate, linear, negative relationship between the number of hours worked and the number of hours for recreation.

Question 4
A scatterplot shows the relationship between two numerical variables. Because gender is nominal rather than numerical, gender cannot be shown on a scatterplot.

Question 5
Response and explanatory variables

Question 6
As the number of hours of study increases, the marks achieved at school increase.

2.7 Estimating and interpreting Pearson's product–moment correlation coefficient

Question 1
A *positive* association means that as the explanatory variable (*HDI*) increases, so too does the response variable (*carbon dioxide emissions*).

Question 2
A scatterplot shows the relationship between two numerical variables. Because gender is nominal rather than numerical, gender cannot be shown on a scatterplot.

Question 3
The data is tightly grouped in a very linear formation. This indicates r is positive and would be close to 1 . Option A is the most representative of this data set.

Question 4
There are many factors involved in the stress levels of people so we cannot conclude what *caused* them to be stressed. However, we can conclude from the study that the older a person is, the higher their stress level.

Question 5
There are many factors involved in people's ability to save money so we cannot conclude what *caused* them to save. However, we can conclude from the study that the more money people spent each week, the less they saved.

2.8 Calculating r and the coefficient of determination, r^2

Question 1
Using a CAS calculator and entering the rate of pay for 1990 as the independent variable and the rate of pay for 2010 as the dependent variable gives $r = 0.9622$, which is closest to 0.96.

Question 2
The value of $r = -0.563$ indicates **moderate** (negative) correlation between the population density and distance from the centre of the city, while the value of $r = 0.357$indicates **weak** (positive) correlation between the house size and distance from the centre of the city. Therefore, population density is more strongly associated with distance from the centre of the city than is house size.

Question 3
We don't know how fit the students were, or whether they played computer games very much or not. There is a negative correlation, which tells us that greater fitness levels tended to occur with lower time playing computer games and vice versa.

Question 4
Performing a linear regression analysis of the data via a calculator generates the r value of 0.8198.

Question 5
The regression equation indicates a negative gradient for the data; therefore r will be negative.
The coefficient of determination represents r^2, while the product–moment correlation is r.
For this data:
$r = -\sqrt{0.8192}$
$\;\; = -0.905$

2.9 Cause and effect

Question 1
Three variables have been listed in this question.
The number of stray cats, the number of stray dogs and the population of the city (which grew in size over the time period data was collected).
Correlation does not imply causality. Option D is the only logical conclusion.

Question 2
There is no causal relationship between the two variables, which makes the correlation value unfounded. There would be other factors affecting the variables. For example, the weather could be one possibility. If the investigation involved data taken during the colder months of the year, a rise in tissue sales would be more than likely as would a rise in hot chocolate sales.

Question 3
Association is not causation, making Option B the incorrect response.

Question 4
High ATAR scores and high university scores – both are linked to a third variable of individual ability and previously attained knowledge. No definite causal relationship.
High red meat intake and higher incidence of prostate cancer – both are linked to a third variable of testosterone. No definite causal relationship.
High divorce rates and a higher incidence of drug use – both are linked to a third variable of a coincidental trend of time. No definite causal relationship.
High mortality rate and low level of personal happiness – result is confounded by other factors such as exercise level and willingness to take prescribed medicines. No definite causal relationship.
High quantity of food consumed in the last hour and low level of hunger – obvious causal relationship between food consumed and level of hunger. Definite causal relationship.

Question 5
The non-causal descriptors are II, III and V

Question 6
Because the correlation is linked to an unobserved third variable, it is categorised as a non-causal explanation.

2.10 Review

Question 1

As we know, 50% of the difference is between the minimum and the median, 50% of the difference is between Q_1 and Q_3, and the other 50% could also be between the median and the maximum.
Based on the parallel boxplots, we can clearly see that the median of Pond B is greater than the maximum of Pond A; it indicates that at least 50% of the fish caught in Pond B are longer than all the fish caught in Pond A.

Question 2

This question is asking for the gradient connecting the two variables. It is investigating the effect that drinking coffee has on sleep, so sleep is the response variable.

$$b = r\frac{s_y}{s_x} = r\frac{s_{sleep}}{s_{coffee}} = -0.770 \times \frac{1.12}{1.56} = -0.55282$$

***VCAA Examination Report* note:**

Students needed to recognise that sleep was the response variable and *coffee* the explanatory variable, and be familiar with the rules connecting a least squares regression line equation to summary statistics.

Question 3

a. Place of capture **[1 mark]**

b. The most frequently occurring value in the forest section of the stem plot is 20, so the modal wingspan is 20 mm. **[1 mark]**

c. The minimum wingspan in the forest is 16 mm. **[1 mark]**
The upper quartile (Q_3) in the grassland is 36 mm. **[1 mark]**

d. $IQR = Q_3 - Q_1$
$= 32 - 20$
$= 12$
The wingspan of 52 mm is at the upper end of the forest values. Students need to show that this value is greater than the upper fence, that is, greater than $Q_3 + 1.5 \times IQR$.
$Q_3 + 1.5 \times IQR$
$= 32 + 1.5 \times 12$
$= 50$
As 52 mm is greater than this upper-fence value of 50 mm, it is an outlier. **[1 mark]**

e. Possible solution:
The wingspan is associated with the place of capture. Those captured in the grassland had a median wingspan of 30 mm, which is greater than the median wingspan of 21 mm of the moths captured in the forest.
[Award **1 mark** for stating that the median wingspan of those captured in grassland is greater than the median wingspan of those captured in the forest. Award **1 mark** for stating the values of the two medians, 30 mm and 21 mm.]
***VCAA Examination Report* note:**
The mean is not part of a stemplot five-number summary and was not appropriate to use because of the outlier of 52 in the forest data.
Students who initially gave the required median comparisons and then went further by quoting comparisons of other irrelevant statistics were not awarded full marks.

Question 4

A *positive* association means that as the explanatory variable (*HDI*) increases, so too does the response variable (*carbon dioxide emissions*).

Question 5

a. Reading off Australia's column, the black section ends at 19. Therefore 19% of people were aged 0–14 years old in 2010. **[1 mark]**

b. Reading off Japan's column, the grey section starts at 77, so the percentage of people in Japan 65 years and over is 100 – 77 = 23%.

$\frac{23}{100} \times 128\,000\,000 = 29\,440\,000$ **[1 mark]**

***VCAA Assessment Report* note:**

The answer is expected to be written in full and not, for example using technology syntax such as 2.944E7, a technology representation of scientific form. Many students gave the percentage as the answer rather than the required number. Technology syntax is not to be used in providing answers; standard mathematical notation is to be used.

c. 15 – 64 age group:

Australia's population percentage = 86 − 19 = 67%

India's population percentage = 95 − 31 = 64%

Japan's population percentage = 77 − 13 = 64%

As the percentage of people aged 15 – 64 is almost the same for all three countries, there is no association between these percentages and the country in which they live.

[1 mark for appropriate calculations and correct reference to them in explanation]

***VCAA Assessment Report* note:**

Most students were able to explain that, because the percentages were all close to each other in the 15 – 64 age group, there was no association between the percentage of people in this age group and the country in which they lived. However, some students contradicted the given statement and claimed that 'there **was** an association because…'

Question 6

Since there are 2 categorical variables, age and *uses public transport*, a segmented bar chart is the best choice for representing the data.

Question 7

There are 11 adults with high blood pressure who are also under 50 years of age. There are a total of 58 adults who are under 50 years of age.

$$\frac{11}{58} \times 100 = 18.97\%$$
$$= 19\%$$

Question 8

a. $r = -\sqrt{0.141}$

$= -0.376$

Strength = weak

Direction = negative

Form = linear

[**1 mark** – all three must be correct]

***VCAA Assessment Report* note:**

The most common error by students who got the correct strength and form was to assume the direction was positive. The scatterplot is trending downwards as area increases. Another common, incorrect answer offered a discussion about the coefficient of determination and the percentage of the variation in population. Others referred to skewness, which is not applicable to a bivariate data plot.

b. i. $z = \dfrac{x - \overline{x}}{s_x}$

$= \dfrac{3082 - 4370}{1560}$

$= -0.8$ **[1 mark]**

ii. $\overline{x} \pm 2s_x \approx 95\%$

$area \geq \overline{x} \pm 2s_x$

$\approx \dfrac{100 - 95}{2}$

$\approx 2.5\%$

2.5% of $38 = 0.95$

≈ 1 **[1 mark]**

c. $\overline{x} + 2s_x = 3.4 + 2 \times 1.6$

$= 6.6$

Scanning the grid horizontally, there are two suburbs that actually have an area more than two standard deviations above the mean. One has an area of about 6.9 km^2 and the other an area of approximately 8.4 km^2.

[1 mark]

***VCAA Assessment Report* note:**

Very few students gave the correct answer, which relied upon calculating the area that was 'two standard deviations or more above the mean', found from $3.4 + 2 \times 1.6$. This then meant counting the suburbs (points) on the graph that had an area greater than or equal to 6.6 km^2. Some incorrect answers were greater than the number of suburbs in the city.

Question 9

The number of mothers whose height is classified as medium $= 10 + 14 + 7 = 31$.

Question 10

Number of tall mothers $= 3 + 11 + 8 = 22$

Percentage of tall mothers with short daughters $= \dfrac{3}{22} \times 100\% = 13.6364\% \approx 14\%$

***VCAA Assessment Report* note:**

The other common, but **incorrect**, response was 4% (option B). This response was obtained because the wrong base was used in calculating the percentage, as shown below.

$\dfrac{number\ of\ tall\ mothers\ with\ short\ daughters}{total\ number\ of\ mothers} \times 100\% = \dfrac{3}{82} \times 100\% = 3.65\ ...\ \%$This answer rounds to 4% (option B).

Question 11

Calculate 46 first to complete the 3rd column. Then calculate 20 to finish the second row.

Participation in sport	Age group			
	0 – 10	11 – 20	21 – 30	Total
Never	8	3	1	
Sometimes	6	20	46	82
Often	25		15	
Total			62	150

Question 12

The number of people who had their hair colour recorded as red was 13.

The percentage of these with green eyes was $\frac{4}{13} \times 100\% = 31\%$

Question 13

A back-to-back stem plot cannot be used to display the relationship between the weights and ages of females attending a gym because back-to-back stem plots display bivariate data with a single numerical variable and a categorical variable with two categories. In this case, weight and age are both numerical variables and there is only one category; that is, females.

Question 14

The median for females is 68. The median for males is 59.
The range for females is 65. The range for males is 64.
There are 14 females and 15 males.
The *IQR* for females is 40. The *IQR* for males is 27.
The mean for females is 65.21. The mean for males is 60.6.
Therefore, statement E is true.

Question 15

Percentage of male SUV owners $= \frac{15}{103} \times 100\% = 15\%$

3 Investigating and modelling linear associations

Topic	3	Investigating and modelling linear associations
Subtopic	3.2	Least squares line of best fit

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Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 1 (1 mark)

A least squares line of the form $y = a + bx$ is fitted to a scatterplot.

Which one of the following is always true?

A. As many of the data points in the scatterplot as possible will lie on the line.

B. The data points in the scatterplot will be divided so that there are as many data points above the line as there are below the line.

C. The sum of the squares of the shortest distances from the line to each data point will be a minimum.

D. The sum of the squares of the horizontal distances from the line to each data point will be a minimum.

E. The sum of the squares of the vertical distances from the line to each data point will be a minimum.

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.9; © VCAA*

Question 2 (1 mark)

A least squares line is used to model the relationship between the monthly *average temperature* and *latitude* recorded at seven different weather stations. The equation of the least squares line is found to be

$\textit{average temperature} = 42.9842 - 0.877447 \times \textit{latitude}$

When the numbers in this equation are correctly rounded to three significant figures, the equation will be

A. $\textit{average temperature} = 42.984 - 0.877 \times \textit{latitude}$

B. $\textit{average temperature} = 42.984 - 0.878 \times \textit{latitude}$

C. $\textit{average temperature} = 43.0 - 0.878 \times \textit{latitude}$

D. $\textit{average temperature} = 42.9 - 0.878 \times \textit{latitude}$

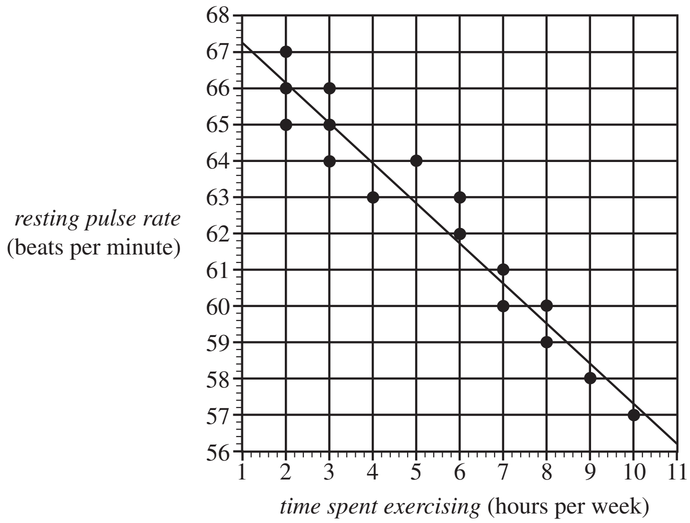
E. $\textit{average temperature} = 43.0 - 0.877 \times \textit{latitude}$

Topic 3 | Subtopic 3.2 Least squares line of best fit

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.8; © VCAA*

Question 3 (1 mark)

The scatterplot below displays the *resting pulse rate*, in beats per minute, and the *time spent exercising*, in hours per week, of 16 students. A least squares line has been fitted to the data.



The equation of this least squares line is closest to

a. *resting pulse rate* $= 67.2 - 0.91 \times$ *time spent exercising*

b. *resting pulse rate* $= 67.2 - 1.10 \times$ *time spent exercising*

c. *resting pulse rate* $= 68.3 - 0.91 \times$ *time spent exercising*

d. *resting pulse rate* $= 68.3 - 1.10 \times$ *time spent exercising*

e. *resting pulse rate* $= 67.2 + 1.10 \times$ *time spent exercising*

***Source:** VCE 2019, Further Mathematics Exam 1, Section A, Q.11; © VCAA*

Question 4 (1 mark)

A study was conducted to investigate the effect of drinking coffee on sleep.

In this study, the amount of *sleep*, in hours, and the amount of *coffee* drunk, in cups, on a given day were recorded for a group of adults.

The following summary statistics were generated.

	Sleep **(hours)**	***Coffee*** **(cups)**
Mean	7.08	2.42
Standard deviation	1.12	1.56
Correlation coefficient (r)	−0.770	

On average, for each additional cup of coffee drunk, the amount of *sleep*

A. decreased by 0.55 hours.

B. decreased by 0.77 hours.

C. decreased by 1.1 hours.

D. increased by 1.1 hours.

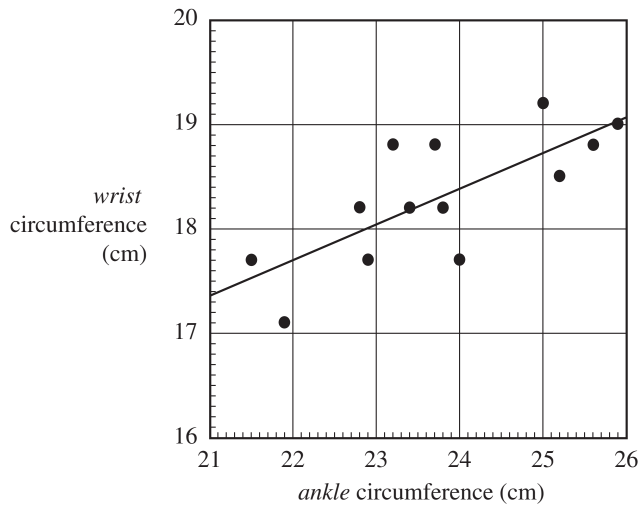
E. increased by 2.3 hours.

Topic 3 | Subtopic 3.2 Least squares line of best fit

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Core, Q.8; © VCAA*

Question 5 (1 mark)

The scatterplot below shows the *wrist* circumference and *ankle* circumference, both in centimetres, of 13 people. A least squares line has been fitted to the scatterplot with *ankle* circumference as the explanatory variable.



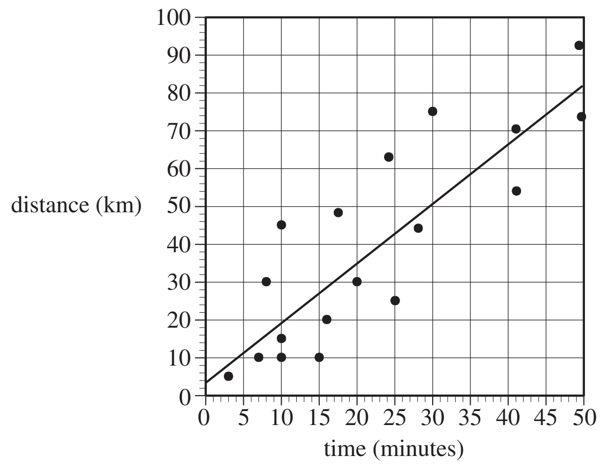
The equation of the least squares line is closest to

i. $ankle = 10.2 + 0.342 \times wrist$
ii. $wrist = 10.2 + 0.342 \times ankle$
iii. $ankle = 17.4 + 0.342 \times wrist$
iv. $wrist = 17.4 + 0.342 \times ankle$
v. $wrist = 17.4 + 0.731 \times ankle$

Topic 3 | Subtopic 3.2 Least squares line of best fit

Source: *VCE 2015, Further Mathematics Exam 1, Section A, Q.9; © VCAA*

Question 6 (1 mark)



A least squares regression line has been fitted to the scatterplot above to enable distance, in kilometres, to be predicted from time, in minutes.
The equation of this line is closest to

A. *distance* = 3.5 + 1.6 × *time*
B. *time* = 3.5 + 1.6 × *distance*
C. *distance* = 1.6 + 3.5 × *time*
D. *time* = 1.8 + 3.5 × *distance*
E. *distance* = 3.5 + 1.8 × *time*

Source: *VCE 2015, Further Mathematics Exam 1, Section A, Q.10; © VCAA*

Question 7 (1 mark)

For a set of bivariate data that involves the variables x and y:

$r = -0.47, \quad \bar{x} = 1.8, \quad s_x = 1.2, \quad \bar{y} = 7.2, \quad s_y = 0.85$

Given the information above, the least squares regression line predicting y from x is closest to

A. $y = 8.4 - 0.66x$
B. $y = 8.4 + 0.66x$
C. $y = 7.8 - 0.33x$
D. $y = 7.8 + 0.33x$
E. $y = 1.8 + 5.4x$

Source: VCE 2014, Further Mathematics Exam 1, Section A, Q.9; © VCAA

Question 8 (1 mark)

The equation of a least squares regression line is used to predict the fuel consumption, in kilometres per litre of fuel, from a car's weight, in kilograms.

This equation predicts that a car weighing 900 kg will travel 10.7 km per litre of fuel, while a car weighing 1700 kg will travel 6.7 km per litre of fuel.

The slope of this least squares regression line is closest to

A. −250
B. −0.005
C. −0.004
D. 0.005
E. 200

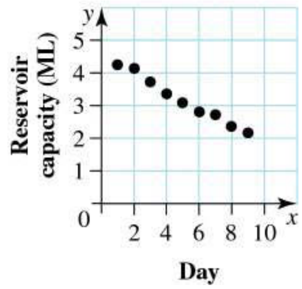
Question 9 (1 mark)

Which of the following statements about using least-squares regression to fit a straight line to data is false?

A. The least-squares regression method is used when there are no obvious outliers.
B. The vertical distance between the data value and the line is minimised.
C. The least-squares regression line can only be calculated by using a calculator.
D. The least-squares regression equation minimises the average deviation of the points in the data set from the line of best fit.
E. The least-squares regression equation can be determined arithmetically.

Question 10 (1 mark)

The graph shown below plots the daily water level in a reservoir during a drought.



If a line of best fit was fitted by eye, it would have a

A. positive gradient.

B. y-intercept at 0.

C. negative gradient.

D. y-intercept at 14.

E. a gradient of 0.

Topic	**3**	**Investigating and modelling linear associations**
Subtopic	**3.3**	**Interpretation, interpolation and extrapolation**

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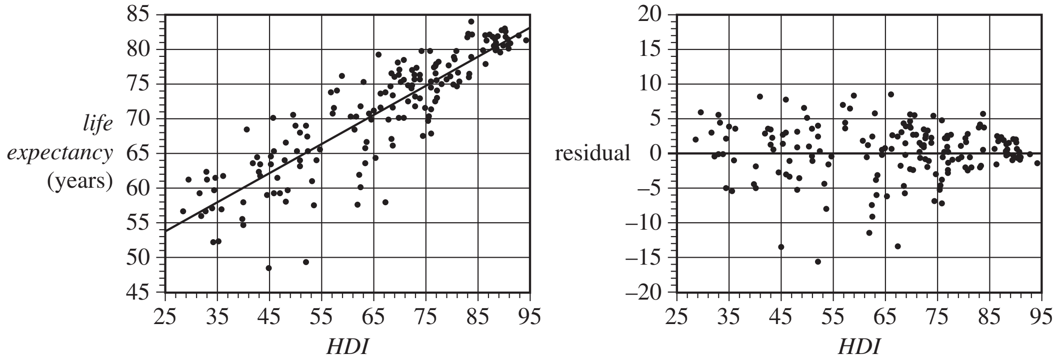
Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.9; © VCAA*

Question 1 (1 mark)

The scatterplot below shows life expectancy in years *(life expectancy)* plotted against the Human Development Index *(HDI)* for a large number of countries in 2011.

A least squares line has been fitted to the data and the resulting residual plot is also shown.



Data: Gapminder

The equation of this least squares line is

life expectancy $= 43.0 + 0.422 \times$ HDI

The coefficient of determination is $r^2 = 0.875$

Given the information above, which one of the following statements is not true?

A. The value of the correlation coefficient is close to 0.94.

B. 12.5% of the variation in life expectancy is not explained by the variation in the Human Development Index.

C. On average, life expectancy increases by 43.0 years for each 10-point increase in the Human Development Index.

D. Ignoring any outliers, the association between life expectancy and the Human Development Index can be described as strong, positive and linear.

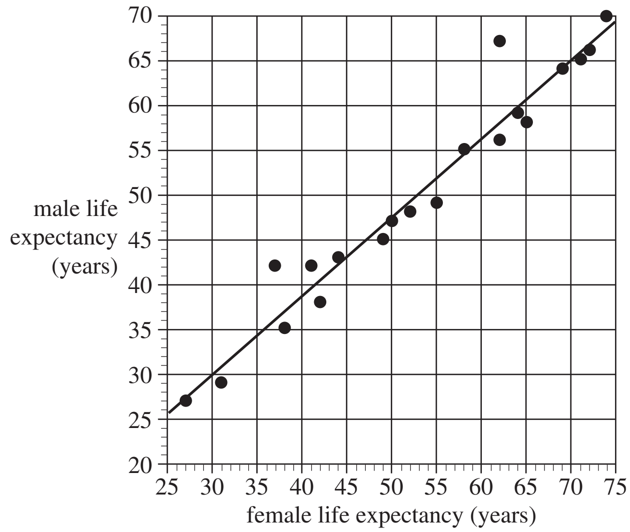
E. Using the least squares line to predict the life expectancy in a country with a Human Development Index of 75 is an example of interpolation.

Topic 3 Subtopic 3.3 Interpretation, interpolation and extrapolation

Source: *VCE 2015, Further Mathematics Exam 2, Section A, Q.3; © VCAA*

Question 2 (3 mark)

The scatterplot below plots male life expectancy (*male*) against female life expectancy (*female*) in 1950 for a number of countries. A least squares regression line has been fitted to the scatterplot as shown.



The slope of this least squares regression line is 0.88.

a. Interpret the slope in terms of the variables *male* life expectancy and *female* life expectancy. **(1 mark)**

The equation of this least squares regression line is

$male = 3.6 + 0.88 \times female$

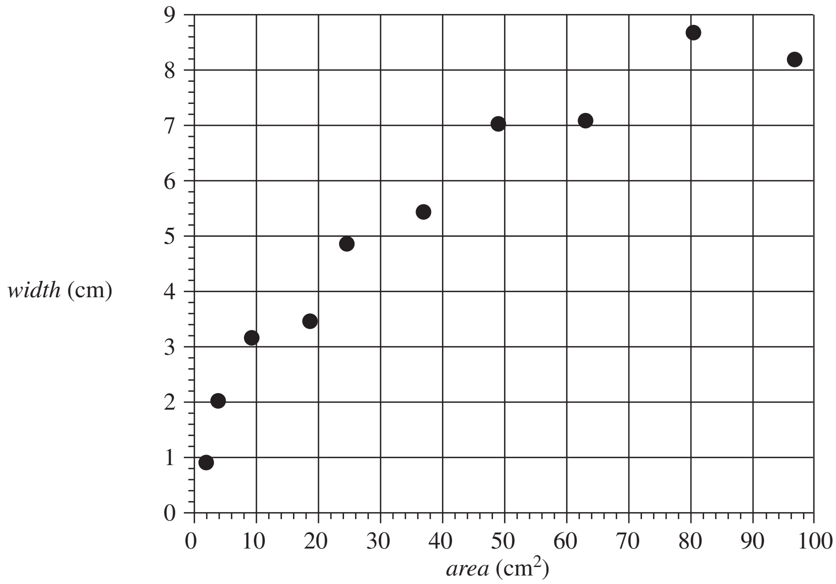
b. In a particular country in 1950, female life expectancy was 35 years.
Use the equation to predict male life expectancy for that country. **(1 mark)**

c. The coefficient of determination is 0.95
Interpret the coefficient of determination in terms of male life expectancy and female life expectancy. **(1 mark)**

Source: *VCE 2013, Further Mathematics Exam 1, Section A, Q.10; © VCAA*

Question 3 (1 mark)

The data in the scatterplot below shows the *width*, in cm, and the surface area, in cm^2, of leaves sampled from 10 different trees. The scatterplot is non-linear.



To linearise the scatterplot, $(width)^2$ is plotted against area and a least squares regression line is then fitted to the linearised plot.

The equation of this least squares regression line is

$(width)^2 = 1.8 + 0.8 \times area$

Using this equation, a leaf with a surface area of 120 cm^2 is predicted to have a width, in cm, closest to

A. 9.2

B. 9.9

C. 10.6

D. 84.6

E. 97.8

Topic 3 | Subtopic 3.3 Interpretation, interpolation and extrapolation

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.10; © VCAA*

Question 4 (1 mark)

In a study of the association between a person's *height,* in centimetres, and *body surface area*, in square metres, the following least squares line was obtained.

$\textit{body surface area} = -1.1 + 0.019 \times \textit{height}$

Which one of the following is a conclusion that can be made from this least squares line?

A. An increase of 1 m^2 in *body surface area* is associated with an increase of 0.019 cm in *height*.
B. An increase of 1 cm in *height* is associated with an increase of 0.019 m^2 in *body surface area*.
C. The correlation coefficient is 0.019
D. A person's *body surface area*, in square metres, can be determined by adding 1.1 cm to their *height*.
E. A person's *height*, in centimetres, can be determined by subtracting 1.1 from their *body surface area*, in square metres.

Question 5 (1 mark)

The relationship between the age of a group of 10 people and their arm span is shown in the table below.

Age (years)	Arm span (cm)
12	134
13	132
15	140
15	144
18	156
19	163
22	175
25	189
26	157
28	168

Find the equation of the least squares line for this data.

From this equation it can be concluded that, on average, for these people, their arm span

A. decreases by 104.22 cm with each increase of one year in age
B. increases by 104.22 cm with each increase of one year in age
C. decreases by 2.67 cm with each increase of one year in age
D. increases by 2.67 cm with each increase of one year in age
E. increases by 0.82 cm with each increase of one year in age

Question 6 (1 mark)
A survey of 15 students collected information related to the time they spent completing practice questions and the errors they made on a linked assessment task. The table below shows the results.

Practice question time	Errors
1	12
2	10
2	8
3	9
3	7
4	8
4	8
5	6
5	5
6	6
7	2
8	2
8	3
9	1
9	2

From the least squares equation, the interpretation of the vertical axis intercept is
A. Zero errors would be made if 12 hours was spent completing practice questions.
B. 1.2 errors would be made very every hour spent completing practice questions.
C. 1 error would be made for every hour spent completing practice questions.
D. 1.2 hours is spent on practice questions for each error made.
E. 12 errors would be made if zero time was spent completing practice questions.

Question 7 (1 mark)
A study was conducted to find the relationship between the *height of plants* (measured in centimetres) and the *hours of daylight* they are exposed to. The following least squares equation was found to model the data.
$\textit{Height of plant} = 28 + 1.5 \times \textit{hours of daylight}$
Which of the following conclusions drawn from the results is false?
A. The plants grew 1.5 cm for every hour of daylight.
B. The plants were 28 cm in height at the beginning of the study.
C. The plants grew 28 cm for every 1.5 hours of daylight.
D. The plants were 29.5 cm tall after one hour of daylight.
E. The relationship between the height of the plants and the number of hours of daylight is a positive one.

Question 8 (1 mark)

The relationship between life expectancy and birth rate (per 1000 people) is linear. It can also be modelled by the line

Life expectancy $= 102 - 1.5 \times$ *birth rate*

Which of the following statements is false?

A. On average, life expectancy will decrease by 1.5 for every 1 birth per thousand.
B. On average, the life expectancy in countries with a zero birth rate is 102 years.
C. On average, as the birth rate increases, life expectancy decreases.
D. On average, life expectancy will decrease by 1.5 for every 102 births per thousand.
E. On average, life expectancy decreases as birth rates increase.

Topic	**3**	**Investigating and modelling linear associations**
Subtopic	**3.4**	**Residual analysis**

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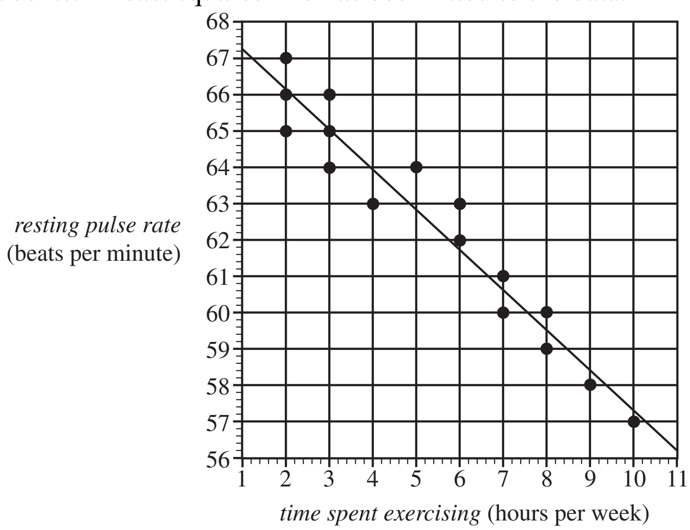
***Source:** VCE 2018, Further Mathematics Exam 1, Section A, Q.7; © VCAA*

Question 1 (1 mark)

The scatterplot below displays the *resting pulse rate*, in beats per minute, and the *time spent exercising*, in hours per week, of 16 students. A least squares line has been fitted to the data.



Using this least squares line to model the association between *resting pulse rate* and *time spent exercising*, the residual for the student who spent four hours per week exercising is closest to

A. −2.0 beats per minute.

B. −1.0 beats per minute.

C. −0.3 beats per minute.

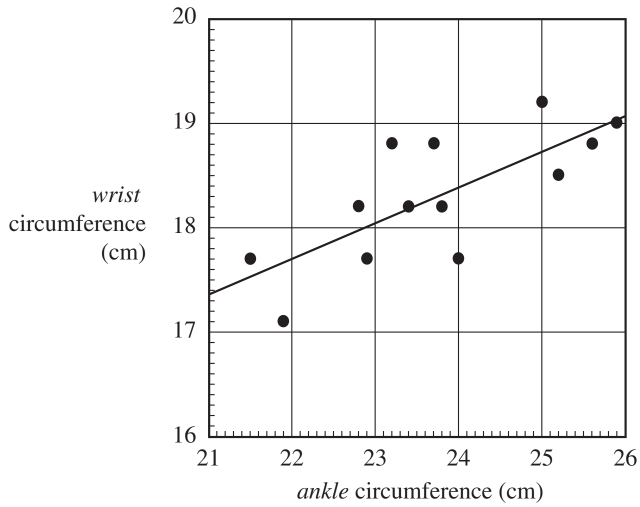
D. 1.0 beats per minute.

E. 2.0 beats per minute.

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.9; © VCAA*

Question 2 (1 mark)

The scatterplot below shows the wrist circumference and ankle circumference, both in centimetres, of 13 people. A least squares line has been fitted to the scatterplot with ankle circumference as the explanatory variable.



When the least squares line on the scatterplot is used to predict the wrist circumference of the person with an ankle circumference of 24 cm, the residual will be closest to

a. −0.7
b. −0.4
c. −0.1
d. 0.4
e. 0.7

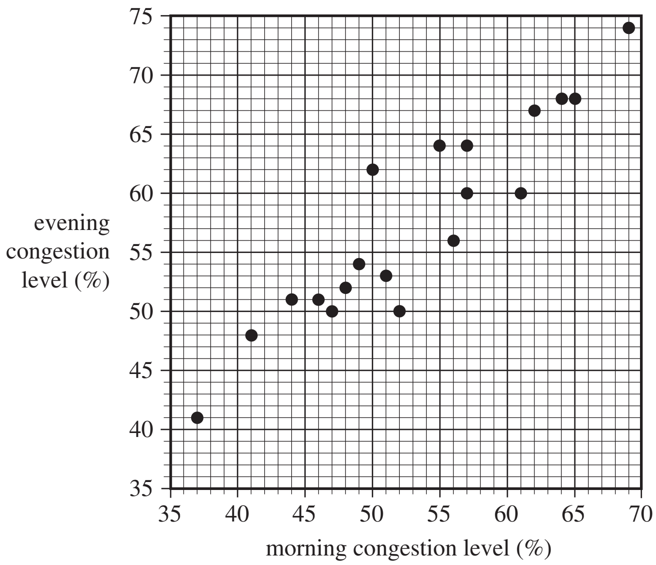
***Source:** VCE 2018, Further Mathematics Exam 2, Section A, Q2; © VCAA*

Question 3 (7 marks)

The congestion level in a city can also be recorded as the percentage increase in travel time due to traffic congestion in peak periods (compared to non-peak periods).

This is called the percentage congestion level.

The percentage congestion levels for the morning and evening peak periods for 19 large cities are plotted on the scatterplot below.



a. Determine the median percentage congestion level for the morning peak period and the evening peak period.

Write your answers in the appropriate boxes provided below. **(2 marks)**

Median percentage congestion level for morning peak period ☐%

Median percentage congestion level for evening peak period ☐%

A least squares line is to be fitted to the data with the aim of predicting evening congestion level from morning congestion level.

The equation of this line is

evening congestion level = 8.48 + 0.922 × *morning congestion level*

b. Name the response variable in this equation. **(1 mark)**

c. Use the equation of the least squares line to predict the evening congestion level when the morning congestion level is 60%. **(1 mark)**

d. Determine the residual value when the equation of the least squares line is used to predict the evening congestion level when the morning congestion level is 47%.
Round your answer to one decimal place. **(2 marks)**

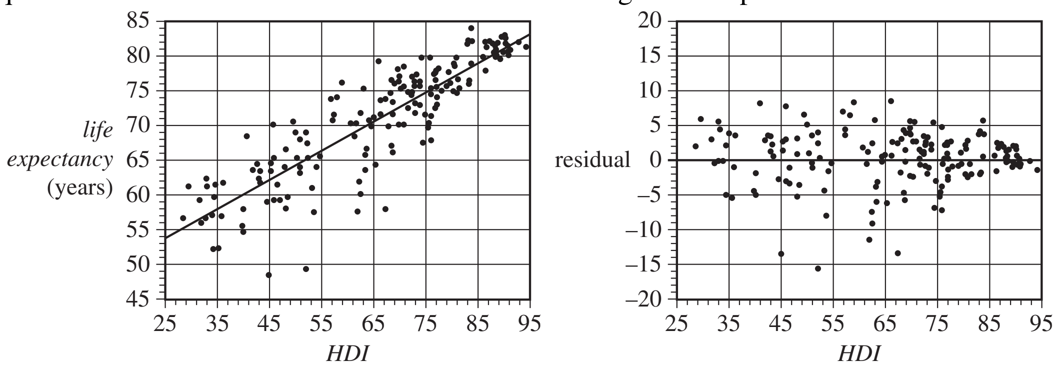
e. The value of the correlation coefficienta r is 0.92.
What percentage of the variation in the evening congestion level can be explained by the variation in the morning congestion level?
Round your answer to the nearest whole number. **(1 mark)**

***Source:** VCE 2016, Further Mathematics Exam 1, Section A, Q.10; © VCAA*

Question 4 (1 mark)

The scatterplot below shows life expectancy in years (life expectancy) plotted against the Human Development Index (HDI) for a large number of countries in 2011.

A least squares line has been fitted to the data and the resulting residual plot is also shown.



Data: Gapminder

The equation of this least squares line is

life expectancy $= 43.0 + 0.422 \times$ *HDI*

The coefficient of determination is $r^2 = 0.875$

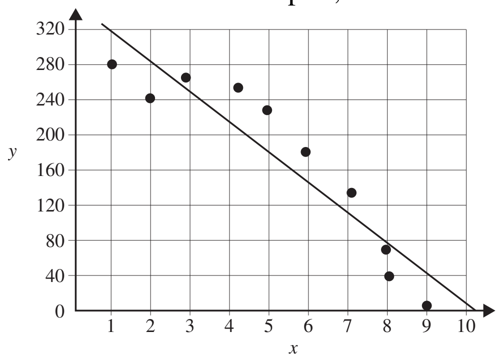
In 2011, life expectancy in Australia was 81.8 years and the Human Development Index was 92.9 When the least squares line is used to predict life expectancy in Australia, the residual is closest to

A. −0.6
B. −0.4
C. 0.4
D. 11.1
E. 42.6

Source: *VCE 2013, Further Mathematics Exam 1, Section A, Q.11; © VCAA*

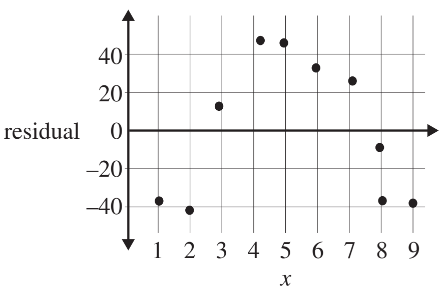
Question 5 (1 mark)

A least squares regression line is fitted to data in a scatterplot, as shown below.

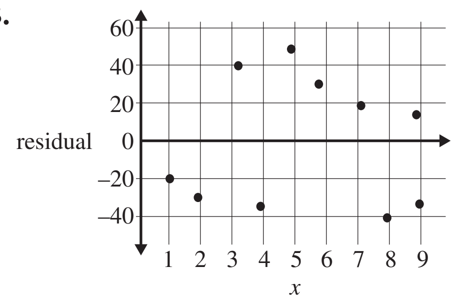


The corresponding residual plot is closest to

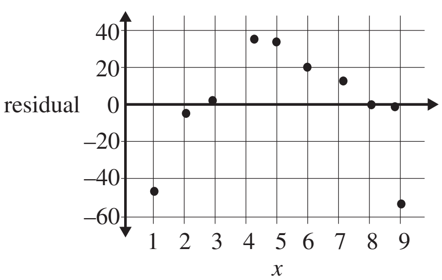
A.



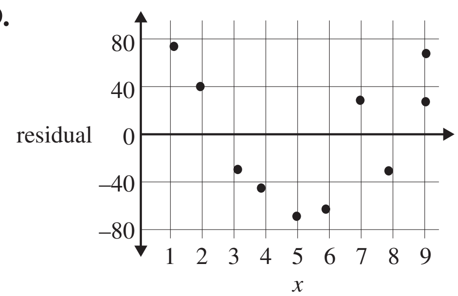
B.



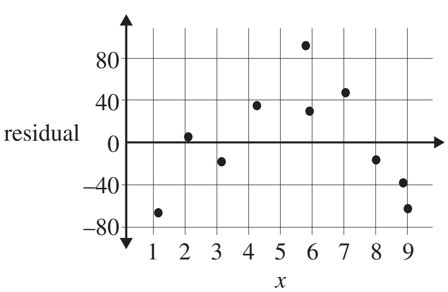
C.



D.



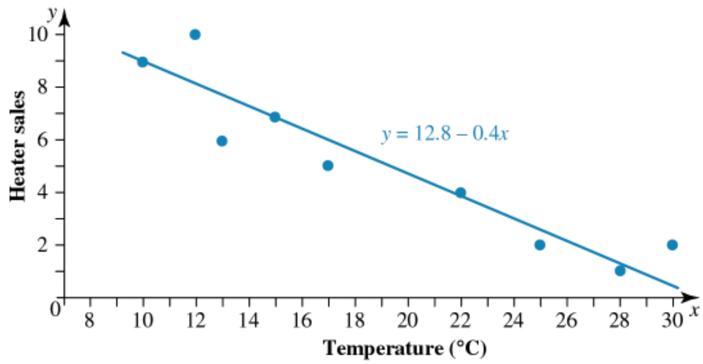
E.



Question 6 (1 mark)

The following scatterplot shows the relationship between heater sales and air temperature. The least squares line of best fit is found to be:

Heater sales $= 12.8 - 0.4 \times$ *temperature*



The data point that gives a residual value of -1 is

A. $(30, 2)$
B. $(17, 5)$
C. $(13, 6)$
D. $(10, 9)$
E. $(12, 10)$

Question 7 (1 mark)

A line of best fit for the association between the variables of average rainfall and temperature range is:

Average rainfall $= 210 - 10 \times$ temperature range

A residual value of 30 was obtained using an actual average rainfall value of 160. The temperature range value used was

A. 5
B. 6
C. 7
D. 8
E. 9

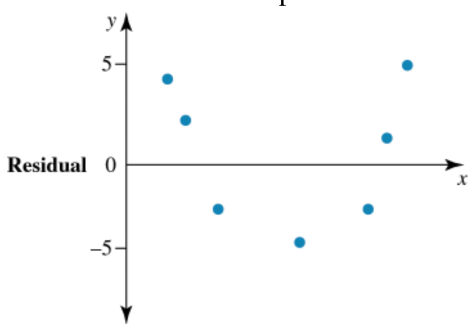
Question 8 (1 mark)

The least squares line of best fit for a set of data is $y = 0.78x + 267$. A point in the data set is $(6, 260)$.
What is the residual value?

A. -11.68
B. 11.68
C. 0.78
D. 267
E. 7

Question 9 (1 mark)

Which of the following statements is true for the residual plot shown below?



A. A linear model for the data is the most appropriate.
B. There are an equal number of dots above and below the x-axis and therefore a linear model is appropriate.
C. A linear model for the data is not appropriate.
D. A residual plot must always have exactly the same number of dots above and below the x-axis.
E. No information about the form of the original data can be deduced from this residual plot.

Question 10 (1 mark)

The following data produces a residual plot that indicates

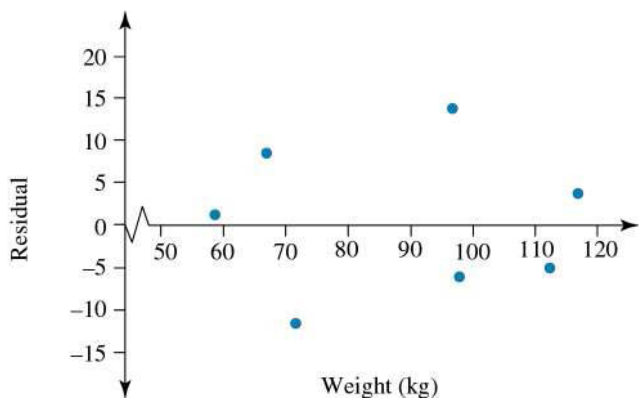
Temperature	Heater sales
30	2
28	1
25	2
22	4
17	5
15	7
13	6
12	10
10	9

A. a linear model for the data is the most appropriate due to the randomness of the points.
B. there are an equal number of points above and below the horizontal axis indicating a linear model is appropriate.
C. a linear model for the data is not appropriate.
D. a residual plot must always have the same number of points above and below the horizontal axis.
E. No information about the form of the original data can be deduced from this residual plot.

Question 11 (1 mark)

The table below shows the relationship between two variables and the linked residual plot.

Weight	Height
72	74
67	89
59	75
97	120
112	114
98	101
89	94
117	126



The value missing from the residual plot is

A. (72, −10.8)
B. (89, −5.2)
C. (72, −5.2)
D. (89, −10.8)
E. (72, 10.8)

Question 12 (1 mark)

Which of the following statements about linearity is true?

A. For linearity to exist between variables, the residual plot must show a random pattern with points above and below zero.
B. For linearity to exist between variables, the residual plot must show a clear pattern.
C. For linearity to exist between variables, the residual plot must show a clear linear pattern.
D. For linearity to exist between variables, the residual plot must show both variables on its axes.
E. For linearity to exist between variables, the residual plot must show an equal number of points above and below the horizontal axis.

Topic	3	**Investigating and modelling linear associations**
Subtopic	3.5	**Transforming to linearity**

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Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.14; © VCAA*

Question 1 (1 mark)

In a study, the association between the number of tasks completed on a test and the time allowed for the test, in hours, was found to be non-linear.

The data can be linearised using a $\log_{10}$ transformation applied to the variable number of tasks.

The equation of the least squares line for the transformed data is

$\log_{10}(\textit{number of tasks}) = 1.160 + 0.03617 \times \textit{time}$

This equation predicts that the number of tasks completed when the time allowed for the test is three hours is closest to

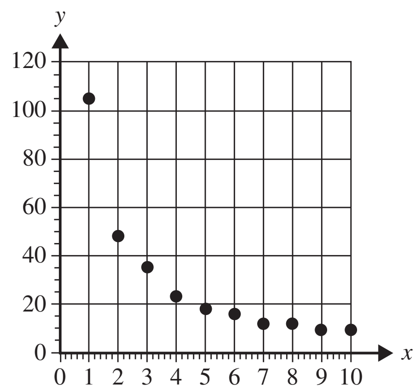
A. 13
B. 16
C. 19
D. 25
E. 26

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.11; © VCAA*

Question 2 (1 mark)

Freya uses the following data to generate the scatterplot below.

x	1	2	3	4	5	6	7	8	9	10
y	105	48	35	23	18	16	12	12	9	9



The scatterplot shows that the data is non-linear.
To linearise the data, Freya applies a reciprocal transformation to the variable y.
She then fits a least squares line to the transformed data.
With x as the explanatory variable, the equation of this least squares line is closest to

A. $\frac{1}{y} = -0.0039 + 0.012x$

B. $\frac{1}{y} = -0.025 + 1.1x$

C. $\frac{1}{y} = 7.8 - 0.082x$

D. $y = 45.3 + 59.7 \times \frac{1}{x}$

E. $y = 59.7 + 45.3 \times \frac{1}{x}$

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.11; © VCAA*

Question 3 (1 mark)

A $\log_{10}(y)$ transformation was used to linearise a set of non-linear bivariate data.

A least squares line was then fitted to the transformed data.

The equation of this least squares line is

$\log_{10}(y) = 3.1 - 2.3x$

This equation is used to predict the value of y when $x = 1.1$

The value of y is closest to

A. -0.24

B. 0.57

C. 0.91

D. 1.6

E. 3.7

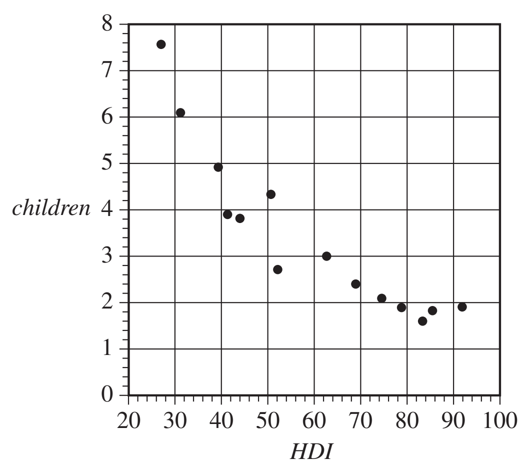
Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.11; © VCAA*

Question 4 (1 mark)

The table below gives the Human Development Index (*HDI*) and the mean number of children per woman (children) for 14 countries in 2007.

A scatterplot of the data is also shown.

HDI	***Children***
27.3	7.6
31.3	6.1
39.5	4.9
41.6	3.9
44.0	3.8
50.8	4.3
52.3	2.7
62.5	3.0
69.1	2.4
74.6	2.1
78.9	1.9
85.6	1.8
92.0	1.9
83.4	1.6



Data: Gapminder

The scatterplot is non-linear.

A log transformation applied to the variable childrenacan be used to linearise the scatterplot.

With *HDI* as the explanatory variable, the equation of the least squares line fitted to the linearised data is closest to

A. $\log(children) = 1.1 - 0.0095 \times HDI$

B. $children = 1.1 - 0.0095 \times \log(HDI)$

C. $\log(children) = 8.0 - 0.77 \times HDI$

D. $children = 8.0 - 0.77 \times \log(HDI)$

E. $\log(children) = 21 - 10 \times HDI$

Source: *VCE 2015, Further Mathematics Exam 1, Section A, Q.11; © VCAA*

Question 5 (1 mark)

A log transformation is used to linearise the relationship between the weight of a mouse, in grams, and its age, in weeks.

When a least squares regression line is fitted to the transformed data, its equation is

$weight = -7 + 30\log_{10}(age)$

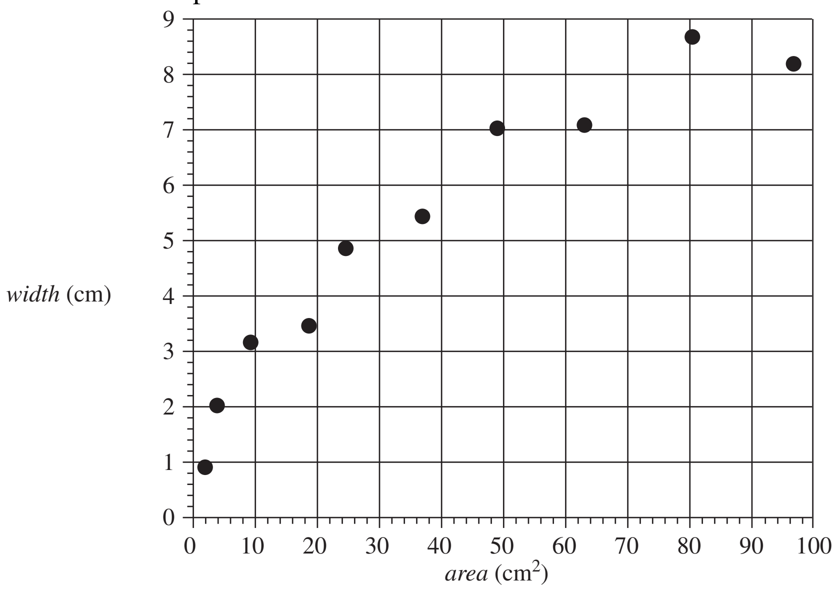
This equation predicts that a mouse aged five weeks has a weight, in grams, that is closest to

A. 14
B. 21
C. 23
D. 41
E. 143

Source: *VCE 2013, Further Mathematics Exam 1, Section A, Core, Q.10; © VCAA*

Question 6 (1 mark)

The data in the scatterplot below shows the *width*, in cm, and the *surface area*, in cm^2, of leaves sampled from 10 different trees. The scatterplot is non-linear.



To linearise the scatterplot, $(width)^2$ is plotted against area and a least squares regression line is then fitted to the linearised plot.

The equation of this least squares regression line is

$(width)^2 = 1.8 + 0.8 \times area$

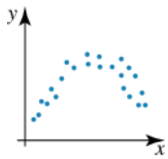
Using this equation, a leaf with a surface area of $120\,cm^2$ is predicted to have a width, in cm, closest to

A. 9.2
B. 9.9
C. 10.6
D. 84.6
E. 97.8

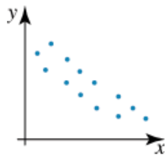
Question 7 (1 mark)

Which of the following data sets is likely to be linear?

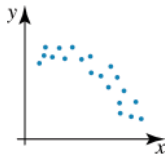
I



II



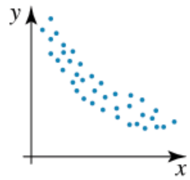
III



A. I, II and III
B. none of them
C. II only
D. I and III only
E. II and III only

Question 8 (1 mark)

The relationship between the two variables, x and y, in the scatterplot shown below is non-linear. Which of the following transformations is most likely to linearise this data?



1. an x^2 transformation
2. a y^2 transformation
3. a $\log(y^2)$ transformation
4. a $\frac{1}{x}$ transformation
5. a $\log(x^2)$ transformation

Topic	**3**	**Investigating and modelling linear associations**
Subtopic	**3.6**	**Review**

online only

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Source: *VCE 2020, Further Mathematics Exam 2, Section A, Q.4; © VCAA*

Question 1 (1 mark)

The *age*, in years, *body density*, in kilograms per litre, and *weight*, in kilograms, of a sample of 12 men aged 23 to 25 years are shown in the table below.

Age **(years)**	***Body density*** **(kg/litre)**	***Weight*** **(kg)**
23	1.07	70.1
23	1.07	90.4
23	1.08	73.2
23	1.08	85.0
24	1.03	84.3
24	1.05	95.6
24	1.07	71.7
24	1.06	95.0
25	1.07	80.2
25	1.09	87.4
25	1.02	94.9
25	1.09	65.3

a. For these 12 men, determine

i. their median *age*, in years **(1 mark)**

ii. the mean of their *body density*, in kilograms per litre. **(1 mark)**

b. A least squares line is to be fitted to the data with the aim of predicting *body density* from *weight*.

i. Name the explanatory variable for this least squares line. **(1 mark)**

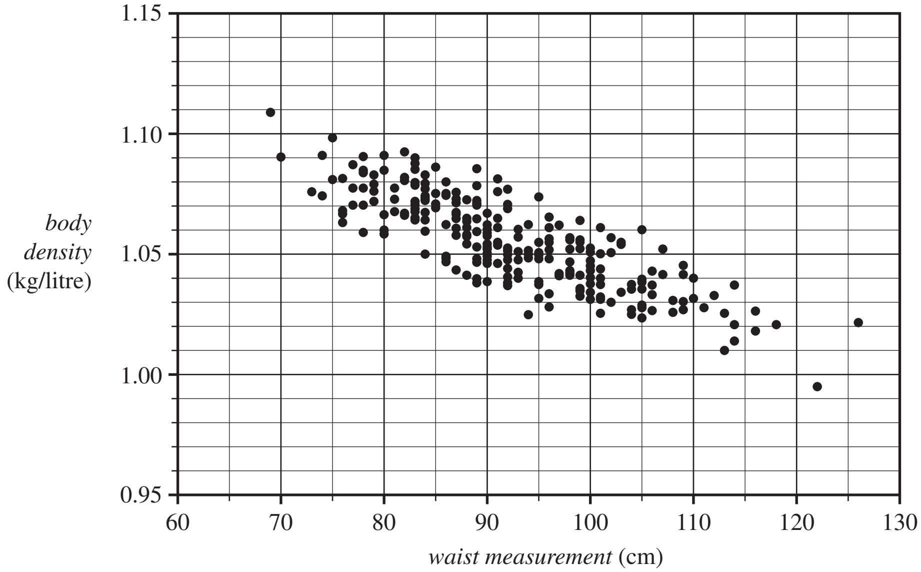
ii. Determine the slope of this least squares line.
Round your answer to three significant figures. **(1 mark)**

c. What percentage of the variation in *body density* can be explained by the variation in *weight*?
Round your answer to the nearest percentage. **(1 mark)**

Source: *VCE 2020, Further Mathematics Exam 2, Section A, Q.5; © VCAA*

Question 2 (1 mark)

The scatterplot below shows *body density*, in kilograms per litre, plotted against *waist measurement*, in centimetres, for 250 men.



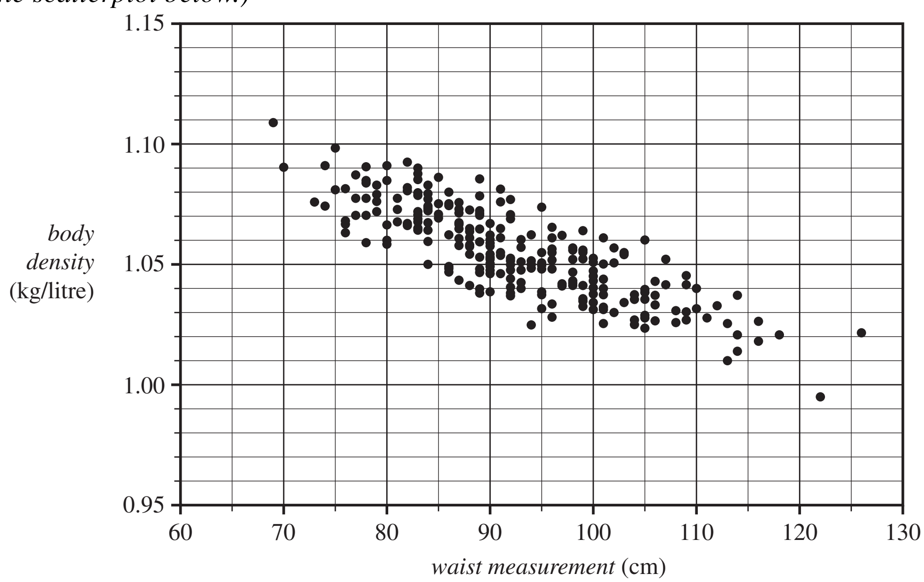
Data: RW Johnson, 'Fitting percentage of body fat to simple body measurements', *Journal of Statistics Education*, 4:1, 1996, <https://doi.org/10.1080/10691898.1996.11910505>

When a least squares line is fitted to the scatterplot, the equation of this line is

body density = 1.195 − 0.001512 × *waist measurement*

a. Draw the graph of this least squares line on the **scatterplot below**. **(1 mark)**

(Answer on the scatterplot below.)



b. Use the equation of this least squares line to predict the *body density* of a man whose *waist measurement* is 65 cm.
Round your answer to two decimal places. **(1 mark)**

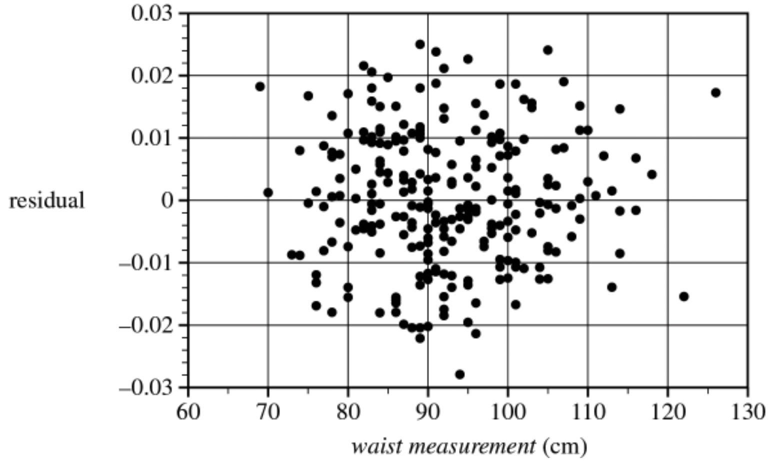
c. When using the equation of this least squares line to make the prediction in **part b**., are you extrapolating or interpolating? **(1 mark)**

d. Interpret the slope of this least squares line in terms of a man's *body density* and *waist measurement*. **(1 mark)**

e. In this study, the body density of the man with a waist measurement of 122 cm was 0.995 kg/litre.
Show that, when this least squares line is fitted to the scatterplot, the residual, rounded to two decimal places, is −0.02 **(1 mark)**

f. The coefficient of determination for this data is 0.6783
Write down the value of the correlation coefficient r. Round your answer to three decimal places. **(1 mark)**

g. The residual plot associated with fitting a least squares line to this data is shown below. **(1 mark)**



Does this residual plot support the assumption of linearity that was made when fitting this line to this data? Briefly explain your answer.

Source: *VCE 2020, Further Mathematics Exam 2, Section A, Q.6 (adapted); © VCAA*

Question 3 (1 mark)

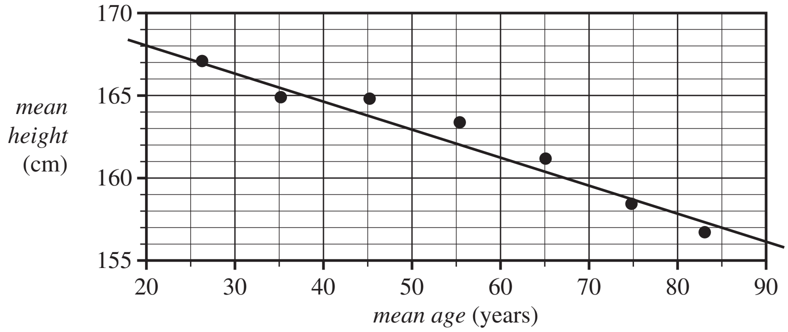
The table below shows the *mean age,* in years, and the *mean height*, in centimetres, of 648 women from seven different age groups.

	Age group						
	Twenties	**Thirties**	**Forties**	**Fifties**	**Sixties**	**Seventies**	**Eighties**
***Mean age* (years)**	26.3	35.2	45.2	55.3	65.1	74.8	83.1
***Mean height* (cm)**	167.1	164.9	164.8	163.4	161.2	158.4	156.7

Data: J Sorkin et al., 'Longitudinal change in height of men and women: Implications for interpretation of the body mass index', *American Journal of Epidemiology*, vol. 150, no. 9, 1999, p. 971.

a. What was the difference, in centimetres, between the mean height of the women in their twenties and the mean height of the women in their eighties? **(1 mark)**

A scatterplot displaying this data shows an association between the *mean height* and the *mean age* of these women. In an initial analysis of the data, a line is fitted to the data by eye, as shown.

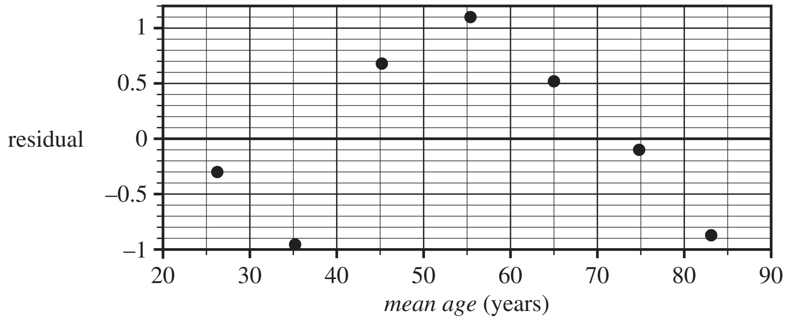


b. Describe this association in terms of strength and direction. **(1 mark)**

c. The line on the scatterplot passes through the points (20, 168) and (85, 157).
Using these two points, determine the equation of this line. Write the values of the intercept and the slope in the appropriate spaces below.
Round your answers to three significant figures. **(1 mark)**

mean height = ____________ + ____________ × *mean age*

d. In a further analysis of the data, a least squares line was fitted.
The associated residual plot that was generated is shown below. **(1 mark)**



The residual plot indicates that the association between the *mean height* and the *mean age* of women is non-linear.
The data presented in the table is repeated below. It can be linearised by applying an appropriate transformation to the variable *mean age*.

***Mean age* (years)**	26.3	35.2	45.2	55.3	65.1	74.8	83.1
Mean height (cm)	167.1	164.9	164.8	163.4	161.2	158.4	156.7

Apply an appropriate transformation to the variable *mean age* to linearise the data. Fit a least squares line to the transformed data and write its equation below.
Round the values of the intercept and the slope to four significant figures.

Source: *VCE 2019, Further Mathematics Exam 2, Section A, Q.4; © VCAA*

Question 4 (3 marks)

The relative humidity (%) at 9 am and 3 pm on 14 days in November 2017 is shown in the table below.

Relative humidity (%)	
9 am	3 pm
100	87
99	75
95	67
63	57
81	57
94	74
96	71
81	62
73	53
53	54
57	36
77	39
51	30
41	32

Data: Australian Government, Bureau of Meteorology, <www.bom.gov.au/>

A least squares line is to be fitted to the data with the aim of predicting the relative humidity at 3 pm (*humidity* 3 *pm*) from the relative humidity at 9 am (*humidity* 9 *am*).

a. Name the explanatory variable. **(1 mark)**

b. Determine the values of the intercept and the slope of this least squares line. Round both values to three significant figures and write them in the appropriate boxes provided. **(1 mark)**

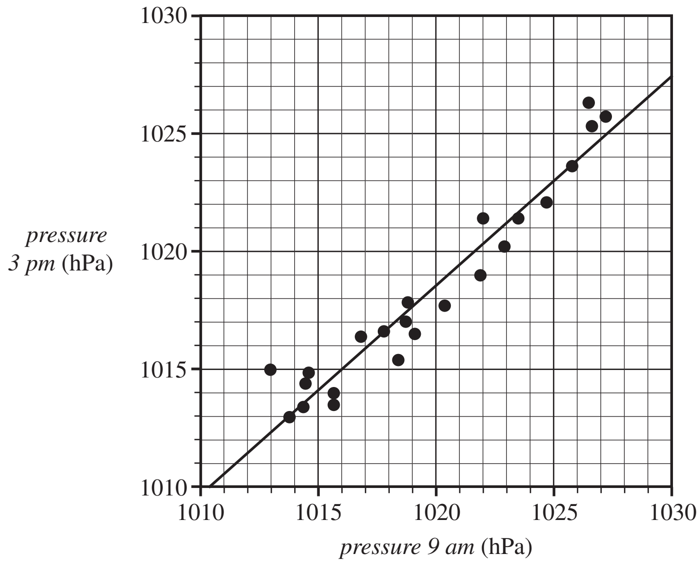
$$humidity\ 3\ pm = \square + \square \times humidity\ 9\ am$$

c. Determine the value of the correlation coefficient for this data set. Round your answer to three decimal places. **(1 mark)**

Source: *VCE 2019, Further Mathematics Exam 2, Section A, Q.5; © VCAA*

Question 5 (1 mark)

The scatterplot below shows the atmospheric pressure, in hectopascals (hPa), at 3 pm (*pressure* 3 *pm*) plotted against the atmospheric pressure, in hectopascals, at 9 am (*pressure* 9 *am*) for 23 days in November 2017 at a particular weather station.



Data: Australian Government, Bureau of Meterology, <www.bom.gov.au/>

A least squares line has been fitted to the scatterplot as shown.

The equation of this line is

pressure 3 *pm* = 111.4 + 0.8894 × *pressure* 9 *am*

a. Interpret the slope of this least squares line in terms of the atmospheric pressure at this weather station at 9 am and at 3 pm. **(1 mark)**

b. Use the equation of the least squares line to predict the atmospheric pressure at 3 pm when the atmospheric pressure at 9 am is 1025 hPa.
Round your answer to the nearest whole number. **(1 mark)**

c. Is the prediction made in **part b.** an example of extrapolation or interpolation? **(1 mark)**

d. Determine the residual when the atmospheric pressure at 9 am is 1013 hPa.
Round your answer to the nearest whole number. **(1 mark)**

e. The mean and the standard deviation of *pressure* 9 *am* and *pressure* 3 *pm* for these 23 days are shown in Table 4 below.

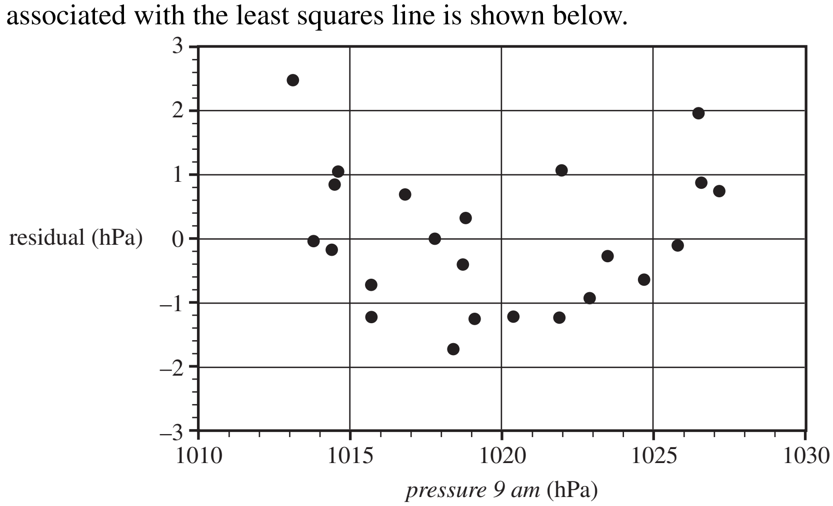
Table 4

	Pressure 9 ***am***	***Pressure*** 3 ***pm***
Mean	1019.7	1018.3
Standard deviation	4.5477	4.1884

i. Use the equation of the least squares line and the information in Table 4 to show that the correlation coefficient for this data, rounded to three decimal places, is $r = 0.966$ **(1 mark)**

ii. What percentage of the variation in *pressure* 3 *pm* is explained by the variation in *pressure* 9 *am*?
Round your answer to one decimal place. **(1 mark)**

f. The residual plot associated with the least squares line is shown below.



i. The residual plot above can be used to test one of the assumptions about the nature of the association between the atmospheric pressure at 3 pm and the atmospheric pressure at 9 am.
What is this assumption? **(1 mark)**

ii. The residual plot above does not support this assumption.
Explain why. **(1 mark)**

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.12; © VCAA*

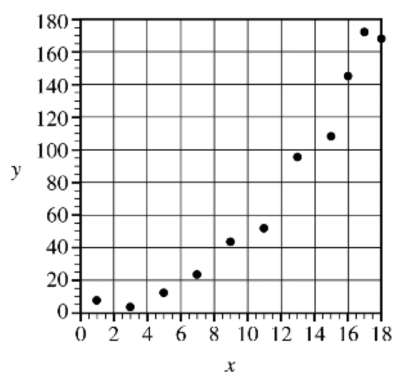
Question 6 (1 mark)

The table below shows the values of two variables x and y.

The associated scatterplot is also shown.

The explanatory variable is x.

x	y
1	7.6
3	3.4
5	12.1
7	23.4
9	43.6
11	51.8
13	95.4
15	108
16	145
17	172
18	168



The scatterplot is non-linear.

A squared transformation applied to the variable x can be used to linearise the scatterplot.

The equation of the least squares line fitted to the linearised data is closest to

A. $y = -1.34 + 0.546x$
B. $y = -1.34 + 0.546x^2$
C. $y = 3.93 - 0.00864x^2$
D. $y = 34.6 - 10.5x$
E. $y = 34.6 - 10.5x^2$

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.12; © VCAA*

Question 7 (1 mark)

A least squares line is used to model the relationship between the monthly *average temperature and latitude* recorded at seven different weather stations. The equation of the least squares line is found to be

$average\ temperature = 42.9842 - 0.877447 \times latitude$

The coefficient of determination was calculated to be 0.893743

The value of the correlation coefficient, rounded to three decimal places, is

A. -0.945
B. -0.898
C. 0.806
D. 0.898
E. 0.945

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 8 (1 mark)

The statistical analysis of a set of bivariate data involving variables x and y resulted in the information displayed in the table below.

Mean	$\bar{x} = 27.8$	$\bar{y} = 33.4$
Standard deviation	$s_x = 2.33$	$s_y = 3.24$
Equation of the least squares line	$y = -2.84 + 1.31x$	

Using this information, the value of the correlation coefficient r for this set of bivariate data is closest to

A. 0.88
B. 0.89
C. 0.92
D. 0.94
E. 0.97

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.14; © VCAA*

Question 9 (1 mark)

A least squares line is fitted to a set of bivariate data.

Another least squares line is fitted with response and explanatory variables reversed.

Which one of the following statistics will **not** change in value?

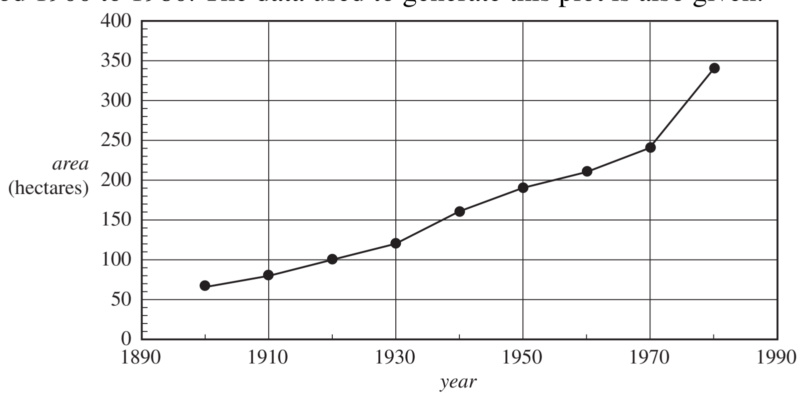
A. the residual values
B. the predicted values
C. the correlation coefficient r
D. the slope of the least squares line
E. the intercept of the least squares line

Source: *VCE 2017, Further Mathematics Exam 2, Section A, Q.4; © VCAA*

Question 10 (4 marks)

The eggs laid by the female moths hatch and become caterpillars.

The following time series plot shows the total area, in hectares, of forest eaten by the caterpillars in a rural area during the period 1900 to 1980. The data used to generate this plot is also given.



Year	1900	1910	1920	1930	1940	1950	1960	1970	1980
***Area* (hectares)**	66	80	100	120	160	190	210	240	340

The association between *area* of forest eaten by the caterpillars and *year* is non-linear.

A $\log_{10}$ transformation can be applied to the variable *area* to linearise the data.

a. When the equation of the least squares line that can be used to predict $\log_{10}(area)$ from year is determined, the slope of this line is approximately 0.0085385
Round this value to three significant figures. **(1 mark)**

b. Perform the $\log_{10}$ transformation to the variable area and determine the equation of the least squares line that can be used to predict $\log_{10}(area)$ from year.
Write the values of the intercept and slope of this least squares line in the appropriate spaces provided below.
Round your answers to three significant figures. **(1 mark)**
$\log_{10}(area) =$ ____________ $+$ ____________ $\times$ *year*

c. Answer the following.
i. The least squares line predicts that the $\log_{10}(area)$ of forest eaten by the caterpillars by the year 2020 will be approximately 2.85 **(1 mark)**
Using this value of 2.85, calculate the expected area of forest that will be eaten by the caterpillars by the year 2020.
Round your answer to the nearest hectare.

ii. Give a reason why this prediction may have limited reliability. **(1 mark)**

Source: *VCE 2017, Further Mathematics Exam 2, Section A, Q.3; © VCAA*

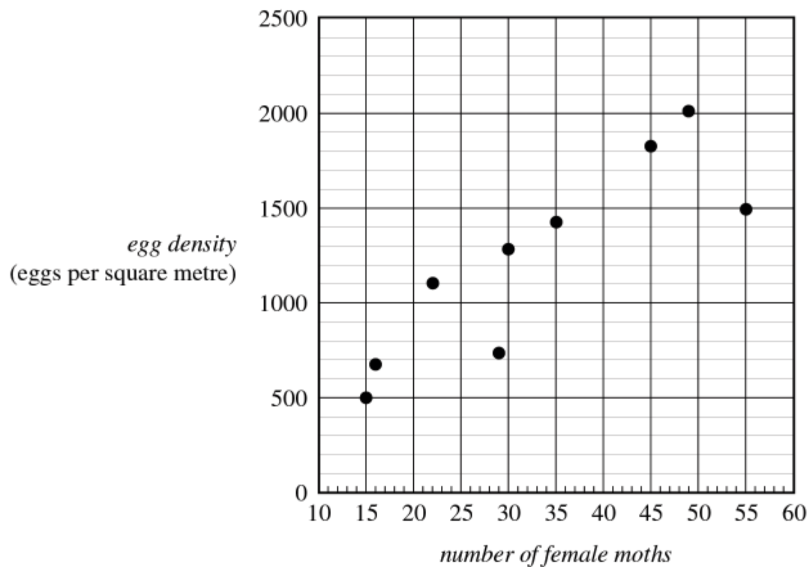
Question 11 (6 marks)
The *number of male moths* caught in a trap set in a forest and the *egg density* (eggs per square metre) in the forest are shown in the table below.

Number of male moths	35	37	45	49	65	74	77	86	95
***Egg density* (eggs per square metre)**	471	635	664	997	1350	1100	2010	1640	1350

a. Determine the equation of the least squares line that can be used to predict the *egg density* in the forest from the *number of male moths* caught in the trap.
Write the values of the intercept and slope of this least squares line in the appropriate spaces provided below.
Round your answers to one decimal place. **(2 marks)**
egg density $=$ __________ $+$ __________ $\times$ *number of male moths*

b. The number of female moths caught in a trap set in a forest and the *egg density* (eggs per square metre) in the forest can also be examined.

A scatterplot of the data is shown below.



The equation of the least squares line is

egg density $= 191 + 31.3 \times$ *number of female moths*

i. Draw the graph of this least squares line on the scatterplot above. **(1 mark)**

ii. Interpret the slope of the regression line in terms of the variables *egg density* and *number of female moths* caught in the trap. **(1 mark)**

iii. The *egg density* is 1500 when the *number of female moths* caught is 55.
Determine the residual value if the least squares line is used to predict the *egg density* for this number of female moths. **(1 mark)**

iv. The correlation coefficient is $r = 0.862$
Determine the percentage of the variation in *egg density* in the forest explained by the variation in the *number of female moths* caught in the trap.
Round your answer to one decimal place. **(1 mark)**

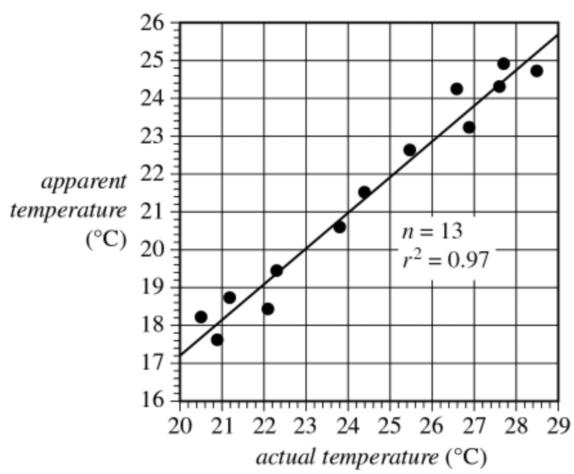
Source: *VCE 2016, Further Mathematics Exam 2, Q.3; © VCAA*

Question 12 (8 marks)

The data in the table below shows a sample of actual temperatures and apparent temperatures recorded at the weather station. A scatterplot of the data is also shown.

The data will be used to investigate the association between the variables *apparent temperature* and *actual temperature*.

Apparent temperature **(°C)**	***Actual temperature*** **(°C)**
24.7	28.5
24.3	27.6
24.9	27.7
23.2	26.9
24.2	26.6
22.6	25.5
21.5	24.4
20.6	23.8
19.4	22.3
18.4	22.1
17.6	20.9
18.7	21.2
18.2	20.5



a. Use the scatterplot to describe the association between *apparent temperature* and *actual temperature* in terms of strength, direction and form. **(1 mark)**

b. Answer the following.

i. Determine the equation of the least squares line that can be used to predict the *apparent temperature* from the *actual temperature*.

Write the values of the intercept and slope of this least squares line in the appropriate spaces provided below.

Round your answers to two significant figures. **(3 marks)**

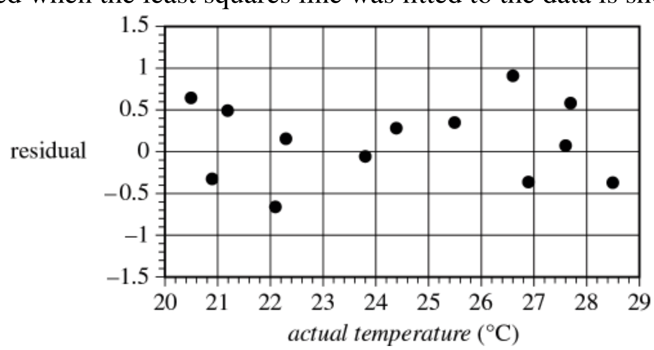
apparent temperature = __________ + __________ × *actual temperature*

ii. Interpret the intercept of the least squares line in terms of the variables *apparent temperature* and *actual temperature*. **(1 mark)**

c. The coefficient of determination for the association between the variables apparent temperature and actual temperature is 0.97

Interpret the coefficient of determination in terms of these variables. **(1 mark)**

d. The residual plot obtained when the least squares line was fitted to the data is shown below.



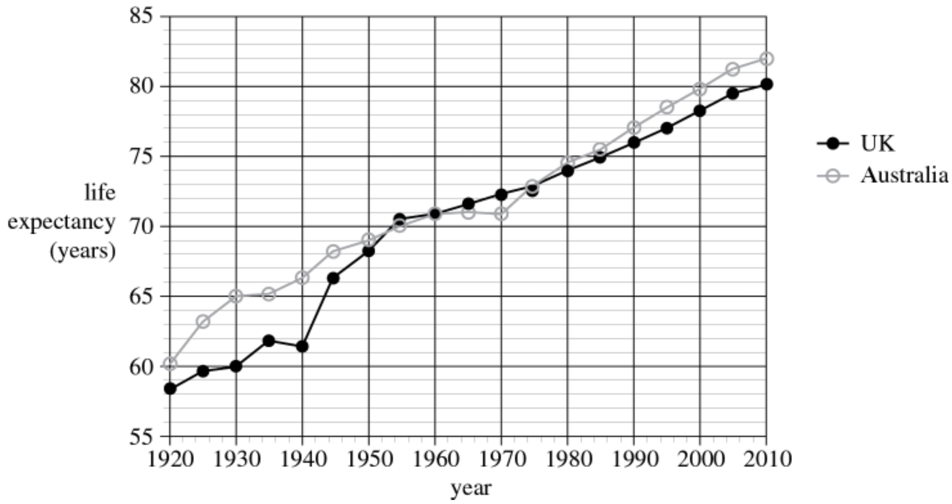
i. A residual plot can be used to test an assumption about the nature of the association between two numerical variables. What is this assumption? **(1 mark)**

ii. Does the residual plot above support this assumption? Explain your answer. **(1 mark)**

Source: *VCE 2015, Further Mathematics Exam 2, Q.5; © VCAA*

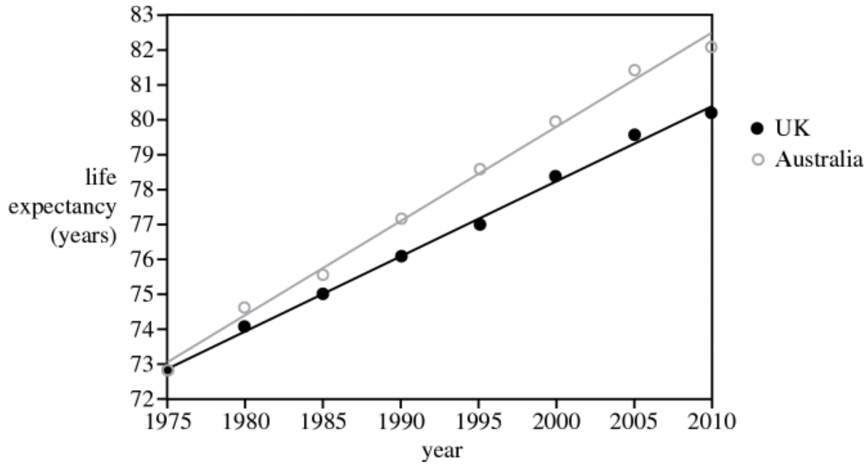
Question 13 (4 marks)

The time series plot below displays the *life expectancy*, in years, of people living in Australia and the United Kingdom (UK) for each year from 1920 to 2010.



a. By how much did *life expectancy* in Australia increase during the period 1920 to 2010?
Write your answer correct to the nearest year. **(1 mark)**

b. In 1975, the life expectancies in Australia and the UK were very similar.
From 1975, the gap between the life expectancies in the two countries increased, with people in Australia having a longer life expectancy than people in the UK.
To investigate the difference in life expectancies, least squares regression lines were fitted to the data for both Australia and the UK for the period 1975 to 2010.
The results are shown below.



The equations of the least squares regression lines are as follows.
Australia: *life expectancy* = −451.7 + 0.2657 × *year*
UK: *life expectancy* = −350.4 + 0.2143 × *year*

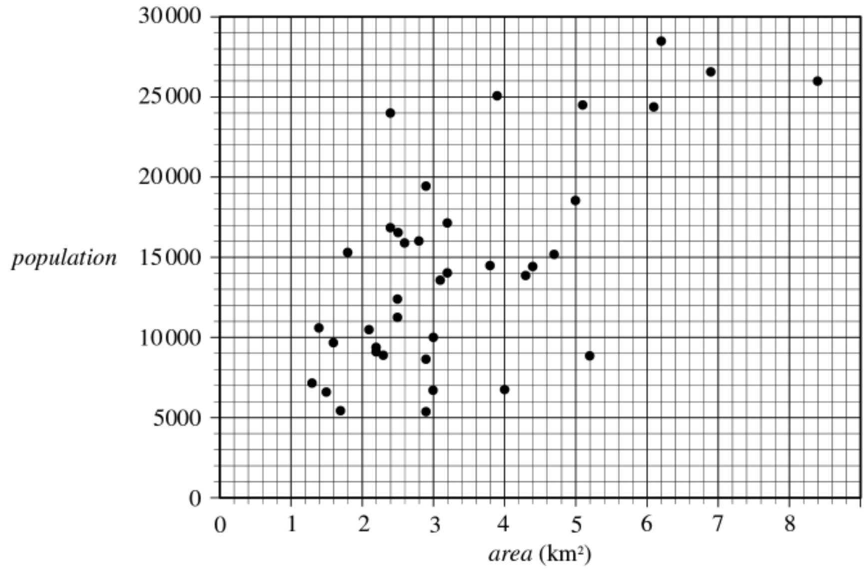
i. Use these equations to predict the difference between the life expectancies of Australia and the UK in 2030.
Give your answer correct to the nearest year. **(2 marks)**

ii. Explain why this prediction may be of limited reliability. **(1 mark)**

Source: *VCE 2014, Further Mathematics Exam 2, Q.2; © VCAA*

Question 14 (6 marks)

The scatterplot below shows the population and area (in square kilometres) of a sample of inner suburbs of a large city.



The equation of the least squares regression line for the data in the scatterplot is

population = 5330 + 2680 × *area*

a. Write down the dependent variable. **(1 mark)**

b. Draw the least squares regression line on the **scatterplot above**. **(1 mark)**

(*Answer on the scatterplot above.*)

c. Interpret the slope of this least squares regression line in terms of the variables area and population. **(2 marks)**

d. Wiston is an inner suburb. It has an area of 4 km^2 and a population of 6690.
The correlation coefficient, r, is equal to 0.668.

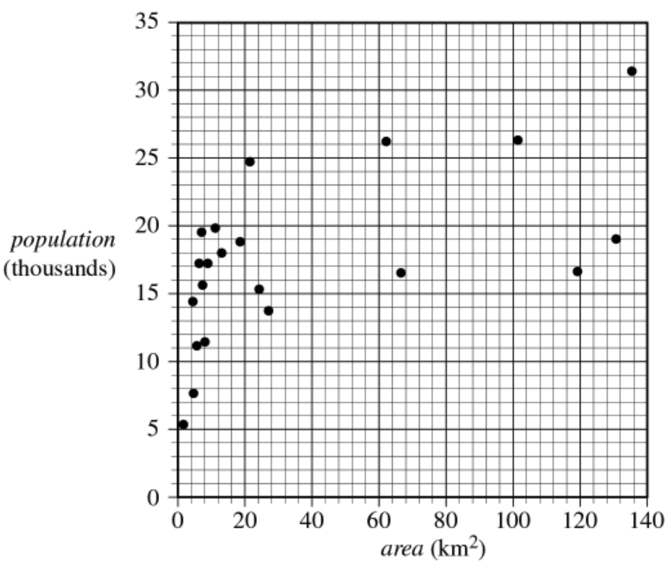
i. Calculate the residual when the least squares regression line is used to predict the population of Wiston from its area. **(1 mark)**

ii. What percentage of the variation in the population of the suburbs is explained by the variation in area?
Write your answer, correct to one decimal place. **(1 mark)**

Source: *VCE 2014, Further Mathematics Exam 2, Q.3; © VCAA*

Question 15 (2 marks)

The scatterplot and table below show the population, in thousands, and the area, in square kilometres, for a sample of 21 outer suburbs of the same city.



Area (km^2)	*Population* (thousands)
1.6	5.2
4.4	14.3
4.6	7.5
5.6	11.0
6.3	17.1
7.0	19.4
7.3	15.5
8.0	11.3
8.8	17.1
11.1	19.7
13.0	17.9
18.5	18.7
21.3	24.6
24.2	15.2
27.0	13.6
62.1	26.1
66.5	16.4
101.4	26.2
119.2	16.5
130.7	18.9
135.4	31.3

In the outer suburbs, the relationship between *population* and *area* is non-linear.

A ***log*** transformation can be applied to the variable area to linearise the scatterplot.

a. Apply the log transformation to the data and determine the equation of the least squares regression line that allows the population of an outer suburb to be predicted from the logarithm of its area.
Write the slope and intercept of this regression line in the equation provided below.
Write your answers, correct to one decimal place. **(1 mark)**

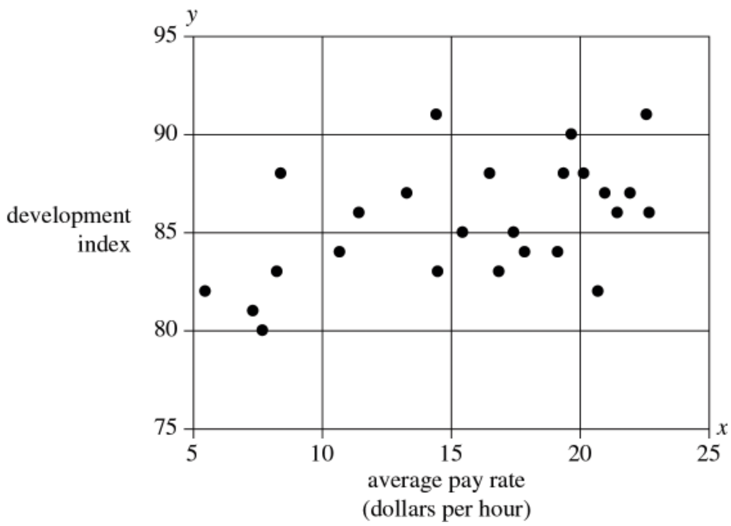
$$population = \boxed{\quad} + \boxed{\quad} \times \log_{10}(area)$$

b. Use this regression equation to predict the population of an outer suburb with an area of 90 km^2. Write your answer, correct to the nearest one thousand people. **(1 mark)**

Source: *VCE 2013, Further Mathematics Exam 2, Q.3; © VCAA*

Question 16 (5 marks)

The development index and the average pay rate for workers, in dollars per hour, for a selection of 25 countries are displayed in the scatterplot below.



The table below contains the values of some statistics that have been calculated for this data.

Statistic	Average pay rate(x)	Development index(y)
mean	$\bar{x} = 15.7$	$\bar{y} = 85.6$
standard deviation	$s_x = 5.37$	$s_y = 2.99$
correlation coefficient	$r = 0.488$	

a. Determine the standardised value of the development index (z score) for a country with a development index of 91. Write your answer, correct to one decimal place. **(1 mark)**

b. Use the information in the table to show that the equation of the least squares regression line for a country's development index, y, in terms of its average pay rate, x, is given by **(2 marks)**

$y = 81.3 + 0.272x$

c. The country with an average pay rate of \$14.30 per hour has a development index of 83.
Determine the residual value when the least squares regression line given in **part b**. is used to predict this country's development index.
Write your answer, correct to one decimal place. **(2 marks)**

Question 17 (1 mark)

A survey of 15 students collected information related to the time they spent completing practice questions and the errors they made on a linked assessment task. The table below shows the results..

Practice question time	Errors
1	12
2	10
2	8
3	9
3	7
4	8
4	8
5	6
5	5
6	6
7	2
8	2
8	3
9	1
9	2

Using the least squares line of best fit, the number of hours predicted to be spent completing practice questions if 8 errors are made is

A. 2.0
B. 2.4
C. 3.0
D. 3.3
E. 4.0

Question 18 (1 mark)

The relationship between the age of a group of 10 people and their arm span is shown in the table below.

Age (years)	Arm span (cm)
12	134
13	132
15	140
15	144
18	156
19	163
22	175
25	189
26	157
28	168

The least squares line of best fit for this data is:

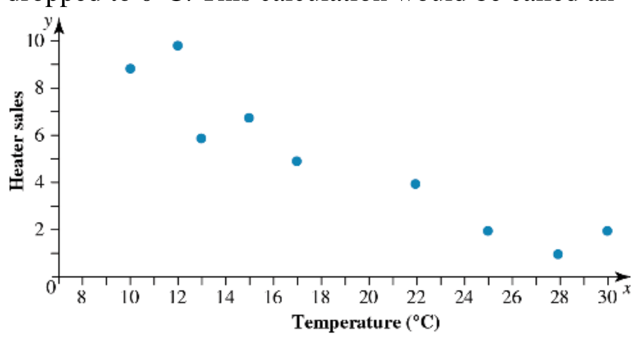
Arm span $= 104.22 + 2.67 \times$ age

This equation could be used to

A. interpolate an age for an arm span of 180 cm.
B. extrapolate an age for an arm span of 180 cm.
C. interpolate an arm span for an age of 30 years.
D. extrapolate an arm span for an age of 20 years.
E. interpolate an age for an arm span of 120 cm.

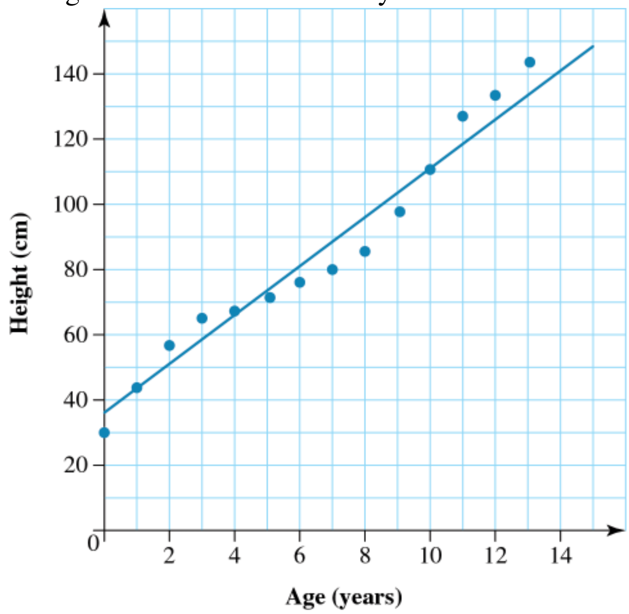
Question 19 (1 mark)

The scatterplot shown and calculated least squares equation are used to predict the heater sales that would be made if the temperature dropped to 0°C. This calculation would be called an



A. interpolation.
B. interpretation.
C. explanation.
D. extrapolation.
E. inference.

Question 20 (1 mark)
The graph shows the height of a girl who is measured each year.



The graph could be used to

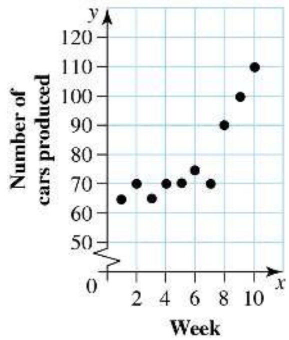
A. interpolate the girl's weight at 6 years old.
B. extrapolate the girl's height at 80 years old.
C. interpolate the girl's height at 3 years old.
D. extrapolate the girl's weight at 16 years old.
E. interpolate the girl's height at 20 years old.

Question 21 (1 mark)
Which of the following statements about interpolation or extrapolation is true?

A. Interpolation is the use of the least squares line of best fit to predict values smaller than the smallest value or larger than the largest value in the data set.
B. Extrapolation is the use of the least squares line of best fit to predict values 'in between' two values already in the data set.
C. Interpolation is the use of the least squares line of best fit to predict values smaller than the smallest value or larger than the largest value in the data set and to predict values 'in between' two values already in the data set.
D. Extrapolation is the use of the least squares line of best fit to predict values smaller than the smallest value or larger than the largest value in the data set.
E. Interpolation is the use of the least squares line of best fit to predict values larger than the smallest value or larger than the largest value in the data set.

Question 22 (1 mark)

Productivity at a car factory has been improving steadily since the introduction of robots.
The scatterplot below shows the number of cars produced each week over a 10-week period.

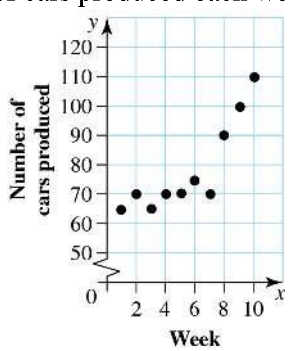


If a line of best fit was fitted by eye, it would have

A. a gradient of 60.
B. a y-intercept at 70.
C. a negative gradient.
D. a y-intercept at 0.
E. a positive gradient.

Question 23 (1 mark)

Productivity at a car plant has been improving steadily since the introduction of robots.
The scatterplot below shows the number of cars produced each week over a 10-week period.



If a least squares line of best fit was calculated, which of the following equations would it be closest to?

A. $\textit{Number of cars produced} = 5 \times \textit{week} + 70$
B. $\textit{Number of cars produced} = 6 \times \textit{week} + 50$
C. $\textit{Number of cars produced} = -5 \times \textit{week} + 70$
D. $\textit{Number of cars produced} = -6 \times \textit{week} + 50$
E. $\textit{Number of cars produced} = 6 \times \textit{week} + 70$

Answers and marking guide

3.2 Least squares line of best fit

Question 1

This is the definition of how a least squares line is calculated.

Question 2

42.9842 will become 43.0 (Normal rounding still applies, and the 0 after the decimal place is significant.).
−0.877447 will become −0.877 (The 0 before the decimal place is a place holder. Significant figures start from the first non-zero number.).

Question 3

Note that the scales do not start at 0, so the vertical axis intercept is not 67.2.
Choose any two points along the line, say (1, 67.2) and (4, 64).

Gradient: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{64 - 67.2}{4 - 1} = \dfrac{-3.3}{3} = -1.10$

Equation:

$\textit{resting pulse rate} = a + b \times \textit{time spent exercising}$

$\textit{resting pulse rate} = a - 1.10 \times \textit{time spent exercising}$

Passes through (4, 64):

$64 = a - 1.1 \times 4$

$68.4 = a$

So the closest equation is $\textit{resting pulse rate} = 68.3 - 1.10 \times \textit{time spent exercising}$

***VCAA Examination Report* note:**

Students were asked to identify the equation of the least squares line drawn on a graph that contained 16 points. While it was not possible to determine exact slope and intercept values from the graph, students should have been able to approximate these values. Many students incorrectly assumed that the intercept value of the line was 67.2, read directly from the graph; however, this is only possible if the horizontal axis begins at value zero. Students are encouraged to look carefully at graphs before choosing what might seem to be the obvious answer.

Question 4

This question is asking for the gradient connecting the two variables. It is investigating the effect that drinking coffee has on sleep, so sleep is the response variable.

$b = r\dfrac{s_y}{s_x} = r\dfrac{s_{sleep}}{s_{coffee}} = -0.770 \times \dfrac{1.12}{1.56} = -0.55282$

***VCAA Examination Report* note:**

Students needed to recognise that *sleep* was the response variable and *coffee* the explanatory variable, and be familiar with the rules connecting a least squares regression line equation to summary statistics.

Question 5

Note that the scales do not start at 0, so the vertical-axis intercept is not 17.4.
Choose any two points along the line, say (21, 17.4) and (26, 19.1).
Gradient:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{19.1 - 17.4}{26 - 21} \\ &= 0.34 \end{aligned}$$

Equation: $\textit{wrist} = a + b \times \textit{ankle}$

$\textit{wrist} = a + 0.34 \times \textit{ankle}$ Passes through $(21, 17.4)$

$17.4 = a + 0.34 \times 21$

$a = 10.26$

So, the closest equation is wrist $= 10.2 + 0.342 \times ankle$

***VCAA Examination Report* note:**

Students were asked to identify the equation of the least squares line drawn on a graph that contained 13 points. While it was not possible to determine exact slope and intercept values from the graph, students should have been able to approximate these values. Many students incorrectly assumed that the intercept value of the line was 17.4, read directly from the graph; however, this is only possible if the horizontal axis begins at value zero. Students are encouraged to look very carefully at graphs before choosing what might seem to be the obvious answer.

Question 6

The equation of a least squares regression line is $y = a + bx$ or in this case $distance = a + b\,(time)$.
The y-intercept, a, is at approximately 3.5 and the gradient, b, can be calculated approximately to be
$b = \frac{\text{rise}}{\text{run}} = \frac{80}{50} = 1.6$
The equation is therefore $distance = 3.5 + 1.6 \times time$.

Question 7

$b = r \times \frac{s_y}{s_x} = -0.47 \times \frac{0.85}{1.2} = -0.332\,916\,7$
$a = \overline{y} - b\overline{x} = 7.2 - (-0.3329)(1.8) = 7.799\,25$
$\therefore y = 7.8 - 0.33x$

Question 8

Two points: (900, 10.7) and (1700, 6.7)
Method 1: Use CAS to find the regression line.
$y = a + bx$
$\quad = 15.2 - 0.0005x$
Therefore the slope = –0.005.
Method 2:

$$\begin{aligned}\text{Slope} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{6.7 - 10.7}{1700 - 900}\\ &= -0.005\end{aligned}$$

Question 9

The least-squares regression line can only be calculated by using a calculator.

Question 10

If a line of best fit was fitted by eye, it would have a negative gradient.

3.3 Interpretation, interpolation and extrapolation

Question 1

The gradient is 0.422, which means that as the HDI increases by 10 units, life expectancy increases by 4.22 years, NOT 43.0 years as suggested by option C.

Question 2

a. The male life expectancy increases by 0.88 years with each year of increase in the female life expectancy. **[1 mark]**

***VCAA Assessment Report* note:**

This question was not well answered, with many students describing the scatterplot instead of interpreting the slope of the line as asked.

The increase in *male* life expectancy needed to be related to one unit (year) increase in *female* life expectancy. Answers such as 'On average, *male* life expectancy increased by 0.88 years for every increase in *female* life expectancy' were not accepted.

Some students incorrectly interpreted slope = 0.88 as $r = 0.88$

b. $male = 3.6 + 0.88 \times 35 = 34.4$ years **[1 mark]**

c. About 95% of the variability in the male life expectancy is explained by the variability of the female life expectancy using the linear least squares model. **[1 mark]**

***VCAA Assessment Report* note:**

A common error was to interpret 'The coefficient of determination is 0.95' as meaning $r = 0.95$ and then using $r^2 = 0.9025 = 90.25\%$ for the interpretation.

Question 3

$$\begin{aligned}(\text{width})^2 &= 1.8 + 0.8 \times 120\\ &= 97.8\\ \text{width} &= \sqrt{97.8}\\ &= 9.889\,39\\ &\approx 9.9\end{aligned}$$

Question 4

The association $body\ surface\ area = -1.1 + 0.019 \times height$ has a y-intercept $a = -1.1$ and a gradient $b = 0.019$.

The slope $b = 0.019$ indicates that the *body surface area* increased by $0.019\ \text{m}^2$ for every 1 cm added to a person's height.

Question 5

The least squares equation is: Armspan = 104.22 + 2.67 × age

Therefore, the arm span increases by 2.67 cm with each increase of one year in age.

Question 6

The least squares line calculated for this data is:

Errors = 12.0 − 1.2 × time on questions.

The vertical intercept can be interpreted as: 12 errors would be made if no time was spent completing practice questions.

Question 7

The following conclusion is false and therefore it is the correct choice:

The plants grew 28 cm for every 1.5 hours of daylight.

Question 8

The following statement is false and therefore it is the correct choice:

On average, life expectancy will decrease by 1.5 for every 102 births per thousand.

3.4 Residual analysis

Question 1

Residual value = actual value − predicted value

For the student who spends 4 hours exercising per week, their actual resting pulse rate is 63 beats per minute and their predicted resting pulse rate is 64 beats per minute.

$63 - 64 = -1$ beats per minute

Question 2

Residual $= y_a - y_p$ $\quad y_a = 17.8$ (from the scatterplot, when ankle circumference is 24 cm)

$$y_p = 10.2 + 0.342 \times 24$$
$$= 18.408$$

Residual $= 17.8 - 18.408$

$$= -0.608$$

Closest to -0.7

Question 3

a. Morning median 52%, Evening median 56%

b. Evening congestion level

c. Evening congestion level $= 8.48 + 0.922 \times 60 = 63.8$, i.e. 63.8%.

d. Residual $= 50 - (8.48 + 0.922 \times 47) = -1.8$, i.e. $-$ 1.8%

e. $0.92^2 = 0.85 = 85\%$

Question 4

$\text{LE} = 43.0 + 0.422 \times 92.9$

$= 82.2038$

Residual = actual − predicted

$= 81.8 - 82.2$

$= -0.4$

Question 5

The first 2 and last 3 data points are below the regression line, which means the corresponding residuals will be negative. The 3rd, 4th, 5th, 6th and 7th data points are above the regression line, which means the corresponding residuals will be positive. This matches with the residual plot shown in A.

Question 6

Testing the provided option points in the calculated regression equation finds (17, 5) as the point needed.

Predicted sales $= 12.8 - 0.4 \times 17 = 6$

Residual $= 5 - 6 = -1$

Question 7

Residual = Actual − predicted

$$\begin{aligned}\text{Residual} &= \text{Actual} - \text{predicted}\\ 30 &= 160 - \text{predicted}\\ \text{predicted} &= 130 \text{ (average rainfall)}\\ 130 &= 210 - 10 \times \text{temperature range}\end{aligned}$$

temperature range $= 8$

Question 8

$y = 0.78x + 267$

Actual value when $x = 6$ is 260

Predicted value when $x = 6$ is $y = 0.78 \times 6 + 267 = 271.68$

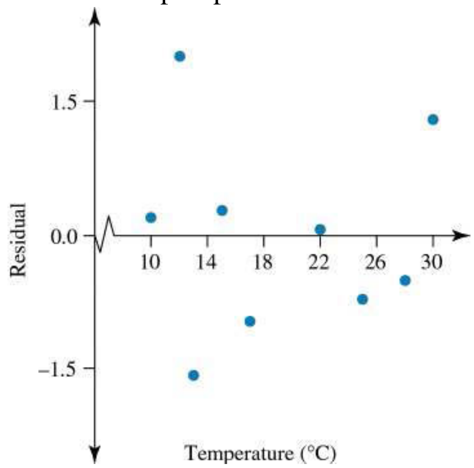
$$\begin{aligned}\text{Residual value} &= \text{actual} - \text{predicted}\\ &= 260 - 271.68\\ &= -11.68\end{aligned}$$

Question 9

There is a clear pattern in the residual plot. Therefore, a linear model for the data is not appropriate.

Question 10

The residual plot produced from this data is:



The randomness in the residual plot indicates a linear model is appropriate for this data set.

Question 11

The regression equation for the data is: height = 23.84 + 0.8471 × weight.

The residual calculations produce the results in the table shown.

Weight	Height
72	–10.8
67	8.4
59	1.2
97	14.0
112	–4.7
98	–5.9
89	–5.2
117	3.0

The missing residual is (89, –5.2).

Question 12

For linearity to exist between variables, the residual plot must show a random pattern with points above and below zero.

3.5 Transforming to linearity

Question 1

$$\log_{10} N = 1.160 + 0.03617 \times 3$$

$$\begin{aligned}\log_{10} N &= 1.126851\\ N &= 10^{1.126851}\\ N &= 18.557\\ &= 19\end{aligned}$$

Question 2

The reciprocal of y is $\frac{1}{y}$. The data used to create the linear graph is:

x	1	2	3	4	5	6	7	8	9	10
$\frac{1}{y}$	0.0095	0.020	0.028	0.043	0.056	0.063	0.083	0.083	0.11	0.11

The gradient is positive, and is approximately $\frac{0.11 - 0.0095}{9} = 0.011$.

The only equation with a positive gradient that is approximately 0.011 is $\frac{1}{y} = -0.0039 + 0.012x$.

(*Note:* Technology may be used to calculate the exact equation.)

Question 3

Substitute $x = 1.1$ into the equation.

$$\begin{aligned}\log_{10}(y) &= 3.1 - 2.3(1.1)\\ &= 0.57\\ y &\approx 3.7\end{aligned}$$

Question 4

Use a CAS calculator to perform a log transformation on the variable *children*, then display a linear regression equation.

$\log(children) = 1.1 - 0.0095 \times HDI$

Question 5

$$\begin{aligned}age &= 5\\ weight &= -7 + 30\log_{10}(5)\\ weight &= 13.9691\\ &\approx 14\text{ g}\end{aligned}$$

Question 6

$$\begin{aligned}(\text{width})^2 &= 1.8 + 0.8 \times 120\\ &= 97.8\\ \text{width} &= \sqrt{97.8}\\ &= 9.889\,39\\ &\approx 9.9\end{aligned}$$

Question 7

II only

Question 8

To linearise the data, it should be compressed at the upper end of the x-scale and to do this use a $\frac{1}{x}$ transformation

3.6 Review

Question 1

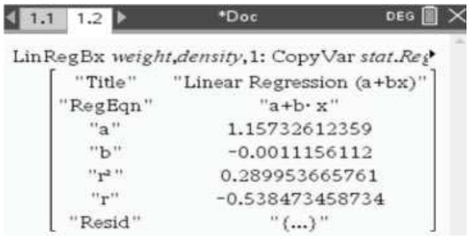
a. i. Median age is 24 years. **[1 mark]**

ii. $\dfrac{\sum \text{body} \quad \text{density}}{12} = 1.065$ **[1 mark]**

b.

i. Body density is being predicted *from* weight; therefore, *weight* is the explanatory variable. **[1 mark]**

ii. Using your CAS calculator, perform a least squares analysis.



Slope = –0.00112 correct to 3 significant figures. **[1 mark]**

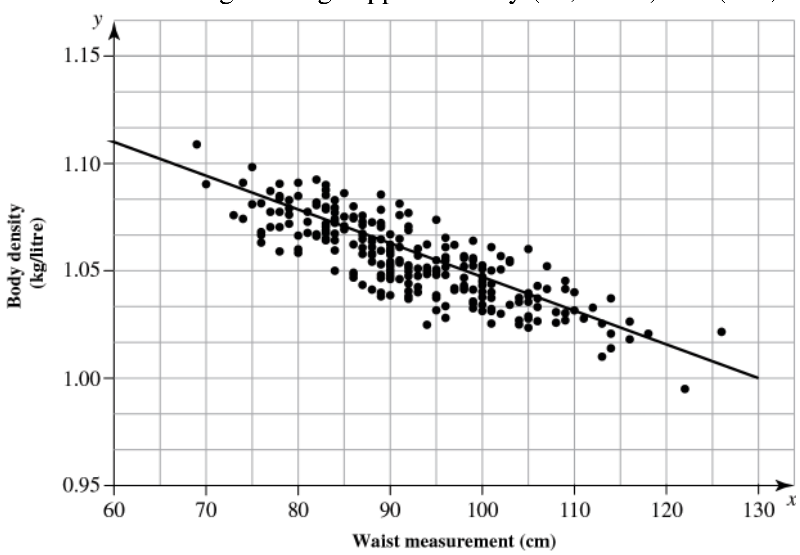
c. This question is referring to the coefficient of determination (r^2), which is found on the least squares analysis screen on CAS.

$r^2 = 0.289\,95\ldots$

Therefore, 29% of the variation in body density may be explained by the variation in weight. **[1 mark]**

Question 2

a. The line needs to go through approximately (60, 1.104) and (130, 0.998).



b. body density = 1.195 – 0.001512 × 65 = 1.09672 = 1.10 kg/L **[1 mark]**

c. A waist measurement of 65 cm is outside of the original data set, therefore the prediction in **part b** is *extrapolated*. **[1 mark]**

d. For every 1 cm increase in waist measurement, body density will decrease by 0.001512 kg/litre. **[1 mark]**

e. predicted body density $= 1.195 - 0.001\,512 \times 122 = 1.010\,536$

$$\begin{aligned}\text{residual} &= \text{actual} - \text{predicted}\\ &= 0.995 - 1.010\,536\\ &= -0.015\,536\\ &= -0.02\end{aligned}$$

Award **1 mark** for the residual calculation.

f. If $r^2 = 0.6783$ and the data is decreasing (i.e. a negative gradient), then
$r = -\sqrt{0.6783} = -0.824$. **[1 mark]**

g. Yes, because there is no pattern to the residual plot. The residuals are randomly scattered above and below the horizontal axis. **[1 mark]**

Question 3

a. Difference in the mean heights $= 167.1 - 156.7 = 10.4$ cm. **[1 mark]**

b. The association is strong and negative. **[1 mark]**

c. $\text{slope} = \dfrac{157 - 168}{85 - 20} \approx -0.169$

intercept $= 168 - (-0.169 \times 20) = 171.38 = 171$

Therefore, the equation will be $\textit{mean height} = 171 - 0.169 \times \textit{mean age}$.

Award **1 mark** for both values.

d. The most appropriate transformation of the explanatory variable would be x^2.

Use a CAS calculator to perform the transformation and the associated least squares line of best fit.

$\textit{mean height} = 167.9 - 0.001621 \times (\textit{mean age})^2$

Award **1 mark** for the values to 4 significant figures.

Award **1 mark** for the transformation.

Question 4

a. Relative humidity at 9 am **[1 mark]**

b. Perform a least squares calculation on your CAS calculator:

humidity 3 pm $= -1.26 + 0.765 \times$ humidity 9 am

Award 1 mark for both numbers correct to three significant figures.

c. From CAS: $r = 0.871$ **[1 mark]**

Question 5

a. On average, for each 1 hPa increase in pressure at 9 am, the pressure at 3 pm increases by 0.8894 hPa. **[1 mark]**

***VCAA Examination Report* note:**

Students had to be careful when answering this interpretative question. Many gave a response that was almost correct but failed to reference the one-unit increase in *pressure* 9 *am*.

Interpreting the slope in terms of the given variables required identifying the correct constant and then describing it.

Describing both constants was to ignore the first step; specific knowledge was required, not various statements provided in the hope of including something relevant.

b. Pressure 3 pm $= 111.4 + 0.8894 \times 1025 = 1023.035 = 1023$ hpa **[1 mark]**

c. Interpolation **[1 mark]**

The value 1025 hPa is within the data range for pressure at 9 am. **[1 mark]**

d. When pressure 9 am = 1013 hPa, Predicted pressure 3 pm

$= 111.4 + 0.8894 \times 1013 = 1012.3622$ hpa.

According to the graph, actual pressure at 3 pm is 1015 hPa.

Residual = actual − predicted = 1015 − 1012 = 3 hPa. **[1 mark]**

***VCAA Examination Report* note:**

This question was poorly done. Many found the predicted value correctly but then subtracted from 1013 rather than from the *pressure* 3 *pm* value of 1015.

e. i. $b = r\dfrac{s_y}{s_x}$ or rearrange to be $r = b\dfrac{s_x}{s_y}$

$r = 0.8894 \times \dfrac{4.5477}{4.1884} = 0.96569678 = 0.966$ **[1 mark]**

ii. $r^2 = 0.966^2 = 0.933156$

93.3% of the variation in pressure at 3 pm may be explained by the variation in pressure at 9 am.

[1 mark]

***VCAA Examination Report* note:**

This was quite often given as 96.6% by students who did not square the r value.

f. i. That the data follows a linear relationship between the two variables. **[1 mark]**

***VCAA Examination Report* note:**

A one-word answer was all that was required. Some gave long answers that contradicted what was expected of them in **part f. ii.**

ii. There is a curved pattern in the residual plot. It is not random enough. A transformation may be required. **[1 mark]**

***VCAA Examination Report* note:**

This question was poorly done. A common mistake was to ignore the given statement that the residuals did not support the assumption and to state that linearity was confirmed.

Many students had difficulty with responses to **part f. i.** and **part f. ii.**, often contradicting themselves or writing definitions directly from a bound reference without any link to the question. Some confused residuals and the correlation coefficient.

Question 6

Input data into the Spreadsheet function on your CAS.

Square the explanatory variable, x^2, and then perform a least-squares regression using x^2 and y.

The resulting equation is $y = -1.34 + 0.546x^2$ (don't forget to include the transformation).

Question 7

$r^2 = 0.893743$

$$\begin{aligned} r &= \sqrt{0.893743} \\ &= 0.9453798 \end{aligned}$$

The gradient of the regression equation is negative; therefore the correlation coefficient is negative.

$r = -0.945$

Question 8

The gradient for the least squares line is $b = r \times \frac{s_y}{s_x}$. This can be rearranged to be $r = b \times \frac{s_x}{s_y}$.

Substitute $b = 1.31$, $s_y = 3.24$ and $s_x = 2.33$ into $r = b \times \frac{s_x}{s_y}$.

$$r = 1.31 \times \frac{2.33}{3.24} = 0.942 \approx 0.94$$

***VCAA Examination Report* note:**

Students needed to be familiar with the rules required for finding a least squares line equation from summary statistics.

Question 9

Given the equation $y = a + bx$, if y and x were switched and then the equation was solved for y, the equation would become $y = -\frac{a}{b} + \frac{1}{b}x$.

Looking at the two equations, the slopes and intercepts are different. Since the slopes are different, the predicted values and thus the residual values will also be different.

The only statistic that will not change is the correlation coefficient r, as s_x and s_y are not affected by the least squares line.

***VCAA Examination Report* note:**
Students needed to be aware that reversing the two variables will give a different equation.
The slope and intercept will therefore both change in value, the predictions the line gives will change and hence the residual values will also change.
The correlation coefficient will not change in value as the degree of scattering of the points remains unchanged – i.e. the scattering of 'y' values relative to 'x' is the same as the scattering of 'x' values relative to 'y'.

Question 10

a. 0.008 538 5, rounded to three significant figures is 0.008 54 **[1 mark]**

b. Students can use their CAS to enter the data and perform a regression analysis. From the CAS Linear Regression $(a + bx)$

$a \simeq -14.39600$ (intercept) and $b \simeq 0.00853852$ (slope)

Rounding to three significant figures, the equation of the least squares line is:

$\log_{10}(area) = -14.4 + 0.00854 \times year$

Award 1 mark for −14.4 and 1 mark for 0.00854

***VCAA Examination Report* note:**
Some students did not recognise that the rounded slope value from **part a.** was a required value in **part b.**
Some students appeared to have renumbered the values of the variable year as 1, 2, 3, etc. This was not asked for and was inappropriate given the increments of 10 in the given values.

c. i. $\log_{10}(area) = 2.85$

$$\begin{aligned} area &= 10^{2.85} \\ &= 707.9458 \\ &= 708 \text{ hectares (to the nearest hectare)} \end{aligned}$$ **[1 mark]**

ii. The year 2020 is outside of the data range that was used to generate the regression line. That is, it involved extrapolation. **[1 mark]**

***VCAA Examination Report* note:**
Many students recognised that extrapolation leads to limited reliability of predictions.

Question 11

a. Students can use their CAS to enter the data and perform a regression analysis. From the CAS Linear Regression $(a + bx)$

$a = -46.8486$ (intercept) and $b = 18.8963$ (slope)

Rounding to one decimal place, the equation of the least squares line is:

$egg\ density = -46.8 + 18.9 \times number\ of\ male\ moths$

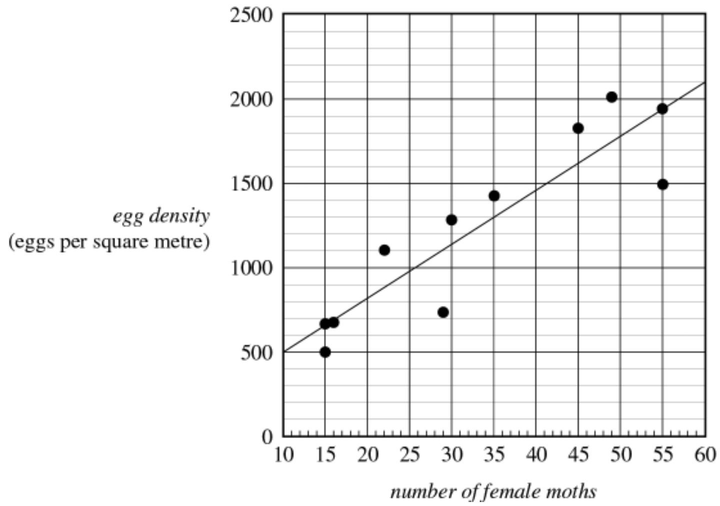
Award 1 mark for −46.8 and 1 mark for 18.9

b. i. Choose any two points and plot them onto the scatterplot.

Let $x = number\ of\ female\ moths$ and let $y = egg\ density$

$$\text{When } x = 15, y = 191 + 31.3 \times 15$$
$$= 660.5$$
$$\text{When } x = 55, y = 191 + 31.3 \times 55$$
$$= 1912.5$$

Plot the line passing through the coordinates (15, 660.5) and (55, 1912.5) as shown in the figure.
[1 mark]



***VCAA Examination Report* note:**
This question was not answered well. Many students did not realise that the scale on the horizontal axis started at 10 and not 0. It was clear that some students did not take a ruler into the examination.

ii. The slope value is 31.3. On average, for each increase of 1 female moth caught in the trap, the egg density increases by 31.3 eggs per square metre. **[1 mark]**

VCAA Examination Report note:
Students were not awarded marks when their response did not clearly refer to the increase in egg density for every one-unit increase in female moths.

iii. From **part i**, the predicted egg density when the *number of female moths* caught is 55 is 1912.5.

$$\text{Residual} = y_a - y_p$$
$$= 1500 - 1912.5$$
$$= -412.5 \ \textbf{[1 mark]}$$

***VCAA Examination Report* note:**
The most common incorrect calculation led to a positive residual of 412.5.
Rounding did not apply to this question, so −412.5 was the only acceptable answer.

iv. $r = 0.862$
$\therefore r = (0.862)^2$
$= 0.743044$
As a percentage to one decimal place, 74.3%. **[1 mark]**

Question 12

a. Strong, positive, linear [1 mark]
***VCAA Assessment Report* note:**
A relatively common incorrect response was ‘Strong, linear and positively skewed’.

b. i. Here you can either use your CAS to enter the data and perform a regression analysis, or you can use the graph to find an approximate equation.
Using CAS: $\textit{Apparent temp} = -1.7 + 0.94 \times \textit{actual temp}$

Using the graph: $m = \dfrac{25.7 - 17.2}{29 - 20} = 0.94$ **[1 mark]**

$y - 17.2 = 0.94\,(x - 20)$
$y = -1.6 + 0.94x$
$\text{Apparent temp} = -1.6 + 0.94 \times \text{actual temp}$ **[1 mark]**

***VCAA Assessment Report* note:**
The first column contained the response variable rather than the explanatory variable. Students who did not notice this gave their answer as (2.4, 1.0).

There was much evidence of confusion between decimal point rounding and significant figure rounding. Many students rounded to two decimal places rather than two significant figures as required. The number 0.9 has only one significant figure, whereas 0.90 has two significant figures.

ii. The apparent temperature is about 1.7 degrees lower than the actual temperature. **[1 mark]**

***VCAA Assessment Report* note:**

A common incorrect answer confused intercept and slope. Another common incorrect answer mixed up the response variable (apparent temperature) and the explanatory variable (actual temperature).

c. Approximately 97% of the variation in the apparent temperature can be explained by the variation in the actual temperature. The other 3% is due to other contributing factors. **[1 mark]**

***VCAA Assessment Report* note:**

Some students mixed up the response variable and the explanatory variable in this statement and wrote, ‘ 97% of the variation in actual temperature can be explained by the variation in apparent temperature’. Other common unacceptable answers used terms that suggested variation in actual temperature caused the variation in apparent temperature. An example was, ‘97% of the variation in actual temperature was due to the variation in apparent temperature’.

d. i. The assumption is that there is a linear relationship between the two variables. **[1 mark]**

ii. Yes, because there is a random pattern. **[1 mark]**

***VCAA Assessment Report* note:**

Reference to a lack of pattern or the randomness of the plot was required. An unacceptable answer was that the ‘… points are all scattered evenly above and below …’ with no mention of randomness.

Question 13

a. In 1920 the life expectancy in Australia was 60 and in 2010 it was 82. That is an increase of 22 years. **[1 mark]**

b. i. Australia $= -451.7 + 0.2657 \times 2030 = 87.671$

UK $= -350.4 + 0.2143 \times 2030 = 84.629$

$\therefore$ difference $= 87.671 - 84.629 = 3.042 \simeq 3$ years

Award **1 mark** for each of the Australia and UK calculations, and **1 mark** for the final answer.

***VCAA Assessment Report* note:**

Some students correctly calculated the numbers for Australia and the United Kingdom but then did not find the required difference.

Some students found negative values for life expectancies. Negative values are non-consistent with the context.

ii. There are many unknown factors that may arise in the next 20 years, such as advances in health care, which may make this prediction of limited reliability. **[1 mark]**

***VCAA Assessment Report* note:**

This question required a response related to the given statistical data rather than any sociological possibilities. Many students wrote about the likely impact of future technological advances, wars, famines, viruses or advances in medicine, or simply stated ‘You don’t know what’s going to happen in the future’. Such responses were not accepted.

Question 14

a. The dependent variable is always the variable on the vertical axis. Therefore it is population. **[1 mark]**

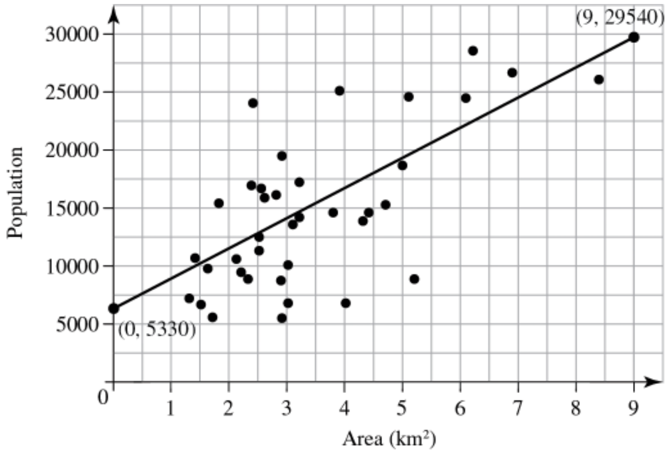
b. Need two points:

$area = 0, population = 5330$

$area = 9, population = 5330 + 2680 \times 9$

$= 29\,450$

Therefore $(0, 0)$ and $(9, 29\,450)$ are the two points to be plotted and joined.



***VCAA Assessment Report* note:**

This question was not answered well by many students. Many seemed to draw the line by eye, with no reference to the least squares regression equation given in the question. Of those who did use the equation, some drew the line through two points that were needlessly close to each other with a resulting inaccurate line.

Students are encouraged to use the whole grid when plotting a straight line, plotting a point at *area* = 0 and *area* = 9.

The grid interval on the population axis seemed to cause problems for a number of students, when they attempted to plot the line with an intercept of 5330.

c. Slope = 2680 **[1 mark]**

For every 1 km^2 increase in the area, the population increases by 2680 people. **[1 mark]**

***VCAA Assessment Report* note:**

Most students were able to find the gradient of 2680 from the equation, although many were unable to explain what it meant. A common unacceptable answer was 'The population increases by 2680 people for every increase in area'. Instead, the slope represented the increase in population for a specific increase in area of **one unit** or 1 km^2.

Other students drew their own line and then used it to calculate a slope that was usually incorrect. Some students inappropriately referred to the data being skewed. The term 'skewed' only applies to univariate data plots, whereas this question referred to a bivariate data plot.

d. i. Predicted value when the area is 4 km^2

$$\text{population} = 5330 + 2680 \times 4$$
$$= 16\ 050$$

$$\text{residual value} = \text{actual value} - \text{predicated value} \quad \textbf{[1 mark]}$$
$$= 6690 - 16\ 050$$
$$= -9360$$

ii. $r^2 = 0.668^2$
$$= 0.446$$
$$= 44.6\%$$

[1 mark]

***VCAA Assessment Report* note:**

Many students simply wrote an answer of 45% without showing any working. This was not correct to one decimal place as required by the question and the mark was not awarded. Some students wrote 44.6% = 45%, thus giving their final answer as 45%. Other common, incorrect answers were 40% and 66.8%.

Question 15

a. Using CAS and finding the regression line between the variables log(*area*) and population gives:

$y = a + bx$

$= 7.7 + 7.7x$

Therefore: $population = 7.7 + 7.7 \times \log_{10}(area)$ **[1 mark]**

VCAA Assessment Report **note:**

A log transformation, using base 10, was required, as indicated in the question. Students need to distinguish between use of the base 10 logarithm and the natural logarithm when using technology to carry out computations.

b. Area 90 km^2

$population = 7.7 + 7.7 \times \log_{10}(90)$

$= 22.7$

$\approx 23\,000$

[1 mark]

VCAA Assessment Report **note:**

Of those students who performed the log transformation to find the correct values in **part a**., many then failed to use the log function in this final calculation and calculated $7.7 + 7.7 \times 90 = 700.7$

Question 16

a. $z = \dfrac{y - \bar{y}}{s_y} = \dfrac{91 - 85.6}{2.99} = 1.8$ (to 1 decimal place) **[1 mark]**

VCAA Assessment Report **note:**

Most students applied the correct rule to find the z score, but many used the wrong variable. The rule on the formula sheet was given as $z = \dfrac{x - \bar{x}}{s_x}$ and applied to the z score for the pay rate (independent variable, x).

However, the question asked for the z score for the development index (the dependent variable, y) for which $z = \dfrac{y - \bar{y}}{s_y}$ must be used.

Many students failed to use brackets to evaluate $\dfrac{91 - 85.6}{2.99}$, which must be entered as $(91 - 85.6) \div 2.99$.

Without brackets, this calculation becomes $91 - \dfrac{85.6}{2.99} = 62.371\ldots$, which was a common, incorrect answer.

b. $y = a + bx$

$b = r \times \dfrac{s_y}{s_x}$

$= 0.488 \times \dfrac{2.99}{5.37}$

$= 0.272$

$a = \bar{y} - b \times \bar{x}$

$= 85.6 - 0.272 \times 15.7$

$= 81.3$ **[1 mark]**

Therefore, the equation of the least squares regression line for a country's development index in terms of its average pay rate is given by $y = 81.3 + 0.272x$. **[1 mark]**

c. Predicted value for the development index:

$y_{\text{predicted}} = 81.3 + 0.272 \times 14.3 = 85.2$ **[1 mark]**

Residual $= y_{\text{observed}} - y_{\text{predicted}} = 83 - 85.2 = -2.2$ **[1 mark]**

***VCAA Assessment Report* note:**
Many students correctly found the predicted value but did not continue to find the residual value as required.

Question 17
The least-squares regression equation is:
$\text{Errors} = 12.0 - 1.2 \times \text{time on questions}$
Replace Errors with 8 and solve the equation.

$$8 = 12.0 - 1.2 \times \text{time on questions}$$

$$1.2 \times \text{time on questions} = 4$$

$$\text{time on questions} = \frac{4}{1.2}$$

$$= 3.3 \text{ hours}$$

Question 18
The arm span length of 180 cm is within the bounds of the listed data and therefore represents an interpolation.

Question 19
The prediction is made on a value outside the current domain and range of the graphed data. Therefore, it is an extrapolation.

Question 20
The graph could be used to interpolate the girl's height at 3 years old.

Question 21
Extrapolation is the use of the least squares line of best fit to predict values smaller than the smallest value or larger than the largest value in the data set.

Question 22
If a line of best fit was fitted by eye, it would have a positive gradient.

Question 23
Entering values into your calculator generates an equation $y = 6x + 50$ (approximately).

4 Investigating and modelling time series data

Topic	4	Investigating and modelling time series data
Subtopic	4.2	Time series plots and trends

online only

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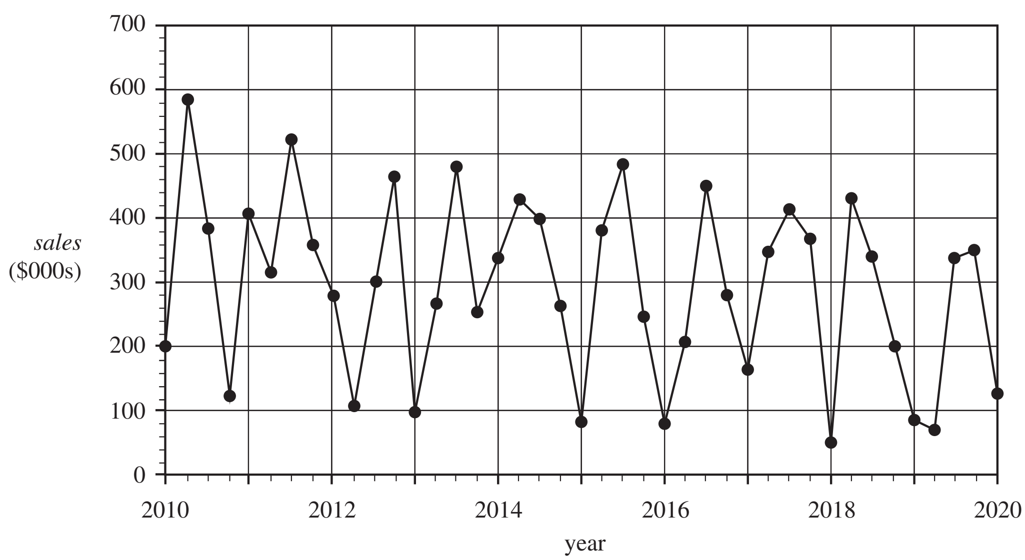
Source: *VCE 2021, Further Mathematics Exam 1, Section A, Q.12; © VCAA*

Question 1 (1 mark)

The time series plot below shows the quarterly *sales*, in thousands of dollars, of a small business for the years 2010 to 2020 .



The time series plot is best described as having

A. seasonality only.

B. irregular fluctuations only.

C. seasonality with irregular fluctuations.

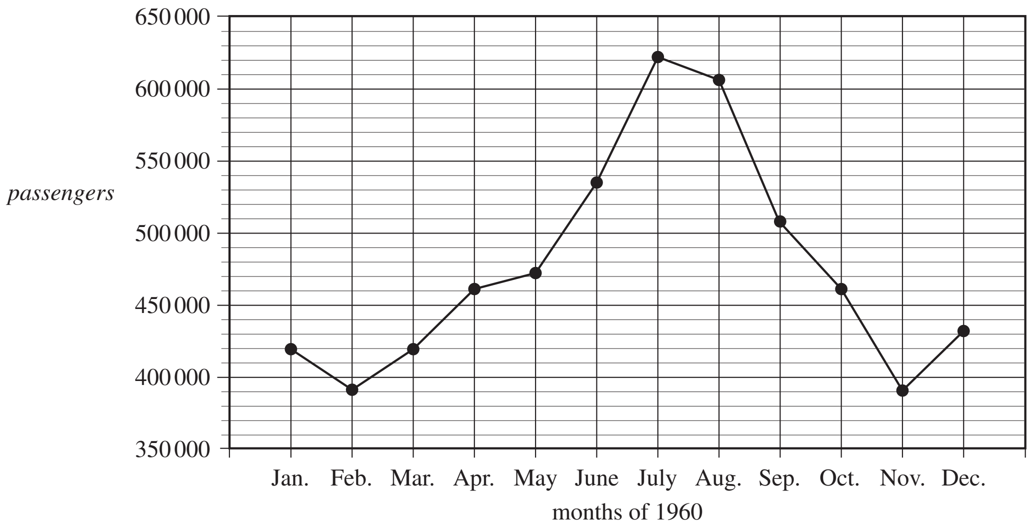
D. a decreasing trend with irregular fluctuations.

E. a decreasing trend with seasonality and irregular fluctuations.

Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.19; © VCAA*

Question 2 (1 mark)

The time series plot below displays the number of airline passengers, in thousands, each month during the period January to December 1960.



Data: GEP Box and GM Jenkins, *Time Series Analysis: Forecasting and Control*, Holden-Day, San Francisco, 1970, p. 531

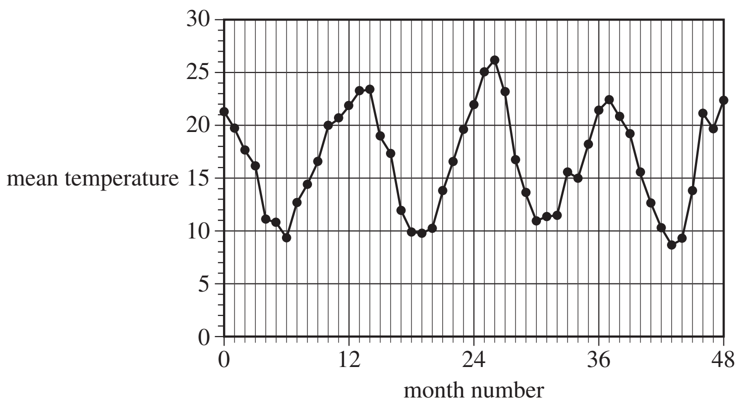
During 1960, the median number of monthly airline *passengers* was closest to

A. 461 000
B. 465 000
C. 471 000
D. 573 000
E. 621 000

Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 3 (1 mark)

Consider the time series plot below.



Data: Commonwealth of Australia, Bureau of Meteorology

The pattern in the time series plot shown above is best described as having

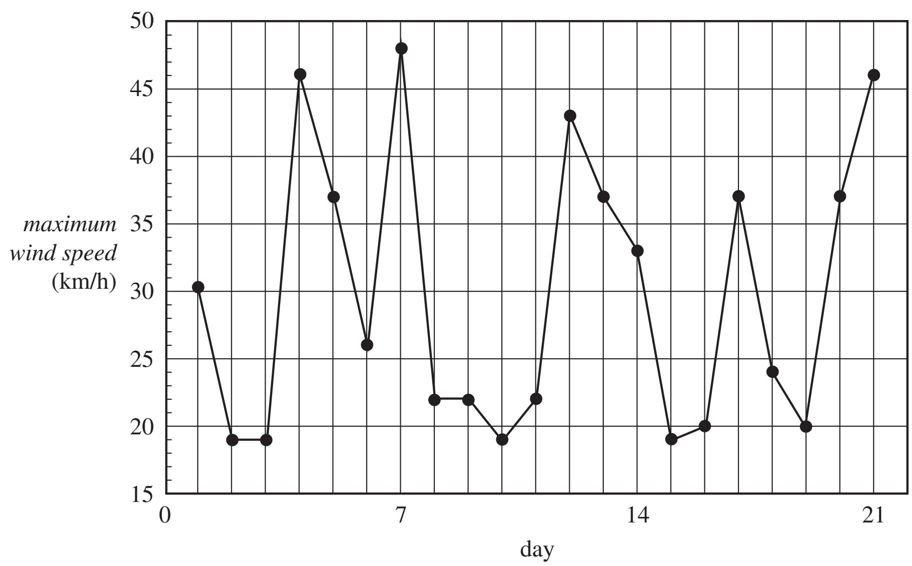
A. irregular fluctuations only.
B. an increasing trend with irregular fluctuations.
C. seasonality with irregular fluctuations.
D. seasonality with an increasing trend and irregular fluctuations.
E. seasonality with a decreasing trend and irregular fluctuations.

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 4 (1 mark)

The wind speed at a city location is measured throughout the day.

The time series plot below shows the daily *maximum wind speed*, in kilometres per hour, over a three-week period.



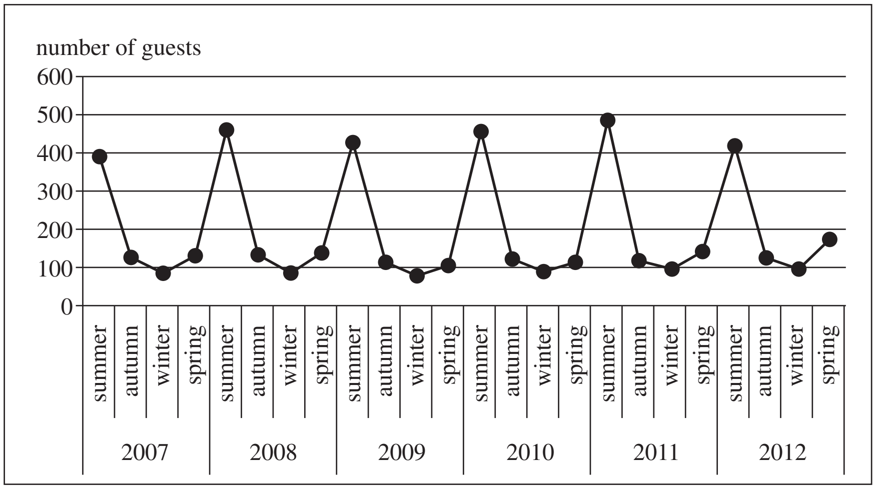
The time series is best described as having

A. seasonality only.
B. irregular fluctuations only.
C. seasonality with irregular fluctuations.
D. a decreasing trend with irregular fluctuations.
E. an increasing trend with irregular fluctuations.

Source: *VCE 2013, Further Mathematics Exam 1, Section A, Q.12; © VCAA*

Question 5 (1 mark)

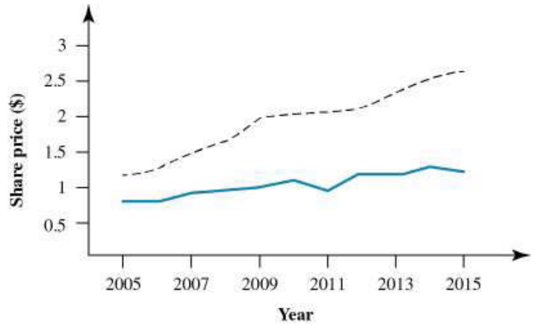
The time series plot below displays the number of guests staying at a holiday resort during summer, autumn, winter and spring for the years 2007 to 2012 inclusive.



Which one of the following best describes the pattern in the time series?

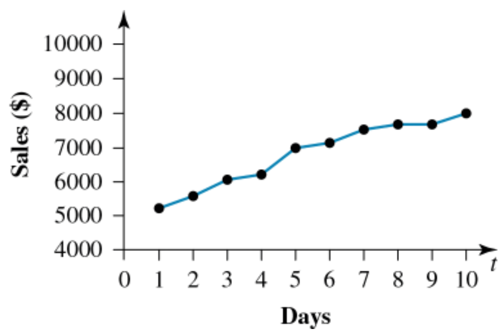
A. random variation only
B. decreasing trend with seasonality
C. seasonality only
D. increasing trend only
E. increasing trend with seasonality

Question 6 (1 mark)
From the plot it can be concluded that over the time 2005–2015 the **difference** in share prices between the two stocks has shown



A. an increasing trend.
B. a decreasing trend.
C. no trend.
D. seasonal variations.
E. irregular fluctuations.

Question 7 (1 mark)
Which type of trend is apparent in the graph shown below?



A. long term
B. seasonal
C. cyclic
D. random
E. negative

Question 8 (1 mark)

Which of the following statements about a long term trend is false?

A. A long term trend is one that steadily increases or decreases over time with no major changes in direction.
B. A long term trend is best viewed over a long period of time.
C. A long term trend shows peaks and troughs.
D. A long term trend relates to time series data.
E. A long term trend is different from a seasonal trend.

Question 9 (1 mark)

Which of the following trends is most likely to be evident over the long term?

A. The amount of monthly rainfall in Victoria.
B. The number of personnel in the Australian navy, measured annually.
C. The share price of BHP, measured monthly.
D. The number of people living in Australia, measured annually.
E. The number of seats held by the Australian Labor Party in the Victorian Parliament.

Question 10 (1 mark)

The table provides information about interest rates over an 11-year period of time.

Year	Interest rate
2005	5.6
2006	6.2
2007	6.8
2008	7.2
2009	4
2010	3.8
2011	4.6
2012	4.2
2013	3
2104	2.6
2015	2.6

Let 2005 = 1, 2006 = 2, 2007 = 3 as a time code to allow the data to be plotted.

The long term trend of this information predicts interest rates in 2020 to be

A. 842.1%
B. 6.8%
C. 2.6%
D. 0.4%
E. 0.0%

Topic	4	Investigating and modelling time series data
Subtopic	4.3	Fitting the least squares line and forecasting

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Source: *VCE 2021, Further Mathematics Exam 1, Section A, Q.10; © VCAA*

Question 1 (1 mark)

Oscar walked for nine consecutive *days*. The *time*, in minutes, that Oscar spent walking on each day is shown in the table below.

Day	1	2	3	4	5	6	7	8	9
Time	46	40	45	34	36	38	39	40	33

A least squares line is fitted to the data.

The equation of this line predicts that on day 10 the time Oscar spends walking will be the same as the time he spent walking on

A. day 3
B. day 4
C. day 6
D. day 8
E. day 9

Source: *VCE 2021, Further Mathematics Exam 1, Section A, Q.11; © VCAA*

Question 2 (1 mark)

The table below shows the weight, in kilograms, and the height, in centimetres, of 10 adults.

Weight **(kg)**	***Height*** **(cm)**
59	173
67	180
69	184
84	195
64	173
74	180
76	192
56	169
58	164
66	180

A least squares line is fitted to the data.

The least squares line enables an adult's *weight* to be predicted from their *height*.

The number of times that the predicted value of an adult's *weight* is greater than the actual value of their *weight* is

A. 3
B. 4
C. 5
D. 6
E. 7

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.14; © VCAA*

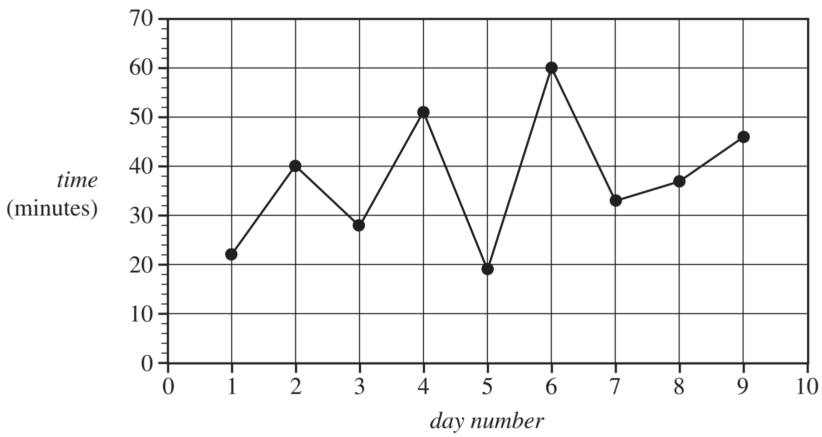
Question 3 (1 mark)

The *time*, in minutes, that Liv ran each day was recorded for nine days.

These times are shown in the table below.

Day number	1	2	3	4	5	6	7	8	9
Time (minutes)	22	40	28	51	19	60	33	37	46

The time series plot below was generated from this data.



A least squares line is to be fitted to the time series plot shown above.

The equation of this least squares line, *with day* number as the explanatory variable, is closest to

A. $day\ number = 23.8 + 2.29 \times time$

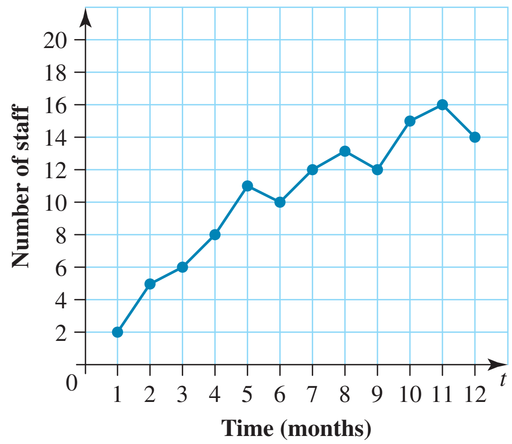
B. $day\ number = 28.5 + 1.77 \times time$

C. $time = 23.8 + 1.77 \times day\ number$

D. $time = 23.8 + 2.29 \times day\ number$

E. $time = 28.5 + 1.77 \times day\ number$

Question 4 (1 mark)
When Tally Ho Books started, it had only one employee. Over time, that number has increased, as shown on the graph below.



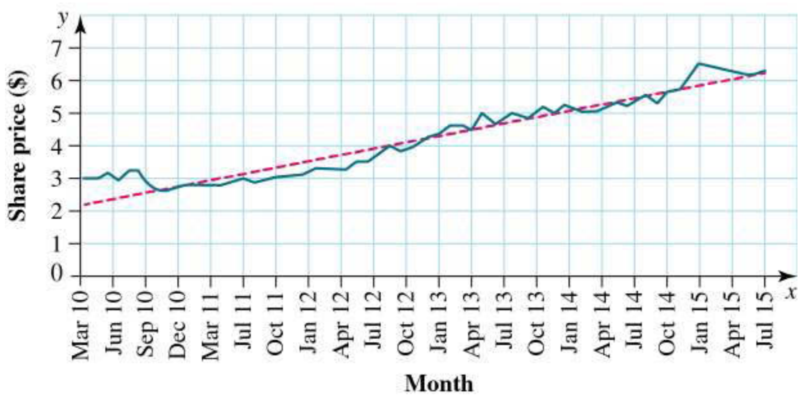
When a 3-median trend line is fitted to this data, what is its equation closest to?

A. $n = 1.1t + 3.2$
B. $n = -1.1t + 3.2$
C. $n = 1.1t + 8$
D. $n = 1.1t$
E. $n = -1.1t$

Topic 4 | Subtopic 4.3 Fitting the least squares line and forecasting

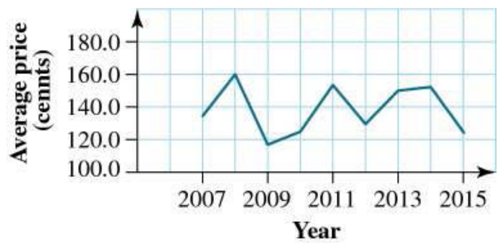
Question 5 (1 mark)
A trend line is fitted to data that shows red wine sales (in thousands of litres) over time.
The equation of the trend line is:
Wine sales $= 500 + 500 \times$ *number of years*
The number of litres predicted to be sold after 6 years is
A. 3500 L
B. 6500 L
C. 3 500 000 L
D. 6 500 000 L
E. 350 000 L

Question 6 (1 mark)
The share price of a particular stock over time is displayed (along with the trend line) in the image below:



Using the time code for the month variable of Jan 10 = 1, Feb 10 = 2, Mar 10 = 3 … the long term trend line equation is:
Share price $= 2.22 + 0.06 \times$ month.
The least squares line of best fit predicts the share price for Jan 16 as
A. $6.48
B. $6.54
C. $6.60
D. $6.66
E. $6.72

Question 7 (1 mark)
The following time series graph shows the average price of petroleum over the years 2007–2015.



A least squares line fitted to this data predicts the average price of petrol in 2016 to be
A. 44.5 cents.
B. 132.5 cents.
C. 140.9 cents.
D. 141.8 cents.
E. 147.9 cents.

Topic	4	Investigating and modelling time series data
Subtopic	4.4	Smoothing using the moving mean with an odd number of points

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Source: *VCE 2021, Further Mathematics Exam 1, Section A, Q.14; © VCAA*

Question 1 (1 mark)

A garden centre sells garden soil.

The table below shows the daily *quantity* of garden soil sold, in cubic metres, over a one-week period.

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Quantity (m^3)	234	186				346	346

The *quantity* of garden soil sold on Wednesday, Thursday and Friday is not shown.

The five-mean smoothed *quantity* of garden soil sold on Thursday is 206 m^3.

The three-mean smoothed *quantity* of garden soil sold on Thursday, in cubic metres, is

A. 143

B. 166

C. 206

D. 239

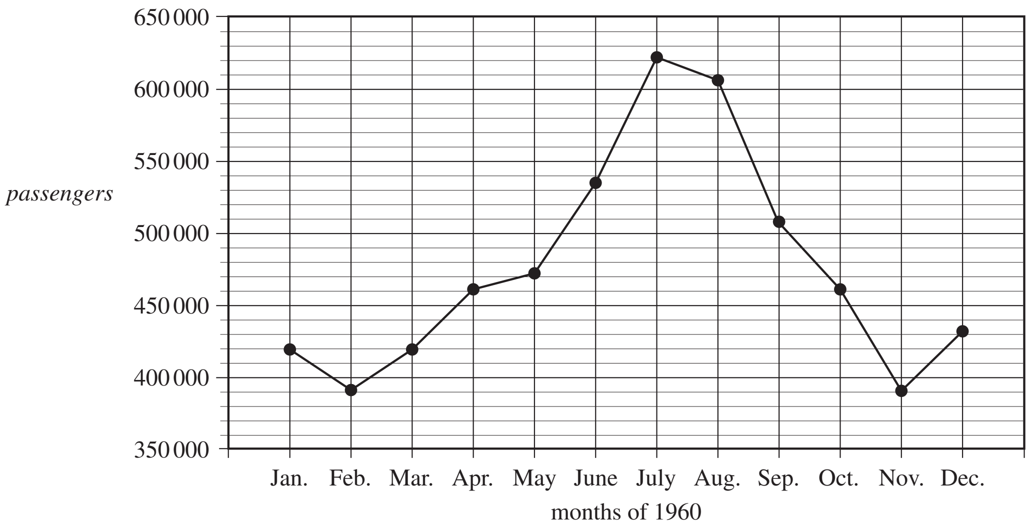
E. 403

Topic 4 Subtopic 4.4 Smoothing using the moving mean with an odd number of points

Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.20; © VCAA*

Question 2 (1 mark)

The time series plot below displays the number of airline passengers, in thousands, each month during the period January to December 1960.



Data: GEP Box and GM Jenkins, *Time Series Analysis: Forecasting and Control*, Holden-Day, San Francisco, 1970, p. 531

During the period January to May 1960, the total number of airline passengers was 2 160 000.

The five-mean smoothed number of passengers for March 1960 is

A. 419 000

B. 424 000

C. 430 000

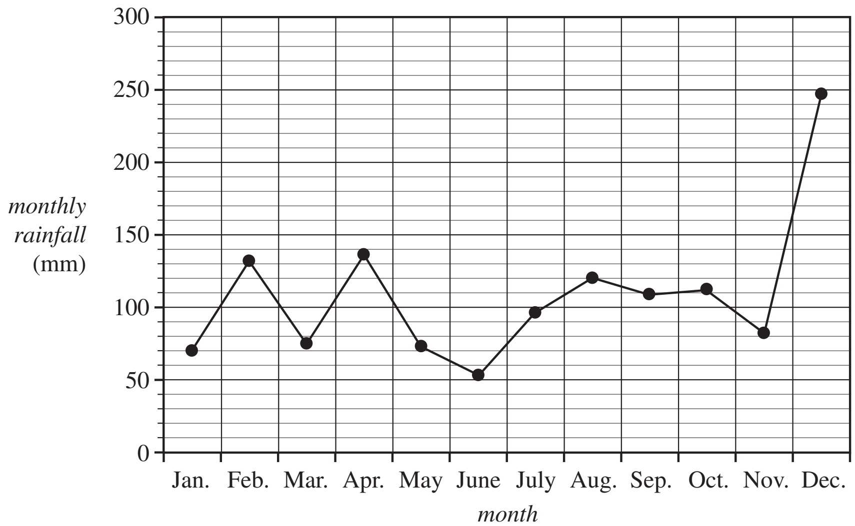
D. 432 000

E. 434 000

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.16; © VCAA*

Question 3 (1 mark)

The time series plot below shows the *monthly rainfall* at a weather station, in millimetres, for each *month* in 2017.



If seven-mean smoothing is used to smooth this time series plot, the number of smoothed data points would be

A. 3
B. 5
C. 6
D. 8
E. 10

Topic 4 | Subtopic 4.4 Smoothing using the moving mean with an odd number of points

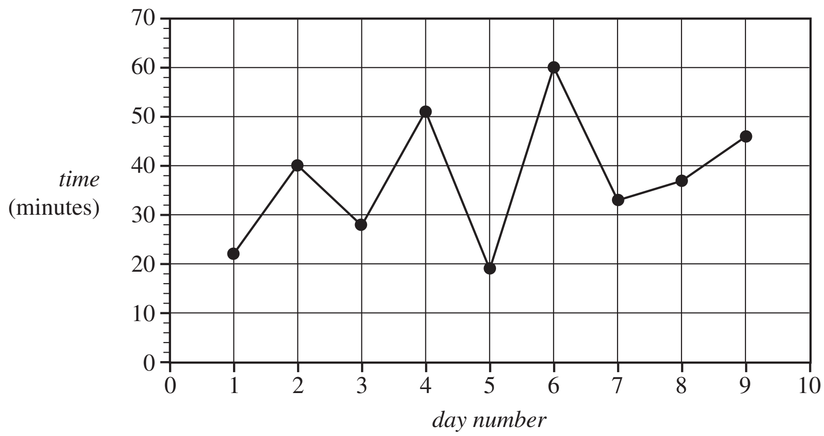
Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 4 (1 mark)

The *time*, in minutes, that Liv ran each day was recorded for nine days.
These times are shown in the table below.

Day number	1	2	3	4	5	6	7	8	9
Time (minutes)	22	40	28	51	19	60	33	37	46

The time series plot below was generated from this data.



Both three-median smoothing and five-median smoothing are being considered for this data.
Both of these methods result in the same smoothed value on *day number*

A. 3
B. 4
C. 5
D. 6
E. 7

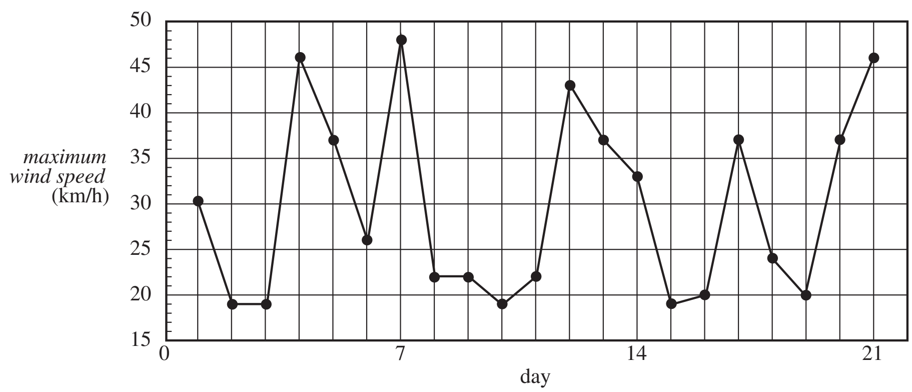
Topic 4 | Subtopic 4.4 Smoothing using the moving mean with an odd number of points

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.14; © VCAA*

Question 5 (1 mark)

The wind speed at a city location is measured throughout the day.

The time series plot below shows the daily *maximum wind* speed, in kilometres per hour, over a three-week period.



The seven-median smoothed *maximum wind* speed, in kilometres per hour, for day 4 is closest to

A. 22
B. 26
C. 27
D. 30
E. 32

Topic	4	Investigating and modelling time series data
Subtopic	4.5	Smoothing using the moving mean with an even number of points

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Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.18; © VCAA*

Question 1 (1 mark)

Table 4 below shows the monthly rainfall for 2019, in millimetres, recorded at a weather station, and the associated long-term seasonal indices for each month of the year.

Table 4

	Jan.	Feb.	Mar	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov
Mean rainfall (mm)	18.4	17.6	46.8	23.6	92.6	77.2	80.0	86.8	93.8	55.2	97.3
Seasonal index	0.728	0.734	0.741	0.934	1.222	0.973	1.024	1.121	1.159	1.156	1.138

Data: Adapted from © Commonwealth of Australia 2020, Bureau of Meteorology, <www.bom.gov.au/>

The six-mean smoothed monthly rainfall with centring for August 2019 is closest to

A. 67.8 mm
B. 75.9 mm
C. 81.3 mm
D. 83.4 mm
E. 86.4 mm

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.15; © VCAA*

Question 2 (1 mark)

The table below shows the monthly profit, in dollars, of a new coffee shop for the first nine months of 2018.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.
Profit($)	2890	1978	2402	2456	4651	3456	2823	2678	2345

Using four-mean smoothing with centring, the smoothed profit for May is closest to

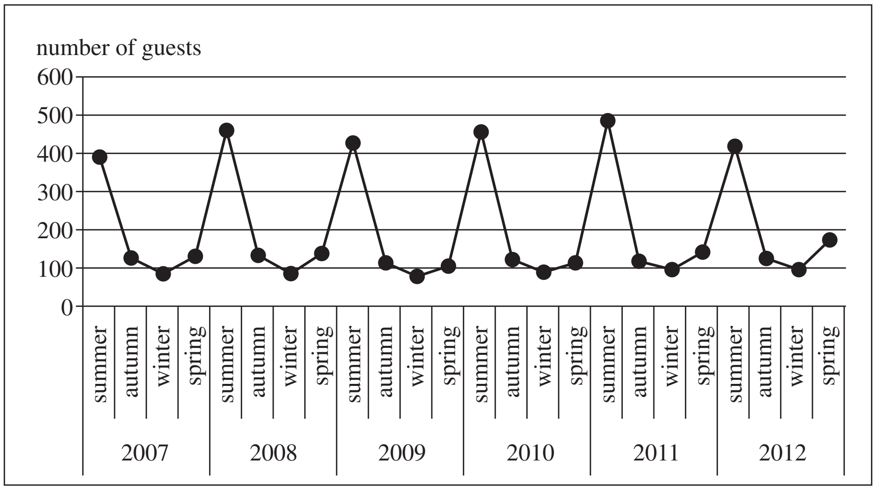
A. $2502
B. $3294
C. $3503
D. $3804
E. $4651

Topic 4 | Subtopic 4.5 Smoothing using the moving mean with an even number of points

Source: *VCE 2013, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 3 (1 mark)

The time series plot below displays the number of guests staying at a holiday resort during summer, autumn, winter and spring for the years 2007 to 2012 inclusive.



The table below shows the data from the times series plot for the years 2007 and 2008.

Year	Season	Number of guests
2007	summer	390
	autumn	126
	winter	85
	spring	130
2008	summer	460
	autumn	136
	winter	86
	spring	142

Using four-mean smoothing with centring, the smoothed number of guests for winter 2007 is closest to

A. 85
B. 107
C. 183
D. 192
E. 200

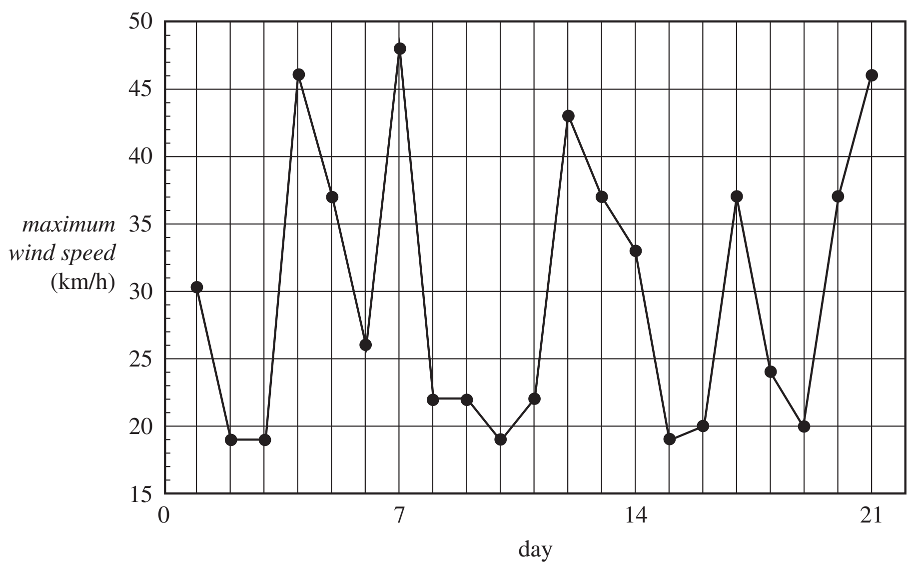
Topic 4 Subtopic 4.5 Smoothing using the moving mean with an even number of points

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.15; © VCAA*

Question 4 (1 mark)

The wind speed at a city location is measured throughout the day.

The time series plot below shows the *daily maximum wind speed*, in kilometres per hour, over a three-week period.



The table below shows the daily *maximum wind speed*, in kilometres per hour, for the days in week 2.

Day	8	9	10	11	12	13	14
Maximum wind speed (km/h)	22	22	19	22	43	37	33

A four-point moving mean with centring is used to smooth the time series data above.

The smoothed *maximum wind speed*, in kilometres per hour, for day 11 is closest to

A. 22
B. 24
C. 26
D. 28
E. 30

Topic	4	Investigating and modelling time series data
Subtopic	4.6	Median smoothing from a graph

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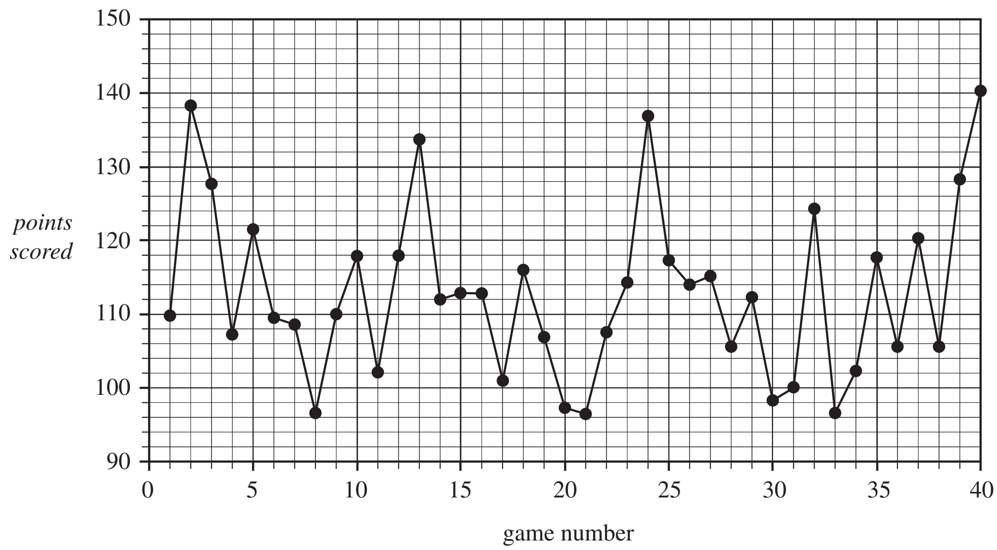
Source: *VCE 2021, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 1 (1 mark)

The time series plot below shows the *points scored* by a basketball team over 40 games.



The nine-median smoothed *points scored* for game number 10 is closest to

A. 102
B. 108
C. 110
D. 112
E. 117

Topic 4 Subtopic 4.6 Median smoothing from a graph

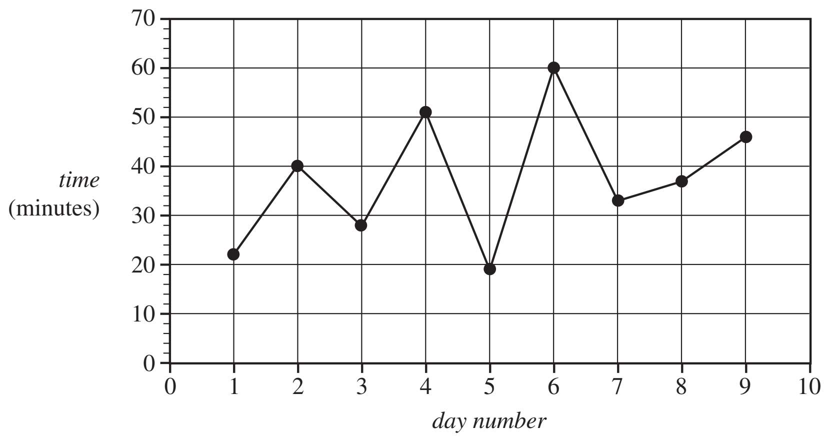
Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 2 (1 mark)

The *time*, in minutes, that Liv ran each day was recorded for nine days.
These times are shown in the table below.

Day number	1	2	3	4	5	6	7	8	9
Time (minutes)	22	40	28	51	19	60	33	37	46

The time series plot below was generated from this data.



Both three-median smoothing and five-median smoothing are being considered for this data.
Both of these methods result in the same smoothed value on *day number*

A. 3
B. 4
C. 5
D. 6
E. 7

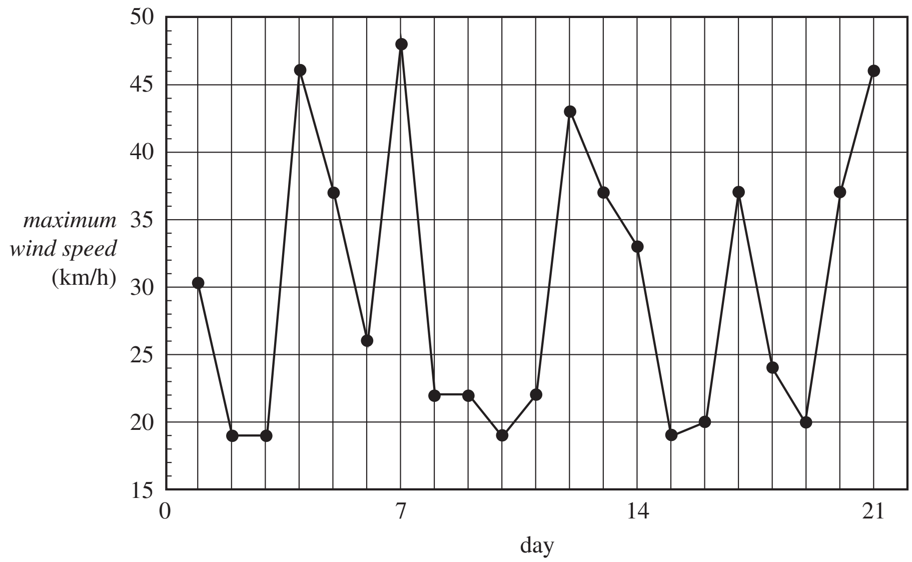
Topic 4 Subtopic 4.6 Median smoothing from a graph

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.14; © VCAA*

Question 3 (1 mark)

The wind speed at a city location is measured throughout the day.

The time series plot below shows the *daily maximum wind speed*, in kilometres per hour, over a three-week period.



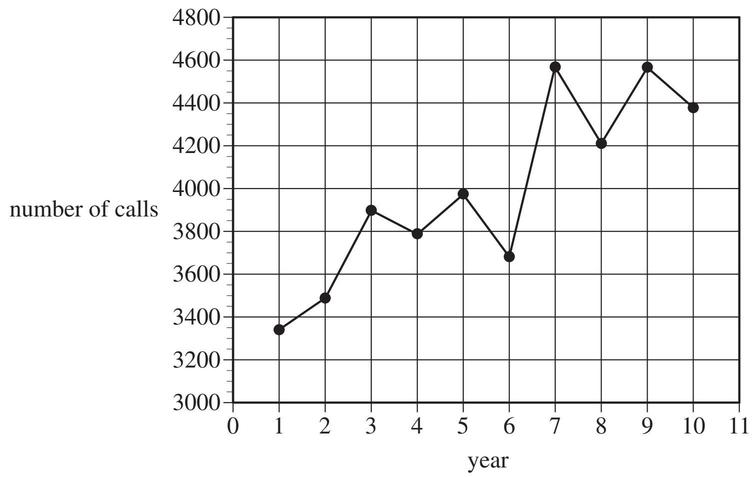
The seven-median smoothed *maximum wind speed*, in kilometres per hour, for day 4 is closest to

A. 22
B. 26
C. 27
D. 30
E. 32

Source: *VCE 2015, Further Mathematics Exam 1, Section A, Q.12; © VCAA*

Question 4 (1 mark)

The time series plot below charts the number of calls per year to a computer help centre over a 10-year period.



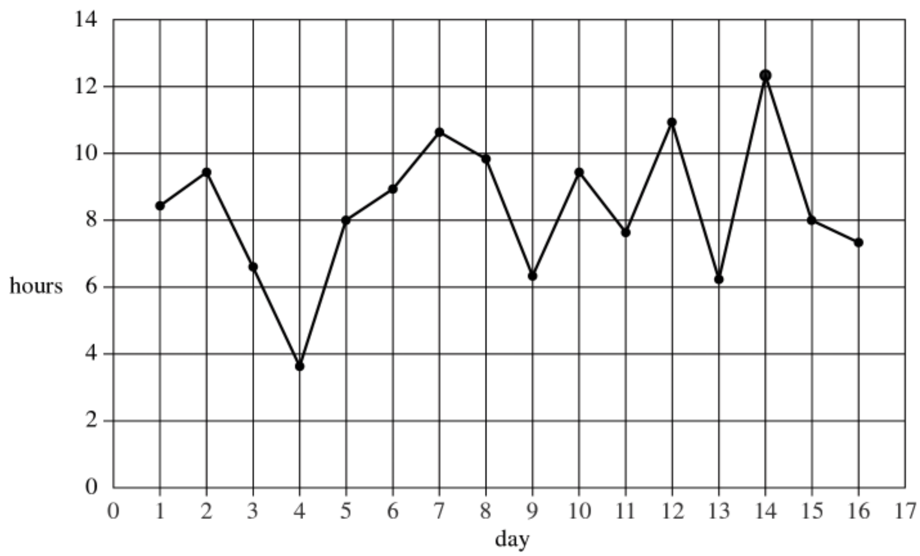
Using five-median smoothing, the smoothed number of calls in year 6 was closest to

A. 3500
B. 3700
C. 3800
D. 4000
E. 4200

Source: *VCE 2014, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 5 (1 mark)

The time series plot below shows the hours of sunshine per day at a particular location for 16 consecutive days.



The three median method is used to fit a trend line to the data.

The slope of this trend line will be closest to

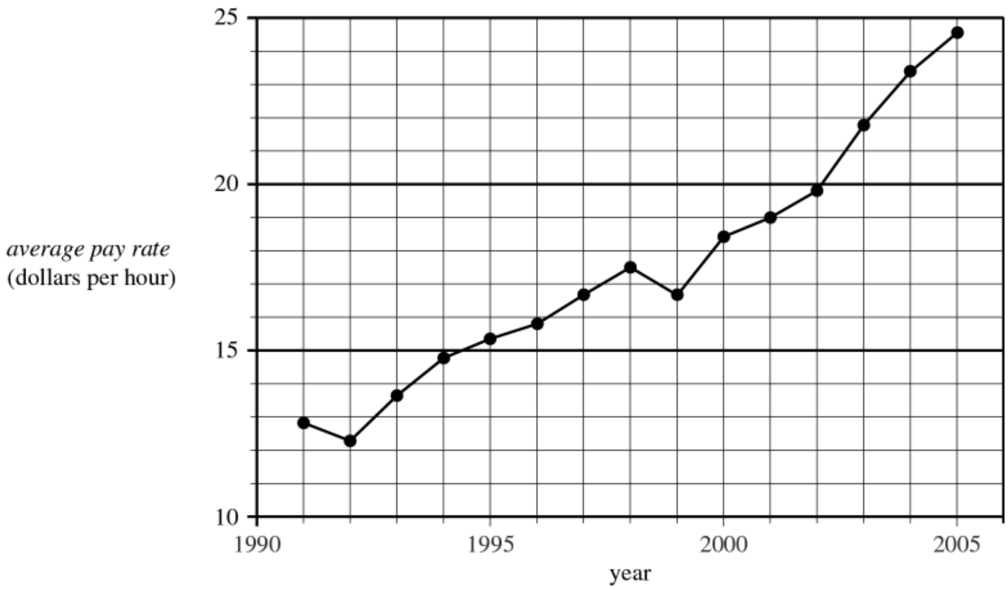
A. −0.7
B. −0.2
C. 0.0
D. 0.2
E. 0.7

Topic 4 Subtopic 4.6 Median smoothing from a graph

Source: *VCE 2013, Further Mathematics Exam 2, Q.4; © VCAA*

Question 6 (3 marks)

The time series plot below shows the average pay rate, in dollars per hour, for workers in a particular country for the years 1991 to 2005.



A three median line will be used to model the increasing trend in the average pay rate shown in this time series.

The independent variable to be used is *year*.

a. Three medians will be used to draw the three median line.

i. On the time series plot, mark the location of each of the three medians with a cross (X). **(2 marks)**

ii. Draw the three median line on the time series plot. **(1 mark)**

Question 7 (1 mark)

The table below shows the number of muggings on a notorious Melbourne street over a number of years.

Year	No. of muggings
2000	4
2001	6
2002	3
2003	8
2004	5
2005	9
2006	4
2007	5
2008	2

Using three-moving median smoothing, what is the smoothed value of the number of muggings in 2006?

A. 9
B. 4
C. 5
D. 6
E. 7

Topic	4	Investigating and modelling time series data
Subtopic	4.7	Seasonal adjustment

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Source: *VCE 2021, Further Mathematics Exam 1, Section A, Core, Q.15; © VCAA*

Question 1 (1 mark)

The table below shows the number of visitors to an art gallery during the summer, autumn, winter and spring quarters for the years 2017 to 2019.

The quarterly average is also shown for each of these years.

Season	2017	2018	2019
summer quarter	29 685	25 420	31 496
autumn quarter	27 462	23 320	29 874
winter quarter	25 564	21 097	27 453
spring quarter	26 065	22 897	28 149
Quarterly average	27 194.0	23 183.5	29 243.0

The seasonal index for summer is closest to

A. 1.077
B. 1.081
C. 1.088
D. 1.092
E. 1.096

Source: *VCE 2021, Further Mathematics Exam 1, Section A, Q.16; © VCAA*

Question 2 (1 mark)

The number of visitors to a regional animal park is seasonal.
Data is collected and deseasonalised before a least squares line is fitted.
The equation of the least squares line is

$$\textit{deseasonalised number of visitors} = 2349 - 198.5 \times \textit{month number}$$

where *month number* 1 is January 2020.
The seasonal indices for the 12 months of 2020 are shown in the table below.

Month number	1	2	3	4	5	6	7	8	9	10	11	12
Seasonal index	1.10	1.25	1.15	0.95	0.85	0.75	0.80	0.85	0.95	1.10	1.15	1.10

The actual number of visitors predicted for February 2020 was closest to

A. 1562
B. 1697
C. 1952
D. 2245
E. 2440

Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.17; © VCAA*

Question 3 (1 mark)

The table below shows the monthly rainfall for 2019, in millimetres, recorded at a weather station, and the associated long-term seasonal indices for each month of the year.

	Jan.	**Feb.**	**Mar**	**Apr.**	**May**	**June**	**July**	**Aug.**	**Sep.**	**Oct.**	**Nov**
Mean rainfall (mm)	18.4	17.6	46.8	23.6	92.6	77.2	80.0	86.8	93.8	55.2	97.3
Seasonal index	0.728	0.734	0.741	0.934	1.222	0.973	1.024	1.121	1.159	1.156	1.138

Data: adapted from © Commonwealth of Australia 2020, Bureau of Meteorology,
The deseasonalised rainfall for May 2019 is closest to

A. 71.3 mm
B. 75.8 mm
C. 86.1 mm
D. 88.1 mm
E. 113.0 mm

Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.15; © VCAA*

Question 4 (1 mark)

The table below shows the long-term mean rainfall, in millimetres, recorded at a weather station, and the associated long-term seasonal indices for each month of the year.

The long-term mean rainfall for December is missing.

	Jan.	Feb.	Mar	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov	Dec.
Mean rainfall (mm)	51.9	52.3	52.8	66.6	87.1	69.4	73.0	79.9	82.6	82.4	81.1	
Seasonal index	0.728	0.734	0.741	0.934	1.222	0.973	1.024	1.121	1.159	1.156	1.138	1.072

Data: Adapted from © Commonwealth of Australia 2020, Bureau of Meteorology,

To correct the rainfall in March for seasonality, the actual rainfall should be, to the nearest per cent

A. decreased by 26%
B. increased by 26%
C. decreased by 35%
D. increased by 35%
E. increased by 74%

Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.16; © VCAA*

Question 5 (1 mark)

The table below shows the long-term mean rainfall, in millimetres, recorded at a weather station, and the associated long-term seasonal indices for each month of the year.

The long-term mean rainfall for December is missing.

	Jan.	Feb.	Mar	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov	Dec.
Mean rainfall (mm)	51.9	52.3	52.8	66.6	87.1	69.4	73.0	79.9	82.6	82.4	81.1	
Seasonal index	0.728	0.734	0.741	0.934	1.222	0.973	1.024	1.121	1.159	1.156	1.138	1.072

Data: Adapted from © Commonwealth of Australia 2020, Bureau of Meteorology,

The long-term mean rainfall for December is closest to

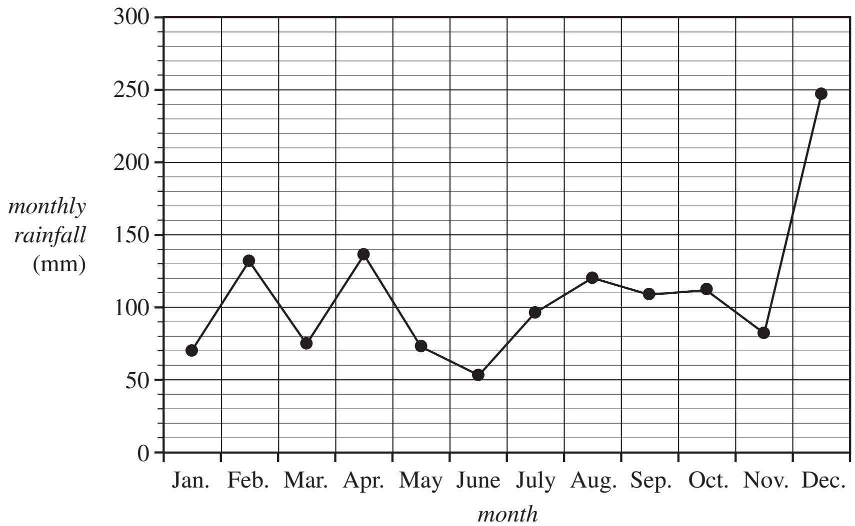
A. 64.7 mm
B. 65.1 mm
C. 71.3 mm
D. 76.4 mm
E. 82.0 mm

Topic 4 Subtopic 4.7 Seasonal adjustment

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.15; © VCAA*

Question 6 (1 mark)

The time series plot below shows the *monthly rainfall* at a weather station, in millimetres, for each *month* in 2017.



The median *monthly rainfall* for 2017 was closest to

A. 53 mm
B. 82 mm
C. 96 mm
D. 103 mm
E. 111 mm

Topic 4 Subtopic 4.7 Seasonal adjustment

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.16; © VCAA*

Question 7 (1 mark)

The quarterly sales figures for a large suburban garden centre, in millions of dollars, for 2016 and 2017 are displayed in the table below.

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2016	1.73	2.87	3.34	1.23
2017	1.03	2.45	2.05	0.78

Using these sales figures, the seasonal index for Quarter 3 is closest to

A. 1.28
B. 1.30
C. 1.38
D. 1.46
E. 1.48

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.11; © VCAA*

Question 8 (1 mark)

Which one of the following statistics can never be negative?

A. the maximum value in a data set
B. the value of a Pearson correlation coefficient
C. the value of a moving mean in a smoothed time series
D. the value of a seasonal index
E. the value of the slope of a least squares line fitted to a scatterplot

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.16; © VCAA*

Question 9 (1 mark)

The seasonal index for the sales of cold drinks in a shop in January is 1.6

To correct the January sales of cold drinks for seasonality, the actual sales should be

A. reduced by 37.5%
B. reduced by 40%
C. reduced by 62.5%
D. increased by 60%
E. increased by 62.5%

Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.15; © VCAA*

Question 10 (1 mark)

The table below shows the long-term average of the number of meals served each day at a restaurant. Also shown is the daily seasonal index for Monday through to Friday.

	Day of the week						
	Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**	**Saturday**	**Sunday**
Long-term average	89	93	110	132	145	190	160
Seasonal index	0.68	0.71	0.84	1.01	1.10		

Last Tuesday, 108 meals were served in the restaurant.

The deseasonalised number of meals served last Tuesday was closest to

A. 98
B. 100
C. 110
D. 131
E. 152

Source: *VCE 2014, Further Mathematics Exam 1, Section A, Q.12; © VCAA*

Question 11 (1 mark)

The seasonal index for heaters in winter is 1.25.

To correct for seasonality, the actual heater sales in winter should be

A. reduced by 20%.
B. increased by 20%.
C. reduced by 25%.
D. increased by 25%.
E. reduced by 75%.

Question 12 (1 mark)

A set of seasonal indices was calculated each fortnight throughout the year for a manufacturing and distribution centre. The sum of all the seasonal indices for this full year of calculations should be

A. Cannot be determined without knowing the specific seasonal values.
B. 1
C. 4
D. 12
E. 26

Question 13 (1 mark)

The seasonal indices for four seasons in a particular year for a retail outlet are shown in the table below.

	Season			
	Summer	Autum	Winter	Spring
Seasonal indices	1.2	0.95	1.02	0.83

A seasonal index of 0.83 in Spring typically means that the sales in Spring are

A. 83% above the yearly average.
B. 83% below the yearly average.
C. 17% above the yearly average.
D. 17% below the yearly average.
E. 8.3% below the yearly average.

Question 14 (1 mark)

The seasonal indices for the Quarters 1,2 and 4 are shown in the table below.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Seasonal index	1.07	0.86		1.12

What is the seasonal index for Quarter 3?

A. 1.02
B. 0.95
C. 1.01
D. 0.97
E. 1.00

Question 15 (1 mark)

The table below shows the long-term average temperature for Summer, Autumn, Winter and Spring. Also shown are the seasonal indices for Summer and Winter. The seasonal indices for Autumn and Spring are missing.

	Season			
	Summer	**Autum**	**Winter**	**Spring**
Long term average temperature	28.5	22.6	18.3	24.2
Seasonal indices	1.22		0.78	0.83

The seasonal index for Autumn is closest to

A. 0.97
B. 1.00
C. 1.03
D. 1.12
E. 2.00

Topic	4	Investigating and modelling time series data
Subtopic	4.8	Review

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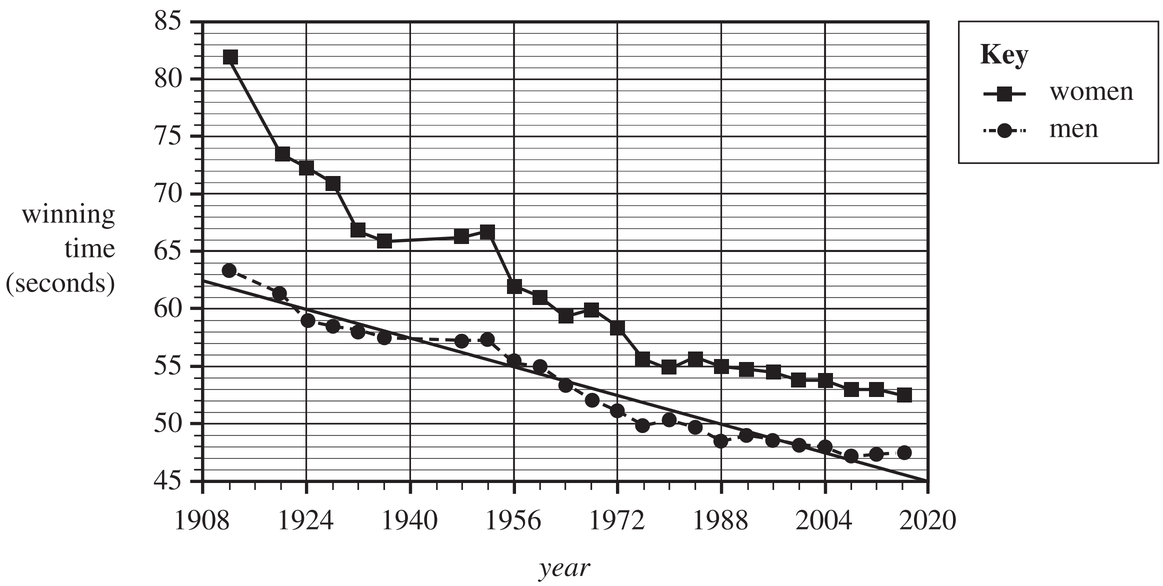
Source: *VCE 2021, Further Mathematics Exam 2, Section A, Q.4 (adapted); © VCAA*

Question 1 (5 marks)

The time series plot below shows that the winning time for both men and women in the 100 m freestyle swim in the Olympic Games has been decreasing during the period 1912 to 2016.



Data: International Olympic Committee, <https://olympics.com/en/olympic-games/olympic-results>

Least squares lines are used to model the trend for both men and women.

The least squares line for the men's winning time has been drawn on the time series plot above.

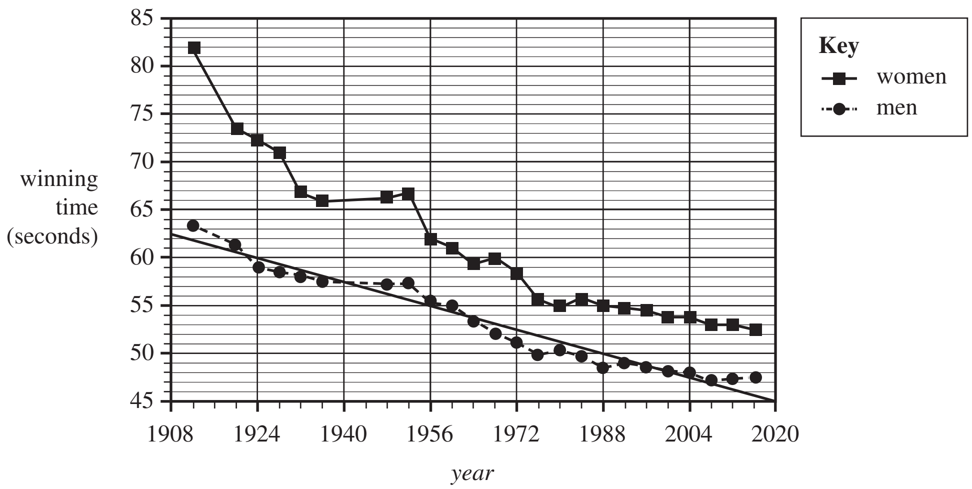
The equation of the least squares line for men is

winning time men $= 356.9 - 0.1544 \times$ *year*

The equation of the least squares line for women is

winning time women $= 538.9 - 0.2430 \times$ *year*

a. Draw the least squares line for *winning time women* on the **time series plot**.
(*Answer on the time series plot below*.) **(1 mark)**



b. The difference between the women's predicted winning time and the men's predicted winning time can be calculated using the formula

difference = *winning time women* – *winning time men*

Use the equations of the least squares lines and the formula above to calculate the *difference* predicted for the 2024 Olympic Games.
Round your answer to one decimal place. **(2 marks)**

c. The Olympic Games are held every four years. The next Olympic Games will be held in 2024, then 2028, 2032 and so on.
In which **Olympic year** do the two least squares lines predict that the winning time for women will first be faster than the winning time for men in the 100 m freestyle? **(2 marks)**

Source: *VCE 2019, Further Mathematics Exam 2, Section A, Q.6; © VCAA*

Question 2 (3 marks)

The total rainfall, in millimetres, for each of the four seasons in 2015 and 2016 is shown in the table below.

	Total rainfall (mm)			
Year	**Summer**	**Autumn**	**Winter**	**Spring**
2015	142	156	222	120
2016	135	153	216	96

a. The seasonal index for winter is shown in the next table below.
Use the values in the previous table to find the seasonal indices for summer, autumn and spring.
Write your answers in the table below, rounded to two decimal places.

	Summer	**Autumn**	**Winter**	**Spring**
Seasonal index			1.41	

(2 marks)

b. The total rainfall for each of the four seasons in 2017 is shown in the table below.

	Total rainfall (mm)			
Year	**Summer**	**Autumn**	**Winter**	**Spring**
2017	141	156	262	120

Use the appropriate seasonal index from the previous table to deseasonalise the total rainfall for winter in 2017.
Round your answer to the nearest whole number. **(1 mark)**

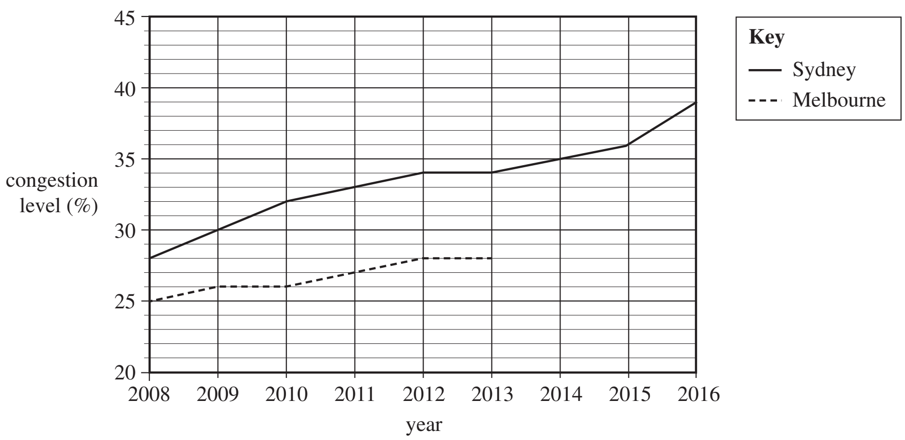
Source: *VCE 2018, Further Mathematics Exam 2, Section A, Q.3 (adapted); © VCAA*

Question 3 (9 marks)

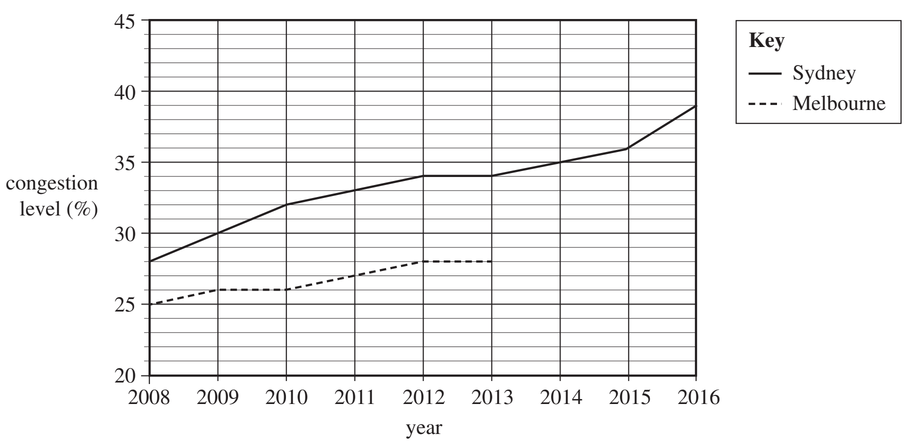
The table below shows the yearly average traffic congestion levels in two cities, Melbourne and Sydney, during the period 2008 to 2016. Also shown is a time series plot of the same data.

The time series plot for Melbourne is incomplete.

	Congestion level(%)								
Year	2008	2009	2010	2011	2012	2013	2014	2015	2016
Melbourne	25	26	26	27	28	28	28	29	33
Sydney	28	30	32	33	34	34	35	36	39



a. Use the data in the table above to complete the **time series plot** for Melbourne.

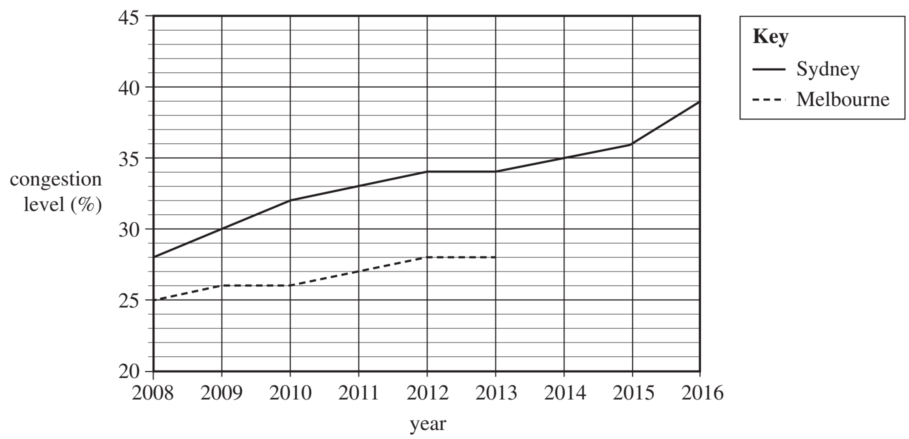


(1 mark)

b. A least squares line is used to model the trend in the time series plot for Sydney. The equation is

congestion level $= -2280 + 1.15 \times$ *year*

i. Draw this least squares line on the **time series plot**.



(1 mark)

ii. Use the equation of the least squares line to determine the average rate of increase in percentage congestion level for the period 2008 to 2016 in Sydney.

Write your answer in the box provided below.

☐ % per year **(1 mark)**

iii. Use the least squares line to predict when the percentage congestion level in Sydney will be 43%. **(1 mark)**

c. The yearly average traffic congestion level data for Melbourne is repeated in the following table.

	Congestion level(%)								
Year	2008	2009	2010	2011	2012	2013	2014	2015	2016
Melbourne	25	26	26	27	28	28	28	29	33

When a least squares line is used to model the trend in the data for Melbourne, the intercept of this line is approximately -1514.75556
Round this value to four significant figures. **(1 mark)**

d. Use the data in the table in part **c** to determine the equation of the least squares line that can be used to model the trend in the data for Melbourne. The variable *year* is the explanatory variable.
Write the values of the intercept and the slope of this least squares line in the appropriate spaces provided below.
Round both values to four significant figures.
congestion level $=$ ____________ $+$ ____________ $\times$ *year* **(2 marks)**

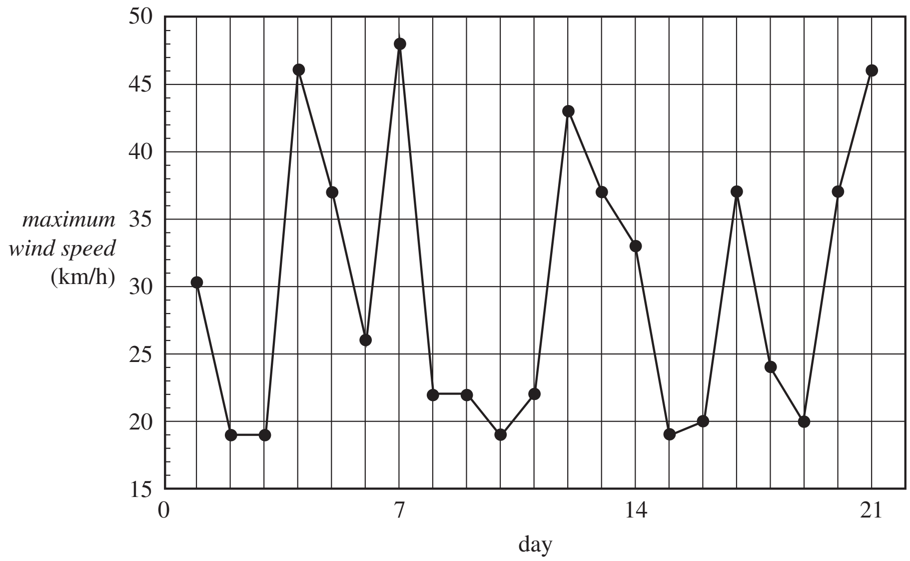
e. Since 2008, the equations of the least squares lines for Sydney and Melbourne have predicted that future traffic congestion levels in Sydney will always exceed future traffic congestion levels in Melbourne.
Explain why, quoting the values of appropriate statistics. **(2 marks)**

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.15; © VCAA*

Question 4 (1 mark)

The wind speed at a city location is measured throughout the day.

The time series plot below shows the daily *maximum wind speed*, in kilometres per hour, over a three-week period.



The table below shows the daily *maximum wind speed*, in kilometres per hour, for the days in week 2.

Day	8	9	10	11	12	13	14
***Maximum wind speed* (km/h)**	22	22	19	22	43	37	33

A four-point moving mean with centring is used to smooth the time series data above.

A. 22
B. 24
C. 26
D. 28
E. 30

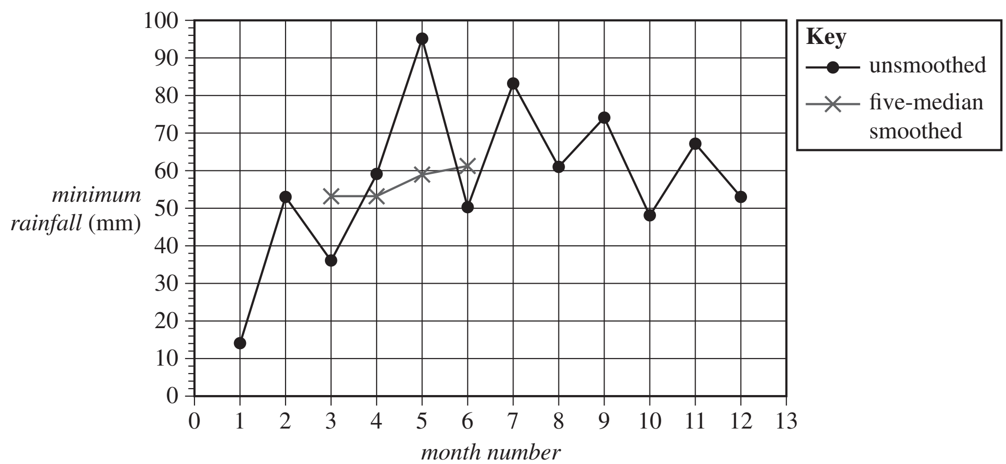
Source: *VCE 2016, Further Mathematics Exam 2, Q.4 (adapted); © VCAA*

Question 5 (4 marks)

The time series plot below shows the *minimum rainfall* recorded at the weather station each month plotted against the *month number* (1 = January, 2 = February, and so on).

Rainfall is recorded in millimetres.

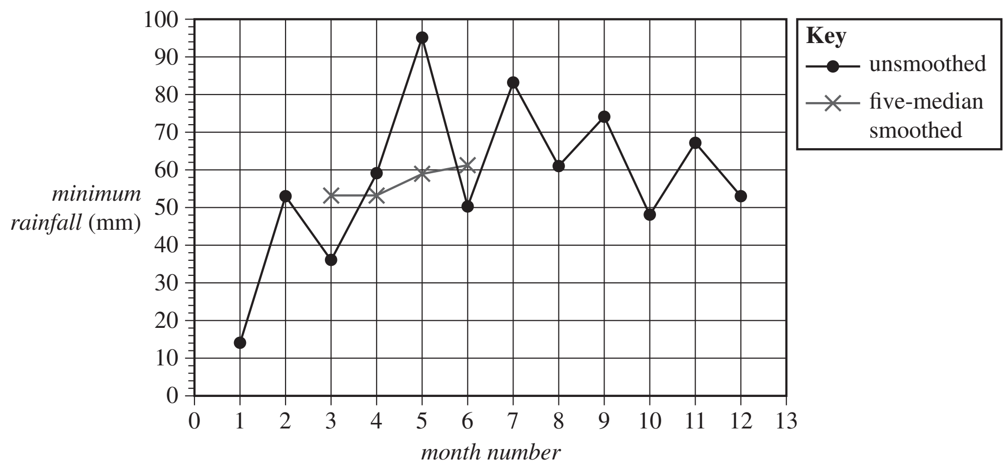
The data was collected over a period of one year.



a. Five-median smoothing has been used to smooth the time series plot above.

The first four smoothed points are shown as crosses (x).

Complete the five-median smoothing by marking smoothed values with crosses (x) on the **time series plot**.

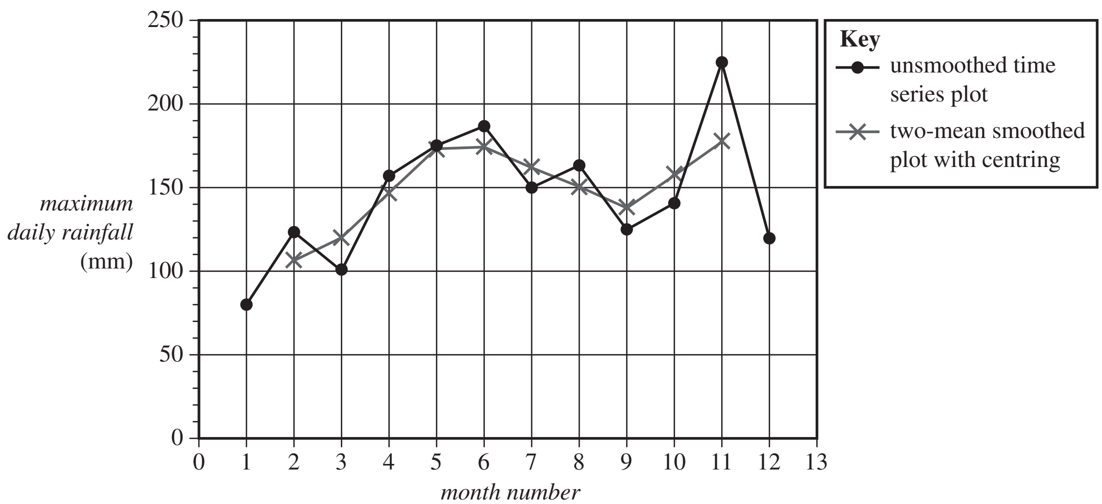


(2 marks)

The maximum daily rainfall each month was also recorded at the weather station.
The table below shows the maximum *daily rainfall* each month for a period of one year.

Month	**Jan.**	**Feb.**	**Mar.**	**Apr.**	**May**	**June**	**July**	**Aug.**	**Sep.**	**Oct.**	**Nov.**	**Dec.**
Month number	1	2	3	4	5	6	7	8	9	10	11	12
Maximum daily rainfall (mm)	79	123	100	156	174	186	149	162	124	140	225	119

The data in the table has been used to plot *maximum daily rainfall* against month number in the time series plot below.



b. Two-mean smoothing with centring has been used to smooth the time series plot above.
The smoothed values are marked with crosses (x).
Using the data given in the table, show that the two-mean smoothed rainfall centred on October is 157.25 mm. **(2 marks)**

Source: *VCE 2021, Further Mathematics Exam 2, Section A, Q.5; © VCAA*

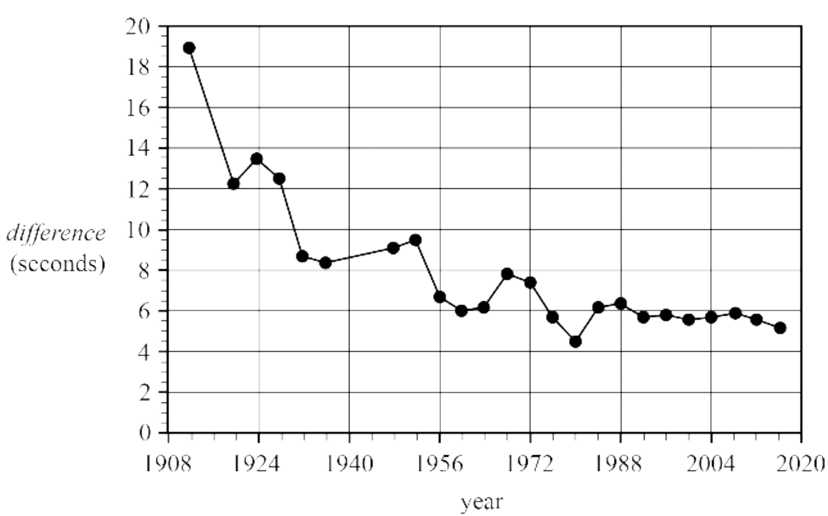
Question 6 (2 marks)

A method for predicting future time differences in the 100 m freestyle swim is to use the formula

difference = winning time women – winning time men

The resulting data and time series plot are shown below. The plot is clearly non-linear.

Year	Difference (seconds)
1912	18.8
1920	12.2
1924	13.4
1928	12.4
1932	8.6
1936	8.3
1948	9.0
1952	9.4
1956	6.6
1960	6.0
1964	6.1
1968	7.8
1972	7.4
1976	5.7
1980	4.4
1984	6.1
1988	6.3
1922	5.6
1996	5.8
2000	5.5
2004	5.7
2008	5.9
2012	5.5
2016	5.1



Note: No Olympic Games were held in 1916 , 1940 and 1944.

a. Apply a reciprocal transformation to the variable *difference* to linearise the data. Fit a least squares line to the transformed data and write its equation below.
Round the values of the intercept and the slope to four significant figures. **(1 mark)**

b. Use the equation from part **a** to predict, in seconds, the *difference* between the women's and men's winning times in the year 2032.
Round your answer to one decimal place. **(1 mark)**

Source: *VCE 2021, Further Mathematics Exam 2, Section A, Q.2; © VCAA*

Question 7 (3 marks)

The two running events in the heptathlon are the 200 m run and the 800 m run. The times taken by the athletes in these two events, *time*200 and *time*800, are linearly related.
When a least squares line is fitted to the data, the equation of this line is found to be
$time800 = 0.03931 + 5.2756 \times time200$

a. Round the values for the intercept and the slope to three significant figures. Write your answers in the boxes provided.
$time800 =$ ___________ $+$ ___________ $\times\ time200$ **(1 mark)**

b. The mean and the standard deviation for each variable, *time*200 and *time*800, are shown in the table below.

Statistic	*Time*200(seconds)	*Timee*800(seconds)
mean	24.6492	136.054
standard deviation	0.96956	8.2910

The equation of the least squares line is
$time800 = 0.03931 + 5.2756 \times time200$
Use this information to calculate the coefficient of determination as a percentage.
Round your answer to the nearest percentage. **(2 marks)**

Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.14; © VCAA*

Question 8 (1 mark)

The table below shows the long-term average of the number of meals served each day at a restaurant. Also shown is the daily seasonal index for Monday through to Friday.

	Day of the week						
	Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**	**Saturday**	**Sunday**
Long-term average	89	93	110	132	145	190	160
Seasonal index	0.68	0.71	0.84	1.01	1.10		

The seasonal index for Wednesday is 0.84

This tells us that, on average, the number of meals served on a Wednesday is

A. 16% less than the daily average.
B. 84% less than the daily average.
C. the same as the daily average.
D. 16% more than the daily average.
E. 84% more than the daily average.

Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.16; © VCAA*

Question 9 (1 mark)

The table below shows the long-term average of the number of meals served each day at a restaurant. Also shown is the daily seasonal index for Monday through to Friday.

	Day of the week						
	Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**	**Saturday**	**Sunday**
Long-term average	89	93	110	132	145	190	160
Seasonal index	0.68	0.71	0.84	1.01	1.10		

The seasonal index for Saturday is closest to

A. 1.22
B. 1.31
C. 1.38
D. 1.45
E. 1.49

Source: *VCE 2015, Further Mathematics Exam 1, Section A, Q.13; © VCAA*

Question 10 (1 mark)

The quarterly seasonal indices for tractor sales for a supplier are displayed in Table 1.

Table 1

Quarter number	1	2	3	4
Seasonal index	1.6	0.6	0.7	1.1

The quarterly tractor sales in 2014 for this supplier are displayed in Table 2.

Table 2

Quarter number	1	2	3	4
Sales(tractors sold)	2800	1032	875	759

The sales data in Table 2 is to be deseasonalised before a least squares regression line is fitted.

The equation of this least squares regression line is closest to

A. $\textit{deseasonalised sales} = 0.32 + 910 \times \textit{quarter number}$

B. $\textit{deseasonalised sales} = 370 - 2300 \times \textit{quarter number}$

C. $\textit{deseasonalised sales} = 910 + 0.32 \times \textit{quarter number}$

D. $\textit{deseasonalised sales} = 2300 - 370 \times \textit{quarter number}$

E. $\textit{deseasonalised sales} = 2300 - 0.32 \times \textit{quarter number}$

Source: *VCE 2014, Further Mathematics Exam 1, Section A, Q.10; © VCAA*

Question 11 (1 mark)

The seasonal indices for the first 11 months of the year, for sales in a sporting equipment store, are shown in the table below.

Month	Jan.	Feb.	Mar	Apr.	May.	June.	July.	Aug.	Sep.	Oct.	Nov.	Dec.
Seasonal index	1.23	0.96	1.12	1.08	0.89	0.98	0.86	0.76	0.76	0.95	1.12	

The seasonal index for December is

A. 0.89

B. 0.97

C. 1.02

D. 1.23

E. 1.29

Source: *VCE 2014, Further Mathematics Exam 1, Section A, Q.11; © VCAA*

Question 12 (1 mark)

The seasonal indices for the first 11 months of the year, for sales in a sporting equipment store, are shown in the table below.

Month	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sep.	Oct.	Nov.	Dec.
Seasonal index	1.23	0.96	1.12	1.08	0.89	0.98	0.86	0.76	0.76	0.95	1.12	

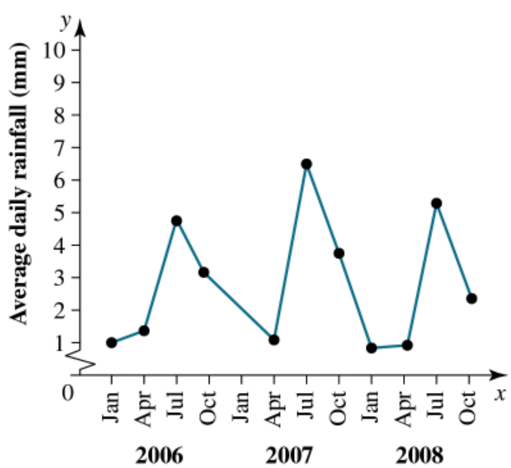
In May, the store sold \$213 956 worth of sporting equipment.

The deseasonalised value of these sales was closest to

A. \$165 857
B. \$190 420
C. \$209 677
D. \$218 322
E. \$240 400

Question 13 (1 mark)

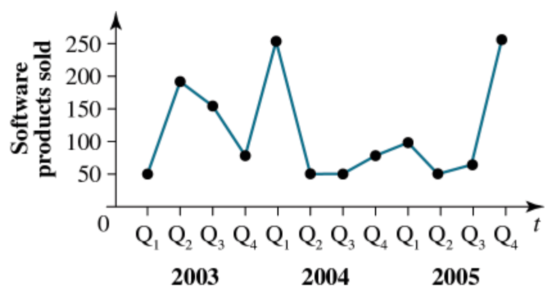
Which type of trend does the graph shown below have?



A. long term
B. seasonal
C. cyclic
D. irregular
E. negative

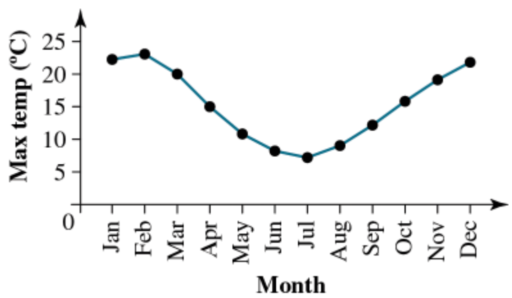
Question 14 (1 mark)
Which of the following statements about seasonal trends is false?
A. A seasonal trend relates only to the seasons of the year; that is, spring, summer, autumn and winter.
B. A seasonal trend is best viewed over a long period of time.
C. A seasonal trend can show peaks and troughs.
D. A seasonal trend relates to time series data.
E. A seasonal trend is different from a secular trend.

Question 15 (1 mark)
Which type of trend is apparent in the graph shown below?



A. long term
B. seasonal
C. cyclic
D. irregular
E. positive

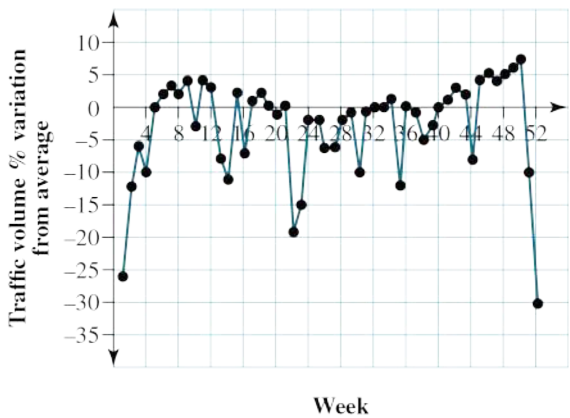
Question 16 (1 mark)
The provided time series graph shows the fluctuating nature of the median monthly maximum temperature in a regional township measured across one year.



If this graph is assumed to display a seasonal trend, the absolute lowest maximum temperature in 5 years from now should occur in the month of

A. June.
B. July.
C. August.
D. September.
E. October.

Question 17 (1 mark)
The time series plot below shows the % variation in traffic volumes (compared to the average weekly traffic volume) over the course of a year.



If this time series plot shows seasonal trends, the % traffic volume that would be predicted for week 22 of the following year would be

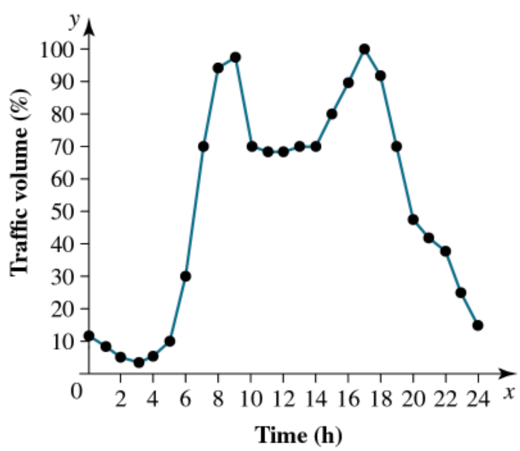
A. 19%
B. 0%
C. −10%
D. −19%
E. −25%

Question 18 (1 mark)
Which of the following statements about a cyclic trend is false?

A. A cyclic trend is one that steadily increases or decreases over time with no major changes in direction.
B. A cyclic trend is best viewed over a long period of time.
C. A cyclic trend shows peaks and troughs.
D. A cyclic trend relates to time series data.
E. A cyclic trend is different from a seasonal trend.

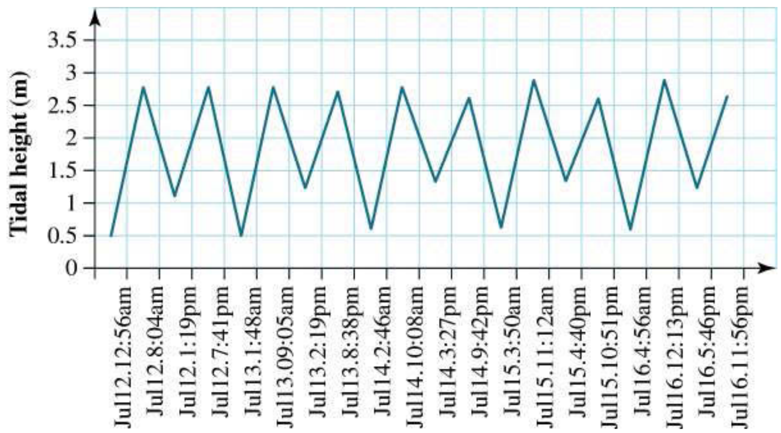
Question 19 (1 mark)
Which of the following trends is most likely to be cyclic?
A. The amount of monthly rainfall in Victoria.
B. The number of personnel in the Australian navy, measured annually.
C. The share price of BHP, measured monthly.
D. The number of people living in Australia, measured annually.
E. The number of seats held by the Australian Labor Party in the Victorian Parliament.

Question 20 (1 mark)
The time series plot below shows the cyclic fluctuations of traffic on a busy freeway during the day.



A least squares regression line fitted to this scatterplot indicates
A. a fluctuating trend.
B. a decreasing trend.
C. an increasing trend.
D. no overall trend.
E. a cyclic trend.

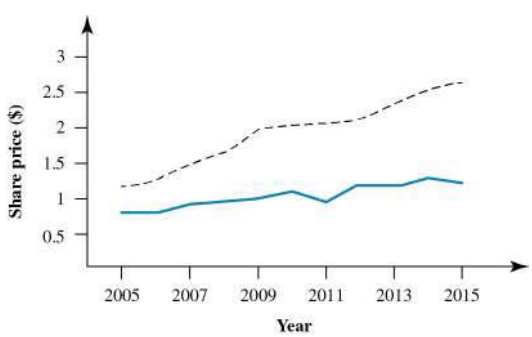
Question 21 (1 mark)
The cyclic image below is to be smoothed using a 3-moving median technique.



The smoothed tidal height value that will be found for Jul 13 at 2:19 pm is

A. 0.5
B. 1.1
C. 1.2
D. 2.7
E. 2.8

Question 22 (1 mark)
Which type of trend is apparent in the graph shown below?



A. long term
B. seasonal
C. cyclic
D. irregular
E. positive

Question 23 (1 mark)

Which of the following statements about trends is false?

A. There is no obvious way to predict the direction of an irregular trend as the direction can change unexpectedly.

B. An irregular trend is best viewed over a long period of time.

C. An irregular trend has no obvious pattern.

D. A cyclic trend relates to time series data.

E. An irregular trend is impossible to detect.

Question 24 (1 mark)

Which of the following trends is most likely to be irregular?

A. The amount of monthly rainfall in Victoria.

B. The number of personnel in the Australian navy, measured annually.

C. The share price of BHP, measured monthly.

D. The number of people living in Australia, measured annually.

E. The number of seats held by the Australian Labor Party in the Victorian Parliament.

Question 25 (1 mark)

The provided data gives monthly information about the internet download traffic for a particular company.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
Internet	195	201	221	198	180	223	207	199	185	197	230	190

The times series graph shows:

A. irregular fluctuations throughout and no overall trend

B. irregular fluctuations throughout and an increasing trend

C. irregular fluctuations throughout and a decreasing trend

D. seasonal fluctuations

E. a long term decreasing trend

Question 26 (1 mark)
The seasonal index is 0.85 and the actual sales figure for a particular month is $102 000.
What is the deseasonalised figure?
A. $96 900
B. $86 700
C. $107 368.42
D. $102 000
E. $120 000

Question 27 (1 mark)
The deseasonalised sales figure for a particular month is $7 809 and the seasonal index is 0.82.
What is the actual sales figure?
A. $9 523.17
B. $6 403.38
C. $7 809.82
D. $10 000
E. $6 403

Question 28 (1 mark)
The seasonal indices for the Quarters 1,2 and 4 are shown in the table below.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Seasonal index	1.07	0.86		1.12

What is the seasonal index for Quarter 3?
A. 1.02
B. 0.95
C. 1.01
D. 0.97
E. 1.00

Answers and marking guide

4.2 Time series plots and trends

Question 1
There is evidence of a decreasing trend and irregular fluctuations.
However, there is no evidence of a seasonality, as the peaks and troughs do not occur at regular intervals.

Question 2
Calculate the median of the vertical axis ('passengers'). There are 12 data values, so the median will be in the 6th spot counting from the bottom (or the top).

Question 3
Temperature is seasonal, which we can see from the time series plot. However, there are irregular fluctuations in the data.

Question 4
The low points and the high points are not evenly spaced, so seasonality is not a feature of the time series. Overall there is no decreasing nor increasing trend, so the plot can be described as having irregular fluctuations only.
***VCAA Examination Report* note:**
This question presented students with a time series graph and required them to identify the correct description. Around half of the students answered this question correctly; however, many others incorrectly identified the time series graph as showing seasonality. The existence of 'peaks' and 'troughs' is not enough information to determine seasonality; they must exist with regular intervals of time between them. Students needed to observe that the peaks did not occur with this regularity and thus reject the existence of seasonality.

Question 5
The seasonal pattern repeats every year with the highest values occurring in summer and the lowest in winter. There is no increasing or decreasing trend. Therefore, the time series plot shows seasonality only.

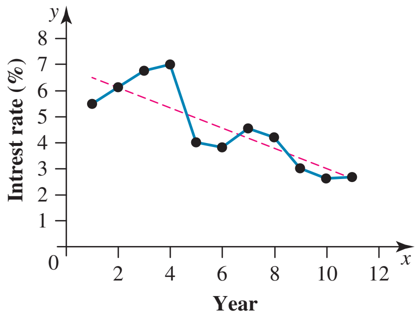
Question 6
The difference between the two stocks is gradually increasing over time. Therefore, the stock prices are showing an increasing trend over the time period.

Question 7
long term

Question 8
'A long term trend shows peaks and troughs.' is a false statement. Therefore, this is the correct answer to the question.

Question 9
The number of people living in Australia, measured annually.

Question 10
The time series graph and trend line show:

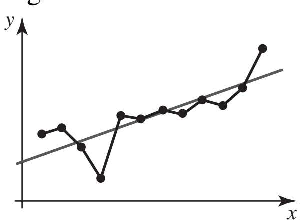


The regression equation for this data is: Interest rate = 7.12 − 0.42 × Year
For the year 2020, year 16, Interest rate = 7.12 − 0.42 × 16 = 0.4%

4.3 Fitting the least squares line and forecasting

Question 1
On your CAS, enter the data and find the LSR equation: *time* = 44 − *day*
When *day* = 10, *time* = 44 − 10 = 34, which is the same as day 4.

Question 2
On your CAS calculator, enter the data, generate the scatterplot and add the LSR line. 6 values lie below the regression line.



Question 3
Enter the original data into the CAS calculator and perform a least-squares regression.
$y = 28.5 + 1.767x$

Question 4
The gradient is approximately 1.1 and the y-intercept is approximately 3.2.
So, $n = 1.1t + 3.2$

Question 5

$$\begin{aligned}\textit{Wine sales} &= 500 + 500 \times \textit{number of years}\\ &= 500 + 500 \times 6\\ &= 3\,500(1000s\,\text{L})\\ &= 3\,500\,000\,\text{L}\end{aligned}$$

Question 6
The time code for Jan 16 is 73 (12 months per year, 6 years to the end of 2015, then the start of a new year, Jan 16)
The least squares prediction for Jan 16 is:
Share price = 2.22 + 0.06 × 73 = \$6.60

Question 7
The least-squares regression line for this data is: Average price = 44.09 + 0.048 × year.
In 2016 the average price is predicted to be: 44.09 + 0.048 × 2016 = 140.9 cents.

4.4 Smoothing using the moving mean with an odd number of points

Question 1

The five-mean smoothed value on Thursday is:

$$\frac{186 + x + y + z + 346}{5} = 206$$

$$x + y + z = 205 \times 5 - 186 - 346$$

The three-mean smoothed value on Thursday will be:

$$\frac{x + y + z}{3} = \frac{205 \times 5 - 186 - 346}{3}$$

$$= 166$$

Question 2

$$\frac{2\,160\,000}{5} = 432\,000$$

Question 3

For seven-mean smoothing, you would lose three data points at the beginning and another three at the end. Six data points would be lost in total, which means that six smoothed data points would be left.

Question 4

Quickly perform three and five-median smoothing.

Day	2	3	4	5	6	7	8
Three-median	28	40	28	51	33	37	37
Five-median		28	40	33	37	37	

Day number 7 is the same value.

VCAA Examination Report **note:**

Students needed to be able to apply graphical smoothing of a time series plot using moving medians involving an odd number of points.

Question 5

The seven values centred on day 4 are: 30, 19, 19, 46, 37, 26 and 48.

Order these to find the middle value: 19, 19, 26, 30, 37, 46 and 48

The middle value (which is centred on day 4) is 30 (km/h).

4.5 Smoothing using the moving mean with an even number of points

Question 1

Six-mean smoothed value to the left of August:

$$\frac{92.6 + 77.2 + 80.0 + 86.8 + 93.8 + 55.2}{6} = 80.9333...$$

Six-mean smoothed value to the right of August:

$$\frac{77.2 + 80.0 + 86.8 + 93.8 + 55.2 + 97.3}{6} = 81.71666...$$

Centered on August: $\dfrac{80.93333 + 81.716\,67}{2} = 81.325$

Question 2

For the 4-mean smoothing centring on May, the profits from March, April, May and June must be used in one calculation, and the profits from April, May, June and July must be used in the second calculation.

Left of May: $\dfrac{March + April + May + June}{4} = \dfrac{2402 + 2456 + 4651 + 3456}{4} = \$3241.25$

Right of May: $\dfrac{April + May + June + July}{4} = \dfrac{2456 + 4651 + 3456 + 2823}{4} = \$3346.5$

Centring on May: $\frac{Left\ of\ May + Right\ of\ May}{2} = \frac{3241.25 + 3346.5}{2} = \$3293.875 \approx \$3294$

Question 3

The first 4-point mean $= \frac{390 + 126 + 85 + 130}{4} = 182.75$. (This value corresponds to the time between autumn and winter 2007.)

The second 4-point mean $= \frac{126 + 85 + 130 + 460}{4} = 200.25$. (This value corresponds to the time between winter and spring 2007.)

The centred value for winter 2007 $= \frac{182.75 + 200.25}{2} = 191.5 \cong 192$.

Question 4

The first four values are 22, 19, 22 and 43. They will be centred between days 10 and 11.

Mean $= \frac{22 + 19 + 22 + 43}{4} = 26.5$

The second four values are 19, 22, 43 and 37. They will be centred between days 11 and 12.

Mean $= \frac{19 + 22 + 43 + 37}{4} = 30.25$

$$\begin{aligned}\text{Smoothed value for day } 11 &= \frac{26.5 + 30.25}{2} \\ &= 28.375\ (\text{km/h})\end{aligned}$$

This is closest to 28 km/h.

4.6 Median smoothing from a graph

Question 1

Centre the median on game 10. A nine-median smoothing would be 4 values either side of game 10.
The nine-median smoothed value will be the points scored at game number 9, which is 110.

Question 2

Quickly perform three and five-median smoothing.

Day	2	3	4	5	6	7	8
Three-median	28	40	28	51	33	37	37
Five-median		28	40	33	37	37	

Day number 7 is the same value.

***VCAA Examination Report* note:**

Students needed to be able to apply graphical smoothing of a time series plot using moving medians involving an odd number of points.

Question 3

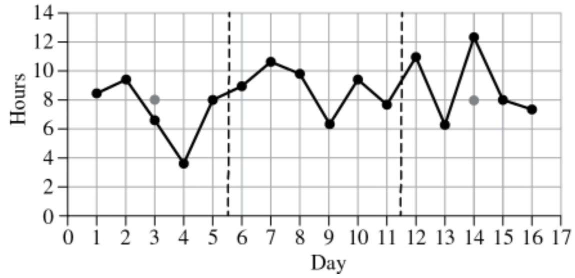
The seven values centred on day 4 are: 30, 19, 19, 46, 37, 26 and 48.
Order these to find the middle value: 19, 19, 26, 30, 37, 46 and 48
The middle value (which is centred on day 4) is 30 (km/h).

Question 4

To perform a 5-median smoothing of the data for year 6, we need to use the data for years 4, 5, 6, 7 and 8. This gives the number of calls as approximately 3800, 4000, 3700, 4450 and 4200 respectively. The median is therefore 4000.

Question 5

Split the 16 points into three groups $(5 - 6 - 5)$, then find the medians of the outer groups. The slope of the line is found by using the two outer medians.

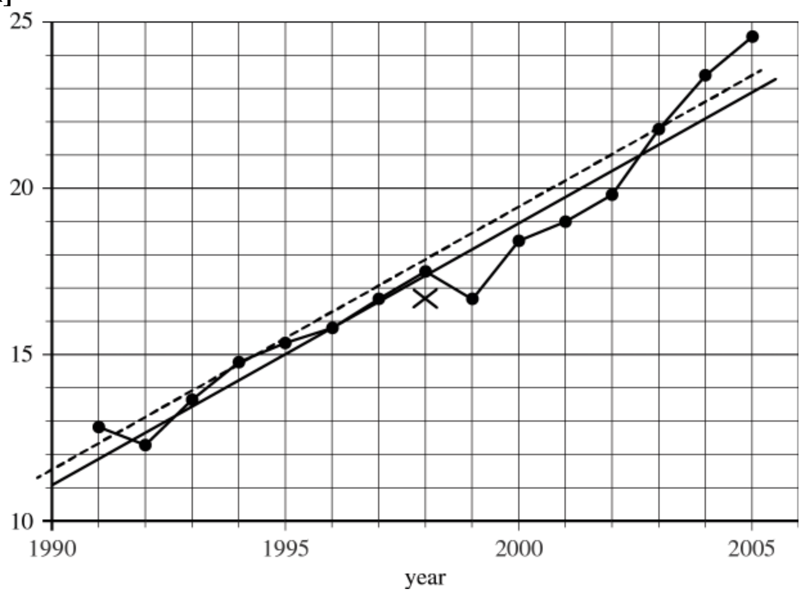


Median 1 = (3, 8) Median 2 = (14, 8)
As a horizontal line joins the two medians, the gradient is zero.

Question 6

a. i. There are 15 data points; therefore, divide the plot into 3 groups of 5 points. **[1 mark]**
For each group, locate the median (third highest) x- and y-values and mark them with a cross. **[1 mark]**
VCAA Assessment Report **note:**
Most students found the two correct outer points but few found the correct middle point. Commonly, the middle point was incorrectly located on the dot for 1999 and was in line with the two outer points at 1992 and 2003. A parallel movement of the line that connected the two outer points one third of the way towards the middle point could not then be demonstrated.

ii. Join the outer medians with a straight line and move it one-third of the way towards the middle median. **[1 mark]**



Question 7
Calculate the median for 2004, 2005 and 2006.
The median of 5, 9, and 4 is 5.

4.7 Seasonal adjustment

Question 1

$$\left(\frac{29685}{27194.0}+\frac{25420}{23183.5}+\frac{31496}{29243.0}\right)\div 3 = 1.088$$

Question 2
February 2020 is when $n = 2$
Deseasonalised number of visitors = 2349 − 198.5 × 2 = 1952

Actual number of visitors = deseasonalised value $\times SI = 1952 \times 1.25 = 2440$

Question 3

$$\frac{92.6}{1.222} = 75.777$$

Question 4

$$\text{deseasonalised value} = \frac{\text{actual}}{\text{SI}}$$

$$\text{deseasonalised value} = \frac{\text{actual}}{0.741}$$

deseasonalised value $= 1.349 \times$ actual

An increase of 35%

Question 5

$$\frac{1.072}{1.138} \times 81.1 = 76.4$$

Question 6

Find the median by listing the approximate values from the graph and selecting the middle value (between the 6th and 7th measurements):

54, 70, 73, 75, 82, 98, 110, 112, 120, 133, 137, 248

The median will fall between 98 and 110, therefore 103 is the best approximation.

Question 7

The seasonal index for Quarter 3 must be calculated for both 2016 and 2017, then averaged to find the seasonal index for Quarter 3 across both years.

Average sales in 2016: $\dfrac{1.73 + 2.87 + 3.34 + 1.23}{4} = 2.2925$

Seasonal index of Quarter 3 in 2016: $\dfrac{3.34}{2.2925} = 1.4569..$

Average sales in 2017: $\dfrac{1.03 + 2.45 + 2.05 + 0.78}{4} = 1.5775$

Seasonal index of Quarter 3 in 2017: $\dfrac{2.05}{1.5775} = 1.2995..$

Seasonal index of Quarter 3: $\dfrac{1.4569 + 1.2295}{2}$-1.38

Question 8

A seasonal index can be used to explore seasonal variation in a time series and does not take negative values. For the examples indicated by the other four options, it is possible for them to take negative values.

Question 9

$$\begin{aligned}
\text{Deseasonalised value} &= \frac{\text{actual value}}{\text{seasonal index}} \\
&= \frac{\text{actual value}}{1.6} \\
&= 0.625 \times \text{actual value} \qquad (1 \div 1.6 = 0.625) \\
&= 62.5\% \text{ of actual value}
\end{aligned}$$

The deseasonalised value for January will be 62.5% of the actual value. So, to correct for seasonality, the actual sales should be reduced by 37.5%(100% − 62.5%).

***VCAA Examination Report* note:**

This question required students to analyse the percentage change when adjusting a value for seasonality. While some students answered this question correctly, many had difficulty completing this analysis. A sales figure would be divided by the seasonal index of 1.6, which is the equivalent of multiplying by 0.625, since $\frac{1}{1.6} = 0.625$. The deseasonalised value will be 62.5% of the original value, which means the original value has been reduced by $100\% - 62.5\% = 37.5\%$

Question 10

$$\begin{aligned}\text{Deseasonalised value} &= \frac{\text{actual value}}{\text{season index}}\\ &= \frac{108}{0.71}\\ &= 152.11\\ &= 152\end{aligned}$$

Question 11

$$\begin{aligned}\text{Deseasonalised value} &= \frac{\text{sales}}{\text{index}}\\ &= \frac{\text{sales}}{1.25}\\ &= \text{sales} \div \frac{5}{4}\\ &= \text{sales} \times \frac{4}{5}\\ &= 80\% \times \text{sales}\end{aligned}$$

Multiplying sales by 80% is the same as reducing sales by 20%.

***VCAA Assessment Report* note:**

In this question, students were asked to determine the percentage by which an actual sales figure would change when the seasonal index is 1.25 . Few students answered this question correctly. The majority of students chose one of the two options with a 25% in the answer. This was not a question that could be answered by inspection and a calculation was required.

From the formula sheet:

$$\text{seasonal index} = \frac{\text{actual figure}}{\text{deseasonalised figure}}$$

Making the deseasonalised sales the subject of the formula:

$$\text{deseasonalised figure} = \frac{\text{actual figure}}{\text{seasonal index}}$$

Here, SI = 1.25, so

$$\text{deseasonalised figure} = \frac{\text{actual figure}}{1.25} = 0.80 \times \text{actual figure}$$

Thus, to obtain the deseasonalised sales for summer, the actual sales figure must be decreased by 20%.

Question 12

The sum of the seasonal indices = the sum of the seasons. The seasons here are the fortnights.

As there are 26 fortnights in a year, the total sum of all seasonal indices for this calculated year will be 26.

Question 13

A seasonal index of 0.83 indicates the sales are 17% below the yearly average.

Question 14

The sum of the seasonal indices is equal to the number of seasons. In this case that is 4.

$$\begin{aligned}S_{\text{Quarter 3}} &= 4 - (1.07 + 0.86 + 1.12)\\ &= 0.95\end{aligned}$$

Question 15

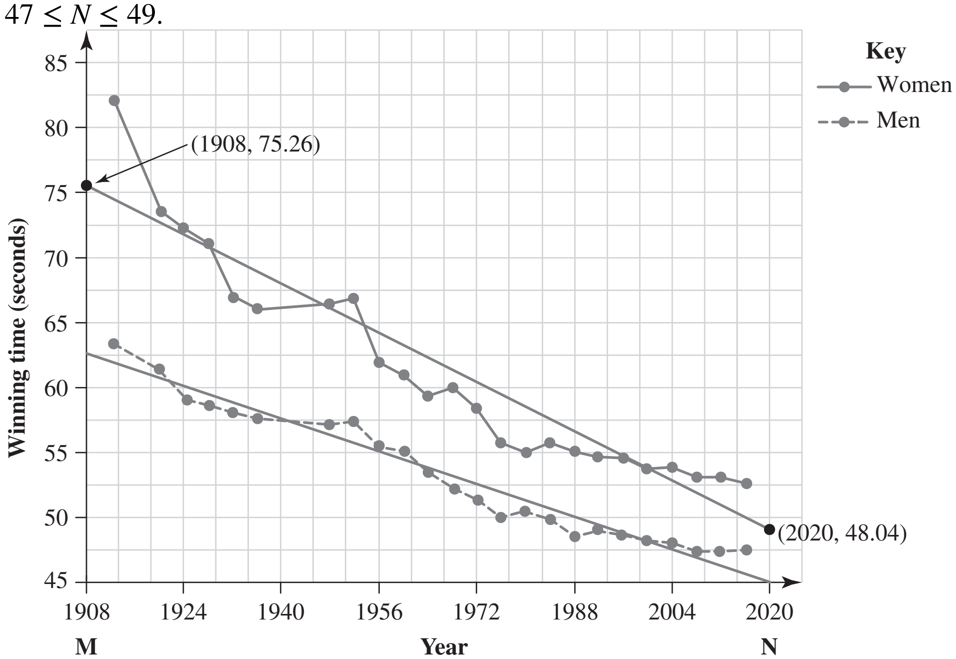
$$\text{Seasonal average} = \frac{28.5 + 22.6 + 18.3 + 24.2}{4} = 23.4$$

$$\text{Seasonal index (Autumn)} = \frac{22.6}{23.4} = 0.97$$

4.8 Review

Question 1

a. The end point at 1908 needs to be between $75 \leq M \leq 76$ and the endpoint at 2020 needs to be between $47 \leq N \leq 49$.



[Award **1 mark** for the line correctly drawn]

b. winning time men $= 356.9 - 0.1544 \times 2024 = 44.3944$
winning time women $= 538.9 - 0.2430 \times 2024 = 47.068$ [Award **1 mark** if one of these is correct]
difference $= 47.068 - 44.3944 \simeq 2.7$sec [Award **1 mark** — note that rounding applies here]

c. Solve on your CAS: winning time women < winning time men
$538.9 - 0.2430x < 356.9 - 0.1544x$
Therefore, year $= 2054.176$. [Award 1 mark for an answer rounding to 2054]
So the year in which the women's time will first be faster than the men's is 2056. **[1 mark]**

Question 2

a. First, calculate the yearly averages:

$$2015: \frac{142 + 156 + 222 + 120}{4} = 160$$

$$2016: \frac{135 + 153 + 216 + 96}{4} = 150$$

Divide each row by its yearly average:

Year	Summer	Autumn	Winter	Spring
2015	0.8875	0.975	1.3875	0.75
2016	0.9	1.02	1.44	0.64

Lastly, find the average of the season to get the seasonal index:

$$\text{Summer}: \frac{0.8875 + 0.9}{2} = 0.89375 = 0.89$$

Autumn: $\dfrac{0.975 + 1.02}{2} = 0.9975 = 1.00$

Spring: $\dfrac{0.75 + 0.64}{2} = 0.695 = 0.70$

Award 2 marks for all correct; 1 mark for 2 correct, 0 marks for 1 or none correct.

***VCAA Examination Report* note:**

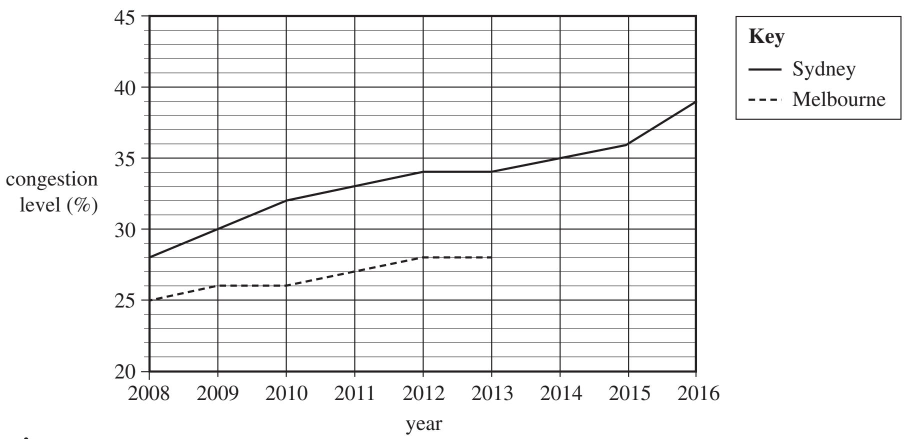
Student calculations were generally reasonable, but rounding was not. In particular, the calculated value for autumn of 0.9975 was often rounded to the nearest whole number rather than to two decimal places.

b. $\dfrac{262}{1.41} = 185.8156 = 186\,\text{mm}$ **[1 mark]**

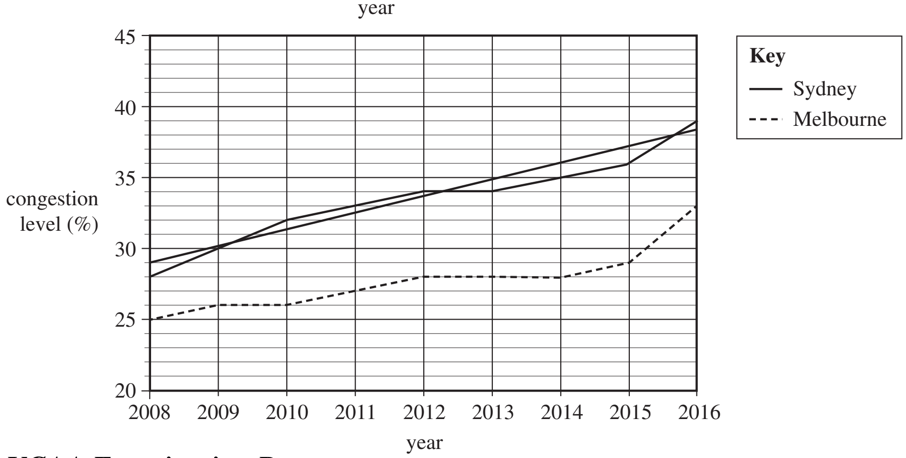
***VCAA Examination Report* note:**

This question was quite well done. Many students recognised that they could still answer this part even if unsuccessful in **part a.**

Question 3

a. 

[1 mark]

b. i. 

[1 mark]

***VCAA Examination Report* note:**

A significant number of students did not answer this question. Those who answered correctly typically used a ruler and often also wrote down the coordinates of the two endpoints as (2008, 29.2) and (2016, 38.4).

ii. The average rate of increase in percentage congestion level from 2008 to 2016 in Sydney is equal to the graph's gradient. This is 1.15% per year on average. **[1 mark]**

***VCAA Examination Report* note:**

This question was not well answered as many students did not realise that the slope of the least squares line was a measure of the average rate of increase. Some seemed to get confused with the R value used in financial mathematics and gave 15%.

iii. Substitute 43 for *congestion level* in the equation and solve for *year*. The predicted year that Sydney will have a 43% congestion level is 2020. **[1 mark]**

c. -1514.75556 rounded to 4 significant figures is -1515. **[1 mark]**

***VCAA Examination Report* note:**

This question was reasonably well answered; however, some gave -1514 or gave an answer rounded to four decimal places.

d. Using the CAS calculator to find the line of least squares, we get:

congestion level $= -1515 + 0.7667 \times$ *year* [**1 mark** for -1515, 1 mark for 0.7667]

***VCAA Examination Report* note:**

Some students did not recognise the link with **part c.** and wrote a different number in the first box. Some who correctly rounded in **part c**. used -1514.7556 in **part d.**

e. Sydney's traffic congestion in 2008 (28%) is greater than Melbourne's traffic congestion in 2008 (25%), and the gradient for Sydney's least squares line is 1.15, which is greater than Melbourne's least squares gradient of 0.7669. Therefore, Sydney's traffic congestion will increase faster than Melbourne's, meaning it will always exceed future traffic congestion levels in Melbourne. [**1 mark** for appropriate statistics, **1 mark** for proving why Sydney's congestion will be greater]

***VCAA Examination Report* note:**

Only a small proportion of students answered the question entirely correctly.

Many students focused on the starting values for Sydney and Melbourne from the table rather than the least squares lines. Some gave gradient figures for Sydney and Melbourne but did not clearly say that Sydney's slope was greater.

Again some students seemed to get confused with the R value used in financial mathematics and described Melbourne's gradient 0.7667 as a decrease of about 23%.

Question 4

The first four values are 22, 19, 22 and 43. They will be centred between days 10 and 11.

$$\text{Mean} = \frac{22 + 19 + 22 + 43}{4} = 26.5$$

The second four values are 19, 22, 43 and 37. They will be centred between days 11 and 12.

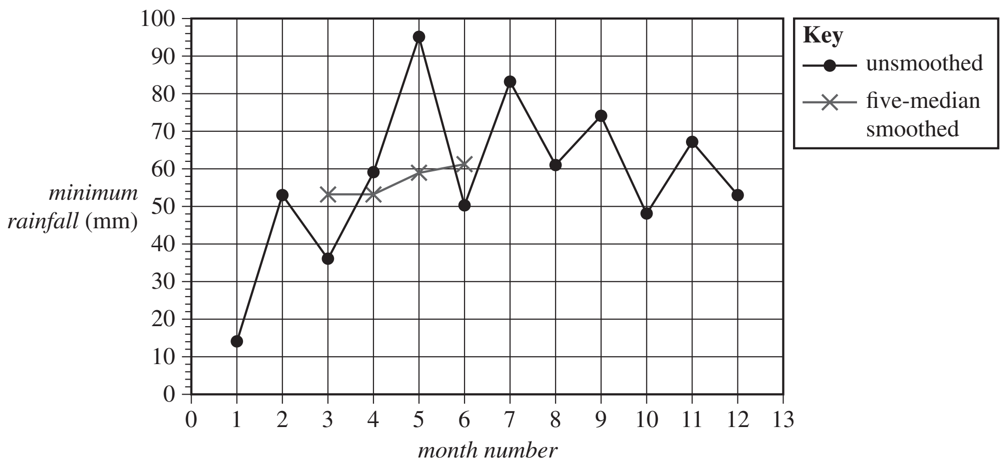
$$\text{Mean} = \frac{19 + 22 + 43 + 37}{4} = 30.25$$

$$\text{Smoothed value for day } 11 = \frac{26.5 + 30.25}{2}$$
$$= 28.375\ (\text{km/h})$$

This is closest to 28 km/h.

Question 5

a.



Award **1 mark** for the correct placement of the points and **1 mark** for joining them.

***VCAA Assessment Report* note:**

Median smoothing is a graphical technique and requires some accuracy in the correct placement of crosses or dots. Reading the values from points on the graph is unlikely to produce accurate enough placement of points and should be discouraged.

b. Mean of September and October $= \dfrac{124 + 140}{2} = 132$

Mean of October and November $= \dfrac{140 + 225}{2} = 182.5$

Centering on October $= \dfrac{132 + 182.5}{2} = 157.25$

Award **1 mark** for the two means.

Award **1 mark** for the centering.

***VCAA Assessment Report* note:**

The data needed for these calculations should have been taken from the table rather than from the graph.

Question 6

a. $\dfrac{1}{difference} = -2.234 + 0.001209 \times year$

[Award **1 mark** for two correct numbers rounded to 4 significant figures]

[Award **1 mark** for full correct equation, including the transformation]

b. $\dfrac{1}{difference} = -2.234 + 0.001209 \times 2032$

$difference = 4.4905877...$

$\simeq 4.5$ seconds

[**1 mark** — rounding to 1 decimal place applies]

Question 7

a. $time800 = 0.0393 + 5.28 \times time200$ [**1 mark** — note the rounding applies here]

b. If $b = r\dfrac{s_y}{s_x}$, $r = \dfrac{b \times s_x}{s_y} = \dfrac{5.2756 \times 0.96956}{8.2910} = 0.6169...$ then **[1 mark]**

So, $r^2 = (0.6169...)^2 = 0.3806...$

As a percentage: $r^2 = 0.3806... \times 100\% = 38\%$ [**1 mark** — note that rounding applies here]

Question 8

If the daily average is 100%, then a seasonal index of 0.84 is 16% *below* the daily average.

Question 9

$$\frac{89 + 93 + 110 + 132 + 145 + 190 + 160}{7} = \frac{919}{7}$$

$$\begin{aligned} \text{SI for Saturday} &= \frac{190}{919/7} \\ &= 1.4472 \\ &= 1.45 \end{aligned}$$

Question 10

Deseasonalise Table 2 using the formula $deseasonalise = \dfrac{actual}{SI}$

Quarter number	1	2	3	4
Deseasonalised sales	1750	1720	1250	690

Perform a least squares regression on CAS to give: $deseasonalised\ sales = 2265 - 365 \times quarter\ number$, which is closest to D.

Question 11

Sum of the indices = 12(12 months)

$$\begin{aligned}\text{December index} &= 12 - (1.23 + 0.96 + 1.12 + 1.08 + 0.89 + 0.98 + 0.86 + 0.76 + 0.76 + 0.95 + 1.12)\\ &= 1.29\end{aligned}$$

Question 12

$$\begin{aligned}\text{Deseasonalised value} &= \frac{213956}{0.89}\\ &= \$240\,400\end{aligned}$$

Question 13

The pattern repeats itself at roughly the same time (season) each year and so the data is seasonal.

Question 14

A seasonal trend does not relate only to the seasons of the year.

Question 15

There is no consistent pattern linked to the same quarter in respective years of each set of four quarters over one year.

Question 16

The time series shows a seasonal trend and the absolute lowest maximum temperature should occur in July, just as it does in this graph.

Question 17

As the time series plot is showing seasonal trends, a similar % variation to the current week 22 value would be expected the following year. The closest % to the current value is −19%. Option D

Question 18

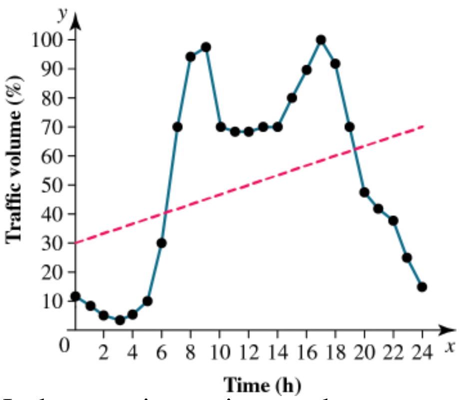
A cyclic trend is one that steadily increases or decreases over time with no major changes in direction.

Question 19

The number of seats held by the Australian Labor Party in the Victorian Parliament.

Question 20

The trend line fitted to the data is:



It shows an increasing trend.

Question 21

The three values needed to calculate the smoothed value for Jul 13 at 2:19 pm are 2.8, 1.2 and 2.7.
The median value is 2.7.

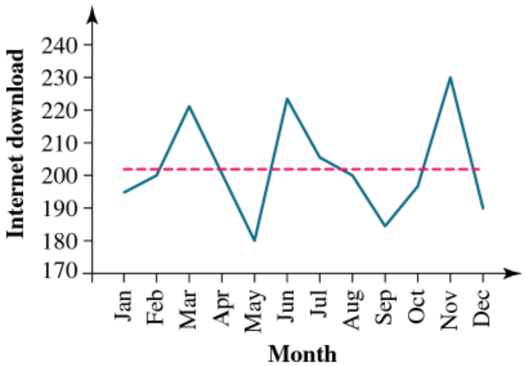
Question 22

irregular

Question 23
An irregular trend is impossible to detect.

Question 24
The share price of BHP, measured monthly.

Question 25
The time series graph with fitted trend line is:



The graph shows irregular fluctuations and no overall trend.

Question 26

$$\text{Deseasonalised figure} = \frac{\text{actual figure}}{\text{seasonal index}}$$
$$= \frac{102\,000}{0.85} = \$120\,000$$

Question 27

$$\text{Deseasonalised figure} = \frac{\text{actual figure}}{\text{seasonal index}}$$
$$7\,809 = \frac{\text{actual figure}}{0.82}$$
$$\text{actual figure} = 7\,809 \times 0.82$$
$$\text{actual figure} = \$6\,403.38$$

Question 28
The sum of the seasonal indices is equal to the number of seasons. In this case that is 4.

$$S_{\text{Quarter 3}} = 4 - (1.07 + 0.86 + 1.12)$$
$$= 0.95$$

5 Modelling depreciation of assets using recursion

Topic	5	Modelling depreciation of assets using recursion
Subtopic	5.2	A first-order linear recurrence relation

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Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.17; © VCAA*

Question 1 (1 mark)

Consider the recurrence relation shown below.

$A_0 = 3, \quad A_{n+1} = 2A_n + 4$

The value of A_3 in the sequence generated by this recurrence relation is given by

A. $2 \times 3 + 4$
B. $2 \times 4 + 4$
C. $2 \times 10 + 4$
D. $2 \times 24 + 4$
E. $2 \times 52 + 4$

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.18; © VCAA*

Question 2 (1 mark)

The first five terms of a sequence are $2,\ 6,\ 22,\ 86,\ 342\ \ldots$

The recurrence relation that generates this sequence could be

A. $P_0 = 2, \quad P_n + 1 = P_n + 4$
B. $P_0 = 2, \quad P_n + 1 = 2P_n + 2$
C. $P_0 = 2, \quad P_n + 1 = 3P_n$
D. $P_0 = 2, \quad P_n + 1 = 4P_n - 2$
E. $P_0 = 2, \quad P_n + 1 = 5P_n - 4$

Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.17; © VCAA*

Question 3 (1 mark)

Consider the recurrence relation below.

$A_0 = 2, \; A_{n+1} = 3A_n + 1$

The first four terms of this recurrence relation are

A. 0, 2, 7, 22 ...
B. 1, 2, 7, 22 ...
C. 2, 5, 16, 49 ...
D. 2, 7, 18, 54 ...
E. 2, 7, 22, 67 ...

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 1, Q.4; © VCAA*

Question 4 (1 mark)

The amount added to a new savings account each month follows a geometric sequence.
In the first month, $64 was added to the account.
In the second month, $80 was added to the account.
In the third month, $100 was added to the account.
Assuming this sequence continues, the total amount that will have been added to this savings account after five months is closest to

A. $155
B. $195
C. $370
D. $400
E. $525

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 1, Q.5; © VCAA*

Question 5 (1 mark)

A family bought a country property.
At the end of the first year, there were two thistles per hectare on the property.
At the end of the second year, there were six thistles per hectare on the property.
At the end of the third year, there were 18 thistles per hectare on the property.
Assume the number of thistles per hectare continues to follow a geometric pattern of growth.
At the end of the seventh year, the number of thistles per hectare is expected to be

A. 972
B. 1458
C. 2916
D. 4374
E. 8748

Source: *VCE 2014, Further Mathematics Exam 1, Section B, Module 1, Q.4; © VCAA*

Question 6 (1 mark)

On day 1, Vikki spends 90 minutes on a training program.

On each following day, she spends 10 minutes less on the training program than she did the day before.

Let t_n be the number of minutes that Vikki spends on the training program on day n.

A difference equation that can be used to model this situation for $1 \leq n \leq 10$ is

A. $t_{n+1} = 0.90t_n \quad t_1 = 90$

B. $t_{n+1} = 1.10t_n \quad t_1 = 90$

C. $t_{n+1} = t_n - 0.10 \quad t_1 = 90$

D. $t_{n+1} = 1 - 10t_n \quad t_1 = 90$

E. $t_{n+1} = t_n - 10 \quad t_1 = 90$

Question 7 (1 mark)

Given the recurrence relation: $t_{n+1} = 2t_n + 3$, $t_1 = 4$, the first five terms of this sequence are

A. 4, 7, 10, 13, 16

B. 4, 8, 16, 32, 64

C. 4, 9, 14, 19, 24

D. 4, 11, 25, 53, 109

E. 3, 8, 13, 18, 23

Question 8 (1 mark)

The first three terms of a series are $t_1 = 4$, $t_2 = 7$ and $t_3 = 10$.

What is the value of t_5?

A. 5

B. 10

C. 11

D. 16

E. 28

Question 9 (1 mark)
The first three terms of a series are $t_1 = 7,\ t_2 = 11$ and $t_3 = 15$.
What is the value of t_7?
A. 7
B. 26
C. 31
D. 33
E. 105

Question 10 (1 mark)
The first three terms of a series are $t_1 = 5,\ t_2 = 15$ and $t_3 = 45$.
What is the value of t_5
A. 60
B. 65
C. 135
D. 225
E. 405

Question 11 (1 mark)
The following recurrence relation can generate a sequence of numbers.
$T_0 = 10,\ T_{n+1} = T_n + 3$
The number 13 appears in this sequence as
A. T_1
B. T_2
C. T_3
D. T_{10}
E. T_{13}

Question 12 (1 mark)
Which one of the following sequences shows the first five terms of a first–order recurrence relation?
A. $1, 3, 6, 10, 15$
B. $1, 3, 8, 19, 42$
C. $-10, -5, 5, 10, 20$
D. $-4, -1, 2, 5, 8$
E. $1, 3, 8, 15, 24$

Question 13 (1 mark)
Which of the following equations **cannot** be a rule for a first order difference equation?
A. $t_{n+1} = 3n$
B. $t_{n+1} = 3 - t_n$
C. $t_{n+1} = 4t_n + 4$
D. $t_n = t_{n-1} + 10$
E. $t_{n+1} = 3t_n - 6$

Question 14 (1 mark)
What are the first five terms of the first order difference equation where $t_{n+1} = 3 + 2t_n$ and $t_1 = 5$
A. $5, 8, 11, 14, 17$
B. $5, 10, 20, 40, 80$
C. $5, 13, 29, 61, 125$
D. $5, 10, 15, 20, 25$
E. $5, 16, 38, 82, 170$

Question 15 (1 mark)
The value of a car decreases by 15% per year. The current value of the car is \$14 000. Where C_n is the value of the car in the n^{th} year.
Which of the following difference equation describes this?
A. $C_{n+1} = 0.15C_n$ where $C_1 = \$14\,000$
B. $C_{n+1} = 0.85C_n$ where $C_1 = \$14\,000$
C. $C_{n+1} = C_n + 15$ where $C_1 = \$14\,000$
D. $C_{n+1} = 0.85C_n + 15$ where $C_1 = \$14\,000$
E. $C_{n+1} = 0.85C_n - 15$ where $C_1 = \$14\,000$

Topic	**5**	**Modelling depreciation of assets using recursion**
Subtopic	**5.3**	**Modelling flat rate depreciation with a recurrence relation**

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Source: *VCE 2020, Further Mathematics Exam 2, Section A, Q.7; © VCAA*

Question 1 (4 marks)

Samuel owns a printing machine.

The printing machine is depreciated in value by Samuel using flat rate depreciation.

The value of the machine, in dollars, after n years, V_n, can be modelled by the recurrence relation

$V_0 = 120\,000, \qquad V_{n+1} = V_n - 15\,000$

a. By what amount, in dollars, does the value of the machine decrease each year? **(1 mark)**

b. Showing recursive calculations, determine the value of the machine, in dollars, after two years. **(1 mark)**

c. What annual flat rate percentage of depreciation is used by Samuel? **(1 mark)**

d. The value of the machine, in dollars, after n years, V_n, could also be determined using a rule of the form $V_n = a + bn$.

Write down this rule for V_n. **(1 mark)**

Topic 5 Subtopic 5.3 Modelling flat rate depreciation with a recurrence relation

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.19; © VCAA*

Question 2 (1 mark)

Geoff purchased a computer for \$4500. He will depreciate the value of his computer by a flat rate of 10% of the purchase price per annum.

A recurrence relation that Geoff can use to determine the value of the computer after years, V_n, is

A. $V_0 = 4500, \quad V_{n+1} = V_n - 450$

B. $V_0 = 4500, \quad V_{n+1} = V_n + 450$

C. $V_0 = 4500, \quad V_{n+1} = 0.9V_n$

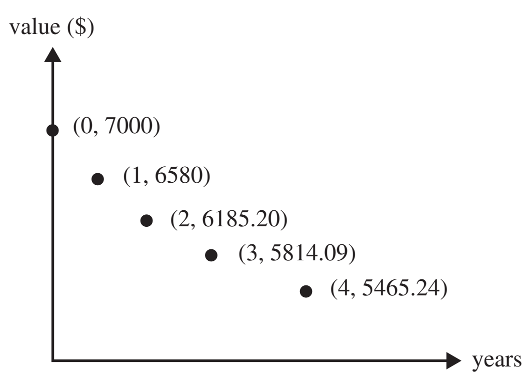
D. $V_0 = 4500, \quad V_{n+1} = 1.1V_n$

E. $V_0 = 4500, \quad V_{n+1} = 0.1\,(V_n - 450)$

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.22; © VCAA*

Question 3 (1 mark)

Consider the graph below.



This graph could show the value of

A. a piano depreciating at a flat rate of 6% per annum.

B. a car depreciating with a reducing balance rate of 6% per annum.

C. a compound interest investment earning interest at the rate of 6% per annum.

D. a perpetuity earning interest at the rate of 6% per annum.

E. an annuity investment with additional payments of 6% of the initial investment amount per annum.

Topic 5 Subtopic 5.3 Modelling flat rate depreciation with a recurrence relation

***Source:** VCE 2021, Further Mathematics Exam 1, Section A, Q.17; © VCAA*

Question 4 (1 mark)

The following recurrence relation can generate a sequence of numbers.

$L_0 = 37, \ L_{n+1} = L_n + C$

The value of L_2 is 25.

The value of C is

A. -6
B. -4
C. 4
D. 6
E. 37

***Source:** VCE 2020, Further Mathematics Exam 1, Section A, Q.22; © VCAA*

Question 5 (1 mark)

An asset is purchased for $2480.

The value of this asset after n time periods, V_n, can be determined using the rule

$V_n = 2480 + 45n$

A recurrence relation that also models the value of this asset after n time periods is

A. $V_0 = 2480, \quad V_{n+1} = V_n + 45n$
B. $V_n = 2480, \quad V_{n+1} = V_n + 45n$
C. $V_0 = 2480, \quad V_{n+1} = V_n + 45$
D. $V_1 = 2480, \quad V_{n+1} = V_n + 45$
E. $V_n = 2480, \quad V_{n+1} = V_n + 45$

***Source:** VCE 2016, Further Mathematics Exam 1, Section A, Q.20; © VCAA*

Question 6 (1 mark)

Consider the recurrence relation below.

$V_0 = 10\,000, \quad V_{n+1} = 1.04V_n + 500$

This recurrence relation could be used to model

A. a reducing balance depreciation of an asset initially valued at $10 000.
B. a reducing balance loan with periodic repayments of $500.
C. a perpetuity with periodic payments of $500 from the annuity.
D. an annuity investment with periodic additions of $500 made to the investment.
E. an interest-only loan of $10 000.

Topic	5	Modelling depreciation of assets using recursion
Subtopic	5.4	Modelling reducing balance depreciation with a recurrence relation

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***Source:** VCE 2020, Further Mathematics Exam 1, Section A, Q.23; © VCAA*

Question 1 (1 mark)

Consider the following four recurrence relations representing the value of an asset after n years, V_n.

- $V_0 = 20\,000,\ V_{n+1} = V_n + 2500$
- $V_0 = 20\,000,\ V_{n+1} = V_n - 2500$
- $V_0 = 20\,000,\ V_{n+1} = 0.875V_n$
- $V_0 = 20\,000,\ V_{n+1} = 1.125V_n - 2500$

How many of these recurrence relations indicate that the value of an asset is depreciating?

A. 0
B. 1
C. 2
D. 3
E. 4

***Source:** VCE 2017, Further Mathematics Exam 2, Section A, Q.5; © VCAA*

Question 2 (5 marks)

Alex is a mobile mechanic.

He uses a van to travel to his customers to repair their cars.

The value of Alex's van is depreciated using the flat rate method of depreciation.

The value of the van, in dollars, after n years, V_n, can be modelled by the recurrence relation shown below.

$V_0 = 75\,000, \quad V_{n+1} = V_n - 3375$

a. Recursion can be used to calculate the value of the van after two years.

Complete the calculations below by writing the appropriate numbers in the spaces provided.

$V_0 = 75\,000$

$V_1 = 75\,000 - \_\_\_\_\_\_ = 71\,625$

$V_2 = \_\_\_\_\_\_ - \_\_\_\_\_\_ = \_\_\_\_\_\_$

(2 marks)

b. i. By how many dollars is the value of the van depreciated each year? **(1 mark)**

ii. Calculate the annual flat rate of depreciation in the value of the van.
Write your answer as a percentage. **(1 mark)**

c. The value of Alex's van could also be depreciated using the reducing balance method of depreciation.The value of the van, in dollars, after $n$years, R_n, can be modelled by the recurrence relation shown below.

$R_0 = 75\,000, \quad R_{n+1} = 0.943R_n$

At what annual percentage rate is the value of the van depreciated each year? **(1 mark)**

***Source:** VCE 2016, Further Mathematics Exam 1, Section A, Q.19; © VCAA*

Question 3 (1 mark)

The purchase price of a car was \$26 000.
Using the reducing balance method, the value of the car is depreciated by 8% each year.
A recurrence relation that can be used to determine the value of the car after n years, C_n

A. $C_0 = 26\,000, \ C_{n+1} = 0.92C_n$
B. $C_0 = 26\,000, \ C_{n+1} = 1.08C_n$
C. $C_0 = 26\,000, \ C_{n+1} = C_n + 8$
D. $C_0 = 26\,000, \ C_{n+1} = C_n - 8$
E. $C_0 = 26\,000, \ C_{n+1} = 0.92C_n - 8$

***Source:** VCE 2016, Further Mathematics Exam 1, Section A, Q.18; © VCAA*

Question 4 (1 mark)

The value of an annuity, V_n, after n monthly payments of \$555 have been made, can be determined using the recurrence relation

$V_0 = 100\,000, \qquad V_n + 1 = 1.0025\,V_n - 555$

The value of the annuity after five payments have been made is closest to Responses

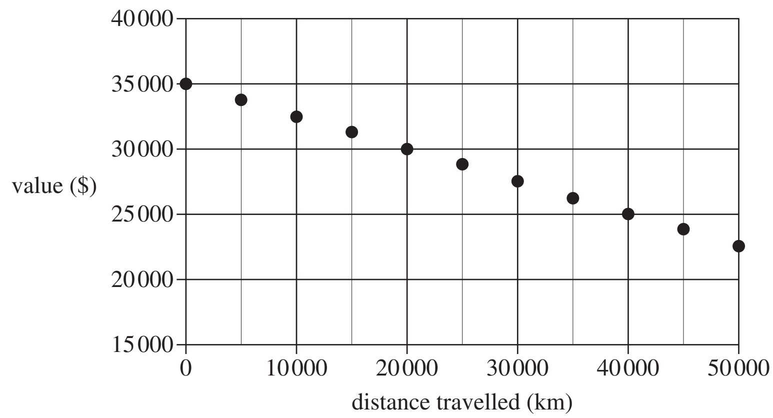
A. \$97 225
B. \$98 158
C. \$98 467
D. \$98 775
E. \$110 224

Topic 5 | Subtopic 5.4 Modelling reducing balance depreciation with a recurrence relation

***Source:** VCE 2015, Further Mathematics Exam 1, Section B, Module 4, Q.7; © VCAA*

Question 5 (1 mark)

The following graph shows the depreciating value of a van.



The graph could represent the van being depreciated using

A. flat rate depreciation with an initial value of $35 000 and a depreciation rate of $25 per year.

B. flat rate depreciation with an initial value of $35 000 and a depreciation rate of 25 cents per year.

C. reducing balance depreciation with an initial value of $35 000 and a depreciation rate of 2.5% per annum.

D. unit cost depreciation with an initial value of $35 000 and a depreciation rate of 25 cents per kilometre travelled.

E. unit cost depreciation with an initial value of $35 000 and a depreciation rate of $25 per kilometre travelled.

Question 6 (1 mark)

Office equipment for a new business was purchased for $18 000. For tax purposes, the owner applies a flat rate depreciation method to the equipment. If the quoted flat rate of interest to be used is 6.5%, use a recurrence relation to calculate and record the depreciated value at the end of the first three years of use.

n	V_n	Depreciation: $V_{n+1} = V_n - \left(18\,000 \times \frac{6.5}{100}\right)$
0	$V_0 = 18\ 000$	
1		
2		

Question 7 (1 mark)

A machine is purchased for $42 000. Its value depreciates at a rate of $0.18 for each unit it produces. On average, the machine produces 18 000 units a year. Using the unit cost method of depreciation, create a table showing the depreciated value at the end of the first five years of its use.

n	V_n	Depreciation: $V_{n+1} = V_n - (0.18 \times 18\,000)$
0		
1		
2		
3		
4		

Topic	**5**	**Modelling depreciation of assets using recursion**
Subtopic	**5.5**	**Modelling unit cost depreciation with a recurrence relation**

online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Source: *VCE 2021, Further Mathematics Exam 2, Section A, Q.7; © VCAA*

Question 1 (3 marks)

Sienna owns a coffee shop.

A coffee machine, purchased for \$12 000, is depreciated in value using the unit cost method.

The rate of depreciation is \$0.05 per cup of coffee made.

The recurrence relation that models the year-to-year value, in dollars, of the coffee machine is

$M_0 = 12\ 000, \quad M_{n+1} = M_n - 1440$

a. Calculate the number of cups of coffee that the machine produces per year. **(1 mark)**

b. The recurrence relation above could also represent the value of the coffee machine depreciating at a flat rate.

What annual flat rate percentage of depreciation is represented? **(1 mark)**

c. Complete the rule below that gives the value of the coffee machine, M_n, in dollars, after n cups have been produced. **(1 mark)**

$$M_n = \boxed{} + \boxed{} \times n$$

Topic 5 | Subtopic 5.5 Modelling unit cost depreciation with a recurrence relation

Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.29; © VCAA*

Question 2 (1 mark)

The value of a van purchased for \$45 000 is depreciated by k% per annum using the reducing balance method.

After three years of this depreciation, it is then depreciated in the fourth year under the unit cost method at the rate of 15 cents per kilometre.

The value of the van after it travels 30 000 km in this fourth year is \$26 166.24

The value of k is

A. 9
B. 12
C. 14
D. 16
E. 18

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.22; © VCAA*

Question 3 (1 mark)

A machine is purchased for \$30 000.

It produces 24 000 items each year.

The value of the machine is depreciated using a unit cost method of depreciation.

After three years, the value of the machine is \$18 480.

A rule for the value of the machine after n units are produced, V_n, is

A. $V_n = 0.872n$
B. $V_n = 24\ 000n - 3840$
C. $V_n = 30\ 000 - 24\ 000n$
D. $V_n = 30\ 000 - 0.872n$
E. $V_n = 30\ 000 - 0.16n$

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Q.21; © VCAA*

Question 4 (1 mark)

A printer was purchased for $680.

After four years the printer has a value of $125.

On average, 1920 pages were printed every year during those four years.

The value of the printer was depreciated using a unit cost method of depreciation.

The depreciation in the value of the printer, per page printed, is closest to

A. 3 cents.
B. 4 cents.
C. 5 cents.
D. 6 cents.
E. 7 cents.

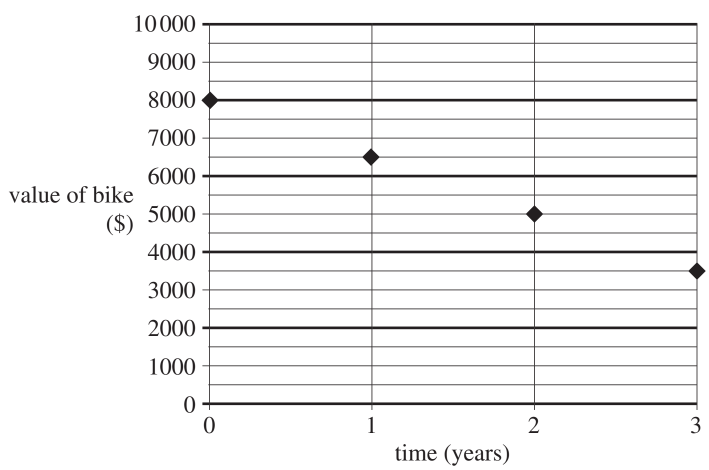
Source: *VCE 2013, Further Mathematics Exam 2, Module 4, Q.1; © VCAA*

Question 5 (4 marks)

Hugo is a professional bike rider.

The value of his bike will be depreciated over time using the flat rate method of depreciation.

The graph below shows his bike's initial purchase price and its value at the end of each year for a period of three years.



a. What was the initial purchase price of the bike? **(1 mark)**

b. i. Show that the bike depreciates in value by $1500 each year. **(1 mark)**

ii. Assume that the bike's value continues to depreciate by $1500 each year.
Determine its value five years after it was purchased. **(1 mark)**

c. The unit cost method of depreciation can also be used to depreciate the value of the bike.
In a two-year period, the total depreciation calculated at $0.25 per kilometre travelled will equal the depreciation calculated using the flat rate method of depreciation as described above.
Determine the number of kilometres the bike travels in the two-year period. **(1 mark)**

Topic	5	Modelling depreciation of assets using recursion
Subtopic	5.6	Review

online only

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Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.22; © VCAA*

Question 1 (1 mark)

An asset is purchased for \$2480.

The value of this asset after n time periods, V_n, can be determined using the rule

$V_n = 2480 + 45n$

A recurrence relation that also models the value of this asset after n time periods is

A. $V_0 = 2480, \quad V_{n+1} = V_n + 45n$
B. $V_n = 2480, \quad V_{n+1} = V_n + 45n$
C. $V_0 = 2480, \quad V_{n+1} = V_n + 45$
D. $V_1 = 2480, \quad V_{n+1} = V_n + 45$
E. $V_n = 2480, \quad V_{n+1} = V_n + 45$

Source: *VCE 2019, Further Mathematics Exam 2, Section A, Q.7; © VCAA*

Question 2 (4 marks)

Phil is a builder who has purchased a large set of tools.

The value of Phil's tools is depreciated using the reducing balance method.

The value of the tools, in dollars, after n years, V_n, can be modelled by the recurrence relation shown below.

$V_0 = 60\,000, \quad V_{n+1} = 0.9V_n$

a. Use recursion to show that the value of the tools after two years, V_2, is \$48 600. **(1 mark)**

b. What is the annual percentage rate of depreciation used by Phil? **(1 mark)**

c. Phil plans to replace these tools when their value first falls below \$20 000.
After how many years will Phil replace these tools? **(1 mark)**

d. Phil has another option for depreciation. He depreciates the value of the tools by a flat rate of 8% of the purchase price per annum.
Let V_n be the value of the tools after n years, in dollars.
Write down a recurrence relation, in terms of V_0, V_{n+1} and V_n, that could be used to model the value of the tools using this flat rate depreciation. **(1 mark)**

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Core, Q.21; © VCAA*

Question 3 (1 mark)

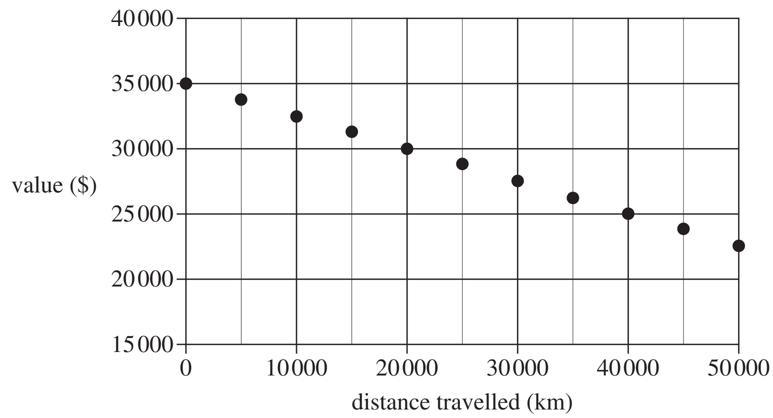
A printer was purchased for $680.
After four years the printer has a value of $125.
On average, 1920 pages were printed every year during those four years.
The value of the printer was depreciated using a unit cost method of depreciation.
The depreciation in the value of the printer, per page printed, is closest to

A. 3 cents.
B. 4 cents.
C. 5 cents.
D. 6 cents.
E. 7 cents.

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 4, Q.7; © VCAA*

Question 4 (1 mark)

The following graph shows the depreciating value of a van.



The graph could represent the van being depreciated using

A. flat rate depreciation with an initial value of \$35 000 and a depreciation rate of \$25 per year.
B. flat rate depreciation with an initial value of \$35 000 and a depreciation rate of 25 cents per year.
C. reducing balance depreciation with an initial value of \$35 000 and a depreciation rate of 2.5% per annum.
D. unit cost depreciation with an initial value of \$35 000 and a depreciation rate of 25 cents per kilometre travelled.
E. unit cost depreciation with an initial value of \$35 000 and a depreciation rate of \$25 per kilometre travelled.

Source: *VCE 2014, Further Mathematics Exam 1, Section B, Module 4, Q.7; © VCAA*

Question 5 (1 mark)

New furniture was purchased for an office at a cost of \$18 000.

Using flat rate depreciation, the furniture will be valued at \$5000 after four years.

The expression that can be used to determine the value of the furniture, in dollars, after one year is

A. $18\,000 - (4 \times 5000)$

B. $18\,000 - \left(\dfrac{18\,000 - 5000}{4}\right)$

C. $18\,000 - \dfrac{5000}{4}$

D. $\dfrac{18\,000}{4} - 5000$

E. $18\,000 \times 0.726$

Source: *VCE 2021, Further Mathematics Exam 1, Section A, Core, Q.20; © VCAA*

Question 6 (1 mark)

Sammy purchased a boat for \$72 000.

The value of the boat is depreciated each year by 10% using the reducing balance method.

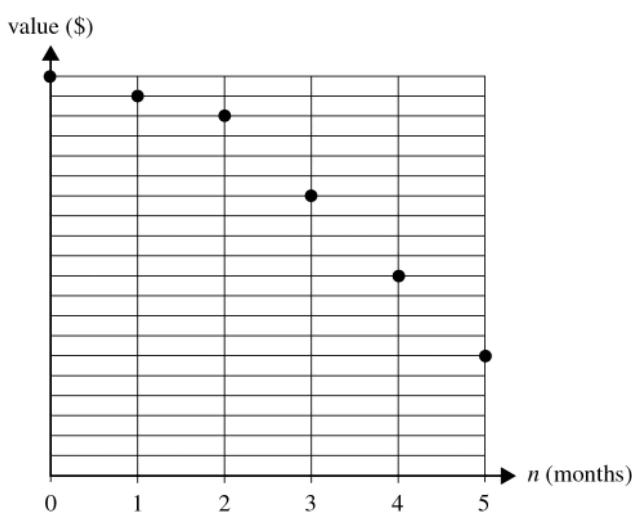
In the third year, the boat will depreciate in value 10% by of

A. \$47 239.20
B. \$52 488.00
C. \$58 320.00
D. \$64 800.00
E. \$72 000.00

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.20; © VCAA*

Question 7 (1 mark)

The graph below shows the value, V_n, of an asset as it depreciates over a period of five months.



Which one of the following depreciation situations does this graph best represent?

A. flat rate depreciation with a decrease in depreciation rate after two months
B. flat rate depreciation with an increase in depreciation rate after two months
C. unit cost depreciation with a decrease in units used per month after two months
D. reducing balance depreciation with an increase in the rate of depreciation after two months
E. reducing balance depreciation with a decrease in the rate of depreciation after two months

Source: *VCE 2018, Further Mathematics Exam 2, Section A, Q.5; © VCAA*

Question 8 (3 marks)

After three years, Julie withdraws $14 000 from her account to purchase a car for her business.

For tax purposes, she plans to depreciate the value of her car using the reducing balance method.

The value of Julie's car, in dollars, after n years, C_n, can be modelled by the recurrence relation shown below.

$C_0 = 14000, \quad C_{n+1} = R \times Cn$

a. For each of the first three years of reducing balance depreciation, the value of R is 0.85

What is the annual rate of depreciation in the value of the car during these three years? **(1 mark)**

b. For the next five years of reducing balance depreciation, the annual rate of depreciation in the value of the car is changed to 8.6%.

What is the value of the car eight years after it was purchased?

Round your answer to the nearest cent. **(2 marks)**

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 1, Q.6; © VCAA*

Question 9 (1 mark)

Miki is competing as a runner in a half-marathon.

After 30 minutes, his progress in the race is modelled by the difference equation

$K_{n+1} = 0.99K_n + 250 \quad K_{30} = 7550$

where $n \geq 30$ and K_n is the total distance Miki has run, in metres, after n minutes.

Using this difference equation, the total distance, in metres, that Miki is expected to have run 32 minutes after the start of the race is closest to

A. 7650
B. 7725
C. 7800
D. 7900
E. 8050

***Source:** VCE 2015, Further Mathematics Exam 2, Module 4, Q.2; © VCAA*

Question 10 (3 marks)

The sound system used by a business was initially purchased at a cost of \$3800.

After two years, the value of the sound system had depreciated to \$3150.

a. Assuming the flat rate method of depreciation was used, show that the value of the sound system was depreciated by \$325 each year. **(1 mark)**

b. The value of the sound system will continue to depreciate by \$325 each year.

How many years will it take, after the initial purchase, for the sound system to have a value of \$550? **(1 mark)**

c. The recording equipment used by the business was initially purchased at a cost of \$2100.

After five years, the value of the recording equipment had depreciated to \$1040 using the reducing balance method.

Find the annual percentage rate by which the value of this recording equipment depreciated. Write your answer correct to two decimal places. **(1 mark)**

***Source:** VCE 2015, Further Mathematics Exam 1, Section B, Module 1, Q.3; © VCAA*

Question 11 (1 mark)

A town has a population of 200 people when a company opens a large mine.

Due to the opening of the mine, the town's population is expected to increase by 50% each year.

Let P_n be the population of the town n years after the mine opened.

The expected growth in the town's population can be modelled by

A. $P_{n+1} = P_n + 100 \quad P_0 = 200$

B. $P_{n+1} = P_n + 100 \quad P_1 = 300$

C. $P_{n+1} = 0.5P_n \quad P_0 = 200$

D. $P_{n+1} = 1.5P_n \quad P_0 = 300$

E. $P_{n+1} = 1.5P_n \quad P_1 = 300$

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 1, Q.9; © VCAA*

Question 12 (1 mark)

Paul has to replace 3000 m of fencing on his farm.

Let F_n be the length, in metres, of fencing left to replace after n weeks.

The difference equation

$F_{n+1} = 0.95F_n + a \quad F_0 = 3000$

can be used to calculate the length of fencing left to replace after n weeks.

In this equation, a is a constant.

After one week, Paul still has 2540 m of fencing left to replace.

After three weeks, the length of fencing, in metres, left to replace will be closest to

A. 1310
B. 1380
C. 1620
D. 1690
E. 2100

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 4, Q.1; © VCAA*

Question 13 (1 mark)

Fong's gas bill is $368.40. If he pays this bill on time, it will be reduced by 5%.

In this case, the bill would be reduced by

A. $1.84
B. $5.00
C. $18.42
D. $184.20
E. $349.98

Source: VCE 2014, Further Mathematics Exam 2, Module 1, Q.1; © VCAA

Question 14 (4 marks)

Land in a wildlife reserve contains both grassland and desert.

Each year, some grassland becomes desert.

Let L_n be the expected area of grassland in the wildlife reserve, in square kilometres, at the end of year n.

The change in the area of grassland in the wildlife reserve, from year to year, is modelled by the difference equation

$L_{n+1} = 0.99L_n \quad L_{2014} = 20\,000$

a. How many square kilometres of grassland are expected to be in the wildlife reserve at the end of 2014? **(1 mark)**

b. What percentage of grassland is expected to become desert each year? **(1 mark)**

c. Show that 19 800 km^2 of grassland are expected to be in the wildlife reserve at the end of 2015. **(1 mark)**

d. What area of grassland, in square kilometres, is expected to become desert between the end of 2015 and 2016? **(1 mark)**

Answers and marking guide

5.2 A first-order linear recurrence relation

Question 1

$A_0 = 3$

$A_1 = 2 \times A_0 + 4 = 2 \times 3 + 4 = 10$
$A_2 = 2 \times A_1 + 4 = 2 \times 10 + 4 = 24$
$A_3 = 2 \times A_2 + 4 = 2 \times 24 + 4$

Question 2

Note that the sequence does not increase by a constant amount so it is not linear. Eliminate clear distractors and test the remaining options with the given sequence.
Option D:
2,
$4 \times (2) - 2 = 6$,
$4 \times (6) - 2 = 22$,
$4 \times (22) - 2 = 86$,
$4 \times (86) - 2 = 342$,

Question 3

$A_0 = 2$

$A_1 = 3\,(2) + 1 = 7$
$A_2 = 3\,(7) + 1 = 22$
$A_3 = 3\,(22) + 1 = 67$

Question 4

$a = 64$ and $r = \dfrac{80}{64} = \dfrac{100}{80} = 1.25$

$S_n = \dfrac{a\,(1 - r^n)}{1 - r} = \dfrac{64\left(1 - 1.25^n\right)}{1 - 1.25}$

So after five months, $S_5 = \dfrac{64\left(1 - 1.25^5\right)}{-0.25} = 525.25$

Question 5

$a = 2$ and $r = \dfrac{6}{2} = \dfrac{18}{6} = 3$

$t_n = ar^{n-1} = 2 \times 3^{n-1}$

So at the end of the seventh year, $t_7 = 2 \times 3^{7-1} = 1458$

Question 6

$t_{n+1} = t_n - 10$

The right-hand side indicates that the next day, t_{n+1}, is 10 minutes less training time than the previous day, t_n.

Question 7

The first term of this sequence is 4 and the sequence says to: double the current value (t_n) then add 3. Starting with 4, doubling and adding 3 gives 11. Doubling this value and adding 3 give 25. Doubling this value and adding 3 gives 53.
Sequence is: 4, 11, 25, 53, 109

Question 8

The sequence goes up by three. 4, 7, 10, 13, 16

Question 9
The sequence goes up by four. $7,\ 11,\ 15,\ 19,\ 23,\ 27,\ 31$

Question 10
Each term of the sequence is 3 times bigger than the last term.
$5,\ 15,\ 45,\ 135,\ 405$

Question 11
$$\begin{aligned} T_1 &= T_0 + 3 \\ &= 10 + 3 \\ &= 13 \end{aligned}$$

Question 12
The only sequence that is generated using the previous term in some way is D.
The recurrence relation would be $t_{n+1} = t_n + 3$, $t_1 = -4$

Question 13
The difference equation must link to a successive term in the sequence.

Question 14
The sequence is
$t_1 = 5$
$t_2 = 3 + 2 \times (5) = 13$
$t_3 = 3 + 2 \times (13) = 29$
$t_4 = 3 + 2 \times (29) = 61$
$t_5 = 3 + 2 \times (61) = 125$

Question 15
A decrease of 15% is used to find 85% of the cars value in the previous year.

5.3 Modelling flat rate depreciation with a recurrence relation

Question 1
a. \$15 000 [1 mark]
b. $V_1 = V_0 - 15\,000 = 120\,000 - 15\,000 = \$105\,000$
$V_2 = V_1 - 15\,000 = 105\,000 - 15\,000 = \$90\,000$ **[1 mark]**
c. $\dfrac{15\,000}{120\,000} \times 100 = 12.5\%$ [1 mark]
d. $V_n = 120\,000 - 15\,000n$, where $n = 0,\ 1,\ 2...$ **[1 mark]**

Question 2
10% of the purchase price is $10\% \times \$4500 = \$450$
Geoff will depreciate his computer by \$450 per year, so $V_{n+1} = V_n - 450$

Question 3
The decrease in the values is not linear and is not a growth, so some of the options can be eliminated.
The graph highlights a geometric decay with a reducing value of 6% p.a.
Test the values with the reducing-balance recurrence relation:
$V_0 = 7000$, $\quad V_{n+1} = (1 - 0.06) \times V_n$

Question 4
$L_1 = 37 + C$

$$L_2 = L_1 + C$$
$$25 = 37 + C + C$$
$$-12 = 2C$$
$$C = -6$$

Question 5

$$V_{n+1} = 2480 + 45(n + 1)$$
$$= 2480 + 45n + 45$$

Note that $V_n = 2480 + 45n$ and $V_0 = 2480$.
Therefore, $V_0 = 2480, \quad V_{n+1} = V_n + 45$

Question 6

The +500 indicates that there are periodic payments into an account or investment.
The 1.04 indicates that the account or investment is growing by 4% each period.

5.4 Modelling reducing balance depreciation with a recurrence relation

Question 1

Only the 2nd and the 3rd recurrence relations indicate depreciation:

- the 2 nd because it is subtracting 2500
- the 3rd because V_n is being multiplied by a number less than zero.

That is, 2 of the given recurrence relations indicate depreciation.

Question 2

a. $V_0 = 75\,000$

$V_1 = 75\,000 - 3375 = 71\,625$ **[1 mark]**
$V_2 = 71\,625 - 3375 = 68\,250$ **[1 mark]**

***VCAA Examination Report* note:**
This question was answered well. A few students made careless transcription errors.

b. **i.** The constant decrease amount from the recurrence relation is the annual depreciation amount. $3375 **[1 mark]**

ii. Let x represent the annual flat rate of depreciation.

$x\%$ of $\$75\,000 = \$3375$

$$\frac{x}{100} \times 75\,000 = 3375$$

Solving for x, $x = 4.5\%$ p.a. **[1 mark]**

c. From the recurrence relation, $R = 0.943$

For the reducing balance method of depreciation, $R = 1 - \frac{r}{100}$, where $r$is the annual rate of depreciation

$$0.943 = 1 - \frac{r}{100}, \text{ solving for } r$$
$$r = 5.7\% \text{ p.a.}$$ **[1 mark]**

Question 3

If the value of a car is reduced by 8% each year, then the value of a car each year will be 100% − 8% = 92%
$\therefore C_{n+1} + 0.92C_n$, where $C_0 = 26\,000$

Question 4

$$V_1 = 1.0025(100\,000) - 555 = 99\,695$$
$$V_2 = 1.0025(99\,695) - 555 = 99\,389.2375$$
$$V_3 = 1.0025(99\,389.2375) - 555 = 99\,082.71\,059$$
$$V_4 = 1.0025(99\,082.710\,59) - 555 = 98\,775.41\,737$$
$$V_5 = 1.0025(98\,775.41\,737) - 555 = 98\,467.3559$$

Question 5

It is linear depreciation, so it must be flat rate or unit cost depreciation.

The gradient is in dollars per kilometre travelled and is worked out using the coordinates (0, 35 000) and (20 000, 30 000):

$$m = \frac{35\,000 - 30\,000}{0 - 20\,000}$$
$$= -\$0.25/\text{km}$$

Question 6

The recurrence relation relevant for this is:

$$V_{n+1} = V_n - 18\,000 \times \frac{6.5}{100}, \; V_0 = 18\,000$$

n	V_n	Depreciation: $V_{n+1} = V_n - \left(18\,000 \times \frac{6.5}{100}\right)$
0	$V_0 = 18\,000$	$V_1 = 18\,000 - \left(18\,000 \times \frac{6.5}{100}\right) = 16\,830$
1	$V_1 = 16\,830$	$V_2 = 16\,830 - \left(18\,000 \times \frac{6.5}{100}\right) = 15\,660$
2	$V_2 = 15\,660$	$V_3 = 15\,660 - \left(18\,000 \times \times \frac{6.5}{100}\right) = 14\,490$

Question 7

The recurrence relation relevant for this is:

$$V_{n+1} = V_n - (0.18 \times 18\,000), \qquad V_0 = 42\,000$$

n	V_n	Depreciation: $V_{n+1} = V_n - (0.18 \times 18\,000)$
0	$V_0 = 42\,000$	$V_1 = 42\,000 - (0.18 \times 18\,000) = 38\,760$
1	$V_1 = 38\,760$	$V_2 = 38\,760 - (0.18 \times 18\,000) = 35\,520$
2	$V_2 = 35\,520$	$V_3 = 35\,520 - (0.18 \times 18\,000) = 32\,280$
3	$V_3 = 32\,280$	$V_4 = 32\,280 - (0.18 \times 18\,000) = 29\,040$
4	$V_4 = 29\,040$	$V_5 = 29\,040 - (0.18 \times 18\,000) = 25\,800$

5.5 Modelling unit cost depreciation with a recurrence relation

Question 1

a. $\frac{1440}{0.05} = 28\,800$ **[1 mark]**

b. $\frac{1440}{12\,000} \times 100 = 12\%$ **[1 mark]**

c. $M_n = 12\,000 - 0.05 \times n$ **[1 mark]**

Question 2

depreciation by reducing balance – depreciation by unit cost = 26 166.24

$$45\,000 \times \left(1 - \frac{k}{100}\right)^3 - (30\,000 \times 0.15) = 26\,166.24$$

$k = 12$

Question 3

The rule for unit cost depreciation is $V_n = V_0 - nd$, which gives the value of the asset after n outputs. d is the depreciation per output.

Here: $V_0 = 30\,000$, and when $n = 3$, $V_3 = 18\,480$

$d = \dfrac{30\,000 - 18\,480}{3 \times 24\,000}$
$d = 0.16$
Therefore: $V_n = 30\,000 - 0.16n$

Question 4
In the 4 years, the printer prints $1920 \times 4 = 7680$ pages.
Let $V_n = 680 - 7680_n$,
where n represents the depreciation amount per page
Solve $125 = 680 - 7680n$ (use CAS)
$n = 0.072\,265$ (\$ per page)
$\simeq 7$ cents
VCAA Examination Report **note:**
Although this was a standard application of unit-cost depreciation, almost half of the students were unable to determine the depreciation per page printed as required.

Question 5
a. From the graph, the purchase price was \$8000. **[1 mark]**
b. i. From the graph, the initial price is \$8000; the price at the end of 3 years is 3500.

$$\begin{aligned}\text{Therefore, depreciation per year} &= \frac{8000 - 3500}{3}\\ &= \frac{4500}{3}\\ &= \$1500 \quad \textbf{[1 mark]}\end{aligned}$$

ii. $\begin{aligned}\text{At the end of 5 years, value} &= 8000 - 5 \times 1500\\ &= \$500 \quad \textbf{[1 mark]}\end{aligned}$

c. $0.25x = 2 \times 1500$
$x = 12\,000$
Therefore, the bike travels 12 000 km in the two-year period. **[1 mark]**

5.6 Review

Question 1

$$\begin{aligned}V_{n+1} &= 2480 + 45\,(n + 1)\\ &= 2480 + 45n + 45\end{aligned}$$

Note that $V_n = 2480 + 45n$ and $V_0 = 2480$.
Therefore, $V_0 = 2480$, $V_{n+1} = V_n + 45$.

Question 2
a. $V_1 = 0.9 \times 60\,000 = 54\,000$
$V_2 = 0.9 \times 54\,000 = \$48\,600$ **[1 mark]**
b. $1 - 0.9 = 0.1$ or 10% [1 mark]
c. $0.9^n \times 60\,000 < 20\,000$
Solve on your CAS calculator:
$n = 10.427...$
Phil will replace these tools after 10 years, or sometime in the 11th year. **[1 mark]**
d. $8\% \times 60\,000 = 4800$
$V_{n+1} = V_n - 4800$, where $V_0 = 60\,000$
It could also be written as $V_{n+1} = V_n - 0.08V_0$, where $V_0 = 60\,000$ **[1 mark]**
VCAA Examination Report **note:**
Responses generally used the notation well, but many wrote a recurrence relation reflective of reducing balance depreciation rather than flat rate.

A few wrote both a recurrence relation and a rule with V_n in terms of n which could not be accepted.

Question 3

In the 4 years, the printer prints $1920 \times 4 = 7680$ pages.
Let $V_n = 680 - 7680_n$,
where n represents the depreciation amount per page
Solve $125 = 680 - 7680n$ (use CAS)
$n = 0.072265$(\$ per page)
$\simeq 7$ cents

VCAA Examination Report **note:**

Although this was a standard application of unit-cost depreciation, almost half of the students were unable to determine the depreciation per page printed as required.

Question 4

It is linear depreciation, so it must be flat rate or unit cost depreciation.
The gradient is in dollars per kilometre travelled and is worked out using the coordinates (0, 35 000) and (20 000, 30 000):

$$m = \frac{35\,000 - 30\,000}{0 - 20\,000}$$
$$= -\$0.25/\text{km}$$

Question 5

$$\text{Amount lost in four years} = 18\,000 - 5000$$
$$= 13\,000$$

Therefore, each year $\$\frac{13\,000}{4}$ was lost in value.
After one year:

$$\text{Value} = 18\,000 - \frac{13\,000}{4}$$
$$= 18\,000 - \left(\frac{18\,000 - 5000}{4}\right)$$

Question 6

In the first year, Sammy's boat depreciated by: $0.9 \times 72\,000 = \$64\,800$
Second year: $0.9 \times 64\,800 = \$58\,320$
So, in the third year, the boat will be depreciated in value by 10% of \$58 320.

Question 7

The first two months have a constant negative slope before becoming a steeper slope. Since both slopes are constant, the graph represents a flat rate depreciation. To become a steeper slope, the rate of depreciation must increase.
This graph represents a flat rate depreciation with an increase in depreciation rate after two months.

VCAA Examination Report **note:**

Students needed to recognise that the rate of reduction in value is constant in both phases of depreciation, indicating flat-rate depreciation, not reducing balance depreciation.

Question 8

a. The annual rate of depreciation is $(1 - 0.85) \times 100 = 15\%$. **[1 mark]**
VCAA Examination Report note:
This question was not well answered. Some students gave the R value of 0.85.

b. The value of R is 0.85 for the first three recursions, then the value of R becomes $1 - 0.086 = 0.914$ for the remaining years.

$$C_3 = C_0 \times R_1^3$$
$$= \$14\,000 \times 0.85^3 = \$8597.75 \quad \textbf{[1 mark]}$$
$$C_8 = C_3 \times R_2^5$$
$$= \$8597.75 \times 0.914^5 = \$5484.23$$

The final value of the car eight years after it was purchased is \$5484.23. **[1 mark]**

***VCAA Examination Report* note:**

Some students obtained one mark for correctly finding the value after three years.
Many depreciated by 8.6% for the entire eight years.

Question 9

This is a difference equation, so we will need to use iteration to work out the answer.

$$K_{31} = 0.99K_{30} + 250 = 7724.5$$
$$K_{32} = 0.99K_{31} + 250 = 7897.255 \approx 7900$$

Question 10

a. The sound system depreciated by $3800 - 3150 = \$650$ over two years.
Therefore, the sound system depreciated by \$325 each year. **[1 mark]**

b. $BV = P - dT$ **[1 mark]**

$$550 = 3800 - 325T$$
$$T = 10 \quad \text{years}$$

c. $BV = P\left(1 - \dfrac{r}{100}\right)^T$ **[1 mark]**

$$1040 = 2100\left(1 - \frac{r}{100}\right)^5$$
$$r = 13.11\% \text{ p.a.}$$

Question 11

The initial population is $P_0 = 200$ and the rate of increase is $100\% + 50\% = 150\%$.
From this information, we can work out that $P_1 = 1.5 \times P_0 = 1.5 \times 200 = 300$.
Therefore $P_{n+1} = 1.5P_n$, $P_1 = 300$.

Question 12

$$F_1 = 2540$$
$$2540 = 0.95\,(3000) + a$$
$$a = -310$$
$$\therefore F_{n+1} = 0.95F_n - 310$$
$$F_2 = 0.95\,(2540) - 310 = 2103$$
$$F_3 = 0.95\,(2103) - 310 = 1687.85 \approx 1690$$

Question 13

$0.05 \times \$368.40 = \$18.42$

Question 14

a. As stated in the question, $L_{2014} = 20\,000$, therefore 20 000 square kilometres of grassland are expected to be in the wildlife reserve at the end of 2014. **[1 mark]**

***VCAA Assessment Report* note:**

It was evident that a number of students did not understand that L_n was the area at the end of year n and gave $19\,800\,\text{km}^2$.

b. $L_{n+1} = 0.99L_n$

The 0.99 indicates that 99% of the previous year's grassland area remains as grassland the next year. Therefore 1% becomes desert. **[1 mark]**

***VCAA Assessment Report* note:**

A common, incorrect answer was 99%.

c. $L_{2015} = 0.99 \times L_{2014}$

$= 0.99 \times 20\,000$

$= 19\,800$

[**1 mark** – must have second line of working. Preferable to have first and second line.]

d. $L_{2016} = 0.99 \times L_{2015}$

$= 0.99 \times 19\,800$

$= 19\,602$

Area of desert = amount of grassland lost **[1 mark]**

$= 19\,800 - 19\,602$

$= 198 \text{ km}^2$

6 Modelling compound interest investments and loans using recursion

Topic	6	Modelling compound interest investments and loans using recursion
Subtopic	6.2	Simple interest

online only

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Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.24; © VCAA*

Question 1 (1 mark)

Manu invests $3000 in an account that pays interest compounding monthly.

The balance of his investment after n months, B_n, can be determined using the recurrence relation

$B_0 = 3000, B_{n+1} = 1.0048 \times B_n$

The total interest earned by Manu's investment after the first five months is closest to

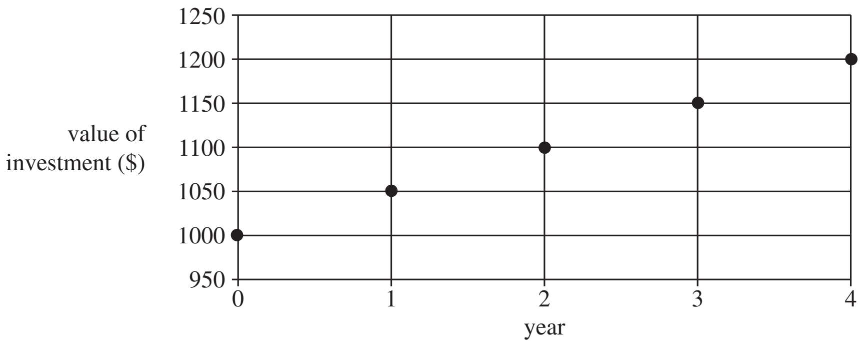
A. \$57.60
B. \$58.02
C. \$72.00
D. \$72.69
E. \$87.44

Topic 6 Subtopic 6.2 Simple interest

Source: *VCE 2013, Further Mathematics Exam 1, Section B, Module 4, Q.7; © VCAA*

Question 2 (1 mark)

The graph below shows the growth in value of a $1000 investment over a period of four years.



A different amount of money is invested under the same investment conditions for eight years.

In total, the amount of interest earned on this investment is $600.

The amount of money invested is

A. $500
B. $600
C. $1500
D. $2000
E. $2400

Question 3 (1 mark)

Tayla wants to earn $1800 interest on a 5-year investment at 4.5% per annum simple interest. The amount she needs to invest is closest to

A. $405
B. $1800
C. $7310
D. $8000
E. $9000

Source: *VCE 2014, Further Mathematics Exam 2, Module 4, Q.1; © VCAA*

Question 4 (5 marks)

The adult membership fee for a cricket club is $150.

Junior members are offered a discount of $30 off the adult membership fee.

a. Write down the discount for junior members as a percentage of the adult membership fee. **(1 mark)**

b. Adult members of the cricket club pay $15 per match in addition to the membership fee of $150.
If an adult member played 12 matches, what is the total this member would pay to the cricket club? **(1 mark)**

c. If a member does not pay the membership fee by the due date, the club will charge simple interest at the rate of 5% per month until the fee is paid.
Michael paid the $150 membership fee exactly two months after the due date.
Calculate, in dollars, the interest that Michael will be charged. **(1 mark)**

d. The cricket club received a statement of the transactions in its savings account for the month of January 2014.
The statement is shown below.

Date	Details	Deposit	Withdrawal	Balance
01 Jan. 2014	Brought Forward			$58 950.00
08 Jan. 2014	Match Fees	$750.00		$59 700.00
17 Jan. 2014	Withdrawal		$	$42 700.00
23 Jan. 2014	Membership	$4500.00		$47 200.00
31 Jan. 2014	Interest	$125.12		$47 325.12

i. Calculate the amount of the withdrawal on 17 January 2014. **(1 mark)**

ii. Interest for this account is calculated on the minimum balance for the month and added to the account on the last day of the month.
What is the annual rate of interest for this account?
Write your answer, correct to one decimal place. **(1 mark)**

Source: *VCE 2014, Further Mathematics Exam 1, Section B, Module 4, Q.1; © VCAA*

Question 5 (1 mark)

This month, a business charges \$1500 to install a water tank.

Next month, the charge will increase by 3.5%.

The charge next month will be

A. \$45.00

B. \$52.50

C. \$1545.00

D. \$1552.50

E. \$1950.00

Question 6 (1 mark)

Joshua invested \$32 000 for 30 months and earned \$2160 in simple interest. The annual interest rate for the investment is

A. 0.027%

B. 2.5%

C. 2.7%

D. 3.2%

E. 4.5%

Topic	6	Modelling compound interest investments and loans using recursion
Subtopic	6.3	Compound interest as a geometric recurrence relation

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Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.18; © VCAA*

Question 1 (1 mark)

The value of a compound interest investment, in dollars, after n years, V_n, can be modelled by the recurrence relation shown below.

$V_0 = 100\,000, \quad V_{n+1} = 1.01V_n$

The interest rate, per annum, for this investment is

A. 0.01%
B. 0.101%
C. 1%
D. 1.01%
E. 101%

Source: *VCE 2017, Further Mathematics Exam 2, Section A, Core, Q.6; © VCAA*

Question 2 (4 marks)

Alex sends a bill to his customers after repairs are completed.

If a customer does not pay the bill by the due date, interest is charged.

Alex charges interest after the due date at the rate of 1.5% per month on the amount of an unpaid bill.

The interest on this amount will compound monthly.

a. Alex sent Marcus a bill of \$200 for repairs to his car. **(1 mark)**

Marcus paid the full amount one month after the due date.

How much did Marcus pay?

b. Alex sent Lily a bill of \$428 for repairs to her car. **(2 marks)**

Lily did not pay the bill by the due date.

Let A_n be the amount of this bill n months after the due date.

Write down a recurrence relation, in terms of A_0, A_{n+1} and A_n, that models the amount of the bill.

c. Lily paid the full amount of her bill four months after the due date. **(1 mark)**

How much interest was Lily charged?

Round your answer to the nearest cent.

Topic 6 Subtopic 6.3 Compound interest as a geometric recurrence relation

Question 3 (2 marks)

Using the recurrence relation of $V_{n+1} = V_n\left(1 + \dfrac{4.8}{100}\right)$, $V_0 = \$25\,000$, calculate the value of V_3.

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Core, Q.19; © VCAA*

Question 4 (1 mark)

Shirley would like to purchase a new home. She will establish a loan for \$225 000 with interest charged at the rate of 3.6% per annum, compounding monthly.

Each month, Shirley will pay only the interest charged for that month.

After three years, the amount that Shirley will owe is

A. \$73 362
B. \$170 752
C. \$225 000
D. \$239 605
E. \$245 865

Source: *VCE 2013, Further Mathematics Exam 1, Section B, Module 4, Q.6; © VCAA*

Question 5 (1 mark)

A worker has received an annual salary increase of 3% for the past two years.

This year, the worker's annual salary is \$46 500.

Two years ago, her salary was closest to

A. \$42 315
B. \$43 750
C. \$43 830
D. \$45 140
E. \$49 330

Topic	6	Modelling compound interest investments and loans using recursion
Subtopic	6.4	Compound interest using a rule

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Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 4, Q.4; © VCAA*

Question 1 (1 mark)

Mary invests $1200 for two years.

Interest is calculated at the rate of 3.35% per annum, compounding monthly.

The amount of interest she earns in two years is closest to

A. $6.71
B. $40.82
C. $80.40
D. $81.75
E. $83.03

Source: *VCE 2014, Further Mathematics Exam 1, Section B, Module 4, Q.8; © VCAA*

Question 2 (1 mark)

Robert invested $6000 at 4.25% per annum with interest compounding quarterly.

Immediately after interest is paid at the end of each quarter, he adds $500 to his investment.

The value of Robert's investment at the end of the third quarter, after his $500 has been added, is closest to

A. $6193
B. $7569
C. $7574
D. $7709
E. $8096

Question 3 (1 mark)

If an investment is worth $7035.04 after two months of compounding interest (paid monthly) at the rate of 3% per annum, find the initial amount invested.

A. $7017.50
B. $7000
C. $6990
D. $7010
E. $7200

Topic 6 › Subtopic 6.4 Compound interest using a rule

Question 4 (1 mark)

Consider the information given below.

- Principal value of \$12 000
- Interest rate of 3.5% per annum
- Compound interest paid quarterly

The recurrence relation that would best display the information is

A. $V_{x+1} = V_x\left(1 + \dfrac{3.5}{100}\right), V_0 = \$12\,000$

B. $V_{x+1} = V_x\left(\dfrac{0.875}{100}\right), V_0 = \$12\,000$

C. $V_{x+1} = V_x\left(1 + \dfrac{0.875}{100}\right), V_0 = \$12\,000$

D. $V_{x+1} = V_x\left(\dfrac{3.5}{100}\right), V_0 = \$12\,000$

E. $V_{x+1} = V_x\left(1 + \dfrac{14}{100}\right), V_0 = \$12\,000$

Question 5 (2 marks)

\$14 500 is invested in an account earning 4% interest compounding quarterly.

Complete the following table to record the interest earned each month in the first three months.

V_n	V_{n+1}	Interest $= V_{n+1} - V_n$

Question 6 (1 mark)

An investment of \$25 000 is invested in an account earning 4.7% interest with bi-annual compounding periods.

The difference between the interest earned in the first and the second investment periods is

A. \$13.81

B. \$15.47

C. \$587.50

D. \$601.31

E. \$25 587.50

Topic	6	Modelling compound interest investments and loans using recursion
Subtopic	6.5	Calculating rate or time for compound interest

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Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.27; © VCAA*

Question 1 (1 mark)

Gen invests \$10 000 at an interest rate of 5.5% per annum, compounding annually.

After how many years will her investment first be more than double its original value?

A. 12
B. 13
C. 14
D. 15
E. 16

Source: *VCE 2020, Further Mathematics Exam 2, Section A, Q.9; © VCAA*

Question 2 (3 marks)

Samuel opens a savings account.

Let B_n be the balance of this savings account, in dollars, n months after it was opened.

The month-to-month value of B_n can be determined using the recurrence relation shown below.

$B_0 = 5000, \qquad B_{n+1} = 1.003B_n$

a. Write down the value of B_4, the balance of the savings account after four months. Round your answer to the nearest cent. **(1 mark)**

b. Calculate the monthly interest rate percentage for Samuel's savings account. **(1 mark)**

c. After one year, the balance of Samuel's savings account, to the nearest dollar, is \$5183.
If Samuel had deposited an additional \$50 at the end of each month immediately after the interest was added, how much extra money would be in the savings account after one year?
Round your answer to the nearest dollar. **(1 mark)**

Topic 6 | Subtopic 6.5 Calculating rate or time for compound interest

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.18; © VCAA*

Question 3 (1 mark)

The value of a compound interest investment, in dollars, after n years, V_n, can be modelled by the recurrence relation shown below.

$V_0 = 100\,000, \qquad V_{n+1} = 1.01V_n$

The interest rate, per annum, for this investment is

A. 0.01%
B. 0.101%
C. 1%
D. 1.01%
E. 101%

Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.26; © VCAA*

Question 4 (1 mark)

Ray deposited \$5000 in an investment account earning interest at the rate of 3% per annum, compounding quarterly.

A rule for the balance, R_n, in dollars, after n years is given by

A. $R_n = 5000 \times 0.03^n$
B. $R_n = 5000 \times 1.03^n$
C. $R_n = 5000 \times 0.03^{4n}$
D. $R_n = 5000 \times 1.0075^n$
E. $R_n = 5000 \times 1.0075^{4n}$

Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.30; © VCAA*

Question 5 (1 mark)

Twenty years ago, Hector invested a sum of money in an account earning interest at the rate of 3.2% per annum, compounding monthly.

After 10 years, he made a one-off extra payment of \$10 000 to the account.

For the next 10 years, the account earned interest at the rate of 2.8% per annum, compounding monthly.

The balance of his account today is \$686 904.09

The sum of money Hector originally invested is closest to

A. \$355 000
B. \$370 000
C. \$377 000
D. \$384 000
E. \$385 000

Topic	6	Modelling compound interest investments and loans using recursion
Subtopic	6.6	Nominal and effective annual interest rate

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Source: *VCE 2020, Further Mathematics Exam 1, Section A, Q.28; © VCAA*

Question 1 (1 mark)

The nominal interest rate for a loan is 8% per annum.

When rounded to two decimal places, the effective interest rate for this loan is **not**

A. 8.33% per annum when interest is charged daily.
B. 8.32% per annum when interest is charged weekly.
C. 8.31% per annum when interest is charged fortnightly.
D. 8.30% per annum when interest is charged monthly.
E. 8.24% per annum when interest is charged quarterly.

Question 2 (1 mark)

A comparison of two loans was made with the following details:

Loan A: Nominal rate of 8% with interest compounding daily (standard year).

Loan B: Nominal rate of 8.1% with interest compounding biannually.

Which loan has the greater effective interest and by what margin?

A. Loan B, Margin = 8.33%
B. Loan B, Margin = 0.07%
C. Loan A, Margin = 8.33%
D. Loan A, Margin = 0.07%
E. Loan B, Margin = 8.26%

Question 3 (1 mark)
A compound interest rate of 5.85% p.a. is charged on a hire-purchase item, with monthly repayments over 6 years.
What is the effective interest rate closest to?
A. 6.0%
B. 8.9%
C. 9.3%
D. 10.6%
E. 11.53%

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.24; © VCAA*

Question 4 (1 mark)
Millie invested $20 000 in an account at her bank with interest compounding monthly.
After one year, the balance of Millie's account was $20 732.
The difference between the rate of interest per annum used by her bank and the effective annual rate of interest for Millie's investment is closest to
A. 0.04%
B. 0.06%
C. 0.08%
D. 0.10%
E. 0.12%

Topic 6 | Subtopic 6.6 Nominal and effective annual interest rate

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.19; © VCAA*

Question 5 (1 mark)

Daniel borrows \$5000, which he intends to repay fully in a lump sum after one year.

The annual interest rate and compounding period for five different compound interest loans are given below:

- Loan I – 12.6% per annum, compounding weekly
- Loan II – 12.8% per annum, compounding weekly
- Loan III – 12.9% per annum, compounding weekly
- Loan IV – 12.7% per annum, compounding quarterly
- Loan V – 13.2% per annum, compounding quarterly

When fully repaid, the loan that will cost Daniel the least amount of money is

A. Loan I.
B. Loan II.
C. Loan III.
D. Loan IV.
E. Loan V.

Source: *VCE 2014, Further Mathematics Exam 1, Section B, Module 4, Q.6; © VCAA*

Question 6 (1 mark)

A loan of \$1000 is to be repaid with six payments of \$180 per month.

The effective annual rate of interest charged is closest to

A. 8.0%
B. 13.7%
C. 16.0%
D. 27.4%
E. 30.9%

Question 7 (1 mark)

A comparison of two loans was made with the following details:

Loan *A*: Nominal rate of 8% with interest compounding daily (standard year).

Loan *B*: Nominal rate of 8.1% with interest compounding biannually.

Which loan has the greater effective interest and by what margin?

A. Loan *B*, Margin = 8.33%
B. Loan *B*, Margin = 0.07%
C. Loan *A*, Margin = 8.33%
D. Loan *A*, Margin = 0.07%
E. Loan *B*, Margin = 8.26%

Topic 6 | Subtopic 6.6 Nominal and effective annual interest rate

Question 8 (1 mark)
An effective interest rate for a loan of $180 00 paid off in 48 months, with compound interest calculated monthly, is 9.25%. What would be the nominal interest rate for this loan if the compound interest was calculated weekly?
A. 10.65%
B. 10.15%
C. 9.65%
D. 9.25%
E. 8.88%

Question 9 (1 mark)
A compound interest rate of 10% p.a. is charged on a hire-purchase item, with weekly repayments over a year.
What is the effective interest rate closest to?
A. 10.5%
B. 10.9%
C. 12.3%
D. 19.6%
E. 20.4%

Question 10 (1 mark)
A compound interest rate of 5.85% p.a. is charged on a hire-purchase item, with monthly repayments over 6 years.
What is the effective interest rate closest to?
A. 6.0%
B. 8.9%
C. 9.3%
D. 10.6%
E. 11.53%

Topic	6	Modelling compound interest investments and loans using recursion
Subtopic	6.7	Review

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Source: *VCE 2021, Further Mathematics Exam 2, Section A, Core, Q.8; © VCAA*

Question 1 (2 marks)

For renovations to the coffee shop, Sienna took out a reducing balance loan of \$570 000 with interest calculated fortnightly.

The balance of the loan, in dollars, after n fortnights, S_n, can be modelled by the recurrence relation

$S_0 = 570\,000, \qquad S_{n+1} = 1.001S_n - 1193$

a. Calculate the balance of this loan after the first fortnightly repayment is made. **(1 mark)**

b. Show that the compound interest rate for this loan is 2.6% per annum. **(1 mark)**

Source: *VCE 2017, Further Mathematics Exam 1, Section A, Core, Q.19; © VCAA*

Question 2 (1 mark)

Shirley would like to purchase a new home. She will establish a loan for \$225 000 with interest charged at the rate of 3.6% per annum, compounding monthly.

Each month, Shirley will pay only the interest charged for that month.

After three years, the amount that Shirley will owe is

A. \$73 362

B. \$170 752

C. \$225 000

D. \$239 605

E. \$245 865

Source: *VCE 2016, Further Mathematics Exam 2, Core, Q.5; © VCAA*

Question 3 (5 marks)

Ken has opened a savings account to save money to buy a new caravan.

The amount of money in the savings account after n years, V_n, can be modelled by the recurrence relation shown below.

$V_0 = 15\,000,\ V_{n+1} = 1.04 \times V_n$

a. How much money did Ken initially deposit into the savings account? **(1 mark)**

b. Use recursion to write down calculations that show that the amount of money in Ken's savings account after two years, V_2, will be \$16 224. **(1 mark)**

c. What is the annual percentage compound interest rate for this savings account? **(1 mark)**

d. The amount of money in the account after n years, V_n, can also be determined using a rule.

i. Complete the rule below by writing the appropriate numbers in the equation below. **(1 mark)**

$V_n = \boxed{}^{\,n} \times \boxed{}$

ii. How much money will be in Ken's savings account after 10 years? **(1 mark)**

Source: *VCE 2014, Further Mathematics Exam 2, Module 4, Q.3; © VCAA*

Question 4 (4 marks)

The cricket club had invested \$45 550 in an account for four years.

After four years of compounding interest, the value of the investment was \$60 000.

a. How much interest was earned during the four years of this investment? **(1 mark)**

b. Interest on the account had been calculated and paid quarterly. **(1 mark)**

What was the annual rate of interest for this investment?

Write your answer, correct to one decimal place.

c. The \$60 000 was re-invested in another account for 12 months.

The new account paid interest at the rate of 7.2% per annum, compounding monthly.

At the end of each month, the cricket club added an additional \$885 to the investment.

i. The equation below can be used to determine the account balance at the end of the first month, immediately after the \$885 was added. **(1 mark)**

Complete the equation by filling in the spaces.

Account balance $= 60\,000 \times (1 + \_\_\_\_\_\_\_\_\_\_) + \_\_\_\_\_\_\_\_\_\_$

ii. What was the account balance at the end of 12 months? **(1 mark)**

Write your answer, correct to the nearest dollar.

Source: *VCE 2013, Further Mathematics Exam 2, Module 4, Q.3; © VCAA*

Question 5 (6 marks)

Hugo paid $7500 for a second bike under a hire-purchase agreement.
A flat interest rate of 8% per annum was charged.
He will fully repay the principal and the interest in 24 equal monthly instalments.

a. Determine the monthly instalment that Hugo will pay.
Write your answer in dollars, correct to the nearest cent. **(2 marks)**

b. Find the effective rate of interest per annum charged on this hire-purchase agreement.
Write your answer as a percentage, correct to two decimal places. **(1 mark)**

c. Explain why the effective interest rate per annum is higher than the flat interest rate per annum. **(1 mark)**

d. The value of his second bike, purchased for $7500, will be depreciated each year using the reducing balance method of depreciation.
One year after it was purchased, this bike was valued at $6375.
Determine the value of the bike five years after it was purchased.
Write your answer, correct to the nearest dollar. **(2 marks)**

Source: *VCE 2021, Further Mathematics Exam 1, Section A, Core, Q.21; © VCAA*

Question 6 (1 mark)

Enrico invests \$3000 in an account that pays interest compounding monthly.

After four years, the balance of the account is \$3728.92.

The effective annual interest rate for this investment, rounded to two decimal places, is

A. 5.45%
B. 5.52%
C. 5.56%
D. 5.59%
E. 5.60%

Source: *VCE 2021, Further Mathematics Exam 2, Section A, Core, Q.7; © VCAA*

Question 7 (3 marks)

Sienna owns a coffee shop.

A coffee machine, purchased for \$12 000, is depreciated in value using the unit cost method.

The rate of depreciation is \$0.05 per cup of coffee made.

The recurrence relation that models the year-to-year value, in dollars, of the coffee machine is

$M_0 = 12\ 000, \quad M_{n+1} = M_n - 1440$

a. Calculate the number of cups of coffee that the machine produces per year. **(1 mark)**

b. The recurrence relation above could also represent the value of the coffee machine depreciating at a flat rate.
What annual flat rate percentage of depreciation is represented? **(1 mark)**

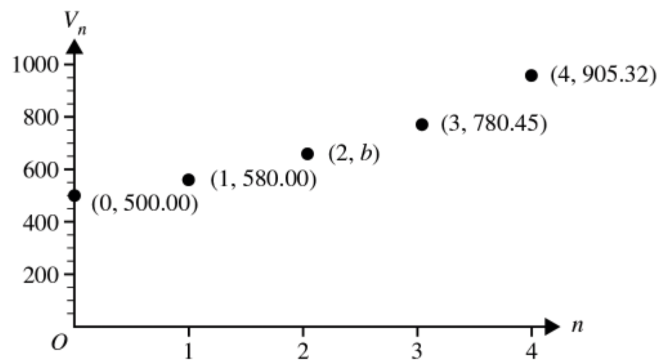
c. Complete the rule below that gives the value of the coffee machine, M_n, in dollars, after n cups have been produced. **(1 mark)**

$M_n = \boxed{} + \boxed{} \times n$

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.21; © VCAA*

Question 8 (1 mark)

The graph below shows the value, in dollars, of a compound interest investment after n compounding periods, V_n, for a period of four compounding periods.



The coordinates of the point where $n = 2$ are $(2, b)$.

The value of b is

A. 660.00
B. 670.00
C. 672.80
D. 678.40
E. 685.60

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 4, Q.2; © VCAA*

Question 9 (1 mark)

An investment property was purchased for $600 000.

Over a 10-year period, its value increased to $850 000.

The increase in value, as a percentage of the purchase price, is closest to

A. 4.2%
B. 25.0%
C. 29.4%
D. 41.7%
E. 70.6%

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 4, Q.3; © VCAA*

Question 10 (1 mark)

The closing price of a share on Wednesday was $160.

The closing price of the same share on Thursday was 3% less than its closing price on Wednesday.

The closing price of the same share on Friday was 4.5% more than its closing price on Thursday.

The closing price of the share on Friday is closest to

A. $157.38
B. $161.98
C. $162.18
D. $162.40
E. $172.22

Source: *VCE 2015, Further Mathematics Exam 2, Module 1, Q.1; © VCAA*

Question 11 (5 marks)

A crop of capsicums is being harvested from a field on a large farm.

In week 1 of the harvest, 2000 kg of capsicums were picked.

In week 2 of the harvest, 2150 kg of capsicums were picked.

In week 3 of the harvest, 2300 kg of capsicums were picked.

The weight of the capsicums, in kilograms, picked each week continues in this pattern for the first eight weeks of the harvest.

The weight of the capsicums, in kilograms, picked each week forms the terms of an arithmetic sequence, as shown below.

2000, 2150, 2300 . . .

a. Write down a calculation that shows that the common difference for this sequence is 150. **(1 mark)**

b. How many kilograms of capsicums will be picked in week 5? **(1 mark)**

c. How many kilograms of capsicums in total will have been picked in the first eight weeks of the harvest? **(1 mark)**

d. In which week will the total weight of the harvest first exceed 14 000 kg? **(1 mark)**

e. The weight of the capsicums, C_n, picked in week n, is modelled by a difference equation. Write down the rule for this difference equation in the box provided below. **(1 mark)**

$C_{n+1} = \boxed{}$ $\quad C_1 = 2000$

Source: *VCE 2013, Further Mathematics Exam 1, Section B, Module 4, Q.4; © VCAA*

Question 12 (1 mark)

The purchase price of a car is $15 000.

A deposit of $3000 is paid and the balance will be repaid with 36 monthly payments of $400.

The annual flat rate of interest charged is closest to

A. 1.3%

B. 4.0%

C. 5.3%

D. 6.7%

E. 20.0%

Answers and marking guide

6.2 Simple interest

Question 1

Interest $= \left(1.0048^5 \times 3000\right) - 3000 = \$72.69$

Question 2

From the graph, the value of investment increases by the same amount each year.

This is simple interest investment with $P = 1000$, $I = 200$ and $t = 4$.

$$r = \frac{100I}{Pt} = \frac{100 \times 200}{1000 \times 4} = 5\% \text{ p.a.}$$

For the other investment, $r = 5,\ t = 8$ and $I = 600$.

$$r = \frac{100I}{rt} = \frac{100 \times 600}{5 \times 8} = \$1500$$

Question 3

$$I = \frac{V_0 rn}{100}$$

where $n = 5$, $I = 1800$ and $r = 4.5$

$$1800 = 5 \times V_0 \times \frac{4.5}{100}$$

$$V_0 = \$8000$$

Question 4

a. $\frac{30}{150} \times 100 = 20\%$ **[1 mark]**

b. Total $= 150 + 12 \times 15$

$= \$330$ **[1 mark]**

c. $V_0 = 150, r = 5, n = 2$

$$\begin{aligned} I &= \frac{V_0 rn}{100} \\ &= \frac{150 \times 5 \times 2}{100} \\ &= \$15 \end{aligned}$$ **[1 mark]**

***VCAA Assessment Report* note:**

Many students misread the 5% rate of interest as an annual rate, rather than a monthly rate and obtained the incorrect answer of $1.25.

d. i. $59\,700 - 42\,700 = \$17\,000$ **[1 mark]**

ii. minimum balance $= \$42\,700$

$$\begin{aligned} I &= \frac{V_0 rn}{100} \\ 125.12 &= \frac{42\,700 \times r \times \frac{1}{12}}{100} \\ r &= \$3.5\% \end{aligned}$$ **[1 mark]**

Question 5

$$\begin{aligned} \text{Charge} &= 1500\left(1 + \frac{3.5}{100}\right) \\ &= \$1552.50 \end{aligned}$$

Question 6

$$I = \frac{V_0 rn}{100}$$

where $n = 2.5,\ I = 2160$ and $V_0 = 32\,000$

$2160 = 2.5 \times 32\,000 \times \frac{r}{100}$

Solving for r using a calculator gives:

$r = 2.7\%$

6.3 Compound interest as a geometric recurrence relation

Question 1

The interest rate is the number found in front of V_n.

1.01 means 101% = 100% + 1%, so the interest rate, per annum, is 1%.

Question 2

a. Marcus pays the $200 owing plus 1.5% of the amount owed as interest.

$$\begin{aligned}\text{Payment} &= 200 + \frac{1.5}{100} \times 200 \\ &= 200 + 3 \\ &= \$203\end{aligned}$$ **[1 mark]**

***VCAA Assessment Report* note:**

Many students did not read that the given interest rate was per month, so $200.25 was a common incorrect answer.

b. $A_0 = 428 \;\; A_{n+1} = R \times A_n,$ where $R = 1 + \frac{r}{100}$

$$\begin{aligned}R &= 1 + \frac{1.5}{100} \\ &= 1.015\end{aligned}$$

$A_0 = 428, \;\; A_{n+1} = 1.015 \times A_n$

Award 1 mark for stating $A_0 = 428$, 1 mark for $A_{n+1} = 1.015 \times A_n$

***VCAA Assessment Report* note:**

Many students did not write a recurrence relation in the required form.

A recurrence relation has the initial value written first.

The name of the variable needed to be consistent in the recurrence relation.

Some students did not recognise the difference between the recurrence relation above and the rule $A_n = 428 \times 1.015^n$.

c. $A_0 = 428$

$$\begin{aligned}A_1 &= 1.015 \times 428 \\ &= 434.42 \\ A_2 &= 1.015 \times 434.42 \\ &= 440.9363 \\ A_3 &= 1.015 \times 440.9363 \\ &= 447.5503445 \\ A_4 &= 1.015 \times 447.5503445 \\ &\simeq 454.2635997\end{aligned}$$

After 4 months, Lily pays $454.26.

The interest charged is 454.26 − 428 = $26.26. **[1 mark]**

***VCAA Assessment Report* note:**

Some students who answered parts a. and b. correctly gave $454.26 as the answer here, which was the total amount Lily was charged rather than the interest. An answer to the nearest (one) cent was required, not the nearest five or 10 cents.

Question 3

$V_0 = \$25\,000$

$V_1 = V_0\left(1 + \frac{4.8}{100}\right) = 25\,000\left(1 + \frac{4.8}{100}\right) = \$26\,200$

$V_2 = V_1\left(1 + \frac{4.8}{100}\right) = 26\,200\left(1 + \frac{4.8}{100}\right) = \$27\,457.60$

$V_3 = V_2\left(1 + \frac{4.8}{100}\right) = 27\,457.60\left(1 + \frac{4.8}{100}\right) = \$28\,775.56$

Question 4

The interest rate of 3.6% p.a. is equivalent to $\frac{3.6}{12}$% or 0.3% p.m.

Shirley has an interest-only loan, so her balance will remain at \$225 000 after she pays each monthly interest charge.

Question 5

Let the salary 2 years ago = x.

$x \times 1.03 \times 1.03 = 46\,500$

$$x = \frac{46\,500}{1.03 \times 1.03}$$
$$x = 43\,830.71$$

Therefore, 2 years ago her salary was closest to \$43 830.

6.4 Compound interest using a rule

Question 1

$$A = P\left(1 + \frac{r}{100}\right)^n = 1200\left(1 + \frac{0.279\,166\,7}{100}\right)^{2\times12} = \$1283.03$$

$$\begin{aligned} I &= A - P \\ &= \$1283.03 - \$1200 \\ &= \$83.03 \end{aligned}$$

Question 2

$$\begin{aligned} V_n \text{ after 1 quarter} &= V_0R^n \\ &= 6000\left(1 + \frac{4.25/4}{100}\right)^1 + 500 \\ &= 6563.75 \\ V_n \text{ after 2nd quarter} &= V_0R^n \\ &= 6563.75\left(1 + \frac{4.25/4}{100}\right)^1 + 500 \\ &= 7133.49 \\ V_n \text{ after 3rd quarter} &= V_0R^n \\ &= 7133.49\left(1 + \frac{4.25/4}{100}\right)^1 + 500 \\ &= 7709.28 \end{aligned}$$

Question 3

Substituting the known values, the recurrence relation becomes:

$$\begin{aligned} V_2 &= V_1\left(1 + \frac{3/12}{100}\right) \\ &= V_1(1.0025) \\ \Rightarrow 7035.04 &= V_1(1.0025) \end{aligned}$$

Rearranging finds the value of V_1:

$$V_1 = \frac{7035.04}{(1.0025)}$$
$$= 7017.496$$

Repeating the process allows us to find V_0:

$$V_1 = V_0\,(1.0025)$$
$$\Rightarrow 7017.496 = V_0\,(1.0025)$$
$$V_0 = \frac{7017.496}{1.0025}$$
$$= \$7000$$

Therefore, the initial investment was $7000

Question 4

As the principal is $12 000, it is the starting value. Therefore $V_0 = 12\,000$

An interest rate of 3.5% per annum equates to $\frac{3.5}{4} = 0.875\%$ per quarter.

$$V_{x+1} = V_x\left(1 + \frac{0.875}{100}\right),\ V_0 = \$12\,000$$

Question 5

Using the recurrence relation: $V_{n+1} = V_n\left(1 + \frac{\frac{4}{4}}{100}\right),\ V_0 = \$14\ 500$

This simplifies to: $V_{n+1} = V_n\,(1.01),\ V_0 = \$14\ 500$

V_n	V_{n+1}	Interest $= V_{n+1} - V_n$
$V_0 = 14\,500$	$V_1 = 14\,500 \times (1.01)$ $= 14\,645$	$14 645 − $14 500 = $145
$V_1 = 14\,645$	$V_2 = 14\,645 \times (1.01)$ $= 14\,791.45$	$14 791.45 − $14 645 = $146.45
$V_2 = 14\,791.45$	$V_3 = 14\,791.45 \times (1.01)$ $= 14\,939.36$	$14 939.36 − $14 791 = $147.91

Question 6

$$V_{n+1} = V_n\left(1 + \frac{4.7/2}{100}\right),\ V_0 = \$25\,000$$
$$V_1 = 25\,000 \times \left(1 + \frac{4.7/2}{100}\right)$$
$$= 25\,587.50$$

First period interest is $25\,587.50 - 25\,000 = \$587.50$

$$V_2 = 25\,587.50 \times \left(1 + \frac{4.7/2}{100}\right)$$
$$= 26\,188.81$$

Second period interest is $26\,188.81 - 25\,587.50 = \$601.31$

Difference in payments is $601.31 - 587.50 = \$13.81$

6.5 Calculating rate or time for compound interest

Question 1

Solve on CAS

$$\begin{aligned}10\,000 \times 1.055^n &= 20\,000\\ n &= 12.946\\ &= 13\end{aligned}$$

Question 2

a. $B_1 = 1.003 \times B_0 = 1.003 \times 5000 = 5015$

$B_2 = 1.003 \times B_1 = 1.003 \times 5015 = 5030.045$

$B_3 = 1.003 \times 5030.045 = 5045.14$

$B_4 = 1.003 \times 5045.14 = \$5060.27$ **[1 mark]**

b. Monthly interest rate is 0.3% **[1 mark]**

c. Use the rule $B_{n+1} = 1.003 \times B_n + 50$ to get $B_{12} = \$5793$.

Therefore, Samuel would have $\$5793 - \$5183 = \$610$ extra money. **[1 mark]**

Question 3

The interest rate is the number found in front of V_n.

1.01 means 101% = 100% + 1%, so the interest rate, per annum, is 1%.

Question 4

$$\begin{aligned}R_n &= R_0\left(1+\frac{r}{100}\right)^n\\ &= 5000\left(1+\frac{3}{4\times 100}\right)^{4\times n}\\ &= 5000(1.0075)^{4n}\end{aligned}$$

Question 5

$$\left[A\left(1+\frac{3.2}{1200}\right)^{120} + 10\,000\right]\left[1+\frac{2.8}{1200}\right]^{120} = 686\,904.09$$

$A = 370\ 000$

6.6 Nominal and effective annual interest rate

Question 1

$$r_{effective} = \left[\left(1+\frac{8}{100n}\right)^n - 1\right] \times 100\%$$

Go through each option to find which one is *not* correct.

When interest is charged fortnightly:

$r_{effective} = \left[\left(1+\dfrac{8}{100\times 26}\right)^{26} - 1\right] \times 100\% = 8.3154 = 8.32\%$, which does not match the percentage given.

Question 2

Loan A: Nominal rate: 8%. Effective rate is $\left(1+\dfrac{\frac{8}{365}}{100}\right)^{365} - 1 = 0.08328 = 8.33\%$

Loan B: Nominal rate: 8.1%. Effective rate is $\left(1+\dfrac{\frac{8.1}{2}}{100}\right)^{2} - 1 = 0.08264 = 8.26\%$

Loan A has the greater effective interest rate, 8.33%, by a margin of 0.07%.

Question 3

$n = 12$

$$\text{Effective rate} = \left(1 + \frac{i}{n}\right)^n - 1$$
$$= \left(1 + \frac{0.0585}{12}\right)^{12} - 1$$
$$= 0.060$$
$$\Rightarrow \text{Rate} = 6.0\%$$

Question 4

Interest rate used by bank: $20\,000\left(1 + \dfrac{r}{12 \times 100}\right)^{12} = 20732$, where $r = 3.6\%$ *p.a.*

Effective interest rate: $r_{eff} = \left[\left(1 + \dfrac{3.6}{100 \times 12}\right)^{12} - 1\right] \times 100\% = 3.66\%$

Difference $= 3.66 - 3.6 = 0.06\%$

Question 5

The smallest effective interest rate for the loans will cost Daniel the least amount of money.
Since Loans I, II and III are all compounding weekly, the lowest rate of the three will be Loan I with 12.6%.
Loans IV and V are both compounding quarterly, so the lowest rate out of them will be Loan IV with 12.7%.

Effective interest rate of Loan I: $r = \left(1 + \dfrac{i}{n}\right)^n - 1 = \left(1 + \dfrac{0.126}{52}\right)^{52} - 1 \approx 13.41\%$ annually

Effective interest rate of Loan IV: $r = \left(1 + \dfrac{i}{n}\right)^n - 1 = \left(1 + \dfrac{0.127}{4}\right)^{4} - 1 \approx 13.32\%$ annually

Since Loan IV has a lower effective interest rate over the year, it will cost Daniel the least amount of money.

***VCAA Assessment Report* note:**

This question required an understanding that both the interest rate and compounding period in combination determine the overall cost of the loan. Students who selected option A seemed to simply select the lowest interest rate without considering the compounding period.

Question 6

$$\text{Payments} = 6 \times 180$$
$$= 1080$$
$$I = 1080 - 1000$$
$$= 80$$
$$I = \frac{PRT}{100}$$
$$80 = \frac{1000 \times r \times 6}{100}$$
$$r = 1.3\% \text{ month}$$
$$= 16\% \text{ per annum}$$
$$\text{Effective interest rate} = \frac{2 \times 6}{6 + 1} \times 16$$
$$= 27.4\%$$

***VCAA Assessment Report* note:**

The majority of students did not correctly answer this question on determining an effective interest rate.
Calculating an effective interest problem is a two-step process.
Step 1 involved calculating the flat rate of interest.

interest paid = total amount repaid − amount owed = 61 × 80 − 1000 = \$1080 = \$80

$$\text{flat rate of interest } (r_f) = \left(\frac{\text{interest paid}}{\text{amount owed} \times \text{time in years}}\right) \times 100 = \left(\frac{80}{1000 \times 0.5}\right) \times 100 = 16\%$$

Step 2 involved converting this flat rate of interest into an effective rate of interest.
For n payments,

$$\text{effective interest} = \left(\frac{2n}{n+1}\right) \times r_f = \left(\frac{2 \times 6}{6+1}\right) \times 16 = 27.42...\% \text{ (option D)}$$

Question 7

Loan A: Nominal rate: 8%. Effective rate is $\left(1 + \frac{\frac{8}{365}}{100}\right)^{365} - 1 = 0.08328 = 8.33\%$

Loan B: Nominal rate: 8.1%. Effective rate is $\left(1 + \frac{\frac{8.1}{2}}{100}\right)^{2} - 1 = 0.08264 = 8.26\%$

Loan A has the greater effective interest rate, 8.33%, by a margin of 0.07%.

Question 8

Effective rate = 9.25% = 0.0925

$$\left(1 + \frac{\frac{r}{12}}{100}\right)^{12} - 1 = 0.0925$$

Solving on a calculator finds $r = 8.88$
Therefore, the nominal interest rate is 8.88%.

Question 9

$$\begin{aligned}\text{Effective rate} &= \left(1 + \frac{i}{n}\right)^{n} - 1 \\ &= \left(1 + \frac{0.1}{52}\right)^{52} - 1 \\ &= 0.105 \\ &\Rightarrow \text{Rate} = 10.5\%\end{aligned}$$

Question 10

$$\begin{aligned}\text{Effective rate} &= \left(1 + \frac{i}{n}\right)^{n} - 1 \\ &= \left(1 + \frac{0.0585}{72}\right)^{72} - 1 \\ &= 0.060 \\ &\Rightarrow \text{Rate} = 6.0\%\end{aligned}$$

6.7 Review

Question 1

a. $S_1 = 1.001 \times 570\,000 - 1193 = \$569\,377$ **[1 mark]**
b. $0.001 \times 26 \times 100 = 2.6\%$ **[1 mark]** (note that 0.1×26 and 0.001×2600 were accepted)

Question 2

The interest rate of 3.6% p.a. is equivalent to $\frac{3.6}{12}\%$ or 0.3% p.m.

Shirley has an interest-only loan, so her balance will remain at \$225 000 after she pays each monthly interest charge.

Question 3

a. \$15 000 **[1 mark]**

b. $V_0 = 15\,000$

$V_1 = 1.04 \times 15\,000 = \$15\,600$

$V_2 = 1.04 \times 15\,600 = \$16\,224$ **[1 mark]**

***VCAA Assessment Report* note:**

Some students used only the explicit rule $V_n = R^n V_0$ to write $V_2 = 1.04^2 \times 15\,600 = 16\,224$, but this was not using recursion as required.

c. There is an increase of 4% each compounding period. **[1 mark]**

d. i. $V_n = 1.04^n \times 15\,000$ **[1 mark]**

ii. $V_{10} = 1.04^{10} \times 15\,000 = \$22\,203.66$ **[1 mark]**

***VCAA Assessment Report* note:**

Many students only wrote an answer of \$22 203.7, which might arise from a calculator setting that restricts the number of displayed digits.

Some students appeared to round all currency answers to the nearest five or ten cents. A common incorrect answer was \$22 203.70, which had clearly been rounded from a written \$22 203.664... This was assessed as a rounding error.

Question 4

a. $I = V_n - V_0$

$= 60\,000 - 45\,550$

$= \$14\,450$ **[1 mark]**

b.
$$V_n = V_0 R^n$$
$$= V_0\left(1 + \frac{r}{100}\right)^n$$
$$60\,000 = 45\,550\left(1 + \frac{r/4}{100}\right)^{4\times 4}$$

Solve on CAS

$r = 6.9\%$ **[1 mark]**

***VCAA Assessment Report* note:**

A common error was to treat this as a simple interest question, despite the question stating that there were 'four years of compounding interest'.

c. i. Account balance $= 60\,000\left(1 + \dfrac{7.2/12}{100}\right)^1 + 885$

$= 60\,000\,(1 + 0.006) + 885$ **[1 mark]**

***VCAA Assessment Report* note:**

Answers equivalent to 0.006 were accepted in the first box, commonly $\dfrac{7/100}{12}$ or $\dfrac{7/12}{100}$. However, many students wrote only $\dfrac{7}{100}$ or $\dfrac{7}{12}$.

A number of students added a power to complete a formula to find the account balance at the end of 12 months, rather than the first month as required.

ii. Using the previous equation and working out each month's balance, after 12 months the balance is \$75 443. **[1 mark]**

Question 5

a. Interest $= \dfrac{7500 \times 8 \times 2}{100} = \$1200$ **[1 mark]**

Total to be repaid $= 1200 + 7500 = \$8700$

Monthly instalment $= 8700 \div 24 = \$362.50$ **[1 mark]**

***VCAA Assessment Report* note:**
Students often confuse simple (flat) interest with compound interest calculations. Many students inappropriately treated this as a reducing balance problem. Students should understand that the term ‘flat interest rate’ refers to simple interest.

b. Effective rate of interest:

$$r_e = \frac{100I}{Pt} \times \frac{2n}{n+1}$$

$$r_e = \frac{100 \times 1200}{7500 \times 2} \times \frac{2 \times 24}{24+1}$$

$r_e = 15.36\%$ **[1 mark]**

c. A flat rate is charged on the initial borrowed amount (for the whole duration of the hire-purchase agreement). An effective rate is charged on the reduced monthly balance. Therefore, an effective rate must be higher than the flat interest rate to result in the same amount of interest. **[1 mark]**

d. First, calculate the rate of depreciation. For reducing balance depreciation:

$$V = P \times \left(1 - \frac{r}{100}\right)^t$$

$$6375 = 7500 \times \left(1 - \frac{r}{100}\right)^1$$

$r = 15\%$ p.a **[1 mark]**

After 5 years:

$$V = P \times \left(1 - \frac{r}{100}\right)^t$$

$$V = 7500 \times \left(1 - \frac{15}{100}\right)^5$$

$V = 3327.79$

Therefore, the value of the bike after 5 years is \$3328. **[1 mark]**

Question 6

Use Finance Solver to find the nominal interest rate:

N: 48
I(%): ??
PV: −3000
Pmt: 0
FV: 3728.92
PpY: 12
CpY: 12

The nominal rate is 5.45%.

So, the effective interest rate is:

$$r_{eff} = \left[\left(1 + \frac{5.45}{100 \times 12}\right)^{12} - 1\right] \times 100\%$$

$$= 5.59\%$$

Question 7

a. $\frac{1440}{0.05} = 28\,800$ **[1 mark]**

b. $\frac{1440}{12\,000} \times 100 = 12\%$ **[1 mark]**

c. $M_n = 12\,000 - 0.05 \times n$ **[1 mark]**

Question 8

The investment is in compound interest, so there will be a common ratio of r.

$500 \times r = 580$

$r = 1.16$

Therefore:

$b = 580 \times 1.16$

$b = 672.80$

Question 9

$\$850\,000 - \$600\,000 = \$250\,000$ increase in the property value.

$$\begin{aligned}\frac{250\,000}{600\,000} \times 100\% &= 41.666\,67\% \\ &\approx 41.7\%\end{aligned}$$

Question 10

Thursday: $0.97 \times 160 = \$155.20$

Friday: $1.045 \times 155.20 = \$162.184$

Question 11

a. $2150 - 2000 = 150$ **[1 mark]**

b. Add 150 to each term until you reach week 5, so the sequence will be 2000, 2150, 2300, 2450, 2600.
So in week 5, there will be 2600 kg of capsicums picked. **[1 mark]**

c. You can either continue the sequence to week 8 and add them up, or use the series formula.

$$\begin{aligned}S_8 &= \frac{8}{2}\left[2\,(2000) + (8-1)\,(150)\right] \\ &= 20\,200\text{ kg}\end{aligned}$$ **[1 mark]**

d. Using solve on CAS:

$$\begin{aligned}14\,000 &= \frac{n}{2}\left[2\,(2000) + (n-1)\,(150)\right] \\ n &= 5.9113\end{aligned}$$

Therefore, the harvest will first exceed 14 000 kg in the 6th week. **[1 mark]**

e. The difference each week is 150 kg, so $C_{n+1} = C_n + 150, \quad C_1 = 2000$. **[1 mark]**

Question 12

Total amount paid for the car $= 3000 + 36 \times 400 = \$17\,400$

Interest, $I = 15\,000 - 17\,400 = \$2400$

Principal owing after the deposit had been paid, $P = 15\,000 - 3000 = \$12\,000$

Number of years of repayments, $t = 36 \div 12 = 3$

Flat interest rate per annum, $r_f = \dfrac{100I}{Pt} = \dfrac{100 \times 2400}{12\,000 \times 3} = 6.6666...\% \approx 6.7\%$

7 Modelling reducing balance loans, annuities and perpetuities using recursion

Topic	7	Modelling reducing balance loans, annuities and perpetuities using recursion
Subtopic	7.2	Modelling reducing balance loans with recurrence relations

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Source: *VCE 2019, Further Mathematics 1, Section A, Q.20; © VCAA*

Question 1 (1 mark)

Consider the following amortisation table for a reducing balance loan.

Payment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	300 000.00
1	1050.00	900.00	150.00	299 850.00
2	1050.00	899.55	150.45	299 699.55
3	1050.00	899.10	150.90	299 548.65

The annual interest rate for this loan is 3.6%.
Interest is calculated immediately before each payment.
For this loan, the repayments are made

A. weekly.
B. fortnightly.
C. monthly.
D. quarterly.
E. yearly.

Source: *VCE 2018, Further Mathematics 1, Section A, Q.22; © VCAA*

Question 2 (1 mark)

Adam has a home loan with a present value of $175 260.56

The interest rate for Adam's loan is 3.72% per annum, compounding monthly.

His monthly repayment is $3200.

The loan is to be fully repaid after five years.

Adam knows that the loan cannot be exactly repaid with 60 repayments of $3200.

To solve this problem, Adam will make 59 repayments of $3200. He will then adjust the value of the final repayment so that the loan is fully repaid with the 60th repayment.

The value of the 60th repayment will be closest to

A. $368.12

B. $2831.88

C. $3200.56

D. $3557.09

E. $3568.12

Source: *VCE 2017, Further Mathematics 1, Section A, Core, Q.17; © VCAA*

Question 3 (1 mark)

The value of a reducing balance loan, in dollars, after n months, V_n, can be modelled by the recurrence relation shown below.

$V_0 = 26\,000, \quad V_{n+1} = 1.003V_n - 400$

What is the value of this loan after five months?

A. $24 380.31

B. $24 706.19

C. $25 031.10

D. $25 355.03

E. $25 678.00

Source: *VCE 2019, Further Mathematics Exam 1, Section A, Q.23; © VCAA*

Question 4 (1 mark)

Joseph borrowed $50 000 to buy a new car.
Interest on this loan is charged at the rate of 7.5% per annum, compounding monthly.
Joseph will fully repay this loan with 60 monthly repayments over five years.
Immediately after the 59th repayment is made, Joseph still owes $995.49
The value of his final repayment, to the nearest cent, will be

A. $995.49
B. $998.36
C. $1001.71
D. $1001.90
E. $1070.15

Source: *VCE 2016, Further Mathematics 1, Section A, Q.22; © VCAA*

Question 5 (1 mark)

The first three lines of an amortisation table for a reducing balance home loan are shown below.
The interest rate for this home loan is 4.8% per annum compounding monthly.
The loan is to be repaid with monthly payments of $1500

Payment number	Payment	Interest	Principal reduction	Balance of loan
0	0	0	0	250 000.00
1	1500	1000.00	500.00	249 500.00
2	1500			

The amount of payment number 2 that goes towards reducing the principal of the loan is

A. $486
B. $502
C. $504
D. $996
E. $998

Question 6 (1 mark)
Jane and Hashim have been paying $2500 in monthly instalments towards their home loan. At the start of the month, there is $70 000 still outstanding on their 6.3% p.a. loan.
How much of the principal will still be outstanding at the end of the first month?
A. $2132.50
B. $65 590.00
C. $67 500.00
D. $67 867.50
E. $69 632.50

Question 7 (1 mark)
Interest is charged on a loan of $25 000 at a rate of 0.8% per month. The monthly repayments are $225.
How much of the principal is still outstanding after 2 months?
A. $24 949.80
B. $24 959.60
C. $24 965.00
D. $24 975.00
E. $24 987.50

Question 8 (1 mark)
Interest is charged on a loan of $75 000 at a rate of 0.7% per month. The monthly repayments are $725.
How much of the principal is still outstanding after 2 months?
A. $74 949.80
B. $74 859.00
C. $74 965.00
D. $74 800.00
E. $74 598.60

Topic	7	Modelling reducing balance loans, annuities and perpetuities using recursion
Subtopic	7.3	Solving reducing balance loan problems using finance solver

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Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.22; © VCAA*

Question 1 (1 mark)

Joanna deposited $12 000 in an investment account earning interest at the rate of 2.8% per annum, compounding monthly.

She would like this account to reach a balance of $25 000 after five years.

To achieve this balance, she will make an extra payment into the account each month, immediately after the interest is calculated.

The minimum value of this payment is closest to

A. $113.85
B. $174.11
C. $580.16
D. $603.22
E. $615.47

Source: *VCE 2015, Further Mathematics 1, Section B, Module 4, Q.8; © VCAA*

Question 2 (1 mark)

Cindy took out a reducing balance loan of $8400 to finance an overseas holiday.

Interest was charged at a rate of 9% per annum, compounding quarterly.

Her loan is to be fully repaid in six years, with equal quarterly payments.

After three years, Cindy will have reduced the balance of her loan by approximately

A. 9%
B. 35%
C. 43%
D. 50%
E. 57%

Topic 7 Subtopic 7.3 Solving reducing balance loan problems using finance solver

Source: *VCE 2014, Further Mathematics 1, Section B, Module 4, Q.5; © VCAA*

Question 3 (1 mark)

A bank approves a $90 000 loan for a customer.
The loan is to be repaid fully over 20 years in equal monthly payments.
Interest is charged at a rate of 6.95% per annum on the reducing monthly balance.
To the nearest dollar, the monthly payment will be

A. $478
B. $692
C. $695
D. $1409
E. $1579

Source: *VCE 2016, Further Mathematics 1, Section A, Q.23; © VCAA*

Question 4 (1 mark)

Sarah invests $5000 in a savings account that pays interest at the rate of 3.9% per annum compounding quarterly. At the end of each quarter, immediately after the interest has been paid, she adds $200 to her investment.
After two years, the value of her investment will be closest to

A. $5805
B. $6600
C. $7004
D. $7059
E. $9285

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 4, Q.9; © VCAA*

Question 5 (1 mark)

Ravi borrowed $160 000 at an interest rate of 6.18% per annum.
Interest is calculated monthly on the reducing balance of the loan.
The loan will be fully repaid with monthly payments of $1950.
Which one of the following statements is **not** true?

A. His first payment reduces the loan by less than $1950.
B. His second payment reduces the loan by more than the first payment.
C. Repaying more than $1950 per month will reduce the term of the loan.
D. His final payment will be less than $1760.
E. His final payment includes interest.

Source: *VCE 2014, Further Mathematics Exam 1, Section B, Module 4, Q.9; © VCAA*

Question 6 (1 mark)

Leslie borrowed \$35 000 from a bank.

Interest is charged at the rate of 4.75% on the reducing monthly balance.

The loan is to be repaid with 47 monthly payments of \$802.00 and a final payment that is to be adjusted so that the loan will be fully repaid after exactly 48 monthly payments.

Correct to the nearest cent, the amount of the final payment will be

A. \$0.39
B. \$3.57
C. \$802.00
D. \$802.39
E. \$805.57

Source: *VCE 2013, Further Mathematics Exam 2, Module 4, Q.4; © VCAA*

Question 7 (2 marks)

Hugo took out a reducing balance loan of \$25 000 to compete in road races overseas.

Interest was charged at a rate of 12% per annum compounding quarterly.

His loan is to be repaid fully in four years with equal quarterly payments.

After two years, how much of the \$25 000 will Hugo have repaid?

Write your answer, correct to the nearest dollar.

Topic	7	Modelling reducing balance loans, annuities and perpetuities using recursion
Subtopic	7.4	The effect of rate and repayment changes on reducing balance loans

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Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.24; © VCAA*

Question 1 (1 mark)

Bob borrowed $400 000 to buy an apartment.

The interest rate for this loan was 3.14% per annum, compounding monthly.

A scheduled monthly repayment that allowed Bob to fully repay the loan in 20 years was determined.

Bob decided, however, to make interest-only repayments for the first two years.

After these two years the interest rate changed. Bob was still able to pay off the loan in the 20 years by repaying the scheduled amount each month.

The interest rate, per annum, for the final 18 years of the loan was closest to

A. 1.85%
B. 2.21%
C. 2.79%
D. 3.14%
E. 4.07%

Source: *VCE 2017, Further Mathematics 1, Section A, Core, Q.24; © VCAA*

Question 2 (1 mark)

Xavier borrowed $245 000 to pay for a house.

For the first 10 years of the loan, the interest rate was 4.35% per annum, compounding monthly.

Xavier made monthly repayments of $1800.

After 10 years, the interest rate changed.

If Xavier now makes monthly repayments of $2000, he could repay the loan in a further five years.

The new annual interest rate for Xavier's loan is closest to

A. 0.35%
B. 4.1%
C. 4.5%
D. 4.8%
E. 18.7%

Topic 7 Subtopic 7.4 The effect of rate and repayment changes on reducing balance loans

Source: *VCE 2013, Further Mathematics 1, Section B, Module 4, Q.9; © VCAA*

Question 3 (1 mark)

The following information relates to the repayment of a home loan of \$300 000.

- The loan is to be repaid fully with monthly payments of \$2500.
- After the first monthly payment has been made, the amount owing on the loan is \$299 000.
- Interest compounds monthly.

Which one of the following statements is true?

A. After two months, \$297 995 is still owing on the loan.
B. \$1000 of interest has been paid in the first month.
C. The loan will be fully repaid in less than 15 years.
D. Halfway through the term of the loan, the amount still owing will be \$150 000.
E. Payments of \$2750 rather than \$2500 per month will reduce the time to repay the loan fully by more than three years.

Source: *VCE 2015, Further Mathematics 1, Section B, Module 4, Q.8; © VCAA*

Question 4 (1 mark)

Cindy took out a reducing balance loan of \$8400 to finance an overseas holiday.
Interest was charged at a rate of 9% per annum, compounding quarterly.
Her loan is to be fully repaid in six years, with equal quarterly payments.
After three years, Cindy will have reduced the balance of her loan by approximately

A. 9%
B. 35%
C. 43%
D. 50%
E. 57%

Topic	7	Modelling reducing balance loans, annuities and perpetuities using recursion
Subtopic	7.5	Annuities and perpetuities

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Source: *VCE 2020, Further Mathematics 2, Section A, Q.10; © VCAA*

Question 1 (3 marks)

Samuel now invests \$500 000 in an annuity from which he receives a regular monthly payment. The balance of the annuity, in dollars, after n months, A_n, can be modelled by a recurrence relation of the form

$A_0 = 500\,000, \qquad A_{n+1} = kA_n - 2000$

a. Calculate the balance of this annuity after two months if $k = 1.0024$ **(1 mark)**

b. Calculate the annual compound interest rate percentage for this annuity if $k = 1.0024$ **(1 mark)**

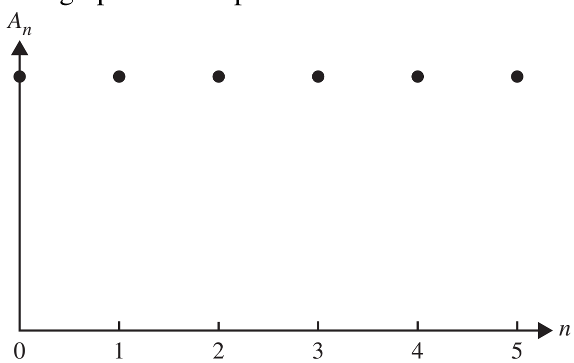
c. For what value of k would this investment act as a simple perpetuity? **(1 mark)**

Topic 7 Subtopic 7.5 Annuities and perpetuities

Source: *VCE 2020, Further Mathematics 1, Section A, Q.25; © VCAA*

Question 2 (1 mark)

The graph below represents the value of an annuity investment, A_n, in dollars, after n time periods.



A recurrence relation that could match this graphical representation is

A. $A_0 = 200\,000, A_{n+1} = 1.015A_n - 2500$
B. $A_0 = 200\,000, A_{n+1} = 1.025A_n - 5000$
C. $A_0 = 200\,000, A_{n+1} = 1.03A_n - 5500$
D. $A_0 = 200\,000, A_{n+1} = 1.04A_n - 6000$
E. $A_0 = 200\,000, A_{n+1} = 1.05A_n - 8000$

Source: *VCE 2016, Further Mathematics 1, Section A, Q.24; © VCAA*

Question 3 (1 mark)

Mai invests in an annuity that earns interest at the rate of 5.2% per annum compounding monthly.
Monthly payments are received from the annuity.
The balance of the annuity will be \$130 784.93 after five years.
The balance of the annuity will be \$66 992.27 after 10 years.
The monthly payment that Mai receives from the annuity is closest to

A. \$1270
B. \$1400
C. \$1500
D. \$2480
E. \$3460

Topic 7 | Subtopic 7.5 Annuities and perpetuities

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.21; © VCAA*

Question 4 (1 mark)

Which one of the following recurrence relations could be used to model the value of a perpetuity investment, P_n, after n months?

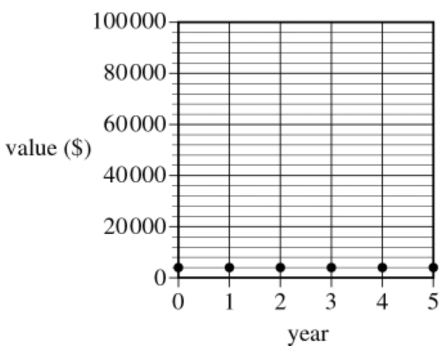
A. $P_0 = 120\,000, \quad P_{n+1} = 1.0029 \times P_n - 356$
B. $P_0 = 180\,000, \quad P_{n+1} = 1.0047 \times P_n - 846$
C. $P_0 = 210\,000, \quad P_{n+1} = 1.0071 \times P_n - 1534$
D. $P_0 = 240\,000, \quad P_{n+1} = 0.0047 \times P_n - 2232$
E. $P_0 = 250\,000, \quad P_{n+1} = 0.0085 \times P_n - 2125$

Source: *VCE 2016, Further Mathematics Exam 1, Section A, Q.21; © VCAA*

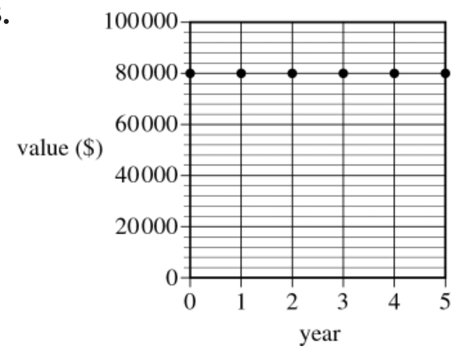
Question 5 (1 mark)

Juanita invests $80 000 in a perpetuity that will provide $4000 per year to fund a scholarship at a university. The graph that shows the value of this perpetuity over a period of five years is

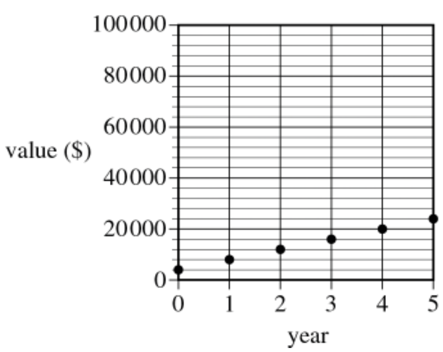
A.



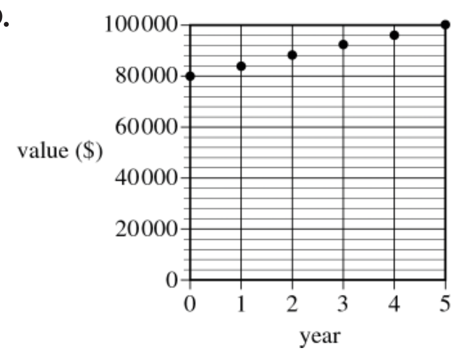
B.



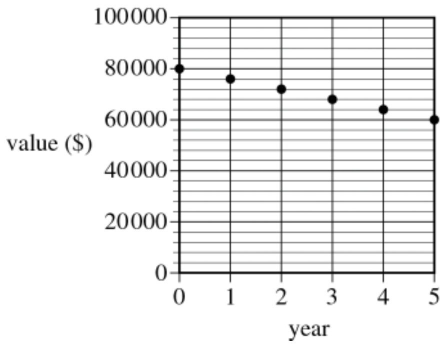
C.



D.



E.



Question 6 (2 marks)

Using the information below:

Principal (V_0) = $30 000, Interest ($r$) = 5% per annum, Compounding period (n) = monthly,
Repayment (d) = $200

Complete the following table showing the repayment made, interest charged, reduction in Principal and the loan balance.

Month	Repayment	Interest	Principal paid	Loan balance
1	$200	$125	$75	$29 925
2	$200			
3	$200			

Question 7 (1 mark)

Using the annuities recurrence relation of $V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d,\ V_0 = \text{Principal}$

and the information below:

Principal (V_0) = $25 000, Interest = 6.3% per annum, d = $450, Compounding periods = monthly,
the value of V_3 is

A. $23 714.92
B. $24 038.72
C. $24 360.83
D. $24 681.00
E. $25 000.00

Question 8 (1 mark)
Basil takes a 20 year loan of $210 000 at 6.5% per annum, compounding monthly and with monthly repayments. To fully repay the loan in 20 years, the amount he must repay each month is
A. $1320.43
B. $18 757.16
C. $11 088.41
D. $1232.26
E. $1565.70

Question 9 (1 mark)
Using the annuities recurrence relation of $V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d,\ \ V_0 =$ Principal and the information below:
A_3 = $150 000, Interest = 6.2% per annum, d = $1100, Compounding periods = monthly.
The value of V_2 is
A. $13 977.50
B. $15 000.00
C. $150 323.33
D. $150 645.00
E. $14 528.00

Question 10 (1 mark)
Interest is paid monthly into an account at a rate of 3.5% per annum. Each month, immediately after the interest is paid, the account is debited $10 in fees. No other money is taken from the account. The initial amount of money in the account is $15 500.
After all the interest has been paid and fees debited, the balance in the account at the end of **two** months is
A. $15 555.20
B. $15 610.58
C. $15 500.00
D. $15 535.21
E. $15 570.52

Question 11 (1 mark)
An ex-student would like to provide an annual trophy to their former school by setting up an account which will pay for the trophy each year.
If the annual interest is 3%, what is the minimum amount that the ex-student should invest so that a $100 trophy can be bought in perpetuity?
A. $103.00
B. $1000.00
C. $1300.00
D. $2222.22
E. $3333.33

Question 12 (1 mark)
A grandmother sets up an account for her first granddaughter so that she will receive a $200 birthday present each year. If the grandmother invests $4000, what is the minimum interest required so that the granddaughter's present continues for perpetuity?
A. 5%
B. 7.5%
C. 10%
D. 12.5%
E. 15%

Question 13 (1 mark)
A benefactor left an amount of $115 000 to the local bowls club. The club's committee decides to invest the money at a rate of 6% p.a.
What is the maximum amount the bowls club could remove annually if the total is to remain at $115 000 for perpetuity?
A. $6000.00
B. $6509.43
C. $6900.00
D. $7124.37
E. $8315.07

Topic	7	Modelling reducing balance loans, annuities and perpetuities using recursion
Subtopic	7.6	Annuity investments

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Source: *VCE 2017, Further Mathematics 1, Section A, Core, Q.23; © VCAA*

Question 1 (1 mark)

Four lines of an amortisation table for an annuity investment are shown below.

The interest rate for this investment remains constant, but the payment value may vary.

Payment number	Payment	Interest	Principal reduction	Balance
17	100.00	27.40	127.40	6977.50
18	100.00	27.91	127.91	7105.41
19	100.00	28.42	128.42	7233.83
20				7500.00

The balance of the investment after payment number 20 is $7500.

The value of payment number 20 is closest to

A. $29

B. $100

C. $135

D. $237

E. $295

Source: *VCE 2017, Further Mathematics 2, Section A, Core, Q.7; © VCAA*

Question 2 (3 marks)

Alex sold his mechanics' business for $360 000 and invested this amount in a perpetuity.

The perpetuity earns interest at the rate of 5.2% per annum.

Interest is calculated and paid monthly.

a. What monthly payment will Alex receive from this investment? **(1 mark)**

b. Later, Alex converts the perpetuity to an annuity investment.
This annuity investment earns interest at the rate of 3.8% per annum, compounding monthly.
For the first four years Alex makes a further payment each month of $500 to his investment.
This monthly payment is made immediately after the interest is added.
After four years of these regular monthly payments, Alex increases the monthly payment.
This new monthly payment gives Alex a balance of $500 000 in his annuity after a further two years.
What is the value of Alex's new monthly payment?
Round your answer to the nearest cent. **(2 marks)**

Source: *VCE 2015, Further Mathematics 2, Module 4, Q.3; © VCAA*

Question 3 (2 marks)

Jane and Michael decide to set up an annual music scholarship.
To fund the scholarship, they invest in a perpetuity that pays interest at a rate of 3.68% per annum.
The interest from this perpetuity is used to provide an annual $460 scholarship.

a. Determine the minimum amount they must invest in the perpetuity to fund the scholarship. **(1 mark)**

b. For how many years will they be able to provide the scholarship? **(1 mark)**

Source: *VCE 2018, Further Mathematics Exam 1, Section A, Q.17; © VCAA*

Question 4 (1 mark)

The value of an annuity investment, in dollars, after n years, V_n, can be modelled by the recurrence relation shown below.

$V_0 = 46\,000, \quad V_{n+1} = 1.0034V_n + 500$

What is the value of the regular payment added to the principal of this annuity investment?

A. \$34.00
B. \$156.40
C. \$466.00
D. \$500.00
E. \$656.40

Source: ***VCE 2018, Further Mathematics Exam 1, Section A, Q.18; © VCAA***

Question 5 (1 mark)

The value of an annuity investment, in dollars, after n years, V_n, can be modelled by the recurrence relation shown below.

$V_0 = 46\,000, \quad V_{n+1} = 1.0034V_n + 500$

Between the second and third years, the increase in the value of this investment is closest to

A. \$656
B. \$658
C. \$661
D. \$1315
E. \$1975

Source: *VCE 2018, Further Mathematics 1, Section A, Q.24; © VCAA*

Question 6 (1 mark)

Mariska plans to retire from work 10 years from now.

Her retirement goal is to have a balance of $600 000 in an annuity investment at that time.

The present value of this annuity investment is $265 298.48, on which she earns interest at the rate of 3.24% per annum, compounding monthly.

To make this investment grow faster, Mariska will add a $1000 payment at the end of every month.

Two years from now, she expects the interest rate of this investment to fall to 3.20% per annum, compounding monthly. It is expected to remain at this rate until Mariska retires.

When the interest rate drops, she must increase her monthly payment if she is to reach her retirement goal.

The value of this new monthly payment will be closest to

A. $1234
B. $1250
C. $1649
D. $1839
E. $1854

Question 7 (1 mark)

Blair pays $450 into a superannuation account each month which earns 5.8% interest per annum, compounded monthly. His initial balance at the beginning of 2016 is $12 000. Calculate the balance of the superannuation account after three months of payments.

A. $12 508.00
B. $12 554.48
C. $12 601.10
D. $13 018.45
E. $13 018.45

Question 8 (1 mark)

Using the information below:

Initial balance = $\$30\,000$, Interest ($r$) = 5% per annum,

Compounding interest period (n) = monthly, Payment (d) = $800

Complete the following table showing the payment made, interest earned and account balance.

Month	Payment	Interest	Account balance
1	$800	$125	$30 925
2	$800		
3	$800		

Topic	7	Modelling reducing balance loans, annuities and perpetuities using recursion
Subtopic	7.7	Review

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Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.18; © VCAA*

Question 1 (1 mark)

Deepa invests \$500 000 in an annuity that provides an annual payment of \$44 970.55

Interest is calculated annually.

The first five lines of the amortisation table are shown below.

Payment number	Payment (\$)	Interest (\$) P	Principal reduction (\$)	Balance (\$)
0	0.00	0.00	0.00	500 000.00
1	44 970.55	20 000.00	24 970.55	475 029.45
2	44 970.55	19 001.18	25 969.37	449 060.08
3	44 970.55	17 962.40		422 051.93
4	44 970.55	16 882.08	28 088.47	393 963.46

The principal reduction associated with payment number 3 is

A. \$17 962.40

B. \$25 969.37

C. \$27 008.15

D. \$28 088.47

E. \$44 970.55

Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.19; © VCAA*

Question 2 (1 mark)

Deepa invests \$500 000 in an annuity that provides an annual payment of \$44 970.55

Interest is calculated annually.

The first five lines of the amortisation table are shown below.

Payment number	Payment (\$)	Interest (\$) P	Principal reduction (\$)	Balance (\$)
0	0.00	0.00	0.00	500 000.00
1	44 970.55	20 000.00	24 970.55	475 029.45
2	44 970.55	19 001.18	25 969.37	449 060.08
3	44 970.55	17 962.40		422 051.93
4	44 970.55	16 882.08	28 088.47	393 963.46

The number of years, in total, for which Deepa will receive the regular payment of \$44 970.55 is closest to

A. 12
B. 15
C. 16
D. 18
E. 20

Source: *VCE 2020, Further Mathematics 2, Section A, Q.8; © VCAA*

Question 3 (4 marks)

Samuel has a reducing balance loan.

The first five lines of the amortisation table for Samuel's loan are shown below.

Payment number	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance (\$)
0	0.00	0.00	0.00	320 000.00
1	1600.55	960.00	640.00	319 360.00
2	1600.55	958.08	641.92	318 718.08
3	1600.55	956.15		318 074.23
4	1600.55			

Interest is calculated monthly and Samuel makes monthly payments of \$1600.

Interest is charged on this loan at the rate of 3.6% per annum.

a. Using the values in the amortisation table **(2 marks)**

i. calculate the principal reduction associated with payment number 3 **(1 mark)**

ii. calculate the balance of the loan after payment number 4 is made. **(1 mark)**
Round your answer to the nearest cent.

b. Let S_n be the balance of Samuel's loan after n months. **(2 marks)**
Write down a recurrence relation, in terms of S_0, S_{n+1} and S_n, that could be used to model the month-to-month balance of the loan.

Source: *VCE 2018, Further Mathematics 1, Section A, Q.23; © VCAA*

Question 4 (1 mark)

Five lines of an amortisation table for a reducing balance loan with monthly repayments are shown below.

Repayment number	Repayment	Interest	Principal reduction	Balance of loan
25	\$2200.00	\$972.24	\$1227.76	\$230 256.78
26	\$2200.00	\$967.08	\$1232.92	\$229 023.86
27	\$2200.00	\$961.90	\$1238.10	\$227 785.76
28	\$2200.00	\$1002.26	\$1197.74	\$226 588.02
29	\$2200.00	\$996.99	\$1203.01	\$225 385.01

The interest rate for this loan changed immediately before repayment number 28.
This change in interest rate is best described as

A. an increase of 0.24% per annum.
B. a decrease of 0.024% per annum.
C. an increase of 0.024% per annum.
D. a decrease of 0.0024% per annum.
E. an increase of 0.00024% per annum.

Source: *VCE 2016, Further Mathematics 2, Core, Q.7; © VCAA*

Question 5 (1 mark)

Ken has borrowed $70 000 to buy a new caravan.

He will be charged interest at the rate of 6.9% per annum, compounding monthly.

a. Answer the following. **(2 marks)**

i. Find the amount that Ken will owe on his loan after he has made 12 repayments. **(1 mark)**

ii. What is the total interest that Ken will have paid after 12 repayments? **(1 mark)**

b. After three years, Ken will make a lump sum payment of $$L$ in order to reduce the balance of his loan. **(2 marks)**

This lump sum payment will ensure that Ken's loan is fully repaid in a further three years.

Ken's repayment amount remains at $800 per month and the interest rate remains at 6.9% per annum, compounding monthly.

What is the value of Ken's lump sum payment, $$L$?

Round your answer to the nearest dollar.

Source: *VCE 2021, Further Mathematics 1, Section A, Core, Q.23; © VCAA*

Question 6 (1 mark)

Bimal has a reducing balance loan.

The balance, in dollars, of the loan from month to month, B_n, is modelled by the recurrence relation below.

$B_0 = 450\,000, \qquad B_{n+1} = RB_n - 2633$

Given that the loan will be fully repaid in 20 years, the value of R is closest to

A. 1.003
B. 1.0036
C. 1.03
D. 1.0368
E. 1.36

Source: *VCE 2021, Further Mathematics 2, Section A, Core, Q.9; © VCAA*

Question 7 (3 mark)

Sienna invests \$152 431 into an annuity from which she will receive a regular monthly payment of \$900 for 25 years. The interest rate for this annuity is 5.1% per annum, compounding monthly.

a. Let V_n be the balance of the annuity after n monthly payments. A recurrence relation written in terms of V_0, V_{n+1} and V_n can model the value of this annuity from month to month. **(2 marks)**
Showing recursive calculations, determine the value of the annuity after two months.
Round your answer to the nearest cent.

b. After two years, the interest rate for this annuity will fall to 4.6%. **(1 mark)**
To ensure that she will still receive the same number of \$900 monthly payments, Sienna will add an extra one-off amount into the annuity at this time.
Determine the value of this extra amount that Sienna will add.
Round your answer to the nearest cent.

Source: *VCE 2020, Further Mathematics Exam 2, Section A, Q.11; © VCAA*

Question 8 (1 mark)

Later, Samuel took out a new reducing balance loan.

The interest rate for this loan was 4.1% per annum, compounding monthly.

The balance of the loan after four years of monthly repayments was \$329 587.25.

The balance of the loan after seven years of monthly repayments was \$280 875.15.

Samuel will continue to make the same monthly repayment.

To ensure the loan is fully repaid, to the nearest cent, the required final repayment will be lower.

In the first seven years, Samuel made 84 monthly repayments.

From this point on, how many more monthly repayments will Samuel make to fully repay the loan?

Source: *VCE 2019, Further Mathematics Exam 2, Section A, Q.8; © VCAA*

Question 9 (1 mark)

Phil invests \$200 000 in an annuity from which he receives a regular monthly payment.

The balance of the annuity, in dollars, after n months, A_n, can be modelled by the recurrence relation

$A_0 = 200\,000, \quad A_{n+1} = 1.0035A_n - 3700$

a. What monthly payment does Phil receive? **(1 mark)**

b. Show that the annual percentage compound interest rate for this annuity is 4.2%. **(1 mark)**

c. At some point in the future, the annuity will have a balance that is lower than the monthly payment amount. **(1 mark)**

What is the balance of the annuity when it first falls below the monthly payment amount?

Round your answer to the nearest cent.

d. If the payment received each month by Phil had been a different amount, the investment would act as a simple perpetuity. **(1 mark)**

What monthly payment could Phil have received from this perpetuity?

Source: *VCE 2019, Further Mathematics Exam 2, Section A, Q.9; © VCAA*

Question 10 (4 marks)

Phil would like to purchase a block of land.

He will borrow \$350 000 to make this purchase.

Interest on this loan will be charged at the rate of 4.9% per annum, compounding fortnightly.

After three years of equal fortnightly repayments, the balance of Phil's loan will be \$262 332.33

a. What is the value of each fortnightly repayment Phil will make?
Round your answer to the nearest cent. **(1 mark)**

b. What is the total interest Phil will have paid after three years?
Round your answer to the nearest cent. **(1 mark)**

c. Over the next four years of his loan, Phil will make monthly repayments of \$3517.28 and will be charged interest at the rate of 4.8% per annum, compounding monthly.
Let B_n be the balance of the loan n months after these changes apply.
Write down a recurrence relation, in terms of B_0, B_{n+1} and B_n, that could be used to model the balance of the loan over these four years. **(2 marks)**

Source: *VCE 2018, Further Mathematics Exam 2, Section A, Q.4; © VCAA*

Question 11 (5 marks)

Julie deposits some money into a savings account that will pay compound interest every month.

The balance of Julie's account, in dollars, after n months, V_n, can be modelled by the recurrence relation shown below.

$V_0 = 12\,000, \qquad V_{n+1} = 1.0062\, V_n$

a. How many dollars does Julie initially invest? **(1 mark)**

b. Recursion can be used to calculate the balance of the account after one month. **(1 mark)**

i. Write down a calculation to show that the balance in the account after one month, V_1, is \$12 074.40 **(1 mark)**

ii. After how many months will the balance of Julie's account first exceed \$12 300? **(1 mark)**

c. A rule of the form $V_n = a \times b^n$ can be used to determine the balance of Julie's account after n months.

i. Complete this rule for Julie's investment after n months by writing the appropriate numbers in the spaces provided below. **(1 mark)**

balance = ________ × ________^{n}

ii. What would be the value of n if Julie wanted to determine the value of her investment after three years? **(1 mark)**

Source: *VCE 2018, Further Mathematics Exam 2, Section A, Q.6; © VCAA*

Question 12 (4 marks)

Julie has retired from work and has received a superannuation payment of \$492 800.

She has two options for investing her money.

Option 1

Julie could invest the \$492 800 in a perpetuity. She would then receive \$887.04 each fortnight for the rest of her life.

a. At what annual percentage rate is interest earned by this perpetuity? **(1 mark)**

b. Option 2

Julie could invest the \$492 800 in an annuity, instead of a perpetuity.

The annuity earns interest at the rate of 4.32% per annum, compounding monthly.

The balance of Julie's annuity at the end of the first year of investment would be \$480 242.25 **(3 marks)**

i. What monthly payment, in dollars, would Julie receive? **(1 mark)**

ii. How much interest would Julie's annuity earn in the second year of investment?
Round your answer to the nearest cent. **(1 mark)**

Source: *VCE 2016, Further Mathematics Exam 2, Core, Q.6; © VCAA*

Question 13 (3 marks)

Ken's first caravan had a purchase price of \$38 000.

After eight years, the value of the caravan was \$16 000.

a. Show that the average depreciation in the value of the caravan per year was \$2750. **(1 mark)**

b. Let C_n be the value of the caravan n years after it was purchased. **(1 mark)**
Assume that the value of the caravan has been depreciated using the flat rate method of depreciation.
Write down a recurrence relation, in terms of C_{n+1} and C_n, that models the value of the caravan.

c. The caravan has travelled an average of 5000 km in each of the eight years since it was purchased. **(1 mark)**
Assume that the value of the caravan has been depreciated using the unit cost method of depreciation.
By how much is the value of the caravan reduced per kilometre travelled?

Source: *VCE 2015, Further Mathematics Exam 1, Section B, Module 4, Q.5; © VCAA*

Question 14 (1 mark)

The purchase price of a car is \$20 000.
A deposit of \$5000 is paid.
The balance will be repaid with 60 monthly repayments of \$400.
The total amount of interest charged is

A. \$1000
B. \$4000
C. \$9000
D. \$19 000
E. \$24 000

Source: *VCE 2015, Further Mathematics Exam 2, Module 4, Q.4; © VCAA*

Question 15 (1 mark)

As their business grows, Jane and Michael decide to invest some of their earnings.
They each choose a different investment strategy.
Jane opens an account with Red Bank, with an initial deposit of \$4000.
Interest is calculated at a rate of 3.6% per annum, compounding monthly.

a. Determine the amount in Jane's account at the end of six months. **(1 mark)**
Write your answer correct to the nearest cent.

b. Michael decides to open an account with Blue Bank, with an initial deposit of \$2000. **(1 mark)**
At the end of each quarter, he adds an additional \$200 to his account.
Interest is compounded at the end of each quarter.
The equation below can be used to determine the balance of Michael's account at the end of the first quarter.
account balance = $2000 \times (1 + 0.008) + 200$
Show that the annual compounding rate of interest is 3.2%.

c. Determine the amount in Michael's account, after the \$200 has been added, at the end of five years. **(1 mark)**
Write your answer correct to the nearest cent.

Source: *VCE 2014, Further Mathematics 2, Module 4, Q.4; © VCAA*

Question 16 (2 marks)

The cricket club borrowed \$400 000 to build a clubhouse.
Interest is calculated at the rate of 4.5% per annum, compounding monthly.
The cricket club will make monthly repayments of \$2500.
After a number of monthly repayments, the balance of the loan will be reduced to \$143 585.33.
What percentage of the next monthly repayment will reduce the balance of the loan?
Write your answer, correct to the nearest percentage.

Source: *VCE 2014, Further Mathematics Exam 2, Module 4, Q.2; © VCAA*

Question 17 (4 marks)

A sponsor of the cricket club has invested \$20 000 in a perpetuity.
The annual interest from this perpetuity is \$750.
The interest from the perpetuity is given to the best player in the club every year, for a period of 10 years.

a. What is the annual rate of interest for this perpetuity investment? **(1 mark)**

b. After 10 years, how much money is still invested in the perpetuity? **(1 mark)**

c. The average rate of inflation over the next 10 years is expected to be 3% per annum. **(1 mark)**

i. Michael was the best player in 2014 and he considered purchasing cricket equipment that was valued at $750. **(1 mark)**
What is the expected price of this cricket equipment in 2015?

ii. What is the 2014 value of cricket equipment that could be bought for $750 in 2024? **(1 mark)**
Write your answer, correct to the nearest dollar.

Source: *VCE 2013, Further Mathematics Exam 1, Section B, Module 4, Q.5; © VCAA*

Question 18 (1 mark)

$100 000 is invested in a perpetuity at an interest rate of 6% per annum.
After 10 quarterly payments have been made, the amount of money that remains invested in the perpetuity is

A. $15 000
B. $40 000
C. $85 000
D. $94 000
E. $100 000

Source: *VCE 2013, Further Mathematics 2, Module 4, Q.2; © VCAA*

Question 19 (3 marks)

Hugo won \$5000 in a road race and invested this sum at an interest rate of 4.8% per annum compounding monthly.

a. What is the value of Hugo's investment after 12 months? **(1 mark)**
Write your answer in dollars, correct to the nearest cent.

b. Answer the following.

i. Suppose instead that at the end of each month Hugo added \$200 to his initial investment of \$5000. Find the value of this investment immediately after the 12th monthly payment of \$200 is made.
Write your answer in dollars, correct to the nearest cent. **(1 mark)**

ii. Assume Hugo follows the investment that is described in **part b. i**.
Determine the total interest he would earn over the 12-month period.
Write your answer in dollars, correct to the nearest cent. **(1 mark)**

Question 20 (1 mark)

Using the annuities investment recurrence relation of $V_{n+1} = V_n\left(1 + \frac{r}{100}\right) + d$, $V_0 =$ Initial balance, and the following information:

$V_3 = \$28\,000$, Interest $= 7.5\%$ per annum, $d = \$1100$, Compounding periods = monthly,

the value of V_2 is

A. \$26 732.92
B. \$29 275.00
C. \$25 023.26
D. \$31 200.00
E. \$28 740.14

Question 21 (1 mark)

Using the annuities investment recurrence relation of $V_{n+1} = V_n \left(1 + \frac{r}{100}\right) + d, \quad V_0 = P$ and the following information:

Initial balance (P) = \$220 000, Interest = 5.2% per annum, d = \$450, Compounding periods = weekly, the value of V3 is

A. \$22 0670.00
B. \$221 340.67
C. \$222 000.00
D. \$222 012.01
E. \$222 684.02

Question 22 (1 mark)

A comparison of two superannuation accounts was being made, both with an initial balance of \$100 000.
Plan A: Interest = 6.8% compounding monthly, Extra payments = \$800 every month
Plan B: Interest = 5% compounding monthly, Extra payments = \$950 every month
For how many months is the balance of each account the same (not including the initial balance)?

A. 0
B. 1
C. 2
D. 3
E. 4

Question 23 (1 mark)

Peter has an investment account of \$80 000 that includes monthly contributions of \$350, earning an interest rate of 7.7% p.a., debited quarterly.
How much will be in Peter's account in 10 years' time?

A. \$229 128.57
B. \$231 274.23
C. \$233 826.43
D. \$235 327.34
E. \$234 321.38

Question 24 (1 mark)

Sienna has a loan of \$125 000 that includes monthly repayments of \$1264.41, with an interest rate of 3.2% p.a., debited monthly.
How much of the loan will still be outstanding in 15 years' time?

A. \$35 234.74
B. \$36 420.78
C. \$37 412.09
D. \$38 675.63
E. \$39 744.32

Question 25 (1 mark)

Voula has an investment of \$8000 that includes monthly contributions of \$250 and interest payments.
If the amount in the account in 9 years' time is \$50 593.31, what was the per annum interest rate closest to?

A. 5.1%
B. 5.4%
C. 5.8%
D. 6.4%
E. 7.3%

Answer

7.2 Modelling reducing balance loans with recurrence relations

Question 1

Use the initial value of $300 000 to calculate the interest of $900.

$$\frac{3.6}{100 \times n} = 300\,000 = 900$$

$$\frac{3.6}{100 \times n} = 0.003$$

$3.6 = 0.3n$

$n = 12$

Question 2

Calculate with financial solver on a CAS calculator using $N = 60,\ I = 3.72,\ PV = 175\,260.56,$ $Pmt = -3200$ and $Ppy = 12$. The final value will be $-368.116\ldots$

Since the final value is negative, Adam must still pay this amount.

Therefore, Adam's final payment will be $3200 + 368.116 \approx \$3568.12$

Question 3

$$\begin{aligned} V_1 &= 1.003 \times (26\,000) - 400 \\ &= 25\,678 \\ V_2 &= 1.003 \times (25\,678) - 400 \\ &= 25\,355.034 \\ V_3 &= 1.003 \times (25\,355.034) - 400 \\ &= 25\,031.0991 \\ V_4 &= 1.003 \times (25\,031.0991) - 400 \\ &= 24\,706.1924 \\ V_5 &= 1.003 \times (24\,706.1924) - 400 \\ &= 24\,380.3110 \end{aligned}$$

Question 4

Joseph's final payment will be the $995.49 plus any interest accumulated for the month.

$$\text{interest} = \frac{7.5}{12 \times 100} \times 995.49 = 6.22$$

$\$995.49 + \$6.22 = \$1001.71$

Question 5

Payment number	Payment	Interest	Principal reduction	Balance of loan
2	1500	$I = 249\,500 \times \frac{4.8}{100 \times 12}$ $= \$998$	$1500 - 998$ $= \$502$	$248 998

Question 6

Interest rate per month: $6.3\% \div 12 = 0.525\%$

$70\,000 \times 0.525\% = 70\,000 \times 0.005\,25 = \$367.50$

Repayment interest$= 2500 - 367.50 = \$2132.50$

Principal outstanding$= 70\,000 - 2132.50 = \$67\,867.50$

Question 7

End of month	Interest	Repayment	Balance
1	200	225	24 975
2	199.80	225	24 949.80

Question 8

End of month	Interest	Repayment	Balance
1	525	725	74 800
2	523.6	725	74 598.60

7.3 Solving reducing balance loan problems using finance solver

Question 1

Use Finance Solver:
N: 60
I(%): 2.8
PV: −12000
Pmt: ??
FV: 25000
PpY: 12
CpY: 12
Therefore, the minimum value of the payment that Joanna needs to make is $174.11.

Question 2

$P = \$8400, \quad n = 24 \text{ quarters}, \quad r = \dfrac{9\%}{4} = 2.25\%, \quad R = 1 + \dfrac{2.25}{100} = 1.0225$

Substitute into the annuities formula to find the regular quarterly repayments:

$$
\begin{aligned}
Q &= \frac{PR^n(R-1)}{R^n - 1} \\
&= \frac{(8400)(1.0225)^{24}(1.0225-1)}{1.0225^{24} - 1} \\
&= \$456.793\,922\,8
\end{aligned}
$$

3 years means 12 quarterly payments, so balance remaining after 3 years (12 quarterly payments) = $4757.41 (using the finance solver).

$$
\begin{aligned}
\text{Balance reduced} &= 8400 - 4757.41 \\
&= 3642.59
\end{aligned}
$$

$$
\begin{aligned}
\text{Percentage reduced by} &= (3642.59/8400) \times 100 \\
&= 43.4\%
\end{aligned}
$$

The correct answer is **C**.

Question 3

Via TVM solver:
N = 240
I = 6.95
PV = 90 000
Pmt = ?
FV = 0
PpY = 12
CpY = 12
Therefore, monthly payment = $695.09 ≈ $695.

Question 4
Use Finance Solver on CAS:
N: 2 years × 4 quarters = 8
I(%): 3.9p.a.
PV: −5000
Pmt: −200
PpY/CpY: 4
Therefore, the final value (FV) is $7059.25.

Question 5
Using the finance solver on the CAS:
N = unknown
$I(\%) = 6.18$
$PV = 160\,000$
$Pmt = -1950$
$FV = 0$
$PpY = 12$
$CpY = 12$
Therefore $N = 106.906\,93$
The final payment will therefore be $1950 \times 0.906\,93 = \$1768.51$.
So option D is incorrect.

Question 6
Using TVM solver:
N = 48
I = 4.75
PV = 35 000
Pmt = −802
FV = ?
PpY = 12
CpY = 12
Therefore FV = −3.57.
If 48 payments of $802 were made, the balance would still be $3.57 owing.
Therefore the final payment must be 802 + 3.57 = $805.57
***VCAA Assessment Report* note:**
This question asked students to calculate the amount of the final payment for a reducing balance loan. this was a two-step problem.
Step 1 involved calculating the future value of the loan after 47 payments.
Using a financial solver, this amount is found to be $802.39...
Step 2 required adding a month's interest to this amount to find the final payment.
final payment $= 802.3911... + 802.3911... \times \left(\dfrac{4.75}{12 \times 100}\right) = \$805.57$ to the nearest cent (option E)
While most students could correctly use their financial solver to find the amount still owed after the second last payment had been made ($802.39), they apparently failed to realise that this amount would attract interest during the last month of the loan.

Question 7
First, calculate monthly repayments using Finance solver on CAS.
N = 4 × 4 = 16; I = 12; PV = 25 000; FV = 0; P/Y = 4; C/Y = 4
Solving for Pmt gives −1990.27. Therefore, the quarterly payment is $1990.27. **[1 mark]**

Next, calculate the amount owed after 2 years.
$N = 2 \times 4 = 8$; $I = 12$; $PV = 25\,000$; $Pmt = -1990.27$; $P/Y = 4$ and $C/Y = 4$.
Solving for FV gives −13 971.09.
Therefore, after 2 years the amount still owed is \$13 971.09.

$$\begin{aligned}\text{Hence, the amount repaid} &= \$25\,000 - \$13\,971.09\\ &= \$11\,028.91\\ &= \$11\,029 \text{ (to the nearest dollar) } [\mathbf{1\ mark}]\end{aligned}$$

***VCAA Assessment Report* note:**
Students needed to read this question carefully to identify the three steps needed for the solution. Writing and labelling such steps can be very helpful in organising thoughts, but very few students showed any TVM input or other working.

7.4 The effect of rate and repayment changes on reducing balance loans

Question 1
First, use finance solver to find the value of the payment if it is paid off over 20 years:
N: 240
I(%): 3.14
PV: 400 000
Pmt: ??
FV: 0
PpY: 12
CpY: 12
The monthly payment will be \$2246.53.
However, Bob makes only interest-only payments for 2 years, which means that at the end of 2 years he still owes \$400 000.
Use finance solver a second time to calculate the new interest rate for the final 18 years of the loan.
N: 216
I(%): ??
PV: 400 000
Pmt: −2246.53
FV: 0
PpY: 12
CpY: 12
The interest rate will be 2.21%.

Question 2
Use Finance Solver on CAS:
N: 10 years 12 months = 120
I (%): 4.35 p. a.
PV: 245 000
Pmt: −1800
FV: −108 219.161 1158
PpY/CpY: 12
After 10 years, Xavier still owes \$108 219.16
N: 5 years 12 months = 60
I (%): 4.142 758 985 . . . p. a.
PV: 108 219.16
Pmt: −2000

FV: 0
PpY/CpY: 12
To be paid off in 5 more years, the annual interest rate is closest to 4.1%.
***VCAA Assessment Report* note:**
Understanding of the sign convention for TVM use is very important, as is the careful tracking of values used in subsequent calculations.

Question 3
Using Finance solver in CAS, first find the interest rate per annum.
Enter N = 1, PV = 300 000, Pmt = 2500, FV = –299 000, PpY = 12, CpY = 12.
Solving for I gives I = 6%.
Check whether option A is correct:
Enter N = 2, PV = 300 000, Pmt = 2500, PpY = 12, CpY = 12
Solving for FV gives FV = –297 995.
Therefore, option A (‘After 2 months, $297 995 is still owing on the loan’) is correct.
***VCAA Assessment Report* note:**
In this question, students needed to test the truth of five statements relating to the repayment of a reducing balance loan. Many students struggled with this question. The key to answering this question was to use a TVM to work out the interest rate applying to the loan. This knowledge, with the aid of a TVM solver, could then be used to test the truth of each of the five statements.

Question 4
$P = \$8400$, $n = 24$ quarters, $r = \dfrac{9\%}{4} = 2.25\%$, $R = 1 + \dfrac{2.25}{100} = 1.0225$
Substitute into the annuities formula to find the regular quarterly repayments:
$$Q = \frac{PR^n (R-1)}{R^n - 1}$$
$$= \frac{(8400)(1.0225)^{24}(1.0225-1)}{1.0225^{24}-1}$$
$$= \$456.793\,922\,8$$
3 years means 12 quarterly payments, so $\$456.793\,922\,8 \times 12 = \$5481.527\,07$
Therefore $\$8400 - 5481.527\,07 = \$2918.472.93$ balance left, which is $\dfrac{2918.472\,93}{8400} \times 100 = 34.74\% \approx 35\%$

7.5 Annuities and perpetuities

Question 1
a. After one month: $A_1 = 1.0024 \times 500\,000 - 2000 = \$499\,200$
After two months: $A_2 = 1.0024 \times 499\,200 - 2000 = \$498\,398.08$
b. $1.0024 = 1 + \dfrac{r}{1200}$, so $r = 2.88\%$ **[1 mark]**
c. Perpetuity means that the balance will always remain the same, i.e. $A_{n+1} = A_n$
$500\,000 = k \times 500\,000 - 2000$
$k = 1.004$ **[1 mark]**

Question 2
The value of A is not changing, so the recurrence relation is subtracting the same value as the interest earned.
$0.025 \times 200\,000 = 5000$

Question 3
Use Finance Solver on CAS:
N: 5 years × 12 months = 60

I (%): 5.2 p. a.
PV: −130 784.93
FV: 66 992.27
PpY/CpY: 12
Therefore, the payment (Pmt) each month is \$1500.
***VCAA Assessment Report* note:**
A common error was to choose option E, which corresponded to both the PV and FV being incorrectly allocated the same sign.

Question 4
Since it is a perpetuity account, the balance must remain the same after each iteration. Therefore, $d = RV_0$
Options D and E are not perpetuity accounts as they are reduced before each payment (both have a negative interest rate).
Option A: $d = 356,\ RV_0 = 0.0029 \times 120\,000 = 348 \neq 356$
Option B: $d = 846, RV_0 = 0.0047 \times 180\,000 = 846 = 846$
Option C: $d = 1534, RV_0 = 0.0071 \times 210\,000 = 1491 \neq 1534$
Option B shows the increase in value is equal to the payment out; therefore, it is a perpetuity account.

Question 5
For a perpetuity, funds last for an indefinite period of time as long as the amount paid out is the same as the interest earned on the initial amount. Therefore, the amount in the account does not change each year from the initial \$80 000.

Question 6
Interest is calculated using the rule: Balance $\times \dfrac{r}{100}$
Principal paid = Repayment − Interest
Loan balance = Current balance − Principal paid

Month	Repayment	Interest	Principal paid	Loan balance
1	\$200	\$125	\$75	\$29 925
2	\$200	$29\,925 \times \dfrac{\frac{5}{12}}{100} = \$124.69$	200 − 124.69 = \$75.31	29 925 − 75.31 = \$29 849.69
3	\$200	$29\,849.69 \times \dfrac{\frac{5}{12}}{100} = \$124.37$	200 − 124.37 = \$75.63	29 849.69 − 75.63 = \$29 774.06

1 mark – any 3 correct values
1 mark – all other values correct

Question 7
Substituting the relevant values:

$$\begin{aligned} V_1 &= V_0\left(1 + \frac{\frac{6.3}{12}}{100}\right) - 450 \\ &= 25\,000 \times \left(1 + \frac{\frac{6.3}{12}}{100}\right) - 450 \\ &= \$24\,681.25 \end{aligned}$$

$$V_2 = V_1\left(1+\frac{\frac{6.3}{12}}{100}\right)-450$$
$$= 24\,681.25 \times \left(1+\frac{\frac{6.3}{12}}{100}\right)-450$$
$$= \$24\,360.83$$
$$V_3 = V_2\left(1+\frac{\frac{6.3}{12}}{100}\right)-450$$
$$= 24\,360.83 \times \left(1+\frac{\frac{6.3}{12}}{100}\right)-450$$
$$= \$24\,038.72$$

Question 8

A:

Using the annuities formula $V_n = V_0R^n - \dfrac{d\,(R^n-1)}{R-1}$

$$R = \left(1+\frac{\frac{i}{n}}{100}\right) = \left(1+\frac{\frac{6.5}{12}}{100}\right) = \left(1+\frac{6.5}{1200}\right)$$

$n = 20 \times 12 = 240$

$$V_{240} = 0 = 210\,000\left(1+\frac{6.5}{1200}\right)^{240} - \frac{d\left(\left(1+\frac{6.5}{1200}\right)^{240}-1\right)}{\left(1+\frac{6.5}{1200}\right)-1}$$

Use a calculator to solve for Q.

$d = \$1565.70$

B:

TVM Solver:

N: $20 \times 12 = 240$

I %: 6.5

PV: −210 000

FV: 0

P/Y: 12

C/Y: 12

Solve for PMT

Question 9

Substituting the relevant values:

$$V_3 = V_2\left(1+\frac{r}{100}\right)-d,\ \ r = \frac{6.2}{12} \approx 0.5167$$

$$150\,000 = V_2\left(1+\frac{\frac{6.2}{100}}{100}\right)-1100$$

Then using a CAS to solve for V_2:

$V_2 = \$150\,323.33$

Question 10

$V_1 = V_0\left(1+\dfrac{r}{100}\right) - d,\ r = \dfrac{3.5}{12} \approx 0.2917,\ V_0 = 15\,500,\ d = 10$

$V_1 = 15\,500\left(1+\dfrac{\frac{3.5}{12}}{100}\right) - 10 = 15\,535.21$

$V_2 = 15\,535.21\left(1+\dfrac{\frac{3.5}{12}}{100}\right) - 10 = 15\,570.52$

Question 11

$100 = 0.03 \times x$

$x = 3333.33$

Question 12

$200 = \left(1+\dfrac{r}{100}\right) \times 4000$

$r = 5$

Question 13

$0.6 \times 115\,000 = \$6900$

7.6 Annuity investments

Question 1

Use the first balance and second interest amount to calculate the interest rate per period (x).

x% of 6977.50 = 27.91 Solving for x,

$x = 0.4\%$ per period

The interest earned for payment number 20 is 0.4% of \$7233.83 = \$28.94

The principal addition for payment number 20 is \$7500 − \$7233.83 = \$266.17

So, interest + payment = principal addition

$$\begin{aligned} \text{Payment} + 28.94 &= 266.17 \\ \text{Payment} &= 266.17 - 28.94 \\ &= \$237.23 \end{aligned}$$

***VCAA Assessment Report* note:**

This question required the interpretation of an amortisation table and the calculation of a missing payment value. Many students seemed to assume that the payment value would be constant at \$100 (option B), despite the question stating that the payment may vary.

Question 2

a. 5.2% p.a. is equivalent to $\dfrac{5.2}{12}$% per month.

$\dfrac{\frac{5.2}{12}}{100} \times 360\,000 = \$1560$ per month. **[1 mark]**

***VCAA Assessment Report* note:**

Some students gave the annual payment of \$18 20, while others knew the correct method but rounded $\dfrac{5.2}{1000}$ to two decimal places before multiplying by \$360 000, giving \$1 548.

b. Use Finance Solver on CAS:

N: 4 years 12 months = 48

I(%): 3.8 p.a.

PV: –360 000

Pmt: –500

FV: **444 872.9444992**
PpY/CpY: 12
After 4 years, Alex's investment grows to $444 872.94 **[1 mark]**
N: 2 years 12 months = 24
I3.8
PV: –444 872.94
Pmt: –805.65070094875
FV: 500 000
PpY/CpY: 12
To grow to $500 000 in a further two years, Alex's new monthly payment will be $805.65. **[1 mark]**
***VCAA Assessment Report* note:**
Some students entered the $500 payment as a positive into their finance solver, giving $393 121.15 as the four-year value. A few students tried to use a formula but made little progress.

Question 3

a. $Q = \dfrac{P \times r}{100}$

$460 = \dfrac{P \times 3.68}{100}$ **[1 mark]**

$P = \$12\,500$

b. A perpetuity is an investment that provides regular payments that continue forever. Therefore, they will be able to provide the scholarship for an infinite number of years. **[1 mark]**
***VCAA Assessment Report* note:**
Many students did not know that perpetuities pay out only the interest earned, while the principal remains unchanged.
The most common incorrect answer was $\dfrac{12\,500}{460} \approx 27$ years

Question 4

The regular payment is d in $V_{n+1} = V_n\left(1 + \dfrac{r}{100}\right) + d$, so the payment in $V_{n+1} = 1.0034V_n + 500$ is $500.

Question 5

The difference in the value of the annuity investment between $n = 2$ and $n = 3$ is equal to $V_3 - V_2$.
$V_0 = 46\,000$
$V_1 = 1.0034V_0 + 500 = 1.0034 \times 46\,000 + 500 = \$46\,656.4$
$V_2 = 1.0034V_1 + 500 = 1.0034 \times 46\,656.4 + 500 = \$47\,315.03$
$V_3 = 1.0034V_2 + 500 = 1.0034 \times 47\,315.03 + 500 = \$47\,975.902$
Difference between $n = 2$ and $n = 3$: $V_3 - V_2 = 47\,975.902 - 47\,315.03 \approx \$661$

Question 6

Use the financial solver on a CAS calculator to find the value of the investment after two years (24 months) compounding at 3.24% monthly ($307 794.499). Then use that value and the final value of $600 000 to calculate the payment needed over 8 years (96 months) compounding at 3.2% per month. The answer of $1854.05 is closest to option E, $1854.
***VCAA Assessment Report* note:**
This question required the use of a finance solver application. Some students were able to do so successfully, although it seemed.
Understanding of the sign convention for the finance solver is very important, as is the careful tracking of values used in subsequent calculations.

Question 7

$$V_1 = V_0\left(1+\frac{r}{100}\right)+d,\ r=\frac{5.8}{12}\approx 0.4833,\quad V_0 = 12\,000,\quad d = 450$$

$$V_1 = 12\,000\left(1+\frac{\frac{5.8}{12}}{100}\right)+450 = \$12\,508$$

$$V_2 = 12\,508\left(1+\frac{\frac{5.8}{12}}{100}\right)+450 = \$13\,018.46$$

$$V_3 = 13\,018.46\left(1+\frac{\frac{5.8}{12}}{100}\right)+450 = \$13\,531.38$$

Question 8

Account balance = Current balance + Interest + Payment

Month	Payment	Interest	Account balance
1	\$800	\$125	\$30 925
2	\$800	$30\,925 \times \frac{\frac{5}{12}}{100} = \$128.25$	30 925 + 800 + 128.85 = \$31 853.85
3	\$800	$31\,853.85 \times \frac{\frac{5}{12}}{100} = \$132.72$	31 853.85 + 800 + 132.72 = \$32 786.57

1 mark – any 3 correct values
1 mark – all other values correct

7.7 Review

Question 1

\$449 060.08 − \$422 051.93 = \$27 008.15

Question 2

Use Finance Solver on CAS:
N: ??
I(%): 4
PV: −500 000
Pmt: 44 970.55
FV: 0
PpY: 1
CpY: 1
Therefore, N = 15 years.

Question 3

a. **i** Principal reduction \$318 718.08 − \$318 074.23 = \$643.85 **[1 mark]**

ii Monthly interest $\frac{3.6}{12}\% \times \$318\,074.23 = \$954.22$

Therefore, the balance after payment 4 will be
\$318 074.23 + \$954.22 − \$1600 = \$317 428.45 **[1 mark]**

b. Note that the monthly interest rate will be $\frac{3.6}{12} = 3\%$

$S_0 = 320\,000,\quad S_{n+1} = 1.003 \times S_n - 1600$ **[1 mark]**

Question 4

Change in rate: $\frac{L_{28}}{V_{27}} - \frac{L_{27}}{V_{26}} = \frac{1002.26}{227\,785.76} - \frac{961.90}{229\,023.86} \approx +0.02\%$ per month

$+0.02 \times 12 = +0.24\%$ per annum.

The interest rate increased by 0.24% per annum.

Question 5

a. i. Use Finance Solver on CAS:

N: 12
I(%): 6.9
PV: 70 000
Pmt: −800
PpY/Cpy: 12

Therefore, Ken will owe \$65 076.22 after 12 months. **[1 mark]**

***VCAA Assessment Report* note:**

Rather than use a financial solver to answer the question above, a number of students adopted a formulaic approach, almost always unsuccessfully, based on the compound interest formula.

ii. Total interest after 12 payments = $800 \times 12 - (70\,000 - 65\,076.22) = \$4676.22$ **[1 mark]**

***VCAA Assessment Report* note:**

A common incorrect answer was \$4923.78, which is the reduction in the principal over the year. This failed to take into account the \$9600 total of repayments made in the year.

b. After 3 years, the value of the loan is \$54 151.60.

Using Finance Solver on CAS
N: 36
I(%):6.9
Pmt: −800
FV: 0
PpY/Cpy: 12

The principal amount after a further 3 years would be \$25947.58. **[1 mark]**

Ken's lump sum payment, $\$L$, will be: $L = 54\,151.60 - 25\,947.58 = \$28\,204$. **[1 mark]**

***VCAA Assessment Report* note:**

Correct tables of input values for the financial solver may have illustrated working out to qualify for a method mark even if the final answer was incorrect.

Question 6

Firstly, use Finance Solver to find the annual interest rate:
N: 240
I(%): ??
PV: 450000
Pmt: −2633
FV: 0
PpY: 12
CpY: 12

The annual interest rate is 3.6%.

Therefore, $R = 1 + \frac{3.6}{12 \times 100} = 1.003$

Question 7

a. Note that this question specifically asks you to show recursive calculations – so it must be shown.

Monthly interest rate is $\dfrac{5.1\%}{12} = 0.425\%$

$V_{n+1} = 1.00425 \times V_n - 900$

$V_1 = 1.00425 \times 152\,431 - 900 = \$152\,178.83$

$V_2 = 1.00425 \times 152\,178.83 - 900 = \$151\,925.59$

[Award **1 mark** for evidence of using 1.00425 in a recurrence relation]

[Award **1 mark** for both recursive calculations shown, and final answer correct]

b. After 2 years:

N: 24

I: 5.1

PV: −152 431

Pmt: 900

FV: ??

PpY: 12

CpY: 12

The value of the investment will be $146 073.74

For the next 23 years:

N: 276

I: 4.6

PV: ??

Pmt: 900

FV: 0

PpY: 12

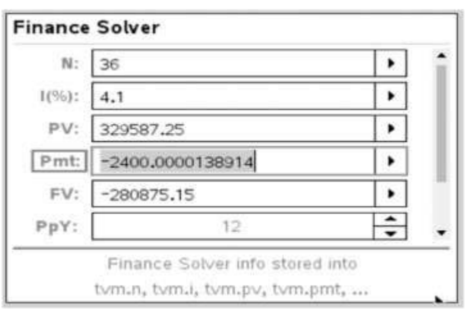
CpY: 12

The principal value would need to be $153 112.94.

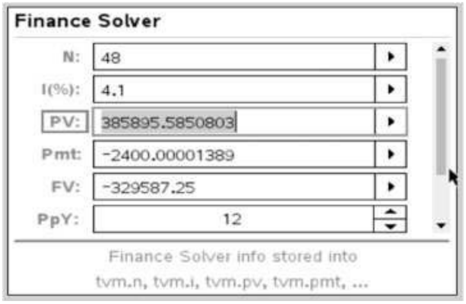
So the extra money that Sienna needs to pay is $153 112.94 − $146 073.74 = $7039.20 [**1 mark** – rounding applies]

Question 8

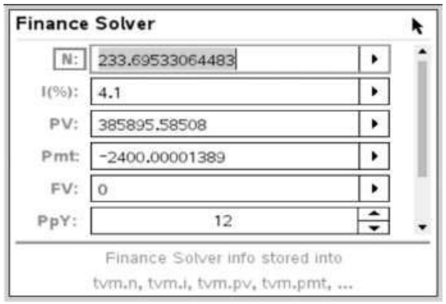
From the information given, we know that there are 36 payments (3 years) between the two balances. Using Finance Solver on the CAS, calculate the monthly payment to be $2400.



Now backtrack to find the principal value of the loan to be \$385 895.59.



Now backtrack to find the principal value of the loan to be \$385 895.59.



But the question says that the final payment will be lower, so we need 234 payments.
We are told that 84 payments have already been made, so we will need $234 - 84 = 150$ more payments.
Award 1 mark for the number of payments and 1 mark for the correct calculation of the additional payments required.

Question 9

a. \$3700 **[1 mark]**

***VCAA Assessment Report* note:**

The question was straightforward but often misunderstood. Some gave the answer to part d. while others gave \$3000 after calculating A_1.

b. $0.0035 \times 12 = 0.042$, which is 4.2% **[1 mark]**

***VCAA Assessment Report* note:**

Responses needed to show the calculation using basic arithmetic that resulted in an answer of 4.2%, not use the 4.2% and attempt to verify it.

c. Use the finance solver and allow I = 4.2, PV = 200 000, Pmt = −3700, FV = 0 and PpY/CpY = 12 to give N = 60.02495

Now set N = 60 to find the FV = \$92.15

After 60 months, the value of the annuity is \$92.15. **[1 mark]**

d. $0.35\% \times 200\,000 = \$700$ **[1 mark]**

Question 10

a. Use the finance solver to allow N = 78, I = 4.9, PV = −350 000, FV = 262 332.33 and PpY/CpY = 26 to find Pmt = \$1704.03

Each fortnightly repayment is \$1704.03 **[1 mark]**

***VCAA Assessment Report* note:**

\$7954.54 was a common incorrect answer obtained by entering the FV in technology as a positive rather than as −262 332.33.

This very large fortnightly repayment should have been a signal that something had been entered incorrectly.

b. $1704.03 \times 78 - (350000 - 262332.33) = \$45\,246.67$ **[1 mark]**

***VCAA Assessment Report* note:**

Some students followed through with the incorrect response they obtained for part a. but did not show the working required for the consequential mark.

c. $1 + \dfrac{4.8}{1200} = 1.004$

$B_{n+1} = 1.004\,B_n - 3517.28$ where $B_0 = 262\,332.33$

Award **1 mark** for correct B_0.

Award **1 mark** for the correct equation.

***VCAA Assessment Report* note:**

The question was done reasonably well, but some responses gave B_0 as 350 000, or R as 1.048, and added rather than subtracted the 3517.28.

Question 11

a. Julie initially invests V_0, which is \$12 000. **[1 mark]**

b. i. $V_1 = 1.0062 \times V_0 = 1.0062 \times 12\,000 = \$12\,074.40$ **[1 mark]**

***VCAA Examination Report* note:**

This question was answered well, although some students used methods other than simple recursion, which were not acceptable.

ii. $V_1 = 1.0062 \times V_0 = 1.0062 \times 12\,000 = \$12\,074.40$

$V_2 = 1.0062 \times V_1 = 1.0062 \times 12\,074.4 \approx \$12\,149.26$

$V_3 = 1.0062 \times V_2 = 1.0062 \times 12\,149.26 \approx \$12\,224.59$

$V_4 = 1.0062 \times V_3 = 1.0062 \times 12\,224.59 \approx \$12\,300.38$

The answer is 4 months. **[1 mark]**

c. i. a is equal to the principle value of \$12 000 and b is equal to 1 plus the percentage increase.

The equation is $balance = 12\,000 \times 1.0062^n$. **[1 mark]**

***VCAA Assessment Report* note:**

A common error was for students to write the two numbers in the incorrect boxes.

ii. Since n is in months, there are $3 \times 12 = 36$ months in 3 years. Therefore, $n = 36$. **[1 mark]**

Question 12

a. Since the account is a perpetuity, the interest earned on the initial amount is equal to the payout. The payout is fortnightly, so there are 26 payments.

$492\,800 \times R = 887.04 \times 26$

$R = 0.0468$

$R = 4.68\%$ per annum **[1 mark]**

***VCAA Examination Report* note:**

A common error was 0.18%, which did not account for the fortnightly payments. Perpetuities was an area that requires improvement for many students.

b. i. Use the financial solver on your CAS calculator with the inputs $n = 12$, $I = 4.32$, $PV = -492\,800$, $FV = 480\,242.25$, $Ppy = 12$, and solve for Pmt. The payment is equal to \$2800.00. **[1 mark]**

ii. The second year of investment is from $n = 12$ to $n = 24$.

Using the financial solver with inputs

$n = 12,\ I = 4.32,\ PV = -480\,242.25,\ Ppy = 12$ and $Pmt = 2800$, we can solve for FV.

$FV = 467\,131.13$ **[1 mark]**

$\textit{Interest earned} = \textit{payout} - \textit{change in balance} = 2800 \times 12 - (480\,242.25 - 467\,131.13) =$

\$20 488.88 **[1 mark]**

***VCAA Examination Report* note:**
Many students did not attempt this question. Many found the future value after two years of \$467 131 but could not proceed further.
A few students tried unsuccessfully to use the annuities formula.

Question 13

a. Average depreciation per year $= \dfrac{38\,000 - 16\,000}{8} = \$2750$ **[1 mark]**

***VCAA Assessment Report* note:**
'Show that' questions give the answer to a basic and appropriate calculation, and students must write that calculation. The given number in a 'show that' question is sometimes needed in a following question. With this number, students can attempt the following question, even if they cannot complete the 'show that' question.
The given number in a 'show that' question is not to be used in a calculation; it must be the result of a calculation. The following calculations use the 2750 to show a different result than what is required and were not acceptable for this question:
- $38\,000 - 8 \times 2750 = 16\,000$ – shows how to find the depreciated value after eight years
- $16\,000 + 8 \times 2750 = 38\,000$ – shows how to find the initial value

b. $C_{n+1} = C_n - 2750,\ C_0 = 38\,000$ **[1 mark]**

***VCAA Assessment Report* note:**
Common errors included:
- failure to include the initial value, C_0
- writing the initial value as C_n, not C_0
- using different symbols for different parts of the recurrence relation, e.g. $V_0 = 38\,000,\ C_{n+1} = C_n - 2750$
- using a rule for the nth term $C_n = 38\,000 - 2750n$ instead of the recurrence rule.

c. In 8 years, the caravan depreciates by $38\,000 - 16\,000 = 22\,000$.
$\therefore$ depreciation $= \dfrac{22\,000}{5000 \times 8} = \$0.55$ per km travelled. **[1 mark]**

***VCAA Assessment Report* note:**
A very common incorrect answer of \$4.40 was found by ignoring the eight years over which the depreciation occurred.

Question 14

Total repaid $= 60 \times \$400 = \$24\,000$

Interest paid $= \$24\,000 - \$15\,000 = \$9000$

Question 15

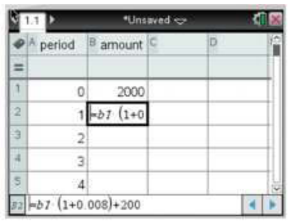
a. Jane's investment uses compound interest:

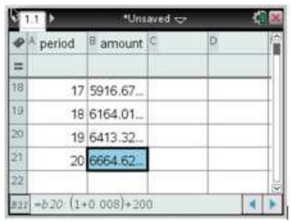
$$\begin{aligned} V_n &= V_0\left(1 + \frac{r}{100}\right)^n \\ &= 4000\left(1 + \frac{3.6/12}{100}\right)^6 \quad \textbf{[1 mark]} \\ &= \$4072.54 \end{aligned}$$

***VCAA Assessment Report* note:**
Answers rounded to \$4072.55 or \$4072.50 were not accepted.

b. From the equation, we know that $\dfrac{r}{100} = 0.008$ which will be 0.8% per quarter (as the investment compounds quarterly). So the percentage rate per year is $0.8 \times 4 = 3.2\%$. **[1 mark]**

c. Use CAS – a spreadsheet is best.





Therefore, the amount that Michael has in his account at the end of 5 years is \$6664.63. **[1 mark]**

***VCAA Assessment Report* note:**

Answers of \$6664.60 or \$6664.65 were not accepted.

Question 16

Balance = \$143 585.33

After next month:

$$V_n = V_0R^n - \text{repayment}$$

$$= 143\,585.33\left(1 + \frac{4.5/12}{100}\right)^1 - 2500$$

$$= \$141\,623.77$$

Amount loan has been reduced by = 143 585.33 − 141 623.77

= \$1961.56 **[1 mark]**

Percentage of next monthly repayment $= \dfrac{1961.56}{2500} \times 100$

$= 78.46\%$

$= 78\%$ **[1 mark]**

***VCAA Assessment Report* note:**

Many students made poor attempts at this question, gave a single number as the answer or did not attempt it at all. A common incorrect answer was 22%, which was the interest component of the next repayment. A method mark was available for an incorrect answer only if the working out of a significant step towards a solution could be followed, but this was rare.

Question 17

a. $d = \dfrac{V_0 r}{100}$

$$750 = \frac{20\,000 \times r}{100}$$

$r = 3.75\%$ **[1 mark]**

***VCAA Assessment Report* note:**

A common, unacceptable answer was 0.0375%.

b. As only the interest is given away each year, the principal remains unchanged. Therefore the amount still invested = \$20 000. **[1 mark]**

***VCAA Assessment Report* note:**

Many students did not understand what a perpetuity is. A perpetuity balance remains constant since only the interest earned is withdrawn in each compounding period.

c. i. $\text{Cost} = 750 + \dfrac{3}{100} \times 750$

$= 750\,(1 + 0.03)$

$= 772.50$ **[1 mark]**

***VCAA Assessment Report* note:**

The answer must have been written correct to the nearest cent and so the zero at the second decimal place was required in this question.

ii. Cost in 2014 = x

Cost in 2024 = 750

$750 = x(1 + 0.03)^{10}$

$x = 558.07$

Therefore the value in 2014 is \$558. **[1 mark]**

Question 18

The amount of money invested in perpetuity always remains the same. In this case, it is \$100 000.

Question 19

a. The value of the investment is:

$$V = 5000 \times \left(1 + \frac{4.8 \div 12}{100}\right)^{12}$$

$$V = 5000 \times 1.004^{12}$$

$V = \$5245.35$ **[1 mark]**

b. i. Using Finance Solver on CAS:

N = 12, I% = 4.8, PV = −5000, PMT = −200, P/Y = 12, C/Y = 12.

Solving for future value gives \$7698.86.

Therefore, the value of the investment is \$7698.86. **[1 mark]**

ii. Interest = value of investment − principal − extra payments

Interest = 7698.86 − 5000 − 12 × 200

Interest = \$298.86 **[1 mark]**

***VCAA Assessment Report* note:**

Many students did not allow for the \$200 that Hugo had added each month for a year.

Question 20

Substituting the relevant values:

$$V_3 = V_2\left(1 + \frac{r}{100}\right) + d, \quad r = \frac{7.5}{12} = 0.625$$

$$28\,000 = V_2\left(1 + \frac{\frac{7.5}{12}}{100}\right) + 1100$$

Using a CAS to solve for V_2,

$V_2 = \$26\,732.92$

Question 21

$V_1 = V_0\left(1 + \frac{r}{100}\right) + d,\ \ r = \frac{5.2}{52} = 0.1,\ V_0 = 220\,000,\ \ d = 450$

$V_1 = 220\,000\left(1 + \frac{0.1}{100}\right) + 450 = \$220\,670$

$V_2 = 220\,670\left(1 + \frac{0.1}{100}\right) + 450 = \$221\,340.67$

$V_3 = 221\,340.67\left(1 + \frac{0.1}{100}\right) + 450 = \$222\,012.01$

Question 22

Creating a table of values and using the annuities investment rule:

$V_{n+1} = V_n\left(1 + \frac{r}{200}\right) + d,\ \ V_0 = 100\,000$

Month	Plan A	Plan B
0	100 000	100 000
1	$V_1 = 100\,000\left(1 + \frac{\frac{6.8}{12}}{100}\right) + 800 = \$101\,366.67$	$V_1 = 100\,000\left(1 + \frac{\frac{5}{12}}{100}\right) + 950 = \$101\,366.67$
2	$V_2 = 101\,366.67\left(1 + \frac{\frac{6.8}{12}}{100}\right) + 800 = \$102\,741.08$	$V_2 = 101\,366.67\left(1 + \frac{\frac{5}{12}}{100}\right) + 950 = \$102\,739.03$

The accounts remain at the same balance for one month only.

Question 23

$N = 120$
$I\% = 7.7$
$PV = 80000$
$PMT = 350$
$FV = ?$
$P/Y = 12$
$C/Y = 4$
Therefore, $FV = \$234\,321.38$.

Question 24

$N = 180$
$I\% = 3.2$
$PV = 125\,000$
$PMT = 1264.41$
$FV = ?$
$P/Y = 12$
$C/Y = 12$
Therefore, $FV = \$39\,744.32$.

Question 25
$N = 108$
$1\% = ?$
$PV = 8\ 000$
$PMT = 250$
$FV = 50\ 593.31$
$P/Y = 12$
$C/Y = 12$
Therefore the per annum interest rate = 6.4%.

8 Matrices

Topic	8	Matrices
Subtopic	8.2	Matrix representation

online only

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Source: *VCE 2021, Further Mathematics 1, Section B, Module 1, Q.1; © VCAA*

Question 1 (1 mark)

If matrix $M = \begin{bmatrix} 3 & 2 \\ 8 & 9 \\ 13 & 7 \end{bmatrix}$, then its transpose, M^T, is

A. $\begin{bmatrix} 2 & 3 \\ 9 & 8 \\ 7 & 13 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 9 & 7 \\ 3 & 8 & 13 \end{bmatrix}$

C. $\begin{bmatrix} 7 & 9 & 2 \\ 13 & 8 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 8 & 13 \\ 2 & 9 & 7 \end{bmatrix}$

E. $\begin{bmatrix} 13 & 8 & 3 \\ 7 & 9 & 2 \end{bmatrix}$

***Source:** VCE 2017, Further Mathematics 1, Section B, Module 1, Q.1; © VCAA*

Question 2 (1 mark)

Kai has a part-time job.

Each week, he earns money and saves some of this money.

The matrix below shows the amounts earned (E) and saved (S), in dollars, in each of three weeks.

$$\begin{array}{l} \\ week\ 1 \\ week\ 2 \\ week\ 3 \end{array} \begin{array}{c} \begin{array}{cc} E & S \end{array} \\ \begin{bmatrix} 300 & 100 \\ 270 & 90 \\ 240 & 80 \end{bmatrix} \end{array}$$

How much did Kai save in week 2?

A. $80& lt; /p>
B. $90
C. $100
D. $170
E. $270

***Source:** VCE 2016, Further Mathematics 1, Section B, Module 1, Q.5; © VCAA*

Question 3 (1 mark)

Let $M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

The element in row i and column j of M is m_{ij}.

The elements of M are determined by the rule

A. $m_{ij} = i + j - 1$
B. $m_{ij} = 2i - j + 1$
C. $m_{ij} = 2i + j - 2$
D. $m_{ij} = i + 2j - 2$
E. $m_{ij} = i + j + 1$

Topic 8 Subtopic 8.2 Matrix representation

Source: *VCE 2020, Further Mathematics 1, Section B, Module 1, Q.6; © VCAA*

Question 4 (1 mark)

The element in row i and column j of matrix M is m_{ij}.

M is a 3×3 matrix. It is constructed using the rule $m_{ij} = 3i + 2j$.

M is

A. $\begin{bmatrix} 5 & 7 & 9 \\ 7 & 9 & 11 \\ 11 & 13 & 15 \end{bmatrix}$

B. $\begin{bmatrix} 5 & 7 & 9 \\ 8 & 10 & 12 \\ 11 & 13 & 15 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 7 & 10 \\ 8 & 10 & 13 \\ 11 & 13 & 16 \end{bmatrix}$

D. $\begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{bmatrix}$

E. $\begin{bmatrix} 5 & 8 & 11 \\ 8 & 11 & 14 \\ 11 & 14 & 17 \end{bmatrix}$

***Source:** VCE 2020, Further Mathematics 1, Section B, Module 1, Q.8; © VCAA*

Question 5 (1 mark)

The table below shows information about three matrices: A, B and C.

Matrix	Order
A	2×4
B	2×3
C	3×4

The transpose of matrix A, for example, is written as A^T.

What is the order of the product $C^T \times (A^T \times B)^T$?

A. 2×3

B. 3×4

C. 4×2

D. 4×3

E. 4×4

***Source:** VCE 2016, Further Mathematics 1, Section B, Module 1, Q.1; © VCAA*

Question 6 (1 mark)

The transpose of $\begin{bmatrix} 2 & 7 & 10 \\ 13 & 19 & 8 \end{bmatrix}$ is

A. $\begin{bmatrix} 13 & 19 & 8 \\ 2 & 7 & 10 \end{bmatrix}$

B. $\begin{bmatrix} 10 & 7 & 2 \\ 8 & 19 & 13 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 13 \\ 7 & 19 \\ 10 & 8 \end{bmatrix}$

D. $\begin{bmatrix} 13 & 2 \\ 19 & 7 \\ 8 & 10 \end{bmatrix}$

E. $\begin{bmatrix} 8 & 10 \\ 19 & 7 \\ 13 & 2 \end{bmatrix}$

Source: *VCE 2015, Further Mathematics 1, Section B, Module 6, Q.1; © VCAA*

Question 7 (1 mark)

Matrix B below shows the number of photography (P), art (A) and cooking (C) books owned by Steven (S), Trevor (T), Ursula (U), Veronica (V) and William (W).

$$B = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} P & A & C \end{array} \\ \begin{bmatrix} 8 & 5 & 4 \\ 1 & 4 & 5 \\ 3 & 3 & 4 \\ 4 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ S \\ T \\ U \\ V \\ W \end{array}$$

The element in row i and column j of matrix B is b_{ij}. The element b_{32} is the number of

A. art books owned by Trevor.
B. art books owned by Ursula.
C. art books owned by Veronica.
D. cooking books owned by Ursula.
E. cooking books owned by Trevor.

Question 8 (1 mark)

Which of the following matrices has an order of (3×2)?

A. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 45 \\ 67 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$

Question 9 (1 mark)

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 2 & 4 & 5 \\ 1 & 3 & 4 & 5 \\ 1 & 2 & 4 & 1 \end{bmatrix}$$

What is element a_{23} in the above matrix?

A. 1
B. 2
C. 3
D. 4
E. 5

Question 10 (1 mark)

Which of the following matrices is a square matrix?

A. $\begin{bmatrix} 15 & 4 & 8 \\ 4 & 55 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 4 & 9 \\ 16 & 25 & 36 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 4 & 9 & 16 \end{bmatrix}$

Question 11 (1 mark)

Which of the following is a diagonal matrix?

A. $\begin{bmatrix} 15 & 4 & 8 \\ 4 & 55 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 4 & 9 \\ 16 & 25 & 36 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 4 & 9 & 16 \end{bmatrix}$

Question 12 (1 mark)

If matrix S is $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$, then S^T is given by:

A. $\begin{bmatrix} 2 & 1 \\ -2 & -1 \\ 3 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 3 \\ -1 & -2 \\ 1 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 2 \\ -2 & -1 \\ 2 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 3 \\ -1 & -2 \\ 2 & 2 \end{bmatrix}$

Topic	8	Matrices
Subtopic	8.3	Addition, subtraction and scalar operations with matrices

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Source: *VCE 2019, Further Mathematics 1, Section B, Module 1, Q.1; © VCAA*

Question 1 (1 mark)

Consider the following four matrix expressions.

$$\begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 0 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 8 & 0 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

How many of these four matrix expressions are defined?

A. 0
B. 1
C. 2
D. 3
E. 4

Source: *VCE 2019, Further Mathematics 1, Section B, Module 1, Q.3; © VCAA*

Question 2 (1 mark)

Consider the matrix P, where $P = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}$

The element in row i and column j of matrix P is p_{ij}.
The elements in matrix P are determined by the rule

A. $p_{ij} = 4 - j$
B. $p_{ij} = 2i + 1$
C. $p_{ij} = i + j + 1$
D. $p_{ij} = i + 2j$
E. $p_{ij} = 2i - j + 2$

Source: *VCE 2017, Further Mathematics 1, Section B, Module 1, Q.6 ;© VCAA*

Question 3 (1 mark)

The table below shows information about two matrices, A and B.

Matrix	Order	Rule
A	3×3	$aij = 2i + j$
B	3×3	$bij = i - j$

The element in row i and column j of matrix A is a_{ij}.
The element in row i and column j of matrix B is b_{ij}.
The sum $A + B$ is

A. $\begin{bmatrix} 5 & 7 & 9 \\ 8 & 10 & 12 \\ 11 & 13 & 15 \end{bmatrix}$

B. $\begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 6 & 9 \\ 3 & 6 & 9 \\ 3 & 6 & 9 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 9 & 9 & 9 \end{bmatrix}$

E. $\begin{bmatrix} 3 & 6 & 3 \\ 6 & 3 & 9 \\ 3 & 9 & 3 \end{bmatrix}$

Topic 8 > Subtopic 8.3 Addition, subtraction and scalar operations with matrices

Source: *VCE 2015, Further Mathematics 1, Section B, Module 6, Q.8 ; © VCAA*

Question 4 (1 mark)

The order of matrix X is 2×3.

The element in row i and column j of matrix X is x_{ij} and it is determined by the rule

$xij = i - j$

Which one of the following calculations would result in matrix X?

A. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$

Topic 8 | Subtopic 8.3 Addition, subtraction and scalar operations with matrices

Source: *VCE 2014, Further Mathematics 1, Section B, Module 6, Q.6; © VCAA*

Question 5 (1 mark)

The order of matrix X is 3×2.

The element in row i and column j of matrix X is x_{ij} and it is determined by the rule $x_{ij} = i + j$.

The matrix X is

A. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}$

E. $\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$

Question 6 (1 mark)

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

Then $A - B$ equals

A. $\begin{bmatrix} -4 & -3 \\ -5 & -4 \end{bmatrix}$

B. $\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

C. $\begin{bmatrix} 7 & 7 \\ 10 & 12 \end{bmatrix}$

D. $\begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{bmatrix}$

Question 7 (1 mark)

Which of the following matrices is equal to the matrix $\begin{bmatrix} 24 & 36 \\ 48 & 18 \end{bmatrix}$?

A. $12\begin{bmatrix} 2 & 6 \\ 4 & 1 \end{bmatrix}$

B. $\frac{1}{12}\begin{bmatrix} 2 & 6 \\ 4 & 1 \end{bmatrix}$

C. $\frac{1}{6}\begin{bmatrix} 4 & 6 \\ 8 & 3 \end{bmatrix}$

D. $6\begin{bmatrix} 4 & 6 \\ 8 & 3 \end{bmatrix}$

E. $2\begin{bmatrix} 12 & 10 \\ 22 & 9 \end{bmatrix}$

Topic	8	Matrices
Subtopic	8.4	Multiplying matrices

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***Source**: VCE 2020, Further Mathematics 1, Section B, Module 1, Q.1; © VCAA*

Question 1 (1 mark)

The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is an example of

A. a binary matrix.
B. an identity matrix.
C. a triangular matrix.
D. a symmetric matrix.
E. a permutation matrix.

***Source**: VCE 2020, Further Mathematics 1, Section B, Module 1, Q.2; © VCAA*

Question 2 (1 mark)

Matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 0 \\ 4 & 5 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 2 & 0 & 3 & 1 \\ 4 & 5 & 2 & 0 \end{bmatrix}$

Matrix $Q = A \times B$.
The element in row i and column j of matrix Q is q_{ij}.
Element q_{41} is determined by the calculation

A. $0 \times 0 + 3 \times 5$
B. $1 \times 1 + 2 \times 0$
C. $1 \times 2 + 2 \times 4$
D. $4 \times 1 + 5 \times 0$
E. $4 \times 2 + 5 \times 4$

***Source**: VCE 2019, Further Mathematics 1, Section B, Module 1, Q.2; © VCAA*

Question 3 (1 mark)

There are two rides called The Big Dipper and The Terror Train at a carnival.
The cost, in dollars, for a child to ride on each ride is shown in the table below.

Ride	**Cost** ($)
The Big Dipper	7
The Terror Train	8

Six children ride once only on The Big Dipper and once only on The Terror Train.
The total cost of the rides, in dollars, for these six children can be determined by which one of the following calculations?

A. $[6] \times [7 \quad 8]$

B. $[6] \times \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

C. $\begin{bmatrix} 6 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \end{bmatrix}$

D. $\begin{bmatrix} 6 & 6 \end{bmatrix} \times \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

E. $\begin{bmatrix} 6 \\ 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \end{bmatrix}$

***Source**: VCE 2020, Further Mathematics 1, Section B, Module 1, Q.3 © VCAA*

Question 4 (1 mark)

Matrices P and W are defined below.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad W = \begin{bmatrix} A \\ S \\ T \\ O \\ R \end{bmatrix}$$

If $P^n \times W = \begin{bmatrix} A \\ S \\ T \\ O \\ R \end{bmatrix}$, the value of n could be

A. 1
B. 2
C. 3
D. 4
E. 5

***Source**: VCE 2018, Further Mathematics 1, Section B, Module 1, Q.2; © VCAA*

Question 5 (1 mark)

The matrix product $\begin{bmatrix} 4 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 4 \\ 12 \\ 8 \end{bmatrix}$ is equal to

A. $[144]$

B. $\begin{bmatrix} 16 \\ 24 \\ 0 \end{bmatrix}$

C. $4 \times \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 12 \\ 8 \end{bmatrix}$

D. $2 \times \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$

E. $4 \times \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$

***Source**: VCE 2016, Further Mathematics 1, Section B, Module 1, Q.2; © VCAA*

Question 6 (1 mark)

The matrix product $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \text{L} \\ \text{E} \\ \text{A} \\ \text{P} \\ \text{S} \end{bmatrix}$ is equal to

A. $\begin{bmatrix} \text{L} \\ \text{A} \\ \text{P} \\ \text{S} \\ \text{E} \end{bmatrix}$

B. $\begin{bmatrix} \text{L} \\ \text{E} \\ \text{A} \\ \text{P} \\ \text{S} \end{bmatrix}$

C. $\begin{bmatrix} \text{P} \\ \text{L} \\ \text{E} \\ \text{A} \\ \text{S} \end{bmatrix}$

D. $\begin{bmatrix} \text{P} \\ \text{A} \\ \text{L} \\ \text{E} \\ \text{S} \end{bmatrix}$

E. $\begin{bmatrix} \text{P} \\ \text{E} \\ \text{A} \\ \text{L} \\ \text{S} \end{bmatrix}$

***Source**: VCE 2016, Further Mathematics 1, Section B, Module 1, Q.4; © VCAA*

Question 7 (1 mark)

The table below shows the number of each type of coin saved in a moneybox.

Coin	5 cent	10 cent	20 cent	50 cent
Number	15	32	48	24

The matrix product that displays the total number of coins and the total value of these coins is

A. $\begin{bmatrix} 5 & 10 & 20 & 50 \end{bmatrix} \begin{bmatrix} 15 \\ 32 \\ 48 \\ 24 \end{bmatrix}$

B. $\begin{bmatrix} 15 & 32 & 48 & 24 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 1 & 10 \\ 1 & 20 \\ 1 & 50 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 10 & 20 & 50 \end{bmatrix} \begin{bmatrix} 1 & 15 \\ 1 & 32 \\ 1 & 48 \\ 1 & 24 \end{bmatrix}$

D. $\begin{bmatrix} 15 & 32 & 48 & 24 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 20 \\ 50 \end{bmatrix}$

E. $\begin{bmatrix} 5 & 10 & 20 & 50 \\ 15 & 32 & 48 & 24 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

***Source**: VCE 2015, Further Mathematics 1, Section B, Module 6, Q.2; © VCAA*

Question 8 (1 mark)

Four matrices are shown below.

$$W = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \quad X = \begin{bmatrix} 4 & 1 & 5 \\ 2 & 0 & 6 \end{bmatrix} \quad Y = \begin{bmatrix} 7 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} 8 & 5 & 0 \\ 1 & 9 & 3 \\ 4 & 2 & 7 \end{bmatrix}$$

Which one of the following matrix products is **not** defined?

A. $W \times Y$

B. $X \times W$

C. $Y \times X$

D. $Z \times W$

E. $Z \times Y$

***Source**: VCE 2014, Further Mathematics 1, Section B, Module 6, Q.9; © VCAA*

Question 9 (1 mark)

A and B are square matrices such that $AB = BA = I$, where I is an identity matrix.

Which one of the following statements is **not** true?

A. $ABA = A$

B. $AB^2A = I$

C. B must equal A

D. B is the inverse of A

E. both A and B have inverses

Source: *VCE 2014, Further Mathematics 1, Section B, Module 6, Q.5; © VCAA*

Question 10 (1 mark)

Students from Year 7 and Year 8 in a school sold trees to raise funds for a school trip.

The number of large, medium and small trees that were sold by each year group is shown in the table below.

Year group	Large	Medium	Small
7	52	78	61
8	45	56	81

The large trees were sold for \$32 each, the medium trees were sold for \$26 each and the small trees were sold for \$18 each.

A matrix product that can be used to calculate the amount, in dollars, raised by each year group by selling trees is

A. $\begin{bmatrix} 52 & 78 & 61 \\ 32 & 26 & 18 \end{bmatrix}\begin{bmatrix} 45 \\ 56 \\ 81 \end{bmatrix}$

B. $\begin{bmatrix} 7 & 52 & 78 & 61 \\ 8 & 45 & 56 & 81 \end{bmatrix}\begin{bmatrix} 32 \\ 26 \\ 18 \\ 0 \end{bmatrix}$

C. $\begin{bmatrix} 32 & 26 & 18 \end{bmatrix}\begin{bmatrix} 52 & 45 \\ 78 & 56 \\ 61 & 81 \end{bmatrix}$

D. $\begin{bmatrix} 32 & 26 & 18 \end{bmatrix}\begin{bmatrix} 52 & 78 & 61 \\ 45 & 56 & 81 \\ 0 & 0 & 0 \end{bmatrix}$

E. $\left(\begin{bmatrix} 52 \\ 78 \\ 61 \end{bmatrix} \times \begin{bmatrix} 45 \\ 56 \\ 81 \end{bmatrix}\right) \times \begin{bmatrix} 32 & 26 & 18 \end{bmatrix}$

***Source**: VCE 2014, Further Mathematics 1, Section B, Module 6, Q.1; © VCAA*

Question 11 (1 mark)

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ is equal to

A. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & 4 & 0 \\ 4 & 1 & 1 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

***Source**: VCE 2013, Further Mathematics 1, Section B, Module 6, Q.2; © VCAA*

Question 12 (1 mark)

Matrix A has three rows and two columns.

Matrix B has four rows and three columns.

Matrix $C = B \times A$ has

A. two rows and three columns.

B. three rows and two columns.

C. three rows and three columns.

D. four rows and two columns.

E. four rows and three columns.

***Source**: VCE 2013, Further Mathematics 1, Section B, Module 6, Q.7; © VCAA*

Question 13 (1 mark)

A school has three computer classes, A, B and C. There are 15 students in each class.

Each student is given a mark out of 100 based on their performance in a test.

Matrix M below displays the marks obtained by these 45 students, listed by class.

$$M = \begin{array}{c} \begin{bmatrix} 56 & 78 & 79 & 43 & 67 & 56 & 80 & 85 & 75 & 89 & 55 & 64 & 95 & 34 & 63 \\ 90 & 45 & 56 & 65 & 76 & 79 & 27 & 45 & 69 & 73 & 70 & 63 & 65 & 34 & 59 \\ 76 & 76 & 89 & 47 & 50 & 66 & 68 & 89 & 88 & 90 & 45 & 67 & 78 & 45 & 87 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \ \text{class} \\ C \end{array}$$

Two other matrices, S and R, are defined below.

$$S = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Which one of the following matrix expressions can be used to generate a matrix that displays the mean mark obtained for each class?

A. $\frac{1}{45}M$

B. $\frac{1}{3}R \times M$

C. $\frac{1}{3}R \times M \times S$

D. $\frac{1}{15}M \times S$

E. $\frac{1}{15}S \times R \times M$

Question 14 (1 mark)

If $A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & -8 & 17 \\ 5 & 14 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 3 \\ -5 & 1 & 0 \end{bmatrix}$, the order of the matrix AB is

A. (3×2)
B. (2×3)
C. (3×3)
D. (2×2)
E. not defined

Topic	8	Matrices
Subtopic	8.5	The inverse of a matrix and its determinant

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Source: *VCE 2021, Further Mathematics 1, Section B, Module 1, Q.4; © VCAA*

Question 1 (1 mark)

Ramon and Norma are names that contain the same letters but in a different order.

The permutation matrix that can change $\begin{bmatrix} R \\ A \\ M \\ O \\ N \end{bmatrix}$ into $\begin{bmatrix} N \\ O \\ R \\ M \\ A \end{bmatrix}$ is

A. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

Source: *VCE 2021, Further Mathematics 1, Section B, Module 1, Q.3; © VCAA*

Question 2 (1 mark)

$ax + 4y = 10$

$18x + by = 6$

The set of simultaneous linear equations above does not have a unique solution when

A $a = 2,\ b = 36$
B $a = 3,\ b = 22$
C $a = 4,\ b = 20$
D $a = 5,\ b = 12$
E $a = 6,\ b = 14$

Source: *VCE 2015, Further Mathematics 1, Section B, Module 6, Q.7; © VCAA*

Question 3 (1 mark)

Matrix P has inverse matrix P^{-1}.

Matrix P is multiplied by the scalar $w\ (w \neq 0)$ to form matrix Q.

Matrix Q^{-1} is equal to

A. $\frac{1}{w}P^{-1}$
B. $\frac{1}{w^2}P^{-1}$
C. wP^{-1}
D. w^2P^{-1}
E. P^{-1}

Source: *VCE 2018, Further Mathematics 1, Section B, Module 1, Q.1; © VCAA*

Question 4 (1 mark)

Which one of the following matrices has a determinant of zero?

A. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

E. $\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$

Source: *VCE 2016, Further Mathematics 1, Section B, Module 1, Q.3; © VCAA*

Question 5 (1 mark)

The matrix equation below represents a pair of simultaneous linear equations.

$$\begin{bmatrix} 12 & 9 \\ m & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

These simultaneous linear equations have no unique solution when m is equal to

A. -4

B. -3

C. 0

D. 3

E. 4

Question 6 (1 mark)

$A = \begin{bmatrix} 6 & -8 \\ 5 & 14 \end{bmatrix}$

The inverse of A equals

A. $\begin{bmatrix} 14 & 8 \\ -5 & 6 \end{bmatrix}$

B. $\begin{bmatrix} 7 & 2 \\ 5 & 3 \end{bmatrix}$

C. $\begin{bmatrix} 0.11 & 0.06 \\ -0.04 & 0.05 \end{bmatrix}$

D. $\frac{1}{124}$

E. not defined

Question 7 (1 mark)

The determinant of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is

A. $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

B. $-\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

C. $-\frac{1}{2}$

D. -2

E. not defined

Question 8 (1 mark)

The inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ is

A. $\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$

B. $\begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}$

C. $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

D. 0

E. not defined

Topic	8	Matrices
Subtopic	8.6	Dominance and communication matrices

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Source: *VCE 2020, Further Mathematics 1, Section B, Module 1, Q.9; © VCAA*

Question 1 (1 mark)

Five competitors, Andy (A), Brie (B), Cleo (C), Della (D) and Eddie (E), participate in a darts tournament. Each competitor plays each of the other competitors once only, and each match results in a winner and a loser.

The matrix below shows the results of this darts tournament.

There are still two matches that need to be played.

$$\begin{array}{cc} & \textit{loser} \\ & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{r} A \\ B \\ \textit{winner}\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & ... & 0 & 1 & 0 \\ ... & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & ... & 1 \\ 0 & 1 & ... & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

A ‘1’ in the matrix shows that the competitor named in that row defeated the competitor named in that column.

For example, the ‘1’ in row 2, column 3 shows that Brie defeated Cleo.

A ‘. . .’ in the matrix shows that the competitor named in that row has not yet played the competitor named in that column.

The winner of this darts tournament is the competitor with the highest sum of their one-step and two-step dominances.

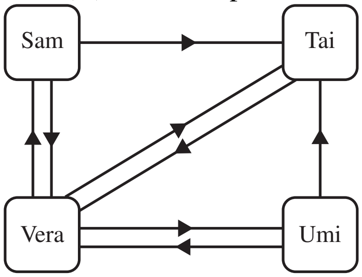
Which player, by winning their remaining match, will ensure that they are ranked first by the sum of their one-step and two-step dominances?

A. Andy
B. Brie
C. Cleo
D. Della
E. Eddie

Source: *VCE 2020, Further Mathematics 1, Section B, Module 1, Q.5; © VCAA*

Question 2 (1 mark)

The diagram below shows the direct communication links that exist between Sam (S), Tai (T), Umi (U) and Vera (V). For example, the arrow from Umi to Vera indicates that Umi can communicate directly with Vera.



A communication matrix can be used to convey the same information.

In this matrix:

a '1' indicates that a direct communication link exists between a sender and a receiver

a '0' indicates that a direct communication link does not exist between a sender and a receiver.

The communication matrix could be

A.
$$\textit{sender}\quad \begin{array}{c} \\ S \\ T \\ U \\ V \end{array}\begin{array}{c} \textit{receiver} \\ \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

B.
$$\textit{sender}\quad \begin{array}{c} \\ S \\ T \\ U \\ V \end{array}\begin{array}{c} \textit{receiver} \\ \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

C.
$$\textit{sender}\quad \begin{array}{c} \\ S \\ T \\ U \\ V \end{array}\begin{array}{c} \textit{receiver} \\ \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

D.
$$\textit{sender}\quad \begin{array}{c} \\ S \\ T \\ U \\ V \end{array}\begin{array}{c} \textit{receiver} \\ \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

E.
$$\textit{sender}\quad \begin{array}{c} \\ S \\ T \\ U \\ V \end{array}\begin{array}{c} \textit{receiver} \\ \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix} \end{array}$$

Topic 8 | Subtopic 8.6 Dominance and communication matrices

Source: *VCE 2019, Further Mathematics 1, Section B, Module 1, Q.7; © VCAA*

Question 3 (1 mark)

The communication matrix below shows the direct paths by which messages can be sent between two people in a group of six people, U to Z.

$$\begin{array}{cc} & \textit{receiver} \\ & \begin{array}{cccccc} U & V & W & X & Y & Z \end{array} \\ \textit{sender} \begin{array}{c} U \\ V \\ W \\ X \\ Y \\ Z \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

A ‘1’ in the matrix shows that the person named in that row can send a message directly to the person named in that column.

For example, the ‘1’ in row 4, column 2 shows that X can send a message directly to V.

In how many ways can Y get a message to W by sending it directly to one other person?

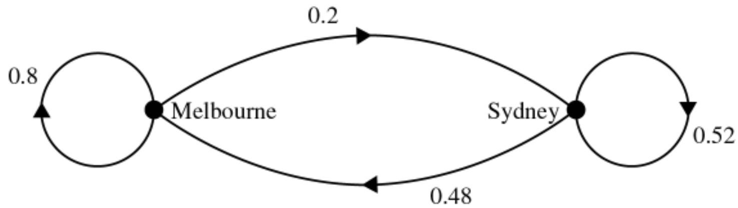
A. 0
B. 1
C. 2
D. 3
E. 4

Source: *VCE 2019, Further Mathematics 1, Section B, Module 1, Q.8; © VCAA*

Question 4 (1 mark)

An airline parks all of its planes at Sydney airport or Melbourne airport overnight.

The transition diagram below shows the change in the location of the planes from night to night.



There are always *m* planes parked at Melbourne airport.

There are always *s* planes parked at Sydney airport.

Of the planes parked at Melbourne airport on Tuesday night, 12 had been parked at Sydney airport on Monday night.

How many planes does the airline have?

A. 25
B. 37
C. 62
D. 65
E. 85

Source: *VCE 2018, Further Mathematics 1, Section B, Module 1, Q.5; © VCAA*

Question 5 (1 mark)

Liam cycles, runs, swims and walks for exercise several times a month.
Each time he cycles, Liam covers a distance of c kilometres.
Each time he runs, Liam covers a distance of r kilometres.
Each time he swims, Liam covers a distance of s kilometres.
Each time he walks, Liam covers a distance of w kilometres.
The number of times that Liam cycled, ran, swam and walked each month over a four-month period, and the total distance that Liam travelled in each of those months, are shown in the table below.

	Number of times in a month				
	Cycle	**Run**	**Swim**	**Walk**	**Total distance for a month (km)**
Month 1	5	7	6	8	160
Month 2	8	6	9	7	172
Month 3	7	8	7	6	165
Month 4	8	8	5	5	162

The matrix that contains the distance each time Liam cycled, ran, swam and walked, $\begin{bmatrix} c \\ r \\ s \\ w \end{bmatrix}$, is

A. $\begin{bmatrix} 8 \\ 6 \\ 7 \\ 9 \end{bmatrix}$

B. $\begin{bmatrix} 8 \\ 6 \\ 1 \\ 9 \end{bmatrix}$

C. $\begin{bmatrix} 8 \\ 6 \\ 7 \\ 9 \end{bmatrix}$

D. $\begin{bmatrix} 8 \\ 8 \\ 9 \\ 8 \end{bmatrix}$

E. $\begin{bmatrix} 4290 \\ 4931 \\ 4623 \\ 4291 \end{bmatrix}$

Source: *VCE 2018, Further Mathematics 1, Section B, Module 1, Q.4; © VCAA*

Question 6 (1 mark)

Matrix P is a 4×4 permutation matrix.

Matrix W is another matrix such that the matrix product $P \times W$ is defined.

This matrix product results in the entire first and third rows of matrix W being swapped.

The permutation matrix P is

A. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Source: *VCE 2018, Further Mathematics 1, Section B, Module 1, Q.7; © VCAA*

Question 7 (1 mark)

A study of the antelope population in a wildlife park has shown that antelope regularly move between three locations, east (E), north (N) and west (W).

Let An be the state matrix that shows the population of antelope in each location n months after the study began.

The expected population of antelope in each location can be determined by the matrix recurrence rule

$A_{n+1} = TA_n - \text{D}$

where

$$T = \begin{array}{c} \textit{this month} \\ \begin{array}{ccc} E & N & W \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.2 \\ 0.3 & 0.6 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ E \\ N \\ W \end{array} \quad \textit{next month}$$

and

$$D = \begin{bmatrix} 50 \\ 50 \\ 50 \end{bmatrix} \begin{array}{c} E \\ N \\ W \end{array}$$

The state matrix, A_3, below shows the population of antelope three months after the study began.

$$A_3 = \begin{bmatrix} 1616 \\ 2800 \\ 2134 \end{bmatrix} \begin{array}{c} E \\ N \\ W \end{array}$$

The number of antelope in the **west** (W) location two months after the study began, as found in the state matrix A_2, is closest to

A. 2060
B. 2130
C. 2200
D. 2240
E. 2270

Source: *VCE 2018, Further Mathematics 1, Section B, Module 1, Q.3; © VCAA*

Question 8 (1 mark)

Five people, India (I), Jackson (J), Krishna (K), Leanne (L) and Mustafa (M), competed in a table tennis tournament.

Each competitor played every other competitor once only.

Each match resulted in a winner and a loser.

The matrix below shows the tournament results.

$$\begin{array}{cc} & \textit{loser} \\ & \begin{array}{ccccc} I & J & K & L & M \end{array} \\ \textit{winner} \begin{array}{c} I \\ J \\ K \\ L \\ M \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

A '1' in the matrix shows that the competitor named in that row defeated the competitor named in that column.

For example, the '1' in the fourth row shows that Leanne defeated Jackson.

A '0' in the matrix shows that the competitor named in that row lost to the competitor named in that column.

There is an error in the matrix. The winner of one of the matches has been incorrectly recorded as a '0'.

This match was between

A. India and Mustafa.
B. India and Krishna.
C. Krishna and Leanne.
D. Leanne and Mustafa.
E. Jackson and Mustafa.

Topic 8 Subtopic 8.6 Dominance and communication matrices

Source: *VCE 2017, Further Mathematics 1, Section B, Module 1, Q.4; © VCAA*

Question 9 (1 mark)

A permutation matrix, P, can be used to change $\begin{bmatrix} F \\ E \\ A \\ R \\ S \end{bmatrix}$ into $\begin{bmatrix} S \\ F \\ A \\ E \\ R \end{bmatrix}$.

Matrix P is

A. $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Source: *VCE 2017, Further Mathematics 1, Section B, Module 1, Q.2; © VCAA*

Question 10 (1 mark)

The matrix below shows how five people, Alan (A), Bevan (B), Charlie (C), Drew (D) and Esther (E), can communicate with each other.

$$\begin{array}{cc} & \textit{receiver} \\ & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \textit{sender} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

A ‘1’ in the matrix shows that the person named in that row can send a message directly to the person named in that column.

For example, the ‘1’ in row 3 and column 4 shows that Charlie can send a message directly to Drew.

Esther wants to send a message to Bevan.

Which one of the following shows the order of people through which the message is sent?

A. Esther – Bevan
B. Esther – Charlie – Bevan
C. Esther – Charlie – Alan – Bevan
D. Esther – Charlie – Drew – Bevan
E. Esther – Charlie – Drew – Alan – Bevan

Source: *VCE 2016, Further Mathematics 1, Section B, Module 1, Q.8; © VCAA*

Question 11 (1 mark)

The matrix below shows the result of each match between four teams, A, B, C and D, in a bowling tournament. Each team played each other team once and there were no draws.

$$\begin{array}{cc} & \textit{loser} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \textit{winner} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

In this tournament, each team was given a ranking that was determined by calculating the sum of its one-step and two-step dominances. The team with the highest sum was ranked number one (1). The team with the second-highest sum was ranked number two (2), and so on.

Using this method, team C was ranked number one (1).

Team A would have been ranked number one (1) if the winner of one match had lost instead.

That match was between teams

A. A and B.

B. A and D.

C. B and C.

D. B and D.

E. C and D.

Question 12 (1 mark)

Four teams, A, B, C and D, competed in a round-robin competition where each team played each of the other teams once. There were no draws.

The results are shown in the matrix below.

$$\begin{array}{cc} & \textit{loser} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \textit{winner} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & f & 1 \\ 1 & 0 & 0 & 0 \\ 1 & g & 0 & 1 \\ 0 & 1 & 0 & h \end{bmatrix} \end{array}$$

A ‘ 1’ in the matrix shows that the team named in that row defeated the team named in that column.

For example, the ‘ 1’ in row 2 shows that team B’ in defeated team A.

In this matrix, the values of f, g and h are

A. $f = 0, \quad g = 1, \quad h = 0$

B. $f = 0, \quad g = 1, \quad h = 1$

C. $f = 1, \quad g = 0, \quad h = 0$

D. $f = 1, \quad g = 1, \quad h = 0$

E. $f = 1, \quad g = 1, \quad h = 1$

Question 13 (1 mark)

The following transition matrix shows the order in which warehouses (*A, B, C, D*) are visited by a particular courier company.

$$\begin{array}{ccc} & & \text{from} \\ & & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{to} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

It can be determined from this matrix, a correct order of visitation to the warehouses could be

A. C – D – A – B
B. B – A – C – D
C. D – C – A – B
D. D – A – C – B
E. A – B – C – D

Question 14 (1 mark)

On a recent holiday around Australia the following towns were visited in the order mentioned: Esperance (E), Broome (B), Townsville (T), Mildura (M) and back to Esperance (E).

A transition matrix showing this sequence of town visits is

A.

$$\begin{array}{cc} & \text{from} \\ & \begin{array}{cccc} B & E & M & T \end{array} \\ \text{to} \begin{array}{c} B \\ E \\ M \\ T \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

B.

$$\begin{array}{cc} & \text{from} \\ & \begin{array}{cccc} B & E & M & T \end{array} \\ \text{to} \begin{array}{c} B \\ E \\ M \\ T \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

C.

$$\begin{array}{cc} & \text{from} \\ & \begin{array}{cccc} B & E & M & T \end{array} \\ \text{to} \begin{array}{c} B \\ E \\ M \\ T \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

D.

$$\begin{array}{cc} & \text{from} \\ & \begin{array}{cccc} B & E & M & T \end{array} \\ \text{to} \begin{array}{c} B \\ E \\ M \\ T \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

E.

$$\begin{array}{cc} & \text{from} \\ & \begin{array}{cccc} B & E & M & T \end{array} \\ \text{to} \begin{array}{c} B \\ E \\ M \\ T \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Question 15 (1 mark)

An ice-cream vendor sells his special creation filled with flavours delivered in the order of the matrix below. The flavours are Rocky Round (R), Chocolate (C), Liquorice (L) and Vanilla (V).

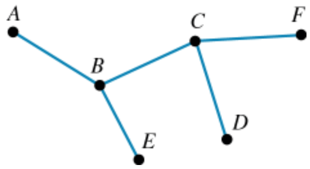
$$\begin{array}{cc} & \text{from} \\ & \begin{array}{cccc} C & L & R & V \end{array} \\ \text{to} \begin{array}{c} C \\ L \\ R \\ V \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The correct order of ice-cream delivery, if Rocky Round starts, is

A. R – C – L – V
B. R – V – L – C
C. R – L – C – V
D. R – L – V – C
E. R – C – V – L

Question 16 (1 mark)

A communication network is represented by the diagram shown.



The matrix that best represents this communication matrix is

A.
$$\begin{array}{c} \\ A\\B\\C\\D\\E\\F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array}\\ \begin{bmatrix} 0&1&0&0&0&0\\1&0&1&0&1&0\\0&1&0&1&0&1\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0 \end{bmatrix} \end{array}$$

B.
$$\begin{array}{c} \\ A\\B\\C\\D\\E\\F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array}\\ \begin{bmatrix} 0&1&0&1&0&0\\1&0&1&0&1&0\\0&1&0&1&0&1\\1&0&1&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0 \end{bmatrix} \end{array}$$

C.
$$\begin{array}{c} \\ A\\B\\C\\D\\E\\F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array}\\ \begin{bmatrix} 0&1&0&0&0&0\\1&0&1&0&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\\0&1&0&1&0&0\\0&0&1&1&0&0 \end{bmatrix} \end{array}$$

D.
$$\begin{array}{c} \\ A\\B\\C\\D\\E\\F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array}\\ \begin{bmatrix} 0&1&0&1&0&0\\1&0&1&0&1&0\\0&1&0&1&0&1\\1&0&1&0&1&0\\0&1&0&1&0&1\\0&0&1&0&1&0 \end{bmatrix} \end{array}$$

E.
$$\begin{array}{c} \\ A\\B\\C\\D\\E\\F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array}\\ \begin{bmatrix} 0&1&0&0&0&0\\1&0&1&0&1&0\\0&1&0&1&0&1\\0&0&1&0&0&0\\0&1&0&0&0&1\\0&0&1&0&1&0 \end{bmatrix} \end{array}$$

Question 17 (1 mark)

Which of the following would not be representative of a matrix display of a communication network, where communications are in both directions between vertices?

A. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Topic	**8**	**Matrices**
Subtopic	**8.7**	**Transition matrices and Leslie matrices**

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Source: *VCE 2021, Further Mathematics 1, Section B, Module 1, Q.2; © VCAA*

Question 1 (1 mark)

Every Friday, the same number of workers from a large office building regularly purchase their lunch from one of two locations: the deli, D, or the cafe, C.

It has been found that:

- of the workers who purchase lunch from the deli on one Friday, 65% will return to purchase from the deli on the next Friday
- of the workers who purchase lunch from the cafe on one Friday, 55% will return to purchase from the cafe on the next Friday.

A transition matrix that can be used to describe this situation is

A.
$$\begin{array}{cc} & \textit{this Friday} \\ & \begin{array}{cc} D & C \end{array} \\ & \begin{bmatrix} 0.55 & 0.35 \\ 0.45 & 0.65 \end{bmatrix} \begin{array}{c} D \\ C \end{array} \; \textit{next Friday} \end{array}$$

B.
$$\begin{array}{cc} & \textit{this Friday} \\ & \begin{array}{cc} D & C \end{array} \\ & \begin{bmatrix} 0.65 & 0.45 \\ 0.45 & 0.55 \end{bmatrix} \begin{array}{c} D \\ C \end{array} \; \textit{next Friday} \end{array}$$

C.
$$\begin{array}{cc} & \textit{this Friday} \\ & \begin{array}{cc} D & C \end{array} \\ & \begin{bmatrix} 0.65 & 0.55 \\ 0.45 & 0.55 \end{bmatrix} \begin{array}{c} D \\ C \end{array} \; \textit{next Friday} \end{array}$$

D.
$$\begin{array}{cc} & \textit{this Friday} \\ & \begin{array}{cc} D & C \end{array} \\ & \begin{bmatrix} 0.65 & 0.45 \\ 0.35 & 0.55 \end{bmatrix} \begin{array}{c} D \\ C \end{array} \; \textit{next Friday} \end{array}$$

E.
$$\begin{array}{cc} & \textit{this Friday} \\ & \begin{array}{cc} D & C \end{array} \\ & \begin{bmatrix} 0.65 & 0.55 \\ 0.35 & 0.45 \end{bmatrix} \begin{array}{c} D \\ C \end{array} \; \textit{next Friday} \end{array}$$

Source: *VCE 2020, Further Mathematics 1, Section B, Module 1, Q.7; © VCAA*

Question 2 (1 mark)

A small shopping centre has two coffee shops: Fatima's (F) and Giorgio's (G).

The percentage of coffee-buyers at each shop changes from day to day, as shown in the transition matrix T.

$$\begin{array}{cc} & \textit{today} \\ & \begin{array}{cc} F & G \end{array} \\ T = & \begin{bmatrix} 0.85 & 0.35 \\ 0.15 & 0.65 \end{bmatrix} \begin{array}{l} F \\ G \end{array} \textit{tomorrow} \end{array}$$

On a particular Monday, 40% of coffee-buyers bought their coffees at Fatima's.

The matrix recursion relation $S_{n+1} = TS_n$ is used to model this situation.

The percentage of coffee-buyers who are expected to buy their coffee at Giorgio's on Friday of the same week is closest to

A. 31%

B. 32%

C. 34%

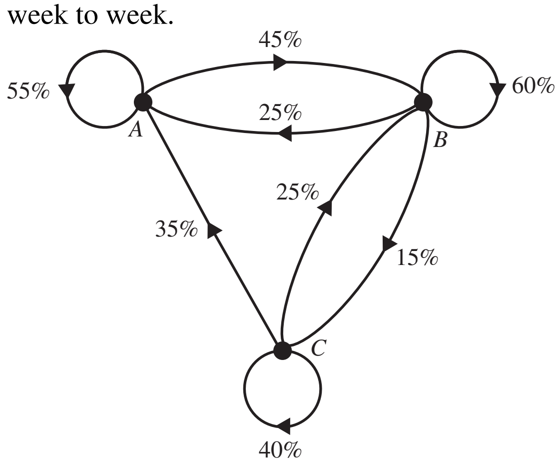
D. 45%

E. 68%

Source: *VCE 2020, Further Mathematics 1, Section B, Module 1, Q.4; © VCAA*

Question 3 (1 mark)

In a particular supermarket, the three top-selling magazines are Angel (A), Bella (B) and Crystal (C). The transition diagram below shows the way shoppers at this supermarket change their magazine choice from week to week.



A transition matrix that provides the same information as the transition diagram is

A.

$$\begin{array}{cc} & \textit{this week} \\ & \begin{array}{ccc} A & B & C \end{array} \\ & \begin{bmatrix} 0.55 & 0.70 & .035 \\ 0.70 & 0.60 & 0.40 \\ 0.35 & 0.40 & 0.40 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \ \textit{next week} \end{array}$$

B.

$$\begin{array}{cc} & \textit{this week} \\ & \begin{array}{ccc} A & B & C \end{array} \\ & \begin{bmatrix} 0.55 & 0.60 & .025 \\ 0.45 & 0.15 & 0.35 \\ 0 & 0.25 & 0.40 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \ \textit{next week} \end{array}$$

C.

$$\begin{array}{cc} & \textit{this week} \\ & \begin{array}{ccc} A & B & C \end{array} \\ & \begin{bmatrix} 0.55 & 0.25 & .035 \\ 0.45 & 0.60 & 0.25 \\ 0 & 0.15 & 0.40 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \ \textit{next week} \end{array}$$

D.

$$\begin{array}{cc} & \textit{this week} \\ & \begin{array}{ccc} A & B & C \end{array} \\ & \begin{bmatrix} 0.55 & 0.25 & 0.35 \\ 0.45 & 0.60 & 0.25 \\ 0.35 & 0.15 & 0.40 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \ \textit{next week} \end{array}$$

E.

$$\begin{array}{cc} & \textit{this week} \\ & \begin{array}{ccc} A & B & C \end{array} \\ & \begin{bmatrix} 0.55 & 0.25 & 0 \\ 0.45 & 0.60 & 0.25 \\ 0 & 0.15 & 0.75 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \ \textit{next week} \end{array}$$

Source: *VCE 2020, Further Mathematics 1, Section B, Module 1, Q.10; © VCAA*

Question 4 (1 mark)

Consider the matrix recurrence relation below.

$$S_0 = \begin{bmatrix} 30 \\ 20 \\ 40 \end{bmatrix}, \quad S_{n+1} = TS_n \text{ where } T = \begin{bmatrix} j & 0.3 & l \\ 0.2 & m & 0.3 \\ 0.4 & 0.2 & n \end{bmatrix}$$

Matrix T is a regular transition matrix.

Given the information above and that 1, which one of the following is true?

A. $m > l$

B. $j + l = 0.7$

C. $j = n$

D. $j > m$

E. $l = m + n$

Source: *VCE 2019, Further Mathematics 1, Section B, Module 1, Q.6; © VCAA*

Question 5 (1 mark)

A water park is open from 9 am until 5 pm.

There are three activities, the pool (P), the slide (S) and the water jets (W), at the water park.

Children have been found to change their activity at the water park each half hour, as shown in the transition matrix, T, below.

$$\begin{array}{c} \\ \\ T = \end{array} \begin{array}{c} \textit{this half hour} \\ \begin{array}{ccc} P & S & W \end{array} \\ \begin{bmatrix} 0.80 & 0.20 & 0.40 \\ 0.05 & 0.60 & 0.10 \\ 0.15 & 0.20 & 0.50 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ P \\ S \quad \textit{next half hour} \\ W \end{array}$$

A group of children has come to the water park for the whole day.

The percentage of these children who are expected to be at the slide (S) at closing time is closest to

A. 14%

B. 20%

C. 24%

D. 25%

E. 62%

Source: *VCE 2018, Further Mathematics 1, Section B, Module 1, Q.8; © VCAA*

Question 6 (1 mark)

A public library organised 500 of its members into five categories according to the number of books each member borrows each month.

These categories are

J = no books borrowed per month

K = one book borrowed per month

L = two books borrowed per month

M = three books borrowed per month

N = four or more books borrowed per month

The transition matrix, T, below shows how the number of books borrowed per month by the members is expected to change from month to month.

$$\begin{array}{cc} & \textit{this month} \\ & \begin{array}{ccccc} J & K & L & M & N \end{array} \\ T = & \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0 & 0 \\ 0.5 & 0.2 & 0.3 & 0.1 & 0 \\ 0.3 & 0.3 & 0.4 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.1 & 0.6 & 0.3 \\ 0 & 0.1 & 0 & 0.2 & 0.5 \end{bmatrix} \begin{array}{l} J \\ K \\ L \quad \textit{next month} \\ M \\ N \end{array} \end{array}$$

In the long term, which category is expected to have approximately 96 members each month?

A. J

B. K

C. L

D. M

E. N

Topic 8 Subtopic 8.7 Transition matrices and Leslie matrices

Source: *VCE 2017, Further Mathematics 1, Section B, Module 1, Q.7; © VCAA*

Question 7 (1 mark)

At a fish farm:

- young fish (Y) may eventually grow into juveniles (J) or they may die (D)
- juveniles (J) may eventually grow into adults (A) or they may die (D)
- adults (A) eventually die (D).

The initial state of this population, F_0, is shown below.

$$F_{n+1} = \begin{bmatrix} 0.65 & 0 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0.20 & 0.95 & 0 \\ 0.10 & 0.05 & 0.05 & 1 \end{bmatrix} F_n + B$$

Every month, fish are either sold or bought so that the number of young, juvenile and adult fish in the farm remains constant.

The population of fish in the fish farm after n months, F_n, can be determined by the recurrence rule where

$$F_{n+1} = \begin{bmatrix} 0.65 & 0 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0.20 & 0.95 & 0 \\ 0.10 & 0.05 & 0.05 & 1 \end{bmatrix} F_n + B$$

where B is a column matrix that shows the number of young, juvenile and adult fish bought or sold each month and the number of dead fish that are removed.

Each month, the fish farm will

A. sell 1650 adult fish.
B. buy 1750 adult fish.
C. sell 17 500 young fish.
D. buy 50 000 young fish.
E. buy 10 000 juvenile fish.

Source: *VCE 2017, Further Mathematics 1, Section B, Module 1, Q.8; © VCAA*

Question 8 (1 mark)

Consider the matrix recurrence relation below.

$$S_0 = \begin{bmatrix} 40 \\ 15 \\ 20 \end{bmatrix}, S_{n+1} = TS_n \text{ where } T = \begin{bmatrix} 0.3 & 0.2 & V \\ 0.2 & 0.2 & W \\ X & Y & Z \end{bmatrix}$$

Matrix T is a regular transition matrix.

Given the above and that $S_1 = \begin{bmatrix} 29 \\ 13 \\ 33 \end{bmatrix}$, which one of the following expressions is **not** true?

A. $W > Z$
B. $Y > X$
C. $V > Y$
D. $V + W + Z = 1$
E. $X + Y + Z > 1$

Source: *VCE 2016, Further Mathematics 1, Section B, Module 1, Q.7; © VCAA*

Question 9 (1 mark)

Each week, the 300 students at a primary school choose art (A), music (M) or sport (S) as an afternoon activity.

The transition matrix below shows how the students' choices change from week to week.

$$\begin{array}{cc} & \textit{this week} \\ & \begin{array}{ccc} A & M & S \end{array} \\ T = & \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{bmatrix} \begin{array}{c} A \\ M \\ S \end{array} \; \textit{next week} \end{array}$$

Based on the information above, it can be concluded that, in the long term

A. no student will choose sport.
B. all students will choose to stay in the same activity each week.
C. all students will have chosen to change their activity at least once.
D. more students will choose to do music than sport.
E. the number of students choosing to do art and music will be the same.

Source: *VCE 2015, Further Mathematics 1, Section B, Module 6, Q.5; © VCAA*

Question 10 (1 mark)

Wendy buys one type of flower each day.
She chooses from tulips (T), roses (R), carnations (C), irises (I) and daisies (D).
The type of flower she buys on one day depends on the type of flower she bought the previous day, according to a transition matrix.
Today, Wendy bought tulips.
The transition matrix that, starting tomorrow, ensures Wendy buys flowers in alphabetical order (C, D, I, R, T) is

A.
$$\begin{array}{c} \text{today} \\ \begin{array}{ccccc} T & R & C & I & D \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ T \\ R \\ C \text{ tomorrow} \\ I \\ D \end{array}$$

B.
$$\begin{array}{c} \text{today} \\ \begin{array}{ccccc} T & R & C & I & D \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ T \\ R \\ C \text{ tomorrow} \\ I \\ D \end{array}$$

C.
$$\begin{array}{c} \text{today} \\ \begin{array}{ccccc} T & R & C & I & D \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ T \\ R \\ C \text{ tomorrow} \\ I \\ D \end{array}$$

D.
$$\begin{array}{c} \text{today} \\ \begin{array}{ccccc} T & R & C & I & D \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ T \\ R \\ C \text{ tomorrow} \\ I \\ D \end{array}$$

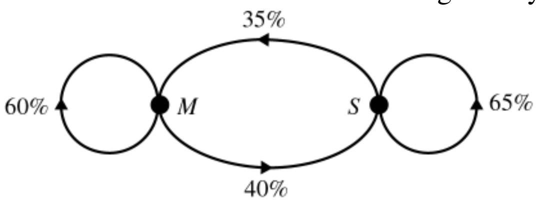
E.
$$\begin{array}{c} \text{today} \\ \begin{array}{ccccc} T & R & C & I & D \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ T \\ R \\ C \text{ tomorrow} \\ I \\ D \end{array}$$

Source: *VCE 2014, Further Mathematics 1, Section B, Module 6, Q.4; © VCAA*

Question 11 (1 mark)

Two hundred and fifty people buy bread each day from a corner store. They have a choice of two brands of bread: Megaslice (M) and Superloaf (S).

The customers' choice of brand changes daily according to the transition diagram below.



On a given day, 100 of these people bought Megaslice bread while the remaining 150 people bought Superloaf bread.

The number of people who are expected to buy each brand of bread the next day is found by evaluating the matrix product

A. $\begin{array}{cc} M & S \end{array}$
$$\begin{bmatrix} 0.60 & 0.40 \\ 0.35 & 0.65 \end{bmatrix} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{array}{c} M \\ S \end{array}$$

B. $\begin{array}{cc} M & S \end{array}$
$$\begin{bmatrix} 0.60 & 0.40 \\ 0.65 & 0.35 \end{bmatrix} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{array}{c} M \\ S \end{array}$$

C. $\begin{array}{cc} M & S \end{array}$
$$\begin{bmatrix} 0.60 & 0.35 \\ 0.40 & 0.65 \end{bmatrix} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{array}{c} M \\ S \end{array}$$

D. $\begin{array}{cc} M & S \end{array}$
$$\begin{bmatrix} 0.65 & 0.40 \\ 0.35 & 0.60 \end{bmatrix} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{array}{c} M \\ S \end{array}$$

E. $\begin{array}{cc} M & S \end{array}$
$$\begin{bmatrix} 0.60 & 0.65 \\ 0.40 & 0.35 \end{bmatrix} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{array}{c} M \\ S \end{array}$$

Source: *VCE 2014, Further Mathematics Exam 1, Section B, Module 6, Q.8; © VCAA*

Question 12 (1 mark)

Wendy will have lunch with one of her friends each day of this week.
Her friends are Angela (A), Betty (B), Craig (C), Daniel (D) and Edgar (E).
On Monday, Wendy will have lunch with Craig.
Wendy will use the transition matrix below to choose a friend to have lunch with for the next four days of the week.

$$T = \begin{array}{c} \text{today} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ A \\ B \\ C \\ D \\ E \end{array} \text{ tomorrow}$$

The order in which Wendy has lunch with her friends for the next four days is

A. Angela, Betty, Craig, Daniel
B. Daniel, Betty, Angela, Craig
C. Daniel, Betty, Angela, Edgar
D. Edgar, Angela, Daniel, Betty
E. Edgar, Daniel, Betty, Angela

Source: *VCE 2014, Further Mathematics Exam 1, Section B, Module 6, Q.7; © VCAA*

Question 13 (1 mark)

A transition matrix, T, and a state matrix, S_2, are defined as follows.

$$T = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \quad S_2 = \begin{bmatrix} 300 \\ 200 \\ 100 \end{bmatrix}$$

If $S_2 = TS_1$, the state matrix S_1 is

A. $\begin{bmatrix} 200 \\ 250 \\ 150 \end{bmatrix}$

B. $\begin{bmatrix} 300 \\ 200 \\ 100 \end{bmatrix}$

C. $\begin{bmatrix} 300 \\ 0 \\ 300 \end{bmatrix}$

D. $\begin{bmatrix} 400 \\ 0 \\ 200 \end{bmatrix}$

E. undefined

Source: *VCE 2014, Further Mathematics 1, Section B, Module 6, Q.3; © VCAA*

Question 14 (1 mark)

Regular customers at a hairdressing salon can choose to have their hair cut by Shirley, Jen or Narj.

The salon has 600 regular customers who get their hair cut each month.

In June, 200 customers chose Shirley (S) to cut their hair, 200 chose Jen (J) to cut their hair and 200 chose Narj (N) to cut their hair.

The regular customers' choice of hairdresser is expected to change from month to month as shown in the transition matrix, T, below.

$$\begin{array}{cc} & \text{this month} \\ & \begin{array}{ccc} S & J & N \end{array} \\ T = & \begin{bmatrix} 0.75 & 0.10 & 0.10 \\ 0.10 & 0.75 & 0.15 \\ 0.15 & 0.15 & 0.75 \end{bmatrix} \begin{array}{l} S \\ J \\ N \end{array} \text{ next month} \end{array}$$

In the long term, the number of regular customers who are expected to choose Shirley is closest to

A. 150
B. 170
C. 185
D. 195
E. 200

Source: *VCE 2013, Further Mathematics 1, Section B, Module 6, Q.8; © VCAA*

Question 15 (1 mark)

The matrix $Sn+1$ is determined from the matrix Sn using the rule $S_{n+1} = TS_n - C$ where T, S_0 and C are defined as follows.

$$T = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}, \ S_0 = \begin{bmatrix} 100 \\ 250 \end{bmatrix} \text{ and } C = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

Given this information, the matrix S_2 equals

A. $\begin{bmatrix} 100 \\ 250 \end{bmatrix}$

B. $\begin{bmatrix} 148 \\ 122 \end{bmatrix}$

C. $\begin{bmatrix} 170 \\ 140 \end{bmatrix}$

D. $\begin{bmatrix} 180 \\ 130 \end{bmatrix}$

E. $\begin{bmatrix} 190 \\ 160 \end{bmatrix}$

Source: *VCE 2013, Further Mathematics 1, Section B, Module 6, Q.3; © VCAA*

Question 16 (1 mark)

A coffee shop sells three types of coffee, Brazilian (B), Italian (I) and Kenyan (K). The regular customers buy one cup of coffee each per day and choose the type of coffee they buy according to the following transition matrix, T.

$$\begin{array}{cc} & \text{choose today} \\ & \begin{array}{ccc} B & I & K \end{array} \\ T = & \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.8 \end{bmatrix} \begin{array}{c} B \\ I \\ K \end{array} \quad \text{choose tomorrow} \end{array}$$

On a particular day, 84 customers bought Brazilian coffee, 96 bought Italian coffee and 81 bought Kenyan coffee.

If these same customers continue to buy one cup of coffee each per day, the number of these customers who are expected to buy each of the three types of coffee in the long term is

A. Brazilian 85
Italian 85
Kenyan 91

B. Brazilian 87
Italian 58
Kenyan 116

C. Brazilian 88
Italian 86
Kenyan 87

D. Brazilian 89
Italian 89
Kenyan 83

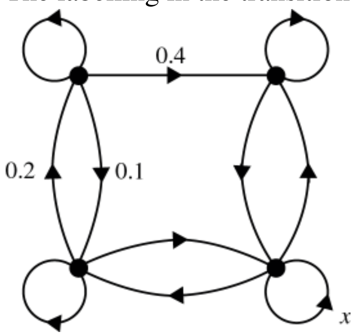
E. Brazilian 116
Italian 89
Kenyan 58

Question 17 (1 mark)

A transition matrix, V, is shown below.

The transition diagram below has been constructed from the transition matrix V.

The labelling in the transition diagram is not yet complete.



The proportion for one of the transitions is labelled x.

The value of x is

A. 0.2

B. 0.5

C. 0.6

D. 0.7

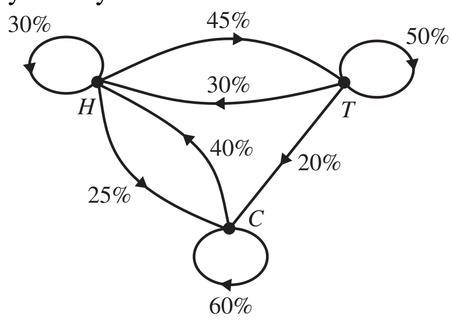
E. 0.8

Question 18 (1 mark)

Families in a country town were asked about their annual holidays.

Every year, these families choose between staying at home (H), travelling (T) and camping (C).

The transition diagram below shows the way families in the town change their holiday preferences from year to year.



A transition matrix that provides the same information as the transition diagram is

A.
$$\begin{array}{c c} & \textit{from} \\ & \begin{array}{ccc} H & T & C \end{array} \\ & \begin{bmatrix} 0.30 & 0.75 & 0.65 \\ 0.75 & 0.50 & 0.20 \\ 0.65 & 0.20 & 0.60 \end{bmatrix} \begin{array}{l} H \\ T\ \textit{to} \\ C \end{array} \end{array}$$

B.
$$\begin{array}{c c} & \textit{from} \\ & \begin{array}{ccc} H & T & C \end{array} \\ & \begin{bmatrix} 0.30 & 0.30 & 0.40 \\ 0.45 & 0.50 & 0 \\ 0.25 & 0.20 & 0.60 \end{bmatrix} \begin{array}{l} H \\ T\ \textit{to} \\ C \end{array} \end{array}$$

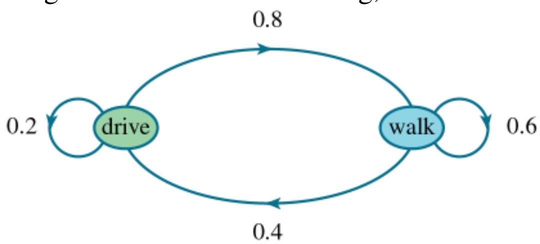
C.
$$\begin{array}{c c} & \textit{from} \\ & \begin{array}{ccc} H & T & C \end{array} \\ & \begin{bmatrix} 0.30 & 0.30 & 0.40 \\ 0.45 & 0.50 & 0.20 \\ 0.25 & 0.20 & 0.60 \end{bmatrix} \begin{array}{l} H \\ T\ \textit{to} \\ C \end{array} \end{array}$$

D.
$$\begin{array}{c c} & \textit{from} \\ & \begin{array}{ccc} H & T & C \end{array} \\ & \begin{bmatrix} 0.30 & 0.30 & 0.40 \\ 0.45 & 0.50 & 0.20 \\ 0.25 & 0.20 & 0.40 \end{bmatrix} \begin{array}{l} H \\ T\ \textit{to} \\ C \end{array} \end{array}$$

E.
$$\begin{array}{c c} & \textit{from} \\ & \begin{array}{ccc} H & T & C \end{array} \\ & \begin{bmatrix} 0.30 & 0.45 & 0.25 \\ 0.30 & 0.50 & 0.20 \\ 0.40 & 0 & 0.60 \end{bmatrix} \begin{array}{l} H \\ T\ \textit{to} \\ C \end{array} \end{array}$$

Question 19 (1 mark)

To get to school in the morning, Tessa and her sister either walk (w) or are driven by their father (d).



The diagram below shows the probability that the girls will walk or their father will drive them to school tomorrow based on how they got to school today.

Which of the following transition matrices represents the diagram above?

A.
$$T = \begin{array}{c} \begin{array}{cc} d & w \end{array} \\ \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ d \\ w \end{array}$$

B.
$$T = \begin{array}{c} \begin{array}{cc} d & w \end{array} \\ \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ d \\ w \end{array}$$

C.
$$T = \begin{array}{c} \begin{array}{cc} d & w \end{array} \\ \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \begin{array}{c} \\ d \\ w \end{array}$$

D.
$$T = \begin{array}{c} \begin{array}{cc} d & w \end{array} \\ \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ d \\ w \end{array}$$

E.
$$T = \begin{array}{c} \begin{array}{cc} d & w \end{array} \\ \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \end{array} \begin{array}{c} \\ d \\ w \end{array}$$

Question 20 (1 mark)

The transition matrix below is missing values 'Assume the population of the initial state remains the same throughout the process'.

$$T = \begin{bmatrix} 0.2 & 0.1 & \\ 0.5 & & 0.3 \\ & 0.6 & 0.4 \end{bmatrix}$$

Which of the following matrices is the completed transition matrix?

A. $T = \begin{bmatrix} 0.2 & 0.1 & 0.7 \\ 0.5 & 0.2 & 0.3 \\ 0 & 0.6 & 0.4 \end{bmatrix}$

B. $T = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.5 & 0.2 & 0.3 \\ 0.7 & 0.6 & 0.4 \end{bmatrix}$

C. $T = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.5 & 0.2 & 0.3 \\ 0.7 & 0.6 & 0.4 \end{bmatrix}$

D. $T = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.5 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.4 \end{bmatrix}$

E. $T = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.5 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.4 \end{bmatrix}$

Question 21 (1 mark)

There are 3 brands of instant coffee that dominate the market: Brand A, Brand B, and Brand C. Consumers change from one brand of coffee to another all the time.

If consumers use Brand A this week, next week the probability that they will change to Brand B is 0.1, the probability that they will change to brand C is 0.3 and the probability that they will continue to use Brand A is 0.6.

If consumers use Brand B this week, next week the probability that they will change to Brand A is 0.5, the probability that they will change to Brand C is 0.1 and the probability that they will stay with Brand B is 0.4.

If consumers use Brand C this week, next week the probability that they will change to Brand A is 0.2, the probability that they will change to Brand B is 0.3, and the probability that they will stay with Brand C is 0.5.

Which of the following transition matrices represents this situation?

A.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.5 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

B.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.5 & 0.1 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

C.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.6 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.1 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

D.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.2 \\ 0.3 & 0.1 & 0.3 \\ 0.6 & 0.4 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

E.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.5 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

Question 22 (1 mark)

When buying fruit in winter, people alternate between oranges (O) and apples (A). 75% of those who buy apples will buy oranges next time and 55% of the people who buy oranges will buy apples next time. On the first day of winter, 85 apples and 120 oranges are sold.

The initial state matrix for this situation is

A. $\begin{array}{cc} & \begin{array}{cc} A & O \end{array} \\ \begin{array}{c} A \\ O \end{array} & \begin{bmatrix} 0.75 & 0.55 \\ 0.25 & 0.45 \end{bmatrix} \end{array}$

B. $\begin{array}{cc} & \begin{array}{cc} A & O \end{array} \\ \begin{array}{c} A \\ O \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ 0.55 & 0.45 \end{bmatrix} \end{array}$

C. $\begin{array}{c} \begin{array}{cc} A & O \end{array} \\ \begin{bmatrix} 85 & 120 \end{bmatrix} \end{array}$

D. $\begin{array}{c} A \\ O \end{array} \begin{bmatrix} 85 \\ 120 \end{bmatrix}$

E. $\begin{array}{cc} & \begin{array}{cc} A & O \end{array} \\ \begin{array}{c} A \\ O \end{array} & \begin{bmatrix} 75 & 25 \\ 55 & 45 \end{bmatrix} \end{array}$

Question 23 (1 mark)

When buying fruit in winter people alternate between oranges (O) and apples (A). 75% of those who buy apples will buy oranges next time and 55% of the people who buy oranges will buy apples next time.
On the first day of winter, 85 apples and 120 oranges are sold.
The transition matrix for this situation is

A.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.5 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

B.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.5 & 0.1 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

C.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.6 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.1 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

D.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.2 \\ 0.3 & 0.1 & 0.3 \\ 0.6 & 0.4 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

E.
$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.5 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$$

Question 24 (1 mark)

If an initial state matrix is $S_0 = \begin{bmatrix} 60 \\ 75 \end{bmatrix}$ and the transition matrix is $T = \begin{bmatrix} 0.65 & 0.45 \\ 0.35 & 0.55 \end{bmatrix} = \begin{bmatrix} 60 \\ 75 \end{bmatrix}$, then s_1 is

A. $\begin{bmatrix} 72.75 \\ 62.25 \end{bmatrix}$

B. $\begin{bmatrix} 73 \\ 62 \end{bmatrix}$

C. $\begin{bmatrix} 60 \\ 75 \end{bmatrix}$

D. $\begin{bmatrix} 75 \\ 60 \end{bmatrix}$

E. not defined

Question 25 (1 mark)

If $s_4 = \begin{bmatrix} 15 \\ 27 \end{bmatrix}$ and the transition matrix is $T = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix}$, then s_5 is

A. $\begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix}$

B. $\begin{bmatrix} 15 \\ 27 \end{bmatrix}$

C. $\begin{bmatrix} 19 \\ 23 \end{bmatrix}$

D. $\begin{bmatrix} 19.2 \\ 22.8 \end{bmatrix}$

E. $\begin{bmatrix} 19.3 \\ 22.9 \end{bmatrix}$

Question 26 (1 mark)

To calculate s_5 using the matrix for s_3 and T

A. multiply s_3 by T.
B. multiply s_3 by T^2.
C. multiply s_3 by T^3.
D. multiply s_3 by T^4.
E. multiply s_3 by T^5.

Question 27 (1 mark)

A rehabilitation training program includes the activities of gymnasium work and running, where 30 min is the running time. The time spent on these two activities varies from week to week according to the transition matrix $\begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$. Initially the weekly time (in minutes) intended to be dedicated to each activity is $[S_0] = \begin{bmatrix} 120 \\ 30 \end{bmatrix}$. At the end of each week more minutes are added according to the matrix $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$. A model for this rehabilitation program is $[S_{n+1}] = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix} [S_n] + \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

The number of minutes, correct to the nearest whole number, dedicated to running after 4 weeks of the program is

Topic	8	Matrices
Subtopic	8.8	Review

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Source: *VCE 2021, Further Mathematics 2, Section B, Module 1, Q.2; © VCAA*

Question 1 (3 marks)

The main computer system in Elena's office has broken down.

The five staff members, Alex (A), Brie (B), Chai (C), Dex (D) and Elena (E), are having problems sending information to each other.

Matrix M below shows the available communication links between the staff members.

$$M = \textit{sender}\ \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \textit{receiver} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

In this matrix:

- the '1' in row A, column B indicates that Alex can send information to Brie
- the '0' in row D, column C indicates that Dex cannot send information to Chai.

a. Which two staff members can send information directly to each other? **(1 mark)**

b. Elena needs to send documents to Chai.
What is the sequence of communication links that will successfully get the information from Elena to Chai? **(1 mark)**

c. Matrix M^2 below is the square of matrix M and shows the number of two-step communication links between each pair of staff members.

$$M^2 = \textit{sender}\ \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \textit{receiver} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Only one pair of individuals has two different two-step communication links.
List each two-step communication link for this pair. **(1 mark)**

Source: *VCE 2020, Further Mathematics 2, Section B, Module 1, Q.1; © VCAA*

Question 2 (6 marks)

The three major shopping centres in a large city, Eastmall (E), Grandmall (G) and Westmall (W), are owned by the same company.

The total number of shoppers at each of the centres at 1.00 pm on a typical day is shown in matrix V.

$$\begin{array}{cc} & \begin{array}{ccc} E & G & W \end{array} \\ V = & \begin{bmatrix} 2300 & 2700 & 2200 \end{bmatrix} \end{array}$$

a. Write down the order of matrix V. **(1 mark)**

Each of these centres has three major shopping areas: food (F), clothing (C) and merchandise (M).

The proportion of shoppers in each of these three areas at 1.00 pm on a typical day is the same at all three centres and is given in matrix P below.

$$P = \begin{bmatrix} 0.48 \\ 0.27 \\ 0.25 \end{bmatrix} \begin{array}{c} F \\ C \\ M \end{array}$$

b. Grandmall's management would like to see 700 shoppers in its merchandise area at 1.00 pm.
If this were to happen, how many shoppers, in total, would be at Grandmall at this time? **(1 mark)**

c. The matrix $Q = P \times V$ is shown below. Two of the elements of this matrix are missing.

$$\begin{array}{cc} & \begin{array}{ccc} E & G & W \end{array} \\ Q = & \begin{bmatrix} 1104 & \_\_\_ & 1056 \\ 621 & \_\_\_ & 594 \\ 575 & 675 & 550 \end{bmatrix} \begin{array}{c} F \\ C \\ M \end{array} \end{array}$$

i. Complete matrix Q above by filling in the missing elements. **(1 mark)**

ii. The element in row i and column j of matrix Q is q_{ij}.
What does the element q_{23} represent? **(1 mark)**

The average daily amount spent, in dollars, by each shopper in each of the three areas at Grandmall in 2019 is shown in matrix A_{2019} below.

$$A_{2019} = \begin{array}{c} \begin{bmatrix} 21.30 \\ 34.00 \\ 14.70 \end{bmatrix} \end{array} \begin{array}{c} F \\ C \\ M \end{array}$$

On one particular day, 135 shoppers spent the average daily amount on food, 143 shoppers spent the average daily amount on clothing and 131 shoppers spent the average daily amount on merchandise.

d. Write a matrix calculation, using matrix A_{2019}, showing that the total amount spent by all these shoppers is \$9663.20 **(1 mark)**

e. In 2020, the average daily amount spent by each shopper was expected to change by the percentage shown in the table below.

Area	food	clothing	merchandise
Expected change	increase by 5%	decrease by 15%	decrease by 1%

The average daily amount, in dollars, expected to be spent in each area in 2020 can be determined by forming the matrix product $A_{2020} = K \times A_{2019}$

Write down matrix K.

$K =$ **(1 mark)**

Source: *VCE 2020, Further Mathematics 1, Section B, Module 1, Q.8; © VCAA*

Question 3 (1 mark)

The table below shows information about three matrices: A, B and C.

Matrix	**Order**
A	2 × 4
B	2 × 3
C	3 × 4

The transpose of matrix A, for example, is written as A^T.

What is the order of the product $C^T \times (A^T \times B)^T$?

A. 2 × 3
B. 3 × 4
C. 4 × 2
D. 4 × 3
E. 4 × 4

Source: *VCE 2020, Further Mathematics 2, Section B, Module 1, Q.2; © VCAA*

Question 4 (3 marks)

The preferred number of cafes (x) and sandwich bars (y) in Grandmall's food court can be determined by solving the following equations written in matrix form.

$$\begin{bmatrix} 5 & -9 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

a. The value of the determinant of the 2×2 matrix is 1.
Use this information to explain why this matrix has an inverse. **(1 mark)**

b. Write the three missing values of the inverse matrix that can be used to solve these equations. **(1 mark)**

$$\begin{bmatrix} \_\_ & 9 \\ \_\_ & \_\_ \end{bmatrix}$$

c. Determine the preferred number of sandwich bars for Grandmall's food court. **(1 mark)**

Source: *VCE 2019, Further Mathematics 2, Section B, Module 1, Q.1; © VCAA*

Question 5 (5 marks)

The car park at a theme park has three areas, A, B and C.

The number of empty (E) and full (F) parking spaces in each of the three areas at 1 pm on Friday are shown in matrix Q below.

$$\begin{array}{c} \\ Q = \end{array} \begin{array}{c} \begin{array}{cc} E & F \end{array} \\ \begin{bmatrix} 70 & 50 \\ 30 & 20 \\ 40 & 40 \end{bmatrix} \end{array} \begin{array}{l} \\ A \\ B \;\; area \\ C \end{array}$$

a. What is the order of matrix Q? **(1 mark)**

b. Write down a calculation to show that 110 parking spaces are full at 1 pm. **(1 mark)**

c. Drivers must pay a parking fee for each hour of parking.

Matrix P, below, shows the hourly fee, in dollars, for a car parked in each of the three areas.

$$\begin{array}{cc} & \textit{area} \\ & \begin{array}{ccc} A & B & C \end{array} \\ P = & \begin{bmatrix} 1.30 & 3.50 & 1.80 \end{bmatrix} \end{array}$$

The total parking fee, in dollars, collected from these 110 parked cars if they were parked for one hour is calculated as follows.

$P \times L = [207.00]$

where matrix L is a 3×1 matrix.

Write down matrix L.

$L =$ **(1 mark)**

d. The number of whole hours that each of the 110 cars had been parked was recorded at 1 pm. Matrix R, below, shows the number of cars parked for one, two, three or four hours in each of the areas A, B and C.

$$\begin{array}{ccc} & \textit{area} & \\ & \begin{array}{ccc} A & B & C \end{array} & \\ R = & \begin{bmatrix} 3 & 1 & 1 \\ 6 & 10 & 3 \\ 22 & 7 & 10 \\ 19 & 2 & 26 \end{bmatrix} & \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \textit{hours} \end{array}$$

Matrix R^T is the transpose of matrix R.

Complete the matrix R^T below. **(1 mark)**

$R^T = \begin{bmatrix} \quad \end{bmatrix}$

e. The number of whole hours that each of the 110 cars had been parked was recorded at 1 pm. Matrix R, below, shows the number of cars parked for one, two, three or four hours in each of the areas A, B and C.

$$\begin{array}{ccc} & \textit{area} & \\ & \begin{array}{ccc} A & B & C \end{array} & \\ R = & \begin{bmatrix} 3 & 1 & 1 \\ 6 & 10 & 3 \\ 22 & 7 & 10 \\ 19 & 2 & 26 \end{bmatrix} & \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \textit{hours} \end{array}$$

Explain what the element in row 3, column 2 of matrix R^T represents. **(1 mark)**

Source: *VCE 2021, Further Mathematics 2, Section B, Module 1, Q.4; © VCAA*

Question 6 (2 marks)

Five staff members in Elena's office played a round-robin video game tournament, where each employee played each of the other employees once. In each game there was a winner and a loser.

A table of their one-step and two-step dominances was prepared to summarise the results.

Staff member	One-step dominance	Two-step dominance
Ike (*I*)	3	5
Joelene (*J*)	3	4
Katie (*K*)	1	1
Leslie (*L*)	1	2
Mikki (*M*)	2	4

Consider the results matrix shown below.

A '1' in this matrix shows that the player named in that row defeated the player named in that column.

A '0' in this matrix shows that the player named in that row lost to the player named in that column.

Use all of the information provided to complete the results matrix.

$$\begin{array}{ccc} & & \textit{loser} \\ & & \begin{array}{ccccc} I & J & K & L & M \end{array} \\ & \begin{array}{c} I \\ J \\ K \\ L \\ M \end{array} & \begin{bmatrix} 0 & \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & 0 & \_\_ & \_\_ & \_\_ \\ 0 & 0 & 0 & 1 & 0 \\ \_\_ & \_\_ & \_\_ & 0 & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ & 0 \end{bmatrix} \end{array}$$

winner

Source: *VCE 2021, Further Mathematics 1, Section B, Module 1, Q.5; © VCAA*

Question 7 (1 mark)

A is a 7 × 7 matrix.

B is a 10 × 7 matrix.

Which one of the following matrix expressions is defined?

A. $AB - 2B$

B. $A(BA)^{-1}$

C. AB^2

D. AB^2

E. $A(B^T)$

Source: *VCE 2021, Further Mathematics 1, Section B, Module 1, Q.6; © VCAA*

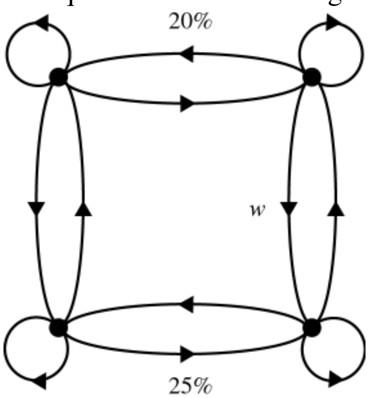
Question 8 (1 mark)

A fitness centre offers four different exercise classes: aerobics (A), boxfit (B), cardio (C) and dance (D).

A customer's choice of fitness class is expected to change from week to week according to the transition matrix P, shown below.

$$\begin{array}{cc} & \textit{this week} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ P = & \begin{bmatrix} 0.65 & 0 & 0.20 & 0.10 \\ 0 & 0.65 & 0.10 & 0.30 \\ 0.20 & 0.10 & 0.70 & 0 \\ 0.15 & 0.25 & 0 & 0.60 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array} \quad \textit{next} \quad \textit{week} \end{array}$$

An equivalent transition diagram has been constructed below, but the labelling is not complete.



The proportion for one of the transitions is labelled w.

The value of w is

A. 10%

B. 15%

C. 20%

D. 25%

E. 30%

Source: *VCE 2021, Further Mathematics 1, Section B, Module 1, Q.7; © VCAA*

Question 9 (1 mark)

The matrix S_{n+1} is determined from the matrix S_n using the recurrence relation $S_{n+1} = T \times S_n - C$, where

$$T = \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.5 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 21 \\ 51 \\ 31 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 24.0 \\ 54.3 \\ 20.7 \end{bmatrix}$$

and C is a column matrix.

Matrix S_2 is equal to

A. $\begin{bmatrix} 23.04 \\ 55.78 \\ 16.18 \end{bmatrix}$

B. $\begin{bmatrix} 25.34 \\ 56.28 \\ 17.38 \end{bmatrix}$

C. $\begin{bmatrix} 26.04 \\ 54.78 \\ 18.18 \end{bmatrix}$

D. $\begin{bmatrix} 28.34 \\ 55.28 \\ 19.38 \end{bmatrix}$

E. $\begin{bmatrix} 29.04 \\ 53.78 \\ 20.18 \end{bmatrix}$

Source: *VCE 2021, Further Mathematics 1, Section B, Module 1, Q.8; © VCAA*

Question 10 (1 mark)

A new colony of endangered marsupials is established on a remote island.

For one week, the marsupials can feed from only one of three feeding stations: A, B or C.

On Monday, 50% of the marsupials were observed feeding at station A and 50% were observed feeding at station B. No marsupials were observed feeding at station C.

The marsupials are expected to change their feeding stations each day this week according to the transition matrix T.

$$\begin{array}{c} \text{this day} \\ \begin{array}{ccc} A & B & C \end{array} \end{array}$$

$$T = \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.2 & 0.5 & 0.2 \\ 0.4 & 0.4 & 0.6 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \quad \text{next day}$$

Let S_n represent the state matrix showing the percentage of marsupials observed feeding at each feeding station n days after Monday of this week.

The matrix recurrence rule $S_{n+1} = TS_n$ is used to model this situation.

From Tuesday to Wednesday, the percentage of marsupials who are **not** expected to change their feeding location is

A. 44.5%

B. 45%

C. 50%

D. 51.5%

E. 52%

Source: *VCE 2021, Further Mathematics 2, Section B, Module 1, Q.1; © VCAA*

Question 11 (2 marks)

Elena imports three brands of olive oil: Carmani (C), Linelli (L) and Ohana (O).

The number of 1 litre bottles of these oils sold in January 2021 is shown in matrix J below.

$$J = \begin{bmatrix} 2800 \\ 1700 \\ 2400 \end{bmatrix} \begin{array}{c} C \\ L \\ O \end{array}$$

a. What is the order of matrix J? **(1 mark)**

b. Elena expected that in February 2021 the sales of all three brands of olive oil would increase by 5%. She multiplied matrix J by a scalar value, k, to determine the expected volume of sales for February.

Write down the value of the scalar k.

$k =$ **(1 mark)**

Source: *VCE 2021, Further Mathematics 2, Section B, Module 1, Q.3; © VCAA*

Question 12 (5 marks)

A market research study of shoppers showed that the buying preferences for the three olive oils, Carmani (C), Linelli (L) and Ohana (O), change from month to month according to the transition matrix T below.

$$T = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \textit{this month} \\ \begin{array}{ccc} C & L & O \end{array} \\ \begin{bmatrix} 0.85 & 0.10 & 0.05 \\ 0.05 & 0.80 & 0.05 \\ 0.10 & 0.10 & 0.90 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ C \\ L \quad \textit{next month} \\ O \end{array}$$

The initial state matrix S_0 below shows the number of shoppers who bought each brand of olive oil in July 2021.

$$S_0 = \begin{bmatrix} 3200 \\ 2000 \\ 2800 \end{bmatrix} \begin{array}{l} C \\ L \\ O \end{array}$$

Let S_n represent the state matrix describing the number of shoppers buying each brand n months after July 2021.

a. How many of these 8000 shoppers bought a different brand of olive oil in August 2021 from the brand bought in July 2021? **(1 mark)**

b. Using the rule $S_{n+1} = T \times S_n$, complete the matrix S_1 below. **(1 mark)**

$$S_1 = \begin{bmatrix} 3060 \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix} \begin{array}{l} C \\ L \\ O \end{array}$$

c. Consider the shoppers who were expected to buy Carmani olive oil in August 2021.
What percentage of these shoppers also bought Carmani olive oil in July 2021? Round your answer to the nearest percentage. **(1 mark)**

d. Write a calculation that shows that Ohana olive oil is the brand bought by 50% of these shoppers in the long run. **(1 mark)**

e. Further research suggests more shoppers will buy olive oil in the coming months.

A rule to model this situation is $R_{n+1} = T \times R_n + B$, where R_n represents the state matrix describing the number of shoppers n months after July 2021.

$$T = \begin{array}{c} \textit{this month} \\ \begin{array}{ccc} C & L & O \end{array} \\ \begin{bmatrix} 0.85 & 0.10 & 0.05 \\ 0.05 & 0.80 & 0.05 \\ 0.10 & 0.10 & 0.90 \end{bmatrix} \end{array} \begin{array}{l} C \\ L \\ O \end{array} \textit{next month}, \quad B = \begin{bmatrix} 200 \\ 100 \\ k \end{bmatrix} \begin{array}{l} C \\ L \\ O \end{array}, \quad R_0 = \begin{bmatrix} 3200 \\ 2000 \\ 2800 \end{bmatrix} \begin{array}{l} C \\ L \\ O \end{array}$$

k represents the extra number of shoppers expected to buy Ohana olive oil each month.

If $R_2 \begin{bmatrix} 3333 \\ 2025 \\ 3642 \end{bmatrix}$, what is the value of k? **(1 mark)**

Source: *VCE 2020, Further Mathematics 2, Section B, Module 1, Q.3; © VCAA*

Question 13 (4 marks)

An offer to buy the Westmall shopping centre was made by a competitor.

One market research project suggested that if the Westmall shopping centre were sold, each of the three centres (Westmall, Grandmall and Eastmall) would continue to have regular shoppers but would attract and lose shoppers on a weekly basis.

Let S_n be the state matrix that shows the expected number of shoppers at each of the three centres n weeks after Westmall is sold.

A matrix recurrence relation that generates values of S_n is

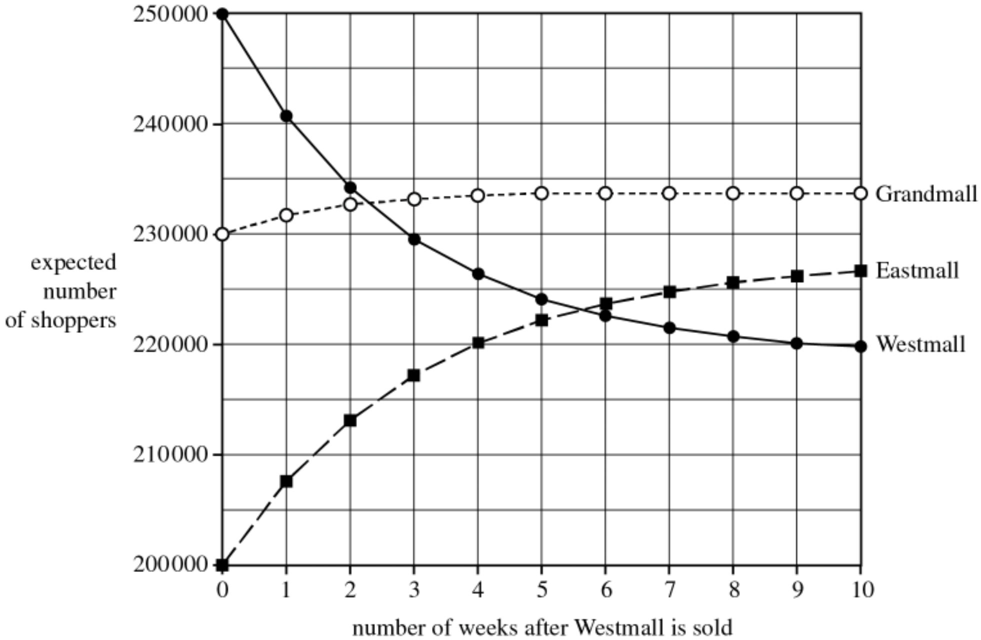
$S_{n+1} = T \times S_n$

where
$$T = \begin{array}{c} \textit{this week} \\ \begin{array}{ccc} W & G & E \end{array} \\ \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.79 & 0.10 \\ 0.08 & 0.12 & 0.80 \end{bmatrix} \end{array} \begin{array}{l} W \\ G \\ E \end{array} \textit{next week}, \quad S_0 = \begin{bmatrix} 250\,000 \\ 230\,000 \\ 200\,000 \end{bmatrix} \begin{array}{l} W \\ G \\ E \end{array}$$

a. Calculate the state matrix, S_1, to show the expected number of shoppers at each of the three centres one week after Westmall is sold. **(1 mark)**

$$S_1 = \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix} \begin{array}{l} W \\ G \\ E \end{array}$$

Using values from the recurrence relation above, the graph below shows the expected number of shoppers at Westmall, Grandmall and Eastmall for each of the 10 weeks after Westmall is sold.



b. What is the difference in the expected weekly number of shoppers at Westmall from the time Westmall is sold to 10 weeks after Westmall is sold?
Give your answer correct to the nearest thousand. **(1 mark)**

c. Grandmall is expected to achieve its maximum number of shoppers sometime between the fourth and the tenth week after Westmall is sold.
Write down the week number in which this is expected to occur. **(1 mark)**

d. In the long term, what is the expected weekly number of shoppers at Westmall?
Round your answer to the nearest whole number. **(1 mark)**

Source: *VCE 2020, Further Mathematics 2, Section B, Module 1, Q.4; © VCAA*

Question 14 (2 marks)

A second market research project also suggested that if the Westmall shopping centre were sold, each of the three centres (Westmall, Grandmall and Eastmall) would continue to have regular shoppers but would attract and lose shoppers on a weekly basis.

Let R_n be the state matrix that shows the expected number of shoppers at each of the three centres n weeks after Westmall is sold.

A matrix recurrence relation that generates values of R_n is

$R_{n+1} = TR_n + B$

$$\text{where } T = \begin{array}{c} \textit{this week} \\ \begin{array}{ccc} W & G & E \end{array} \\ \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.79 & 0.10 \\ 0.08 & 0.12 & 0.80 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ W \\ G \\ E \end{array} \;\textit{next week}, \quad B = \begin{bmatrix} -400 \\ 700 \\ 500 \end{bmatrix} \begin{array}{l} W \\ G \\ E \end{array}$$

The matrix R_2 is the state matrix that shows the expected number of shoppers at each of the three centres in the second week after Westmall is sold.

a. Determine the expected number of shoppers at Westmall in the third week after it is sold. **(1 mark)**

b. Determine the expected number of shoppers at Westmall in the first week after it is sold. **(1 mark)**

Source: *VCE 2019, Further Mathematics 2, Section B, Module 1, Q.3; © VCAA*

Question 15 (3 marks)

On Sunday, matrix V is used when calculating the expected number of visitors at each location every hour after 10 am. It is assumed that the park will be at its capacity of 2000 visitors for all of Sunday.

Let L_0 be the state matrix that shows the number of visitors at each location at 10 am on Sunday.

The number of visitors expected at each location at 11 am on Sunday can be determined by the matrix product

$$V \times L_0 \qquad \text{where} \qquad L_0 = \begin{bmatrix} 500 \\ 600 \\ 500 \\ 400 \end{bmatrix} \begin{matrix} A \\ F \\ G \\ W \end{matrix} \text{ and } \quad V = \begin{array}{c} \textit{this hour} \\ \begin{matrix} A & \;\; F & \;\; G & \;\; W \end{matrix} \\ \begin{bmatrix} 0.3 & 0.4 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.4 \end{bmatrix} \end{array} \begin{matrix} A \\ F \\ G \\ W \end{matrix} \; \textit{next hour}$$

a. Safety restrictions require that all four locations have a maximum of 600 visitors. **(1 mark)**
Which location is expected to have more than 600 visitors at 11 am on Sunday?

b. Whenever more than 600 visitors are expected to be at a location on Sunday, the first 600 visitors can stay at that location and all others will be moved directly to Ground World (G).
State matrix R_n contains the number of visitors at each location n hours after 10 am on Sunday, after the safety restrictions have been enforced.
Matrix R_1 can be determined from the matrix recurrence relation

$$R_0 = \begin{bmatrix} 500 \\ 600 \\ 500 \\ 400 \end{bmatrix} \begin{matrix} A \\ F \\ G \\ W \end{matrix}, \qquad R_1 = V \times R_0 + B_1$$

where matrix B_1 shows the required movement of visitors at 11 am.

i. Determine the matrix B_1. **(1 mark)**

$B_1 =$

ii. State matrix R_2 can be determined from the new matrix rule **(1 mark)**

$R_2 = VR_1 + B_2$

where matrix B_2 shows the required movement of visitors at 12 noon.

Determine the state matrix R_2.

$R_2 =$

Source: *VCE 2019, Further Mathematics 2, Section B, Module 1, Q.2 © VCAA*

Question 16 (4 marks)

The theme park has four locations, Air World (A), Food World (F), Ground World (G) and Water World (G).

The number of visitors at each of the four locations is counted every hour.

By 10 am on Saturday the park had reached its capacity of 2000 visitors and could take no more visitors.

The park stayed at capacity until the end of the day.

The state matrix, S_0, below, shows the number of visitors at each location at 10 am on Saturday.

$$S_0 = \begin{bmatrix} 600 \\ 600 \\ 400 \\ 400 \end{bmatrix} \begin{matrix} A \\ F \\ G \\ W \end{matrix}$$

a. What percentage of the park's visitors were at Water World (W) at 10 am on Saturday? **(1 mark)**

b. Let S_n be the state matrix that shows the number of visitors expected at each location n hours after 10 am on Saturday. **(1 mark)**

The number of visitors expected at each location n hours after 10 am on Saturday can be determined by the matrix recurrence relation below.

$$S_1 = \begin{bmatrix} 600 \\ 600 \\ 400 \\ 400 \end{bmatrix}, \quad S_{n+1} + T \times S_n \text{ where } \quad T = \begin{array}{c} \textit{this hour} \\ \begin{array}{cccc} A & F & G & W \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.4 \end{bmatrix} \end{array} \begin{matrix} A \\ F \\ G \\ W \end{matrix} \; \textit{next hour}$$

Complete the state matrix, S_1, below to show the number of visitors expected at each location at 11 am on Saturday.

$$S_1 = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ 300 \\ \_\_\_\_ \end{bmatrix} \begin{matrix} A \\ F \\ G \\ W \end{matrix}$$

c. Of the 300 visitors expected at Ground World (G) at 11 am, what percentage was at either Air World (A) or Food World (F) at 10 am? **(1 mark)**

d. The proportion of visitors moving from one location to another each hour on Sunday is different from Saturday. **(1 mark)**

Matrix V, below, shows the proportion of visitors moving from one location to another each hour after 10 am on Sunday.

$$\begin{array}{cc} & \textit{this hour} \\ & \begin{array}{cccc} A & F & G & W \end{array} \\ V = & \begin{bmatrix} 0.3 & 0.4 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.4 \end{bmatrix} \begin{array}{c} A \\ F \\ G \\ W \end{array} \textit{next hour} \end{array}$$

Matrix V is similar to matrix T but has the first two rows of matrix T interchanged. The matrix product that will generate matrix V from matrix T is

$V = M \times T$

where matrix M is a binary matrix.

Write down matrix M.

$$M = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Question 17 (3 marks)

Source: *VCE 2018, Further Mathematics 2, Section B, Module 1, Q.1; © VCAA*

A toll road is divided into three sections, E, F and G.

The cost, in dollars, to drive one journey on each section is shown in matrix C below.

$$C = \begin{bmatrix} 3.58 \\ 2.22 \\ 2.87 \end{bmatrix} \begin{array}{c} E \\ F \\ G \end{array}$$

a. What is the cost of one journey on section G? **(1 mark)**

b. Write down the order of matrix C. **(1 mark)**

c. One day Kim travels once on section E and twice on section G. **(1 mark)**

His total toll cost for this day can be found by the matrix product $M \times C$.

Write down the matrix M.

$M =$

Source: *VCE 2018, Further Mathematics 2, Section B, Module 1, Q.2; © VCAA*

Question 18 (2 mark)

The Westhorn Council must prepare roads for expected population changes in each of three locations: main town (M), villages (V) and rural areas (R).

The population of each of these locations in 2018 is shown in matrix P_{2018} below.

$$P_{2018} = \begin{array}{c} \begin{bmatrix} 2100 \\ 1800 \\ 1700 \end{bmatrix} \begin{matrix} M \\ V \\ R \end{matrix} \end{array}$$

The expected annual change in population in each location is shown in the table below.

Location	main town	villages	rural areas
Annual change	increase by 4%	decrease by 1%	decrease by 2%

a. Write down matrix P_{2019}, which shows the expected population in each location in 2019. **(1 mark)**

$P_{2019} =$

b. The expected population in each of the three locations in 2019 can be determined from the matrix product **(1 mark)**

$P_{2019} = F \times P_{2018}$

where F is a diagonal matrix.

Write down matrix F.

$F =$

Source: *VCE 2018, Further Mathematics 2, Section B, Module 1, Q.3; © VCAA*

Question 19 (5 marks)

The Hiroads company has a contract to maintain and improve 2700 km of highway.

Each year sections of highway must be graded (G), resurfaced (R) or sealed (S).

The remaining highway will need no maintenance (N) that year.

Let S_n be the state matrix that shows the highway maintenance schedule for the nth year after 2018.

The maintenance schedule for 2018 is shown in matrix S_0 below.

$$S_0 = \begin{bmatrix} 700 \\ 400 \\ 200 \\ 1400 \end{bmatrix} \begin{matrix} G \\ R \\ S \\ N \end{matrix}$$

The type of maintenance in sections of highway varies from year to year, as shown in the transition matrix, T, below.

$$\begin{array}{c} \textit{this year} \\ \begin{matrix} G & \quad R & \quad S & \quad N \end{matrix} \end{array}$$

$$T = \begin{bmatrix} 0.2 & 0.1 & 0.0 & 0.2 \\ 0.1 & 0.1 & 0.0 & 0.2 \\ 0.2 & 0.1 & 0.2 & 0.1 \\ 0.5 & 0.7 & 0.8 & 0.5 \end{bmatrix} \begin{matrix} G \\ R \\ S \\ N \end{matrix} \quad \textit{next year}$$

a. Of the length of highway that was graded (G) in 2018, how many kilometres are expected to be resurfaced (R) the following year? **(1 mark)**

---

b. Show that the length of highway that is to be graded (G) in 2019 is 460 km by writing the appropriate numbers in the boxes below. **(1 mark)**

$\square \times 700 + \square \times 400 + \square \times 200 + \square \times 1400 = 460$

---

c. The state matrix describing the highway maintenance schedule for the nth year after 2018 is given by **(1 mark)**

$S_{n+1} = TS_n$

Complete the state matrix, S_1, below for the highway maintenance schedule for 2019 (one year after 2018).

$$S_1 = \begin{bmatrix} 460 \\ \underline{\qquad} \\ \underline{\qquad} \\ 1490 \end{bmatrix} \begin{matrix} G \\ R \\ S \\ N \end{matrix}$$

---

d. In 2020, 1536 km of highway is expected to require no maintenance (N). **(1 mark)**
Of these kilometres, what percentage is expected to have had no maintenance (N) in 2019?
Round your answer to one decimal place.

e. In the long term, what percentage of highway each year is expected to have no maintenance (N)? **(1 mark)**
Round your answer to one decimal place.

Source: *VCE 2018, Further Mathematics 2, Section B, Module 1, Q.4; © VCAA*

Question 20 (2 marks)

Beginning in the year 2021, a new company will take over maintenance of the same 2700 km highway with a new contract.

Let M_n be the state matrix that shows the highway maintenance schedule of this company for the nth year after 2020.

The maintenance schedule for 2020 is shown in matrix M_0 below.

For these 2700 km of highway, the matrix recurrence relation shown below can be used to determine the number of kilometres of this highway that will require each type of maintenance from year to year.

$M_{n+1} = TM_n + B$

where

$$M_0 = \begin{bmatrix} 500 \\ 400 \\ 300 \\ 1500 \end{bmatrix} \begin{array}{l} G \\ R \\ S \\ N \end{array}, \quad T = \begin{array}{c} \textit{this year} \\ \begin{array}{cccc} G & R & S & N \end{array} \\ \begin{bmatrix} 0.2 & 0.1 & 0.0 & 0.2 \\ 0.1 & 0.1 & 0.0 & 0.2 \\ 0.2 & 0.1 & 0.2 & 0.1 \\ 0.5 & 0.7 & 0.8 & 0.5 \end{bmatrix} \end{array} \begin{array}{l} G \\ R \\ S \\ N \end{array} \textit{next year}, \quad B = \begin{bmatrix} k \\ 20 \\ 10 \\ -60 \end{bmatrix}$$

a. Write down the value of k in matrix B. **(1 mark)**

b. How many kilometres of highway are expected to be graded (G) in the year 2022? **(1 mark)**

Source: *VCE 2017, Further Mathematics 2, Section B, Module 1, Q.1; © VCAA*

Question 21 (4 marks)

A school canteen sells pies (P), rolls (R) and sandwiches (S).

The number of each item sold over three school weeks is shown in matrix M.

$$M = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} P & R & S \end{array} \\ \begin{bmatrix} 35 & 24 & 60 \\ 28 & 32 & 43 \\ 32 & 30 & 56 \end{bmatrix} \end{array}\begin{array}{l} \\ week\ 1 \\ week\ 2 \\ week\ 3 \end{array}$$

a. In total, how many sandwiches were sold in these three weeks? **(1 mark)**

b. The element in row i and column j of matrix M is m_{ij}. **(1 mark)**

What does the element m_{12} indicate?

c. Consider the matrix equation

$$\begin{bmatrix} 35 & 24 & 60 \\ 28 & 32 & 43 \\ 32 & 30 & 56 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 491.55 \\ 428.00 \\ 487.60 \end{bmatrix}$$

where $a = \text{cost of one pie}$, $b = \text{cost of one roll}$ and $c = \text{cost of one sandwich}$

i. What is the cost of one sandwich? **(1 mark)**

ii. The matrix equation below shows that the total value of all rolls and sandwiches sold in these three weeks is \$915.60 **(1 mark)**

$$L \times \begin{bmatrix} 491.55 \\ 428.00 \\ 487.60 \end{bmatrix} = [915.60]$$

Matrix L in this equation is of order 1×3.

Write down matrix L

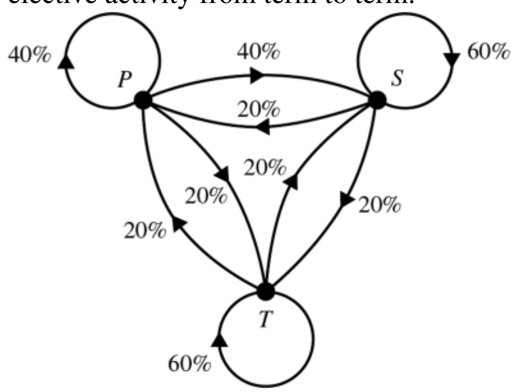
***Source:** VCE 2017, Further Mathematics 2, Section B, Module 1, Q.2; © VCAA*

Question 22 (3 marks)

Junior students at this school must choose one elective activity in each of the four terms in 2018.

Students can choose from the areas of performance (P), sport (S) and technology (T).

The transition diagram below shows the way in which junior students are expected to change their choice of elective activity from term to term.



a. Of the junior students who choose performance(P) in one term, what percentage are expected to choose sport (S) the next term? **(1 mark)**

b. Matrix J_1 lists the number of junior students who will be in each elective activity in Term 1. **(1 mark)**

$$J_1 = \begin{array}{c} \begin{bmatrix} 300 \\ 240 \\ 210 \end{bmatrix} \begin{array}{l} P \\ S \\ T \end{array} \end{array}$$

306 junior students are expected to choose sport (S) in Term 2.

Complete the calculation below to show this.

$300 \times \square + 240 \times \square + 210 \times \square = 306$

c. in Term 4, how many junior students in total are expected to participate in performance (P) or sport (S) or technology (T)? **(1 mark)**

Source: *VCE 2017, Further Mathematics 2, Section B, Module 1, Q.3; © VCAA*

Question 23 (5 marks)

Senior students at this school will also choose one elective activity in each of the four terms in 2018.
Their choices are communication (C), investigation (I), problem-solving (P) and service (S).
The transition matrix T shows the way in which senior students are expected to change their choice of elective activity from term to term.

$$T = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \textit{this term} \\ \begin{array}{cccc} C & I & P & S \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.3 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ C \\ I \\ P \\ S \end{array} \quad \textit{next term}$$

Let S_n be the state matrix for the number of senior students expected to choose each elective activity in Term n.

For the given matrix S_1, a matrix rule that can be used to predict the number of senior students in each elective activity in Terms 2, 3 and 4 is

$$S_1 = \begin{bmatrix} 300 \\ 200 \\ 200 \\ 300 \end{bmatrix}, \quad S_n + 1 = TS_n$$

a. How many senior students will **not** change their elective activity from Term 1 to Term 2? **(1 mark)**

b. Complete S_2, the state matrix for Term 2, below. **(1 mark)**

$$S_2 = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} \begin{array}{c} C \\ I \\ P \\ S \end{array}$$

c. Of the senior students expected to choose investigation (I) in Term 3, what percentage chose service (S) in Term 2? **(2 marks)**

d. What is the maximum number of senior students expected in investigation (I) at any time during 2018? **(1 mark)**

Source: VCE 2016, Further Mathematics 2, Module 1, Q.1; © VCAA

Question 24 (3 marks)

A travel company arranges flight (F), hotel (H), performance(P) and tour (T) bookings.
Matrix C contains the number of each type of booking for a month.

$$C = \begin{array}{c} \begin{bmatrix} 85 \\ 38 \\ 24 \\ 43 \end{bmatrix} \end{array} \begin{array}{l} F \\ H \\ P \\ T \end{array}$$

a. Write down the order of matrix C. **(1 mark)**

b. A booking fee, per person, is collected by the travel company for each type of booking.
Matrix G contains the booking fees, in dollars, per booking.

$$\begin{array}{cc} & \begin{array}{cccc} F & H & P & T \end{array} \\ G = & \begin{bmatrix} 40 & 25 & 15 & 30 \end{bmatrix} \end{array}$$

i. Calculate the matrix product $J = G \times C$. **(1 mark)**

ii. what does matrix J represent? **(1 mark)**

Source: *VCE 2016, Further Mathematics 2, Module 1, Q.2; © VCAA*

Question 25 (2 marks)

The travel company has five employees, Amara (A), Ben (B), Cheng (C), Dana (D) and Elka (E).

The company allows each employee to send a direct message to another employee only as shown in the communication matrix G beiow.

The matrix G^2 is also shown below.

$$G = \textit{sender} \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\!\!\begin{array}{c} \textit{receiver} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad G^2 = \textit{sender} \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\!\!\begin{array}{c} \textit{receiver} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The '1' in row E, column D of matrix G indicates that Elka (*sender*) can send a direct message to Dana (*receiver*).

The '0' in row E, column C of matrix G indicates that Elka cannot send a direct message to Cheng.

a. To whom can Dana send a direct message? **(1 mark)**

b. Cheng needs to send a message to Elka, but cannot do this directly.
Write down the names of the employees who can send the message from Cheng directly to Elka. **(1 mark)**

***Source:** VCE 2016, Further Mathematics 2, Module 1, Q.3; © VCAA*

Question 26 (7 marks)

The travel company is studying the choice between air (A), land (L), sea (S) or no (N) travel by some of its customers each year.

Matrix T, shown below, contains the percentages of customers who are expected to change their choice of travel from year to year.

$$T = \begin{array}{c} \quad\quad\quad\textit{this year} \\ \begin{array}{cccc} A & L & S & N \end{array} \\ \begin{bmatrix} 0.65 & 0.25 & 0.25 & 0.50 \\ 0.15 & 0.60 & 0.20 & 0.15 \\ 0.05 & 0.10 & 0.25 & 0.20 \\ 0.15 & 0.05 & 0.30 & 0.15 \end{bmatrix} \end{array} \begin{array}{c} A \\ L \\ S \\ N \end{array} \; \textit{next year}$$

Let S_n be the matrix that shows the number of customers who choose each type of travel n years after 2014.

Matrix S_0 below shows the number of customers who chose each type of travel in 2014.

$$S_0 = \begin{bmatrix} 520 \\ 320 \\ 80 \\ 80 \end{bmatrix} \begin{array}{c} A \\ L \\ S \\ N \end{array}$$

Matrix S_1 below shows the number of customers who chose each type of travel in 2015.

$$S_1 = TS_0 = \begin{bmatrix} 478 \\ d \\ e \\ f \end{bmatrix} \begin{array}{c} A \\ L \\ S \\ N \end{array}$$

a. Write the values missing from matrix $S_1(d, e, f)$ in the boxes provided below. **(1 mark)**

$d = \square \; e = \square \; f = \square$

b. Write a calculation that shows that 478 customers were expected to choose air travel in 2015. **(1 mark)**

c. Consider the customers who chose sea travel in 2014. **(1 mark)**

How many of these customers were expected to choose sea travel in 2015?

d. Consider the customers who were expected to choose air travel in 2015. **(1 mark)**
What percentage of these customers had also chosen air travel in 2014?
Round your answer to the nearest whole number.

e. In 2016, the number of customers studied was increased to 1360.
Matrix R_{2016}, shown below, contains the number of these customers who chose each type of travel in 2016.

$$R_{2016} = \begin{bmatrix} 646 \\ 465 \\ 164 \\ 85 \end{bmatrix} \begin{array}{l} A \\ L \\ S \\ N \end{array}$$

The company intends to increase the number of customers in the study in 2017 and in 2018.
The matrix that contains the number of customers who are expected to choose each type of travel in 2017(R_{2017}) and 2018(R_{2018}) can be determined using the matrix equations shown below.

$R_{2017} = TR_{2016} + \text{B}$ $\quad$ $R_{2018} = TR_{2017} + B$

Where

$$T = \begin{array}{c} \textit{this year} \\ \begin{array}{cccc} A & L & S & N \end{array} \\ \begin{bmatrix} 0.65 & 0.25 & 0.25 & 0.50 \\ 0.15 & 0.60 & 0.20 & 0.15 \\ 0.05 & 0.10 & 0.25 & 0.20 \\ 0.15 & 0.05 & 0.30 & 0.15 \end{bmatrix} \end{array} \begin{array}{l} A \\ L \\ S \\ N \end{array} \textit{next year} \qquad B = \begin{bmatrix} 80 \\ 80 \\ 40 \\ -80 \end{bmatrix} \begin{array}{l} A \\ L \\ S \\ N \end{array}$$

i. The element in the fourth row of matrix B is -80 **(1 mark)**
Explain this number in the context of selecting customers for the studies in 2017 and 2018.

ii. Determine the number of customers who are expected to choose sea travel in 2018. **(2 marks)**
Round your answer to the nearest whole number.

***Source:** VCE 2015, Further Mathematics 1, Section B, Module 6, Q.9; © VCAA*

Question 27 (1 mark)

A fast-food stand at the football sells pies (P) and chips (C).

Each week, 300 customers regularly buy either a pie or chips, but not both, from this stand.

For the first five weeks, the customers' choice of pie or chips is expected to change weekly according to the transition matrix T_1, where

$$\begin{array}{cc} & \text{this week} \\ & \begin{array}{cc} P & C \end{array} \\ T_1 = & \begin{bmatrix} 0.65 & 0.25 \\ 0.35 & 0.75 \end{bmatrix} \begin{array}{c} P \\ C \end{array} \text{next week} \end{array}$$

After the first five weeks, due to expected cold weather, the customers' choice of pie or chips is expected to change weekly according to the transition matrix T_2, where

$$\begin{array}{cc} & \text{this week} \\ & \begin{array}{cc} P & C \end{array} \\ T_2 = & \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix} \begin{array}{c} P \\ C \end{array} \text{next week} \end{array}$$

In week 1, 150 customers bought a pie and 150 customers bought chips.

Let S_1 be the state matrix for week 1.

The number of customers expected to buy a pie or chips in week 8 can be found by evaluating

A. $T_2^7 S_1$
B. $T_1^8 S_1$
C. $T_2^3 \left(T_1^4 S_1\right)$
D. $T_1^3 \left(T_2^4 S_1\right)$
E. $T_1^3 \left(T_2^5 S_1\right)$

***Source:** VCE 2015, Further Mathematics 2, Module 6, Q.3; © VCAA*

Question 28 (7 marks)

A new model for the number of students in the school after each assessment takes into account the number of students who are expected to leave the school after each assessment.

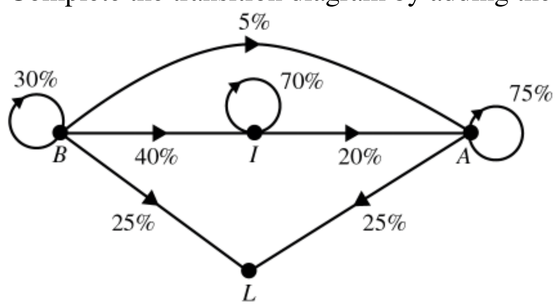
After each assessment, students are classified as beginner (B), intermediate (I), advanced (A) or left the school (L).

Let matrix T_2 be the transition matrix for this new model.

Matrix T_2, shown below, contains the percentages of students who are expected to change their ability level or leave the school after each assessment.

$$\begin{array}{cc} & \text{before assessment} \\ & \begin{array}{cccc} B & I & A & L \end{array} \\ T_2 = & \begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix} \begin{array}{c} B \\ I \\ A \\ L \end{array} \quad \text{after assessment} \end{array}$$

a. An incomplete transition diagram for matrix T_2 is shown below. **(2 marks)**
Complete the transition diagram by adding the missing information.



The number of students at each level, immediately before the first assessment of the year, is shown in matrix R_0 below.

$$R_0 = \begin{array}{c} \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} B \\ I \\ A \\ L \end{array}$$

Matrix T_2, repeated below, contains the percentages of students who are expected to change their ability level or leave the school after each assessment.

$$\begin{array}{c} \text{before assessment} \\ \begin{array}{cccc} B & I & A & L \end{array} \end{array}$$

$$T_2 = \begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix} \begin{array}{c} B \\ I \\ A \\ L \end{array} \quad \text{after assessment}$$

b. What percentage of students is expected to leave the school after the first assessment? **(1 mark)**

c. How many advanced-level students are expected to be in the school after two assessments? **(1 mark)**
Write your answer correct to the nearest whole number.

d. After how many assessments is the number of students in the school, correct to the nearest whole number, first expected to drop below 50? **(1 mark)**

Another model for the number of students in the school after each assessment takes into account the number of students who are expected to join the school after each assessment.
Let R_n be the state matrix that contains the number of students in the school immediately after n assessments.
Let V be the matrix that contains the number of students who join the school after each assessment.
Matrix V is shown below.

$$V = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix} \begin{matrix} B \\ I \\ A \\ L \end{matrix}$$

The expected number of students in the school after n assessments can be determined using the matrix equation

$$R_{n+1} = T_2 \times R_n + V$$

where

$$R_0 = \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} \begin{matrix} B \\ I \\ A \\ L \end{matrix}$$

e. Consider the intermediate-level students expected to be in the school after three assessments. **(2 marks)**
How many are expected to become advanced-level students after the next assessment?
Write your answer correct to the nearest whole number.

Source: *VCE 2015, Further Mathematics 1, Section B, Module 6, Q.4; © VCAA*

Question 29 (1 mark)

The numbers of adult and child tickets purchased for five performances of a stage show are shown in the table below.

Performance	Adult	Child
1	143	24
2	128	31
3	89	24
4	104	18
5	115	23

Which one of the following matrix calculations can be used to determine both the total number of adult tickets and the total number of child tickets purchased for all five performances?

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 142 & 128 & 89 & 104 & 115 \\ 24 & 31 & 24 & 18 & 23 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 142 & 128 & 89 & 104 & 115 \\ 24 & 31 & 24 & 18 & 23 \end{bmatrix}$

C. $\begin{bmatrix} 142 & 128 & 89 & 104 & 115 \\ 24 & 31 & 24 & 18 & 23 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 142 & 128 & 89 & 104 & 115 \\ 24 & 31 & 24 & 18 & 23 \end{bmatrix}$

E. $\begin{bmatrix} 142 & 128 & 89 & 104 & 115 \\ 24 & 31 & 24 & 18 & 23 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Source: *VCE 2015, Further Mathematics 2, Module 6, Q.1; © VCAA*

Question 30 (5 marks)

Students in a music school are classified according to three ability levels: beginner (B), intermediate (I) or advanced (A).

Matrix S_0, shown below, lists the number of students at each level in the school for a particular

$$S_0 = \begin{array}{c} \begin{bmatrix} 20 \\ 60 \\ 40 \end{bmatrix} \end{array} \begin{array}{c} B \\ I \\ A \end{array}$$

a. How many students in total are in the music school that week? **(1 mark)**

b. The music school has four teachers, David (D), Edith (E), Flavio (F) and Geoff (G). Each teacher will teach a proportion of the students from each level, as shown in matrix P below.

$$\begin{array}{cc} & \begin{array}{cccc} D & E & F & G \end{array} \\ P = & \begin{bmatrix} 0.25 & 0.5 & 0.15 & 0.1 \end{bmatrix} \end{array}$$

The matrix product, $Q = S_0P$, can be used to find the number of students from each level taught by each teacher.

i. Complete matrix QQ, shown below, by writing the missing elements in the shaded boxes. **(1 mark)**

$$Q = \begin{bmatrix} 5 & \boxed{} & 3 & 2 \\ 15 & 30 & \boxed{} & 6 \\ 10 & 20 & 6 & 4 \end{bmatrix}$$

ii. How many intermediate students does Edith teach? **(1 mark)**

c. The music school pays the teachers \$15 per week for each beginner student, \$25 per week for each intermediate student and \$40 per week for each advanced student.

These amounts are shown in matrix C below.

$$\begin{array}{cc} & \begin{array}{ccc} B & I & A \end{array} \\ C = & \begin{bmatrix} 15 & 25 & 40 \end{bmatrix} \end{array}$$

The amount paid to each teacher each week can be found using a matrix calculation.

i. Write down a matrix calculation in terms of Q and C that results in a matrix that lists the amount paid to each teacher each week. **(1 mark)**

ii. How much is paid to Geoff each week? **(1 mark)**

Question 31 (3 marks)

The ability level of the students is assessed regularly and classified as beginner (B), intermediate (I) or advanced (A).

After each assessment, students either stay at their current level or progress to a higher level.

Students cannot be assessed at a level that is lower than their current level.

The expected number of students at each level after each assessment can be determined using the transition matrix, T_1, shown below.

$$\begin{array}{c} \text{before assessment} \\ \begin{array}{ccc} B & I & A \end{array} \end{array}$$

$$T_1 = \begin{bmatrix} 0.50 & 0 & 0 \\ 0.48 & 0.80 & 0 \\ 0.02 & 0.20 & 1 \end{bmatrix} \begin{matrix} B \\ I \\ A \end{matrix} \quad \text{after assessment}$$

a. The element in the third row and third column of matrix T_1 is the number 1 **(1 mark)**

Explain what this tells you about the advanced-level students.

b. Let matrix S_n be a state matrix that lists the number of students at beginner, intermediate and advanced levels after n assessments.

The number of students in the school, immediately before the first assessment of the year, is shown in matrix S_0 below.

$$S_0 = \begin{bmatrix} 20 \\ 60 \\ 40 \end{bmatrix} \begin{matrix} B \\ I \\ A \end{matrix}$$

i. Write down the matrix S_1 that contains the expected number of students at each level after one assessment. **(1 mark)**

Write the elements of this matrix correct to the nearest whole number.

ii. How many intermediate-level students have become advanced-level students after one assessment? **(1 mark)**

Source: *VCE 2014, Further Mathematics 2, Module 6, Q.1;* © VCAA

Question 32 (6 marks)

A small city is divided into four regions: Northern (N), Eastern (E), Southern (S) and Western (W).

The number of adult males (M) and the number of adult females (F) living in each of the regions in 2013 is shown in matrix V below.

$$V = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cc} M & F \end{array}$$

$$V = \begin{bmatrix} 1360 & 1460 \\ 1680 & 1920 \\ 900 & 1060 \\ 1850 & 1770 \end{bmatrix} \begin{array}{l} N \\ E \\ S \\ W \end{array}$$

a. Write down the order of matrix V **(1 mark)**

b. How many adult males lived in the Western region in 2013? **(1 mark)**

c. In terms of the population of the city, what does the sum of the elements in the second column of matrix V represent? **(1 mark)**

d. An election is to be held in the city. **(1 mark)**

All of the adults in each of the regions of the city will vote in the election.

One of the election candidates, Ms Aboud, estimates that she will receive 45% of the male votes and 55% of the female votes in the election.

This information is shown in matrix P below.

$$P = \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix} \begin{array}{l} M \\ F \end{array}$$

Explain, in terms of rows and columns, why the matrix product $V \times P$ is defined.

e. The product of matrices V and P is shown below. **(1 mark)**

$$V \times P = \begin{bmatrix} 1360 & 1460 \\ 1680 & 1920 \\ 900 & 1060 \\ 1850 & 1770 \end{bmatrix} \times \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix} = \begin{bmatrix} w \\ 1812 \\ 988 \\ 1806 \end{bmatrix}$$

Using appropriate elements from the matrix product $V \times P$, write a calculation to show that the value of w is 1415.

f. How many votes does Ms Aboud expect to receive in the election? **(1 mark)**

***Source:** VCE 2014, Further Mathematics 2, Module 6, Q.3; © VCAA*

Question 33 (3 marks)

Mr Choi may need to withdraw from the election at the end of May.

Matrix T, shown below, shows the percentage of voters who change their preferred candidate, from month to month, **before** Mr Choi would withdraw from the election.

$$\begin{array}{cc} & \text{this month} \\ & \begin{array}{ccc} A & B & C \end{array} \\ T = & \begin{bmatrix} 0.75 & 0.10 & 0.20 \\ 0.05 & 0.80 & 0.40 \\ 0.20 & 0.10 & 0.40 \end{bmatrix} \begin{array}{l} A \\ B \quad \text{next month} \\ C \end{array} \end{array}$$

Matrix T_1, shown below, shows the percentage of voters who change their preferred candidate, from May to June, after Mr Choi would withdraw from the election.

$$\begin{array}{cc} & \text{May} \\ & \begin{array}{ccc} A & B & C \end{array} \\ T_1 = & \begin{bmatrix} 0.75 & 0.15 & 0.6 \\ 0.25 & 0.85 & 0.4 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} A \\ B \quad \text{June} \\ C \end{array} \end{array}$$

Consider the voters who preferred Mr Broad (B) in May and who were expected to prefer Mr Choi (C) in June.

a. What percentage of these voters are now expected to prefer Mr Broad in June? **(1 mark)**

b. The state matrix that indicates the number of voters who are expected to have a preference for each candidate in January, S_1, is given below. **(2 marks)**

$$S_1 S_1 = \begin{array}{c} \begin{bmatrix} 6000 \\ 3840 \\ 2160 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

If Mr Choi withdraws, how many votes is Mr Broad expected to receive in the election in June? Write your answer, correct to the nearest vote.

Source: *VCE 2014, Further Mathematics 2, Module 6, Q.2; © VCAA*

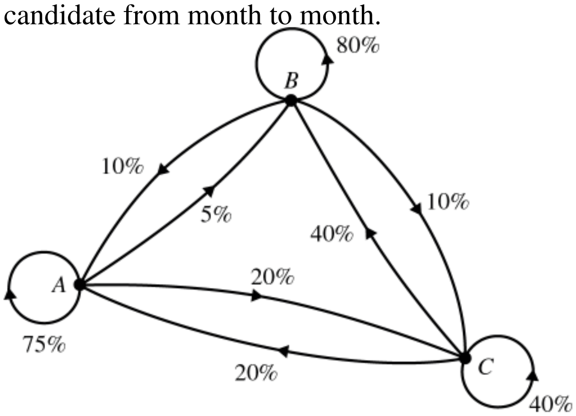
Question 34 (6 marks)

There are three candidates in the election: Ms Aboud (A), Mr Broad (B) and Mr Choi (C).

The election campaign will run for six months, from the start of January until the election at the end of June.

A survey of voters found that voting preference can change from month to month leading up to the election.

The transition diagram below shows the percentage of voters who are expected to change their preferred candidate from month to month.



a. i. Of the voters who prefer Mr Choi this month, what percentage are expected to prefer Ms Aboud next month? **(1 mark)**

ii. Of the voters who prefer Ms Aboud this month, what percentage are expected to change their preferred candidate next month? **(1 mark)**

b. In January, 12 000 voters are expected in the city. The number of voters in the city is expected to remain constant until the election is held in June.
The state matrix that indicates the number of voters who are expected to have a preference for each candidate in January, $S_1, isgivenbelow.$

$$S_1 = \begin{array}{c} \begin{bmatrix} 6000 \\ 3840 \\ 2160 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \end{array}$$

How many voters are expected to change their preference to Mr Broad in February? **(1 mark)**

c. The information in the transition diagram has been used to write the transition matrix, T, shown below.

$$T = \begin{array}{c} \text{this month} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.75 & 0.10 & 0.20 \\ 0.05 & 0.80 & 0.40 \\ 0.20 & 0.10 & 0.40 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ A \\ B \ \text{next month} \\ C \end{array}$$

i. Evaluate the matrix $S_3 = T^2 S_1$ and write it down in the space below. **(1 mark)**
Write the elements, correct to the nearest whole number.

$$S_3 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

ii. What information does matrix S_3 contain? **(1 mark)**

d. Using matrix T, how many votes would the winner of the election in June be expected to receive? **(1 mark)**
Write your answer, correct to the nearest whole number.

***Source:** VCE 2013, Further Mathematics 1, Section B, Module 6, Q.5; © VCAA*

Question 35 (1 mark)

Five students, Richard (R), Brendon (B), Lee (L), Arif (A) and Karl (k), were asked whether they played each of the following sports, football (F), golf (G), soccer (S) or tennis (T). Their responses are displayed in the table below.

Student	**Sport played**			
	Football(F)	**Golf**(G)	**Soccer**(S)	**Tennis**(T)
R	yes	no	no	yes
B	yes	yes	yes	no
L	no	no	no	yes
A	no	yes	no	yes
K	yes	no	no	yes

If 1 is used to indicate that the student plays a particular sport and 0 is used to indicate that the student does not play a particular sport, which one of the following matrices could be used to represent the information in the table?

A. $\begin{array}{c c} \begin{matrix} R & B & L & A & K \end{matrix} & \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} F \\ G \\ S \\ T \end{matrix} \end{array}$

B. $\begin{array}{c c} \begin{matrix} F & G & S & T \end{matrix} & \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} & \begin{matrix} R \\ B \\ L \\ A \\ K \end{matrix} \end{array}$

C. $\begin{array}{c c} \begin{matrix} R & B & L & A & K \end{matrix} & \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} & \begin{matrix} F \\ G \\ S \\ T \end{matrix} \end{array}$

D. $\begin{array}{c c} \begin{matrix} F & G & S & T \end{matrix} & \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} R \\ B \\ L \\ A \\ K \end{matrix} \end{array}$

E. $\begin{array}{c c} \begin{matrix} F & G & S & T \end{matrix} & \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} & \begin{matrix} R \\ B \\ L \\ A \\ K \end{matrix} \end{array}$

Source: *VCE 2013, Further Mathematics 1, Section B, Module 6, Q.1; © VCAA*

Question 36 (1 mark)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2 \times \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$ equals

A. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$

C. $\begin{bmatrix} 2 \\ -2 \\ -2 \\ 2 \end{bmatrix}$

D. $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix}$

E. $\begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

***Source:** VCE 2013, Further Mathematics 1, Section B, Module 6, Q.6; © VCAA*

Question 37 (1 mark)

A worker can assemble 10 bookcases and four desks in 360 minutes, and eight bookcases and three desks in 280 minutes.

If each bookcase takes b minutes to assemble and each desk takes d minutes to assemble, the matrix $\begin{bmatrix} b \\ d \end{bmatrix}$ will be given by

A. $\begin{bmatrix} -1.5 & 2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 360 \\ 280 \end{bmatrix}$

B. $\begin{bmatrix} 10 & 4 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} 360 \\ 280 \end{bmatrix}$

C. $\begin{bmatrix} 3 & -4 \\ -8 & 10 \end{bmatrix} \begin{bmatrix} 360 \\ 280 \end{bmatrix}$

D. $\begin{bmatrix} 5 & -2 \\ -4 & 1.5 \end{bmatrix} \begin{bmatrix} 360 \\ 280 \end{bmatrix}$

E. $\begin{bmatrix} 10 \\ 4 \end{bmatrix} [360] + \begin{bmatrix} 8 \\ 3 \end{bmatrix} [280]$

***Source:** VCE 2013, Further Mathematics 1, Section B, Module 6, Q.9; © VCAA*

Question 38 (1 mark)

P, Q, R and S are matrices such that the matrix product $P = QRS$ is defined.

Matrix Q and matrix S are square, non-zero matrices for which $Q + S$ is **not defined.**

Which one of the following matrix expressions is **defined**?

A. $R - S$

B. $Q + R$

C. P^2

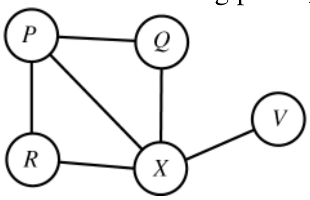
D. R^{-1}

E. $P \times S$

***Source:** VCE 2013, Further Mathematics 2, Module 6, Q.1; © VCAA*

Question 39 (3 marks)

Five trout-breeding ponds, P, Q, R, X and V , are connected by pipes, as shown in the diagram below.



The matrix W is used to represent the information in this diagram.

$$W = \begin{array}{c} \begin{array}{ccccc} P & Q & R & X & V \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ P \\ Q \\ R \\ X \\ V \end{array}$$

In matrix W

- the 1 in column 1, row 2, for example, indicates that a pipe directly connects pond P and pond Q
- the 0 in column 1, row 5, for example, indicates that pond P and pond V are not directly connected by a pipe.

a. Find the sum of the elements in row 3 of matrix W. **(1 mark)**

b. In terms of the breeding ponds described, what does the sum of the elements in row 3 of matrix W represent? **(1 mark)**

c. The pipes connecting pond P to pond R and pond P to pond X are removed. Matrix N will be used to show this situation. However, it has missing elements. **(1 mark)**

Complete matrix N below by filling in the missing elements in row 1 and column 1.

$$N = \begin{array}{c} \begin{array}{ccccc} P & Q & R & X & V \end{array} \\ \left[\begin{array}{ccccc} 0 & \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & 0 & 0 & 1 & 0 \\ \_\_ & 0 & 0 & 1 & 0 \\ \_\_ & 1 & 1 & 0 & 1 \\ \_\_ & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ P \\ Q \\ R \\ X \\ V \end{array}$$

Source: *VCE 2013, Further Mathematics 2, Module 6, Q.2; © VCAA*

Question 40 (12 marks)

10 000 trout eggs, 1000 baby trout and 800 adult trout are placed in a pond to establish a trout population.

In establishing this population

- eggs (E) may die (D) or they may live and eventually become baby trout (B)
- baby trout (B) may die (D) or they may live and eventually become adult trout (A)
- adult trout (A) may die (D) or they may live for a period of time but will eventually die.

From year to year, this situation can be represented by the transition matrix T, where

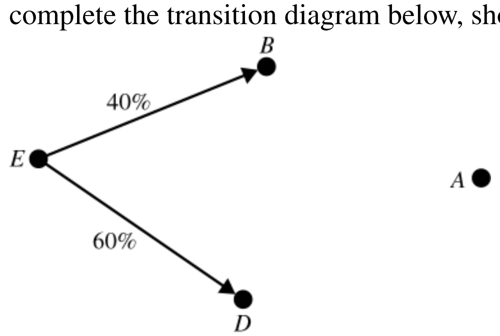
$$\begin{array}{c} \textit{this year} \\ \begin{array}{cccc} E & B & A & D \end{array} \end{array}$$

$$T = \begin{array}{c} \\ \end{array}\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{array}\right]\begin{array}{c} E \\ B \\ A \\ D \end{array} \; \textit{next year}$$

a. Use the information in the transition matrix T to

i. determine the number of eggs in this population that die in the first year. **(1 mark)**

ii. complete the transition diagram below, showing the relevant percentages. **(2 marks)**



b. The initial state matrix for this trout population, S_0, can be written as

$$S_0 = \begin{array}{c} \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} E \\ B \\ A \\ D \end{array}$$

Let S_n represent the state matrix describing the trout population after n years.

Using the rule $S_n = TS_n - 1$, determine each of the following.

i. S_1 **(1 mark)**

ii. the number of adult trout predicted to be in the population after four years. **(1 mark)**

c. The transition matrix T predicts that, in the long term, all of the eggs, baby trout and adult trout will die.

i. How many years will it take for all of the adult trout to die (that is, when the number of adult trout in the population is first predicted to be less than one)? **(1 mark)**

ii. What is the largest number of adult trout that is predicted to be in the pond in any one year? **(1 mark)**

d. Determine the number of eggs, baby trout and adult trout that, if added to or removed from the pond at the end of each year, will ensure that the number of eggs, baby trout and adult trout in the population remains constant from year to year. **(2 marks)**

e. The rule $S^n = TS_{n-1}$ that was used to describe the development of the trout in this pond does not take into account new eggs added to the population when the adult trout begin to breed.
To take breeding into account, assume that 50% of the adult trout lay 500 eggs each year. The matrix describing the population after one year, S_1, is now given by the new rule
$S_1 = TS_0 + 500MS_0$ where

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.40 & 0 & 0 & 0 \\ 0 & 0.25 & 0.50 & 0 \\ 0.60 & 0.75 & 0.50 & 1.0 \end{bmatrix}, \; M = \begin{bmatrix} 0 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix}$$

i. Use this new rule to determine S_1. **(1 mark)**

ii. This pattern continues so that the matrix describing the population after n years, S_n, is given by the rule **(2 marks)**
$S_n = TS_n - 1 + 500MS_n - 1$
Use this rule to determine the number of eggs in the population after two years.

Question 41 (1 mark)

The order of matrix A is

$$A = \begin{bmatrix} 1 & 23 & 42 & 8 \\ 20 & 12 & 14 & 45 \\ 1 & 13 & 84 & 19 \end{bmatrix}$$

A. (3×4)
B. (4×3)
C. (2×3)
D. (4×2)
E. (2×3)

Question 42 (1 mark)

Which of the following is a null matrix?

A. $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}$

Question 43 (1 mark)

The order of matrix A is 3×3. The element in row i and column j of matrix A is a_{ij} and it is found using the rule $a_{ij} = 2i - 3j$.

The matrix A is:

A. $\begin{bmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$

B. $\begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & 0 & -3 \end{bmatrix}$

C. $\begin{bmatrix} -1 & 1 & 3 \\ -4 & -2 & 0 \\ -7 & 5 & -3 \end{bmatrix}$

D. $\begin{bmatrix} -1 & -4 & 3 \\ -4 & -2 & -5 \\ -7 & 0 & -3 \end{bmatrix}$

E. $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

Question 44 (1 mark)

A 3×2 matrix is defined using the rules: $a_{ij} = 3$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$. The matrix that matches these rules is:

A. $\begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$

Question 45 (1 mark)

The transpose of an upper triangular matrix will always produce

A. a zero matrix.
B. another upper triangular matrix.
C. the identity.
D. a lower triangular matrix.
E. an identical matrix.

Question 46 (1 mark)

Given $A^T = A$, which of the following is not always true?

A. A must be a symmetrical matrix
B. $A^T A = A^2$
C. $\left(A^T\right)^T = A$
D. $A^T A^T = I$
E. A must be a square matrix

Question 47 (1 mark)

Which matrix represents a lower triangular matrix?

A. $\begin{bmatrix} 3 & 3 & 0 \\ 2 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 1 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix}$

Question 48 (1 mark)

Given the 3×3 matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which of the following are true of this matrix?

I. I. It is an identity matrix.
II. It is a symmetrical matrix.
III. It is a lower triangular matrix.
IV. It is an upper triangular matrix

A. I and II
B. I, II and III
C. II and III
D. II, III and IV
E. I, II, III and IV

Question 49 (1 mark)

If $A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & -8 & 17 \\ 5 & 14 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 3 \\ -5 & 1 & 0 \end{bmatrix}$, the product of BA is

A. $\begin{bmatrix} 13 & 43 & 13 \\ -4 & -3 & -8 \end{bmatrix}$

B. $\begin{bmatrix} 13 & -4 \\ 43 & -3 \\ 13 & -8 \end{bmatrix}$

C. $\begin{bmatrix} -4 & -3 & -8 \\ 13 & 43 & 13 \end{bmatrix}$

D. $\begin{bmatrix} -4 & 13 \\ -3 & 43 \\ -8 & 13 \end{bmatrix}$

E. not defined

Question 50 (1 mark)

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Then A^3 is

A. $\begin{bmatrix} 1 & 6 \\ 9 & 12 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 8 \\ 27 & 64 \end{bmatrix}$

C. $\begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix}$

D. $\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

E. not defined

Question 51 (1 mark)

$$\begin{array}{ccc} & & \text{to} \\ & & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{from} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ & 0 & 1 & 0 \\ & & & \\ & 1 & 1 & \end{bmatrix} \end{array}$$

Which of the following matrices is the completed dominance matrix above?

A.

$$\begin{array}{ccc} & & \text{to} \\ & & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{from} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

B.

$$\begin{array}{ccc} & & \text{to} \\ & & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{from} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

C.

$$\begin{array}{ccc} & & \text{to} \\ & & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{from} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

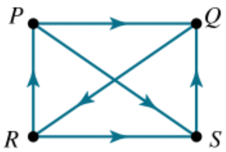
D.

$$\begin{array}{ccc} & & \text{to} \\ & & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{from} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

E.

$$\begin{array}{ccc} & & \text{to} \\ & & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{from} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Question 52 (1 mark)
The digraph below shows the results of a tennis competition. An arrow from P to Q means that P defeated Q.



Which of the following matrices show one-step dominance?

A.
$$\begin{array}{ccccc} & & \text{to} & & \\ ; & P & Q & R & S \\ P & 0 & 0 & 0 & 1 \\ Q & 1 & 0 & 1 & 0 \\ R & 1 & 0 & 0 & 1 \\ S & 1 & 1 & 0 & 0 \end{array}$$
from

B.
$$\begin{array}{ccccc} & & \text{to} & & \\ & P & Q & R & S \\ P & 0 & 1 & 1 & 0 \\ Q & 1 & 0 & 0 & 1 \\ R & 1 & 0 & 0 & 0 \\ S & 0 & 1 & 0 & 0 \end{array}$$
from

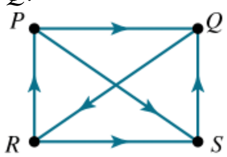
C.
$$\begin{array}{ccccc} & & \text{to} & & \\ & P & Q & R & S \\ P & 0 & 1 & 1 & 0 \\ Q & 0 & 0 & 0 & 1 \\ R & 1 & 0 & 0 & 0 \\ S & 1 & 0 & 1 & 1 \end{array}$$
from

D.
$$\begin{array}{ccccc} & & \text{to} & & \\ & P & Q & R & S \\ P & 0 & 1 & 0 & 1 \\ Q & 0 & 0 & 1 & 0 \\ R & 1 & 0 & 0 & 1 \\ S & 0 & 1 & 1 & 0 \end{array}$$
from

E.
$$\begin{array}{ccccc} & & \text{to} & & \\ ; & P & Q & R & S \\ P & 0 & 1 & 0 & 1 \\ Q & 0 & 0 & 1 & 0 \\ R & 1 & 0 & 0 & 1 \\ S & 0 & 1 & 0 & 0 \end{array}$$
from

Question 53 (1 mark)

The digraph below shows the results of a tennis competition. An arrow from P to Q means that P defeated Q.



The most dominant player overall after both one- and two-step dominance is taken into account is

A. P
B. Q
C. R
D. S
E. a tie between P and R

Question 54 (1 mark)

The communication matrix below shows a number of connections between vertices.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

The total number of connections is

A. 1
B. 3
C. 5
D. 7
E. 10

Question 55 (1 mark)

The communication matrix, $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, shows the connections existing between four vertices A, B, C and D respectively.

A path between vertices that is not valid for this communication network is

A. A – B – C – D
B. A – C – B – D
C. C – A – D – B
D. C – A – B – D
E. B – D – C – A

Question 56 (1 mark)

A study is conducted into the use of two brands of toothpaste: Brand *A* and Brand *B*. The proportion of people changing from one brand of toothpaste to the other brand of toothpaste is shown in the transition matrix below.

$$\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 1 & 0.1 \\ 0 & 0.9 \end{bmatrix} \end{array}$$

Assuming that this pattern of change continues, in the long term, the number of people using Brand *B* toothpaste will

A. not change from what it currently is.
B. gradually increase to 100%.
C. gradually decrease to zero.
D. eventually settle at around 30%.
E. eventually settle at around 70%.

Question 57 (1 mark)

If an initial state matrix is $s_0 = \begin{bmatrix} 60 \\ 75 \end{bmatrix}$ and the transition matrix is $T = \begin{bmatrix} 0.65 & 0.45 \\ 0.35 & 0.55 \end{bmatrix}$, then the steady state matrix is

A. $\begin{bmatrix} 72.75 \\ 62.25 \end{bmatrix}$

B. $\begin{bmatrix} 75 \\ 60 \end{bmatrix}$

C. $\begin{bmatrix} 76 \\ 59 \end{bmatrix}$

D. $\begin{bmatrix} 75.9375 \\ 59.0625 \end{bmatrix}$

E. not defined

Question 58 (1 mark)

There are 3 brands of chocolate that dominate the market: Brand A, Brand B, and Brand C. People change from one brand of chocolate to another all the time.

The transition matrix to represent this situation is shown below.

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ T = & \begin{bmatrix} 0.6 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.1 & 0.5 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

If there are initially 300 people using Brand A, 250 people using Brand B and 425 people using Brand C, in the long term the number of people using each brand will be

A. $\begin{bmatrix} 438.75 \\ 227.5 \\ 308.75 \end{bmatrix}$

B. $\begin{bmatrix} 439 \\ 228 \\ 309 \end{bmatrix}$

C. $\begin{bmatrix} 438 \\ 227 \\ 308 \end{bmatrix}$

D. $\begin{bmatrix} 440 \\ 230 \\ 310 \end{bmatrix}$

E. $\begin{bmatrix} 438.7 \\ 227.5 \\ 308.75 \end{bmatrix}$

Question 59 (1 mark)

When buying fruit in winter, people alternate between oranges (O) and apples (A). 75% of the people who buy apples will buy oranges next time and 55% of the people who buy oranges will buy apples next time.

On the first day of winter, 85 apples and 120 oranges are sold.

In the long term, the number of oranges sold each day is

A. 86.73

B. 118.27

C. 87

D. 118

E. 119

Answers and marking guide

8.2 Matrix representation

Question 1

In a transpose matrix, the rows are swapped to the columns, and vice versa.

If $M = \begin{bmatrix} 3 & 2 \\ 8 & 9 \\ 13 & 7 \end{bmatrix}$, then $M^T = \begin{bmatrix} 3 & 8 & 13 \\ 2 & 9 & 7 \end{bmatrix}$

Question 2

The amount saved in week 2 is given by the element in row 2, column 2. That is, 90.

Question 3

Select an element, say 5 in row 2, column 3, $m_{2,3} = 5$, and try all the options.

A:$2 + 3 - 1 \neq 5$

B:$2(2) - 3 + 1 \neq 5$

C:$2(2) + 3 - 2 = 5$ This is the correct answer.

D:$2 + 2(3) - 2 \neq 5$

E:$2 + 3 + 1 \neq 5$

Question 4

Check an element that is different is all matrices, for example row 2, column 3:

$m_{23} = 3 \times 2 + 2 \times 3 = 12$

Question 5

$$\begin{aligned} C^T \times \left(A^T \times B\right)^T &= (3 \times 4)^T \times \left((2 \times 4)^T \times (2 \times 3)\right)^T \\ &= (4 \times 3) \times ((4 \times 2) \times (2 \times 3))^T \\ &= (4 \times 3) \times (4 \times 3)^T \\ &= (4 \times 3) \times (3 \times 4) \\ &= (4 \times 4) \end{aligned}$$

Question 6

To *transpose* a matrix, swap the rows into columns.

So if $A = \begin{bmatrix} 2 & 7 & 10 \\ 13 & 19 & 8 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & 13 \\ 7 & 19 \\ 10 & 8 \end{bmatrix}$.

Question 7

Row 3 means Ursula and column 2 is art books, so element b_{32} is the number of art books owned by Ursula.

Question 8

The order of the matrix is determined by *number of rows* $\times$ *number of columns*.

Question 9

Element a_{23} is found in the second row of the third column.

Question 10

A square matrix has the same number of rows and columns.

Question 11

A diagonal matrix has all non-zero elements on the leading diagonal.

Question 12

S^T is the transpose of matrix S. S^T is found by swapping the rows for columns.
The first row of matrix S becomes the first column of S^T.
The second row of matrix S becomes the second column of S^T.

Therefore $S^T = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 3 \end{bmatrix}$

8.3 Addition, subtraction and scalar operations with matrices

Question 1

To add matrices together, all matrices must be of the same order. Therefore, only $\begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 8 & 0 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ are possible.

Question 2

Start by choosing an element, say 5, which is $i = 2, j = 1$
A: $P_{2,1} = 4 - 1 \neq 5$
B: $P_{2,1} = 2(2) + 1 = 5 -$ check with another element, $P_{2,2} = 2(2) + 1 \neq 4$, which is the element in row 2, column 2.
C: $P_{2,1} = 2 + 1 + 1 \neq 5$
D: $P_{2,1} = 2 + 2(1) \neq 5$
E: $P_{2,1} = 2(2) - 1 + 2 = 5 -$ check with another element, $P_{2,2} = 2(2) - 2 + 2 = 4$, which is correct.

Question 3

Using the given information, $A = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

$$A + B = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 9 & 9 & 9 \end{bmatrix}$$

Question 4

$i =$ row number and $j =$ column number

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{bmatrix} = \begin{bmatrix} 1-1 & 1-2 & 1-3 \\ 2-1 & 2-2 & 2-3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

***VCAA Assessment Report* note:**

To answer this question, students needed to first use the rule $x_{ij} = i - j$ for ijth term of the matrix X to determine the matrix and then equate it to one of five given matrix expressions.

Question 5

$$X = \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \begin{bmatrix} 1+1 & 1+2 \\ 2+1 & 2+2 \\ 3+1 & 3+2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$$

Question 6

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} - \begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = \begin{bmatrix}1-5 & 2-6\\3-7 & 4-8\end{bmatrix}$$
$$= \begin{bmatrix}-4 & -4\\-4 & -4\end{bmatrix}$$

Question 7

There is a common factor of 6 in all the elements. The common factor is written at the front of the matrix containing the remaining factors.

8.4 Multiplying matrices

Question 1

It is a binary matrix, as the elements all only 0 or 1.
It does not fit the form of the other special binary matrices (identity or permutation).

Question 2

Matrixes are multiplied by *rows* $\times$ *columns*
So we need to multiply the 4th row of A by the 1st column of B.
$4 \times 2 + 5 \times 4$

Question 3

Remember: Rows by Columns

$$(6 \times 7) + (6 \times 8) = \begin{bmatrix}6 & 6\end{bmatrix}\begin{bmatrix}7\\8\end{bmatrix}$$

Question 4

$$p^4 \times W = \begin{bmatrix}A\\S\\T\\O\\R\end{bmatrix} \text{ where } p^4 = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Question 5

Remember: Rows by Columns $\begin{bmatrix}4 & 2 & 0\end{bmatrix} \times \begin{bmatrix}4\\12\\8\end{bmatrix} = [4 \times 4 + 2 \times 12 + 0 \times 8] = [40]$

A. $[144] \neq [40]$

B. $\begin{bmatrix}16\\24\\0\end{bmatrix} \neq [40]$

C. $4 \times \begin{bmatrix}1 & 2 & 0\end{bmatrix} \times \begin{bmatrix}1\\12\\8\end{bmatrix} = [4 \times 1 \times 1 + 4 \times 2 \times 12 + 4 \times 0 \times 8] = [100] \neq [40]$

D. $2 \times \begin{bmatrix}2 & 1 & 0\end{bmatrix} \times \begin{bmatrix}2\\6\\4\end{bmatrix} = [2 \times 2 \times 2 + 2 \times 1 \times 6 + 2 \times 0 \times 4] = [20] \neq [40]$

E. $4 \times \begin{bmatrix}2 & 1 & 0\end{bmatrix} \times \begin{bmatrix}2\\6\\4\end{bmatrix} = [4 \times 2 \times 2 + 4 \times 1 \times 6 + 4 \times 0 \times 4] = [40]$

The answer is E.

Question 6

To multiply matrices, multiply the rows by the columns. With only 0's and 1's in the first matrix, look at the position of the 1's and match them to the position in the second matrix.

$$\begin{bmatrix} 1 \times P \\ 1 \times A \\ 1 \times L \\ 1 \times E \\ 1 \times S \end{bmatrix}$$

Question 7

Remember to multiply rows by columns, and we want two answers (total number of coins and their total value).

The number of coins must have a row matrix first, and the second matrix must be made up of two columns.

$$\begin{bmatrix} 15 & 32 & 48 & 24 \end{bmatrix} \times \begin{bmatrix} 1 & 5 \\ 1 & 10 \\ 1 & 20 \\ 1 & 50 \end{bmatrix}$$

Question 8

For a matrix product to be defined, the number of columns in the first matrix must equal to the number of rows in the second.

As Z has an order of (3×3) and Y has the order (1×2), this matrix product is not defined.

Question 9

C is incorrect. If $BA = AB = I$, this indicates that A and B are inverses of each other; however, it is not true to say that they must be equal to each other.

Question 10

As the **total** amount raised per year group is wanted, the final product must be either a $[2 \times 1]$ or $[1 \times 2]$.

Option A:

$[2 \times 3] \times [3 \times 1]$

$= [2 \times 1]$

But this is incorrect, as the first matrix is a mixture of the number of trees sold by Year 7s and the costs. The matrices can't be mixed up like that.

Option B:

$[2 \times 4] \times [4 \times 1]$

$= [2 \times 1]$

This is incorrect, though, as the actual category values '7' and '8' shouldn't be part of the first matrix.

Option C:

$[1 \times 3] \times [3 \times 2]$

$= [1 \times 2]$

This is correct, as the number of trees sold by year groups 7 and 8 are contained in the same matrix.

Option D:

$[1 \times 3] \times [3 \times 3]$

$= [1 \times 3]$

This is incorrect, as the order of the final product is wrong.

Option E:

This is incorrect as it first adds up the total number of trees sold by both year groups (without keeping the figures separate).

Question 11

Multiplying the two matrices on CAS (or by hand) gives:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 12

A is (3×2); B is (4×3).
$C = B \times A = (4 \times 3) \times (3 \times 2) = (4 \times 2)$

Question 13

To find the mean mark for each class, all marks (in each class) must be added together, and each total must be divided by 15 (the number of students in each class).

The product $M \times S$ will give the total marks for each class. This will happen because:

- all scores in the first row are multiplied by 1 (which leaves the scores unchanged) and added together; that is, $56 \times 1 + 78 \times 1 + 79 \times 1 + \cdots + 63 \times 1$, giving the total of all scores in class A
- all scores in the first row are multiplied by 1 (which leaves the scores unchanged) and added together; that is $90 \times 1 + 45 \times 1 + 56 \times 1 + \cdots + 59 \times 1$, , giving the total of all scores in class B
- all scores in the first row are multiplied by 1 (which leaves the scores unchanged) and added together; that is $76 \times 1 + 76 \times 1 + 89 \times 1 + \cdots + 87 \times 1$, , giving the total of all scores in class C.

Thus, the product $M \times S$ will be a 3×1 matrix, showing the total marks for each class.

Each score then needs to be divided by 15. This can be achieved by multiplying the resultant matrix by $\frac{1}{15}$.

Therefore, the required expression is $\frac{1}{15} M \times S$.

Question 14

The order of $AB = (3 \times \boldsymbol{3}) \times (\boldsymbol{2} \times 3)$.

As the middle two numbers are different, the number of columns in the first matrix and the number of rows in the second matrix are not equal and the product is not defined.

8.5 The inverse of a matrix and its determinant

Question 1

In the first position, we need an N, but that is located in the last position in RAMON, so the first row will be $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

In the second position, we need an O, which is located in the fourth position in RAMON, so the second row will be $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

In the third position, we need an R, which is located in the first position in RAMON, so the third row will be $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

In the fourth position, we need an M, which is located in the third position in RAMON, so the fourth row will be $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

And, finally, in the fifth position, we need the A, which is in the second position in RAMON, so the fifth row will be $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ A \\ M \\ O \\ N \end{bmatrix} = \begin{bmatrix} N \\ O \\ R \\ M \\ A \end{bmatrix}$$

Question 2

Write as a matrix equation: $\begin{bmatrix} a & 4 \\ 18 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$

There will not be a unique solution when the determinant of the coefficient matrix, $\begin{bmatrix} a & 4 \\ 18 & b \end{bmatrix}$, equals to zero.

$(a \times b) - (4 \times 18) = 0$
$ab = 72$
Going through each of the options, the only factors of 72 are when $a = 2$ and $b = 36$.

Question 3

$$\begin{aligned} Q &= wP \\ QP^{-1} &= w \\ P^{-1} &= Q^{-1}w \\ \frac{P^{-1}}{w} &= Q^{-1} \\ \therefore Q^{-1} &= \frac{1}{w}P^{-1} \end{aligned}$$

Question 4

In the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinant is $ad - bc$.
A: $0 \times 0 - 1 \times 1 = -1 \neq 0$
B: $1 \times 1 - 0 \times 0 = 1 \neq 0$
C: $1 \times 6 - 2 \times -3 = 12 \neq 0$
D: $3 \times 4 - 2 \times 6 = 0$
E: $4 \times -2 - 0 \times 0 = -8 \neq 0$
The answer is D.

Question 5

$\det = (12 \times 3) - (9 \times m) = 36 - 9m$
There is no solution when $\det = 0$.

$$\begin{aligned} 36 - 9m &= 0 \\ -9m &= -36 \\ m &= 4 \end{aligned}$$

Question 6

Use a calculator to find $\begin{bmatrix} 6 & -8 \\ 5 & 14 \end{bmatrix}^{-1}$

Question 7

The determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$

$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1 \times 4 - 3 \times 2 = 4 - 6 = -2$

Question 8

$\det\left(\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\right) = 0$
Hence the inverse is not defined.

8.6 Dominance and communication matrices

Question 1

There are four scenarios that need to be considered: Brie losing to Andy or Andy losing to Brie, and Cleo losing to Della or Della losing to Cleo.
When the sum of one-step and two-step dominances is calculated (use CAS to do), we find that Brie will win with the highest dominance of 9.

Question 2

Use the diagram to set up a communication matrix.

$$\begin{array}{cc} & \textit{receiver} \\ & \begin{array}{cccc} S & T & U & V \end{array} \\ \textit{sender} \begin{array}{c} S \\ T \\ U \\ V \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Question 3

Y cannot send directly to U or V, only directly to W or Z. However, Z does not communicate directly with W. Therefore the answer is 0 – Y cannot send a message to W by sending it via one other person.

Question 4

$m = 0.8\,m + 0.48\,s$ and $0.48\,s = 12$ (for Tuesday night)
Solve simultaneously on your CAS calculator to give $m = 60$ and $s = 25$
Therefore, the airline has 85 planes.

Question 5

The table can be rearranged to become simultaneous equations in a matrix.

$$\begin{bmatrix} 5 & 7 & 6 & 8 \\ 8 & 6 & 9 & 7 \\ 7 & 8 & 7 & 6 \\ 8 & 8 & 5 & 5 \end{bmatrix} \times \begin{bmatrix} c \\ r \\ s \\ w \end{bmatrix} = \begin{bmatrix} 160 \\ 172 \\ 165 \\ 162 \end{bmatrix}$$

$$\begin{bmatrix} 5c + 7r + 6s + 8w \\ 8c + 6r + 9s + 7w \\ 7c + 8r + 7s + 6w \\ 8c + 8r + 5c + 5w \end{bmatrix} = \begin{bmatrix} 160 \\ 172 \\ 165 \\ 162 \end{bmatrix}$$

Through substituting the options for their respective pronumerals, only B gives the correct answers.

Question 6

As P is a permutation matrix, there must be one 1 in each column and row. The only matrix that satisfies these conditions is in option B.

Question 7

Since the equation is $A_{n+1} = TA_n - D$, substitute A_3 for A_{n+1} and A_2 for A_n, then solve to find A_n.
$A_3 = TA_2 - D$
$T^{-1}(A_3 + D) = A_2$

This will give you the matrix $A_2 = \begin{bmatrix} 1630 \\ 2800 \\ 2270 \end{bmatrix}$. The West location is 2270, which is option E.

Question 8
Every pair must have a 1 and a 0.MI and IM both have a 0, which means neither India nor Mustafa won the game against each other. Therefore, the match between India and Mustafa has an incorrect record.

Question 9

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} F \\ E \\ A \\ R \\ S \end{bmatrix} = \begin{bmatrix} S \\ A \\ F \\ E \\ R \end{bmatrix}$$

Alternatively, use your CAS to check each option.

Question 10
Starting in row E, move across to the 1 (which represents the receiver – Charlie (C) in this case). Next, start at row C and move across to the 1 (ignore the second 1 here as it represents E again) and continue this process until B is reached.
Esther – Charlie – Drew – Alan – Bevan.

Question 11
When C is ranked number 1:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} A = 3 \\ B = 4 \\ C = 5 \\ D = 2 \end{matrix}$$

However, if B had lost to A:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} A = 5 \\ B = 2 \\ C = 4 \\ D = 3 \end{matrix}$$

Therefore, A would have been ranked number one.

Question 12
f : Represents the result of the $A - C$ match. We know that C defeated A (1 in row 3, column 1), so $f = 0$.
g : Represents the result of the $B - C$ match. We know that B lost to C (0 in row 2, column 3), so $g = 1$.
h :The position of the h on the diagonal shows that D does not play itself, so $h = 0$.

Question 13
Following a cyclic visitation order from the matrix:
Starting at C, the next warehouse is A, then from A to B, then from B to D, then from D to C and the cycle continues.
Any of the warehouses could be the starting point, making the order D – C – A – B a correct answer.

Question 14
The only transition matrix that follows the given order is C.
Answer A has the order: E – B – M – T – E
Answer B has the order: E – E, M – B – M and T – T on a cycling loop
Answer C is correct: E – B – T – M – E.
Answer D has the order: E – E on a cycling loop. (same cycling loop for other towns if you begin with them).
Answer E has the order: E – B – T – E and it missed M

Question 15

Correct order is R – C – L – V. This corresponds to answer A.

Question 16

From the communication network it can be seen there are 5 single connections. As these connections represent communication in both directions of the nodes, the required matrix must be symmetrical about the main diagonal. Option A is the matrix that correctly displays the communications between nodes.
A is only connected to B, so there is only a single 1 in both its row and column matching with B.
B is connected to A, C, E so there are 3 single 1s in both its row and column.
C is connected to B, D, F, so there are 3 single 1s in both its row and column.
D is only connected to C, so there is only a single 1 in both its row and column matching with C.
E is only connected to B, so there is only a single 1 in both its row and column matching with B.
F is only connected to C, so there is only a single 1 in both its row and column matching with C.

Question 17

A communication matrix that allows communication in both directions will be a symmetrical matrix. The only matrix that complies with this condition is option B.

8.7 Transition matrices and Leslie matrices

Question 1

Use the information given to construct the transition matrix, keeping in mind that the columns must add to 1.

The transition matrix will be: $\begin{array}{c} \begin{array}{cc} D & C \end{array} \\ \begin{bmatrix} 0.65 & 0.45 \\ 0.35 & 0.55 \end{bmatrix} \end{array} \begin{array}{c} \\ D \\ C \end{array}$

Question 2

$S_4 = T^4 \times S_0 = \begin{bmatrix} 68.125 \\ 31.875 \end{bmatrix}$

32% of customers bought their coffee at Giorgio's on Friday.

Question 3

Use the transition diagram to set up a transition matrix to identify the correct answer.

$T = \begin{bmatrix} 0.55 & 0.25 & 0.35 \\ 0.45 & 0.60 & 0.25 \\ 0 & 0.15 & 0.40 \end{bmatrix}$

Question 4

$S_1 = T \times S_0 = \begin{bmatrix} 30j + 40l + 6 \\ 20m + 18 \\ 40n + 16 \end{bmatrix} = \begin{bmatrix} 42 \\ 28 \\ 20 \end{bmatrix}$

Therefore:

$20m + 18 = 28$, so $m = 0.5$

$40n + 16 = 20$, so $n = 0.1$

From the transition matrix where the columns must add to 1, $j = 0.4$ and $l = 0.6$.

Now go through the alternatives to determine which one is true.

$$\begin{aligned} m + n &= 0.5 + 0.1 \\ &= 0.6 \\ &= l \end{aligned}$$

Question 5

From 9 am to 5 pm are 16 half-hour increments.

Math input error

By the end of the day, 14% of these children are expected to be at the slide.

VCAA Examination Report **note:**

Students needed to recognise that this question involved 'long term expectation'.

Although the number of half-hours that the group stays is not defined, it makes little difference whether this is taken to be 10,15,20 or higher. There is no initial state needed.

Question 6

To find the long-term distribution of library members, raise the transition matrix to a large power.

$$T^{50} = \begin{bmatrix} 0.0979\ldots & 0.0979\ldots & 0.0979\ldots & 0.0979\ldots & 0.0979\ldots \\ 0.1921\ldots & 0.1921\ldots & 0.1921\ldots & 0.1921\ldots & 0.1921\ldots \\ 0.2485\ldots & 0.2485\ldots & 0.2485\ldots & 0.2485\ldots & 0.2485\ldots \\ 0.3021\ldots & 0.3021\ldots & 0.3021\ldots & 0.3021\ldots & 0.3021\ldots \\ 0.1593\ldots & 0.1593\ldots & 0.1593\ldots & 0.1593\ldots & 0.1593\ldots \end{bmatrix}$$

This shows the proportion of members ending up in each category.

96 members is $\frac{96}{500} = 0.192$ as a proportion. This matches with category K.

In the long term category K is expected to have 96 members.

Question 7

Matrix B must balance what is lost/gained if the numbers are to remain constant.

$$F_1 = T \times F_0$$

$$= \begin{bmatrix} 32500 \\ 20000 \\ 8650 \\ 5850 \end{bmatrix}$$

From this matrix and the initial-state matrix, we can see that the number of young fish has decreased from 50 000 to 32 500; the number of juveniles has increased from 10 000 to 20 000; the number of adults has increased from 7000 to 8650, and 5850 died in total.

So, to balance these changes, 17 500 young fish need to be added each month, 10 000 juveniles need to be sold and 1650 adults need to be sold.

VCAA Examination Report **note:**

Many students struggled to answer this question correctly. The critical information in the question was the fact that the number of each type of fish in the farm remained constant, so that the matrices F_0 and F_1 were the same.

Question 8

Matrix T is a regular transition matrix, so the columns must add to 1.

This means that $X = 0.5$ and $Y = 0.6$.

The recurrence relation becomes $\begin{bmatrix} 0.3 & 0.2 & V \\ 0.2 & 0.2 & W \\ 0.5 & 0.6 & Z \end{bmatrix} \times \begin{bmatrix} 40 \\ 15 \\ 20 \end{bmatrix} = \begin{bmatrix} 29 \\ 13 \\ 33 \end{bmatrix}$

Multiplying out, $0.3 \times 40 + 0.2 \times 15 + V \times 20 = 29$ and solving for V, $V = 0.7$

$0.2 \times 40 + 0.2 \times 15 + W \times 20 = 13$ and solving for W, $W = 0.1$

$0.5 \times 40 + 0.6 \times 15 + Z \times 20 = 33$ and solving for Z, $Z = 0.2$

From $V = 0.7$, $W = 0.1$ and $Z = 0.2$, it can be seen that option A is not true.

***VCAA Examination Report* note:**
Many students seemed to struggle with the complexity of this question. Two of the unknown values in the matrix T could be found from the elementary concept of transition matrices – that is, the column values must add to one.

Question 9

$$\begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{bmatrix}^{70} \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} \approx \begin{bmatrix} 105 \\ 110 \\ 86 \end{bmatrix}$$

In the long term, more students will choose to do music than sport.
Note: 70 weeks has been chosen at random.

Question 10
Follow the transitions. If today she buys carnations, tomorrow she will buy daisies, so there will be a 1 in row 5, column 3. From that we eliminate all options except D and E. So if today she buys daisies, tomorrow she will buy irises, so there will be a 1 in row 4, column 5. That just leaves option D, which is the answer.

Question 11
The loops describe the percentage of customers who stay with a particular brand the next day.

$$S_0 = \begin{array}{c} M \\ S \end{array}\begin{bmatrix} 100 \\ 150 \end{bmatrix}$$

$$T = \begin{array}{c} M \\ S \end{array}\begin{bmatrix} 0.6 & 0.35 \\ 0.4 & 0.65 \end{bmatrix}$$

Question 12
Looking at the matrix, the 1 in the columns signifies the next person Wendy has lunch with.
Starting with C (Craig): E – D – B – A – C
Therefore, the order after Craig is Edgar, Daniel, Betty then Angela.

Question 13

$$\begin{aligned} S_2 &= TS_1 \\ T^{-1}S_2 &= T^{-1}TS_1 \\ S_1 &= T^{-1}S_2 \\ &= \begin{bmatrix} 400 \\ 0 \\ 200 \end{bmatrix} \end{aligned}$$

Question 14

$$\begin{aligned} S_0 &= \begin{bmatrix} 200 \\ 200 \\ 200 \end{bmatrix} \\ \text{long term} &= T^{100} \times S_0 \\ &= \begin{bmatrix} 171.4 \\ 203.6 \\ 225 \end{bmatrix} \end{aligned}$$

Therefore, the number of Shirley's customers is closest to 170 in the long term.

Question 15

$S_1 = TS_0 - C$

$$S_1 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix} \times \begin{bmatrix} 100 \\ 250 \end{bmatrix} - \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 180 \\ 130 \end{bmatrix}$$

$S_2 = TS_1 - C$

$$S_2 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix} \times \begin{bmatrix} 180 \\ 130 \end{bmatrix} - \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 148 \\ 122 \end{bmatrix}$$

Question 16

Find the steady state:

$$S_{50} = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.8 \end{bmatrix}^{50} \times \begin{bmatrix} 84 \\ 96 \\ 81 \end{bmatrix} = \begin{bmatrix} 87 \\ 58 \\ 116 \end{bmatrix}$$

$$S_{51} = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.8 \end{bmatrix}^{51} \times \begin{bmatrix} 84 \\ 96 \\ 81 \end{bmatrix} = \begin{bmatrix} 87 \\ 58 \\ 116 \end{bmatrix}$$

$S_{50} = S_{51}$

Therefore, steady state is achieved. In the long term, 87 customers will buy Brazilian coffee, 58 will buy Italian coffee and 116 will buy Kenyan coffee.

Question 17

The only transition of 0.4 is from M to F. Thus, M is top left and F is top right of the diagram.

From M there is a transition of 0.1 down. The matrix shows a transition of 0.1 from M to T; thus, T is the bottom left vertex.

This leaves L as the bottom right vertex.

x is the transition from L to L. Reading from the matrix, the transition x equals 0.6.

Question 18

Note that there is no transition from C to T, so that element will be 0. Otherwise, set up the matrix and follow the transitions.

Question 19

The transition matrix is then

$$\begin{array}{cc} & \text{today} \\ & \begin{array}{cc} d & w \end{array} \\ T = & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{array}{c} d \\ w \end{array} \text{ tomorrow} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} d & w \end{array} \\ T = & \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix} \begin{array}{c} d \\ w \end{array} \end{array}$$

Question 20

Each of the columns in a transition matrix add to 1.

$$\begin{array}{cc} T = & \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.5 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.4 \end{bmatrix} \\ & \begin{array}{ccc} 1 & 1 & 1 \end{array} \end{array}$$

Question 21

Each of the columns in a transition matrix must sum to 1.

Disregard answers A and B.

The information in the question must be read very carefully and a transition matrix should be created where

$$T = \begin{array}{c} \text{this week} \\ \begin{array}{ccc} A & B & C \end{array} \\ \left[\begin{array}{ccc} \quad & \quad & \quad \\ & & \\ & & \end{array}\right] \end{array} \begin{array}{l} A \\ B \\ C \end{array} \text{ next week}$$

$$T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.6 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.1 & 0.5 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \\ C \end{array}$$

Question 22

The initial state matrix should contain information about the starting state of the number of apples and oranges. Disregard options A, B and E.

Question 23

The transition matrix must show the proportions and not the percentages. Disregard option E.

The transition matrix is set up as

$$\text{to} \begin{array}{c} A \\ O \end{array} \begin{array}{c} \text{from} \\ \begin{array}{cc} A & O \end{array} \\ \left[\begin{array}{cc} \quad & \quad \\ & \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ A \\ O \end{array} \begin{array}{c} \begin{array}{cc} A & O \end{array} \\ \begin{bmatrix} 0.25 & 0.55 \\ 0.75 & 0.45 \end{bmatrix} \end{array}$$

75% of apple purchasers purchase oranges next time. There should be 0.75 from A to O.

Question 24

$$\begin{aligned} s_1 &= T \times s_0 \\ &= \begin{bmatrix} 0.65 & 0.45 \\ 0.35 & 0.55 \end{bmatrix} \begin{bmatrix} 60 \\ 75 \end{bmatrix} \\ &= \begin{bmatrix} 72.75 \\ 62.25 \end{bmatrix} \end{aligned}$$

Question 25

$$\begin{aligned} s_5 &= T \times s_4 \\ &= \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix} \begin{bmatrix} 15 \\ 27 \end{bmatrix} \\ &= \begin{bmatrix} 19.2 \\ 22.8 \end{bmatrix} \end{aligned}$$

Question 26

$$\begin{aligned} s_5 &= T \times s_4 \\ &= T \times T \times s_3 \\ &= T^2 \times s_3 \end{aligned}$$

Question 27

The answer requires repeated calculations of the recursive equation of $\begin{bmatrix} S_{n+1} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix} \begin{bmatrix} S_n \end{bmatrix} + \begin{bmatrix} 5 \\ 7 \end{bmatrix}$.

A calculator should be used to assist and to complete these calculations efficiently.

The first entry into the calculator would be $\begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix} \begin{bmatrix} 120 \\ 30 \end{bmatrix} + \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ giving an answer of $\begin{bmatrix} S_1 \end{bmatrix} = \begin{bmatrix} 102.5 \\ 59.5 \end{bmatrix}$

The second entry would be $\begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix} \begin{bmatrix} 102.5 \\ 59.5 \end{bmatrix} + \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ giving an answer of $\begin{bmatrix} S_2 \end{bmatrix} = \begin{bmatrix} 103.525 \\ 70.475 \end{bmatrix}$

This continues until the fourth entry is made giving an answer of $\begin{bmatrix} S_4 \end{bmatrix} = \begin{bmatrix} 116.00 \\ 82.00 \end{bmatrix}$

Rounded to the nearest whole number, the time spent running after 4 weeks of the program is 82 minutes.

8.8 Review

Question 1

a. Brie and Dex – in the matrix, they both have 1's to send to each other. **[1 mark]**

b. Elena → Dex → Brie → Chai **[1 mark]**

c. Alex → Brie → Dex and Alex → Elena → Dex [Award **1 mark** for both paths]
[Note: The matrix M^2 has been transcribed incorrectly. If students calculated M^2 themselves, they also found that there was 2-step communication possible between Chai and Brie, so the answers of $C \to A \to B$ and $C \to D \to B$ (must have both) were also accepted]

Question 2

a. One row and three columns means the order is 1×3. **[1 mark]**

b. $\dfrac{700}{0.25} = 2800$ shoppers **[1 mark]**

c. i. $Q = \begin{bmatrix} 1104 & 1296 & 1056 \\ 621 & 729 & 594 \\ 575 & 675 & 550 \end{bmatrix}$

Award **1 mark** for both missing numbers.

ii. 594 shoppers were in the clothing area at Westmall. **[1 mark]**

d. Total amount $= \begin{bmatrix} 135 & 143 & 131 \end{bmatrix} \times \begin{bmatrix} 21.30 \\ 34.00 \\ 14.70 \end{bmatrix} = [9663.20]$ **[1 mark]**

e. If A_{2019} is a 3×1 matrix, then A_{2020} must be too.

$A_{2020} = K \times A_{2019}$

$(3 \times 1) = (m \times n) \times (3 \times 1)$

So K will be a 3×3 matrix.

$K = \begin{bmatrix} 1.05 & 0 & 0 \\ 0 & 0.85 & 0 \\ 0 & 0 & 0.99 \end{bmatrix}$ **[1 mark]**

Question 3

$$\begin{aligned} C^T \times \left(A^T \times B\right)^T &= (3 \times 4)^T \times \left((2 \times 4)^T \times (2 \times 3)\right)^T \\ &= (4 \times 3) \times ((4 \times 2) \times (2 \times 3))^T \\ &= (4 \times 3) \times (4 \times 3)^T \\ &= (4 \times 3) \times (3 \times 4) \\ &= (4 \times 4) \end{aligned}$$

Question 4

a. If the determinant of a matrix is zero, there is no inverse. But the determinant is 1; therefore, there is an inverse. **[1 mark]**

b. $\begin{bmatrix} -7 & 9 \\ -4 & 5 \end{bmatrix}$

Award **1 mark** for all three correct numbers.

c. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Therefore, 2 sandwich bars are preferred in Grandmall's food court. **[1 mark]**

Question 5

a. 3×2 **[1 mark]**

b. Add up column 2:$50 + 20 + 40 = 110$ **[1 mark]**

c. $L = \begin{bmatrix} 50 \\ 20 \\ 40 \end{bmatrix}$ **[1 mark]**

d. $R^T = \begin{bmatrix} 3 & 6 & 22 & 19 \\ 1 & 10 & 7 & 2 \\ 1 & 3 & 10 & 26 \end{bmatrix}$ **[1 mark]**

e. The number of cars that parked for two hours in area C. **[1 mark]**

***VCAA Examination Report* note:**

Some students recognised area C and 2 hours but did not refer to the number of cars parked.

Question 6

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

[**1 mark** was awarded for 2 lines correct]

[Award **2 marks** for all correct]

Question 7

You'll need to go through each option.

A: Not defined, because the matrix multiplication AB is not possible because the number of columns in A does not match the number of rows in B.

B: Not defined, because although the product of BA is defined, the resultant matrix is not square, therefore an inverse is not possible.

C: Not defined, because matrix B is not square and so squaring it is not possible.

D: Not defined, because the subtraction of the matrices is not possible as they are of a different order.

E: IS defined because the order of B^T will be 7×10, which means that the number of columns in A is the same as thenumber of rows in B^T, so the product is possible.

Question 8

w is the transition from A to D, located in the fourth row, first column, so $w = 15\%$

Question 9

First, calculate the matrix C:

$S_{n+1} = T \times S_n - C$

$S_1 = T \times S_0 - C$

$$\begin{bmatrix} 24.0 \\ 54.3 \\ 20.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.5 \end{bmatrix} \times \begin{bmatrix} 21 \\ 51 \\ 31 \end{bmatrix} - C$$

$$C = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Therefore, we can now calculate S_2:

$S_2 = T \times S_1 - C$

$$= \begin{bmatrix} 0.6\ 0.1\ 0.3 \\ 0.3\ 0.8\ 0.2 \\ \\ 0.1\ 0.1\ 0.5 \end{bmatrix} \times \begin{bmatrix} 24.0 \\ 54.3 \\ 20.7 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 23.04 \\ 55.78 \\ 16.18 \end{bmatrix}$$

Question 10

State matrix for Monday: $\begin{bmatrix} 50 \\ 50 \\ 0 \end{bmatrix}$

State matrix for Tuesday: $T \times \begin{bmatrix} 50 \\ 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 35 \\ 40 \end{bmatrix}$

The percentage not expected to change from Tuesday to Wednesday is:

$(0.4 \times 25) + (0.5 \times 35) + (0.6 \times 40) = 51.5\%$

Question 11

a. There are 3 rows and 1 column, therefore, the order of matrix J is 3×1 **[1 mark]**

b. $k = 1 + \dfrac{5}{100} = 1.05$ **[1 mark]**

Question 12

a. $0.15 \times 3200 + 0.2 \times 2000 + 0.1 \times 2800 = 1160$ **[1 mark]**

b. $S_1 = \begin{bmatrix} 3060 \\ 1900 \\ 3040 \end{bmatrix}$ **[1 mark]**

c. $\dfrac{0.85 \times 3200}{3060} \times 100\% = 89\%$ [**1 mark** – whole number rounding applies]

d. Note that this is a 'show that' question, so the calculation must be explicitly shown and the answer written. There were two ways to do this:

Method 1: $T^n = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$ for a sufficiently large value of n.

Then stating ' 0.5 indicates 50% for Ohana olive oil'

[Award **1 mark** for showing both matrix and statement]

Method 2: Math input errorfor a sufficiently large value of n.

Then showing the calculation: $\dfrac{4000}{8000} \times 100 = 50\%$

[Award **1 mark** for showing both matrix calculation and ratio]

e. $R_1 = T \times R_0 + B = \begin{bmatrix} 3260 \\ 2000 \\ k + 3040 \end{bmatrix}$

$R_2 = T \times R_1 + B$

$\begin{bmatrix} 3333 \\ 2025 \\ 3642 \end{bmatrix} = T \times \begin{bmatrix} 3260 \\ 2000 \\ k + 3040 \end{bmatrix} + B$

Solve this matrix equation on your CAS to find the value of k, or:

$0.1 \times 3260 + 0.1 \times 2000 + 0.9 \times (k + 3040) = 3642$

$k = 200$ **[1 mark]**

Question 13

a. $S_1 = T \times S_0 = \begin{bmatrix} 240\,700 \\ 231\,700 \\ 207\,600 \end{bmatrix}$ **[1 mark]**

b. 250 000 − 220 000 = 30 000 **[1 mark]**

c. $S_6 = T^6 \times S_0 = \begin{bmatrix} 233\,708 \end{bmatrix} G$, $S_1 = T^7 \times S_0 = \begin{bmatrix} 233\,710 \end{bmatrix} G$, $S_8 = T^8 \times S_0 = \begin{bmatrix} 233\,689 \end{bmatrix} G$

So the maximum number of shoppers will be in the seventh week. **[1 mark]**

d. $S_\infty = T^{50} \times S_0 = \begin{array}{c} \begin{bmatrix} 218884 \\ \\ \\ \end{bmatrix} \end{array} \begin{array}{c} W \\ G \\ E \end{array}$

In the long term, the expected weekly number of shoppers at Westmall is 218 884. **[1 mark]**

Question 14

a. $R_3 = T \times R_2 + B = \begin{array}{c} \begin{bmatrix} 237\,966 \\ \\ \\ \end{bmatrix} \end{array} \begin{array}{c} W \\ G \\ E \end{array}$

The expected number of shoppers at Westmall in the third week is 237 966. **[1 mark]**

b. $R_2 = T \times R_1 + B$

$R_2 - B = T \times R_1$

$R_1 = T^{-1} (R_2 - B)$

$$R_1 = \begin{bmatrix} 0.78 & 0.13 & 0.10 \\ 0.12 & 0.82 & 0.10 \\ 0.10 & 0.05 & 0.80 \end{bmatrix}^{-1} \times \left(\begin{bmatrix} 239\,060 \\ 250\,840 \\ 192\,900 \end{bmatrix} - \begin{bmatrix} -400 \\ 700 \\ 500 \end{bmatrix} \right)$$

$$R_1 = \begin{bmatrix} 241\,000 \\ 246\,000 \\ 195\,000 \end{bmatrix}$$

In the first week, 241 000 shoppers were expected at Westmall. **[1 mark]**

Question 15

a. $\begin{bmatrix} 0.3 & 0.4 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.4 \end{bmatrix} \times \begin{bmatrix} 500 \\ 600 \\ 500 \\ 400 \end{bmatrix} = \begin{bmatrix} 810 \\ 300 \\ 310 \\ 580 \end{bmatrix}$

∴ the answer is location A because it has 810 visitors. **[1 mark]**

b. i. Matrix $B_1 = \begin{bmatrix} -210 \\ 0 \\ 210 \\ 0 \end{bmatrix}$ since, $R_1 = \begin{bmatrix} 810 \\ 300 \\ 310 \\ 580 \end{bmatrix} + \begin{bmatrix} -210 \\ 0 \\ 210 \\ 0 \end{bmatrix} = \begin{bmatrix} 600 \\ 300 \\ 520 \\ 580 \end{bmatrix}$ **[1 mark]**

***VCAA Examination Report* note:**

This question was poorly done, with many students apparently unable to identify what was required.

ii. $VR_1 = \begin{bmatrix} 0.3 & 0.4 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.4 \end{bmatrix} \times \begin{bmatrix} 600 \\ 300 \\ 520 \\ 580 \end{bmatrix} = \begin{bmatrix} 786 \\ 288 \\ 282 \\ 644 \end{bmatrix}$

$R_2 = \begin{bmatrix} 786 \\ 288 \\ 282 \\ 644 \end{bmatrix} + \begin{bmatrix} -186 \\ 0 \\ 230 \\ -44 \end{bmatrix} = \begin{bmatrix} 600 \\ 288 \\ 512 \\ 600 \end{bmatrix}$ **[1 mark]**

***VCAA Examination Report* note:**

Many students did not attempt this question. A few responses gave the matrix B_2 rather than the state matrix R_2.

Question 16

a. $\frac{400}{2000} \times 100\% = 20\%$ **[1 mark]**

b. $S_1 = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.4 \end{bmatrix} \times \begin{bmatrix} 600 \\ 600 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 300 \\ 780 \\ 300 \\ 620 \end{bmatrix}$ **[1 mark]**

c. $0.1 \times 600 + 0.2 \times 600 = 180$

$\frac{180}{300} \times 100 = 60\%$ **[1 mark]**

***VCAA Examination Report* note:**

Many responses were incorrect. A few correctly worked out the individual percentages from *A* or *F* but did not give an overall percentage.

d. $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **[1 mark]**

***VCAA Examination Report* note:**

Many responses were incorrect. Some gave the identity matrix; others gave various assortments of 1s and 0s.

Question 17

a. The cost of one journey on section G is \$2.87. **[1 mark]**

b. Matrix C has an order of 3×1. **[1 mark]**

c. The final matrix has an order of 1×1; therefore, M has an order of 1×3. For E to be travelled along once, G to be travelled along twice, and F not to be travelled on, C must be multiplied by [1 0 2].

$M = [1\ 0\ 2]$[**1 mark**]

***VCAA Examination Report* note:**

Some students gave the correct numbers but in a column matrix.

Question 18

a. $M_{2019} = M_{2018} \times 1.04 = 2184$

$V_{2019} = V_{2018} \times 0.99 = 1782$

$R_{2019} = R_{2018} \times 0.98 = 1666$

$P_{2019} = \begin{bmatrix} 2184 \\ 1782 \\ 1666 \end{bmatrix}$ **[1 mark]**

***VCAA Examination Report* note:**

Some students gave only one or two of the correct elements.

b. F is a diagonal matrix with the values used in **part a** along the diagonals.

$F = \begin{bmatrix} M\% & 0 & 0 \\ 0 & V\% & 0 \\ 0 & 0 & R\% \end{bmatrix} = \begin{bmatrix} 1.04 & 0 & 0 \\ 0 & 0.99 & 0 \\ 0 & 0 & 0.98 \end{bmatrix}$ **[1 mark]**

***VCAA Examination Report* note:**

Most students recognised that a 3×3 matrix was required. Numbers such as $0.04, -0.01 < /span >$ and -0.02 were often included in students' answers.

Question 19

a. The value of 2019's resurfacing requirements based on 2018's graded distance is $0.1 \times 700 = 70$ km. **[1 mark]**

***VCAA Examination Report* note:**

Many students gave 390 as their answer, indicating that they had not read the question fully. Some gave 10% or 0.1 rather than finding the length.

b. The sections to be regraded in 2019 are

$GG \times G_{2018} + RG \times R_{2018} + SG \times S_{2018} + NG \times N_{2018}$

$= 0.2 \times 700 + 0.1 \times 400 + 0.0 \times 200 + 0.2 \times 1400 = 460$ **[1 mark]**

***VCAA Examination Report* note:**

Some students listed the numbers in the G column rather than the row.

c. $S_1 = \begin{bmatrix} 460 \\ 0.1 \times 700 + 0.1 \times 400 + 0 \times 200 + 0.2 \times 1400 \\ 0.2 \times 700 + 0.1 \times 400 + 0.2 \times 200 + 0.1 \times 1400 \\ 1490 \end{bmatrix} = \begin{bmatrix} 460 \\ 390 \\ 360 \\ 1490 \end{bmatrix}$ **[1 mark]**

d. The percentage of no maintenance continuing from 2019 into 2020 is

$\dfrac{NN}{GN + RN + SN + NN} = \dfrac{0.5 \times 1490}{1536} = 48.5\%$. **[1 mark]**

***VCAA Examination Report* note:**

This question was not answered well, with most students giving 97%, which did not take account of the 0.5 change.

e. The matrix at 49 transitions is approximately equal to the matrix at 50 transitions. In the long run, the matrix becomes

$$\begin{bmatrix} 0.160 & 0.160 & 0.160 & 0.160 \\ 0.144 & 0.144 & 0.144 & 0.144 \\ 0.129 & 0.129 & 0.129 & 0.129 \\ 0.567 & 0.567 & 0.567 & 0.567 \end{bmatrix}$$

Therefore, the percentage of highway expected to have no maintenance in the long run is 56.7%. **[1 mark]**

***VCAA Examination Report* note:**

Some students left the answer as 1532.1 and did not convert to the required percentage.

Question 20

a. The total number of kilometres in M_1 must equal 2700. Therefore, matrix B cannot add or remove any kilometres from the total.
This means the sum of the values in B must equal 0.
$k + 20 + 10 - 60 = 0$
$k = 60 - 20 - 10 = 30$ **[1 mark]**
***VCAA Examination Report* note:**
Many students did not recognise that the 2700 km of highway remained constant throughout.

b. Kilometres graded in $0.2 \times 500 + 0.1 \times 400 + 0.2 \times 1500 + 30 = 470$
Kilometres graded in $0.2 \times 470 + 0.1 \times 410 + 0.2 \times 1460 + 30 = 457$ km **[1 mark]**
***VCAA Examination Report* note:**
Students who found k correctly in **part a.** were often able to answer this question correctly. Some wrote the correct matrix for 2022 only and did not extract the number 457 as their answer.

Question 21

a. $60 + 43 + 56 = 159$ sandwiches sold in the three weeks. **[1 mark]**
***VCAA Examination Report* note:**
This question was answered well, although a few students gave 56 as their answer, which was the number sold in week 3.

b. Element m_{12} indicates the value in row 1 and column 2.
That is, 24 rolls were sold in week 1. **[1 mark]**

c. i. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 35 & 24 & 60 \\ 28 & 32 & 43 \\ 32 & 30 & 56 \end{bmatrix}^{-1} \times \begin{bmatrix} 491.55 \\ 428.00 \\ 487.60 \end{bmatrix}$

$= \begin{bmatrix} 4.65 \\ 4.2 \\ 3.8 \end{bmatrix}$

The cost of one sandwich, c, is equal to \$3.80. **[1 mark]**

ii. For the matrix multiplication to be defined, the order of matrix L must be 1×3.
L must be a summing matrix that does not include the values for pies (P).

Question 22

a. $P \rightarrow S = 40\%$ **[1 mark]**

b. $300 \times 0.40 + 240 \times 0.60 + 210 \times 0.20 = 306$ **[1 mark]**

c. The same amounts as for all the terms, that is, $300 + 240 + 210 = 750$ students. **[1 mark]**
VCAA Examination Report note:
This question did not require any working if students realised that the original population of 750 remained unchanged throughout the transition process.
Those who calculated the figures for Term 4 often ended up with a total of 749 due to rounding or did not give the final answer as a total.

Question 23

a. Students who do not change from term to term are on the main diagonal of the transition matrix.
40% of 300+40% of 200+30% of 200+20% of 300
$= 120 + 80 + 60 + 60$
$= 320$ students **[1 mark]**

b.
$S_2 = T \times S_1$

$$= \begin{bmatrix} 0.4 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.3 & 0.2 \end{bmatrix} \times \begin{bmatrix} 300 \\ 200 \\ 200 \\ 300 \end{bmatrix}$$

$$= \begin{bmatrix} 250 \\ 250 \\ 300 \\ 200 \end{bmatrix}$$ **[1 mark]**

***VCAA Examination Report* note:**

This question was answered quite well. Students who answered incorrectly tended to give the state matrix for Term 3.

c.
$S_3 = T \times S_2$

$$= \begin{bmatrix} 0.4 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.3 & 0.2 \end{bmatrix} \times \begin{bmatrix} 250 \\ 250 \\ 300 \\ 200 \end{bmatrix}$$

$$= \begin{bmatrix} 260 \\ 240 \\ 295 \\ 205 \end{bmatrix}$$

240 students are expected to choose investigation (I) in Term 3. **[1 mark]**

In Term 2,200 students chose service (S). From the transition matrix, 30% of students who choose service (S) in one term will change to investigation (I) in the next term.

30% of 200 = 60 students would expect to change from service (S) in Term 2 to investigation (I) in Term 3.

As a percentage, 60 of the 240 students expected choose investigation (I) in Term 3 chose service (S) in Term 2.

$\frac{60}{240} \times 100\%$

$= 25\%$ **[1 mark]**

***VCAA Examination Report* note:**

This question was not answered well. Some students were able to find either the 240 doing investigation or the 60 transitioning from service to investigation but were unable to link these together.

d. $S_1 = \begin{bmatrix} 300 \\ 200 \\ 200 \\ 300 \end{bmatrix}, S_2 = \begin{bmatrix} 250 \\ 250 \\ 300 \\ 200 \end{bmatrix}, S_3 = \begin{bmatrix} 260 \\ 240 \\ 295 \\ 205 \end{bmatrix}, S_4 = \begin{bmatrix} 261 \\ 239 \\ 294.5 \\ 205.5 \end{bmatrix}$

In Term 1, 200 students were in investigation (I). The expected numbers for Terms 2,3 and 4 respectively, are 250,240 and 239.

The maximum number expected is 250 (Term 2). **[1 mark]**

Question 24

a. There are 4 rows and 1 column, so the order is (4×1). **[1 mark]**

b. i. $J = G \times C = \begin{bmatrix} 40 & 25 & 15 & 30 \end{bmatrix} \begin{bmatrix} 85 \\ 38 \\ 24 \\ 53 \end{bmatrix} = [6300]$

***VCAA Assessment Report* note:**
The brackets needed to be included.

ii. Matrix J represents the total amount of booking fees collected that month. **[1 mark]**

Question 25

a. Dana can send a direct message to Ben and Elka, as there are 1's in those position in the matrix G. **[1 mark]**

b. Cheng can send a message through Amara and Dana to Elka. **[1 mark]**

Question 26

a. $S_1 = TS_0 = \begin{bmatrix} 0.65 & 0.25 & 0.25 & 0.50 \\ 0.15 & 0.60 & 0.20 & 0.15 \\ 0.05 & 0.10 & 0.25 & 0.20 \\ 0.15 & 0.05 & 0.30 & 0.15 \end{bmatrix} \begin{bmatrix} 520 \\ 320 \\ 80 \\ 80 \end{bmatrix} = \begin{bmatrix} 478 \\ 298 \\ 94 \\ 130 \end{bmatrix}$

Therefore, $d = 298$, $e = 94$ and $f = 130$. **[1 mark]**

b. $(0.65 \times 520) + (0.25 \times 320) + (0.25 \times 80) + (0.50 \times 80) = 478$ **[1 mark]**

***VCAA Assessment Report* note:**
A matrix product such as TS_0 does not show an understanding of how 478 is calculated.

c. $0.25 \times 80 = 20$ **[1 mark]**

***VCAA Assessment Report* note:**
A common incorrect answer was 94.

d. $\frac{0.65 \times 520}{478} \approx 0.707 \approx 71\%$ **[1 mark]**

***VCAA Assessment Report* note:**
Common incorrect answers were 65% and 94%.

e. i. 80 customers have been removed from the study, presumably because they have not made a booking within the past year or so. **[1 mark]**

***VCAA Assessment Report* note:**
Many students were unable to explain the −80 in matrix B.
The answer needed to refer to the removal from the study of 80 people selected from the no travel group (N) in 2017 and in 2018.
A common unacceptable response was, '80 people chose not to travel each year'.

ii. $R_{2017} = TR_{2016} + B = \begin{bmatrix} 699.65 \\ 501.45 \\ 176.8 \\ 102.1 \end{bmatrix}$

$R_{2018} = TR_{2017} + B = \begin{bmatrix} 755.385 \\ 536.4925 \\ 189.7475 \\ 118.375 \end{bmatrix}$ **[1 mark]**

In 2018, 190 customers are expected to choose sea travel. **[1 mark]**

***VCAA Assessment Report* note:**
Many students found the correct matrix R_{2018} but did not extract the answer 190 customers who chose sea travel.

Some students only found R_{2017}, and some of these students simply called it R_{2018} instead.
A method mark for this two-mark question may have been earned for a correct and labelled matrix R_{2017}, even if the final answer was incorrect.

Question 27

For the first five weeks, the equation will be:
$S_5 = T_1^4 S_1$
But the transition matrix changes after week 5 to T_2.
So in week 8:
$S_8 = T_2^3 \times S_5$
$S_8 = T_2^3 \times \left(T_1^4 \times S_1\right)$

***VCAA Assessment Report* note:**
In answering this question many students did not take into account that the transition matrix changed from T_1 to T_2 after week 5.
The following is one approach to answering this question.
Step 1: Find an expression for S_5 in terms of T_1 and S_1

$S_2 = T_1 S_1$

$S_3 = T_1 S_2 = T_1 (T_1 S_1) = T_1^2 S_1$

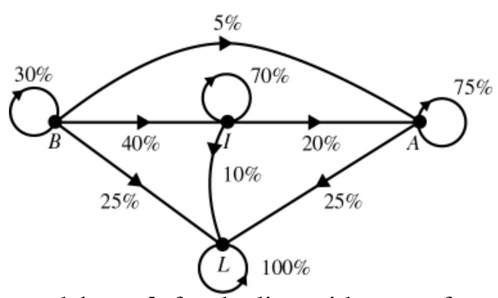
so $S_5 = T_1^4 S_1$
Step 2: The transition matrix changes to T_2 after week 5, so using S_5 as the new starting point:

$S_6 = T_2 S_5$ or $S_6 = T_2 \left(T_1^4 S_1\right)$

so $S_8 = T_2^3 \left(T_1^4 S_1\right)$ (option C)

Question 28

a.



ward **1 mark** for the line with arrow from I to L with 10%. Award **1 mark** for the loop at L with 100%.

***VCAA Assessment Report* note:**
A number of students missed the loop entirely, while others incorrectly labelled it as 1%. Some students incorrectly added a second 5% directed transition from B to A. Others unnecessarily added a lot of 0% transitions in the reverse directions for all shown transitions.

b. After the first assessment:

$$\begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix} \times \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 50 \\ 43 \\ 21 \end{bmatrix}$$

21 students leave the school, which is 17.5%. **[1 mark]**

***VCAA Assessment Report* note:**
Many students did not convert 21 students into the required percentage.

c. After two assessments:

$$\begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix}^2 \times \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 37.4 \\ 42.55 \\ 38.25 \end{bmatrix}$$

There will be 42 advanced level students, to the nearest lower whole number. **[1 mark]**

***VCAA Assessment Report* note:**

Some students rounded 42.55 to 42 despite the question instruction, 'Write your answer correct to the nearest whole number'.

d. Using CAS we can see that after four assessments:

$$\begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix}^4 \times \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.162 \\ 19.046 \\ 35.019 \\ 65.773 \end{bmatrix}$$

There are still approximately 54 students in the school. And after five assessments:

$$\begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix}^5 \times \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0486 \\ 13.397 \\ 30.0815 \\ 76.4727 \end{bmatrix}$$

There will only be approximately 43 students in the music school.

Therefore, the number of students will drop below 50 after five assessments. **[1 mark]**

e. $R_{n+1} = T_2 \times R_n + V$

$$R_1 = \begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix} \times \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 52 \\ 46 \\ 21 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix} \times \begin{bmatrix} 10 \\ 52 \\ 46 \\ 21 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 42.4 \\ 48.4 \\ 40.2 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 42.4 \\ 48.4 \\ 40.2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6.1 \\ 34.48 \\ 48.13 \\ 58.29 \end{bmatrix}$$

After three assessments, there are expected to be 34 intermediate students in the school.

After the next assessment, $0.2 \times 34 = 6.8$ so 7 intermediate students are expected to become advanced.

Award **1 mark** for calculations to three assessments, and **1 mark** for final answer.

***VCAA Assessment Report* note:**

Students were not expected to write out all of these matrix calculations to qualify for a method mark if their final answer was incorrect. The progression in these state matrices could be shown by using the correct equation to write the matrices R_1, R_2 and R_3.

Of those who attempted this question, many inappropriately used $R_3 = (T_2)^3 R_0 + V$, whereas $R_3 = TR_2 + V$ needed to be used instead.

Some students found and interpreted R_4 instead of R_3.

Question 29

To do a matrix multiplication, it must be rows x columns and the number of columns in matrix 1 must be the same as the number of rows in matrix 2. The other condition is that we must add up all the adult tickets separately to the child tickets.

Question 30

a. Add up the column matrix: $20 + 60 + 40 = 120$ music students **[1 mark]**

b. i.

$$Q = S_0 \times P$$

$$Q = \begin{bmatrix} 20 \\ 60 \\ 40 \end{bmatrix} \times \begin{bmatrix} 0.25 & 0.5 & 0.15 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 3 & 2 \\ 15 & 30 & 9 & 6 \\ 10 & 20 & 6 & 4 \end{bmatrix} \quad [1 \text{ mark}]$$

ii. Intermediate students are row 2 and Edith is column 2. So Edith teaches 30 intermediate students. **[1 mark]**

***VCAA Assessment Report* note:**

A common incorrect answer was 60 students.

c. i. amount paid $= C \times Q$ **[1 mark]**

ii. amount paid $= \begin{bmatrix} 15 & 25 & 40 \end{bmatrix} \times \begin{bmatrix} 5 & 10 & 3 & 2 \\ 15 & 30 & 9 & 6 \\ 10 & 20 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 850 & 1700 & 510 & 340 \end{bmatrix}$

Therefore, Geoff is paid \$340 each week. **[1 mark]**

Question 31

a. Students cannot be assessed at a level lower than their current level, so 100% of the advanced students stay at the advanced level. **[1 mark]**.

b. i.

$$S_1 = T_1 \times S_0$$

$$S_1 = \begin{bmatrix} 0.50 & 0 & 0 \\ 0.48 & 0.80 & 0 \\ 0.02 & 0.20 & 1 \end{bmatrix} \times \begin{bmatrix} 20 \\ 60 \\ 40 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 57.6 \\ 52.4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 58 \\ 52 \end{bmatrix} \text{ to the nearest whole number } \textbf{[1 mark]}$$

ii. $0.20 \times 60 = 12$ intermediate students become advanced after one assessment. **[1 mark]**

Question 32

a. order = rows $\times$ columns **[1 mark]**

$= 4 \times 2$

***VCAA Assessment Report* note:**

Some students reversed the two numbers, writing 2×4, which was not accepted.

b. Reading off the matrix in position $V_{4,1} = 1850$ adult males in the Western region. **[1 mark]**

c. The sum of the elements in the second column is the total number of adult females in the city. **[1 mark]**

d. $V \times P$
$= [4 \times 2] \times [2 \times 1]$
$= [4 \times 1]$
The number of columns in the first matrix, V, must be the same as the number of rows in the second matrix, P, in order for a matrix product to be defined.
[**1 mark** – must state the above sentence in some form]

e. $w = 1360 \times 0.45 + 1460 \times 0.55$
$= 1415$
[**1 mark** for first line]
***VCAA Assessment Report* note:**
Some students copied the full given matrix multiplication and replaced the w with 1415. This was not acceptable.

f. No.votes $= 1415 + 1812 + 988 + 1806$
$= 6021$
[1 mark]

Question 33

a. From the original T matrix, 10% of Mr Broad's supporters switched to Mr Choi. In the new T matrix, the percentage of supporters staying with Mr Broad has increased by 5% (80% → 85%).
Therefore half $\left(\frac{5\%}{10\%}\right)$ of the supporters that originally switched to Mr Choi have now decided to continue to support Mr Broad.
$\frac{5}{10} = 50\%$
[1 mark]
***VCAA Assessment Report* note:**
This question concerned the percentage the voters who would have changed their preferred candidate from Mr Broad to Mr Choi. From May to June, this represented 10% of the voters.
If Mr Choi withdrew from the election, the percentage of voters who stayed with Mr Broad as their preferred candidate would rise from 80% to 85%, an increase of 5%, as shown in T_1. This 5% is half (or 50%) of the voters who would have changed from Mr Broad to Mr Choi, had Mr Choi not withdrawn.
Many students seemed unable to identify the sub-group who were expected to change their votes from Mr Broad to Mr Choi. The most common incorrect answer was 5%, which represents the percentage of Mr Broad's total votes that reverted back to Mr Broad.

b. In May: (S_5)
$S_5 = T^4 S_1$
$= \begin{bmatrix} 4454 \\ 5154 \\ 2392 \end{bmatrix}$
[1 mark]
In June:
$S_6 = T^1 S_5$
$= \begin{bmatrix} 5549 \\ 6451 \\ 0 \end{bmatrix}$
Therefore Mr Broad receives 6451 votes.
[1 mark]

***VCAA Assessment Report* note:**
This question was poorly answered, with very few instances seen of working out that might earn a method mark where the final answer was incorrect. The most common error was $S_{\text{June}} = (T_1)^5 \times S_1$ or $S_{\text{June}} = (T_1)^6 \times S_1$ rather than $S_{\text{June}} = T_1 \times S_{\text{May}}$.

Question 34

a. i. Following the line from C to A shows that 20% of people who support Choi one month will switch to Aboud the next. **[1 mark]**

ii. 75% of voters will stay with Ms Aboud; therefore 25% will change each month. **[1 mark]**

b. 5% of Ms Aboud's supporters switch to Mr Broad.
40% of Mr Choi's supporters switch to Mr Broad.
Therefore the number of voters who will change to Mr Broad:
$= 6000 \times \dfrac{5}{100} + 2160 \times \dfrac{40}{100}$
$= 1164$

***VCAA Assessment Report* note:**
Many students were unable to answer this question correctly, often giving an answer without showing any working. The most common incorrect answer was 396, the increase in Mr Broad's total votes from January to February. This number includes an allowance for the loss of votes from Mr Broad to one of the other candidates in the month and is, therefore, fewer than the number of voters whose votes had changed to Mr Broad.

c. i. $S_3 = T^2 S_1$ **[1 mark]**

$= \begin{bmatrix} 4900 \\ 4634 \\ 2466 \end{bmatrix}$

ii. This matrix indicates the number of voters who support the three candidates in March.
4900 people support Ms Aboud.
4634 people support Mr Broad.
2466 people support Mr Choi.
[**1 mark** – only first line is required for the mark]

iii. June $\rightarrow S_6$

$S_6 = T^5 S_1$

$= \begin{bmatrix} 4334 \\ 5303 \\ 2363 \end{bmatrix}$

Therefore, the winner (Mr Broad) receives 5303 votes. **[1 mark]**

***VCAA Assessment Report* note:**
Many students used $S_{\text{June}} = S_6 = T^6 \times S_1$ to find the answer of 5404, which was incorrect.
The state matrix for January was given as S_1. Then,
$S_2 = T \times S_1, S_3 = T^2 \times S_1, S_4 = T^3 \times S_1, \ldots, S_n = T^{n-1} \times S_1$. The power by which the transition matrix must be raised is one less than the number of the required state matrix.
Some students wrote the complete matrix as their answer. This did not demonstrate an understanding of the correct required element in the matrix.

Question 35

From the table:
Richard plays football and tennis, so the entry for R will read 1, 0, 0, 1.
Brendon plays football, golf and soccer, so the entry for B will read 1, 1, 1, 0.
Lee plays tennis, so the entry for L will read 0, 0, 0, 1.
Arif plays golf and tennis, so the entry for A will read 0, 1, 0, 1.

Karl plays football and tennis, so the entry for K will read 1, 0, 0, 1.
The matching matrix is given in option C.
(In the answer options, note that the students are shown in columns and the sports are shown in rows, which is opposite of the layout of the table in the question.)

Question 36
Using a calculator, the correct option is B.

Question 37
$10b + 4d = 360$
$8b + 3d = 280$
Using matrices:

$$\begin{bmatrix} 10 & 4 \\ 8 & 3 \end{bmatrix} \times \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 360 \\ 280 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 8 & 3 \end{bmatrix}^{-1} \times \begin{bmatrix} 360 \\ 280 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -1.5 & 2 \\ 4 & -5 \end{bmatrix} \times \begin{bmatrix} 360 \\ 280 \end{bmatrix}$$

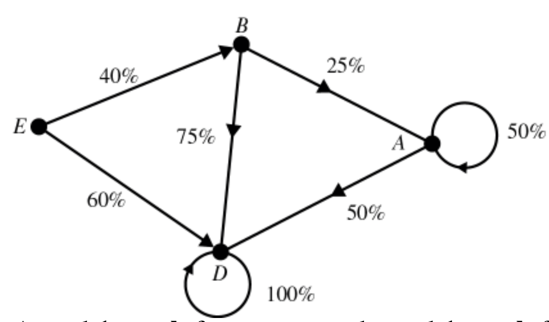
Question 38
Because $Q + S$ is not defined, Q and S must have different dimensions.
Let $Q = (a \times a)$; then R must be $(a \times b)$ and S must be $(b \times b)$, as the product QRS is defined and Q and S are both square matrices of different dimensions. The resultant matrix $P = (a \times b)$. Therefore, the expression $P \times S$ is defined, as the number of columns in P equals the number of rows in S (i.e. b).

Question 39
a. Sum of elements in row 3 = 1 + 0 + 0 + 1 + 0 = 2 **[1 mark]**
b. The sum of elements in row 3 shows that there are 2 pipes connected to pond R. **[1 mark]**
c.

$$N = \begin{array}{ccccc|c} P & Q & R & X & V & \\ 0 & 1 & 0 & 0 & 0 & P \\ 1 & 0 & 0 & 1 & 0 & Q \\ 0 & 0 & 0 & 1 & 0 & R \\ 0 & 1 & 1 & 0 & 1 & X \\ 0 & 0 & 0 & 1 & 0 & V \end{array}$$

[1 mark]

Question 40
a. i. $10\,000 \times 0.6 = 6000$; therefore, 6000 eggs die in the first year. **[1 mark]**
ii.



Award **1 mark** for correct paths and **1 mark** for correct percentages.

***VCAA Assessment Report* note:**
Students should not add edges marked 0% that are in the opposite direction to the given edges. Similarly, loops with 0% at E and B should not be included.

b. i. $S_1 = T \times S_0$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix} \times \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4000 \\ 650 \\ 7150 \end{bmatrix}$$ **[1 mark]**

ii. $S_4 = T^4 \times S_0$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix}^4 \times \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 331.25 \\ 11468.8 \end{bmatrix}$$

Therefore, there are expected to be 331 adult trout after 4 years. **[1 mark]**

c. i. $S_{12} = \begin{bmatrix} 0 \\ 0 \\ 1.29395 \\ 11798.7 \end{bmatrix}$ and $S_{12} = \begin{bmatrix} 0 \\ 0 \\ 0.646973 \\ 11799.4 \end{bmatrix}$

Therefore, the number of adult trout will be less than 1 (i.e. the trout will die out) after 13 years. **[1 mark]**

ii. The largest number of adult trout at any time will be 1325 (after 2 years). **[1 mark]**

***VCAA Assessment Report* note:**
It was not sufficient to assume that the number of adult trout was decreasing without checking the values given by S_2 and S_3.
A common incorrect answer was 800, taken directly from S_0. This may have been based on observing that S_1 had fewer (650) adult trout and S_4 had even fewer (331.25) adult trout. The assumption that this meant there was a decrease from S_0 through to S_4 should have been checked.

d. $$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix} \times \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix} + \begin{bmatrix} e \\ b \\ a \\ d \end{bmatrix} = \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4000 \\ 650 \\ 7150 \end{bmatrix} + \begin{bmatrix} e \\ b \\ a \\ d \end{bmatrix} = \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} e \\ b \\ a \\ d \end{bmatrix} = \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 4000 \\ 650 \\ 7150 \end{bmatrix}$$

$$\begin{bmatrix} e \\ b \\ a \\ d \end{bmatrix} = \begin{bmatrix} 10\,000 \\ -3000 \\ 150 \\ -7150 \end{bmatrix}$$ **[1 mark]**

Therefore, at the end of each year, 10 000 eggs must be added, 3000 babies must be removed and 150 adult trout must be added so that the number of eggs, babies and adult trout in the population remains unchanged. **[1 mark]**

e. i. $S_1 = T \times S_0 + 500M \times S_0$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix} \times \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix} + 500 \times \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 200\,000 \\ 4000 \\ 650 \\ 7150 \end{bmatrix}$$ **[1 mark]**

ii. $S_2 = T \times S_1 + 500M \times S_1$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix} \times \begin{bmatrix} 200\,000 \\ 4000 \\ 650 \\ 7150 \end{bmatrix} + 500 \times \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 200\,000 \\ 4000 \\ 650 \\ 7150 \end{bmatrix}$$

$$= \begin{bmatrix} 162\,500 \\ 80\,000 \\ 1325 \\ 130\,475 \end{bmatrix}$$ **[1 mark]**

Therefore, there will be 162 500 eggs after 2 years. **[1 mark]**

***VCAA Assessment Report* note:**

Using the equation given, the state matrix for each year is calculated by using the state matrix from the year before. Many students instead found $S_2 = T^2S_0 + 500MS_0$ rather than using S_1 to determine S_2.

Of those who did a correct calculation, some left the matrix S_2 as their answer. The number of eggs had to be extracted from matrix S_2 for full marks.

Question 41
The order of a matrix is *number of rows* × *number of columns*.
There are 3 rows and 4 columns.

Question 42
A null matrix contains only zeros.

Question 43
Via substitution of the relevant row and column values the following calculations occur:
$a_{11} = 2(1) - 3(1) = -1,\ a_{12} = 2(1) - 3(2) = -4,\ a_{13} = 2(1) - 3(3) = -7$
$a_{21} = 2(2) - 3(1) = 1,\ a_{22} = 2(2) - 3(2) = -2,\ a_{23} = 2(2) - 3(3) = -5$
$a_{31} = 2(3) - 3(1) = 3,\ a_{32} = 2(3) - 3(2) = 0,\ a_{33} = 2(3) - 3(3) = -3$

Question 44
Since $a_{ij} = 3$, it can be concluded $a_{11} = 3$, $a_{22} = 3$ and $a_{33} = 3$ (the leading diagonal positions)
All other values are zero; therefore, the matrix must be $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Question 45
The transpose of a matrix interchanges the rows and columns.
Upper triangular matrices have zeros in all positions to the right of leading diagonal entries.
When transposed, these zeros will be positioned below the leading diagonal entries.
Therefore the transposed matrix will always be a lower triangular matrix.
$U = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ (upper triangular matrix),
then $U^T = \begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$ (lower triangular matrix)

Question 46
The property of $A^T = A$ exists when A is a symmetrical matrix.
$A^T A = A^2$ is true since $A^T = A$
AAlso $\left(A^T\right)^T = A$ is true since $A^T = A$
A symmetrical matrix can only be created from a square matrix.
Therefore $A^T A^T = I$ is not always true.

Question 47
A lower triangular matrix has zeros below the leading diagonal making $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ the only matrix that satisfies this property.

Question 48
The matrix has all the above properties.

Question 49
The order of $BA = (2 \times 3) \times (3 \times 3)$.
The product will be (2×3). (Two rows and three columns.) Answer possibilities are either A or C.
Use a calculator to multiply the matrices.

Question 50

$A^3 = (A \times A) \times A$

Use a calculator to evaluate

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^3 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix}$$

Question 51

A dominance matrix has only zeros on the main diagonal. Disregard options B, D and E.

Player A is dominant over player B, so there should be a 1 in row A, column B and a 0 in row B, column A. Disregard option A.

Question 52

No player is dominant over themselves, so there should be only zeros on the main diagonal. Disregard option C.

Player P is dominant over player Q, so there should be a 1 in row P, column Q. Disregard option A.

Since player P is dominant over player Q, there should be a 0 in row Q, column P. Disregard option B.

Matrix options D and E differ only in row S. Player S is dominant over player Q only.

Question 53

Determine the adjacency matrix (which shows one-step dominance). Two-step dominance is calculated by squaring the one-step dominance matrix.

$$M^1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, M^2 = \left(M^1\right)^2$$

Add one- and two-step dominance matrices to find overall dominance matrix M.

Calculate the sum of each row to determine the player's dominance.

$M^1 + M^2 = M$

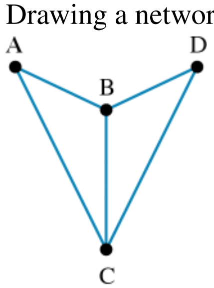
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{array}{c} P \\ Q \\ R \\ S \end{array} \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{array}{c} 4 \\ 3 \\ 5 \\ 2 \end{array}$$

Question 54

The symmetrical nature of the matrix indicates the two-way connection that occurs between each vertex. As such, each connection is recorded twice e.g. A – B is recorded as well as B – A, but this is only one connection. As there are 10 listings in the matrix, there would be 5 connections only in the network.

Question 55

Drawing a network to comply with this communication matrix would look like:



Using the indicated connections in this diagram it is clear the path C – A – D – B does not exist.

Question 56

All of the people using Brand A remain with Brand A. Some of the people using brand B will always change over to Brand A where they will remain. Brand B will gradually decrease to zero.

Question 57

$$s_{50} = T^{50} \times s_0$$
$$= \begin{bmatrix} 0.65 & 0.45 \\ 0.35 & 0.55 \end{bmatrix}^{50} \begin{bmatrix} 60 \\ 75 \end{bmatrix}$$
$$= \begin{bmatrix} 75.9375 \\ 59.0625 \end{bmatrix}$$

Question 58

Test a consecutive pair of high number transitions. If the matrix for both transitions is the same, steady state has been reached.

$$s_{50} = \begin{bmatrix} 0.6 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.1 & 0.5 \end{bmatrix}^{50} \begin{bmatrix} 300 \\ 250 \\ 425 \end{bmatrix} = \begin{bmatrix} 438.75 \\ 227.5 \\ 308.75 \end{bmatrix}$$
$$s_{51} = \begin{bmatrix} 0.6 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.1 & 0.5 \end{bmatrix}^{51} \begin{bmatrix} 300 \\ 250 \\ 425 \end{bmatrix} = \begin{bmatrix} 438.75 \\ 227.5 \\ 308.75 \end{bmatrix}$$

Steady state has been reached.

The matrix represents the number of people and therefore the answers must be whole numbers. Round each of the elements of the matrix to the nearest whole number.

Question 59

Test a consecutive pair of high number transitions. If the matrix for both transitions is the same, steady state has been reached.

$$s_{50} = \begin{bmatrix} 0.25 & 0.55 \\ 0.75 & 0.45 \end{bmatrix}^{50} \begin{bmatrix} 85 \\ 120 \end{bmatrix} = \begin{bmatrix} 86.7308 \\ 118.269 \end{bmatrix}$$
$$s_{51} = \begin{bmatrix} 0.25 & 0.55 \\ 0.75 & 0.45 \end{bmatrix}^{51} \begin{bmatrix} 85 \\ 120 \end{bmatrix} = \begin{bmatrix} 86.7308 \\ 118.269 \end{bmatrix}$$

Steady state has been reached.

The matrix represents the number of people and therefore the answers must be whole numbers. Round each of the elements of the matrix to the nearest whole number.

$$s_n = \begin{bmatrix} 87 \\ 118 \end{bmatrix}$$

The transition matrix is set up as

$$\begin{array}{cc} & \text{from} \\ & \begin{array}{cc} A & O \end{array} \\ \text{to} \begin{array}{c} A \\ O \end{array} & \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$

so the state matrixes will have the same labels for their rows:

$$s_n = [\,] \begin{array}{c} A \\ O \end{array}$$

The number of oranges is 118

9 Undirected graphs, networks and trees

Topic	9	Undirected graphs, networks and trees
Subtopic	9.2	Basic concepts of a network

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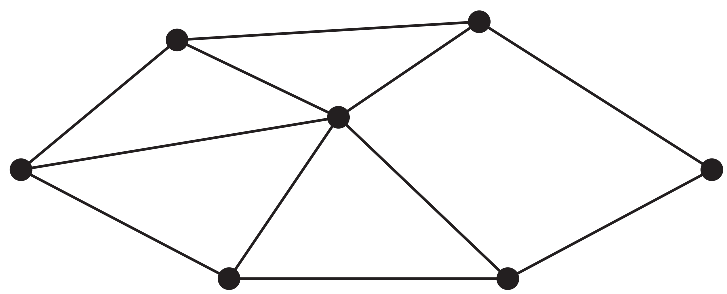
Source: *VCE 2021, Further Mathematics 1, Section B, Module 2, Q.1; © VCAA*

Question 1 (1 mark)

Consider the graph below.



The number of vertices with a degree of 3 is

A. 1
B. 2
C. 3
D. 4
E. 5

Source: *VCE 2020, Further Mathematics 1, Section B, Module 2, Q.8; © VCAA*

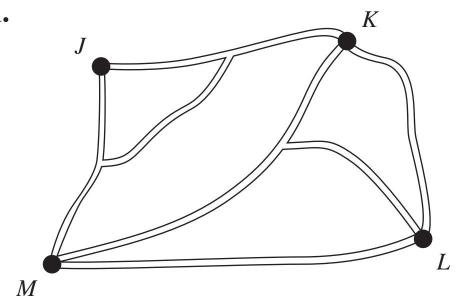
Question 2 (1 mark)

The adjacency matrix below shows the number of pathway connections between four landmarks: J, K, L and M.

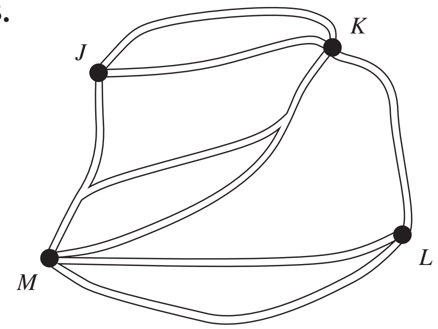
$$\begin{array}{c} \\ J \\ K \\ L \\ M \end{array} \begin{array}{c} \begin{array}{cccc} J & K & L & M \end{array} \\ \begin{bmatrix} 1 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix} \end{array}$$

A network of pathways that could be represented by the adjacency matrix is

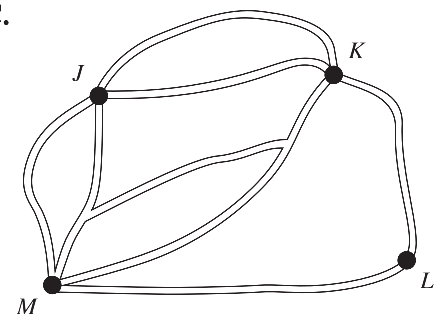
A.



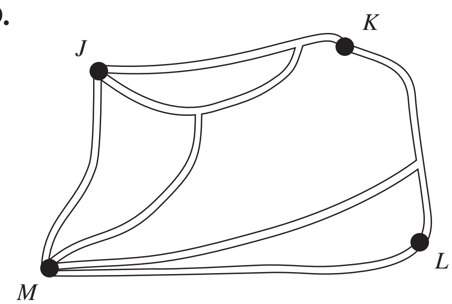
B.



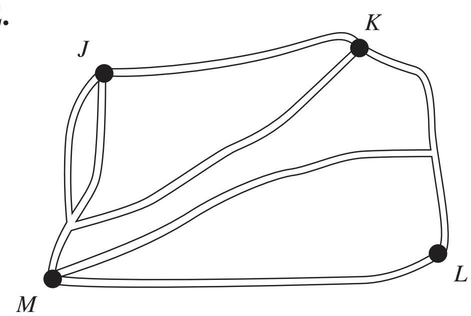
C.



D.

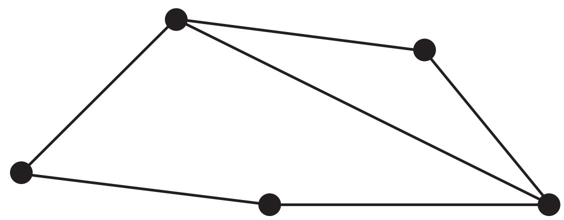


E.



Source: *VCE 2019, Further Mathematics 1, Section B, Module 2, Q.1; © VCAA*

Question 3 (1 mark)



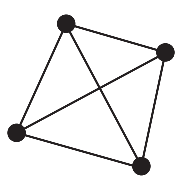
In the graph shown above, the sum of the degrees of the vertices is

A. 5
B. 6
C. 10
D. 11
E. 12

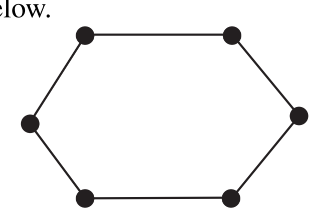
Source: *VCE 2017, Further Mathematics 1, Section B, Module 2, Q.2; © VCAA*

Question 4 (1 mark)

Two graphs, labelled Graph 1 and Graph 2, are shown below.



Graph 1



Graph 2

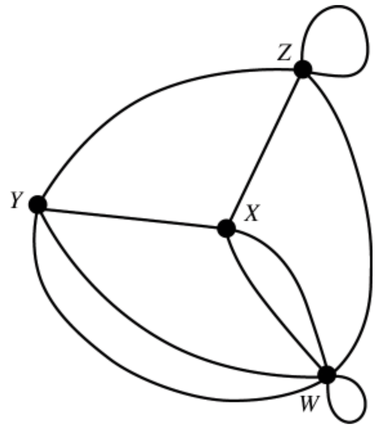
The sum of the degrees of the vertices of Graph 1 is

A. two less than the sum of the degrees of the vertices of Graph 2.
B. one less than the sum of the degrees of the vertices of Graph 2.
C. equal to the sum of the degrees of the vertices of Graph 2.
D. one more than the sum of the degrees of the vertices of Graph 2.
E. two more than the sum of the degrees of the vertices of Graph 2.

Source: *VCE 2017, Further Mathematics 1, Section B, Module 2, Q.3; © VCAA*

Question 5 (1 mark)

Consider the following graph.



The adjacency matrix for this graph, with some elements missing, is shown below.

$$\begin{array}{c} \\ W \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 1 & \_ & \_ & \_ \\ \_ & 0 & \_ & \_ \\ \_ & \_ & 0 & \_ \\ \_ & \_ & \_ & 1 \end{bmatrix} \end{array}$$

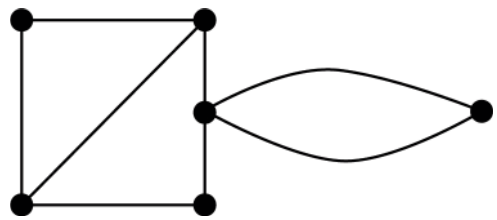
This adjacency matrix contains 16 elements when complete.

Of the 12 missing elements

A. eight are ‘1’ and four are ‘2’.
B. four are ‘1’ and eight are ‘2’.
C. six are ‘1’ and six are ‘2’.
D. two are ‘0’, six are ‘1’ and four are ‘2’.
E. four are ‘0’, four are ‘1’ and four are ‘2’.

Source: *VCE 2015, Further Mathematics 1, Section B, Module 5, Q.1; © VCAA*

Question 6 (1 mark)



In the graph above, the number of vertices of odd degree is

A. 0
B. 1
C. 2
D. 3
E. 4

Topic	9	Undirected graphs, networks and trees
Subtopic	9.3	Planar graphs and Euler's formula

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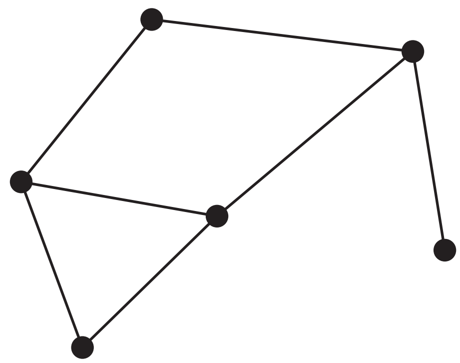
Source: *VCE 2021, Further Mathematics 1, Section B, Module 2, Q.5; © VCAA*

Question 1 (1 mark)



Consider the following five statements about the graph above:
- The graph is planar.
- The graph contains a cycle.
- The graph contains a bridge.
- The graph contains an Eulerian trail.
- The graph contains a Hamiltonian path.

How many of these statements are true?

A. 1
B. 2
C. 3
D. 4
E. 5

Source: *VCE 2020, Further Mathematics 1, Section B, Module 2, Q.1; © VCAA*

Question 2 (1 mark)

A connected planar graph has seven vertices and nine edges.

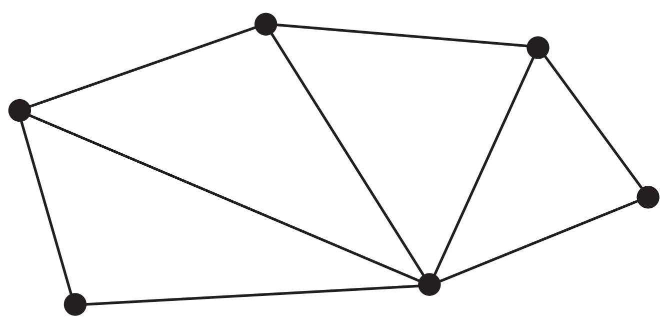
The number of faces that this graph will have is

A. 1
B. 2
C. 3
D. 4
E. 5

Source: *VCE 2019, Further Mathematics 1, Section B, Module 2, Q.2; © VCAA*

Question 3 (1 mark)

Consider the graph below.



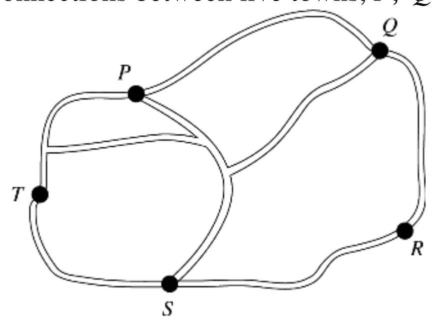
The minimum number of extra edges that are required so that an Eulerian circuit is possible in this graph is

A. 0
B. 1
C. 2
D. 3
E. 4

Source: *VCE 2019, Further Mathematics 1, Section B, Module 2, Q.6; © VCAA*

Question 4 (1 mark)

The map below shows all the road connections between five towns, P, Q, R, S and T.



The road connections could be represented by the adjacency matrix

A. $\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 1 & 3 & 0 & 0 & 1 \\ 3 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 3 \\ 1 & 0 & 0 & 3 & 0 \end{bmatrix} \end{array}$

B. $\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$

C. $\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 & 2 \\ 3 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$

D. $\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$

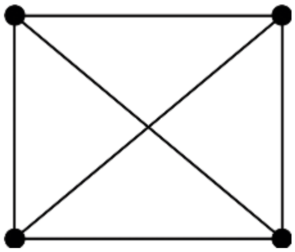
E. $\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$

Source: *VCE 2018, Further Mathematics 1, Section B, Module 2, Q.6; © VCAA*

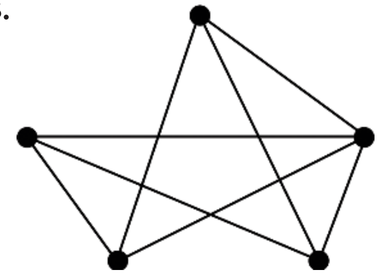
Question 5 (1 mark)

Which one of the following graphs is **not** a planar graph?

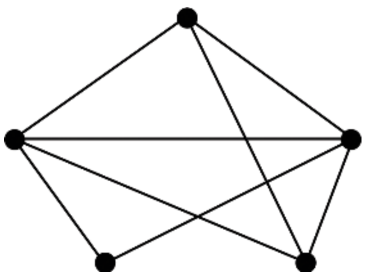
A.



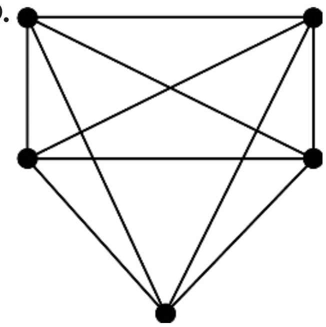
B.



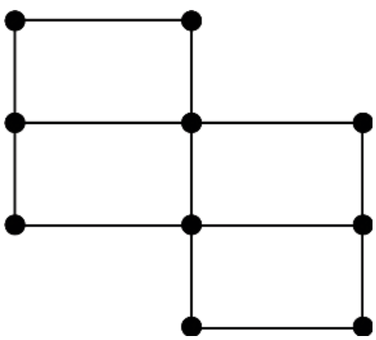
C.



D.



E.



Source: *VCE 2018, Further Mathematics 1, Section B, Module 2, Q.3; © VCAA*

Question 6 (1 mark)

A planar graph has five faces.

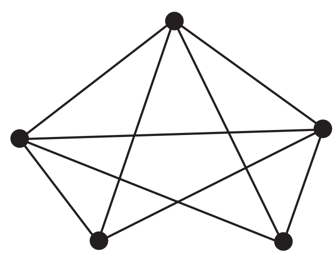
This graph could have

A. eight vertices and eight edges.
B. six vertices and eight edges.
C. eight vertices and five edges.
D. eight vertices and six edges.
E. five vertices and eight edges.

Source: *VCE 2016, Further Mathematics 1, Section B, Module 2, Q.3; © VCAA*

Question 7 (1 mark)

The following graph with five vertices is a complete graph.



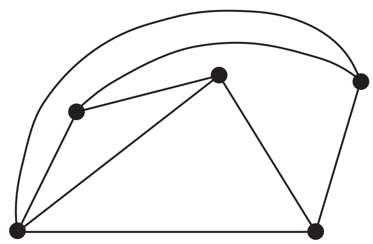
Edges are removed so that the graph will have the minimum number of edges to remain connected. The number of edges that are removed is

A. 4
B. 5
C. 6
D. 9
E. 10

Source: *VCE 2016, Further Mathematics 1, Section B, Module 2, Q.5; © VCAA*

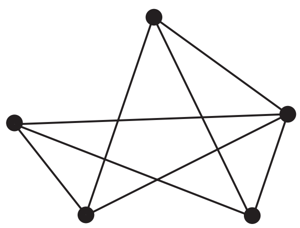
Question 8 (1 mark)

Consider the planar graph below.

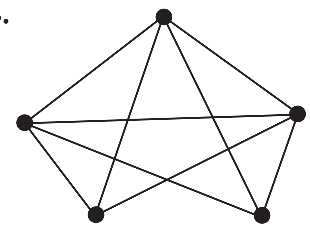


Which one of the following graphs can be redrawn as the planar graph above?

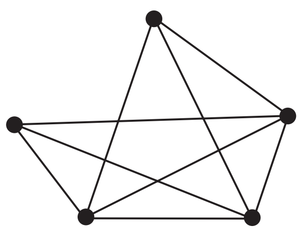
A.



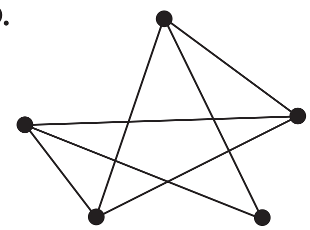
B.



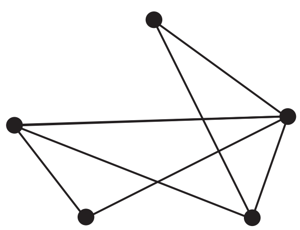
C.



D.



E.



Source: *VCE 2015, Further Mathematics 1, Section B, Module 5, Q.2; © VCAA*

Question 9 (1 mark)

A planar graph has five vertices and six faces.

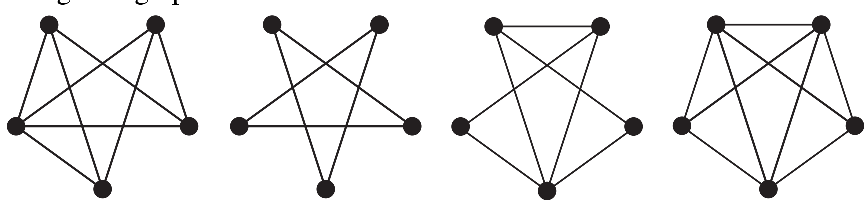
The number of edges is

A. 3
B. 6
C. 9
D. 11
E. 13

Source: *VCE 2014, Further Mathematics 1, Section B, Module 5, Q.7; © VCAA*

Question 10 (1 mark)

Consider the following four graphs.



How many of these four graphs are planar?

A. 0
B. 1
C. 2
D. 3
E. 4

Source: *VCE 2013, Further Mathematics 1, Section B, Module 5, Q.2; © VCAA*

Question 11 (1 mark)

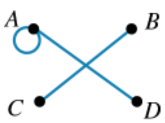
The number of edges needed to make a complete graph with four vertices is

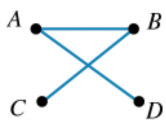
A. 2
B. 3
C. 4
D. 5
E. 6

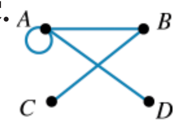
Question 12 (1 mark)

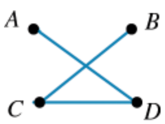
Which of the following networks is represented by the adjacency matrix below?

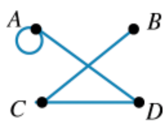
$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

A.


B.


C.


D.


E.


Question 13 (1 mark)

The given adjacency matrix represents a planar graph with 3 vertices.

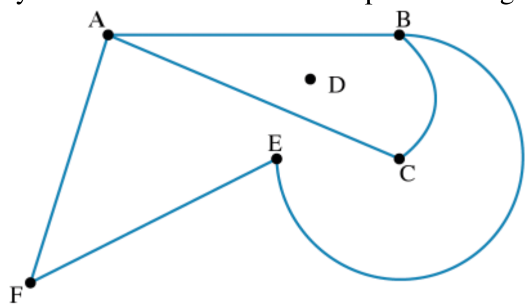
$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

The number of faces (regions) on this planar graph is

A. 3
B. 4
C. 5
D. 6
E. 7

Question 14 (1 mark)

Which of the following adjacency matrices could be used to represent the graph below?



A. $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

Question 15 (1 mark)

Which of the following statements best describes a simple graph?

A. A graph in which each vertex is connected directly to every other vertex.

B. A graph with no edges.

C. A graph with no loops or multiple edges between vertices.

D. A graph with loops and multiple edges.

E. A graph which can be drawn without any edges crossing each other.

Topic	9	Undirected graphs, networks and trees
Subtopic	9.4	Walks, trails, paths, cycles and circuits

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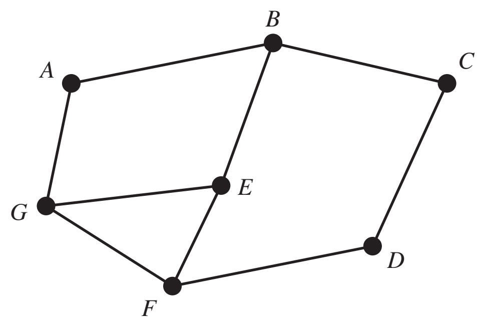
Source: *VCE 2020, Further Mathematics 1, Section B, Module 2, Q.2; © VCAA*

Question 1 (1 mark)

Consider the graph below.



Which one of the following is not a Hamiltonian cycle for this graph?

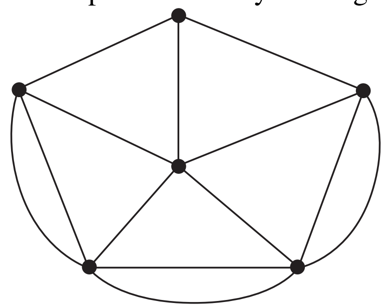
A. *ABCDFEGA*
B. *BAGEFDCB*
C. *CDFEGABC*
D. *DCBAGFED*
E. *EGABCDFE*

Topic 9 Subtopic 9.4 Walks, trails, paths, cycles and circuits

Source: *VCE 2017, Further Mathematics 1, Section B, Module 2, Q.6; © VCAA*

Question 2 (1 mark)

An Eulerian trail for the graph above will be possible if only one edge is removed.



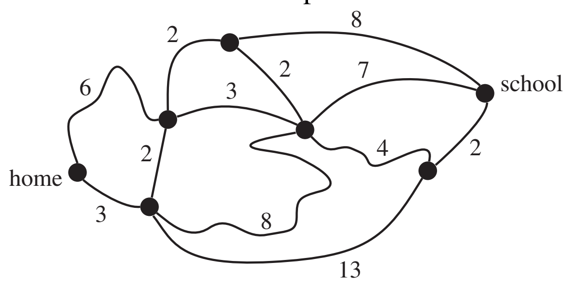
In how many different ways could this be done?

A. 1
B. 2
C. 3
D. 4
E. 5

Source: *VCE 2014, Further Mathematics 1, Section B, Module 5, Q.3; © VCAA*

Question 3 (1 mark)

The diagram below shows the network of roads that Stephanie can use to travel between home and school.



The numbers on the roads show the time, in minutes, that it takes her to ride a bicycle along each road. Using this network of roads, the shortest time that it will take Stephanie to ride her bicycle from home to school is

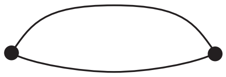
A. 12 minutes
B. 13 minutes
C. 14 minutes
D. 15 minutes
E. 16 minutes

Source: *VCE 2017, Further Mathematics 1, Section B, Module 2, Q.1; © VCAA*

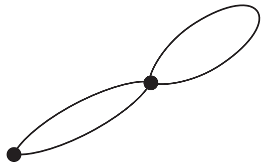
Question 4 (1 mark)

Which one of the following graphs contains a loop?

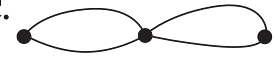
A.



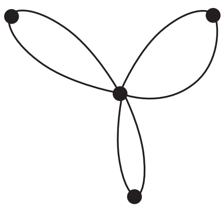
B.



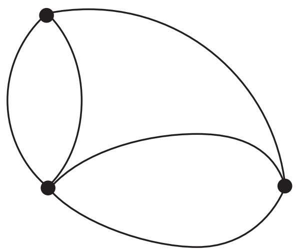
C.



D.



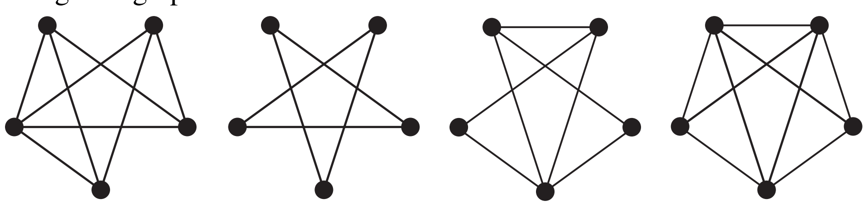
E.



Source: *VCE 2014, Further Mathematics 1, Section B, Module 5, Q.6; © VCAA*

Question 5 (1 mark)

Consider the following four graphs.



How many of these four graphs have an Eulerian circuit?

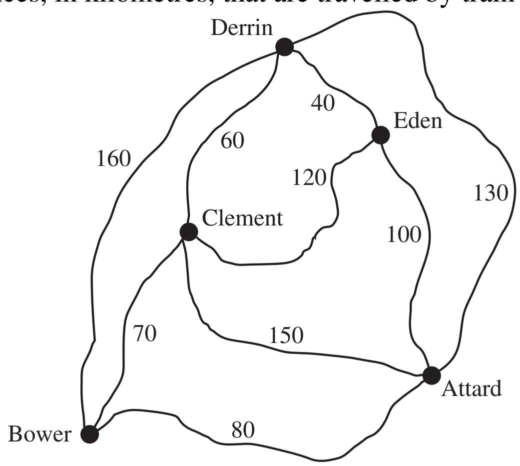
A. 0
B. 1
C. 2
D. 3
E. 4

Source: *VCE 2014, Further Mathematics 2, Module 5, Q.3; © VCAA*

Question 6 (4 marks)

The diagram below shows a network of train lines between five towns: Attard, Bower, Clement, Derrin and Eden.

The numbers indicate the distances, in kilometres, that are travelled by train between connected towns.



Charlie followed an Eulerian path through this network of train lines.

a. Answer the following.

i. Write down the names of the towns at the start and at the end of Charlie's path. **(1 mark)**

ii. What distance did he travel? **(1 mark)**

b. Brianna will follow a Hamiltonian path from Bower to Attard.

What is the shortest distance that she can travel? **(1 mark)**

c. The train line between Derrin and Eden will be removed. If one other train line is removed from the network, Andrew would be able to follow an Eulerian circuit through the network of train lines.

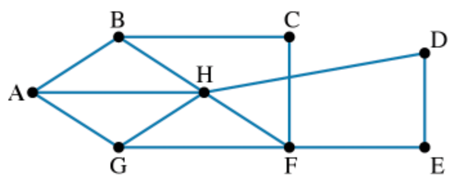
Which other train line should be removed?

In the boxes below, write down the pair of towns that this train line connects. **(1 mark)**

Between [ ] and [ ]

Question 7 (1 mark)
Which of the following statements best describes a Hamiltonian path?
A. A path that begins and ends at the same vertex, and passes through each vertex exactly once.
B. A path that begins and ends at the same vertex and passes through each edge exactly once.
C. A path that begins and ends at different vertices and passes through each vertex exactly once.
D. A path that begins and ends at different vertices and passes through each edge exactly once.
E. A path that begins and ends at different vertices and passes through each vertex and edge exactly once.

Question 8 (1 mark)
The sequence of vertices that represent a trail in the network shown is

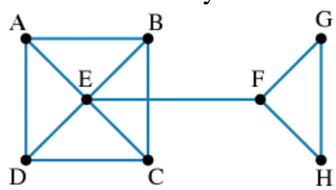


A. BHFGHBA
B. HFEDHFG
C. HBAHGFH
D. AHGAHBC
E. CBHABHG

Question 9 (1 mark)
Which of the following statements is incorrect?
A. A walk uses vertices and edges in an alternating sequence
B. A path uses vertices only once.
C. A trail uses edges only once
D. A cycle finishes on its starting edge.
E. A circuit finishes on its starting vertex.

Question 10 (1 mark)

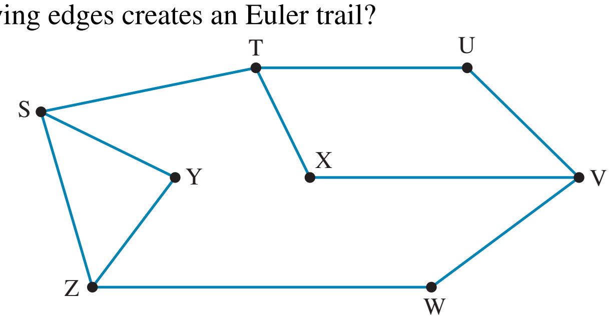
Which of the following options does not define either a cycle or a circuit for the network shown?



A. ADECBEA
B. GFHG
C. AECDAE
D. EBCEADE
E. CEABEDC

Question 11 (1 mark)

In the network below an Euler trail can be created by adding one new edge.
Adding which of the following edges creates an Euler trail?



A. SU
B. YZ
C. SZ
D. ZW
E. UX

Question 12 (1 mark)

Which of the given adjacency matrices represents a network that contains an Eulerian circuit?

A. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 3 & 2 & 0 \\ 3 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

Topic	**9**	**Undirected graphs, networks and trees**
Subtopic	**9.5**	**Trees and their applications**

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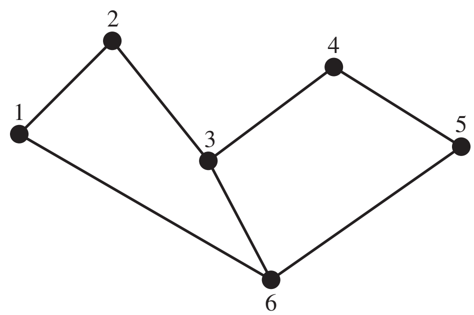
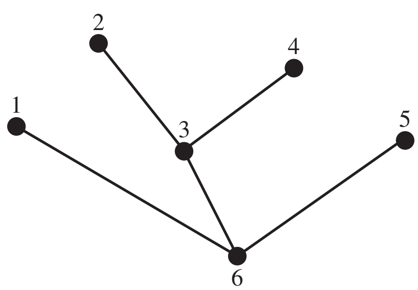
Question 1 (1 mark)

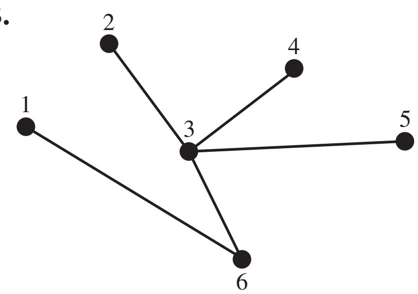


Which one of the following is not a spanning tree for the network above?

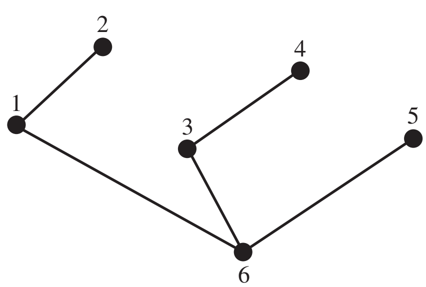
A.



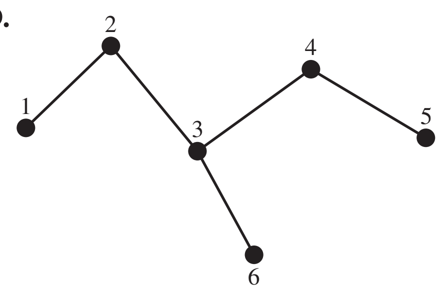
B.



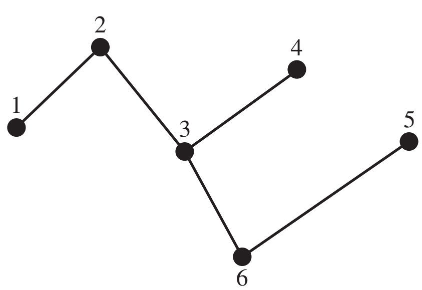
C.



D.



E.

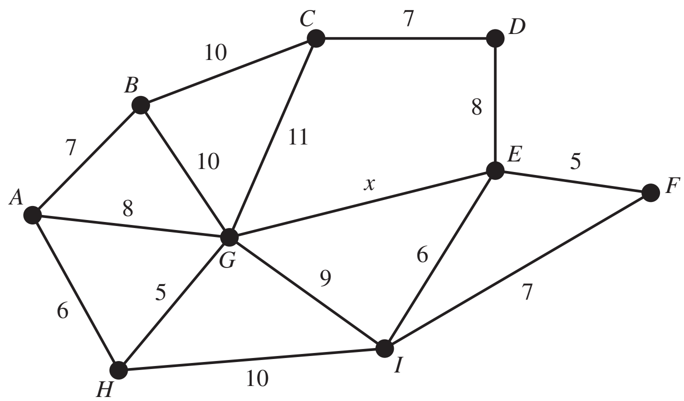


Source: *VCE 2020, Further Mathematics 1, Section B, Module 2, Q.5; © VCAA*

Question 2 (1 mark)

The network below shows the distances, in metres, between camp sites at a camping ground that has electricity.

The vertices A to I represent the camp sites.



The minimum length of cable required to connect all the camp sites is 53 m.

The value of x, in metres, is at least

A. 5

B. 6

C. 8

D. 9

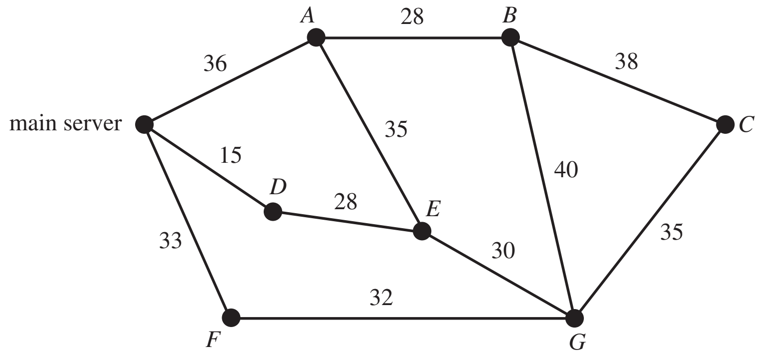
E. 11

Topic 9 | Subtopic 9.5 Trees and their applications

Source: *VCE 2019, Further Mathematics 1, Section B, Module 2, Q.5; © VCAA*

Question 3 (1 mark)

The following diagram shows the distances, in metres, along a series of cables connecting a main server to seven points, *A* to *G*, in a computer network.



The minimum length of cable, in metres, required to ensure that each of the seven points is connected to the main server directly or via another point is

A. 175
B. 203
C. 208
D. 221
E. 236

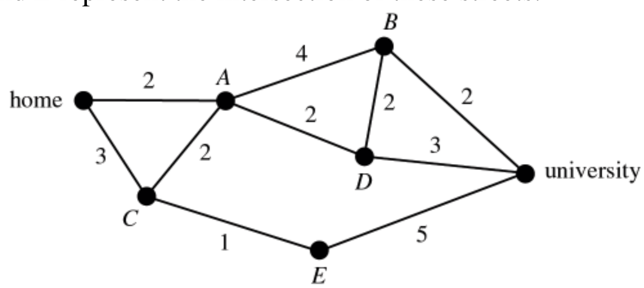
Source: *VCE 2018, Further Mathematics 1, Section B, Module 2, Q.2; © VCAA*

Question 4 (1 mark)

Niko drives from his home to university.

The network below shows the distances, in kilometres, along a series of streets connecting Niko's home to the university.

The vertices A, B, C, D and E represent the intersection of these streets.



The shortest path for Niko from his home to the university could be found using

A. a minimum cut.

B. Prim's algorithm.

C. Dijkstra's algorithm.

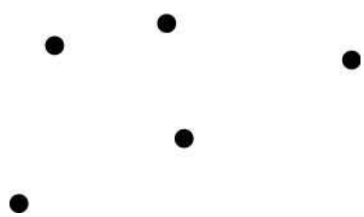
D. critical path analysis.

E. the Hungarian algorithm.

Source: *VCE 2018, Further Mathematics 1, Section B, Module 2, Q.1; © VCAA*

Question 5 (1 mark)

Consider the graph with five isolated vertices shown below.



To form a tree, the minimum number of edges that must be added to the graph is

A. 1

B. 4

C. 5

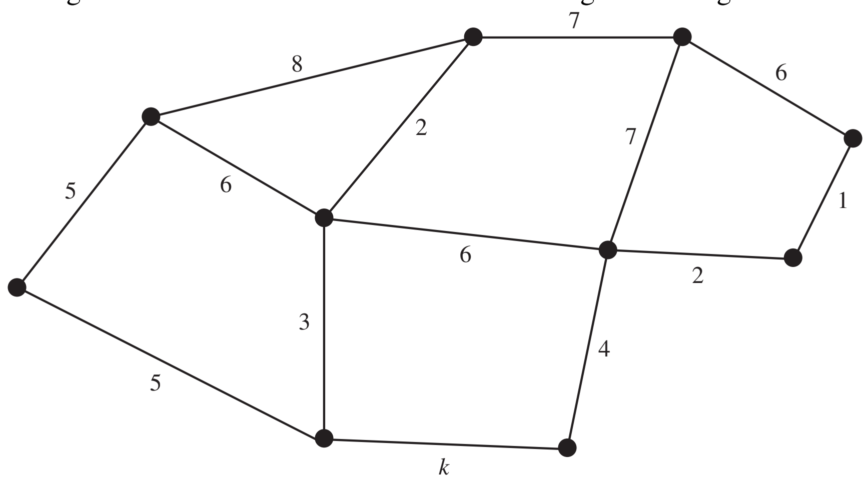
D. 6

E. 10

Source: *VCE 2016, Further Mathematics 1, Section B, Module 2, Q.4; © VCAA*

Question 6 (1 mark)

The minimum spanning tree for the network below includes the edge with weight labelled k.



The total weight of all edges for the minimum spanning tree is 33.

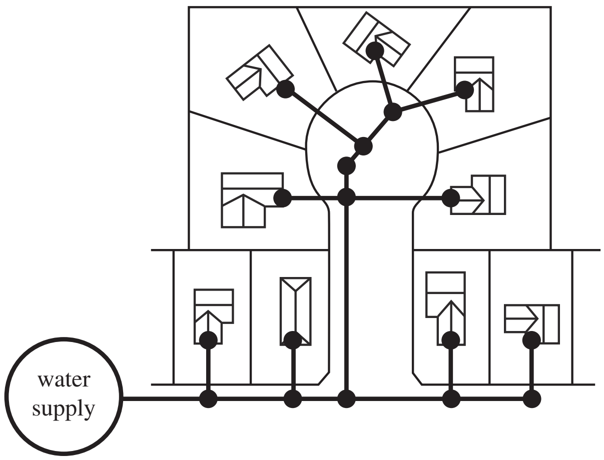
The value of k is

A. 1
B. 2
C. 3
D. 4
E. 5

Source: *VCE 2015, Further Mathematics 1, Section B, Module 5, Q.3; © VCAA*

Question 7 (1 mark)

The plan shows the layout of a section of pipes, drawn in bold lines, that supplies water to nine houses in a new estate.

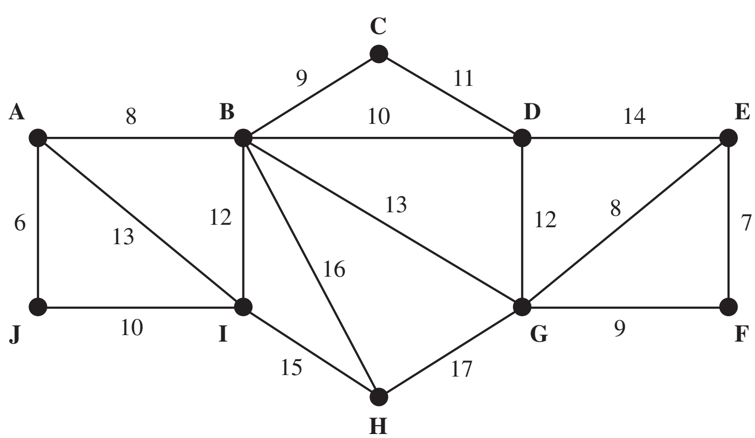


Which one of the following types of graph could be used to represent the layout of water pipe connections to the water supply and these houses?

A. a bipartite graph
B. a complete graph
C. a loop
D. a Hamiltonian path
E. a tree

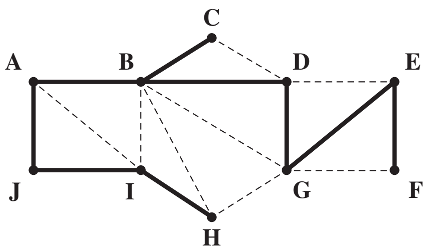
Source: *VCE 2014, Further Mathematics 1, Section B, Module 5, Q.5; © VCAA*

Question 8 (1 mark)

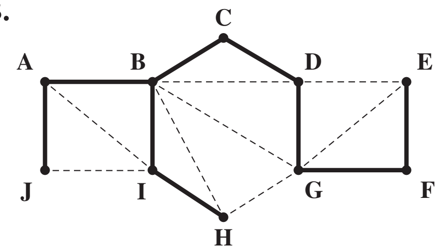


Which one of the following is the minimal spanning tree for the weighted graph shown above?

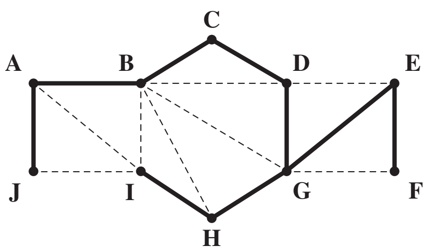
A.


C
A
B
D
E
J
I
G
F
H

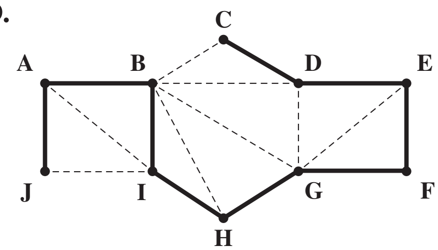
B.


C
A
B
D
E
J
I
G
F
H

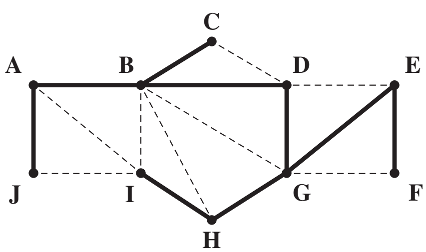
C.


C
A
B
D
E
J
I
G
F
H

D.

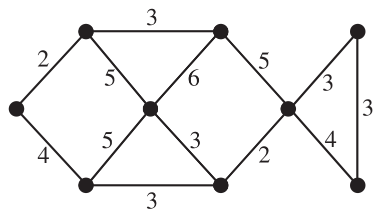

C
A
B
D
E
J
I
G
F
H

E.


C
A
B
D
E
J
I
G
F
H

Source: *VCE 2013, Further Mathematics 1, Section B, Module 5, Q.3; © VCAA*

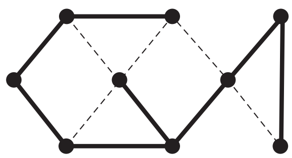
Question 9 (1 mark)



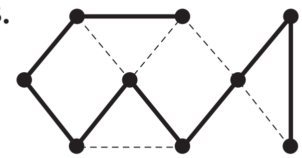
The vertices of the graph above represent nine computers in a building. The computers are to be connected with optical fibre cables, which are represented by edges. The numbers on the edges show the costs, in hundreds of dollars, of linking these computers with optical fibre cables.

Based on the same set of vertices and edges, which one of the following graphs shows the cable layout (in bold) that would link all the computers with optical fibre cables for the minimum cost?

A.



B.



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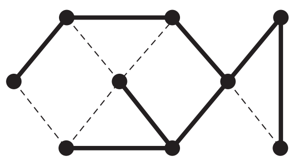
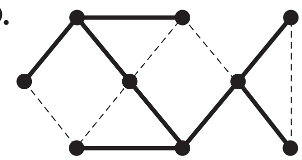
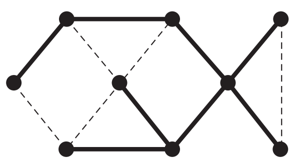


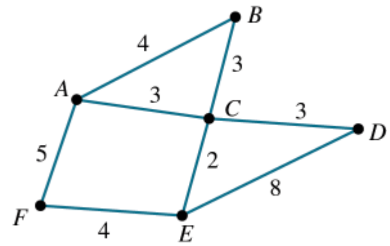
D.



E.



Question 10 (1 mark)



What is the shortest path between D and F in the graph shown above?

A. 12
B. 11
C. 9
D. 8
E. 4

Question 11 (1 mark)

The matrix below shows the roads between four towns K, L, M and N, and the distance in kilometres between the towns along each road. The dash (–) indicates that there is no road directly linking the two towns.

$$\begin{array}{c c} & \begin{array}{cccc} K & L & M & N \end{array} \\ \begin{array}{c} K \\ L \\ M \\ N \end{array} & \begin{bmatrix} 0 & - & - & 15 \\ - & 0 & 15 & 30 \\ - & 15 & 0 & 10 \\ 15 & 30 & 10 & 0 \end{bmatrix} \end{array}$$

What is the shortest route, in kilometres, between K and L?

A. 15
B. 45
C. 25
D. 30
E. 40

Topic	9	Undirected graphs, networks and trees
Subtopic	9.6	Review

online only

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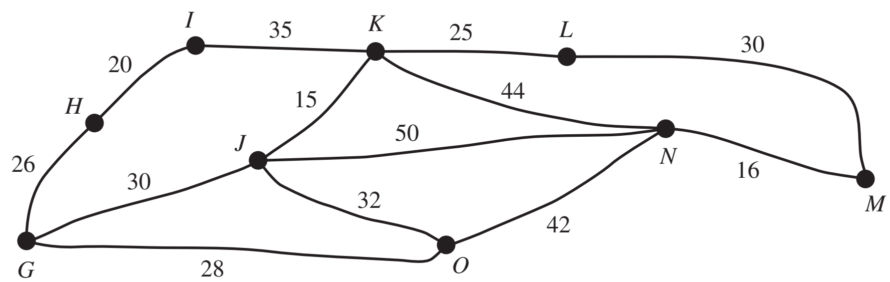
Source: *VCE 2021, Further Mathematics 2, Section B, Module 2, Q.2; © VCAA*

Question 1 (2 marks)

George lives in Town G and Maggie lives in Town M.
The diagram below shows the network of main roads between Town G and Town M.
The vertices G, H, I, J, K, L, M, N and O represent towns.
The edges represent the main roads. The numbers on the edges indicate the distances, in kilometres, between adjacent towns.



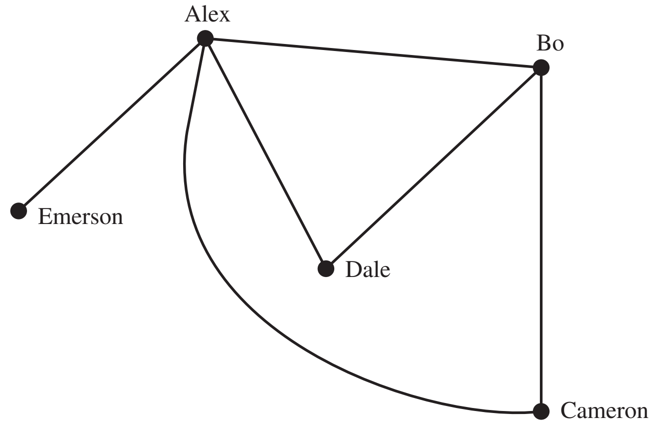
a. What is the shortest distance, in kilometres, between Town G and Town M? **(1 mark)**

b. George plans to travel to Maggie's house. He will pass through all the towns shown above.
George plans to take the shortest route possible.
Which town will George pass through twice? **(1 mark)**

Source: *VCE 2020, Further Mathematics 2, Section B, Module 2, Q.1; © VCAA*

Question 2 (3 marks)

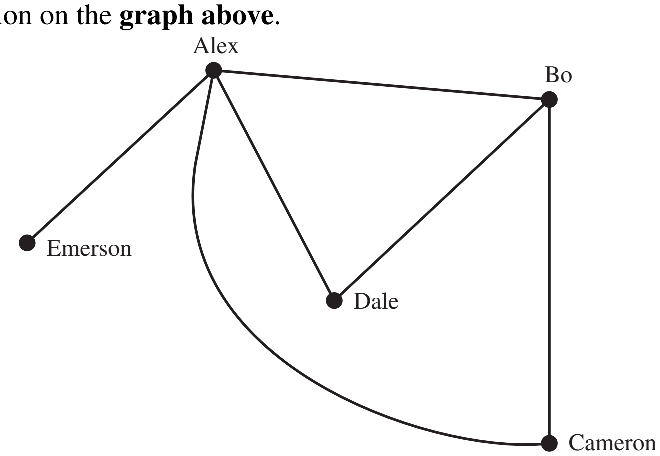
The Sunny Coast Cricket Club has five new players join its team: Alex, Bo, Cameron, Dale and Emerson.
The graph below shows the players who have played cricket together before joining the team.
For example, the edge between Alex and Bo shows that they have previously played cricket together.



a. How many of these players had Emerson played cricket with before joining the team? **(1 mark)**

b. Who had played cricket with both Alex and Bo before joining the team? **(1 mark)**

c. During the season, another new player, Finn, joined the team.
Finn had not played cricket with any of these players before.
Represent this information on the **graph above**. **(1 mark)**



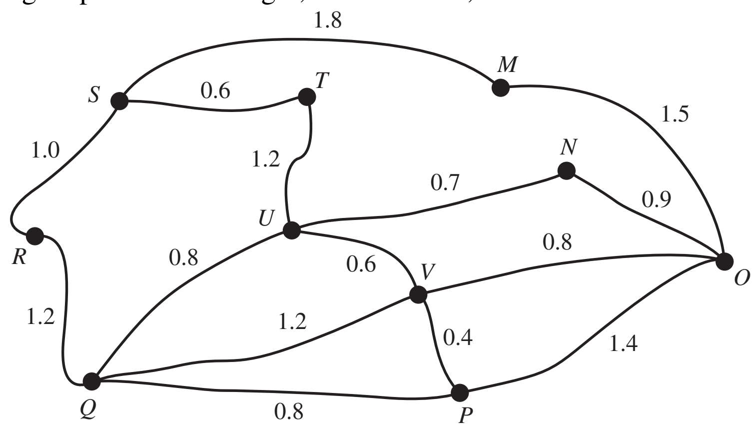
Source: *VCE 2020, Further Mathematics 2, Section B, Module 2, Q.3; © VCAA*

Question 3 (4 marks)

A local fitness park has 10 exercise stations: M to V.

The edges on the graph below represent the tracks between the exercise stations.

The number on each edge represents the length, in kilometres, of each track.



The Sunny Coast cricket coach designs three different training programs, **all starting at exercise station *S***.

Training program number	Training details
1	The team must run to exercise station *O*.
2	The team must run along all tracks just once.
3	The team must visit each exercise station and return to exercise station *S*.

a. What is the shortest distance, in kilometres, covered in training program 1? **(1 mark)**

b. Answer the following

i. What mathematical term is used to describe training program 2? **(1 mark)**

ii. At which exercise station would training program 2 finish? **(1 mark)**

c. To complete training program 3 in the minimum distance, one track will need to be repeated.

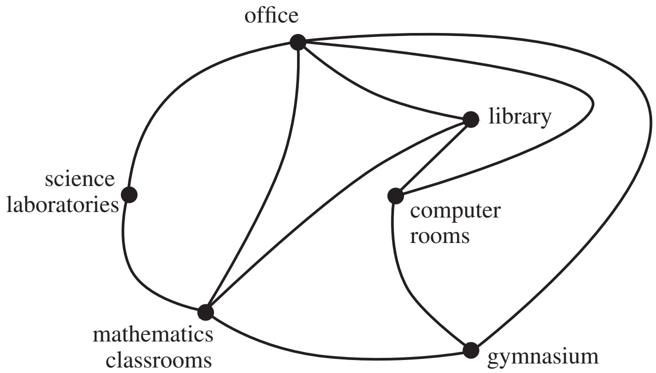
Complete the following sentence by filling in the blanks provided.

This track is between exercise station ________ and exercise station ________ . **(1 mark)**

Source: *VCE 2019, Further Mathematics 2, Section B, Module 2, Q.1; © VCAA*

Question 4 (3 marks)

Fencedale High School has six buildings. The network below shows these buildings represented by vertices. The edges of the network represent the paths between the buildings.

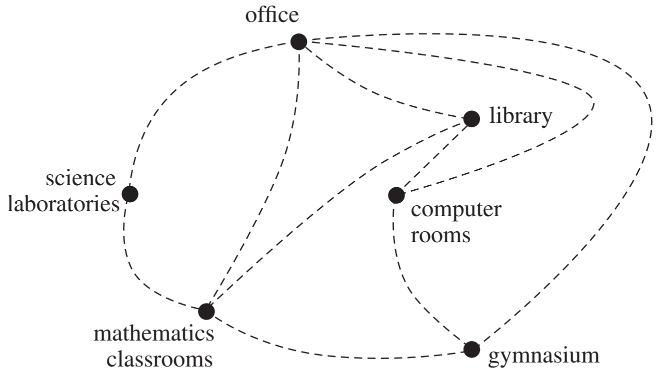


a. Which building in the school can be reached directly from all other buildings? **(1 mark)**

b. A school tour is to start and finish at the office, visiting each building only once.

i. What is the mathematical term for this route? **(1 mark)**

ii. Draw in a possible route for this school tour on the diagram below.

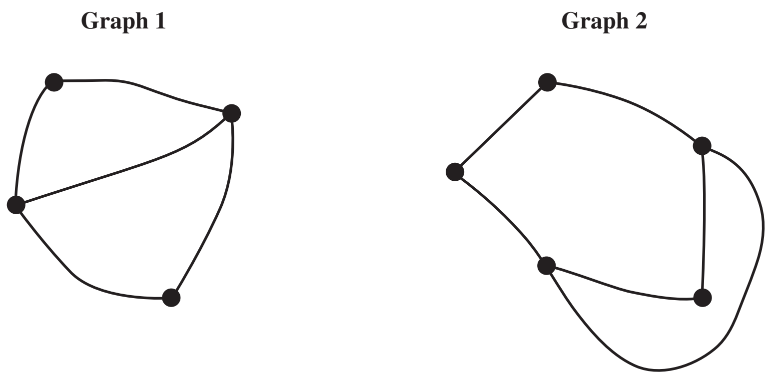


(1 mark)

Source: *VCE 2019, Further Mathematics 1, Section B, Module 2, Q.4; © VCAA*

Question 5 (1 mark)

Two graphs, labelled Graph 1 and Graph 2, are shown below.



Which one of the following statements is **not** true?

A. Graph 1 and Graph 2 are isomorphic.
B. Graph 1 has five edges and Graph 2 has six edges.
C. Both Graph 1 and Graph 2 are connected graphs.
D. Both Graph 1 and Graph 2 have three faces each.
E. Neither Graph 1 nor Graph 2 are complete graphs.

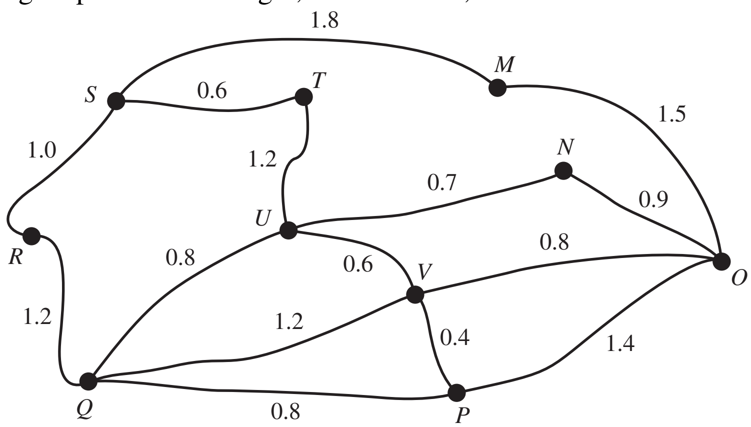
Source: *VCE 2020, Further Mathematics 2, Section B, Module 2, Q.3; © VCAA*

Question 6 (4 marks)

A local fitness park has 10 exercise stations: M to V.

The edges on the graph below represent the tracks between the exercise stations.

The number on each edge represents the length, in kilometres, of each track.



The Sunny Coast cricket coach designs three different training programs, **all starting at exercise station *S***.

Training program number	Training details
1	The team must run to exercise station *O*.
2	The team must run along all tracks just once.
3	The team must visit each exercise station and return to exercise station *S*.

a. What is the shortest distance, in kilometres, covered in training program 1? **(1 mark)**

b. Answer the following.

i. What mathematical term is used to describe training program 2? **(1 mark)**

ii. At which exercise station would training program 2 finish? **(1 mark)**

c. To complete training program 3 in the minimum distance, one track will need to be repeated.

Complete the following sentence by filling in the blanks provided.

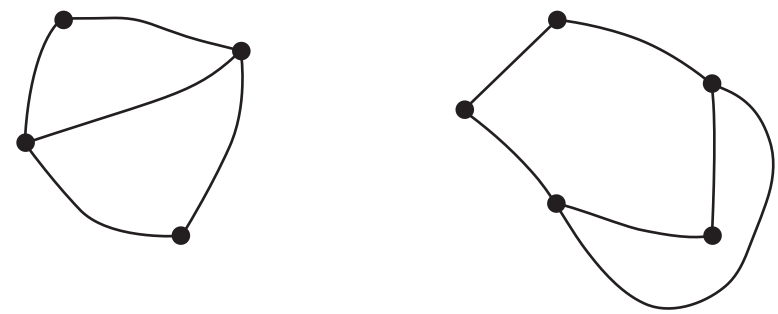
This track is between exercise station ________ and exercise station ________ . **(1 mark)**

Source: *VCE 2019, Further Mathematics 1, Section B, Module 2, Q.4; © VCAA*

Question 7 (1 mark)

Two graphs, labelled Graph 1 and Graph 2, are shown below.

Graph 1 **Graph 2**



Which one of the following statements is **not** true?

A. Graph 1 and Graph 2 are isomorphic.

B. Graph 1 has five edges and Graph 2 has six edges.

C. Both Graph 1 and Graph 2 are connected graphs.

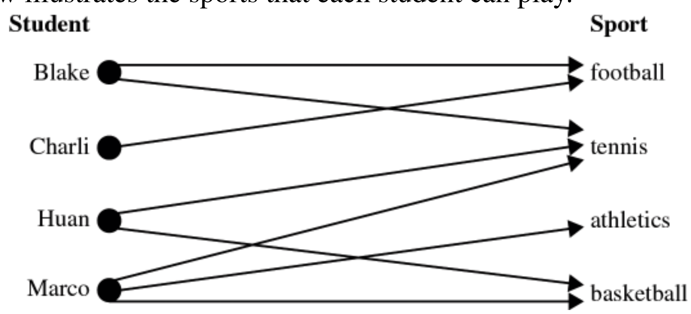
D. Both Graph 1 and Graph 2 have three faces each.

E. Neither Graph 1 nor Graph 2 are complete graphs.

Source: *VCE 2019, Further Mathematics 2, Section B, Module 2, Q.2; © VCAA*

Question 8 (3 marks)

Fencedale High School offers students a choice of four sports, football, tennis, athletics and basketball. The bipartite graph below illustrates the sports that each student can play.



Each student will be allocated to only one sport.

a. Complete the table below by allocating the appropriate sport to each student.

Student	Sport
Blake	
Charli	
Huan	
Marco	

(1 mark)

b. The school medley relay team consists of four students: Anita, Imani, Jordan and Lola.
The medley relay race is a combination of four different sprinting distances: 100 m, 200 m, 300 m and 400 m, run in that order.
The following table shows the best time, in seconds, for each student for each sprinting distance.

	Best time for each sprinting distance (seconds)			
Student	100 m	200 m	300 m	400 m
Anita	13.3	29.6	61.8	87.1
Imani	14.5	29.6	63.5	88.9
Jordan	13.3	29.3	63.6	89.1
Lola	15.2	29.2	61.6	87.9

The school will allocate each student to one sprinting distance in order to minimise the total time taken to complete the race.
To which distance should each student be allocated?
Write your answers in the table below.

Student	**Sprinting distance (m)**
Anita	
Imani	
Jordan	
Lola	

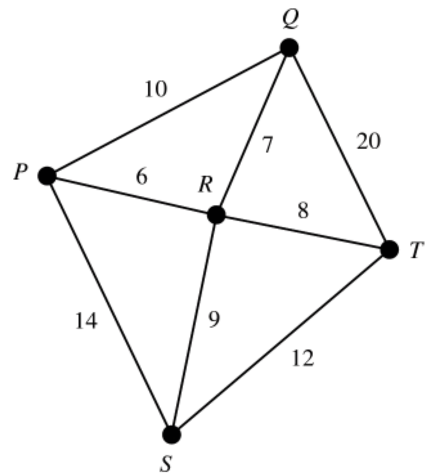
(2 marks)

Source: *VCE 2018, Further Mathematics 2, Section B, Module 2, Q.4; © VCAA*

Question 9 (2 marks)

Parcel deliveries are made between five nearby towns, P to T.

The roads connecting these five towns are shown on the graph below. The distances, in kilometres, are also shown.



A road inspector will leave from town P to check all the roads and return to town P when the inspection is complete. He will travel the minimum distance possible.

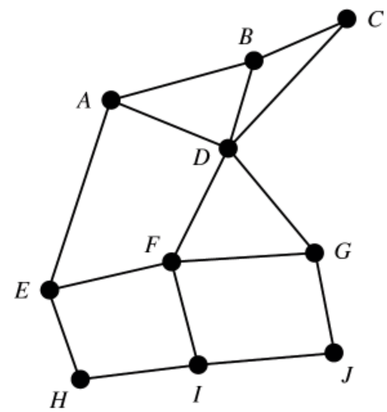
a. How many roads will the inspector have to travel on more than once? **(1 mark)**

b. Determine the minimum distance, in kilometres, that the inspector will travel. **(1 mark)**

Source: *VCE 2018, Further Mathematics 2, Section B, Module 2, Q.2; © VCAA*

Question 10 (3 marks)

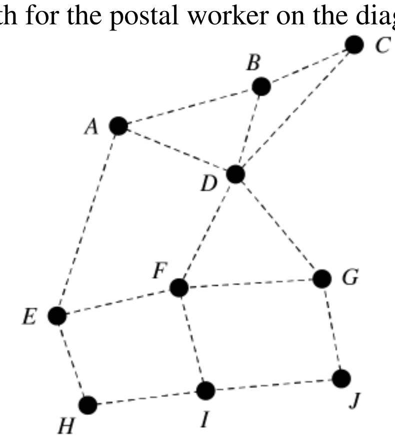
In one area of the town of Zenith, a postal worker delivers mail to 10 houses labelled as vertices A to J on the graph below.



a. Which one of the vertices on the graph has degree 4? **(1 mark)**

b. For this graph, an Eulerian trail does not currently exist.
For an Eulerian trail to exist, what is the minimum number of extra edges that the graph would require? **(1 mark)**

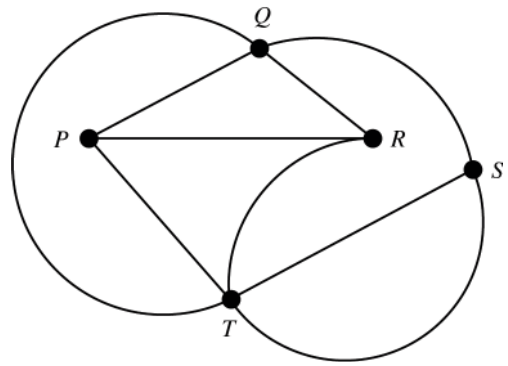
c. The postal worker has delivered the mail at F and will continue her deliveries by following a Hamiltonian path from F.
Draw in a possible Hamiltonian path for the postal worker on the diagram below. **(1 mark)**



Source: *VCE 2018, Further Mathematics 1, Section B, Module 2, Q.4; © VCAA*

Question 11 (1 mark)

Consider the graph below.



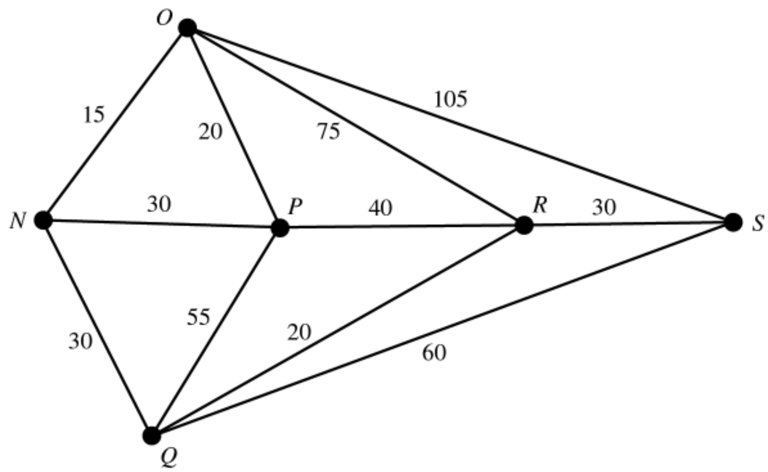
Which one of the following is **not** a path for this graph?

A. *PRQTS*
B. *PQRTS*
C. *PRTSQ*
D. *PTQSR*
E. *PTRQS*

Source: *VCE 2017, Further Mathematics 2, Section B, Module 2, Q.1; © VCAA*

Question 12 (3 marks)

Bus routes connect six towns.
The towns are Northend (N), Opera (O), Palmer (P), Quigley (Q), Rosebush (R)and Seatown (S).
The graph below gives the cost, in dollars, of bus travel along these routes.
Bai lives in Northend (N) and he will travel by bus to take a holiday in Seatown (S).



a. Bai considers travelling by bus along the route Northend (N) – Opera (O) – Seatown (S).
How much would Bai have to pay? **(1 mark)**

b. If Bai takes the cheapest route from Northend (N) to Seatown (S), which other town (s) will he pass through? **(1 mark)**

c. Euler's formula, $v + f = e + 2$, holds for this graph.
Complete the formula by writing the appropriate numbers in the boxes provided below.

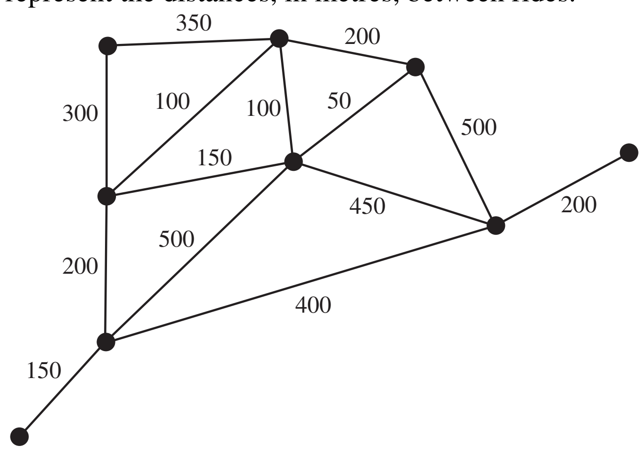
$\boxed{}_{v} + \boxed{}_{f} = \boxed{}_{e} + \boxed{2}$

(1 mark)

Source: *VCE 2017, Further Mathematics 2, Section B, Module 2, Q.3; © VCAA*

Question 13 (2 marks)

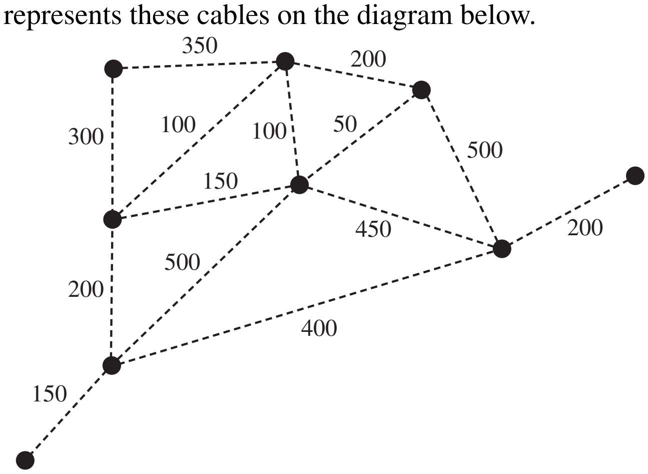
While on holiday, four friends visit a theme park where there are nine rides.
On the graph below, the positions of the rides are indicated by the vertices.
The numbers on the edges represent the distances, in metres, between rides.



a. Electrical cables are required to power the rides.
These cables will form a connected graph.
The shortest total length of cable will be used.

i. Give a mathematical term to describe a graph that represents these cables. **(1 mark)**

ii. Draw in the graph that represents these cables on the diagram below. **(1 mark)**

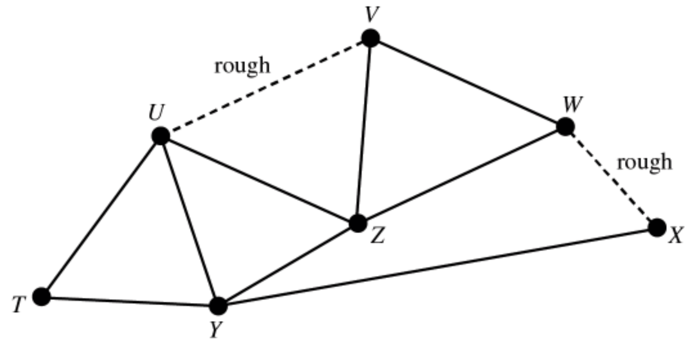


Source: *VCE 2016, Further Mathematics 2, Module 2, Q.2; © VCAA*

Question 14 (3 marks)

The suburb of Alooma has a skateboard park with seven ramps.

The ramps are shown as vertices T, U, V, W, X, Y and Z on the graph below.



The tracks between ramps U and V and between ramps W and X are rough, as shown on the graph above.

a. Nathan begins skating at ramp W and follows an Eulerian trail.
At which ramp does Nathan finish? **(1 mark)**

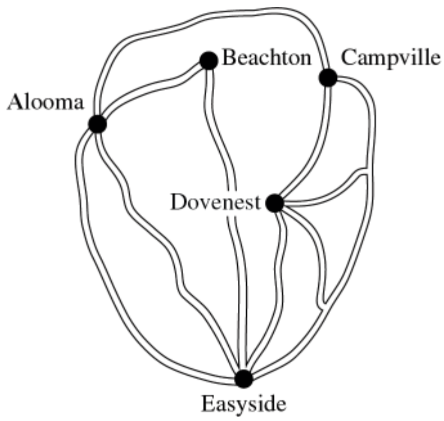
b. Zoe begins skating at ramp X and follows a Hamiltonian path.
The path she chooses does not include the two rough tracks.
Write down a path that Zoe could take from start to finish. **(1 mark)**

c. Birra can skate over any of the tracks, including the rough tracks.
He begins skating at ramp X and will complete a Hamiltonian cycle.
In how many ways could he do this? **(1 mark)**

Source: *VCE 2016, Further Mathematics 2, Module 2, Q.1; © VCAA*

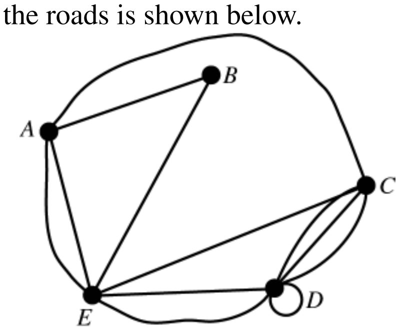
Question 15 (3 marks)

A map of the roads connecting five suburbs of a city, Alooma (A), Beachton (B), Campville (C), Dovenest (D) and Easyside (E), is shown below.



a. Starting at Beachton, which **two** suburbs can be driven to using only one road? **(1 mark)**

b. A graph that represents the map of the roads is shown below.



One of the edges that connects to vertex E is missing from the graph.

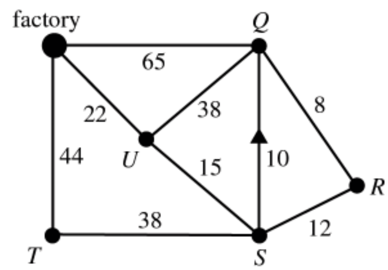
i. Add the missing edge to the **graph above**. **(1 mark)**

ii. Explain what the loop at D represents in terms of a driver who is departing from Dovenest. **(1 mark)**

Source: *VCE 2015, Further Mathematics 2, Module 5, Q.2; © VCAA*

Question 16 (3 marks)

The factory supplies groceries to stores in five towns, Q, R, S, T and U, represented by vertices on the graph below.



The edges of the graph represent roads that connect the towns and the factory.
The numbers on the edges indicate the distance, in kilometres, along the roads.
Vehicles may only travel along the road between towns S and Q in the direction of the arrow due to temporary roadworks.
Each day, a van must deliver groceries from the factory to the five towns.
The first delivery must be to town T, after which the van will continue on to the other four towns before returning to the factory.

a. Answer the following.

i. The shortest possible circuit from the factory for this delivery run, starting with town T, is not Hamiltonian.
Complete the order in which these deliveries would follow this shortest possible circuit.
factory – T – ________________________________ – factory **(1 mark)**

ii. With reference to the town names in your answer to **part a. i.**, explain why this shortest circuit is not a Hamiltonian circuit. **(1 mark)**

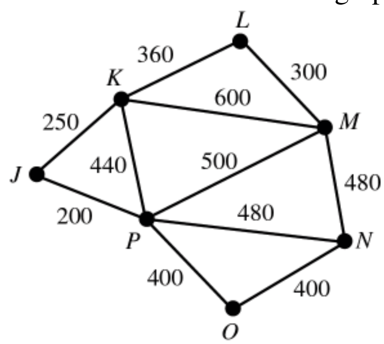
b. Determine the length, in kilometres, of a delivery run that follows a Hamiltonian circuit from the factory to these stores if the first delivery is to town T. **(1 mark)**

Source: *VCE 2015, Further Mathematics 2, Module 5, Q.1; © VCAA*

Question 17 (5 marks)

A factory requires seven computer servers to communicate with each other through a connected network of cables.

The servers, J, K, L, M, N, O and P, are shown as vertices on the graph below.



The edges on the graph represent the cables that could connect adjacent computer servers.

The numbers on the edges show the cost, in dollars, of installing each cable.

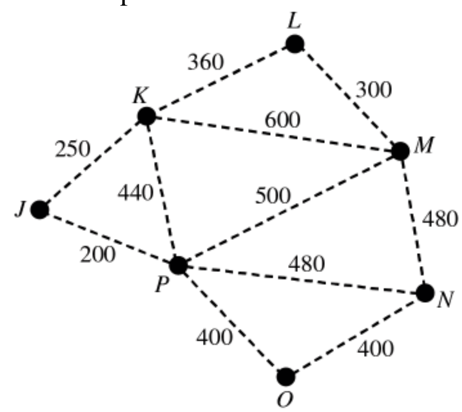
a. What is the cost, in dollars, of installing the cable between server L and server M? **(1 mark)**

b. What is the cheapest cost, in dollars, of installing cables between server K and server N? **(1 mark)**

c. An inspector checks the cables by walking along the length of each cable in one continuous path. To avoid walking along any of the cables more than once, at which vertex should the inspector start and where would the inspector finish? **(1 mark)**

d. The computer servers will be able to communicate with all the other servers as long as each server is connected by cable to at least one other server.

i. The cheapest installation that will join the seven computer servers by cable in a connected network follows a minimum spanning tree.
Draw the minimum spanning tree on the plan below. **(1 mark)**

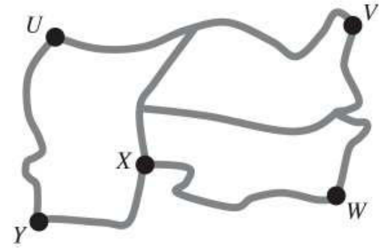


ii. The factory's manager has decided that only six connected computer servers will be needed, rather than seven.
How much would be saved in installation costs if the factory removed computer server P from its minimum spanning tree network? **(1 mark)**

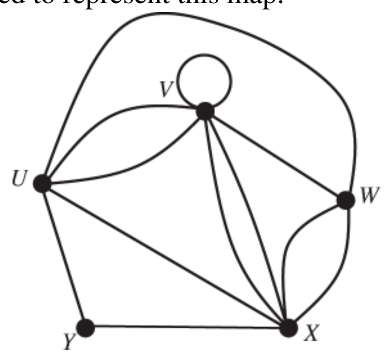
Source: *VCE 2015, Further Mathematics 1, Section B, Module 5, Q.6; © VCAA*

Question 18 (1 mark)

The map below shows all road connections between five towns, U, V, W, X and Y.



A graph, shown below, was constructed to represent this map.



A mistake has been made in constructing this graph.

This mistake can be corrected by

A. drawing another edge between V and W.

B. drawing a loop at W.

C. removing the loop at V.

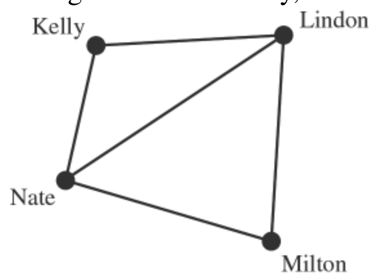
D. removing one edge between U and V.

E. removing one edge between X and V.

Source: *VCE 2014, Further Mathematics 1, Section B, Module 5, Q.1; © VCAA*

Question 19 (1 mark)

The graph below shows the roads connecting four towns: Kelly, Lindon, Milton and Nate.



A bus starts at Kelly, travels through Nate and Lindon, then stops when it reaches Milton.

The mathematical term for this route is

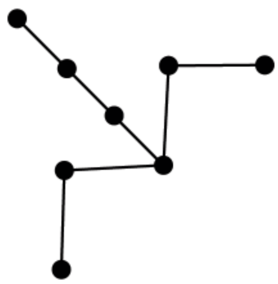
A. a loop.
B. an Eulerian path.
C. an Eulerian circuit.
D. a Hamiltonian path.
E. a Hamiltonian circuit.

Source: *VCE 2013, Further Mathematics 1, Section B, Module 5, Q.1; © VCAA*

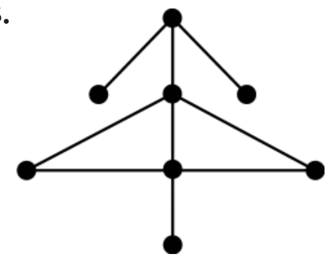
Question 20 (1 mark)

Which one of the following graphs is a tree?

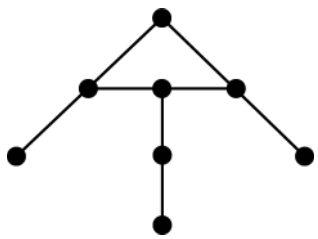
A.



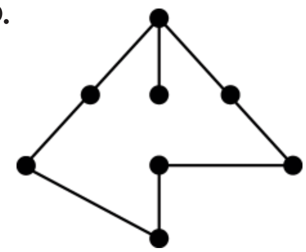
B.



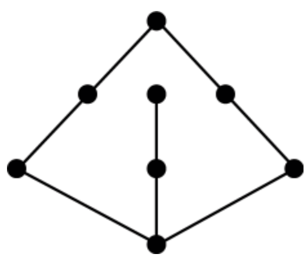
C.



D.



E.



Source: *VCE 2013, Further Mathematics 1, Section B, Module 5, Q.7; © VCAA*

Question 21 (1 mark)

A connected graph consists of five vertices and four edges.

Consider the following five statements.

- The graph is planar.
- The graph has more than one face.
- All vertices are of even degree.
- The sum of the degrees of the vertices is eight.
- The graph cannot have a loop.

How many of these statements are always true for such a graph?

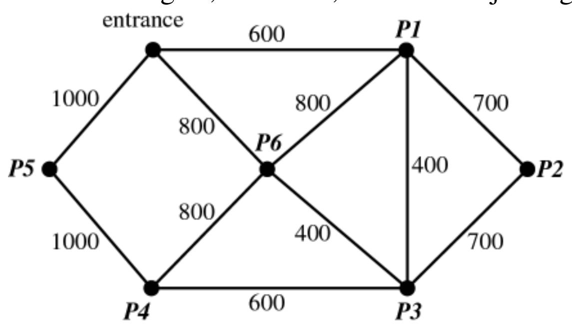
A. 1
B. 2
C. 3
D. 4
E. 5

Source: *VCE 2013, Further Mathematics 2, Module 5, Q.1; © VCAA*

Question 22 (5 marks)

The vertices in the network diagram below show the entrance to a wildlife park and six picnic areas in the park: $P1$, $P2$, $P3$, $P4$, $P5$ and $P6$.

The numbers on the edges represent the lengths, in metres, of the roads joining these locations.



a. In this graph, what is the degree of the vertex at the entrance to the wildlife park? **(1 mark)**

b. What is the shortest distance, in metres, from the entrance to picnic area $P3$? **(1 mark)**

c. A park ranger starts at the entrance and drives along every road in the park once.

i. At which picnic area will the park ranger finish? **(1 mark)**

ii. What mathematical term is used to describe the route the park ranger takes? **(1 mark)**

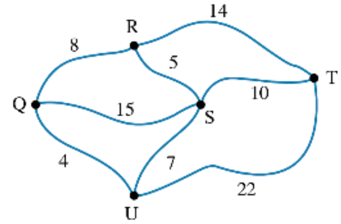
d. A park cleaner follows a route that starts at the entrance and passes through each picnic area once, ending at picnic area $P1$.

Write down the order in which the park cleaner will visit the six picnic areas. **(1 mark)**

Question 23 (1 mark)

The network below shows the roads between five towns, Q, R, S, T and U, and the distances, in kilometres, between the towns.

Use Dijkstra's algorithm to calculate the shortest distance between towns Q and T.



A. 21 km
B. 22 km
C. 24 km
D. 25 km
E. 26 km

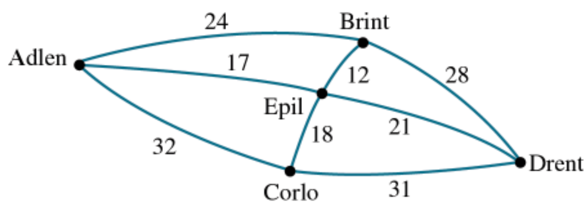
Question 24 (1 mark)

Which one of the following matrices does not represent a connected network?

A. $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ **C.** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

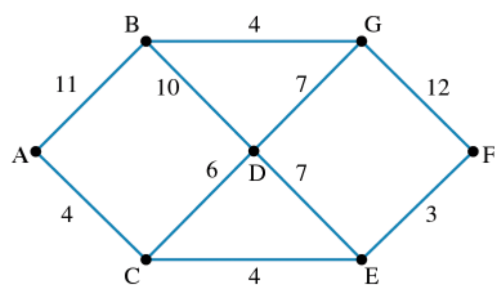
D. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ **E.** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Question 25 (1 mark)
The weighted network shows the distance, in kilometres, between a series of towns, Adlen, Brint, Corlo, Epil and Drent. Which of the following distances is not possible to achieve in travelling from Adlen to Drent?



A. 57 km
B. 90 km
C. 66 km
D. 52 km
E. 40 km

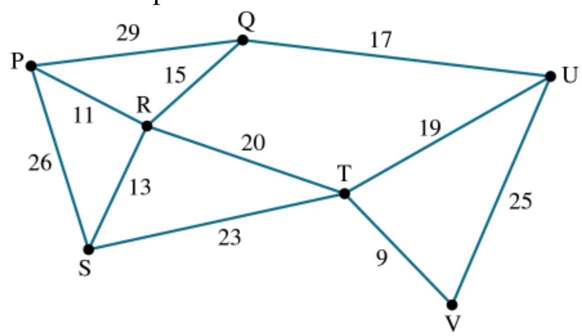
Question 26 (1 mark)
A journey is taken from point A to point G where the numbers on edges indicate distances in kilometres. What is the maximum possible distance that can be travelled if the journey must pass through each vertex only once?



A. 15 km
B. 23 km
C. 46 km
D. 52 km
E. 68 km

Question 27 (1 mark)

The given network shows the position of various checkpoints on a series of paths in a park. The numbers indicate the distances between each checkpoint.

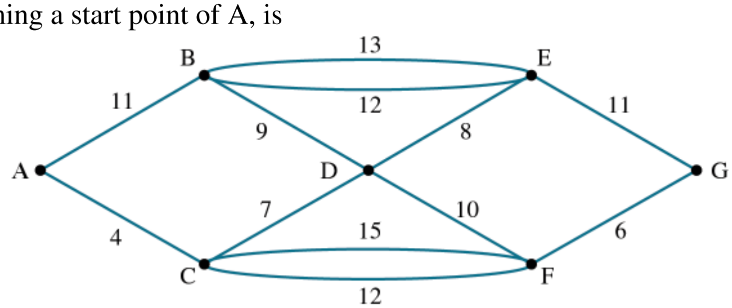


A walker follows a **trail** through the park with a total distance of 157 units. A possible route the walker undertook was

A. PRTUQRS
B. PRQPSTUVT
C. TUVTRST
D. QRSTVUQP
E. TRPQUVU

Question 28 (1 mark)

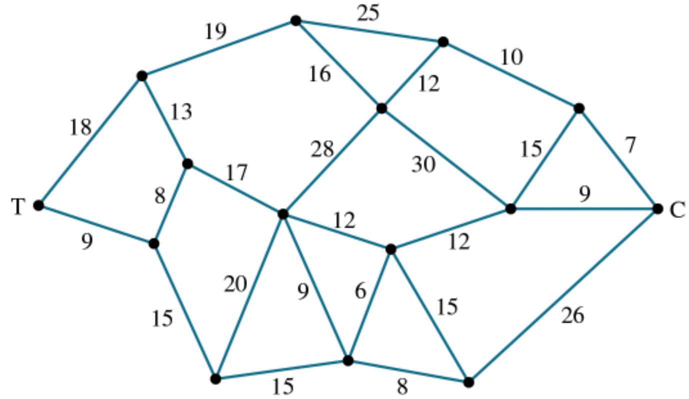
The difference in length between the minimum Hamiltonian cycle and Eulerian circuit in the following weighted graph, assuming a start point of A, is



A. 49
B. 57
C. 69
D. 101
E. 118

Question 29 (1 mark)

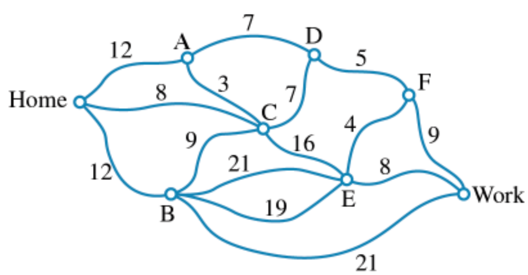
The network diagram shows the distances, in kilometres, between a major regional town, T, and the nearest capital city, C. The area in between is home to many smaller towns. On a particular day a locum doctor leaves the regional town, T, and must visit each small town and the capital city once only, and return home to town T. Calculate the minimum distance he must travel to achieve this task.



A. 195 km
B. 215 km
C. 236 km
D. 245 km
E. 256 km

Question 30 (1 mark)

The network below shows the travel times, in minutes, along a series of roads that connect a man's home and his place of work.



Using Dijkstra's algorithm, the shortest time between the man's home and his work is:

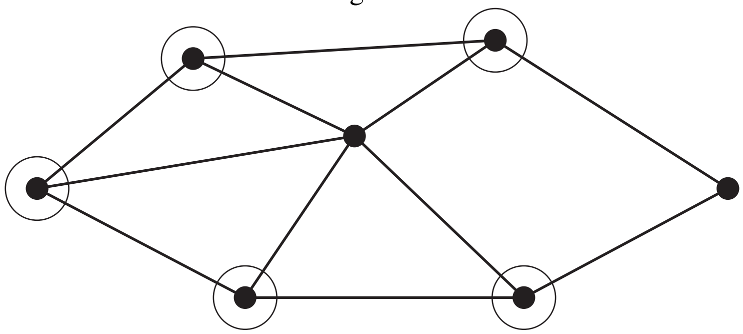
A. 15 mins
B. 20 mins
C. 24 mins
D. 29 mins
E. 33 mins

Answers and marking guide

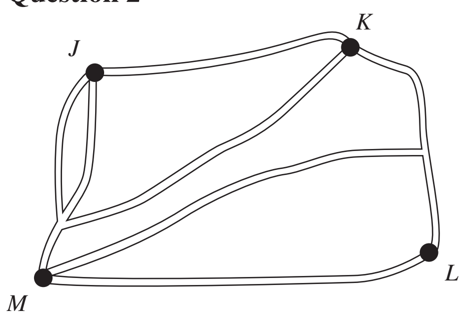
9.2 Basic concepts of a network

Question 1

There are 5 vertices with a degree of 3:



Question 2



Question 3

$2 + 3 + 2 + 3 + 2 = 12$

Question 4

The degree of each vertex in Graph 1 is 3, so the sum of the degrees in this graph is 12.
The degree of each vertex in Graph 2 is 2, so the sum of the degrees in this graph is also 12.

Question 5

The adjacency matrix shows the number of pathways between each of the vertices.

The complete adjacency matrix is $\begin{array}{c} \\ W \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$.

In total, there are eight are '1s' and four are '2s'.

Question 6

Each edge coming out of a vertex is one degree.
So there are two vertices with odd degree (3 each).

9.3 Planar graphs and Euler's formula

Question 1

There is no Eulerian trail, because it is not possible to get to and from the vertex on the right without passing over the edge twice. The other 4 statements are true.

Question 2

$$v + f - e = 2$$
$$7 + f - 9 = 2$$
$$f = 4$$

Question 3

For a Eulerian circuit to be possible, all vertices must be of even degree. There are four vertices with odd degree vertices, so two more edges are needed.

Question 4

The easiest way to find the correct answer to this question is to eliminate the possibilities.
There are three direct routes between P and Q, so this eliminates options B, D and E.
There is a loop from P and P, so this eliminates option C.
Therefore, the answer is A.

Question 5

If a graph is planar, it can be redrawn so that no edges overlap.
A is planar if one diagonal edge goes around the outside of the square
B is planar if the top node is moved below and to the right of the network. And then edges are redrawn to ensure none cross.
C is planar if the bottom right node is moved into the top triangle.
D cannot be redrawn without overlapping edges and is therefore not planar.
E already has no overlapping edges.
***VCAA Examination Report* note:**
Of the five graphs presented, only option D could not be redrawn without intersecting edges. Many students would have found this by trial, but some would have recognised option D as a complete graph with five vertices. Any complete graph with five, or more, vertices is non-planar.

Question 6

The rule for planar graphs is $v - e + f = 2$.
A: $8 - 8 + 5 = 5 \neq 2$
B: $6 - 8 + 5 = 3 \neq 2$
C: $8 - 5 + 5 = 8 \neq 2$
D: $8 - 6 + 5 = 7 \neq 2$
E: $5 - 8 + 5 = 2$

Question 7

There are 10 edges. If 6 are removed, the graph is still connected (i.e. all vertices can be reached).
***VCAA Assessment Report* note:**
The key to answering this question was to recognise that the answer would be a spanning tree, which, for a graph with five vertices, would have four edges. As the original graph had 10 edges, the number of edges to be removed was $10 - 4 = 6$ (option C).

Question 8

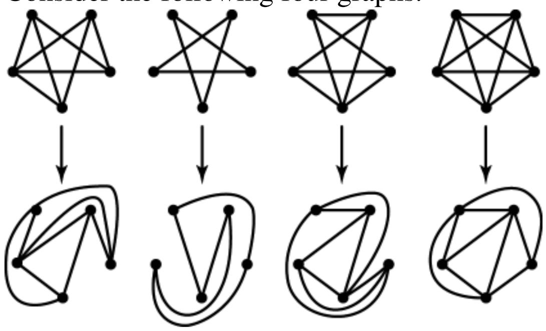
The planar graph has 8 edges, so the original must have 8 edges too.

Question 9

$$v + f - e = 2$$
$$5 + 6 - e = 2$$
$$e = 9$$

Question 10

Consider the following four graphs.



All options can be made into planar graphs.

***VCAA Assessment Report* note:**

The majority of students chose option A, 0. This suggests that most students were unaware that intersecting edges in a graph do not automatically preclude the graph from being planar.

Question 11

For a complete graph with n vertices, the number of edges, e, is:

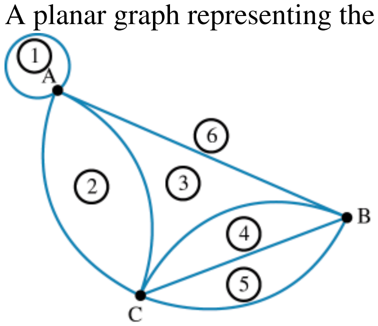
$$e = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$$

Question 12

The 1 on the main diagonal at A represents a loop at A. Disregard options B and D.
The only other connection from A is to D. Disregard C.
There should be a connection from C to D.

Question 13

A planar graph representing the above adjacency matrix is



Number each of the separate regions to find the total number of faces (regions). Make sure you include the outside or infinite region.
There are 6 regions.

Question 14

Create a 6 × 6 matrix labelling successive rows and columns with A, B, C, D, E and F respectively. Allocate numbers to the matrix that represent a connection between two nodes. Continue to allocate numbers until the matrix is complete. The first entry will be in the first row and second column indicating a connection between the vertices of A and B. This will be matched by an entry in the first column and second row that shows the connection from B to A.
Matrix B is the correct representation.

Question 15

A simple graph has no loops or multiple edges between vertices.

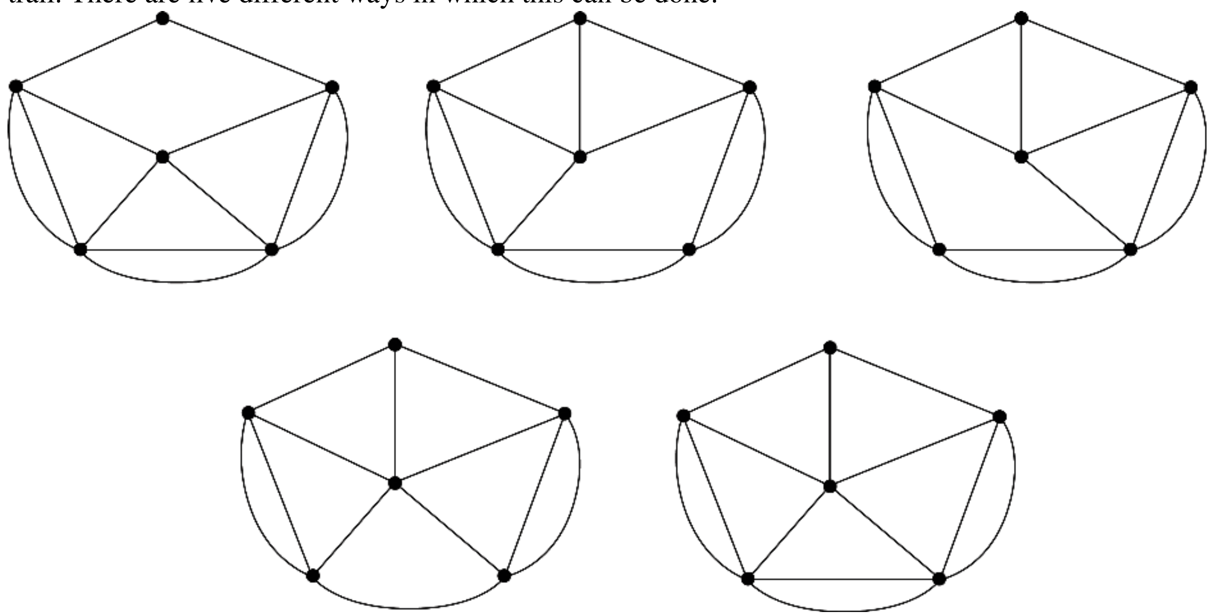
9.4 Walks, trails, paths, cycles and circuits

Question 1

A Hamiltonian cycle passes through each vertex only once and starts and finishes at the same vertex.
There is no edge connecting E and D. Therefore, option D is incorrect.

Question 2

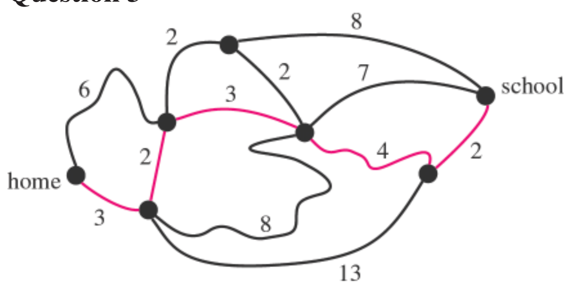
An Eulerian trail exists if the graph has two vertices with an odd degree and the degree of the other vertices are even. The given graph has six vertices of which four are of an odd degree and two are of an even degree. Removing any edge between two vertices that are of an odd degree will change the network to an Eulerian trail. There are five different ways in which this can be done.



***VCAA Examination Report* note:**

Students should be familiar with scenarios involving adding or removing an edge from a graph to enable an Eulerian trail. This question relied on the knowledge that a graph will have an Eulerian trail if exactly two of the vertices of that graph have an odd degree.

Question 3



The diagram is highlighted to show the shortest path.
Time = 3 + 2 + 3 + 4 + 2 = 14 minutes

Question 4

A loop joins a vertex to itself, so option B has the only loop.

Question 5

In an Eulerian circuit, each edge is used once and the path starts and finishes at the same vertex. This means that each vertex must be of even degree.

Graph 1 – All except one vertex are of odd degree.
Graph 2 – All vertices are of even degree.
Graph 3 – Two vertices are of odd degree.
Graph 4 – Two vertices are of odd degree.
Therefore only one graph (Graph 2) contains an Eulerian circuit.

Question 6

a. i. Euler path – every edge is only used once. If any nodes/vertices are of odd degree, the Euler path starts and finishes at those two positions. Therefore, Bower and Eden are at the start and end of Charlie's path as they are both vertices of odd degree. **[1 mark]**

ii. The sum of all the edges
$= 160 + 130 + 80 + 70 + 150 + 100 + 120 + 60 + 40 = 910$ km **[1 mark]**

b. Hamiltonian path – each vertex is used exactly once.
Shortest path from Bower to Attard:
Bower–Clement–Derrin–Eden–Attard
Distance $= 70 + 60 + 40 + 100 = 270$ km. **[1 mark]**

c. An Euler circuit is an Euler path that starts and finishes at the same vertex. For this to occur, all the vertices must be of even degree. If the line DE is gone, this makes Eden an even vertex, but means Derrin is now an odd vertex. The other odd vertex is Bower, so if the line between Bower and Derrin is removed, all the vertices will be even. **[1 mark]**

Question 7

A Hamiltonian path is a path that begins and ends at different vertices and passes through each vertex exactly once.

Question 8

The only sequence of vertices that does not pass over the same edge twice is HBAHGFH.

Question 9

A cycle finishes on its starting vertex, not its starting edge and it can only use vertices only once.

Question 10

Cycles and circuits start and end at the same vertex. The only sequence that does not end where it began is AECDAE. Therefore it does not represent either a cycle or circuit.

Question 11

The network currently contains 4 vertices with odd degrees, S, T, V and Z. If two of these can be changed to even degrees, leaving only two odd vertices, an Euler trail can be created. The only connection that would change two of these odd degree vertices into even degrees is SZ.

Question 12

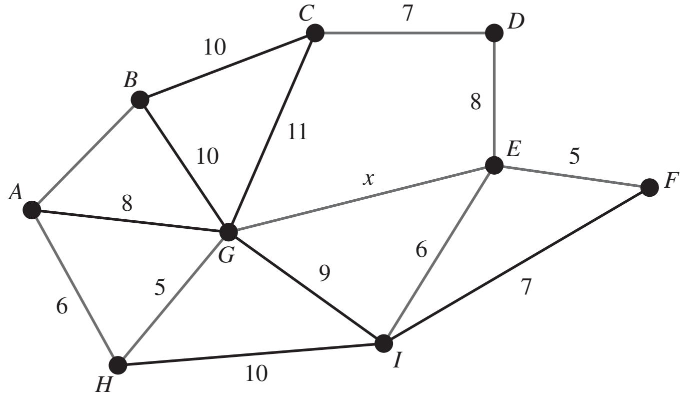
The sum of each row gives the degree of each vertex. Only Matrix E has all rows adding to an even number indicating that all vertices have an even degree. This is the only network that can contain an Eulerian circuit.

9.5 Trees and their applications

Question 1

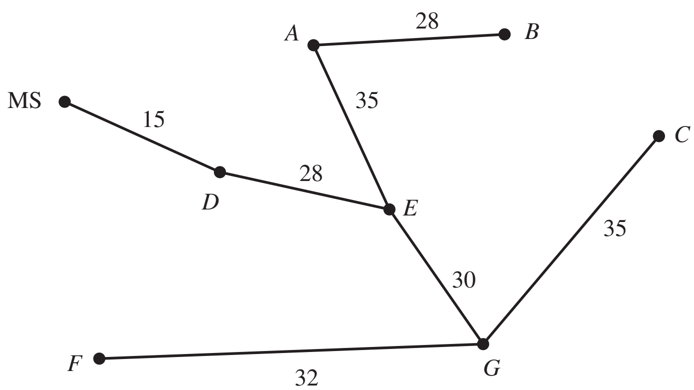
A spanning tree includes all the vertices and some of the edges of the original network, and no loops, multiple edges or cycles.
Option B has an edge connecting vertices 3 and 5 that is not present in the original network.

Question 2
Draw a minimum spanning tree that has a minimum length of 53 m.



$x = 53 - (7 + 6 + 5 + 6 + 5 + 8 + 7)$
$= 53 - 44$
$= 9$

Question 3
A minimum spanning tree is required.

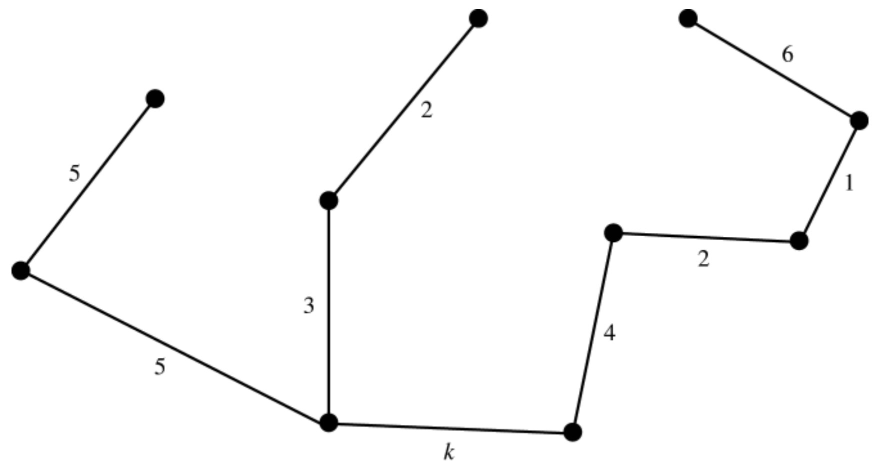


Add up the edges:
$15 + 28 + 30 + 32 + 35 + 35 + 28 = 203$

Question 4
The minimum cut, critical path analysis and the Hungarian algorithm cannot be used as this is an undirected graph. Prim's algorithm will find the minimum spanning tree for the network, but that does not necessarily mean it will be the shortest path from one specific node to another.
The shortest path from one node to another will be found by using Dijkstra's algorithm.

Question 5
A tree with n vertices will have $n - 1$ edges. There are 5 vertices, so $n = 5$; therefore, at least $5 - 1 = 4$ edges must be added.

Question 6
It can be useful to draw the maximum spanning tree.



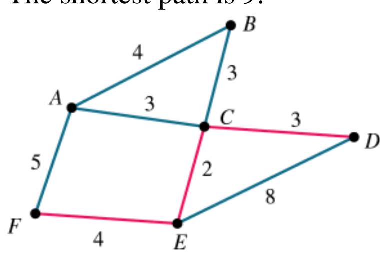
$$5 + 5 + 2 + 3 + k + 4 + 2 + 1 + 6 = 33$$
$$28 + k = 33$$
$$k = 5$$

Question 7
There are no loops, multiple edges or cycles. So this is a tree.

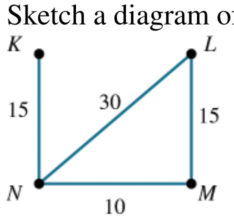
Question 8
Start with the smallest edge AJ (6). The smallest edge off either A or J is AB (8). Then the next smallest edge off any of the vertices A, J or B is BC (9).
Following on: (different orders are possible)
Edge BD (10) , JI (10) , DG (12) , GE (8) , EF (7) , IH (15).
The graph that matches these edges is A.

Question 9
A minimum spanning tree is required. The correct option is A.

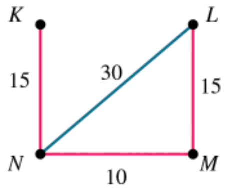
Question 10
The shortest path is 9.



Question 11
Sketch a diagram of the network.



K is only linked to N and hence the shortest distance from K to L is the shortest distance from N to L which is via M.



The shortest path is $15 + 10 + 15 = 40$.

9.6 Review

Question 1

a. Shortest distance is for $G - O - N - M$ for 86 km **[1 mark]**

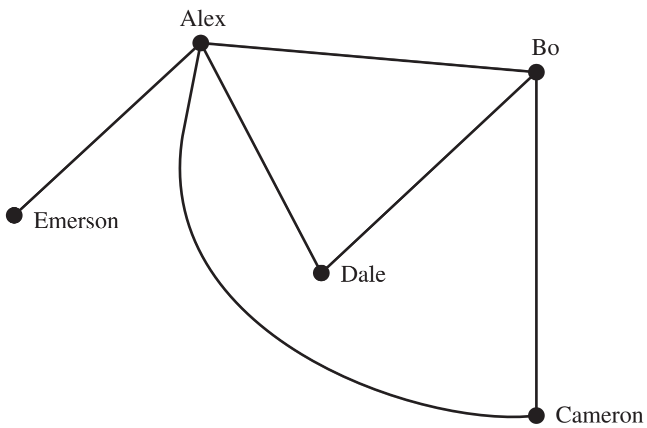
b. George will pass through K twice ($G - H - I - K - L - K - J - O - N - M$) **[1 mark]**

Question 2

a. Emerson has played cricket with only ONE other player (i.e. there is only one edge connecting Emerson). **[1 mark]**

b. Cameron and Dale (edges joining both these players to Alex and Bo) **[1 mark]**

c.



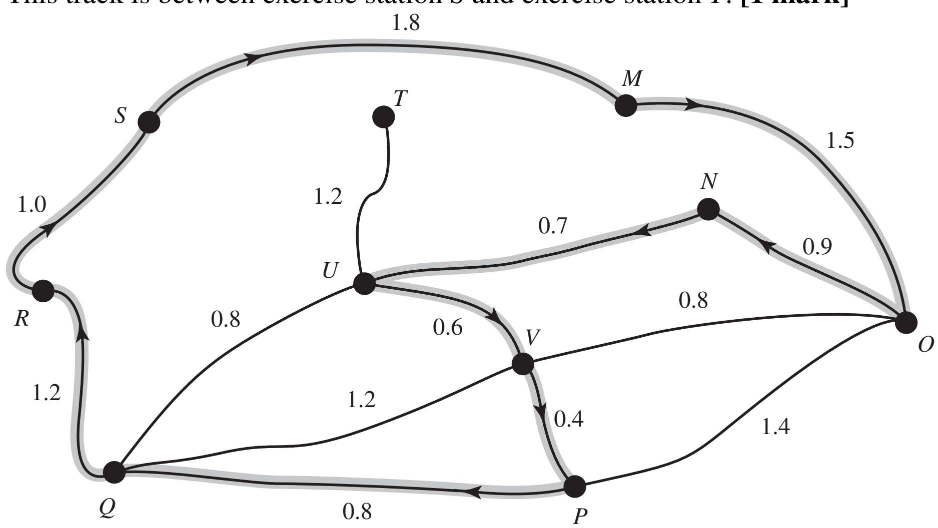
Question 3

a. Shortest distance $= 0.6 + 1.2 + 0.6 + 0.8 = 3.2$ kilometres **[1 mark]**

b. i. Eulerian trail **[1 mark]**

ii. Eulerian trails start and finish at vertices with an odd degree. The training program starts at S, with a degree of 3, and will finish at P, also with a degree of 3. **[1 mark]**

c. This track is between exercise station S and exercise station T. **[1 mark]**



Question 4

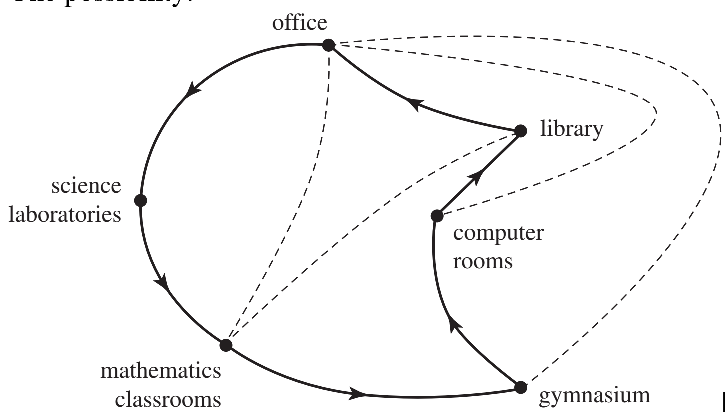
a. The office **[1 mark]**

b. i. Hamiltonian cycle **[1 mark]**

***VCAA Examination Report* note:**

A few responses erroneously named the route as a path or circuit.

ii. One possibility:



[1 mark]

Question 5

Isomorphic means the equivalent number of vertices and edges.

Graph 1 has four vertices and five edges, while Graph 2 has five vertices and six edges. Therefore, Graphs 1 and 2 are not isomorphic.

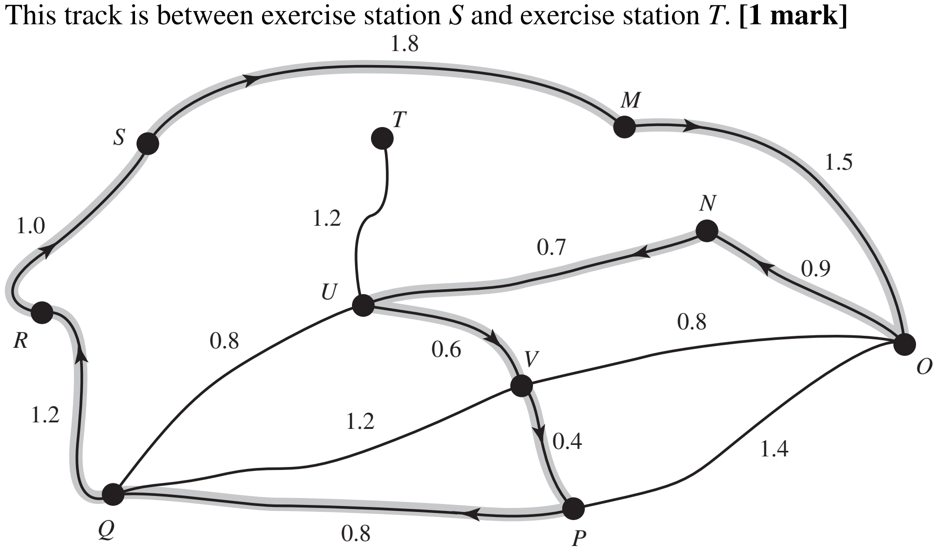
Question 6

a. Shortest distance $= 0.6 + 1.2 + 0.6 + 0.8 = 3.2$ kilometres **[1 mark]**

b. i. Eulerian trail **[1 mark]**

ii. Eulerian trails start and finish at vertices with an odd degree. The training program starts at S, with a degree of 3, and will finish at P, also with a degree of 3. **[1 mark]**

c. This track is between exercise station S and exercise station T. **[1 mark]**



Question 7

Isomorphic means the equivalent number of vertices and edges.

Graph 1 has four vertices and five edges, while Graph 2 has five vertices and six edges. Therefore, Graphs 1 and 2 are not isomorphic.

Question 8

a.

Student	Sport
Blake	**Tennis**
Charli	**Football**
Huan	**Basketball**
Marco	**Athletics**

[1 mark]

b. Using the Hungarian algorithm, start with row reduction:

	Student	100 m	200 m	300 m	400 m
Subtract 13.3 in this row	Anita	0	16.3	48.5	73.8
Subtract 14.5 in this row	Imani	0	15.1	49.0	74.4
Subtract 13.3 in this row	Jordan	0	16.0	50.3	75.8
Subtract 15.2 in this row	Lola	0	14.0	46.4	72.7

All the zeros can be covered with just one line, so we will need to continue with column reduction:
Again, allocation is not possible because all zeros can be covered with just two lines, so we will need to perform the Hungarian algorithm.
The smallest uncovered number from the table above is 1.1.
Add this to all covered numbers. At the intersection of the straight lines, add it twice.

	Subtract 0 (do nothing)	Subtract 14.0	Subtract 46.4	Subtract 72.7
Student	**100 m**	**200 m**	**300 m**	**400 m**
Anita	0	2.3	2.1	1.1
Imani	0	1.1	2.6	1.7
Jordan	0	2.0	3.9	3.1
Lola	0	0	0	0

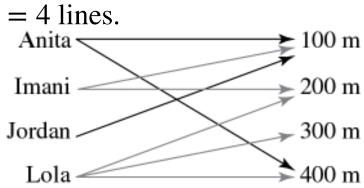
Again, allocation is not possible because all zeros can be covered with just two lines, so we will need to perform the Hungarian algorithm.
The smallest uncovered number from the table above is 1.1.
Add this to all covered numbers. At the intersection of the straight lines, add it twice.

Student	100 m	200 m	300 m	400 m
Anita	1.1	2.3	2.1	1.1
Imani	1.1	1.1	2.6	1.7
Jordan	1.1	2.0	3.9	3.1
Lola	2.2	1.1	1.1	1.1

The overall smallest number is still 1.1. This is now subtracted from all numbers in the table.

Student	100 m	200 m	300 m	400 m
Anita	0	1.2	1	0
Imani	0	0	1.5	0.6
Jordan	0	0.9	2.8	2.0
Lola	1.1	0	0	0

Attempting an optimal allocation again, we now need four lines to cover all the zeros – four options = 4 lines.



The allocations will be:

Student	Sprinting distance (m)
Anita	400 m
Imani	200 m
Jordan	100 m
Lola	300 m

Award **2 marks** for all four correct; **1 mark** for two correct

Question 9

a. There are 4 odd vertices; to make them all even, 2 edges need to be added. The inspector will have to travel along 2 roads more than once. **[1 mark]**

***VCAA Examination Report* note:**

An Eulerian circuit was required; therefore, two extra edges were needed to make all vertices of even degree. Common incorrect answers were 3 and 4.

b. PQ and ST are the shortest edges joining odd vertices, so they are the roads that the inspector will walk twice. This gives a minimum distance of$14 + 20 + 6 + 7 + 8 + 9 + 2 \times 10 + 2 \times 12 = 108$ km **[1 mark]**

***VCAA Examination Report* note:**

This question was not well answered, with many different incorrect responses given, including 109 and 113.

A number of students incorrectly calculated the minimum distance to visit all the towns (Hamiltonian) rather than check all the roads (Eulerian).

Question 10

a. Vertex F has a degree of four as there are 4 edge connections on it. **[1 mark]**

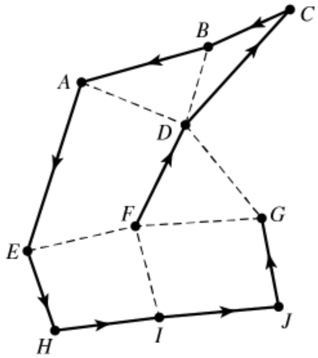
b. For an Eulerian trail to exist, only two vertices can have an odd degree. Currently vertices A, B, D, E, G and I have an odd degree.

The minimum number of edges to make an Eulerian trail possible is 2, provided they are between two of the odd degree vertices. **[1 mark]**

***VCAA Examination Report* note:**

Some students gave a definition of an Eulerian trail rather than stating how many extra edges were required.

c. Several answers are possible. One possibility is shown.



[1 mark]

Question 11

A path uses vertices only once and follows the edges from one vertex to the next.
The only path that is not possible is *PTQSR*, as there is no edge from *T* to *Q* or from *S* to *R*.

Question 12

a. 15 + 105 = \$120 **[1 mark]**

b. The cheapest route is \$80 (N – Q – R – S). The other towns to pass through are Quigley (*Q*) and Rosebush (*R*). **[1 mark]**

***VCAA Examination Report* note:**

This question was generally answered well.

Some gave the path $N - Q - R - S$, which was accepted.

c. $v + f = e + 2$

$6 + 7 = 11 + 2$ **[1 mark]**

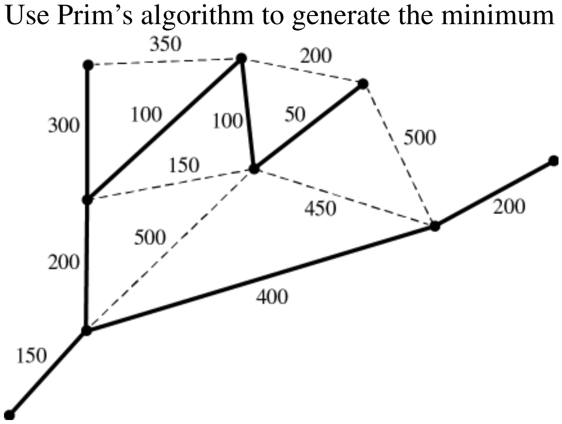
***VCAA Examination Report* note:**

This question was generally answered well, although some students did not seem to check that the values were arithmetically correct, for example, $6 + 6 = 11 + 2$.

Question 13

a. i. Minimum spanning tree **[1 mark]**

ii. Use Prim's algorithm to generate the minimum spanning tree as indicated in this figure.



[1 mark]

Question 14

a. An Eulerian trail means that Nathan starts at a vertex with odd degree (*W*) nd finishes at the other vertex of odd degree (*V*). **[1 mark]**

b. A Hamiltonian path means that Zoe passes over all the vertices (ramps) once: *XYTUZVW*. **[1 mark]**

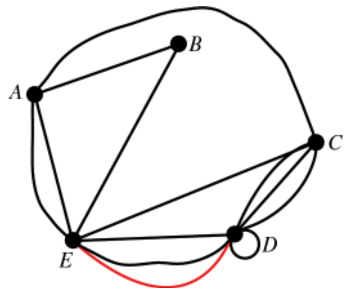
c. A Hamiltonian cycle means that Birra starts and finishes at the same vertex, and passes over all the vertices (ramps).

Birra can do this in 4 ways: *XYTUZVWX*, *XYTUVZWX*, and the same two routes in reverse order. **[1 mark]**

Question 15

a. Alooma and Easyside **[1 mark]**

b. i. **[1 mark]**



ii. The loop at D represents a route that a driver can use to depart from Dovenest and return to Dovenest without passing through any other suburb. **[1 mark]**

Question 16

a. i. Remember that the shortest path is needed. So it will be:
$factory - T - S - Q - R - S - U - factory$ (149 km). **[1 mark]**

ii. The path is not Hamiltonian because it passes through town S twice. (Hamiltonian paths/circuits pass through each vertex only once). **[1 mark]**

b. The Hamiltonian circuit would be $factory - T - S - R - Q - U - factory$ with a distance of 162 km. **[1 mark]**

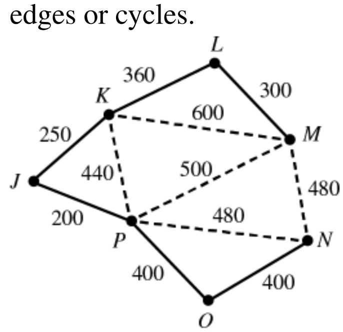
Question 17

a. Reading from the graph, the cost is \$300. **[1 mark]**

b. The cheapest way is from $K - P - N$ at a cost of \$920. **[1 mark]**

c. An Eulerian path would be followed (covering each edge only once), so the inspector would start at either of the odd degree vertices and finish at the other. So could start at P (or N) and finish at N (or P). **[1 mark]**

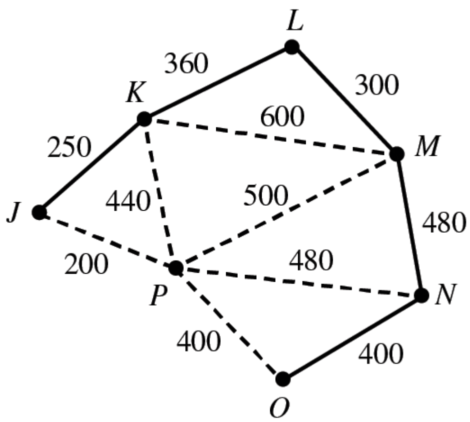
d. i. A minimum spanning tree connects all the vertices together, without creating any loops, multiple edges or cycles.



[1 mark]

***VCAA Assessment Report* note:**
Many students were unable to find this minimum spanning tree.
Common incorrect trees excluded *PO* or *KL* instead of *MN*.
Some students drew complete circuits.

ii. Including computer server P, the minimum installation cost would be \$1910. Excluding P, the minimum spanning tree becomes:



This minimum spanning tree has a cost of \$1790, which is a \$120 saving. **[1 mark]**

Question 18
Check all the paths in the map with the graph and you find that an edge between V and W is missing.

Question 19
Each vertex is used only once so this is a Hamiltonian path (not a circuit as the path doesn't start and end at the same vertex).

Question 20
A tree must not contain any circuits. Therefore, the network in option A is a tree.

Question 21
Using Euler's formula, the number of faces for this graph is $f = 2 - v + e = 2 - 5 + 4 = 1$.
Because there is only 1 face, the graph can always be redrawn so that no edges meet. Therefore the graph is always planar, so statement (1) is true.
As there is only 1 face, statement (2) is not true.
Because the number of edges is 1 less than the number of vertices, at least 2 vertices will not be directly connected by the edge and will hence have degree 1 (odd degree). Therefore, statement (3) is not true.
The sum of degrees of vertices = number of edges $\times 2 = 4 \times 2 = 8$. Therefore, statement (4) is true.
For the graph with v vertices to be connected, the minimum number of edges is $e = v - 1$, which is the case for this graph: $4 = 5 - 1$.
Because the graph has the minimum number of edges, a loop is not possible. (If an edge is used to form a loop, there will not be enough edges left to connect all vertices.) Therefore, statement (5) is true.
Statements (1), (4) and (5) are true. Therefore, the number of statements that are always true for this graph is 3.
***VCAA Assessment Report* note:**
One way of answering this question was to construct all possible different connected graphs with five vertices and four edges.

Question 22
a. The degree of the 'entrance' vertex = 3. **[1 mark]**
b. The shortest distance is via picnic area $P1 : 600 + 400 = 1000\,\text{m}$. **[1 mark]**
c. **i.** The ranger will finish at another vertex of odd degree, at picnic area $P4$. **[1 mark]**
ii. Because each edge is travelled once only and the starting point is not the same as the finishing point, the route the ranger takes is an Euler path. **[1 mark]**
d. Entrance, $P5$, $P4$, $P6$, $P3$, $P2$, $P1$ **[1 mark]**

Question 23
Dijkstra's algorithm produces the following summary of paths and distances.

Processed (solved) node	R	S	T	U	Path	Length
Q	8_Q	15_Q	∞	$\boxed{4_Q}$	Q – U	4 km.
U	$\boxed{8_Q}$	$\boxed{11_U}$	26_U		Q – R	8 km.
R		$\boxed{11_U}$	22_R		Q – U – S	11 km.
S			$\boxed{21_S}$		Q – U – S – T	21 km.

The shortest distance is 21 km following the path Q – U – S – T

Question 24
The matrix that shows a single row and matching column of zeros indicates no connection between that vertex and all other vertices in the network. The only matrix that exhibits this property is Matrix D. Drawing a network for this matrix would show vertex 1 to have no connections to any other vertex in the network.

Question 25
57 km is travelled using the route AEBD or ABED
90 km is travelled using the route ACEBD
66 km is travelled using the route AECD
52 km is travelled using the route ABD
The distance of 40 km cannot be achieved by any valid route.

Question 26
Starting the journey at A, travelling to B (11), then to D (10), onto C (6), then E (4), followed by F (3) and finishing at G (12).
This route takes in the three largest distances in the network.
Adding the values gives a total of 46 km.

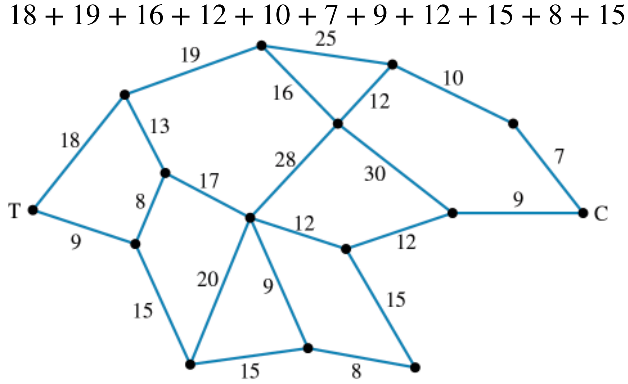
Question 27
A trail uses edges only once. Starting at P then travelling to R (11), then Q (15), then P (29), followed by S (26), then T (23), followed by U (19), then V (25) and finishing at T (9) gives a total of 157 units.

Question 28
Minimum Hamiltonian cycle is ABDEGFCA or ABEGFDCA with a length of
$11 + 9 + 8 + 11 + 6 + 12 + 4 = 61$
The Eulerian circuit uses every edge once with a total distance of 118.
The difference between them is $118 - 61 = 57$

Question 29

The network below shows the route needed. The total distance of this route is

$18 + 19 + 16 + 12 + 10 + 7 + 9 + 12 + 15 + 8 + 15 + 20 + 17 + 8 + 9 = 195$ km



Question 30

Dijkstra's algorithm produces the following summary of paths and distances.

Processed (solved) node	A	B	C	D	E	F	Work	Path	Length
Home	12_H	12_H	$\boxed{8_H}$	∞	∞	∞	∞	Home–C	8
C	$\boxed{11_C}$	12_H		15_C	24_C	∞	∞	Home–C – A	11
A		$\boxed{12_H}$		15_C	24_C	∞	∞	Home–B	12
B				$\boxed{15_C}$	24_C	∞	33_B	H – C – D	15
D					24_C	$\boxed{20_D}$	33_B	H – C – D – F	20
F					$\boxed{24_C}$		29_F	H – C – E	24
E							$\boxed{29_F}$	H – C – D – F – W	29

The shortest time is 29 minutes along the path Home-C – D – F-Work

10 Directed graphs and network flow

Topic	10	Directed graphs and network flow
Subtopic	10.2	Precedence tables and activity networks

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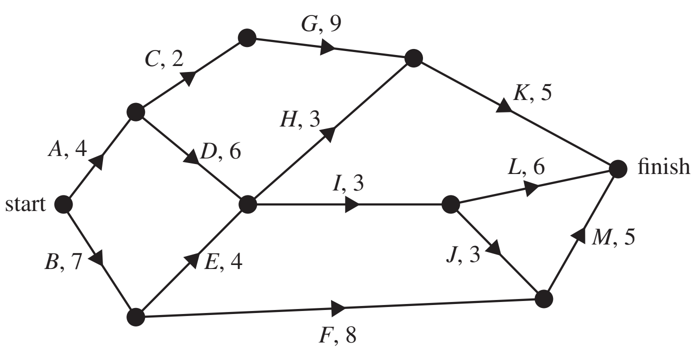
Source: *VCE 2020, Further Mathematics 1, Section B, Module 2, Q.6; © VCAA*

Question 1 (1 mark)

The activity network below shows the sequence of activities required to complete a project.

The number next to each activity in the network is the time it takes to complete that activity, in days.



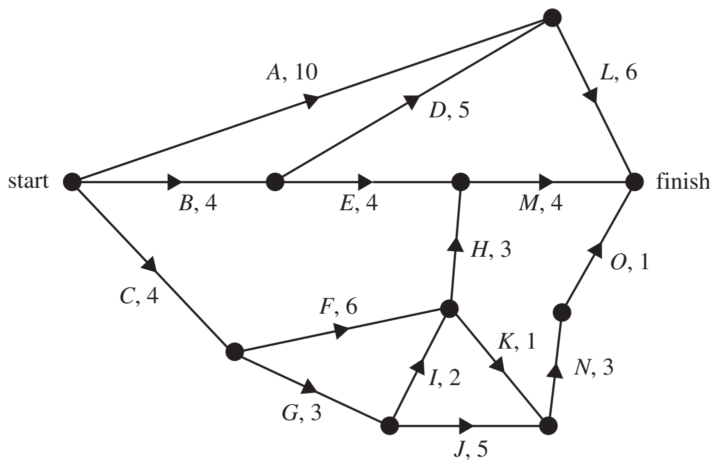
The minimum completion time for this project, in days, is

A. 18
B. 19
C. 20
D. 21
E. 22

Source: *VCE 2017, Further Mathematics 1, Section B, Module 2, Q.4; © VCAA*

Question 2 (1 mark)

The directed graph below shows the sequence of activities required to complete a project.
The time to complete each activity, in hours, is also shown.



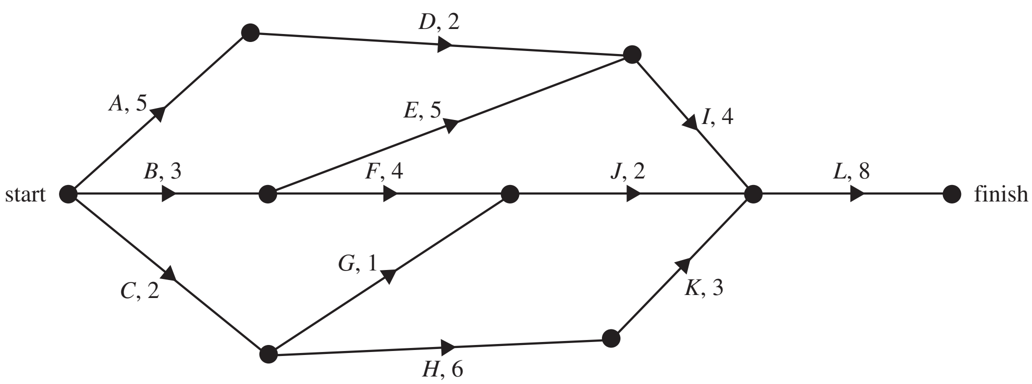
The earliest starting time, in hours, for activity *N* is

A. 3
B. 10
C. 11
D. 12
E. 13

Source: *VCE 2016, Further Mathematics 1, Section B, Module 2, Q.6; © VCAA*

Question 3 (1 mark)

The directed graph below shows the sequence of activities required to complete a project.
All times are in hours.



The number of activities that have exactly two immediate predecessors is

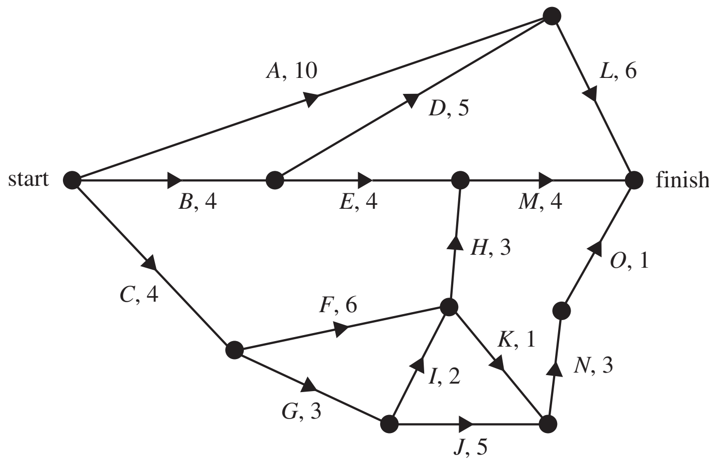
A. 0
B. 1
C. 2
D. 3
E. 4

Topic 10 | Subtopic 10.2 Precedence tables and activity networks

Source: *VCE 2017, Further Mathematics 1, Section B, Module 2, Q.4; © VCAA*

Question 4 (1 mark)

The time to complete each activity, in hours, is also shown.

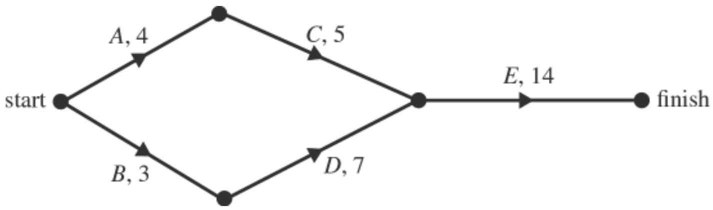


The earliest starting time, in hours, for activity N is

A. 3
B. 10
C. 11
D. 12
E. 13

Source: *VCE 2013, Further Mathematics 1, Section B, Module 5, Q.8 ; © VCAA*

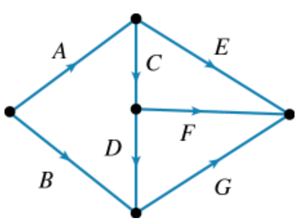
Question 5 (1 mark)



The graph above shows five activities, A, B, C, D and E, that must be completed to finish a project.
The number next to each letter shows the completion time, in hours, for the activity.
Each of the five activities can have its completion time reduced by a maximum of one hour at a cost of $100 per hour.
The least cost to achieve the greatest reduction in the time taken to finish the project is

A. $100
B. $200
C. $300
D. $400
E. $500

Question 6 (1 mark)



Which of the following statements is true about the network shown above?

A. *A* is a predecessor for *E*.
B. *B* is a predecessor for *E*.
C. *C* is a predecessor of *E*.
D. *B* is a predecessor of *D*.
E. *F* is a predecessor of *G*.

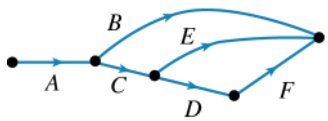
Question 7 (1 mark)

For a particular project, there are 6 activities that need to be completed in a specific order. The activities and their immediate predecessors are shown in the table below.

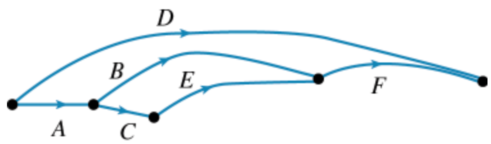
Activity	Immediate Predecessors
A	-
B	*A*
C	*A*
D	-
E	*C*
F	*B*, *E*

Which of the following directed graphs could represent this project?

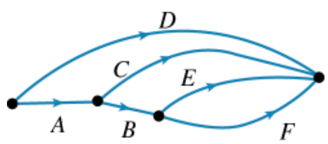
A.



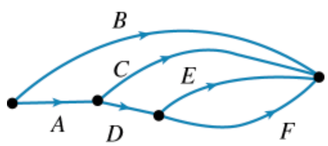
B.



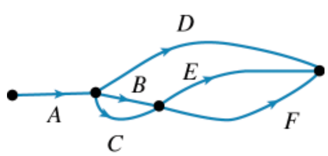
C.



D.



E.



Question 8 (1 mark)
The table below shows the activities involved in a project.

Activity	Immediate predecessor	Duration (hours)
A	-	5
B	*A*	3
C	*A*	2
D	*C*	5

The earliest start time (in hours) for Activity D is
A. 5
B. 7
C. 13
D. 8
E. 15

Topic	10	Directed graphs and network flow
Subtopic	10.3	Critical path analysis with backward scanning and crashing

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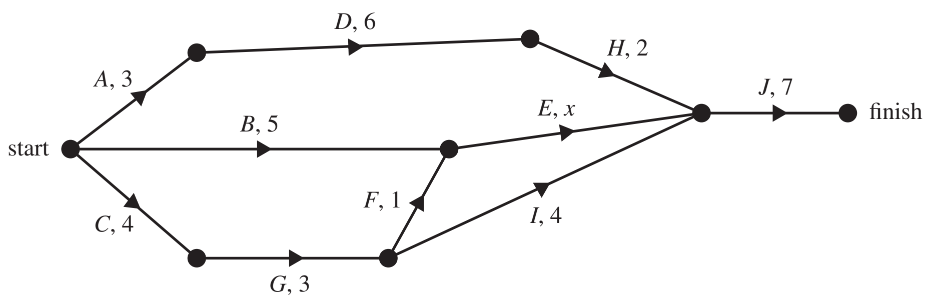
Source: *VCE 2021, Further Mathematics 1, Section B, Module 2, Q.6;* © VCAA

Question 1 (1 mark)

The directed graph below shows the sequence of activities required to complete a project.
The time taken to complete each activity, in hours, is also shown.



The minimum completion time for this project is 18 hours.
The time taken to complete activity E is labelled x.
The maximum value of x is

A. 2
B. 3
C. 4
D. 5
E. 6

Source: *VCE 2019, Further Mathematics 1, Section B, Module 2, Q.7*; © VCAA

Question 2 (1 mark)

A project involves nine activities, *A* to *I*.

The immediate predecessor(s) of each activity is shown in the table below.

Activity	Immediate predecessor(s)
A	-
B	*A*
C	*A*
D	*B*
E	*B*, *C*
F	*D*
G	*D*
H	*E*, *F*
I	*G*, *H*

A directed network for this project will require a dummy activity.

The dummy activity will be drawn from the end of

A. activity *B* to the start of activity *C*.

B. activity *B* to the start of activity *E*.

C. activity *D* to the start of activity *E*.

D. activity *E* to the start of activity *H*.

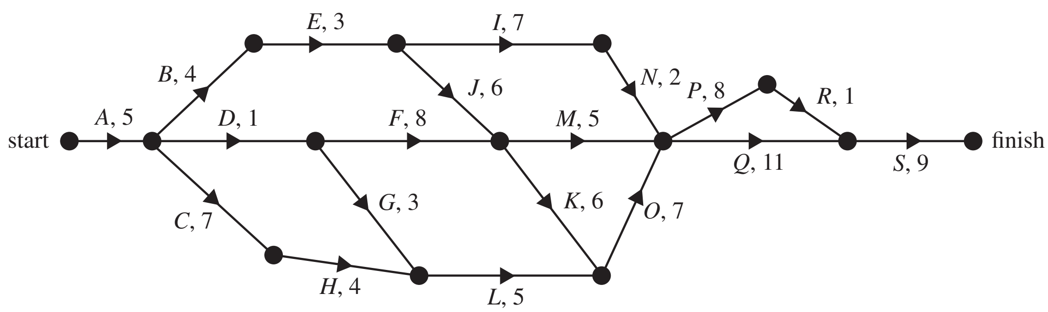
E. activity *E* to the start of activity *F*.

Source: *VCE 2019, Further Mathematics 1, Section B, Module 2, Q.8*; © VCAA

Question 3 (1 mark)

The directed network below shows the sequence of activities, A to S, that is required to complete a manufacturing process.

The time taken to complete each activity, in hours, is also shown.



The number of activities that have a float time of 10hours is

A. 0
B. 1
C. 2
D. 3
E. 4

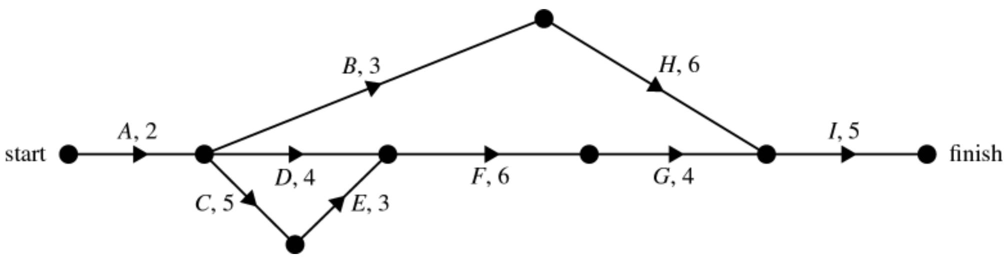
Topic 10 Subtopic 10.3 Critical path analysis with backward scanning and crashing

Source: *VCE 2020, Further Mathematics 1, Section B, Module 2, Q.10*; © VCAA

Question 4 (1 mark)

The directed network below shows the sequence of activities, A to I, that is required to complete an office renovation.

The time taken to complete each activity, in weeks, is also shown.



The project manager would like to complete the office renovation in less time.

The project manager asks all the workers assigned to activity H to also work on activity F.

This will reduce the completion time of activity F to three weeks.

The workers assigned to activity H cannot work on both activity H and activity F at the same time.

No other activity times will be changed.

This change to the network will result in a change to the completion time of the office renovation. Which one of the following is correct?

A. The completion time will be reduced by one week if activity F is completed before activity H is started.

B. The completion time will be reduced by three weeks if activity F is completed before activity H is started.

C. The completion time will be reduced by one week if activity H is completed before activity F is started.

D. The completion time will be reduced by three weeks if activity H is completed before activity F is started.

E. The completion time will be increased by three weeks if activity H is completed before activity F is started.

Source: *VCE 2018, Further Mathematics 1, Section B, Module 2, Q.7*; © VCAA

Question 5 (1 mark)

A project requires nine activities ($A - 1$) to be completed. The duration, in hours, and the immediate predecessor(s) of each activity areshown in the table below.

Activity	Duration (hours)	Immediate predecessor(s)
A	4	-
B	3	*A*
C	7	*A*
D	2	*A*
E	5	*B*
F	2	*C*
G	4	*E*, *F*
H	5	*D*
I	3	*G*, *H*

The minimum completion time for this project, in hours, is

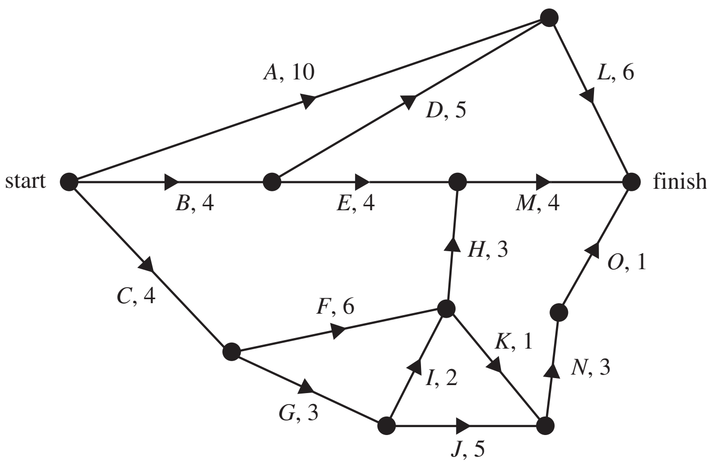
A. 14
B. 19
C. 20
D. 24
E. 35

Topic 10 Subtopic 10.3 Critical path analysis with backward scanning and crashing

Source: *VCE 2017, Further Mathematics 1, Section B, Module 2, Q.5*; © VCAA

Question 6 (1 mark)

The directed graph below shows the sequence of activities required to complete a project.
The time to complete each activity, in hours, is also shown.



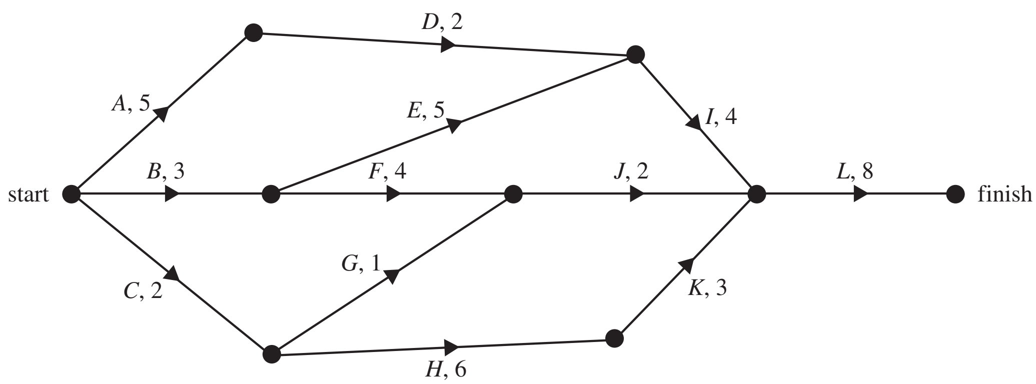
To complete the project in minimum time, some activities cannot be delayed.
The number of activities that cannot be delayed is

A. 2
B. 3
C. 4
D. 5
E. 6

Source: *VCE 2016, Further Mathematics 1, Section B, Module 2, Q.6*; © VCAA

Question 7 (1 mark)

The directed graph below shows the sequence of activities required to complete a project.
All times are in hours.



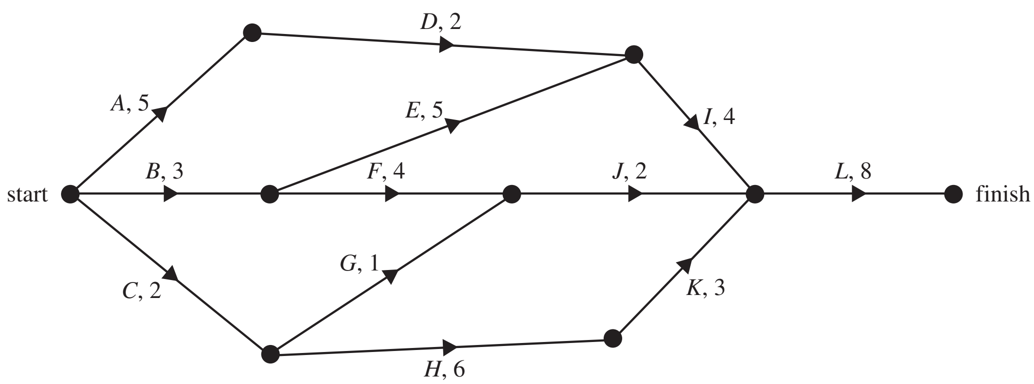
The number of activities that have exactly two immediate predecessors is

A. 0
B. 1
C. 2
D. 3
E. 4

Source: *VCE 2016, Further Mathematics 1, Section B, Module 2, Q.7*; © VCAA

Question 8 (1 mark)

The directed graph below shows the sequence of activities required to complete a project.
All times are in hours.



There is one critical path for this project.
Three critical paths would exist if the duration of activity

A. *I* were reduced by two hours.
B. *E* were reduced by one hour.
C. *G* were increased by six hours.
D. *K* were increased by two hours.
E. *F* were increased by two hours.

Source: *VCE 2015, Further Mathematics 1, Section B, Module 5, Q.9*; © VCAA

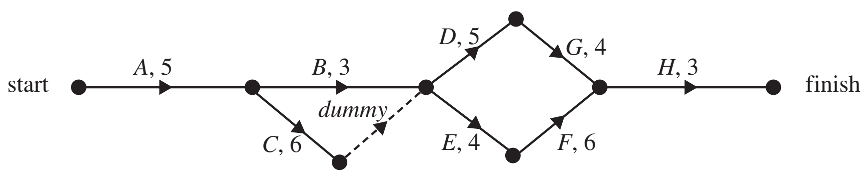
Question 9 (1 mark)

The table below shows, in minutes, the duration, the earliest starting time (EST) and the latest starting time (LST) of eight activitiesneeded to complete a project

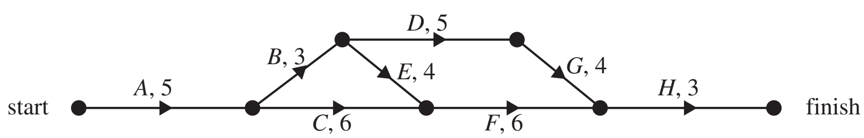
Activity	Duration	EST	LST
A	5	0	0
B	3	5	5
C	6	5	6
D	5	8	9
E	4	8	8
F	6	12	12
G	4	13	14
H	3	18	18

Which one of the following directed graphs shows the sequence of these activities?

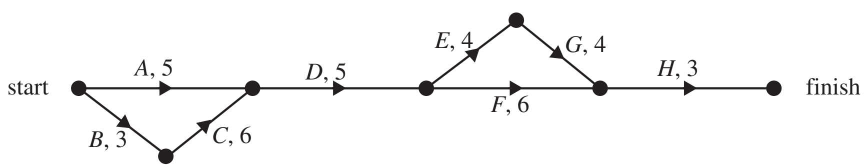
A.



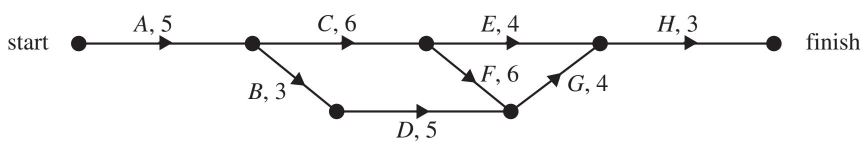
B.



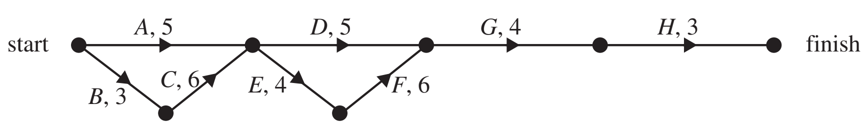
C.



D.



E.



Source: *VCE 2014, Further Mathematics 1, Section B, Module 5, Q.8*; © VCAA

Question 10 (1 mark)

Which one of the following statements about critical paths is true?

A. There can be only one critical path in a project.

B. A critical path always includes at least two activities.

C. A critical path will always include the activity that takes the longest time to complete.

D. Reducing the time of any activity on a critical path for a project will always reduce the minimum completion time for the project.

E. If there are no other changes, increasing the time of any activity on a critical path will always increase the completion time of a project.

Question 11 (1 mark)

The table below shows the activities involved in a project.

Activity	Immediate predecessor	Duration (hours)
A	-	5
B	A	3
C	A	2
D	C	5

For the network showing the relationships between the activities, a dummy activity is required before

A. Activity A.

B. Activity B.

C. Activity A and Activity B.

D. Activity A and Activity C.

E. Activity D.

Topic	10	Directed graphs and network flow
Subtopic	10.4	Flow problems

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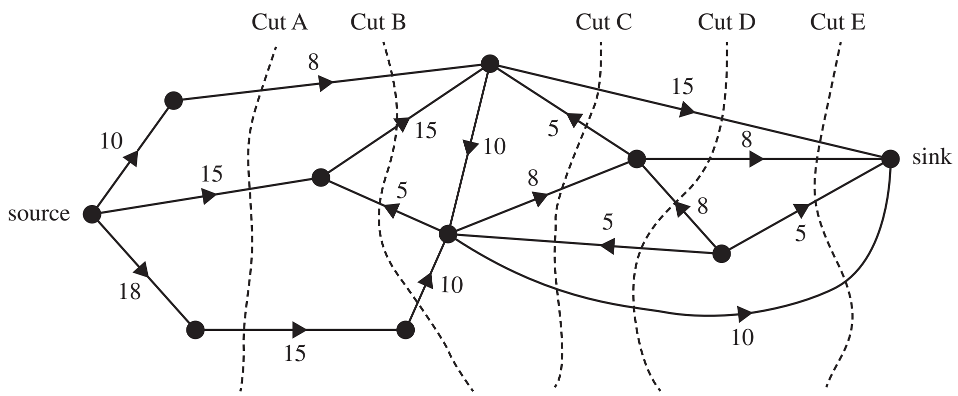
Source: *VCE 2020, Further Mathematics 1, Section B, Module 2, Q.9;* © VCAA

Question 1 (1 mark)

The flow of liquid through a series of pipelines, in litres per minute, is shown in the directed network below.



Five cuts labelled A to E are shown on the network.

The number of these cuts with a capacity equal to the maximum flow of liquid from the source to the sink, in litres per minute, is

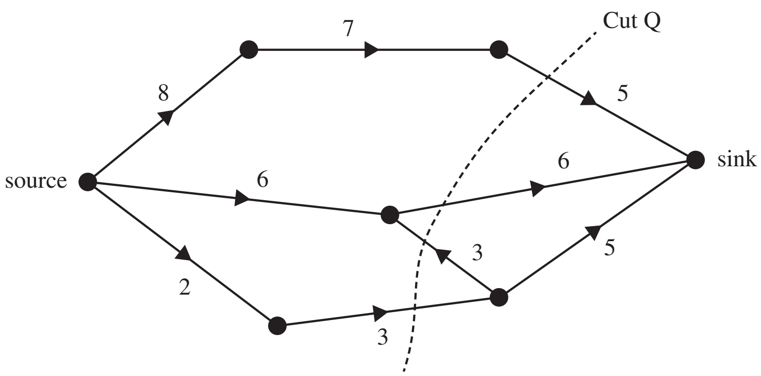
A. 1
B. wq 2
C. wq 3
D. wq 4
E. wq 5

Source: *VCE 2019, Further Mathematics 1, Section B, Module 2, Q.3;* © VCAA

Question 2 (1 mark)

The flow of water through a series of pipes is shown in the network below.

The numbers on the edges show the maximum flow through each pipe in litres per minute.



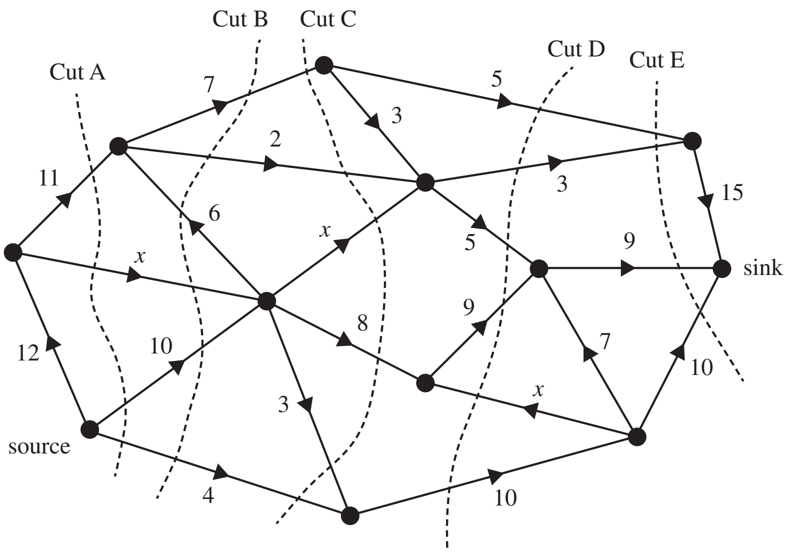
The capacity of Cut Q, in litres per minute, is

A. 11
B. 13
C. 14
D. 16
E. 17

Source: *VCE 2017, Further Mathematics 1, Section B, Module 2, Q.8;* © VCAA

Question 3 (1 mark)

The flow of oil through a series of pipelines, in litres per minute, is shown in the network below.



The weightings of three of the edges are labelled x .

Five cuts labelled A–E are shown on the network.

The maximum flow of oil from the source to the sink, in litres per minute, is given by the capacity of

A. Cut A if $x = 1$

B. Cut B if $x = 2$

C. Cut C if $x = 2$

D. Cut D if $x = 3$

E. Cut E if $x = 3$

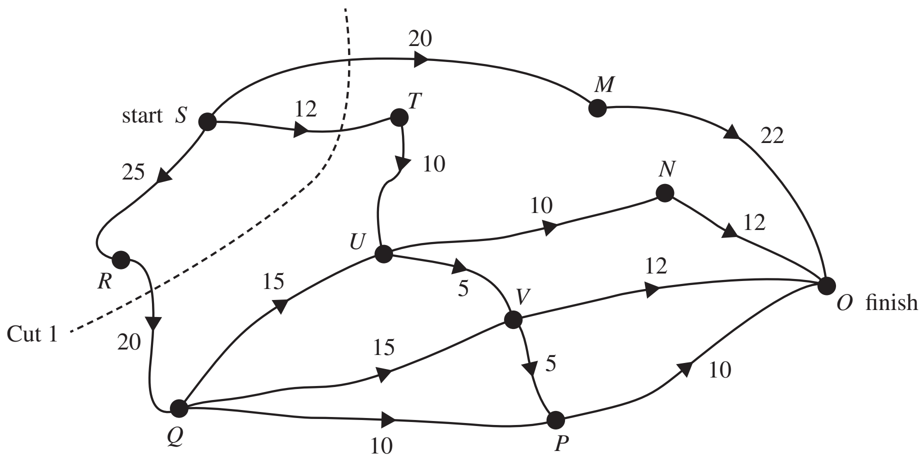
Source: *VCE 2020, Further Mathematics 2, Section B, Module 2, Q.4;* © VCAA

Question 4 (3 mark)

Training program 1 has the cricket team starting from exercise station S and running to exercise station O.

For safety reasons, the cricket coach has placed a restriction on the maximum number of people who can use the tracks in the fitness park.

The directed graph below shows the capacity of the tracks, in number of people per minute.



a. How many different routes from S to O are possible? **(1 mark)**

b. When considering the possible flow of people through this network, many different cuts can be made. Determine the capacity of Cut 1, shown above. **(1 mark)**

c. What is the maximum flow from S to O, in number of people per minute? **(1 mark)**

Topic 10 Subtopic 10.4 Flow problems

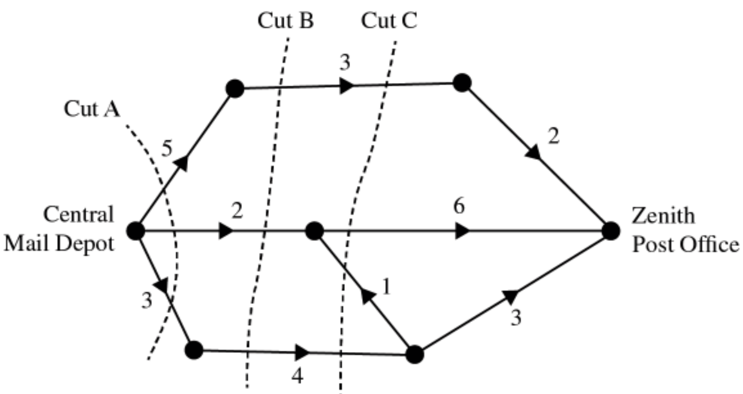
Source: *VCE 2018, Further Mathematics 2, Section B, Module 2, Q.1;* © VCAA

Question 5 (3 marks)

The graph below shows the possible number of postal deliveries each day between the Central Mail Depot and the Zenith Post Office.

The unmarked vertices represent other depots in the region.

The weighting of each edge represents the maximum number of deliveries that can be made each day.



a. Cut A, shown on the graph, has a capacity of 10.
Two other cuts are labelled as Cut B and Cut C.

i. Write down the capacity of Cut B. **(1 mark)**

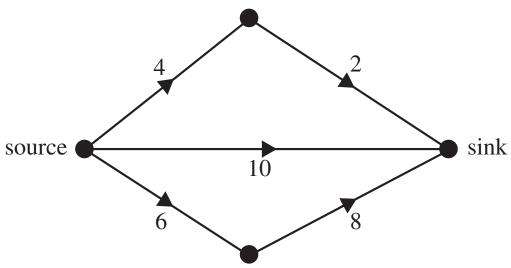
ii. Write down the capacity of Cut C. **(1 mark)**

b. Determine the maximum number of deliveries that can be made each day from the Central Mail Depot to the Zenith Post Office. **(1 mark)**

Source: *VCE 2016, Further Mathematics 1, Section B, Module 2, Q.2;* © VCAA

Question 6 (1 mark)

The following directed graph shows the flow of water, in litres per minute, in a system of pipes connecting the source to the sink.



The maximum flow, in litres per minute, from the source to the sink is

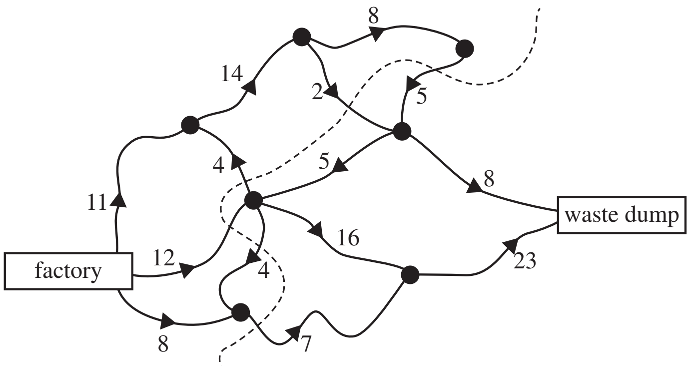
A. 10
B. 14
C. 18
D. 20
D. 22

Source: *VCE 2015, Further Mathematics 1, Section B, Module 5, Q.4;* © VCAA

Question 7 (1 mark)

The arrows on the diagram below show the direction of the flow of waste through a series of pipelines from a factory to a waste dump.

The numbers along the edges show the number of megalitres of waste per week that can flow through each section of pipeline.



The minimum cut is shown as a dotted line.

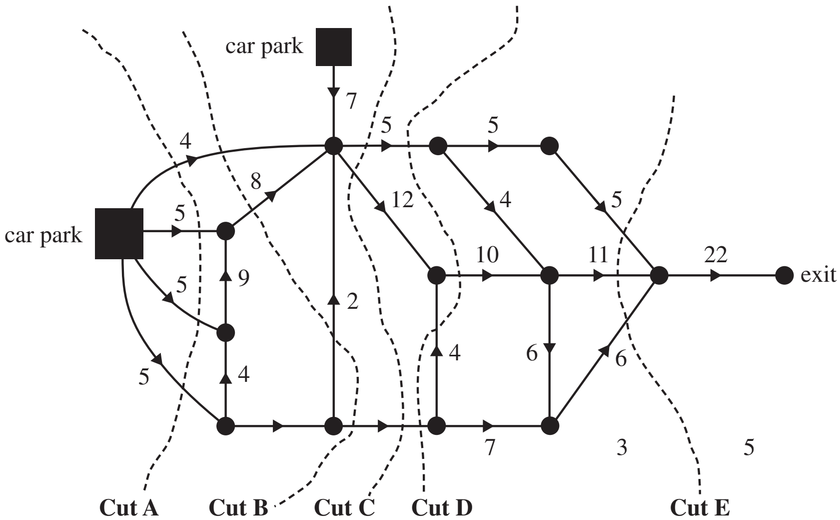
The capacity of this cut, in megalitres of waste per week, is

A. 6
B. 18
C. 26
D. 32
E. 34

Source: *VCE 2014, Further Mathematics 1, Section B, Module 5, Q.9;* © VCAA

Question 8 (1 mark)

A network of tracks connects two car parks in a festival venue to the exit, as shown in the directed graph below.



The arrows show the direction that cars can travel along each of the tracks and the numbers show each track's capacity in cars per minute.

Five cuts are drawn on the diagram.

The maximum number of cars per minute that will reach the exit is given by the capacity of

A. Cut A
B. Cut B
C. Cut C
D. Cut D
E. Cut E

Topic	**10**	**Directed graphs and network flow**
Subtopic	**10.5**	**Bipartite graphs and allocation problems**

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Source: *VCE 2020, Further Mathematics 1, Section B, Module 2, Q.7; © VCAA*

Question 1 (1 mark)

Four friends go to an ice-cream shop.
Akiro chooses chocolate and strawberry ice cream.
Doris chooses chocolate and vanilla ice cream.
Gohar chooses vanilla ice cream.
Imani chooses vanilla and lemon ice cream.
This information could be presented as a graph.
Consider the following four statements:

- The graph would be connected.
- The graph would be bipartite.
- The graph would be planar.
- The graph would be a tree.

How many of these four statements are true?

A. 0
B. 1
C. 2
D. 3
E. 4

Source: *VCE 2020, Further Mathematics 2, Section B, Module 2, Q.2; © VCAA*

Question 2 (1 mark)

A cricket team has 11 players who are each assigned to a batting position.

Three of the new players, Alex, Bo and Cameron, can bat in position 1, 2 or 3.

The table below shows the average scores, in runs, for each player for the batting positions 1, 2 and 3.

		Batting position		
		1	2	3
Player	**Alex**	22	24	24
	Bo	25	25	21
	Cameron	24	25	19

Each player will be assigned to one batting position.

To which position should each player be assigned to **maximise** the team's score? Write your answer in the table below.

Player	**Batting position**
Alex	
Bo	
Cameron	

Source: *VCE 2016, Further Mathematics 1, Section B, Module 2, Q.8; © VCAA*

Question 3 (1 mark)

Five children, Alan, Brianna, Chamath, Deidre and Ewen, are each to be assigned a different job by their teacher. The table below shows the time, in minutes, that each child would take to complete each of the five jobs.

	Alan	**Brianna**	**Chamath**	**Deidre**	**Ewen**
Job 1	5	8	5	8	7
Job 2	5	7	6	7	4
Job 3	9	5	7	5	9
Job 4	7	7	9	8	5
Job 5	4	4	4	4	3

The teacher wants to allocate the jobs so as to minimise the total time taken to complete the five jobs.

In doing so, she finds that two allocations are possible.

If each child starts their allocated job at the same time, then the first child to finish could be either

A. Alan or Brianna.

B. Brianna or Deidre.

C. Chamath or Deidre.

D. Chamath or Ewen.

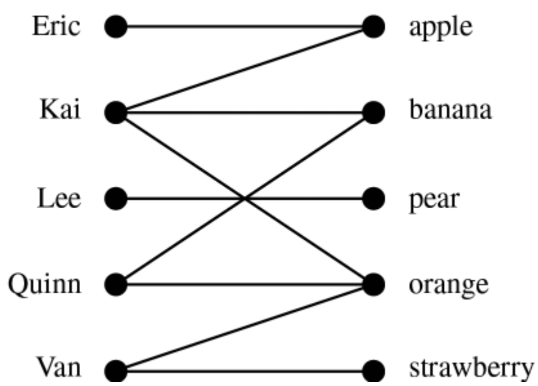
E. Deidre or Ewen.

Source: *VCE 2021, Further Mathematics 1, Section B, Module 2, Q.2; © VCAA*

Question 4 (1 mark)

Five friends ate fruit for morning tea.

The bipartite graph below shows which types of fruit each friend ate.



Which one of the following statements is **not** true?

A. Only Lee ate pear.

B. Eric and Kai each ate apple.

C. Van ate only strawberry.

D. Quinn and Kai each ate banana.

E. Orange was the most eaten type of fruit.

Source: *VCE 2018, Further Mathematics 1, Section B, Module 2, Q.8*; © *VCAA*

Question 5 (1 mark)

Annie, Buddhi, Chuck and Dorothy work in a factory.

Today each worker will complete one of four tasks, 1, 2, 3 and 4.

The usual completion times for Annie, Chuck and Dorothy are shown in the table below.

	Task 1	**Task** 2	**Task** 3	**Task** 4
Annie	7	3	8	2
Buddhi	k	k	3	k
Chuck	5	6	9	2
Dorothy	4	8	5	3

Buddhi takes 3 minutes for Task 3.

He takes k minutes for each other task.

Today the factory supervisor allocates the tasks as follows:

- Task 1 to Dorothy
- Task 2 to Annie
- Task 3 to Buddhi
- Task 4 to Chuck

This allocation will achieve the minimum total completion time if the value of k is at least

A. 0

B. 1

C. 2

D. 3

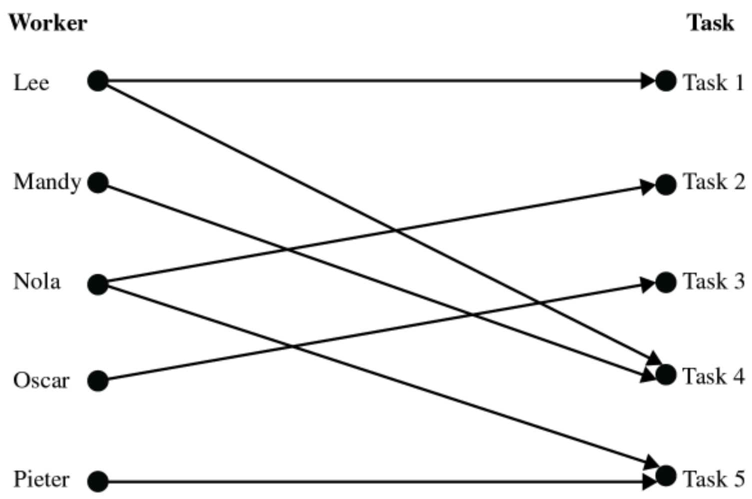
E. 4

Topic 10 | Subtopic 10.5 Bipartite graphs and allocation problems

Source: *VCE 2016, Further Mathematics 1, Section B, Module 2, Q.1; © VCAA*

Question 6 (1 mark)

Lee, Mandy, Nola, Oscar and Pieter are each to be allocated one particular task at work.
The bipartite graph below shows which task(s), 1–5, each person is able to complete.



Each person completes a different task.
Task 4 must be completed by

A. Lee.
B. Mandy.
C. Nola.
D. Oscar.
E. Pieter.

Source: *VCE 2015, Further Mathematics 1, Section B, Module 5, Q.7; © VCAA*

Question 7 (1 mark)

Four people, Abe, Bailey, Chris and Donna, are each to be allocated one of four tasks. Each person can complete each of the four tasks in a set time. These times, in minutes, are shown in the table below.

	Abe	Bailey	Chris	Donna
Task 1	80	100	95	90
Task 2	95	95	100	90
Task 3	125	125	125	120
Task 4	60	60	75	65

If each person is allocated a different task, the minimum total time for these four people to complete these four tasks is

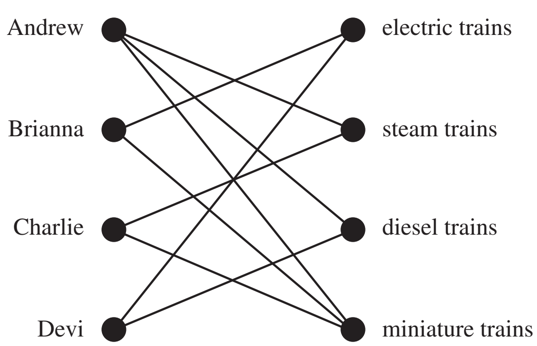
A. 260 minutes
B. 355 minutes
C. 360 minutes
D. 365 minutes
E. 375 minutes

Source: *VCE 2014, Further Mathematics 2, Module 5, Q.1; © VCAA*

Question 8 (2 marks)

Four members of a train club, Andrew, Brianna, Charlie and Devi, have joined one or more interest groups for electric, steam, diesel or miniature trains.

The edges of the bipartite graph below show the interest groups that these four train club members have joined.



a. How many of these four members have joined the steam trains interest group? **(1 mark)**

b. Which interest group have both Brianna and Charlie joined? **(1 mark)**

Source: *VCE 2013, Further Mathematics 1, Section B, Module 5, Q.4; © VCAA*

Question 9 (1 mark)

Kate, Lexie, Mei and Nasim enter a competition as a team. In this competition, the team must complete four tasks, W, X, Y and Z, as quickly as possible.

The table shows the time, in minutes, that each person would take to complete each of the four tasks.

	Kate	Lexie	Mei	Nasim
W	6	3	4	6
X	4	3	5	5
Y	5	7	9	6
Z	3	2	3	2

If each team member is allocated one task only, the minimum time in which this team would complete the four tasks is

A. 10 minutes
B. 12 minutes
C. 13 minutes
D. 14 minutes
E. 15 minutes

Question 10 (1 mark)

The time in minutes for a team of people to complete a series of tasks is given in the following table.

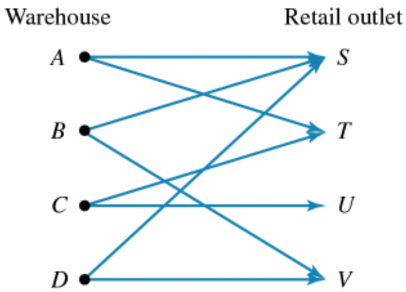
	T1	T2	T3	T4	T5
P1	12	9	11	8	15
P2	12	15	13	12	6
P3	9	11	8	11	14
P4	8	12	9	13	14
P5	12	15	13	15	13

Determine the allocation of tasks to each individual to ensure all tasks are completed in the minimum time possible.

A. 38 mins
B. 39 mins
C. 42 mins
D. 45 mins
E. 54 mins

Question 11 (1 mark)

The bipartite graph below represents the roadway connections between four warehouses and their retail outlets.



The matrix representation for this task allocation is

A.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

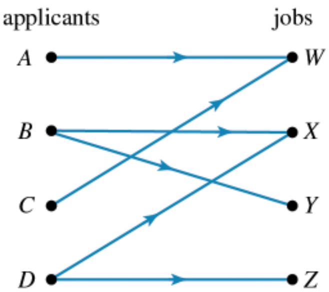
B.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

C.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

D.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

E.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Question 12 (1 mark)

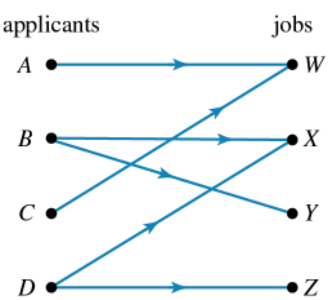


Which of the following statements is true about the bipartite graph above?

A. Person A must do job W.
B. Person B must do job Z.
C. Person C cannot do job W.
D. It is possible to fill all the positions.
E. Person D can do jobs W, X or Y.

Question 13 (1 mark)

Consider the bipartite graph below.

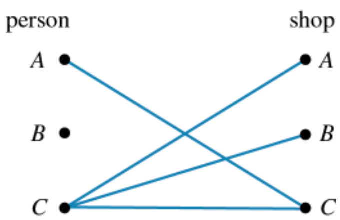


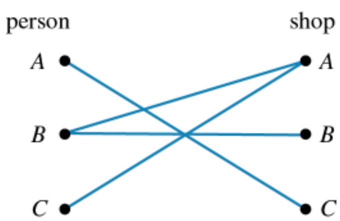
It would be possible to allocate each person a job if

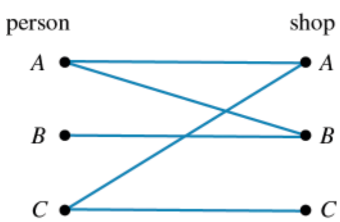
A. person A could also do job X.
B. person B could also do job W.
C. person D could only do job Z.
D. person B could only do job Y.
E. person D could also do job Y.

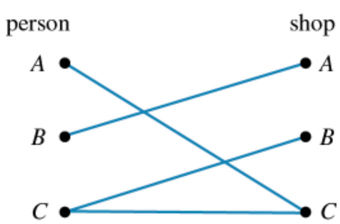
Question 14 (1 mark)

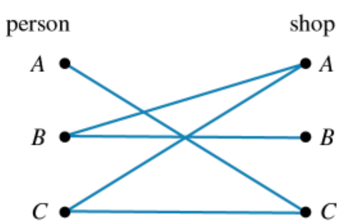
Mr A will only shop at Coles (C).
Mrs B is happy to shop at either Aldi (A) or Bi-lo (B).
Miss C shops at both Coles (C) and Aldi (A).
Which of the following bipartite graphs represents the information above?

A.


B.


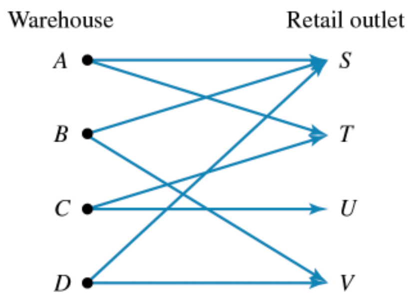
C.


D.


E.


Question 15 (1 mark)

The bipartite graph below represents the roadway connections between four warehouses and their retail outlets



If each warehouse is to be allocated one retail outlet only, then a feasible task allocation is

A. $A-S$
$B-T$
$C-U$
$D-V$

B. $A-T$
$B-S$
$C-U$
$D-V$

C. $A-U$
$B-S$
$C-T$
$D-V$

D. $A-V$
$B-U$
$C-S$
$D-T$

E. $A-S$
$B-U$
$C-T$
$D-V$

Question 16 (1 mark)

Four people are to be each allocated one of four tasks (A, B, C, D). The table shows the time, in hours, that each person takes to complete the tasks.

$$\begin{array}{r c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{r} Jonas \\ Brit \\ Anh \\ Nina \end{array} & \begin{bmatrix} 14 & 22 & 99 & 10 \\ 10 & 7 & 10 & 15 \\ 99 & 12 & 9 & 14 \\ 7 & 99 & 9 & 11 \end{bmatrix} \end{array}$$

The tasks must be completed in the least possible total amount of time.

If no person can help another, Anh should be allocated which task?

A. A

B. B

C. C

D. D

E. Cannot be determined

Question 17 (1 mark)

Four call centre representatives are assigned four tasks to complete with minimum completion times, in hours, listed in the following table:

Call centre representative	T1	T2	T3	T4
P1	12	10	8	9
P2	6	9	8	8
P3	6	4	7	6
P4	2	3	4	5

Which task should each person be assigned for a minimum completion time?

A. P1 – T1, P2 – T2, P3 – T3, P4 – T4

B. P1 – T4, P2 – T3, P3 – T2, P4 – T1

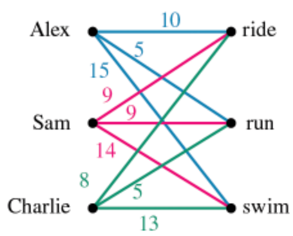
C. P1 – T3, P2 – T4, P3 – T1, P4 – T2

D. P1 – T3, P2 – T1, P3 – T4, P4 – T2

E. P1 – T3, P2 – T4, P3 – T2, P4 – T1

Question 18 (1 mark)

Three friends are planning to compete in a team triathlon, where each of them will complete one leg of the triathlon. The numbers represents each person's average time (in minutes) to complete each event.



Their best chance of winning will occur if

A. Alex rides, Sam runs and Charlie swims.
B. Sam rides, Alex runs and Charlie swims.
C. Charlie rides, Sam swims and Alex runs.
D. Alex rides, Sam runs and Charlie swims.
E. Alex rides, Sam swims and Charlie runs.

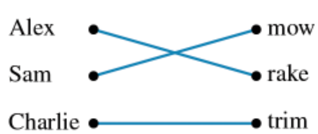
Question 19 (1 mark)

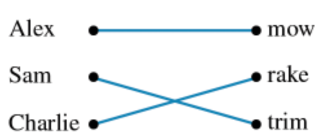
Three friends are earning money doing gardening work over the summer break. The average time taken by each person to complete each of the tasks is shown in the table below.

	mow	rake	trim
Alex	1	3	3
Sam	1	9	6
Charlie	6	5	9

Which of the following bipartite graphs represent the optimal allocation for completing the tasks as quickly as possible?

A.


B.


C.


D.

E.

Question 20 (1 mark)

Delivery trucks must transport produce from four market places to four individual fruit shops. The following table records the distances, in kilometres, between the market places and fruit shops. The owner of the fruit shops wants his trucks to make the deliveries ensuring they cover the minimum possible distance.

Market Place	F1	F2	F3	F4
M1	26	32	24	34
M2	27	20	27	30
M3	18	22	24	16
M4	22	21	19	20

A. 78 km
B. 79 km
C. 81 km
D. 82 km
E. 91 km

Question 21 (1 mark)
A manager has 4 people to complete 4 tasks in the smallest amount of time possible. The time (in hours) it takes for each person to complete each of the tasks is shown in the table below.

	Task 1	Task 2	Task 3	Task 4
Person A	14	5	8	7
Person B	2	12	6	5
Person C	7	8	3	9
Person D	2	4	6	10

What is the minimum possible time needed (in hours) to complete all of the tasks?
A. 12
B. 13
C. 14
D. 15
E. 16

Topic	10	Directed graphs and network flow
Subtopic	10.6	Review

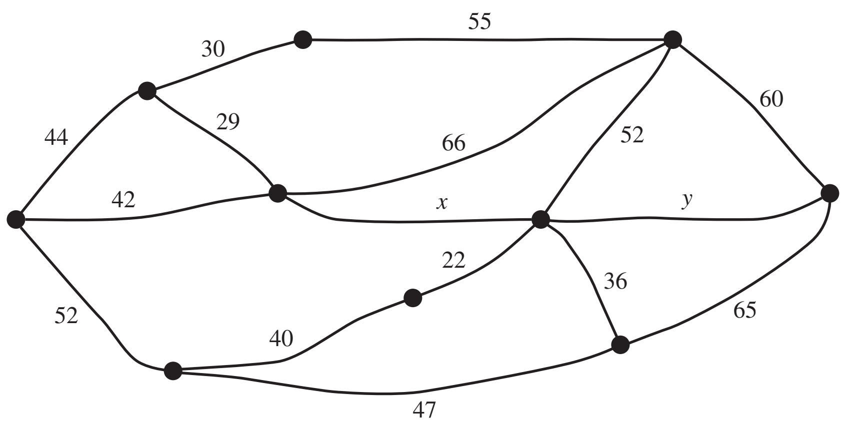
To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au.**

Source: *VCE 2021, Further Mathematics 1, Section B, Module 2, Q.8;* © VCAA

Question 1 (1 mark)

A network of roads connecting towns in an alpine region is shown below.

The distances between neighbouring towns, represented by the vertices, are given in kilometres.



The region receives a large snowfall, leaving all roads between the towns closed to traffic.

To ensure each town is accessible by car from every other town, some roads will be cleared.

The minimum total length of road, in kilometres, that needs to be cleared is

A. 361 if $x = 50$ and $y = 55$

B. 361 if $x = 50$ and $y = 60$

C. 366 if $x = 55$ and $y = 55$

D. 366 if $x = 55$ and $y = 60$

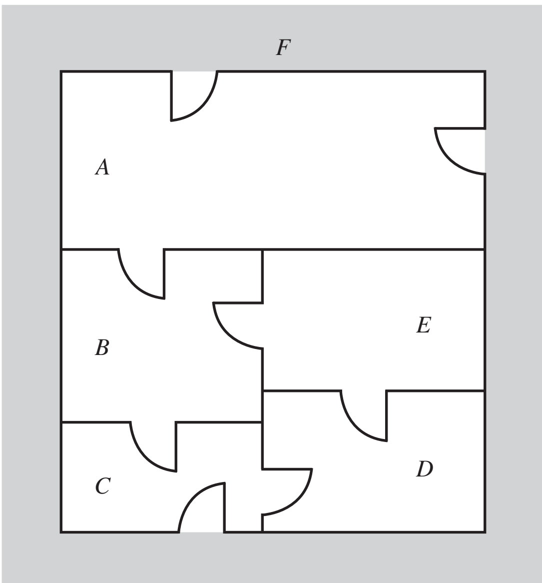
E. 371 if $x = 55$ and $y = 65$

Source: *VCE 2021, Further Mathematics 2, Section B, Module 2, Q.1;* © VCAA

Question 2 (4 marks)

Maggie's house has five rooms, A, B, C, D and E, and eight doors.

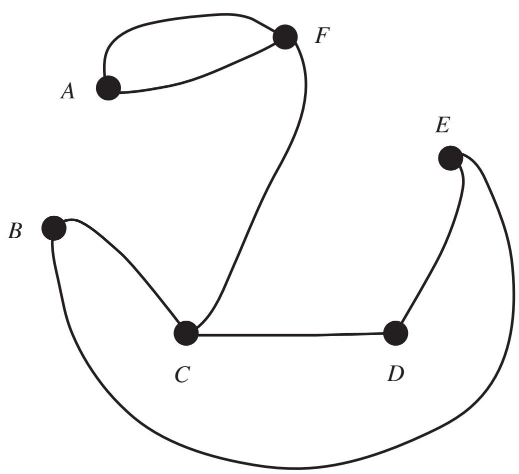
The floor plan of these rooms and doors is shown below. The outside area, F, is shown shaded on the floor plan.



The floor plan is represented by the graph below.

On this graph, vertices represent the rooms and the outside area. Edges represent direct access to the rooms through the doors.

One edge is missing from the graph.



a. On the **graph above**, draw the missing edge. **(1 mark)**
(*Redraw the graph in the space given below.*)

b. What is the degree of vertex E? **(1 mark)**

c. Maggie hires a cleaner to clean the house.
It is possible for the cleaner to enter the house from the outside area, F , and walk through each room only once, cleaning each room as he goes and finishing in the outside area, F

i. Complete the following to show **one** possible route that the cleaner could take. **(1 mark)**

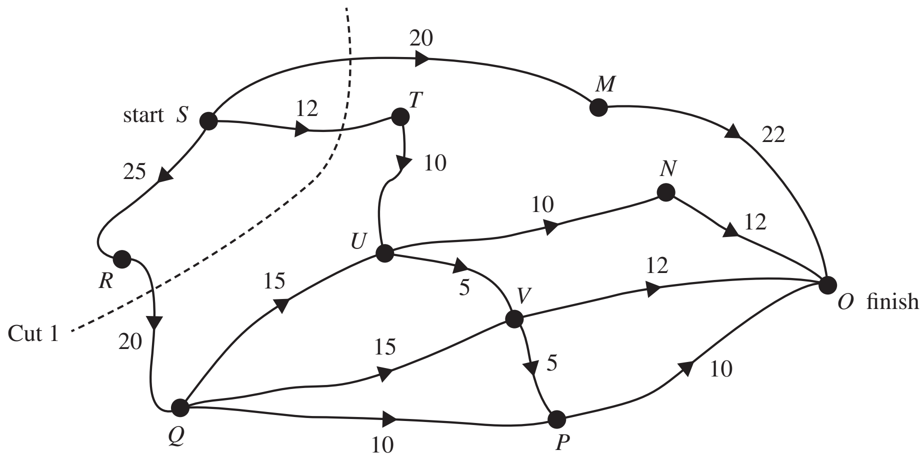
$F - \square - \square - \square - \square - \square - F$

ii. What is the mathematical term for such a journey? **(1 mark)**

Source: *VCE 2020, Further Mathematics 2, Section B, Module 2, Q.4;* © VCAA

Question 3 (2 marks)

Training program 1 has the cricket team starting from exercise station S and running to exercise station O.
For safety reasons, the cricket coach has placed a restriction on the maximum number of people who can use the tracks in the fitness park.
The directed graph below shows the capacity of the tracks, in number of people per minute.



a. How many different routes from S to O are possible? **(1 mark)**

b. When considering the possible flow of people through this network, many different cuts can be made. Determine the capacity of Cut 1, shown above. **(1 mark)**

Source: *VCE 2020, Further Mathematics 2, Section B, Module 2, Q.5*; © VCAA

Question 4 (4 marks)

The Sunny Coast cricket clubroom is undergoing a major works project.

This project involves nine activities: A to I.

The table below shows the earliest start time (EST) and duration, in months, for each activity.

The immediate predecessor(s) is also shown.

The duration for activity C is missing.

Activity	EST	Duration	Immediate predecessor(s)
A	0	2	–
B	0	5	–
C	5		A, B
D	7	7	C
E	7	9	C
F	5	3	B
G	14	4	D
H	8	9	F
I	18	2	E, G, H

The information in the table above can be used to complete a directed network.

This network will require a dummy activity.

a. Complete the following sentence by filling in the spaces provided.

This dummy activity could be drawn as a directed edge from the end of activity ________ to the start of activity ________. **(1 mark)**

b. What is the duration, in months, of activity C? **(1 mark)**

c. Name the four activities that have a float time. **(1 mark)**

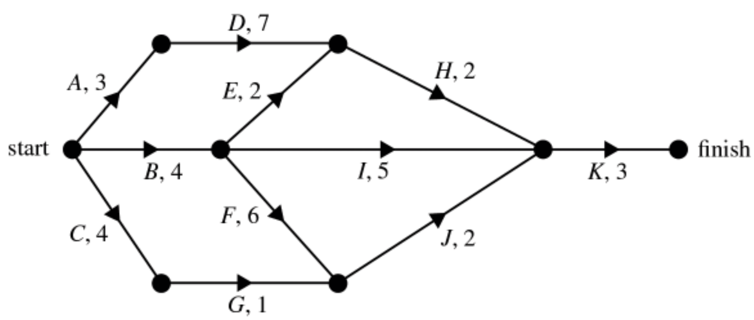
d. The project is to be crashed by reducing the completion time of one activity only.

What is the minimum time, in months, that the project can be completed in? **(1 mark)**

Source: *VCE 2018, Further Mathematics 1, Section B, Module 2, Q.5*; © VCAA

Question 5 (1 mark)

The directed network below shows the sequence of 11 activities that are needed to complete a project. The time, in weeks, that it takes to complete each activity is also shown.



How many of these activities could be delayed without affecting the minimum completion time of the project?

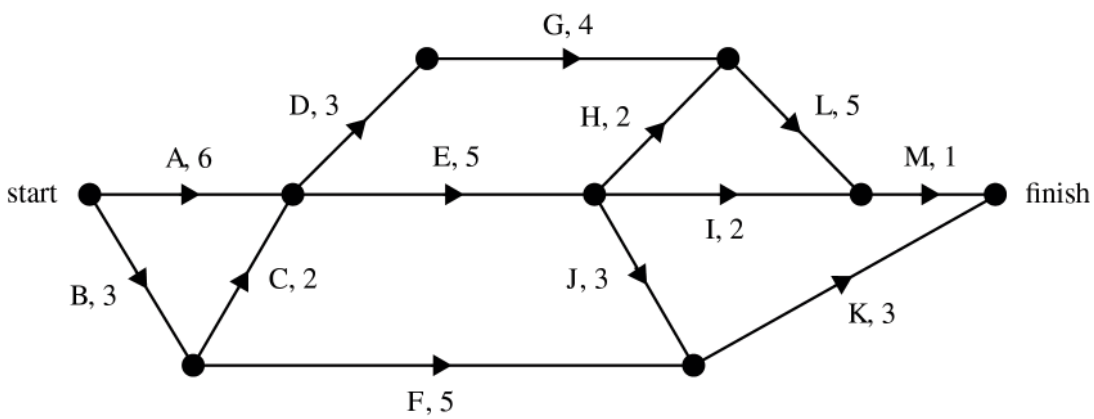
A. 3
B. 4
C. 5
D. 6
E. 7

Source: *VCE 2021, Further Mathematics 2, Section B, Module 2, Q.4;* © VCAA

Question 6 (3 marks)

Roadworks planned by the local council require 13 activities to be completed.

The network below shows these 13 activities and their completion times in weeks.



a. What is the earliest start time, in weeks, of activity K? **(1 mark)**

b. How many of these activities have zero float time? **(1 mark)**

c. It is possible to reduce the completion time for activities A, E, F, L and K.

The reduction in completion time for each of these five activities will incur an additional cost.

The table below shows the five activities that can have their completion time reduced and the associated weekly cost, in dollars.

Activity	Weekly cost ($)
A	140000
E	100000
F	100000
L	120000
K	80000

The completion time for each of these five activities can be reduced by a maximum of two weeks.

The overall completion time for the roadworks can be reduced to 16 weeks.

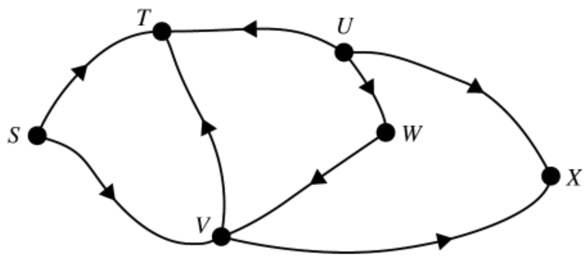
What is the minimum cost, in dollars, of this change in completion time? **(1 mark)**

Source: *VCE 2020, Further Mathematics 1, Section B, Module 2, Q.4;* © VCAA

Question 7 (1 mark)

The directed graph below represents a series of one-way streets.

The vertices represent the intersections of these streets.



The number of vertices that can be reached from S is

A. 1
B. 2
C. 3
D. 4
E. 5

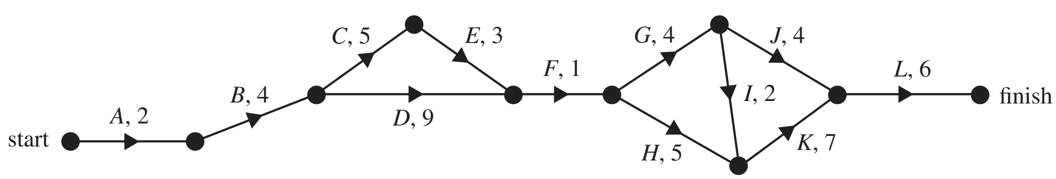
Source: *VCE 2019, Further Mathematics 2, Section B, Module 2, Q.3*; © VCAA

Question 8 (6 marks)

Fencedale High School is planning to renovate its gymnasium.

This project involves 12 activities, A to L.

The directed network below shows these activities and their completion times, in weeks.



The minimum completion time for the project is 35 weeks.

a. How many activities are on the critical path? **(1 mark)**

b. Determine the latest start time of activity E. **(1 mark)**

c. Which activity has the longest float time? **(1 mark)**

It is possible to reduce the completion time for activities C, D, G, H and K by employing more workers. The completion time for each of these five activities can be reduced by a maximum of two weeks.

d. What is the minimum time, in weeks, that the renovation project could take?

It is possible to reduce the completion time for activities C, D, G, H and K by employing more workers. The reduction in completion time for each of these five activities will incur an additional cost to the school.

The table below shows the five activities that can have their completion times reduced and the associated weekly cost, in dollars.

Activity	**Weekly cost** ($)
C	3000
D	2000
G	2500
H	1000
K	4000

The completion time for each of these five activities can be reduced by a maximum of two weeks.

Fencedale High School requires the overall completion time for the renovation project to be reduced by four weeks at minimum cost. **(1 mark)**

e. Complete the table below, showing the reductions in individual activity completion times that would achieve this. **(2 mark)**

Activity	**Reduction in completion time (0, 1 or 2 weeks)**
C	
D	
G	
H	
K	

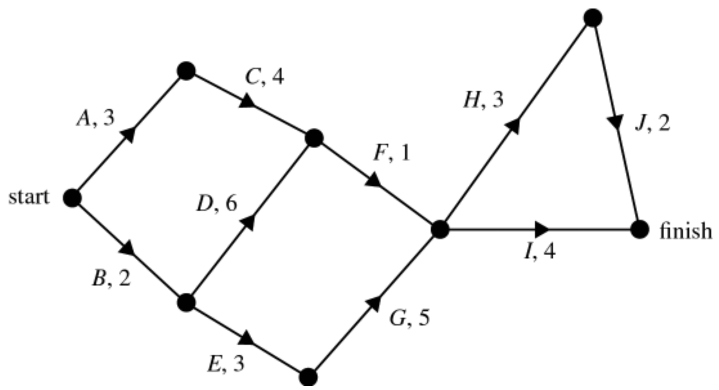
Source: *VCE 2018, Further Mathematics 2, Section B, Module 2, Q.3*; © VCAA

Question 9 (4 marks)

At the Zenith Post Office all computer systems are to be upgraded.

This project involves 10 activities, A to J.

The directed network below shows these activities and their completion times, in hours.



a. Determine the earliest starting time, in hours, for activity I. **(1 mark)**

b. The minimum completion time for the project is 15 hours.
Write down the critical path. **(1 mark)**

c. Two of the activities have a float time of two hours.
Write down these two activities. **(1 mark)**

d. For the next upgrade, the same project will be repeated but one extra activity will be added.
This activity has a duration of one hour, an earliest starting time of five hours and a latest starting time of 12 hours.
Complete the following sentence by filling in the blanks.
The extra activity could be represented on the network above by a directed edge from the **(1 mark)**

end of activity ______ to the start of activity ______

Source: *VCE 2017, Further Mathematics 2, Section B, Module 2, Q.2;* © VCAA

Question 10 (2 marks)

Bai joins his friends Agatha, Colin and Diane when he arrives for the holiday in Seatown.
Each person will plan one tour that the group will take.
Table 1 shows the time, in minutes, it would take each person to plan each of the four tours.

Table 1

	Agatha	Bai	Colin	Diane
Tour 1	13	7	13	12
Tour 2	14	9	8	7
Tour 3	19	25	21	18
Tour 4	10	7	11	10

The aim is to minimise the total time it takes to plan the four tours.
Agatha applies the Hungarian algorithm to Table 1 to produce Table 2.
Table 2 shows the final result of all her steps of the Hungarian algorithm.

Table 2

	Agatha	Bai	Colin	Diane
Tour 1	3	0	3	3
Tour 2	6	4	0	0
Tour 3	0	9	2	0
Tour 4	0	0	1	1

a. In Table 2 there is a zero in the column for Colin.
When all values in the table are considered, what conclusion about minimum total planning time can be made from this zero? **(1 mark)**

b. Determine the minimum total planning time, in minutes, for all four tours. **(1 mark)**

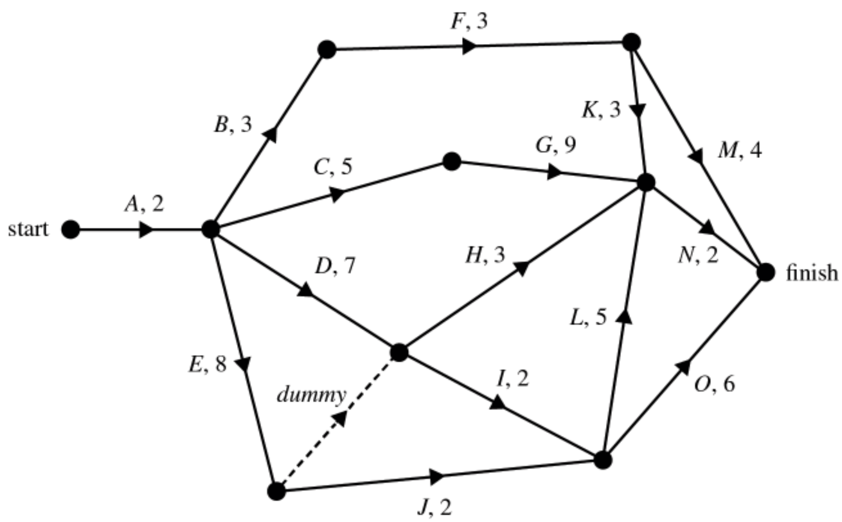
Source: *VCE 2017, Further Mathematics 2, Section B, Module 2, Q.4*; © VCAA

Question 11 (5 marks)

The rides at the theme park are set up at the beginning of each holiday season.

This project involves activities A to O.

The directed network below shows these activities and their completion times in days.



a. Write down the two immediate predecessors of activity I. **(1 mark)**

b. The minimum completion time for the project is 19 days.

i. There are two critical paths. One of the critical paths is $A - E - J - L - N$.
Write down the other critical path. **(1 mark)**

ii. Determine the float time, in days, for activity F. **(1 mark)**

c. The project could finish earlier if some activities were crashed.
Six activities, B, D, G, I, J and L, can all be reduced by one day.
The cost of this crashing is \$1000 per activity.

i. What is the minimum number of days in which the project could now be completed? **(1 mark)**

ii. What is the minimum cost of completing the project in this time? **(1 mark)**

Source: *VCE 2017, Further Mathematics 1, Section B, Module 2, Q.7;* © VCAA

Question 12 (1 mark)

A graph with six vertices has no loops or multiple edges.
Which one of the following statements about this graph is **not** true?

A. If the graph is a tree it has five edges.
B. If the graph is complete it has 15 edges.
C. If the graph has eight edges it may have an isolated vertex.
D. If the graph is bipartite it will have a minimum of nine edges.
E. If the graph has a cycle it will have a minimum of three edges.

Topic 10 > Subtopic 10.6 Review

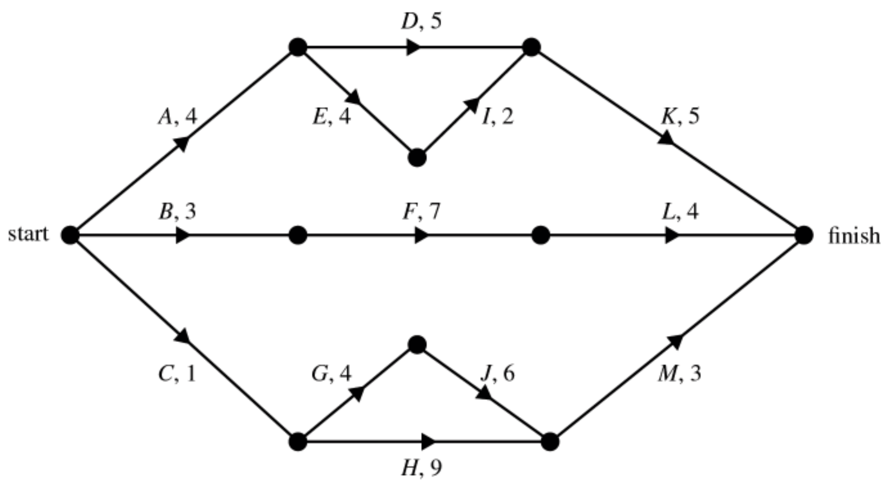
Source: *VCE 2016, Further Mathematics 2, Module 2, Q.3*; © VCAA

Question 13 (6 marks)

A new skateboard park is to be built in Beachton.

This project involves 13 activities, A to M.

The directed network below shows these activities and their completion times in days.



a. Determine the earliest start time for activity M. **(1 mark)**

b. The minimum completion time for the skateboard park is 15 days.
Write down the critical path for this project. **(1 mark)**

c. Which activity has a float time of two days? **(1 mark)**

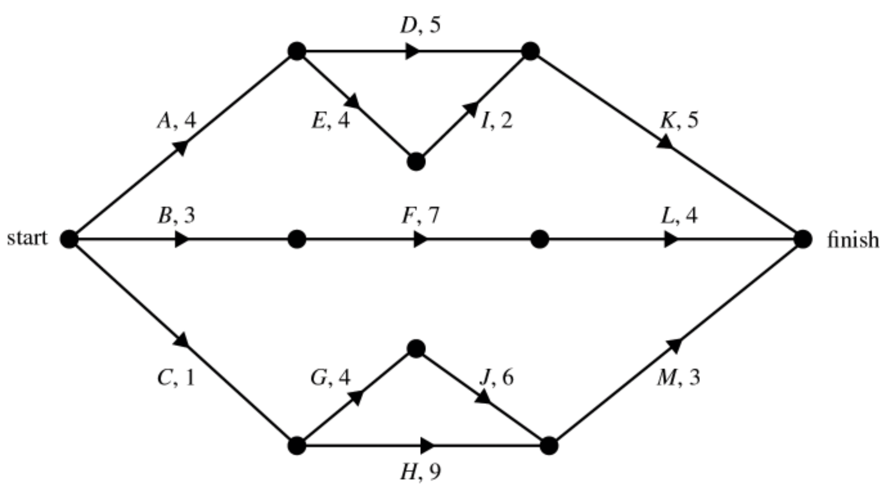
d. The completion times for activities E, F, G, I and J can each be reduced by one day.
The cost of reducing the completion time by one day for these activities is shown in the table below.

Activity	**Cost** ($)
E	3000
F	1000
G	5000
I	2000
J	4000

What is the minimum cost to complete the project in the shortest time possible? **(1 mark)**

e. The skateboard park project will be repeated at Campville, but with the addition of one extra activity.
The new activity, N, will take six days to complete and has a float time of one day.
Activity N will finish at the same time as the project.

i. Add activity N to the network below. **(1 mark)**



ii. What is the latest start time for activity N? **(1 mark)**

Source: *VCE 2015, Further Mathematics 2, Module 5, Q.3*; © VCAA

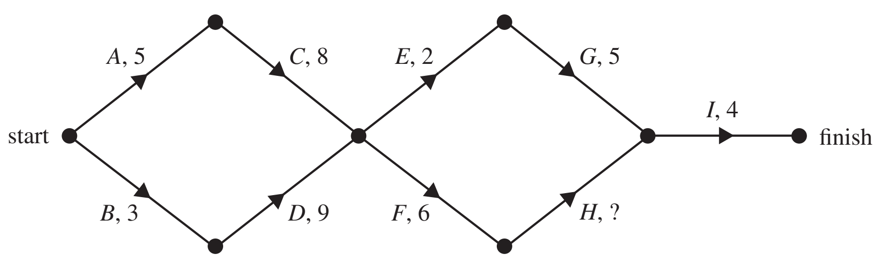
Question 14 (7 marks)

Nine activities are needed to prepare a daily delivery of groceries from the factory to the towns.

The duration, in minutes, earliest starting time (EST) and immediate predecessors for these activities are shown in the table below.

Activity	Duration	EST	Predecessor(s)
A	5	0	-
B	3	0	-
C	8	5	*A*
D	9		*B*
E	2	13	*C*, *D*
F	6	13	*C*, *D*
G	5	15	*E*
H		19	*F*
I	4	22	*G*, *H*

The directed network that shows these activities is shown below.



All nine of these activities can be completed in a minimum time of 26 minutes.

a. What is the EST of activity *D*? **(1 mark)**

b. What is the latest starting time (LST) of activity *D*? **(1 mark)**

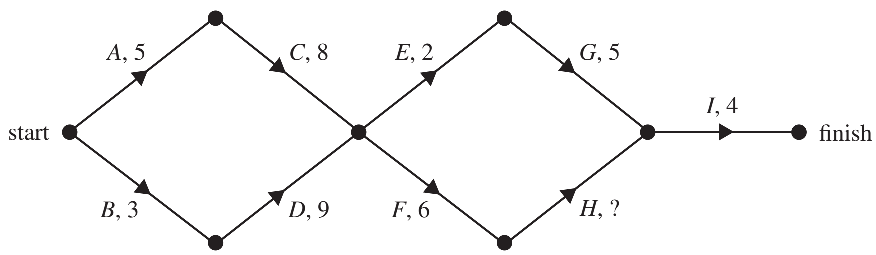
c. Given that the EST of activity *I* is 22 minutes, what is the duration of activity *H*? **(1 mark)**

d. Write down, in order, the activities on the critical path. **(1 mark)**

e. Activities C and D can only be completed by either Cassie or Donna.
One Monday, Donna is sick and both activities C and D must be completed by Cassie. Cassie must complete one of these activities before starting the other.
What is the least effect of this on the usual minimum preparation time for the delivery of groceries from the factory to the five towns? **(1 mark)**

f. Every Friday, a special delivery to the five towns includes fresh seafood. This causes a slight change to activity G, which then cannot start until activity F has been completed.

i. On the directed graph below, show this change without duplicating any activity. **(1 mark)**



ii. What effect does the inclusion of seafood on Fridays have on the usual minimum preparation time for deliveries from the factory to the five towns? **(1 mark)**

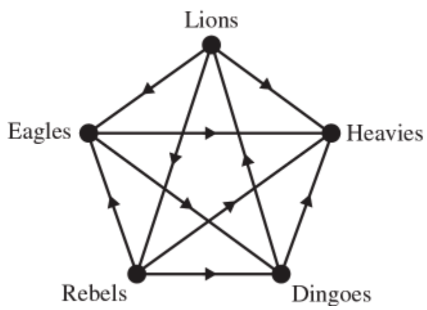
Source: *VCE 2015, Further Mathematics 1, Section B, Module 5, Q.8*; © VCAA

Question 15 (1 mark)

There are five teams in a table tennis competition.

Every team played one match against every other team, and each match had a winner and a loser.

The results of the matches are summarised in the directed graph below. For example, an arrow from Lions to Eagles indicates that Lions defeated Eagles.



In determining the ranking of these teams, the total of each team's one-step dominances and two-step dominances will be calculated.

The team with the highest total will be ranked first.

The team with the next highest total will be ranked second, and so on.

The ranking of these five teams from first to last is

A. Lions, Rebels, Dingoes, Eagles, Heavies
B. Lions, Rebels, Eagles, Dingoes, Heavies
C. Rebels, Lions, Dingoes, Eagles, Heavies
D. Rebels, Lions, Eagles, Dingoes, Heavies
E. Eagles, Lions, Rebels, Dingoes, Heavies

Source: *VCE 2014, Further Mathematics 2, Module 5, Q.2;* © VCAA

Question 16 (4 marks)

Planning a train club open day involves four tasks.

Table 1 shows the number of hours that each club member would take to complete these tasks.

Table 1

Task	Andrew	Brianna	Charlie	Devi
publicity	13	12	10	10
finances	9	10	11	11
equipment	8	12	11	10
catering	9	10	11	8

The Hungarian algorithm will be used to allocate the tasks to club members so that the total time taken to complete the tasks is minimised.

The first step of the Hungarian algorithm is to subtract the smallest element in each row of Table 1 from each of the elements in that row.

The result of this step is shown in Table 2 below.

a. Complete Table 2 by filling in the missing numbers for Andrew. **(1 mark)**

Table 2

Task	Andrew	Brianna	Charlie	Devi
publicity	3	2	0	0
finances		1	2	2
equipment		4	3	2
cateing		10	3	0

After completing Table 2, Andrew decided that an allocation of tasks to minimise the total time taken was not yet possible using the Hungarian algorithm.

b. Explain why Andrew made this decision. **(1 mark)**

c. Table 3 shows the final result of all steps of the Hungarian algorithm.

Table 3

Task	Andrew	Brianna	Charlie	Devi
publicity	4	2	0	1
finances	0	0	1	2
equipment	0	3	2	2
catering	1	1	2	0

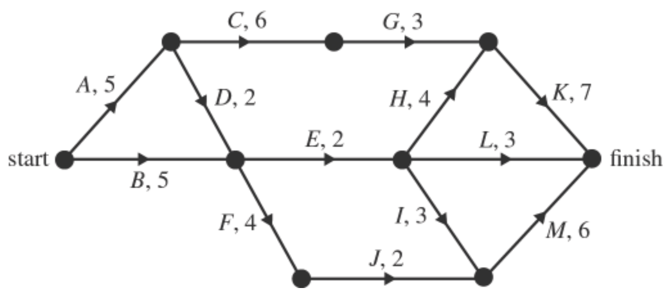
i. Which task should be allocated to Andrew? **(1 mark)**

ii. How many hours in total are used to plan for the open day? **(1 mark)**

Source: *VCE 2014, Further Mathematics 2, Module 5, Q.4*; © VCAA

Question 17 (5 marks)

To restore a vintage train, the members of a train club need to complete 13 activities.
The network below shows these 13 activities and their completion times in hours.



a. Determine the earliest starting time of activity *F*. **(1 mark)**

The minimum time in which all 13 activities can be completed is 21 hours.

b. What is the latest starting time of activity *L*? **(1 mark)**

c. What is the float time of activity *J*? **(1 mark)**

Just before they started restoring the train, the members of the club needed to add another activity, *X*, to the project.
Activity *X* will take seven hours to complete.
Activity *X* has no predecessors, but must be completed before activity *G* starts.

d. What is the latest starting time of activity *X* if it is not to increase the minimum completion time of the project? **(1 mark)**

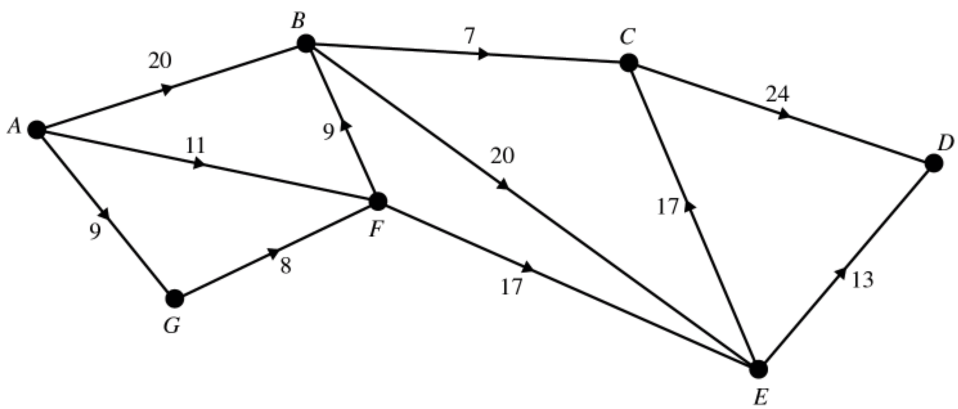
Activity *A* can be crashed by up to four hours at an additional cost of $90 per hour.
This may reduce the minimum completion time for the project, including activity *X*.

e. Determine the least cost of crashing activity *A* to give the greatest reduction in the minimum completion time of the project. **(1 mark)**

Source: *VCE 2013, Further Mathematics 2, Module 5, Q.3;* © VCAA

Question 18 (4 marks)

The rangers at the wildlife park restrict access to the walking tracks through areas where the animals breed. The edges on the directed network diagram below represent one-way tracks through the breeding areas. The direction of travel on each track is shown by an arrow. The numbers on the edges indicate the maximum number of people who are permitted to walk along each track each day.



a. Starting at A, how many people, in total, are permitted to walk to D each day? **(1 mark)**

b. One day, all the available walking tracks will be used by students on a school excursion. The students will start at A and walk in four separate groups to D. Students must remain in the same groups throughout the walk

i. Group 1 will have 17 students. This is the maximum group size that can walk together from A to D. Write down the path that group 1 will take. **(1 mark)**

ii. Groups 2, 3 and 4 will each take different paths from A to D. Complete the six missing entries shaded in the table below. **(2 marks)**

Group	Maximum group size	Path taken from A to D
1	17	**answered in part b.i.**
2		
3		
4		

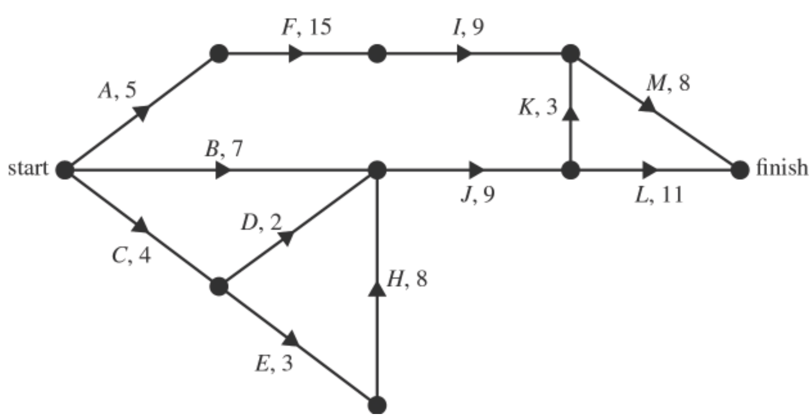
Source: *VCE 2013, Further Mathematics 2, Module 5, Q.2*; © VCAA

Question 19 (5 marks)

A project will be undertaken in the wildlife park. This project involves the 13 activities shown in the table below. The duration, in hours, and predecessor(s) of each activity are also included in the table.

Activity	Duration (hours)	Predessor(s)
A	5	–
B	7	–
C	4	–
D	2	C
E	3	C
F	15	A
G	4	B, D, H
H	8	E
I	9	F, G
J	9	B, D, H
K	3	J
L	11	J
M	8	I, K

Activity G is missing from the network diagram for this project, which is shown below.



a. Complete the network diagram above by inserting activity G. **(1 mark)**

b. Determine the earliest starting time of activity H. **(1 mark)**

c. Given that activity G is not on the critical path

i. find the latest starting time for activity D. **(1 mark)**

d. Consider the following statement.

'If just one of the activities in this project is crashed by one hour, then the minimum time to complete the entire project will be reduced by one hour.'

Explain the circumstances under which this statement will be true for this project. **(1 mark)**

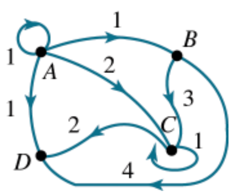
e. Assume activity F is crashed by two hours.

What will be the minimum completion time for the project? **(1 mark)**

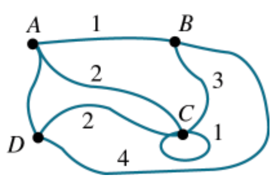
Question 20 (1 mark)

An adjacency matrix for a digraph is $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A possible network to match this matrix is:

A.

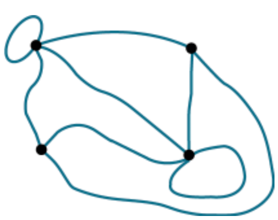


B.

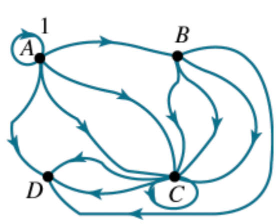


C.

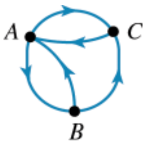
D.



E.



Question 21 (1 mark)



The digraph above can be represented by which of the following adjacency matrices?

A.
$$\begin{array}{cc} & \text{To} \\ & \begin{matrix} A & B & C \end{matrix} \\ \text{From}\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

B.
$$\begin{array}{cc} & \text{To} \\ & \begin{matrix} A & B & C \end{matrix} \\ \text{From}\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

C.
$$\begin{array}{cc} & \text{To} \\ & \begin{matrix} A & B & C \end{matrix} \\ \text{From}\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

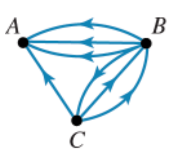
D.
$$\begin{array}{cc} & \text{To} \\ & \begin{matrix} A & B & C \end{matrix} \\ \text{From}\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$$

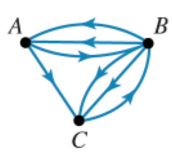
E.
$$\begin{array}{cc} & \text{To} \\ & \begin{matrix} A & B & C \end{matrix} \\ \text{From}\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \end{array}$$

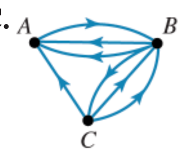
Question 22 (1 mark)

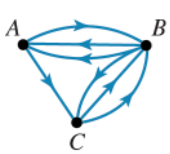
$$\begin{array}{cc} & \text{To} \\ & \begin{matrix} A & B & C \end{matrix} \\ \text{From}\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

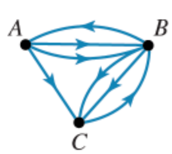
Which of the following digraphs can be represented by the adjacency matrix shown above?

A.


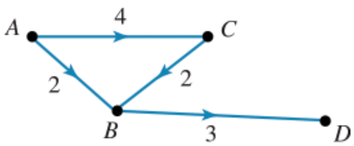
B.


C.


D.


E.


Question 23 (1 mark)



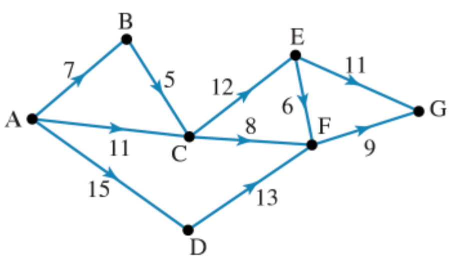
Which of the following adjacency matrices represents the directed graph shown above?

A. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **C.** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$ **E.** $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Question 24 (1 mark)

The shown network indicates the distances, in kilometres, between a series of towns. Which weighted matrix below best represents this network? *Note*: A dash (–) indicates that there is no direct connection between two particular towns.



A. $\begin{bmatrix} 0 & 0 & 0 & 0 & - & - & - \\ 7 & 0 & 0 & - & - & - & - \\ 11 & 5 & 0 & - & 0 & 0 & - \\ 15 & - & - & 0 & - & 0 & - \\ - & - & 12 & - & 0 & 0 & 0 \\ - & - & 8 & 13 & 6 & 0 & 0 \\ - & - & - & - & 11 & 9 & 0 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & 1 & 1 & 1 & - & - & - \\ 1 & 0 & 1 & - & - & - & - \\ 1 & 1 & 0 & - & 1 & 1 & - \\ 1 & - & - & 0 & - & 1 & - \\ - & - & 1 & - & 0 & 1 & 1 \\ - & - & 1 & 1 & 1 & 0 & 1 \\ - & - & - & - & 1 & 1 & \end{bmatrix}$ **C.** $\begin{bmatrix} 0 & 7 & 11 & 15 & - & - & - \\ 0 & 0 & 5 & - & - & - & - \\ 0 & 0 & 0 & - & 12 & 8 & - \\ 0 & - & - & 0 & - & 13 & - \\ - & - & 0 & - & 0 & 6 & 11 \\ - & - & 0 & 0 & 0 & 0 & 9 \\ - & - & - & - & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 7 & 11 & 15 & 0 & 0 & 0 \\ 7 & 0 & 5 & 0 & 0 & 0 & 0 \\ 11 & 5 & 0 & 0 & 12 & 8 & 0 \\ 15 & 0 & 0 & 0 & 0 & 13 & 0 \\ 0 & 0 & 12 & 0 & 0 & 6 & 11 \\ 0 & 0 & 8 & 13 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 11 & 9 & 0 \end{bmatrix}$ **E.** $\begin{bmatrix} 0 & 1 & 1 & 1 & - & 0 & 1 \\ 1 & 0 & 1 & - & - & - & - \\ 1 & 0 & 0 & - & 1 & 1 & - \\ 1 & - & - & 0 & - & 1 & - \\ - & 1 & 1 & 0 & 0 & 1 & 1 \\ - & - & 1 & 1 & 1 & 0 & 0 \\ - & - & - & 2 & 1 & 1 & 0 \end{bmatrix}$

Answers and marking guide

10.2 Precedence tables and activity networks

Question 1

Minimum completion time $= 7 + 4 + 3 + 3 + 5 = 22$

Question 2

Activity	EST	LST	Float
A	0	1	1
B	0	2	2
C	0	0	0
D	4	6	2
E	4	9	5
F	4	4	0
G	4	5	1
H	10	10	0
I	7	8	1
J	7	8	1
K	10	12	2
L	10	11	1
M	13	13	0
N	12	13	1
O	15	16	1

Forward and backward scanning through the activity network results in the following information.
The earliest starting time for activity N is 12 hours after the start of the project.
Alternatively, look for the longest path to the start of activity N.
***VCAA Examination Report* note:**
The critical path analysis in this question involved standard forward-scanning calculations. While there was some complexity of the activity network, students should be able to apply standard routine calculations to graphs such as this with care.

Question 3

I and J have immediate predecessors.

Question 4

Activity	EST	LST	Float
A	0	1	1
B	0	2	2
C	0	0	0
D	4	6	2
E	4	9	5
F	4	4	0
G	4	5	1
H	10	10	0
I	7	8	1
J	7	8	1
K	10	12	2
L	10	11	1
M	13	13	0
N	12	13	1
O	15	16	1

Forward and backward scanning through the activity network results in the following information.
The earliest starting time for activity N is 12 hours after the start of the project.
Alternatively, look for the longest path to the start of activity N.
***VCAA Examination Report* note:**
The critical path analysis in this question involved standard forward-scanning calculations. While there was some complexity of the activity network, students should be able to apply standard routine calculations to graphs such as this with care.

Question 5
The critical path for this project is $B - D - E$. If each activity along this path is reduced by 1 hour, total completion time will be reduced to 21 h $(2 + 6 + 13)$ at a cost of \$300. However, the other path, $A - C - E$. will now take 22 h $(4 + 5 + 13)$. Therefore, either activity A or activity C needs to be reduced by 1 h to get the total completion time down to 21 h. This will cost an additional \$100. Thus, the least cost to achieve the greatest reduction (of 3 hours) is \$400.

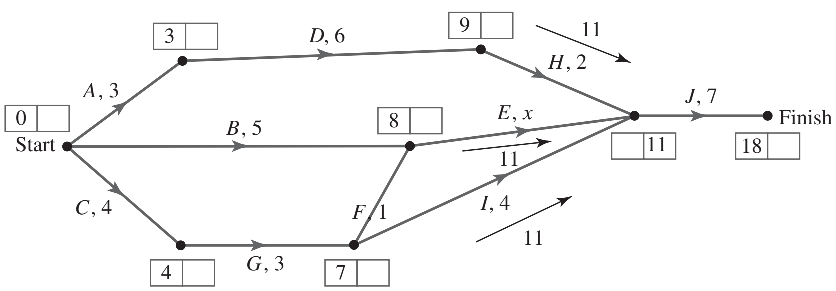
Question 6
A must occur before E can occur.

Question 7
Activities A and D have no immediate predecessors. Disregard answers A, D and E.
Activity C is a predecessor for activity E.

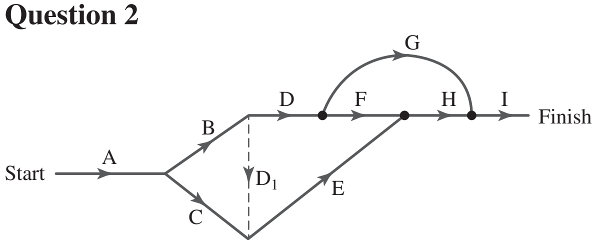
Question 8
Predecessors for D are:
C, 2 hours
A, 5 hours.
Activity D cannot start until Activities A and C are complete.

10.3 Critical path analysis with backward scanning and crashing

Question 1
When forward scanning is done, the latest start time of activity J is 11 (as the minimum completion time is 18 hours). To not affect the completion time, x must be a value less than 3 – i.e. $8 + x = 11$, so $x = 11 - 8 = 3$.



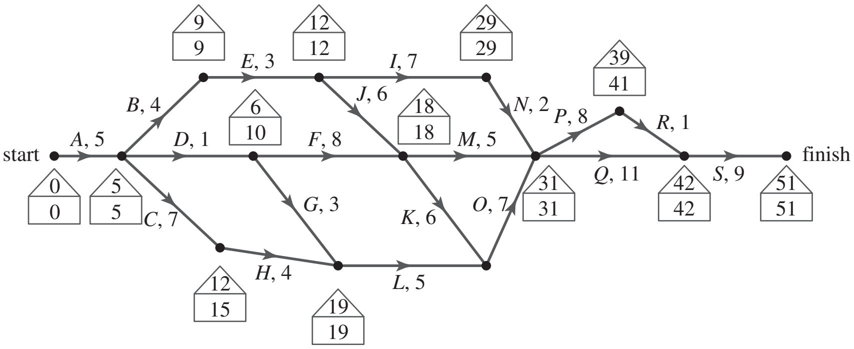
Question 2



***VCAA Examination Report* note:**
Students could have answered the question without sketching the entire network.

Question 3

Start by completing forwards and backwards scanning.



Float time = Latest Starting Time − Earliest Starting Time − Duration of Activity.

The activities with a float time of 10 hours are:

19 − 6 − 3 = 10

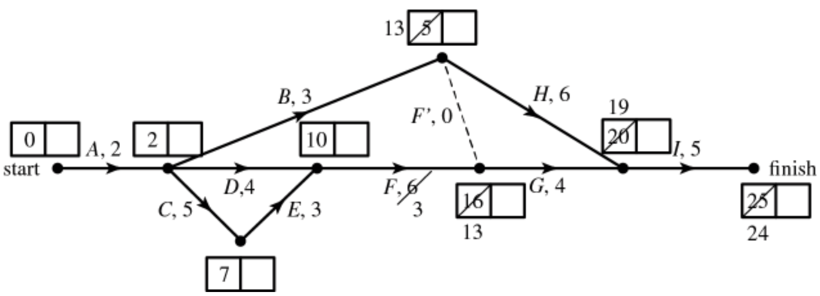
29 − 12 − 7 = 10

31 − 19 − 2 = 10

Therefore, there are three activities with a float time of 10 hours.

Question 4

A dummy activity needs to be drawn so that *H* does not start until *F* is complete.



The completion time will be reduced by 1 week if activity *F* is completed before activity *H* is started.

Question 5

The critical path for this network is ACFGI, which takes a minimum of 20 hours to complete.

Question 6

Activity	EST	LST	Float
A	0	1	1
B	0	2	2
C	0	0	0
D	4	6	2
E	4	9	5
F	4	4	0
G	4	5	1
H	10	10	0
I	7	8	1
J	7	8	1
K	10	12	2
L	10	11	1
M	13	13	0
N	12	13	1
O	15	16	1

Forward and backward scanning through the activity network results in the following information.
The activities that cannot be delayed are the critical activities. That is, the activities with a float time of 0.
The critical activities are C, F, H and M. So, there are 4 activities that cannot be delayed.

Question 7
I and J have immediate predecessors.

Question 8

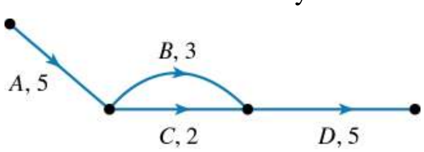
activity	EST	LST	Float
A	0	1	1
B	0	0	0
C	0	1	1
D	5	6	1
E	3	3 4	0 1
F	3	6	3
G	2	9	7
H	2	3	1
I	8	8	0
J	7	10	3
K	8	9	1
L	12	12	0

The critical path for this project is B, E, I, L.
However, if E were reduced by one hour, there would be 3 critical paths.

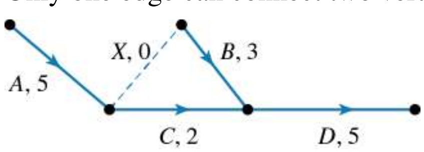
Question 9
A process of elimination can be used. The earliest starting time for both activities B and C is 5 minutes, so this must be directly after activity A has finished. The EST for both activities D and E is 8 minutes, so they can only start after activity B.
Therefore, option B is the answer.

Question 10
E is true. The minimum completion time of a project is determined by the total time each critical activity (activity on the critical path) takes. So if one of these activities increases in time, the overall completion time of the project also increases.

Question 11
The network described by the table is shown below.



Only one edge can connect two vertices.



A dummy activity is required before either Activity B or Activity C.

10.4 Flow problems

Question 1

Maximum flow is 33 L/min.

Cuts B, C and D have the same capacity as the maximum flow.

Question 2

The flow must be heading towards the sink.

$3 + 6 + 5 = 14$

Question 3

Exploring all the options for the three given values for x. shows the following.

When $x = 1$: The capacity of the minimum cut is 24 – Cut B. This is not stated.

When $x = 2$: The capacity of the minimum cut is 25 – Cut B.

When $x = 3$: The capacity of the minimum cut is 26 – Cut B. This is not stated.

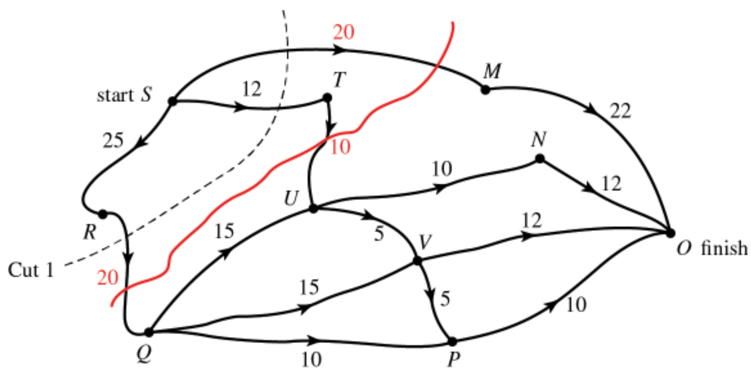
The maximum flow is Cut B if $x = 2$.

Question 4

a By counting the number of different routes, you will find that there are 10 different routes possible. **[1 mark]**

b Capacity of cut $1 = 20 + 12 + 20 = 52$ **[1 mark]**

c Remember that the minimum cut = maximum flow.



Using trial and error, the minimum cut is shown in red above.

Therefore, the maximum flow is 50 people per minute. **[1 mark]**

Question 5

a. **i.** Cut B has a capacity of $4 + 2 + 3 = 9$. **[1 mark]**

ii. Cut C has a capacity of $4 + 6 + 3 = 13$. **[1 mark]**

***VCAA Examination Report* note:**

Some gave an answer of 14 by not allowing for the 1 against the flow.

b. The maximum number of deliveries is $2 + 2 + 3 = 7$ deliveries each day. **[1 mark]**

***VCAA Examination Report* note:**

This question was not answered well. The answer of 9 from **part a.** was often repeated.

Question 6

For maximum flow, you must use a minimum cut.

minimum cut $= 6 + 10 + 2 = 18$

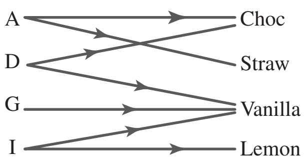
Question 7

$5 + 2 + 12 + 7 = 26$

Question 8
Cut A and B don't cut the two sources from the sink, so they aren't viable options.
$C = 5 + 12 + 3 = 20$
$D = 5 + 10 + 0 + 3 = 18$
$E = 5 + 11 + 6 = 22$
Therefore, cut D has the lowest value (which gives the maximum number of cars per minute).
***VCAA Assessment Report* note:**
A flow network with two sources (two car parks) and a single sink (the exit) was provided. While Cut C had the minimum capacity, neither it nor Cut A separated both sources (car parks) from the sink. Thus, neither of these cuts could be used to determine the minimum flow. Of the three cuts that did separate both sources from the sink, Cut D had the minimum capacity and hence determined the maximum flow.

10.5 Bipartite graphs and allocation problems

Question 1
The graph would be bipartite (two parts) and planar (no edges which cross).
(The graph would not be 'connected' as not all vertices connect to each other. It is not a 'tree' as the vertices are not connected to each other.)



(Note that the lines from A to Strawberry and from D to Chocolate do NOT have to be drawn as crossed. Hence, the graph is planar.)

Question 2
Looking at batting position number 1, Bo's average score is the highest, so he would be placed in the first position
In batting position number 2, we can see that both Bo and Cameron have an equal highest average score, but Bo has already been used, so we will assign Cameron to this position.
This leaves Alex in batting position number 3 (he also has the highest position 3 batting average).

Player	Batting position
Alex	3
Bo	1
Cameron	2

[1 mark]

Question 3
Using the Hungarian algorithm, the minimum total time is 24 minutes.
This is produced when:
Job 1 = Chamath
Job 2 = Alan
Job 3 = Deidre
Job 4 = Ewen
Job 5 = Brianna
OR
Job 1 = Chamath
Job 2 = Alan

Job 3 = Brianna
Job 4 = Ewen
Job 5 = Deidre.

Question 4
You'll need to go through each of the statements to determine if they are true.
Van ate both strawberry and orange.

Question 5
Using the Hungarian algorithm, the optimal allocation is Annie to Task 2 and Chuck to Task 4. Buddhi and Dorothy can be chosen for either Task 1 or 3 depending on what k is equal to.
The minimum time with the current allocation (Dorothy doing Task 1) is 12 minutes.
If Dorothy was doing Task 3, the minimum time would be $10 + k$ minutes.
For the current allocation to be the minimum time, $10 + k$ must be equal to or greater than 12.
Therefore, k must be equal to or greater than 2.

Question 6
Mandy must complete Task 4 as that is the only task she can complete.

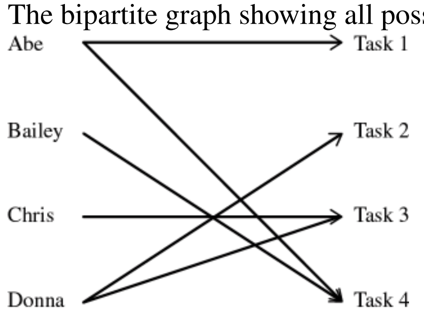
Question 7
Start with row reduction (subtract the smallest value from each row):

0	20	15	10
5	5	10	0
5	5	5	0
0	0	15	5

This does not give an optimal allocation, so do a column reduction:

0	20	10	10
5	5	5	0
5	5	0	0
0	0	10	5

The bipartite graph showing all possible allocations is:



There is only one possible allocation. Abe completes Task 1 in 80 mins; Bailey completes Task 4 in 60 mins; Chris completes Task 3 in 125 mins; and Donna completes Task 2 in 90 mins. This gives a minimum total time of 355 minutes.

Question 8

a Andrew and Charlie both have lines connecting them to steam trains, therefore the answer is 2. **[1 mark]**

b Brianna joined electric trains and miniature trains, while Charlie joined steam trains and miniature trains. Therefore the common interest group is miniature trains. **[1 mark]**

Question 9

Using the Hungarian algorithm, perform row reduction first:

3 0 1 3
1 0 2 2
0 2 4 1
1 0 1 0

Allocation is not possible as all zeros can be crossed with 3 lines.
Perform column reduction:

2 0 0 3
1 0 1 2
0 2 3 1
1 0 0 0

Allocation is now possible.
Perform allocation:
First allocate tasks to Kate and Nasim, as these columns have only one 0. Kate will do task Y (5 minutes) and Nasim will do task Z (2 minutes).
Next, allocate a task to Mei. Options for Mei are task W or task Z. Because Z has been allocated, Mei will do task W (4 minutes). The remaining task, X. will be done by Lexie (3 minutes).
Therefore, the minimum total time $= 5 + 2 + 4 + 3 = 14$ minutes.

Question 10

Using the Hungarian algorithm.

1 Subtracting the lowest value in each row from all values in that row gives:

	T1	T2	T3	T4	T5
P1	4	0	3	0	7
P2	6	8	7	6	0
P3	1	2	0	3	1
P4	0	3	1	5	6
P5	0	2	1	3	1

2 Subtracting the lowest value in each column form all other values in the column.

	T1	T2	T3	T4	T5
P1	4	0	3	0	7
P2	6	8	7	6	0
P3	1	2	0	3	1
P4	0	3	1	5	6
P5	0	2	1	3	1

3 Cover the zeros with the minimum number of lines.

	T1	T2	T3	T4	T5
P1	4	0	3	0	7
P2	6	8	7	6	0
P3	1	2	0	3	1
P4	0	3	1	5	6
P5	0	2	1	3	1

4 Covering the zeros would take a minimum of 4 lines (we need 5 as there are 5 rows). Therefore subtract the lowest value from all uncovered values and add the lowest value to any places where two covering lines intersect.

	T1	T2	T3	T4	T5
P1	5	0	3	0	8
P2	6	7	6	5	0
P3	2	2	0	3	2
P4	0	2	0	4	6
P5	0	1	0	2	1

5 Cover the zeros with the minimum number of lines.

	T1	T2	T3	T4	T5
P1	5	0	3	0	8
P2	6	7	6	5	0
P3	2	2	0	3	2
P4	0	2	0	4	6
P5	0	1	0	2	1

6 Covering the zeros would take a minimum of 4 lines (we need 5, as there are 5 rows). Therefore, subtract the lowest value from all uncovered values and add the lowest value to any places where two covering lines intersect.

	T1	T2	T3	T4	T5
P1	6	0	4	0	9
P2	6	6	6	4	0
P3	2	1	0	2	2
P4	0	1	0	3	6
P5	0	0	0	1	1

7 Covering the zeros would take a minimum of 5 lines; therefore, allocations can now occur.

	T1	T2	T3	T4	T5
P1	6	0	4	0	9
P2	6	6	6	4	0
P3	2	1	0	2	2
P4	0	1	0	3	6
P5	0	0	0	1	1

Allocation of tasks to people for minimum time is:
P1 to T4, P2 to T5, P3 to T3, P4 to T1 and P5 to T2. For these allocations, the times are
$8 + 6 + 8 + 8 + 15 = 45$ minutes

Question 11

Looking at the direct connections between warehouse and retail outlet the correct representation is Matrix A.

All other options involve at least one connection that does not exist between the warehouse and retail outlet; this eliminates them from the list of correct possibilities.

Question 12
The only job that person A can do is job W.

Question 13
The allocation of tasks is a problem because persons A and C can only do the same job. To allocate each person a job, alternative tasks must be found for either person A or person C..

Question 14
Person A must be connected to shop C. Disregard answer C.
Person B must be connected to shop A and B. Disregard answers A and D.
Person C must be connected to shop A and C. Disregard answer B.

Question 15
Option A lists a connection between B and T that does not exist.
Option C lists a connection between A and U that does not exist.
Option D lists a connection between A and V that does not exist.
Option E lists a connection between B and U that does not exist.
This leaves Option B as the only possible allocation.

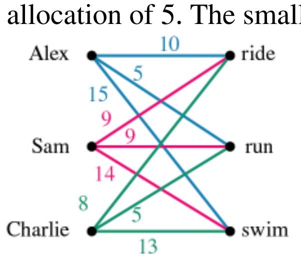
Question 16
Eliminating the greatest values that certainly would not form part of a 'minimum' solution gives the matrix:

$$\begin{array}{c} \\ Jonas \\ Brint \\ Anh \\ Nina \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 14 & 22 & & 10 \\ 10 & 7 & 10 & 15 \\ & 12 & 9 & 14 \\ 7 & & 9 & 11 \end{bmatrix} \end{array}$$

Looking at the values that are left and focusing on the least valued entries the minimum allocation would suggest:
Nina should take task A
Brit should take task B
Anh should take task C and
Jonas should take task D
This would result in a minimum time allocation of $7 + 7 + 9 + 10 = 33$ hours.

Question 17
Allocation of $\text{P1} - \text{T3}, \text{P2} - \text{T4}, \text{P3} - \text{T2}, \text{P4} - \text{T1}$ gives minimum times of $8 + 8 + 4 + 2 = 22$ hours.
All other possibilities give greater times than 22 hours.

Question 18
The smallest number is 5. It is on two allocations. Begin with the next smallest allocation of 8 and highlight it. The allocation of 8 removes one of the possible allocations with a weight of 5. Highlight the remaining allocation of 5. The smallest allocation for Sam is 14. Highlight the 14.



Their best chance of winning occurs if the highlighted allocations are made.

Question 19

The smallest number is 1. It is on two allocations. Begin with the next smallest allocation 3.
3 is also on two allocations, move onto the next smallest allocation 5 and highlight it. The allocation of 5 removes one of the possible allocations with a weight of 3. Highlight the remaining allocation of 3. The allocation of 3 removes one of the possible allocations with a weight of 1.

Question 20

Using the Hungarian algorithm:

1 Subtract the lowest value in each row from all values in that row.

Market Place	F1	F2	F3	F4	
M1	26	32	24	34	-24
M2	27	20	27	30	-20
M3	18	22	24	16	-16
M4	22	21	19	20	-19

3 Subtract the lowest value in each column from all other values in the column.

	F1	F2	F3	F4
M1	2	8	0	10
M2	7	0	7	10
M3	2	6	8	0
M4	3	2	0	1
	-2	-0	-0	-0

5 Covering the zeros takes a minimum of 4 lines, therefore allocations can now occur.

	F1	F2	F3	F4
M1	0	8	0	10
M2	5	0	7	10
M3	0	6	8	0
M4	1	2	0	1

6

7 M1 to F1, M2 to F2, M3 to F4 and M4 to F3. For these journeys the minimum time travelled is $26 + 20 + 16 + 19 = 81$ kilometres.

Question 21

Use the Hungarian algorithm on the matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} T1 & T2 & T3 & T4 \end{array} \\ \begin{bmatrix} 14 & 5 & 8 & 7 \\ 2 & 12 & 6 & 5 \\ 7 & 8 & 3 & 9 \\ 2 & 4 & 6 & 10 \end{bmatrix} \end{array} \begin{array}{c} \\ -5 \\ -2 \\ -3 \\ -2 \end{array} \longrightarrow \begin{array}{c} \begin{array}{cccc} T1 & T2 & T3 & T4 \\ & & & -2 \end{array} \\ \begin{bmatrix} 9 & 0 & 3 & 2 \\ 0 & 10 & 4 & 3 \\ 4 & 5 & 0 & 6 \\ 0 & 2 & 4 & 8 \end{bmatrix} \end{array} \longrightarrow \begin{array}{c} \begin{array}{cccc} T1 & T2 & T3 & T4 \end{array} \\ \begin{bmatrix} 9 & \boxed{0} & 3 & 0 \\ 0 & 10 & 4 & 1 \\ 4 & 5 & \boxed{0} & 4 \\ \boxed{0} & 2 & 4 & 6 \end{bmatrix} \end{array} \text{only 3 possible allocations}$$

Cover all rows and columns containing zeros, and then subtract the lowest uncovered value from every uncovered row and column.

$$\begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{bmatrix} 10 & 0 & 4 & 0 \\ 0 & 9 & 4 & 0 \\ 4 & 4 & 0 & 3 \\ 0 & 1 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 10 & \boxed{0} & 4 & 0 \\ 0 & 9 & 4 & \boxed{0} \\ 4 & 4 & \boxed{0} & 3 \\ \boxed{0} & 1 & 4 & 5 \end{bmatrix}$$

The corresponding positions on the original table are

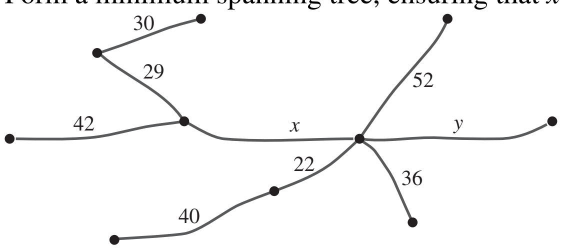
	Task 1	Task 2	Task 3	Task 4
Person A	14	5	8	7
Person B	2	12	6	5
Person C	7	8	3	9
Person D	2	4	6	10

Minimum allocation of time is 15 hours.

10.6 Review

Question 1

Form a minimum spanning tree, ensuring that x and y are included:



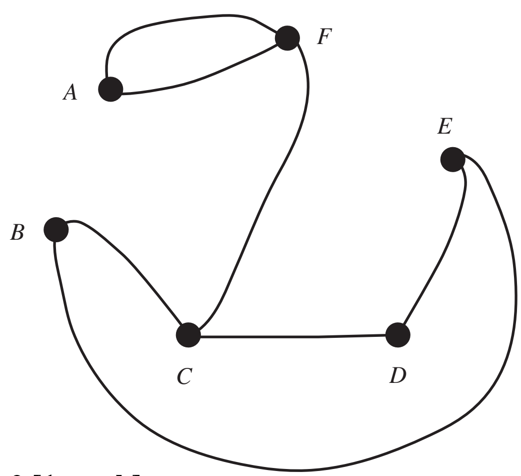
Adding up the lengths:

$30 + 29 + 42 + 40 + 22 + 36 + 52 + x + y = 251 + x + y$

The answer that matches the options is when $x = 50$ and $y = 60$ for a minimum length of 361 km.

Question 2

a **[1 mark]**



b 2 **[1 mark]**

c **i.** $C - D - E - B - A$ or the other way $A - B - E - D - C$ **[1 mark]**

ii. Hamiltonian Cycle **[1 mark]**

Question 3

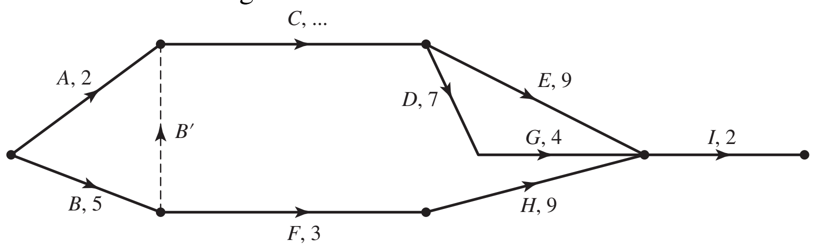
a By counting the number of different routes, you will find that there are 10 different routes possible. **[1 mark]**

b Capacity of cut 1 $= 20 + 12 + 20 = 52$ **[1 mark]**

Question 4

a This dummy activity could be drawn as a directed edge from the end of activity B to the start of activity C. **[1 mark]**

See network drawing below.



b As activity C has an EST of 7 and activity D has an EST of 7, and activity C is a predecessor of activity D, then the duration of activity C is 2 months. **[1 mark]**

c Float = LST − EST

A, F, H and E. **[1 mark]**

d Reduce the completion of time of activity B from 5 to 2 months, so that the project will be completed in a minimum time of 17 months. **[1 mark]**

Question 5

This directed network has two critical paths, ADHK and BFJK. This means the activities that are not on the critical path can be delayed.

These activities are C, E, G and I. Therefore, 4 activities can be delayed without affecting the minimum completion time of the project.

Question 6

a 14 weeks (path $B - C - E - J$) **[1 mark]**

b 7 activities have zero float time (there are 2 critical paths) **[1 mark]**

c Reduce activity A by 1 week and activity L by 2 weeks, so the minimum cost will be $380 000 **[1 mark]**

Question 7

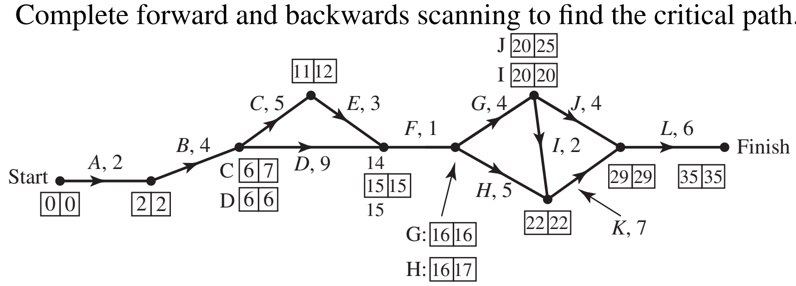
T and V can be reached directly from S.

X can be reached via V.

The number of vertices that can be reached from S is 3.

Question 8

a Complete forward and backwards scanning to find the critical path.



The critical path is $A - B - D - F - G - I - K - L$. Therefore there are eight activities. **[1 mark]**

***VCAA Examination Report* note:**

This question was reasonably well done. Some responses attempted to give the critical path rather than state the number of activities.

b The latest start time of activity E is after 12 weeks. **[1 mark]**

c Activity J with 5 weeks. **[1 mark]**

***VCAA Examination Report* note:**

Students should be aware that if they did not leave their answer as J but further engaged with the question and then stated an incorrect float time, they could not be awarded marks.

d Activities C and H are not on the critical path (initially) so there is no point reducing them.
If activity D is reduced by one week, D is now equivalent to activities C to E, therefore it can be reduced by one week.
Reduce activity D again by one week and activity C is also reduced by one week, because both are equal parts on the critical path, therefore they can be reduced by one week.
Activity D is now reduced by the maximum time. Activity C cannot be reduced anymore because it would then not be on the critical path.
Reducing activity G by one week, meaning activities $GIK = 12$. creates an equivalent critical path for activities H to K. Therefore it is reduced by one week.
If you reduce activity G again by one week then activity H must also reduce by one week. Therefore it is reduced by one week.
Activity G is now reduced by the maximum time. Activity H cannot be reduced anymore because it would then not be on the critical path.
Activity K can be reduced by the maximum of two weeks.
Therefore, there is a reduction of six weeks overall, so now the project can be completed in 29 weeks. **[1 mark]**
VCAA Examination Report note:
Many failed to take two weeks off each of the critical activities.

e

Activity	Reduction in completion time (0, 1 or 2 weeks)
C	0 weeks
D	1 week
G	2 weeks
H	1 week
K	1 week

Award 1 mark for correct table.
Award 1 mark for correct values.
VCAA Examination Report note:
A method mark was available for one of the following answers that also reduced the overall time by four weeks and did not involve unnecessary wastage.

Question 9

a The earliest start time for activity I is 10 hours, as activities B, E and G must finish before it can start. **[1 mark]**
***VCAA Examination Report* note:**
The most common incorrect answer given by students was 8, which indicated that they did not recognise that the longest path to I was required.

b The critical path is BEGHJ. **[1 mark]**

c Activities A and C have a float time of 2 hours. **[1 mark]**
VCAA Examination Report note:
This question was not answered well, with many combinations of incorrect answers given by students.

d To have an earliest start time of 5 hours, the activity must be from the end of activity E. To have a latest start time of 12 hours, the activity must be just before the start of activity J. **[1 mark]**
VCAA Examination Report note:
This question was not answered well. Some students could identify E as the starting point of the required edge but could not identify J as the end point.

Question 10

a Colin will plan Tour 2. **[1 mark]**

***VCAA Examination Report* note:**
This question was not well answered. Students needed to state that Colin must plan Tour 2 (or equivalent).
Many gave additional information usually from values in Table 1; however, Table 1 did not need to be considered at all.
An answer such as 'Colin will plan Tour 2 and take 8 minutes to do it' was acceptable as the extra information was correct and did not negate the first part. However, an answer such as 'Colin will plan Tour 2 because he is the fastest' was not correct as Diane could plan Tour 2 more quickly than Colin.
The algorithm gives the best overall allocation taking all values into account.

b From Table 2, the optimum allocation is Agatha will plan Tour 4, Bai will plan Tour 1, Colin will plan Tour 2 and Diane will plan Tour 3.
Minimum total planning time is $10 + 7 + 8 + 18 = 43$ minutes. **[1 mark]**
VCAA Examination Report note:
Some responses showed evidence of the correct allocation but did not give the total planning time.

Question 11

a D and E (via the dummy edge) are the two immediate predecessors of Activity I. **[1 mark]**
***VCAA Examination Report* note:**
This question was answered quite well. The dummy is not an activity and hence writing it as an additional predecessor could not be accepted.
Look for the activities (edges) that lead into Activity I.

b **i.**

Activity	EST	LST	Float
A	0	0	0
B	2	8	6
C	2	3	1
D	2	3	1
E	2	2	0
F	5	11	6
G	7	8	1
H	10	14	4
I	10	10	0
J	10	10	0
K	8	14	4
L	12	12	0
M	8	15	7
N	17	17	0
O	12	13	1

The other critical path is $A - E - I - L - N$. **[1 mark]**

ii. Float time for activity $F = LST - EST$
$$= 11 - 5$$
$$= 6 \text{ days}$$ **[1 mark]**

VCAA Examination Report note:
This question was not answered well, with 7 and 5 the most common incorrect responses.

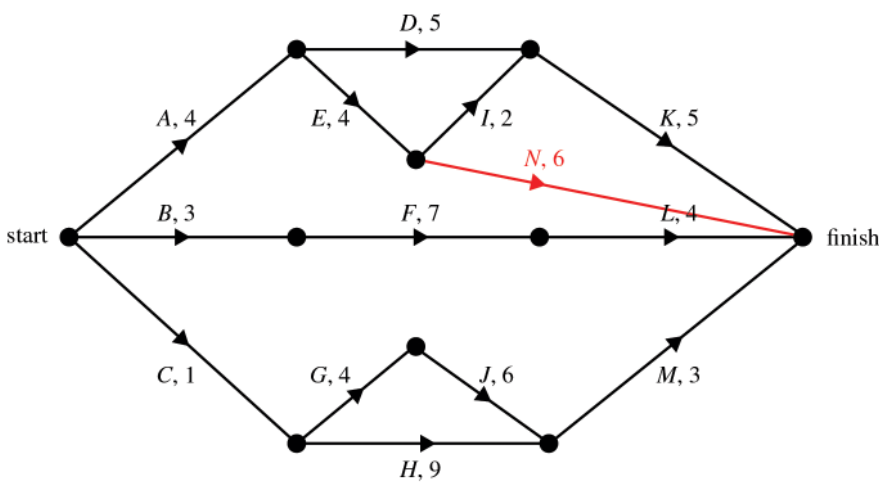
c **i.** Activities B and D will not save any time by being crashed.
The project can be reduced by 2 days to 17 days if each of the activities, G, I, J and L are crashed by one day. **[1 mark]**

ii. To complete the project in 17 days, the four activities G, I, J and L are crashed by one day each at a cost of \$1000 per activity. The minimum cost is $4 \times \$1000 = \$4000$. **[1 mark]**

***VCAA Examination Report* note:**
This question was not answered well. Some students recognised the need to reduce the critical activities (I, J and L) but did not realise that activity G also had to be reduced as it was part of a new critical path formed.

Question 12
The number of edges in any tree is one less than the number of vertices, so option A is true.
A complete graph with 6 vertices has $\frac{6\times5}{2}$ = 15 edges, so option B is true.
It is possible that the graph *may* have an isolated vertex, so option C is true.
Given that the graph has no loops or multiple edges, if a cycle exists it will have a minimum of three edges, so option E is true.
There is no rule to state that the bipartite graph will have a minimum number of nine edges, so option D is not true.
***VCAA Examination Report* note:**
This question was not completed well. An effective way to solve this question was to check each of the options separately, with the aid of a diagram.

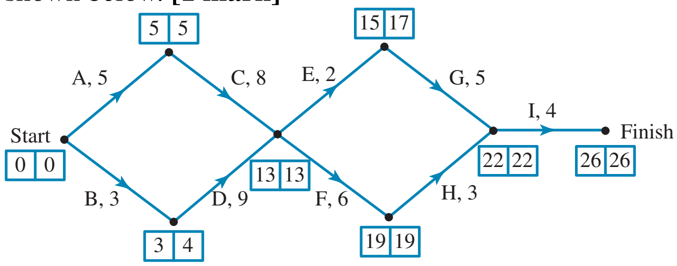
Question 13
a $1 + 4 + 6 = 11$ days **[1 mark]**
b $A - E - I - K$ **[1 mark]**
c H (float $= LST - EST = 3 - 1 = 2$ days) **[1 mark]**
VCAA Assessment Report note:
A common incorrect answer was activity M.
d By reducing either E or I by 1 day, as they are on the critical path, the shortest time possible to complete the project is 14 days.
Reducing I by 1 day will incur the least cost of \$2000. **[1 mark]**
e i. **[1 mark]**



ii. Latest start time $= 15 - 6 = 9$ days **[1 mark]**

Question 14
a Activity D 's immediate predecessor is activity B, so D cannot start until B has finished. This is at 3 minutes.**[1 mark]**
b Activity D is the immediate predecessor to activities E and F. To get to the same starting point, we also need activities A and C to have finished, which takes 13 minutes in total. This means that the LST of activity D is 4 minutes (float time of 1 minute). **[1 mark]**
c The EST of activity H is 19 minutes and I is 22 minutes. That makes the duration of activity H 3 minutes. **[1 mark]**

d The activities on the critical path are those with a float time of zero. That is activities A, C, F, H and I, as shown below. **[1 mark]**

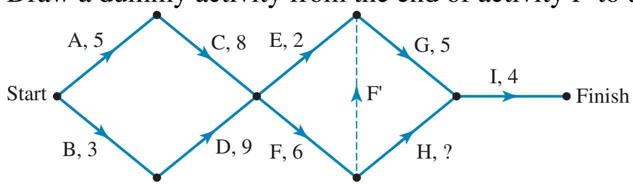


e It will add 7 minutes to the usual minimum preparation time. **[1 mark]**

***VCAA Assessment Report* note:**

It was insufficient to simply write an answer of '7 minutes' or '33 minutes'. Reference to an 'increase' or 'new critical path' was expected.

f i. Draw a dummy activity from the end of activity F to the start of activity G.



Award 1 mark for dummy activity.

***VCAA Assessment Report* note:**

The dummy activity needed to indicate that F has become a prerequisite for G. This required an arrow on the line. Then, this connection needed to be identified as having a duration of zero or be labelled as *dummy*.

ii. It will add 2 minutes to the usual minimum preparation time. **[1 mark]**

***VCAA Assessment Report* note:**

A number of students incorrectly stated that 'there is no effect since a dummy takes zero time', while others stated that 'there is no effect since G is not on the critical path'.

Question 15

The adjacency matrix for one-step dominance is:

$$\begin{array}{c} L \\ E \\ H \\ R \\ D \end{array}\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

So one-step dominance plus two-step dominance is:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 2 & 3 & 1 & 2 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 2 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix}$$

The sum for Lions is 8; the sum for Eagles is 4; the sum for Heavies is 0; the sum for Rebels is 7; and the sum for Dingoes is 5.

Therefore, the final ranking is Lions, Rebels, Dingoes, Eagles, Heavies.

Question 16

a Filling down Andrew's column:
finances: $9 - 9 = 0$ (smallest $= 9$)
equipment: $8 - 8 = 0$ (smallest $= 8$)
catering: $9 - 8 = 1$ (smallest $= 8$) **[1 mark]**

b

Task	Andrew	Brianna	Charlie	Devi
publicity	3	2	0	0
finances	0	1	2	2
equipment	0	4	3	2
catering	1	2	3	0

The number of rows = 4. The minimum number of lines to cover all the zeros is 3. As the number of lines is not greater than or equal to the number of rows, an allocation of tasks isn't possible yet.
[1 mark – must mention number of rows and lines covering zeros]

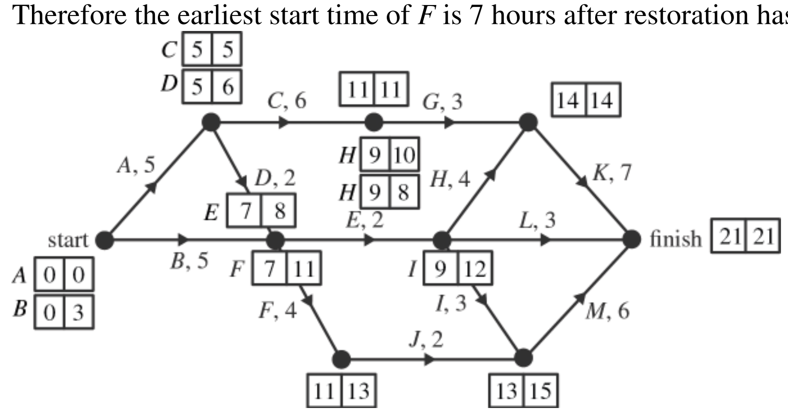
***VCAA Assessment Report* note:**
Students' explanations needed to refer to the stage in the process of the Hungarian algorithm. This required reference to the required minimum number of lines though zeroes.
Some students simply stated that there was 'no clear allocation to Brianna'. Such reference to an 'allocation' does not explain how allocations might be attempted at this first stage of the algorithm. Further, if there is 'no clear allocation to Brianna', it also follows that there are no clear allocations to anybody at this stage of the algorithm.
Another common but unacceptable answer was 'there are not enough zeroes'. This answer does not indicate how many zeroes might be enough or how the zeroes would be used. Even if the table had up to 12 zeroes in three lines or columns, the Hungarian algorithm indicates that at least one further step is needed.

c i. Andrew should be allocated either finances or equipment (as indicated by the zeros); however, as Brianna can only do finances (her only zero), Andrew should be allocated equipment. **[1 mark]**

ii. Andrew (equipment) = 8 hours
Brianna (finance) = 10 hours
Charlie (publicity) = 10 hours
Devi (catering) = 8 hours
Therefore the total number of hours = 36. **[1 mark]**

Question 17

a Activity F follows activities B and D. B takes 5 hours. D (with A preceding it) takes $5 + 2 = 7$ hours. Therefore the earliest start time of F is 7 hours after restoration has started. **[1 mark]**



Setting up the diagram first will help to answer all the questions.

b LST of activity L $= 21 - -3 = 18$ hours after the restoration has started. **[1 mark]**

c $\text{LST}_J - -\text{EST}_J = 13 - -11 = 2$ hours (after backward scanning) **[1 mark]**

d LST of $G = 11$ hours

Therefore LST of $X = 11 - 7 = 4$ hours after restoration has started. **[1 mark]**

e Critical path = ACGK

If A → 4 hours, ACGK is still the critical path (now 20 hours).

If A → 3 hours, ACGK is still the critical path (now 19 hours).

If A → 2 hours, ACGK is still the critical path (now 18 hours), but BEHK is now also critical (18 hours).

There would be no point in reducing A further as BEHK would still remain the critical path at 18 hours.

Total cost $= 3 \times 90 = \$270$ **[1 mark]**

***VCAA Assessment Report* note:**

Reducing any path that includes A below 18 hours is pointless since $B - E - H - K$ becomes a critical path at 18 hrs. The critical path $A - C - G - K$ can be reduced to 18 hours if A is reduced by three hours.

Max. reduction = 3 hrs × \$90 = \$270

Question 18

a 37 people

VCAA Examination Report note:

The numbers on the edges of the directed network gave the maximum number of people who are permitted to walk along any one of the tracks.

The question required the maximum number of people permitted to walk from *A* to *D* each day. Many students did not see this question as a minimum cut/maximum flow problem.

Maximum flow = minimum cut of 37 through *CD* and *ED* or through *AB*, *FB* and *FE* or through *BC*, *EC* and *ED*.

b **i.** $A - B - E - C - D$ **[1 mark]**

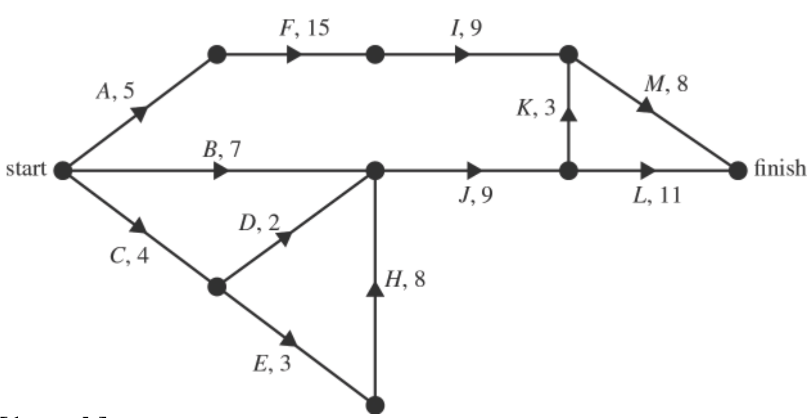
ii.

Group	Maximum group size	Path from *A* to *D*
1	17	$A - B - E - C - D$
2	11	$A - F - E - D$
3	7	$A - G - F - B - C - D$
4	2	$A - B - E - D$

Award 1 mark for correct group sizes and 1 mark for correct paths.

Question 19

a



[1 mark]

b The earliest starting time of activity H is $4 + 3 = 7$ h. **[1 mark]**

c i. The latest finish time for activity D is equal to the latest starting time for activity G, which is 16 h (as G has to finish by the time A and F finish, which is at 20h, and the duration of G is 4). The duration of activity D is 2 h. Therefore, the latest starting time for activity D is 14h. **[1 mark]**

***VCAA Assessment Report* note:**

A common incorrect answer was 15 hours.

d This statement will be true if the activity that is being crashed is one of the activities on the critical path (that is,A, F, I or M). **[1 mark]**

e If activity F is crashed by 2 hours, the new critical path emerges:

$C - E - H - G - I - M$.

The minimum completion time is then $4 + 3 + 8 + 4 + 9 + 8 = 36\,\text{h}$. **[1 mark]**

Question 20

Looking at each of the network displays (and given it is an adjacency matrix, not a weighted matrix) the numbers in the matrix indicate the number of directed connections. The only matrix that matches the directions indicated and the number of connections involved is E.

Question 21

There are no loops in the network. Disregard any matrix with a 1 on the main diagonal.

There is only one edge between each pair of vertices with the same direction. Disregard any matrix containing the number 2.

Question 22

There is one path from A to C. Disregard answers A and C.

There are two paths from C to B. Disregard answers B and E.

Question 23

There are no loops in the directed graph. Therefore, there should be only zeros on the main diagonal. Disregard answer E.

There are only single paths between each vertex. The adjacency matrix shows only the number of paths not the weight of the paths.

Disregard answers B and D.

The direction of the path is from the row to the column. There is a path from A to B, therefore a 1 should appear in the row for A (the first row) and the column for B (the second column). Disregard answer C.

Question 24

The required matrix will not be symmetrical due to the directional connections as these lead to only one entry in the matrix.

Linking connections to matrix entries gives Matrix D as the only valid option. A 0 is placed where the arrow is directed away from the vertex.