ATARNotes

Maths Methods

ATARNotes September Lecture Series

Presented by:
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About Me

- Hey, everyone my name is Manjot Bhullar
- Bachelor of Biomedical Science at Monash
- Maths and Science tutor at Tutesmart
- The subjects I did throughout VCE
    - Chemistry
    - Maths Methods
    - Specialist Maths
    - English
    - Biology
    - Further Maths

Overview

- Function, Relations and Matrices
- Transformations
- Logarithms and Exponents
- Circular Functions
- Differentiation and Application
- Integration and Application
- Probability Tools and Definitions
- The Binomial Distribution
- The Normal Distribution

- Function, Relations and Matrices
  - Understanding the basic properties of functions and their notation
  - To represent functions in matrix form
  - Matrix operations
- Transformations
  - Understanding the geometric representation of transformations
  - Computing transformations in function and matrix notation
- Logarithms and Exponents
  - Understanding the geometric representation of the functions
  - Properties, rules and application of logarithms and exponentials
- Circular Functions
  - Understanding the geometric representation of the functions
  - Properties of complementary and supplementary angles

- Notation is super important!
- There are some simple things to look out for to get full marks

- Trig and log functions always need brackets, even if it's just $\sin(x)$
- Make sure each bracket has a friend, no lonely brackets!
- Think of integral symbols like brackets, an $\int f(x)$ needs to be followed by a dx

- Always give answers in exact form, unless a number of decimal places is specified
- Leave technology in radians, only switch if the question specifies degrees
- Don’t put tech jargon in your working out (e.g. don’t write menu, algebra, solve)

- If your function is $y =$, then use $\frac{dy}{dx}$
- If your function is $f(x) =$, then use $f'(x)$
- If you're just differentiating an expression (e.g. $\sin(x)$), then use $\frac{d}{dx}$(sin(x))=

- Make sure you use $\log_e(x)$ not $\log e(x)$
- Also note that $\ln(x)$ means $\log_e(x)$ not $\log_{10}(x)$
- Always use brackets

- If a question defines a function as $f(x)$, you can refer to it as $f(x)$ throughout instead of writing out the whole function

- Make sure your square root covers what you intend it to cover

- $f^{-1} \neq \frac{1}{f}$

- $\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$

- Be careful when manipulating:
  - Fractions
  - Negatives
  - Square roots
  - Inequalities

- It's really easy to make mistakes with these, so avoid making too many manipulations in one line

Function Notation

$$f: [a, b] \to R, f(x) = y$$

- The domain is explicated stated
- The range is derived from restricting y, by the domain

- Basic understanding is required for the exam
- Be careful when labelling the function name and dependent variable
- Look at marks to see if domain is required in your answer

Implied Domain

- Assume implied or maximal domain if not explicitly stated

$f(x) = \frac{a}{d(x)} + g(x)$	$d(x) \neq 0$ $\text{Dom}(f): \text{Dom}(g) \backslash \{x: d(x) = 0\}$
$f(x) = \sqrt{x} + g(x)$	$\text{Dom}(f): \text{Dom}(g) \backslash (-\infty, 0)$
$f(x) = \frac{a}{\sqrt{x}} + g(x)$	$\text{Dom}(f): \text{Dom}(g) \backslash (-\infty, 0]$

- Questions usually do not explicitly state their domains, you must know the assumptions when giving your answer

Implied Domain

Solve $\sqrt{x} = 2 - x$, for x.

$$\rightarrow x = (2-x)^2$$
$$x = 4 - 4x - x^2$$
$$0 = x^2 - 5x + 4$$
$$0 = (x-4)(x-1)$$
$$x = 1, x = 4$$

Solve $\sqrt{x} = 2 - x$, for x.

$$\rightarrow x = (2 - x)^2$$
$$x = 4 - 4x - x^2$$
$$0 = x^2 - 5x + 4$$
$$0 = (x - 4)(x - 1)$$
$$x = 1, x = 4$$

(POLL)

$\sqrt{1} = 2 - 1$

$\sqrt{4} \neq 2 - 4$

Why?

$$\sqrt{x} \geq 0 \quad \rightarrow \quad 2 - x \geq 0$$
$$\text{dom}(x): \ [0,2]$$

Sum and Product Functions

- Adding and multiplying functions gives us special properties

$$h(x) = f(x) + g(x)$$
$$h(x) = f(x)g(x)$$

$$\text{Dom}(h): \text{Dom}(f) \cap \text{Dom}(g)$$

- You can then sketch them use addition or multiplication of ordinates

- Unlikely to appear as a stand alone question in exam
- In exam 1, will often ask students to explicitly state domain and determine range.

Composite Functions

- Essentially a function within a function

$$h(x) = f\big(g(x)\big)$$

$$\text{Dom}(h) = \text{Dom}(g)$$

If, and only if:
$\text{Ran}(g) \in \text{Dom}(f)$

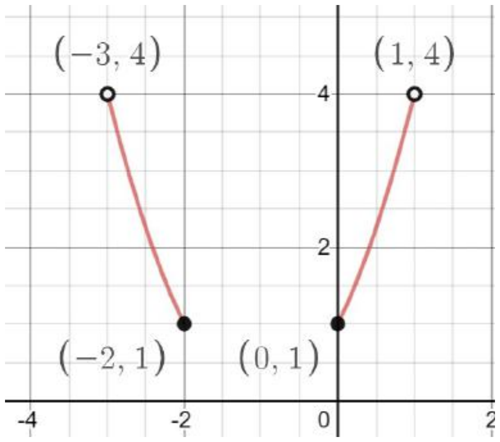
- The reason for this is because the function g, essentially becomes the x values of f. Therefore, as the range of g becomes the domain of f it must fit within the domain of f.

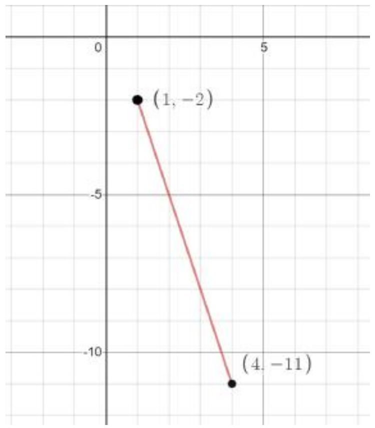
- This is likely to appear in conjunction with sum, product and inverse functions. They usually ask for compatibility.

$$f: R \to R, f(x) = (x+1)^2 \text{ and } g: [1,4] \to R, g(x) = -3x + 1$$

Do the functions $f(g(x))$ and $g(f(x))$ exist?

	dom	*ran*
f	R	$[0,\infty)$
g	[1,4)	(-11,2]





to exist, $ran\ inside \subseteq dom\ outside$

so $f(g(x))$ exists, but $g(f(x))$ needs restriction

So for $g(f(x))$ to exist, $f \in (-3,-2] \cup [0,1)$

- A function reflected in the line $y = x$

$$f(y) = x$$
$$\text{Dom}(f^{-1}) = \text{Ran}(f) \quad \text{Ran}(f^{-1}) = \text{Dom}(f)$$

- Swapping x and y and solving for y gives $y^{-1} = f^{-1}(x)$.

- Most schools, and VCAA, require a systematic process that. must be followed for full marks

- Often appears as a multiple choice question involving the visual representation of inverse functions

For $f(x) = \sqrt{x-2} + 1$, find $f^{-1}(x)$.

(1) Let $y = f(x)$

(2) For inverse, swap x and y

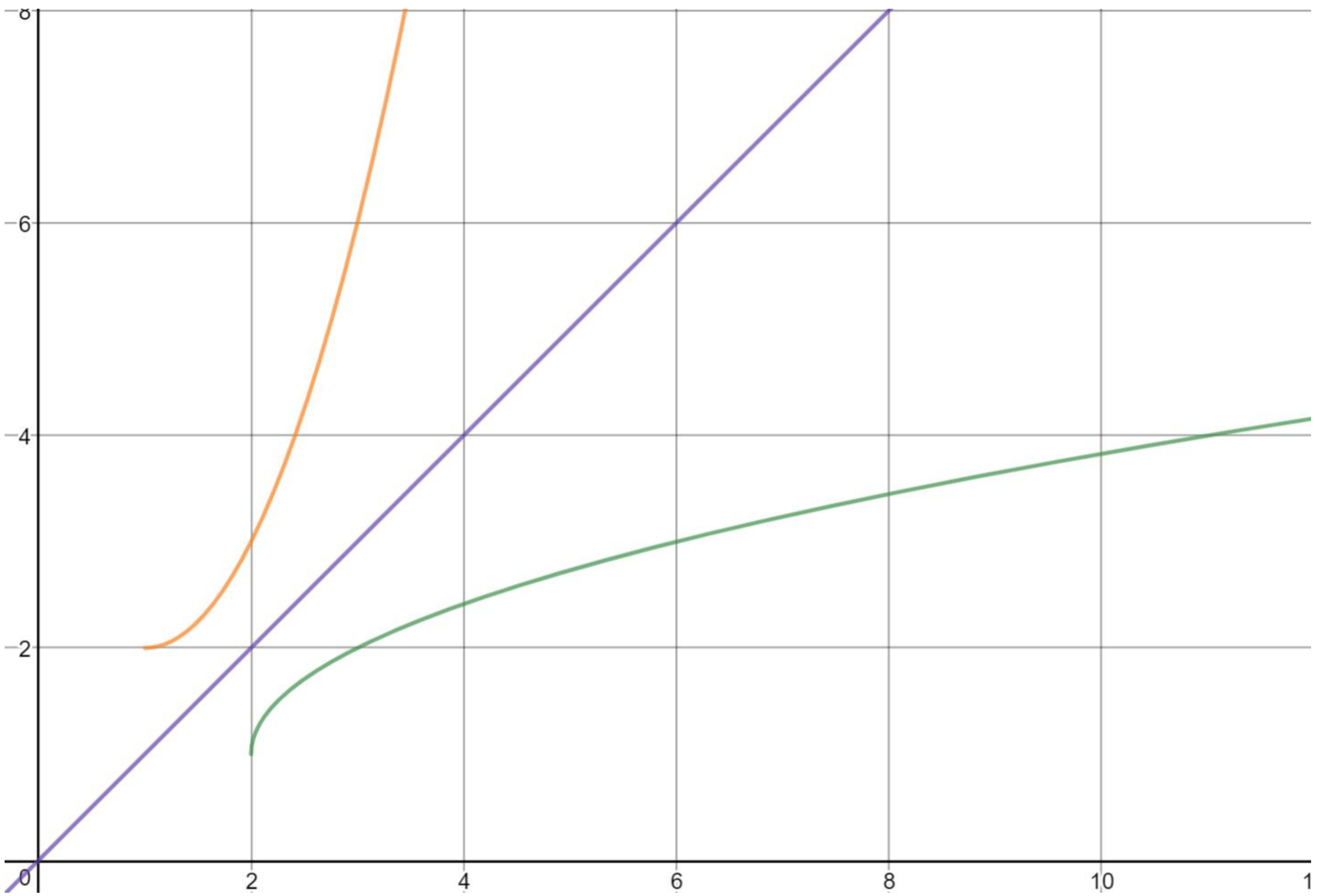
(3) Solve for y, $y = x^2 - 2x + 3$

(4) Apply domain if necessary $x \geq 1$

(5) Write your answer in required form:

$$f^{-1}: [1, \infty) \to R, f^{-1}(x) = x^2 - 2x + 3$$

Inverse Functions



- We apply transformations to *stretch, squeeze,* or *shift* the graphs of functions.
- Need to memorise both **definitions** and **procedures** for transforming functions ☺

- **Reflections** ⇒ **flipping** a graph
- **Dilations** ⇒ **stretching** a graph
- **Translations** ⇒ **shifting** a graph

Dilations from the x-axis

Dilation by factor k away from the x-axis is described by

$$(x, y) \to (x, ky)$$

Replace y with y/k

$$y = f(x) \quad \to \quad y = kf(x)$$

Dilations from the y-axis

Dilation by factor k away from the y-axis is described by

$$(x, y) \to (kx, y)$$

Replace x with x/k

$$y = f(x) \quad \to \quad y = f\left(\frac{x}{k}\right)$$

Reflection in the x-axis

$$(x, y) \to (x, -y)$$

Replace y with $-y$

$$y = f(x) \quad \to \quad -y = f(x)$$

Reflection in the y-axis

$$(x, y) \to (-x, y)$$

Replace x with $-x$

$$y = f(x) \quad \to \quad y = f(-x)$$

Vertical translations

- Translation by k units in the positive direction of the y-axis

$$(x, y) \rightarrow (x, y + k)$$

Replace $\boldsymbol{y}$ with $(\boldsymbol{y} - \boldsymbol{k})$

$$y = f(x) \quad \rightarrow \quad y - k = f(x)$$

Horizontal translations

- Translation by h units in the positive direction of the x-axis

$$(x, y) \rightarrow (x + h, y)$$

Replace $\boldsymbol{x}$ with $(\boldsymbol{x} - \boldsymbol{h})$

$$y = f(x) \quad \rightarrow \quad y = f(x - h)$$

- $h/k > 0 \Rightarrow$ shift in the positive direction
- $h/k < 0 \Rightarrow$ shift in the negative direction

Example

The graph of $g(x)$ has been shifted left 3 units, then dilated by a factor of 2 from the $y\ axis$. If $g(x) = \log_e(3x - 1)$ find $f(x)$, the transformed function.

$$g(\boldsymbol{x} + 3) = \log_e(3(x + 3) - 1)$$

$$g\left(\frac{\boldsymbol{x}}{\boldsymbol{2}} + 3\right) = \log_e\left(3\left(\frac{x}{2} + 3\right) - 1\right)$$

$$\therefore f(x) = \log_e\left(\frac{3x}{2} + 8\right)$$

$$\text{Let } g(x) = 3\left(\frac{x}{2} - 1\right)^2 + 1 \text{ and } f(x) = x^2$$

Determine the set of transformations that take $f(x)$ to $g(x)$.

$$y' = 3\left(\frac{x'}{2} - 1\right)^2 + 1 \qquad y = x^2$$

Rearrange to replace

$$\frac{1}{3}(y' - 1) = \left(\frac{x'}{2} - 1\right)^2 \qquad y = (x)^2$$

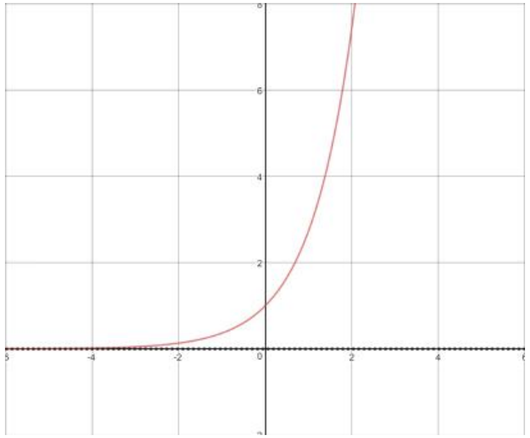
so $\quad y = \frac{1}{3}(y' - 1) \quad$ and $\quad x = \frac{x'}{2} - 1$

$$y' = 3y + 1 \qquad x' = 2x + 1$$

- Introducing Euler's constant $e = 2.71828182846 \ldots$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

- Has some special properties and application in calculus and finance
- Base function $f(x) = e^x$
- Has a horizontal asymptote
- Special property: $\frac{d}{dx}(e^x) = e^x$



$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$a^m = e^{m \log_e(a)}$$

Application of these rules tend to appear in exam one questions which require students to solve equations after finding a common base.

$$\text{Solve } 3\times16^{3x}-1+2\times4^{3x}=0$$

$$3(4^2)^{3x}-1+2(4^{3x})=0$$

$$3(4^{3x})^2+2(4^{3x})-1=0$$

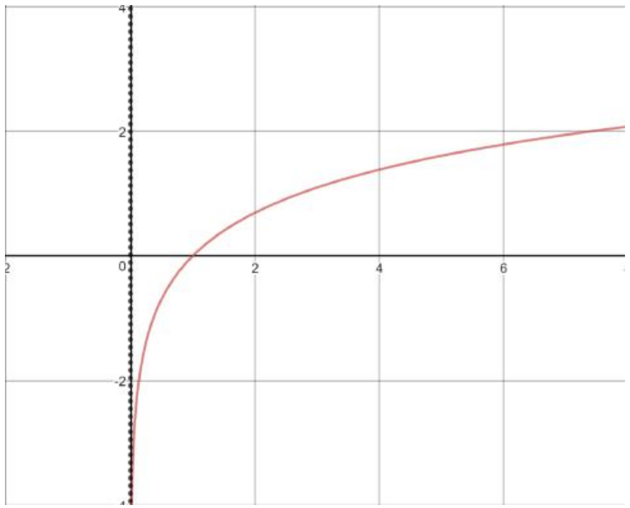
$$(3(4^{3x})-1)(4^{3x}+1)=0$$

$$(3(4^{3x})-1)=0 \text{ and } (4^{3x}+1)=0$$

$$x=-\frac{1}{3}\log_4(3)$$

- Essentially the inverse function of the exponent

$$\log_b(b^a) = a$$



- Base function $f(x) = \log_e(x) = \ln(x)$
- Has a vertical asymptote

Expect to see solve questions, similar to exponentials. Properties are more diverse, students often make mistakes with the implied domain

Logarithmic Laws

$$\log_a(m) + \log_a(n) = \log_a(mn)$$

$$\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$$

$$\log_a(m^n) = n\log_a(m)$$

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

The change of base rule, whilst uncommon, is often forgotten.

- Our second irrational constant $\pi = 3.14159265359 \ldots$

$$\pi^c = 180^\circ$$

- We have three basic functions
  - $\sin(x)$
  - $\cos(x)$
  - $\tan(x)$

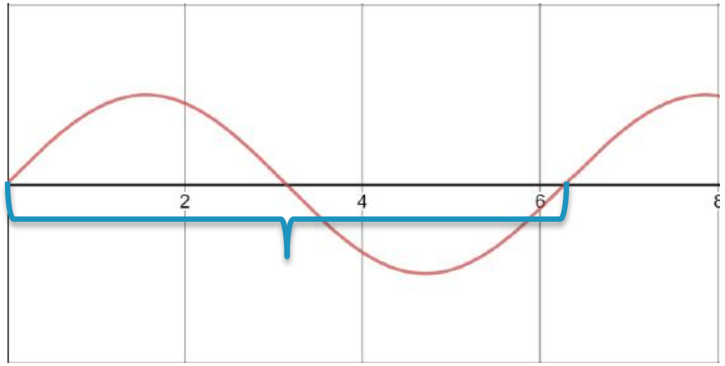
- It is important to understand that $\sin(x)$ and $\cos(x)$ are the same curve at different points, therefore they can be mapped onto each other (transformations)

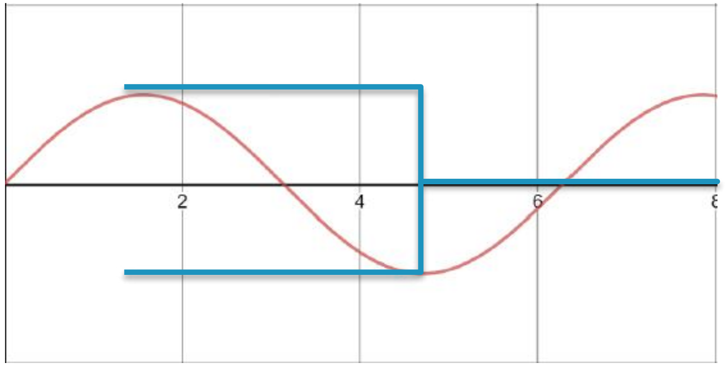
Exact Values

$f(x) \downarrow x \rightarrow$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undef

Circular Functions

	a $\sin(nx)$	a $\cos(nx)$	a $\tan(nx)$
Amplitude	a	a	
Period	$\frac{2\pi}{n}$	$\frac{2\pi}{n}$	$\frac{\pi}{n}$



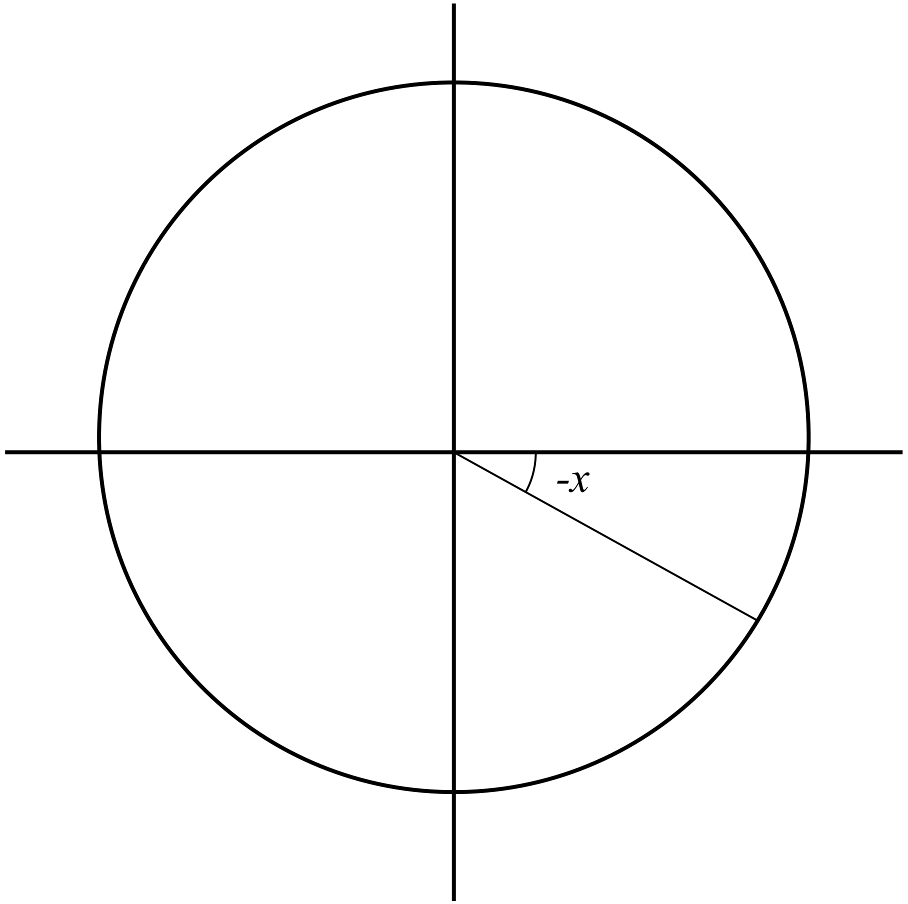


- Common exam one and multiple choice question, usually asked at the beginning for free marks.

Circular Functions

$\sin(-x)$	=	$-\sin(x)$
$\cos(-x)$	=	$\cos(x)$
$\tan(-x)$	=	$-\tan(x)$
$\sin(x+\pi)$	=	$-\sin(x)$
$\cos(x+\pi)$	=	$-\cos(x)$
$\tan(x+\pi)$	=	$\tan(x)$
$\sin\left(x+\frac{\pi}{2}\right)$	=	$\cos(x)$
$\cos\left(x+\frac{\pi}{2}\right)$	=	$-\sin(x)$
$\tan\left(x+\frac{\pi}{2}\right)$	=	$-\frac{1}{\tan(x)}$
$\sin^2(x)+\cos^2(x)=1$		

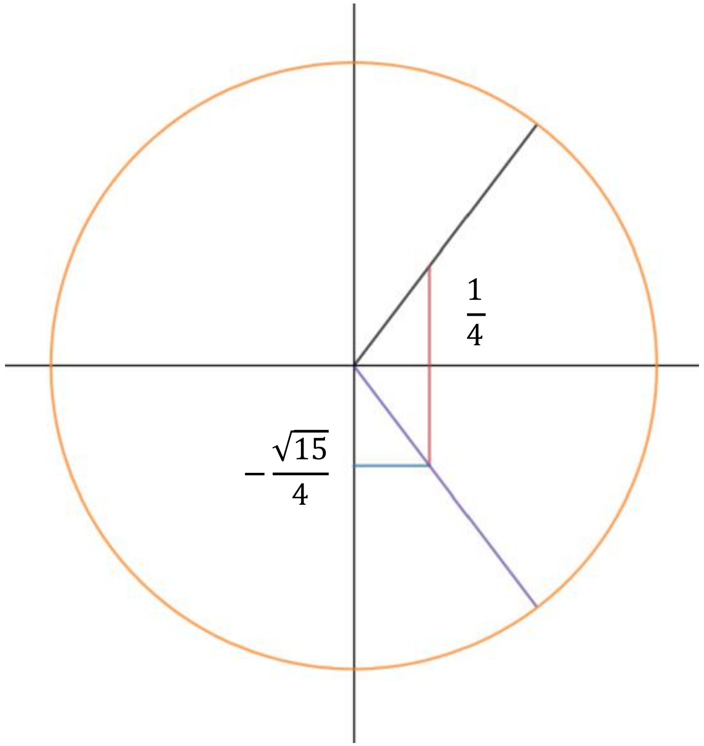
Circular Functions



Given that $\cos(a) = \frac{1}{4}$, find $\sin(a)$ and $\sin\left(a + \frac{\pi}{2}\right)$ for $a \in (-\frac{\pi}{2}, 0)$

$$\sin(a) = -\sqrt{1 - \cos^2(a)} = -\frac{\sqrt{15}}{4}$$

$$\sin\left(a + \frac{\pi}{2}\right) = \cos(a) = \frac{1}{4}$$



Be careful, these values are positive in 2 quadrants, make sure you know which one

Solving can be quite tricky – so it’s best to adopt some systematic steps

- Rearrange for a simple equation, e.g. $\sin(ax + b) = c$
- Find the ***basic angle***
- Determine the sign of the angle and the next solution
- Solve for x
- Add as many periods as required

Solve:

$$\sin(2\theta) = -\frac{1}{2}, \theta \in [0, 2\pi]$$

$$b. a = \frac{\pi}{6}$$

In 3rd or 4th quadrant

$$2\theta = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$\theta = \frac{7\pi}{12} \pm np, \frac{11\pi}{12} \pm np$$

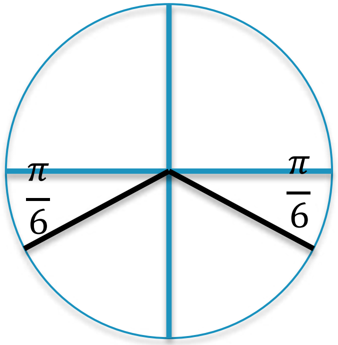
$$n \in Z$$

Solve

$$\sin(2\theta) = -\frac{1}{2}, \theta \in [0, 2\pi]$$

$$\theta = \frac{7\pi}{12} \pm np, \frac{11\pi}{12} \pm np$$

$$p = \pi$$



$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

As $\theta \in [0,2\pi]$

Solve $\sin^3(x) + \cos^2(x)\sin(x) = 0$ for $x \in [2\pi, 4\pi]$

$$\sin(x)(\sin^2(x) + \cos^2(x)) = 0$$

$$\sin(x) = 0$$

$$x = 2\pi, 3\pi, 4\pi$$

What if $x \in R$?

$$x = n\pi, \qquad n \in Z$$

- When the domain is unspecified or when specifically asked, the general solution is required
- When you arrive at the stage $f(x) = a$, then:

$$x = 2n\pi \pm \cos^{-1}(a)$$

$$x = n\pi + \tan^{-1}(a)$$

$$x = 2n\pi + \sin^{-1}(a) \text{ and } x = (2n + 1)\pi - \sin^{-1}(a)$$

$$x = n\pi + (-1)^n \sin^{-1}(a)$$

$$n \in Z$$

This can be quite tricky – so it's best to adopt some systematic steps

- Rearrange for a simple equation, e.g. $\sin(ax + b) = c$
- Find the ***basic angle***
- Determine the sign of the angle and the next solution
- Solve for x
- Add n periods where $n \in Z$

Solve $3\sin\left(\frac{x}{2}-1\right)=\frac{3}{2}$ for x.

$$\sin\left(\frac{x}{2}-1\right)=\frac{1}{2}$$

$$\frac{x}{2}-1=\frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$x=2\left(\frac{\pi}{6}+1\right) or\ 2\left(\frac{5\pi}{6}+1\right)$$

$$\text{Period} = \frac{2\pi}{1/2}=4\pi$$

$$x=2\left(\frac{\pi}{6}+1\right)+4n\pi \text{ or } x=2\left(\frac{5\pi}{6}+1\right)+4n\pi$$

$$n\in Z$$

- Sketching graphs is significantly more important in exam one, examiners are looking for scale, curvature, and asymptotic behaviour when applicable.
- Exam 1 questions will very rarely require you to sketch a complicated trigonometric function
- Important part is getting the domain and general shape correct
- There are many ways to sketch trigonometry graphs, use the method you are comfortable with

Sketching Tan function

For $f(x) = a\times\tan(nx + c)$:

First Asymptote at $nx + c = \frac{\pi}{2}$

First x-int at $nx + c = 0$

Range is R

Period is $\frac{\pi}{n}$

The function must approach the asymptotes as they tend towards them, the general shape must be curvy and you must have the correct end points.

Sketch $y = \tan(x - \pi)$ for $x \in [0, 2\pi]$

1. Asymptotes

$$x - \pi = \frac{\pi}{2}$$

$$x = \frac{3\pi}{2}$$

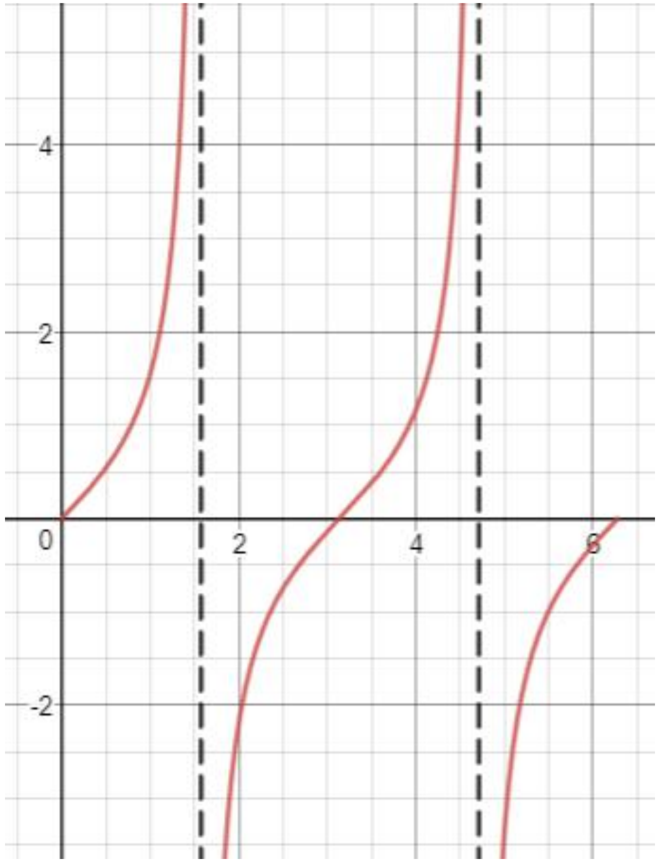
Period = $\frac{\pi}{1} = \pi$

$$x = \frac{3\pi}{2} + \pi = \frac{5\pi}{2}$$

$$x = \frac{3\pi}{2} - \pi = \frac{\pi}{2}$$

Sketch $y = \tan(x - \pi)$ for $x \in [0, 2\pi]$

1. Asymptotes: $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$
2. Draw in the curves



- **Sketching**
  - Shape
  - Scale
  - Asymptotic behaviour
  - Labelling equations of asymptotes and coordinates
- **Solving**
  - Consider underlying domains
  - Do not put equal signs between quantities that are not equal
- **Transformations**
  - Carefully consider which is the original function and which is the image
  - The numbers used in function notation are inconsistent with matrix notation and mapping

Calculus

- Differentiation and Application
  - Understanding the implications of a derivative
  - Derivation and basic derivatives in function and Leibnitz notation
  - Chain, product and quotient rules
  - Tangents and normals
  - Rates of change and stationary points
- Integration and Applications
  - Understanding the implications of an integral
  - Basic antiderivatives and the fundamental theorem of calculus
  - Area bounded by curves and average value

- Differentiation allows us to find a machine that produces the rate of change of a function
- First principles:

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- It allows us to determine the marginal change in y given a point in x
- This is called the instantaneous rate of change

Function Notation	Leibnitz notation
$f(x)$	y
$f'(x)$	
$f'(a)$	

Calculus

Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$(ax+b)^n$	$an(ax+b)^{n-1}$
e^{ax}	ae^{ax}
$e^{g(x)}$	$g'(x)e^{g(x)}$
$\log_e(x)$	$\frac{1}{x}$
$\log_e(g(x))$	$\frac{g'(x)}{g(x)}$
$\sin(ax)$	$a\sin(ax)$
$\cos(ax)$	$-a\cos(ax)$
$\tan(ax)$	$a\sec^2(ax)$

$$\text{Let } y = e^{4x^2} \text{ find } \frac{dy}{dx}.$$

$$\frac{dy}{dx} = \frac{d}{dx}(4x^2) \times e^{4x^2}$$

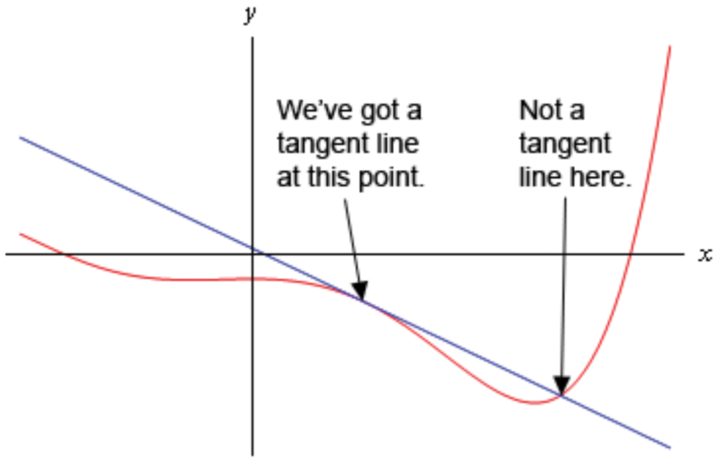
$$= 8xe^{4x^2}$$

These types of questions, in conjunction with our rules have appeared on exam one every single year for the past decade, it is safe to expect it to do so again.

Function Notation	Leibnitz notation
$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

A tangent to a function is a line that shares a common point and gradient at said point

If $y = f(x)$



Then:

$$y_{tangent|x=a} = f'(a)(x - a) + f(a)$$

Note: You can also find the tangent by manually solving for the gradient and then subbing in $(a, f(a))$.

A normal to a function is a line that shares a point and is perpendicular to the tangent at said point

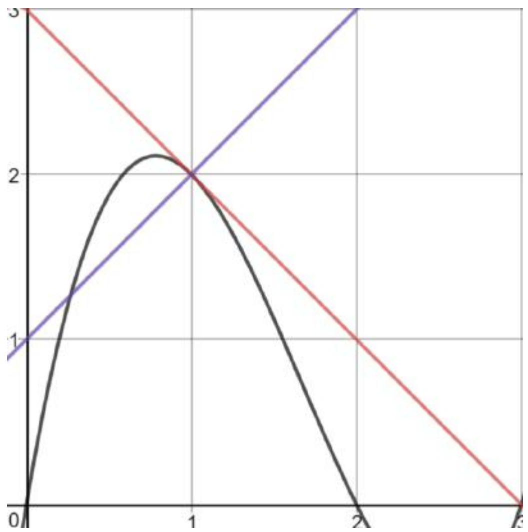
$$\text{If } y = f(x)$$

Then:

$$y_{normal|x=a} = \frac{-1}{f'(a)}(x - a) + f(a)$$

Recently, extended response questions involving tangents and normal have been prominent, they are often used to draw images. In exam ones they are usually 2 mark standalones.

Find the tangent and normal to $f(x) = x(x-2)(x-3)$ at $x = 1$.



$$y_{tangent|x=1} = 3 - x$$

$$y_{normal|x=1} = x + 1$$

It is just as important to know the formula, as it is to know the geometric representation.

Consider $y = f(x)$

Instantaneous rate of change at $x = a$

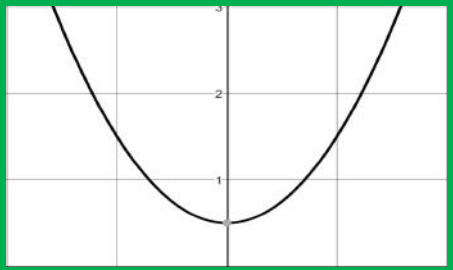
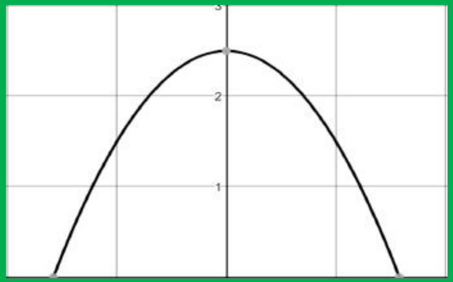
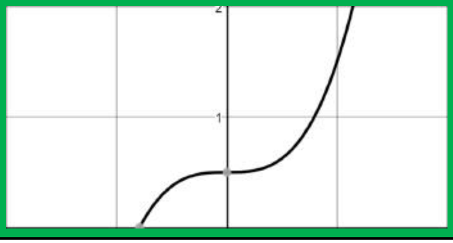
$$f'(a)$$

Average rate of change between $x = a$ and $x = b$

$$\frac{f(b) - f(a)}{b - a}$$

These tend to appear together in questions, it is important to distinguish between the two.

Stationary points occur at $x = a$ if and only if $f'(a) = 0$

Nature	Graph
Local Minima	
Local Maxima	
Stationary Points of Inflection	

The absolute maximum occurs at $x = a$ when $f(a) > f(b)$ for all acceptable values of b

The absolute minimum occurs at $x = a$ when $f(a) < f(b)$ for all acceptable values of b

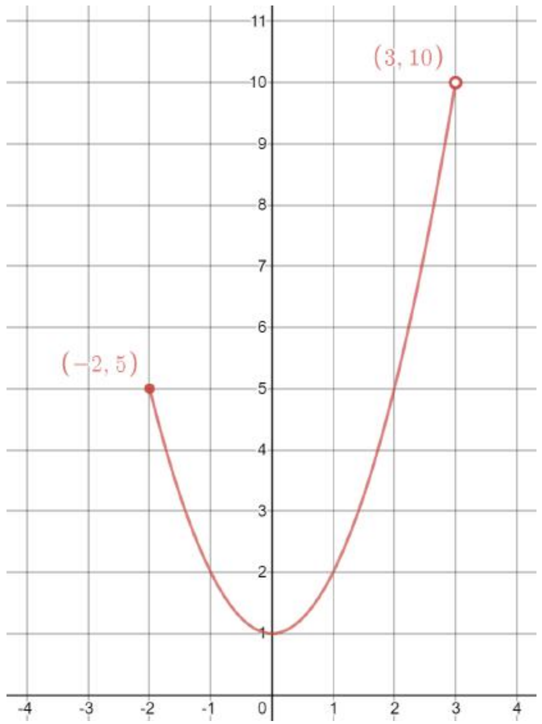
The absolute maximum/minimum will **ALWAYS** occur at a stationary point or at an endpoint.

These two are usually mixed together in questions that ask for the range of a function. Not always will the maximum or minimum of a function occur at a stationary point

Absolute Max and Min

Let $f: [-2,3) \to R, f(x) = x^2 + 1$

Find the range of f



$$f'(x) = 2x$$

$$f'(x) = 0 \text{ when } x = 0$$

$$f(0) = 1$$

$$f(-2) = 5 \text{ and } f(3) = 10$$

Therefore range is $f \in [1, 10)$

- Antidifferentiation is the opposite of differentiation, it allows us to determine areas under the graph
- That is:

$$\int \frac{d}{dx}(f(x))dx = f(x) + c$$

- The dx at the end indicates the variable in question
- It is *integral* to include both the symbol and the dx
- Remember the "plus c"

What is the derivative of $2x$?

$$\frac{d(2x)}{dx} = 2$$

What is the derivative of $2x + 1$?

$$\frac{d}{dx}(2x + 1) = 2$$

So what is the antiderivative of 2?

Antidifferentiation

$f(x)$	$\int f(x)dx$
x^n	$\frac{1}{n+1}x^{n+1} + C, n \neq -1$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1} + C, n \neq -1$
e^{ax}	$\frac{1}{a}e^{ax} + C$
$\frac{1}{x}$	$\log_e(x) + C, x > 0$
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + C$
$\cos(ax)$	$\frac{1}{a}\sin(ax) + C$

For a function $f(x)$ with antiderivative $F(x)$, then:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

The rule is important as it is applied in every integration scenario but is specifically important to Maths Methods as it has been appearing more consistently over the years.

Important to note that even when the function is unknown all rules of integration still apply.

$$\text{Let } \int_0^5 f(x)dx = 5$$

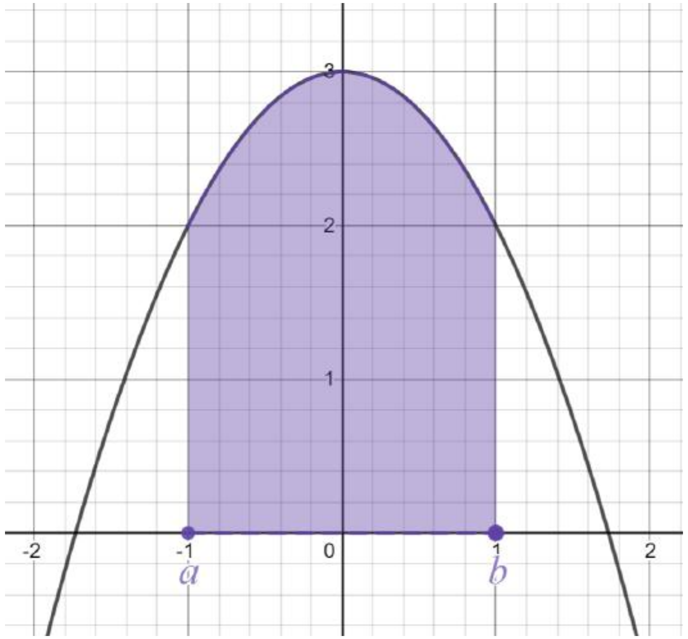
$$\text{Find } \int_0^1 f(5x)dx$$

$$\int_0^5 f(x)dx = F(5) - F(0) = 5$$

$$\int_0^1 f(5x)dx = \frac{1}{5}\big(F(5) - F(0)\big) = \frac{1}{5}(5) = 1$$

As a checking mechanism or last resort, substituting $f(x) = ax$ and solving normally will yield the same answer

For any function, the integral provides the signed area enclosed by the function and the x axis.



$$\text{Area}_{\text{purple}} = \int_a^b f(x)dx = F(b) - F(a)$$

Definite Integrals

$$\int_a^a f(x)dx = 0$$

$$\int_a^b af(x)dx = a\int_a^b f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Manipulations using integration rules pop up on every exam 2 in the MC section

$$\text{Area} = \int_a^b f(x) - g(x)dx$$

Where $f(a) > g(a)$ for all a.

Remember that the x axis is the curve $y = 0$

Sometimes you will need to split the curves into sections

Simple areas may appear in exam one, complex functions appear in the extended response section and general areas, those without numbers, generally appear in multiple choice.

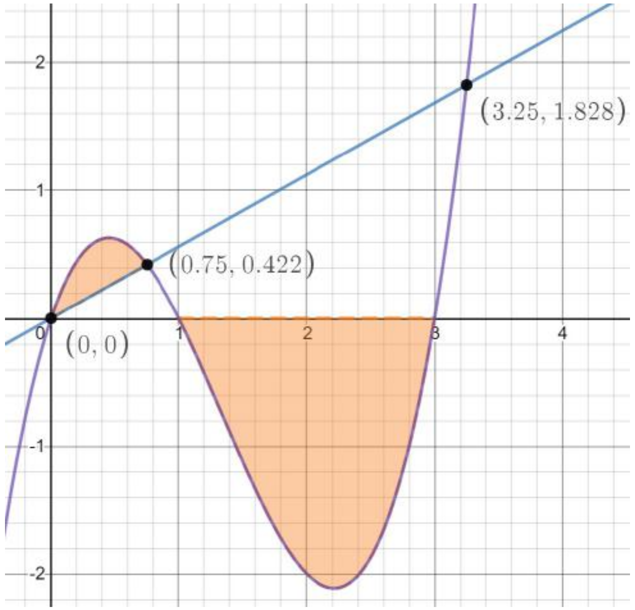
Let $f(x) = x(x-1)(x-3)$ and $g(x) = \frac{9x}{16}$, find the shaded area.

$$f(x) = g(x) \text{ for } x = 0, \frac{3}{4}, \frac{13}{4}$$

$$\text{Area}_1 = \int_0^{\frac{3}{4}} f(x) - g(x)dx$$

$$\text{Area}_2 = \int_1^3 0 - f(x)dx$$

$$\text{Area} = \frac{207}{1024} + \frac{8}{3} \approx 2.87$$



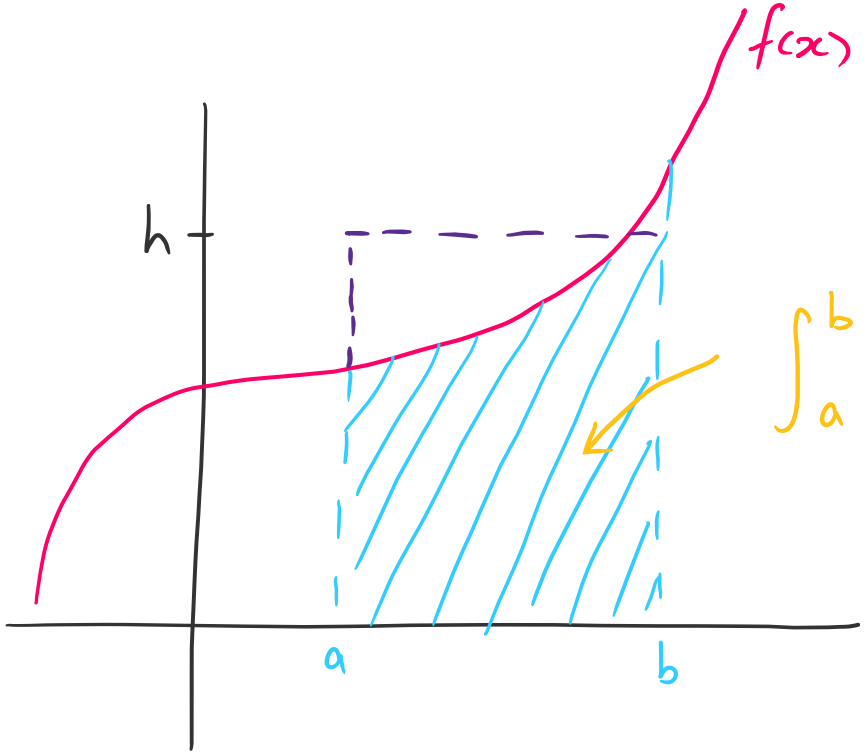
In general the function will be the upper function subtract the lower one, your answer will always be positive.

The average value is the height of a rectangle with the same length and area as the integral.

$$\text{AV} = \frac{\text{Area}}{\text{Length}} = \frac{1}{b-a}\int_a^b f(x)dx$$

It is easier to remember where the average value formula comes from, rather than what it is. It is important that average value is not confused with average rate of change.

Average Value



$$\int_a^b f(x)\,dx = h(b-a)$$

this is the average value

$$\text{so } h = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Key Take Aways

- Your notation must be consistent with the question and the remainder of the working out
- The dx determines the variable in question, this changes when your question changes
- Omitting dx from your integrand or $+C$ will be penalised
- Understand the geometric meaning behind each process

Overview

- ## The Binomial Distribution
  - Understanding the properties of Bernoulli sequences and the relationship with the binomial distribution
  - Calculating probabilities, measure of centre and measures of spread
  - Utilising technology
- ## The Normal Distribution
  - Understanding the underling properties of the normal distribution

- Probability is, in essence, the chance that an event occurs

$$\text{Chance of Success} = \frac{\#\text{Success States}}{\#\text{Total}}$$

$$\Pr(X = x) = \frac{n_x}{n_t}$$

- In technical terms, the probability is measured by the ratio of favourable outcomes to the whole number of cases possible
- These are called marginal or simple probabilities

Probability

$0 \leq \Pr(X = x) \leq 1$	$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$
$\sum \Pr(X = x) = 1$	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Probability Distributions

- A probability distribution is a “list” of all possible outcomes and their respective probabilities
- We denote them with capital letters such as X
- Each possibility is denoted with a lower case letter such as x
- And they must meet:

$$0 \leq \Pr(X = x) \leq 1$$

and

$$\sum \Pr(X = x) = 1$$

Intersection

- An intersection between two variables are the elements that are common between them

$$A \cap B$$

- The probability of an intersection between two variables is the chance that an element chosen is common between them

$$\Pr(A \cap B)$$

Union

- An union between two variables are all the elements that exist between them, that is, either or in both

$$A \cup B$$

- The probability of a union between two variables is the chance that an element chosen is within either set

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Conditional Probability

- A condition is a set of criteria that is assumed to have been satisfied, it is used within other fields of mathematics as well

$$\frac{dy}{dx}|x = a$$

- In probability, it is the chance that a certain event occurs assuming that the condition has been met

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Independence

- When two events are independent, the events of one variable do not affect the events of the other, that is

$$\Pr(A|B) = \Pr(A)$$

- From here we can deduce that

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

- This is key to working with multi stage problems, such as the use of tree diagrams, as it allows us to multiply across branches as we understand that each stage is independent of the others

Mutual Exclusivity

- When two variables are mutual exclusive they have no common elements

$$\Pr(A \cap B) = 0$$

- Thus, from our addition rule

$$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$

- Students tend to confuse mutually exclusive events with independent events. It is *somewhat* important that you don't do this

Binomial Distribution

A distribution is said to be binomial if it meets the following criteria:

- Has n independent trials
- Each trial has only two outcomes, success or failure
- each trial has probability of success p

- It is important to define your distribution before you start using the notation in your working out

$$X \sim Bi(n, p)$$

- Which ever letter you use to denote your variable you must use in further working

- You cannot us a letter more than once within the question for different variables

Number of branches multiplied by the **probability of one branch.**

$$\Pr(X = x) = {}^{n}C_{x} \times p^{x} \times (1 - p)^{n-x}$$

Where, x is number of success, p is the probability of success and n is the number of trials.

$${}^{n}C_{x} = \binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

Every time Shania does her homework, there is a 25% chance that she will require some help. She does homework every weeknight for 1 hr at a time.

What is the probability that she doesn't need help a single time from Monday to Sunday. (POLL)

$$X \sim Bi\left(5, \frac{1}{4}\right)$$

$$\Pr(X = 0) = \binom{5}{0} \times \left(\frac{1}{4}\right)^{0} \times \left(\frac{3}{4}\right)^{5}$$

Prob. of success

Prob. of failure

$$X \sim Bi\left(3, \frac{1}{4}\right)$$

$$\Pr(X = 3 | X \geq 1) = \frac{\Pr(X = 3)}{\Pr(X \geq 1)}$$

$$\Pr(X = 3) = \binom{3}{3} \times \left(\frac{1}{4}\right)^3 \times \left(\frac{3}{4}\right)^0$$

$$\Pr(X = 3) = \left(\frac{1}{4}\right)^3$$

For **Binomial Distributions** we have two main calculator functions;

- **Binomial PDF** – Particular x value
- **Binomial CDF** – Cumulative x value

$$X \sim Bi(4, 0.703)$$

Let's say we want to find:

$$\Pr(X = 2)$$

On the CAS – **Binomial PDF**:

Num Trials, n: 4

Prob Success, p: 0.703

X Value: 2

$$\Pr(X = 2) = 0.262$$

$$X \sim Bi(4, 0.703)$$

Let’s say we want to find:

$$\Pr(X > 2)$$

On the CAS – **Binomial CDF**:

Num Trials, n: 4

Prob Success, p: 0.703

Lower Bound: 3

Upper Bound: 4

$$\Pr(X > 2) = 0.657$$

Each year, a detailed study is conducted on a random sample of 36 Lorenz birdwing butterflies in Town A. A Lorenz birdwing butterfly is considered to be **very large** if its wingspan is greater than 17.5 cm. The probability that the wingspan of any Lorenz birdwing butterfly in Town A is greater than 17.5 cm is 0.0527, correct to four decimal places.

f. **i.** Find the probability that three or more of the butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large**, correct to four decimal places. 1 mark

$X \sim Bi(36, 0.0527)$ $n = 36$ $p = 0.0527$

$Pr(X \geq 3) = 0.2947$ lb $= 3$ ub $= 36$

ii. The probability that n or more butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large** is less than 1%.

Find the smallest value of n, where n is an integer. 2 marks

$Pr(X \geq n) < 0.01$ lb $= n$, ub $= 36$

Trial + Error

$binomCdf(36, 0.0527, n, 36) \rightarrow f(n)$

$f(7) = 0.002$ $n = 7$

Expected Value (Mean)

- The expected value, or mean, is the expected amount of successes you expect to get when conducting the trials

- It is the multiple of the probability of success and number of trials

$$E(X) = np$$

Binomial questions are the hardest as they tend to be the ones that are most disguised. Often extended response questions in exam 2 but it is important you understand the all formulae for exam 1 anomalies.

- The variance is the measure of spread from the mean, or the expected value

$$\sigma^2 = Var(X) = E\left[\left(X - E(X)\right)^2\right]$$
$$= np(1-p)$$

- Standard deviation, as always, is the square root of the variance

$$\sigma = \sqrt{Var(X)} = \sqrt{np(1-p)}$$

Normal Distribution

- The normal distribution is a specific type of continuous function that is defined over all real values and has the following properties

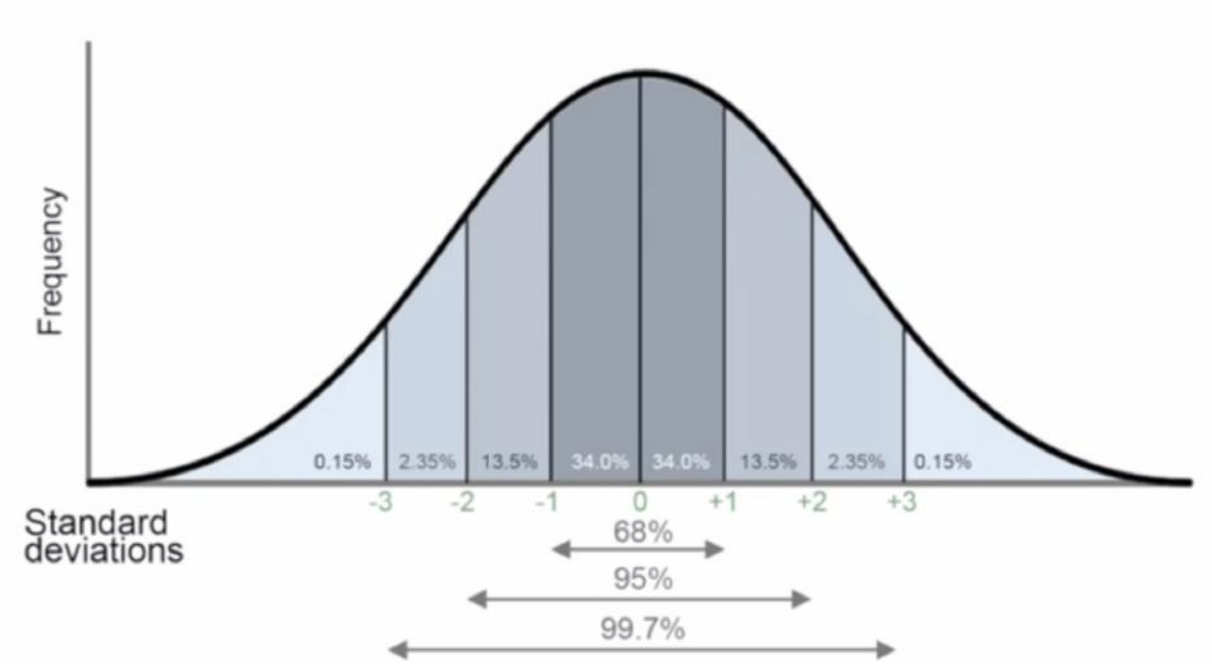
$$\mu = median = mode$$

- We write them as such

$$X \sim N(\mu, \sigma^2)$$

Remember write the variance NOT the standard deviation.

- We use the standard deviation in our calculations, why? Because σ and μ are in the same units.
- Let's take a look at some approximations



- The bell curve you just saw was one of many, it was the ***standard normal distribution*** and has special properties

$$Z \sim N(0, 1)$$

- We can "**standardise**" any normal distribution:

$$z = \frac{X - \mu}{\sigma}$$

- The standardised value tells us the **number of standard deviations** the x value lies from the mean

Exam Tips

- Use reading time ***effectively***

- Identify the really easy questions

- Carefully read the long-worded questions to understand what you are being asked (especially Prob)

Exam Tips

- Begin with questions you immediately know how to do

- Go back and do those that you might know how to do

- Then start on the ones you're not sure of

- DO NOT THINK YOU WILL USE YOUR NOTES

THANK YOU AND GOOD LUCK!