

# Chapter 1 – Functions and relations

## Solutions to Exercise 1A

1 a  $\{8, 11\}$

b  $\{8, 11\}$

c  $\{1, 3, 8, 11, 18, 22, 23, 24, 25, 30\}$

d  $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$

e  $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$

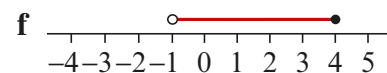
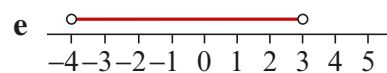
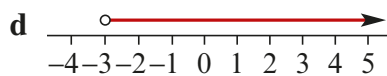
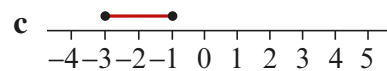
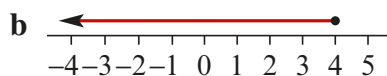
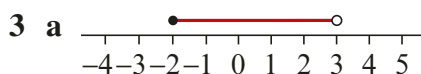
f  $\{1, 8, 11, 25, 30\}$

2 a  $\{3, 18, 22, 23, 24\}$

b  $\{25, 30, 32\}$

c  $\{3, 18, 22, 23, 24\}$

d  $\{1, 25, 30\}$



4 a  $X \cap Y = \{7, 9\}$

b  $X \cap Y \cap Z = \{7, 9\}$

c  $X \cup Y = \{2, 3, 5, 7, 9, 11, 15, 19, 23\}$

d  $X \setminus Y = \{2, 3, 5, 11\}$

e  $Z \setminus Y = \{2\}$

f  $X \cap Z = \{2, 7, 9\}$

g  $[-2, 8] \cap X = \{2, 3, 5, 7\}$

h  $(-3, 8] \cap Y = \{7\}$

i  $(2, \infty) \cap Y = \{7, 9, 15, 19, 23\}$

j  $(3, \infty) \cup Y = (3, \infty)$

5 a  $X \cap Y = \{a, e\}$

b  $X \cup Y = \{a, b, c, d, e, i, o, u\}$

c  $X \setminus Y = \{b, c, d\}$

d  $Y \setminus X = \{i, o, u\}$

6 a  $B \cap C = \{6\}$

b  $B \setminus C = \{2, 4, 8, 10\}$

c  $A \setminus B = \{1, 3, 5, 7, 9\}$

d  $A \setminus B = \{1, 3, 5, 7, 9\}$   
 $A \setminus C = \{2, 4, 5, 7, 8, 10\}$   
 $(A \setminus B) \cup (A \setminus C) = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$

e  $B \cap C = \{6\}$   
 $A \setminus (B \cap C) = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$

f  $A \setminus B = \{1, 3, 5, 7, 9\}$   
 $A \setminus C = \{2, 4, 5, 7, 8, 10\}$

$$(A \setminus B) \cap (A \setminus C) = \{5, 7\}$$

**g**  $B \cup C = \{1, 2, 3, 4, 6, 8, 9, 10\}$

$$A \setminus (B \cup C) = \{5, 7\}$$

**h**  $A \cap B \cap C = \{6\}$

**7 a**  $[-3, 1)$

**b**  $(-4, 5]$

**c**  $(-\sqrt{2}, 0)$

**d**  $(-\frac{1}{\sqrt{2}}, \sqrt{3})$

**e**  $(-\infty, -3)$

**f**  $(0, \infty)$

**g**  $(-\infty, 0)$

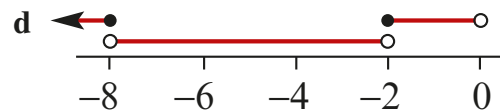
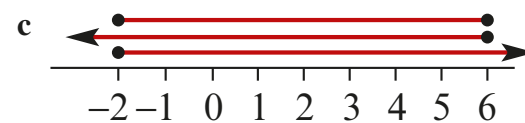
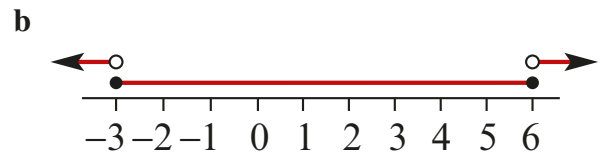
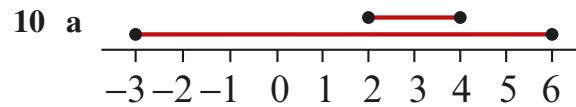
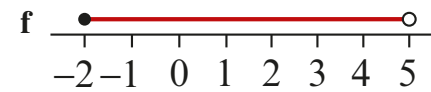
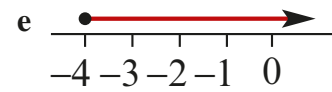
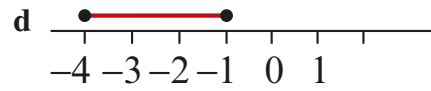
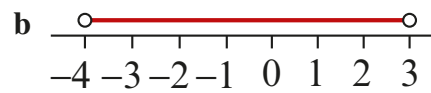
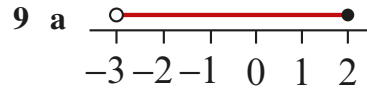
**h**  $[-2, \infty)$

**8 a**  $(-2, 3)$

**b**  $[-4, 1)$

**c**  $[-1, 5]$

**d**  $(-3, 2]$



## Solutions to Exercise 1B

**1 a** Domain =  $\mathbb{R}$   
range =  $[-2, \infty)$

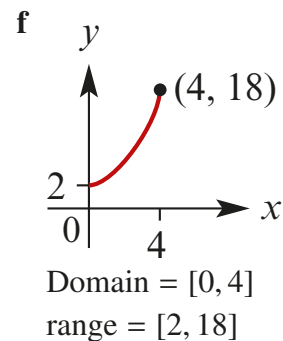
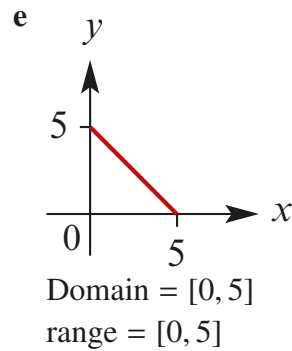
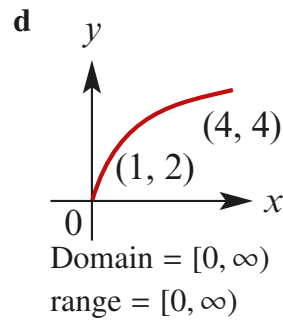
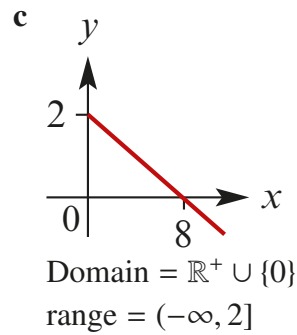
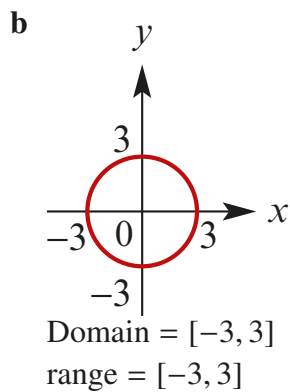
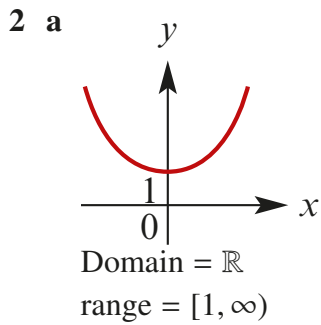
**b** Domain =  $(-\infty, 2]$   
range =  $\mathbb{R}$

**c** Domain =  $(-2, 3)$   
range =  $[0, 9)$

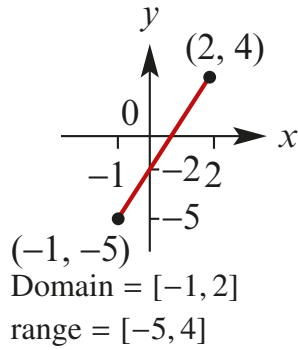
**d** Domain =  $(-3, 1)$   
range =  $(-6, 2)$

**e** Domain =  $[-4, 0]$   
range =  $[0, 4]$

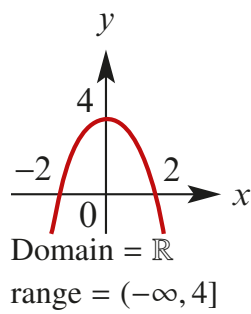
**f** Domain =  $\mathbb{R}$   
range =  $(-\infty, 2)$



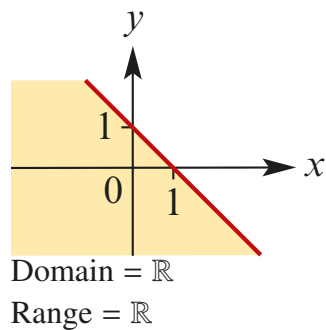
**g**



**h**



**i**



**3 a** not a function

Domain =  $\{-1, 1, 2, 3\}$   
range =  $\{1, 2, 3, 4\}$

**b** function

Domain =  $\{-2, -1, 0, 1, 2\}$   
range =  $\{-4, -1, 0, 3, 5\}$

**c** not a function

Domain =  $\{-2, -1, 2, 4\}$   
range =  $\{-2, 1, 2, 4, 6\}$

**d** function

Domain =  $\{-1, 0, 1, 2, 3\}$

range =  $\{4\}$

**4 a** function Domain =  $\mathbb{R}$

range =  $\{4\}$

**b** not a function

Domain =  $\{2\}$   
range =  $\mathbb{Z}$

**c** function

Domain =  $\mathbb{R}$   
range =  $\mathbb{R}$

**d** not a function

Domain =  $\mathbb{R}$   
range =  $\mathbb{R}$

**e** not a function

Domain =  $[-4, 4]$   
range =  $[-4, 4]$

**5**  $f(x) = 2x^2 + 4x;$

$g(x) = 2x^3 + 2x - 6$

**a**  $f(-1) = 2(-1)^2 + 4(-1) = -2$

$f(2) = 2(2)^2 + 4(2) = 16$

$f(-3) = 2(-3)^2 + 4(-3) = 6$

$f(2a) = 2(2a)^2 + 4(2a) = 8a^2 + 8a$

**b**

$g(-1) = 2(-1)^3 + 2(-1) - 6 = -10$

$g(2) = 2(2)^3 + 2(2) - 6 = 14$

$g(3) = 2(3)^3 + 2(3) - 6 = 54$

$g(a - 1) = 2(a - 1)^3 + 2(a - 1) - 6$

$= 2(a^3 - 3a^2 + 3a - 1) + 2a - 8$

$= 2a^3 - 6a^2 + 8a - 10$

**6**  $g(x) = 3x^2 - 2$

**a**  $g(-2) = 3(-2)^2 - 2 = 10$

$g(4) = 3(4)^2 - 2 = 46$

**b i**  $g(-2) = 3(-2)^2 - 2 = 12x^2 - 2$

**ii**  $g(x-2)^2 = 3(x-2)^2 - 2 = 3x^2 - 12x + 10$

**iii**  $g(x+2)^2 = 3(x+2)^2 - 2 = 3x^2 + 12x + 10$

**iv**  $g(x^2) = 3(x^2)^2 - 2 = 3x^4 - 2$

**7**  $f(x) = 2x - 3$

**a**  $f(3) = 2(3) - 3 = 3$

**b**  $f(x) = 11$

$11 = 2x - 3x = 7$

**c**  $f(x) = 4x$

$4x = 2x - 3$

$2x = -3$

$x = \frac{-3}{2}$

**d**  $f(x) > x$

$2x - 3 > x$

$x > 3$

**8**  $g(x) = 6x + 7$   $h(x) = 3x - 2$

**a**  $6x + 7 = 3x - 2$

$3x = -9$

$x = -3$

**b**  $6x + 7 > 3x - 2$

$3x > -9$

$x > -3$

**c**  $3x - 2 = 0$

$x = \frac{2}{3}$

**9 a**  $f: \mathbb{R} \rightarrow \mathbb{R}$  where  $f(x) = 2x + 3$

**b**  $3y + 4x = 12$

$3y = 12 - 4x$

$y = 4 - \frac{4x}{3}$

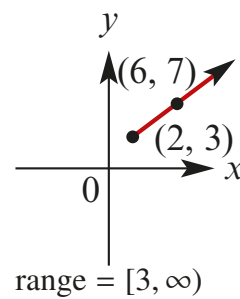
$f: \mathbb{R} \rightarrow \mathbb{R}$  where  $f(x) = \frac{-4x}{3} + 4$

**c**  $f: [0, \infty) \rightarrow \mathbb{R}$  where  $f(x) = 2x - 3$

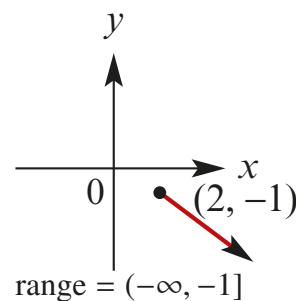
**d**  $f: \mathbb{R} \rightarrow \mathbb{R}$  where  $f(x) = x^2 - 9$

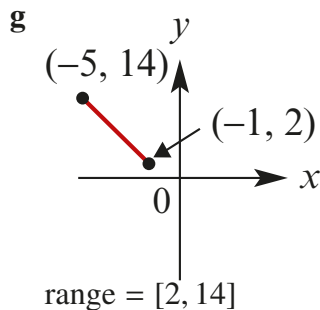
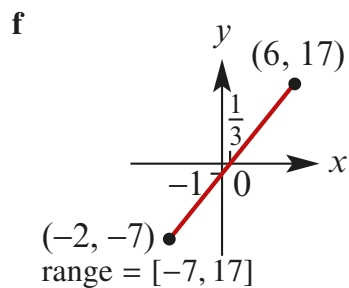
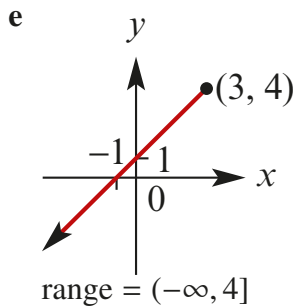
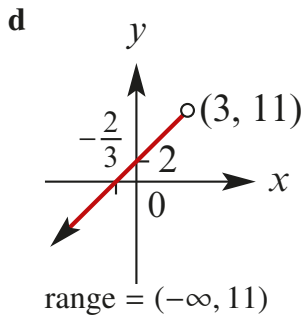
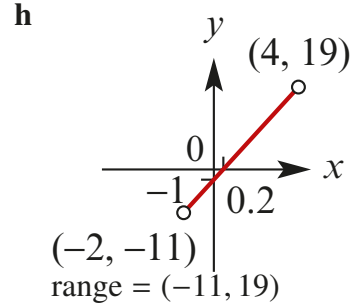
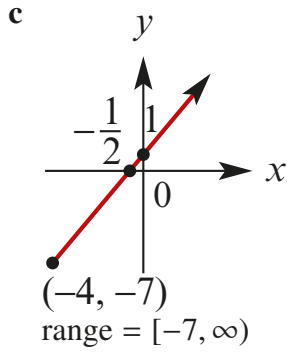
**e**  $f: [0, 2] \rightarrow \mathbb{R}$  where  $f(x) = 5x - 3$

**10 a**



**b**





**11**  $f(x) = 2x^2 - 6x + 1$ ;  $g(x) = 3 - 2x$

**a**  $f(2) = 2(2)^2 - 6(2) + 1 = -3$   
 $f(-3) = 2(-3)^2 - 6(-3) + 1 = 37$   
 $f(-2) = 2(-2)^2 - 6(-2) + 1 = 21$

**b**  $g(-2) = 3 - 2(-2) = 7$   
 $g(1) = 3 - 2(1) = 1$   
 $g(-3) = 3 - 2(-3) = 9$

**c i**  $f(a) = 2a^2 - 6a + 1$   
**ii**  $f(a + 2) = 2(a + 2)^2 - 6(a + 2) + 1$   
 $= 2a^2 + 2a - 3$

**iii**  $g(-a) = 3 + 2a$

**iv**  $g(2a) = 3 - 4a$

**v**  $f(5 - a) = 2(5 - a)^2 - 6(5 - a) + 1$   
 $= 2a^2 - 14a + 21$

**vi**  $f(2a) = 8a^2 - 12a + 1$

**vii**  $g(a) + f(a) = (2a^2 - 6a + 1) + (3 - 2a)$   
 $= 2a^2 - 8a + 4$

viii  $g(a) - f(a) = (3 - 2a)$   
 $- (2a^2 - 6a + 1)$   
 $= -2a^2 + 4a + 2$

12  $f(x) = 3x^2 + x - 2$

a  $f(x) = 0$   
 $3x^2 + x - 2 = 0$   
 using the quadratic formula  
 $x = \frac{-1 \pm \sqrt{(1)^2 - 4(3)(-2)}}{2(3)}$   
 $x = \frac{-1 \pm \sqrt{25}}{6}$   
 $x = -1, \frac{2}{3}$   
 in set notation  
 $\left\{-1, \frac{2}{3}\right\}$

b  $f(x) = x$   
 $3x^2 + x - 2 = x$   
 $3x^2 = 2$   
 $x^2 = \frac{2}{3}$   
 $x = \pm \sqrt{2/3}$   
 in set notation  
 $\left\{-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right\}$

c  $f(x) = -2$   
 $3x^2 + x - 2 = -2$   
 $3x^2 + x = 0$   
 $x(3x + 1) = 0$   
 $\therefore$  either  $x = 0$  or  $3x + 1 = 0$   
 $x = 0, \frac{-1}{3}$   
 in set notation  
 $\left\{0, \frac{-1}{3}\right\}$

d  $f(x) > 0$   
 $3x^2 + x - 2 > x$   
 from (a), the  $x$ -intercepts are  $-1, \frac{2}{3}$   
 as the coefficient of  $x^2 > 0$   
 the shape of the graph  $y = f(x)$  is



$\therefore f(x) > 0$  for  
 $x \in (-\infty, -1) \cup \left(\frac{2}{3}, \infty\right)$

e  $f(x) > x$   
 $3x^2 + x - 2 > x$   
 $3x^2 - 2 > 0$   
 $x^2 > \frac{2}{3}$   
 $x \in \left(-\infty, -\frac{\sqrt{2}}{\sqrt{3}}\right) \cup \left(\frac{\sqrt{2}}{\sqrt{3}}, \infty\right)$

f  $f(x) \leq -2$   
 $3x^2 + x - 2 \leq -2$   
 from (c), the  $x$ -intercepts are  $-\frac{1}{3}, 0$   
 as the coefficient of  $x^2 > 0$   
 the shape of the graph  $y = f(x)$  is



$\therefore f(x) \leq -2$  for  $x \in \left[\frac{-1}{3}, 0\right]$

13  $f(x) = x^2 + x$

a  $f(-2) = (-2)^2 + (-2) = 2$

b  $f(2) = (2)^2 + (2) = 6$

c  $f(-a) = (-a)^2 + (-a) = a^2 - a$

d  $f(a) + f(-a) = (a^2 + a) + (a^2 - a)$   
 $= 2a^2$

$$\begin{aligned} \mathbf{e} \quad f(a) - f(-a) &= (a^2 + a) - (a^2 - a) \\ &= 2a \end{aligned}$$

$$\mathbf{f} \quad f(a^2) = (a^2)^2 + (a^2) = a^4 + a^2$$

$$\mathbf{14} \quad g(x) = 3x - 2$$

$$\mathbf{a} \quad g(x) = 4$$

$$3x - 2 = 4$$

$$x = 2$$

$$\mathbf{b} \quad g(x) > 4$$

$$3x - 2 > 4$$

$$x > 2$$

in set notation

$$\{x : x > 2\}$$

$$\mathbf{c} \quad g(x) = a$$

$$3x - 2 = a$$

$$x = \frac{a + 2}{3}$$

$$\mathbf{d} \quad g(-x) = 6$$

$$-3x - 2 = 6$$

$$x = \frac{-8}{3}$$

$$\mathbf{e} \quad g(2x) = 4$$

$$6x - 2 = 4$$

$$x = 1$$

$$\mathbf{f} \quad \frac{1}{g(x)} = 6$$

$$1 = 6g(x)$$

$$1 = 6(3x - 2)$$

$$1 = 18x - 12$$

$$18x = 13$$

$$x = \frac{13}{18}$$

$$\mathbf{15} \quad \mathbf{a} \quad f(x) = kx - 1$$

$$3 = 3k - 1$$

$$k = \frac{4}{3}$$

$$\mathbf{b} \quad f(x) = x^2 - k$$

$$3 = 9 - k$$

$$k = 6$$

$$\mathbf{c} \quad f(x) = x^2 + kx + 1$$

$$3 = 9 + 3k + 1$$

$$k = \frac{-7}{3}$$

$$\mathbf{d} \quad f(x) = \frac{k}{x}$$

$$3 = \frac{k}{3}$$

$$k = 9$$

$$\mathbf{e} \quad f(x) = kx^2$$

$$3 = 9k$$

$$k = \frac{1}{3}$$



$$\mathbf{f} \quad f(x) = 1 - kx^2$$

$$3 = 1 - 9k$$

$$9k = -2$$

$$k = \frac{-2}{9}$$

$$\mathbf{16 a} \quad 5x - 4 = 2$$

$$x = \frac{6}{5}$$

$$\mathbf{b} \quad \frac{1}{x} = 5$$

$$x = \frac{1}{5}$$

$$\mathbf{c} \quad \frac{1}{x^2} = 9$$

$$x = \pm \frac{1}{3}$$

$$\mathbf{d} \quad x = \frac{1}{x} = 2$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0$$

$$x = 1$$

$$\mathbf{e} \quad (x + 1)(x - 2) = 2$$

$$\therefore \text{either } x + 1 = 0 \text{ or } x - 2 = 0$$

$$x = -1 \quad x = 2$$

$$\therefore x = -1, 2$$

## Solutions to Exercise 1C

1 a The functions which are one - to - one are **b** and **c**

2 a The functions which are one - to - one are **b,d** and **f**

3 a The graphs of functions are **i, iii, iv, vi, vii,** and **viii.**

b The graphs of one - to - one functions are **iii,** and **vii.**

4  $y^2 = x + 2, x \geq -2$

$$y = \pm \sqrt{x + 2}$$

two possible functions  $f$  and  $g$  are

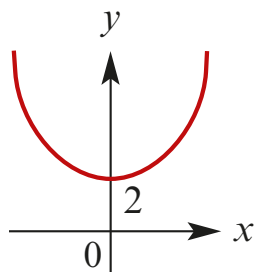
$$f: [-2, \infty) \rightarrow \mathbb{R} \quad f(x) = \sqrt{x + 2}$$

$$\text{range of } f: [0, \infty) = \mathbb{R}^+ \cup \{0\}$$

$$g: [-2, \infty) \rightarrow \mathbb{R} \quad g(x) = -\sqrt{x + 2}$$

$$\text{range of } g: (-\infty, 0] = \mathbb{R}^- \cup \{0\}$$

5 a



b two possible functions are the right half

$$g_1: [0, \infty) \rightarrow \mathbb{R} \quad g_1(x) = x^2 + 2$$

and the left half

$$g_2: (-\infty, 0) \rightarrow \mathbb{R} \quad g_2(x) = x^2 + 2$$

6 a Domain:  $\mathbb{R}$  range:  $\mathbb{R}$

b Domain:  $\mathbb{R}^+ \cup \{0\}$  range:  $\mathbb{R}^+ \cup \{0\}$

c Domain:  $\mathbb{R}$  range:  $[-2, \infty)$

d Domain:  $[-4, 4]$  range:  $[0, 4]$

e Domain:  $\mathbb{R} \setminus \{0\}$  range:  $\mathbb{R} \setminus \{0\}$

f Domain:  $\mathbb{R}$  range:  $(-\infty, 4]$

g Domain  $[3, \infty)$  range:  $[0, \infty)$

7 a Domain:  $\mathbb{R}$  range:  $\mathbb{R}$

b Domain:  $\mathbb{R}$  range:  $[-2, \infty)$

c Domain  $[-3, 3]$  range:  $[0, 3]$

d Domain:  $\mathbb{R} \setminus \{1\}$  range:  $\mathbb{R} \setminus \{0\}$

8 a  $\mathbb{R} \setminus \{3\}$

b  $(-\infty, -\sqrt{3}) \cup [\sqrt{3}, \infty)$

c  $\mathbb{R}$

d  $[4, 11]$

e  $\mathbb{R} \setminus \{-1\}$

f  $h(x) = \sqrt{(x+1)(x-2)}$   
Domain:  $(-\infty, -1] \cup [2, \infty)$

g  $\mathbb{R} \setminus \{-1, 2\}$

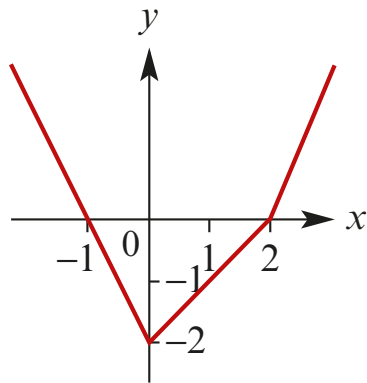
h Domain:  $(-\infty, -2) \cup [1, \infty)$

i  $f(x) = \sqrt{x(1-3x)}$  Domain:  $\left[0, \frac{1}{3}\right]$

j  $[-5, 5]$

k [3, 12]

9 a



b  $[-2, \infty)$

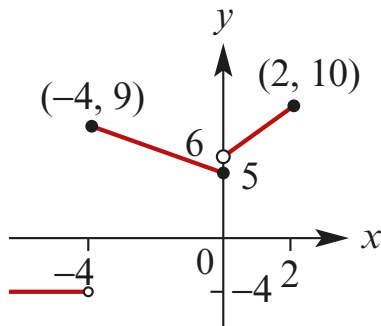
10 Domain:  $(-3, 0] \cup [1, 3)$

range:  $[-2, 3)$

11 Domain:  $[-5, 4]$

range:  $[-4, 0) \cup [2, 5]$

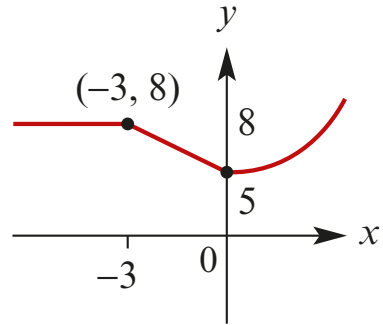
12 a



b Domain =  $(-\infty, 2]$

range =  $[5, 10] \cup \{-4\}$

13 a



b range =  $[5, \infty)$

14  $f(x) = \begin{cases} \frac{1}{x}, & x > 3 \\ 2x, & x \leq 3 \end{cases}$

a  $f(-4) = 2(-4) = -8$

b  $f(0) = 2(0) = 0$

c  $f(4) = \frac{1}{(4)} = \frac{1}{4}$

d  $f(a+3) = \begin{cases} \frac{1}{a+3}, & a > 0 \\ 2a+6, & a \leq 0 \end{cases}$

e  $f(2a) = \begin{cases} \frac{1}{2a}, & a > \frac{3}{2} \\ 4a, & a \leq \frac{3}{2} \end{cases}$

f  $f(a-3) = \begin{cases} \frac{1}{a-3}, & a > 6 \\ 2a-6, & a \leq 6 \end{cases}$

15 a  $f(0) = 4$

b  $f(3) = \sqrt{(3)-1} = \sqrt{2}$

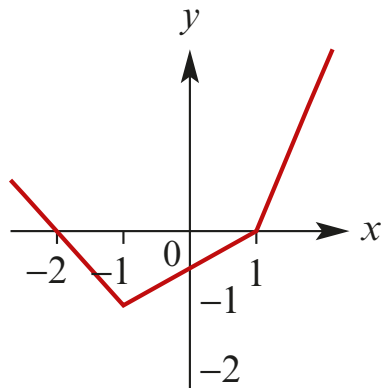
c  $f(8) = \sqrt{(8)-1} = \sqrt{7}$

d  $f(a+1) = \begin{cases} \sqrt{a}, & a \geq 0 \\ 4, & a < 0 \end{cases}$

e  $f(a-1) =$

$$\begin{cases} \sqrt{a-2}, & a-1 > 1 \Rightarrow a \geq 2 \\ 4, & a-0 < 1 \Rightarrow a < 2 \end{cases}$$

16 a



range =  $[-1, \infty)$

17 
$$f(x) = \begin{cases} ax + b, & x < -2 \\ cx + d, & -2 \leq x \leq 3 \\ ex + f, & x > 3 \end{cases}$$
  
using the points given

$$f(x) = \begin{cases} -x - 4, & x < -2 \\ \frac{1}{2}x - 1, & -2 \leq x \leq 3 \\ -\frac{1}{2}x + 2, & x > 3 \end{cases}$$

18 a Even

- b Odd
- c Neither
- d Even
- e Odd
- f Neither

- 19 a Even
- b Even
  - c Odd
  - d Odd
  - e Neither
  - f Even
  - g Neither
  - h Neither
  - i Even

## Solutions to Exercise 1D

**1 a**  $(f + g)(x) = 3x + x + 2$

$$= 4x + 2$$

Domain:  $\mathbb{R}$

$$(fg)(x) = 3x(x + 2)$$

$$= 3x^2 + 6x$$

Domain:  $\mathbb{R}$

**b**  $(f + g)(x) = 1 - x^2 + x^2 = 1$

Domain:  $(0, 2]$

(from  $\text{Domain}(g) \cap \text{Domain}(f)$ )

$$(fg)(x) = (1 - x^2)x^2$$

$$= x^2 - x^4$$

Domain:  $(0, 2]$

(from  $\text{Domain}(g) \cap \text{Domain}(f)$ )

**c**  $(f + g)(x) = \sqrt{x} + \frac{1}{\sqrt{x}} = \frac{x + 1}{\sqrt{x}}$

Domain:  $[1, \infty)$  (from  $g$ )

$$(fg)(x) = \frac{\sqrt{x}}{\sqrt{x}}$$

$$= 1$$

Domain:  $[1, \infty)$  (from  $g$ )

**d**  $(f + g)(x) = x^2 + \sqrt{4 - x}$

Domain:  $[0, 4]$  (from  $g$ )

$$(fg)(x) = x^2 \sqrt{4 - x}$$

Domain:  $[0, 4]$  (from  $g$ )

**2 a** functions  $f$  and  $h$  are even,  $g$  and  $k$  are odd

**b**  $(f + h)(x) = x^2 + 1 + \frac{1}{x^2}, x \in \mathbb{R} \setminus \{0\}$

it is even

$$(fh)(x) = 1 + \frac{1}{x^2}, x \in \mathbb{R} \setminus \{0\}$$

it is even

$$(g + k)(x) = x + \frac{1}{x}, x \in \mathbb{R} \setminus \{0\}$$

it is odd

$$(gk)(x) = 1, x \in \mathbb{R} \setminus \{0\}$$

it is even

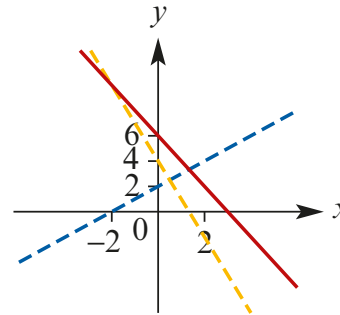
$$(f + g)(x) = x^2 + x + 1, x \in \mathbb{R}$$

it is neither odd nor even

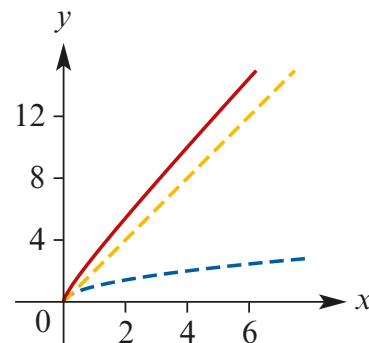
$$(fg)(x) = x^3 + x, x \in \mathbb{R}$$

it is odd

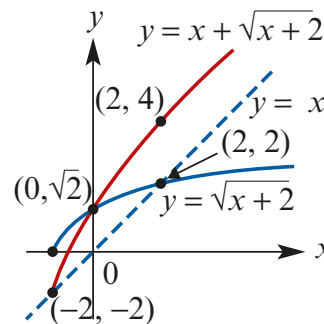
**3**

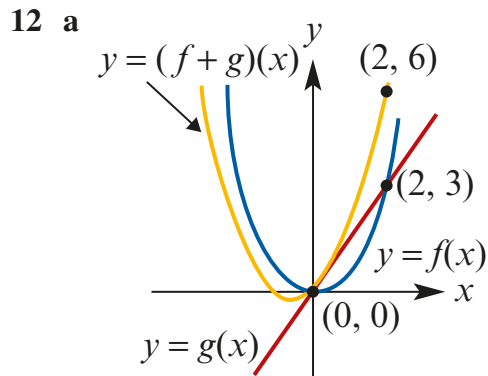
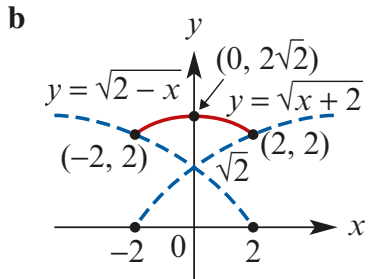
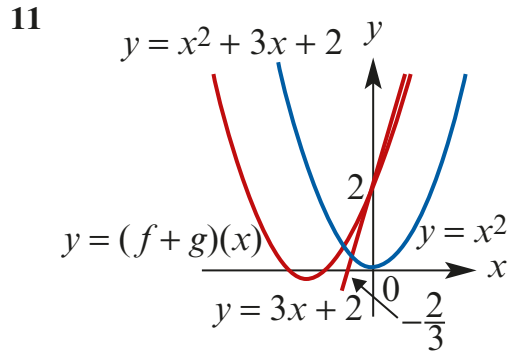
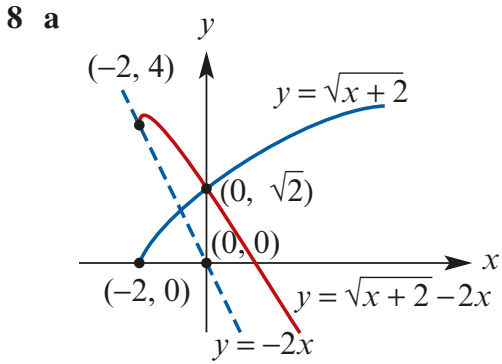
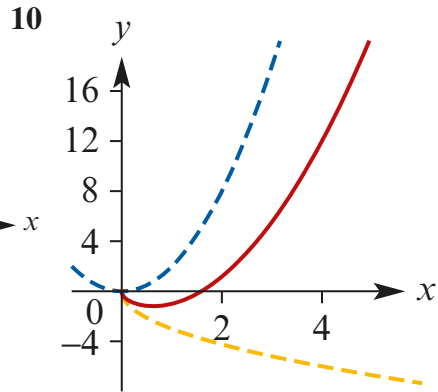
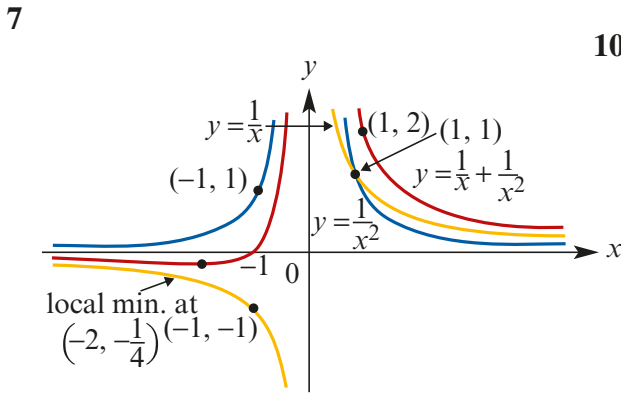
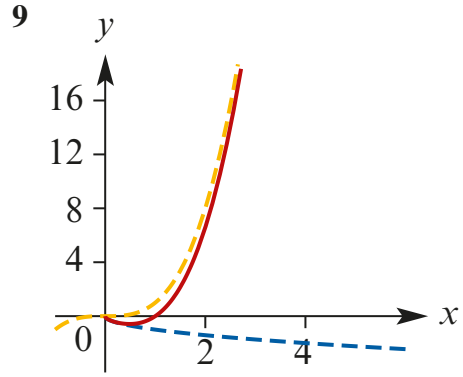
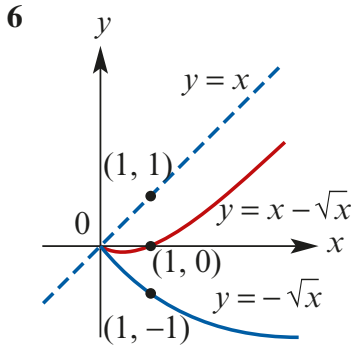


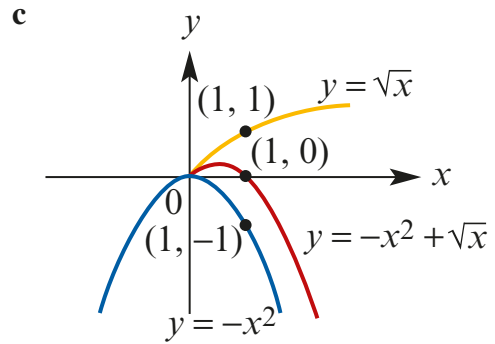
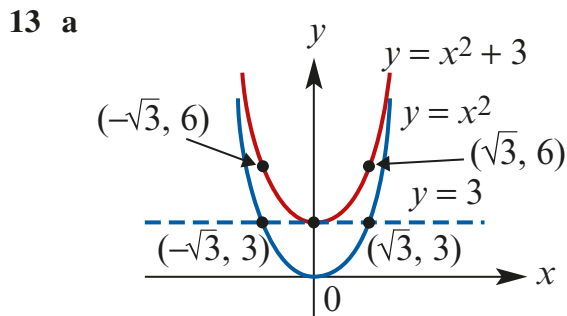
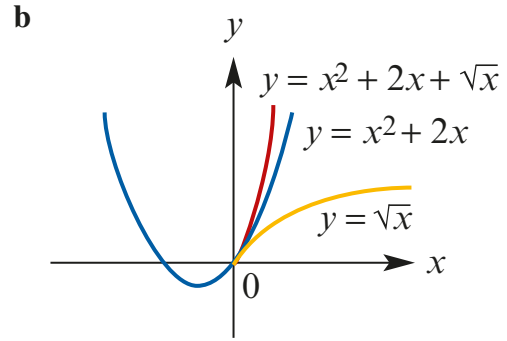
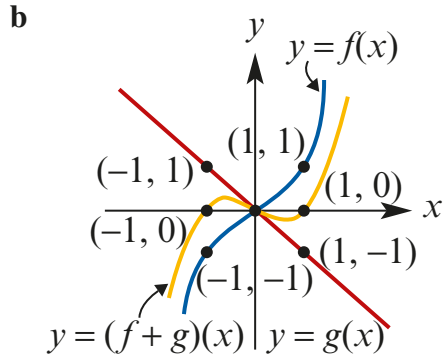
**4**



**5**





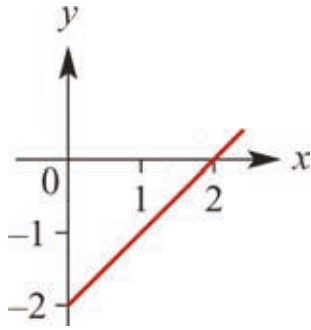


## Solutions to Exercise 1E

- 1 a**  $f(g(x)) = 2(2x) - 1 = 4x - 1$   
 $g(f(x)) = 2(2x - 1) = 4x - 2$
- b**  $f(g(x)) = 4(2x + 1) + 1 = 8x + 5$   
 $g(f(x)) = 2(4x + 1) + 1 = 8x + 3$
- c**  $f(g(x)) = 2(2x - 3) - 1 = 4x - 7$   
 $g(f(x)) = 2(2x - 1) - 3 = 4x - 5$
- d**  $f(g(x)) = 2(x^2) - 1 = 2x^2 - 1$   
 $g(f(x)) = (2x - 1)^2 = 4x^2 - 4x + 1$
- e**  $f(g(x)) = 2(x - 5)^2 + 1$   
 $= 2x^2 - 20x + 51$   
 $g(f(x)) = (2x^2 + 1) - 5$   
 $= 2x^2 - 4$
- f**  $f(g(x)) = 2(x^2) + 1 = 2x^2 + 1$   
 $g(f(x)) = (2x + 1)^2$
- 2 a**  $f \circ h(x) = 2(3x + 2) - 1 = 6x + 3$
- b**  $h(f(x)) = 3(2x - 1) + 2 = 6x - 1$
- c**  $f \circ h(2) = 6(2) + 3 = 15$
- d**  $h \circ f(2) = 6(2) - 1 = 11$
- e**  $f(h(3)) = 6(3) + 3 = 21$
- f**  $h(f(-1)) = 6(-1) - 1 = -7$
- g**  $f \circ h(0) = 6(0) + 3 = 3$
- 3 a**  $f \circ h(x) = (3x + 1)^2 + 2(3x + 1)$   
 $= 9x^2 + 12x + 3$
- b**  $h \circ f(x) = 3(x^2 + 2x) + 1 = 3x^2 + 6x + 1$
- c**  $f \circ h(3) = 9(3)^2 + 12(3) + 3 = 120$
- d**  $h \circ f(3) = 3(3)^2 + 6(3) + 1 = 46$
- e**  $f \circ h(0) = 9(0)^2 + 12(0) + 3 = 3$
- f**  $h \circ f(0) = 3(0)^2 + 6(0) + 1 = 1$
- 4 a**  $h \circ g: \mathbb{R}^+ \rightarrow \mathbb{R}, h \circ g(x) = \frac{1}{(3x + 2)^2}$
- b**  $g \circ h: \mathbb{R} \setminus \{0\}, g \circ h(x) = \frac{3}{x^2} + 2$
- c**  $h \circ g(1) = \frac{1}{(3(1) + 2)^2} = \frac{1}{25}$
- d**  $g \circ h(1) = \frac{3}{(1)^2} + 2 = 5$
- 5 a**  $\text{range}(f) = [-4, \infty)$   
 $\text{range}(g) = \mathbb{R}^+ \cup \{0\}$
- b**  $f \circ g: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}, f \circ g(x) = x - 4$   
 $\text{range}(f \circ g) = [-4, \infty)$
- c**  $g \circ f$  does not exist because the range of  $f$  is not a subset of the Domain of  $g$
- 6 a**  $f(g(x)) = \frac{1}{2}(2x) = x$   
 $f \circ g: \mathbb{R} \setminus \left\{\frac{1}{2}\right\} \rightarrow \mathbb{R}, f \circ g(x) = x$   
 $\text{Range}: \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$
- b**  $g \circ f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, g \circ f(x) = x$   
 $\text{Range}: \mathbb{R} \setminus \{0\}$
- 7 a** the range of  $f$  is  $[-2, \infty)$ , which is not a subset of the Domain of  $g$ ,  $\therefore g \circ f$  does not exist.



**b**  $f \circ g: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}, f \circ g(x) = x - 2$



**8 a** the range of  $g$  is  $[-1, \infty)$ , which is not a subset of the Domain of  $f((-\infty, 3])$ ,  
 $\therefore f \circ g$  does not exist.

**b** the range of  $g^*$  needs to be  $[-1, 3]$  at most.

$$g^*: [-2, 2] \rightarrow \mathbb{R}, g^*(x) = x^2 - 1$$

$$f \circ g^*: [-2, 2] \rightarrow \mathbb{R}, f \circ g^*(x) = 4 - x^2$$

**9 a** The range of  $g$  is  $\mathbb{R}$ , which is not a subset of the Domain of  $f$ ,  
 $f \circ g$  does not exist.

**b** the range of  $g$  needs to be  $\mathbb{R}^+$  at most.

$$\therefore \text{let } g_1: \{x: x < 3\} \rightarrow \mathbb{R},$$

$$g_1(x) = 3 - x$$

$$\text{then } f \circ g_1: \{x: x < 3\} \rightarrow \mathbb{R},$$

$$f \circ g_1(x) = \frac{1}{\sqrt{3-x}}$$

**10 a** the Domain of  $f$  is  $\mathbb{R}$ , the range of  $g$  is

$$\mathbb{R}^+ \cup \{0\}$$

$\therefore f \circ g$  exists.

**b** Range of  $f$  is  $\mathbb{R}^+ \cup \{0\}$

Domain of  $g$  is  $(-\infty, 3]$

The range of  $f$  is not a subset of the Domain of  $g$

$\therefore g \circ f$  does not exist.

**11 a**  $S$  is the maximal Domain of  $f$ ,  
 $\therefore S = [-2, 2]$

**b** Range of  $f = [0, 2]$   
 range of  $g = [1, \infty)$

**c**  $f \circ g$  is not defined as the range of  $g$  is not a subset of the Domain of  $f$ .  
 $g \circ f$  is defined as the range of  $f$  is a subset of the Domain of  $g$ .

**12** For both  $f \circ g$  and  $g \circ f$  to exist, the range of  $g$  must be a subset of the Domain of  $f$  and the range must be a subset of the Domain of  $g$ .

$$\text{Domain of } f: [2, \infty); \text{ Range of } f: (-\infty, a - 2]$$

$$\text{Domain of } g: (-\infty, 1]; \text{ range of } g: [a, \infty)$$

$$\text{So } a \geq 2 \text{ from } f \circ g$$

$$\& a - 2 \leq 1 \text{ from } g \circ f$$

$$\therefore 2 \leq a \leq 3$$

## Solutions to Exercise 1F

**1 a** Let  $y = f^{-1}(x)$  then

$$x = 2y + 3$$

$$y = \frac{x-3}{2}$$

$$f^{-1}(x) = \frac{x-3}{2}$$

**b** Let  $y = f^{-1}(x)$  then

$$x = 4 - 3y$$

$$y = \frac{4-x}{3}$$

$$f^{-1}(x) = \frac{4-x}{3}$$

**c** Let  $y = f^{-1}(x)$  then

$$x = 4y + 3$$

$$y = \frac{x-3}{4}$$

$$f^{-1}(x) = \frac{x-3}{4}$$

**2 a** Let  $y = f^{-1}(x)$  then

$$x = y - 4$$

$$f^{-1}(x) = y = x + 4$$

**b** Let  $y = f^{-1}(x)$  then

$$x = 2y$$

$$f^{-1}(x) = y = \frac{x}{2}$$

**c** Let  $y = f^{-1}(x)$  then

$$x = \frac{3}{4}y$$

$$f^{-1}(x) = y = \frac{4}{3}x$$

**d** Let  $y = f^{-1}(x)$  then

$$x = \frac{3y-2}{4}$$

$$3y = 4x + 2$$

$$f^{-1}(x) = y = \frac{4x+2}{3}$$

**3 a** Let  $y = f^{-1}(x)$  then

$$x = 2y - 4$$

$$f^{-1}(x) = y = \frac{x+4}{2}$$

$$\text{Domain}(f^{-1}) = \text{range}(f) = [-8, 8]$$

$$\therefore f^{-1} : [-8, 8] \rightarrow \mathbb{R}, f^{-1}(x) = \frac{x+4}{2}$$

$$\text{range}(f^{-1}) = \text{Domain}(f) = [-2, 6]$$

**b** let  $g^{-1}(x) = y$  then

$$x = \frac{1}{9-y}$$

$$9-y = \frac{1}{x}$$

$$g^{-1}(x) = y = 9 - \frac{1}{x}$$

$$\text{Domain}(g^{-1}) = \text{range}(g) = \mathbb{R}^-$$

$$\therefore g^{-1} : \mathbb{R}^- \rightarrow \mathbb{R}, g^{-1}(x) = 9 - \frac{1}{x}$$

$$\text{range}(g^{-1}) = \text{Domain}(g) = (9, \infty)$$

**c** Let  $h^{-1}(x) = y$ . Then

$$x = y^2 + 2$$

$$y^2 = x - 2$$

$$y = \pm \sqrt{x-2}$$

$$\text{but range}(h^{-1}) = \text{Domain}(h)$$

$$= \mathbb{R}^+ \cup \{0\}$$

$$\therefore h^{-1}(x) = y = \sqrt{x-2}$$

$$\text{Domain}(h^{-1}) = \text{range}(h) = [2, \infty)$$

$$\therefore h^{-1} : [2, \infty) \rightarrow \mathbb{R}, h^{-1}(x) = \sqrt{x-2}$$

$$\text{range}(h^{-1}) = [0, \infty)$$

**d** Let  $f^{-1}(x) = y$ . Then

$$x = 5y - 2$$

$$f^{-1}(x) = y = \frac{x+2}{5}$$

$$\text{Domain}(f^{-1}) = \text{range}(f) = [-17, 28]$$

$$\therefore f^{-1}[-17, 28] \rightarrow \mathbb{R}, f^{-1}(x) = \frac{x+2}{5}$$

$$\text{range}(f^{-1}) = \text{Domain}(f) = [-3, 6]$$

**e** Let  $g^{-1}(x) = y$ . Then  $x = y^2 - 1$

$$y^2 = x + 1$$

$$y = \pm \sqrt{x+1}$$

$$\text{but range}(g^{-1}) = \text{Domain}(g) = (1, \infty)$$

$$\therefore g^{-1}(x) = \sqrt{x+1}$$

$$\text{Domain}(g^{-1}) = \text{range}(g) = (0, \infty)$$

$$\therefore g^{-1}(0, \infty) \rightarrow \mathbb{R}, g^{-1}(x) = \sqrt{x+1}$$

$$\text{range}(g^{-1}) = (1, \infty)$$

**f** Let  $h^{-1}(x) = y$ . Then  $x = \sqrt{y}$

$$h^{-1}(x) = y = x^2$$

$$\text{Domain}(h^{-1}) = \text{range}(h) = \mathbb{R}^+$$

$$\therefore h^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R}, h^{-1}(x) = x^2$$

$$\text{range}(h^{-1}) = \text{Domain}(h) = \mathbb{R}^+$$

**4 a** Interchange  $x$  and  $y$

$$x = y^2 + 2y$$

Completing the square:

$$(y+1)^2 - x - 1 = 0$$

$$y+1 = \pm \sqrt{1+x}$$

$$y = -1 \pm \sqrt{1+x}$$

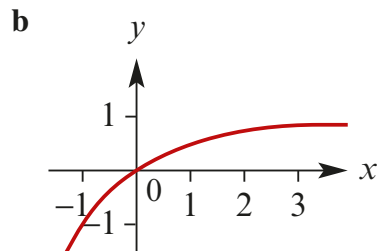
$$\text{but range}(g^{-1}) = \text{Domain}(g) = [-1, \infty)$$

$$\therefore g^{-1}(x) = y = \sqrt{1+x} - 1$$

$$\text{Domain}(g^{-1}) = \text{range}(g) = [-1, \infty)$$

$$g^{-1}[-1, \infty) \rightarrow \mathbb{R}, g^{-1}(x) = \sqrt{1+x} - 1$$

$$\text{range}(g^{-1}) = [-1, \infty)$$



**5** Let  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \frac{1}{x} - 3$

Let  $y = f^{-1}(x)$ . Then we can write

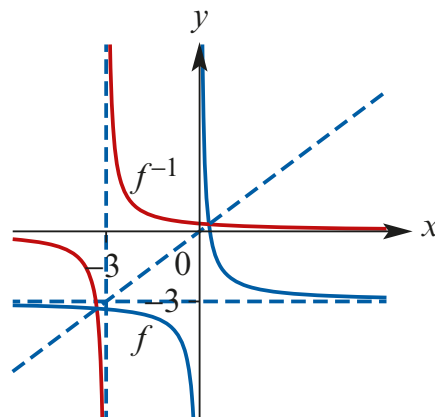
$$x = \frac{1}{y} - 3$$

$$\text{Hence } y = \frac{1}{x+3}.$$

$$\text{That is } f^{-1}(x) = \frac{1}{x+3}.$$

The Domain of  $f^{-1}$  is  $\mathbb{R} \setminus \{-3\}$

$$f^{-1}: \mathbb{R} \setminus \{-3\} \rightarrow \mathbb{R}, f^{-1}(x) = \frac{1}{x+3}$$



**6 a** to find  $f^{-1}(2)$ , use  $f(x) = 2$

$$2 = 3 - 2x$$

$$f^{-1}(2) = x = \frac{1}{2}$$

$$\text{Domain } f^{-1} = \text{range}(f) = [-3, 3]$$

**7 a** Let  $f^{-1}(x) = y$

$$x = 2y$$

$$f^{-1}(x) = y = \frac{x}{2}$$

$$\text{Domain } f^{-1} = \text{range}(f) = [-2, 6]$$

$$\text{range } f^{-1} = \text{Domain}(f) = [-1, 3]$$

$$\therefore f^{-1}[-2, 6] \rightarrow \mathbb{R}, f^{-1}(x) = \frac{x}{2}$$

**b** Let  $f^{-1}(x) = y$

$$x = 2y^2 - 4$$

$$y^2 = \frac{(x+4)}{2}$$

$$y = \pm \sqrt{\frac{(x+4)}{2}}$$

but  $\text{range } f^{-1} = \text{Domain}(f) = [-0, \infty)$

$$\therefore f^{-1}(x) = y = \sqrt{\frac{(x+4)}{2}}$$

$$\text{Domain } f^{-1} = \text{range}(f) = [-4, \infty)$$

$$\therefore f^{-1}[-4, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = \sqrt{\frac{(x+4)}{2}}$$

$$\text{range } f^{-1} = [0, \infty)$$

**c**  $\{(4, 2), (6, 1), (8, 3), (11, 5)\}$

$$\text{Domain} = \{4, 6, 8, 11\}$$

$$\text{range} = \{1, 2, 3, 5\}$$

**d** Let  $h^{-1}(x) = y$ . Then

$$x = \sqrt{-y}$$

$$h^{-1}(x) = y = -x^2$$

$$\text{Domain } h^{-1} = \text{range}(h) = \mathbb{R}^+$$

$$\therefore h^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R}, h^{-1}(x) = -x^2$$

$$\text{range}(h^{-1}) = \mathbb{R}^-$$

**e** Let  $f^{-1}(x) = y$ . Then

$$x = y^3 + 1$$

$$f^{-1}(x) = y = (x-1)^{\frac{1}{3}}$$

$$\text{Domain}(f^{-1}) = \text{range}(f) = \mathbb{R}$$

$$f^{-1}: \mathbb{R} \rightarrow \mathbb{R}, f^{-1}(x) = (x-1)^{\frac{1}{3}}$$

$$\text{range}(f^{-1}) = \mathbb{R}$$

**f** Let  $g^{-1}(x) = y$ . Then

$$x = (y+1)^2$$

$$y = \pm \sqrt{x} - 1$$

but  $\text{range}(g^{-1}) = \text{Domain}(g) = (-1, 3)$

$$\therefore g^{-1}(x) = y = \sqrt{x} - 1$$

$$\text{Domain } g^{-1} = \text{range}(g) = (0, 16)$$

$$\therefore g^{-1}: (0, 16) \rightarrow \mathbb{R}, g^{-1}(x) = \sqrt{x} - 1$$

$$\text{range}(g^{-1}) = (-1, 3)$$

**g** Let  $g^{-1}(x) = y$ . Then

$$x = \sqrt{y-1}$$

$$g^{-1}(x) = y = x^2 + 1$$

$$\text{Domain } g^{-1} = \text{range}(g) = [0, \infty)$$

$$\therefore g^{-1}: [0, \infty) \rightarrow \mathbb{R}, g^{-1}(x) = x^2 + 1$$

$$\text{range } g^{-1} = [1, \infty)$$

**h** Let  $h^{-1}(x) = y$ . Then

$$x = \sqrt{4-y^2}$$

$$y^2 = 4-x^2$$

$$y = \pm \sqrt{4-x^2}$$

but  $\text{range}(h^{-1}) = \text{Domain}(h) = [0, 2]$

$$\therefore h^{-1}(x) = y = \sqrt{4-x^2}$$

$$\text{Domain}(h^{-1}) = \text{range}(h) = [0, 2]$$

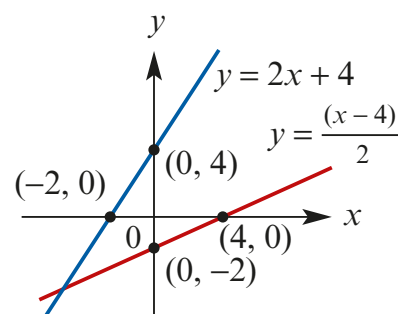
$$\therefore h^{-1}: [0, 2] \rightarrow \mathbb{R}, h^{-1}(x) = \sqrt{4-x^2}$$

$$\text{range}(h^{-1}) = [0, 2]$$

**8 a**  $x = 2y + 4$

$$y = \frac{x-4}{2}$$

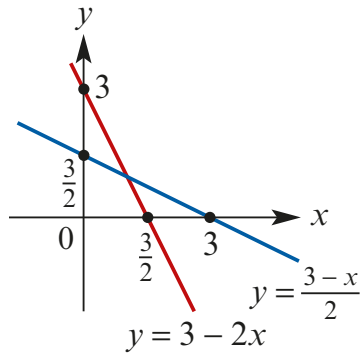
implied Domain:  $\mathbb{R}$  and range:  $\mathbb{R}$



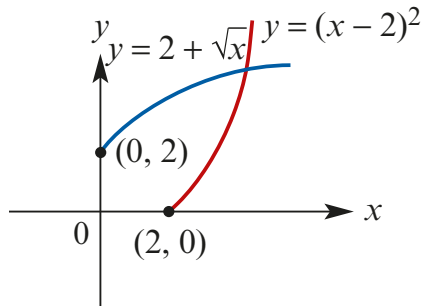
**b**  $x = \frac{3-f^{-1}(x)}{2}$

$$f^{-1}(x) = 3 - 2x$$

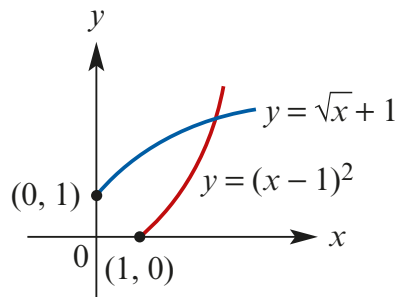
implied Domain:  $\mathbb{R}$   
and range:  $\mathbb{R}$



- c**  $x = (f^{-1}(x) - 2)^2$   
 $\pm \sqrt{x} + 2 = f^{-1}(x)$   
 but range  $(f^{-1}) = \text{dom}(f) = [2, \infty)$   
 $\therefore f^{-1}(x) = \sqrt{x} + 2$   
 Domain:  $[0, \infty)$   
 range:  $[2, \infty)$

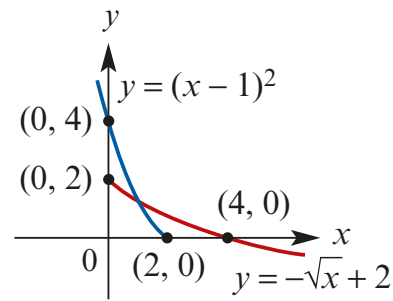


- d**  $x = (f^{-1}(x) - 1)^2$   
 $f^{-1}(x) = \sqrt{x} + 1$   
 Domain:  $[0, \infty)$   
 range:  $[1, \infty)$

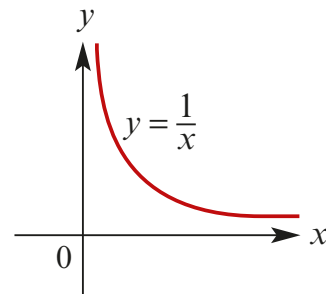


- e** similar to (c)  
 but  $f^{-1}(x) = -\sqrt{x} + 2$   
 Domain:  $[0, \infty)$

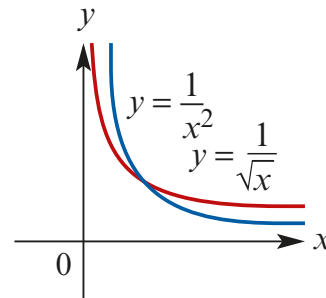
range:  $(-\infty, 2]$



- f**  $f^{-1}(x) = \frac{1}{x}$   
 Domain:  $\mathbb{R}^+$   
 range:  $\mathbb{R}^+$

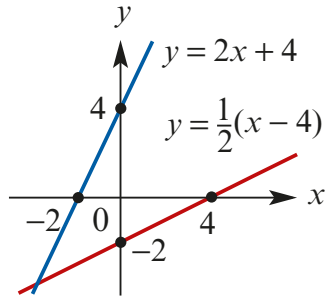


- g**  $x = \frac{1}{(f^{-1}(x))^2}$   
 $f^{-1}(x) = \pm \frac{1}{\sqrt{x}}$   
 but range  $f^{-1}(x) = \text{Domain}(f) = \mathbb{R}^+$   
 $\therefore f^{-1}(x) = \frac{1}{\sqrt{x}}$   
 Domain:  $\mathbb{R}^+$   
 range:  $\mathbb{R}^+$

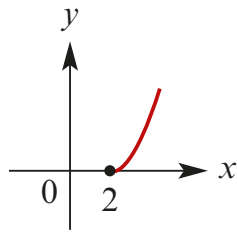


- h**  $x = \frac{1}{2}(h^{-1}(x) - 4)$   
 $h^{-1}(x) = 2x + 4$

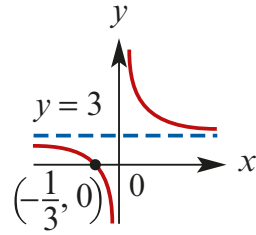
implied Domain:  $\mathbb{R}$   
and range:  $\mathbb{R}$



**9 a**  $x = \sqrt{f^{-1}(x) + 2}$   
 $(x - 2)^2 = f^{-1}(x)$   
 $f^{-1}(x) = x^2 - 4x + 4$   
 $f^{-1}(x) = (x - 2)^2$   
 Therefore,  
 $f^{-1}: [2, \infty) \rightarrow \mathbb{R}$ ,  
 $f^{-1}(x) = (x - 2)^2$

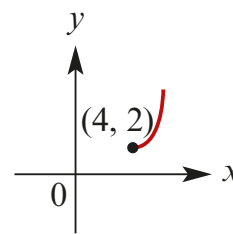


**b**  $x = \frac{1}{f^{-1}(x) - 3}$   
 $f^{-1}(x) - 3 = \frac{1}{x}$   
 $f^{-1}(x) = \frac{1}{x} + 3$   
 Therefore,  
 $f^{-1}: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ ,  
 $f^{-1}(x) = \frac{1}{x} + 3$



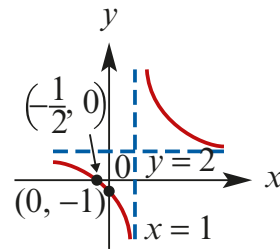
**c**  $x = \sqrt{f^{-1}(x) - 2} + 4$   
 $f^{-1}(x) - 2 = (x - 4)^2$   
 $f^{-1}(x) = x^2 - 8x + 18$   
 $f^{-1}(x) = (x - 4)^2 + 2$

Therefore,  
 $f^{-1}: [4, 4 + \sqrt{6}) \rightarrow \mathbb{R}$ ,  
 $f^{-1}(x) = (x - 4)^2 + 2$



**d**  $x = \frac{3}{f^{-1}(x) - 2} + 1$   
 $f^{-1}(x) - 2 = \frac{3}{x - 1}$   
 $f^{-1}(x) = \frac{3}{x - 1} + 2$

Therefore,  
 $f^{-1}: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$ ,  
 $f^{-1}(x) = \frac{3}{x - 1} + 2$



$$\mathbf{e} \quad x = \frac{5}{f^{-1}(x) - 1} - 1$$

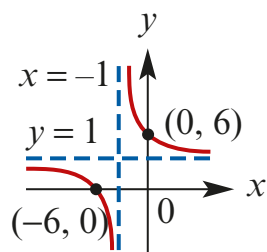
$$f^{-1}(x) - 1 = \frac{5}{x + 1}$$

$$f^{-1}(x) = \frac{5}{x + 1} + 1$$

Therefore,

$$f^{-1}: \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R},$$

$$f^{-1}(x) = \frac{5}{x + 1} + 1$$



$$\mathbf{f} \quad x = \sqrt{2 - f^{-1}(x)} + 1$$

$$(x - 1)^2 = 2 - f^{-1}(x)$$

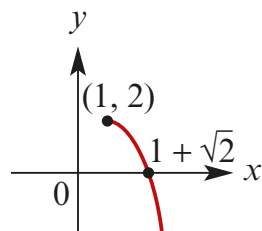
$$f^{-1}(x) = 2 - (x - 1)^2$$

$$f^{-1}(x) = -x^2 + 2x + 1$$

Therefore,

$$f^{-1}: [1, \infty) \rightarrow \mathbb{R},$$

$$f^{-1}(x) = 2 - (x - 1)^2$$



$$\mathbf{10 a} \quad f(x) = 1 + \frac{2}{x - 1}$$

$$x = 1 + \frac{2}{f^{-1}(x) - 1}$$

$$x - 1 = \frac{2}{f^{-1}(x) - 1}$$

$$f^{-1}(x) - 1 = \frac{2}{x - 1}$$

$$f^{-1}(x) = 1 + \frac{2}{x - 1}$$

$$f^{-1}(x) = \frac{x + 1}{x - 1}$$

$$\mathbf{b} \quad f(x) = \sqrt{x - 2}$$

$$x = \sqrt{f^{-1}(x) - 2}$$

$$x^2 = f^{-1}(x) - 2$$

$$f^{-1}(x) = x^2 + 2$$

$$\mathbf{c} \quad f(x) = \frac{2x + 3}{3x - 2}$$

$$= \frac{\frac{2}{3}(3x - 2) + \frac{4}{3} + 3}{3x - 2}$$

$$= \frac{\frac{2}{3} + \frac{13}{3}}{3x - 2}$$

$$= \frac{2}{3} + \frac{13}{9x - 6}$$

$$x = \frac{2}{3} + \frac{13}{9f^{-1}(x) - 6}$$

$$x - \frac{2}{3} = \frac{13}{9f^{-1}(x) - 6}$$

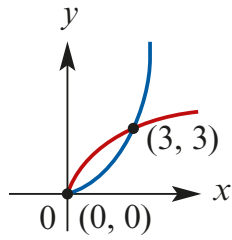
$$9f^{-1}(x) - 6 = \frac{13}{x - \frac{2}{3}}$$

$$3f^{-1}(x) - 2 = \frac{13}{3x - 2}$$

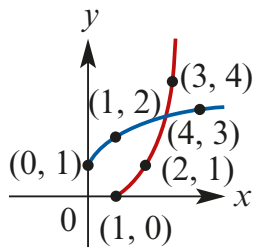
$$3f^{-1}(x) = \frac{13 + 6x - 4}{3x - 2}$$

$$f^{-1}(x) = \frac{2x + 3}{3x - 2}$$

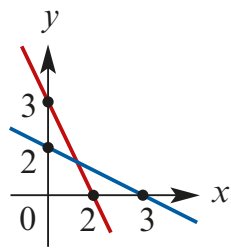
11 a



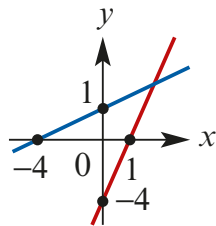
b



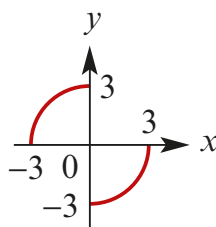
c



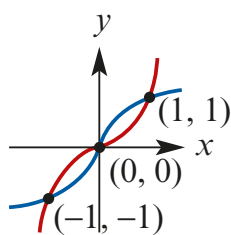
d



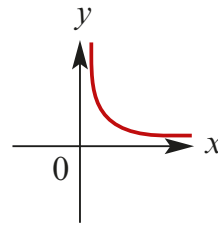
e



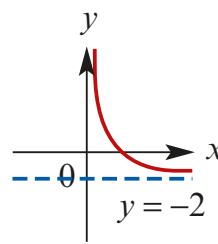
f



g



h



12 a C

b B

c D

d A

13 a  $3 - x \geq 0$

$$x \leq 3$$

$$\therefore A = (-\infty, 3]$$

b minimum  $b$  is at the turning point

$$\text{i.e. } b = 0$$

$$\text{let } g^{-1}(x) = y$$

$$x = 1 - y^2$$

$$y = \pm \sqrt{1 - x}$$

$$\text{,but range}(g^{-1}) = \text{Domain}(g) = [0, 2]$$

$$\therefore y = \sqrt{1 - x}$$

$$\text{Domain}(g^{-1}) = \text{range}(g) = [-3, 1]$$

$$\therefore g^{-1} : [-3, 1] \rightarrow R, g^{-1}(x) = \sqrt{1 - x}$$

$$14 \quad b = -2, \quad g^{-1}(x) = -2 + \sqrt{x+4}$$

$$15 \quad a = 3, \quad f^{-1}(x) = 3 - \sqrt{x+9}$$



**16 a**  $x = \frac{3}{g^{-1}(x)}$

$$g^{-1}(x) = \frac{3}{x}$$

Domain =  $\mathbb{R} \setminus \{0\}$

**b**  $x = \sqrt[3]{g^{-1}(x) + 2} - 4$

$$(x + 4)^3 = g^{-1}(x) + 2$$

$$g^{-1}(x) = (x + 4)^3 - 2$$

Domain =  $\mathbb{R}$

$$x = 2 - \sqrt{h^{-1}(x)}$$

**c**  $\sqrt{h^{-1}(x)} = 2 - x$

$$h^{-1}(x) = (x - 2)^2$$

Domain ( $h^{-1}$ ) = range ( $h$ ) =  $(-\infty, 2]$

$$x = \frac{3}{f^{-1}(x)} + 1$$

**d**  $f^{-1}(x) = \frac{3}{x - 1}$

Domain =  $\mathbb{R} \setminus \{1\}$

**e**  $x = 5 - \frac{2}{(h^{-1}(x) - 6)^3}$

$$\frac{2}{5 - x} = (h^{-1}(x) - 6)^3$$

$$h^{-1}(x) = \sqrt[3]{\frac{2}{5 - x}} + 6$$

Domain =  $\mathbb{R} \setminus \{5\}$

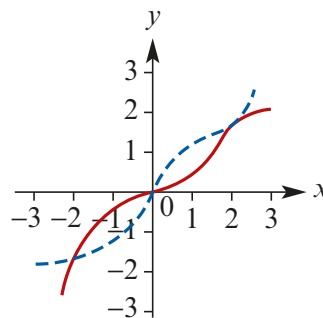
**f**  $x = \frac{1}{(g^{-1}(x) - 1)^{\frac{3}{4}}} + 2$

$$(g^{-1}(x) - 1)^{\frac{3}{4}} = \frac{1}{x - 2}$$

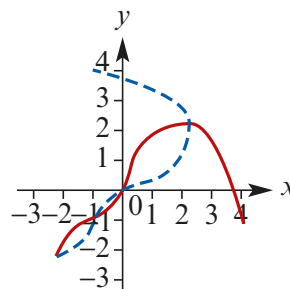
$$g^{-1}(x) = \frac{1}{(x - 2)^{\frac{4}{3}}} + 1$$

Domain =  $(2, \infty)$

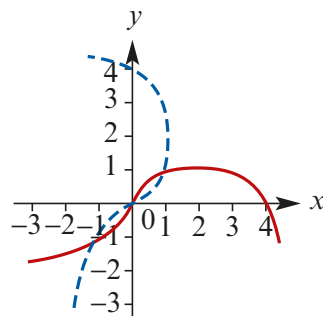
**17 a** Inverse is a function



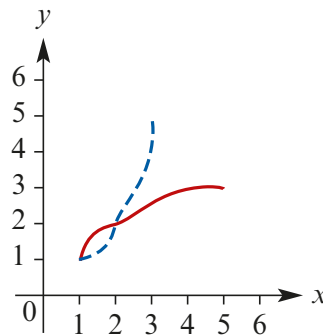
**b** Inverse is not a function



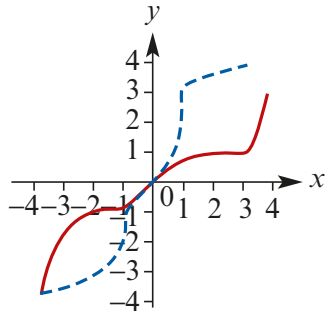
**c** Inverse is not a function



**d** Inverse is a function



**e** Inverse is not a function



**18 a**  $f(x) = \frac{x+3}{2x-1}$

Domain =  $\mathbb{R} \setminus \left\{ \frac{1}{2} \right\}$

$$\begin{aligned} f(x) &= \frac{\frac{1}{2}(2x-1) + \frac{7}{2}}{2x-1} \\ &= \frac{1}{2} + \frac{7}{2(2x-1)} \end{aligned}$$

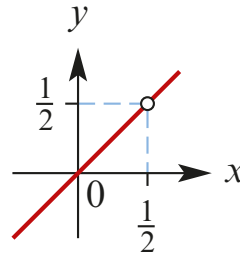
range =  $\mathbb{R} \setminus \left\{ \frac{1}{2} \right\}$

Since range( $f$ ) = Domain( $f$ )

$f \circ f$  is defined.

**b** 
$$\begin{aligned} f \circ f(x) &= \frac{1}{2} + \frac{\frac{7}{2}}{2 \frac{(2x+3)}{(2x-1)} - 1} \\ &= \frac{1}{2} + \frac{7}{2 \left( \frac{2x+6-2x+1}{2x-1} \right)} \\ &= \frac{1}{2} + \frac{7(2x-1)}{14} \\ &= \frac{1}{2} + x - \frac{1}{2} \end{aligned}$$

$f \circ f(x) = x, x \in \mathbb{R} \setminus \left\{ \frac{1}{2} \right\}$



**c** Since  $f \circ f(x) = x$  and  $f^{-1} \circ f(x) = x$   
 $f^{-1} = f = \frac{x+3}{2x-1}, x \in \mathbb{R} \setminus \left\{ \frac{1}{2} \right\}$

## Solutions to Exercise 1G

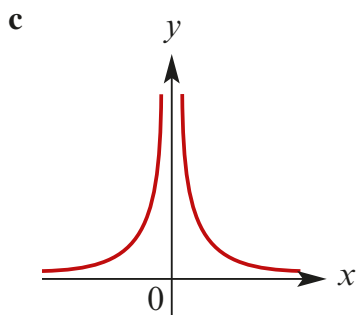
1 a Maximal Domain =  $\mathbb{R} \setminus \{0\}$ ;  
Range =  $\mathbb{R}^+$

b i  $\frac{1}{16}$

ii  $\frac{1}{16}$

iii 16

iv 16



2 a Odd

b Even

c Odd

d Odd

e Even

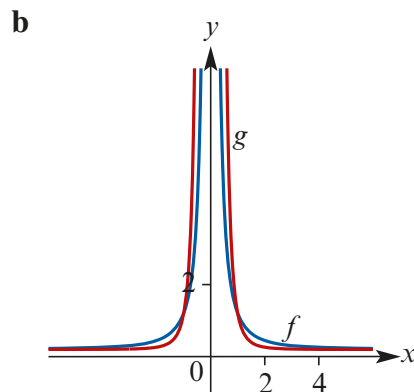
f Odd

3 a  $f(x) = g(x)$

$$x^{-2} = x^{-4}$$

$$x^2 = 1$$

$$x = 1 \text{ or } x = -1$$



4 a  $f(x) = g(x)$

$x = 0$  is one solution.

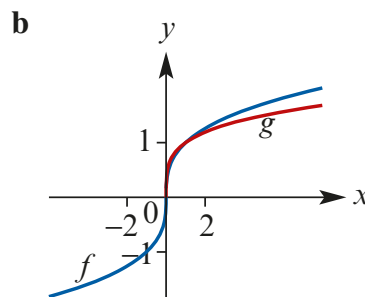
Now assume  $x \neq 0$

$$x^{\frac{1}{3}} = x^{\frac{1}{4}}$$

$$x^{\frac{1}{3} - \frac{1}{4}} = 1$$

$$x^{\frac{1}{12}} = 1$$

$$\therefore x = 1 \text{ or } x = 0$$



5 a  $x = (f^{-1}(x))^7$

$$f^{-1}(x) = x^{\frac{1}{7}}$$

Domain of  $f^{-1}$  = range of  $f$  =  $\mathbb{R}$

$$f^{-1}: \mathbb{R} \rightarrow \mathbb{R}, f^{-1}(x) = x^{\frac{1}{7}}$$

b  $x = (f^{-1}(x))^6$

$$f^{-1}(x) = x^{\frac{1}{6}}$$

Domain of  $f^{-1}$  = range of  $f = [0, \infty)$   
 $f^{-1}: [0, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = -x^{\frac{1}{6}}$

**c**  $x = 27(f^{-1}(x))^3$

$$\frac{x}{27} = (f^{-1}(x))^3$$

$$f^{-1}(x) = \left(\frac{x}{27}\right)^{\frac{1}{3}} = \frac{1}{3}x^{\frac{1}{3}}$$

Domain of  $f^{-1}$  = range of  $f = \mathbb{R}$

$$f^{-1}: \mathbb{R} \rightarrow \mathbb{R}, f^{-1}(x) = \frac{1}{3}x^{\frac{1}{3}}$$

**d**  $x = 16(f^{-1}(x))^4$

$$\frac{x}{16} = (f^{-1}(x))^4$$

$$f^{-1}(x) = \left(\frac{x}{16}\right)^{\frac{1}{4}} = \frac{1}{2}x^{\frac{1}{4}}$$

Domain of  $f^{-1}$  = range of

$$f = (16, \infty)$$

$$f^{-1}: (16, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = \frac{1}{2}x^{\frac{1}{4}}$$

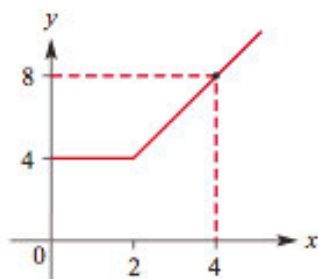
## Solutions to Exercise 1H

- 1** For  $0 \leq x \leq 2$ , the cost is \$ 4. For  $x > 2$ , the cost is \$4 + \$2 for each extra km over 2 km, i.e.  $\$2(x - 2)$ .

Hence:

$$f(x) = \begin{cases} 4 & \text{if } 0 \leq x \leq 2 \\ 4 + 2(x - 2) & \text{if } x > 2 \end{cases}$$

$$= \begin{cases} 4 & \text{if } 0 \leq x \leq 2 \\ 2x & \text{if } x > 2 \end{cases}$$



- 2** The box has length  $(36 - 2x)$  cm, width  $(20 - 2x)$  cm and height  $x$  cm. So the volume  $V$  cm<sup>3</sup> is given by

$$V = x(20 - 2x)(36 - 2x)$$

$$= 4x(10 - x)(18 - x)$$

where  $x > 0$  and  $x < 10$  for a box to exist.

The Domain is  $[0, 10]$ .

- 3 a** Perimeter =  $2x + 2y = 160$ , so  $y = 80 - x$ . The area can be found by subtracting a rectangle of dimensions 12 by  $(y - 20)$  from a rectangle of dimensions  $x$  by  $y$ :

$$A = xy - 12(y - 20)$$

$$= x(80 - x) - 12(60 - x)$$

$$= -x^2 + 80x + 12x - 720$$

$$= -x^2 + 92x - 720$$

- b**  $x > 12$ ; also  $y > 20$  implies  $80 - x > 20$  so that  $x < 60$ . The Domain is  $[12, 60]$ .

- c** The function is a quadratic with (non-included) endpoints where  $x = 12, 60$ . When  $x = 12, A = 240$ ; when  $x = 60, A = 1200$ . Endpoints are  $(12, 240)$  and  $(60, 1200)$ .

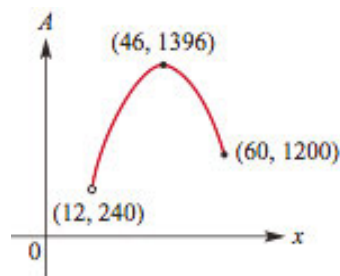
There is a turning point where

$$x = -\frac{b}{2a}$$

$$= -\frac{92}{-2} = 46$$

Then  $A = 1396$ .

The graph is shown here.



- d** The maximum area is 1396 m<sup>2</sup> and it occurs for  $x = 46$  and  $y = 80 - 46 = 34$ .

- 4 a i**  $S = 2x^2 + 2 \leftrightarrow 2x \leftrightarrow h + 2 \leftrightarrow$   
 $x \leftrightarrow h$   
 $= 2x^2 + 6xh$

**ii**  $V = 2x^2h$  where  $h = \frac{V}{2x^2}$

$$S = 2x^2 + 6x \leftrightarrow \frac{V}{2x^2}$$

$$= 2x^2 + \frac{3V}{x}$$

- b**  $x > 0$ , so maximal Domain is  $(0, \infty)$ .

- c  $V = 1000$  so  $S = 2x^2 + \frac{3000}{x}$ .  
A sketch using a CAS calculator shows that there is an endpoint maximum where  $x = 2$ . Then  $S = 1508 \text{ m}^2$ .

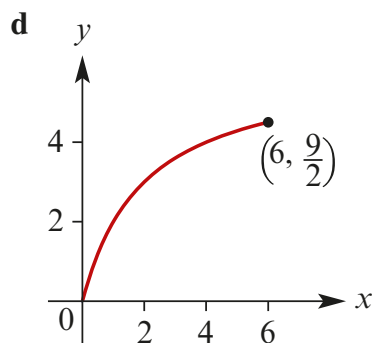
- 5 Let  $x$  be the width of the rectangle and  $y$  be the length of the rectangle.  
The diagonal has length  $2a$ .  
 $\therefore x^2 + y^2 = 4a^2$   
 $\therefore y^2 = 4a^2 - x^2$   
 $\therefore y = \sqrt{4a^2 - x^2}$   
 $\therefore \text{Area} = xy = x(\sqrt{4a^2 - x^2})$   
The Domain is clearly  $[0, 2a]$ .

- 6 The coordinates of  $C$  are  $\left(a, \frac{6}{a+2}\right)$

a  $\text{Area} = a \times \frac{6}{a+2} = \frac{6a}{a+2}$

b Domain =  $[0, 6]$ ; Range =  $\left[0, \frac{9}{2}\right]$

c Maximum value =  $\left[0, \frac{9}{2}\right]$



- 7 a Distance is speed by time, so during the first 45 minutes, the man runs a distance of  $\frac{2}{60}t = \frac{1}{30}t$  km; after 45 minutes, he has run  $\frac{3}{2}$  km and there-  
after adds a distance of  $\frac{4}{60}t = \frac{1}{15}t$  during the next 30 minutes. Hence:

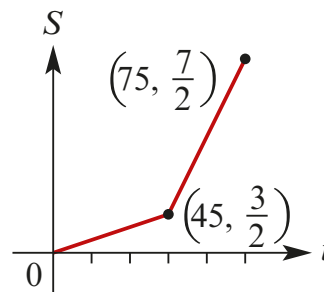
$$S(t) = \begin{cases} \frac{1}{30}t & \text{if } 0 \leq t \leq 45 \\ \frac{3}{2} + \frac{1}{15}(t-45) & \text{if } 45 < t \leq 75 \end{cases}$$

$$= \begin{cases} \frac{1}{30}t & \text{if } 0 \leq t \leq 45 \\ \frac{1}{15}t - \frac{3}{2} & \text{if } 45 < t \leq 75 \end{cases}$$

$$a = \frac{1}{30}, \quad b = \frac{1}{15}, \quad c = 45,$$

$$d = -\frac{3}{2}, \quad e = 75$$

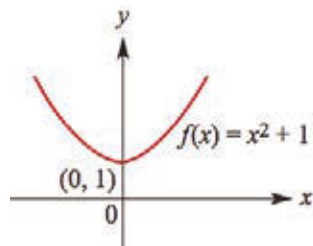
- b The graph comprises two line segments as shown here.



- c The range is  $\left[0, \frac{7}{2}\right]$ .

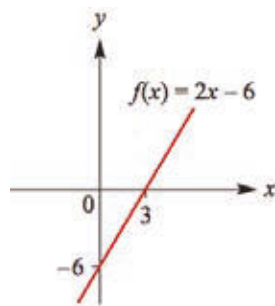
## Solutions to technology-free questions

1 a



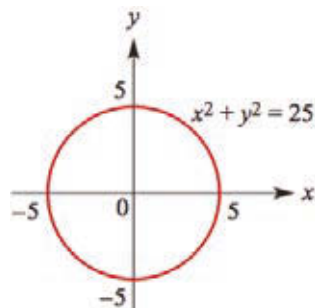
Domain =  $\mathbb{R}$ , range =  $[1, \infty)$

b



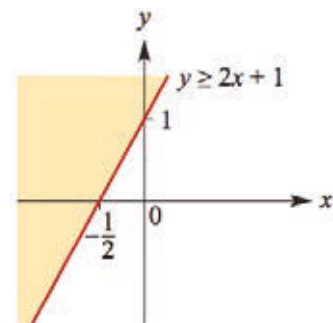
Domain =  $\mathbb{R}$ , range =  $\mathbb{R}$

c



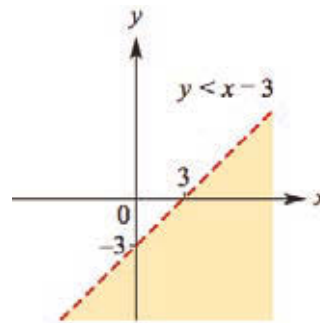
Domain =  $[-5, 5]$ , range =  $[-5, 5]$

d



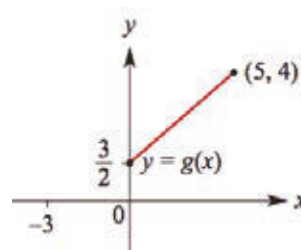
Domain =  $\mathbb{R}$ , range =  $\mathbb{R}$

e



Domain =  $\mathbb{R}$ , range =  $\mathbb{R}$

2 a



b range =  $[1.5, 4]$

c Interchange  $x$  and  $y$  and solve for  $y$ :

$$x = \frac{y+3}{2}$$

$$y+3 = 2x$$

$$y = 2x - 3$$

$$g^{-1} : [1.5, 4] \rightarrow \mathbb{R}, g^{-1}(x) = 2x - 3$$

Domain =  $[1.5, 4]$ , range =  $[0, 5]$

d  $g(x) = 4$

$$\frac{x+3}{2} = 4$$

$$x+3 = 8$$

$$x = 5$$

$$\{x : g(x) = 4\} = \{5\}$$

e If  $g^{-1}(x) = 4$ , then  $x = g(4) = 3.5$ .

$$\{x : g^{-1}(x) = 4\} = \{3.5\}$$

(Alternatively, solve the equation

$$2x - 3 = 4 \text{ for } x.)$$

**3 a**  $5x + 1 = 2$

$$5x = 1$$

$$x = \frac{1}{5}$$

$$\{x : g(x) = 2\} = \left\{ \frac{1}{5} \right\}$$

**b** If  $g^{-1}(x) = 2$ , then  $x = g(2) = 11$ .

$$\{x : g(x) = 2\} = \{11\}$$

**c**  $\frac{1}{5x+1} = 2$

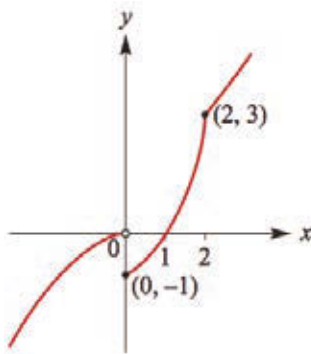
$$5x + 1 = \frac{1}{2}$$

$$5x = -\frac{1}{2}$$

$$x = -\frac{1}{10}$$

$$\left\{ x : \frac{1}{g(x)} = 2 \right\} = \left\{ -\frac{1}{10} \right\}$$

**4**



**5 a**  $2x - 6 \neq 0$ , so  $x \neq 3$

$$\text{Domain} = \mathbb{R} \setminus \{3\}$$

**b**  $x^2 - 5 > 0$

$$(x - \sqrt{5})(x + \sqrt{5}) > 0$$

$$x < -\sqrt{5} \text{ or } x > \sqrt{5}$$

$$\text{Domain} = \mathbb{R} \setminus [-\sqrt{5}, \sqrt{5}]$$

**c**  $(x - 1)(x + 2) \neq 0$ , so  $x \neq 1, -2$

$$\text{Domain} = \mathbb{R} \setminus \{1, -2\}$$

**d**  $25 - x^2 \geq 0$

$$(5 - x)(5 + x) \geq 0$$

$$-5 \leq x \leq 5$$

$$\text{Domain} = [-5, 5]$$

**e**  $x - 5 \geq 0$  and  $15 - x \geq 0$

$$5 \leq x \leq 15$$

$$\text{Domain} = [5, 15]$$

**f**  $3x - 6 \neq 0$ , so  $x \neq 2$

$$\text{Domain} = \mathbb{R} \setminus \{2\}$$

**6**  $(f + g)(x) = (x + 2)^2 + x - 3$

$$= x^2 + 4x + 4 + x - 3$$

$$= x^2 + 5x + 1$$

$$(fg)(x) = (x - 3)(x + 2)^2$$

**7**

$$(f + g)(x) = (x - 1)^2 + 2x$$

$$= x^2 + 1$$

$$(f + g): [1, 5] \rightarrow \mathbb{R}, (f + g)(x) = x^2 + 1$$

$$(fg)(x) = 2x(x - 1)^2$$

$$(fg): [1, 5] \rightarrow \mathbb{R}, (fg)(x) = 2x(x - 1)^2$$

**8**  $f(3) = 8$ , so range of  $f$  is  $[8, \infty)$  (the graph of  $y = f(x)$  is increasing for  $x \geq 3$ ).

Hence Domain of  $f^{-1}$  is  $[8, \infty)$  and the range is  $[3, \infty)$ .



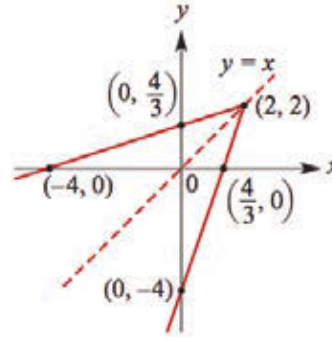
Interchange  $x$  and  $y$  and solve for  $y$ :

$$x = y^2 - 1$$

$$y^2 = x + 1$$

$$y = \sqrt{x + 1} \text{ (as } y > 0\text{)}$$

$$f^{-1} : [8, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = \sqrt{x + 1}$$



**9 a**  $(f + g)(x) = -x^2 + 2x + 3$

**b**  $(fg)(x) = -x^2(2x + 3)$

**c**  $(f + g)(x) = 0$   
 $\{x: (f + g)(x) = 0\}$   
 $= \{-1, 3\}$   
 $-x^2 + 2x + 3 = 0$   
 $-(x^2 - 2x - 3) = 0$   
 $-(x + 1)(x - 3) = 0$   
 $x = -1, 3$

**10**  $f(2) = 2$ , so range of  $f$  is  $(-\infty, 2]$  (the graph of  $y = f(x)$  is a straight line with endpoint at  $(2, 2)$ ).

Interchange  $x$  and  $y$  and solve for  $y$ :

$$x = 3y - 4$$

$$3y = x + 4$$

$$y = \frac{x + 4}{3}$$

$$f^{-1} : (-\infty, 2] \rightarrow \mathbb{R}, f^{-1}(x) = \frac{x + 4}{3}$$

The graphs are straight lines, reflections of each other in the line  $y = x$ , each with endpoint  $(2, 2)$ .

The graph of  $y = f(x)$  has axes intercepts  $(\frac{4}{3}, 0)$ ,  $(0, -4)$ . The graph of  $y = f^{-1}(x)$  has axes intercepts  $(-4, 0)$ ,  $(0, \frac{4}{3})$ .

**11 a**  $x = 8(f^{-1}(x))^3$   
 $\frac{x}{8} = (f^{-1}(x))^3$

$$f^{-1}(x) = \left(\frac{x}{8}\right)^{\frac{1}{3}} = \frac{1}{2}x^{\frac{1}{3}}$$

Domain of  $f^{-1}$  = range of  $f = \mathbb{R}$

**b**  $x = 32(f^{-1}(x))^5$   
 $\frac{x}{32} = (f^{-1}(x))^5$

$$f^{-1}(x) = \left(\frac{x}{32}\right)^{\frac{1}{5}} = \frac{1}{2}x^{\frac{1}{5}}$$

Domain of  $f^{-1}$  = range of  $f = (-\infty, 0]$

**c**  $x = 64(f^{-1}(x))^6$   
 $\frac{x}{64} = (f^{-1}(x))^6$

$$f^{-1}(x) = \left(\frac{x}{64}\right)^{\frac{1}{6}} = \frac{1}{2}x^{\frac{1}{6}}$$

Domain of  $f^{-1}$  = range of  $f = [0, \infty)$

**d**  $x = 10\,000(f^{-1}(x))^4$   
 $\frac{x}{10\,000} = (f^{-1}(x))^4$

$$f^{-1}(x) = \left(\frac{x}{10\,000}\right)^{\frac{1}{4}} = \frac{1}{10}x^{\frac{1}{4}}$$

Domain of  $f^{-1}$  = range of  $f = (10\,000, \infty)$

$$\begin{aligned} \mathbf{12\ a} \quad f \circ g(x) &= f(-x^3) \\ &= -2x^3 + 3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad g \circ f(x) &= g(2x + 3) \\ &= -(2x + 3)^3 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad g \circ g(x) &= g(-x^3) \\ &= (-x^3)^3 \\ &= -x^9 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad f \circ f(x) &= f(2x + 3) \\ &= 2(2x + 3) + 3 \\ &= 4x + 9 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad f \circ (f + g)(x) &= f(f + g(x)) \\ &= f(-x^3 + 2x + 3) \\ &= 2(-x^3 + 2x + 3) + 3 \\ &= -2x^3 + 4x + 9 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad f \circ (f - g)(x) &= f(f - g(x)) \\ &= f(2x + 3 + x^3) \\ &= 2(2x + 3 + x^3) + 3 \\ &= 2x^3 + 4x + 9 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad f \circ (f \cdot g)(x) &= f(f \cdot g(x)) \\ &= f(-2x^4 - 3x^3) \\ &= 2(-2x^4 - 3x^3) + 3 \\ &= -4x^4 - 6x^3 + 3 \end{aligned}$$

$$\mathbf{13} \quad x \geq -1 \text{ or } x \leq -9$$

$$\mathbf{14} \quad h^{-1}(x) = \left( \frac{x - 64}{2} \right)^{\frac{1}{5}}$$

## Solutions to multiple-choice questions

**1 E**  $6 - 2x \geq 0$   
 $6 \geq 2x$   
 $3 \geq x$   
 $\therefore (-\infty, 3]$

**2 B**  $f : [-1, 3) \rightarrow R, f(x) = -x^2$   
 $f(3) = -9$ ; maximum 0 at  $x = 0$   
 $\therefore (-9, 0]$ .

**3 E**  $f(x) = 3x^2 + 2x$   
 $f(2a) = 3(2a)^2 + 2(2a)$   
 $f(2a) = 12a^2 + 4a$

**4 C**  $f(x) = 2x - 3$   
 $\text{let } f(x) = 2(f^{-1}(x)) - 3$   
 $f(x) + 3 = 2(f^{-1}(x))$   
 $f^{-1}(x) = \frac{f(x) + 3}{2}$   
 $f^{-1}(x) = \frac{x + 3}{2}$   
 $f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$

**5 E**  $f : (a, b] \rightarrow R, f(x) = 10 - x, a < b$   
 The minimum is:  
 $f(b) = 10 - b$   
 The maximum is:  
 $f(a) = 10 - a$   
 $\therefore [10 - b, 10 - a)$

**6 C** As  $a$  is a negative real number:  
 $f(a + 3) = -(a + 3) + 6$   
 $f(a + 3) = -a + 3$

**7 D**  $f(x) = (x + 3)^2 - 6$  Graph must be one to one to have an inverse function.

Turning point of function is at  
 $(-3, -6)$   
 Domain must be a sub set of either:  
 $(-\infty, -3]$  or  $[-3, \infty)$   
 $\therefore [6, \infty)$

**8 B** An inverse only exists if the function is one to one.  
 $g : [-4, 4] \rightarrow R, g(x) = \sqrt{16 - x^2}$   
 Is not one to one for the specified Domain.

**9 B** The asymptote is at  $x = -2$  therefore the asymptote of the inverse is at  $y = -2$ .

**10 C**  $f(x) = \frac{2x + 1}{x - 1} = 2 + \frac{3}{x - 1}$   
 Therefore asymptotes  $x = 1$  and  $y = 2$ .

**11 B**  $f(x) = 3x^2$  and  $g(x) = 2x + 1$   
 $\therefore f(g(x)) = 3(2x + 1)^2$   
 $f(g(x)) = 12x^2 + 12x + 3$   
 $\therefore f(g(a)) = 12a^2 + 12a + 3$

**12 E**  $f(x) = x^2 + 2x - 6 = (x + 1)^2 - 7$   
 $\therefore$  vertex has coordinates  $(-1, -7)$   
 $f(-2) = (-2)^2 + 2(-2) - 6 = 4 - 4 - 6 = -6$   
 $f(4) = (4)^2 + 2(4) - 6 = 18$   
 $\therefore$  range =  $[-7, 18)$

**13 C** If  $a > b$  then  $a^{\frac{1}{5}} > b^{\frac{1}{5}}$

**14 C** Maximal Domain  
 $= (-1, \infty) \cap (-\infty, 4] = (-1, 4]$

**15 A** Domain of  $f^{-1} =$  Range of  $f = (\sqrt{7}, \infty)$

$$x = \sqrt{2f^{-1}(x) + 3}$$

$$\therefore f^{-1}(x) = \frac{1}{2}(x^2 - 3)$$

- 16 B**  $5 - x = -2 \Rightarrow x = 7$   
 $5 - x = 3 \Rightarrow x = 2 \therefore$  Domain of  
 $f = (2, 7]$

- 17 A**  $g : R \setminus \{3\} \rightarrow R, g(x) = \frac{1}{x-3} + 2$   
Let  $x = \frac{1}{g^{-1}(x) - 3} + 2$   
 $g^{-1}(x) - 3 = \frac{1}{x-2}$   
 $g^{-1}(x) = \frac{1}{x-2} + 3$   
 $x \neq 2$   
 $\therefore \text{dom } g^{-1}(x) = R \setminus \{2\}$

**18 B**

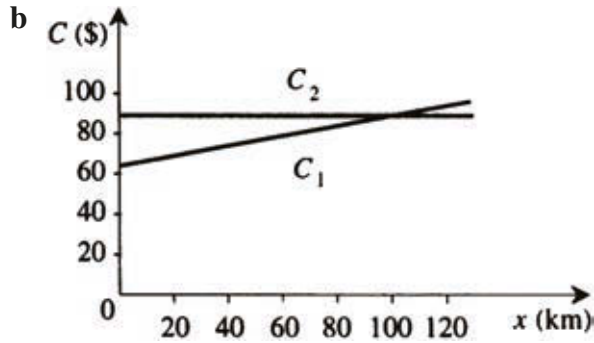
- 19 B** Asymptotes of  $y(x)$  occur at  
 $x + 1 = 0$   
 $\therefore x = -1$   
And at  $y = -2$   
 $\therefore$  Asymptotes of  $y^{-1}(x)$  occur at:

$$y = -1 \text{ and } x = -2$$

- 20 C** Asymptotes of  $\frac{-2}{(x+3)^4} - 5$  occur  
when  $x + 3 = 0$   
 $\therefore x = -3$   
And when  $y = -5$
- 21 A**  $f : [0, \infty) \rightarrow R, f(x) = (x-2)^2 f(x)$   
does not have an inverse function as  
it is not a one to one function.
- 22 D** Note that the graph of  $y = \frac{1}{x^4}$  will  
be like that of  $y = \frac{1}{x^2}$ , but 'steeper'.  
Looking at the alternatives, D  
stands out: its Domain runs from  
negative to positive numbers with  
0 removed. for numbers close to  
0, the value of  $y$  will be very large.  
As  $x \rightarrow 0, f(x) \rightarrow \infty$ . Its range is  
actually  $[1, \infty)$ . (Checking each of  
the remaining alternatives shows that  
the range is correct in each case.)

## Solutions to extended-response questions

1 a  $C_1(x) = 0.25x + 64$   
 $C_2(x) = 89$



c From the graph or using the inequality  
 $0.25x + 64 > 89$   
 $0.25x > 25$   
 $x > 100$

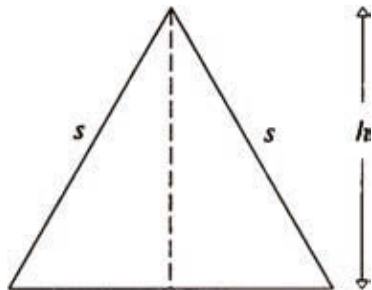
Method 2 is cheaper than Method 1 if the distance travelled is greater than 100 km.

From this it can be seen that Method 2 is cheaper than Method 1 if the distance travelled is more than 100 km.

2 a Area of each face =  $x^2$   
 $\therefore$  the total surface area,  $S = 6x^2$

b The volume,  $V = x^3$   
 $\therefore x = \sqrt[3]{\frac{V}{6}}$   
 and  $S = 6\sqrt[3]{\frac{V}{6}}$

3 a The triangle is equilateral.  
 Area  $A = \frac{1}{2}s^2 \sin 60^\circ$  (Area of triangle =  $\frac{1}{2}bc \sin A$ )  
 $= \frac{\sqrt{3}}{4}s^2 \dots \langle 1 \rangle$



**b** By Pythagoras' Theorem,  $h^2 = s^2 - \frac{s^2}{4}$

$$= \frac{3s^2}{4}$$

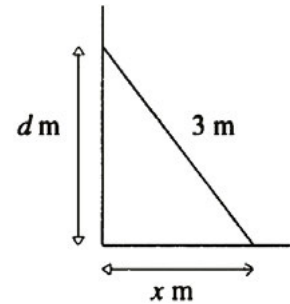
$$\therefore h = \frac{\sqrt{3}s}{2} \quad \text{and} \quad s = \frac{2h}{\sqrt{3}}$$

by (1)  $A = \frac{\sqrt{3}}{4} \left( \frac{2h}{\sqrt{3}} \right)^2 = \frac{\sqrt{3}}{4} \times \frac{4h^2}{3} = \frac{\sqrt{3}h^2}{3}$

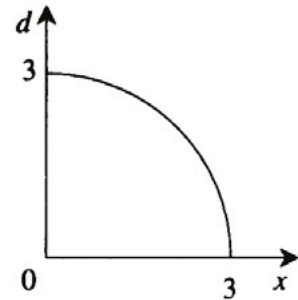
**4 a** By Pythagoras' Theorem  $d^2 = 9 - x^2$

$$\therefore d = \sqrt{9 - x^2}$$

**b** maximal Domain =  $[0, 3]$   
 The range of the function is  $[0, 3]$



**5** Let  $d$  km be the distance travelled.  
 The time taken for journey travelling at 80 km per hour  
 $= \frac{d}{2} \div 80 = \frac{d}{160}$



The time taken for journey travelling at  $x$  km per hour  
 $= \frac{d}{2} \div x = \frac{d}{2x}$

$$\therefore \text{Total time taken} = \frac{d}{160} + \frac{d}{2x} = \frac{d}{2} \left( \frac{1}{80} + \frac{1}{x} \right) = \frac{d}{2} \left( \frac{x + 80}{80x} \right)$$

Average speed =  $\frac{\text{distance travelled}}{\text{total time taken}}$

$$\therefore S(x) = d \div \frac{d}{2} \left( \frac{x + 80}{80x} \right)$$

$$= d \times \frac{2}{d} \times \frac{80x}{x + 80}$$

$$= \frac{160x}{x + 80}$$

Domain of  $S$  is  $[0, \infty)$

6 Volume of cylinder =  $\pi r^2 h$

a The diameter has length 12 cm.

By Pythagoras' Theorem

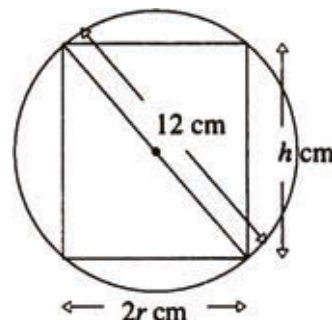
$$12^2 = h^2 + 4r^2 \dots \langle 1 \rangle$$

$$\therefore r^2 = \frac{12^2 - h^2}{4}$$

$$V_1(h) = \frac{\pi}{4}(144 - h^2)h$$

$$= \pi \left( 36 - \frac{h^2}{4} \right) h$$

As  $V_1 > 0$ ,  $h > 0$  and  $r > 0$  Domain of  $V_1 = (0, 12)$



b by  $\langle 1 \rangle$

$$h^2 = 144 - 4r^2$$

$$\therefore h = \sqrt{144 - 4r^2} = 2\sqrt{36 - r^2}$$

$$\therefore V_2(r) = \pi r^2 \times 2\sqrt{36 - r^2}$$

$$= 2\pi r^2 \sqrt{36 - r^2}$$

Domain of  $V_2 = (0, 6)$

	Domain	range
7 a $f$	$R$	$R$
$g$	$R$	$R$

$\text{ran } f = \text{dom } g$   $g$  of exists,

$$g \circ f(x) = g(x + 1) = 2 + (1 + x)^3$$

b  $g \circ f$  is a one-to-one function

$\therefore (g \circ f)^{-1}$  is defined,

Solve the equation  $g \circ f(x) = 10$

$$2 + (1 + x)^3 = 10$$

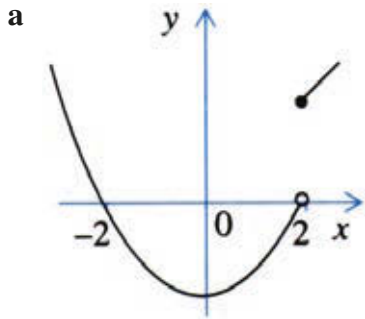
$$\therefore (1 + x)^3 = 8$$

$$\therefore 1 + x = 2$$

$$\therefore x = 1$$

$\therefore (g \circ f)^{-1}$  is defined,  $(g \circ f)^{-1}(10) = 1$

$$8 \quad f(x) = \begin{cases} x^2 - 4 & \text{if } x \in (-\infty, 2) \\ x & \text{if } x \in [2, \infty) \end{cases}$$



**b i**  $f(-1) = 1 - 4 = -3$  as  $-1 \in (-\infty, 2)$

**ii**  $f(3) = 3$  as  $3 \in [2, \infty)$

**c**  $S = (-\infty, 0]$  as  $f$  is one to one for this interval. and  $-1 \in S$ .

**d**  $h(x) = 2x$ , then  $f(h(x)) = f(2x)$

$$f(2x) = \begin{cases} (2x)^2 - 4 & \text{if } 2x \in (-\infty, 2) \\ 2x & \text{if } 2x \in [2, \infty) \end{cases}$$

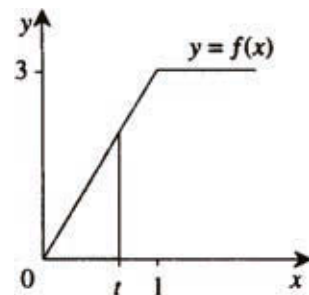
$$\text{Therefore } f \circ h(x) = \begin{cases} 4x^2 - 4 & \text{if } x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$$

$$\text{Now } h \circ f(x) = h \left( \begin{cases} x^2 - 4 & \text{if } x \in (-\infty, 2) \\ x & \text{if } x \in [2, \infty) \end{cases} \right)$$

$$h \circ f(x) = \begin{cases} 2x^2 - 8 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$$

**9** For  $0 \leq t \leq 1$

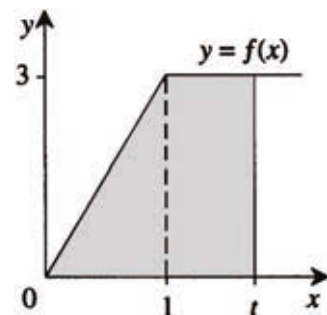
$$\begin{aligned} \text{Area} &= \frac{1}{2} \times t \times 3t \\ &= \frac{3}{2}t^2 \end{aligned}$$



For  $t > 1$

Area = area of triangle  $\Delta$  + area of rectangle  $\square$

$$\begin{aligned} &= \frac{1}{2} \times 1 \times 3 + 3(t-1) \\ &= \frac{3}{2} + 3t - 3 \\ &= 3t - \frac{3}{2} \end{aligned}$$





$$A(t) = \begin{cases} \frac{3}{2}t^2 & \text{for } 0 \leq t \leq 1 \\ 3t - \frac{3}{2} & \text{for } t > 1 \end{cases}$$

Domain of  $A = [0, \infty)$

Range of  $A = [0, \infty)$

**10 a** Let  $x = \frac{ay + b}{cy + d}$

$$\therefore x(cy + d) = ay + b$$

$$\text{and } xcy - ay = b - xd$$

$$y(xc - a) = b - xd$$

$$\therefore y = \frac{b - xd}{xc - a}$$

$$\text{Hence } f^{-1} : \mathbb{R} \setminus \left\{ \frac{a}{c} \right\} \rightarrow \mathbb{R}, f^{-1}(x) = \frac{b - xd}{xc - a}$$

$$\text{For the range of } f \text{ note: } f(x) = \frac{ax + b}{cx + d} = \frac{a}{c} + \frac{cb - da}{c(cx + d)} \text{ (by division)}$$

$$\therefore \text{range of } f = \mathbb{R} \setminus \left\{ \frac{a}{c} \right\}$$

$$\text{and Domain of } f^{-1} = \text{range of } f = \mathbb{R} \setminus \left\{ \frac{a}{c} \right\}$$

$$\text{range of } f^{-1} = \text{Domain of } f = \mathbb{R} \setminus \left\{ \frac{-d}{c} \right\}$$

**b i** For  $f(x) = \frac{3x + 2}{3x + 1}$   
 $a = 3, b = 2, c = 3, d = 1$

$$\text{and } f^{-1}(x) = \frac{2 - x}{3x - 3}; \text{ Domain of } f^{-1} = \mathbb{R} \setminus \{1\}$$

**ii** For  $f(x) = \frac{3x + 2}{2x - 3}$   
 $a = 3, b = 2, c = 2, d = -3$

$$\text{and } f^{-1}(x) = \frac{3x + 2}{2x - 3}; \text{ Domain of } f^{-1} = \mathbb{R} \setminus \left\{ \frac{3}{2} \right\}$$

**iii** For  $f(x) = \frac{x - 1}{-x - 1}$   
 $f^{-1}(x) = \frac{x - 1}{-x - 1} = \frac{1 - x}{x + 1}; \text{ Domain of } f^{-1} = \mathbb{R} \setminus \{-1\}$

**iv** For  $f(x) = \frac{-x + 1}{x + 1}$   
 $f^{-1}(x) = \frac{1 - x}{x + 1}; \text{ Domain of } f^{-1} = \mathbb{R} \setminus \{-1\}$

**c** If  $f^{-1} = f$  then Domain of  $f^{-1} = \text{Domain of } f$

$$\therefore \frac{a}{c} = \frac{-d}{c} \text{ (we will assume } c \neq 0)$$

$$\begin{aligned} \therefore a &= -d \\ \text{As } f(x) &= \frac{ax+b}{cx+d} \\ \text{and } f^{-1}(x) &= \frac{b-xd}{xc-a} \\ \text{If } a &= -d \quad f^{-1}(x) = \frac{ax+b}{cx+d} = f(x) \\ \therefore \text{For } c \neq 0 \quad f^{-1} &= f \Leftrightarrow a = -d \end{aligned}$$

**11 a i**  $YB = r$  cm (sides of square)

**ii**  $ZB = r$  cm (sides of square)

**iii**  $AZ = (x - r)$  cm

**iv**  $CY = (3 - r)$  cm

**b**  $CY = CX = 3 - r$  (tangents from a point)

$AX = AZ = x - r$  (tangents from a point)

Therefore  $AC = AX + XC = x - r + 3 - r = x + 3 - 2r$

Using Pythagoras' Theorem for triangle  $ABC$

$$x^2 + 9 = (x + 3 - 2r)^2$$

$$\text{i.e. } x^2 + 9 = (x + 3)^2 - 4r(x + 3) + 4r^2$$

$$\therefore x^2 + 9 = x^2 + 6x + 9 - 4rx - 12r + 4r^2$$

$$\therefore 0 = 6x - 4rx - 12r + 4r^2$$

$$\therefore 0 = 2r^2 - 2r(x + 3) + 3x$$

$$\begin{aligned} \therefore r &= \frac{2(x+3) \pm \sqrt{4(x+3)^2 - 24x}}{4} \\ &= \frac{2x+6 \pm \sqrt{4(x^2+6x+9) - 24x}}{4} \\ &= \frac{2x+6 \pm \sqrt{4x^2+36}}{4} \\ &= \frac{x+3 \pm \sqrt{x^2+9}}{2} \end{aligned}$$

$$\text{But } r < \frac{x+3}{2}$$

$$\therefore r = \frac{x+3 - \sqrt{x^2+9}}{2}$$

When  $x = 4$ ,

**c i** 
$$r = \frac{7 - \sqrt{25}}{2}$$

i.e.  $r = 1$

ii When  $r = \frac{1}{2}$

$$\frac{1}{2} = \frac{(x+3) - \sqrt{x^2+9}}{2}$$

$$\therefore -2 - x = -\sqrt{x^2+9}$$

$$\therefore 4 + 4x + x^2 = x^2 + 9$$

$$\therefore 4x = 5$$

$$x = \frac{5}{4} \text{ (Note this must be tested because of squaring)}$$

12  $f(x) = \frac{px+q}{x+r}$   $x \in R \setminus \{-r, r\}$  for  $x \in R \setminus \{-r, r\}$

a  $f(x) = f(-x)$

implies

$$\frac{px+q}{x+r} = \frac{-px+q}{-x+r}$$

$$\therefore (-x+r)(px+q) = (-px+q)(x+r)$$

$$\therefore -px^2 - qx + pxr + qr = -px^2 - pxr + qx + qr$$

$$\therefore 2pxr = 2qx$$

$$\therefore pr = q$$

$$\therefore f(x) = \frac{px+pr}{x+r}$$

$$\therefore f(x) = p$$

b  $f(-x) = -f(x)$

implies

$$\frac{-px+q}{-x+r} = \frac{-px-q}{x+r}$$

$$\therefore -px^2 + qx - prx + qr = px^2 + qx - pxr - qr$$

$$\therefore 2px^2 - 2qr = 0$$

$$\text{i.e. } px^2 = qr$$

$$\therefore p = \frac{qr}{x^2} \text{ since } x \neq 0.$$

Substitute for  $p$  in  $f(x) = \frac{px+q}{x+r}$ :

$$\begin{aligned}
 f(x) &= \frac{\frac{qr}{x} + q}{x+r} \\
 &= \frac{qr + qx}{x(x+r)} \\
 &= \frac{q(x+r)}{x(x+r)} \\
 &= \frac{q}{x} \text{ (make that } x \neq -r)
 \end{aligned}$$

**c i** If  $p = 3$ ,  $q = 8$  and  $r = -3$

$$f(x) = \frac{3x+8}{x-3}$$

$$\text{Consider } x = \frac{3y+8}{y-3}$$

$$yx - 3x = 3y + 8$$

$$\therefore yx - 3y = 3x + 8$$

$$\therefore y(x-3) = 3x+8$$

$$\therefore y = \frac{3x+8}{x-3}$$

$$f(x) \text{ Hence } f^{-1}(x) = \frac{3x+8}{x-3}$$

$$\text{Domain of } f^{-1} = R \setminus \{3\}$$

**ii**  $x = \frac{3x+8}{x-3}$

$$3x+8 = x^2 - 3x$$

$$0 = x^2 - 6x - 8$$

$$\therefore x = \frac{6 \pm \sqrt{36+32}}{2}$$

$$= \frac{6 \pm 2\sqrt{9+8}}{2}$$

$$= 3 \pm \sqrt{17}$$

**13 a**  $f : R \setminus \{1\} \rightarrow R, f(x) = \frac{x+1}{x-1}$

Note: For this function  $f = f^{-1}$  from Question 10.

**i**  $f(2) = \frac{2+1}{2-1} = 3$

$$f(f(2)) = f(3) = \frac{3+1}{3-1} = 2$$

$$f(f(f(2))) = f(2) = 3$$

ii  $f(f(x)) = x$  for all  $x$

$$\begin{aligned} \mathbf{b} \quad f : R \setminus \{-1\} &\rightarrow R, & f(x) &= \frac{x-3}{x+1} \\ f(f(x)) &= f\left(\frac{x-3}{x+1}\right) \\ &= \frac{\frac{x-3}{x+1} - 3}{\frac{x-3}{x+1} + 1} \\ &= \frac{x-3-3x-3}{x-3+x+1} \\ &= \frac{-x-3}{x-1} \\ f(f(f(x))) &= f\left(\frac{-x-3}{x-1}\right) \\ &= \frac{\frac{-x-3}{x-1} - 3}{\frac{-x-3}{x-1} + 1} \\ &= \frac{-x-3-3x+3}{-x-3+x-1} \\ &= \frac{-4x}{-4} \\ &= x \end{aligned}$$

$$\text{i.e. : } f(fx) = f^{-1}(x)$$

# Chapter 2 – Coordinate geometry and matrices

## Solutions to Exercise 2A

**1 a**  $3x - 4 = 2x + 6$   
 $x = 10$

**b**  $8x - 4 = 3x + 1$   
 $5x = 5$   
 $x = 1$

**c**  $3(2 - x) - 4(3 - 2x) = 14$   
 $6 - 3x - 12 + 8x = 14$   
 $5x - 20 = 0$   
 $x = 4$

**d**  $\frac{3x}{4} - 4 = 17$   
 $\frac{x}{4} = 7$   
 $x = 28$

**e**  $6 - 3y = 5y - 62$   
 $8y = 68$   
 $y = \frac{17}{2}$

**f**  $\frac{2}{3x-1} = \frac{3}{7}$   
 $14 = 9x - 3$   
 $x = \frac{17}{9}$

**g**  $\frac{2x-1}{3} = \frac{x+1}{4}$   
 $8x - 4 = 3x + 3$   
 $5x = 7$   
 $x = \frac{7}{5}$

**h**  $\frac{2(x-1)}{3} - \frac{(x+4)}{2} = \frac{5}{6}$   
 $4x - 4 - 3x - 12 = 5$   
 $x = 21$

**i**  $4y - \frac{3y+4}{2} + \frac{1}{3} = \frac{5(4-y)}{3}$   
 $24y - 9y - 12 + 2 = 40 - 10y$   
 $25y = 50$   
 $y = 2$

**j**  $\frac{x+1}{2x-1} = \frac{3}{4}$   
 $4x + 4 = 6x - 3$   
 $2x = 7$   
 $x = \frac{7}{2}$

**2 a**  $x - 4 = y \dots (1)$   
 $4y - 2x = 8 \dots (2)$   
 $(2) + 2 \times (1) \Rightarrow 4y - 8 = 8 + 2y$   
 $2y = 16$   
 $y = 8$   
 $\Rightarrow x = 12$

**b**  $9x + 4y = 13 \dots (1)$   
 $2x + y = 2 \dots (2)$   
 $(1) - 4 \times (2) \Rightarrow x = 5$   
 $\Rightarrow 10 + y = 2$   
 $y = -8$

**c**  $7x - 3y = 18 \dots (1)$   
 $22x + 5y = 11 \dots (2)$   
 $5 \times (1) + 3 \times (2) \Rightarrow 41x = 123$   
 $x = 3$

$$\Rightarrow 6 + 5y = 11$$

$$y = 1$$

**d**  $5x + 3y = 13 \dots (1)$   
 $7x + 2y = 16 \dots (2)$   
 $3 \times (2) - 2 \times (1) \Rightarrow 11x = 22$   
 $x = 2$   
 $\Rightarrow 10 + 3y = 13$   
 $y = 1$

**e**  $19x + 17y = 0 \dots (1)$   
 $2x - y = 53 \dots (2)$   
From (1)  $y = \frac{-19}{17}x$   
 $\Rightarrow 2x + \frac{19}{17}x = 53$   
 $53x = 17 \times 53$   
 $x = 17$   
 $\Rightarrow 34 - y = 53$   
 $y = -19$

**f**  $\frac{x}{5} + \frac{y}{2} = 5 \dots (1)$   
 $x - y = 4 \dots (2)$   
 $(2) + 2 \times (1) \Rightarrow \frac{7x}{5} = 14$   
 $x = 10$   
 $\Rightarrow 10 - y = 4$   
 $y = 6$

**3**  $l = w + 4 \dots (1)$   
 $2(l - 5) + 2(w - 2) = 18 \dots (2)$   
Substitute from (1) into (2)  
 $w - 1 + w - 2 = 9$   
 $w = 6 \text{ cm}$   
 $\Rightarrow l = 10 \text{ cm}$

**4** Let  $g$  represent the number of goals scored, and  $t$  the number of throws.  
 $t_0 = t_j \dots (1)$   
 $g_0 = 2g_j \dots (2)$   
 $t_j + 2g_j = 11 \dots (3)$   
 $t_j + 2g_0 = 19 \dots (4)$

Subtract Equation(3) from Equation(4)

$$\Rightarrow 2g_0 - 2g_j = 8$$

Substitute from Equation (2)

$$4g_j - 2g_j = 8$$

$$g_j = 4$$

$$g_0 = 8$$

John scored 4 goals and David scored 8.

**5 a**  $w = 800 + 20n$

**b**  $w = 800 + 20(30)$   
 $\$w = \$1400$

**c**  $1620 = 800 + 20n$   
 $20n = 820$   
 $n = 41 \text{ units}$

**6 a**  $V = 250 + 15t$

**b**  $V = 250 + 15(60)$   
 $V = 1150\text{L}$

**c**  $5000 = 250 + 15t$   
 $t = \frac{4750}{15} = \frac{950}{3} \text{ min}$   
 $t = 5\text{h } 16 \text{ min } 40\text{s}$

**7 a**  $V = 10000 - 10t$

**b**  $V = 10000 - 10(60)$   
 $= 9400 \text{ L}$

**c**  $0 = 10000 - 10t$   
 $t = 1000 \text{ min}$   
 $t = 16 \text{ h } 40 \text{ min}$

**8**  $\frac{x}{240} + \frac{x}{320} = \frac{35}{60}$

$$\frac{7x}{12} = 80 \times \frac{7}{12}$$
$$x = 80 \text{ km}$$

**9**  $\frac{x}{48} + \frac{x}{4.8} = 24 - 2$

$$11x = 22 \times 48$$
$$x = 96 \text{ km}$$

**10 a**  $C = 100 + 25t$

**b i**  $C = 100 + 252$

$$\$C = \$150$$

**ii**  $C = 100 + 25(2.5)$

$$\$C = \$162.50$$

**c i**  $375 = 100 + 25t$

$$t = 11 \text{ h}$$

**ii**  $400 = 100 + 25t$

$$t = 12 \text{ h}$$



## Solutions to Exercise 2B

**1 a**  $ax + n = m$

$$x = \frac{m - n}{a}$$

**b**  $ax + b = bx$

$$x = \frac{b}{b - a}$$

**c**  $\frac{ax}{b} + c = 0$

$$x = \frac{-bc}{a}$$

**d**  $px = qx + 5$

$$x = \frac{5}{p - q}$$

**e**  $mx + n = nx - m$

$$(m - n)x = -(n + m)$$

$$x = \frac{n + m}{n - m}$$

**f**  $\frac{1}{x + a} = \frac{b}{x}$

$$x = b(x + a)$$

$$(1 - b)x = ba$$

$$x = \frac{ba}{1 - b}$$

**g**  $\frac{b}{x - a} = \frac{2b}{x + a}$

$$bx + ab = 2bx - 2ab$$

$$bx = 3ab$$

$$x = 3a, b \neq 0$$

**h**  $\frac{x}{m} + n = \frac{x}{n} + m$

$$nx + n^2m = mx + m^2n$$

$$(n - m)x = nm(m - n)$$

$$x = -mn$$

**i**  $-b(ax + b) = a(bx - a)$

$$-bax - b^2 = abx - a^2$$

$$2abx = a^2 - b^2$$

$$x = \frac{a^2 - b^2}{2ab}$$

**j**  $p^2(1 - x) - 2pqx = q^2(1 + x)$

$$p^2 - (p^2 + 2pq)x = q^2 + q^2x$$

$$p^2 - q^2 = (p + q)^2x$$

$$x = \frac{p - q}{p + q}$$

**k**  $bx - ab = ax + 2b$

$$(b - a)x = 3ab$$

$$x = \frac{3ab}{b - a}$$

**l**  $\frac{x}{a - b} + \frac{2x}{a + b} = \frac{1}{a^2 - b^2}$

$$x(a + b) + 2x(a - b) = 1$$

$$x(a + b + 2a - 2b) = 1$$

$$x = \frac{1}{3a - b}$$

**m**  $\frac{p - qx}{t} + p = \frac{qx - t}{p}$

$$p^2 - qpx + p^2t = qtx - t^2$$

$$qtx + qpx = p^2 + p^2t + t^2$$

$$x = \frac{p^2 + p^2t + t^2}{q(t + p)}$$

**n**  $\frac{1}{x + a} + \frac{1}{x + 2a} = \frac{2}{x + 3a}$   
 $(x + 2a)(x + 3a) + (x + a)(x + 3a)$   
 $= 2(x + a)(x + 2a)$

$$2x^2 + 9ax + 9a^2 = 2x^2 + 6ax + 4a^2$$

$$3ax = -5a^2$$

$$x = \frac{-5a}{3}$$

**2 a**  $ax + y = c \dots (1)$

$$x + by = d \dots (2)$$

$$(1) - a \times (2)$$

$$\Rightarrow y(1 - ab) = c - ad$$

$$y = \frac{c - ad}{1 - ab}$$

$$\text{Equation (2)} - b \times \text{Equation (1)}$$

$$\Rightarrow x(1 - ab) = d - bc$$

$$x = \frac{d - bc}{1 - ab}$$

**b**  $ax - by = a^2 \dots (1)$

$$bx - ay = b^2 \dots (2)$$

$$b \times \text{Equation (1)} - a \times \text{Equation (2)}$$

$$\Rightarrow (-b^2 + a^2)y = a^2b - b^2a$$

$$y = \frac{ab(a - b)}{a^2 - b^2}$$

$$y = \frac{ab}{a + b}$$

$$a \times \text{Equation (1)} - b \times \text{Equation (2)}$$

$$\Rightarrow (a^2 - b^2)x = a^3 - b^3$$

$$x = \frac{a^3 - b^3}{a^2 - b^2}$$

$$x = \frac{a^2 + ab + b^2}{a + b}$$

**c**  $ax + by = t \dots (1)$

$$ax - by = s \dots (2)$$

$$(1) + (2) \Rightarrow 2ax = t + s$$

$$x = \frac{t + s}{2a}$$

$$\text{Equation (1)} - \text{Equation (2)}$$

$$\Rightarrow 2by = t - s$$

$$y = \frac{t - s}{2b}$$

**d**  $ax + by = a^2 + 2ab - b^2 \dots (1)$

$$bx + ay = a^2 + b^2 \dots (2)$$

$$a \times (1) - b \times (2)$$

$$\Rightarrow (a^2 - b^2)x$$

$$= a^3 + 2a^2b - ab^2 - a^2b - b^3$$

$$x = \frac{a^3 + a^2b - ab^2 - b^3}{a^2 - b^2}$$

$$x = \frac{(a + b)(a^2 - b^2)}{a^2 - b^2}$$

$$x = a + b$$

$$\text{Substitute into (2)}$$

$$\Rightarrow b(a + b) + ay = a^2 + b^2$$

$$ay = a^2 + b^2 - ba - b^2$$

$$y = a - b$$

**e**  $(a + b)x + cy = bc \dots (1)$

$$(b + c)y + ax = -ab \dots (2)$$

$$a \times (1) - (a + b) \times (2)$$

$$\Rightarrow (ac - ab - b^2 - ac - bc)y$$

$$= abc + a^2b + b^2 \quad y = \frac{ab(c + a + b)}{-b(a + b + c)}$$

$$y = -a$$

$$\text{Substitute into (2)}$$

$$\Rightarrow (-ab - ac) + ax = -ab$$

$$x = -b + b + c$$

$$x = c$$

**f**  $3(x - a) - 2(y + a) = 5 - 4a \dots (1)$

$$\Rightarrow 3x - 2y = 5 + a$$

$$2(x + a) + 3(y - a) = 4a - 1 \dots (2)$$

$$\Rightarrow 2x + 3y = 5a - 1$$

$$3 \times (1) + 2 \times (2)$$

$$\Rightarrow 13x = 15 + 3a + 10a - 2$$

$$13x = 13 + 13a$$

$$x = 1 + a$$

$$\text{Substitute into (1)}$$

$$\Rightarrow 3 + 3a - 2yt = 5 + a$$

$$y = -1 + a$$

$$y = a - 1$$

$$\begin{aligned} \mathbf{3 a} \quad s &= a(2a + 1) \\ s &= 2a^2 + a \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad h &= a(2 + h) \\ h &= 2a + ah \\ (1 - a)h &= 2a \\ h &= \frac{2a}{1 - a} \\ s &= a\left(\frac{2a}{1 - a}\right) \\ s &= \frac{2a^2}{1 - a} \\ h &= \frac{1}{1 + a} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad as &= a + \frac{1}{1 + a} \\ s &= 1 + \frac{1}{a + a^2} \\ s &= 1 + \frac{1}{a + a^2} \\ s &= \frac{a^2 + a + 1}{a^2 + a} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad ah &= a + h \\ (a - 1)h &= a \\ h &= \frac{a}{a - 1} \\ as &= s + \frac{a}{a - 1} \\ (a - 1)s &= \frac{a}{a - 1} \\ s &= \frac{a}{(a - 1)^2} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad s &= (3a^2)^2 + a(3a^2) \\ s &= 9a^4 + 3a^3 \\ s &= 3a^3(3a + 1) \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad as &= a + 2(a - s) \\ as &= a + 2a - 2s \end{aligned}$$

$$\begin{aligned} (a + 2)s &= 3a \\ s &= \frac{3a}{a + 2} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad s &= 2 + a\left(a - \frac{1}{a}\right) + \left(a - \frac{1}{a}\right)^2 \\ s &= 2 + a^2 - 1 + a^2 - 2 + \frac{1}{a^2} \\ s &= 2a^2 - 1 + \frac{1}{a^2} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad 3s - ah &= a^2 \dots (1) \\ as + 2h &= 3a \dots (2) \\ 2 \times \text{Eq}(1) + a \times \text{Eq}(2) &\Rightarrow (6 + a^2)s = 5a^2 \\ s &= \frac{5a^2}{6 + a^2} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad ax + by &= p \dots (1) \\ bx - ay &= q \dots (2) \\ a \times (1) + b \times (2) &\Rightarrow (a^2 + b^2)x = pa + bq \\ x &= \frac{ap + bq}{a^2 + b^2} \\ b \times (1) - a \times (2) &\Rightarrow (b^2 + a^2)y = bp - aq \\ y &= \frac{bp - aq}{a^2 + b^2} \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad bx + ay &= ab \dots (1) \\ ax + by &= ab \dots (2) \\ a \times (1) - b \times (2) &\Rightarrow (a^2 - b^2)y = ab(a - b) \\ y &= \frac{ab}{a + b} \\ b \times (1) - a \times (2) &\Rightarrow (b^2 - a^2)x = ab(b - a) \\ x &= \frac{ab}{a + b} \end{aligned}$$

## Solutions to Exercise 2C

1 a  $\sqrt{205}$

b  $(1, -\frac{1}{2})$

c  $-\frac{13}{6}$

d  $13x + 6y = 10$

e  $13x + 6y = 43$

f  $13y - 6x = -\frac{25}{2}$

2 a  $(3, \frac{15}{2})$

b  $(\frac{-5}{2}, -2)$

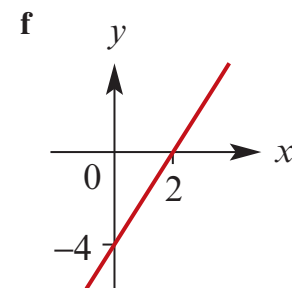
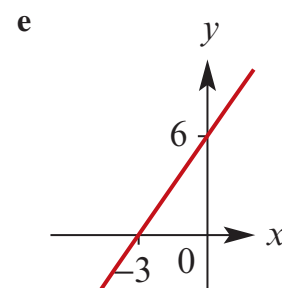
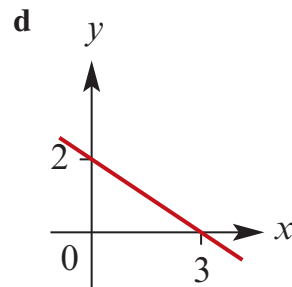
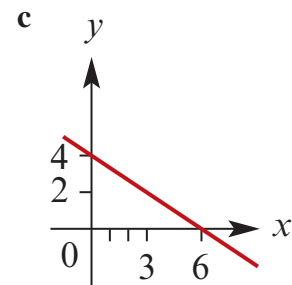
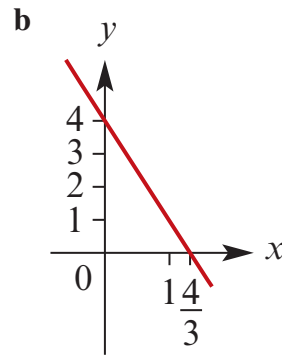
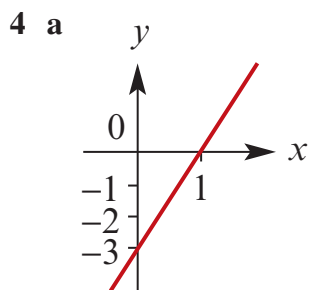
c  $(\frac{3}{2}, \frac{1}{2})$

3 a (4, 7)

b (5, -2)

c (2, 19)

d (-2, -9)



5 a  $y - 2 = 2(x - 4)$   
 $y = 2x - 6$

b  $y - 4 = -3(x + 3)$   
 $y = -3x - 5$   
 $m = \frac{4}{3}$

c  $y - 3 = \frac{4}{3}(x - 1)$   
 $y = \frac{4}{3}x + \frac{5}{3}$  or  $3y - 4x = 5$

d  $m = 2$   
 $y - 5 = 2(x - 2)$   
 $y = 2x + 1$

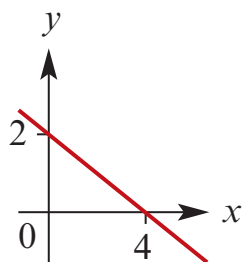
6 a  $\frac{\frac{x}{-3} + \frac{y}{2}}{\frac{y}{2} - \frac{x}{3}} = 1$

b  $\frac{x}{4} + \frac{y}{6} = 1$

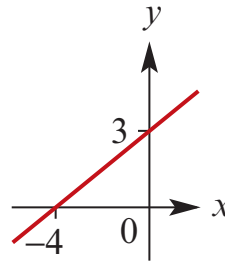
c  $\frac{\frac{x}{-4} + \frac{y}{-3}}{\left(-\frac{x}{4} - \frac{y}{3} = 1\right)} = 1$

d  $\frac{\frac{x}{6} + \frac{y}{-2}}{\frac{x}{6} - \frac{y}{2}} = 1$

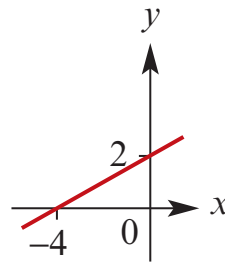
7 a  $\frac{x}{4} + \frac{y}{2} = 1$



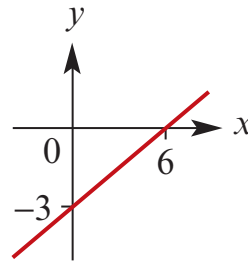
b  $\frac{y}{3} - \frac{x}{4} = 1$



c  $\frac{y}{2} - \frac{x}{4} = 1$



d  $\frac{x}{6} - \frac{y}{3} = 1$



8 (600,35) (800,46)

$$m = \frac{11}{200}$$

$$C - 35 = \frac{11}{200}(n - 600)$$

$$C = \frac{11}{200}n + 2$$

$$C(1000) = \frac{11}{200}(1000) + 2$$

$$\$C = \$57$$

9 a (120,775) (160,975)

$$m = 5$$

$$C - 775 = 5(n - 120)$$

$$C = 5n + 175$$

**b** yes

**c** \$175

**10 a**  $d = \sqrt{1^2 + 2^2}$

$$= \sqrt{5} \approx 2.236$$

**b**  $d = \sqrt{1^2 + 1^2}$

$$= \sqrt{2} \approx 1.414$$

**c**  $d = \sqrt{5^2 + 2^2}$

$$= \sqrt{29} \approx 5.385$$

**d**  $d = \sqrt{2^2 + 18^2}$

$$= \sqrt{326}$$

$$= 2\sqrt{82} \approx 18.111$$

**e**  $d = \sqrt{4^2 + 2^2}$

$$= \sqrt{20}$$

$$= 2\sqrt{5} \approx 4.472$$

**f**  $d = \sqrt{3^2 + 4^2}$

$$= 5$$

**11 a i**  $y - 6 = 2(x + 1)$

$$y = 2x + 4$$

**ii**  $y - 6 = \frac{-1}{2}(x - 1)$

$$y = \frac{-1}{2}x + \frac{13}{2} \text{ or } 2y + x = 13$$

**b i**  $y - 3 = -2(x - 2)$

$$y = -2x + 7$$

**ii**  $y - 3 = \frac{1}{2}(x - 2)$

$$y = \frac{1}{2}x + 2$$

$$\text{or } 2y - x = 4$$

**12**  $(3, 3), m = \frac{-1}{-3/6} = 2$

$$y - 3 = 2(x - 3)$$

$$y = 2x - 3$$

**13**  $5 = \sqrt{3^2 + (y + 1)^2}$

$$25 = 9 + (y + 1)^2$$

$$y + 1 = \pm 4$$

$$y = -1 \pm 4$$

$$y = -5, 3$$

**14**  $10 = \sqrt{8^2 + (y - 6)^2}$

$$100 = 64 + (y - 6)^2$$

$$y - 6 = \pm 6$$

$$y = 6 \pm 6$$

$$y = 0, 12$$

**15**  $26 = \sqrt{10^2 + (y - 8)^2}$

$$676 = 100 + (y - 8)^2$$

$$y - 8 = \pm 576$$

$$y = 8 \pm 24$$

$$y = -16, 32$$

$$16 \text{ a i } y - 3 = \frac{-2}{5}(x + 1)$$

$$y = \frac{-2}{5}x + \frac{13}{5} \text{ or } 5y + 2x = 13$$

$$\text{ii } y - 3 = \frac{-4}{5}(x + 1)$$

$$y = \frac{-4}{5}x + \frac{11}{5} \text{ or } 5y + 4x = 11$$

$$\text{b i } y - 3 = \frac{5}{2}(x + 1)$$

$$y = \frac{5}{2}x + \frac{11}{2} \text{ or } 2y - 5x = 11$$

$$\text{ii } y - 3 = \frac{5}{4}(x + 1)$$

$$y = \frac{5}{4}x + \frac{17}{4} \text{ or } 4y - 5x = 17$$

$$17 \text{ a } m = \frac{6 - 1}{4 + 4} = \frac{5}{8}$$

$$\theta = \tan^{-1}\left(\frac{5}{8}\right) = 32.01^\circ$$

$$\text{b } m = \frac{-1}{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{2}\right) = 153.43^\circ$$

$$\text{c } m = \frac{3}{2}$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.31^\circ$$

$$\text{d } m = \frac{-10}{6}$$

$$\theta = \tan^{-1}\left(\frac{-5}{3}\right) = 120.96^\circ$$

$$18 \text{ } m_1 = 2$$

$$m_2 = -3$$

$$\theta_1 = 63.43^\circ$$

$$\theta_2 = 108.43^\circ$$

$$\alpha = \theta_2 - \theta_1 = 45^\circ$$

$$19 \text{ } \sqrt{(-2 - a)^2 + (-2)^2} = 2\sqrt{5^2 + 1^2}$$

square both sides

$$4 + 4a + a^2 + 4 = 4(26)$$

$$a^2 + 4a - 96 = 0$$

$$(a + 2)^2 - 100 = 0$$

$$a + 2 = \pm 10$$

$$a = -12, 8$$

$$20 \text{ a } \frac{5 - 7}{7 - 1} = \frac{-1}{3}$$

$$m = 3$$

midpoint = (4, 6)

$$y - 6 = 3(x - 4)$$

$$y = 3x - 6$$

**b**  $\overrightarrow{BC}$  has  $m = 1$

$$1y = x - 2$$

$$2y = 3x - 6$$

$$x = 2, y = 0$$

point of intersection: (2, 0)

**21**  $k = h + 1 \dots (1)$   
 $\sqrt{h^2 + (2 - k)^2} = 5 \dots (2)$   
 $\Rightarrow h^2 + (k - 2)^2 = 25$   
Substitute in (1)  
 $\Rightarrow h^2 + (h - 1)^2 = 25$   
 $2h^2 - 2h + 1 = 25$   
 $2h^2 - 2h - 24 = 0$   
 $h^2 - h - 12 = 0$   
 $(h + 3)(h - 4) = 0$   
 $h = -3, 4$   
Substitute in (1)  
 $\Rightarrow k = -2, 5$   
 $(h, k) = (-3, -2) \text{ or } (4, 5)$

**22**  $P = (3, 0), Q = (0, 2)$

**a**  $\overrightarrow{QR} : y - 2 = \frac{1}{2}x$   
 $y = \frac{1}{2}x + 2$   
if  $x = 2a$ ,  
 $y = a + 2$   
 $R = (2a, a + 2)$

**b**  $\overrightarrow{PR}$ : has  $m = \frac{a + 2}{2a - 3}$   
but  $m = -2$   
 $\therefore -2 = \frac{a + 2}{2a - 3}$   
 $6 - 4a = a + 2$   
 $4 = 5a$   
 $a = \frac{4}{5}$

**23 a**  $AB$  has gradient  $-3m$   
 $-3m = \frac{1 - 4}{1 + 1} = \frac{-3}{2}$   
 $m = \frac{1}{2}$

**b**  $AC : 1y = \frac{3}{2}x - \frac{1}{2}$   
 $BC : y - 4 = \frac{1}{2}x + \frac{1}{2}$   
 $2y = \frac{1}{2}x + \frac{9}{2}$   
 $1 - 20 = x - 5$   
 $x = 5$   
 $\Rightarrow 2y = \frac{5}{2} + \frac{9}{2} = 7$   
 $C = (5, 7)$

**c**  $AC = \sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$   
 $AB = \sqrt{2^2 + 3^2} = \sqrt{13}$   
 $AC = 2AB$   
*QED*

**24 a**  $BC$  has gradient  $= \frac{-1}{\text{grad}(AB)} = \frac{1}{3}$   
 $y - 8 = \frac{1}{3}(x - 2)$   
 $BC : y = \frac{1}{3}x + \frac{22}{3}$  or  $3y - x = 22$

**b**  $1y = \frac{1}{3}x + \frac{22}{3}$   
 $2y = x - 2$   
 $2 - 10 = \frac{2x}{3} + \frac{28}{3}$   
 $x = 14$   
 $\Rightarrow 2y = 12$   
 $C = (14, 12)$

**c** because it is a rectangle



$$\begin{aligned}
 D &= C - (B - A) \\
 &= (14, 12) - ((2, 8) - (4, 2)) \\
 &= (14, 12) - (-2, 6) \\
 &= (16, 6)
 \end{aligned}$$

**d** Area =  $AB \times BC$

$$\begin{aligned}
 &= \sqrt{2^2 + 6^2} \times \sqrt{12^2 + 4^2} \\
 &= \sqrt{40} \times \sqrt{160} \\
 &= 2\sqrt{10} \times 4\sqrt{10} \\
 &= 80 \text{ square units}
 \end{aligned}$$

**25 a** (2,3)

**b**  $BD$  has  $m = -5$

$$y - 3 = -5(x - 2)$$

$$y = -5x + 13$$

**c i**  $BC$  has  $m = \frac{1}{\text{grad}(AC)} = \frac{3}{2}$

$$y - 1 = \frac{3}{2}(x - 5)$$

$$y = \frac{3}{2}x - \frac{13}{2} \text{ or } 2y = 3x - 13$$

**ii**  $1y = \frac{3}{2}x - \frac{13}{2}$

$$2y = -5x + 13$$

$$1 - 20 = \frac{13x}{2} - \frac{39}{2}$$

$$x = 3$$

$$\Rightarrow 2y = -2$$

$$B = (3, -2)$$

**iii**  $D = A + (C - B)$

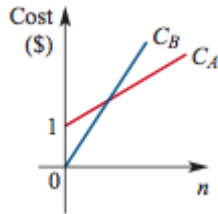
$$D = (-1, 5) + (5, 1) - (3, -2)$$

$$D = (1, 8)$$

## Solutions to Exercise 2D

1 a  $C_A = 0.4n + 1; C_B = 0.6n$

b The graphs are straight lines as shown here.



c  $C_A = C_B$

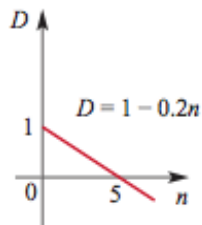
$$0.4n + 1 = 0.6n$$

$$0.2n = 1$$

$$n = 5$$

The charge is the same for 5 km.

d  $D = C_A - C_B = 1 - 0.2n$ , so the graph is a straight line with intercept  $(0, 1)$  and gradient  $-0.2$ .



$D$  gives the difference in charges of the two firms in terms of the distance travelled.

2 a Since the journey lasts 4 hours,  $4 - T$  hours are spent on country roads.

b i 90 km/h for  $T$  hours gives a distance of  $90T$  km.

ii 70 km/h for  $4 - T$  hours gives a distance of  $70(4 - T)$  km.

c i  $90T + 70(4 - T) = 300$

$$20T = 20$$

$$T = 1$$

ii 90 km on the freeway,  
 $70 \times 3 = 210$  km.

3 Let  $L = at + b$ , since a constant rate of decrease means the relation is linear.

a  $t = 20, L = 3000: 20a + b = 3000 \quad \dots 1$

$t = 35, L = 1200: 35a + b = 1200 \quad \dots 2$

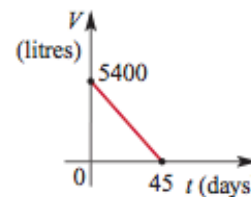
$2 - 1: 15a = -1800, \text{ so } a = -120$

Sub in 1:  $b = 3000 - 20(-120) = 5400$

$$L = -120t + 5400$$

b 5400 litres (at  $t = 0$  when it was filled)

c The tank will be empty when  $-120t + 5400 = 0 \Rightarrow t = 45$



d From c, the domain is  $[0, 45]$

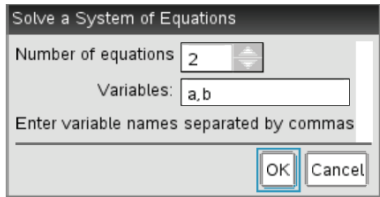
e 45 days

f The coefficient of  $t$ , i.e.  $-120$ , in the linear relation represents the rate of 'increase'. So the water is decreasing, or leaving the tank, at 120 litres/day

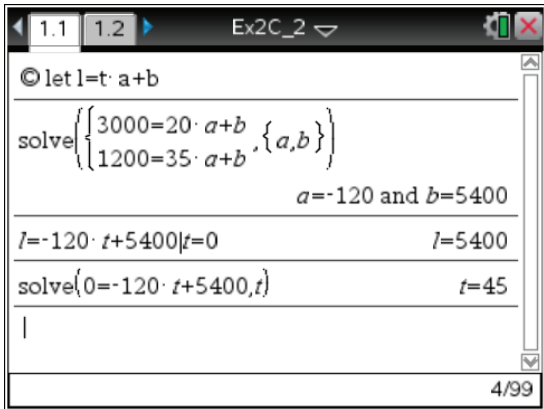
**Graphic calculator techniques for**

**question**

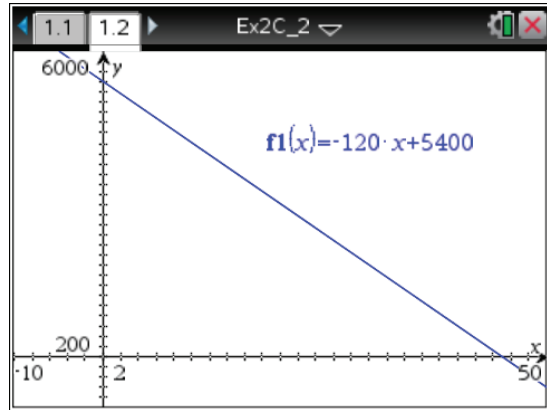
In a **Calculator** page insert the linear equation template (b>**Algebra**>**Solve System of Equations**>**Solve System of Equations**) and complete the dialogue box as shown.



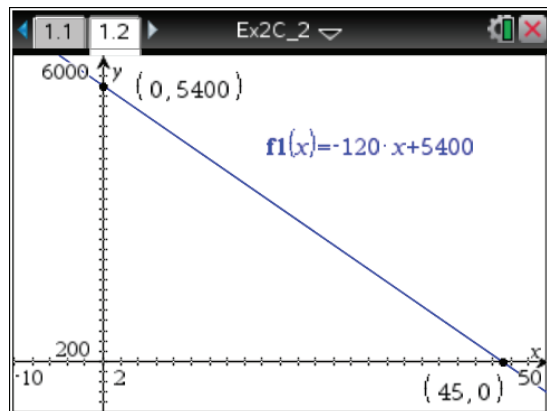
Defining the rule(b>**Actions**>**Define**) allows you to use the rule elsewhere by just typing in *l*. To find when the tank will be empty use the solve command (b>**Algebra**>**Solve**) with the equation equal to zero.



Insert a **Graphs** page (/ + **I**) and type in the rule  $f1(x) = -120x + 5400$   
Use b>**Window/Zoom**>**Window Settings** to set an appropriate window.



Note: the default digit display for **Graphs** is float3. You can increase this either in the settings or by placing the cursor over the designated value and pressing the + key. Hence 5.4E+3 now displays as 5400.

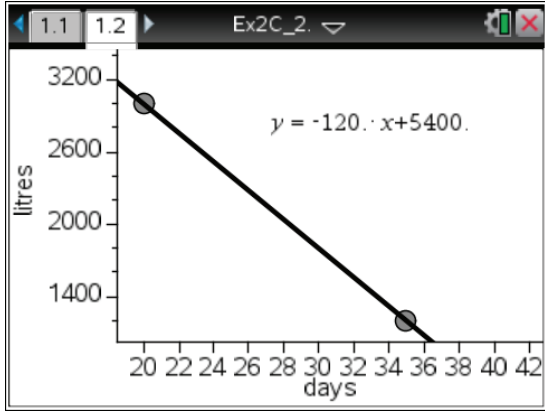


Alternative method as is current example  
In a Lists & Spreadsheet page enter the list data as shown.

	litres	days
1	3000	20
2	1200	35
3		
4		
5		
6		

Insert a **Data & Statistics** page (/ + I) and plot the data as shown.

A linear regression can be obtained using **b>Analyze>Regression>Show Linear (mx+b)**



$$4 \text{ a } y = \frac{22.5}{10}x$$

$$= \frac{9}{4}x$$

$$b \text{ } OA = \sqrt{10^2 + 22.5^2}$$

$$= 24.622 \text{ km}(24622 \text{ m})$$

$$c \text{ } y - 9 = \frac{9 - 22.5}{23 - 10}(x - 23)$$

$$y = -\frac{27}{26}x + \frac{855}{26}$$

d The midpoint of  $AB$  has coordinates  $\left(\frac{33}{2}, \frac{61}{4}\right)$  and the perpendicular has gradient  $\frac{26}{27}$ , so its equation is:

$$y - \frac{61}{4} = \frac{26}{27}\left(x - \frac{33}{2}\right)$$

As the port has  $x$ -coordinate 52, substitute to find the  $y$ -coordinate:

$$y - \frac{61}{4} = \frac{26}{27}\left(52 - \frac{33}{2}\right)$$

$$y = \frac{5339}{108}$$

$$5 \text{ a } i \text{ grad } AB = \frac{2 - 4}{2 - 1.5} = -4$$

$$ii \text{ grad } AD = \text{grad } BC = \frac{6 - 4}{6 - 1.5}$$

$$= \frac{4}{9}$$

$$b \text{ } i \text{ } y - 4 = \frac{4}{9}(x - 1.5)$$

$$y = \frac{4}{9}x + \frac{10}{3}$$

$$ii \text{ } y - 6 = d(x - 6)$$

$$y = -4x + 30$$

c Notice that  $A$  is '0.5 across and 2 down' from  $B$ , so  $D$  is '0.5 across and 2 down' from  $C$ , so its coordinates are  $(6.5, 4)$ . As  $B$  and  $D$  have the same  $y$ -coordinate, the equation of  $BD$  is  $y = 4$ .

The equation of  $AC$  is given by:

$$y - 6 = \frac{6 - 2}{6 - 2}(x - 6)$$

$$y = x$$

d The diagonals intersect at  $(4, 4)$ .

6 a  $M$  is the midpoint of  $AC$

Coordinates of  $M\left(\frac{5 + 9}{2}, \frac{0 + 10}{2}\right) = M(7, 5)$

Coordinates of  $N\left(\frac{13 + 9}{2}, \frac{0 + 10}{2}\right) = M(11, 5)$

b i Gradient of  $AC = \frac{10}{4} = \frac{5}{2}$

$\therefore$  Equation of line  $AC$  is

$$y - 0 = \frac{5}{2}(x - 5)$$

ii Gradient of  $BC = \frac{10}{-4} = -\frac{5}{2}$

$\therefore$  Equation of line  $BC$  is  
 $y - 0 = -\frac{5}{2}(x - 13)$

**iii** Gradient of  $MN = 0$   
 $\therefore$  Equation of line  $MN$  is  
 $y = 5$

**c** Line perpendicular to  $AC$  has  
gradient  $-\frac{2}{5}$   
 $\therefore$  equation of line passing through  $M$   
perpendicular to  $AC$  has equation  
 $y - 5 = -\frac{2}{5}(x - 7)$   
Line perpendicular to  $BC$  has

gradient  $\frac{2}{5}$   
 $\therefore$  equation of line passing through  $N$   
perpendicular to  $BC$  has equation

$y - 5 = \frac{2}{5}(x - 11)$

Intersect when

$-\frac{2}{5}(x - 7) = \frac{2}{5}(x - 11)$

$7 - x = x - 11$

$18 = 2x$

$x = 9$

When  $x = 9, y = \frac{21}{5}$

## Solutions to Exercise 2E

$$1 \text{ a } \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$$

$$1 \text{ b } \begin{bmatrix} 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$2 \text{ } 2\mathbf{A} + \mathbf{X} = \mathbf{B}$$

$$\mathbf{X} = \mathbf{B} - 2\mathbf{A}$$

$$\mathbf{X} = \mathbf{B} - 2\mathbf{A}$$

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} 1 & -4 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ -4 & 8 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -10 \\ 1 & -2 \end{bmatrix} \end{aligned}$$

$$3 \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 24 \end{bmatrix}$$

$$4 \text{ a } \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 6 & 4 \end{bmatrix}$$

$$4 \text{ b } \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$$

$$4 \text{ c } \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ 12 & 7 \end{bmatrix}$$

$$4 \text{ d } \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}$$

$$4 \text{ e } k \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2k & k \\ 3k & 2k \end{bmatrix}$$

4 f

$$\begin{aligned} 2 \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + 3 \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix} &= \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix} + \begin{bmatrix} -6 & -6 \\ 9 & 6 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -4 \\ 15 & 10 \end{bmatrix} \end{aligned}$$

5 g

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - 2 \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -4 & -4 \\ 6 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 5 \\ -3 & -2 \end{bmatrix} \end{aligned}$$

$$5 \text{ a } 2 \begin{bmatrix} 3 & 4 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ -6 & -6 \end{bmatrix}$$

$$5 \text{ b } 3 \begin{bmatrix} 0 & -4 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -12 \\ 15 & 3 \end{bmatrix}$$

5 c

$$\begin{aligned} 2 \begin{bmatrix} 3 & 4 \\ -3 & -3 \end{bmatrix} + 3 \begin{bmatrix} 0 & -4 \\ 5 & 1 \end{bmatrix} &= \begin{bmatrix} 6 & 8 \\ -6 & -6 \end{bmatrix} + \begin{bmatrix} 0 & -12 \\ 15 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -4 \\ 9 & -3 \end{bmatrix} \end{aligned}$$

5 d

$$\begin{aligned} 3 \begin{bmatrix} 0 & -4 \\ 5 & 1 \end{bmatrix} - 2 \begin{bmatrix} 3 & 4 \\ -3 & -3 \end{bmatrix} &= \begin{bmatrix} 0 & -12 \\ 15 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ -6 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -6 & -20 \\ 21 & 9 \end{bmatrix} \end{aligned}$$

$$6 \text{ a } \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 5 & -4 \end{bmatrix}$$

6 b

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} + 3 \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} -3 & -12 \\ 15 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -12 \\ 15 & -4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 2 \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 1 & 1 \end{bmatrix} \\
 & = \begin{bmatrix} 2 & 0 \\ 0 & -8 \end{bmatrix} - \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 1 & 1 \end{bmatrix} \\
 & = \begin{bmatrix} 3 & 0 \\ -4 & -7 \end{bmatrix}
 \end{aligned}$$

$$7 \quad 2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$$

$$\text{Therefore, } \mathbf{X} = \frac{1}{3}(2\mathbf{A} - \mathbf{B})$$

$$\text{Therefore, } \mathbf{X} = \begin{bmatrix} 2 & 3 \\ -\frac{4}{3} & -3 \end{bmatrix}$$

$$3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$$

$$\text{Therefore, } \mathbf{Y} = \frac{1}{2}(2\mathbf{B} - 3\mathbf{A})$$

$$\text{Therefore } \mathbf{Y} = \begin{bmatrix} -\frac{9}{2} & -8 \\ \frac{5}{2} & 7 \end{bmatrix}$$

$$8 \quad \mathbf{AX} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$$\mathbf{BX} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$\mathbf{IX} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\mathbf{IB} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 3 & -2 \\ -3 & 3 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 3 & -6 \\ -1 & 3 \end{bmatrix}$$

$$\mathbf{A}^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$$

$$\mathbf{B}^2 = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

## Solutions to Exercise 2F

**1 a**  $3x + 2y = 6 \dots (1)$

$$x - y = 7 \dots (2)$$

Multiply (2) by 2 and add to (1)

$$5x = 20$$

$$x = 4$$

$$\therefore y = -3$$

**b**  $2x + 6y = 0 \dots (1)$

$$y - x = 2 \dots (2)$$

Multiply (2) by 2 and add to (1)

$$8y = 4$$

$$y = \frac{1}{2}$$

$$\therefore x = -\frac{3}{2}$$

**c**  $4x - 2y = 7 \dots (1)$

$$5x + 7y = 1 \dots (2)$$

Multiply (2) by 4, (1) by 5 and subtract

$$38y = -31$$

$$y = -\frac{31}{38}$$

$$\therefore x = \frac{51}{38}$$

**d**  $2x - y = 6 \dots (1)$

$$4x - 7y = 5 \dots (2)$$

Multiply (1) by 2, and subtract

$$-5y = -7$$

$$y = \frac{7}{5}$$

$$\therefore x = \frac{37}{10}$$

**2 a** one solution

**b** infinitely many solutions

**c** no solutions

**3** The two corresponding lines are parallel but not equal, and have no intersection.

**4**  $x - y = 6$

$$y = x - 6$$

Let  $x = \lambda, y = \lambda - 6$

$$y = \lambda - 6$$

**5** Lines are parallel. The gradients are the same and the lines have a common point.

$$3x + my = 5 \dots (1)$$

$$(m + 2)x + 5y = m \dots (2)$$

$$\text{Gradient of (1)} = -\frac{3}{m}$$

$$\text{Gradient of (2)} = -\frac{m + 2}{5}$$

For the lines to coincide:

$$\frac{3}{m} = \frac{m + 2}{5}$$

$$m^2 + 2m - 15 = 0$$

$$(m - 3)(m + 5) = 0$$

$$m = 3 \text{ or } m = -5$$

**a** If  $m = 3$  the equations are

$$3x + 3y = 5 \dots (1)$$

$$5x + 5y = 5 \dots (2)$$

The lines don't coincide.

If  $m = -5$  the equations are

$$3x - 5y = 5 \dots (1)$$

$$-3x + 5y = -5 \dots (2)$$

The lines coincide.



- b** A unique solution if  $m \in \mathbb{R} \setminus \{3, -5\}$   
 No solution if  $m = 3$

**Alternative method**

$$\begin{bmatrix} 3 & m \\ m+2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ m \end{bmatrix}$$

$$\begin{bmatrix} 3 & m \\ m+2 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & m \\ m+2 & 5 \end{bmatrix} = 15 - m^2 - 2m$$

$$m^2 + 2m - 15 = 0$$

$$(m+5)(m-3) = 0$$

$$m = -5, 3$$

substitute  $m = -5$

$$3x - 5y = 5$$

$$-3x + 5y = -5$$

infinite solutions

Substitute  $m = 3$

$$3x + 3y = 5$$

$$5x + 5y = 3$$

No solutions

$$\therefore \mathbf{a} \ m = -5 \quad \mathbf{b} \ m = 3$$

**6**  $m = 9$

**7 a**  $mx + 2y = 8 \dots (1)$

$$4x - (2 - m)y = 2m \dots (2)$$

$$4 \times (1) - m \times (2)$$

$$\Rightarrow (8 + m(m - 2))y = 32 - 2m^2$$

$$y = \frac{32 - 2m^2}{m^2 - 2m + 8}$$

$$y = \frac{2(m+4)(m-4)}{(m-4)(m+2)}$$

$$y = \frac{2(m+4)}{(m+2)}$$

$$m \neq 4, -2$$

Substitute in (1)

$$\Rightarrow mx + 4 \frac{(m-4)(m+2)}{(m-4)(m+2)} = 8$$

$$x = \frac{8(m-4)(m+2) - 4(m+4)(m-4)}{m(m-4)(m+2)}$$

$$x = \frac{4(m-4)(2m+4-m-4)}{m(m-4)(m+2)}$$

$$x = \frac{4(m-4)m}{m(m-4)(m+2)}$$

$$x = \frac{4}{m+2}, \quad m \neq -2, 0, 4,$$

- b** Values to test:  $m = -2, 0, 4$

$$m = -2$$

$$-2x + 2y = 8 \dots (1)$$

$$4x - 4y = -4 \dots (2)$$

No solutions

$$m = 0$$

$$10 + 2y = 8$$

$$y = 4$$

$$2 - 4x - 2y = 0$$

$$4x - 8 = 0$$

$$x = 2 \text{ unique solution}$$

(Note: this value for  $m$  would not have appeared if equation 2 had been used to find  $x$ .)

$$m = 4$$

$$4x + 2y = 8 \dots (1)$$

$$4x + 2y = 8 \dots (2)$$

infinite solutions

$\therefore$

**i**  $m = -2$

**ii**  $m = 4$

**8 a**  $2x - 3y = 4 \dots (1)$

$$x + ky = 2 \dots (2)$$

$$(1) - 2 \times (2)$$

$$\Rightarrow (3 - 2k)y = 0$$

$$y = 0, \quad k \neq \frac{-3}{2}$$

$$\Rightarrow 2x = 4$$

$$x = 2$$

$$\mathbf{b} \quad k = \frac{-3}{2}$$

$$\mathbf{9} \quad x + 5y = 4 \dots (1)$$

$$2x + by = c \dots (2)$$

$$\text{Gradient of (1)} = -\frac{1}{5}$$

$$\text{Gradient of (2)} = -\frac{2}{b}$$

For the lines to be parallel  
or coincide:

$$\frac{1}{5} = \frac{2}{b}$$

$$b = 10$$

**a** A unique solution for  $b \in \mathbb{R} \setminus \{10\}$

**b** If  $b = 10$  and  $c = 8$  the corresponding lines coincide and there are infinitely many solutions.

**c** If  $b = 10$  and  $c \neq 8$  the corresponding lines are parallel and there are no solutions

## Solutions to Exercise 2G

**1 a**  $2x + 3y - z = 12 \dots (1)$

$$2y + z = 7 \dots (2)$$

$$2y - z = 5 \dots (3)$$

Add (2) and (3)

$$4y = 12$$

$$y = 3$$

$$\therefore z = 1$$

Substitute in (1) to find  $x$

$$x = 2$$

**b**  $x + 2y + 3z = 13 \dots (1)$

$$-x - y + 2z = 2 \dots (2)$$

$$-x + 3y + 4z = 26 \dots (3)$$

Add (1) and (2)

$$y + 5z = 15 \dots (4)$$

Subtract (2) from (3)

$$4y + 2z = 24$$

$$2y + z = 12 \dots (5)$$

Multiply (4) by 2 and

subtract from (4)

$$-9z = -18$$

$$z = 2$$

$$\therefore y = 5$$

$$\therefore x = -3$$

**c**  $x + y = 5 \dots (1)$

$$y + z = 7 \dots (2)$$

$$z + x = 12 \dots (3)$$

Subtract (2) from (3)

$$x - y = 5 \dots (4)$$

Add (1) and (4)

$$\therefore x = 5$$

$$\therefore y = 0$$

$$\therefore z = 7$$

**d**  $x - y - z = 0 \dots (1)$

$$5x + 20z = 50 \dots (2)$$

$$10y - 20z = 30 \dots (3)$$

Simplify (2) and (3)

$$x + 4z = 10 \dots (4)$$

$$y - 2z = 3 \dots (5)$$

Subtract (5) from (4)

$$(x - y) + 6z = 7 \dots (6)$$

Subtract (1) from (6)

$$7z = 7$$

$$\therefore z = 1$$

$$\therefore x = 6$$

$$\therefore y = 5$$

**2 a**  $y - 4z = -2$

$$y = 4z - 2$$

**b**  $z = \lambda$

$$y = 4\lambda - 2$$

$$\therefore x + 8\lambda - 4 - 3\lambda = 4$$

$$x = 8 - 5\lambda$$

**3 a**  $-y + 5z = 15$  (2) + (1)  
 $-y + 5z = 15$  (3) - (2)

**b** They are the same

**c**  $z = \lambda$   
 $-y + 5\lambda = 15$   
 $y = 5\lambda - 15$

**d**  $x + 10\lambda - 30 + 3\lambda = 13$   
 $x = 43 - 13\lambda$

**4 a** (1) + (2)  
 $\Rightarrow 2z = 10$   
 $z = 5$   
 Substitute into (1)  
 $\Rightarrow x - y + 5 = 4$   
 let  $y = \lambda$   
 $x = \lambda - 1$

**b** Let  $z = \lambda$   
 Substitute in (2)  
 $\Rightarrow x = 3 + \lambda$   
 Substitute into (1)  
 $\Rightarrow 6 + 2\lambda - y + \lambda = 6$   
 $y = 3\lambda$

**c** (1) + 2 × (2)  
 $6x + 3z = 14$   
 Let  $z = \lambda$   
 $6x = 14 - 3\lambda$   
 $x = \frac{14 - 3\lambda}{6}$   
 Substitute into (2)  
 $\frac{14 - 3\lambda}{6} + y + \lambda = 4$   
 $y = 4 - \frac{14 + 3\lambda}{6}$   
 $y = \frac{10 - 3\lambda}{6}$

**5**  $x + y + z + w = 4 \dots (1)$   
 $x + 3y + 3z = 2 \dots (2)$   
 $x + y + 2z - w = 6 \dots (3)$   
 (3) - (1)

$\Rightarrow z - 2w = 2$   
 Let  $z = t, t \in \mathbb{R}$   
 $w = \frac{t - 2}{2} = \frac{1}{2}t - 1$

(2) - (3) gives  
 $2y + z + w = -4$   
 $2y + t + \frac{1}{2}t - 1 = -4$

$2y = -3 - \frac{3}{2}t$   
 $y = -\frac{3}{2} - \frac{3}{4}t$   
 Substitute into (1)  
 $x - \frac{3}{2} - \frac{3}{4}t + t + \frac{1}{2}t - 1 = 4$   
 $x = 6\frac{1}{2} - \frac{3}{4}t = \frac{26 - 3t}{4}$

when  $w = 6, \frac{t - 2}{2} = 6$  so  $t = z = 14$   
 $y = \frac{-3(14 + 2)}{4} = -12$   
 $x = \frac{26 - 42}{4} = -4$

**6 a**  $3x - y + z = 4 \dots (1)$   
 $x + 2y - z = 2 \dots (2)$   
 $-x + y - z = -2 \dots (3)$   
 (2) - (3)  $\Rightarrow 2x + y = 4$   
 (3) + (1)  $\Rightarrow 2x = 2$   
 $x = 1$   
 $y = 2$   
 Substitute into (3)  
 $\Rightarrow -1 + 2 - z = -2$   
 $z = 3$

**b**  $x - y - z = 0 \dots (1)$

$3y + 3z = -5 \dots (2)$

$3 \times (1) + (2)$

$\Rightarrow 3x = -5$

$x = \frac{-5}{3}$

$3y = -5 - 3z$

Let  $z = \lambda$

$y = \frac{-5 - 3\lambda}{3}$

**c**  $12x - y + z = 0$

$2y + 2z = 2$

Let  $z = \lambda$

$y = 2 - 2\lambda$

$2x - 2 + 2\lambda + \lambda = 0$

$2x = 2 - 3\lambda$

$x = \frac{2 - 3\lambda}{2}$

## Solutions to technology-free questions

**1 a**  $3x - 2 = 4x + 6$

$$4x - 3x = -2 - 6$$

$$x = -8$$

**b**  $\frac{x+1}{2x-1} = \frac{4}{3}$

$$3(x+1) = 4(2x-1)$$

$$3x+3 = 8x-4$$

$$8x-3x = 3+4$$

$$5x = 7$$

$$x = \frac{7}{5}$$

**c**  $\frac{3x}{5} - 7 = 11$

$$\frac{3x}{5} = 18$$

$$3x = 90$$

$$x = 30$$

**d**  $\frac{2x+1}{5} = \frac{x-1}{2}$

$$2(2x+1) = 5(x-1)$$

$$4x+2 = 5x-5$$

$$x = 7$$

**2 a**  $y = x + 4$  ... 1

$$5y + 2x = 6$$
 ... 2

Substitute 1 into 2:

$$5(x+4) + 2x = 6$$

$$5x + 20 + 2x = 6$$

$$7x = -14$$

$$x = -2$$

Substitute into 1:

$$y = -2 + 4 = 2$$

**b**  $\frac{x}{4} - \frac{y}{3} = 2$  ... 1  
 $y - x = 5$  ... 2

Multiply 1 by 12:

$$3x - 4y = 24$$
 ... 3

Multiply 2 by 3:

$$3y - 3x = 15$$
 ... 4

3 + 4 gives  $-y = 39$ , so  $y = -39$

Substitute into 2:

$$-39 - x = 5, \text{ so } x = -44$$

**3 a**  $\frac{n+m}{b}$

**b**  $\frac{b}{c+b}$

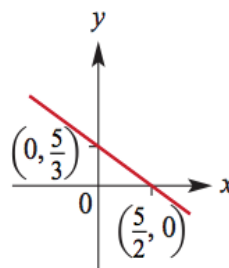
**c**  $d$

**d**  $\frac{6}{q-p}$

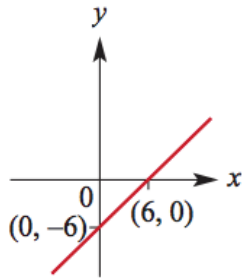
**e**  $\frac{m+n}{m-n}$

**f**  $\frac{a^2}{a-1}$

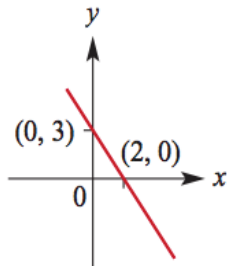
**4 a** intercepts  $(\frac{5}{2}, 0), (0, \frac{5}{3})$



**b** intercepts  $(6, 0), (0, -6)$



c intercepts (2, 0), (0, 3)



5 a  $y - 3 = -2(x - 1)$   
 $y = -2x + 5$

b  $m = \frac{8 - 4}{3 - 1} = 2$   
 $y - 4 = 2(x - 1)$   
 $y = 2x + 2$

c  $y = -2x + 6$  has gradient  $m_1 = -2$ ;  
 for the gradient  $m_2$  of a perpendicular  
 line:  $m_1 m_2 = -1$   
 $m_2 = \frac{-1}{-2} = \frac{1}{2}$   
 $y - 1 = \frac{1}{2}(x - 1)$   
 $y = \frac{1}{2}x + \frac{1}{2}$

d  $y = 6 - 2x$  has gradient  $-2$ ; a parallel  
 line has the same gradient:  
 $y - 1 = -2(x - 1)$   
 $y = -2x + 3$

6 distance =  $\sqrt{(2 - (-1))^2 + (4 - 6)^2}$   
 $= \sqrt{3^2 + (-2)^2}$   
 $= \sqrt{9 + 4}$   
 $= \sqrt{13}$

7 Midpoint =  $\left(\frac{4 + (-2)}{2}, \frac{6 + 8}{2}\right) = (1, 7)$

8 a Let  $(x, y)$  be the coordinates of  $Y$ .  
 $\left(\frac{x + (-6)}{2}, \frac{y + 2}{2}\right) = (8, 3)$   
 $\therefore \frac{x - 6}{2} = 8$  and  $\frac{y + 2}{2} = 3$   
 $\therefore x = 22$  and  $y = 4$

b Let  $(x, y)$  be the coordinates of  $Y$ .  
 $\left(\frac{x + (-1)}{2}, \frac{y + (-4)}{2}\right) = (2, -8)$   
 $\therefore \frac{x - 1}{2} = 2$  and  $\frac{y - 4}{2} = -8$   
 $\therefore x = 5$  and  $y = -12$

9 a  $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 4 \end{bmatrix}$

b  $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 14 & 4 \end{bmatrix}$

c  $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

d  $\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$

e  $3 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$

f  $\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 8 & 5 \end{bmatrix}$

$$\mathbf{g} \quad \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\mathbf{h} \quad k \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2k & k \\ 3k & 2k \end{bmatrix}$$

$$\mathbf{i} \quad 2 \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + 3 \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix} + \begin{bmatrix} 12 & 0 \\ 3 & 6 \end{bmatrix} \\ = \begin{bmatrix} 16 & 2 \\ 9 & 10 \end{bmatrix}$$

$$\mathbf{j} \quad \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - 2 \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 2 & 4 \end{bmatrix} \\ = \begin{bmatrix} -6 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\mathbf{10} \quad \sqrt{(10-5)^2 + (y-12)^2} = 13 \\ 25 + (y-12)^2 = 169 \\ (y-12)^2 = 144 \\ y-12 = \pm 12 \\ y = 0 \text{ or } y = 24$$

$$\mathbf{11} \quad mx - 4y = m + 3 \dots (1) \\ 4x + (m+10)y = -2 \dots (2) \\ \text{Gradient of (1)} = \frac{m}{4} \\ \text{Gradient of (2)} = -\frac{4}{m+10} \\ \text{Infinitely many or no solutions} \\ \text{when the gradients are the same.} \\ -\frac{m}{4} = \frac{4}{m+10} \\ m^2 + 10m + 16 = 0 \\ (m+8)(m+2) = 0 \\ m = -8 \text{ or } m = -2$$

**a** Checking back in the equations there are infinitely many solutions when  $m = -2$ .

Equation (1) becomes  $-2x - 4y = 1$

Equation (2) becomes  $4x + 8y = -2$

**b** There is a unique solution for  $m \in \mathbb{R} \setminus \{-2, -8\}$

$$\mathbf{12} \quad \mathbf{a} \quad 2x - 3y + z = 6 \dots (1)$$

$$-2x + 3y + z = 8 \dots (2)$$

Add (1) and (2)

$$2z = 14$$

$$z = 7$$

Substitute in (1)

$$2x - 3y = -1$$

$$\therefore y = \frac{2x+1}{3}$$

Let  $x = \lambda$

The solution is  $(\lambda, \frac{2\lambda+1}{3}, 7)$  where  $\lambda \in \mathbb{R}$

$$\mathbf{b} \quad x - z + y = 6 \dots (1)$$

$$2x + z = 4 \dots (2)$$

Let  $z = \lambda$

$$\text{Then } x = \frac{4-\lambda}{2}$$

Substitute in (1)

$$\frac{4-\lambda}{2} - \lambda + y = 6$$

$$\therefore y = \frac{8+3\lambda}{2}$$

The solution is  $(\frac{4-\lambda}{2}, \frac{3\lambda+8}{2}, \lambda)$  where  $\lambda \in \mathbb{R}$



## Solutions to multiple-choice questions

**1 E**  $y = mx + c$

The  $m$  (gradient) value is  $-\frac{1}{2}$ ,

It passes through the point  $(1, 4)$

$$4 = -\frac{1}{2} \cdot 1 + c$$

$$\therefore c = \frac{9}{2}$$

$$\therefore y = -\frac{1}{2}x + \frac{9}{2}$$

**2 E**  $y = -2x + 4$

Point  $(a, 3)$

$$3 = -2a + 4$$

$$a = \frac{1}{2}$$

**3 D** Line passes through the points

$(-2, 0)$  and  $(0, -1)$

$$\therefore \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 0}{0 - -2}$$

$$m \text{ (gradient)} = -\frac{1}{2}$$

$$\text{Perpendicular line} = -\frac{1}{m}$$

$$\therefore -\frac{1}{-\frac{1}{2}} = 2$$

**4 C** Midpoint at  $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$

$$= \left(-\frac{2}{2}, \frac{17}{2}\right)$$

$$= (-1, 8.5)$$

**5 B**  $2ax - 10by = 22$

$$+ \quad + \quad +$$

$$4ax + 10by = 2$$

$$\therefore 6ax = 24$$

$$\therefore x = \frac{4}{a}$$

Do not need to solve for  $y$  as there is only one possible option.

**6 A** Line passes through the points

$(3, -2)$  and  $(-1, 10)$

$$\therefore \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - -2}{-1 - 3} = \frac{12}{-4} = -3$$

**7 A** Eqn 1:  $y = 2x + 3$

$$\text{Eqn 2: } y = \frac{ax}{3} + \frac{4}{3}$$

To be parallel gradients must be the same.

$$\therefore \frac{a}{3} = 2$$

$$\therefore a = 6$$

**8 C**  $y = mx + c$

$$m = \frac{10 - -2}{3 - -1}$$

$$m = 3$$

Passes through the point  $(3, 10)$

$$\therefore 10 = 9 + c$$

$$\therefore c = 1$$

$$\therefore y = 3x + 1$$

**9 B** Distance between  $x$  points

$$= |x_2 - x_1|$$

$$= |5 - 1|$$

$$= 4$$

Distance between  $y$  points

$$= |y_2 - y_1|$$

$$= |-2 - 4|$$

$$= 6$$

Using Pythagoras

$$\sqrt{4^2 + 6^2}$$

$$= \sqrt{52}$$

$$= 2\sqrt{13}$$

**10 C**  $y = mx + c$

Passes through points (4, 0) and

(0, -3)

$$m = \frac{-3 - 0}{0 - 4}$$

$$m = \frac{3}{4}$$

Y intercept = -3

$$\therefore c = -3$$

$$\therefore f(x) = \frac{3}{4}x - 3$$

**11 D**  $bx + 3y = 0 \dots (1)$

$$4x + (b + 1)y = 0 \dots (2)$$

$$\text{Gradient of (1)} = -\frac{b}{3}$$

$$\text{Gradient of (2)} = -\frac{4}{b+1}$$

Infinitely many solutions when the gradients are the same.

$$\frac{b}{3} = \frac{4}{b+1}$$

$$b^2 + b - 12 = 0$$

$$(b + 4)(b - 3) = 0$$

$$b = -4 \text{ or } b = 3$$

**12 A**

$$(a - 1)x + 5y = 7 \dots (1)$$

$$3x + (a - 3)y = 0 \dots (2)$$

$$\text{Gradient of (1)} = -\frac{a - 1}{5}$$

$$\text{Gradient of (2)} = -\frac{3}{a - 3}$$

Infinitely many or no solutions

when the gradients are the same.

$$\frac{a - 1}{5} = \frac{3}{a - 3}$$

$$a^2 - 4a - 12 = 0$$

$$(a - 6)(a + 2) = 0$$

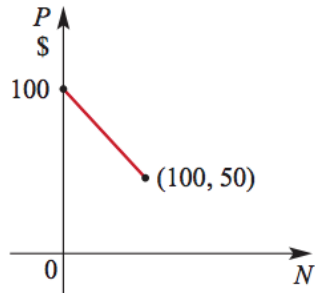
$$a = -2 \text{ or } a = 6$$

**13 D**  $\left(\frac{0+4}{2}, \frac{d-6}{2}\right) = \left(2, \frac{d-6}{2}\right)$

**14 C** Gradient of line segment joining (3, 0) and (0, -6) is  $\frac{6}{3} = 2$  Gradient of line perpendicular to this is  $-\frac{1}{2}$

## Solutions to extended-response questions

- 1 a Graph is a straight line passing through (100, 50) and (50, 75).



Note that extending it back to the  $P$  axis shows that the intercept is (0, 100); this is confirmed in part **b** below.

- b** Relationship is linear:  $P = aN + b$   
 $P = 50, N = 100: 50 = 100a + b \quad \dots 1$   
 $P = 75, N = 50: 75 = 50a + b \quad \dots 3$   
 $1 - 2: 50a = -25$

$$a = -\frac{1}{2}$$

which implies  $b = 100$

$$\text{Hence } P = -\frac{1}{2}N + 100.$$

- c i**  $N = 88: P = -\frac{1}{2} \times 88 + 100$   
 $= 56$

So the price would be \$56.

- ii**  $P = 60: 60 = -\frac{1}{2}N + 100$

$$\frac{1}{2}N = 40$$

$$N = 80$$

So the number of jackets would be 80.

- 2 a** The rule is of the form  $p = at + b$   
 When  $t = 3, p = 12000$  and when  $t = 8, p = 19240$

Therefore the equations

$$12000 = 3a + b1$$

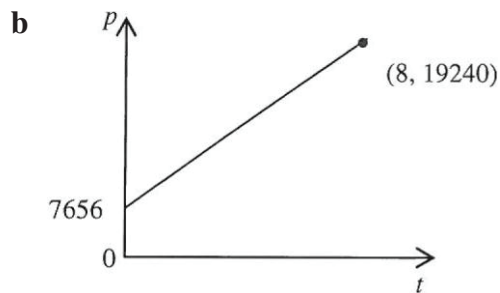
and  $19240 = 8a + b2$  are satisfied.

Subtract 1 from 2 to give  $5a = 7240$ .

Hence  $a = 1448$

Substitute in 1 to find that  $b = 7656$

Therefore  $p = 1448t + 7656$



The  $p$  axis intercept gives the initial population.

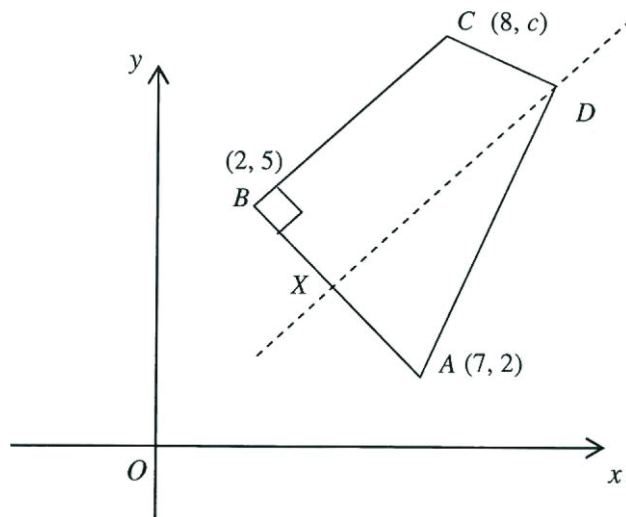
**c** When  $t = 10$ ,

$$p = 14480 + 7656$$

$$= 22136$$

**d** The average rate of growth is the gradient. The growth rate is 1448 people per year.

**3**



**a** Midpoint of  $AB = \left( \frac{7+2}{2}, \frac{5+2}{2} \right) = \left( \frac{9}{2}, \frac{7}{2} \right)$

$$\text{Gradient of } AB = \frac{5-2}{2-7} = -\frac{3}{5}$$

Therefore equation of perpendicular bisector of  $AB$  is

$$y - \frac{7}{2} = \frac{5}{3} \left( x - \frac{9}{2} \right)$$

$$\text{Therefore } y = \frac{5}{3}x - 4$$

- b** Solving the equations  $y = 4x - 26$  and  $y = \frac{5}{3}x - 4$  simultaneously for  $x$  and  $y$  will give the coordinates of  $D$

$$\text{Consider } 4x - 26 = \frac{5}{3}x - 4$$

$$\frac{7x}{3} = 22$$

$$x = \frac{66}{7}$$

$$\text{Substitute } x = \frac{66}{7} \text{ in the equation } y = 4x - 26 \text{ to give } y = \frac{82}{7}$$

$$\text{Coordinates of } D \text{ are } \left(\frac{66}{7}, \frac{82}{7}\right)$$

- c** Line  $BC$  is perpendicular to line  $AB$ . Therefore gradient of  $BC$  is  $\frac{5}{3}$

- d**  $B(2, 5)$  and  $C(8, c)$ . The gradient of  $BC$  can also be written as  $\frac{5 - c}{-6}$

$$\text{Therefore } \frac{5 - c}{-6} = \frac{5}{3}$$

$$\text{Solving for } c \text{ gives } c = 15$$

- e** The area will be found by calculating the area of triangle  $DXA$  and trapezium  $BCDX$ . Let  $X$  be the midpoint of  $AB$ . From the above the coordinates of  $X$  are  $\left(\frac{9}{2}, \frac{7}{2}\right)$

$$\text{Distance } XD = \sqrt{\left(\frac{66}{7} - \frac{9}{2}\right)^2 + \left(\frac{82}{7} - \frac{7}{2}\right)^2}$$

$$= \sqrt{\frac{8993}{98}}$$

$$= \frac{23\sqrt{34}}{14}$$

$$\text{Distance } XA = \text{distance } XB = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$$

$$= \sqrt{\frac{17}{2}}$$

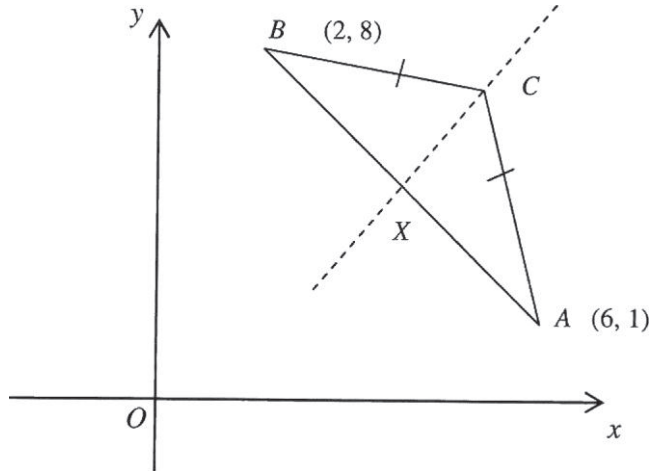
Area = area of triangle  $DXA$  + area of trapezium  $BCDX$ .

$$= \frac{1}{2}XA \times XB + \frac{1}{2}BX(BC + XD)$$

$$= \frac{1}{2}AX(BX + 2XD)$$

$$= \frac{629}{14}$$

4



a Midpoint of  $BC = \left(4, \frac{9}{2}\right)$

Gradient of  $AB = \frac{8-1}{2-6} = -\frac{7}{4}$

Therefore gradient of perpendicular bisector  $= \frac{4}{7}$

The equation of the perpendicular bisector is

$$y - \frac{9}{2} = \frac{4}{7}(x - 4)$$

Therefore  $y = \frac{4}{7}x + \frac{31}{14}$

b The perpendicular bisector passes through  $C$  as the triangle is isosceles.

When  $x = 3.5$ ,  $y = \frac{4}{7} \times 3.5 + \frac{31}{14} = \frac{59}{14}$

The coordinates of  $C$  are  $\left(\frac{7}{2}, \frac{59}{14}\right)$

c The length of  $AB = \sqrt{(6-2)^2 + (1-8)^2} = \sqrt{65}$

d The area of the triangle  $= \frac{1}{2} \times \sqrt{65} \times XC$  where  $X$  is the midpoint of  $AB$

$$XC = \sqrt{\left(\frac{7}{2} - 4\right)^2 + \left(\frac{59}{14} - \frac{9}{2}\right)^2} = \frac{\sqrt{65}}{14}$$

Therefore area  $= \frac{1}{2} \times \sqrt{65} \times \frac{\sqrt{65}}{14} = \frac{65}{28}$  square units.

5  $A(-4, 6)$  and  $B(6, -7)$

a Midpoint  $= \left(\frac{-4+6}{2}, \frac{6+(-7)}{2}\right) = \left(1, -\frac{1}{2}\right)$

b/c The length  $AB = \sqrt{(-7-6)^2 + (6-(-4))^2} = \sqrt{269}$  = the distance between  $A$  and  $B$

d gradient of  $AB = \frac{6 - -7}{-4 - 6}$   
 $= -\frac{13}{10}$

The equation of  $AB$  is  $y - 6 = -\frac{13}{10}(x + 4)$

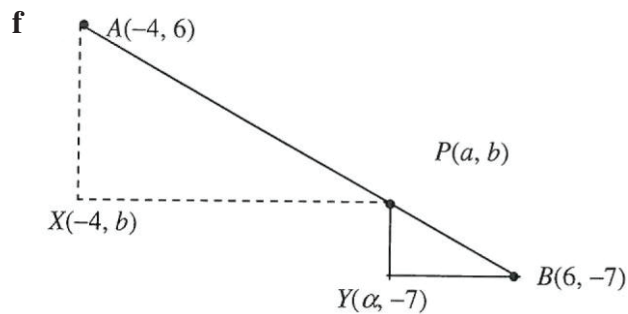
Rearranging gives

$$y = -\frac{13}{10}x + \frac{4}{5}$$

e The perpendicular bisector has gradient  $\frac{10}{13}$

The equation is  $y + \frac{1}{2} = \frac{10}{13}(x - 1)$

Therefore  $y = \frac{10}{13}x - \frac{33}{26}$



Triangles  $AXP$  and  $PYB$  are similar with scale factor 3.

$$AX : PY = 3 : 1$$

Therefore  $\frac{6 - b}{b + 7} = \frac{3}{1}$

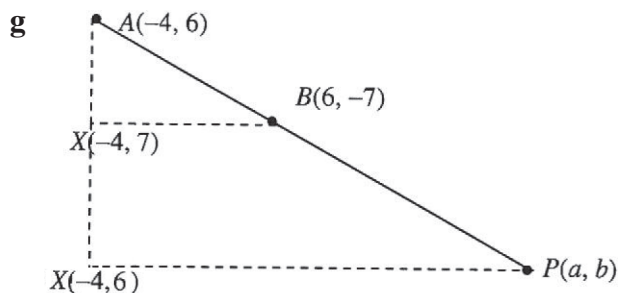
Therefore  $b = -\frac{15}{4}$

Also  $XP : YB = 3$

$$\frac{a + 4}{6 - a} = 3$$

$$a = \frac{7}{2}$$

coordinates of  $P$  are  $\left(\frac{7}{2}, -\frac{15}{4}\right)$



Triangles  $AXB$  and  $AYP$  are similar with scale factor 3.

Therefore

$$\frac{a+4}{10} = \frac{3}{1}$$

$$a = 26$$

$$\text{Also } \frac{b-6}{-7-6} = 3$$

$$b = -33$$

The coordinates of  $P$  are  $(26, -33)$

**6 a** 25% of 500 = 125

125 litres of acid is required to produce 500 litres of a 25% acid solution.

**b** Let  $x$  denote the amount of 30% solution.

Let  $y$  denote the amount of 18% solution.

$$\therefore x + y = 500 \quad 1$$

$$0.3x + 0.18y = 125 \quad 2$$

From 1  $y = 500 - x$ . Substitute in 2

$$\therefore 0.3x + 0.18(500 - x) = 125$$

$$\therefore (0.3 - 0.18)x + 90 = 125$$

$$\therefore 0.12x = 35$$

$$\therefore x = \frac{875}{3}$$

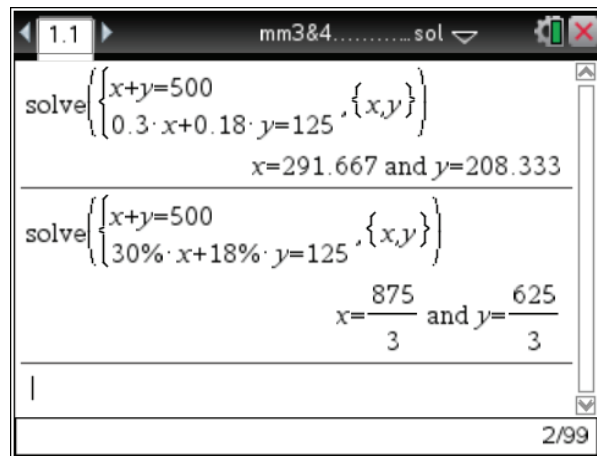
$$\text{Substitute in 1 } y = 500 - \frac{875}{3} = \frac{625}{3}$$

$\frac{875}{3}$  litres of the 30% solution and  $\frac{625}{3}$  litres of the 18% solution are required.

### Graphical Calculator techniques for Question 6.

In a **Calculator** page use **Algebra>Solve System of Equations>Solve System of Equations**.

For exact answers enter decimal inputs as fractions such as 30/100 or 30 as shown.





# Chapter 3 – Transformations

## Solutions to Exercise 3A

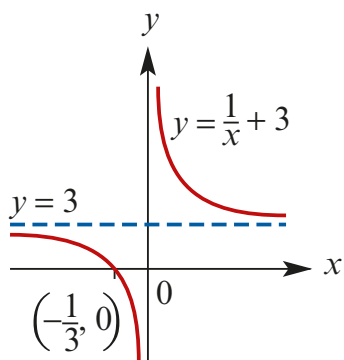
- 1 a  $(-2, 5) \rightarrow (-2 + 1, 5 - 2) = (-1, 3)$   
 b  $(-2, 5) \rightarrow (-2 - 3, 5 + 5) = (-5, 10)$   
 c  $(-2, 5) \rightarrow (-2 - 1, 5 - 6) = (-3, -1)$   
 d  $(-2, 5) \rightarrow (-2 - 3, 5 + 2) = (-5, 7)$   
 e  $(-2, 5) \rightarrow (-2 - 1, 5 + 1) = (-3, 6)$

2 a  $y = \frac{1}{x-2} - 3$

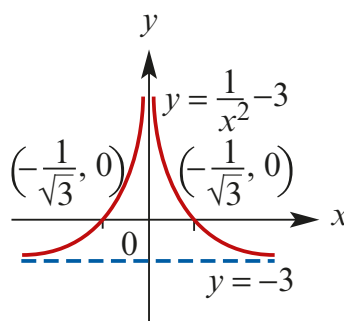
b  $y = \frac{1}{x+2} + 3$

c  $y = \frac{1}{x - \frac{1}{2}} + 4 = \frac{2}{2x - 1} + 4$

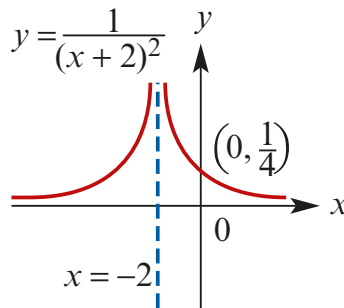
- 3 a Domain =  $\mathbb{R} \setminus \{0\}$   
 Range =  $\mathbb{R} \setminus \{3\}$



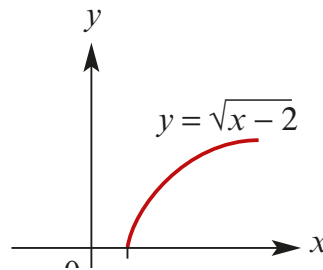
- b Domain =  $\mathbb{R} \setminus \{0\}$   
 Range =  $(-3, \infty)$



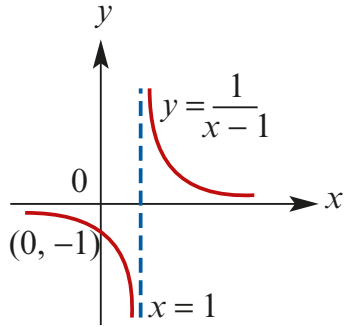
- c Domain =  $\mathbb{R} \setminus \{-2\}$   
 Range =  $\mathbb{R}^+$



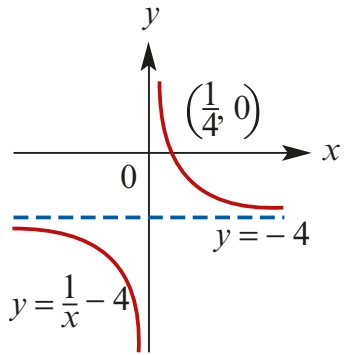
- d Domain =  $[2, \infty)$   
 Range =  $\mathbb{R}^+ \cup \{0\}$



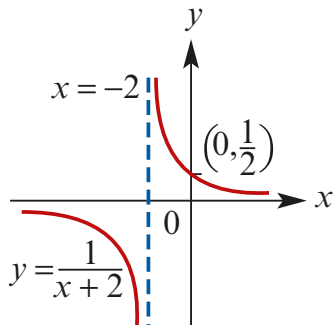
- e Domain =  $\mathbb{R} \setminus \{1\}$   
 Range =  $\mathbb{R} \setminus \{0\}$



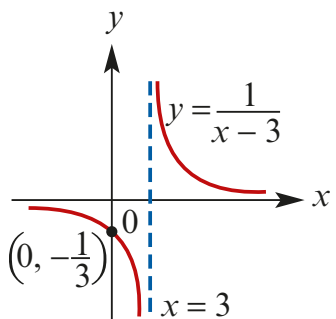
- f** Domain =  $\mathbb{R} \setminus \{0\}$   
Range =  $\mathbb{R} \setminus \{-4\}$



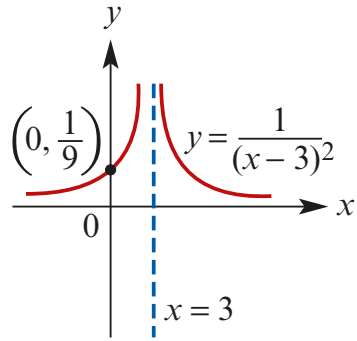
- g** Domain =  $\mathbb{R} \setminus \{-2\}$   
Range =  $\mathbb{R} \setminus \{0\}$



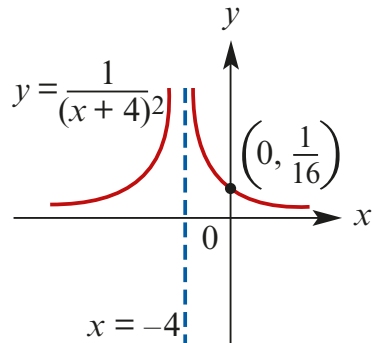
- h** Domain =  $\mathbb{R} \setminus \{3\}$   
Range =  $\mathbb{R} \setminus \{0\}$



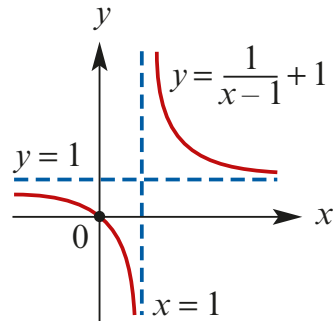
- i** Domain =  $\mathbb{R} \setminus \{3\}$   
Range =  $\mathbb{R}^+$



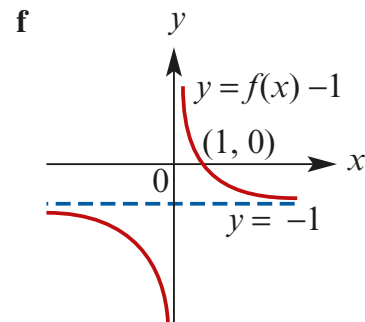
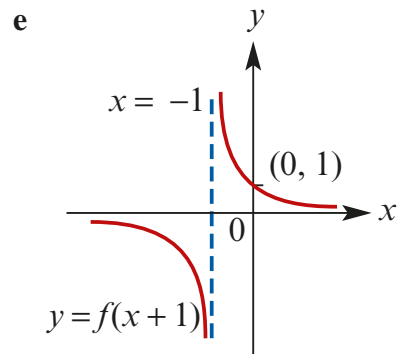
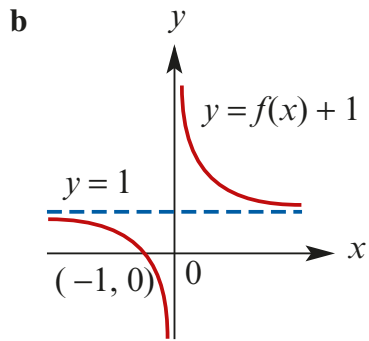
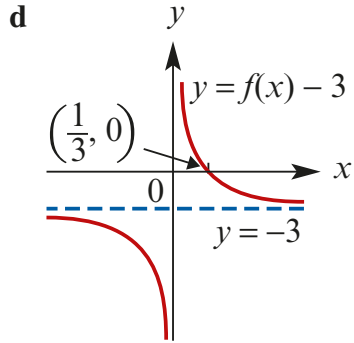
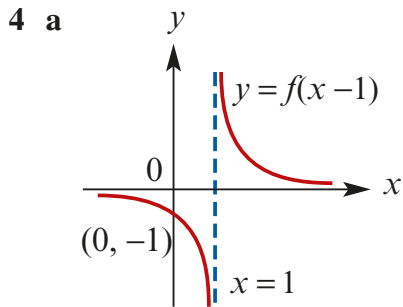
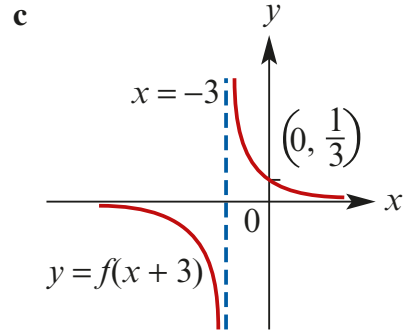
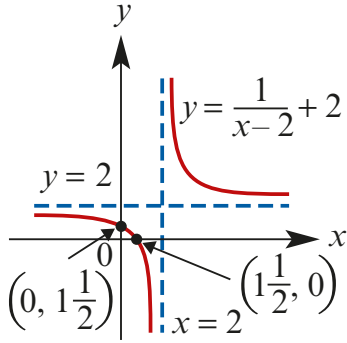
- j** Domain =  $\mathbb{R} \setminus \{-4\}$   
Range =  $\mathbb{R}^+$



- k** Domain =  $\mathbb{R} \setminus \{1\}$   
Range =  $\mathbb{R} \setminus \{1\}$



- l** Domain =  $\mathbb{R} \setminus \{2\}$   
Range =  $\mathbb{R} \setminus \{2\}$



**5 a** Translation of 5 to the left;  
 $(x, y) \rightarrow (x - 5, y)$

**b** Translation of 2 up;  $(x, y) \rightarrow (x, y + 2)$

**c** Translation of 4 up;  $(x, y) \rightarrow (x, y + 4)$

**d** Translation  $(x, y) \rightarrow (x, y + 3)$

**e** Translation  $(x, y) \rightarrow (x - 3, y)$

**6 a i**  $y - 1 = (x - 7)^{\frac{1}{4}} ; y = (x - 7)^{\frac{1}{4}} + 1$

**ii**  $y + 6 = (x + 2)^{\frac{1}{4}} ; y = (x + 2)^{\frac{1}{4}} - 6$

**iii**  $y + 3 = (x - 2)^{\frac{1}{4}} ; y = (x - 2)^{\frac{1}{4}} - 3$

**iv**  $y - 4 = (x + 1)^{\frac{1}{4}} ; y = (x + 1)^{\frac{1}{4}} + 4$

**b i**  $y = \sqrt[3]{(x - 7)} + 1$

**ii**  $y = \sqrt[3]{(x + 2)} - 6$

**iii**  $y = \sqrt[3]{(x - 2)} - 3$

**iv**  $y = \sqrt[3]{(x + 1)} + 4$

**c i**  $y = \frac{1}{(x - 7)^3} + 1$

**ii**  $y = \frac{1}{(x + 2)^3} - 6$

**iii**  $y = \frac{1}{(x - 2)^3} - 3$

**iv**  $y = \frac{1}{(x + 1)^3} + 4$

**d i**  $y = \frac{1}{(x - 7)^4} + 1$

**ii**  $y = \frac{1}{(x + 2)^4} - 6$

**iii**  $y = \frac{1}{(x - 2)^4} - 3$

**iv**  $y = \frac{1}{(x + 1)^4} + 4$

**7 a**  $x' = x - 3$  and  $y' = y + 2$

$\therefore x = x' + 3$  and  $y = y' - 2$

$\therefore y = (x - 2)^2 + 3$  maps to

$y' - 2 = (x' + 3 - 2)^2 + 3$

The image is  $y = (x + 1)^2 + 5$

**b**  $x' = x + 3$  and  $y' = y - 3$

$\therefore x = x' - 3$  and  $y = y' + 3$

$\therefore y = 2(x + 3)^2 + 3$  maps to

$y' + 3 = 2(x' - 3 + 3)^2 + 3$

The image is  $y = 2x^2$

**c**  $x' = x + 4$  and  $y' = y - 2$

$\therefore x = x' - 4$  and  $y = y' + 2$

$\therefore y = \frac{1}{(x - 2)^2} + 3$  maps to

$y' + 2 = \frac{1}{(x' - 4 - 2)^2} + 3$

The image is  $y = \frac{1}{(x - 6)^2} + 1$

**d**  $x' = x - 1$  and  $y' = y + 1$

$\therefore x = x' + 1$  and  $y = y' - 1$

$\therefore y = (x + 2)^3 + 1$  maps to

$y' - 1 = (x' + 1 + 2)^3 + 1$

The image is  $y = (x + 3)^3 + 2$

**e**  $x' = x - 1$  and  $y' = y + 1$

$\therefore x = x' + 1$  and  $y = y' - 1$

$\therefore y = \sqrt[3]{x - 3} + 2$  maps to

$y' - 1 = \sqrt[3]{x' + 1 - 3} + 2$

The image is  $y = \sqrt[3]{x - 2} + 3$

**8 a** Write

$y = \frac{1}{x^2}$  and  $y' = \frac{1}{(x' - 2)^2} + 3$

Therefore, choose:

$y = y' - 3$  and  $x = x' - 2$

$\therefore y' = y + 3$  and  $x' = x + 2$

That is,  $(x, y) \rightarrow (x + 2, y + 3)$

**b** Write

$y = \frac{1}{x}$  and  $y' = \frac{1}{(x' + 2)} - 3$

Therefore, choose:

$$y = y' + 3 \text{ and } x = x' + 2$$

$$\therefore y' = y - 3 \text{ and } x' = x - 2$$

$$\text{That is, } (x, y) \rightarrow (x - 2, y - 3)$$

**c** Write

$$y = \sqrt{x} \text{ and } y' = \sqrt{x' + 4} + 2$$

Therefore, choose:

$$y = y' - 2 \text{ and } x = x' + 4$$

$$\therefore y' = y + 2 \text{ and } x' = x - 4$$

$$\text{That is, } (x, y) \rightarrow (x - 4, y + 2)$$

## Solutions to Exercise 3B

**1 a**  $x' = x, y' = 3y$

$$\therefore y = \frac{1}{x} \text{ maps to } \frac{y'}{\frac{3}{3}} = \frac{1}{x'}$$

The image is  $y = \frac{3}{x}$

**b**

**c**  $x' = 3x, y' = y$

$$\therefore y = \frac{1}{x} \text{ maps to } y' = \frac{1}{\frac{x'}{3}}$$

The image is  $y = \frac{3}{x}$

**2 a**  $x' = x, y' = 2y$

$$\therefore y = \frac{1}{x^2} \text{ maps to } \frac{y'}{\frac{2}{2}} = \frac{1}{(x')^2}$$

The image is  $y = \frac{2}{x^2}$

**b**

**c**  $x' = 2x, y' = y$

$$\therefore y = \frac{1}{x^2} \text{ maps to } y' = \frac{1}{\left(\frac{x'}{2}\right)^2}$$

The image is  $y = \frac{4}{x^2}$

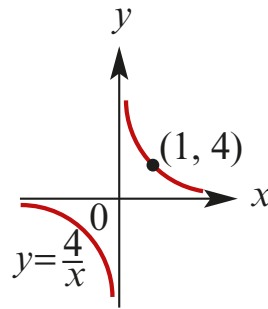
**3 a**  $y = 2\sqrt{x}$

**b**  $y = \sqrt{\frac{x}{2}}$

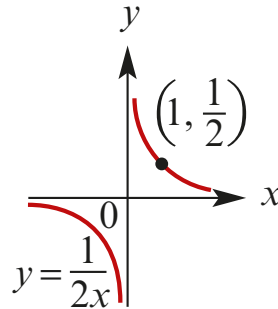
**4 a**  $y = 2x^3$

**b**  $y = \frac{x^3}{8}$

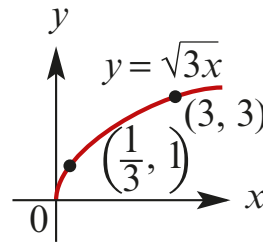
**5 a**



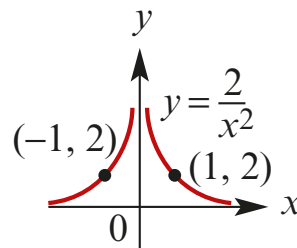
**b**



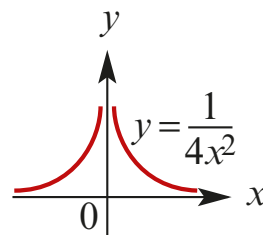
**c**

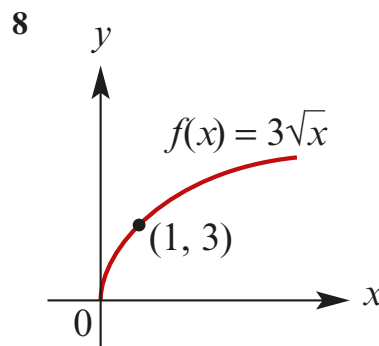
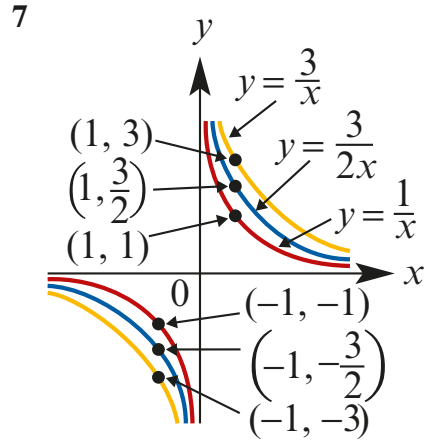
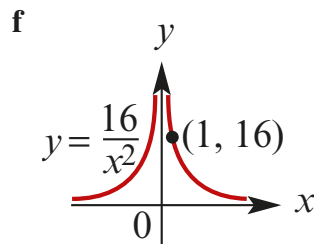
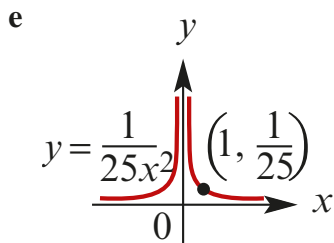
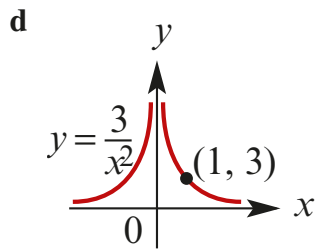
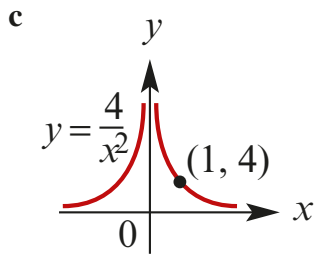
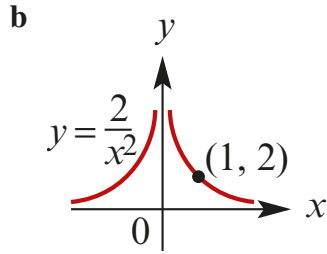


**d**



**6 a**





**9 a** Dilation factor  $\frac{1}{5}$  from the  $y$ -axis

**b** Dilation factor  $\sqrt{5}$  from the  $x$ -axis

**10 a** Let  $y = f(x) = \frac{1}{x^2}$  and

$$y' = f_1(x') = \frac{1}{(x')^2}$$

Then rewrite as  $y = \frac{1}{x^2}$  and

$$\frac{y'}{5} = \frac{1}{(x')^2}$$

Choose  $\frac{y'}{5} = y$  and  $x = x'$ .

One transformation is  $y' = 5y$  and  $x' = x$

A dilation of factor 5 from the  $x$ -axis.

**b** Let  $y = f(x) = \sqrt{x}$  and

$$y' = f_1(x') = 4\sqrt{x'}$$

Then rewrite as  $y = \sqrt{x}$  and

$$\frac{y'}{4} = \sqrt{x'}.$$

Choose  $\frac{y'}{4} = y$  and  $x = x'$ .

One transformation is  $y' = 4y$  and  $x' = x$

A dilation of factor 4 from the  $x$ -axis.

**c** Let  $y = f(x) = \sqrt{x}$  and

$$y' = f_1(x') = \sqrt{5x'}$$

Then rewrite as  $y = \sqrt{x}$  and

$$y' = \sqrt{5x'}.$$

Choose  $y' = y$  and  $x = 5x'$ .

One transformation is  $y' = y$  and

$$x' = \frac{1}{5}x$$

A dilation of factor  $\frac{1}{5}$  from the  $y$ -axis.

**d** Let  $y = f(x) = \sqrt{\frac{x}{3}}$  and

$$y' = f_1(x') = \sqrt{x'}$$

Then rewrite as  $y = \sqrt{\frac{x}{3}}$  and

$$y' = \sqrt{x'}.$$

Choose  $y' = y$  and  $\frac{x}{3} = x'$ .

One transformation is  $y' = y$  and

$$x' = \frac{1}{3}x$$

A dilation of factor  $\frac{1}{3}$  from the  $y$ -axis.

**e** Let  $y = f(x) = \frac{1}{4x^2}$  and

$$y' = f_1(x') = \frac{1}{(x')^2}$$

Then rewrite as  $y = \frac{1}{(2x)^2}$  and

$$y' = \frac{1}{(x')^2}.$$

Choose  $y' = y$  and  $2x = x'$ .

One transformation is  $y' = 5y$  and

$$x' = 2x$$

A dilation of factor 2 from the  $y$ -axis.

**11 a i**  $y = 4x^2$

**ii**  $y = \frac{2}{3}x^2$

**iii**  $y = (2x)^2 = 4x^2$

**iv**  $y = \left(\frac{x}{5}\right)^2 = \frac{1}{25}x^2$

**b i**  $y = \frac{4}{x^2}$

**ii**  $y = \frac{2}{3x^2}$

**iii**  $y = \frac{1}{4x^2}$

**iv**  $y = \frac{25}{x^2}$

**c i**  $y = 4\sqrt[3]{x}$

**ii**  $y = \frac{2}{3} \times \sqrt[3]{x}$

**iii**  $y = \sqrt[3]{2x}$

**iv**  $y = \sqrt[3]{\frac{x}{5}}$

**d i**  $y = \frac{4}{x^3}$

**ii**  $y = \frac{2}{3x^3}$

**iii**  $y = \frac{1}{8x^3}$

**iv**  $y = \frac{125}{x^3}$

**e i**  $y = \frac{4}{x^4}$



$$\text{ii } y = \frac{2}{3x^4}$$

$$\text{iii } y = \frac{1}{16x^4}$$

$$\text{iv } y = \frac{625}{x^4}$$

$$\text{f i } y = 4\sqrt[4]{x}$$

$$\text{ii } y = \frac{2}{3} \times \sqrt[4]{x}$$

$$\text{iii } y = \sqrt[4]{2x}$$

$$\text{iv } y = \sqrt[4]{\frac{x}{5}}$$

$$\text{g i } y = 4x^{\frac{1}{5}}$$

$$\text{ii } y = \frac{2}{3}x^{\frac{1}{5}}$$

$$\text{iii } y = (2x)^{\frac{1}{5}}$$

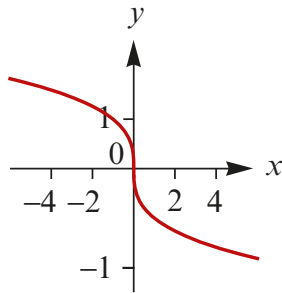
$$\text{iv } y = \left(\frac{x}{5}\right)^{\frac{1}{5}}$$

## Solutions to Exercise 3C

1 a  $y = -(x - 1)^2$

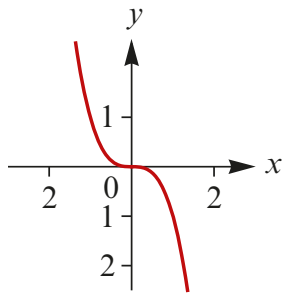
b  $y = (x + 1)^2$

2 a



Domain =  $\mathbb{R}$

b



Domain =  $\mathbb{R}$

3 Reflection in the y-axis

4 a i  $y = -x^3$

ii  $y = -x^3$

b i  $y = -\sqrt[3]{x}$

ii  $y = -\sqrt[3]{x}$

c i  $y = \frac{-1}{x^3}$

ii  $y = \frac{-1}{x^3}$

d i  $y = \frac{-1}{x^4}$

ii  $y = \frac{1}{x^4}$

e i  $y = -x^{\frac{1}{3}}$

ii  $y = -x^{\frac{1}{3}}$

f i  $y = -x^{\frac{1}{5}}$

ii  $y = -x^{\frac{1}{5}}$

g i  $y = -x^{\frac{1}{4}}$

ii  $y = (-x)^{\frac{1}{4}}$

## Solutions to Exercise 3D

1 Part a will be done with the method.

a i The mapping is

$$(x, y) \rightarrow (x + 2, 2y - 3) = (x', y')$$

$$\text{Hence } x' = x + 2 \text{ and } y' = 2y - 3$$

This implies  $x = x' - 2$  and

$$y = \frac{y' + 3}{2}$$

$\therefore y = x^2$  maps to

$$\frac{y' + 3}{2} = (x' - 2)^2$$

$$\text{That is, } y = 2(x - 2)^2 - 3$$

ii The mapping is

$$(x, y) \rightarrow (3x - 2, y - 4) = (x', y')$$

$$\text{Hence } x' = 3x - 2 \text{ and } y' = y - 4$$

This implies  $x = \frac{x' + 2}{3}$  and

$$y = y' + 4$$

$\therefore y = x^2$  maps to

$$y' + 4 = \left(\frac{x' + 2}{3}\right)^2$$

$$\text{That is, } y = \left(\frac{x + 2}{3}\right)^2 - 4$$

iii The mapping is

$$(x, y) \rightarrow (3x - 2, y - 4) = (x', y')$$

$$\text{Hence } x' = -x \text{ and } y' = 2y$$

This implies  $x = -x'$  and  $y = \frac{y'}{2}$

$$\therefore y = x^2 \text{ maps to } y = 2x^2$$

b i  $y = 2\sqrt[3]{x - 2} - 3$

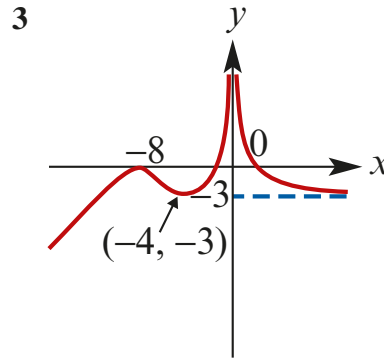
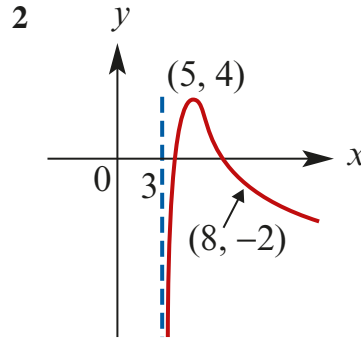
ii  $y = \sqrt[3]{\frac{x + 2}{3}} - 4$

iii  $y = -2\sqrt[3]{x}$

c i  $y = \frac{2}{(x - 2)^2} - 3$

ii  $y = \frac{9}{(x + 2)^2} - 4$

iii  $y = \frac{2}{x^2}$



4 a i  $y = -2(x - 3)^2 - 4$

ii  $y = -(2(x - 3)^2) - 4$   
 $y = -2(x - 3)^2 + 4$

iii  $y = 2(-(x - 3)^2) - 4$   
 $y = -2(x - 3)^2 - 4$

iv  $y = 2(-(x - 3)^2 - 4)$   
 $y = -2(x - 3)^2 - 8$

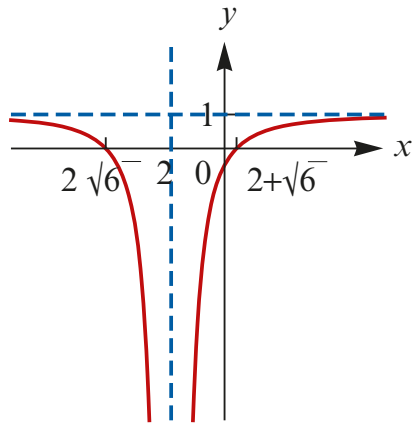
v  $y = -2((x - 3)^2 - 4)$   
 $y = -2(x - 3)^2 + 8$

vi  $y = 2(-((x - 3)^2 - 4))$   
 $y = -2(x - 3)^2 + 8$

b i  $y = -2\sqrt[3]{x - 3} - 4$

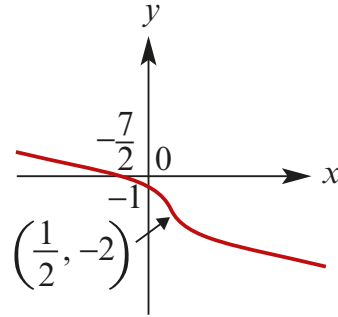
$$\begin{array}{ll}
 \text{ii} & y = -2\sqrt[3]{x-3} + 4 \\
 \text{iii} & y = -2\sqrt[3]{x-3} - 4 \\
 \text{iv} & y = -2\sqrt[3]{x-3} - 8 \\
 \text{v} & y = -2\sqrt[3]{x-3} + 8 \\
 \text{vi} & y = -2\sqrt[3]{x-3} + 8 \\
 \text{c} & \text{i} \quad y = \frac{-2}{(x-3)^2} - 4 \\
 & \text{ii} \quad y = \frac{-2}{(x-3)^2} + 4 \\
 & \text{iii} \quad y = \frac{-2}{(x-3)^2} - 4 \\
 & \text{iv} \quad y = \frac{-2}{(x-3)^2} - 8 \\
 & \text{v} \quad y = \frac{-2}{(x-3)^2} + 8 \\
 & \text{vi} \quad y = \frac{-2}{(x-3)^2} + 8 \\
 \text{d} & \text{i} \quad y = -2(x-3)^4 - 4 \\
 & \text{ii} \quad y = -2(x-3)^4 + 4 \\
 & \text{iii} \quad y = -2(x-3)^4 - 4 \\
 & \text{iv} \quad y = -2(x-3)^4 - 8 \\
 & \text{v} \quad y = -2(x-3)^4 + 8 \\
 & \text{vi} \quad y = -2(x-3)^4 + 8 \\
 \text{e} & \text{i} \quad y = \frac{-2}{(x-3)^3} - 4 \\
 & \text{ii} \quad y = \frac{-2}{(x-3)^3} + 4 \\
 & \text{iii} \quad y = \frac{-2}{(x-3)^3} - 4 \\
 \text{iv} & y = \frac{-2}{(x-3)^3} - 8 \\
 \text{v} & y = \frac{-2}{(x-3)^3} + 8 \\
 \text{vi} & y = \frac{-2}{(x-3)^3} + 8 \\
 \text{f} & \text{i} \quad y = \frac{-2}{(x-3)^4} - 4 \\
 & \text{ii} \quad y = \frac{-2}{(x-3)^4} + 4 \\
 & \text{iii} \quad y = \frac{-2}{(x-3)^4} - 4 \\
 & \text{iv} \quad y = \frac{-2}{(x-3)^4} - 8 \\
 & \text{v} \quad y = \frac{-2}{(x-3)^4} + 8 \\
 & \text{vi} \quad y = \frac{-2}{(x-3)^4} + 8 \\
 \text{g} & \text{i} \quad y = \frac{-2}{(x-3)^2} - 4 \\
 & \text{ii} \quad y = \frac{-2}{(x-3)^2} + 4 \\
 & \text{iii} \quad y = \frac{-2}{(x-3)^2} - 4 \\
 & \text{iv} \quad y = \frac{-2}{(x-3)^2} - 8 \\
 & \text{v} \quad y = \frac{-2}{(x-3)^2} + 8 \\
 & \text{vi} \quad y = \frac{-2}{(x-3)^2} + 8 \\
 \text{5} & y = -\sqrt{\frac{x+12}{3}}
 \end{array}$$

6 a



b  $y = 1 - \frac{6}{(x+2)^2}$

7 a



b  $y = (1 - 2x)^{\frac{1}{3}} - 2$

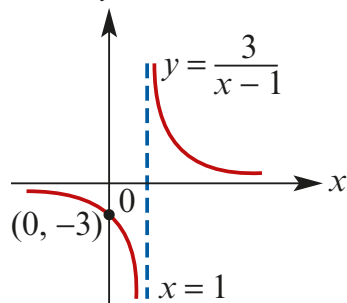
## Solutions to Exercise 3E

- 1 a i** Dilation of factor 2 from the  $x$ -axis, then translation 1 unit to the right and 3 units up
- ii** Reflection in the  $x$ -axis, then translation 1 unit to the left and 2 units up
- iii** Dilation of factor  $\frac{1}{2}$  from the  $y$ -axis, then translation  $\frac{1}{2}$  unit to the left and 2 units down
- b i** Dilation of factor 2 from the  $x$ -axis, then translation 3 units to the left
- ii** Translation 3 units to the left and 2 units up
- iii** Translation 3 units to the right and 2 units down
- c i** Translation 3 units to the left and 2 units up
- ii** Dilation of factor  $\frac{1}{3}$  from the  $y$ -axis and dilation of factor 2 from the  $x$ -axis
- iii** Reflection in the  $x$ -axis, then translation 2 units up
- 2 a** Translation 1 unit to the left and 6 units down
- b** Dilation of factor  $\frac{1}{2}$  from the  $x$ -axis, then translation  $\frac{3}{2}$  units up and 1 unit to the left
- c** Translation 1 unit to the left and 6 units up
- d** Dilation of factor  $\frac{1}{2}$  from the  $x$ -axis, then translation  $\frac{5}{2}$  units up and 1 unit to the left
- e** Dilation of factor 2 from the  $y$ -axis, then translation of 1 unit to the left and 6 units down
- 3 a** Dilation of factor  $\frac{1}{5}$  from the  $x$ -axis, then translation  $\frac{7}{5}$  units up and 3 units to the left
- b** Dilation of factor 3 from the  $y$ -axis, then translation 2 units to the right and 5 units down
- c** Reflection in the  $x$ -axis, dilation of factor  $\frac{1}{3}$  from the  $x$ -axis, translation  $\frac{7}{3}$  units up, dilation of factor 3 from the  $y$ -axis, translation 1 unit to the right
- d** Reflection in the  $y$ -axis, translation 4 units to the right, dilation of factor  $\frac{1}{2}$  from the  $x$ -axis
- e** Reflection in the  $y$ -axis, translation 4 units to the right, reflection in the  $x$ -axis, dilation of factor  $\frac{1}{2}$  from the  $x$ -axis, translation  $\frac{15}{2}$  units up

- 4 a** Dilation of factor 2 from the  $x$ -axis, then translation 1 unit to the right and 3 units up
- b** Dilation of factor 2 from the  $x$ -axis, then translation 4 units to the left and 7 units down
- c** Reflection in the  $y$ -axis and dilation of factor 4 from the  $x$ -axis (in either order), then translation 1 unit to the right and 5 units down
- d** Reflection in the  $x$ -axis, then translation 1 unit to the left and 2 units up
- e** Reflection in the  $y$ -axis and dilation of factor 2 from the  $x$ -axis (in either order), then translation 3 units up
- f** Translation 3 units to the left and 4 units down, then reflection in either axis and dilation of factor  $\frac{1}{2}$  from the  $x$ -axis (in either order)

## Solutions to Exercise 3F

1 a  $f(x) = \frac{3}{x-1}$



**Asymptotes**

$$y = 0$$

$$x - 1 = 0$$

$$x = 1$$

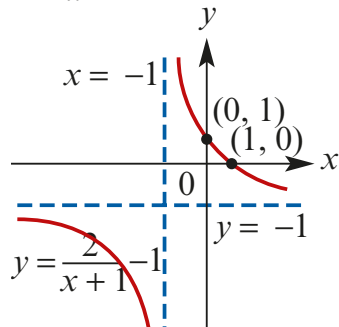
**Axis intercepts**

$$y = \frac{3}{0-1} = -3$$

$$0 = \frac{3}{x-1} \therefore \text{no } x\text{-axis intercept.}$$

$$\text{Range } \mathbb{R} \setminus \{0\}$$

b  $y = \frac{2}{x-1} - 1$



**Asymptotes**

$$y = 0 - 1 = -1$$

$$x + 1 = 0$$

$$x = -1$$

**Axis intercepts**

$$0 = \frac{2}{x+1} - 1$$

$$x + 1 = 2$$

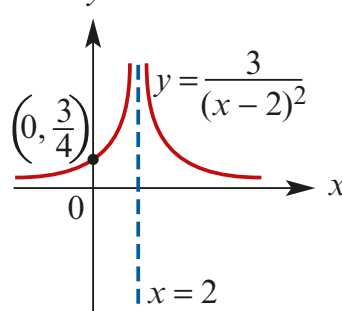
$$x = 1$$

$$y = \frac{2}{0+1} - 1$$

$$y = 1$$

$$\text{Range } \mathbb{R} \setminus \{-1\}$$

c  $y = \frac{3}{(x-2)^2}$



**Asymptotes**

$$y = 0$$

$$x - 2 = 0$$

$$x = 2$$

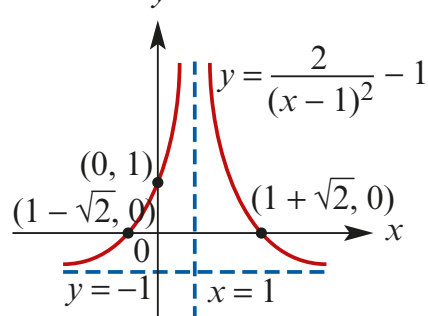
**Axis intercepts**

$$0 = \frac{3}{(x-2)^2} \therefore \text{no } x\text{-axis intercept.}$$

$$y = \frac{3}{(-2)^2} = \frac{3}{4}$$

$$\text{Range} = \mathbb{R}^+$$

d  $y = \frac{2}{(x-1)^2} - 1$



**Asymptotes**



$$y = 0 - 1 = -1$$

$$x - 1 = 0$$

$$x = 1$$

**Axis intercepts**

$$0 = \frac{2}{(x-1)^2} - 1$$

$$x - 1 = \pm \sqrt{2}$$

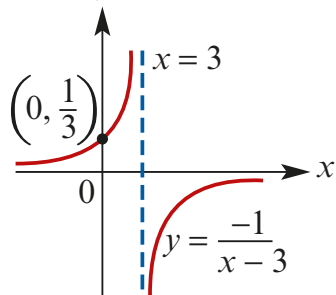
$$x = 1 \pm \sqrt{2}$$

$$y = \frac{2}{(0-1)^2} - 1$$

$$y = 1$$

$$\text{Range} = (-1, \infty)$$

e  $y = \frac{-1}{x-3}$



**Asymptotes**

$$y = 0$$

$$x - 3 = 0$$

$$x = 3$$

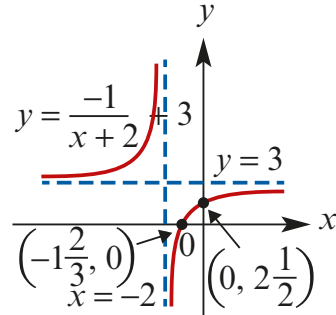
**Axis intercepts**

$$0 = \frac{-1}{x-3} \therefore \text{no } x\text{-axis intercept.}$$

$$y = \frac{-1}{0-3} = \frac{1}{3}$$

$$\text{Range} = \mathbb{R} \setminus \{0\}$$

f  $y = \frac{-1}{x+2} + 3$



**Asymptotes**

$$y = 0 + 3 = 3$$

$$x + 2 = 0$$

$$x = -2$$

**Axis intercepts**

$$y = \frac{-1}{0+2} + 3 = \frac{5}{2}$$

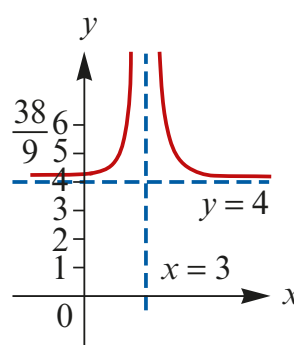
$$0 = \frac{-1}{x+2} + 3$$

$$3x + 6 = 1$$

$$x = \frac{-5}{3}$$

$$\text{Range} = \mathbb{R} \setminus \{3\}$$

g  $y = \frac{2}{(x-3)^2} + 4$



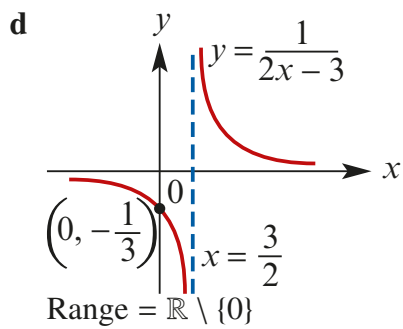
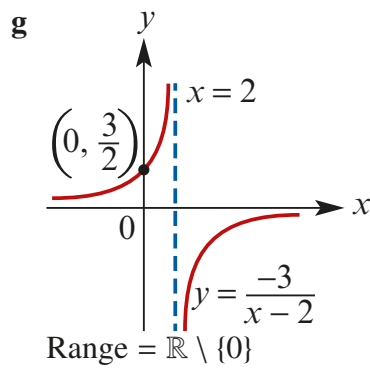
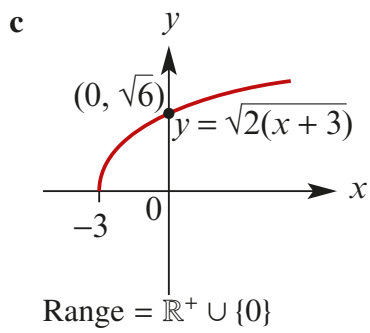
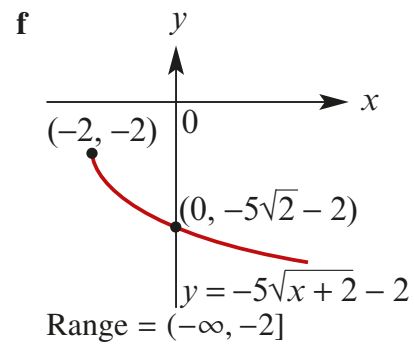
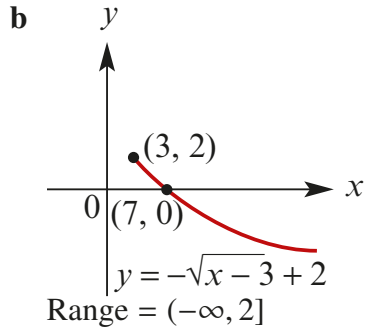
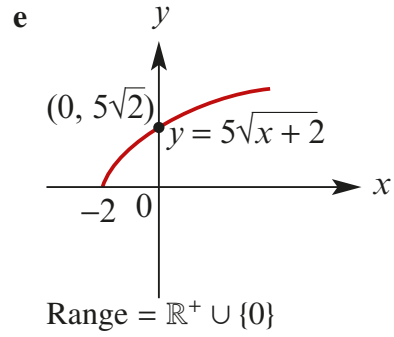
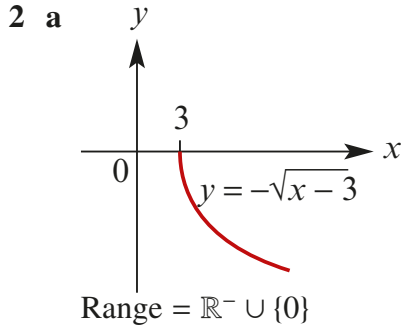
**Asymptotes**

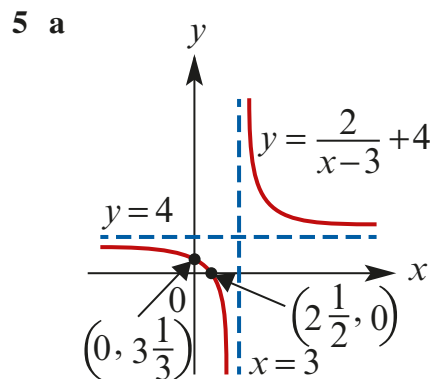
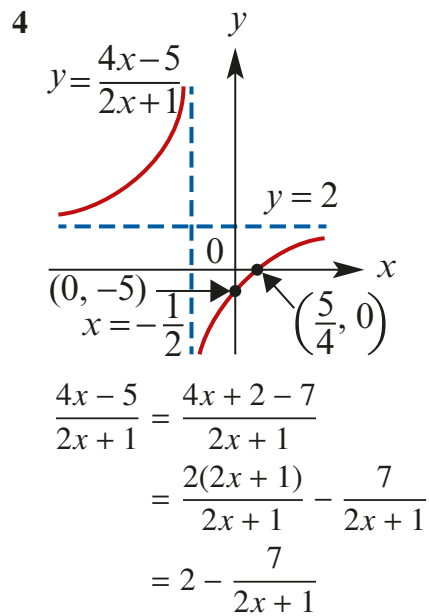
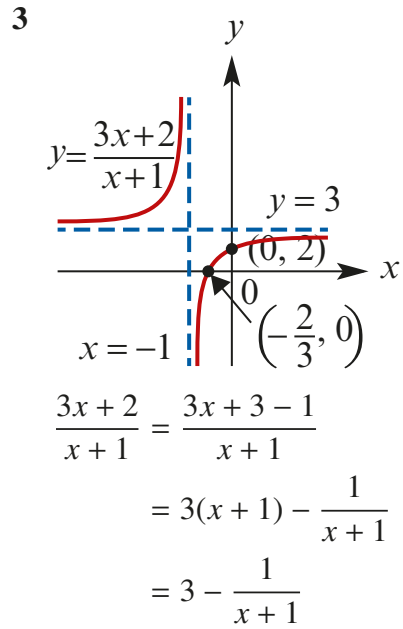
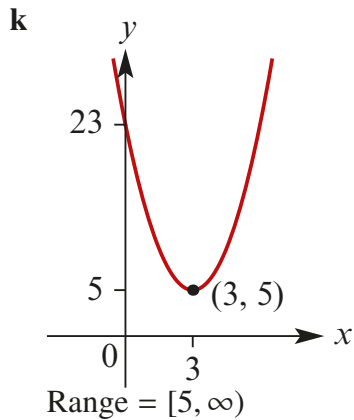
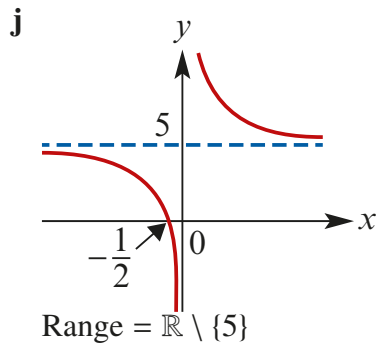
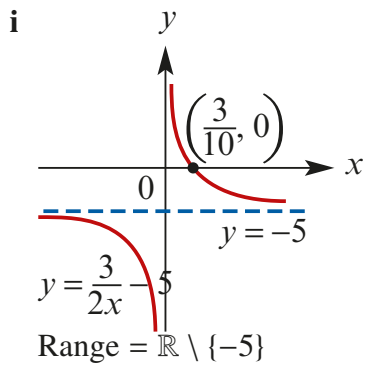
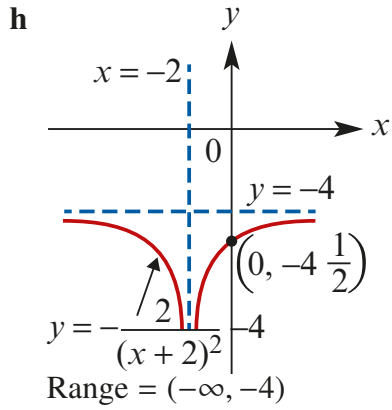
$$x - 3 = 0$$

$$x = 3$$

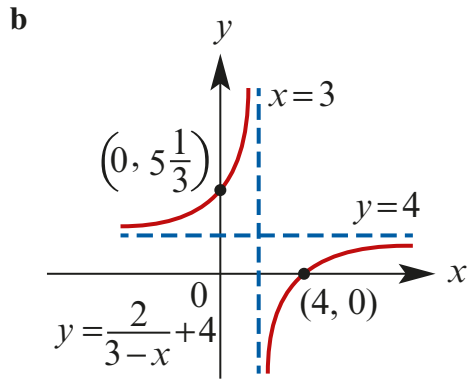
$$y = 0 + 4 = 4$$

$$\text{Range} = (4, \infty)$$

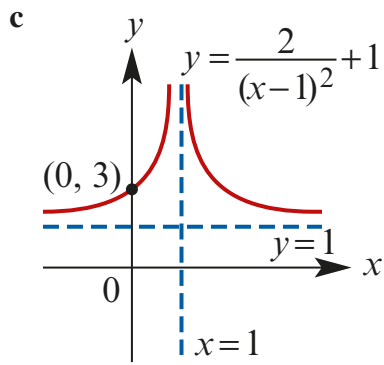




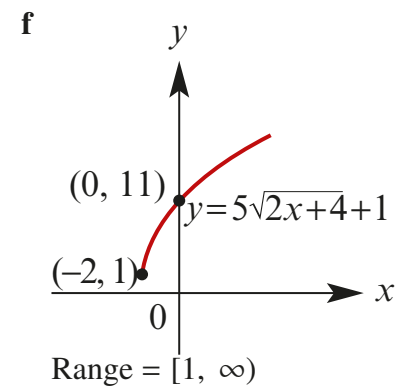
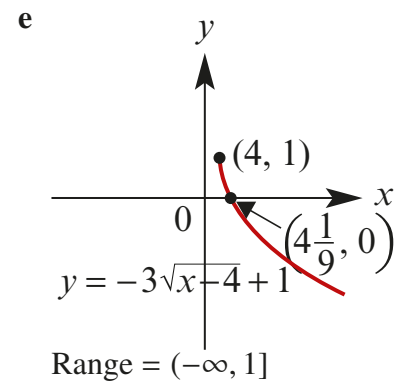
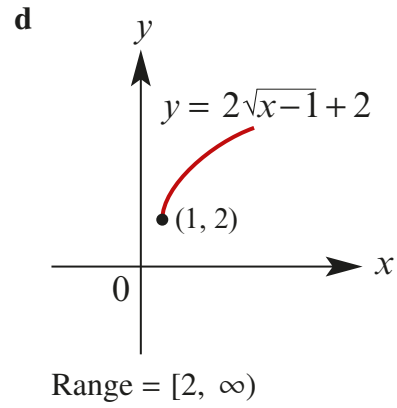
Range =  $\mathbb{R} \setminus \{4\}$



Range =  $\mathbb{R} \setminus \{4\}$

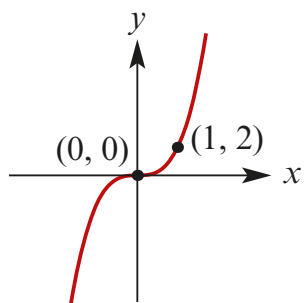


Range =  $(1, \infty)$

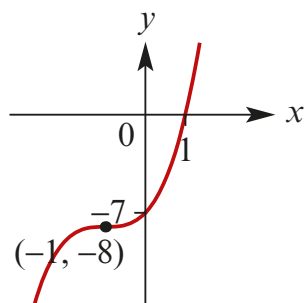


## Solutions to Exercise 3G

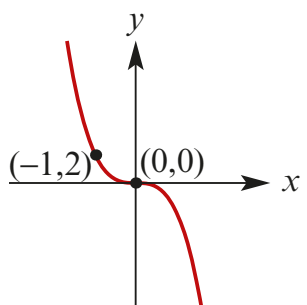
1 a



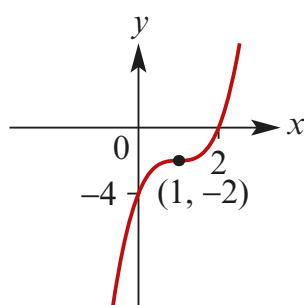
e



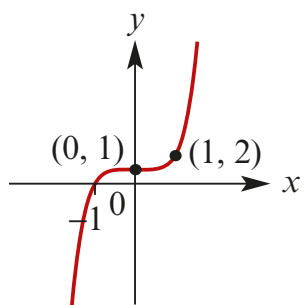
b



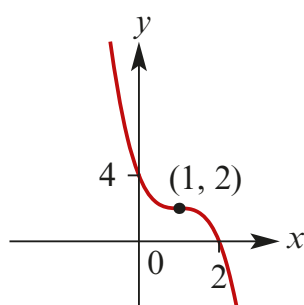
f



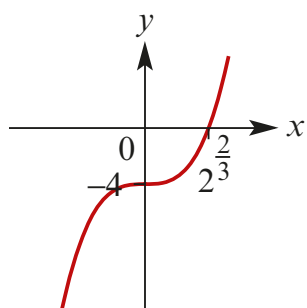
c

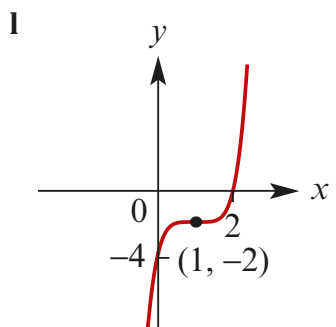
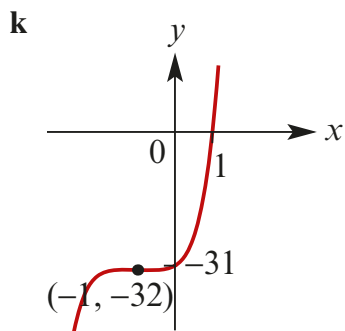
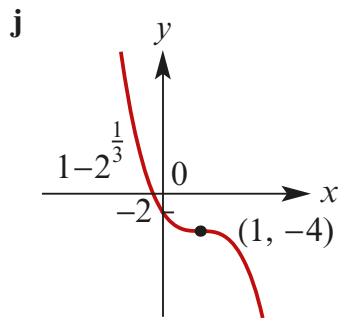
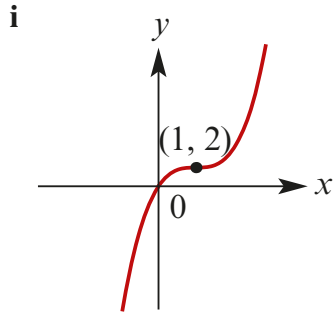
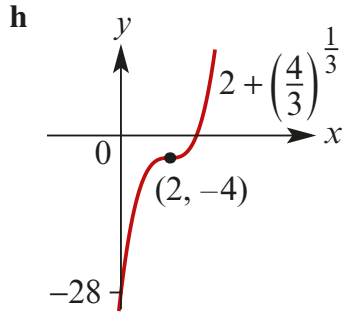


g



d





**2**  $h = 0$  and  $k = 4$   
 $y = ax^3 + 4$   
 When  $x = 1, y = 1$   
 $\therefore 1 = a + 4$   
 $\therefore a = -3$   
 $\therefore y = -3x^3 + 4$

**3 a**  $y = 3x^3$

**b**  $y = (x + 1)^3 + 1$

**c**  $y = -(x - 2)^3 - 3$

**d**  $y = 2(x + 1)^3 - 2$

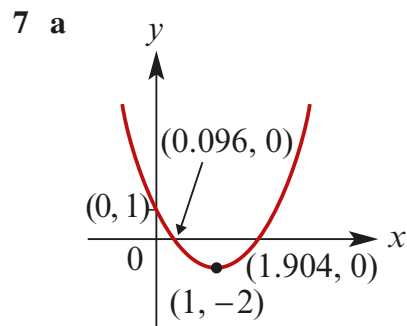
**e**  $y = \frac{x^3}{27}$

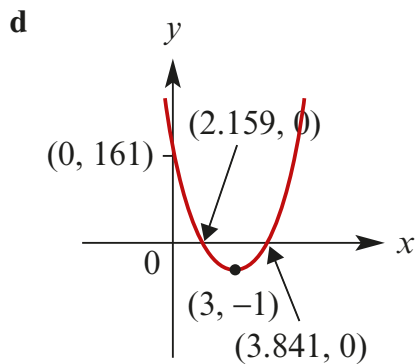
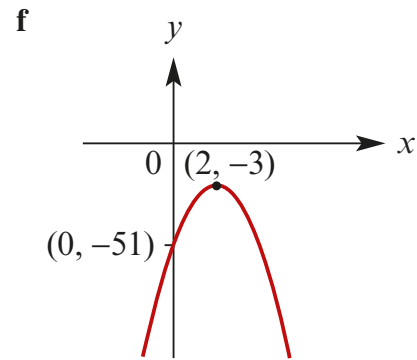
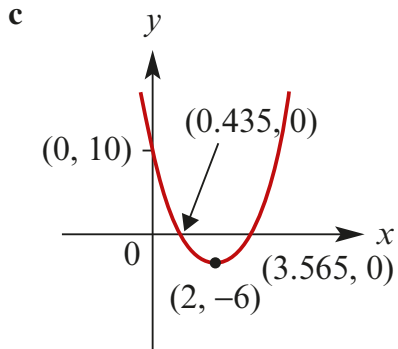
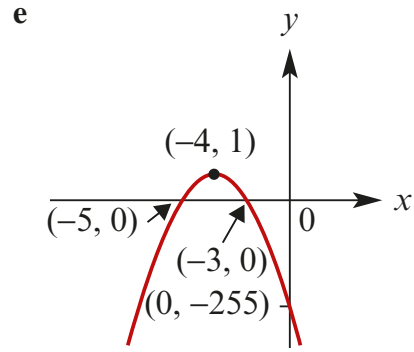
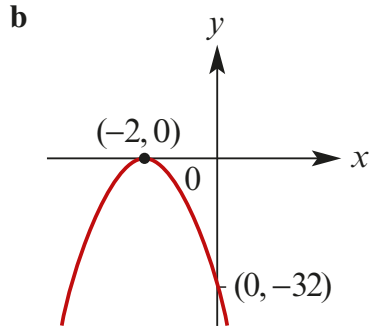
**4 a**  $y = \frac{(3 - x)^3}{27} + 1$

**b** Dilation of factor 3 from the  $x$ -axis, reflection in the  $x$ -axis, then translation 1 unit to the left and 4 units up

**5**  $y = \frac{(x + 2)^4}{16} - 1$

**6** Dilation of factor 3 from the  $x$ -axis, reflection in the  $x$ -axis, then translation 1 unit to the right and 5 units up





**8**  $h = -2$  and  $k = 3$   
 Passes through  $(0, -6)$   
 $y = a(x + 2)^4 + 3$   
 $-6 = 16a + 3$   
 $a = -\frac{9}{16}$   
 $y = -\frac{9}{16}(x + 2)^4 + 3$

**9**  $h = 1$  and  $k = 7$   
 Passes through  $(0, 23)$   
 $y = a(x - 1)^4 + 7$   
 $23 = a + 7$   
 $a = 16$   
 $y = 16(x - 1)^4 + 7$

## Solutions to Exercise 3H

$$1 \quad 4 = a + b \dots (1)$$

$$1 = \frac{a}{3} + b \dots (2)$$

Equation (1) – Equation (2)

$$3 = \frac{2a}{3}$$

$$a = \frac{9}{2}$$

From (1)

$$b = -\frac{1}{2}$$

$$2 \quad \text{Asymptotes: } x = 1, y = 2$$

$$x + b = 0 \quad y = B$$

$$1 + b = 0 \quad B = 2$$

$$b = -1$$

**Point:** (0, 1)

$$1 = \frac{A}{-1} + 2$$

$$A = 1$$

$$3 \quad 1 = A + B \dots (1)$$

$$6 = 3A + B \dots (2)$$

Equation (2) – Equation (1)

$$5 = 2A$$

$$A = \frac{5}{2}$$

From (1)

$$B = -\frac{3}{2}$$

$$4 \quad y = A\sqrt{x} + B$$

$$5 = A\sqrt{1} + B \dots (1)$$

$$= A + B$$

$$11 = A\sqrt{16} + B \dots (2)$$

$$= 4A + B$$

Equation (2) – Equation (1)

$$\Rightarrow 3A = 6$$

$$A = 2$$

From (1)

$$\Rightarrow 5 = 2 + B$$

$$B = 3$$

$$5 \quad y = \frac{A}{x^2} + B$$

$$1 = \frac{A}{1^2} + B \dots (1)$$

$$= A + B$$

$$7 = \frac{A}{0.5^2} + B \dots (2)$$

$$= 4A + B$$

Equation (2) – Equation (1)

$$\Rightarrow 3A = 6$$

$$A = 2$$

From (1)

$$1 = 2 + B$$

$$B = -1$$

$$6 \quad y = \frac{A}{(x+b)^2} + B$$

**Asymptotes**

$$x = -2 \quad y = -3$$

$$x + b = 0 \quad y = 0 + B$$



$$-2 + b = 0 \quad B = -3$$

$$b = 2$$

**Point:** (0, -1)

$$-1 = \frac{A}{2^2} - 3$$

$$A = 8$$

7

$$y = \frac{a}{x^3} + b$$

$$-1 = \frac{a}{1^3} + b = a + b \dots (1)$$

$$\frac{3}{4} = \frac{a}{2^3} + b = \frac{1}{8}a + b \dots (2)$$

Equation (2) - Equation (1)

$$\Rightarrow \frac{-7}{8}a = \frac{7}{4}$$

$$a = -2$$

From (1)

$$\Rightarrow -1 = -2 + b$$

$$b = 1$$

8

$$y = ax^{\frac{1}{3}} + b$$

$$-8 = a + b \dots (1)$$

$$4 = -a + b \dots (2)$$

Equation (2) + Equation (1)

$$2b = -4$$

$$b = -2$$

From (2)

$$a = -6$$

## Solutions to Exercise 3I

$$1 \text{ a } \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$\therefore (1, -2) \rightarrow (1, -6)$$

$$1 \text{ b } \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\therefore (1, -2) \rightarrow (2, -2)$$

$$1 \text{ c } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore (1, -2) \rightarrow (1, 2)$$

$$1 \text{ d } \begin{bmatrix} 11 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\therefore (1, -2) \rightarrow (-1, -2)$$

$$1 \text{ e } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\therefore (1, -2) \rightarrow (-2, 1)$$

$$2 \text{ a } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2 \text{ b } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$2 \text{ c } \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$2 \text{ d } \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$2 \text{ e } \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$3 \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\therefore a = 2 \text{ and } c = -1$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$\therefore b = -2 \text{ and } d = 5$$

$$\begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 7 \end{bmatrix}$$

$$4 \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\therefore a = 2 \text{ and } c = -1$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\therefore b = 4 \text{ and } d = -2$$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

$$6 \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \left( \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x' - 2 \\ y' - 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' - 2 \\ \frac{y' - 1}{3} \end{bmatrix}$$

$$x = x' - 2, \quad y = \frac{y' - 1}{3}$$

$$7 \quad \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x\frac{1}{4} \\ y\frac{1}{2} \end{bmatrix}$$

$$x = -x\frac{1}{4}, \quad y = y\frac{1}{2}$$

$$\text{but } y = x^2 + x + 2$$

$$\frac{y'}{2} = \frac{(x')^2}{16} - \frac{x'}{4} + 2$$

$$y' = \frac{(x')^2}{8} - \frac{x'}{2} + 4$$

$$8 \quad \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x = x', \quad y = \frac{-y'}{2}$$

$$\text{but } y = x^3 + 2x$$

$$\frac{-y'}{2} = (x')^3 + 2x'$$

$$y' = -2(x')^3 - 4x'$$

$$9 \quad \begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x = \frac{-y'}{2}, \quad y = \frac{x'}{3}$$

$$\text{but } y = 2x + 3$$

$$\frac{x'}{3} = -y' + 3$$

$$y' = 3 - \frac{x'}{3}$$

$$10 \quad \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{-y'}{2} \\ \frac{x'}{2} \end{bmatrix}$$

$$x = \frac{-y'}{3}, \quad y = \frac{x'}{2}$$

$$\text{but } y = -2x + 4$$

$$\frac{x'}{2} = \frac{2y'}{3} + 4$$

$$\frac{2y'}{3} = \frac{x'}{2} - 4$$

$$y' = \frac{3x'}{4} - 6$$

$$11 \quad \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} \left( \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}y' + 1 \\ x' - 2 \end{bmatrix}$$

$$x = \frac{-1}{3}y' + 1, \quad y = x' - 2$$

$$\text{but } y = -2x + 6$$

$$x' - 2 = \frac{2y'}{3} - 2 + 6$$

$$\frac{2y'}{3} = x' - 6$$

$$y' = \frac{3x'}{2} - 9$$

**12**  $\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \left( \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left( \begin{bmatrix} x' + 3 \\ y' - 2 \end{bmatrix} \right)$$

$$x = y' - 2, \quad y = \frac{-1}{3}(x' + 3)$$

but  $y = -2x + 6$

$$-\frac{1}{3}(x' + 3) = -2y' + 4 + 6$$

$$-2y' = \frac{-x'}{3} - 1 - 10$$

$$y' = \frac{x'}{6} + \frac{11}{2}$$

**13**  $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left( \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x' + 1}{4} \\ \frac{y' - 4}{2} \end{bmatrix}$$

$$x = \frac{x' + 1}{4}, \quad y = \frac{y' - 4}{2}$$

but  $y = -2x^3 + 6x$

$$\frac{y' - 4}{2} = -2 \left( \frac{x' + 1}{4} \right)^3 + 6 \left( \frac{x' + 1}{4} \right)$$

$$14 \quad \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & -1 \\ 0 & -1 \\ & 2 \end{bmatrix} \left( \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-(x'+1)}{3} \\ \frac{-(y'-2)}{2} \end{bmatrix}$$

$$x = \frac{-x'-1}{3}, \quad y = \frac{2-y'}{2}$$

$$\text{but } y = -2x^3 + 6x^2 + 2$$

$$\frac{2-y'}{2} = -2 \left( \frac{-(x'+1)}{3} \right)^3 + 6 \left( \frac{-(x'+1)}{3} \right)^2 + 2$$

$$\frac{2-y'}{2} = 2 \left( \frac{(x'+1)}{3} \right)^3 + 6 \left( \frac{(x'+1)}{3} \right)^2 + 2$$

$$15 \quad \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\therefore x' = 4x + 3 \text{ and } y' = -2y - 1$$

$$\therefore x = \frac{x'-3}{4} \text{ and } y = -\frac{y'+1}{2}$$

$$\therefore y = x^2 \text{ maps to } -\frac{y'+1}{2} = \left( \frac{x'-3}{4} \right)^2$$

$$\text{The image is: } y = -\frac{1}{8}(x-3)^2 - 1$$

$$16 \quad \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\therefore x' = 3x - 2 \text{ and } y' = -y - 1$$

$$\therefore x = \frac{x'+2}{3} \text{ and } y = -y' - 1$$

$$\therefore y = \frac{1}{x^2} \text{ maps to } -y' - 1 = \frac{1}{\left( \frac{x'+2}{3} \right)^2}$$

$$\text{The image is: } y = -\frac{9}{(x+2)^2} - 1$$

$$17 \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\therefore x' = 2x + 3 \text{ and } y' = y + 4$$

$$\therefore x = \frac{x'-3}{2} \text{ and } y = y' - 4$$

$$\therefore y = 3(x-2)^2 - 4 \text{ maps to } y' - 4 = 3 \left( \frac{x'-3}{2} - 2 \right)^2 - 4$$

$$\text{The image is: } y = \frac{3}{4}(x-7)^2$$

## Solutions to Exercise 3J

1 a 1

$$\mathbf{b} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

c 2

$$\mathbf{d} \begin{bmatrix} 1 & 1 \\ -\frac{3}{2} & -1 \end{bmatrix}$$

2 a  $\det = -3 + 4 = 1$

$$\text{inverse} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$$

b  $\det = 12 + 2 = 14$

$$\text{inverse} = \begin{bmatrix} \frac{2}{7} & \frac{-1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$$

c  $\det = k$

$$\text{inverse} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$$

3 a  $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$ ,  $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b

$$AB = \begin{bmatrix} 2+3 & 0+1 \\ 0-3 & 0-1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$$

$$\det(AB) = -5 + 3 = -2$$

$$(AB)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -3 & -5 \end{bmatrix}$$

$$\begin{aligned} \mathbf{c} \quad A^{-1}B^{-1} &= \frac{1}{2} \begin{bmatrix} 1-3 & 0+1 \\ 0+6 & 0-2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -2 & 1 \\ 6 & -2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B^{-1}A^{-1} &= \frac{1}{2} \begin{bmatrix} 1+0 & 1+0 \\ -3+0 & -3-2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -3 & -5 \end{bmatrix} \end{aligned}$$

$$(AB)^{-1} = B^{-1}A^{-1} \neq A^{-1}B^{-1}$$

4 a  $\det(A) = 4 - 6 = -2$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$$

$$\mathbf{b} \quad AX = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$IX = A^{-1} \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$X = \frac{1}{2} \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$X = \frac{1}{2} \begin{bmatrix} 0 & 14 \\ 2 & 16 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$$

$$\mathbf{c} \quad YA = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$YI = \frac{1}{2} \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$$

$$Y = \frac{1}{2} \begin{bmatrix} -3+8 & 9-16 \\ -1+12 & 3-24 \end{bmatrix}$$

$$Y = \frac{1}{2} \begin{bmatrix} 5 & -7 \\ 11 & -21 \end{bmatrix}$$

5 a  $AX + B = C$

$$AX = C - B$$

$$X = A^{-1}(C - B)$$

$$\det A = 18 - 2 = 16$$

$$A^{-1} = \frac{1}{16} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$X = \frac{1}{16} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix} \left( \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix} \right)$$

$$X = \frac{1}{16} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$X = \frac{1}{16} \begin{bmatrix} (6 \times -1) + (-2 \times 0) & (6 \times 5) + (-2 \times 4) \\ (-1 \times -1) + (3 \times 0) & (-1 \times 5) + (3 \times 4) \end{bmatrix}$$

$$= \frac{1}{16} \begin{bmatrix} -6 & 22 \\ 1 & 7 \end{bmatrix}$$

**b**  $YA = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$

$$Y = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix} \times \frac{1}{16} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$Y = \frac{1}{16} \begin{bmatrix} -11 & 17 \\ -4 & 12 \end{bmatrix}$$

**6**  $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\therefore x = -\frac{1}{3}x' \text{ and } y = y'$$

$$\therefore y = x^2 + 2x \text{ maps to } y' = \frac{1}{9}(x')^2 - \frac{2}{3}x'$$

$$\text{The image is } y = \frac{x^2}{9} - \frac{2x}{3}$$

**7**  $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\therefore x = \frac{1}{3}x' \text{ and } y = -\frac{1}{3}y'$$

$$\therefore y = x^3 + 4 \text{ maps to } -\frac{1}{3}y' = \frac{1}{27}(x')^3 + 4$$

$$\text{The image is } y = -\frac{x^3}{9} - 12$$

**8**  $\begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 0 & -4 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\therefore x = -\frac{1}{2}y' \text{ and } y = \frac{1}{4}x'$$

$$\therefore y = 5x + 3 \text{ maps to } \frac{1}{4}x' = -5\left(\frac{1}{2}y'\right) + 3$$

$$\text{The image is } x + 10y = 12$$

**9**  $\begin{bmatrix} 0 & 5 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{15} \begin{bmatrix} 0 & -5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\therefore x = \frac{1}{3}y' \text{ and } y = -\frac{1}{5}x'$$

$$\therefore y = -2x + 7 \text{ maps to}$$

$$-\frac{1}{5}x' = -2\left(\frac{1}{3}y'\right) + 7$$

$$\text{The image is } 3x - 10y = 105$$

**10**  $\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{X}'$

Therefore

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{X}' - \mathbf{B}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\therefore x = -\frac{1}{4}y' + 2$$

$$\text{and } y = -\frac{1}{2}x' - 2$$

Hence  $y = -2x + 8$  is mapped to

$$-\frac{1}{2}x' - 2 = -2\left(-\frac{1}{4}y' + 2\right) + 8$$

$$\text{The image is } x + y = -12$$

**11**  $\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \left( \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left( \begin{bmatrix} x' + 3 \\ y' - 2 \end{bmatrix} \right)$$

$$x = y' - 2, \quad y = \frac{-1}{3}(x' + 3)$$

$$\text{but } y = -2x + 6$$

$$-\frac{1}{3}(x' + 3) = -2y' + 4 + 6$$

$$-2y' = \frac{-x'}{3} - 1 - 10$$

$$y' = \frac{x'}{6} + \frac{11}{2}$$

$$12 \quad \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left( \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x' + 1}{4} \\ \frac{y' - 4}{2} \end{bmatrix}$$

$$x = \frac{x' + 1}{4}, \quad y = \frac{y' - 4}{2}$$

$$\text{but } y = -2x^3 + 6x$$

$$\frac{y' - 4}{2} = -2 \left( \frac{x' + 1}{4} \right)^3 + 6 \left( \frac{x' + 1}{4} \right)$$

$$13 \quad \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \left( \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-(x' + 1)}{3} \\ \frac{-(y' - 2)}{2} \end{bmatrix}$$

$$x = \frac{-x' - 1}{3}, \quad y = \frac{2 - y'}{2}$$

$$\text{but } y = -2x^3 + 6x^2 + 2$$

$$\frac{2 - y'}{2} = -2 \left( \frac{-(x' + 1)}{3} \right)^3$$

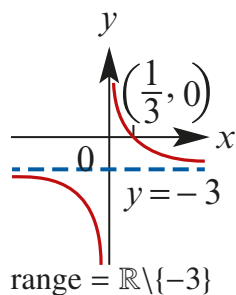
$$+ 6 \left( \frac{-(x' + 1)}{3} \right)^2 + 2$$

$$\frac{2 - y'}{2} = 2 \left( \frac{(x' + 1)}{3} \right)^3 + 6 \left( \frac{(x' + 1)}{3} \right)^2 + 2$$

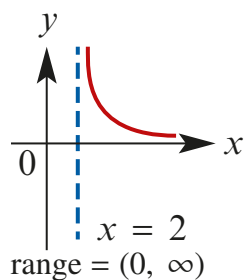


## Solutions to technology-free questions

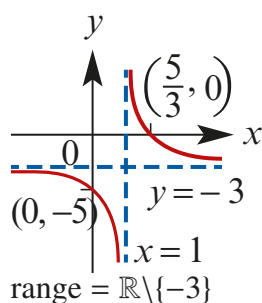
- 1 a**  $y = \frac{1}{x} - 3, x \neq 0$   
 no  $y$  intercept  
 $y = 0: x = \frac{1}{3}$   
 asymptotes:  $x = 0$  &  $y = -3$



- b**  $y = \frac{1}{x-2}, x > 2$   
 no intercepts  
 asymptotes:  $x = 2$  &  $y = 0$

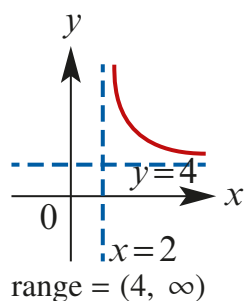


- c**  $y = \frac{2}{x-1} - 3, x \neq 1$   
 $x = 0: y = -5$   
 $y = 0: \frac{2}{x-1} = 3 \Rightarrow x = \frac{5}{3}$   
 asymptotes:  $x = 1$  &  $y = -3$

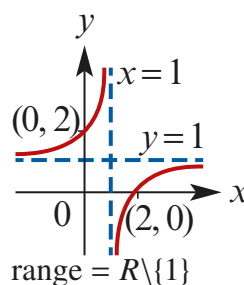


- d**  $y = -\frac{3}{2-x} + 4, x > 2$   
 no intercepts

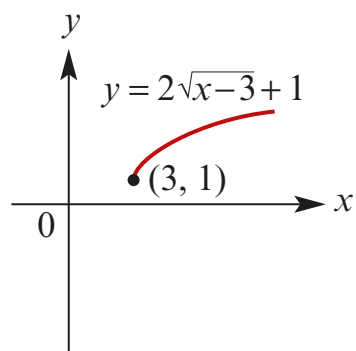
asymptotes:  $x = 2$  &  $y = 4$



- e**  $y = 1 - \frac{1}{x-1}, x \neq 1$   
 $x = 0: y = 2$   
 $y = 0: \frac{1}{x-1} = 1 \Rightarrow x = 2$   
 asymptotes:  $x = 1$  &  $y = 1$

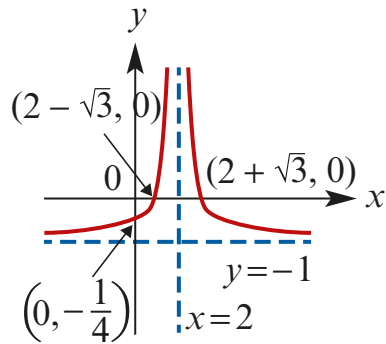


- 2 a**  $y = 2\sqrt{x-3} + 1$   
 $x \geq 3; y \geq 1$ ; endpoint  $(3, 1)$



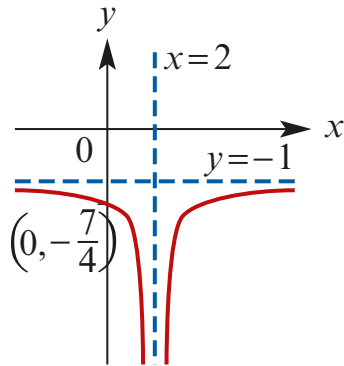
- b**  $y = \frac{3}{(x-2)^2} - 1$   
 $x = 0: y = -\frac{1}{4}$   
 $y = 0: \frac{3}{(x-2)^2} = 1$

$(x - 2)^2 = 3 \Rightarrow x = 2 \pm \sqrt{3}$   
 asymptotes:  $x = 2$  &  $y = -1$

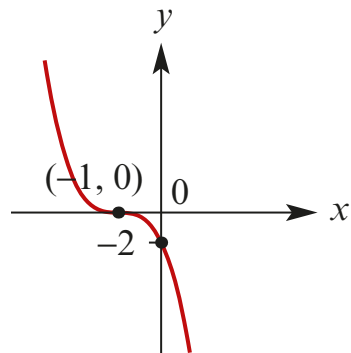


**c**  $y = \frac{-3}{(x - 2)^2} - 1$

This is a reflection in the line  $y = -1$  of the graph in part **b** above. There are no  $x$  intercepts, the  $y$  intercept is at  $y = -\frac{7}{4}$  and the asymptotes are the same.

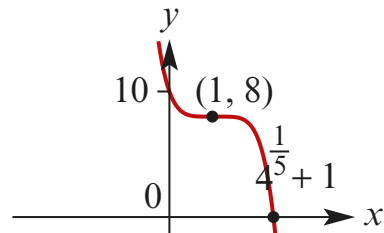


**3 a**



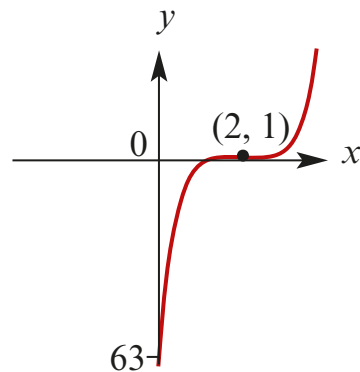
Point of zero gradient  $(-1, 0)$ ;  
 Axis intercepts  $(-1, 0), (0, -2)$

**b**



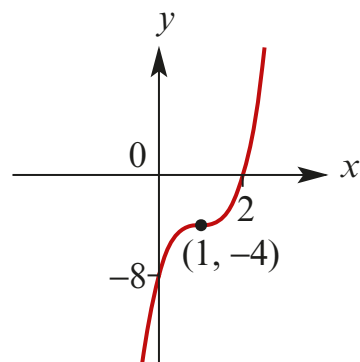
Point of zero gradient  $(1, 8)$ ;  
 Axis intercepts  $(4^{\frac{1}{5}}, 0), (0, 10)$

**c**



Point of zero gradient  $(2, 1)$ ;  
 Axis intercepts  $(-\frac{1}{2})^{\frac{1}{5}} + 2, 0), (0, -63)$

**d**



Point of zero gradient  $(1, -4)$ ;  
 Axis intercepts  $(2, 0), (0, -8)$

**4**  $y = a\sqrt{x} + b$

$(1, 6)$  and  $(16, 12)$  lie on the curve

$$6 = a + b \dots (1)$$

$$12 = 4a + b \dots (2)$$

Subtract (1) from (2)

$$6 = 3a$$

$$a = 2$$

$$\therefore b = 4$$

$$5 \quad \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \therefore x' = x - 4$$

$$\text{and } y' = -2y - 1 \therefore x = x' + 4$$

$$\text{and } y = -\frac{y' + 1}{2} \therefore \text{the image of}$$

$$y = \sqrt{x} \text{ under this transformation is}$$

$$-\frac{y' + 1}{2} = \sqrt{x' + 4}$$

$$\text{The image is } y = -2\sqrt{x + 4} - 1$$

$$6 \quad \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\therefore x' = 3x - 4 \text{ and } y' = -y - \frac{1}{2}$$

$$\therefore x = \frac{x' + 4}{3} \text{ and } y = -y' - \frac{1}{2}$$

$\therefore$  the image of  $y = 2\sqrt{x - 4} + 3$  under this transformation is

$$-y' - \frac{1}{2} = 2\sqrt{\frac{x' + 4}{3} - 4} + 3$$

$$\text{The image is } y = -2\sqrt{\frac{x - 8}{3}} - \frac{7}{2}$$

$$7 \quad (1, 3): 3 = a + b \dots (1)$$

$$(3, 7): 7 = \frac{a}{3} + b \dots (2)$$

Subtract (1) from (2):

$$\frac{a}{3} - a = 4$$

$$-\frac{2a}{3} = 4$$

$$a = -6$$

Substitute into (1):  $b = 9$

$$8 \quad \mathbf{a} \quad (x, y) \rightarrow (-x, y) \rightarrow (-2x, y) \rightarrow (-2x + 4, y + 6) = (x', y')$$

$$\therefore x' = -2x + 4 \text{ and } y' = y + 6$$

$$\therefore x = \frac{4 - x'}{2} \text{ and } y = y' - 6.$$

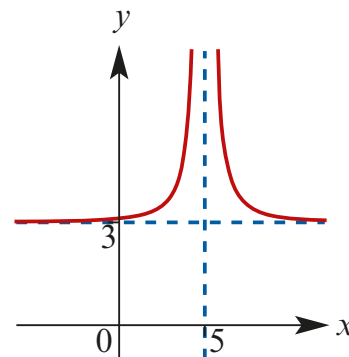
$$\therefore y = -x^2 \text{ maps to}$$

$$y' - 6 = -\left(\frac{4 - x'}{2}\right)^2 \text{ That is to}$$

$$y = -\left(\frac{x - 4}{2}\right)^2 + 6$$

**b** Reflection in the  $x$ -axis, dilation of factor 4 from the  $x$ -axis, then translate 1 unit to the left and 6 units up

**9** Dilation of factor 3 from the  $x$ -axis, then translation 5 units to the right and 3 units up



Asymptotes  $x = 5, y = 3$ ; Intercept  $(0, \frac{78}{25})$

**10** Dilation of factor  $\frac{1}{2}$  from the  $x$ -axis, then translation  $\frac{3}{2}$  units up

**11** Dilation of factor  $\frac{1}{2}$  from the  $x$ -axis, then translation 3 units to the left and 2 units down

## Solutions to multiple-choice questions

**1 B**  $(3, -4) \rightarrow (3, -1) \rightarrow (3, 1)$

**2 B**  $y = x^3 + 4 \rightarrow y = x^3 + 1 \rightarrow y = (x - 2)^3 + 1$

**3 B**

**4 E**  $y = x^2 \rightarrow y = -x^2 \rightarrow y = -(x + 4)^2 - 3$

**5 D** Asymptotes at  $x = -3$  and  $y = -2$   
 $\therefore b = 3$  and  $c = -2$

**6 A** Let  $y = x^{\frac{1}{3}}$  Reflection in the  $y$ -axis:  
 $y = -x^{\frac{1}{3}}$  Dilation by a factor of 5  
 units from the  $x$ -axis:  $y = -5x^{\frac{1}{3}}$

**7 D**  $x' = 3x - 2$  and  $y' = -y - 1$   
 $\therefore x = \frac{x' + 2}{3}$  and  $y = -y' - 1$

The image is  $-y' - 2 = \sqrt[3]{\frac{x' + 2}{3}}$

$$\therefore y = -\sqrt[3]{\frac{x' + 2}{3}} - 2$$

**8 A** Rearranging  $\frac{y + 4}{3} = \frac{1}{2x + 1}$

Choose  $x = 2x' + 1$  and  $y = \frac{y' + 4}{3}$

$$\therefore x' = \frac{1}{2}x - \frac{1}{2} \text{ and } y' = 3y - 4$$

**9 A** Rearrange  $-\frac{y - 3}{5} = \frac{1}{2x - 1}$   
 Choose  $x' = 2x - 1$  and

$$y' = -\frac{y - 3}{5} = -\frac{y}{5} + \frac{3}{5}$$

**10 A**  $g(f(x)) = (3x - 2)^2 - 4(3x - 2) + 2.$

Therefore  $x = \frac{3x' - 2}{3}$   
 $\therefore x' = \frac{x + 2}{3} = \frac{x}{3} + \frac{2}{3}$

## Solutions to extended-response questions

1 a  $\mathbb{R} \setminus \{-2\}$

- b
- dilation of factor 24 from the  $x$ -axis
  - translation of 2 in the negative direction of the  $x$ -axis
  - translation of 6 in the negative direction of the  $y$ -axis

c  $f(0) = \frac{24}{2} - 6 = 12 - 6 = 6$   
 $\therefore$   $y$  axis intercept is 6:  $(0, 6)$

$f(x) = 0$  implies

$$\frac{24}{x+2} - 6 = 0$$

$$\therefore 24 = 6(x+2)$$

$$\therefore 24 = 6x + 12$$

$$\therefore x = 2$$

$y = f(x)$  intercepts with the  $x$  axis at  $(2, 0)$

d  $g: (-2, \infty) \rightarrow \mathbb{R}, g(x) = \frac{24}{x+2} - 6$

Consider  $x = \frac{24}{y+2} - 6$

i.e.  $(y+2)x = 24 - 6(y+2)$

$$\therefore yx + 6y = 24 - 12 - 2x$$

$$y(x+6) = 12 - 2x$$

$$\therefore y = \frac{12-2x}{x+6}$$

$$\therefore g^{-1}(x) = \frac{12-2x}{x+6} = -2 + \frac{24}{x+6}$$

e  $\therefore$  domain of  $g^{-1} =$  range of  $g = (-6, \infty)$

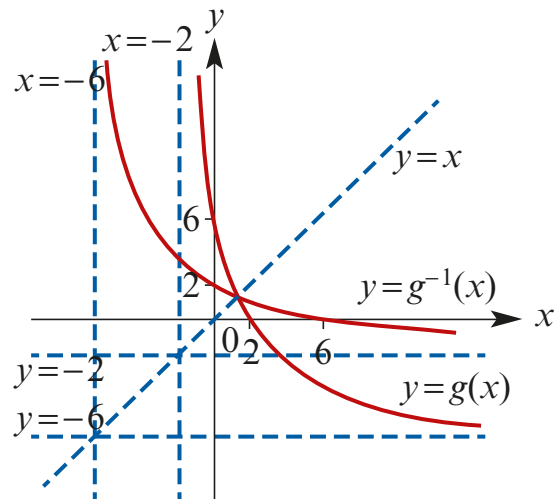
$$\begin{aligned}
 & \text{For } g(x) = x \\
 & \frac{24}{x+2} - 6 = x \\
 \text{f,g} \quad & \text{i.e. } \frac{12-2x}{x+6} = x \\
 & \therefore 12-2x = x^2 + 6x \\
 & \therefore x^2 + 8x - 12 = 0 \\
 & \therefore x = \frac{-8 \pm \sqrt{64+48}}{2} \\
 & = \frac{-8 \pm \sqrt{112}}{2} \\
 & = \frac{-8 \pm 4\sqrt{7}}{2} \\
 & = -4 \pm 2\sqrt{7}
 \end{aligned}$$

But  $x \in (-6, \infty)$

Therefore  $x = -4 + 2\sqrt{7}$

The graphs of  $y = g(x)$  and  $y = g^{-1}(x)$  intersect where  $y = x$

$\therefore$  they intersect at  $x = -4 + 2\sqrt{7}$



2  $f : D \rightarrow R, f(x) = 4 - 2\sqrt{2x+6}$

a  $2x + 6 \geq 0$

i.e.  $x \geq -3$

$\therefore$  domain is  $[-3, \infty)$

b • dilation of factor  $\frac{1}{2}$  from the y axis

• dilation of factor 2 from the x axis

• reflection in the x axis

• translation 3 units in the negative direction of the x axis

• translation 4 units in the positive direction of the y axis

c  $f(0) = 4 - 2\sqrt{6}$

$\therefore y = f(x)$  cuts the y axis at  $(0, 4 - 2\sqrt{6})$

When  $4 - 2\sqrt{2x+6} = 0$

$$4 = 2\sqrt{2x+6}$$

i.e.  $2 = \sqrt{2x+6}$

$$\therefore 4 = 2x + 6$$

$$\therefore x = -1$$

$\therefore y = f(x)$  cuts the x axis at  $(-1, 0)$

**d** Consider  $x = 4 - 2\sqrt{2y + 6}$

$$\text{Then } 2\sqrt{2y + 6} = 4 - x$$

Squaring both sides yields

$$4(2y + 6) = (4 - x)^2$$

$$\therefore 8y + 24 = 16 - 8x + x^2$$

$$\therefore y = \frac{1}{8}(x^2 - 8x - 8)$$

$$= \frac{1}{8}(x^2 - 8x + 16 - 24)$$

$$= \frac{1}{8}(x - 4)^2 - 3$$

$$\text{i.e. } f^{-1}(x) = \frac{1}{8}(x - 4)^2 - 3$$

**e** The domain of  $f^{-1} = \text{range of } f = (-\infty, 4]$

**f,g**  $f(x) = x$

$$4 - 2\sqrt{2x + 6} = x$$

implies  $2\sqrt{2x + 6} = 4 - x$

$$\therefore 4(2x + 6) = 16 - 8x + x^2$$

$$\therefore 8x + 24 = 16 - 8x + x^2$$

$$\therefore x^2 - 16x - 8 = 0$$

$$\therefore x = \frac{16 \pm \sqrt{256 + 32}}{2} = \frac{16 \pm \sqrt{288}}{2} = \frac{16 \pm 12\sqrt{2}}{2}$$

$$x = 8 \pm 6\sqrt{2}$$

and the required solution is  $x = 8 - 6\sqrt{2}$

The curves intersect at two other points

$$\text{Consider } 4 - 2\sqrt{2x + 6} = \frac{1}{8}(x^2 - 8x - 8)$$

Use a CAS calculator to find the other solutions.

It can be shown that they intersect on the line

$$y = -x$$

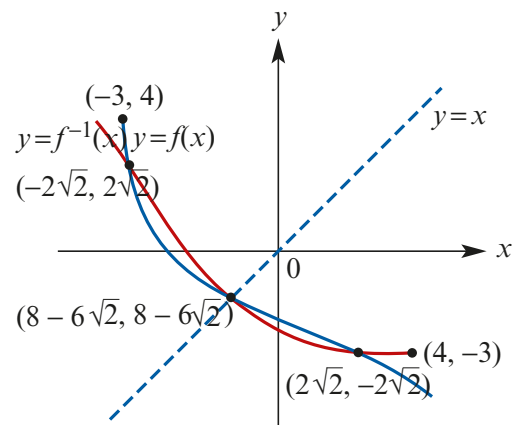
$$4 - 2\sqrt{2x + 6} = -x$$

$$-2\sqrt{2x + 6} = -x - 4$$

$$4(2x + 6) = 16 + 8x + x^2$$

$$\therefore x^2 = 8$$

$$x = \pm 2\sqrt{2}$$





**3 a i**  $(x, y) \rightarrow (x, ky)$ , so  $(25, 625) \rightarrow (25, 15)$

$$\therefore k = \frac{15}{625} = \frac{3}{125}$$

Dilation of factor  $\frac{3}{125}$  from the  $x$  axis

**ii**  $(x, y) \rightarrow (x, -y)$

**iii**  $(x, y) \rightarrow (x + 25, y + 15)$

**iv**  $(x, y) \rightarrow \left(x + 25, \frac{-3}{125}y + 15\right)$

**b i**  $y = \frac{-3}{125}(x - 25)^2 + 15$

**ii**  $(x, y) \rightarrow (x + 50, y)$

**iii**  $y = \frac{-3}{125}(x - 75)^2 + 15$

**c i** Dilation factor from the  $x$  axis

$(x, y) \rightarrow (x, ky)$

$\left(\frac{m}{2}, \frac{m^2}{4}\right) \rightarrow \left(\frac{m}{2}, n\right)$

$$\therefore k = \frac{n}{\frac{m^2}{4}}$$

$$= \frac{4n}{m^2}$$

reflection in  $x$  axis  $(x, y) \rightarrow (x, -y)$

translation  $(x, y) \rightarrow \left(x + \frac{m}{2}, y + n\right)$

overall  $(x, y) \rightarrow \left(x + \frac{m}{2}, \frac{-4n}{m^2}y + n\right)$

**ii**  $y = \frac{-4n}{m^2}\left(x - \frac{m}{2}\right)^2 + n$

**iii**  $y = \frac{-4n}{m^2}\left(x - \frac{3m}{2}\right)^2 + n$

**4 a**  $\mathbb{R} \setminus \left\{\frac{4}{3}\right\}$

**b**  $a = \frac{4}{3}$

c Consider

$$x = \frac{3}{(3y-4)^2} + 6$$

$$x - 6 = \frac{3}{(3y-4)^2}$$

$$\frac{x-6}{3} = \frac{1}{(3y-4)^2}$$

$$\therefore (3y-4)^2 = \frac{3}{x-6}$$

$$\therefore y = \frac{1}{3} \sqrt{\frac{3}{x-6}} + \frac{4}{3} \quad \text{as range of } f^{-1} = \text{domain of } f = \left(\frac{4}{3}, \infty\right)$$

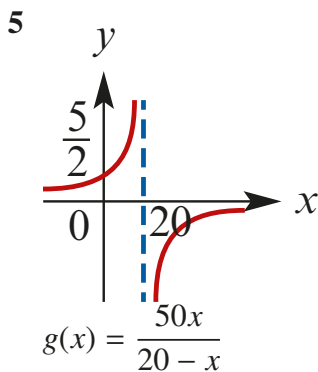
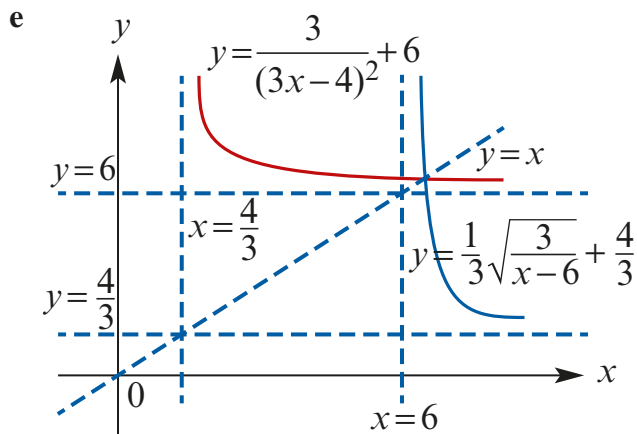
d Consider  $\frac{3}{(3x-4)^2} + 6 = x$  as  $y = f(x)$  and  $y = f^{-1}(x)$  intersect on the line  $y = x$

$$\frac{3}{(3x-4)^2} = x - 6$$

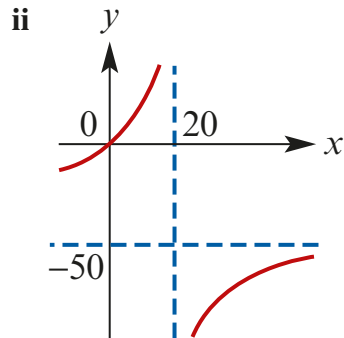
$$3 = (x-6)(3x-4)^2$$

$$x = 6.015$$

(Solve the equation with the 'solve' command of a CAS calculator.)



$$\begin{aligned}
 \text{a} \quad & \frac{1000}{20-x} - 50 \\
 &= \frac{1000 - 50(20-x)}{20-x} \\
 &= \frac{1000 - 1000 + 50x}{20-x} \\
 &= \frac{50x}{20-x} = g(x)
 \end{aligned}$$



$$\begin{aligned}
 \text{iii} \quad & 20f(x) - 50 = 20\left(\frac{50}{20-x}\right) - 50 \\
 &= \frac{1000 - 50(20-x)}{20-x} \\
 &= \frac{1000 - 1000 + 50x}{20-x} \\
 &= \frac{50x}{20-x} = g(x)
 \end{aligned}$$

b Consider  $x = \frac{50y}{20-y}$

$$\begin{aligned}
 (20-y)x &= 50y \\
 \therefore 20x &= 50y + yx \\
 \therefore 20x &= y(50+x) \\
 y &= \frac{20x}{50+x} \\
 \therefore g^{-1}(x) &= \frac{20x}{50+x}
 \end{aligned}$$

- 6 a i  $(x, y) \rightarrow (x+3, y+5) \rightarrow (y+5, x+3)$   
 $(x, y)$  maps to a unique point  $(x', y')$   
Hence  $x' = y+5$  and  $y' = x+3$   
Hence  $y = x' - 5$  and  $x = y' - 3$   
Therefore the graph of  $y = f(x)$  maps to the graph of  $x' - 5 = f(y' - 3)$

The inverse function exists and therefore

$$y' = f^{-1}(x' - 5) + 3$$

**ii**  $(x, y) \rightarrow (y, x) \rightarrow (y + 3, x + 5)$

$(x, y)$  maps to a unique point  $(x', y')$

Hence  $x' = y + 3$  and  $y' = x + 5$

Hence  $y = x' - 3$  and  $x = y' - 5$

Therefore the graph of  $y = f(x)$  maps to the graph of  $x' - 3 = f(y' - 5)$

The inverse function exists and therefore

$$y' = f^{-1}(x' - 3) + 5$$

**iii**  $(x, y) \rightarrow (5x, 3y) \rightarrow (3y, 5x)$

$(x, y)$  maps to a unique point  $(x', y')$

Hence  $x' = 3y$  and  $y' = 5x$

Hence  $y = \frac{x'}{3}$  and  $x = \frac{y'}{5}$

Therefore the graph of  $y = f(x)$  maps to the graph of  $\frac{x'}{3} = f\left(\frac{y'}{5}\right)$

The inverse function exists and therefore

$$y' = 5f^{-1}\left(\frac{x'}{3}\right)$$

**iv**  $(x, y) \rightarrow (y, x) \rightarrow (5y, 3x)$

$(x, y)$  maps to a unique point  $(x', y')$

Hence  $x' = 5y$  and  $y' = 3x$

Hence  $y = \frac{x'}{5}$  and  $x = \frac{y'}{3}$

Therefore the graph of  $y = f(x)$  maps to the graph of  $\frac{x'}{5} = f\left(\frac{y'}{3}\right)$

The inverse function exists and therefore

$$y' = 3f^{-1}\left(\frac{x'}{5}\right)$$

**b**  $x' = ay + b$  and  $y' = cx + d$

Therefore

$$\text{Therefore } y = \frac{x' - b}{a} \text{ and } x = \frac{y' - d}{c}$$

The graph of  $y = f(x)$  maps to the graph of  $\frac{x' - b}{a} = f\left(\frac{y' - d}{c}\right)$

Therefore as the inverse function exists  $y' = cf^{-1}\left(\frac{x' - b}{a}\right) + d$

From

$$x' = ay + b \text{ and } y' = cx + d:$$

the graph of  $y = f(x)$  has undergone the following sequence of transformations:

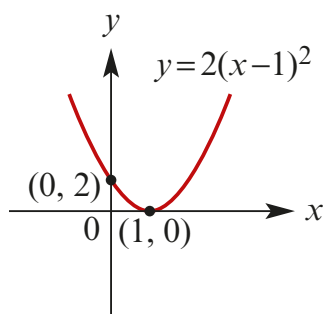
A reflection in the line  $y = x$ , then a dilation of factor  $c$  from the  $x$  axis and factor  $a$  from the  $y$  axis, and a translation of  $b$  units in the positive direction of the  $x$  axis and

$d$  units in the positive direction of the  $y$  axis.

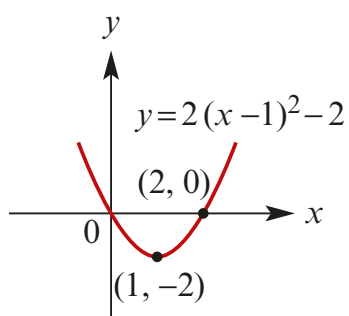
# Chapter 4 – Polynomial functions

## Solutions to Exercise 4A

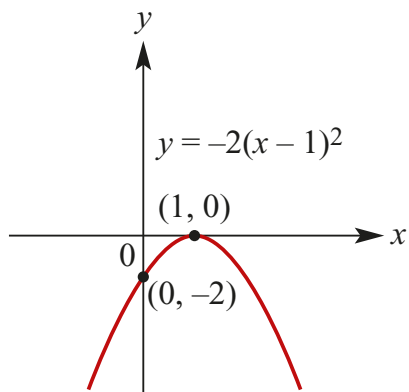
1 a



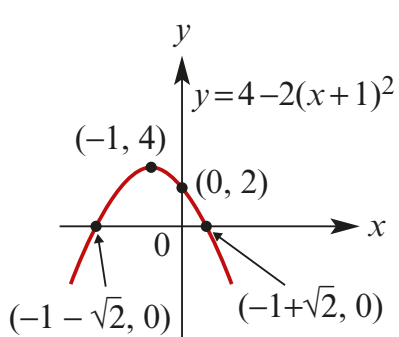
b



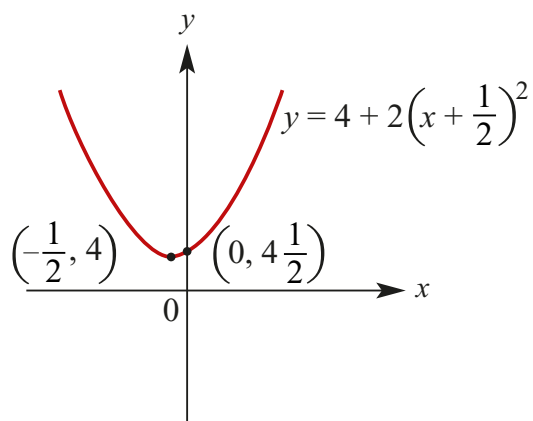
c



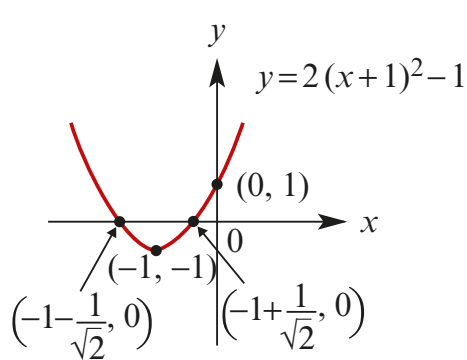
d



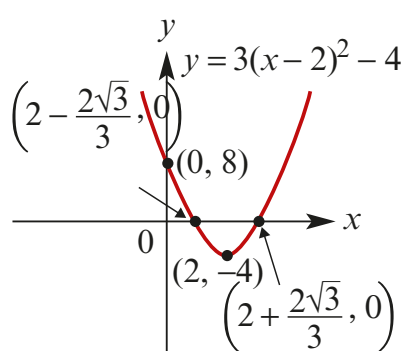
e

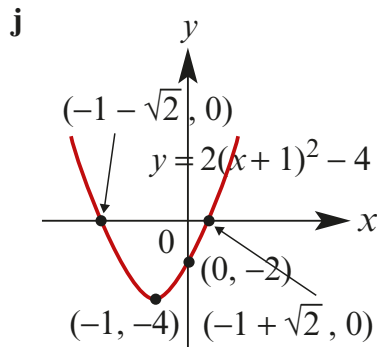
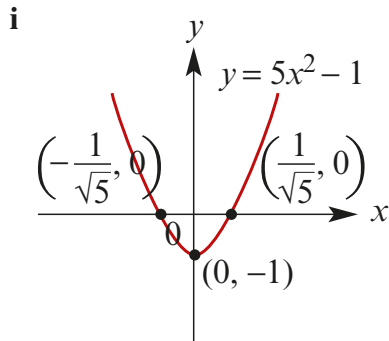
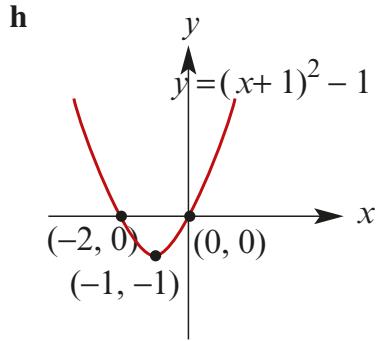


f



g





**2 a**  $f(x) = x^2 + 3x - 2$

$$= x^2 + 3x + 2.25 - 2.25 - 2$$

$$= (x + 1.5)^2 - 4.25$$

Minimum =  $-4.25$  and the range is  $[-4.25, \infty)$

**b**  $f(x) = x^2 - 6x + 8$

$$= x^2 - 6x + 9 - 9 + 8$$

$$= (x - 3)^2 - 1$$

Minimum =  $-1$  and the range is  $[-1, \infty)$

**c**  $f(x) = 2x^2 + 8x - 6$

$$= 2(x^2 + 4x - 3)$$

$$= 2(x^2 + 4x + 4) - 14$$

$$= 2(x + 2)^2 - 14$$

Minimum =  $-14$  and the range is  $[-14, \infty)$

**d**  $f(x) = 4x^2 + 8x - 7$

$$= 4(x^2 + 2x) - 7$$

$$= 4(x^2 + 2x + 1) - 4 - 7$$

$$= 4(x + 1)^2 - 11$$

Minimum =  $-11$  and the range is  $[-11, \infty)$

**e**  $f(x) = 2x^2 - 5x$

$$= 2\left(x^2 - \frac{5}{2}x\right)$$

$$= 2\left(x^2 - \frac{5}{2}x + \frac{25}{16}\right) - \frac{25}{8}$$

$$= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8}$$

Minimum =  $-\frac{25}{8}$  and the range is  $\left[-\frac{25}{8}, \infty\right)$

**f**  $f(x) = -3x^2 - 2x + 7$

$$= -3\left(x^2 - \frac{2}{3}x\right) + 7$$

$$= -3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + \frac{1}{3} + 7$$

$$= -3\left(x - \frac{1}{3}\right)^2 + \frac{22}{3}$$

maximum =  $\frac{22}{3}$  and the range is  $\left(-\infty, \frac{22}{3}\right]$

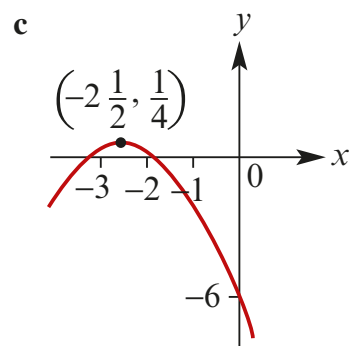
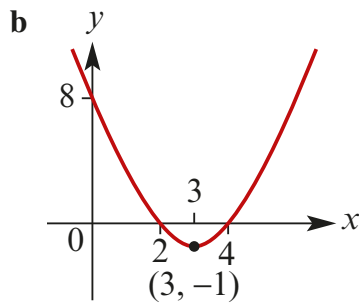
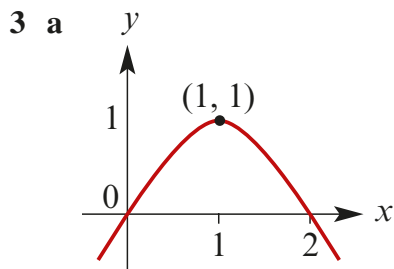
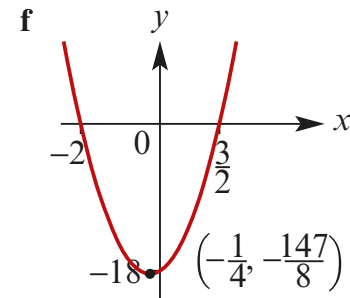
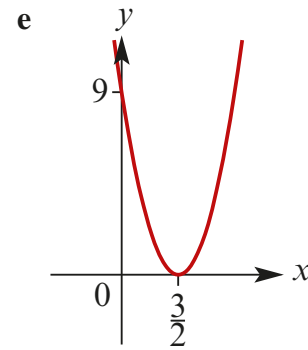
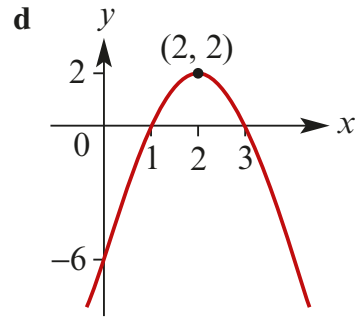
**g**  $f(x) = -2x^2 + 9x + 11$

$$= -2\left(x^2 - \frac{9}{2}x\right) + 11$$

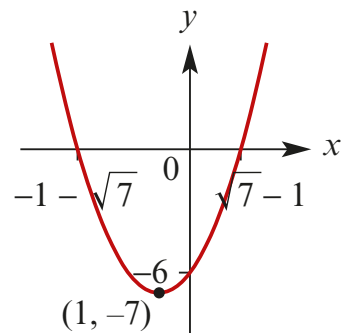
$$= -2\left(x^2 - \frac{9}{2}x + \frac{81}{16}\right) + \frac{81}{8} + 11$$

$$= -2\left(x - \frac{9}{4}\right)^2 + \frac{169}{8}$$

maximum =  $\frac{169}{8}$  and the range is  $\left(-\infty, \frac{169}{8}\right]$

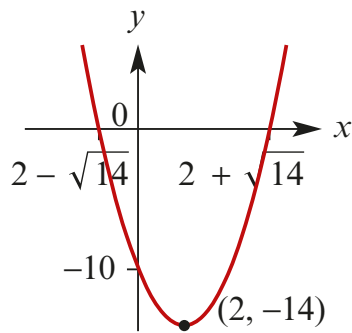


**4 a**  $y = (x + 1)^2 - 7$

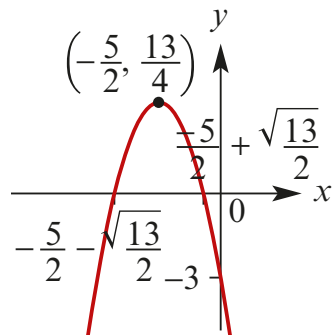




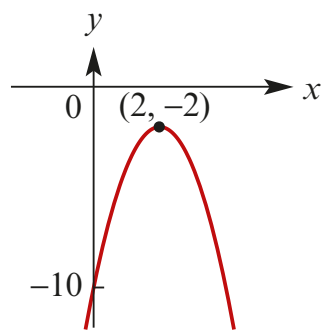
**b**  $y = (x - 2)^2 - 14$



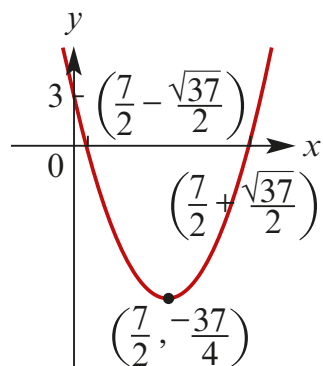
**c**  $y = \frac{13}{4} - \left(x + \frac{5}{2}\right)^2$



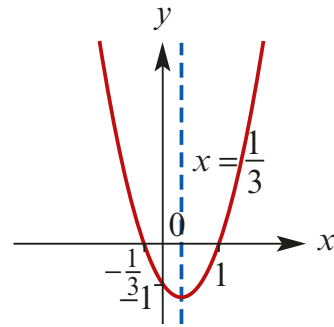
**d**  $y = -2(x - 2)^2 - 2$



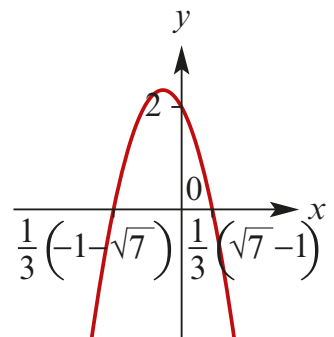
**e**  $y = \left(x - \frac{7}{2}\right)^2 - \frac{37}{4}$



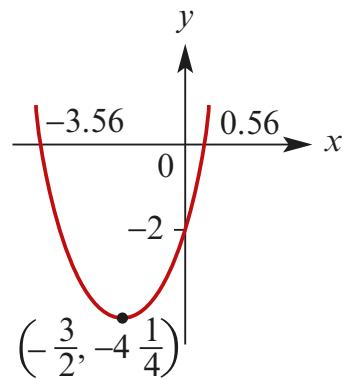
**5**



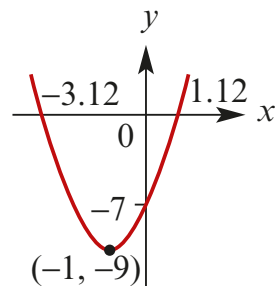
**6**

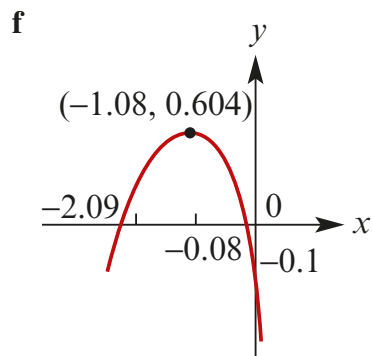
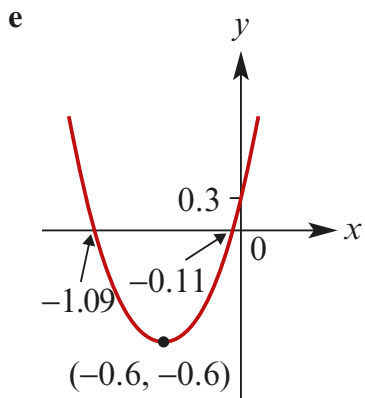
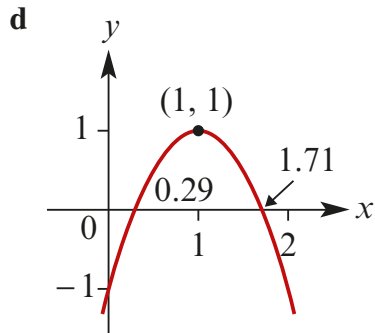
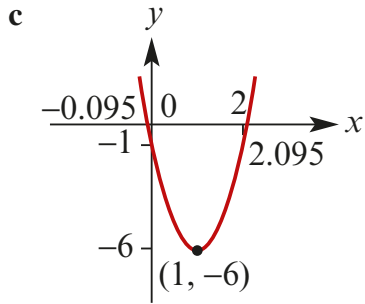


**7 a**



**b**





**8 a B**

**b D**

by looking at turning point

coordinates

**9 a C**  $x$ -axis intercepts

**b B** turning point  $x$ -value

**c D** turning point coordinates

**d A** turning point  $x$ -value

**10 a**  $b^2 - 4ac = 25 - 8$

$$> 0$$

$\therefore$  it crosses the  $x$ -axis

**b**  $b^2 - 4ac = 4 - 4 \times -4 - 1$

$$= 4 - 16$$

$$< 0$$

$\therefore$  it does not intersect the  $x$ -axis

**c**  $b^2 - 4ac = 36 - 36 = 0$

$\therefore$  it touches the  $x$ -axis

**d**  $b^2 - 4ac = 9 - 4 \times 8 \times -2$

$$= 9 + 64$$

$$> 0$$

$\therefore$  it crosses the  $x$ -axis

**e**  $b^2 - 4ac = 4 - 60 < 0$

$\therefore$  it does not intersect the  $x$ -axis

**f**  $b^2 - 4ac = 1 - 4$

$$< 0$$

$\therefore$  it does not intersect the  $x$ -axis

**11**  $mx^2 - 2mx + 3 = 0$

$$b^2 - 4ac = 4m^2 - 12m$$

$$= 4m(m - 3)$$

**a**  $4m(m - 3) > 0$

$$m < 0 \text{ or } m > 3$$

**b**  $4m(m - 3) = 0$

$$m = 3$$

( $m = 0$ , is not a solution as it gives  $3 = 0$ )

**12**  $\Delta = 36m^2 - 16(4m + 1)$

$$= 36m^2 - 64m - 16$$

$$= 4(9m^2 - 16m - 4)$$

$$= 4(9m + 2)(m - 2)$$

Perfect square if  $\Delta = 0$

$$\therefore m = -\frac{2}{9} \text{ or } m = 2$$

**13**  $\Delta = 4a^2 - 4(a + 2)(a - 3)$

$$= 4a^2 - 4(a^2 - a - 6)$$

$$= 4a + 24$$

No solutions if  $\Delta < 0$

$$\therefore a < -6$$

**14**  $\Delta = (a + 1)^2 - 4(a - 2)$

$$= a^2 + 2a + 1 - 4a + 8$$

$$= a^2 - 2a + 9$$

$$= (a - 1)^2 + 8$$

$\therefore \Delta > 0$  for all values of  $a$

**15 a**  $(k + 1)x^2 - 2x - k = 0$

$$b^2 - 4ac = 4 + k(k + 1)$$

need to show

$$4 + k(k + 1) > 0$$

$$\text{i.e. } k(k + 1) > -4$$

$$LHS = k^2 + k$$

$$= k^2 + k + \frac{1}{4} - \frac{1}{4}$$

$$= \left(k + \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$> -4$$

which is what is required

**16**  $\Delta = 4k^2 + 20k$

$$= 4k(k + 5)$$

**a**  $\Delta > 0 \Leftrightarrow k \in (-\infty, -5) \cup (0, \infty)$ .

**b**  $\Delta = 0 \Leftrightarrow k = 0 \text{ or } k = -5$

**17**  $\Delta = 4k^2 - 4(k + 2)(k - 3)$

$$= 4k^2 - 4(k^2 - k - 6)$$

$$= 4(k + 6)$$

**a** Two solutions if  $k > -6$

**b** One solution if  $k = -6$

**18 a**  $ax^2 - (a + b)x + b = 0$

$$(a + b)^2 - 4ab = a^2 + 2ab + b^2 - 4ab$$

$$= (a - b)^2 \geq 0$$

$\therefore$  the equation always has at least one solution

## Solutions to Exercise 4B

**1**  $y = k(x + 3)(x + 2)$

When  $x = 1, y = -24$

$$\therefore -24 = k(4)(3)$$

$$\therefore k = -2$$

$$\therefore y = -2(x + 3)(x + 2)$$

**2**  $y = k(x + 3)(2x + 3)$

When  $x = 1, y = 20$

$$\therefore 20 = k(4)(5)$$

$$\therefore k = 1$$

$$\therefore y = (x + 3)(2x + 3)$$

**3**  $y = a(x + 2)^2 + 4$

When  $x = 4, y = 58$

$$\therefore 58 = 36a + 4$$

$$\therefore a = \frac{54}{36} = \frac{3}{2}$$

$$\therefore y = \frac{3}{2}(x + 2)^2 + 4$$

**4**  $y = a(x + 2)^2 - 3$

When  $x = -3, y = -5$

$$\therefore -5 = a - 3$$

$$\therefore a = -2$$

$$\therefore y = -2(x + 2)^2 - 3$$

**5** Passes through (1, 19), (0, 18) and (-1, 7)

The equation has form  $y = ax^2 + bx + 18$

$$19 = a + b + 18 \dots (1)$$

$$7 = a - b + 18 \dots (2)$$

Equation (1) - Equation (2)

$$12 = 2b$$

$$\therefore b = 6$$

$$\therefore a = -5$$

$$\therefore y = -5x^2 + 6x + 18$$

**6** Passes through (2, -14), (0, 10) and (-4, 10)

The equation has form  $y = ax^2 + bx + 10$

$$-14 = 4a + 2b + 10 \dots (1)$$

$$10 = 16a - 4b + 10 \dots (2)$$

$2 \times$  Equation (1) + Equation (2)

$$-18 = 24a + 30$$

$$\therefore a = -2$$

$$\therefore b = -8$$

$$\therefore y = -2x^2 - 8x + 10$$

**7 a**  $y = ax^2 + bx + c$

$c = 4$  (y-intercept)

$b = 0$  (x-value at turning point)

$$y = ax^2 + 4$$

$x = 5, y = 0$

$$0 = 25a + 4$$

$$a = \frac{-4}{25}$$

$$y = \frac{-4}{25}x^2 + 4$$

**b**  $y = a(x + h)^2 + k$

$$y = ax^2$$

$x = 2, y = -4$

$$-4 = 4a$$

$$a = -1$$

$$y = -x^2$$

$$\begin{aligned} \mathbf{c} \quad y &= a(x+b)(x+c) \\ y &= a(x+2)(x+0) \\ y &= ax^2 + 2ax \\ x &= 1, y = 3 \\ 3 &= a + 2a \\ a &= 1 \\ y &= x^2 + 2x \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad y &= a(x+b)(x+c) \\ y &= a(x+0)(x-2) \\ y &= ax^2 - 2ax \\ x &= -1, y = -3 \\ -3 &= a + 2a \\ a &= -1 \\ y &= -x^2 + 2x \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad y &= a(x+b)(x+c) \\ y &= a(x-1)(x-4) \\ y &= ax^2 - 5ax + 4a \\ 4a &= 4(\text{y-intercept}) \\ a &= 1 \\ y &= x^2 - 5x + 4 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad y &= a(x+b)(x+c) \\ y &= a(x+1)(x-5) \\ y &= ax^2 - 4ax - 5a \\ -5a &= -5 \text{ (y-intercept)} \\ a &= 1 \\ y &= x^2 - 4x - 5 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad y &= a(x+h)^2 + k \\ y &= a(x-1)^2 - 2 \\ x &= -1, y = 2 \\ 2 &= 4a - 2 \\ a &= 1 \end{aligned}$$

$$y = (x-1)^2 - 2 = x^2 - 2x - 1$$

$$\begin{aligned} \mathbf{h} \quad y &= a(x+h)^2 + k \\ y &= a(x-2)^2 + 2 \\ x &= 0, y = 6 \\ 6 &= 4a + 2 \\ a &= 1 \\ y &= (x-2)^2 + 2 = x^2 - 4x + 6 \end{aligned}$$

### 8 left hand curve

$$\begin{aligned} -y &= ax^2 + x + c \\ c &= -5 & C \\ x = 4, y = 1 & B \\ 1 &= 16a + 4 - 5 \\ a &= \frac{1}{8} \\ y &= \frac{1}{8}x^2 + x - 5 \end{aligned}$$

### right hand curve

$$\begin{aligned} y &= ax^2 + x + c \\ c &= 1 & D \\ y &= ax^2 + x + 1 \\ x = 4, y = 3 & A \\ 3 &= 16a + 4 + 1 \\ 16a &= -2 \end{aligned}$$

$$\begin{aligned} a &= -\frac{1}{8} \\ y &= -\frac{1}{8}x^2 + x + 1 \end{aligned}$$

$$\begin{aligned} \mathbf{9} \quad f(x) &= A(x+b)^2 + B \\ &= A(x+2)^2 + 4(\text{vertex}) \\ f(0) &= 8 \\ 8 &= 4a + 4 \\ A &= 1, b = 2, B = 4 \\ f(x) &= (x+2)^2 + 4 \end{aligned}$$

## Solutions to Exercise 4C

1 a  $P(1) = 3$

b  $P(-1) = -5$

c  $P(2) = 7$

d  $P(-2) = -21$

e  $P\left(\frac{1}{2}\right) = \frac{17}{8}$

f  $P\left(-\frac{1}{2}\right) = -\frac{9}{8}$

2 a  $P(0) = 6$

b  $P(1) = 6$

c  $P(2) = 18$

d  $P(-1) = 12$

e  $P(a) = a^3 + 3a^2 - 4a + 6$

f  $P(2a) = 8a^3 + 12a^2 - 8a + 6$

3 a  $P(2) = 0$

$$8 + 12 - 2a - 30 = 0$$

$$-2a = 10$$

$$a = -5$$

b  $P(3) = 68$

$$27 + 9a + 15 - 14 = 68$$

$$9a = 40$$

$$a = \frac{40}{9}$$

c  $P(1) = 6$

$$1 - 1 - 2 + c = 6$$

$$c = 8$$

d

$$P(-1) = P(2) = 0$$

$$2 + 5 + a - b + 12 = 0$$

$$a - b = -19 \dots (1)$$

$$128 - 40 + 4a + 2b + 12 = 0$$

$$4a + 2b = -100$$

$$2a + b = -50 \dots (2)$$

$$\text{Equation (1) + Equation(2)}$$

$$3a = -69$$

$$a = -23$$

$$\therefore b = -4$$

e

$$P(3) = P(1) = 0$$

$$3^5 - 2 \times 3^4 + 27a + 9b + 36 - 36 = 0$$

$$81 + 27a + 9b = 0$$

$$3a + b = -9 \dots (1)$$

$$1 - 2 + a + b + 12 - 36 = 0$$

$$a + b = 25 \dots (2)$$

$$\text{Equation (1) - Equation(2)}$$

$$2a = -34$$

$$a = -17$$

$$\therefore b = 42$$

4 a  $2x^3 - x^2 + 2x + 2$

b  $2x^3 + 5x$

c  $2x^3 - x^2 + 4x - 2$

d  $6x^3 - 3x^2 + 9x$

e  $-2x^4 + 5x^3 - 5x^2 + 6x$

**f**  $4x - x^3$

**g**  $2x^3 + 4x + 2$

**h**  $2x^5 + 3x^4 + x^3 + 6x^2$

**5 a**  $x^3 - 5x^2 + 10x - 8$

**b**  $x^3 - 7x^2 + 13x - 15$

**c**  $2x^3 - x^2 - 7x - 4$

**d**  $x^2 + (b + 2)x^2 + (2b + c)x + 2c$

**e**  $2x^3 - 9x^2 - 2x + 3$

**6 a**  $(x + 1)(x^2 + bx + c) =$   
 $x^3 + (b + 1)x^2 + (c + b)x + c$

**b**  $x^3 - x^2 - 6x - 4 =$   
 $x^3 + (b + 1)x^2 + (c + b)x + c$   
for all  $x$ .  $\therefore (b + 1) = -1, c = -4$  and  
 $c + b = -6$   
 $\therefore b = -2$  and  $c = -4$

**c**  $x^3 - x^2 - 6x - 4 = (x + 1)(x^2 - 2x - 4)$   
 $\therefore x^3 - x^2 - 6x - 4 =$   
 $(x + 1)(x + \sqrt{5} - 1)(x - \sqrt{5} - 1)$

**7 a**  $2x^3 - 18x^2 + 54x - 49 =$   
 $a(x^3 - 9x^2 + 27x - 27) + b$   
Equating coefficients  
 $a = 2$  and  $-27a + b = -49$   
 $\therefore a = 2$  and  $b = 5$

**b**  $-2x^3 + 18x^2 - 54x + 52 =$   
 $a(x^3 + 3cx^2 + 3c^2x + c^3) + b$   
Equating coefficients  
 $a = -2$  and  $3ca = 18$  and  $52 = ac^3 + b$   
 $\therefore a = -2, c = -3$  and  $b = -2$

**c**  $x^3 - 5x^2 - 2x + 24 =$   
 $a(x^3 + 3cx^2 + 3c^2x + c^3) + b$   
Equating coefficients: For  $x^3$ :  $a = 1$   
For  $x^2$ :  $-5 = 3c$   
For  $x$ :  $-2 = 3c^2$  which is impossible

**8**  $A(x + 3) + B(x + 2) = 4x + 9$

$(A + B)x + (3A + 2B) = 4x + 9$   
by equating coefficients

(1)  $A + B = 4$

(2)  $3A + 2B = 9$

(2) + 2(1)  $\Rightarrow A = 1$

(1)  $\Rightarrow B = 3$

**9 a**  $x^2 - 4x + 10 = Ax^2 + 2ABx + AB^2 + C$   
by equating coefficients

(1)  $A = 1$

$2AB = 4$

(2)  $AB^2 + C = 10$

(1)  $\Rightarrow = 2B = 4$

$B = -2$

$\Rightarrow 2 \Rightarrow 4 + C = 10$

$C = 6$

**b**  $4x^2 - 12x + 14 = Ax^2 + 2ABx + C$   
by equating coefficients

$A = 4$

(1)  $2AB = -12$

(2)  $AB^2 + C = 14$

$\Rightarrow (1) \Rightarrow 8B = -12$

$B = \frac{-3}{2}$

(2)  $\Rightarrow 4 \times \frac{9}{4} + C = 14$

$C = 5$

**c**  $x^3 - 9x^2 + 27x - 22 = A(x + B)^3 + C$   
 $(x - 3)^3 + 5 = A(x + B)^3 + C$

$$A = 1$$

$$B = -3$$

$$C = 5$$



## Solutions to Exercise 4D

$$\begin{array}{r}
 x^2 - 5x + 6 \\
 \hline
 \mathbf{1 \ a} \quad x + 4 \overline{) x^3 - x^2 - 14x + 24} \\
 \underline{x^3 + 4x^2} \phantom{+ 24} \\
 -5x^2 - 14x \phantom{+ 24} \\
 \underline{-5x^2 - 20x} \phantom{+ 24} \\
 6x + 24 \\
 \underline{6x + 24} \\
 0
 \end{array}$$

$$\begin{array}{r}
 2x^2 + 6x + 14 + \frac{54}{x-3} \\
 \hline
 \mathbf{b} \quad x - 3 \overline{) 2x^3 + 0x^2 - 4x + 12} \\
 \underline{2x^3 - 6x^2} \phantom{+ 12} \\
 6x^2 - 4x \phantom{+ 12} \\
 \underline{6x^2 - 18x} \phantom{+ 12} \\
 14x + 12 \\
 \underline{14x - 42} \\
 54
 \end{array}$$

$$\begin{array}{r}
 2x^2 + 7x - 4 \\
 \hline
 \mathbf{b} \quad x - 3 \overline{) 2x^3 + x^2 - 25x + 12} \\
 \underline{2x^3 - 6x^2} \phantom{+ 12} \\
 7x^2 - 25x \phantom{+ 12} \\
 \underline{7x^2 - 21x} \phantom{+ 12} \\
 -4x + 12 \\
 \underline{-4x + 12} \\
 0
 \end{array}$$

$$\begin{array}{r}
 x^2 - \frac{5}{2}x - \frac{15}{4} + \frac{145}{4(2x+3)} \\
 \hline
 \mathbf{3 \ a} \quad 2x + 3 \overline{) 2x^3 - 2x^2 - 15x + 25} \\
 \underline{2x^3 + 3x^2} \phantom{+ 25} \\
 -5x^2 - 15x \phantom{+ 25} \\
 \underline{-5x^2 - \frac{15}{2}x} \phantom{+ 25} \\
 -\frac{15}{2}x + 25 \\
 \underline{-\frac{15}{2}x - \frac{45}{4}} \\
 \frac{145}{4}
 \end{array}$$

$$\begin{array}{r}
 x^2 - 4x - 3 + \frac{34}{x+3} \\
 \hline
 \mathbf{2 \ a} \quad x + 3 \overline{) x^3 - x^2 - 15x + 25} \\
 \underline{x^3 + 3x^2} \phantom{+ 25} \\
 -4x^2 - 15x \phantom{+ 25} \\
 \underline{-4x^2 - 12x} \phantom{+ 25} \\
 -3x + 25 \\
 \underline{-3x - 9} \\
 34
 \end{array}$$

$$\begin{array}{r}
 2x^2 + 6x + 7 + \frac{33}{2x-3} \\
 \hline
 \mathbf{b} \quad 2x - 3 \overline{) 4x^3 + 6x^2 - 4x + 12} \\
 \underline{4x^3 - 6x^2} \phantom{+ 12} \\
 12x^2 - 4x \phantom{+ 12} \\
 \underline{12x^2 - 18x} \phantom{+ 12} \\
 14x + 12 \\
 \underline{14x - 21} \\
 33
 \end{array}$$

$$\begin{array}{r}
 2x^2 - x + 12 \\
 4 \text{ a } x - 3 \overline{) 2x^3 - 7x^2 + 15x - 3} \\
 \underline{2x^3 - 6x^2} \\
 -x^2 + 15x \\
 \underline{-x^2 + 3x} \\
 12x - 3 \\
 \underline{12x - 36} \\
 33 \\
 \underline{2x^3 - 7x^2 + 15x - 3} \\
 x - 3 \\
 = 2x^2 - x + 12 + \frac{33}{x - 3}
 \end{array}$$

$$\begin{array}{r}
 5x^4 + 8x^3 - 8x^2 + 6x - 6 \\
 \text{b } x + 1 \overline{) 5x^5 + 13x^4 - 2x^2 - 6} \\
 \underline{5x^5 + 5x^4} \\
 8x^4 + 0x^3 \\
 \underline{8x^4 + 8x^3} \\
 -8x^3 - 2x^2 \\
 \underline{-8x^3 - 8x^2} \\
 6x^2 + 0x \\
 \underline{6x^2 + 6x} \\
 -6x - 6 \\
 \underline{5x^5 + 13x^4 - 2x^2 - 6} \\
 x + 1 \\
 = 5x^4 + 8x^3 - 8x^2 + 6x - 6
 \end{array}$$

$$\begin{array}{r}
 x^2 - 9x + 27 \\
 5 \text{ a } x^2 - 2 \overline{) x^4 - 9x^3 + 25x^2 - 8x - 2} \\
 \underline{x^4 - 2x^2} \\
 -9x^3 - 8x \\
 \underline{-9x^3 + 18x} \\
 27x^2 - 2 \\
 \underline{27x^2 - 54} \\
 -26x + 52 \\
 \underline{x^4 - 9x^3 + 25x^2 - 8x - 2} \\
 x^2 - 2 \\
 = x^2 - 9x + 27 - 26\left(\frac{x - 2}{x^2 - 2}\right)
 \end{array}$$

**b**

$$\begin{array}{r}
 x^2 + x + 2 \\
 \text{c } x^2 - 1 \overline{) x^4 + x^3 + x^2 - x - 2} \\
 \underline{x^4 + 0x^3 - x^2} \\
 x^3 + 2x^2 - x \\
 \underline{x^3 + 0x^2 - x} \\
 2x^2 + 0x - 2 \\
 \underline{2x^2 - 2} \\
 0
 \end{array}$$

$$\begin{aligned}
 6 \text{ a remainder} &= P(-2) \\
 &= (-2)^3 + 3(-2) - 2 = -16
 \end{aligned}$$

$$\begin{aligned}
 \text{b } P(x) &= (1 - 2a)x^2 + 5ax \\
 &\quad + (a - 1)(a - 8)
 \end{aligned}$$

$$P(2) = 0$$

$$P(1) \neq 0$$

$$\begin{aligned}
 P(2) &= 4 - 8a + 10a + a^2 - 9a + 8 \\
 &= a^2 - 7a + 12
 \end{aligned}$$

$$(a - 3)(a - 4) = 0$$

$$a = 3, 4$$

$$\begin{aligned}
 P(1) &= 1 - 2a + 5a + a^2 - 9a + 8 \\
 &= a^2 - 6a + 9 \\
 &= (a - 3)^2
 \end{aligned}$$

$$P(1) \neq 0, \quad \therefore a \neq 3, \quad \therefore a = 4$$

$$\begin{aligned}
 7 \text{ a } f(x) &= 6x^3 + 5x^2 - 17x - 6 \\
 f(2) &= 6 \times 8 + 5 \times 4 - 17 \times 2 - 6 \\
 &= 48 + 20 - 34 - 6 \\
 &= 28
 \end{aligned}$$

$$\begin{aligned}
 \text{b } f(-2) &= (6 \times -8) + (5 \times 4) \\
 &\quad - (17 \times -2) - 6 \\
 &= -48 + 20 + 34 - 6 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned} \text{c } f(x) &= (x+2)(6x^2 - 7x - 3) \\ &= (x+2)(3x+1)(2x-3) \end{aligned}$$

$$\begin{aligned} \text{8 a } P(-1) &= -1 + (k-1) - (k-9) - 7 \\ &= -1 + k - 1 - k + 9 - 7 \\ &= 0 \\ \therefore \text{ for any value of } k, P(x) &\text{ is divisible} \\ &\text{by } x+1 \end{aligned}$$

$$\begin{aligned} \text{b } P(2) &= 8 + 4(k-1) + 2(k-9) - 7 \\ P(2) &= 12 \\ 1 + 4k - 4 + 2k - 18 &= 12 \\ 6k - 21 &= 12 \\ 6k &= 33 \\ k &= \frac{11}{2} \end{aligned}$$

$$\text{9 } f(x) = 2x^3 + ax^2 - bx + 3$$

$$\begin{aligned} \text{a } f(-3) = 0 &= -54 + 9a + 3b + 3 \\ 9a + 3b &= 51 \\ 3a + b &= 17 \dots (1) \\ f(2) = 15 &= 16 + 4a - 2b + 3 \\ 4a - 2b &= -4 \\ 2a - b &= -2 \dots (2) \\ (1) + (2) & \\ \Rightarrow 5a &= 15 \\ a &= 3 \\ \text{Sub in (1)} &\Rightarrow b = 8 \end{aligned}$$

$$\begin{aligned} \text{b } f(x) &= (x+3)(2x^2 - 3x + 1) \\ &= (x+3)(2x-1)(x-1) \\ \therefore \text{ the other two factors are } &(2x-1) \\ &\& (x-1) \end{aligned}$$

$$\begin{aligned} \text{10 a } f(x) &= 4x^3 + ax^2 - 5x + b \\ f\left(\frac{3}{2}\right) &= -8 = 4 \times \left(\frac{27}{8}\right) \\ &\quad + a \times \left(\frac{9}{4}\right) - 5 \times \left(\frac{3}{2}\right) + b \\ -8 &= \frac{27}{2} + \frac{9}{4}a - \frac{15}{2} + b \\ -32 &= 54 + 9a - 30 + 4b \\ 9a + 4b &= -56 \dots (1) \\ f(3) = 10 &= 4 \times 27 + a \times 9 - 5 \times 3 + b \\ 10 &= 108 + 9a - 15 + b \\ 2 \qquad \qquad 9a + b &= -83 \\ 1 - 2 \Rightarrow \qquad \qquad 3b &= 27 \\ b &= 9 \\ \text{Sub in 2} \Rightarrow 9a + 9 &= -83 \\ a &= \frac{-92}{9} \end{aligned}$$

$$\begin{aligned} \text{11 } P(2) &= (3)^4 \\ &= 81 \end{aligned}$$

$$\text{12 } P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$$

$$\begin{aligned} \text{a } P1 &= -1 - 3 + 2 - 2 + 3 + 1 \\ &= 2 \neq 0 \\ \therefore (x-1) &\text{ is not a factor} \\ P(-1) &= -1 - 3 - 2 - 2 - 3 + 1 \\ &= -10 \neq 0 \\ \therefore (x+1) &\text{ is not a factor} \end{aligned}$$

$$\begin{array}{r} \text{b } P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1 \\ x^2 - 1 \overline{) x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1} \\ \underline{x^5 - x^3} \phantom{+ 1} \\ -3x^4 + 3x^3 - 2x^2 \phantom{+ 3x + 1} \\ \underline{-3x^4 + 3x^2} \phantom{+ 1} \\ 3x^3 - 5x^2 + 3x \phantom{+ 1} \\ \underline{3x^3 - 3x} \phantom{+ 1} \\ -5x^2 + 6x + 1 \phantom{+ 1} \\ \underline{-5x^2 + 5} \phantom{+ 1} \\ 6x - 4 \phantom{+ 1} \end{array}$$

$$P(x) = (x^3 - 3x^2 + 3x - 5)(x^2 - 1) + 6x - 4$$

$\therefore$  the remainder when  $(x^3 - 3x^2 + 3x - 5)$  is divided by  $(x^2 - 1)$  is  $6x - 4$

$$\begin{aligned} \text{13 } P(-1) &= -2 - 5 + 4 + 3 = \\ &\therefore (x + 1) \text{ is factor} \\ 2x^3 - 5x^2 - 4x + 3 &= (x + 1)(2x^2 - 7x + 3) \\ &= (x + 1)(2x - 1)(x - 3) \end{aligned}$$

$$\begin{aligned} \text{14 a } P(x) &= x^4 + x^3 - x^2 - 3x - 6 \\ P(\sqrt{3}) &= 9 + 3\sqrt{3} - 3 - 3\sqrt{3} - 6 \\ &= 0 \\ P(\sqrt{-3}) &= 9 - 3\sqrt{3} - 3 + 3\sqrt{3} - 6 \\ &= 0 \end{aligned}$$

**b** the quadratic factor is  $(x + \sqrt{3})(x - \sqrt{3}) = (x^2 - 3)$   
 $\therefore P(x) = (x^2 - 3)(x^2 + x + 2)$   
 $\therefore$  an other factor is  $(x^2 + x + 2)$

$$\begin{aligned} \text{15 a } (2a + 3b)(4a^2 - 6ab + 9b^2) \\ \text{b } (4 - a)(a^2 + 4a + 16) \end{aligned}$$

$$\text{c } (5x + 4y)(25x^2 - 20xy + 16y^2)$$

$$\text{d } 2a(a^2 + 3b^2)$$

$$\text{16 a } (2x - 1)(2x + 3)(3x + 2)$$

$$\text{b } (2x - 1)(2x^2 + 3)$$

$$\text{17 a } (2x - 3)(2x^2 + 3x + 6)$$

$$\text{b } (2x - 3)(2x - 1)(2x + 1)$$

$$\text{18 a } x = -4, 2, 3$$

$$\text{b } x = 0, 2$$

$$\text{c } x = \frac{1}{2}, 2$$

$$\text{d } x = -2, 2$$

$$\text{e } x = 0, -2, 2$$

$$\text{f } x = 0, -3, 3$$

$$\text{g } x = 1, -2, \frac{-1}{4}, \frac{1}{3}$$

$$\text{h } x = 1, -2$$

$$\text{i } x = 1, -2, \frac{1}{3}, \frac{3}{2}$$

**19** Use a CAS calculator to solve  $y = 0$  in each case to obtain the  $x$ -axis intercepts.

$$\text{a } (-1, 0), (0, 0), (2, 0)$$

$$\text{b } (-2, 0), (0, 6), (1, 0), (3, 0)$$

$$\text{c } (-1, 0), (0, 6), (2, 0), (3, 0)$$

$$\text{d } \left(\frac{-1}{2}, 0\right), (0, 2), (1, 0), (2, 0)$$

**e**  $(-2, 0), (-1, 0), (0, -2), (1, 0)$

**f**  $(-1, 0), \left(\frac{-2}{3}, 0\right), (0, -6), (3, 0)$

**g**  $(-4, 0), (0, -16), \left(-\frac{2}{5}, 0\right), (2, 0)$

**h**  $\left(\frac{-1}{2}, 0\right), (0, 1), \left(\frac{1}{3}, 0\right), (1, 0)$

**i**  $(-2, 0), \left(-\frac{3}{2}, 0\right), (0, -30), (5, 0)$

**20**  $16p - 10 + q = 0 \dots (1)$   
 $16 - 16 - 4p - 2q - 8 = 0 \dots (2)$   
 $\Rightarrow 2p + q + 4 = 0$   
 $(1) - (2) \Rightarrow 14p - 14 = 0$   
 $p = 1$   
Sub in 1  $\Rightarrow 16 - 10 + q = 0$   
 $q = -6$

**21**  $f(x) = x^4 - x^3 + 5x^2 + 4x - 36$   
 $f(-1) = 1 + 1 + 5 - 4 - 36$   
 $= -33$

**22 a**  $(x - 9)(x - 13)(x + 11)$

**b**  $(x + 11)(x - 9)(x - 11)$

**c**  $(x + 11)(2x - 9)(x - 11)$

**d**  $(x + 11)(2x - 13)(2x - 9)$

**23 a**  $(x - 1)(x + 1)(x - 7)(x + 6)$

**b**  $(x - 3)(x + 4)(x^2 + 3x + 9)$

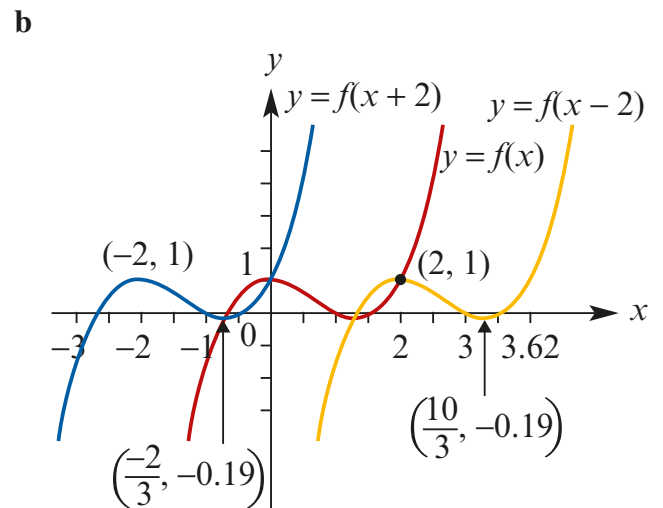
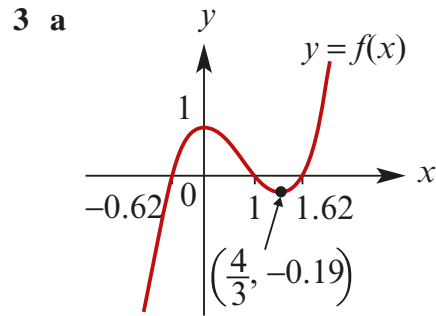
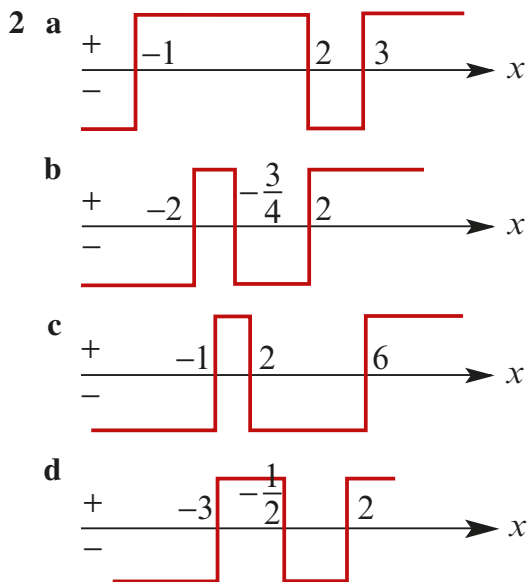
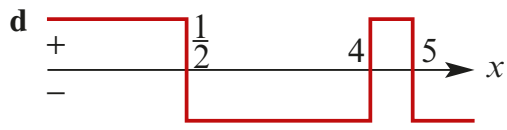
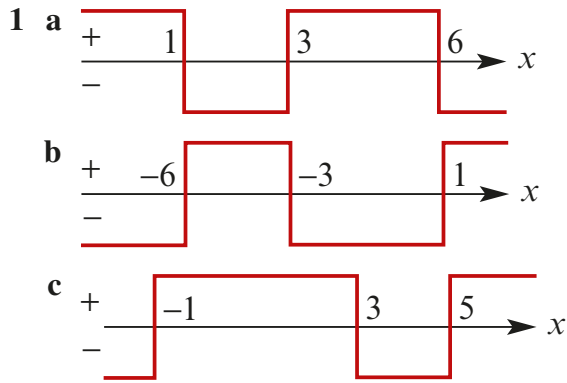
**24 a**  $(x - 9)(x - 5)(2x^2 + 3x + 9)$

**b**  $(x + 5)(x + 9)(x^2 - x + 9)$

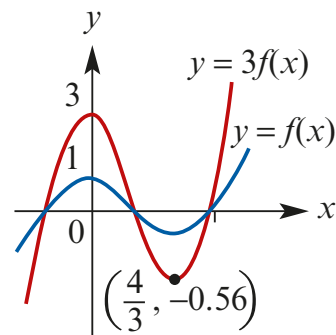
**c**  $(x - 3)(x + 5)(x^2 + x + 9)$

**d**  $(x - 4)(x - 3)(x + 5)(x + 6)$

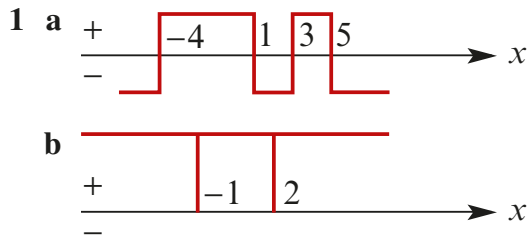
## Solutions to Exercise 4E



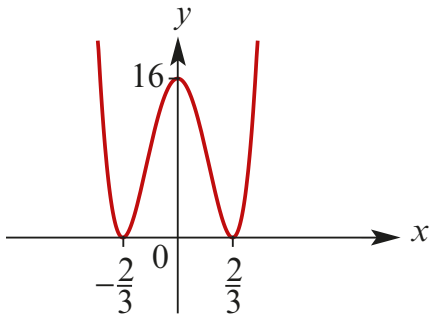
For clarity the graph of  $y = 3f(x)$  is shown on separate axes:



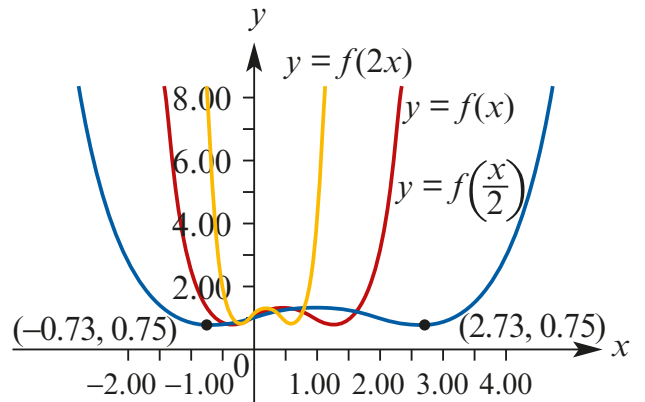
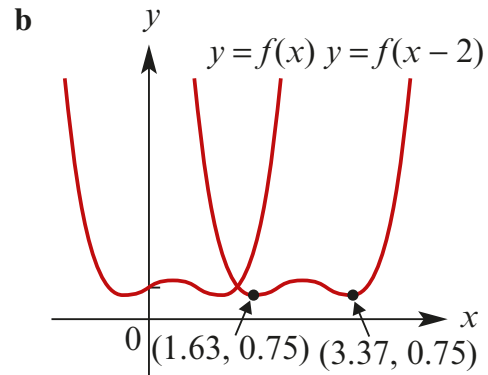
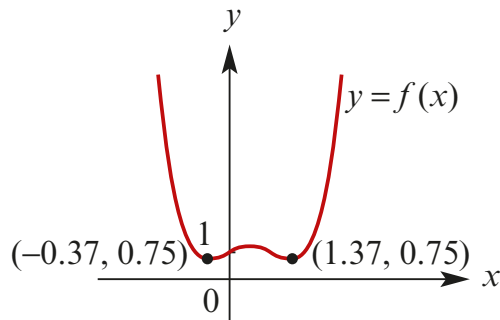
## Solutions to Exercise 4F



- 2** (0, 16)  
 $(\frac{2}{3}, 0)$   
 $(-\frac{2}{3}, 0)$

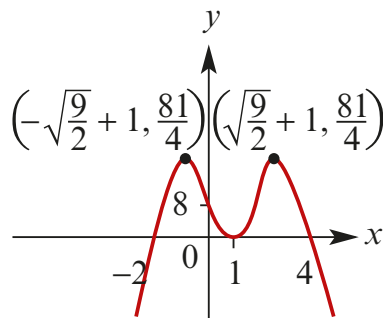


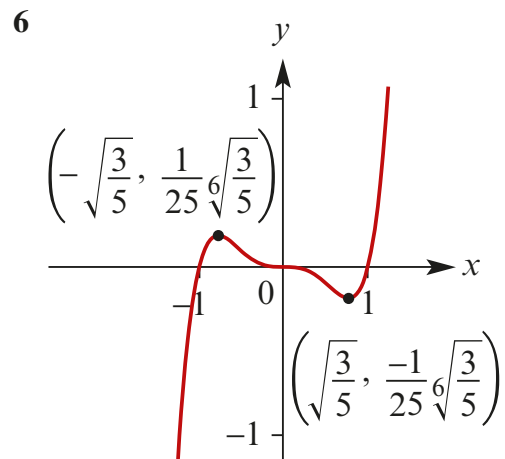
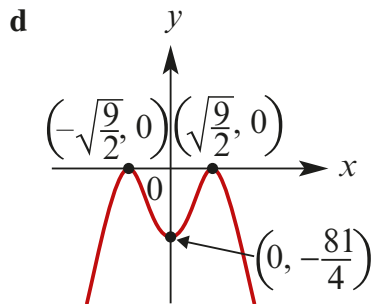
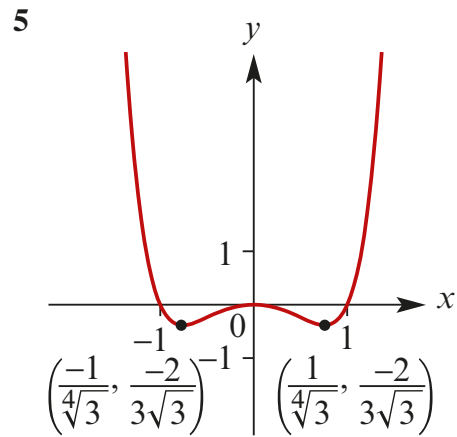
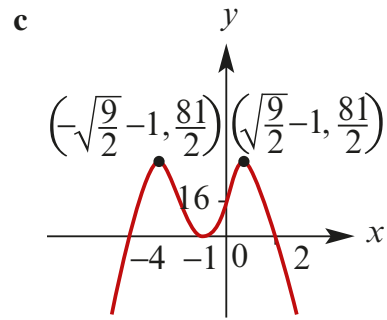
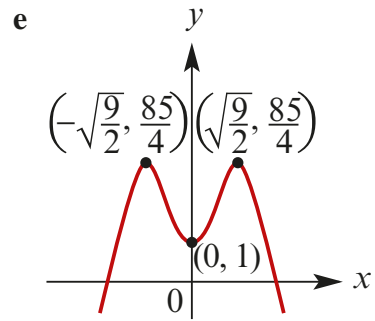
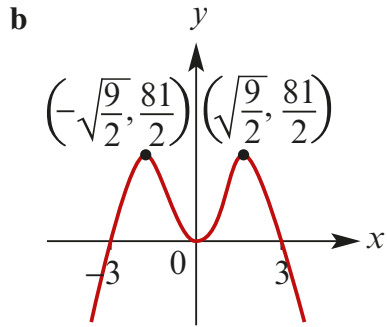
**3 a**



Turning points for  $y = f(2x)$  are at  $(-0.18, 0.75)$  and  $(0.68, 0.75)$

**4 a**







## Solutions to Exercise 4G

**1 a**  $y = a(x - 5)^3 - 2$

When  $x = 4, y = 0$

$$0 = -a - 2$$

$$a = -2$$

**b**  $y = a(x - 1)(x + 1)(x + 2)$

When  $x = 3, y = 120$

$$120 = a(2)(4)(5)$$

$$a = 3$$

**c**  $y = ax^3 + bx$

$(2, -20)$  and  $(-1, 20)$  lie on the graph

$$-20 = 8a + 2b$$

$$-10 = 4a + b \dots (1)$$

$$20 = -a - b \dots (2)$$

Add (1)(2)

$$10 = 3a$$

$$a = \frac{10}{3}$$

$$b = -\frac{70}{3}$$

**2** We know that the  $y$ -intercept is 5.

Consider  $f(x) = ax^3 + bx^2 + cx + 5$

$$f(-1) = 14, \therefore -a + b - c + 5 = 14$$

$$-a + b - c = 9 \dots (1)$$

$$f(1) = 0, \therefore a + b + c + 5 = 0$$

$$a + b + c = -5 \dots (2)$$

$$f(2) = -19, \therefore 8a + 4b + c + 5 = -19$$

$$8a + 4b + c = -24$$

Add (1) and (2)

$$2b = 4$$

$$b = 2$$

$$\therefore a + c = -7$$

and  $4a + c = -16$

$$\therefore 3a = -9$$

$$\therefore a = -3 \text{ and } c = -4$$

**3** Note: A CAS calculator can be used for all questions in this exercise, but should be used for questions 5 and 6.

$$y = a(x - b)(x - c)(x - d)$$

$$b = -5, c = -2, d = 6$$

$$y = a(x + 5)(x + 2)(x - 6)$$

$$x = 0, y = -11$$

$$-11 = -60a$$

$$a = \frac{11}{60}$$

$$y = \frac{11}{60}(x + 5)(x + 2)(x - 6)$$

**4**  $y = a(x - b)(x - c)^2$

$$y = a(x + 1)(x - 3)^2$$

$$x = 0, y = 5$$

$$5 = 9a$$

$$a = \frac{5}{9}$$

$$y = \frac{5}{9}(x + 1)(x - 3)^2$$

**5 a**  $y = ax^3 + bx^2 + cx + d$   
 $(0, 1) \Rightarrow d = 1$   
 $y = ax^3 + bx^2 + cx + 1$   
 $(1, 3) \Rightarrow 3 = a + b + c + 1$   
 $a + b + c = 2 \dots (1)$   
 $(-1, -1) \Rightarrow -1 = -a + b - c + 1$   
 $-a + b - c = 2 \dots (2)$   
 $(1) + (2) \Rightarrow 2b = 0$   
 $b = 0$   
 $(2, 11) \Rightarrow 11 = 8a + 2c + 1$   
 $4a + c = 5 \dots (3)$   
 $(3) + (2) \Rightarrow 3a = 3$   
 $a = 1, c = 1$   
 $y = x^3 + x + 1$

**b**  $y = ax^3 + bx^2 + cx + d$   
 $(0, 1) = d = 1$   
 $(1, 1) = 1 = a + b + c + 1$   
 $a + b + c = 0 \dots (1)$   
 $(-1, 1) = 1 = -a + b - c + 1$   
 $-a + b - c = 0 \dots (2)$   
 $(1) + (2) \Rightarrow b = 0$   
 $(2, 7) \Rightarrow 7 = 8a + 2c + 1$   
 $4a + c = 3 \dots (3)$   
 $3 + 2 \Rightarrow 3a = 3$   
 $a = 1$   
Sub in  $\Rightarrow (1) \Rightarrow c = -1$   
 $y = x^3 - x + 1$

**c**  $y = ax^3 + bx^2 + cx + d$   
 $(0, -2) \Rightarrow d = -2$   
 $(1, 0) \Rightarrow 0 = a + b + c - 2$   
 $a + b + c = 2 \dots (1)$   
 $(-1, -6) \Rightarrow -6 = -a + b - c - 2$   
 $-a + b - c = -4 \dots (2)$   
 $(2, 12) \Rightarrow 12 = 8a + 4b + 2c - 2$   
 $4a + 2b + c = 7 \dots (3)$   
 $(1) + (2) : 2b = -2$   
 $b = -1$   
Sub in  $\Rightarrow (3) \Rightarrow 4a + c = 9$   
Sub in  $\Rightarrow (2) \Rightarrow -a - c = -3$   
 $(3) + (2) \Rightarrow 3a = 6$   
 $a = 2$   
Sub in  $\Rightarrow (3) \Rightarrow c = 1$   
 $y = 2x^3 - x^2 + x - 2$

**6 a**  $y = a(x - b)(x - c)(x - d)$   
 $y = a(2x + 1)(x - 1)(x - 2)$   
 $2 = 2a$   
 $a = 1$   
 $y = (2x + 1)(x - 1)(x - 2)$

**b**  $y = ax^3 + bx^2 + cx$   
 $(1, 0.75) \Rightarrow \frac{3}{4} = a + b + c \dots (1)$   
 $(2, 3) \Rightarrow 3 = 8a + 4b + 2c \dots (2)$   
 $(-2, -3) \Rightarrow -3 = -8a + 4b - 2c \dots (3)$   
 $(2) + (3) \Rightarrow 8b = 0$   
 $b = 0$   
 $(2) - 2(1) \Rightarrow \frac{6}{4} = 6a$   
 $a = \frac{1}{4}$   
Sub in  $(2) \Rightarrow 3 = 2 + 2c$   
 $c = \frac{1}{2}$   
 $y = \frac{1}{4}x^3 + \frac{1}{2}x = \frac{1}{4}x(x^2 + 2)$

$$\begin{aligned} \mathbf{c} \quad y &= a(x-b)(x-c)^2 \\ y &= a(x+1)x^2 \\ 2 &= a(2)1^2 \\ a &= 1 \\ y &= x^2(x+1) \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad y &= a(x-b)(x-c)(x-d) \\ y &= a(x+2)(x+1)(x-1) \\ x &= 0, y = -2 \\ y &= (x+2)(x+1)(x-1) \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad y &= a(x-b)(x-c)^2 \\ y &= a(x+2)(x-3)^2 \end{aligned}$$

$$\begin{aligned} x &= 0, y = 18 \\ y &= (x+2)(x-3)^2 \end{aligned}$$

$$\mathbf{7} \quad \mathbf{a} \quad y = -2x^3 - 25x^2 + 48x + 135$$

$$\mathbf{b} \quad y = 2x^3 - 30x^2 + 40x + 13$$

$$\mathbf{8} \quad \mathbf{a} \quad y = -2x^4 + 22x^3 - 10x^2 - 37x + 40$$

$$\mathbf{b} \quad y = x^4 - x^3 + x^2 + 2x + 8$$

$$\mathbf{c} \quad y = \frac{31}{36}x^4 + \frac{5}{4}x^3 - \frac{157}{36}x^2 - \frac{5}{4}x + \frac{11}{2}$$

## Solutions to Exercise 4H

**1 a**  $kx^2 + x + k = 0$

$$x = \frac{-1 \pm \sqrt{1 - 4k^2}}{2k},$$

$$k \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus \{0\} \text{ since } k^2 \leq \frac{1}{4}.$$

(Note: If  $k = 0$ ,  $x = 0$ )

**b**  $x^3 - 7ax^2 + 12a^2x = 0$

$$\Rightarrow x(x^2 - 7ax + 12a^2) = 0$$

$$x(x - 3a)(x - 4a) = 0$$

$$x = 0, 3a, 4a$$

**c**  $x(x^3 - a) = 0$

$$x = 0, (a)^{\frac{1}{3}}$$

**d**  $x^2 - kx + k = 0$

$$x = \frac{k \pm \sqrt{k^2 - 4k}}{2}, k \leq 0 \text{ a } k \geq 4,$$

$$\text{since } k^2 - 4k \geq 0$$

**e**  $x(x^2 - a) = 0$

$$x = 0, \pm \sqrt{a}, a \geq 0$$

**f**  $x^4 - a^4 = 0$

$$(x^2 + a^2)(x^2 - a^2) = 0$$

$$(x^2 + a^2)(x - a)(x + a) = 0$$

$$x = -a, a$$

**g**  $(x - a)^2(x - b) = 0$

$$x = a, b$$

**h**  $(x - a)^4(a - x^3)(x^2 - a) = 0$

$$x = a, (a)^{\frac{1}{3}}, \pm \sqrt{a} \text{ if } a \geq 0$$

**2 a**  $ax^3 + b = 2c$

$$x^3 = \frac{2c - b}{a}$$

$$x = \left( \frac{2c - b}{a} \right)^{\frac{1}{3}}$$

**b**  $ax^3 - b = c$

$$x^3 = \frac{b + c}{a}$$

$$x = \left( \frac{b + c}{a} \right)^{\frac{1}{3}}$$

**c**  $a - bx^2 = c$

$$x^2 = \frac{a - c}{b}$$

$$x = \left( \frac{a - c}{b} \right)^{\frac{1}{2}}$$

**d**  $x^{\frac{1}{3}} = a$

$$x = a^3$$

**e**  $(x)^{\frac{1}{n}} + c = a$

$$(x)^{\frac{1}{n}} = a - c$$

$$x = (a - c)^n$$

**f**  $a(x - 2b)^3 = c$

$$(x - 2b)^3 = \frac{c}{a}$$

$$x - 2b = \left( \frac{c}{a} \right)^{\frac{1}{3}}$$

$$x = 2b + \left( \frac{c}{a} \right)^{\frac{1}{3}}$$

**g**  $ax^{\frac{1}{3}} = b$

$$x = \left( \frac{b}{a} \right)^3$$

**h**  $x^3 = c + d$

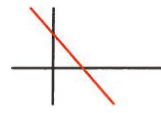
$$x = (c + d)^{\frac{1}{3}}$$

**3 a**  $x^2 = x$   
 $x^2 - x = 0$   
 $x(x - 1) = 0$   
 $x = 0, 1$   
 $y = 0, 1$   
*Pts.*  $(0, 0)$  &  $(1, 1)$

**b**  $2x^2 = x$   
 $2x^2 - x = 0$   
 $x(2x - 1) = 0$   
 $x = 0, \frac{1}{2}$   
*Pts.*  $(0, 0)$   $\left(\frac{1}{2}, \frac{1}{2}\right)$

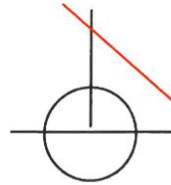
**c**  $y = x^2 - x,$   
 $y = 2x + 1$   
 $\Rightarrow x^2 - x = 2x + 1$   
 $x^2 - 3x - 1 = 0$   
 $x = \frac{3 \pm \sqrt{9 + 4}}{2}$   
 $x = \frac{3 \pm \sqrt{13}}{2}$   
 $y = 2x + 1 = 4 \pm \sqrt{13}$   
*co-ords*  $= \left(\frac{3 - \sqrt{13}}{2}, 4 - \sqrt{13}\right),$   
 $\left(\frac{3 + \sqrt{13}}{2}, 4 + \sqrt{13}\right)$

**4 a**  $y = 16 - x$   
 $y^2 = (16 - x)^2$



$$y^2 + x^2 = 178$$

$$y^2 = 178 - x^2$$



$$178 - x^2 = (16 - x)^2$$

$$178 - x^2 = 256 - 32x + x^2$$

$$2x^2 - 32x + 78 = 0$$

$$x^2 - 16x + 39 = 0$$

$$(x - 3)(x - 13) = 0$$

$$x = 3, 13$$

$$\textit{Pts.} (3, 13), (13, 3)$$

**b**

$$y^2 = 125 - x^2$$

$$y = 15 - x$$

$$\Rightarrow y^2 = 225 - 30x + x^2$$

$$x^2 - 30x + 225 = 125 - x^2$$

$$2x^2 - 30x + 100 = 0$$

$$x^2 - 15x + 50 = 0$$

$$(x - 5)(x - 10) = 0$$

$$x = 5, 10$$

*Pts.* (5, 10), (10, 5)

**c**

$$y^2 = 185 - x^2$$

$$y = x - 3$$

$$y^2 = x^2 - 6x + 9$$

$$x^2 - 6x + 9 = 185 - x^2$$

$$2x^2 - 6x - 176 = 0$$

$$x^2 - 3x - 88 = 0$$

$$(x + 8)(x - 11) = 0$$

$$x = -8, 11$$

*Pts.* (-8, -11), (11, 8)

**d**

$$y^2 = 97 - x^2$$

$$y = 13 - x$$

$$y^2 = 169 - 26x + x^2$$

$$x^2 - 26x + 169 = 97 - x^2$$

$$2x^2 - 26x + 72 = 0$$

$$x^2 - 13x + 36 = 0$$

$$(x - 9)(x - 4) = 0$$

$$x = 4, 9$$

*Pts.* (4, 9), (9, 4)

**e**

$$y^2 = 106 - x^2$$

$$y = x - 4$$

$$y^2 = x^2 - 8x + 16$$

$$x^2 - 8x + 16 = 106 - x^2$$

$$2x^2 - 8x - 90 = 0$$

$$x^2 - 4x - 45 = 0$$

$$(x + 5)(x - 9) = 0$$

$$x = -5, 9$$

*Pts.* (-5, -9), (9, 5)

**5 a**

$$y = 28 - x \dots (1)$$

$$xy = 187 \dots (2)$$

$$\Rightarrow x(28 - x) = 187$$

$$-x^2 + 28x = 187$$

$$x^2 - 28x + 187 = 0$$

$$x = \frac{28 \pm \sqrt{784 - 748}}{2}$$

$$x = \frac{28 \pm 6}{2}$$

$$x = 11, 17$$

$$\Rightarrow \text{pts} = (11, 17), (17, 11)$$

**b**

$$y = 51 - x$$

$$x(51 - x) = 518$$

$$x^2 - 51x + 518 = 0$$

$$x = \frac{51 \pm \sqrt{2601 - 2072}}{2}$$

$$x = \frac{51 \pm \sqrt{529}}{2}$$

$$x = \frac{51 \pm 23}{2}$$

$$x = 14, 37$$

$$\Rightarrow \text{pts} = (14, 37), (37, 14)$$

c

$$y = x - 5$$

$$xy = 126$$

$$x^2 - 5x = 126$$

$$x^2 - 5x - 126 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 504}}{2}$$

$$x = \frac{5 \pm 23}{2}$$

$$x = -9, 14$$

$$\Rightarrow pts = (-9, -14), (14, 9)$$

$$7 \quad x = \frac{1}{x-2} + 3$$

$$x(x-2) = 1 + 3(x-2)$$

$$x^2 - 2x = 1 + 3x - 6$$

$$x^2 - 5x + 5 = 1$$

$$x = \frac{5 \pm \sqrt{25 - 20}}{2}$$

$$x = \frac{5 \pm \sqrt{5}}{2}$$

$$pts = \left( \frac{5 + \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2} \right), \\ \left( \frac{5 - \sqrt{5}}{2}, \frac{5 - \sqrt{5}}{2} \right)$$

6

$$y^2 = 25 - (x-5)^2$$

$$= 25 - x^2 + 10x - 25$$

$$y^2 = -x^2 + 10x \dots (1)$$

$$y = 2x$$

$$y^2 = 4x^2 \dots (2)$$

$$4x^2 = -x^2 + 10x$$

$$x^2 - 2x = 0$$

$$x = 0, 2$$

$$pts = (0, 0), (2, 4)$$

8 a

$$\frac{y}{4} - \frac{x}{5} = 1 \dots (1)$$

$$\Rightarrow y = \frac{4}{5}x + 4$$

$$x^2 + 4x + y^2 = 12 \dots (2)$$

$$\Rightarrow y^2 = 12 - 4x - x^2$$

$$\Rightarrow y^2 = \frac{16}{25}x^2 + \frac{32}{5}x + 16$$

$$16\left(\frac{1}{25}x^2 + \frac{2}{5}x + 1\right) = 12 - 4x - x^2$$

$$16x^2 + 160x + 400 = 300 - 100x - 25x^2$$

$$41x^2 + 260x + 100 = 0$$

$$x = \frac{-260 \pm \sqrt{67600 - 16400}}{82}$$

$$x = \frac{-130 \pm 80\sqrt{2}}{41}$$

Sub in (1)

$$\left( \frac{-130 - 80\sqrt{2}}{41}, \frac{60 - 64\sqrt{2}}{41} \right),$$

$$\left( \frac{-130 + 80\sqrt{2}}{41}, \frac{60 + 64\sqrt{2}}{41} \right)$$

$$\begin{aligned}
 9 \quad -x &= \frac{1}{x+2} - 3 \\
 -x^2 - 2x &= 1 - 3x - 6 \\
 x^2 - x - 5 &= 0 \\
 x &= \frac{1 \pm \sqrt{1+20}}{2} \\
 x &= \frac{1 \pm \sqrt{21}}{2} \\
 \text{pts} &= \left( \frac{1 + \sqrt{21}}{2}, \frac{-1 + \sqrt{21}}{2} \right), \\
 &\quad \left( \frac{1 - \sqrt{21}}{2}, \frac{\sqrt{21} - 1}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 x &= \frac{-12\sqrt{5} \pm \sqrt{144 \times 5 - 20 \times 36}}{10} \\
 x &= \frac{-12\sqrt{5}}{10} \\
 x &= \frac{-6\sqrt{5}}{5} \\
 y &= \frac{-12\sqrt{5}}{10} + 3\sqrt{5} \\
 y &= \frac{3\sqrt{5}}{5} \\
 \text{pts} &= \left( \frac{-6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5} \right)
 \end{aligned}$$

$$\begin{aligned}
 10 \quad y &= \frac{9}{4}x + 1 \\
 y^2 &= \frac{81}{16}x^2 + \frac{9}{2}x + 1 \\
 9x &= \frac{81}{16}x^2 + \frac{9}{2}x + 1 \\
 \frac{81}{16}x^2 - \frac{9}{2}x + 1 &= 0 \\
 x &= \frac{\frac{9}{2} \pm \sqrt{\frac{81}{4} - \frac{81}{4}}}{\frac{81}{8}} \\
 x &= \frac{9}{2} \times \frac{8}{81} \\
 x &= \frac{4}{9} \\
 \text{co ords} &= \left( \frac{4}{9}, 2 \right)
 \end{aligned}$$

$$\begin{aligned}
 11 \quad y^2 &= 9 - x^2 \\
 y &= 2x + 3\sqrt{5} \\
 y^2 &= 4x^2 + 12\sqrt{5}x + 45 \\
 9 - x^2 &= 4x^2 + 12\sqrt{5}x + 45 \\
 5x^2 + 12\sqrt{5}x + 36 &= 0
 \end{aligned}$$

$$\begin{aligned}
 12 \quad \frac{1}{4}x + 1 &= -\frac{1}{x} \\
 \frac{1}{4}x^2 + x + 1 &= 0 \\
 x &= \frac{-1 \pm \sqrt{1-1}}{\frac{1}{2}} \\
 x &= -2 \\
 \text{pt} &= \left( -2, \frac{1}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 13 \quad x - 1 &= \frac{2}{x-2} \\
 (x-1)(x-2) &= 2 \\
 x^2 - 3x + 2 &= 2 \\
 x(x-3) &= 0 \\
 x &= 0, 3 \\
 \text{pts} &= (0, -1), (3, 2)
 \end{aligned}$$



**14 a**

$$5x - 4y = 7$$

$$4y = 5x - 7$$

$$y = \frac{5x - 7}{4}$$

$$xy = 6$$

$$x\left(\frac{5x - 7}{4}\right) = 6$$

$$5x^2 - 7x - 24 = 0$$

$$x = \frac{7 \pm \sqrt{49 + 480}}{10}$$

$$x = \frac{7 \pm \sqrt{529}}{10}$$

$$x = \frac{-8}{5}, 3$$

$$pts = \left(\frac{-8}{5}, \frac{-15}{4}\right), (3, 2)$$

**b**

$$y = \frac{37 - 2x}{3}$$

$$xy = 45$$

$$37x - 2x^2 = 135$$

$$2x^2 - 37x + 135 = 0$$

$$x = \frac{37 \pm \sqrt{1369 - 1080}}{4}$$

$$x = 5, 13.5$$

$$pts = (5, 9), \left(13.5, \frac{10}{3}\right)$$

**c**  $5x - 3y = 18$ 

$$y = \frac{5x - 18}{3}$$

$$xy = 24$$

$$5x^2 - 18x = 72$$

$$x = \frac{18 \pm \sqrt{324 + 1440}}{10}$$

$$x = \frac{18 \pm 42}{10}$$

$$x = -\frac{12}{5}, 6$$

$$pts = \left(-\frac{12}{5}, -10\right), (6, 4)$$

**15**  $x^2 + ax + b$  div by  $x + c$ 

$$(-c)^2 + a(-c) + b = 0$$

$$c^2 - ac + b = 0$$

**16**

$$x + 2 = \frac{160}{x}$$

$$x^2 + 2x - 160 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 640}}{2}$$

$$x = -1 \pm \sqrt{161}$$

$$pts = \left(-1 - \sqrt{161}, 1 - \sqrt{161}\right),$$

$$\left(\sqrt{161} - 1, \sqrt{161} + 1\right)$$

**17**  $y = -7x + 14, y = 5x + 12$ **18**  $m < -7$  or  $m > 1$ **19**  $c = -8$  or  $c = 4$

**20 a**  $mx = \frac{1}{x} + 5$   
 $mx^2 - 5x - 1 = 0$   
 $x = \frac{5 \pm \sqrt{25 + 4m}}{2m}, m \neq 0$   
 Note that if  $m = 0, x = -\frac{1}{5}$ .

**b**  $25 + 4m = 0$   
 $m = \frac{-25}{4}$   
 $x = \frac{5}{\frac{-25}{2}}$   
 $x = \frac{-2}{5}$

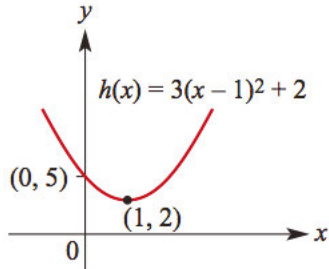
$pt\left(\frac{-2}{5}, \frac{5}{2}\right)$

**c**  $25 + m < 0$   
 $m < \frac{-25}{4}$

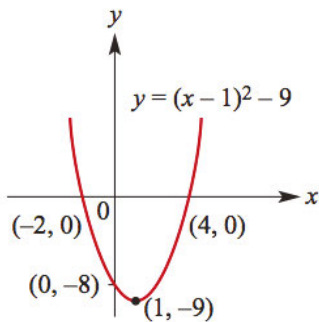
**21**  $y = 3x + 3, y = -x + 3$

## Solutions to technology-free questions

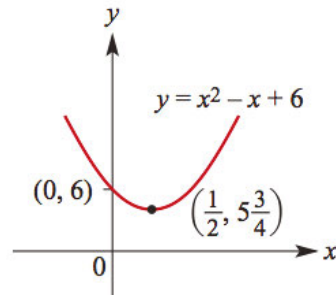
- 1 a**  $h(x) = 3(x-1)^2 + 2$   
 $x = 0: y = 3(-1)^2 + 2 = 5$   
 $y = 0$ : no solutions  
 TP  $(1, 2)$ ; no  $x$  int;  $y$  int  $(0, 5)$



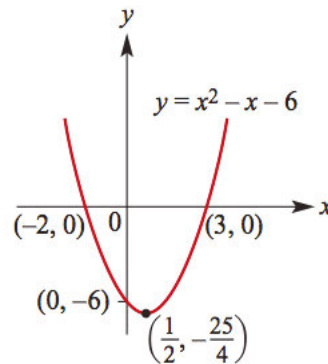
- b**  $h(x) = (x-1)^2 - 9$   
 $x = 0: y = (-1)^2 - 9 = -8$   
 $y = 0: (x-1)^2 - 9 = 0$   
 $x-1 = \pm 3$ , so  $x = -2, 4$   
 TP  $(1, -9)$ ;  $x$  int  $(-2, 0), (4, 0)$ ;  
 $y$  int  $(0, -8)$



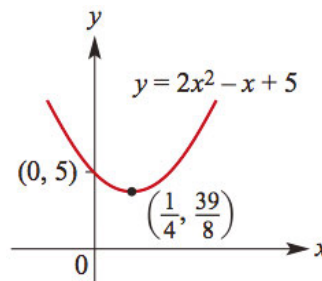
- c**  $f(x) = x^2 - x + 6$   
 $x = 0: y = 6$   
 $y = 0$ : no solutions ( $b^2 - 4ac < 0$ )  
 $x^2 - x + 6 = \left(x - \frac{1}{2}\right)^2 + 5\frac{3}{4}$   
 TP  $\left(\frac{1}{2}, 5\frac{3}{4}\right)$ ; no  $x$  int;  $y$  int  $(0, 6)$



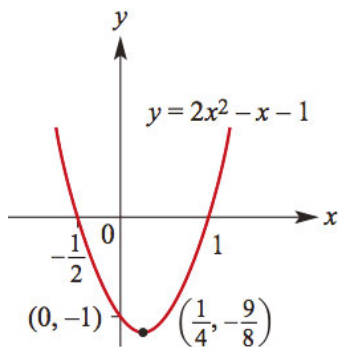
- d**  $f(x) = x^2 - x - 6$   
 $x = 0: y = -6$   
 $y = 0: x^2 - x - 6 = 0$   
 $(x+2)(x-3) = 0$ , so  $x = -2, 3$   
 $x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - 6\frac{1}{4}$   
 TP  $\left(\frac{1}{2}, -6\frac{1}{4}\right)$ ;  $x$  int  $(-2, 0), (3, 0)$ ;  
 $y$  int  $(0, -6)$



- e**  $f(x) = 2x^2 - x + 5$   
 $x = 0: y = 5$   
 $y = 0$ : no solutions ( $b^2 - 4ac < 0$ )  
 $2x^2 - x + 5 = 2\left(x - \frac{1}{4}\right)^2 + 4\frac{7}{8}$   
 TP  $\left(\frac{1}{4}, 4\frac{7}{8}\right)$ ; no  $x$  int;  $y$  int  $(0, 5)$



**f**  $h(x) = 2x^2 - x - 1$   
 $x = 0: y = -1$   
 $y = 0: 2x^2 - x - 1 = 0$   
 $(2x + 1)(x - 1) = 0$ , so  $x = -\frac{1}{2}, 1$   
 $2x^2 - x - 1 = 2\left(x - \frac{1}{4}\right)^2 - 1\frac{1}{8}$   
 TP  $\left(\frac{1}{4}, -1\frac{1}{8}\right)$ ;  $x$  int  $\left(-\frac{1}{2}, 0\right), (1, 0)$ ;  
 $y$  int  $(0, -1)$



**2**  $(1, 1): 1 = a + b$  1

$(2, 5): 5 = 4a + b$  2

Subtract 2 from 1:

$3a = 4$

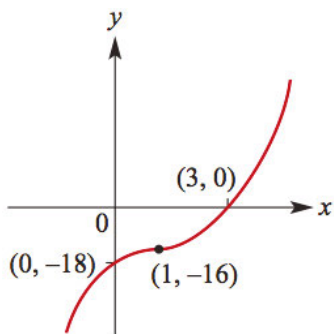
$a = \frac{4}{3}$

Substitute into 1:  $b = -\frac{1}{3}$

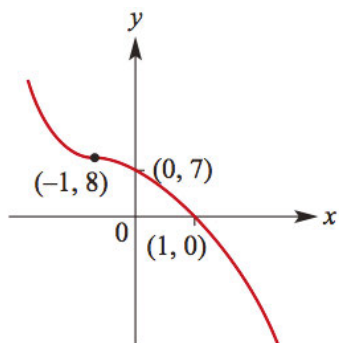
**3**  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$   
 $= \frac{2 \pm \sqrt{4 - 4(3)(-10)}}{6}$   
 $= \frac{2 \pm \sqrt{124}}{6} = \frac{1}{3}(1 \pm \sqrt{31})$

**4 a**  $f(x) = 2(x - 1)^3 - 16$   
 $x = 0: y = 2(-1)^3 - 16 = -18$   
 $y = 0: 2(x - 1)^3 - 16 = 0$   
 $(x - 1)^3 = 8, x - 1 = 2$ , so  $x = 3$   
 zero gradient:  $(1, -16)$

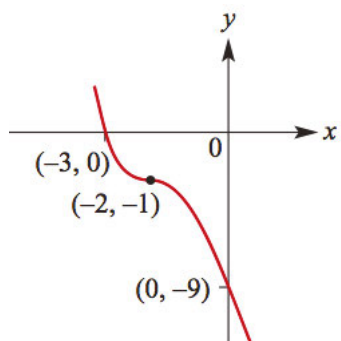
$x$  int  $(3, 0)$ ;  $y$  int  $(0, -18)$



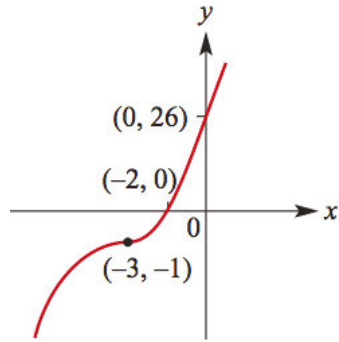
**b**  $g(x) = -(x + 1)^3 + 8$   
 $x = 0: y = -(1)^3 + 8 = 7$   
 $y = 0: -(x + 1)^3 + 8 = 0$   
 $(x + 1)^3 = 8, x + 1 = 2$ , so  $x = 1$   
 zero gradient:  $(-1, 8)$   
 $x$  int  $(1, 0)$ ;  $y$  int  $(0, 7)$



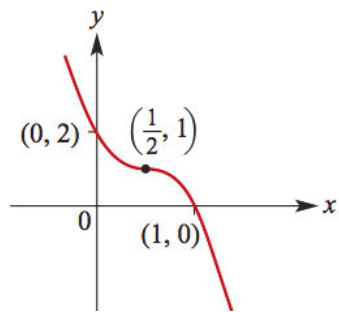
**c**  $h(x) = -(x + 2)^3 - 1$   
 $x = 0: y = -(2)^3 - 1 = -9$   
 $y = 0: -(x + 2)^3 - 1 = 0$   
 $(x + 2)^3 = -1, x + 2 = -1$ , so  $x = -3$   
 zero gradient:  $(-2, -1)$   
 $x$  int  $(-3, 0)$ ;  $y$  int  $(0, -9)$



- d**  $f(x) = (x + 3)^3 - 1$   
 $x = 0: y = (3)^3 - 1 = 26$   
 $y = 0: (x + 3)^3 - 1 = 0$   
 $(x + 3)^3 = 1, x + 3 = 1, \text{ so } x = -2$   
 zero gradient:  $(-3, -1)$   
 $x$  int  $(-2, 0)$ ;  $y$  int  $(0, 26)$

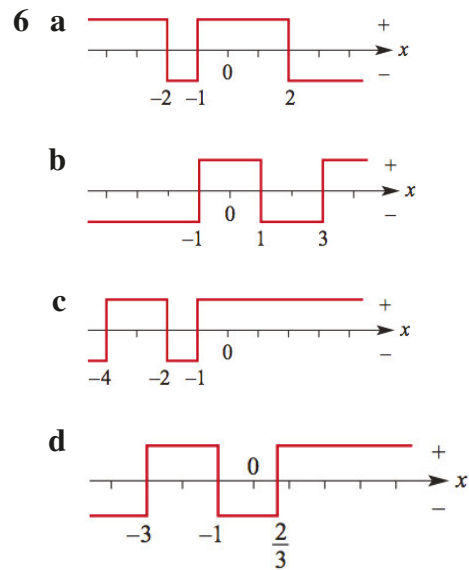


- e**  $f(x) = 1 - (2x - 1)^3$   
 $x = 0: y = 1 - (-1)^3 = 2$   
 $y = 0: 1 - (2x - 1)^3 = 0$   
 $(2x - 1)^3 = 1, 2x - 1 = 1, \text{ so } x = 1$   
 zero gradient:  $(\frac{1}{2}, 1)$   
 $x$  int  $(1, 0)$ ;  $y$  int  $(0, 2)$



- 5 a**  $(x + 2)^2 - 4$   
**b**  $3(x + 1)^2 - 3$   
**c**  $(x - 2)^2 + 2$   
**d**  $2\left(x - \frac{3}{2}\right)^2 - \frac{17}{2}$   
**e**  $2\left(x - \frac{7}{4}\right)^2 - \frac{81}{8}$

**f**  $-\left(x - \frac{3}{2}\right)^2 - \frac{7}{4}$



- 7 a**  $P(x) = x^3 + 3x^2 - 4x + 2$   
 $P(-1) = (-1)^3 + 3(-1)^2 - 4(-1) + 2$   
 $= 8$
- b**  $P(x) = x^3 - 3x^2 - x + 6$   
 $P(2) = 2^3 - 3 \times 2^2 - 2 + 6$   
 $= 0$
- c**  $P(x) = 2x^3 + 3x^2 - 3x - 2$   
 $P(-2) = 2(-2)^3 + 3(-2)^2 - 3(-2) - 2$   
 $= 0$

- 8** From the  $x$  intercepts, the rule must be  
 $y = a(x + 3)(x + 2)(x - 7)$   
 $x = 0: y = a(3)(2)(-7) = -42a$   
 But the  $y$  intercept is  $(0, -42)$  and hence  
 $-42a = -42$ , so  $a = 1$ .  
 Thus  $y = (x + 3)(x + 2)(x - 7)$ .

- 9 a**  $(x - 2)(x + 1)(x + 3)$

**b**  $(x - 1)(x + 1)(x - 3)$

**c**  $(x - 1)(x + 1)(x - 3)(x + 2)$

**d**  $\frac{1}{4}(x - 1)(2x + 3 + \sqrt{13})(2x + 3 - \sqrt{13})$

**10**  $x^2 + 4 = 1 \times (x^2 - 2x + 2) + 2x + 2$

**11**  $a = -6$

**12**  $f(x) = (x + 1)^3(x - 2)$  Note: The tp on the diagrams are incorrect

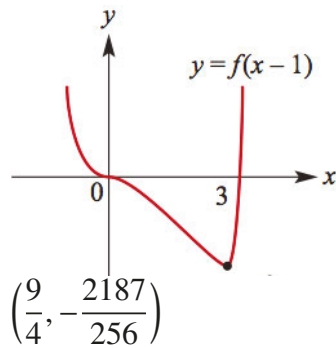
**a**  $y = f(x - 1)$

Translate the given graph 1 unit right.

The new intercepts are  $(0, 0)$ ,  $(3, 0)$ .

The new minimum is at  $(\frac{9}{4}, -\frac{2187}{256})$

since the y value does not change.



**b**  $y = f(x + 1)$

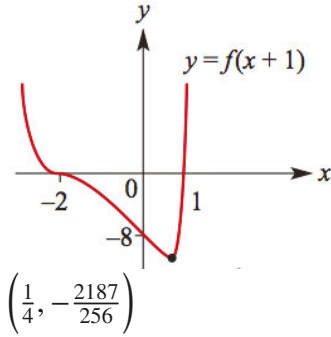
Translate the given graph 1 unit left.

The new x intercepts are  $(-2, 0)$ ,  $(1, 0)$ .

$x = 0: y = f(1) = 2^3(-2) = -16$ , so the new y intercept is  $(0, -16)$ .

The new minimum is at  $(\frac{1}{4}, -\frac{2187}{256})$

since the y value does not change.



**c**  $y = f(2x)$

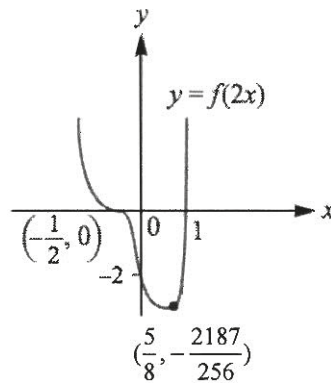
Dilate the given graph  $\frac{1}{2}$  unit from the y axis.

The new x intercepts are  $(1, 0)$ ,

$(-\frac{1}{2}, 0)$ .

The new y intercept stays at  $(0, -2)$ .

The new minimum is at  $(\frac{5}{8}, -\frac{2187}{256})$  since the y value does not change.



**d**  $y = f(x) + 2$

Translate the given graph 2 units up.

This makes the origin an intercept.

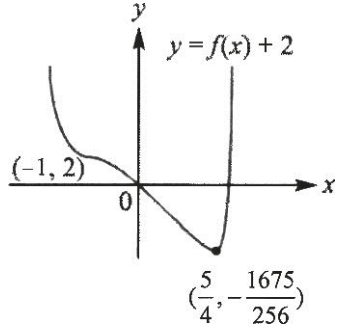
A second x intercept is between  $\frac{5}{4}$  and 2.

The minimum has the same

x value of  $\frac{5}{4}$  and y value of

$$-\frac{2187}{256} + 2 = -\frac{1675}{256}.$$

The new minimum is at  $(\frac{5}{4}, -\frac{1675}{256})$ .



13  $k = \pm 8$

14  $(4, -5), (3, 9)$

15  $a = 3, b = \frac{5}{6}, c = -\frac{13}{12}$

16  $64x^3 + 144x^2 + 108x + 27$

17  $a = 1, b = -1, c = 4$

18  $-2 < p < 6$

19 The rule of the cubic function is of the form  $y = ax^3 + bx^2 + cx + d$ . Since its graph passes through  $(0, 6)$ ,  $d = 6$ . Write the equation as  $y - 6 = ax^3 + bx^2 + cx$ . Use the remaining points to form three simultaneous equations in  $a, b$ , and  $c$ .

$$(1, 1): -5 = a + b + c \quad 1$$

$$(2, 4): -2 = 8a + 4b + 2c \quad 2$$

$$(3, 9): 3 = 27a + 9b + 3c \quad 3$$

$$2 - 21: 6a + 2b = 8 \text{ or equivalently}$$

$$3a + b = 4 \quad 4$$

$$2 - 31: 24a + 6b = 18 \text{ or equivalently}$$

$$4a + b = 3 \quad 5$$

$$5 - 4 \text{ gives } a = -1.$$

$$\text{Substitution into 4 gives } b = 7.$$

$$\text{Substitution into 1 gives } c = -11.$$

$$\text{Hence } a = -1, b = 7, c = -11, d = 6 \text{ and}$$

$$\text{so } y = -x^3 + 7x^2 - 11x + 6.$$

## Solutions to multiple-choice questions

$$\begin{aligned}
 1 \quad \mathbf{E} \quad &= 5x^2 - 10x - 2 \\
 &= 5x^2 - 2x - 2 \\
 &= 5x - 2x + 1 - 1 - 2 \\
 &= 5(x - 2)^2 - 1 - 2 \\
 &= 5(x - 2)^2 - 5 - 2 \\
 &= 5(x - 2)^2 - 7
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \mathbf{D} \quad &\text{There are 2 real roots when the} \\
 &\text{determinant} > 0 \\
 &b^2 - 4ac > 0 \\
 &36 + 12m > 0 \\
 &12m > -36 \\
 &m > -3
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \mathbf{E} \quad &x^3 + 27 \\
 &= x^3 + 3^3 \\
 &a^3 + b^3 = (ax + b)(ax^2 - abx + b^2) \\
 &\text{Where } a = 1 \text{ and } b = 3 \\
 &(x + 3)(x^2 - 3x + 9)
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \mathbf{C} \quad &\text{The equation is a cubic.} \\
 &\text{From null factor theorem:} \\
 &\text{The only possible options are} \\
 &\text{B and C} \\
 &\text{Sub in an } x \text{ value to determine if} \\
 &\text{the graph has a positive or negative} \\
 &y \text{ value:} \\
 &\text{When } x = 2 \\
 &\text{Option C: } y = 16 \times -6 \\
 &\text{Option D: } y = 4 \times 4 \\
 &\text{Therefore it must be option C}
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \mathbf{E} \quad &x - 1 \text{ is a factor} \\
 \therefore \quad &1^3 + 3(1)^2 - 2a + 1 = 0 \\
 &-2a = -5 \\
 &a = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \mathbf{A} \quad &\text{Check by expanding:} \\
 &\text{For option A,} \\
 &(3x + 2y)(2x - 4y) \\
 &= 6x^2 - 12xy + 4xy - 8y^2 \\
 &= 6x^2 - 8xy - 8y^2
 \end{aligned}$$

7 **C** Looking at the part of the graph shown, we can see that at  $x = 1$ , the graph is also showing a turning point. Therefore we can see that the answer must be either D or C, as the  $x$ -intercept points in the other graphs either show points of inflection (i.e.  $f(x) = (x - 1)^3$ ), or an intercept where the graph doesn't change direction (i.e.  $f(x) = x^2(x - 1)$ ). Then substitute values into the equations to check which one of C or D it is. Looking at C, you can see that for all values of  $x$  greater than zero other than 1, the function will be equal to a number less than zero. Looking at D, you can see that for all values of  $x$  greater than zero other than 1, the function will be equal to a number greater than zero.

8 **E** Expand the outer set of brackets to get the function into turning point form for  $m$ . So  $p(x) = 3((x - 2)^2 + 4)$  becomes  $p(x) = 3(x - 2)^2 + 12$ . Therefore the graph is shifted right 2 and up 12 from the origin. The answer is



**9 C** From the graph there is a intercept at  $x = c$  and turning point at  $(b, 0)$  . So the polynomial must have functions  $(x - c)$  and  $(x - b)^2$   
Now  $(x - b)^2$  is the same as  $(b - x)^2$ .  
 $y = (x - c)(b - x)^2$  fits.  
(Note: that option *D* gives a reflection in the  $x$ -axis of the graph given.)

**10 C** We can see immediately by looking at the equation that the function will touch the  $x$ -axis when  $x = b$ , and when  $x = -c$ . The remaining factor of the function is  $(x^2 + a)$  and we know that is a positive real number. When we attempt to solve for  $x$ , we get the following:  $x^2 = -a$ .  
Knowing that  $a$  is a positive real number, we realise that the solutions are not real numbers and hence are not roots.

**11 C**

**12 B**

## Solutions to extended-response questions

1 a The graph passes through the point (15, 20)

$$\therefore 20 = k \times 15^3 \times (20 - 15)$$

$$\therefore k = \frac{4}{15^3}$$

$$\therefore k = \frac{4}{3375} \approx 0.0019$$

$$\therefore R = \frac{4t^3}{3375}(20 - t)$$

b When  $t = 10$   $R = \frac{4 \times 10^3}{3375} \times 10$

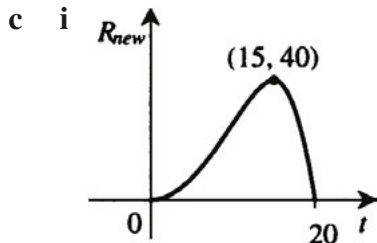
$$= \frac{4 \times 10^4}{3375}$$

$$= \frac{4 \times 80}{27}$$

$$= \frac{320}{27}$$

The rate of flow =  $\frac{320}{27}$  mL/min when  $t = 10$

$$\left(\frac{320}{27} \approx 11.852\right)$$



Note: This graph is given by a dilation of factor 2 from the  $t$ -axis

ii When  $t = 10$

$$R_{new} = 2 \times \frac{4}{3375} \times 10^3 \times 10$$

$$= \frac{640}{27} \text{ mL/min}$$

The rate of flow =  $\frac{640}{27}$  mL/min when  $t = 10$   $\left(\frac{640}{27} \approx 23.704\right)$

- d i** The hint gives that  $R_{out}$  is obtained by a translation of 20 units to the right.

$$\therefore (t, R) \rightarrow (t + 20, R)$$

$$\therefore t' = t + 20 \text{ and } R' = R$$

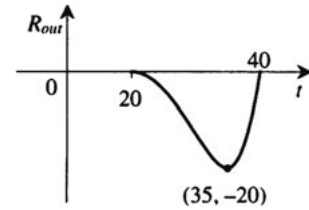
$$\therefore R = kt^3(20 - t) \text{ is transformed to}$$

$$R' = k(t' + 20)^3(20 - (t' - 20))$$

$$= k(t' - 20)^3(40 - t')$$

A reflection in the  $x$ -axis give

$$R_{out} = -k(t - 20)^3(40 - t)$$



- ii** When  $t = 30$ ,  $R_{out} = \frac{-320}{27} \text{ mL/min} \left( -\frac{320}{27} \approx -11.852 \right)$

Note: the simplest way to obtain this is to move  $\left( 10, \frac{320}{7} \right) \rightarrow \left( 30, \frac{-320}{7} \right)$  with this transformation

The rate of flow out is  $\frac{320}{27} \text{ mL/min}$

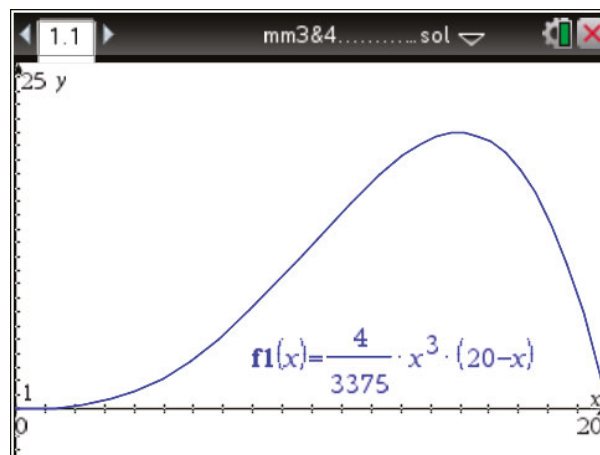
**Calculator technique for question:**

- a** In a Graphs page enter the rule:  $f1(x) = 4/3375x^3(20 - x)$ .

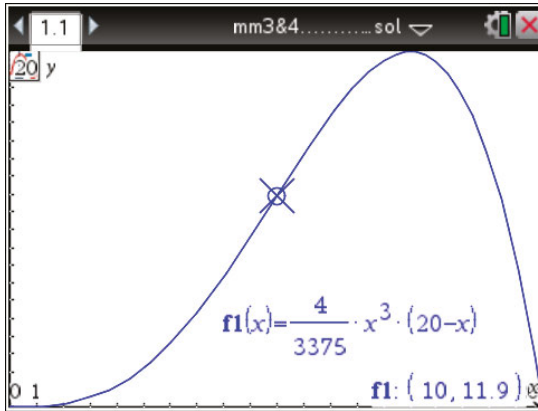
Suitable window settings are:

A screenshot of a calculator's 'Window Settings' menu. The settings are as follows:

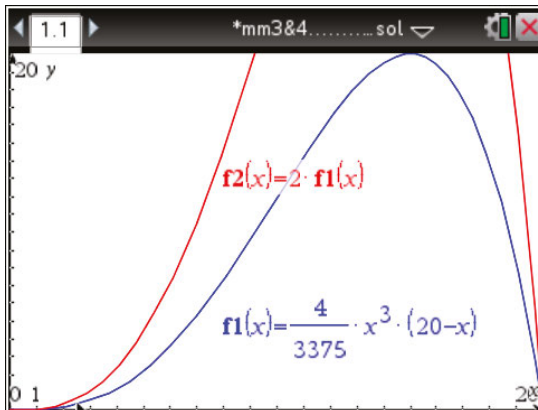
- XMin: 0
- XMax: 20
- XScale: Auto
- YMin: 0
- YMax: 20
- YScale: Auto



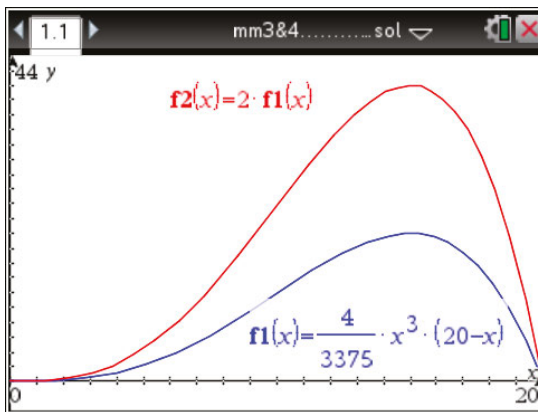
- b The rate of flow when  $t = 10$  is obtained by using Graph Trace from the Trace menu and typing in 10. Press.  
Hint: press d to exit the Graph Trace tool.



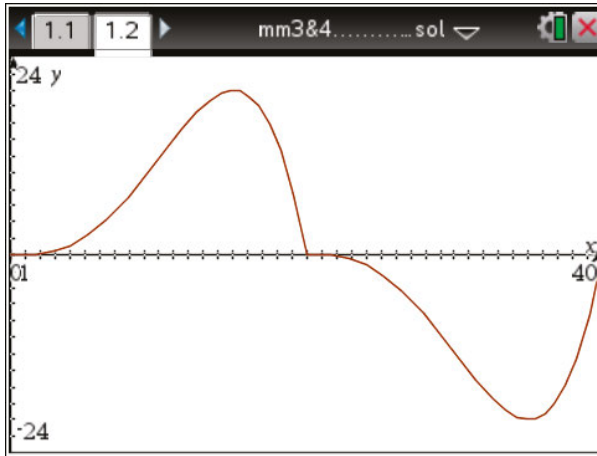
- c The new function is obtained by entering  $f2(x) = 2f1(x)$  in the function entry line (press e or /+G to show the function entry line if required). Press to plot the new graph.



Change the window settings to show key points of both graphs. Hint: use b>Window/Zoom>ZoomFit



In order to see the graphs by R against t and Rout against t a hybrid function must be entered as shown,



$$f_3(x) = \begin{cases} \left(\frac{4}{3375}\right)x^3(20-x) & \text{for } 0 \leq x \leq 20 \\ -\left(\frac{4}{3375}\right)(x-20)^3(40-x) & \text{for } 20 \leq x \leq 40 \end{cases}$$

Insert a new Graphs page (/ + I) From the math templates palette (t)select the piecewise template.



The graph is as shown. For this choose Xmin = 0 and Xmax = 40. Adjust values.

**2 a i** When  $t = 0$ ,  $V = 4 \times 9^3 = 2916$

The volume is  $2916 \text{ m}^3$

**ii** When  $t = 9$ ,  $V = 0$

**b** The volume is  $0 \text{ m}^3$

**c**  $512 = 4(9 - t)^3$

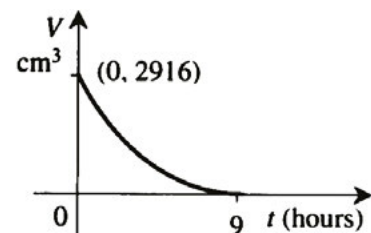
$$128 = (9 - t)^3$$

$$128^{\frac{1}{3}} = 9 - t$$

$$t = 9 - 128^{\frac{1}{3}}$$

$$= 9 - 4 \times 2^{\frac{1}{3}} \approx 3.9603$$

After 3.96 hr the volume is  $512 \text{ m}^3$



$$\begin{aligned}
 \text{3 a i } V &= \frac{1}{3}\pi \times 4 \times (18 - 2) \\
 &= \frac{\pi}{3} \times 4 \times 16 \\
 &= \frac{64\pi}{3}
 \end{aligned}$$

Volume is  $\frac{64\pi}{3} \text{ cm}^3$  when  $x = 2$

$$\begin{aligned}
 \text{ii } V &= \frac{1}{3}\pi \times 3^2 \times (18 - 3) \\
 &= \pi \times 45 \\
 &= 45\pi
 \end{aligned}$$

Volume is  $45\pi \text{ cm}^3$  when  $x = 3$ .

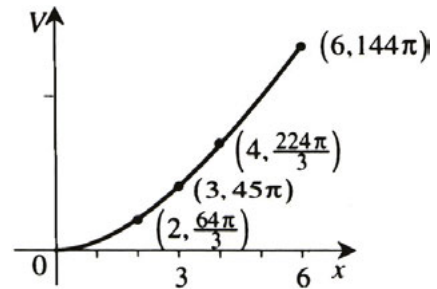
$$\begin{aligned}
 \text{iii } V &= \frac{1}{3}\pi \times 4^2 \times (18 - 4) \\
 &= \frac{\pi}{3} \times 16 \times 14 \\
 &= \frac{224\pi}{3}
 \end{aligned}$$

Volume is  $\frac{224\pi}{3} \text{ cm}^3$  when  $x = 4$ .

**b** When the bowl is full, depth is 6 cm.

$$\begin{aligned}
 \text{When } x = 6, V &= \frac{1}{3}\pi \times 36 \times 12 \\
 &= 144\pi
 \end{aligned}$$

The volume of water is  $144\pi \text{ cm}^3$  when the bowl is full.



$$\text{c If } V = \frac{325\pi}{3}, \quad \frac{325\pi}{3} = \frac{1}{3}\pi x^2(18 - x)$$

which implies  $325 = x^2(18 - x)$

$$\text{and } \therefore x^3 - 18x^2 + 325 = 0$$

$$\text{Let } P(x) = x^3 - 18x^2 + 325$$

$$P(5) = 5^3 - 18 \times 5^2 + 325 = 0$$

which, by the Factor Theorem, implies that  $x - 5$  is a factor.

$$\therefore P(x) = (x - 5)(x^2 - 13x + 65)$$

$$x^2 - 13x - 65 = 0 \text{ implies } x = \frac{13 \pm \sqrt{169 + 4 \times 65}}{2}$$

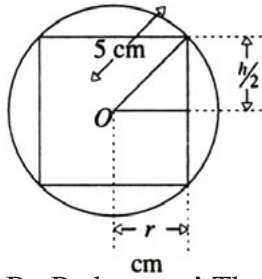
$$= \frac{13 \pm \sqrt{429}}{2}$$

but these two values of  $x$  lie outside the domain of  $V = (0, 6)$

$\therefore x = 5$  is the only solution.

i.e. the depth of the water when  $V = \frac{325\pi}{3}$  is 5 cm.

4 a



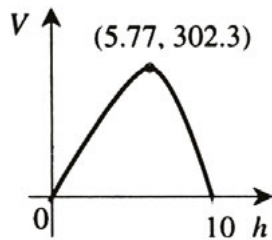
By Pythagoras' Theorem

$$r^2 + \left(\frac{h}{2}\right)^2 = 25$$

$$\therefore r^2 = 25 - \left(\frac{h}{2}\right)^2$$

$$\text{i.e. } r = \frac{1}{2} \sqrt{100 - h^2}$$

b



$$\begin{aligned} \therefore \text{Volume of cylinder} &= \frac{1}{2} \pi r^2 h \\ &= \pi \times \frac{1}{4} (100 - h^2) h \\ &= \frac{1}{4} \pi h (100 - h^2) \end{aligned}$$

$$\text{c } V = \frac{1}{4}\pi h(100 - h^2)$$

When  $h = 6$

$$V = \frac{1}{4} \times \pi \times 6(100 - 36) \\ = 96\pi$$

The volume of the cylinder is  $96\pi\text{cm}^3$

$$\text{d } \quad \quad \quad \text{When } V = 48\pi$$

$$48\pi = \frac{1}{4}\pi h(100 - h^2)$$

$$\therefore 192 = 100h - h^3$$

$$\therefore h^3 - 100h + 192 = 0$$

$$\text{Let } P(h) = h^3 - 100h + 192$$

$$P(2) = 2^3 - 100 \times 2 + 192$$

$$= 0$$

$\therefore h - 2$  is a factor

$$\therefore P(h) = (h - 2)(h^2 + 2h - 96)$$

$$P(h) = 0 \text{ implies } h = 2 \text{ or } h^2 + 2h - 96 = 0$$

$$\therefore h = \frac{-2 \pm \sqrt{4 + 4 \times 96}}{2} \\ = \frac{-2 \pm \sqrt{388}}{2} \\ = -1 \pm \sqrt{97}$$

But  $h > 0$ ,  $\therefore$  the only solutions are  $h = 2$  and  $h = -1 + \sqrt{97}$

When  $h = 2$

$$r = \frac{1}{2} \sqrt{100 - 4}$$

$$= \frac{1}{2} \sqrt{96}$$

$$= 2\sqrt{6}$$

When  $h = -1 + \sqrt{97} \approx 8.849$

$$r \approx \frac{1}{2} \sqrt{100 - 78.30}$$

$$\approx 2.33$$

When the volume of the cylinder is  $48\pi \text{ cm}^3$  the height is 2 cm and the radius

$$2\sqrt{6} \approx 4.899 \text{ cm.}$$

OR the height is  $(-1 + \sqrt{97}) \approx 8.849$  and the radius is  $\approx 2.33$  cm.

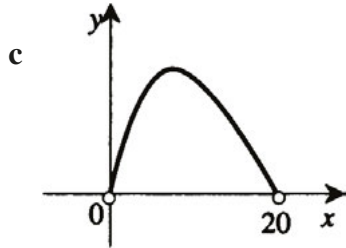


5 a  $V = (84 - 2x)(40 - 2x)x$

b  $84 - 2x > 0$  and  $40 - 2x > 0$  and  $x > 0$

$\therefore x < 42$  and  $x < 20$  and  $x > 0$

$\therefore$  maximal domain =  $(0, 20)$



d i  $5\,760 \text{ cm}^3$

ii  $12\,096 \text{ cm}^3$

iii  $13\,056 \text{ cm}^3$

iv  $12\,800 \text{ cm}^3$

e Use **Intersection** from the **Analyze Graph** menu

$x = 13.50$  or  $x = 4.18$  (answers given correct to two decimal places)

f  $13\,098.71 \text{ cm}^3$  (use **Maximum** from the **Analyze Graph** menu)

6 a i  $A = 2x(16 - x^2)$

ii  $0 < x < 4$

b i  $A = 6(16 - 9)$

$= 42$

ii  $x = 0.82$  or  $x = 3.53$  (use **Intersection** from the **Analyze Graph** menu)

c i  $V = xA$

$= 2x^2(16 - x^2)$

ii  $x = 2.06$  or  $x = 3.43$  (use **Intersection** from the **Analyze Graph** menu)

7 a  $A = yx + \frac{\pi}{2}x^2$

b i  $100 = y + \pi x$

$\therefore y = 100 - \pi x$

$$\begin{aligned}
 \text{ii } A &= (100 - \pi x)x + \frac{\pi}{2}x^2 \\
 &= 100x - \pi x^2 + \frac{\pi}{2}x^2 \\
 &= 100x - \frac{\pi}{2}x^2
 \end{aligned}$$

$$\text{iii } \left(0, \frac{100}{\pi}\right) \text{ as } x > 0 \text{ and } y > 0 \text{ which implies } 100 - \pi x > 0$$

$$\text{c } x = 12.425$$

Intersection from the Analyze Graph menu has been used.

$$\begin{aligned}
 \text{d } \text{i } V &= \frac{x}{50} \left( \frac{\pi}{2}x^2 + yx \right) \\
 &= \frac{x}{50} \left( 100x - \frac{\pi}{2}x^2 \right) \\
 &= \frac{x^2}{50} \left( 100 - \frac{\pi}{2}x \right) \quad x \in \left( 0, \frac{100}{\pi} \right)
 \end{aligned}$$

$$\text{ii } V = 248.5 \text{ m}^3 \text{ using } x = 12.425 \text{ when } A = 1000$$

$$\text{iii } \text{Using Intersection from the Analyze Graph menu gives } x = 18.84$$

**8a** In a **Calculator** page solve the system of equations using **b>Algebra>Solve System of Equations>Solve System of Equations**.

$$\text{solve} \begin{cases} a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = 0 \\ a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d = 1 \\ a \cdot 30^3 + b \cdot 30^2 + c \cdot 30 + d = 2 \\ a \cdot 40^3 + b \cdot 40^2 + c \cdot 40 + d = 3 \end{cases} \{a, b, c, d\}$$

$$a = \frac{1}{12000} \text{ and } b = -\frac{1}{200} \text{ and } c = \frac{17}{120} \text{ and } d = 0$$

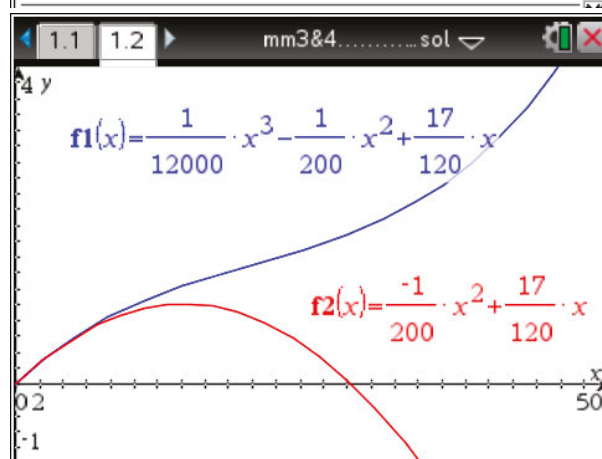
**b** Define the function  $h(x)$   
Find the height when  $x = 1.5$  m

$$\text{Define } h(x) = \frac{1}{12000} \cdot x^3 - \frac{1}{200} \cdot x^2 + \frac{17}{120} \cdot x$$

$$h(1.5) = 0.201531$$

**c** In a **Graphs** page, enter the two functions

The coefficient of  $x^3$ , although small, is clearly influential.



**d** Solve the system of equations.

Hint: to obtain exact (fraction) answers the decimal values in the system of equations can be written as fractions as shown.

$$\text{solve} \begin{cases} a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = 0 \\ a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d = \frac{3}{10} \\ a \cdot 30^3 + b \cdot 30^2 + c \cdot 30 + d = \frac{27}{10} \\ a \cdot 40^3 + b \cdot 40^2 + c \cdot 40 + d = \frac{28}{10} \end{cases} \{a, b, c, d\}$$

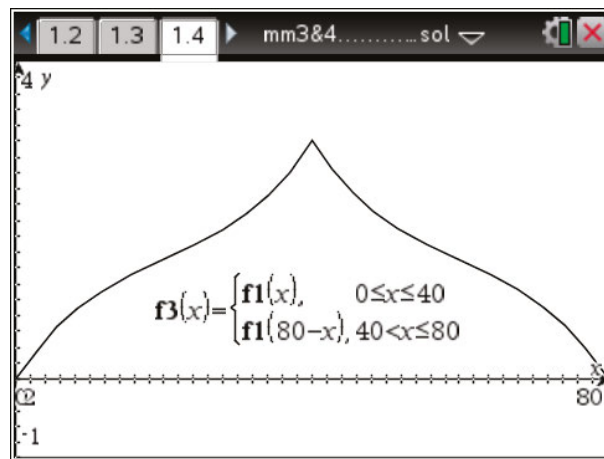
Alternatively use  
**b>Number>Approximate to Fraction** and edit the tolerance to 5.E-5)

$$a = \frac{-1}{6000} \text{ and } b = \frac{29}{3000} \text{ and } c = \frac{-1}{20} \text{ and } d = 0$$

$$h1(x) = \frac{-1}{6000} \cdot x^3 + \frac{29}{3000} \cdot x^2 - \frac{1}{20} \cdot x$$

$$h1(x) = \frac{-x^3}{6000} + \frac{29 \cdot x^2}{3000} - \frac{x}{20}$$

**e (i)** in a **Graphs** page enter the hybrid function using the piecewise template  $\left\{ \begin{matrix} \square & \square \\ \square & \square \end{matrix} \right.$  from the **Math Template** palette (t)



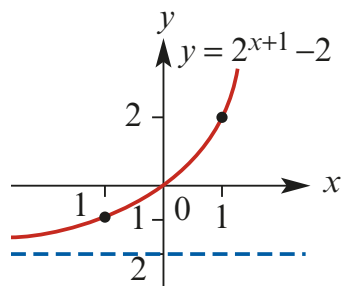
The result is as shown.

**e (ii)** The second section of the graph is formed by a reflection of the graph of  $y = f1(x)$ ,  $x \in (0, 40)$  in the line  $x = 40$

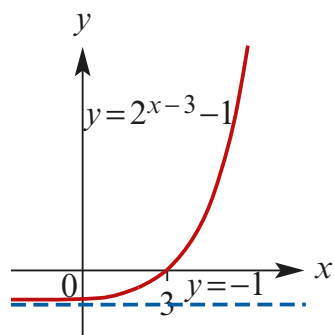
# Chapter 5 – Exponential and logarithmic functions

## Solutions to Exercise 5A

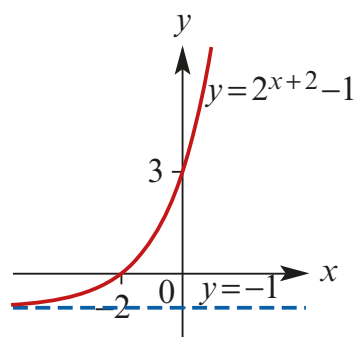
1 a Range =  $(-2, \infty)$



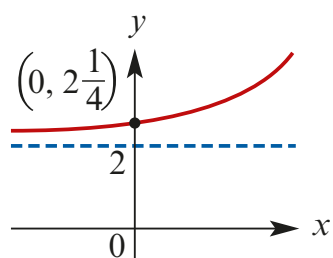
b Range =  $(-1, \infty)$



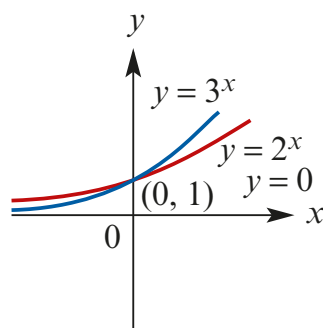
c Range =  $(-1, \infty)$



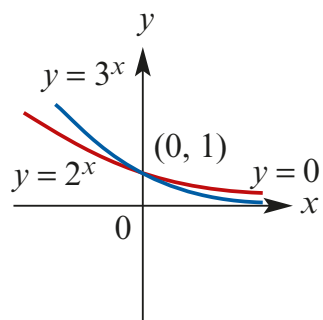
d Range =  $(2, \infty)$



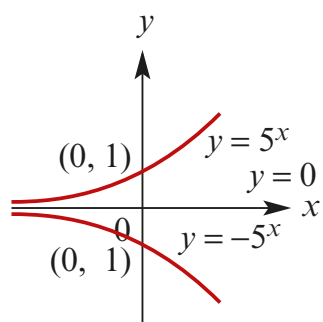
2 a



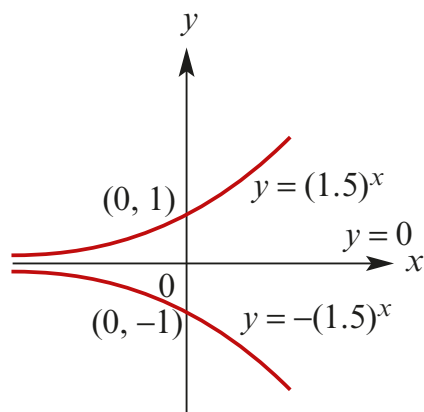
b



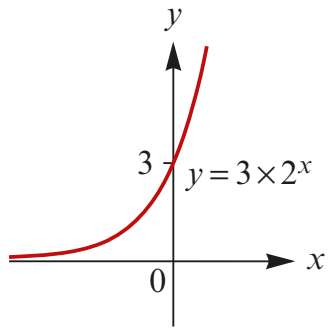
c



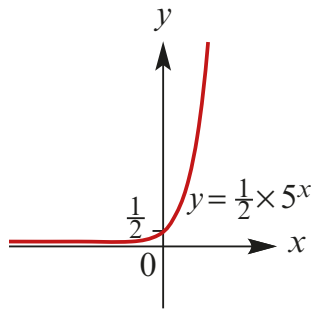
d



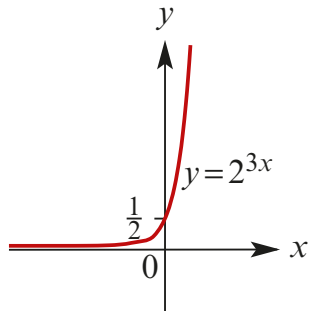
3 a Range =  $(0, \infty)$



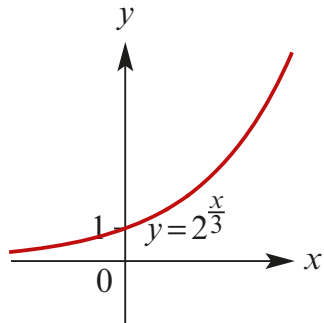
b Range =  $(0, \infty)$



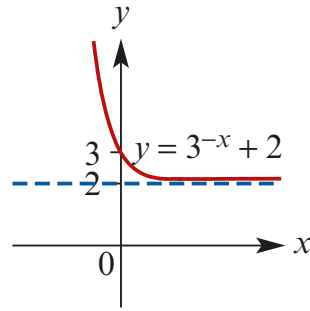
c Range =  $(0, \infty)$



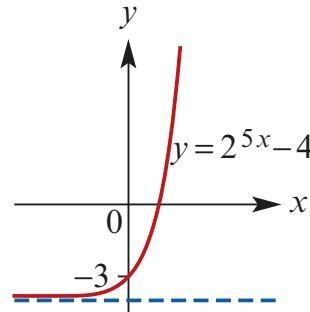
d Range =  $(0, \infty)$



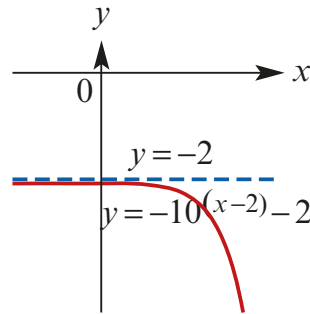
4 a Range =  $(2, \infty)$



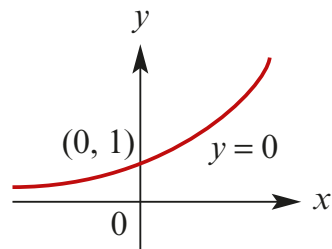
b Range =  $(-4, \infty)$



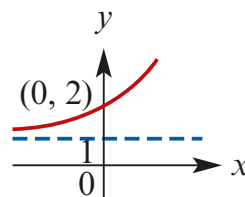
c Range =  $(-\infty, 2)$



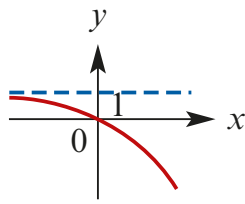
5 a Range =  $\mathbb{R}^+$



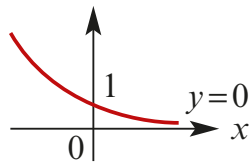
b Range =  $(1, \infty)$



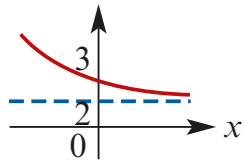
**c** Range =  $(-\infty, 1)$



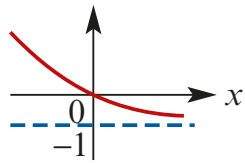
**d** Range =  $\mathbb{R}^+$



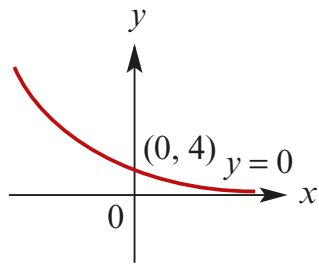
**e** Range =  $(2, \infty)$



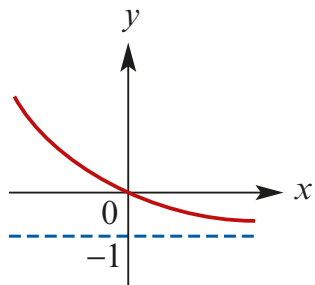
**f** Range =  $(-1, \infty)$



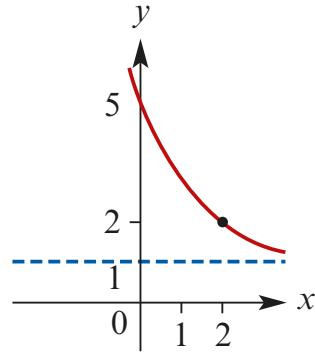
**6 a** Range =  $\mathbb{R}^+$



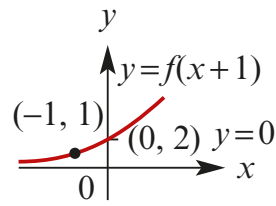
**b** Range =  $(-1, \infty)$



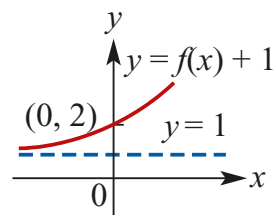
**c** Range =  $(1, \infty)$



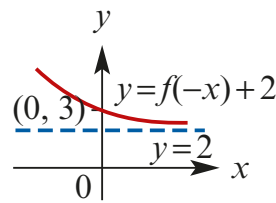
**7 a**



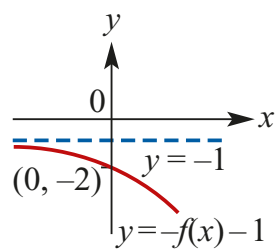
**b**



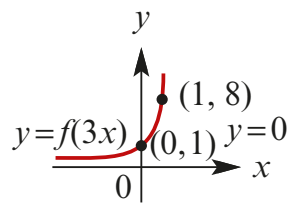
**c**

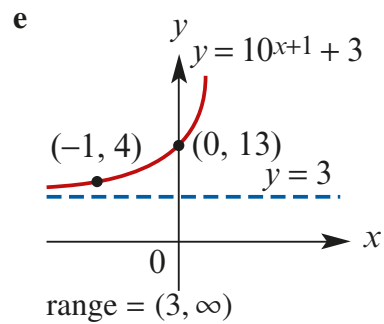
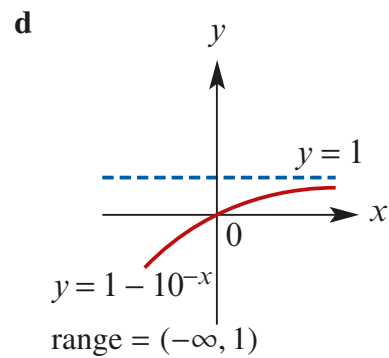
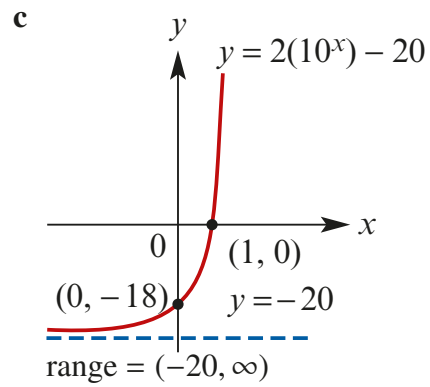
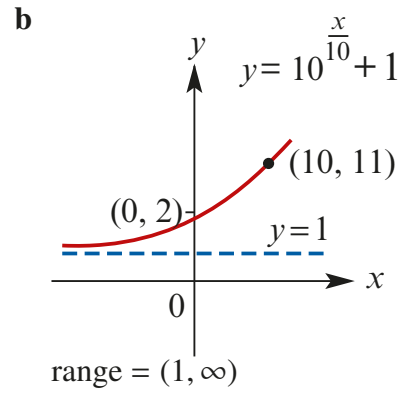
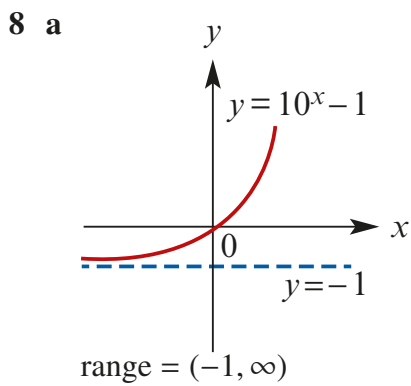
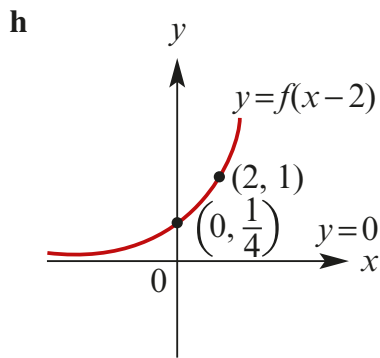
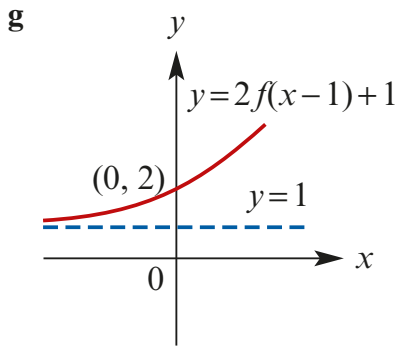
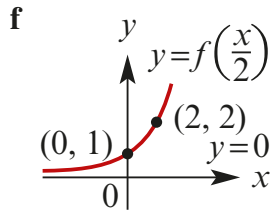


**d**

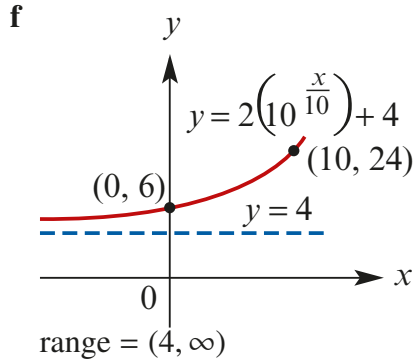


**e**





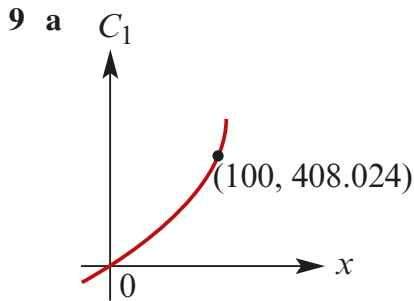




$$C_2 = C_1, x = 301.16$$

$\therefore$  for  $C_2 < C_1$   
minimum  $x = 302$  days

- 10**  $y = 100(1.02)^x$   
what is  $x$  when  $y = 200$ ?  
 $2 = (1.02)^x$   
Use the 'solve' command of a CAS calculator to solve for  $x$ . This gives  $x = 35.003$ . So your money has not quite doubled after 35 days; it will take 36 days.



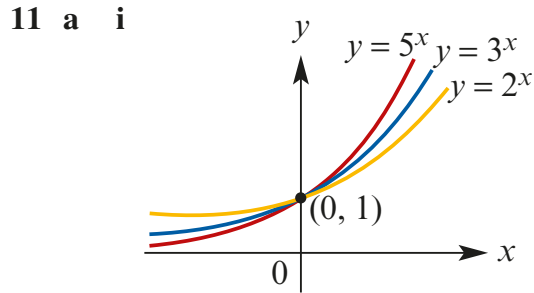
**b**  $C_1 = 10000((1.0004)^x - 1)$

**i**  $C_1 = 10000((1.0004)^{100} - 1)$   
 $= 10000(1.040802 - 1)$   
 $= \$408.02$

**ii**  $C_1 = 10000((1.0004)^{300} - 1)$   
 $= 10000(1.127470 - 1)$   
 $= \$1,274.70$

**c**  $1000 = 10000((1.0004)^x - 1)$   
 $(1.0004)^x = 1.1$   
Use the 'solve' command of a CAS calculator to solve for  $x$ . This gives  $x = 238.32 \dots x = 239$  days  
(you must round up in this case)

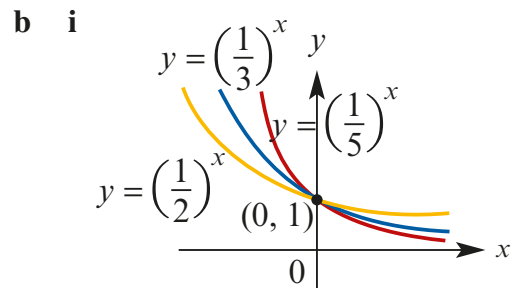
- d i**
- ii** to find  $C_2 < C_1$   
find  $C_2 = C_1$  then round up using the CAS calculator at



**ii**  $x < 0$

**iii**  $x > 0$

**iv**  $x = 0$  (read off graph)

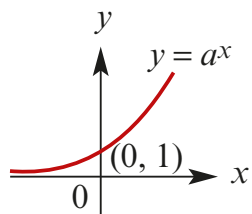


**ii**  $x > 0$

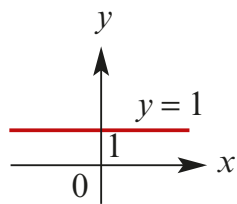
**iii**  $x < 0$

**iv**  $x = 0$  (read off graph)

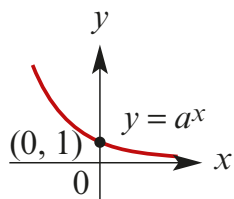
**c i**



**ii**

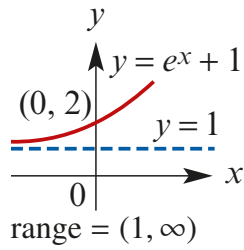


**iii**

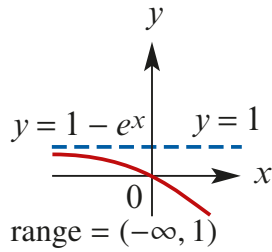


## Solutions to Exercise 5B

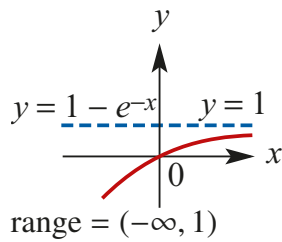
1 a



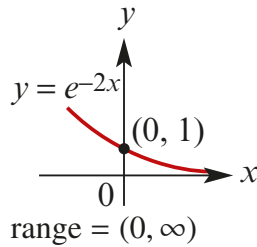
b



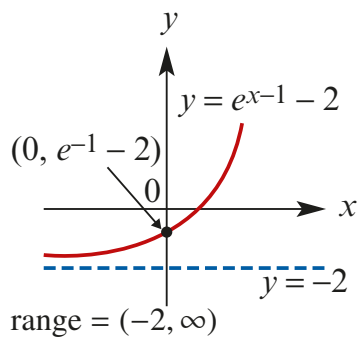
c



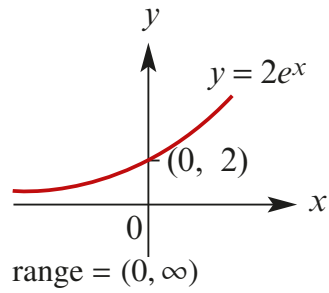
d



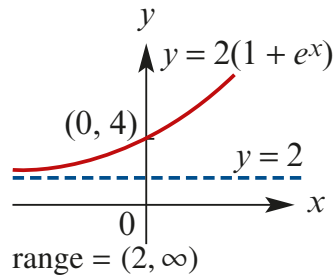
e



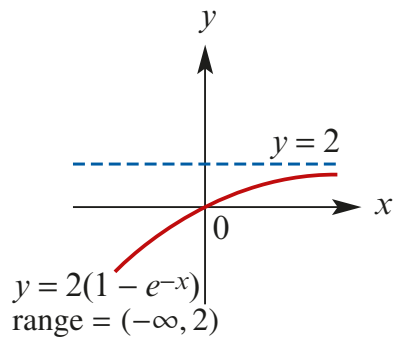
f



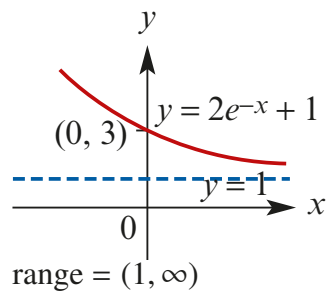
g



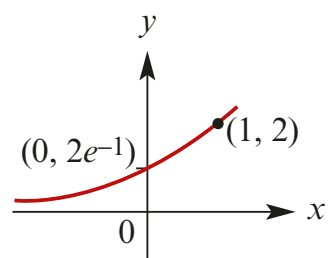
h



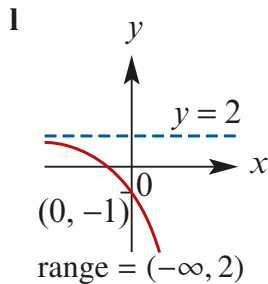
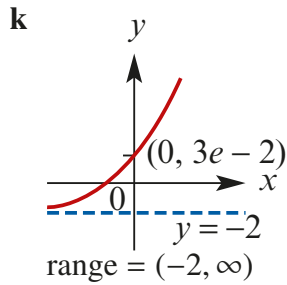
i



j



range =  $(0, \infty)$



- 2 a** Translation 2 units to the left and 3 units down
- b** Dilation of factor 3 from the  $x$ -axis, then translation 1 unit to the left and 4 units down
- c** Dilation of factor 5 from the  $x$ -axis and factor  $\frac{1}{2}$  from the  $y$ -axis, then translation  $\frac{1}{2}$  unit to the left
- d** Reflection in the  $x$ -axis, then translation 1 unit to the right and 2 units up
- e** Dilation of factor 2 from the  $x$ -axis, reflection in the  $x$ -axis, then translation 2 units to the left and 3 units up
- f** Dilation of factor 4 from the  $x$ -axis and factor  $\frac{1}{2}$  from the  $y$ -axis, then translation 1 unit down

**3 a**  $y = -2e^{x-3} - 4$

**b**  $y = 4 - e^{2x-3}$

**c**  $y = -2e^{x-3} - 4$

**d**  $y = -2e^{x-3} - 8$

**e**  $y = -2e^{x-3} + 8$

**f**  $y = -2e^{x-3} + 8$

- 4 a** Translation 2 units to the right and 3 units up

- b** Translation 1 unit to the right and 4 units up, then dilation of factor  $\frac{1}{3}$  from the  $x$ -axis

- c** Translation  $\frac{1}{2}$  unit to the right, then dilation of factor  $\frac{1}{5}$  from the  $x$ -axis and factor 2 from the  $y$ -axis

- d** Translation 1 unit to the left and 2 units down, then reflection in the  $x$ -axis

- e** Translation 2 units to the right and 3 units down, then dilation of factor  $\frac{1}{2}$  from the  $x$ -axis and reflection in the  $x$ -axis

- f** Translation 1 unit up, then dilation of factor  $\frac{1}{4}$  from the  $x$ -axis and factor 2 from the  $y$ -axis

**5 a**  $x = 1.146$  or  $x = -1.841$

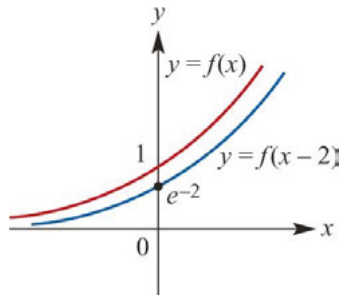
**b**  $x = -0.443$

**c**  $x = -0.703$

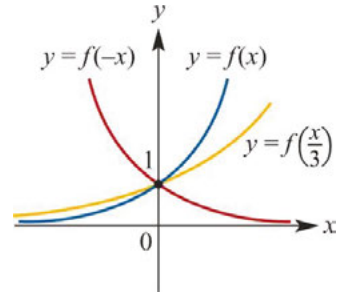
**d**  $x = 1.857$  or  $x = 4.536$

6

a b i



ii iii



## Solutions to Exercise 5C

**1 a**  $3x^2y^2 + 2x^4y^6 = 6x^6y^9$

**b**  $\frac{12x^8}{4x^2} = 3x^6$

**c**  $\frac{18x^2y^3}{3x^4y} = 6x^{-2}y^2$   
 $= \frac{6y^2}{x^2}$

**d**  $(4x^4y^2)^2 + 2(x^2y)^4$   
 $= 16x^8y^4 + 2x^8y^4$   
 $= 18x^8y^4$

**e**  $(4x^0)^2$   
 $= 4^2$   
 $= 16$

**f**  $\frac{15(x^5y^{-2})^4}{3(x^4y)^{-2}} = 5x^{20}y^{-8}x^8y^2$   
 $= 5x^{28}y^{-6}$   
 $= \frac{5x^{28}}{y^6}$

**g**  $\frac{3(2x^2y^3)^4}{2x^3y^2} = \frac{3 * 16x^8y^{12}}{2x^3y^2}$   
 $= 24x^5y^{10}$

**h**  $(8x^3y^6)^{\frac{1}{3}} = 2xy^2$

**i**  $\frac{x^2 + y^2}{x^{-2} + y^{-2}} = \frac{x^2 + y^2}{\frac{1}{x^2} + \frac{1}{y^2}}$   
 $= \frac{x^2 + y^2}{\frac{y^2 + x^2}{x^2y^2}}$   
 $= x^2y^2$

**2 a**  $3^x = 81$

$$3^x = 3^4$$

$$x = 4$$

**b**  $81^x = 9$

$$81^x = 81^{\frac{1}{2}}$$

$$x = \frac{1}{2}$$

**c**  $4^x = 256$

$$4^x = 4^4$$

$$x = 4$$

**d**  $625^x = 5$

$$625^x = 625^{\frac{1}{4}}$$

$$x = \frac{1}{4}$$

**e**  $32^x = 8$

$$2^{5x} = 2^3$$

$$5x = 3$$

$$x = \frac{3}{5}$$

**f**  $5^x = 125$

$$5^x = 5^3$$

$$x = 3$$

**g**  $16^x = 1024$

$$2^{4x} = 2^{10}$$

$$x = \frac{5}{2}$$

$$\begin{aligned} \mathbf{h} \quad 2^{-x} &= \frac{1}{64} \\ 2^{-x} &= 2^{-6} \\ x &= 6 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad 5^{-x} &= \frac{1}{625} \\ 5^{-x} &= 5^{-4} \\ x &= 4 \end{aligned}$$

$$\begin{aligned} \mathbf{3 a} \quad 5^{2n} \times 25^{2n-1} &= 625 \\ 5^{2n} \times 5^{4n-2} &= 5^4 \\ 5^{6n-2} &= 5^4 \\ 6n - 2 &= 4 \\ n &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 4^{2n-2} &= 1 \\ 4^{2n-2} &= 4^0 \\ 2n - 1 &= 0 \\ n &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 4^{2n-1} &= \frac{1}{256} \\ 4^{2n-1} &= 4^{-4} \\ 2n - 2 &= -4 \\ n &= \frac{-3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \frac{3^{n-2}}{9^{2-n}} &= 27 \\ 3^{n-2} \times 3^{-2n-4} &= 3^3 \\ 3^{3n-6} &= 3^3 \\ 3n - 6 &= 3 \\ n &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad 2^{2n-2} \times 4^{-3n} &= 64 \\ 2^{2n-2} \times 2^{-6n} &= 2^6 \\ 2^{-4n-2} &= 2^6 \\ -4n - 2 &= 6 \\ -4n &= 8 \\ n &= -2 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad 2^{n-4} &= 8^{4-n} \\ 2^{n-4} &= 2^{12-3n} \\ n - 4 &= 12 - 3n \\ n &= 4 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad 27^{n-2} &= 9^{3n+2} \\ 3^{3n-6} &= 32^{6n+4} \\ 3n - 6 &= 6n + 4 \\ 3n &= -10 \\ n &= \frac{-10}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad 8^{6n+2} &= 8^{4n-1} \\ 6n + 2 &= 4n - 1 \\ 2n &= -3 \\ n &= \frac{-3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad 125^{4-n} &= 5^{6-2n} \\ 5^{12-3n} &= 5^{6-2n} \\ 12 - 3n &= 6 - 2n \\ n &= 6 \end{aligned}$$

$$\mathbf{j} \quad 2^{n-1} \times 4^{2n+1} = 16$$

$$2^{n-1} \times 2^{4n+2} = 2^4$$

$$2^{5n+1} = 2^4$$

$$5n + 1 = 4$$

$$5n = 3$$

$$n = \frac{3}{5}$$

$$\mathbf{k} \quad (27 \times 3^n)^n = 27^n \times 3^{\frac{1}{4}}$$

$$(3^3 \times 3^n)^n = 3^{3n} \times 3^{\frac{1}{4}}$$

$$(3^{3+n})^n = 3^{3n+\frac{1}{4}}$$

$$3^{3n+n^2} = 3^{3n+\frac{1}{4}}$$

$$3n + n^2 = 3n + \frac{1}{4}$$

$$n^2 = \frac{1}{4}$$

$$n^2 = \pm \frac{1}{2}$$

$$\mathbf{4 a} \quad 3^{2x} - 2(3^x) - 3 = 0$$

$$\Rightarrow (3^x - 3)(3^x + 1) = 0$$

$$3^x = 3; \quad -1$$

$$\therefore x = 1;$$

$$x = 1$$

$$\mathbf{b} \quad 5^{2x} - 23(5^x) - 50 = 0$$

$$\Rightarrow (5^x - 25)(5^x + 2) = 0$$

$$5^x = 25; \quad -2$$

$$\therefore x = 2;$$

$$x = 2$$

$$\mathbf{c} \quad 5^{2x} - 10(5^x) + 25 = 0$$

$$(5^x - 5)^2 = 0$$

$$5^x = 5$$

$$x = 1$$

$$\mathbf{d} \quad 2^{2x} - 6(2^x) + 8 = 0$$

$$\Rightarrow (2^x - 2)(2^x - 4) = 0$$

$$2^x = 2, 4$$

$$x = 1, 2$$

$$\mathbf{e} \quad 8(3^x) - 6 = 2(3^{2x})$$

$$3^{2x} - 4(3^x) + 3 = 0$$

$$(3^x - 3)(x - 1) = 0$$

$$3^x = 3, 1$$

$$x = 1, 0$$

$$\mathbf{f} \quad 2^{2x} - 20(2^x) + 64 = 0$$

$$(2^x - 16)(2^x - 4) = 0$$

$$2^x = 16, 4$$

$$x = 4, 2$$

$$\mathbf{g} \quad 4^{2x} - 5(4^x) + 4 = 0$$

$$(4^x - 4)(4^x - 1) = 0$$

$$4^x = 4, 1$$

$$x = 1, 0$$

$$\mathbf{h} \quad 3(3^{2x}) - 28(3^x) + 9 = 0$$

$$(3(3^x) - 1)(3^x - 9) = 0$$

$$3^x = \frac{1}{3}, 9$$

$$x = -1, 2$$

$$\mathbf{i} \quad 7(7^{2x}) - 8(7^x) + 1 = 0$$

$$(7(7^x) - 1)(7^x - 1) = 0$$

$$7^x = \frac{1}{7}, 1$$

$$x = -1, 0$$



## Solutions to Exercise 5D

1 a 3

b -4

c -3

d 6

e 6

f -7

2 Note: the natural logarithm function  $\log_e x$  is often written  $\ln x$ ; this notation is used here.

a  $\ln 6$

b  $\ln 4$

c  $\ln 10^6 = 6 \ln 10$

d  $\ln 7$

e  $\ln \frac{1}{3 \times 4 \times 5} = \ln \frac{1}{60} = -\ln 60$

f  $\ln(uv \times uv^2 \times uv^3) = \ln u^3 v^6$   
 $= 3 \ln uv^2$

g  $7 \ln x = \ln x^7$

h  $\ln \left( \frac{(x+y)(x-y)}{(x^2-y^2)} \right)$   
 $= \ln 1$   
 $= 0$

3 a  $x = 10^2 = 100$

b  $\log_2 x = 4$   
 $x = 2^4 = 16$

c  $x - 5 = e^0 = 1$   
 $x = 6$

d  $x = 2^6 = 64$

e  $\ln(x+5) = 3$   
 $x+5 = e^3$   
 $x = e^3 - 5 \approx 15.086$

f  $2x = e^0 = 1$   
 $x = \frac{1}{2}$

g  $2x + 3 = e^0 = 1$   
 $2x = -2$   
 $x = -1$

h  $x = 10^{-3}$   
 $= \frac{1}{1000}$

i  $\log_2(x-4) = 5$   
 $x-4 = 2^5 - 32$   
 $x = 36$

4 a  $\log_{10} x = \log_{10} 15$   
 $x = 15$

b  $\ln x = \ln 5$   
 $x = 5$

c  $\ln x = \ln \left( 8^{\frac{2}{3}} \right)$   
 $= \ln 4$   
 $x = 4$

$$\mathbf{d} \quad \ln(2x^2 - x) = 0, x > 0$$

$$2x^2 - x = 1$$

$$2x^2 - x - 1 = 0$$

$$(2x + 1)(x - 1) = 0$$

$$x = -\frac{1}{2}, 1$$

since  $x > 0$ ,  $x = 1$

$$\mathbf{e} \quad \ln x^2 - \ln(x - 1) = \ln(x + 3)$$

$$\ln \frac{x^2}{x - 1} = \ln(x + 3)$$

$$x^2 = (x + 3)(x - 1)$$

$$x^2 = x^2 + 2x - 3$$

$$2x = 3$$

$$x = \frac{3}{2}$$

$$\mathbf{5 a} \quad \log_{10}(3 \times 9) = \log_{10} 27$$

$$\mathbf{b} \quad \log_2\left(\frac{24}{6}\right) = \log_2 4 = 2$$

$$\mathbf{c} \quad \frac{1}{2}(\log_{10} a - \log_{10} b)$$

$$= \frac{1}{2}\left(\log_{10} \frac{a}{b}\right)$$

$$= \log_{10} \sqrt{\frac{a}{b}}$$

$$\mathbf{d} \quad 1 + \log_{10} a - \log_{10}(b)^{\frac{1}{3}}$$

$$= \log_{10} 10 + \log_{10}\left(\frac{a}{b^{\frac{1}{3}}}\right)$$

$$= \log_{10}\left(\frac{10a}{b^{\frac{1}{3}}}\right)$$

$$\mathbf{e} \quad \log_{10} \sqrt[36]{6} - \log_{10}(27)^{\frac{1}{3}} - \log_{10}(64)^{\frac{2}{3}}$$

$$= \log_{10} 6 - \log_{10} 3 - \log_{10} 16$$

$$= \log_{10}\left(\frac{6}{3 \times 16}\right)$$

$$= \log_{10}\left(\frac{1}{8}\right)$$

$$\mathbf{6 a} \quad \log_{10} 10 = 1$$

$$\mathbf{b} \quad \log_{10} 5 + \log_{10} 8 - \log_{10} 4$$

$$= \log_{10} 10$$

$$= 1$$

$$\mathbf{c} \quad \log_2 \sqrt{2} + \log_2 1 + \log_2 4$$

$$= \log_2 4 \sqrt{2}$$

$$= 2\frac{1}{2}$$

$$= \frac{5}{2}$$

$$\mathbf{d} \quad \log_{10} 25 + \log_{10} 4 + \log_{10} 10$$

$$= \log_{10} 1000$$

$$= 3$$

$$\mathbf{e} \quad \log_{10} 16 - \log_{10} 16$$

$$= 0$$

$$\mathbf{7 a} \quad \log_3\left(\frac{1}{3^x}\right) = \log_3(3^{-x})$$

$$= -x \log_3 3$$

$$= -x$$

$$\mathbf{b} \quad \log_2 x - \log_2 y^2 + \log_2(xy^2)$$

$$= \log_2(x^2)$$

$$= 2 \log_2 x$$

$$\begin{aligned} \text{c } \ln(x^2 - y^2) - \ln(x - y) - \ln(x + y) \\ &= \ln\left(\frac{x^2 - y^2}{(x - y)(x + y)}\right) \\ &= \ln 1 = 0 \end{aligned}$$

$$\begin{aligned} \text{8 a } \ln(x^2 - 2x + 8) &= \ln x^2 \\ x^2 - 2x + 8 &= x^2 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

$$\begin{aligned} \text{b } \ln(5x) - \ln(3 - 2x) &= \ln e \\ \ln(5x) &= \ln(e(3 - 2x)) \\ 3e - 2ex &= 5x \\ (5 + 2e)x &= 3e \\ x &= \frac{3e}{5 + 2e} \approx 0.7814 \end{aligned}$$

$$\begin{aligned} \text{9 a } \ln x + \ln(3x + 1) &= \ln e \\ \ln(3x^2 + x) &= \ln e \\ 3x^2 + x - e &= 0 \\ x &= \frac{-1 \pm \sqrt{1 + 12e}}{6} \\ \text{but } x &> \frac{-1}{3} \\ x &= \frac{-1 + \sqrt{1 + 12e}}{6} \\ &\approx 0.7997 \end{aligned}$$

$$\begin{aligned} \text{b } 8e^{-x} - e^x - 2 &= 0 \\ 8 - e^{2x} - 2e^x &= 0 \\ (e^x)^2 + 2e^x - 8 &= 0 \\ (e^x + 4)(e^x - 2) &= 0 \\ e^x &= -4, 2 \\ \text{But } e^x > 0, \text{ so:} \\ e^x &= 2 \\ x &= \ln 2 \approx 0.6931 \end{aligned}$$

$$\begin{aligned} \text{10 a } \log_x 81 &= 4 \\ x^4 &= 81 \\ x &= 3 \end{aligned}$$

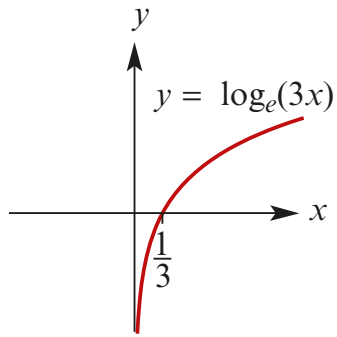
$$\begin{aligned} \text{b } \log_x \frac{1}{32} &= 5 \\ x^5 &= \frac{1}{32} \\ x &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{11 } \ln x^2 + \ln 4 &= \ln(9x - 2) \\ 4x^2 &= 9x - 2 \\ 4x^2 - 9x + 2 &= 0 \\ (4x - 1)(x - 2) &= 0 \\ x &= \frac{1}{4}, 2 \end{aligned}$$

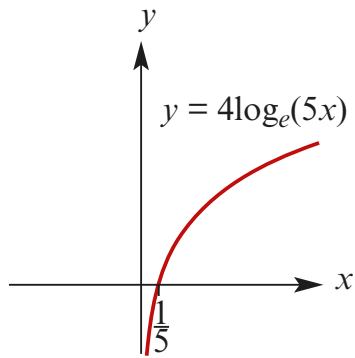
$$\begin{aligned} \text{12 } \log_a N &= \frac{1}{2}(\log_a 24 - \log_a 0.375 - \log_a 729) \\ &= \frac{1}{2}\left(\log_a \frac{64}{729}\right) \\ &= \log_a \frac{8}{27} \\ N &= \frac{8}{27} \end{aligned}$$

## Solutions to Exercise 5E

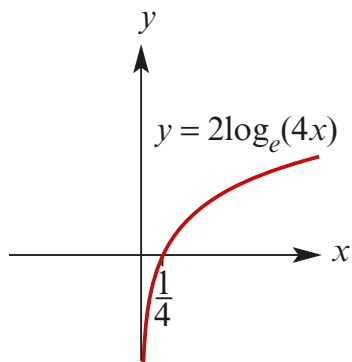
1 a



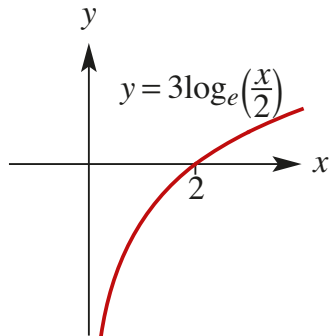
b



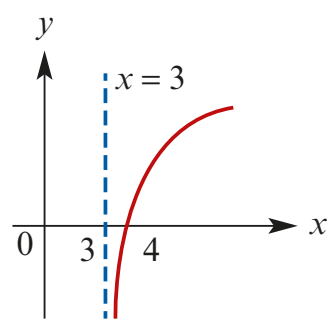
c



d

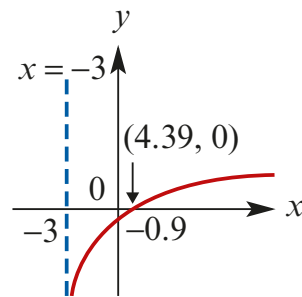


2 a



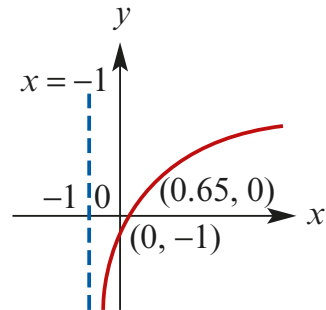
domain =  $(3, \infty)$ , range =  $\mathbb{R}$

b



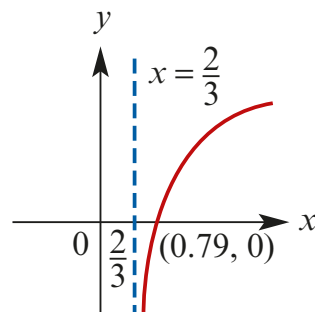
domain =  $(-3, \infty)$ , range =  $\mathbb{R}$

c

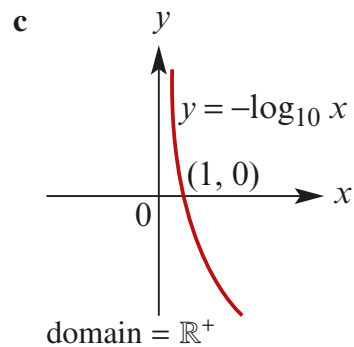
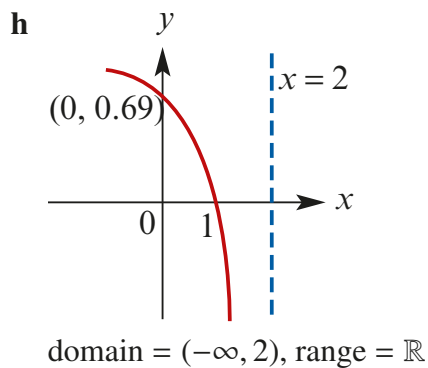
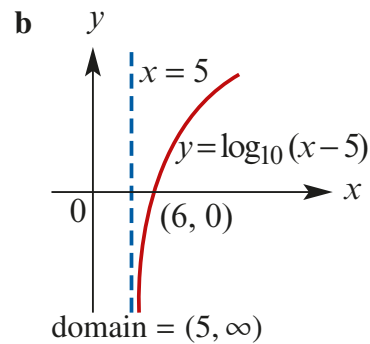
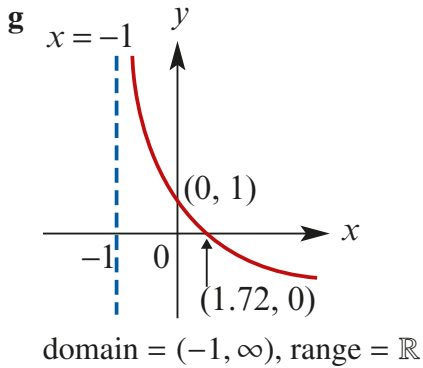
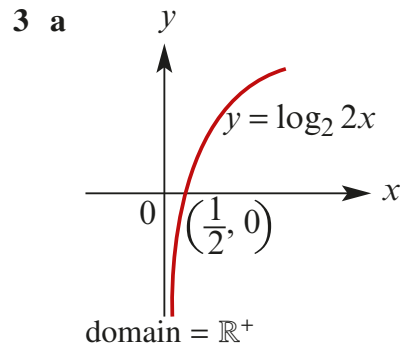
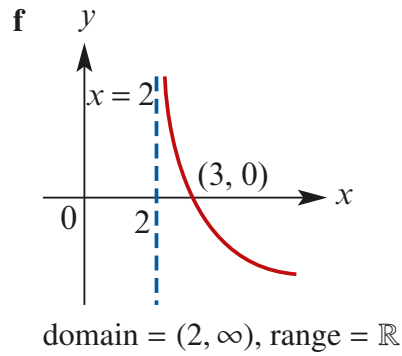
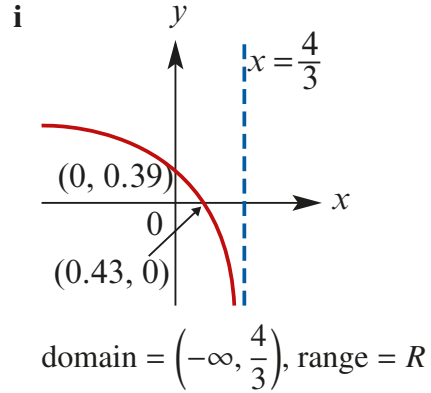
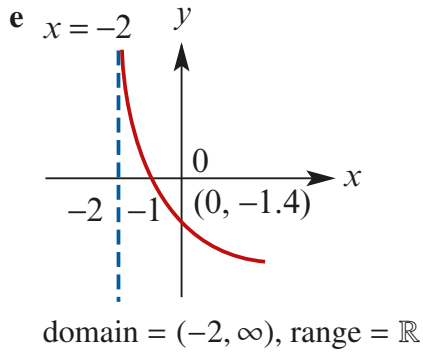


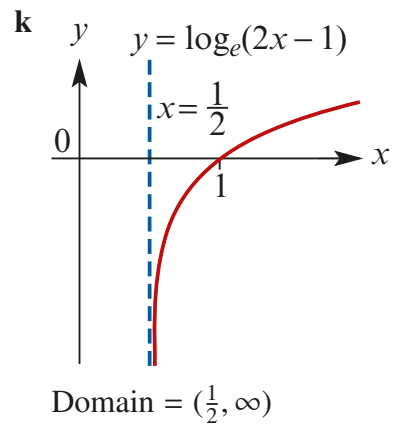
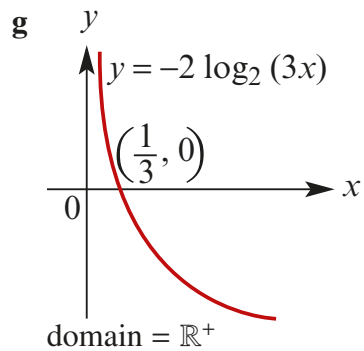
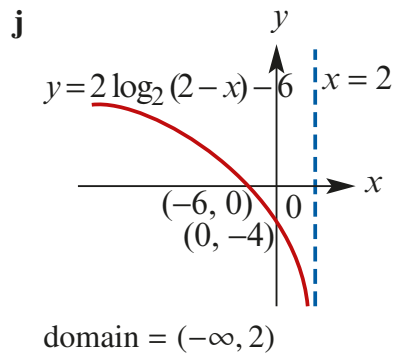
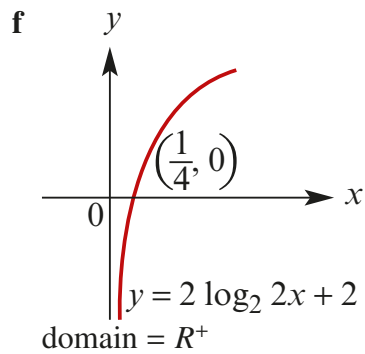
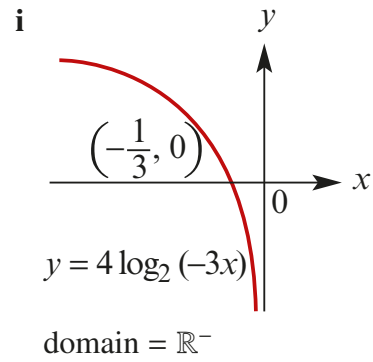
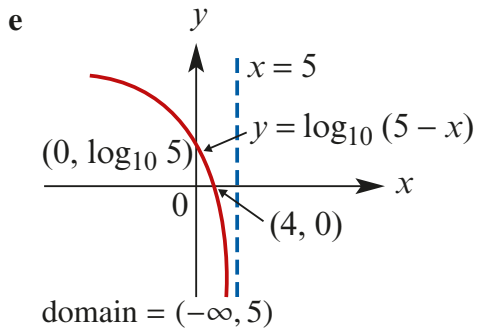
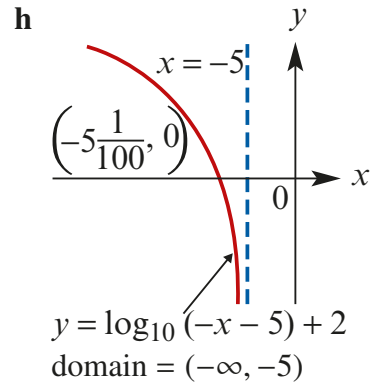
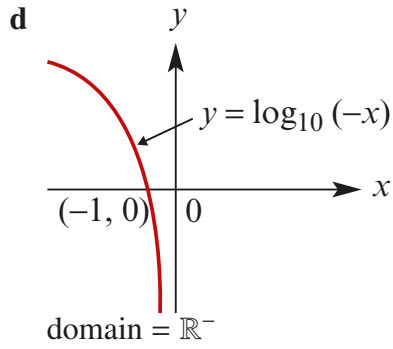
domain =  $(-1, \infty)$ , range =  $\mathbb{R}$

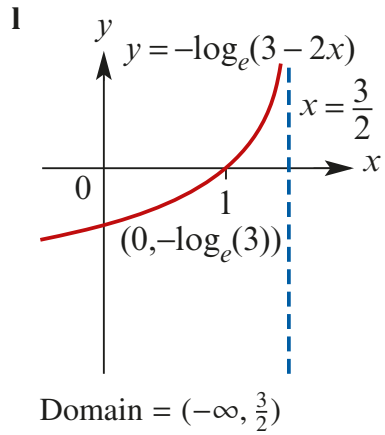
d



domain =  $\left(\frac{2}{3}, \infty\right)$ , range =  $\mathbb{R}$





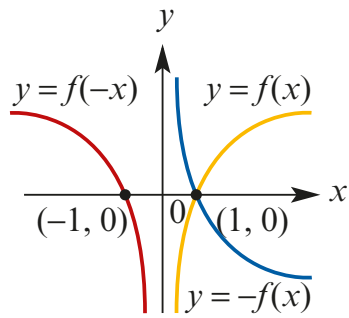


**4 a**  $x = 1.557$

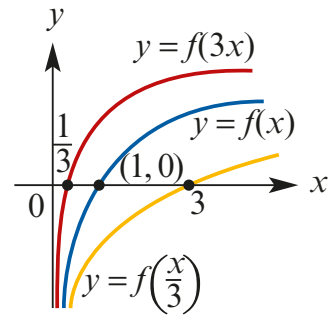
**b**  $x = 1.189$

5 b

i ii



iii iv



6 A dilation of factor  $\log_e 3$  from the y-axis

7 A dilation of factor  $\frac{1}{\log_e 2}$  from the y-axis



## Solutions to Exercise 5F

$$1 \quad a + b = 5 \dots (1)$$

$$ae^4 + b = 11 \dots (2)$$

$$(2) - (1)$$

$$a(e^4 - 1) = 6$$

$$a = \frac{6}{e^4 - 1}$$

$$\therefore b = 5 - \frac{6}{e^4 - 1}$$

$$b = \frac{5e^4 - 11}{e^4 - 1}$$

$$2 \quad a \log_e(5 + b) = 0 \dots (1)$$

$$a \log_e(10 + b) = 2 \dots (2)$$

From (1)

$$\log_e(5 + b) = 0$$

$$5 + b = e^0$$

$$b = -4$$

From (2)

$$\therefore a \log_e 6 = 2$$

$$\therefore a = \frac{2}{\log_e 6}$$

$$3 \quad y = ae^x + b$$

$$x \rightarrow -\infty, y \rightarrow 4$$

$$4 = b$$

$$x = 0, y = 6$$

$$6 = a + b$$

$$= a + 4$$

$$a = 2$$

$$4 \quad y = ae^x + b$$

$$x = 0, y = 0$$

$$0 = a + b$$

$$a = -b$$

$$x = 1, y = 14$$

$$14 = ae + b$$

$$= (e - 1)a$$

$$a = \frac{14}{e - 1} \approx 8.148$$

$$b = \frac{-14}{e - 1} \approx -8.148$$

$$5 \quad y = ae^{-bx}$$

$$x = 3, y = 50$$

$$50 = ae^{-3b} \dots (1)$$

$$x = 6, y = 10$$

$$10 = ae^{-6b} \dots (2)$$

$$\frac{1}{2} \Rightarrow 5 = e^{3b}$$

$$3b = \ln 5$$

$$b = \frac{1}{3} \ln 5$$

$$b = \ln(5)^{\frac{1}{3}}$$

$$y = a \times 5^{\frac{-x}{3}}$$

$$\therefore a = 250$$

$$6 \quad f(x) = ae^{-x} + b$$

$$x \rightarrow \infty, f(x) \rightarrow 500$$

$$500 = b$$

$$x = 0, f(x) = 700$$

$$700 = a + 500$$

$$a = 200$$

$$\text{Sub in equation} \Rightarrow 50 = a \times \frac{1}{5}$$

$$a = 250$$

$$7 \quad y = a \log_2 x + b$$

$$x = 8, y = 10$$

$$10 = 3a + b \dots (1)$$

$$x = 32, y = 14$$

$$14 = 5a + b \dots (2)$$

$$(2) - (1) \Rightarrow a = 2$$

$$\text{Sub in (1)} \Rightarrow 10 = 6 + b$$

$$b = 4$$

$$8 \quad y = a \log_2(x - b)$$

$$x \rightarrow 5, y \rightarrow -\infty$$

$$b = 5$$

$$x = 7, y = 3$$

$$3 = a \log_2(7 - 5)$$

$$a = 3$$

$$9 \quad y = ae^{bx}$$

$$x = 3, y = 10$$

$$x = 6, y = 50$$

$$10 = ae^{3b} \dots (1)$$

$$50 = ae^{6b} \dots (2)$$

$$\frac{(2)}{(1)} \Rightarrow 5 = e^{3b}$$

$$b = \ln(5)^{\frac{1}{3}}$$

$$y = a \times 5^{\frac{x}{3}}$$

$$\text{Sub in (1)} \Rightarrow 10 = a \times 5$$

$$a = 2$$

$$10 \quad y = a \log_2(x - b)$$

$$x = 5, y = 2$$

$$2 = a \log_2(5 - b) \dots (1)$$

$$x = 7, y = 4$$

$$4 = a \log_2(7 - b) \dots (2)$$

$$(2) \div (1) \Rightarrow$$

$$2 \log_2(5 - b) = \log_2(7 - b)$$

$$(5 - b)^2 = 7 - b$$

$$b^2 - 10b + 25 = -b + 7$$

$$b^2 - 9b + 18 = 0$$

$$(b - 6)(b - 3) = 0$$

$$b = 3 \text{ or } 6$$

since  $\log_2(x)$  is only defined for  $x > 0$

and  $\log_2(5 - b)$  is one of the points,

$b = 6$  is impossible

$$\therefore b = 3$$

$$\text{Sub in (1)} \Rightarrow 2 = a \log_2(5 - 3)$$

$$a = 2$$

$$11 \quad y = a \ln(x - b) + c$$

vertical asymptote  $x = 1$ ,  $\therefore b = 1$

$$y = a \ln(x - 1) + c$$

$$x = 3, y = 10$$

$$10 = a \ln 2 + c \dots (1)$$

$$x = 5, y = 12$$

$$12 = a \ln 4 + c \dots (2)$$

$$= 2a \ln 2 + c$$

$$(2) - (1) \Rightarrow a \ln 2 = 2$$

$$a = \frac{2}{\ln 2} \approx 2.885$$

$$y = 2 \log_2(x - 1) + c$$

$$\text{Sub in (1)} \Rightarrow 10 = 2 \log_2 2 + c$$

$$c = 8$$

**12**

$$f(x) = a \ln(-x) + b$$

$$x = -2, f(-2) = 6$$

$$6 = a \ln 2 + b \dots (1)$$

$$x = -4, f(-4) = 8$$

$$8 = a \ln(4) + b \dots (2)$$

$$8 = 2a \ln 2 + b$$

$$(2) - (1) \Rightarrow 2 = a \ln 2$$

$$a = \frac{2}{\ln 2} \approx 2.885$$

$$\text{Sub in 1} \Rightarrow 6 = 2 + b$$

$$b = 4$$

## Solutions to Exercise 5G

**1 a**  $\log_2 8 = k \log_2 7 + 2$

$$3 - 2 = k \log_2 7$$

$$1 = k \log_2 7$$

$$k = \frac{1}{\log_2 7}$$

**b**  $\log_2 7 - x \log_2 7 = 4$

$$(1 - x) \log_2 7 = 4$$

$$1 - x = \frac{4}{\log_2 7}$$

$$x = 1 - \frac{4}{\log_2 7}$$

$$x = \frac{\log_2(7) - 4}{\log_2 7}$$

**c**  $\log_e 7 - x \log_e 14 = 1$

$$\log_e 7 - 1 = x \log_e 14$$

$$x = \frac{\log_e 7 - 1}{\log_e 14}$$

**2 a** 2.58

**b** -0.32

**c** 2.18

**d** 1.16

**e** -2.32

**f** -0.68

**g** -2.15

**h** -1.38

**i** 2.89

**j** -1.7

**k** -4.42

**l** 5.76

**m** -6.21

**n** 2.38

**o** 2.80

**3 a**  $x < 2.81$

**b**  $x > 1.63$

**c**  $x < -0.68$

**d**  $x \leq 3.89$

**e**  $x \geq 0.57$

**4 a**  $x = \log_2 5$

**b**  $2x - 1 = \log_3 8$

$$2x = \log_3(8) + 1$$

$$x = \frac{\log_3(8) + 1}{2}$$

**c**  $3x + 1 = \log_7 20$

$$3x = \log_7(20) - 1$$

$$x = \frac{\log_7(20) - 1}{3}$$

**d**  $x = \log_3 7$

**e**  $x = \log_3 6$

**f**  $x = \log_5 6$

**g** Let  $a = 3^x$

$$a^2 - 9a + 8 = 0$$

$$(a - 8)(a - 1) = 0$$

$$\therefore a = 8 \text{ or } a = 1$$

$$\therefore 3^x = 8 \text{ or } 3^x = 1$$

$$\therefore x = \log_3 8 \text{ or } x = 0$$

**h** Let  $a = 5^x$

$$a^2 - 4a - 5 = 0$$

$$(a - 5)(a + 1) = 0$$

$$\therefore a = 5 \text{ or } a = -1$$

$$\therefore 5^x = 5 \text{ or } 5^x = -1$$

$$\therefore x = 1$$

**5 a**  $7^x > 52 \Leftrightarrow x > \log_7 52$

**b**  $3^{2x-1} < 40 \Leftrightarrow 2x - 1 < \log_3 40$

$$\Leftrightarrow 2x < \log_3(40) + 1$$

$$\Leftrightarrow x < \frac{1}{2}(\log_3(40) + 1)$$

$$= \frac{1}{2}(\log_3(120))$$

**c**  $4^{3x+1} \geq 5 \Leftrightarrow 3x + 1 \geq \log_4 5$

$$\Leftrightarrow 3x \geq \log_4(5) - 1$$

$$\Leftrightarrow x \geq \frac{1}{3} \log_4\left(\frac{5}{4}\right)$$

$$= \frac{1}{6} \log_2\left(\frac{5}{4}\right)$$

**d**  $3^{x-5} \leq 30 \Leftrightarrow x - 5 \leq \log_3 30$

$$\Leftrightarrow 3x \leq \log_3(30) + 5$$

$$\Leftrightarrow x < \frac{1}{3}(\log_3(30) + 5)$$

$$= \log_3(7290)$$

**e**  $3^x < 106 \Leftrightarrow x < \log_3 106$

**f**  $5^x < 0.6 \Leftrightarrow x \leq \log_5 0.6$

**6 a**  $a \log_2 7 = 3 - \log_6 14$

$$a \log_2 7 = \log_6 216 - \log_6 14$$

$$a \log_2 7 = \log_6\left(\frac{108}{7}\right)$$

$$a = \frac{\log_6\left(\frac{108}{7}\right)}{\log_2 7}$$

$$a = \frac{\ln\left(\frac{108}{7}\right)}{\ln 6} \times \frac{\ln 2}{\ln 7}$$

$$a = \frac{2.73622}{1.791759} \times \frac{0.69314}{194591}$$

$$a = 1.5271138 \times 0.356207$$

$$a = 0.544$$

**b**  $\log_3 18 = \log_{11} k$

$$\log_{11} k = \frac{\ln 18}{\ln 3}$$

$$= 2.6309$$

$$k = 11^{2.6309}$$

$$k = 549.3$$

**7**  $\log_r p = q \Rightarrow p = r^q$  (1)

$$\log_q(r) = p \Rightarrow r = q^p$$
 (2)

Raise both sides of (2) to the power  $q$ :

$$r^q = (q^p)^q$$

$$p = q^{pq} \text{ (from (1))}$$

Change to logarithm form:

$$\log_q p = pq$$

**8**  $u = \log_9 x$

**a**  $x = 9^u$

**b**  $\log_9(3x) = \log_9(3 \times 9^u)$

$$= \log_9 9^u + \log_9 3$$

$$= u + \frac{1}{2}$$

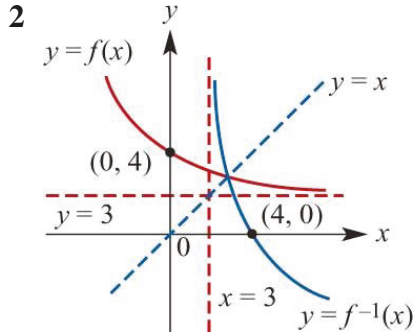
$$\begin{aligned}
 \mathbf{c} \quad & x = 9^u \\
 \Rightarrow & \log_x x = \log_x 9^u \\
 \Rightarrow & 1 = u \log_x 9 \\
 \Rightarrow & \frac{1}{u} = \frac{1}{2} \log_x 81 \\
 \log_x 81 &= \frac{2}{u}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9} \quad & \log_5 x = 16 \log_x 5 \\
 \Rightarrow & \frac{\ln x}{\ln 5} = \frac{16 \ln 5}{\ln x} \\
 (\ln x)^2 &= 16(\ln 5)^2 \\
 \ln x &= \pm 4 \ln 5 \\
 x &= e^{\pm \ln 625} \\
 x &= 625, \frac{1}{625}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad & q^p = 25 \Rightarrow p = \log_q 25 \\
 \log_5 q &= \frac{\log_q q}{\log_q 5} = \frac{1}{\log_q 5} \\
 &= \frac{2}{\log_q 25} = \frac{2}{p}
 \end{aligned}$$

## Solutions to Exercise 5H

1  $f^{-1}: (-2, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = \log_e(x + 2)$



to find  $f^{-1}(x)$ ,

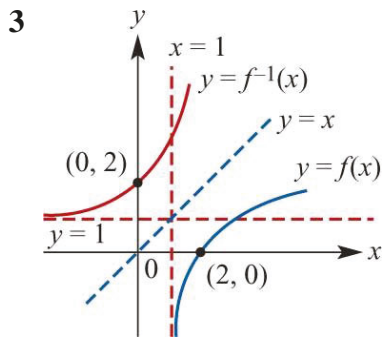
$$f(x) = e^{-x} + 3$$

$$\therefore x = e^{-f^{-1}(x)} + 3$$

$$x - 3 = e^{-f^{-1}(x)}$$

$$-f^{-1}(x) = \ln(x - 3)$$

$$f^{-1}(x) = -\ln(x - 3)$$



to find  $f^{-1}(x)$ ,

$$f(x) = \ln(x - 1)$$

$$x = \ln(f^{-1}(x) - 1)$$

$$e = f^{-1}(x) - 1$$

$$f^{-1}(x) = e^x + 1$$

4  $x = e^{\frac{y+4}{3}}$

5 a domain( $f$ ) =  $\mathbb{R}^+$

$$\text{range}(f) = \mathbb{R}$$

$$\therefore \text{domain}(f^{-1}) = \mathbb{R}$$

$$\text{range}(f^{-1}) = \mathbb{R}^+$$

$$f(x) = \ln 2x$$

$$x = \ln(2f^{-1}(x))$$

$$2f^{-1}(x) = e^x$$

$$f^{-1}(x) = \frac{1}{2}e^x$$

b domain  $f = \mathbb{R}^+$

$$\text{range } f = \mathbb{R}$$

$$\therefore \text{domain } f^{-1} = \mathbb{R}$$

$$\text{range } f^{-1} = \mathbb{R}^+$$

$$f(x) = 3 \ln(2x) + 1$$

$$x = 3 \ln(2f^{-1}(x)) + 1$$

$$\frac{x-1}{3} = \ln(2f^{-1}(x))$$

$$2f^{-1}(x) = e^{\frac{x-1}{3}}$$

$$f^{-1}(x) = \frac{1}{2}e^{\frac{x-1}{3}}$$

c

$$\text{domain}(f) = \mathbb{R}, \text{range}(f) = (2, \infty)$$

$$\therefore \text{domain}(f^{-1}) = (2, \infty), \text{range}(f^{-1}) = \mathbb{R}$$

$$f(x) = e^x + 2$$

$$\therefore x = e^{f^{-1}(x)} + 2$$

$$x - 2 = e^{f^{-1}(x)}$$

$$f^{-1}(x) = \ln(x - 2)$$

**d**  $\text{domain}(f) = \mathbb{R}, \text{range}(f) = \mathbb{R}^+$   
 $\therefore \text{domain}(f^{-1}) = \mathbb{R}^+, \text{range}(f^{-1}) = \mathbb{R}$

$$f(x) = e^{x+2}$$

$$\therefore x = e^{f^{-1}(x)+2}$$

$$\ln x = f^{-1}(x) + 2$$

$$f^{-1}(x) = \ln x - 2$$

**e**  $\text{domain}(f) = \left(-\frac{1}{2}, \infty\right)$ ,  
 $\text{range}(f) = \mathbb{R}$   
 $\therefore \text{domain}(f^{-1}) = \mathbb{R}$ ,

$$\text{range}(f^{-1}) = \left(-\frac{1}{2}, \infty\right)$$

$$f(x) = \ln(2x + 1)$$

$$x = \ln(2f^{-1}(x) + 1)$$

$$e^x = 2f^{-1}(x) + 1$$

$$f^{-1}(x) = \frac{e^x - 1}{2}$$

**f**  $\text{domain}(f) = \left(-\frac{2}{3}, \infty\right)$ ,  
 $\text{range}(f) = \mathbb{R}$   
 $\therefore \text{domain}(f^{-1}) = \mathbb{R}$ ,

$$\text{range}(f^{-1}) = \left(-\frac{2}{3}, \infty\right)$$

$$f(x) = 4 \ln(3x + 2)$$

$$x = 4 \ln(3f^{-1}(x) + 2)$$

$$e^{\frac{x}{4}} = 3f^{-1}(x) + 2$$

$$f^{-1}(x) = \frac{e^{\frac{x}{4}} - 2}{3}$$

**g**  $\text{domain}(f) = (-1, \infty), \text{range}(f) = \mathbb{R}$   
 $\therefore \text{domain}(f^{-1}) = \mathbb{R}, \text{range}(f^{-1}) = (-1, \infty)$

$$f(x) = \log_{10}(x + 1)$$

$$f(x) = \log_{10}(f^{-1}(x) + 1)$$

$$f^{-1}(x) + 1 = 10^x$$

$$f^{-1}(x) = 10^x - 1$$

**h**  $\text{domain}(f) = \mathbb{R}, \text{range}(f) = \mathbb{R}^+$   
 $\therefore \text{domain}(f^{-1}) = \mathbb{R}^+, \text{range}(f^{-1}) = \mathbb{R}$

$$f(x) = 2e^{x-1}$$

$$x = 2e^{(f^{-1}(x)-1)}$$

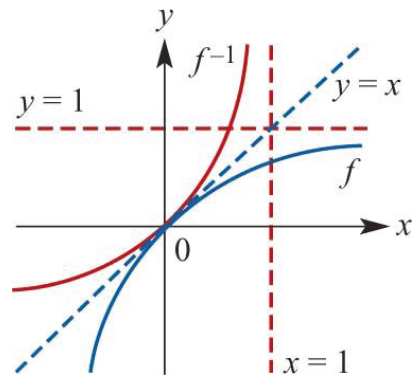
$$\frac{x}{2} = e^{(f^{-1}(x)-1)}$$

$$f^{-1}(x) - 1 = \ln\left(\frac{x}{2}\right)$$

$$f^{-1}(x) = \ln\left(\frac{x}{2}\right) + 1$$

**6**

**a c**



**b**  $\text{range}(f) = (-\infty, 1)$   
 $\therefore \text{domain}(f^{-1}) = (-\infty, 1)$

$$f(x) = 1 - e^{-x}$$

$$x = 1 - e^{-f^{-1}(x)}$$

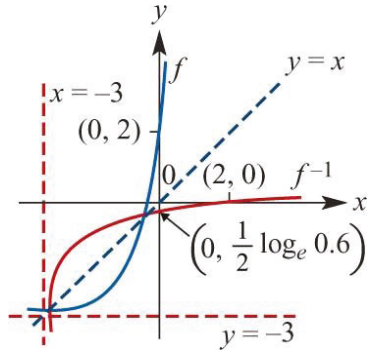
$$-f^{-1}(x) = \ln(1 - x)$$

$$f^{-1}(x) = -\ln(1 - x)$$



7

a c



b

$$f(x) = 5e^{2x} - 3$$

$$x = 5e^{2f^{-1}(x)} - 3$$

$$\frac{x+3}{5} = e^{2f^{-1}(x)}$$

$$2f^{-1}(x) = \ln\left(\frac{x+3}{5}\right)$$

$$f^{-1}(x) = \frac{1}{2} \ln\left(\frac{x+3}{5}\right)$$

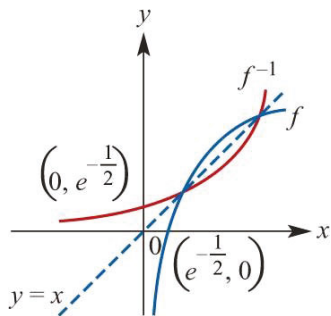
$$\text{domain} = (-3, \infty)$$

$$\therefore f^{-1} : (-3, \infty) \rightarrow \mathbb{R},$$

$$f^{-1}(x) = \frac{1}{2} \ln\left(\frac{x+3}{5}\right)$$

8

a c



b

$$f(x) = 2 \ln x + 1$$

$$x = 2 \ln f^{-1}(x) + 1$$

$$\frac{x-1}{2} = \ln f^{-1}(x)$$

$$f^{-1}(x) = e^{\frac{x-1}{2}}$$

$$\text{range}(f^{-1}) = \text{domain}(f) = \mathbb{R}^+$$

$$9 \quad t = \frac{-1}{k} \log_e\left(\frac{P-b}{A}\right)$$

$$10 \quad a \quad \frac{y-5}{2} = \ln x$$

$$x = e^{\left(\frac{y-5}{2}\right)}$$

b

$$\frac{P}{A} = e^{-6x}$$

$$-6x = \ln\left(\frac{P}{A}\right)$$

$$x = -\frac{1}{6} \ln\left(\frac{P}{A}\right)$$

c

$$\frac{y}{a} = x^n$$

$$n = \log_x\left(\frac{y}{a}\right) = \frac{\ln\left(\frac{y}{a}\right)}{\ln x}$$

d

$$10^x = \frac{y}{5}$$

$$x = \log_{10}\left(\frac{y}{5}\right)$$

e

$$\ln(2x) = \frac{5-y}{3}$$

$$2x = e^{\left(\frac{5-y}{3}\right)}$$

$$x = \frac{1}{2} e^{\left(\frac{5-y}{3}\right)}$$

$$\mathbf{f} \quad \frac{y}{6} = x^{2n}$$

$$2n = \log_x \left( \frac{y}{6} \right)$$

$$n = \frac{1}{2} \log_x \left( \frac{y}{6} \right) = \frac{1}{2} \left( \frac{\ln \left( \frac{y}{6} \right)}{\ln x} \right)$$

$$y = \ln(2x - 1)$$

$$\mathbf{g} \quad 2x - 1 = e^y$$

$$x = \frac{e^y + 1}{2}$$

$$\mathbf{h} \quad y = 5(1 - e^{-x})$$

$$e^{-x} = 1 - \frac{y}{5}$$

$$-x = \ln \left( \frac{5 - y}{5} \right)$$

$$x = -\ln \left( \frac{5 - y}{5} \right) = \ln \left( \frac{5}{5 - y} \right)$$

$$\mathbf{11 a} \quad f(x) = 2e^x - 4$$

$$\frac{x + 4}{2} = e^{f^{-1}(x)}$$

$$f^{-1}(x) = \ln \left( \frac{x + 4}{2} \right)$$

**b** using the CAS calculator

$$(0.895, 0.895), (-3.962, -3.962)$$

$$\mathbf{12 a} \quad f(x) = 2 \ln(x + 3) + 4$$

$$x = 2 \ln(f^{-1}(x) + 3) + 4$$

$$\left( \frac{x - 4}{2} \right) = \ln(f^{-1}(x) + 3)$$

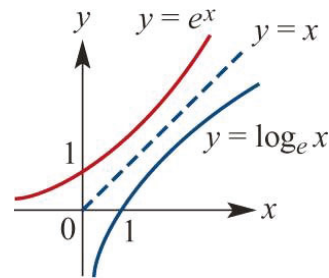
$$f^{-1}(x) + 3 = e^{\left( \frac{x - 4}{2} \right)}$$

$$f^{-1}(x) = e^{\left( \frac{x - 4}{2} \right)} - 3$$

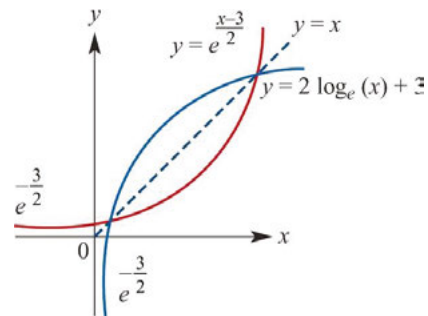
**b** using the CAS calculator

$$(8.964, 8.964), (-2.969, -2.969)$$

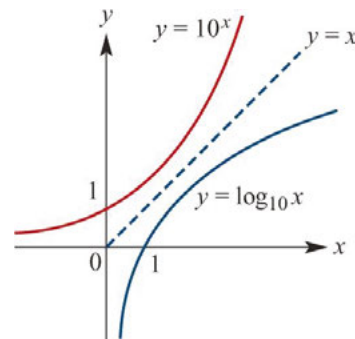
**13 a i**



**ii**



**iii**



**b**  $f(x)$  and  $g(x)$  are inverse functions

## Solutions to Exercise 5I

1 a  $N = 1000 \times 2^{\frac{t}{15}}$

b 50 minutes

2  $d = d_0 10^{mt}$

When  $t = 1$ ,  $d = 52$  cm

When  $t = 3$ ,  $d = 80$

Consider the equations

$$52 = d_0 10^m \quad \dots (1)$$

$$80 = d_0 10^{3m} \quad \dots (2)$$

Divide (2) by (1)

$$\frac{80}{52} = 10^{2m}$$

$$2m = \log_{10}\left(\frac{20}{13}\right)$$

$$m = \frac{1}{2} \log_{10}\left(\frac{20}{13}\right) \approx 0.094$$

Substitute in (1)

$$52 = d_0 10^{\frac{1}{2} \log_{10}\left(\frac{20}{13}\right)}$$

$$\therefore 52 = d_0 10^{\log_{10}\left(\frac{20}{13}\right)^{\frac{1}{2}}}$$

Hence  $52 = \left(\frac{20}{13}\right)^{\frac{1}{2}} 1/2 d_0$

$$\text{and } d_0 = \left(\frac{13}{20}\right)^{\frac{1}{2}} \times 52$$

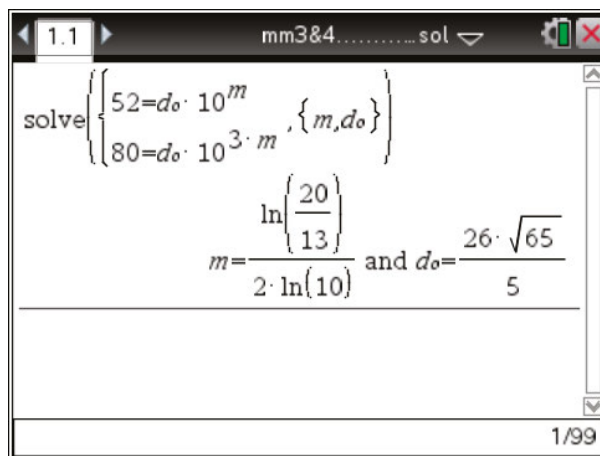
$$\therefore m \approx 0.094 \text{ and } d_0 \approx 41.9237$$

### Graphic calculator techniques for question

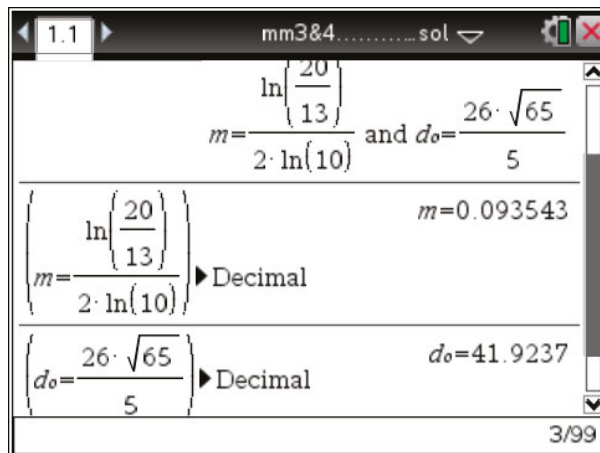
In a Calculator page use:

**b>Algebra>Solve System of Equations>Solve System of Equations** and enter as shown opposite.

Hint: do can be entered using a template from t, otherwise just use d0.



Approximate the solutions using  
**b>Number>Convert to Decimal.**



3 a  $N = N_0 e^{kt}$

i When  $t = 0$ ,  $N = 20\,000$

$$\therefore 20\,000 = N_0 e^0$$

i.e.  $N_0 = 20\,000$

ii  $N = 20\,000 e^{kt}$

When  $t = 1$ ,  $N = 20\,000$  and 20% of 20 000 = 16 000

$$\therefore 16\,000 = 20\,000 \times e^k$$

$$\therefore e^k = 0.8$$

$$\therefore k = \log_e(0.8) \approx -0.223$$

**b** When  $N = 5000$

$$5000 = 20\,000e^{\log_e(0.8)t}$$

$$\therefore 0.25 = 0.8^t$$

$$\text{and } t = \frac{\log_e(0.25)}{\log_e(0.8)}$$

$$\approx 6.2126$$

It takes about 6.2 years for there to be 5000 people infected.

**4**  $M = M_0e^{-kt}$

When  $t = 0, M = 10$

When  $t = 140, M = 5$

**a**  $10 = M_0e^0$

$$\therefore M_0 = 10$$

Also  $5 = 10e^{-140k}$

$$0.5 = e^{-140k}$$

$$k = \frac{-1}{140} \log_e(0.5)$$

$$= \frac{1}{140} \log_e(2) \approx 0.00495$$

$$= 4.95 \times 10^{-3}$$

**b** When  $t = 70$   $M = 10e^{\frac{-1}{140} \log_e(2) \times 70}$

$$= 10e^{-0.5 \log_e 2}$$

$$= 10 \times 2^{-0.5}$$

$$\approx 7.0711$$

The mass is 7.07 g after 70 days.

**c** When  $M = 2$   $2 = 10e^{\frac{-1}{140}(\log_e 2)t}$

$$\therefore 0.2 = 2^{\frac{t}{140}}$$

$$\therefore t = -140 \frac{\log_e(0.2)}{\log_e(2)} \approx 325.07$$

After 325 days the mass remaining is 2g.

**5**

**6**  $A = A_0e^{kt}$

When  $t = 0, A = 20$

$$\therefore A_0 = 20$$

Half life is 24 000 years.

$$\therefore 10 = 20e^{24000k}$$

$$\therefore k = \frac{1}{24000} \log_e\left(\frac{1}{2}\right)$$

When does 20% remain?

$$20e^{kt} = 4$$

$$e^{kt} = \frac{1}{5}$$

$$t = \frac{1}{k} \log_e\left(\frac{1}{5}\right)$$

$$t \approx 55726 \text{ years}$$

**7**  $A = A_0e^{kt}$

$$A = \frac{1}{2}A_0 \text{ when } t = 5730$$

$$\therefore \frac{1}{2} = e^{5730k}$$

$$\therefore k = \frac{1}{5730} \log_e\left(\frac{1}{2}\right)$$

When does 40% remain?

$$e^{kt} = 0.4$$

$$e^{kt} = \frac{2}{5}$$

$$t = \frac{1}{k} \log_e\left(\frac{2}{5}\right)$$

$$t \approx 7575 \text{ years}$$

**8**  $P = P_0e^{kt}$

When  $t = 0, P = 10000$

$$\therefore P_0 = 10000$$

$A = 15000$  when  $t = 13$

$$\therefore \frac{3}{2} = e^{13k}$$

$$\therefore k = \frac{1}{13} \log_e\left(\frac{3}{2}\right)$$

$$P = 10000e^{kt}$$

**a** When  $t = 16$   
 $P = 10000e^{16k}$   
 $\therefore P = 16471$

**b**  $30000 = 10000e^{kt}$   
 $\log_e 3 = kt$   
 $t = \frac{1}{k} \log_e(3)$   
 $t \approx 35$

**9**  $C = C_0(1.12)^n$   
 $M = M_0(0.94)^n$   
 $M_0 = 5C_0$   
 $\therefore M = 5C_0(0.94)^n$   
 $C > M \Leftrightarrow (1.12)^n > 5(0.94)^n$   
 This happens after approximately  
 9.2 years.

**10**  $P(h) = 1000 \times 10^{-0.05428h}$

**a** 607 millibars

**b** 6.389 km

**11**  $P = 500000(1.1)^n$   
 $4000000 = 500000(1.1)^n$   
 $8 = (1.1)^n$   
 $n \approx 21.82$

**12**  $T = T_0e^{-kt}$   
 When  $t = 0, T = 100$   
 $\therefore T_0 = 100$

$T = 40$  when  $t = 5$   
 $\therefore \frac{2}{5} = e^{-5k}$

$\therefore k = -\frac{1}{5} \log_e\left(\frac{2}{5}\right)$

$T = 100e^{-kt}$

When  $t = 15$

$T = 100e^{-15k}$

$\therefore T = 6.4$

**13**  $k = 0.349, N_0 = 50.25$

**14 a**  $k = \log_e\left(\frac{5}{4}\right)$

**b** 7.21 hours

**15 a**  $a = 1000, b = 15^{\frac{1}{5}}$

**b** 3 hours

**c** 13 hours

**d** 664 690

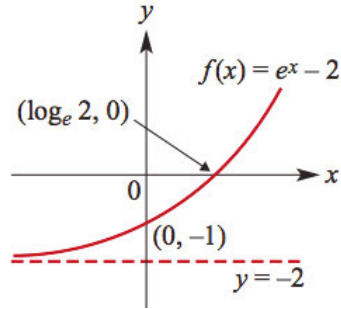
## Solutions to Technology-free questions

**1 a**  $y = e^x - 2$

$x = 0: y = -1$

$y = 0: e^x = 2 \Rightarrow x = \log_e 2$

asymptote:  $y = -2$

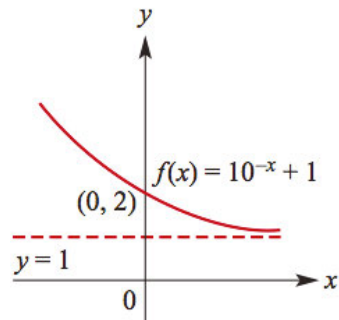


**b**  $y = 10^{-x} + 1$

$x = 0: y = 2$

no  $x$  intercepts as  $y > 1$

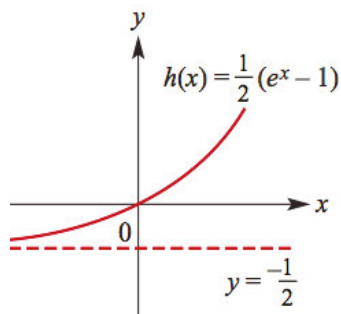
asymptote:  $y = 1$



**c**  $y = \frac{1}{2}(e^x - 1)$

$x = 0: y = 0$

asymptote:  $y = -\frac{1}{2}$

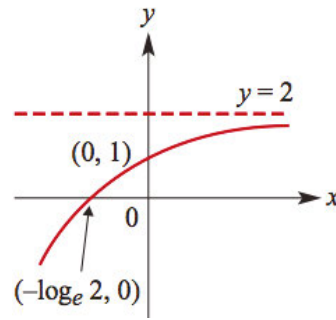


**d**  $y = 2 - e^{-x}$

$x = 0: y = 1$

$y = 0: e^{-x} = 2 \Rightarrow x = -\log_e 2$

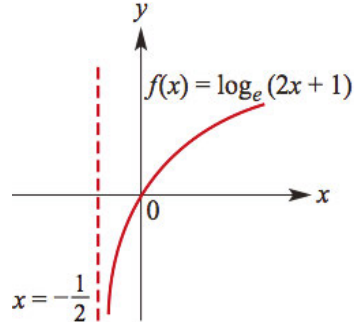
asymptote:  $y = 2$



**e**  $y = \log_e(2x + 1)$

$x = 0: y = 0$

asymptote:  $x = -\frac{1}{2}$

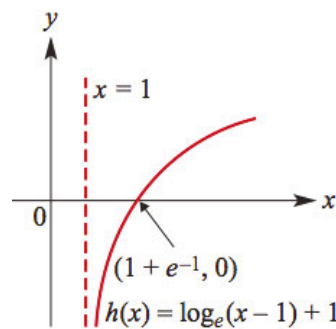


**f**  $y = \log_e(x - 1) + 1$

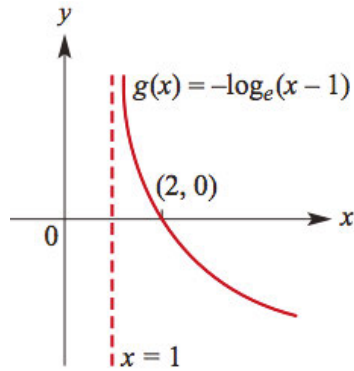
no  $y$  intercepts as  $x > 1$

$y = 0: \log_e(x - 1) = -1 \Rightarrow x = 1 + e^{-1}$

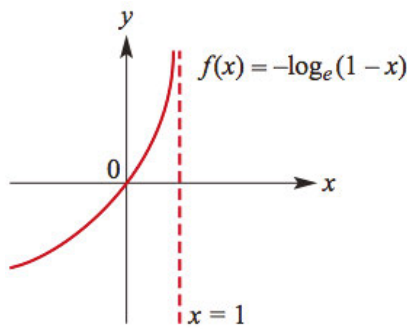
asymptote:  $x = 1$



- g**  $y = -\log_e(x - 1)$   
 no  $y$  intercepts as  $x > 1$   
 $y = 0: -\log_e(x - 1) = 0 \Rightarrow x = 2$   
 asymptote:  $x = 1$



- h**  $y = -\log_e(1 - x)$   
 $x = 0: y = 0$   
 asymptote:  $x = 1$



- 2 a**  $f(x) = e^{2x} - 1$   
 domain =  $\mathbb{R}$ , range =  $(-1, \infty)$   
 The domain of  $f^{-1}$  is  $(-1, \infty)$ .  
 Interchange  $x$  and  $y$  and solve for  $y$ :  
 $x = e^{2y} - 1$   
 $e^{2y} = x + 1$   
 $2y = \log_e(x + 1)$   
 $y = \frac{1}{2} \log_e(x + 1)$   
 $f^{-1}: (-1, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = \frac{1}{2} \log_e(x + 1)$
- b**  $f(x) = 3 \log_e(x - 2)$   
 domain =  $(2, \infty)$ , range =  $\mathbb{R}$

The domain of  $f^{-1}$  is  $\mathbb{R}$ .  
 Interchange  $x$  and  $y$  and solve for  $y$ :  
 $x = 3 \log_e(y - 2)$

$$\log_e(y - 2) = \frac{x}{3}$$

$$y = e^{\frac{x}{3}} + 2$$

$$f^{-1}: \mathbb{R} \rightarrow \mathbb{R}, f^{-1}(x) = e^{\frac{x}{3}} + 2$$

- c**  $f(x) = \log_{10}(x + 1)$   
 domain =  $(-1, \infty)$ , range =  $\mathbb{R}$   
 The domain of  $f^{-1}$  is  $\mathbb{R}$ .  
 Interchange  $x$  and  $y$  and solve for  $y$ :  
 $x = \log_{10}(y + 1)$   
 $y = 10^x - 1$   
 $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}, f^{-1}(x) = 10^x - 1$

- d**  $f(x) = 2^x + 1$   
 domain =  $\mathbb{R}^+$ , range =  $(2, \infty)$   
 The domain of  $f^{-1}$  is  $(2, \infty)$ .  
 Interchange  $x$  and  $y$  and solve for  $y$ :  
 $x = 2^y + 1$   
 $2^y = x - 1$   
 $y = \log_2(x - 1)$   
 $f^{-1}: (2, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = \log_2(x - 1)$

- 3 a**  $\log_e y = \log_e(x) + 2$   
 $= \log_e(x) + \log_e(e^2)$   
 $= \log_e(e^2 x)$   
 $y = e^2 x$

- b**  $\log_{10} y = \log_{10} x + 1$   
 $= \log_{10} x + \log_{10} 10$   
 $= \log_{10} 10x$   
 $y = 10x$

$$\begin{aligned}
 \text{c } \log_2 y &= 3 \log_2 x + 4 \\
 &= \log_2 x^3 + \log_2 2^4 \\
 &= \log_2 16x^3 \\
 y &= 16x^3
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \log_{10} y &= -1 + 5 \log_{10} x \\
 &= -\log_{10} 10 + \log_{10} x^5 \\
 &= \log_{10} \frac{x^5}{10} \\
 y &= \frac{x^5}{10}
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \log_e y &= 3 - \log_e x \\
 &= \log_e e^3 - \log_e x \\
 &= \log_e \frac{e^3}{x} \\
 y &= \frac{e^3}{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \log_e y &= 2x - 3 \\
 y &= e^{2x-3}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a } 3^x &= 11 \\
 x &= \log_3 11 \\
 x &= \frac{\log_e 11}{\log_e 3} \text{ by change of base}
 \end{aligned}$$

(Alternatively, take logarithms to base  $e$  of both sides and simplify, as in part **c** below.)

$$\begin{aligned}
 \text{b } 2^x &= 0.8 \\
 x &= \log_2(0.8) \\
 &= \frac{\log_e(0.8)}{\log_e 2} \text{ by change of base}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } 2^x &= 3^{x+1} \\
 \log_e 2^x &= \log_e 3^{x+1} \\
 x \log_e 2 &= (x+1) \log_e 3 \\
 x \log_e 2 - x \log_e 3 &= \log_e 3 \\
 x(\log_e 2 - \log_e 3) &= \log_e 3 \\
 x &= \frac{\log_e 3}{\log_e 2 - \log_e 3} \\
 &= \frac{\log_e 3}{\log_e \left(\frac{2}{3}\right)}
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a } 2^{2x} - 2^x - 2 &= 0 \\
 (2^x)^2 - 2^x - 2 &= 0 \\
 (2^x - 2)(2^x + 1) &= 0 \\
 2^x &= 2, -1 \\
 \text{But } 2^x > 0 \text{ for all real } x, \text{ so the only} \\
 \text{solution is given by } 2^x = 2, \text{ i.e. } x &= 1.
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \log_e(3x - 1) &= 0 \\
 3x - 1 &= 1 \\
 3x &= 2 \\
 x &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \log_{10}(2x) + 1 &= 0 \\
 \log_{10}(2x) &= -1 \\
 2x &= 10^{-1} \\
 &= \frac{1}{10} \\
 x &= \frac{1}{20}
 \end{aligned}$$



**d**

$$10^{2x} - 7 \times 10^x + 12 = 0$$

$$(10^x)^2 - 7 \times 10^x + 12 = 0$$

$$(10^x - 3)(10^x - 4) = 0$$

$$10^x = 3, 4$$

$$x = \log_{10} 3, \log_{10} 4$$

**6**  $y = 3 \log_2(x + 1) + 2$

$$x = 0: y = 3 \log_2 1 + 2 = 2$$

$y$  intercept:  $(0, 2)$ , so  $b = 2$ .

$$y = 0: 3 \log_2(x + 1) + 2 = 0$$

Solving for  $x$ :

$$3 \log_2(x + 1) = -2$$

$$\log_2(x + 1) = -\frac{2}{3}$$

$$x + 1 = 2^{-\frac{2}{3}}$$

$$x = 2^{-\frac{2}{3}} - 1$$

$$x \text{ intercept: } \left(2^{-\frac{2}{3}} - 1, 0\right) \text{ so } a = 2^{-\frac{2}{3}} - 1.$$

**7**  $f(k) = 5 \log_{10}(k + 1) = 6$ , so solving

for  $k$ :

$$5 \log_{10}(k + 1) = 6$$

$$\log_{10}(k + 1) = \frac{6}{5}$$

$$k + 1 = 10^{\frac{6}{5}}$$

$$k = 10^{\frac{6}{5}} - 1$$

**8**  $4e^{3x} = 287$

$$e^{3x} = \frac{287}{4}$$

$$3x = \log_e \left( \frac{287}{4} \right)$$

$$x = \frac{1}{3} \log_e \left( \frac{287}{4} \right)$$

**9**  $3 \log_a x = 3 + \log_a 8$

$$= 3 + \log_a 2^3$$

$$= 3 + 3 \log_a 2$$

$$= 3(1 + \log_a 2)$$

$$\log_a x = 1 + \log_a 2$$

$$= \log_a a + \log_a 2$$

$$= \log_a 2a$$

$$x = 2a$$

**10** The range of  $f$  is the range of a complete log function, which is  $\mathbb{R}$ . So the domain of  $f^{-1}$  is  $\mathbb{R}$ .

**11**  $y = f(x) = e^{2x} - 3ke^x + 5$

$$(0, 0): 1 - 3k + 5 = 0, \text{ so } k = 2$$

$$\text{Hence } y = e^{2x} - 6e^x + 5.$$

$x \rightarrow -\infty, e^{2x} - 6e^x + 5 \rightarrow 0 + 0 + 5 = 5$ , so the horizontal asymptote is  $y = 5$  and therefore  $b = 5$ .

Now find when  $y = 0$ , i.e.

$$e^{2x} - 6e^x + 5 = 0$$

$$(e^x - 1)(e^x - 5) = 0$$

$$e^x = 1, 5$$

$$x = 0, \log_e 5$$

$x = 0$  corresponds to the intercept

$(0, 0)$ , so  $x = \log_e 5$  corresponds to the intercept  $(a, 0)$ . Thus  $a = \log_e 5$ .

**12 a**  $f^{-1}(x) = \frac{1}{3} \log_e(x + 4)$ ,

$$\text{dom } f^{-1} = (-4, \infty)$$

**b**  $\frac{1}{3x + 4} - 4$

**13 a**  $f(-x) = f(x)$

**b**  $2(e^u + e^{-u})$

**c**  $0$

**d**  $e^{2u} + e^{-2u}$

**e**  $g(-x) = -g(x)$

**f**  $2e^x, 2e^{-x}, e^{2x} - e^{-2x}$

## Solutions to multiple-choice questions

$$\begin{aligned} 1 \quad \mathbf{C} \quad 4 \log_b x^2 &= \log_b 16 + 8 \\ &= \log_b 2^4 + 8 \\ &= 4 \log_b 2 + 8 \end{aligned}$$

$$4 \log_b \frac{(x)^2}{2} = 8$$

$$\log_b \frac{x^2}{2} = 2$$

$$\frac{x^2}{2} = b^2$$

$$x^2 = 2b^2$$

$$x = \pm \sqrt{2}b$$

$$\begin{aligned} 2 \quad \mathbf{D} \quad \log_e 4e^{3x} \\ &= \log_e 4 + \log_e e^{3x} \\ &= \log_e 4 + 3x \end{aligned}$$

$$\begin{aligned} 3 \quad \mathbf{B} \quad 3 \log_3(x-4) \\ &= x-4 \end{aligned}$$

4 **E** The Functions  $g$  and  $h$  here the same domain of  $R \setminus \{-1\}$ , so  $B = C$ . It follows that either option **D** or **E** Must be true.

Now range ( $g$ ) =  $R \setminus \{0\}$ .

Using a CAS calculator to plot the graph of  $h$  shows that range ( $h$ )  $\neq R \setminus \{0\}$ .

$$\begin{aligned} 5 \quad \mathbf{A} \quad \text{As } x = 5 \\ \log_{10}(5k-3) &= 2 \\ 5k-3 &= 10^2 \\ 5k &= 103 \\ k &= \frac{103}{5} \end{aligned}$$

$$\begin{aligned} 6 \quad \mathbf{C} \quad 3^{4 \log_3 x + \log_3 4x} \\ &= 3^{\log_3 x^4 + \log_3 4x} \\ &= 3^{\log_3 4x^5} \\ &= 4x^5 \end{aligned}$$

7 **B** Using the 'solve' command CAS calculator gives  $x = 0.2755 \dots$ , so  $x \approx 0.28$ .

8 **A** The graph is translated 3 units in the negative direction of the  $y$  axis  
 $\therefore b = -3$

When  $x = 0, y = 0$

$$\therefore 0 = ae^0 - 3$$

$$0 = a - 3$$

$$a = 3$$

$$\begin{aligned} 9 \quad \mathbf{C} \quad f : R^+ \rightarrow R, f(x) &= \log_5 x \\ (5, 0) \end{aligned}$$

$$0 \neq \log_5 5$$

$$0 \neq 1$$

The graph does not pass through the point (5,0).

$$10 \quad \mathbf{D} \quad 3 \log_2 x - 7 \log_2(x-1) = 2 + \log_2 y$$

$$\log_2 \frac{x^3}{(x-1)^7} = 2 + \log_2 y$$

$$\log_2 \frac{x^3}{(x-1)^7} - \log_2 y = 2$$

$$\log_2 \frac{x^3}{y(x-1)^7} = 2$$

$$\frac{x^3}{y(x-1)^7} = 2^2$$

$$y = \frac{x^3}{4(x-1)^7}$$

11 **A**

**12**    **C**

**13**    **C**

**14**    **D**

**15**    **B**

**16**    **D**

## Solutions to extended-response questions

- 1 The temperature,  $T^\circ\text{C}$ , of a liquid  $x$  minutes after it begins to cool is given by

$$T = 90(0.98)^x$$

- a When  $x = 10$

$$\begin{aligned} T &= 90(0.98)^{10} \\ &= 73.5366 \end{aligned}$$

- b When  $T = 27$

$$\begin{aligned} 27 &= 90(0.98)^x \\ \frac{27}{90} &= 10.98^x \\ 0.3 &= 0.98^x \end{aligned}$$

$$\therefore \log_e(0.3) = x \log_e(0.98)$$

$$\begin{aligned} \therefore x &= \frac{\log_e(0.3)}{\log_e(0.98)} \\ &= 59.5946 \end{aligned}$$

- 2 Let  $P$  denote the population of the village in years after 1800.

$$P = 240(1.06)^n$$

$$\text{When } n = 0, P = 240$$

- a When  $n = 20$

$$P = 240(1.06)^{20} = 769.71$$

At the beginning of 1820 the population is approximately 770.

- b If  $P = 2500$

$$2500 = 240(1.06)^n$$

$$\frac{2500}{240} = (1.06)^n$$

$$\text{i.e. } \frac{125}{12} = (1.06)^n$$

Taking logarithms of both sides

$$\log_e\left(\frac{125}{12}\right) = n \log_e(1.06)$$

$$\therefore n = \frac{\log_e\left(\frac{125}{12}\right)}{\log_e(1.06)}$$

$$= 40.217$$

The population will reach 2500 in the year 1840.

**3**  $V = ke^{-\lambda t}$

**a** as  $V = 22\,497$  when  $t = 0$

$$k = 22\,497$$

After one year the value of the car is \$18 000

$\therefore$  Take logarithms, base  $e$  of both sides.

$$\log_e 18\,000 = \log_e(22\,497)\lambda$$

$$\therefore \lambda = \log_e\left(\frac{22\,497}{18\,000}\right)$$

$$\approx 0.223$$

$$\approx 0.22 \text{ (correct to two decimal places)}$$

**b**  $V = 22\,497e^{-0.22 \times 3}$

when  $t = 3$

$$V = 22\,497e^{-0.22 \times 3}$$

$$= 11\,627.60$$

The value is \$11 627.6 after 3 years. (This is obtained by taking  $\lambda = 0.22$ )

**4**  $\$M$  is the value of a particular house in a certain area  $t$  years after January 1st 1988.

**a** It is given that  $M = Ae^{-pt}$

and when  $t = 0$ ,  $M = \$65\,000$

$\therefore A = 65\,000$

Furthermore when  $t = 1$ ,  $M = 61\,000$

$$\therefore 61\,000 = 65\,000e^{-p}$$

$$\therefore \frac{61}{65} = e^{-p}$$

and  $-p = \log_e\left(\frac{61}{65}\right)$

i.e.  $p = \log_e\left(\frac{65}{61}\right)$

$$p = 0.635$$

$\therefore A = 65\,000$  and  $p = 0.064$  to two significant figures.

**b**  $M = 65\,000e^{-pt}$

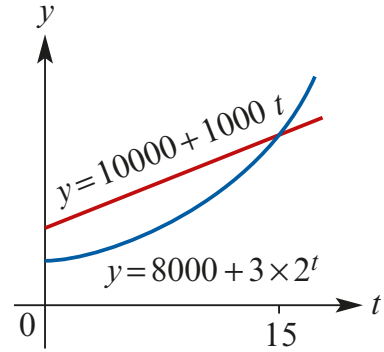
When  $t = 5$

$$M = 65\,000e^{-5p}$$

$$= 47\,199.687$$

To the nearest hundred the value is \$47 200

5 a  $N_A(t) = 10\,000 + 1\,000t$   
 $N_C(t) = 8\,000 + 3 \times 2^t$



b i Using **intersect** from the **CALC** menu the point of intersection of the two graphs has coordinates (12.21, 22209.62)

ii  $t = 12.21$  i.e. on Jan 13

iii 22 210

c i  $10\,000 + 100t = 8\,000 + 3 \times 2^t$   
 $\therefore 2000 + 1000t = 3 \times 2^t$   
 $\therefore \frac{2000 + 1000t}{3} = 2^t$   
 $\therefore \log_{10}\left(\frac{2000 + 1000t}{3}\right) = t \log_{10} 2$   
 $\therefore \log_{10} 1000 + \log_{10}\left(\frac{2 + t}{3}\right) = t \log_{10} 2$   
 $\therefore t = \frac{1}{\log_{10} 2} \left(3 + \log_{10}\left(\frac{2 + t}{3}\right)\right)$

ii (12.21, 12.21) is found by

d  $N_c(15) = N_A(15)$   
 $\therefore 8000 + c \times 2^{15} = 10\,000 + 1000 \times 15$   
 $\therefore c \times 2^{15} = 17\,000$   
 $\therefore c = 0.52$

6  $n = A(1 - e^{-Bt})$

a i When  $t = 2, n = 10\,000$  and when  $t = 4, n = 15\,000$

$$10\,000 = A(1 - e^{-2B}) \quad (1)$$

$$\text{and } 15\,000 = A(1 - e^{-4B}) \quad (2)$$

Divide (2) by (1)

$$\frac{3}{2} = \frac{A(1 - e^{-4B})}{A(1 - e^{-2B})}$$

$$\therefore 3(1 - e^{-2B}) = 2(1 - e^{-4B})$$

$$\therefore 3 - 3e^{-2B} = 2 - 2e^{-4B}$$

$$\therefore 1 + 2e^{-4B} - 3e^{-2B} = 0$$

**ii** Let  $a = e^{-2B}$

$$\text{Then } 1 + 2a^2 - 3a = 0$$

$$\text{i.e. } 2a^2 - 3a + 1 = 0$$

**iii**  $\therefore (2a - 1)(a - 1) = 0$

$$\therefore a = \frac{1}{2} \quad \text{or} \quad a = 1$$

**iv**  $\therefore e^{-2B} = \frac{1}{2} \quad \text{or} \quad e^{-2B} = 1$

$$\therefore -2B = \log_e\left(\frac{1}{2}\right) \quad \text{or} \quad -2B = 0$$

$$\therefore B = \frac{1}{2} \log_e 2 \quad \text{or} \quad B = 0, \text{ and then } A \in R^+ \text{ and } n = 0 \text{ for any } A.$$

**v** Substitute in (1)

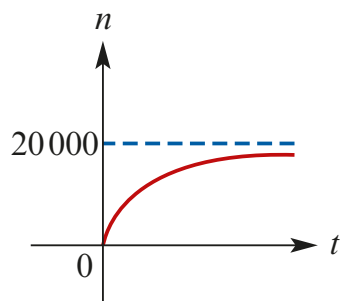
$$10\,000 = A(1 - e^{-\log_e 2})$$

$$10\,000 = A\left(1 - e^{\log_e \frac{1}{2}}\right)$$

$$10\,000 = A\left(\frac{1}{2}\right)$$

$$\therefore A = 20\,000$$

**b**



**c**  $18\,000 = 20\,000\left(1 - e^{\left(-\frac{1}{2} \log_e 2\right)}\right)$

$$18\,000 = 20\,000\left(1 - 2^{-\frac{1}{2}}\right)$$



$$\therefore \frac{9}{10} = 1 - 2^{-\frac{1}{2}}$$

$$\therefore 2^{-\frac{1}{2}} = 0.1$$

$$-\frac{t}{2} \log_e 2 = \log_e 0.1$$

$$\therefore t = \frac{2 \log_e 10}{\log_e 2} \approx 6.644$$

After 6.65 hours the population is 18 000

**7**  $P = 75(10^{-0.15h})$

**a** When  $h = 0, P = 75$

The barometric pressure is 75 cm of mercury when  $h = 0$ .

**b** When  $h = 10, P = 75 \times 10^{-1.5} = 2.3717$

The barometric pressure is 2.37 cm when  $h = 10$ .

**c** When  $P = 60$

$$60 = 75 \times 10^{-0.15h}$$

$$\therefore 0.8 = 10^{-0.15h}$$

$$\therefore \log_{10}(0.8) = -0.15h$$

$$\therefore h = \frac{-1}{0.15} \log_{10}(0.8)$$

$$= 0.646 \text{ km}$$

The barometric pressure is 60 cm of mercury then  $h = 0.646$ .

**8**  $A = A_0 e^{kt}$

When  $t = 1, a = 60.7$

When  $t = 6, a = 5$

Consider the equations

$$60.7 = A_0 e^{kt} \quad (1)$$

$$5 = A_0 e^{kt} \quad (2)$$

Divide 2 by 1

$$\frac{50}{607} = e^{5k}$$

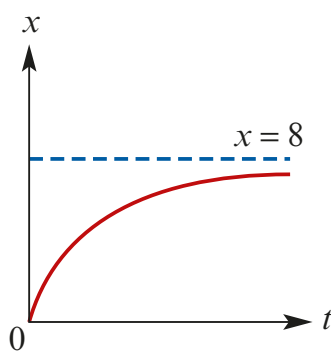
$$\therefore k = \frac{1}{5} \log_e \left( \frac{50}{607} \right) \approx -0.4993 \approx -0.5$$

Substitute in (1)

$$60.7 = A_0 \left( \frac{50}{607} \right)^{\frac{1}{5}}$$

$$\therefore A_0 = 60.7 \times \left( \frac{607}{50} \right)^{\frac{1}{5}} \approx 100.007 \approx 100$$

9 a



Note: When  $t = 0, x = 8(1 - 1) = 0$

As  $t \rightarrow \infty, e^{-0.2t} \rightarrow 0 \therefore x \rightarrow 8$

b i When  $t = 0, x = 8(1 - 1) = 0$  Amount reacted after 0 min is 0 gram

ii When  $t = 2, x = 8(1 - e^{-0.4}) \approx 2.64$  Amount reacted after 2 min is  $\approx 2.64$  gram

iii When  $t = 10, x = 8(1 - e^{-2}) \approx 6.92$  Amount reacted after 10 min is  $\approx 6.92$  gram

c When  $x = 7, 7 = 8(1 - e^{-0.2t})$

$$0.875 = 1 - e^{-0.2t}$$

$$e^{-0.2t} = 0.125$$

$$-0.2t = \log_e(0.125)$$

$$t = -5 \log_e(0.125)$$

$$= 5 \log_e 8$$

$$\approx 10.397$$

After 10.4 minutes there is 7 g of the substance which has reacted.

10  $T - T_s = (T_0 - T_s)e^{-kt}$

$$T_s = 15^\circ$$

$$T_0 = 96^\circ$$

a When  $t = 5, T = 40$

$$\begin{aligned} \therefore 40 - 15 &= (96 - 15)e^{-5k} \\ 25 &= 81e^{-5k} \\ e^{-5k} &= \frac{25}{81} \\ -5k &= \log_e \frac{25}{81} \\ k &= -\frac{1}{5} \log_e \frac{25}{81} \\ &\approx 0.235 \end{aligned}$$

**b** When  $t = 10$

$$\begin{aligned} T - 15 &= (96 - 15)_e^{\frac{1}{5} \left( \log_e \frac{25}{81} \right) \times 10} \\ \text{i.e. } T - 15 &= 81 \times \left( \frac{25}{81} \right)^2 \\ T &= 22.716 \end{aligned}$$

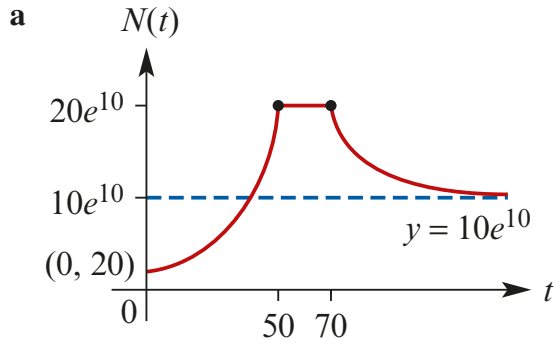
The temperature of the egg is  $22.7^\circ\text{C}$  when  $t = 10$ .

**c** When  $T = 30$

$$\begin{aligned} 30 - 15 &= (96 - 15)_e^{\frac{1}{5} \left( \log_e \frac{25}{81} \right) t} \\ \frac{15}{81} &= \left( \frac{25}{81} \right)^{\frac{t}{5}} \\ \text{i.e. } \frac{5}{27} &= \left( \frac{25}{81} \right)^{\frac{t}{5}} \\ \therefore \frac{\log_e \left( \frac{5}{27} \right)}{\log_e \left( \frac{25}{81} \right)} &= \frac{t}{5} \\ \therefore t &\approx 7.17 \end{aligned}$$

The egg reaches a temperature of  $30^\circ\text{C}$  after 7.17 minutes.

$$\mathbf{11} \quad N(t) = \begin{cases} 20e^{0.2t} & 0 \leq t \leq 50 \\ 20e^{10} & 50 < t \leq 70 \\ 10e^{10}(e^{70-t} + 1) & t > 70 \end{cases}$$



**b i**  $N(10) = 20e^{0.2 \times 10} \quad (0 \leq t \leq 50)$   
 $= 20e^2$   
 $\approx 147.78$

**ii**  $N(40) = 20e^{0.2 \times 40} \quad (0 \leq t \leq 50)$   
 $= 20e^8$   
 $\approx 59\,619.16$

**iii**  $N(60) = 20e^{10} \quad (50 < t \leq 70)$   
 $\approx 440\,529.32$

**iv**  $N(80) = 10e^{10}(e^{70-80} + 1) \quad (t > 70)$   
 $= 10e^{10}(e^{-10} + 1)$   
 $= 10(1 + e^{10})$   
 $\approx 220\,274.66$

**c i** Considering the graph  
 $N = 2968$  for  $0 \leq t \leq 50$   
 $\therefore 2968 = 20e^{0.2t}$   
 $148.4 = e^{0.2t}$   
 $\therefore t = 5 \log_e(148.4)$   
 $= 24.99955$   
 After 25 days the population is 2968.

**ii** For  $N = 21\,932$ ,  $0 \leq t \leq 50$ . This can be seen from the graph above.

$$21\,932 = 20 \cdot e^{0.2t}$$

$$1096.6 = e^{0.2t}$$

$$\therefore t = 5 \log_e(1096.6)$$

$$\approx 34.9998$$

After 35 days the population is 21932.

# Chapter 6 – Circular functions

## Solutions to Exercise 6A

$$\begin{aligned} 1 \text{ a } 50^\circ &= \frac{50}{180}\pi \\ &= \frac{5\pi}{18} \end{aligned}$$

$$\begin{aligned} \text{b } 136^\circ &= \frac{136}{180}\pi \\ &= \frac{34\pi}{45} \end{aligned}$$

$$\begin{aligned} \text{c } 250^\circ &= \frac{250}{180}\pi \\ &= \frac{25\pi}{18} \end{aligned}$$

$$\begin{aligned} \text{d } 340^\circ &= \frac{340}{180}\pi \\ &= \frac{17\pi}{9} \end{aligned}$$

$$\begin{aligned} \text{e } 420^\circ &= \frac{420}{180}\pi \\ &= \frac{7\pi}{3} \end{aligned}$$

$$\begin{aligned} \text{f } 490^\circ &= \frac{490}{180}\pi \\ &= \frac{49\pi}{18} \end{aligned}$$

$$2 \text{ a } \frac{\pi}{3} = \frac{180^\circ}{3} = 60^\circ$$

$$\text{b } \frac{5\pi}{6} = 180^\circ \times \frac{5}{6} = 150^\circ$$

$$\text{c } \frac{4\pi}{3} = 180^\circ \times \frac{4}{3} = 240^\circ$$

$$\text{d } \frac{7\pi}{9} = 180^\circ \times \frac{7}{9} = 140^\circ$$

$$\text{e } 3.5\pi = \frac{7\pi}{2} = \frac{7}{2} \times 180^\circ = 630^\circ$$

$$\text{f } \frac{7\pi}{5} = \frac{7}{5} \times 180^\circ = 252^\circ$$

$$3 \text{ a } 0.8 = \frac{180^\circ}{\pi} \times 0.8 = 45.84^\circ$$

$$\text{b } 1.64 = \frac{180^\circ}{\pi} \times 1.64 = 93.97^\circ$$

$$\text{c } 2.5 = \frac{180^\circ}{\pi} \times 2.5 = 143.24^\circ$$

$$\text{d } 3.96 = \frac{180^\circ}{\pi} \times 3.96 = 226.89^\circ$$

$$\text{e } 4.18 = \frac{180^\circ}{\pi} \times 4.18 = 239.50^\circ$$

$$\text{f } 5.95 = \frac{180^\circ}{\pi} \times 5.95 = 340.91^\circ$$

$$4 \text{ a } 37^\circ = \frac{\pi}{180^\circ} \times 37^\circ = 0.65$$

$$\text{b } 74^\circ = \frac{\pi}{180^\circ} \times 74^\circ = 1.29$$

$$\text{c } 115^\circ = \frac{\pi}{180^\circ} \times 115^\circ = 2.01$$

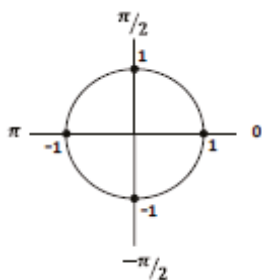
$$\text{d } 122.25^\circ = \frac{\pi}{180^\circ} \times 122.25^\circ = 2.13$$

$$\text{e } 340^\circ = \frac{\pi}{180^\circ} \times 340^\circ = 5.93$$

$$\text{f } 132.5^\circ = \frac{\pi}{180^\circ} \times 132.5^\circ = 2.31$$

## Solutions to Exercise 6B

1



a  $\sin 3\pi = 0$

b  $\cos\left(-\frac{5\pi}{2}\right) = 0$

c  $\sin\left(\frac{7\pi}{2}\right) = -1$

d  $\cos 3\pi = -1$

e  $\sin(-4\pi) = 0$

f  $\tan -\pi = 0$

g  $\tan 2\pi = 0$

h  $\tan -2\pi = 0$

i  $\cos(23\pi) = \cos \pi = -1$

j  $\cos\left(\frac{49\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

k  $\cos(35\pi) = \cos \pi = -1$

l  $\cos\left(\frac{-45\pi}{2}\right) = \cos\left(\frac{-\pi}{2}\right) = 0$

m  $\tan(24\pi) = \tan(0) = 0$

n  $\cos(20\pi) = \cos(0) = 1$

2 a 0.99

b 0.52

c -0.87

d 0.92

e -0.67

f -0.23

g -0.99

h 0.44

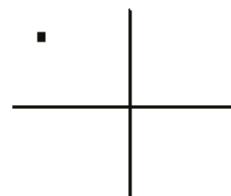
i -34.23

j -2.57

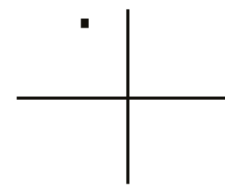
k 0.95

l 0.75

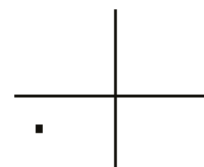
3 a  $\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$



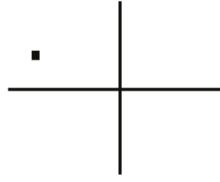
b  $\cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$



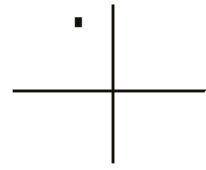
c  $\cos\left(\frac{7\pi}{6}\right) = \frac{-\sqrt{3}}{2}$



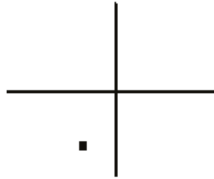
$$\mathbf{d} \quad \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$



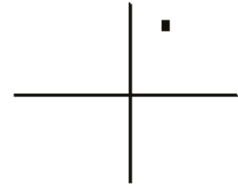
$$\mathbf{j} \quad \sin\left(\frac{200\pi}{3}\right) = \frac{\sqrt{3}}{2}$$



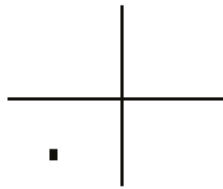
$$\mathbf{e} \quad \cos\left(\frac{4\pi}{3}\right) = \frac{-1}{2}$$



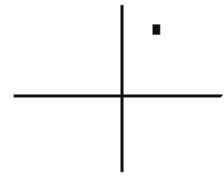
$$\mathbf{k} \quad \cos\left(\frac{-11\pi}{3}\right) = \frac{1}{2}$$



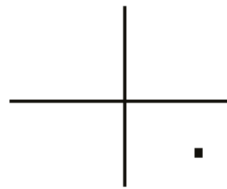
$$\mathbf{f} \quad \sin\left(\frac{5\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$



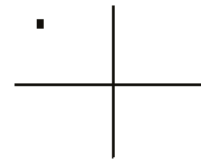
$$\mathbf{l} \quad \sin\left(\frac{25\pi}{3}\right) = \frac{\sqrt{3}}{2}$$



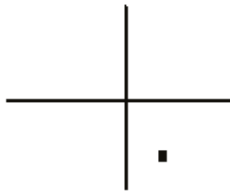
$$\mathbf{g} \quad \sin\left(\frac{7\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$



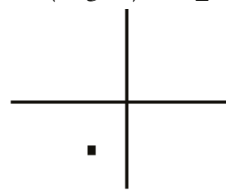
$$\mathbf{m} \quad \sin\left(\frac{-13\pi}{4}\right) = \frac{1}{\sqrt{2}}$$



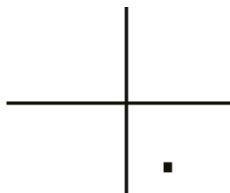
$$\mathbf{h} \quad \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$$



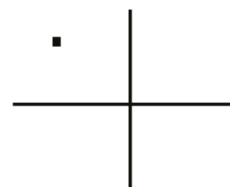
$$\mathbf{n} \quad \cos\left(\frac{-20\pi}{3}\right) = \frac{-1}{2}$$



$$\mathbf{i} \quad \cos\left(\frac{11\pi}{3}\right) = \frac{1}{2}$$

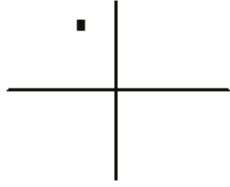


$$\mathbf{o} \quad \sin\left(\frac{67\pi}{4}\right) = \frac{1}{\sqrt{2}}$$





**p**  $\cos\left(\frac{68\pi}{3}\right) = \frac{-1}{2}$



**q**  $\tan\left(\frac{11\pi}{3}\right) = -\sqrt{3}$

**r**  $\tan\left(\frac{200\pi}{3}\right) = -\sqrt{3}$

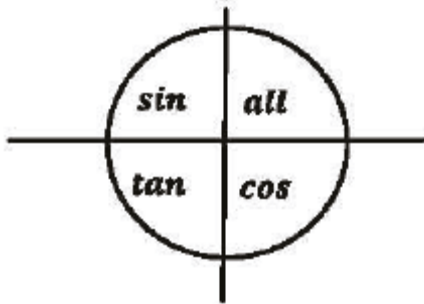
**s**  $\tan\left(\frac{-11\pi}{6}\right) = \frac{1}{\sqrt{3}}$

**t**  $\tan\left(\frac{25\pi}{3}\right) = \sqrt{3}$

**u**  $\tan\left(-\frac{13\pi}{4}\right) = -1$

**v**  $\tan\left(-\frac{25\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

**4**



**a**  $\sin(\pi - \theta) = \sin \theta = 0.52$

**b**  $\cos(\pi + x) = -\cos x = -0.68$

**c**  $\sin(2\pi + \theta) = \sin \theta = 0.52$

**d**  $\tan(\pi + \alpha) = \tan(\alpha) = 0.4$

**e**  $\sin(\pi + \theta) = -\sin \theta = -0.52$

**f**  $\cos(2\pi - x) = \cos(-x) = \cos x = 0.68$

**g**  $\tan(2\pi - \alpha) = \tan(-\alpha)$   
 $= -\tan \alpha$   
 $= -0.4$

**h**  $\cos(\pi - x) = -\cos x = -0.68$

**i**  $\sin(-\theta) = -\sin \theta = -0.52$

**j**  $\cos(-x) = \cos x = 0.68$

**k**  $\tan(-\alpha) = -\tan(\alpha) = -0.4$

**5 a** 0.4

**b** -0.7

**c** 0.4

**d** 1.2

**e** -0.4

**f** 0.7

**g** -1.2

**h** -0.7

**i** -0.4

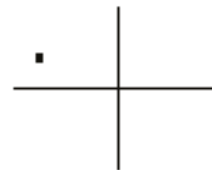
**j** 0.7

**k** -1.2

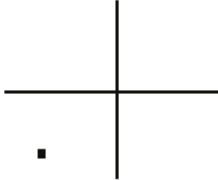
**6 a**  $\sin(150^\circ) = \frac{1}{2}$

$\cos(150^\circ) = \frac{-\sqrt{3}}{2}$

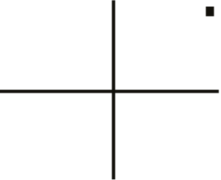
$\tan(150^\circ) = \frac{-1}{\sqrt{3}}$




**b**  $\sin(225^\circ) = \frac{-1}{\sqrt{2}}$   
 $\cos(225^\circ) = \frac{-1}{\sqrt{2}}$   
 $\tan(225^\circ) = 1$




**e**  $\sin(-315^\circ) = \frac{1}{\sqrt{2}}$   
 $\cos(-315^\circ) = \frac{1}{\sqrt{2}}$   
 $\tan(-315^\circ) = 1$



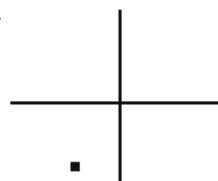
**c**  $\sin(405^\circ) = \frac{1}{\sqrt{2}}$   
 $\cos(405^\circ) = \frac{1}{\sqrt{2}}$   
 $\tan(405^\circ) = 1$



**f**  $\sin(-30^\circ) = \frac{-1}{2}$   
 $\cos(-30^\circ) = \frac{\sqrt{3}}{2}$   
 $\tan(-30^\circ) = \frac{-1}{\sqrt{3}}$



**d**  $\sin(-120^\circ) = \frac{-\sqrt{3}}{2}$   
 $\cos(-120^\circ) = \frac{-1}{2}$   
 $\tan(-120^\circ) = \sqrt{3}$



(ensure calculator is in radians not degrees)

## Solutions to Exercise 6C

1  $\sin x = 0.3$ ,  $\cos x = 0.6$ ,  $\tan x = 0.7$

a  $\cos(-\alpha) = \cos \alpha = 0.6$

b  $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos \alpha = 0.6$

c  $\tan(-\theta) = -\tan \theta = -0.7$

d  $\cos\left(\frac{\pi}{2} - x\right) = \sin x = 0.3$

e  $\sin(-x) = -\sin x = -0.3$

f  $\tan\left(\frac{\pi}{2} - \theta\right) = \cotan(\theta) = \frac{10}{7}$

g  $\cos\left(\frac{\pi}{2} + x\right) = -\sin x = -0.3$

h  $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha = 0.6$

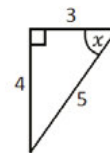
i  $\sin\left(\frac{3\pi}{2} + \alpha\right) = \cos(\pi + \alpha)$   
 $= -\cos \alpha = -0.6$

j  $\cos\left(\frac{3\pi}{2} - x\right) = \cos\left(\frac{-\pi}{2} - x\right)$   
 $= \cos\left(\frac{\pi}{2} + x\right)$   
 $= -\sin x$   
 $= -0.3$

k  $\tan\left(\frac{3\pi}{2} - \theta\right) = \frac{1}{\tan \theta} = \frac{10}{7}$

l  $\cos\left(\frac{5\pi}{2} - \theta\right) = \sin x = 0.3$

a  $\cos x = \frac{3}{5}$   
 $\frac{3\pi}{2} \leq x \leq 2\pi$



Method 1

$$\cos^2 x + \sin^2 x = 1$$

$$\frac{9}{25} + \sin^2 x = 1$$

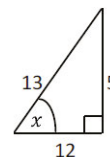
$$\sin^2 x = \frac{16}{25}$$

$$\sin x = \frac{\pm 4}{5}$$

$$\frac{3\pi}{2} \leq x \leq 2\pi, \sin x = \frac{-4}{5}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{-4}{3}$$

b  $\sin x = \frac{5}{13}$   
 $\frac{\pi}{2} < x < \pi$



Method 2

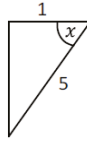
from the triangle,

$$\cos x = \frac{-12}{13} \quad \text{SOH CAH TOA}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{-5}{12}$$

$$\mathbf{c} \quad \cos x = \frac{1}{5}$$

$$\frac{3\pi}{2} < x < 2\pi$$



$$\cos^2 x = \frac{1}{25}$$

$$\therefore \sin^2 x = 1 - \cos^2 x$$

$$\sin^2 x = \frac{24}{25}$$

$$\sin x = \frac{\pm 2\sqrt{6}}{5}$$

$$\text{since } \frac{3\pi}{2} < x < 2\pi, \sin x = \frac{-2\sqrt{6}}{5}$$

$$\tan x = \frac{\sin x}{\cos x} = -2\sqrt{6} \quad \mathbf{f}$$

$$\mathbf{d} \quad \sin x = -\frac{12}{13}$$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

$$= \frac{25}{169}$$

$$\text{Since } \pi < x < \frac{3\pi}{2}$$

$$\cos x = -\frac{5}{13}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{12}{5}$$

$$\mathbf{e} \quad \cos x = \frac{4}{5}$$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

$$= \frac{9}{25}$$

$$\text{Since } \frac{3\pi}{2} < x < 2\pi$$

$$\sin x = -\frac{3}{5}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= -\frac{3}{4}$$

$$\sin x = -\frac{12}{13}$$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

$$= \frac{25}{169}$$

$$\text{Since } \pi < x < \frac{3\pi}{2}$$

$$\cos x = -\frac{5}{13}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{12}{5}$$

$$\mathbf{g} \quad \cos x = \frac{8}{10}$$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

$$= \frac{36}{100}$$

$$\text{Since } \frac{3\pi}{2} < x < 2\pi$$

$$\sin x = -\frac{3}{5}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= -\frac{3}{4}$$

## Solutions to Exercise 6D

1 a  $2\pi, 3$

b  $\frac{2\pi}{3}, 5$

c  $\pi, \frac{1}{2}$

d  $6\pi, 2$

e  $\frac{\pi}{2}, 3$

f  $2\pi, \frac{1}{2}$

g  $4\pi, 3$

h  $3\pi, 2$

2 a Dilation of factor 4 from the  $x$ -axis,  
dilation of factor  $\frac{1}{3}$  from the  $y$ -axis;  
Amplitude = 4; Period =  $\frac{2\pi}{3}$

b Dilation of factor 5 from the  $x$ -axis,  
dilation of factor 3 from the  $y$ -axis;  
Amplitude = 5; Period =  $6\pi$

c Dilation of factor 6 from the  $x$ -axis,  
dilation of factor 2 from the  $y$ -axis;  
Amplitude = 6; Period =  $4\pi$

d Dilation of factor 4 from the  $x$ -axis,  
dilation of factor  $\frac{1}{5}$  from the  $y$ -axis;  
Amplitude = 4; Period =  $\frac{2\pi}{5}$

3 a Dilation of factor 2 from the  $x$ -axis,  
dilation of factor  $\frac{1}{3}$  from the  $y$ -axis;  
Amplitude = 2; Period =  $\frac{2\pi}{3}$

b Dilation of factor 3 from the  $x$ -axis,  
dilation of factor 4 from the  $y$ -axis;

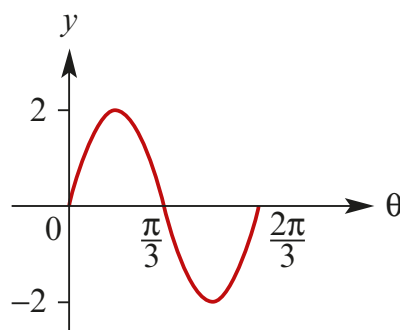
Amplitude = 3; Period =  $8\pi$

c Dilation of factor 6 from the  $x$ -axis,  
dilation of factor 5 from the  $y$ -axis;  
Amplitude = 6; Period =  $10\pi$

d Dilation of factor 3 from the  $x$ -axis,  
dilation of factor  $\frac{1}{7}$  from the  $y$ -axis;  
Amplitude = 3; Period =  $\frac{2\pi}{7}$

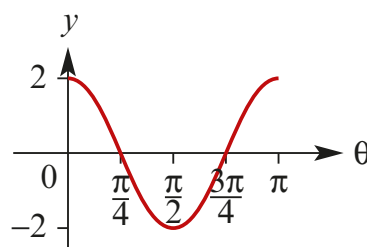
4 a Amplitude = 2

Period =  $\frac{2\pi}{3}$



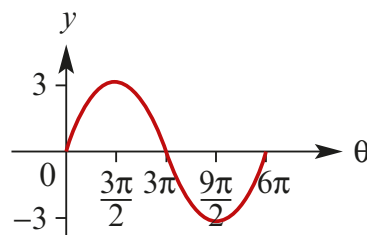
b Amplitude = 2

Period =  $\pi$



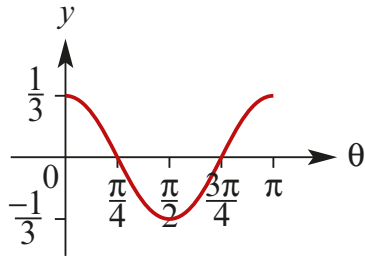
c Amplitude = 3

Period =  $6\pi$



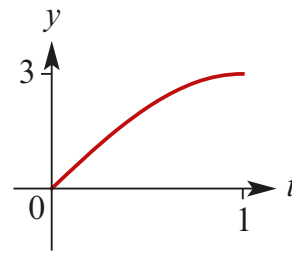
**d** Period =  $\pi$

Amplitude =  $\frac{1}{3}$



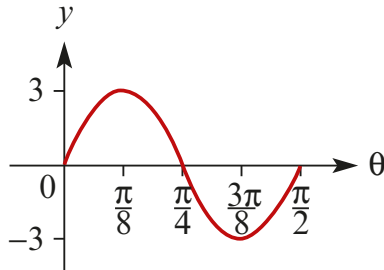
**6** Period =  $= \frac{2\pi}{\frac{\pi}{2}} = 4$

Amplitude = 3



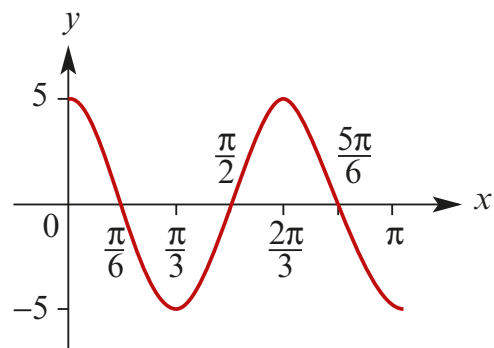
**e** Amplitude = 3

Period =  $\frac{\pi}{2}$



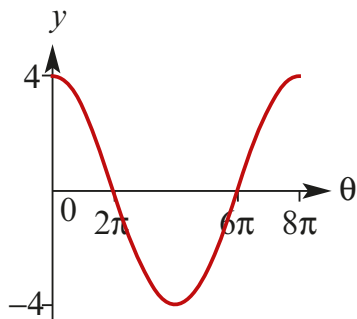
**7** Period =  $= \frac{2\pi}{3}$

Amplitude = 5



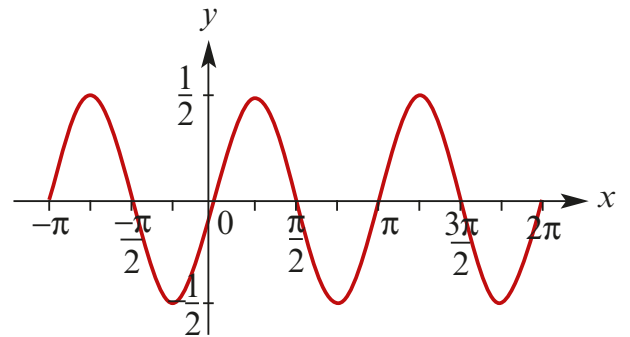
**f** Amplitude = 4

Period =  $8\pi$



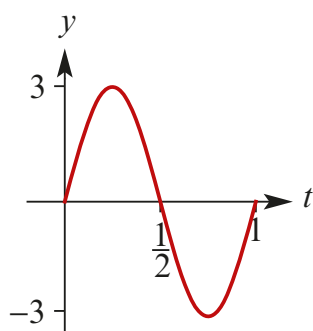
**8** Period =  $= \frac{2\pi}{2} = \pi$

Amplitude =  $\frac{1}{2}$



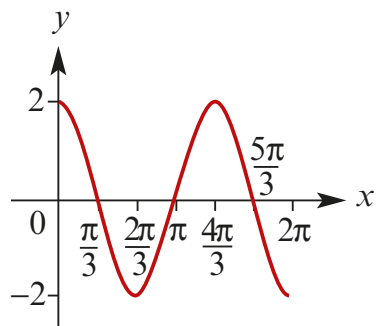
**5** Period =  $\frac{2\pi}{2\pi} = 1$

Amplitude = 3



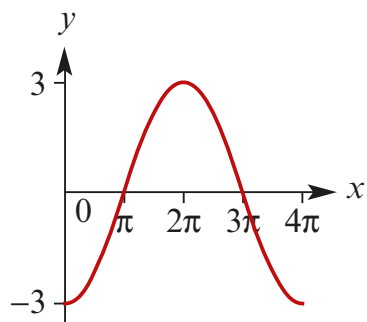
9 Period =  $\frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$

Amplitude = 2



10 Period =  $\frac{2\pi}{\frac{1}{2}} = 4\pi$

Amplitude = 3



11  $y = \sin x$

Dilation of factor 2 from the  $x$ -axis

$\Rightarrow y = 2 \sin x$

Dilation of factor 3 from the  $y$ -axis

$\Rightarrow y = 2 \sin\left(\frac{x}{3}\right)$

12  $y = \cos x$

Dilation of factor  $\frac{1}{2}$  from the  $x$ -axis

$\Rightarrow y = \frac{1}{2} \cos x$

Dilation of factor 3 from the  $y$ -axis

$\Rightarrow y = \frac{1}{2} \cos\left(\frac{x}{3}\right)$

13  $y = \sin x$

Dilation of factor  $\frac{1}{2}$  from the  $x$ -axis

$\Rightarrow y = \frac{1}{2} \sin x$

Dilation of factor 2 from the  $y$ -axis

$\Rightarrow y = \frac{1}{2} \sin\left(\frac{x}{2}\right)$

## Solutions to Exercise 6E

1 a  $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

b  $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$

c  $\frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$

d  $\frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$

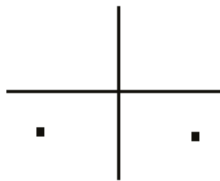
e  $\frac{\pi}{2}, \frac{5\pi}{2}$

f  $\pi, 3\pi$

2 a  $\sin x = \frac{-1}{2}$

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{-5\pi}{6}, \frac{-\pi}{6}$$



b  $\cos x = \frac{\sqrt{3}}{2}$

$$x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

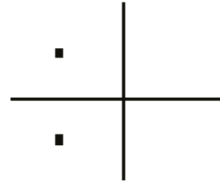
$$x = \frac{-\pi}{6}, \frac{\pi}{6}$$



c  $\cos x = \frac{-\sqrt{3}}{2}$

$$x = \cos^{-1}\frac{-\sqrt{3}}{2}$$

$$x = \frac{-5\pi}{6}, \frac{5\pi}{6}$$



3 a  $\sqrt{2} \sin x = 1$

$$\sin x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

b  $\sqrt{2} \cos x = -1$

$$\cos x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

c  $2 \cos x = -\sqrt{3}$

$$\cos x = -\frac{\sqrt{3}}{2}$$

$$x = \frac{5\pi}{6}, \frac{7\pi}{6}$$

d  $2 \sin x + 1 = 0$

$$\sin x = -\frac{1}{2}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$



**e**  $\sqrt{2} \cos x = 1$   
 $\cos x = \frac{1}{\sqrt{2}}$   
 $x = \frac{\pi}{4}, \frac{7\pi}{4}$

**f**  $4 \cos x = -2$   
 $\cos x = -\frac{1}{2}$   
 $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

**4 a**  $\sin x = 0.6$   
calculator gives  $x_1 \cong 0.6435$   
second answer is  $\pi - x_1 \cong 2.498$

**b**  $\cos x = 0.8$   
calculator gives  $x_1 \cong 0.6435$   
second answer is  $2\pi - x_1 \cong 5.640$

**c**  $\sin x = -0.45$   
calculator gives  $x_1 \cong 5.816$   
second answer is  $\pi - x_1 \cong 3.608$

**d**  $\cos x = -0.2$   
calculator gives  $x_1 \cong 1.772$   
second answer is  $2\pi - x_1 \cong 4.511$

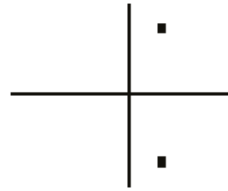
**5 a**  $\sin \theta^\circ = 0.3$   
calculator gives  $\theta_1 = 17.46$   
second answer is  $180^\circ - \theta_1 = 162.54$

**b**  $\cos \theta^\circ = 0.4$   
calculator gives  $\theta_1 = 66.42$   
second answer is  $360^\circ - \theta_1 = 293.58$

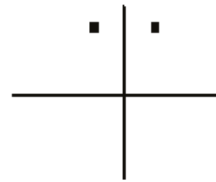
**c**  $\sin \theta^\circ = -0.8$   
calculator gives  $\theta_1 = 306.87$   
second answer is  $180^\circ - \theta_1 = 233.13$

**d**  $\cos \theta^\circ = -0.5$   
 $\theta_1 = 120$   
second answer is  $\theta_2 - 360^\circ - \theta_1 = 240$

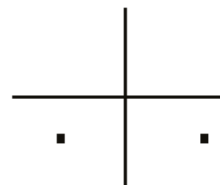
**6 a**  $\cos(\theta^\circ) = \frac{1}{2}$   
 $\theta = \cos^{-1}\left(\frac{1}{2}\right)$   
 $\theta = 60, 300$



**b**  $\sin(\theta^\circ) = \frac{\sqrt{3}}{2}$   
 $\theta^\circ = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$   
 $\theta = 60, 120$



**c**  $\sin(\theta^\circ) = \frac{-1}{\sqrt{2}}$   
 $\theta = 225, 315$

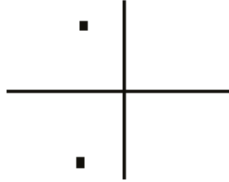


**d**  $2 \cos \theta^\circ + 1 = 0$

$$\cos \theta^\circ = \frac{-1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

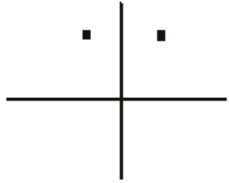
$$\theta = 120, 240$$



**e**  $\sin(\theta^\circ) = \frac{\sqrt{3}}{2}$

$$\theta^\circ = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

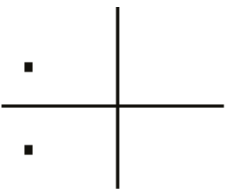
$$\theta = 60, 120$$



**f**  $\cos(\theta^\circ) = \frac{-\sqrt{3}}{2}$

$$\theta^\circ = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

$$\theta = 150, 210$$



**7 a**  $\sin 2\theta = -\frac{1}{2}$

$$2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

**b**  $\cos 2\theta = \frac{\sqrt{3}}{2}$

$$2\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

**c**  $\sin 2\theta = \frac{1}{2}$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

**d**  $\sin 3\theta = -\frac{1}{\sqrt{2}}$

$$3\theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}, \frac{23\pi}{4}$$

$$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12} (= \frac{5\pi}{4}),$$

$$\frac{21\pi}{12} (= \frac{7\pi}{4}), \frac{23\pi}{12}$$

**e**  $\cos 2\theta = -\frac{\sqrt{3}}{2}$

$$2\theta = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$$

$$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$$

**f**  $\sin 2\theta = -\frac{1}{\sqrt{2}}$

$$2\theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$$

$$\theta = \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$$

**8 a**

$$\begin{aligned}\cos 3x &= -\frac{\sqrt{3}}{2} \\ 3x &= \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6} \\ x &= \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}\end{aligned}$$

**b**  $\sin 2x = \frac{1}{2}$

$$\begin{aligned}2x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \\ x &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\end{aligned}$$

**c**  $\cos 3x = \frac{1}{\sqrt{2}}$

$$\begin{aligned}3x &= \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4} \\ x &= \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}\end{aligned}$$

**d**  $\sin 3x = \frac{1}{2}$

$$\begin{aligned}3x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6} \\ x &= \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}\end{aligned}$$

**e**  $\sin 2x = \frac{1}{\sqrt{2}}$

$$\begin{aligned}2x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4} \\ x &= \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}\end{aligned}$$

**f**

$$\cos 3x = -\frac{\sqrt{3}}{2}$$

$$\begin{aligned}3x &= \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6} \\ x &= \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}\end{aligned}$$

**g**  $x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$

**h**  $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

**i**  $x = \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

**9 a** 2.03444, 2.67795, 5.17604, 5.81954

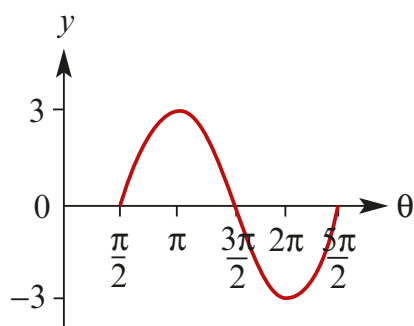
**b** 1.89255, 2.81984, 5.03414, 5.96143

**c** 0.57964, 2.56195, 3.72123, 5.70355

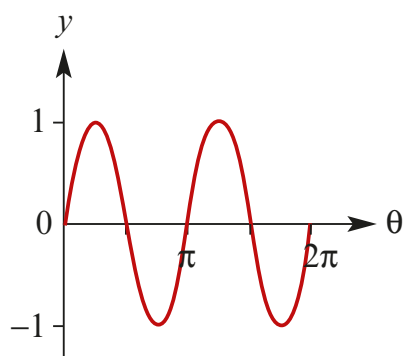
**d** 0.309098, 1.7853, 2.40349, 3.87969, 4.49789, 5.97409

## Solutions to Exercise 6F

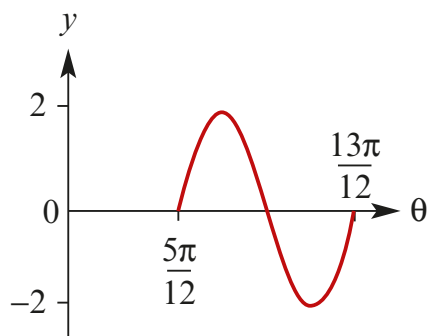
1 a Period =  $2\pi$ ; Amplitude = 3;  $y = \pm 3$



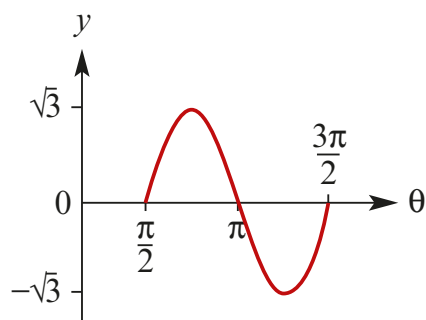
b Period =  $\pi$ ; Amplitude = 1;  $y = \pm 1$



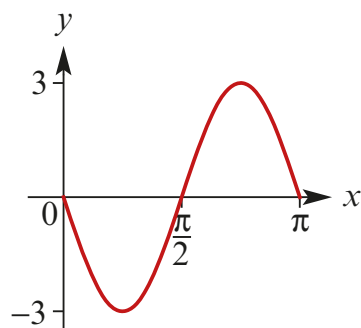
c Period =  $\frac{2\pi}{3}$ ; Amplitude = 2;  $y = \pm 2$



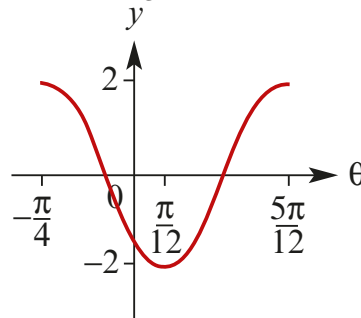
d Period =  $\pi$ ; Amplitude =  $\sqrt{3}$ ;  $y = \pm \sqrt{3}$



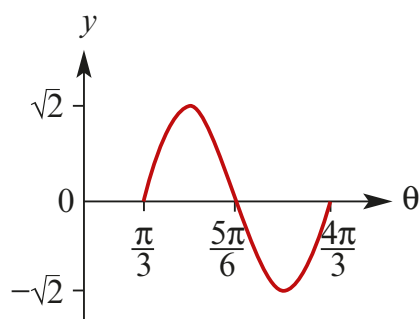
e Period =  $\pi$ ; Amplitude = 3;  $y = \pm 3$



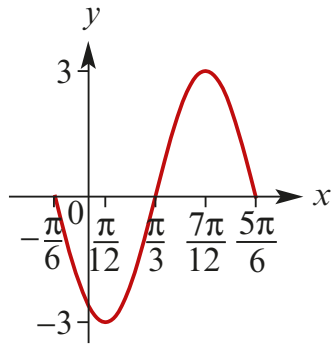
f Period =  $\frac{2\pi}{3}$ ; Amplitude = 2;  $y = \pm 2$



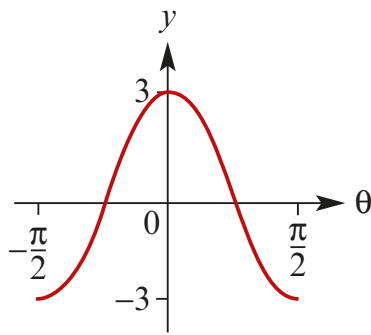
g Period =  $\pi$ ; Amplitude =  $\sqrt{2}$ ;  $y = \pm \sqrt{2}$



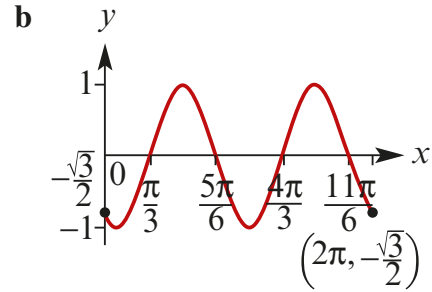
**h** Period =  $\pi$ ; Amplitude = 3;  $y = \pm 3$



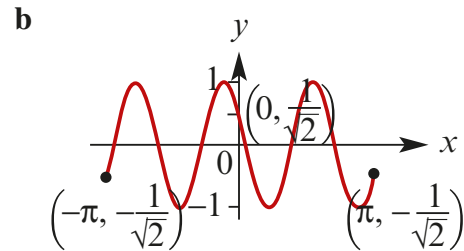
**i** Period =  $\pi$ ; Amplitude = 3;  $y = \pm 3$



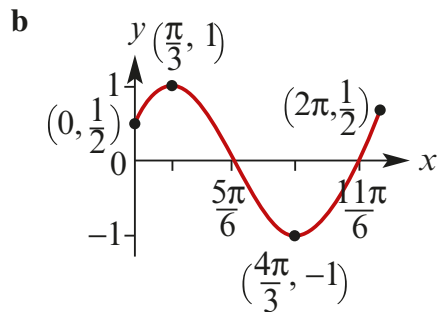
**3 a**  $f(0) = -\frac{\sqrt{3}}{2}$ ,  $f(2\pi) = -\frac{\sqrt{3}}{2}$



**4 a**  $f(-\pi) = -\frac{1}{\sqrt{2}}$ ,  $f(\pi) = -\frac{1}{\sqrt{2}}$



**2 a**  $f(0) = \frac{1}{2}$ ,  $f(2\pi) = \frac{1}{2}$



**5 a**  $y = 3 \sin\left(\frac{x}{2}\right)$

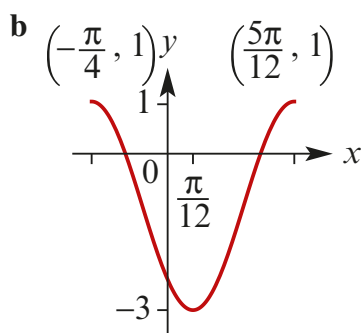
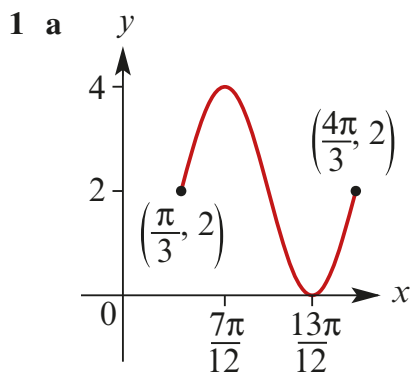
**b**  $y = 3 \sin(2x)$

**c**  $y = 2 \sin\left(\frac{x}{3}\right)$

**d**  $y = \sin 2\left(x - \frac{\pi}{3}\right)$

**e**  $y = \sin \frac{1}{2}\left(x + \frac{\pi}{3}\right)$

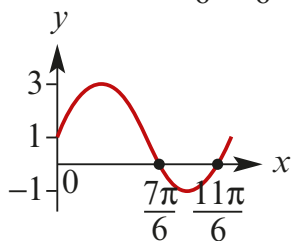
## Solutions to Exercise 6G



**2 a**  $2 \sin x + 1 = 0$

$$\sin x = -\frac{1}{2}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

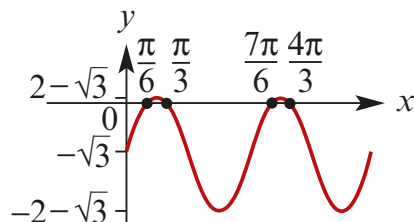


**b**  $2 \sin 2x - \sqrt{3} = 0$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

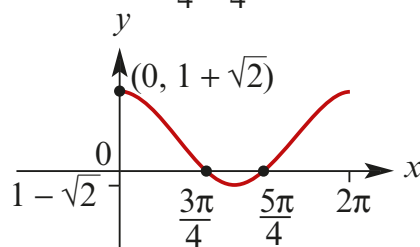
$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$



**c**  $\sqrt{2} \cos x = -1$

$$\cos x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

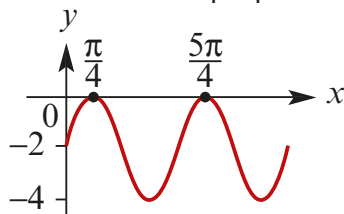


**d**  $2 \sin 2x - 2 = 0$

$$\sin 2x = 1$$

$$2x = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

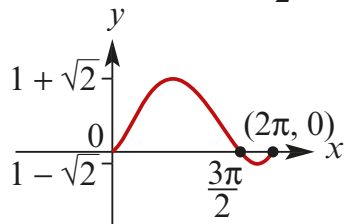


e  $\sqrt{2} \sin(x - \frac{\pi}{4}) = -1$

$$\sin(x - \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$$

$$x - \frac{\pi}{4} = \frac{5\pi}{4}, \frac{7\pi}{4}, -\frac{\pi}{4}$$

$$x = 0, \frac{3\pi}{2}, 2\pi$$



**3 a y-axis intercept**

$$y = -2$$

**x-axis intercepts**

$$2 \sin(3x) = 2$$

$$\sin(3x) = 1$$

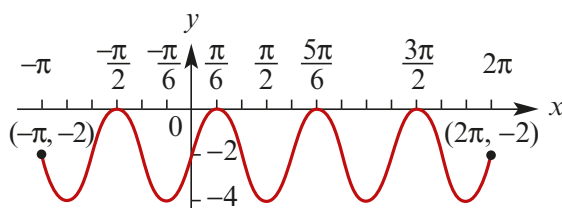
$$3x = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

$$x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

**Endpoints**

When  $x = -\pi, y = -2$

When  $x = 2\pi, y = -2$



**b y-axis intercept**

$$y = -\sqrt{2}$$

**x-axis intercepts**

$$\cos(3(x - \frac{\pi}{4})) = 0$$

$$3(x - \frac{\pi}{4}) = -\frac{7\pi}{2}, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2},$$

$$\frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$$

$$(x - \frac{\pi}{4}) = -\frac{7\pi}{6}, -\frac{5\pi}{6}, -\frac{3\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6},$$

$$\frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}$$

$$(x - \frac{3\pi}{12}) = -\frac{14\pi}{12}, -\frac{10\pi}{12}, -\frac{6\pi}{12}, -\frac{2\pi}{12}, \frac{2\pi}{12}, \frac{6\pi}{12},$$

$$\frac{10\pi}{12}, \frac{14\pi}{12}, \frac{18\pi}{12}$$

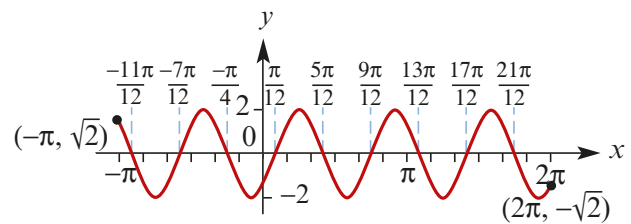
$$x = -\frac{11\pi}{12}, -\frac{7\pi}{12}, -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12},$$

$$\frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}$$

**Endpoints**

When  $x = -\pi, y = \sqrt{2}$

When  $x = 2\pi, y = -\sqrt{2}$



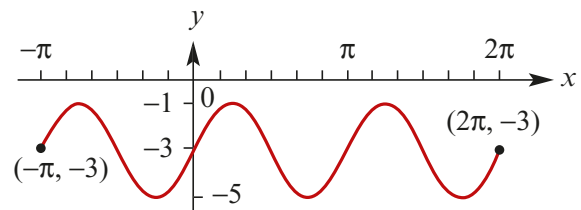
**c y-axis intercept**

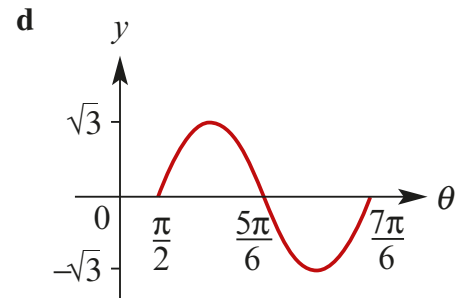
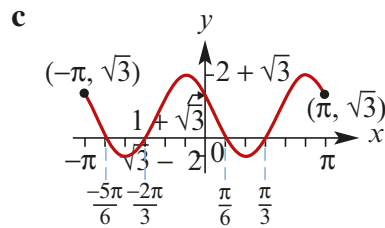
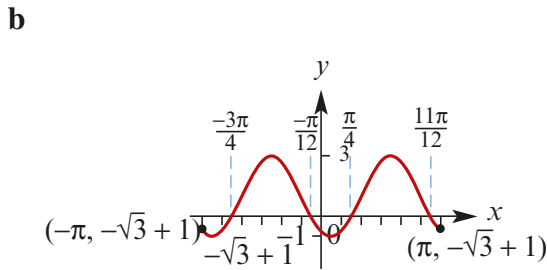
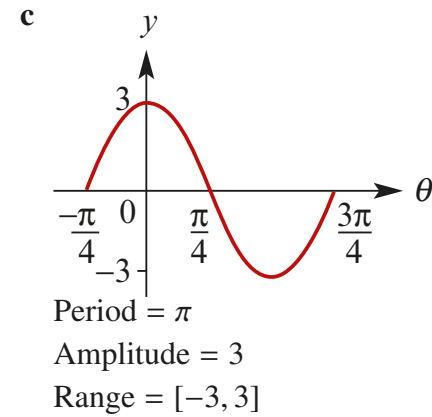
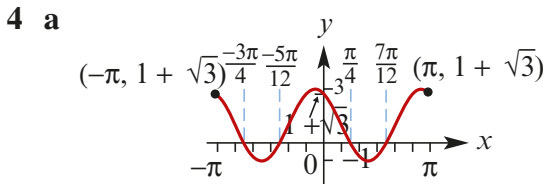
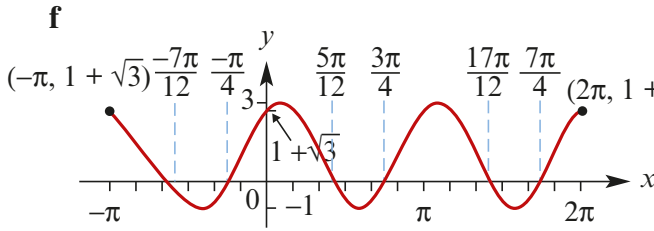
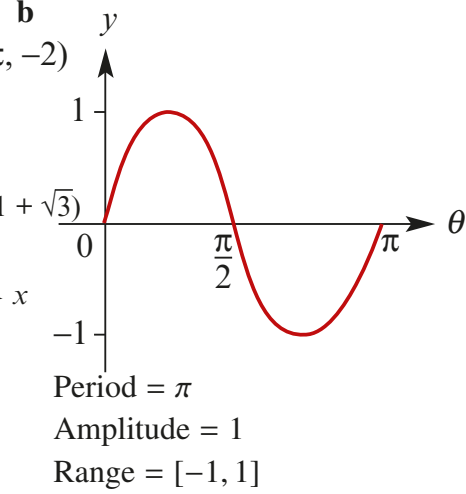
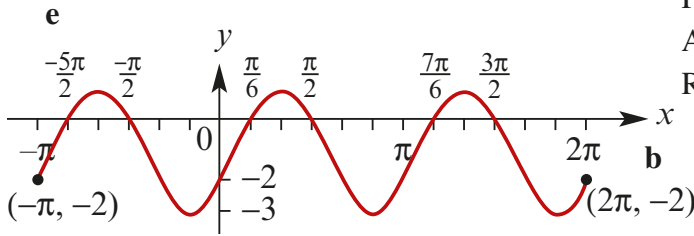
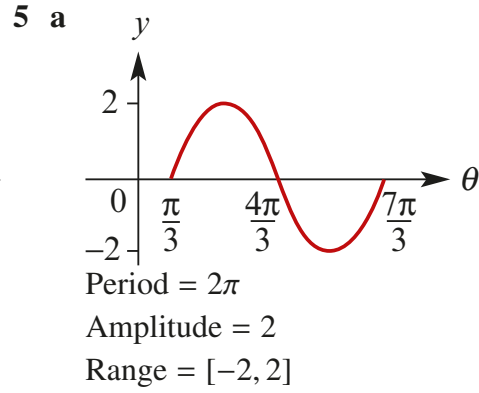
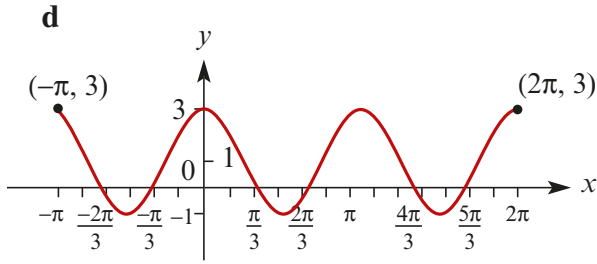
$$y = -3$$

**Endpoints**

When  $x = -\pi, y = -3$

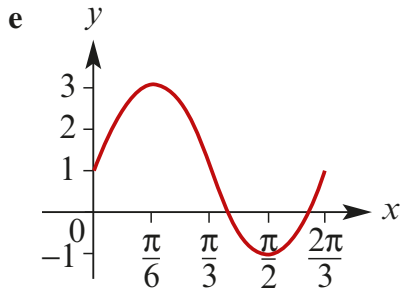
When  $x = 2\pi, y = -3$



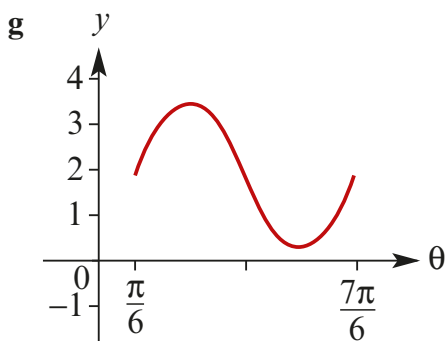
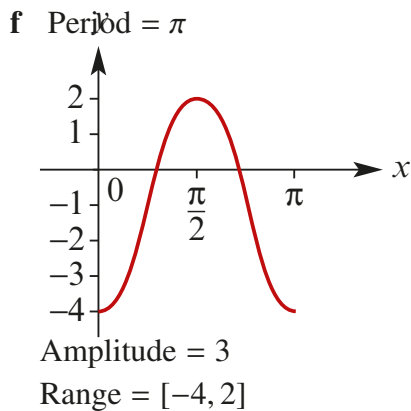




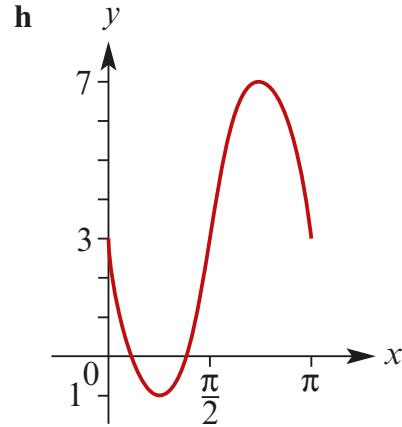
Period =  $\frac{2\pi}{3}$   
 Amplitude =  $\sqrt{3}$   
 Range =  $[-\sqrt{3}, \sqrt{3}]$



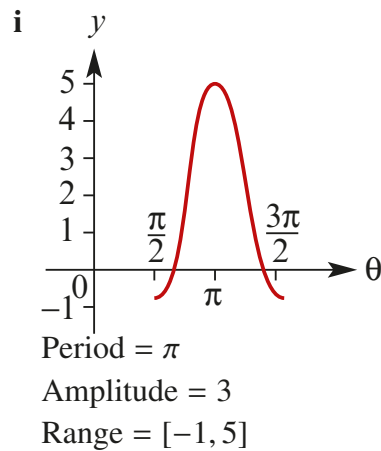
Period =  $\frac{2\pi}{3}$   
 Amplitude = 2  
 Range =  $[-1, 3]$



Period =  $\pi$   
 Amplitude =  $\sqrt{2}$   
 Range =  $[-\sqrt{2} + 2, 2 + \sqrt{2}]$



Period =  $\pi$   
 Amplitude = 4  
 Range =  $[-1, 7]$



**6 a**  $y = \frac{1}{2} \cos\left(\frac{1}{3}\left(x - \frac{\pi}{4}\right)\right)$

**b**  $y = 2 \cos\left(x - \frac{\pi}{4}\right)$

**c**  $y = -\frac{1}{3} \cos\left(x - \frac{\pi}{3}\right)$

- 7 a**
- Dilation of factor 3 from the  $x$ -axis
  - Dilation of factor  $\frac{1}{2}$  from the  $y$ -axis
  - Reflection in the  $x$ -axis
- b**
- Dilation of factor 3 from the  $x$ -axis

- Dilation of factor  $\frac{1}{2}$  from the  $y$ -axis
- Reflection in the  $x$ -axis
- Translation  $\frac{\pi}{3}$  units to the right

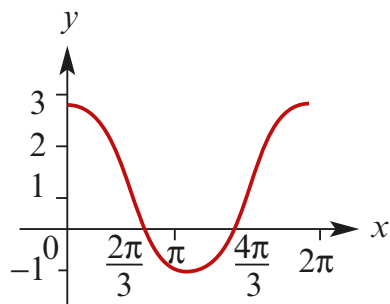
**c** ■ Dilation of factor 3 from the  $x$ -axis

- Dilation of factor  $\frac{1}{2}$  from the  $y$ -axis
- Translation  $\frac{\pi}{3}$  units to the right and 2 units up

**d** ■ Dilation of factor 2 from the  $x$ -axis

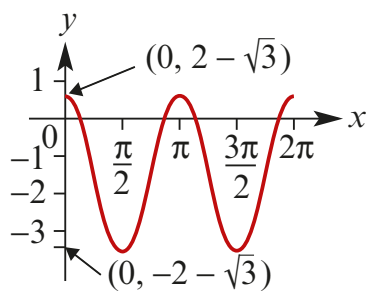
- Dilation of factor  $\frac{1}{2}$  from the  $y$ -axis
- Reflection in the  $x$ -axis
- Translation  $\frac{\pi}{3}$  units to the right and 5 units up

**8 a**



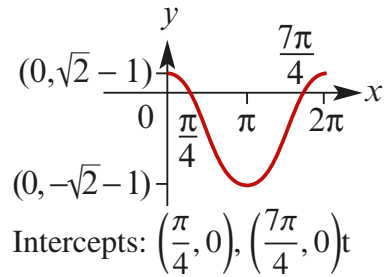
Intercepts:  $(\frac{2\pi}{3}, 0), (\frac{4\pi}{3}, 0)$

**b**



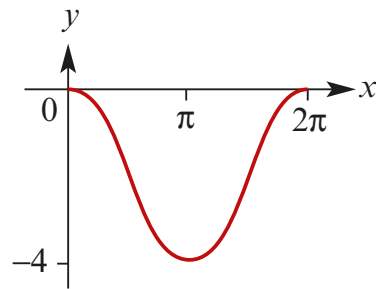
Intercepts:  $(\frac{\pi}{12}, 0), (\frac{11\pi}{12}, 0),$   
 $(\frac{13\pi}{12}, 0), (\frac{23\pi}{12}, 0)$

**c**



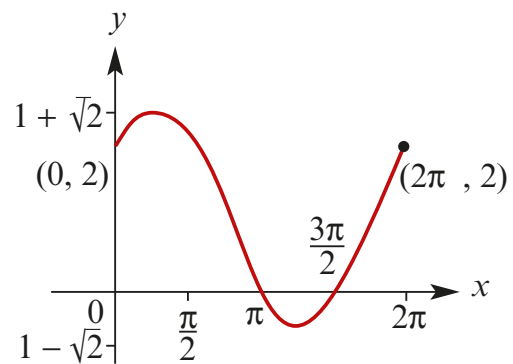
Intercepts:  $(\frac{\pi}{4}, 0), (\frac{7\pi}{4}, 0)$

**d**



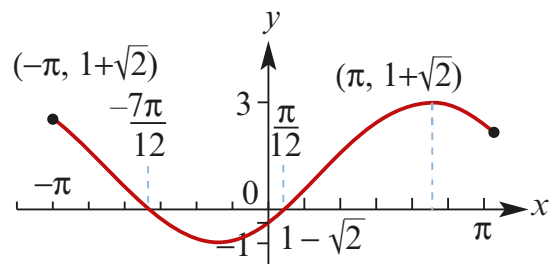
Intercepts:  $(0, 0), (2\pi, 0)$

**e**

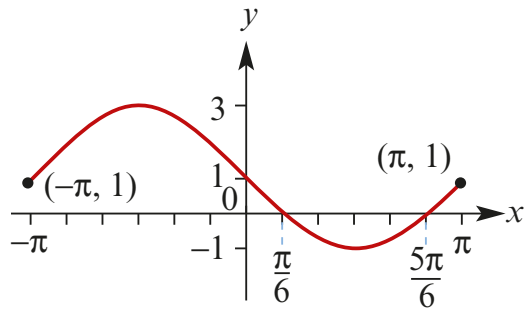


Intercepts:  $(\pi, 0), (\frac{3\pi}{2}, 0)$

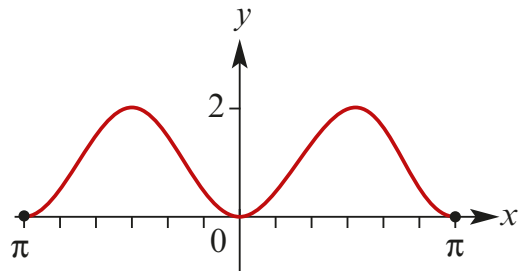
**9 a**



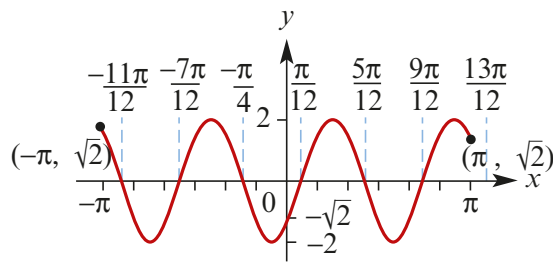
**b**



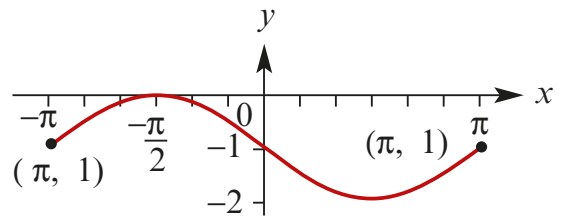
**e**



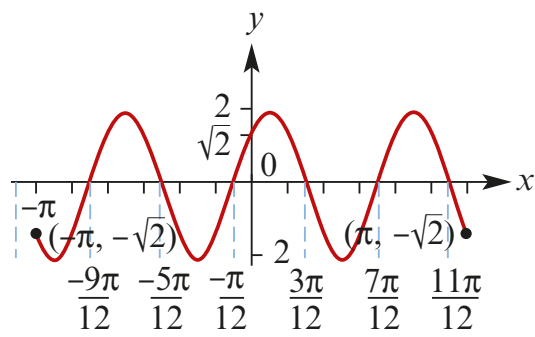
**c**



**f**

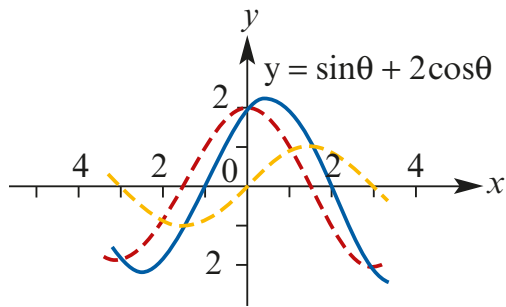


**d**

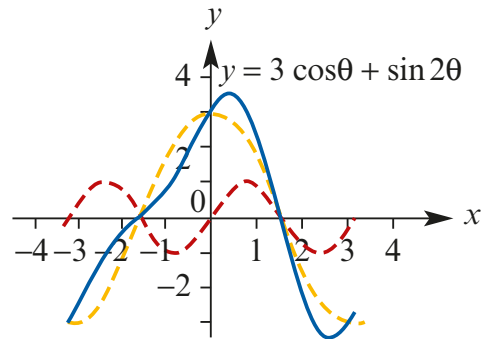


## Solutions to Exercise 6H

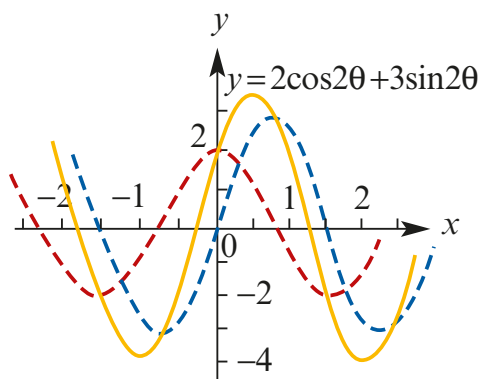
1 a



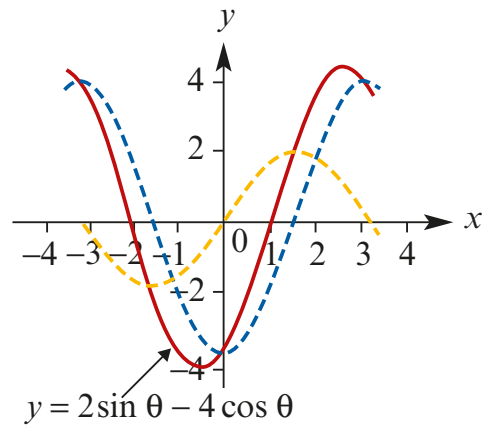
d



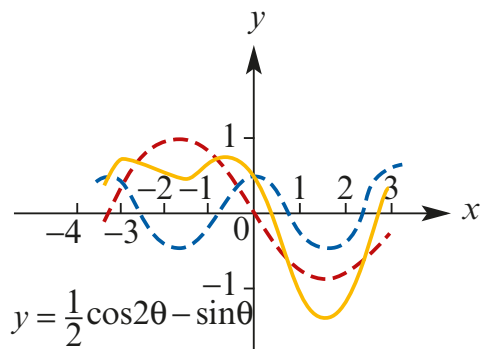
b



e



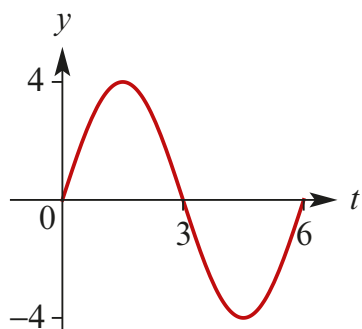
c



## Solutions to Exercise 6I

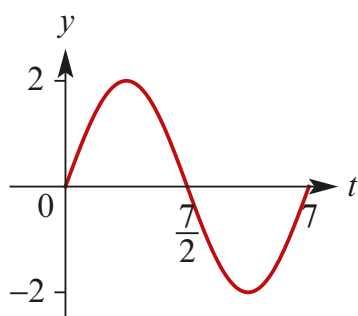
1 a  $n = \frac{\pi}{3}$

$A = 4$



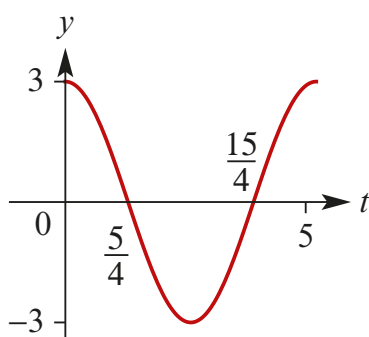
b  $n = \frac{2\pi}{7}$

$A = 2$



c  $n = \frac{2\pi}{5}$

$A = 3$



2  $A = 3, n = \frac{\pi}{4}$

3  $A = -4, n = \frac{\pi}{6}$

4  $A = 0.5, \varepsilon = \frac{-\pi}{3}$

5  $A = 3, n = 3, b = 5$

6  $A = 4$

$$\frac{2\pi}{n} = 8$$

$$n = \frac{\pi}{4}$$

$$t = 2, y = 0$$

$$0 = 4 \sin\left(\frac{\pi}{4} \times 2 + \varepsilon\right)$$

$$\sin\left(\frac{\pi}{2} + \varepsilon\right) = 0$$

$$\varepsilon = -\frac{\pi}{2} + x\pi, x \in \mathbb{Z}$$

(i.e.  $\varepsilon$  can be an infinity of no.s, separated by  $\pi$ )

7  $A = 2$

$$\frac{2\pi}{n} = 6$$

$$n = \frac{\pi}{3}$$

$$t = 1, y = 1$$

$$1 = 2 \sin\left(\frac{\pi}{3} + \varepsilon\right)$$

$$\sin\left(\frac{\pi}{3} + \varepsilon\right) = \frac{1}{2}$$

$$\frac{\pi}{3} + \varepsilon = \left(\frac{\pi}{6}, \frac{5\pi}{6}\right) + 2x\pi$$

$$\varepsilon = \left(\frac{-\pi}{6}, \frac{\pi}{2}\right) + 2x\pi, x \in \mathbb{Z}$$

$$\begin{aligned}
 8 \quad A &= \frac{6 - (-2)}{2} \\
 A &= 4 \\
 d &= 6 - A = 6 - 4 \\
 d &= 2 \\
 \frac{2\pi}{n} &= 8 \\
 n &= \frac{\pi}{4} \\
 t &= 2, y = 2 \\
 2 &= 4 \sin\left(\frac{\pi}{4} \times 2 + \varepsilon\right) + 2 \\
 \sin\left(\frac{\pi}{2} + \varepsilon\right) &= 0 \\
 \frac{\pi}{2} + \varepsilon &= 0 + x\pi \\
 \varepsilon &= -\frac{\pi}{2} + x\pi, x \in \mathbb{Z}
 \end{aligned}$$

$$\begin{aligned}
 9 \quad A &= \frac{4 - 0}{2} = 2 \\
 d &= 4 - A = 2 \\
 \frac{2\pi}{n} &= 6 \\
 n &= \frac{\pi}{3} \\
 t &= 1, y = 3 \\
 3 &= 2 \sin\left(\frac{\pi}{3} \times 1 + \varepsilon\right) + 2 \\
 \sin\left(\frac{\pi}{3} + \varepsilon\right) &= \frac{1}{2} \\
 \frac{\pi}{3} + \varepsilon &= \left(\frac{\pi}{6}, \frac{7\pi}{6}\right) + 2x\pi \\
 \varepsilon &= \left(\frac{-\pi}{6}, \frac{5\pi}{6}\right) + 2x\pi, x \in \mathbb{Z} \\
 \text{note: for } Q1, Q2, Q5, Q6, Q7 \text{ \& } Q8 \\
 \text{A could take the negative of the value} \\
 \text{given if } \varepsilon \text{ is changed to } \varepsilon + \pi
 \end{aligned}$$

## Solutions to Exercise 6J

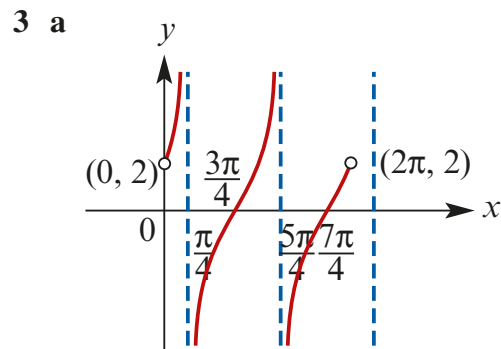
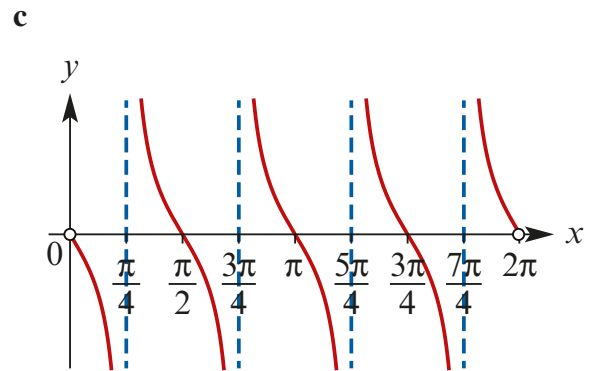
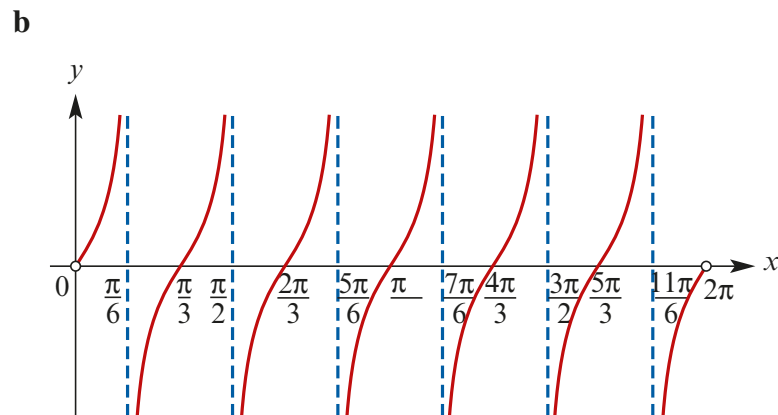
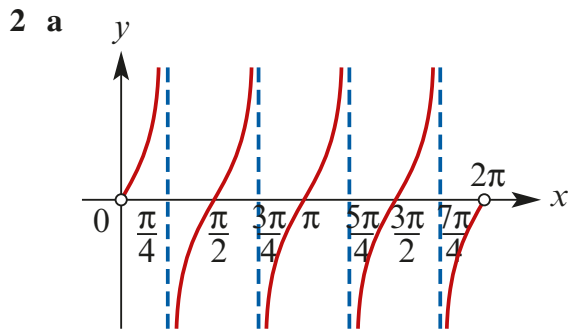
1 a  $T = \frac{\pi}{3}$

b  $T = \frac{\pi}{\frac{1}{2}} = 2\pi$

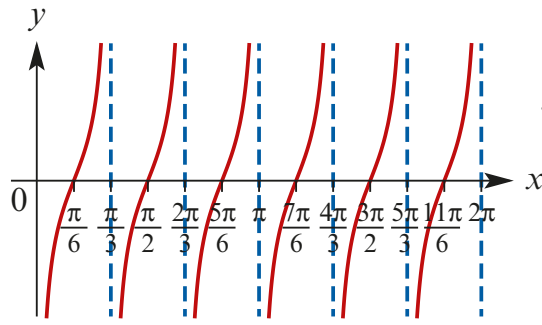
c  $T = \frac{\pi}{\frac{3}{2}} = \frac{2\pi}{3}$

d  $T = \frac{\pi}{\pi} = 1$

e  $T = \frac{\pi}{\frac{\pi}{2}} = 2$



**b**



**e**  $-\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

**5**  $\tan(2(x - \frac{\pi}{3})) = 1$

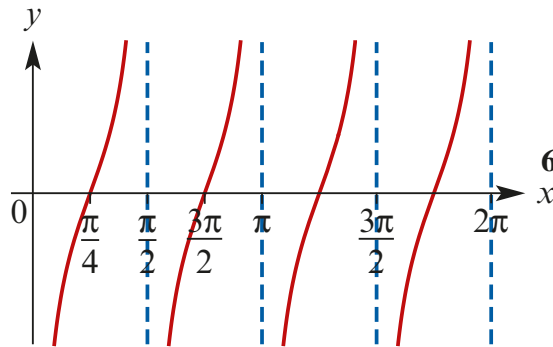
$$(2(x - \frac{\pi}{3})) = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$x - \frac{\pi}{3} = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$$

$$x - \frac{8\pi}{24} = \frac{3\pi}{24}, \frac{15\pi}{24}, \frac{27\pi}{24}, \frac{39\pi}{24}$$

$$x = \frac{11\pi}{24}, \frac{23\pi}{24}, \frac{35\pi}{24}, \frac{47\pi}{24}$$

**c**



**6**  $\tan((x - \frac{\pi}{4})) = \sqrt{3}$

$$(x - \frac{\pi}{4}) = \frac{\pi}{3}, \frac{4\pi}{3}$$

$$x - \frac{3\pi}{12} = \frac{4\pi}{12}, \frac{16\pi}{12}$$

$$x = \frac{7\pi}{12}, \frac{19\pi}{12}$$

**4 a**  $\tan(2x) = 1$

$$2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

**b**  $\tan(2x) = -1$

$$2x = -\frac{\pi}{4}, -\frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$x = -\frac{\pi}{8}, -\frac{5\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$$

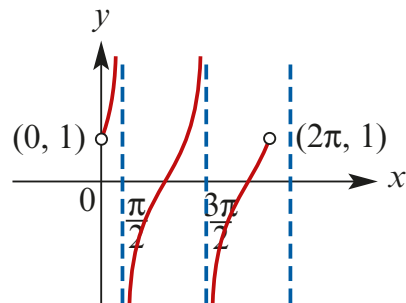
**c**  $\tan(2x) = -\sqrt{3}$

$$2x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$$

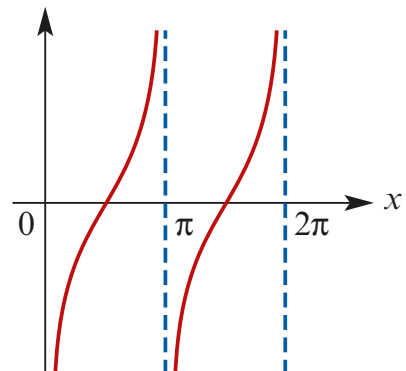
$$x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$$

**d**  $-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

**7 a**

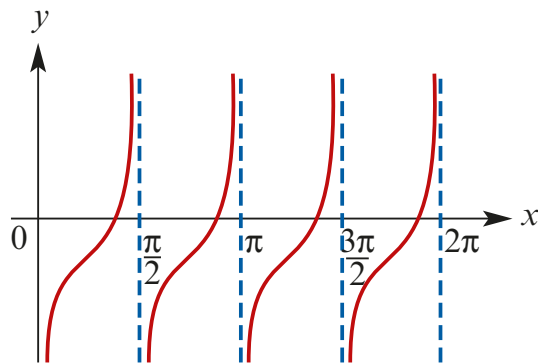


**b**

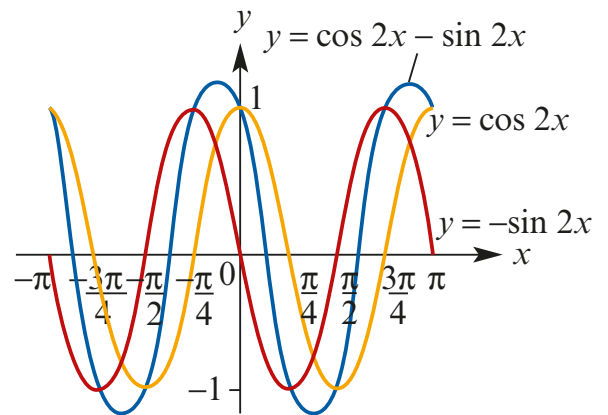




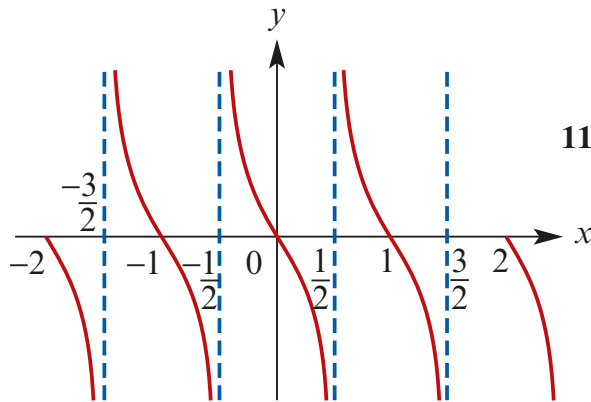
c



10 a c

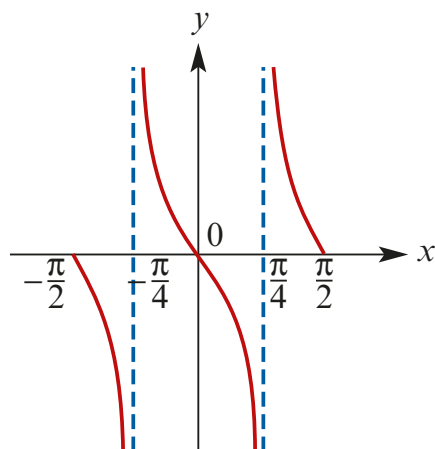


8



b  $\left(\frac{-5\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-\pi}{8}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{7\pi}{8}, \frac{1}{\sqrt{2}}\right)$

9



11 a  $\sqrt{3} \sin x = \cos x$

$\sqrt{3} \tan x = 1$

$\tan x = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{6}, \frac{7\pi}{6}$

b  $\sin(4x) = \cos(4x)$

$\tan(4x) = 1$

$4x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}, \frac{25\pi}{4}, \frac{29\pi}{4}$

$x = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$

c  $\sqrt{3} \sin(2x) = \cos(2x)$

$\tan(2x) = \frac{1}{\sqrt{3}}$

$2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$

**d**  $\tan(2x) = \frac{-1}{\sqrt{3}}$   
 $2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$   
 $x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

**e**  $\sin(3x) = -\cos(3x)$   
 $\tan(3x) = -1$

$3x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$  **12a,c**  
 $x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}$

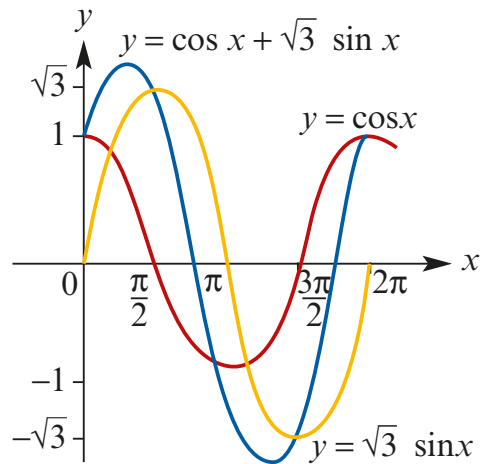
**f**  $\tan x = \frac{1}{2}$   
 using the CAS calculator  $x = 0.4636,$   
 $3.6052$

**g**  $\tan x = 2$   
 using the CAS calculator  $x = 1.1071,$   
 $4.2487$

**h**  $\tan(2x) = -1$   
 $2x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$   
 $x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

**i**  $\sqrt{3} \sin(3x) = \cos(3x)$   
 $\tan(3x) = \frac{1}{\sqrt{3}}$   
 $3x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6},$   
 $\frac{19\pi}{6}, \frac{25\pi}{6}, \frac{31\pi}{6}$   
 $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18},$   
 $\frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$

**j**  $\tan(3x) = \sqrt{3}$   
 $3x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3},$   
 $\frac{10\pi}{3}, \frac{13\pi}{3}, \frac{16\pi}{3}$   
 $x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9},$   
 $\frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$



**b**  $(\frac{\pi}{6}, -\frac{\sqrt{3}}{2}), (\frac{7\pi}{6}, -\frac{\sqrt{3}}{2})$

**13 a**  $\tan(2x - \frac{\pi}{4}) = \sqrt{3}$   
 $2x - \frac{\pi}{4} = \dots \frac{-2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \dots$   
 $2x = \dots \frac{-5\pi}{12}, \frac{7\pi}{12}, \frac{19\pi}{12}, \frac{31\pi}{12}, \frac{43\pi}{12}, \dots$   
 $x = \dots \frac{-5\pi}{24}, \frac{7\pi}{24}, \frac{19\pi}{24}, \frac{31\pi}{24}, \frac{43\pi}{24}, \dots$   
 but  $0 \leq x \leq 2\pi$   
 $\therefore x = \frac{7\pi}{24}, \frac{19\pi}{24}, \frac{31\pi}{24}, \frac{43\pi}{24}$

**b**  $\tan(2x) = \frac{-1}{\sqrt{3}}$   
 $2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$   
 $x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

$$\mathbf{c} \quad \tan\left(3x - \frac{\pi}{6}\right) = -1$$

$$3x - \frac{\pi}{6} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4},$$

$$\frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$$

$$3x = \frac{11\pi}{12}, \frac{23\pi}{12}, \frac{35\pi}{12}, \frac{47\pi}{12},$$

$$\frac{59\pi}{12}, \frac{71\pi}{12}$$

$$x = \frac{11\pi}{36}, \frac{23\pi}{36}, \frac{35\pi}{36},$$

$$\frac{47\pi}{36}, \frac{59\pi}{36}, \frac{71\pi}{36}$$

**14** asymptotes at  $t = (2k + 1)\frac{\pi}{6}$

$$\text{period} = T = \frac{\pi}{3}$$

$$T = \frac{\pi}{n},$$

$$\therefore n = 3$$

$$t = \frac{\pi}{12}, y = 5$$

$$5 = A \tan\left(\frac{\pi}{4}\right) = A \times 1$$

$$A = 5$$

$$\mathbf{15} \quad T = \frac{\pi}{n}$$

$$T = 2$$

$$\therefore n = \frac{\pi}{2}$$

$$t = \frac{1}{2}, y = 6$$

$$6 = A \tan\left(\frac{\pi}{4}\right)$$

$$A = 6$$

## Solutions to Exercise 6K

1 a  $\cos^{-1}(1) = 0$

i  $2\pi \pm 0 = 2\pi$

ii  $4\pi \pm 0 = 4\pi$

iii  $-4\pi \pm 0 = -4\pi$

b  $\cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$

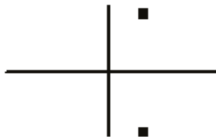
i  $2\pi \pm \frac{2\pi}{3} = \frac{4\pi}{3}, \frac{8\pi}{3}$

ii  $4\pi \pm \frac{2\pi}{3} = \frac{10\pi}{3}, \frac{14\pi}{3}$

iii  $-4\pi \pm \frac{2\pi}{3} = \frac{-14\pi}{3}, \frac{-10\pi}{3}$

2 a  $\cos x = \frac{\sqrt{3}}{2}$

$$x = \left(\frac{\pi}{6}, \frac{-\pi}{6}\right) + 2n\pi \quad n \in \mathbb{Z}$$

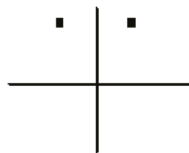


b  $2 \sin 3x = \sqrt{3}$

$$\sin 3x = \frac{\sqrt{3}}{2}$$

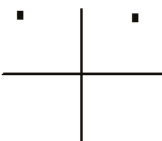
$$3x = \left(\frac{\pi}{3}, \frac{2\pi}{3}\right) + 2n\pi \quad n \in \mathbb{Z}$$

$$x = \left(\frac{\pi}{9}, \frac{2\pi}{9}\right) + \frac{2n\pi}{3} \quad n \in \mathbb{Z}$$

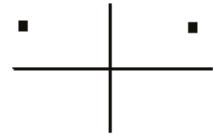


c  $\tan x = \sqrt{3}$

$$x = \frac{\pi}{3} + n\pi, \quad n \in \mathbb{Z}$$



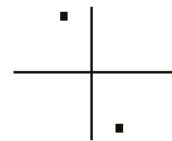
3 a  $\sin x = \frac{1}{2}$   
 $x = \frac{\pi}{6}, \frac{5\pi}{6}$



b  $\cos 2x = \frac{\sqrt{3}}{2}$   
 $2x = \frac{\pi}{6}, \frac{11\pi}{6}$   
 $x = \frac{\pi}{12}, \frac{11\pi}{12}$



c  $\tan 2x = -\sqrt{3}$   
 $2x = \frac{2\pi}{3}, \frac{5\pi}{3}$   
 $x = \frac{\pi}{3}, \frac{5\pi}{6}$



4  $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$   
 $= n\pi + (-1)^n \left(\frac{\pi}{6}\right)$

$$n = -2, x = -2\pi + \frac{\pi}{6} = -\frac{11\pi}{6}$$

$$n = -1, x = -\pi - \frac{\pi}{6} = -\frac{7\pi}{6}$$

$$n = 0, x = \frac{\pi}{6}$$

$$n = 1, x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

...

$$x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$5 \quad x = 2n\pi \pm (\pi = 2x\pi \pm) \cos^{-1}\left(\frac{1}{2}\right)$$

$$= 2n\pi \pm \frac{\pi}{3}$$

$$n = 0, 1 : x = \frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$$

$$6 \quad a \quad 2\left(x + \frac{\pi}{3}\right) = \pm \frac{\pi}{3} + 2n\pi, \quad n \in \mathbb{Z}$$

$$x + \frac{\pi}{3} = n\pi \pm \frac{\pi}{6}$$

$$x = \left(\frac{-\pi}{2}, \frac{-\pi}{6}\right) + n\pi$$

$$b \quad \tan\left(2\left(x + \frac{\pi}{4}\right)\right) = \sqrt{3}$$

$$2\left(x + \frac{\pi}{4}\right) = \frac{\pi}{3} + n\pi, n \in \mathbb{Z}$$

$$x + \frac{\pi}{4} = \frac{\pi}{6} + \frac{n\pi}{2}$$

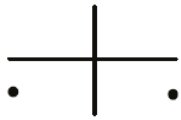
$$x = \frac{-\pi}{12} + \frac{n\pi}{2}$$

$$c \quad \sin\left(x + \frac{\pi}{3}\right) = \frac{-1}{2}$$

$$x + \frac{\pi}{3} = \frac{7\pi}{6}, \frac{11\pi}{6} + 2n\pi, n \in \mathbb{Z}$$

$$x = \left(\frac{5\pi}{6}, \frac{9\pi}{6}\right) + 2n\pi$$

$$= \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right) + 2x\pi$$



$$7 \quad \cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$2x + \frac{\pi}{4} = \pm \frac{\pi}{4} + 2n\pi, n \in \mathbb{Z}$$

$$2x = \left(0, -\frac{\pi}{2}\right) + 2n\pi$$

$$x = \left(0, -\frac{\pi}{4}\right) + n\pi$$

$$x = -2\pi + \frac{3\pi}{4}, -\pi, -\pi + \frac{3\pi}{4},$$

$$0, \frac{3\pi}{4}, \pi, \pi + \frac{3\pi}{4}$$

$$x = \frac{-5\pi}{4}, -\pi, \frac{-\pi}{4}, 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$$

$$8 \quad \tan\left(\frac{\pi}{6} - 3x\right) = \frac{1}{\sqrt{3}}$$

$$\frac{\pi}{6} - 3x = \frac{\pi}{6} + n\pi \quad n \in \mathbb{Z}$$

$$3x - \frac{\pi}{6} = n\pi - \frac{\pi}{6} \quad n \in \mathbb{Z}$$

The  $-\pi$  becomes part of  $n$

$$3x = n\pi$$

$$x = \frac{n\pi}{3} \quad n \in \mathbb{Z}$$

$$x = -\pi, \frac{-2\pi}{3}, \frac{-\pi}{3}, 0$$

9

$$\sin(4\pi x) = \frac{-\sqrt{3}}{2}$$

$$4\pi x = \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right) + 2n\pi \quad n \in \mathbb{Z}$$

$$x = \left(\frac{1}{3}, \frac{5}{12}\right) + \frac{n}{2} \quad n \in \mathbb{Z}$$

$$x = -1 + \frac{1}{3}, -1 + \frac{5}{12}, \frac{-1}{2} + \frac{1}{3},$$

$$\frac{-1}{2} + \frac{5}{12}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2} + \frac{1}{3}, \frac{1}{2} + \frac{5}{12}$$

$$x = \frac{-2}{3}, \frac{-7}{12}, \frac{-1}{6}, \frac{-1}{12},$$

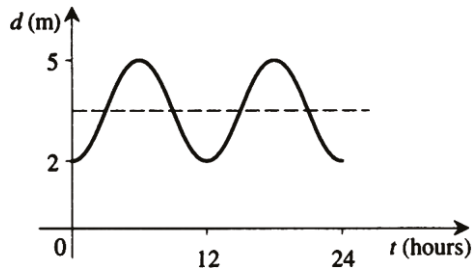
$$\frac{1}{3}, \frac{5}{12}, \frac{5}{6}, \frac{11}{12}$$

## Solutions to Exercise 6L

1 a From the graph

- i The range of the function is  $[2, 5]$  and the amplitude is 1.5.
- ii The period is 12.

iii The function is of the form  
 $d = a \sin(nt + \varepsilon) + b$



The amplitude is 1.5.

Therefore  $a = 1.5$

The period is 12.

Therefore  $\frac{2\pi}{n} = 12$  and  $n = \frac{\pi}{6}$

The centre of motion is at  $d = 3\frac{1}{2}$ .

Therefore  $b = 3\frac{1}{2}$

$$\therefore d = 1.5 \sin\left(\frac{\pi t}{6} + \varepsilon\right) + 3.5$$

When  $t = 0, d = 2$

$$\therefore 2 = 1.5 \sin(\varepsilon) + 3.5$$

$$-1.5 = 1.5 \sin(\varepsilon)$$

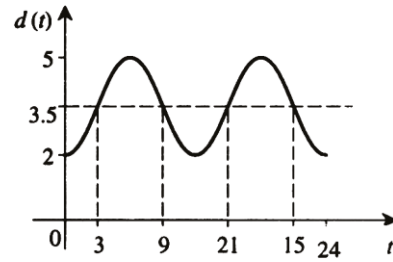
$$\sin(\varepsilon) = -1 \text{ and } \varepsilon = \frac{3\pi}{2}$$

$$\therefore d = 1.5 \sin\left(\frac{\pi t}{6} + \frac{3\pi}{2}\right) + 3.5$$

$$\text{But } \sin\left(\theta + \frac{3\pi}{2}\right) = \cos \theta$$

$$\therefore d = 3.5 - 1.5 \cos\left(\frac{\pi t}{6}\right)$$

- iv The length of the hour hand is 1.5 m. This is given by the amplitude.



b When is  $d(t) \leq 3.5$ ?

Consider  $d(t) = 3.5$

$$3.5 = 3.5 - 1.5 \cos\left(\frac{\pi t}{6}\right)$$

$$\therefore \cos\left(\frac{\pi t}{6}\right) = 0$$

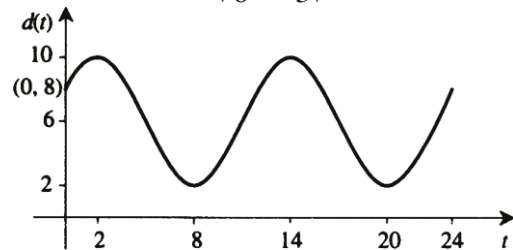
$$\text{And } \frac{\pi t}{6} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{7\pi}{2} \text{ or } \dots$$

$$\therefore t = 3 \text{ or } 9 \text{ or } 15 \text{ or } 21 \text{ or } \dots$$

$\therefore$  From the graph

$$d(t) < 3.5 \text{ for } t \in [0, 3) \cup (9, 15) \cup (21, 24]$$

2 a  $d(t) = 6 + 4 \cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$



Centre:  $d = 6$

Range:  $[6 - 4, 6 + 4] = [2, 10]$

Period:  $2\pi \div \frac{\pi}{6} = 12$

When  $t = 0, d(0) = 6 + 4 \cos\left(\frac{-\pi}{3}\right)$

$$= 6 + 4 \times \frac{1}{2} = 8$$

When  $t = 24,$

$$d(24) = 6 + 4 \cos\left(4\pi - \frac{\pi}{3}\right)$$

$$= 6 + 4 \times \frac{1}{2} = 8$$

b Highest level is 10 m.

Consider  $10 = 6 + 4 \cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$

$$1 = \cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$$

$\therefore \frac{\pi t}{6} = \frac{\pi}{3}$  (No need to consider other solution as question asks for earliest time.)

$$\therefore t = 2$$

The water is first at its highest at 2:00 a.m.

**c**

When  $d(t) = 2$

$$2 = 6 + 4 \cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$$

$$-1 = \cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$$

$$\therefore \frac{\pi t}{6} = \frac{\pi}{3} = \pi \text{ or } 3\pi \text{ or } 5\pi \text{ or } \dots$$

$$\therefore \frac{\pi t}{6} = \frac{4\pi}{3} \text{ or } \frac{10\pi}{3} \text{ or } \frac{16\pi}{3} \text{ or } \dots$$

$$\therefore t = 8 \text{ or } 20 \text{ or } 32 \text{ or } \dots$$

Only 8 and 20 are in the required domain.

$\therefore$  The water is 2 m up the wall at 8:00 a.m. and 8:00 p.m.

**3 a** The time between high tides is 12 hours, so the period = 12

$$\frac{2\pi}{n} = 12 \Rightarrow n = \frac{\pi}{6}$$

The average depth is 5 metres.

Therefore  $b = 5$ .

The high tide is 8 m. Therefore amplitude =  $8 - 5 = 3$  and  $A = 3$

$$\therefore h(t) = 3 \sin\left(\frac{\pi t}{6} + \epsilon\right) + 5$$

When  $t = 0$ ,  $h = 8$  ( $t$  is the number of hours after 12:00 noon.)

$$\therefore 8 = 3 \sin(\epsilon) + 5$$

$$\therefore \sin(\epsilon) = 1 \text{ and } \epsilon = \frac{\pi}{2}$$

$$\therefore h(t) = 3 \sin\left(\frac{\pi t}{6} + \frac{\pi}{2}\right) + 5$$

$$= 3 \cos\left(\frac{\pi t}{6}\right) + 5$$

**b** When  $h = 6$

$$1 = 3 \cos\left(\frac{\pi t}{6}\right)$$

$$\frac{1}{3} = \cos\left(\frac{\pi t}{6}\right)$$

$$\therefore \frac{\pi t}{6} = \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 2\pi - \cos^{-1}\left(\frac{1}{3}\right)$$

$$\times \left(\frac{1}{3}\right) \text{ or } 2\pi + \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 4\pi - \cos^{-1}\left(\frac{1}{3}\right)$$

$$\therefore t = \frac{6}{\pi} \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 12 - \frac{6}{\pi} \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 12$$

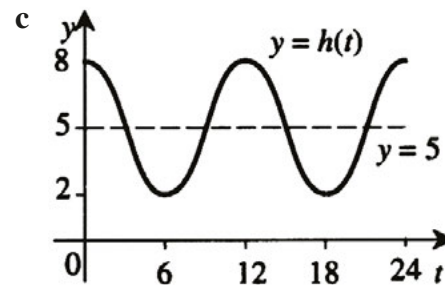
$$+ \frac{6}{\pi} \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 24 - \frac{6}{\pi} \cos^{-1}\left(\frac{1}{3}\right)$$

$$\approx 2.351 \text{ or } 9.649 \text{ or } 14.351 \text{ or } 21.649$$

Depth of the water is 6 metres at the following times (times measured from 12 noon).

2:21 p.m. 9:39 p.m. 2:21 a.m.

9:39 a.m.



**4 a** Greatest distance occurs when  $\sin 3t = 1$

$$\therefore \text{greatest distance} = 3 + 2 = 5\text{m.}$$

**b** Least distance occurs when

$$\sin 3t = -1$$

$$\therefore \text{least distance} = 3 - 2 = 1\text{m.}$$

**c** When  $x = 5$

$$5 = 3 + 2 \sin 3t$$

$$2 = 2 \sin 3t$$

$$1 = \sin 3t$$

$$3t = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \dots$$

$$\therefore t = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{9\pi}{6} \text{ or } \dots$$

For  $0 \leq t \leq 5$  the times are: 0.524 sec,  
2.618 sec, 4.712 sec

**d** When  $x = 3$

$$3 = 3 + 2 \sin 3t$$

$$0 = \sin 3t$$

$$3t = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } \dots$$

$$t = 0 \text{ or } \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ or } \pi \text{ or } \dots$$

For  $0 \leq t \leq 3$  the times are: 0 sec,  
1.047 sec, 2.094 sec

**e** The particle oscillates about the point  
 $x = 3$  from  $x = 1$  to  $x = 5$ .

**5**  $A = 21 - 3 \cos\left(\frac{\pi t}{12}\right)$  for  $0 \leq t \leq 24$  gives  
the temperature inside the house and  
 $B = 22 - 5 \cos\left(\frac{\pi t}{12}\right)$  for  $0 \leq t \leq 24$  gives  
the temperature outside the house.

**a** When  $t = 4$  (time measured from  
4:00 a.m.)

$$A = 21 - 3 \cos \frac{4\pi}{12}$$

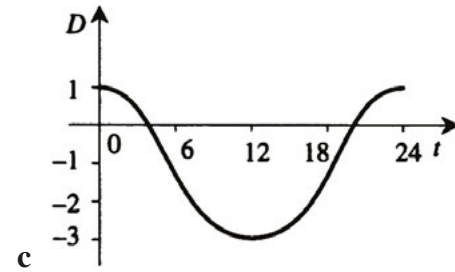
$$= 21 - 3 \cos \frac{\pi}{3}$$

$$= 21 - 1.5$$

$$= 19.5$$

i.e. the temperature outside the house  
is  $19.5^\circ\text{C}$  at 8:00 a.m.

**b**  $D = A - B = 2 \cos\left(\frac{\pi t}{12}\right) - 1$



**d** The inside temperature is less than  
the outside temperature.

This occurs when

$$A < B$$

$$\Leftrightarrow A - B < 0$$

$$\Leftrightarrow D < 0$$

Consider  $D = 0$

$$0 = 2 \cos\left(\frac{\pi t}{12}\right) - 1$$

Implies  $\frac{1}{2} = \cos\left(\frac{\pi t}{12}\right)$

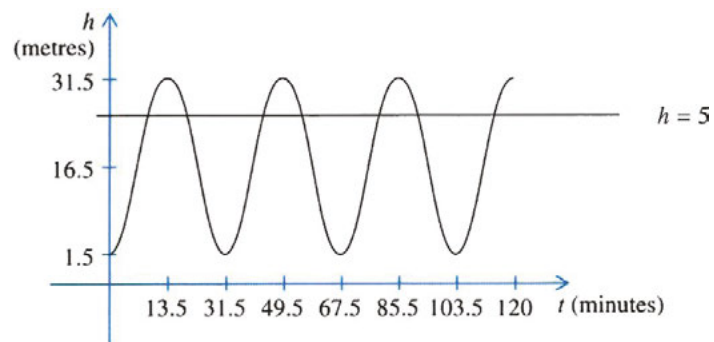
$$\therefore \frac{\pi t}{12} = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \text{ or } \dots$$

$$\therefore t = 4 \text{ or } 20 \text{ or } \dots$$

For  $0 \leq t \leq 24$ ,  $D < 0$  for  $t \in (4, 20)$ ,

i.e.  $4 < t < 20$ .

**6 a**





**b** When  $t = 0$ ,  $h = 15 \sin(-45)^\circ + 16.5$   
 $= 5.89$  m (correct to two decimal places)

**c** Solving the equation  $h(t) = 5$   
 $5 = 15 \sin(10t - 45)^\circ + 16.5$   
 $-11.5 = 15 \sin(10t - 45)^\circ$

$-\frac{23}{30} = \sin(10t - 45)^\circ$   
The first positive solution is  $t = 27.51$   
seconds correct to two decimal places.

**d** There are 6 points of intersection  
with the graph of  $h = 5$

**e** 20 times

**f**  
 $t = 100$ ,  $h(100) = 15 \sin(100 - 45) + 16.5$   
 $= 15 \sin 955^\circ + 16.5$   
 $\approx 4.21$  metres

**g** The phase shift will be different  
for Hamish; the range and the  
period will be the same. Consider  
 $k(t) = 15 \sin(10t + c)^\circ + 16.5$

When  $t = 0$ ,  $k(0) = 1.5$

$$15 \sin(+c)^\circ + 16.5 = 1.5$$

$$\sin c^\circ = -1$$

$$c^\circ = 270^\circ$$

$$k(t) = 15 \sin(10t + 270)^\circ + 16.5$$

$$t = 100,$$

$$k(100) = 15 \sin(1270^\circ) + 16.5$$

$$\approx 13.9 \text{ metres}$$

## Solutions to technology-free questions

1 Note that  $x \in [-\pi, 2\pi]$  throughout.

**a**  $\sin x = \frac{1}{2}$

$x$  is in the first or second quadrant.

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

**b**  $2 \cos x = -1$ , so  $\cos x = -\frac{1}{2}$

$x$  is in the second or third quadrant.

$$x = -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$$

**c**  $2 \cos x = \sqrt{3}$ , so  $\cos x = \frac{\sqrt{3}}{2}$

$x$  is in the first or fourth quadrant.

$$x = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}$$

**d**  $\sqrt{2} \sin x + 1 = 0$ , so  $\sin x = -\frac{1}{\sqrt{2}}$

$x$  is in the third and fourth quadrants.

$$x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

**e**  $4 \sin x + 2 = 0$ , so  $\sin x = -\frac{1}{2}$

$x$  is in the third or fourth quadrant.

$$x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

**f**  $\sin 2x + 1 = 0$ , so  $\sin 2x = -1$

$$2x = -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$$

$$x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

**g**  $\cos 2x = -\frac{1}{\sqrt{2}}$

$2x$  is in the second or third quadrant.

$$2x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$$

$$x = -\frac{5\pi}{8}, -\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$$

**h**  $2 \sin 3x - 1 = 0$ , so  $\sin 3x = \frac{1}{2}$

$3x$  is in the first or second quadrant.

$$3x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

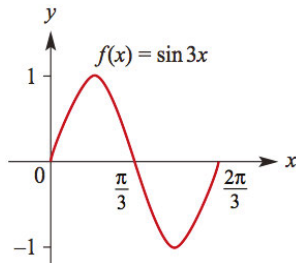
$$\frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$$

$$x = -\frac{11\pi}{18}, -\frac{7\pi}{18}, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}$$

$$\frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$$

**2 a**  $y = \sin 3x$

$y = 0: x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$  for one cycle



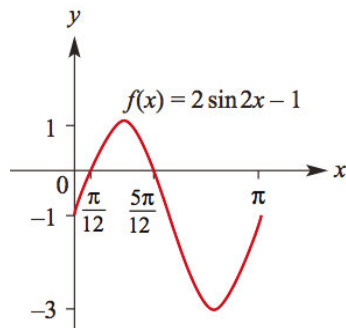
**b**  $y = 2 \sin 2x - 1$

$y = 0: \sin 2x = \frac{1}{2}$

$2x = \frac{\pi}{6}, \frac{5\pi}{6}$  for one cycle

$x = \frac{\pi}{12}, \frac{5\pi}{12}$

$x = 0: y = -1$



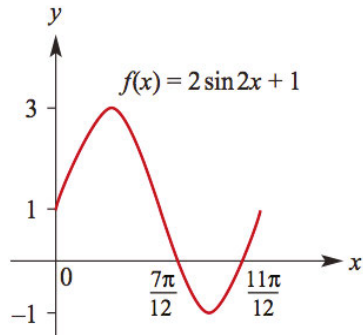
**c**  $y = 2 \sin 2x + 1$

$$y = 0: \sin 2x = -\frac{1}{2}$$

$$2x = \frac{7\pi}{6}, \frac{11\pi}{6} \text{ for one cycle}$$

$$x = \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$x = 0: y = 1$$

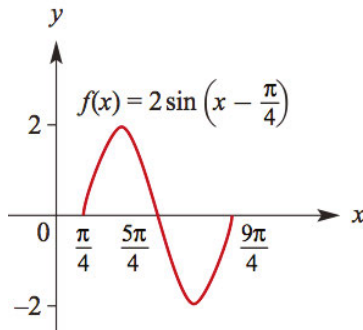


**d**  $y = 2 \sin\left(x - \frac{\pi}{4}\right)$  so translate the graph of  $y = 2 \sin x$  by  $\frac{\pi}{4}$  to the right.

$$y = 0: \sin\left(x - \frac{\pi}{4}\right) = 0$$

$$x - \frac{\pi}{4} = 0, \pi, 2\pi \text{ for one cycle}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$$

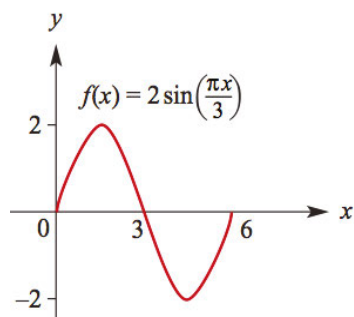


**e**  $y = 2 \sin \frac{\pi x}{3}$

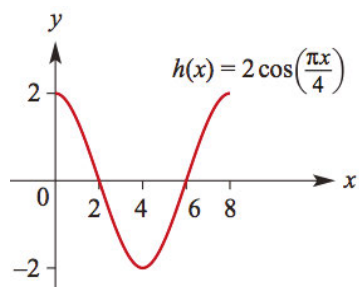
$$y = 0: \sin \frac{\pi x}{3} = 0$$

$$\frac{\pi x}{3} = 0, \pi, 2\pi \text{ for one cycle}$$

$$x = 0, 3, 6$$



**f**  $y = 2 \cos \frac{\pi x}{4}$   
 $y = 0: \cos \frac{\pi x}{4} = 0$   
 $\frac{\pi x}{4} = \frac{\pi}{2}, \frac{3\pi}{2}$  for one cycle  
 $x = 2, 6$   
 $x = 0: y = 2$



**3** Note that  $x \in [0, 360]$  throughout.

**a**  $\sin x^\circ = 0.5$   
 $x^\circ$  is in the first or second quadrant.  
 $x = 30, 150$

**b**  $\cos(2x)^\circ = 0$   
 $2x = 90, 270, 450, 630$   
 $x = 45, 135, 225, 315$

**c**  $2 \sin x^\circ = -\sqrt{3}$ , so  $\sin x^\circ = -\frac{\sqrt{3}}{2}$   
 $x^\circ$  is in the third or fourth quadrant.  
 $x = 240, 300$

**d**  $\sin(2x + 60)^\circ = -\frac{\sqrt{3}}{2}$   
 $(2x + 60)^\circ$  is in the third or fourth quadrant.

$$2x + 60 = 240, 300, 600, 660$$

$$2x = 180, 240, 540, 600$$

$$x = 90, 120, 270, 300$$

**e**  $2 \sin\left(\frac{1}{2}x\right)^\circ = \sqrt{3}$ , so  $\sin\left(\frac{1}{2}x\right)^\circ = \frac{\sqrt{3}}{2}$   
 $\left(\frac{1}{2}x\right)^\circ$  is in the first or second quadrant.  
 $\frac{1}{2}x = 60, 120$

$$x = 120, 240$$

**4 a**  $y = 2 \sin\left(x + \frac{\pi}{3}\right) + 2$

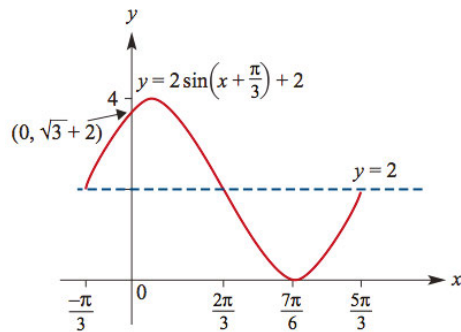
The graph is that of  $y = 2 \sin x$  translated  $\frac{\pi}{3}$  units to the left and 2 units up.

$$y = 0: \sin\left(x + \frac{\pi}{3}\right) = -1$$

$$x + \frac{\pi}{3} = \frac{3\pi}{2} \text{ for one cycle}$$

$$x = \frac{7\pi}{6}$$

$$x = 0: y = \sqrt{3} + 2$$



**b**  $y = -2 \sin\left(x + \frac{\pi}{3}\right) + 1$

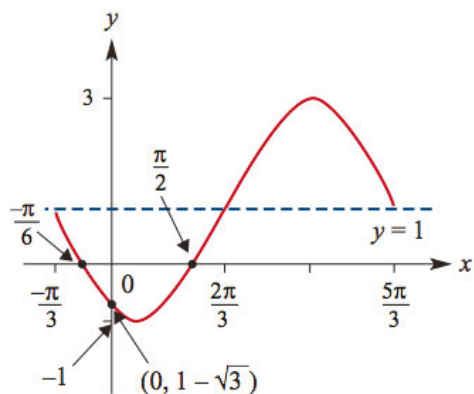
The graph is that of  $y = -2 \sin x$  translated  $\frac{\pi}{3}$  units to the left and 1 unit up.

$$y = 0: \sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6} \text{ for one cycle}$$

$$x = -\frac{\pi}{6}, \frac{\pi}{2}$$

$$x = 0: y = 1 - \sqrt{3}$$



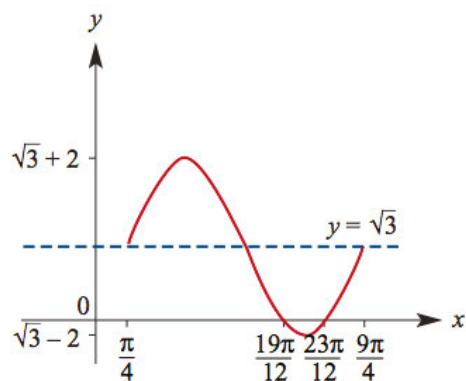
**c**  $y = 2 \sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}$

The graph is that of  $y = 2 \sin x$  translated  $\frac{\pi}{4}$  units to the right and  $\sqrt{3}$  units up.

$$y = 0: \sin\left(x - \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}$$

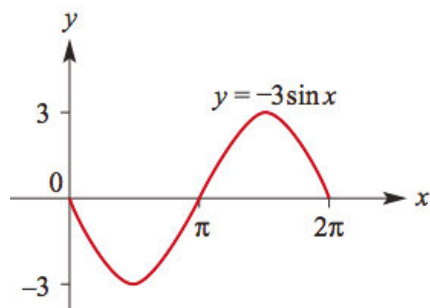
$$x - \frac{\pi}{4} = \frac{4\pi}{3}, \frac{5\pi}{3} \text{ for one cycle}$$

$$x = \frac{19\pi}{12}, \frac{23\pi}{12}$$



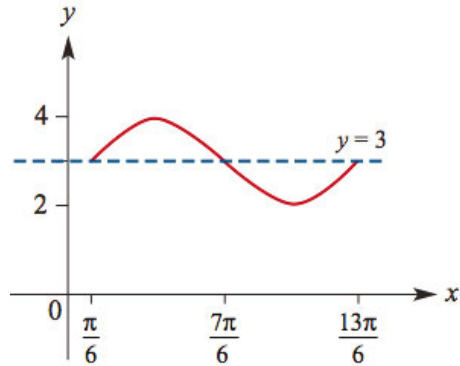
**d**  $y = -3 \sin x$

$$y = 0: x = 0, \pi, 2\pi \text{ for one cycle}$$



e  $y = \sin\left(x - \frac{\pi}{6}\right) + 3$

The graph is that of  $y = \sin x$  translated  $\frac{\pi}{6}$  units to the right and 3 units up, so there are no  $x$  intercepts.



f  $y = 2 \sin\left(x - \frac{\pi}{2}\right) + 1$

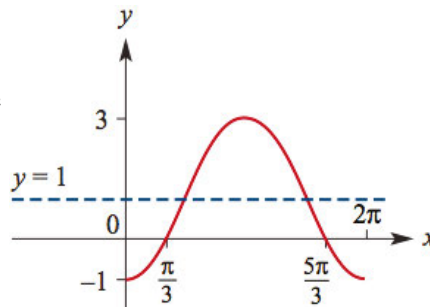
Now  $\sin\left(x - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - x\right) = -\cos x$ , so  $y = -2 \cos x + 1$  is an equivalent form.

The graph is that of  $y = -2 \cos x$  translated 1 unit up.

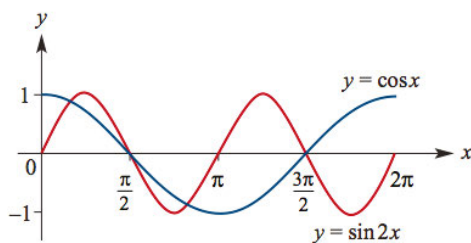
$y = 0: \cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$  for one cycle

$x = 0: y = -1$



5 The graphs are shown below.



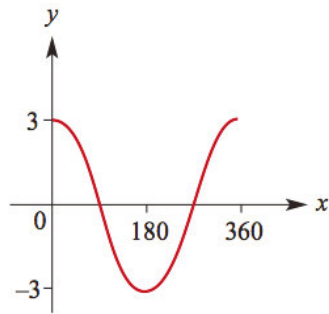
- a The line with equation  $y = 0.6$  cuts the curve with equation  $y = \sin 2x$  four times. The equation has 4 solutions.
- b The curve with equation  $y = \sin 2x$  cuts the curve with equation  $y = \cos x$  four times. The equation has 4 solutions.
- c Rewrite the equation in the form:



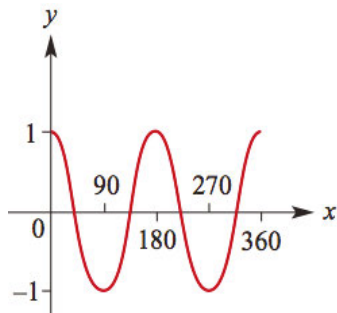
$$\sin 2x - 1 = \cos x$$

The curve with equation  $y = \sin 2x - 1$  is that of  $y = \sin 2x$  translated 1 unit down. Looking at the graphs above, it is clear that translating the sine graph 1 unit down means that two intersections with the cosine graph are lost and only two remain. The equation has 2 solutions.

**6 a**  $y = 3 \cos x^\circ$



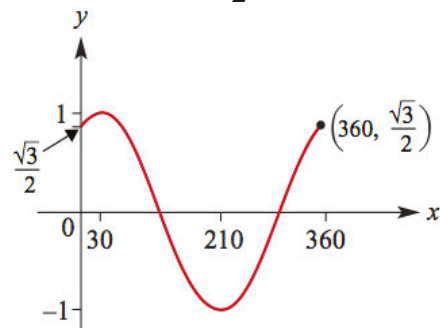
**b**  $y = \cos 2x^\circ$



**c**  $y = \cos(x - 30)^\circ$

The graph is that of  $y = \cos x^\circ$  translated  $30^\circ$  to the right.

$$x = 0, 360: y = \frac{\sqrt{3}}{2}$$



**7** Note that  $x \in [-\pi, \pi]$  throughout.

**a**  $\tan x = \sqrt{3}$

$x$  is in the first or third quadrant.

$$X = -\frac{2\pi}{3}, \frac{\pi}{3}$$

**b**  $\tan x = -1$

$x$  is in the second or fourth quadrant.

$$x = -\frac{\pi}{4}, \frac{3\pi}{4}$$

**c**  $\tan 2x = -1$

$2x$  is in the second or fourth quadrant.

$$2x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$x = -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$$

**d**  $\tan(2x) + \sqrt{3} = 0$ , so  $\tan 2x = -\sqrt{3}$

$2x$  is in the second or fourth quadrant.

$$2x = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$$

$$x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$$

**8**  $\tan(x) = \sqrt{3}$

$$x = -\frac{2\pi}{3}, \frac{\pi}{3}$$

**9 a**  $a \cos \frac{\pi}{6} = \sin \frac{\pi}{6}$

$$\frac{\sqrt{3}a}{2} = \frac{1}{2}$$

$$a = \frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}$$

**b**  $\tan x = \frac{\sqrt{3}}{3}$

$$x = \frac{\pi}{6}, -\frac{5\pi}{6}$$

**10 a**  $\sin 2x = -1$

$$2x = -\frac{\pi}{2} + 2n\pi$$

$$x = -\frac{\pi}{4} + n\pi, n \in \mathbb{Z}$$

**b**  $\cos 2x = 1$

$$3x = 2n\pi$$

$$x = -\frac{2n\pi}{3}, n \in \mathbb{Z}$$

**c**  $\tan x = -1$

$$x = n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$$

## Solutions to multiple-choice questions

**1 C**  $3 \sin\left(\frac{1}{2}x - \pi\right) + 4$   
 Period =  $\frac{2\pi}{n}$ ,  $n = \frac{1}{2}$   
 $= \frac{2\pi}{\frac{1}{2}}$   
 $= 4\pi$

**2 A**  $f(x) = 5 \cos\left(2x - \frac{\pi}{3}\right) - 7$   
 Range =  $[-7 + 5, -7 - 5]$   
 Range =  $[-2, -12]$

**3 E**  $y = \sin x$   
 A dilation of factor  $\frac{1}{2}$  from the  $y$ -axis:  
 $y = \sin(2x)$   
 A translation of  $\frac{\pi}{4}$  in the positive direction of the  $x$  axis:  
 $y = \sin\left(2\left(x - \frac{\pi}{4}\right)\right)$

**4 D**  $f: R \rightarrow R, f(x) = a \sin(bx) + c$   
 Period =  $\frac{2\pi}{b}$

**5 A**  $3 \sin(x) - 1 = b$   
 It is only possible for the equation to have one positive real number solution at the turning point:  
 Max value = 2

**6 C**

**7 C**  $f(x) = p \cos 5x + q, p > 0$   
 $f(x) \leq 0$   
 $0 \geq p \cos 5x + q$   
 Maximum  $y$  value must be negative, this value occurs at  $x = 0$   
 $\therefore 0 \geq p \cos 0 + q$   
 $\therefore p \leq -q$

**8 B** One rotation = period  
 $\frac{2\pi}{6\pi} = \frac{1}{3}$

**9 C**

**10 E**  $y = \cos x$   
 A dilation of factor 2 from the  $x$ -axis:  
 $y = 2 \cos x$   
 A translation of  $\frac{\pi}{4}$  in the positive direction of the  $x$ -axis:  
 $y = 2 \cos\left(x - \frac{\pi}{4}\right)$

**11 D**

**12 B** Period of graph shown:  
 $\frac{2\pi}{n} = 8$   
 $n = \frac{\pi}{4}$   
 Graph is translated 3 units in the positive direction of the  $y$ -axis. As the graph is initially positive it must be a sine function.  
 $\therefore y = 3 + 3 \sin\left(\frac{\pi x}{4}\right)$

## Solutions to extended-response questions

1 The time between high tide and low tide is 6 hours.

Assume the function modelling the river is sinusoidal.

i.e.  $d(t) = a \sin(nt + \varepsilon) + b$

where  $d(t)$  is the depth at time  $t$  (measured from 0)

Period = 12  $\therefore \frac{2\pi}{n} = 12$  and  $n = \frac{\pi}{6}$

Average depth = 4 m i.e.  $d = 4$  is the centre  $\therefore b = 4$

Highest value = 5  $\therefore$  amplitude =  $5 - 4 = 1$

Range =  $[3, 5]$  and  $a = 1$

a  $d(t) = \sin\left(\frac{\pi t}{6} + \varepsilon\right) + 4$

Also there is a high tide at 12:00.

$\therefore$  When  $t = 12, d = 5$

$\therefore 5 = \sin(2\pi + \varepsilon) + 4$

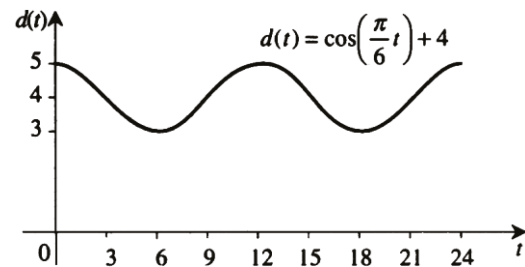
i.e.  $\sin(\varepsilon) = 1$

and  $\therefore \varepsilon = \frac{\pi}{2}$

$\therefore d(t) = \sin\left(\frac{\pi t}{6} + \frac{\pi}{2}\right) + 4$

But  $\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$

$\therefore d(t) = \cos\left(\frac{\pi t}{6}\right) + 4$



b For  $d(t) \geq 4$  consider first  $d = 4$

$$4 = \cos\left(\frac{\pi t}{6}\right) + 4$$

$$\cos\left(\frac{\pi t}{6}\right) = 0$$

$$\therefore \frac{\pi t}{6} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{7\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \dots$$

$$\therefore t = 3 \text{ or } 9 \text{ or } 15 \text{ or } \dots$$

$$\therefore d \geq 4 \text{ for } t \in [0, 3] \cup [9, 15] \cup \dots$$

The boat may enter the harbour after 9:00 a.m. but it must leave by 3:00 p.m.

c For  $d \geq 3.5$

First consider  $d = 3.5$

$$3.5 = \cos\left(\frac{\pi t}{6}\right) + 4$$

$$\therefore -0.5 = \cos\left(\frac{\pi t}{6}\right)$$

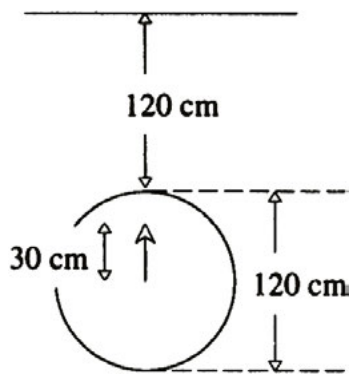
$$\therefore \frac{\pi t}{6} = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \text{ or } \frac{8\pi}{3} \text{ or } \dots$$

$$\therefore t = 4 \text{ or } 8 \text{ or } 16 \text{ or } \dots$$

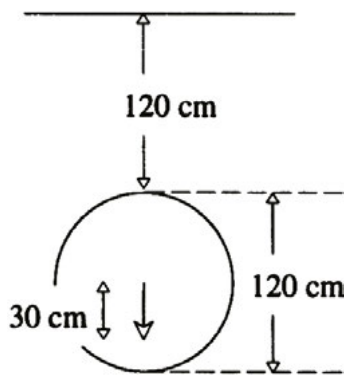
$\therefore d \geq 3.5$  for  $t \in [0, 4] \cup [8, 16] \cup \dots$  (See graph above)

A boat can enter the river after 8:00 a.m. but must leave before 4:00 p.m.

2 a



$$\begin{aligned} \text{The minimum distance} &= 120 + 30 \\ &= 150 \text{ cm} \end{aligned}$$



$$\begin{aligned} \text{The maximum distance} &= 120 + 60 + 30 \\ &= 210 \text{ cm} \end{aligned}$$

$$\text{The mean distance} = 180 \text{ cm}$$

b  $y = A \sin(nt + \varepsilon) + b$

From the above:

Mean distance is 180 cm  $\therefore b = 180$

Range = [150, 210]

$\therefore$  amplitude = 30,  $A = 30$

Period = 12  $\therefore \frac{2\pi}{n} = 12$

i.e.  $n = \frac{\pi}{6}$

$$\therefore y = 30 \sin\left(\frac{\pi t}{6} + \varepsilon\right) + 180$$

When  $t = 0$ , distance is minimum

$$\therefore y = 150$$

$$\therefore 150 = 30 \sin(\varepsilon) + 180$$

$$\therefore \sin(\varepsilon) = -1$$

$$\varepsilon = -\frac{\pi}{2}$$

$$\therefore y = 30 \sin\left(\frac{\pi t}{6} - \frac{\pi}{2}\right) + 180$$

$$= 180 - 30 \cos\left(\frac{\pi t}{6}\right) \text{ Since } \sin\left(\theta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \theta\right) = -\cos \theta.$$

**c i** When  $t = 2$

$$y = 180 - 30 \cos\left(\frac{\pi \times 2}{6}\right)$$

$$= 180 - 30 \cos\left(\frac{\pi}{3}\right)$$

$$= 180 - 15$$

$$= 165$$

The distance from the ceiling to the tip of the hour hand is 165 cm at 2:00.

**ii** When  $t = 23$

$$y = 180 - 30 \cos\left(\frac{\pi \times 23}{6}\right)$$

$$= 180 - 30 \cos\left(\frac{-\pi}{6}\right)$$

$$= 180 - 30 \times \frac{\sqrt{3}}{2}$$

$$= 180 - 15\sqrt{3}$$

$$\approx 154 \text{ cm}$$

The distance from the ceiling to the top of the hour hand is approx. 154 cm at 23:00.

**d** When  $y = 200$

$$200 = 180 - 30 \cos\left(\frac{\pi t}{6}\right)$$

$$\frac{-2}{3} = \cos\left(\frac{\pi t}{6}\right)$$

$$\therefore \frac{\pi t}{6} = \pi - \cos^{-1}\left(\frac{2}{3}\right) \text{ or } \pi + \cos^{-1}\left(\frac{2}{3}\right) \text{ or } \dots$$

$$\therefore t = 6 - \frac{6}{\pi} \cos^{-1}\left(\frac{2}{3}\right) \text{ or } 6 + \frac{6}{\pi} \cos^{-1}\left(\frac{2}{3}\right)$$

$$\therefore t \approx 4.39 \text{ or } 7.61$$

The tip of the hour hand is 200 cm below the ceiling at 7:36 and 4:24.

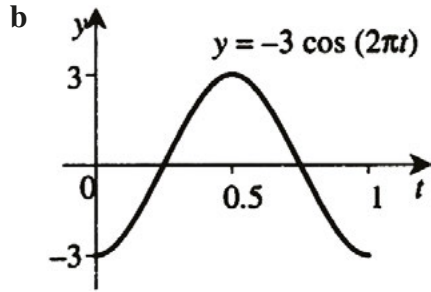
**3 a** Amplitude = 3 When  $t = 0$ ,  $y = -3$  and therefore  $a = -3$

Period = 1  $\therefore \frac{2\pi}{n} = 1$

$$\therefore n = 2\pi$$

$$\therefore a = -3 \text{ and } n = 2\pi$$

$$y = -3 \cos(2\pi t)$$



c i When  $y = 1.5$

$$1.5 = -3 \cos(2\pi t)$$

$$-\frac{1}{2} = \cos(2\pi t)$$

$$\frac{2\pi}{3} = 2\pi t$$

$$\therefore t = \frac{1}{3}$$

The centre of the weight is 1.5 cm above 0 after  $\frac{1}{3}$  second.

ii When  $y = -1.5$

$$-1.5 = -3 \cos(2\pi t)$$

$$\frac{1}{2} = \cos(2\pi t)$$

$$\frac{\pi}{3} = 2\pi t$$

$$\frac{1}{6} = t$$

The centre of the weight is 1.5 cm below 0 after  $\frac{1}{6}$  second.

d When  $y = -1$

$$-1 = -3 \cos(2\pi t)$$

$$\frac{1}{3} = \cos(2\pi t)$$

$$\cos^{-1}\left(\frac{1}{3}\right) = 2\pi t$$

$$\therefore t = \frac{1}{2\pi} \cos^{-1}\left(\frac{1}{3}\right)$$

$$\approx 0.196$$

It reaches a point 1 cm below 0 after 0.196 seconds.



**4 a**  $y = a \sin(nt + \epsilon) + b$

The average inflow is  $100\,000 \text{ m}^3/\text{day} \therefore b = 100\,000$

Minimum flow is  $80\,000$  and maximum  $120\,000$ .

$\therefore$  range =  $[80\,000, 120\,000]$

Amplitude is  $20\,000 \therefore a = 20\,000$

The period is  $365$  days  $\therefore \frac{2\pi}{n} = 365$

and therefore  $n = \frac{2\pi}{365}$

$$\therefore y = 20\,000 \sin\left(\frac{2\pi t}{365} + \epsilon\right) + 100\,000$$

When  $t = 121, y = 120\,000$

$$\therefore 120\,000 = 20\,000 \sin\left(\frac{2\pi \times 121}{365} + \epsilon\right) + 100\,000$$

$$\therefore \sin\left(\frac{2\pi \times 121}{365} + \epsilon\right) = 1$$

$$\frac{2\pi \times 121}{365} + \epsilon = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \dots$$

$$\therefore \epsilon = \frac{\pi}{2} - \frac{2\pi \times 121}{365} \text{ or } \frac{5\pi}{2} - \frac{2\pi \times 121}{365} \text{ or } \frac{9\pi}{2} - \frac{2\pi \times 121}{365} \text{ or } \dots$$

$$\approx -0.512 \text{ or } 5.77 \text{ or } \dots$$

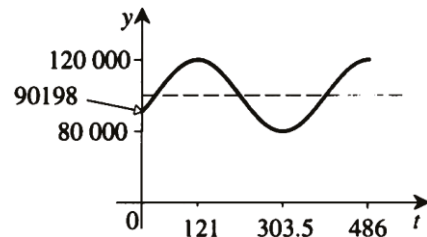
Choose  $\epsilon = 5.77$

$$\therefore y = 20\,000 \sin\left(\frac{2\pi t}{365} + 5.77\right) + 100\,000$$

**b** When  $t = 0,$

$$y = 20\,000 \sin(5.77) + 100\,000$$

$$\approx 90\,198.33$$



**c i** When  $y = 90\,000$

$$90\,000 = 20\,000 \sin\left(\frac{2\pi t}{365} + 5.77\right) + 100\,000$$

$$-\frac{1}{2} = \sin\left(\frac{2\pi t}{365} + 5.77\right)$$

$$\therefore \frac{2\pi t}{365} + 5.77 = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or } \frac{23\pi}{6} \text{ or } \dots$$

$$\therefore \frac{2\pi t}{365} = \frac{7\pi}{6} - 5.77 \text{ or } \frac{11\pi}{6} - 5.77 \text{ or } \frac{19\pi}{6} - 5.77 \text{ or } \frac{23\pi}{6} - 5.77 \text{ or } \dots$$

$$\frac{2\pi t}{365} \approx -2.1058 \text{ or } -0.01141 \text{ or } 4.1773 \text{ or } 6.2717$$

(Negative values are not considered.)

$$\therefore t \approx 242.7 \text{ or } t \approx 364.3$$

i.e. when  $t = 242.7$  and  $t = 364.3$  the inflow per day is  $90\,000 \text{ m}^3/\text{day}$ .

ii When  $y = 110\,000$

$$110\,000 = 20\,000 \sin\left(\frac{2\pi t}{365} + 5.77\right) + 100\,000$$

$$\frac{1}{2} = \sin\left(\frac{2\pi t}{365} + 5.77\right)$$

$$\frac{2\pi t}{365} + 5.77 = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6}$$

$$t = 60.2 \text{ or } t = 181.8$$

(Negative values not considered.)

i.e. when  $t = 60.2$  and  $t = 181.8$  the inflow is  $110\,000 \text{ m}^3/\text{day}$ .

d When  $t = 152$

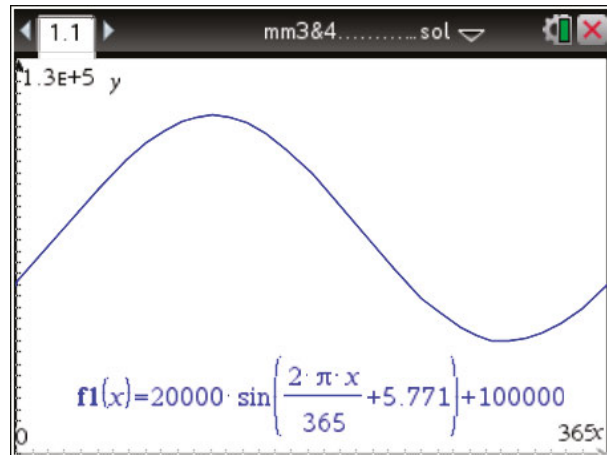
$$\begin{aligned} y &= 20\,000 \sin\left(\frac{2\pi \times 152}{365} + 5.77\right) + 100\,000 \\ &= 117\,219 \end{aligned}$$

The inflow rate is  $117\,219 \text{ m}^3/\text{day}$  on 1 June.

### Graphic calculator techniques for question 7

In a **Graphs** page enter the rule (note that  $x$  must be used here instead of  $t$ ) in the **function entry line**.

Because of the magnitude of the numbers in this problem it is useful to increase the number of display digits using  $b > \text{Settings}$  and change the **Display Digits** to **Auto** Set the **WINDOW** ( $b > \text{Window/Zoom} > \text{Window Settings}$ ) at  $X_{\min} = 0$ ,  $X_{\max} = 365$ ;  $Y_{\min} = 60\,000$ ,  $Y_{\max} = 130\,000$ . The graph appears as shown.



The value of  $Y$  when  $x = 0$  can be found several ways. Use **b>Geometry>Points & Lines>Point On** to place a point on the graph. Press **d** to exit the **Point On** tool. By double clicking on the  $x$ -coordinate you can edit this to 0.

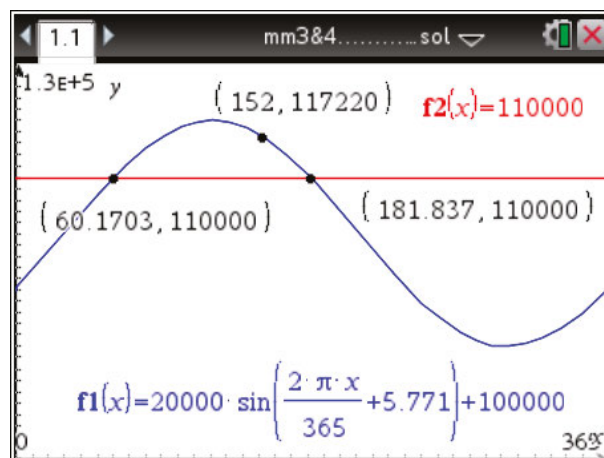
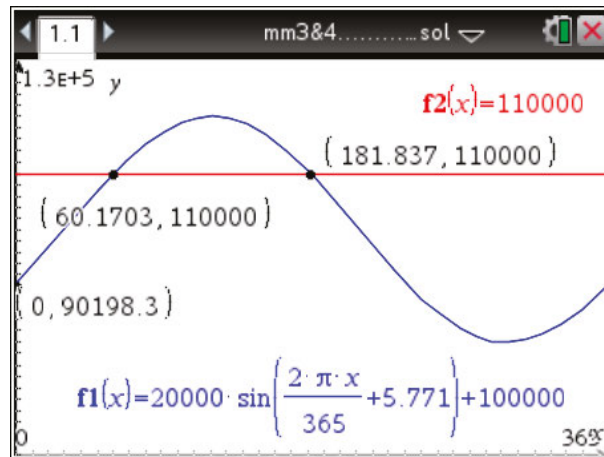
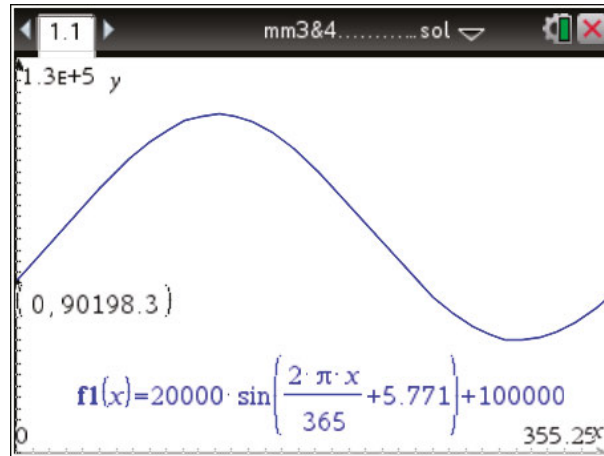
Alternatively, use **b>Trace>Graph Trace** and type 0. (An “ $x =$ ” box will appear as soon as you start typing a value).

Hence  $y = 90198.3$  when  $x = 0$ .

In order to find the  $t$  values for which  $y = 110\,000$ , press **e** or **/+G** to show the function entry line and enter **f2(x) = 110000** (use **Intersection** from the **Analyze Graph** menu to find each of the required values. Hint: using **b>Geometry>Points & Lines>Intersection Point/s** will find all intersections at once.

The value when  $x = 152$  can be found by double clicking on the  $x$ -coordinate of the point found earlier and changing to 152 or using the **b>Trace>Graph Trace** and editing the  $x$ -coordinate to 152.

Alternatively, insert **(/+I)** a **Calculator** page and type in **f1(152)**



$$5 \quad d = 12 + 12 \cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right)$$

**a i** When  $t = 5.7$

$$\begin{aligned} d &= 12 + 12 \cos \frac{\pi}{6}\left(5.7 + \frac{1}{3}\right) \\ &= 1.8276 \times 10^{-3} \approx 1.83 \times 10^{-3} \text{ hours} \end{aligned}$$

**ii** When  $t = 2.7$

$$\begin{aligned} d &= 12 + 12 \cos \frac{\pi}{6}\left(2.7 + \frac{1}{3}\right) \\ &= 11.79 \text{ hours} \end{aligned}$$

**b** When  $d = 5$

$$\begin{aligned} 5 &= 12 + 12 \cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right) \\ \frac{-7}{12} &= \cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right) \\ \therefore \frac{\pi}{6}\left(t + \frac{1}{3}\right) &= 2.1936 \text{ or } 4.089 \end{aligned}$$

$$\therefore t = 3.856 \text{ or } 7.477$$

There will be 5 hours of daylight on 25th April and 14th August.

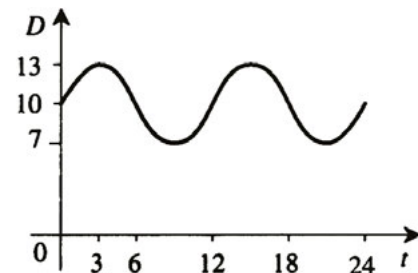
$$\begin{aligned} 6 \quad \mathbf{a} \quad \text{Period} &= 2\pi \div \frac{\pi}{6} \\ &= 12 \end{aligned}$$

$$\text{For } D(t) = 10 + 3 \sin\left(\frac{\pi t}{6}\right)$$

$$D(0) = 10$$

$$D(24) = 10 + 3 \sin\left(\frac{\pi \times 24}{6}\right)$$

$$= 10$$



**b** For  $D(t) \geq 8.5$ , first consider

$$8.5 = 10 + 3 \sin\left(\frac{\pi t}{6}\right)$$

$$-\frac{1}{2} = \sin\left(\frac{\pi t}{6}\right)$$

$$\therefore \frac{\pi t}{6} = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or } \frac{23\pi}{6} \text{ or } \dots$$

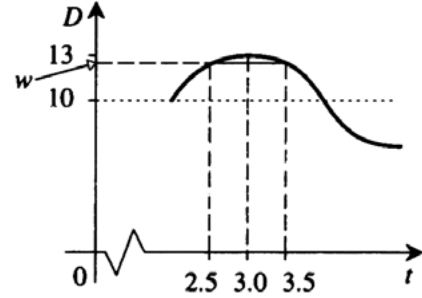
$$\therefore t = 7 \text{ or } 11 \text{ or } 19 \text{ or } 23 \text{ or } \dots$$

From the graph it can be seen that  $D(t) \geq 8.5$  for  $t \in [0, 7] \cup [11, 19] \cup [23, 24]$

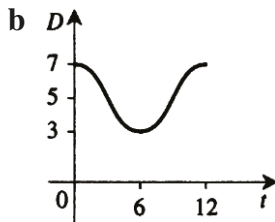
- c The maximum depth is 13 m.  
 From the graph the required period of time is  $[2.5, 3.5]$   
 The largest value of  $w$  occurs for  $t = 2.5$   

$$w = 10 + 3 \sin\left(\frac{2.5\pi}{6}\right)$$

$$= 12.898$$
 The largest value of  $w$  is 12.898.



- 7 a  $D = p + q \cos(rt)^\circ$   
 High tide is 7 m.  
 Low tide is 3 m.  
 Low tide occurs 6 hours after high tide.  
 High tide occurs when  $\cos(rt)^\circ = 1$  and low tide occurs when  $\cos(rt)^\circ = -1$ .  
 $\therefore D = p + q \cos(rt)^\circ$   
 gives  $7 = p + q$  ①  
 and  $3 = p - q$  ②  
 Adding ① and ② gives  $2p = 10$   
 $p = 5$   
 Therefore from ①  $q = 2$   
 Hence  $D = 5 + 2 \cos(rt)^\circ$   
 The period is 12  $\therefore \frac{360}{r} = 12$   
 and  $r = 30$   
 $\therefore D = 5 + 2 \cos(30t)^\circ$



- c Low tide occurs when  $t = 6$ . The depth at low tide is 3 m.  
 $5 + 2 \cos(30t)^\circ = 4$   
 $\cos(30t)^\circ = -\frac{1}{2}$   
 $\therefore 30t = 120$  or  $240$  or ...  
 $\therefore t = 4$  or  $8$  or ...  
 The ship may enter the harbour 2 hours after low tide.

**8 a**  $a = b = 10, \theta = \frac{\pi}{3}$

**i**  $A = \frac{1}{2} \times 100 \times \sin \frac{\pi}{3}$   
 $= 50 \times \frac{\sqrt{3}}{2}$   
 $= 25\sqrt{3}$  square units

**ii**  $P = 10 + 10 + \sqrt{100 + 100 - 200 \cos \frac{\pi}{3}}$   
 $= 20 + \sqrt{200 - 100}$   
 $= 30$

**b**  $P = A$  implies

$$20 + 10\sqrt{2 - 2\cos\theta} = 50\sin\theta$$

$$\Leftrightarrow 2 + \sqrt{2 - 2\cos\theta} = 5\sin\theta$$

Plot the graphs of  $y = 2 + \sqrt{2 - 2\cos\theta}$  and  $y = 5\sin\theta$  to find the point of intersection.

Intersection occurs where  $\theta = 0.53$  or  $\theta = 2.27$

**c** If  $a = b = 6$

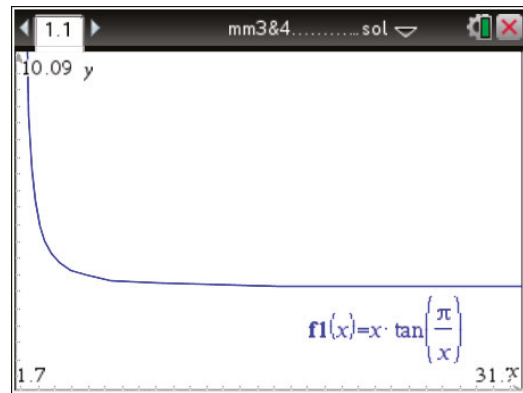
$$A = 18\sin\theta \text{ and}$$

$$P = 12 + \sqrt{72 - 72\cos\theta}$$

$$= 12 + 6\sqrt{2 - 2\cos\theta}$$

Graph  $y = P - A$  for  $\theta \in (0, \pi)$  and note that the minimum  $> 0$ , so

$$P - A > 0 = P > A.$$



**d** If  $\theta = \frac{\pi}{2}$  and  $a = 6$

$A = P$  implies

$$3b = 6 + b + \sqrt{36 + b^2}$$

i.e.  $2b - 6 = \sqrt{36 + b^2}$

$$\therefore 4b^2 - 24b + 36 = 36 + b^2$$

$$\therefore 3b^2 - 24b = 0$$

$$\therefore 3b(b - 8) = 0$$

$$\therefore b = 0 \text{ or } b = 8$$

$b = 0$  does not satisfy the original equation. Therefore  $b = 8$

e If  $a = 10$  and  $b = 6$

$$30 \sin \theta = 16 + \sqrt{136 - 120 \cos \theta}$$

$$15 \sin \theta = 8 + \sqrt{34 - 30 \cos \theta}$$

$\theta = 0.927$  or  $\theta = 1.837$  (from a cas calculator using the 'solve' command)

f If  $a = b$  and  $\theta = \frac{\pi}{3}$

$$A = P$$

implies

$$\frac{a^2}{2} \times \frac{\sqrt{3}}{2} = 2a + \sqrt{2a^2 - a^2}$$

$$\therefore \frac{\sqrt{3}a^2}{4} = 2a + \sqrt{a^2}$$

$$\frac{\sqrt{3}a^2}{4} = 3a$$

$$\therefore \frac{\sqrt{3}a^2}{4} - 3a = 0$$

$$\therefore a\left(\frac{\sqrt{3}}{4}a - 3\right) = 0$$

$$\therefore a = \frac{12}{\sqrt{3}} = 4\sqrt{3} \text{ since } a > 0.$$

9 a The  $n$  sided polygon consists of  $n$  isosceles triangles.

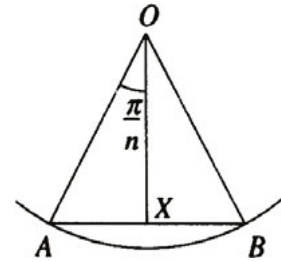
The angle for each triangle at the centre of the circle is

$$\frac{2\pi}{n}$$

Length of  $OX = 1$

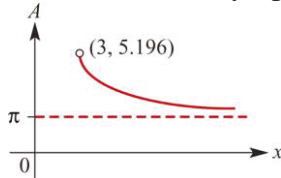
$$\text{Length of } AB = 2 \tan\left(\frac{\pi}{n}\right)$$

$$\therefore \text{Area of triangle} = \frac{1}{2} \times 2 \tan\left(\frac{\pi}{n}\right) \times 1 = \tan\left(\frac{\pi}{n}\right)$$



b Area of the polygon is  $n \tan\left(\frac{\pi}{n}\right)$  ( $n$  triangles).

c The horizontal asymptote is  $y = \pi$



d i  $n = 3$

$$\text{Area of polygon} = 3 \tan\left(\frac{\pi}{3}\right) = 3\sqrt{3} \text{ difference} = 3\sqrt{3} - \pi \approx 2.055$$

**ii**  $n = 4$

Area of polygon =  $4 \tan\left(\frac{\pi}{4}\right) = 4$  difference =  $4 - \pi \approx 0.858$

**iii**  $n = 12$

Area of polygon =  $12 \tan\left(\frac{\pi}{12}\right) \approx 3.215$  difference  $\approx 0.0738$

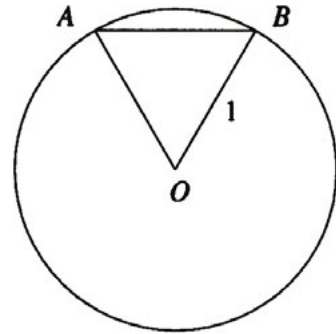
**iv**  $n = 50$

Area of polygon =  $50 \tan\left(\frac{\pi}{50}\right) \approx 3.1457$  difference  $\approx 0.0041$

**e** The circles are similar

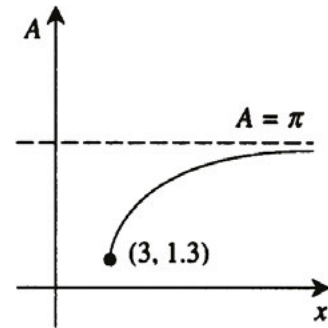
$\therefore$  the area =  $nr \tan\left(\frac{\pi}{n}\right)$

**f i** So the area is  $n \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$



**ii** The polygon consists of  $n$  isosceles triangles.

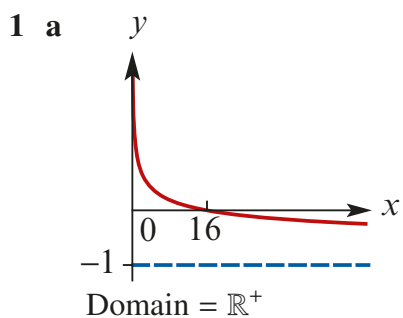
The area of each triangle  
is  $\frac{1}{2} \sin\left(\frac{2\pi}{n}\right) = \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$





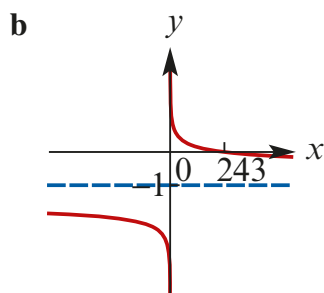
# Chapter 7 – Functions revisited

## Solutions to Exercise 7A



Range =  $(-1, \infty)$

Neither odd nor even



Range =  $\mathbb{R} \setminus \{-1\}$

Neither odd nor even

**2 a**  $32^{\frac{2}{5}} = (32^{\frac{1}{5}})^2 = 4$

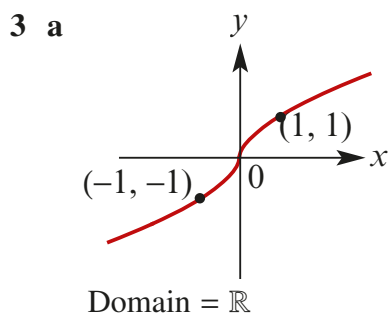
**b**  $(-32)^{\frac{2}{5}} = (-32^{\frac{1}{5}})^2 = 4$

**c**  $32^{\frac{3}{5}} = (32^{\frac{1}{5}})^3 = 8$

**d**  $(-32)^{\frac{3}{5}} = (-32^{\frac{1}{5}})^3 = -8$

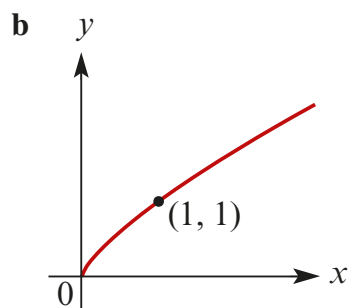
**e**  $(-8)^{\frac{5}{3}} = ((-8)^{\frac{1}{3}})^5 = -32$

**f**  $(-27)^{\frac{4}{3}} = ((-27)^{\frac{1}{3}})^4 = 81$



Range =  $\mathbb{R}$

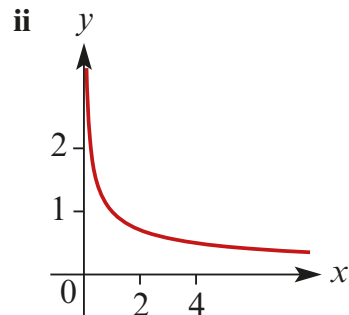
Odd



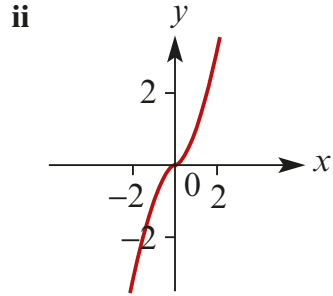
Range =  $\mathbb{R}^+ \cup \{0\}$

Neither

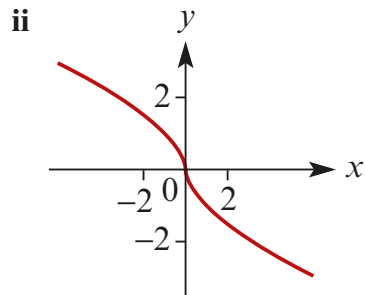
**4 a i** Domain =  $\mathbb{R}^+$ ; Range =  $\mathbb{R}^+$ ;  
Asymptotes:  $x = 0, y = 0$



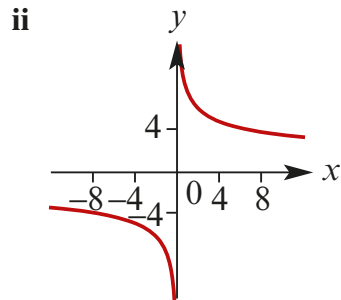
**b i** Domain =  $\mathbb{R}$ ; Range =  $\mathbb{R}$



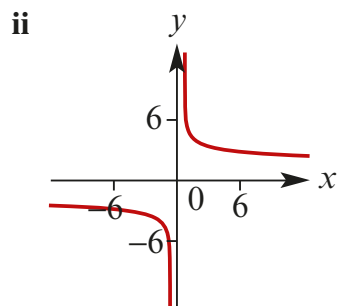
**c i** Domain =  $\mathbb{R}$ ; Range =  $\mathbb{R}$



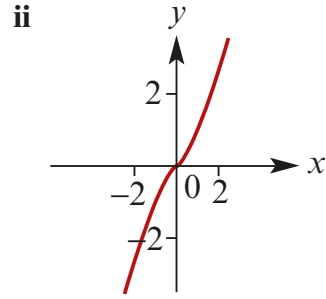
**d i** Domain =  $\mathbb{R} \setminus \{0\}$ ;  
Range =  $\mathbb{R} \setminus \{0\}$ ;  
Asymptotes:  $x = 0, y = 0$



**e i** Domain =  $\mathbb{R} \setminus \{0\}$ ;  
Range =  $\mathbb{R} \setminus \{0\}$ ;  
Asymptotes:  $x = 0, y = 0$



**f i** Domain =  $\mathbb{R}$ ; Range =  $\mathbb{R}$



**5** We can assume  $x \neq 0$  in the this question without affecting the result.

**a**  $x^{\frac{3}{2}} > x^2$   
 $1 > x^{2-\frac{3}{2}}$   
 $1 > x^{\frac{1}{2}}$   
 $x < 1$   
 $\therefore x \in (0, 1)$

**b**  $x^{\frac{3}{2}} < x^{-2}$   
 $x^{2+\frac{3}{2}} < 1$   
 $x^{\frac{5}{2}} < 1$   
 $x < 1$   
 $\therefore x \in (0, 1)$

**6 a** Odd

**b** Even

**c** Odd

**d** Odd

**e** Even

**f** Odd

## Solutions to Exercise 7B

**1 a**  $h(x) = f \circ g(x)$ ,  $f(x) = e^x$ ,  $g(x) = x^3$

**b**  $h(x) = f \circ g(x)$ ,  $f(x) = \sin x$ ,  
 $g(x) = 2x^2$

**c**  $h(x) = f \circ g(x)$ ,  $f(x) = x^n$ ,  
 $g(x) = x^2 - 2x$

**d**  $h(x) = f \circ g(x)$ ,  $f(x) = \cos x$ ,  
 $g(x) = x^2$

**e**  $h(x) = f \circ g(x)$ ,  $f(x) = x^2$ ,  
 $g(x) = \cos x$

**f**  $h(x) = f \circ g(x)$ ,  $f(x) = x^4$ ,  
 $g(x) = x^2 - 1$

**g**  $h(x) = f \circ g(x)$ ,  $f(x) = x^2$ ,  
 $g(x) = \cos(2x)$

**h**  $h(x) = f \circ g(x)$ ,  $f(x) = x^3 - 2x$ ,  
 $g(x) = x^2 - 2x$

**2 a**  $f \circ f^{-1}(x) = x$

$$\Leftrightarrow 4e^{f^{-1}(x)} = x$$

$$\Leftrightarrow e^{f^{-1}(x)} = \frac{x}{4}$$

$$\Leftrightarrow f^{-1}(x) = \log_e \left( \frac{x}{4} \right)$$

Therefore,

$$f^{-1}: (0, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = \frac{1}{3} \log_e \left( \frac{x}{4} \right)$$

**b**  $f \circ g^{-1}(x) = x$

$$\Leftrightarrow \frac{2}{\sqrt[3]{g^{-1}(x)}} = x$$

$$\Leftrightarrow 2 = x \sqrt[3]{g^{-1}(x)}$$

$$\Leftrightarrow g^{-1}(x) = \frac{8}{x^3}$$

Therefore,

$$g^{-1}: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, g^{-1}(x) = \frac{8}{x^3}$$

**c**  $f \circ g: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, f \circ g(x) = 4e^{\frac{6}{\sqrt[3]{x}}}$

**d**  $g \circ f: \mathbb{R} \rightarrow \mathbb{R}, g \circ f(x) = \frac{2}{\sqrt[3]{4e^{3x}}}$

**e** Let  $h(x) = f \circ g(x)$

$$h(x) = 4e^{\frac{6}{\sqrt[3]{x}}}$$

$$h \circ h^{-1}(x) = x$$

$$\Leftrightarrow 4e^{\frac{6}{\sqrt[3]{h^{-1}(x)}}} = x$$

$$\Leftrightarrow e^{\frac{6}{\sqrt[3]{h^{-1}(x)}}} = \frac{x}{4}$$

$$\Leftrightarrow \frac{6}{\sqrt[3]{h^{-1}(x)}} = \log_e \left( \frac{x}{4} \right)$$

$$\Leftrightarrow h^{-1}(x) = \left( \frac{6}{\log_e \left( \frac{x}{4} \right)} \right)^3$$

Therefore,

$$(f \circ g)^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R},$$

$$(f \circ g)^{-1}(x) = \left( \frac{6}{\log_e \left( \frac{x}{4} \right)} \right)^3$$

**f**  $(g \circ f)^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R}, (g \circ f)^{-1}(x) = \frac{1}{3} \log_e \left( \frac{2}{x^3} \right)$

**3 a**  $f^{-1}: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}, f^{-1}(x) = x^{\frac{5}{2}}$

Both  $f$  and  $f^{-1}$  are strictly increasing

**b**  $f^{-1}: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}, f^{-1}(x) = -x^{\frac{5}{2}}$

Both  $f$  and  $f^{-1}$  are strictly decreasing

**c**  $f^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R}, f^{-1}(x) = x^{\frac{2}{5}}$

Both  $f$  and  $f^{-1}$  are strictly increasing

**4 a i**  $f \circ g(x) = 3 \sin(2x^2),$   
 $g \circ f(x) = g(3 \sin 2x)^2 = 9 \sin^2(2x)$

**ii**  $\text{ran}(f \circ g) = [-3, 3],$   
 $\text{dom}(f \circ g) = \mathbb{R},$   
 $\text{ran}(g \circ f) = [0, 9], \text{dom}(g \circ f) = \mathbb{R}$

**b i**  $f \circ g(x) = -2 \cos(2x^2),$   
 $g \circ f(x) = g(-2 \cos 2x) =$   
 $4 \cos^2(2x)$

**ii**  $\text{ran}(f \circ g) = [-2, 2],$   
 $\text{dom}(f \circ g) = \mathbb{R},$   
 $\text{ran}(g \circ f) = [0, 4], \text{dom}(g \circ f) = \mathbb{R}$

**c i**  $f \circ g(x) = e^{x^2}, g \circ f(x) = e^{2x}$

**ii**  $\text{ran}(f \circ g) = (1, \infty),$   
 $\text{dom}(f \circ g) = \mathbb{R},$   
 $\text{ran}(g \circ f) = (0, \infty),$   
 $\text{dom}(g \circ f) = \mathbb{R}$

**d i**  $f \circ g(x) = e^{2x^2} - 1,$   
 $g \circ f(x) = (e^{2x} - 1)^2$

**ii**  $\text{ran}(f \circ g) = [0, \infty),$   
 $\text{dom}(f \circ g) = \mathbb{R},$   
 $\text{ran}(g \circ f) = [0, \infty),$   
 $\text{dom}(g \circ f) = \mathbb{R}$

**e i**  $f \circ g(x) = -2e^{x^2} - 1,$   
 $g \circ f(x) = (2e^x + 1)^2$

**ii**  $\text{ran}(f \circ g) = (-\infty, -3],$   
 $\text{dom}(f \circ g) = \mathbb{R},$   
 $\text{ran}(g \circ f) = (1, \infty),$   
 $\text{dom}(g \circ f) = \mathbb{R}$

**f i**  $f \circ g(x) = \log_e(2x^2),$   
 $g \circ f(x) = (\log_e(2x))^2$

**ii**  $\text{ran}(f \circ g) = \mathbb{R},$   
 $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\},$   
 $\text{ran}(g \circ f) = [0, \infty),$   
 $\text{dom}(g \circ f) = \mathbb{R}^+$

**g i**  $f \circ g(x) = \log_e(x^2 - 1),$   
 $g \circ f(x) = (\log_e(x - 1))^2$

**ii**  $\text{ran}(f \circ g) = \mathbb{R},$   
 $\text{dom}(f \circ g) = \mathbb{R} \setminus [-1, 1],$   
 $\text{ran}(g \circ f) = [0, \infty),$   
 $\text{dom}(g \circ f) = (1, \infty)$

**h i**  $f \circ g(x) = -\log_e(x^2),$   
 $g \circ f(x) = (\log_e x)^2$

**ii**  $\text{ran}(f \circ g) = \mathbb{R},$   
 $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\},$   
 $\text{ran}(g \circ f) = [0, \infty),$   
 $\text{dom}(g \circ f) = \mathbb{R}^+$

**5 a**  $g \circ f(x) = g\left(2x - \frac{\pi}{3}\right) = \sin\left(2x - \frac{\pi}{3}\right)$

**b**  $(x, y) \rightarrow \left(2x' - \frac{\pi}{3}, y'\right)$   
 $\therefore x' = \frac{x + \frac{\pi}{3}}{2} = \frac{1}{2}x + \frac{\pi}{6}$  and  $y' = y$   
 Dilation of factor  $\frac{1}{2}$  from the  $y$ -axis,  
 then translation  $\frac{\pi}{6}$  units to the right

**6 a**  $g \circ f: \left(\frac{1}{3}, \infty\right) \rightarrow \mathbb{R},$   
 $g \circ f(x) = g(3x - 2)$   
 $= \log_e(3x - 2 + 1)$   
 $= \log_e(3x - 1)$

**b** Write  $y' = \log_e(3x' - 1)$  and  
 $y = \log_e(x + 1)$   
 Then choose,  
 $y' = y$  and  $3x' - 1 = x + 1$   
 That is  $y' = y$  and  $x' = \frac{x + 2}{3}$

**7 a**  $[g(x)]^2 - 7g(x) + 12 = 0$   
 $(g(x) - 3)(g(x) - 4) = 0$   
 $\therefore g(x) = 3$  or  $g(x) = 4$

**b**  $[g(x)]^2 - 7xg(x) + 12x^2 = 0$   
 $(g(x) - 3x)(g(x) - 4x) = 0$   
 $\therefore g(x) = 3x$  or  $g(x) = 4x$

**8**  $e^{g(x)} = 2x - 1$   
 $g(x) = \log_e(2x - 1)$   
 $\therefore g(x) = 3x$  or  $g(x) = 4x$

**9**  $f(x) = e^{4x}$ ,  $g(x) = 2\sqrt{x}$

**a**  $g(f(x)) = 2\sqrt{e^{4x}} = 2(e^{4x})^{\frac{1}{2}}$   
 $= 2e^{2x}$

**b**  $x = 2e^{2(g \circ f)^{-1}(x)}$   
 $\ln \frac{x}{2} = 2(g \circ f)^{-1}(x)$   
 $(g \circ f)^{-1}(x) = \frac{1}{2} \ln \frac{x}{2}$

**c**  $x = 2\sqrt{g^{-1}(x)}$   
 $\frac{x}{2} = \sqrt{g^{-1}(x)}$   
 $g^{-1}(x) = \frac{x^2}{4}$   
 $(f \circ g^{-1})(x) = e^{4\left(\frac{x^2}{4}\right)}$   
 $= e^{x^2}$

**10**  $f(x) = e^{-2x}$ ,  $g(x) = x^3 + 1$

**a**  $x = e^{-2f^{-1}(x)}$   
 $\ln x = -2f^{-1}(x)$   
 $f^{-1}(x) = \frac{-1}{2} \ln x$   
 $x = (g^{-1}(x))^3 + 1$   
 $x - 1 = (g^{-1}(x))^3$   
 $g^{-1}(x) = (x - 1)^{\frac{1}{3}}$

**b**  $f \circ g(x) = e^{-2(x^3+1)}$   
 $= e^{-2x^3-2}$   
 $\text{range}(f \circ g) = \mathbb{R}^+$   
 $\text{since range}(-2x^3 - 2) = \mathbb{R}$   
 $\text{and range}(e^x) = \mathbb{R}^+$   
 $g \circ f(x) = (e^{-2x})^3 + 1$   
 $= e^{-6x} + 1$   
 $\text{range}(g \circ f) = (1, \infty)$

**11 a**  $f: (-1, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{x+1}$   
 $\text{domain}(f) = (-1, \infty)$   
 $\therefore \text{range}(f) = \mathbb{R}^+$

$$x = \frac{1}{f^{-1}(x) + 1}$$

$$f^{-1}(x) + 1 = \frac{1}{x}$$

$$f^{-1}(x) = \frac{1}{x} - 1$$

$$\text{range}(f^{-1}) = (-1, \infty),$$

$$\text{domain}(f^{-1}) = \mathbb{R}^+$$

$$\therefore f^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R}, \quad f^{-1}(x) = \frac{1}{x} - 1$$

$$\mathbf{b} \quad f(x) = f^{-1}(x) = x$$

$$f(x) = x$$

$$\frac{1}{x+1} = x$$

$$(x+1)x = 1$$

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1+4}}{2}$$

but  $x > 0$ , (domain( $f^{-1}$ ))

$$\therefore x = \frac{-1 + \sqrt{5}}{2}$$

$$x = \frac{\sqrt{5} - 1}{2}$$

$$\mathbf{12 a} \quad f(x) = \ln(x+1)$$

$$x = \ln(f^{-1}(x) + 1)$$

$$e^x = f^{-1}(x) + 1$$

$$f^{-1}(x) = e^x - 1$$

$$f^{-1}: R \rightarrow R, f^{-1}(x) = e^x - 1$$

$$g(x) = x^2 + 2x, \text{ domain}(g) = (-1, \infty)$$

$$\text{range}(g) = (g(-1), \infty)$$

$$= (-1, \infty)$$

$$x = (g^{-1}(x))^2 + 2g^{-1}(x)$$

$$(g^{-1}(x))^2 + 2(g^{-1}(x)) - x = 0$$

$$(g^{-1}(x)^2 + 1)^2 - x - 1 = 0$$

$$g^{-1}(x) + 1 = \pm \sqrt{x+1}$$

$$g^{-1}(x) + 1 = \pm \sqrt{x+1}$$

$$g^{-1}(x) = -1 \pm \sqrt{x+1}$$

but  $g^{-1}(x) > -1$

$$\therefore g^{-1}(x) = -1 + \sqrt{x+1}$$

$$g^{-1}: (-1, \infty) \rightarrow R, g^{-1}(x) =$$

$$\sqrt{x+1} - 1$$

$$\mathbf{b} \quad f \circ g(x) = \ln(x^2 + 2x + 1)$$

$$= \ln((x+1)^2)$$

$$= 2 \ln(x+1)$$

(Since domain  $g = (-1, \infty)$ )

$$\mathbf{13} \quad f \circ g(x) = \ln\left(\frac{1}{x}\right)$$

$$= -\ln(x)$$

$$f(x) + f \circ g(x) = \ln x - \ln x = 0$$

$$\mathbf{14} \quad h(g(x)) = \sqrt{\frac{(5x^2 + 3) - 3}{5}}$$

$$= \sqrt{\frac{5x^2}{5}}$$

$$= \sqrt{x^2}$$

$$= |x|$$

$$\mathbf{15 a} \quad f(g(x)) = (x^2 - 4 - 4)(x^2 - 4 - 6)$$

$$= (x^2 - 8)(x^2 - 10)$$

$$f(g(x)) = x^4 - 18x^2 + 80$$

$$g(f(x)) = ((x-4)(x-6))^2 - 4$$

$$= (x^2 - 10x + 24)^2 - 4$$

$$= x^4 - 20x^3 + 48x^2 + 100x^2$$

$$- 480x + 576 - 4$$

$$g(f(x)) = x^4 - 20x^3 + 148x^2$$

$$- 480x + 572$$

$$\mathbf{b} \quad g(f(x)) - f(g(x)) = 158$$

$$x^4 - 20x^3 + 148x^2 - 480x + 572$$

$$- x^4 + 18x^2 - 80 = 158$$

$$-20x^3 + 166x^2 - 480x + 334 = 0$$

$$10x^3 - 88x^2 + 240x - 167 = 0$$

CAS calculator gives  $x = 1$  as a solution

$$\Rightarrow (x-1)(10x^2 - 73x + 167) = 0$$

$$x = 1, x = \frac{73 \pm \sqrt{5329 - 6680}}{20}$$

$$x = 1, \frac{73 \pm \sqrt{-1351}}{20}$$

↓

no real solutions

$$\therefore x = 1$$

**16 a**  $f(x) = 4 - x^2$   
 $f(f(x)) = 4 - (4 - x^2)^2$   
 $= 4 - (16 - 8x^2 + x^4)$   
 $= -12 + 8x^2 - x^4$   
 $f(f(x)) = 0$   
 $\Rightarrow x^4 - 8x^2 + 12 = 0$   
 $(x^2)^2 - 8x^2 + 12 = 0$   
 $(x^2 - 6)(x^2 - 2) = 0$   
 $x^2 = 2, 6$   
 $x = \pm\sqrt{2}, \pm\sqrt{6}$

**17**  $f(x) = e^x - e^{-x}$

**a**  $LHS = e^{(-x)} - e^{-(-x)}$   
 $= e^{-x} - e^x$   
 $RHS = -e^x + e^{-x}$   
 $= e^{-x} - e^x$   
 $= LHS \quad QED$

**b**  $RHS = e^{3x} - e^{-3x} - 3e^x + 3e^{-x}$   
 $= e^{3x} - 3e^x + 3e^{-x} - e^{-3x}$   
 $LHS = (e^x - e^{-x})^3$   
 $= e^{3x} - 3e^x + 3e^{-x} - e^{-3x}$   
 $= RHS \quad QED$

**18** Consider,

$$af^{-1}(x) + b = x$$
$$\therefore f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$$

If  $f^{-1}(x) = 6x + 3$   
 $a = \frac{1}{6}$  and  $-\frac{b}{a} = 3$

$$\therefore a = \frac{1}{6} \text{ and } b = -\frac{1}{2}$$

**19**  $\frac{f^{-1}(x) + 2}{f^{-1}(x) - 1} = x$   
 $f^{-1}(x) + 2 = x(f^{-1}(x) - 1)$   
 $f^{-1}(x)(1 - x) = -2 - x$   
 $f^{-1}(x) = \frac{x + 2}{x - 1}$

**20**  $\ln(g(x)) = ax + b$

$$g(x) = e^{ax+b}$$
$$g(0) = 1$$
$$\therefore 1 = e^b$$
$$\therefore b = 0$$

$$g(x) = e^{ax}$$
$$g(1) = e^6$$
$$e^6 = e^a$$
$$a = 6$$
$$g(x) = e^{6x}$$

**21** Let  $y = f^{-1}(x)$

**a**  $\frac{e^y + e^{-y}}{2} = x$   
 $e^y + e^{-y} = 2x$   
 $e^{2y} + 1 = 2xe^y$   
 $e^{2y} - 2xe^y + 1 = 0$

$$e^y = \frac{1}{2}(2x \pm \sqrt{4x^2 - 4})$$

$$y = \log_e(x \pm \sqrt{x^2 - 1})$$

But Range of  $f^{-1}$  = Domain of

$$f = [0, \infty)$$

and Domain of  $f^{-1}$  = Range of

$$f = [1, \infty) \therefore f^{-1}: [1, \infty) \rightarrow \mathbb{R},$$

$$f^{-1}(x) = \log_e(x + \sqrt{x^2 - 1})$$

**b**  $g^{-1}: \mathbb{R} \rightarrow \mathbb{R}, g^{-1}(x) = \log_e(x + \sqrt{x^2 + 1})$

**c** Yes

**d** Yes

**22 a** If  $x > y$  then  $f(x) > f(y)$  and

If  $x > y$  then  $g(x) > g(y)$

Hence  $x > y \Rightarrow f(x) > f(y) \Rightarrow g(f(x)) > g(f(y))$ .

**b** If  $x > y$  then  $f(x) < f(y)$  and

If  $x > y$  then  $g(x) < g(y)$

Hence if  $x > y \Rightarrow f(x) < f(y) \Rightarrow g(f(x)) > g(f(y))$ .

**c** The composite function will be strictly decreasing.



## Solutions to Exercise 7C

1  $f(x) = e^{-2x}, g(x) = -2x$

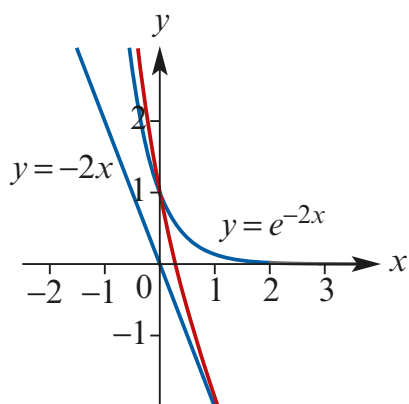
a i  $(f + g)(x) = e^{-2x} - 2x$

ii  $(fg)(x) = -2xe^{-2x}$

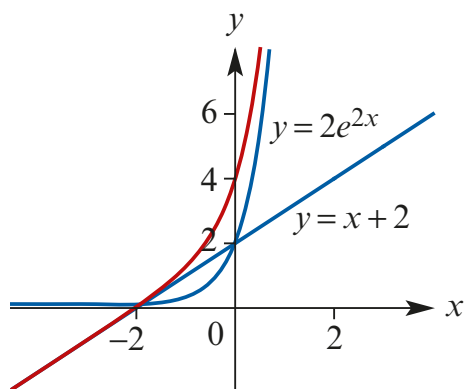
b i  $(f + g)\left(\frac{-1}{2}\right) = e^{-1} + 1$

ii  $(fg)\left(\frac{-1}{2}\right) = 1 \times e^{-1}$   
 $= e^{-1}$

2



3



4  $f(x) = \sin\left(\frac{\pi x}{2}\right), g(x) = -2x$

a i  $(f + g)(x) = \sin\left(\frac{\pi x}{2}\right) - 2x$

ii  $(fg)(x) = -2x \sin\left(\frac{\pi x}{2}\right)$

b i  $(f + g)(1) = \sin\left(\frac{\pi}{2}\right) - 2$   
 $= 1 - 2$   
 $= -1$

ii  $(fg)(1) = -2 \sin\left(\frac{\pi}{2}\right)$   
 $= -2$

5  $f(x) = \cos\left(\frac{\pi x}{2}\right), g(x) = e^x$

a i  $(f + g)(x) = \cos\left(\frac{\pi x}{2}\right) + e^x$

ii  $(fg)(x) = e^x \cos\left(\frac{\pi x}{2}\right)$

b i  $(f + g)(0) = \cos(0) + e^0$   
 $= 1 + 1$   
 $= 2$

ii  $(fg)(0) = 1 \times 1$   
 $= 1$

6 Let  $g(x) = \frac{f(x) + f(-x)}{2}$  and

$h(x) = \frac{f(x) - f(-x)}{2}$

We have,  $g(-x) = g(x)$ . That is  $g(x)$  is even.

We have,  $h(-x) = -h(x)$ . That is  $h(x)$  is odd.

$f(x) = h(x) + g(x)$

## Solutions to Exercise 7D

$$\begin{aligned} 1 \text{ a } f(x - y) &= 2(x - y) \\ &= 2x - 2y \\ &= f(x) - f(y) \end{aligned}$$

$$\begin{aligned} \text{b } f(x - y) &= (x - y - 3) \\ &\neq f(x) - f(y) \end{aligned}$$

$$\begin{aligned} 2 \quad f(x - y) &= k(x - y) \\ &= kx - ky \\ &= f(x) - f(y) \end{aligned}$$

$$\begin{aligned} 3 \quad f(x + y) &= 2(x + y) + 3 \\ &= 2x + 2y + 3 \\ &= 2x + 3 + 2y + 3 - 3 \\ &= f(x) + f(y) - 3 \\ a &= -3 \end{aligned}$$

$$\begin{aligned} 4 \quad f(x) + f(y) &= \frac{3}{x} + \frac{3}{y} \\ &= \frac{3(x + y)}{xy} \\ &= (x + y)f(xy) \end{aligned}$$

$$\begin{aligned} 5 \quad (g(x))^2 &= g(x) \\ (g(x))^2 - g(x) &= 0 \\ g(x)(g(x) - 1) &= 0 \\ g(x) &= 0, 1 \end{aligned}$$

$$\begin{aligned} 6 \quad \frac{1}{g(x)} &= g(x) \\ (g(x))^2 &= 1 \\ g(x) &= \pm 1 \end{aligned}$$

$$\begin{aligned} 7 \quad f(x) &= x^3 \\ f(x + y) &= (x + y)^3 \\ f(x) + f(y) &= x^3 + y^3 \end{aligned}$$

$$\text{Let } x = 1, y = 1$$

$$\begin{aligned} f(x + y) &= 8 \\ f(x) + f(y) &= 2 \end{aligned}$$

$$\begin{aligned} 8 \quad f(x) &= \sin x \\ LHS &= f(x + y) = \sin(x + y) \\ RHS &= f(x) + f(y) = \sin x + \sin y \\ \text{let } x &= \frac{\pi}{2}, y = \frac{\pi}{2} \\ LHS &= \sin(\pi) = 0 \\ RHS &= \sin \frac{\pi}{2} + \sin \frac{\pi}{2} = 2 \neq LHS \text{ QED} \\ &(\text{any non zero numbers would work}) \end{aligned}$$

$$\begin{aligned} 9 \quad f(x) &= \frac{1}{x^2} \\ LHS &= f(x) + f(y) \\ &= \frac{1}{x^2} + \frac{1}{y^2} \\ &= \frac{y^2}{x^2y^2} + \frac{x^2}{x^2y^2} \\ &= (x^2 + y^2) \frac{1}{x^2y^2} \\ &= (x^2 + y^2) \frac{1}{(xy)^2} \\ &= (x^2 + y^2)f(xy) \\ &= RHS \text{ QED} \end{aligned}$$

$$10 \quad h(x) = x^2$$

**a** Let  $x = 1, y = 1$

$$LHS = (1 + 1)^2 = 4$$

$$RHS = (1)^2 + (1)^2 = 2 \neq LHS \text{ QED}$$

(any non zero numbers would work)

**b**  $LHS = (x + y)^2$

$$= x^2 + 2xy + y^2$$

$$RHS = (x)^2 + (y)^2$$

$$= x^2 + y^2$$

$$LHS = RHS + 2xy$$

$$\therefore \text{ given } LHS = RHS$$

$$2xy = 0$$

$$\text{i.e., } x = 0 \text{ or } y = 0 \text{ QED}$$

**11**  $g(x) = 2^{3x}$

$$LHS = 2^{3(x+y)}$$

$$= 2^{3x+3y}$$

$$= 2^{3x} \times 2^{3y}$$

$$= g(x) \times g(y)$$

$$= RHS \text{ QED}$$

**12**  $f(x) = x^n$

$$f(xy) = (xy)^n = x^n y^n$$

(by the indices laws)

$$= f(x)f(y) \text{ QED}$$

$$f\left(\frac{x}{y}\right) = \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

(by the indices laws)

$$= \frac{f(x)}{f(y)} \text{ QED}$$

**13**  $f(x) = ax, \quad a \in R \setminus \{0, 1\}$

$$f(xy) = axy$$

$$= f(x)f(y) = ax \times ay$$

$$= a^2 xy$$

let  $x = 1, \quad y = 1$

(any non zero numbers would work)

$$f(xy) = axy = a$$

$$f(x)f(y) = a^2 xy = a^2$$

$$\text{if } f(x)f(y) = f(xy)$$

$$a^2 = a$$

$$a^2 - a = 0$$

$$a = 0, 1$$

but  $a \neq 0, 1$

$$\therefore f(x)f(y) \neq f(xy)$$

for the case shown

**14**  $f : \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R}, f(x) = \frac{1}{x+1}$ .

$$f(f(x)) = f\left(\frac{1}{x+1}\right) = \frac{1}{\frac{1}{x+1} + 1} =$$

$$\frac{x+1}{x+2}$$

$$f(x+1) = \frac{1}{x+2}$$

$$\therefore f(f(x)) + f(x+1) = \frac{x+1}{x+2} + \frac{1}{x+2}$$

$$= \frac{x+2}{x+2}$$

$$= 1$$

**15** Let  $f(x) = x^2, g(x) = 2, h(x) = 3$

$$f \circ (g + h)(x) = f(5) = 25$$

$$f \circ g(x) + f \circ h(x) = f(2) + f(3) = 13$$

**16**  $g + h \circ f(x) = (g + h)f(x)$

$$= g(f(x)) + h(f(x))$$

$$= g \circ f(x) + h \circ f(x)$$

17 For  $x > 0$

$$\begin{aligned}f(g(x)) - f(x) &= f(xe^x) - \log_e x \\&= \log_e(xe^x) - \log_e x \\&= \log_e x + \log_e(e^x) - \log_e x \\&= \log_e(e^x) \\&= x\end{aligned}$$

$$\begin{aligned}\frac{g(f(x))}{f(x)} &= \frac{g(\log_e x)}{\log_e x} \\&= \frac{\log_e x \times e^{\log_e x}}{\log_e x} \\&= \frac{x \log_e x}{\log_e x} \\&= x\end{aligned}$$

## Solutions to Exercise 7E

**1**  $f(x) = mx - 4, \quad m \in \mathbb{R} \setminus \{0\}$

**a**  $0 = mx - 4$

$$mx = 4$$

$$x = \frac{4}{m}$$

**b**  $\frac{4}{m} \leq 1$

$$\therefore 4 \leq m, \quad m < 0$$

**c**  $x = mf^{-1}(x) - 4$

$$x + 4 = mf^{-1}(x)$$

$$f^{-1}(x) = \frac{x + 4}{m}, \quad \text{domain} = \mathbb{R}$$

**d**  $x = mx - 4$

$$(m - 1)x - 4 = 0$$

$$x = \frac{4}{m - 1}$$

$$\begin{aligned} \text{check } f\left(\frac{4}{m-1}\right) &= \frac{4m}{m-1} - 4 \\ &= \frac{4m - 4m + 4}{m-1} \\ &= \frac{4}{m-1} \end{aligned}$$

$$\text{co-ordinates } \left(\frac{4}{m-1}, \frac{4}{m-1}\right),$$

$$m \in \mathbb{R} \setminus \{0, 1\}$$

**e**  $y = ax + b$

$$a = \frac{-1}{m} \quad (\text{normal line})$$

$$y = \frac{-x}{m} + b$$

$$(0, -4)$$

$$\Rightarrow -4 = b$$

$$y = \frac{-x}{m} - 4$$

**2**  $f(x) = -2x + c$

**a**  $0 = -2x + c$

$$-c = -2x$$

$$x = \frac{c}{2}$$

**b**  $\frac{c}{2} \leq 1$   
 $c \leq 2$

**c**  $x = -2f^{-1}(x) + c$

$$x - c = -2f^{-1}(x)$$

$$f^{-1}(x) = \frac{c - x}{2}, \quad \text{domain} = \mathbb{R}$$

**d**  $x = -2x + c$

$$3x = c$$

$$x = \frac{c}{3}$$

$$y = x$$

$$\text{co-ords} = \left(\frac{c}{3}, \frac{c}{3}\right)$$

**e**  $y = ax + b$

$$(0, c)$$

$$\Rightarrow b = c$$

$$y = ax + c$$

$$\text{normal line}$$

$$\Rightarrow a = \frac{-1}{-2}$$

$$a = \frac{1}{2}$$

$$y = \frac{x}{2} + c$$

**3**  $y = x^2 - bx$

$$\begin{aligned} \mathbf{a} \quad x^2 - bx &= 0 \\ x(x - b) &= 0 \\ x &= 0, b \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad x^2 - bx + \frac{b^2}{4} - \frac{b^2}{4} &= 0 \\ \left(x - \frac{b}{2}\right)^2 - \frac{b^2}{4} &= 0 \\ \text{co-ords: } \left(\frac{b}{2}, \frac{-b^2}{4}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{i} \quad -x &= x^2 - bx \\ x^2 - (b-1)x &= 0 \\ x(x - (b-1)) &= 0 \\ x &= 0, b-1 \\ y &= -x = 0, 1-b \\ \text{co-ords: } (0, 0), (b-1, 1-b) \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad b-1 &= 0, \\ b &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad b-1 &\neq 0, \\ b &\neq 1 \\ b &\in \mathbb{R} \setminus \{1\} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad y &= ax^2 + bx + c \\ \text{When } x = -1, y &= 6 \\ \text{When } x = 1, y &= 4 \\ 6 &= a - b + c \dots (1) \\ 4 &= a + b + c \dots (2) \end{aligned}$$

Equation (1) – Equation (2)

$$\begin{aligned} 2 &= -2b \\ b &= -1 \end{aligned}$$

Substitute in (1)  $6 = a + 1 + c$

$$\begin{aligned} a &= 5 - c \\ y &= (5 - c)x^2 - x + c \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad (1+h)^2 &= 8 \\ 1+h &= \pm 2\sqrt{2} \\ h &= -1 \pm 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (a+1)^2 &= 8 \\ a^2 &= 8 \\ a &= \pm 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad a &= ax^2 + bx \\ y &= a\left(x^2 + \frac{b}{a}x\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} \end{aligned}$$

vertex (1,8)

$$\begin{aligned} \mathbf{(1)} \quad \frac{-b}{2a} &= 1, \quad \mathbf{(2)} \quad -\frac{b^2}{4a} = 8 \\ \mathbf{(2)} &\Rightarrow \frac{b}{2} = 8 \\ \mathbf{(1)} &\Rightarrow \frac{b}{2} = 8 \end{aligned}$$

$$b = 16$$

$$\text{Sub in (1)} \Rightarrow \frac{-16}{2a} = 1$$

$$\frac{-8}{a} = 1$$

$$a = -8$$

check

$$\Rightarrow \mathbf{(2)} \Rightarrow \frac{-(16)^2}{4 \cdot -8} = 8$$

$$\frac{256}{32} = 8$$

$$\frac{2}{2^5} = 2^3 \quad \text{correct}$$

$$a = -8, b = 16$$

$$\mathbf{6} \quad f(x) = \sqrt{2a-x}$$

$$\mathbf{a} \quad 2a - x \geq 0$$

$$2a \geq x, \text{ so domain} = (-\infty, 2a]$$

$$\begin{aligned}
 \mathbf{b} \quad x &= \sqrt{2a-x} \\
 x^2 &= 2a-x \\
 x^2 + x - 2a &= 0 \\
 x &= \frac{-1 \pm \sqrt{1+8a}}{2} \\
 \text{but } y > 0, \quad \therefore x > 0 \\
 \therefore x &= \frac{-1 \pm \sqrt{1+8a}}{2} \\
 y &= x, \\
 \therefore \text{co-ords} &= \left( \frac{-1 + \sqrt{1+8a}}{2}, \frac{-1 + \sqrt{1+8a}}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \frac{-1 + \sqrt{1+8a}}{2} &= 1 \\
 -1 + \sqrt{1+8a} &= 2 \\
 \sqrt{1+8a} &= 3 \\
 1 + 8a &= 9 \\
 8a &= 8 \\
 a &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \frac{-1 + \sqrt{1+8a}}{2} &= 2 \\
 \sqrt{1+8a} &= 5 \\
 1 + 8a &= 25 \\
 8a &= 24 \\
 a &= 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad \frac{-1 + \sqrt{1+8a}}{2} &= c \\
 \sqrt{1+8a} &= 2c + 1 \\
 1 + 8a &= 4c^2 + 4c + 1 \\
 8a &= 4c^2 + 4c \\
 a &= \frac{c^2 + c}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad f(x) &= (x^2 - ax)^2 \\
 \mathbf{a} \quad 0 &= (x^2 - ax)^2 \\
 0 &= x(x-a) \\
 x &= 0, a \\
 \text{co-ords} &= (0, 0), (a, 0)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f(0) &= (0 - 0)^2 \\
 &= 0 \\
 \text{co-ords} &= (0, 0)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad x &\in [0, a] \\
 f(x) &= (ax - x^2)^2 \\
 &= \left( -\left(x^2 - ax + \frac{a^2}{4}\right) + \frac{a^2}{4} \right)^2 \\
 &= \left( -\left(x - \frac{a}{2}\right)^2 + \frac{a^2}{4} \right)^2 \\
 \text{maximum value is } &\frac{a^4}{16}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad f(-1) &= 16 \\
 16 &= ((-1)^2 - a(-1))^2 \\
 \pm 4 &= 1 + a \\
 1 + a &= \pm 4 \\
 a &= -1 \pm 4 \\
 a &= -5, 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad -ae^{bx} + c &= 0 \\
 e^{bx} &= \frac{c}{a} \\
 bx &= \ln \frac{c}{a} \\
 x &= \frac{1}{b} \ln \frac{c}{a}
 \end{aligned}$$

**b**  $c \ln(x + a) = b$

$$\ln(x + a) = \frac{b}{c}$$

$$x + a = e^{\frac{b}{c}}$$

$$x = e^{\frac{b}{c}} - a$$

**c**  $\ln(cx - a) = 0$

$$\ln(cx - a) = 1$$

$$cx = 1 + a$$

$$x = \frac{1 + a}{c}$$

**d**  $e^{ax+b} = c$

$$ax + b = \ln c$$

$$x = \frac{\ln c - b}{a}$$

**9**  $f(x) = c \ln(x - a)$

**a**  $x - a = 0$

$$x = a$$

**b**  $x - a = 1$

$$x = 1 + a$$

$$\text{co-ords} = (1 + a, 0)$$

**c**  $1 = c \ln(x - a)$

$$x - a = e^{\frac{1}{c}}$$

$$x = a + e^{\frac{1}{c}}$$

$$\text{co-ords} = (a + e^{\frac{1}{c}}, 0)$$

**d**  $1 = c \ln(2 - a)$

$$c = \frac{1}{\ln(2 - a)}$$

**a**  $y = 0 - b$

$$y = -b$$

**b**  $0 = e^{x-1} - b$

$$x - 1 = \ln b$$

$$x = \ln b + 1$$

$$\text{co-ords} = (\ln b + 1, 0)$$

**c i**  $\ln b + 1 = 0$

$$\ln b = -1$$

$$b = e^{-1}$$

$$b = \frac{1}{e}$$

**ii**  $\ln b + 1 < 0$

$$\ln b < -1$$

$$b < e^{-1}$$

$$b < \frac{1}{e}$$

but  $b > 0$  as given, else there is no intercept

$$\therefore 0 < b < \frac{1}{e}$$

**11**  $y = ax^3 + bx^2 + cx + d$

$$(-1, 6)$$

$$\Rightarrow \textcircled{1} \quad 6 = -a + b - c + d$$

$$(1, -2)$$

$$\Rightarrow \textcircled{2} \quad -2 = a + b + c + d$$

$$\textcircled{2} + \textcircled{1} \quad 4 = 2b + 2d$$

$$b = 2 - d$$

$$(2, 4)$$

$$\Rightarrow \textcircled{3} \quad 4 = 8a + 4b + 2c + d$$

$$\textcircled{2} - \textcircled{1} \Rightarrow 2a + 2c = -8$$

$$a + c = -4$$

$$a = -4 - c$$

**10**  $f(x) = e^{x-1} - b$



Sub in ③  $\Rightarrow$

$$4 = 8(-4 - c) + 4(2 - d) + 2c + d$$

$$4 = -32 - 8c + 8 - 4d + 2c + d$$

$$4 = -24 - 6c - 3d$$

$$6c + 3d = -28$$

$$6c = -28 - 3d$$

$$c = \frac{-28 - 3d}{6}$$

$$a = -4 - c$$

$$a = 4 - c$$

$$a = \frac{-24 + 28 + 3d}{6}$$

$$a = \frac{3d + 4}{6}$$

$$\begin{aligned} 12 \quad y &= \left(\frac{c-8}{2}\right)x^2 + \left(\frac{20-3c}{2}\right)x + c \\ b^2 - 4ac & \\ &= \left(\frac{20-3c}{2}\right)^2 - 4\left(\frac{c-8}{2}\right)c \\ &= \frac{400 - 120c + 9c^2}{4} - 2c^2 + 16c \\ &= 100 - 30c + \frac{9}{4}c^2 - 2c^2 + 4c \\ &= \frac{1}{4}c^2 - 14c + 100 \end{aligned}$$

**a**  $b^2 - 4ac = 0$

$$c^2 - 56c + 400 = 0$$

$$c = \frac{56 \pm \sqrt{1536}}{2}$$

$$c = 28 \pm 8\sqrt{6}$$

**b**  $b^2 - 4ac > 0$

$$c^2 - 56c + 400 > 0$$

$$c < 28 - 8\sqrt{6} \text{ or } c > 28 + 8\sqrt{6}$$

but  $c \neq 8$  (since if  $c = 8$ , the function

becomes linear

$\therefore c < 8$ , or

$$8 < c < 28 - 8\sqrt{6} \text{ or } c > 28 + 8\sqrt{6}$$

**13 a**  $y = ax^3 + bx^2 + cx + d$

$(-2, 8)$

$$\Rightarrow \textcircled{1} \quad 8 = -8a + 4b - 2c + d$$

$(1, 1)$

$$\Rightarrow \textcircled{2} \quad 1 = a + b + c + d$$

$(3, 4)$

$$\Rightarrow \textcircled{3} \quad \begin{aligned} 4 &= 27a + 9b \\ &\quad + 3c + d \end{aligned}$$

$$\textcircled{3} - 3\textcircled{2} \Rightarrow \textcircled{4} \quad 1 = 24a + 6b - 2d$$

$$2\textcircled{3} + 3\textcircled{1} \Rightarrow \textcircled{5} \quad 32 = 30a + 30b + 5d$$

$$\textcircled{5} \Rightarrow \quad \frac{32}{5} = 6a + 6b + d$$

$$\textcircled{4} - \textcircled{5} \Rightarrow \quad 1 - \frac{32}{5} = 18a - 3d$$

$$-\frac{9}{5} = 6a - d$$

$$a = \frac{5d - 9}{30}$$

$$\text{Sub in } \textcircled{5} \Rightarrow \frac{32}{5} = \frac{5d - 9}{5} + 6b + d$$

$$32 = 5d - 9 + 30b + 5d$$

$$30b = 41 - 10d$$

$$b = \frac{41 - 10d}{30}$$

$$\text{Sub in } \textcircled{2} \Rightarrow 1 = \frac{5d - 9}{30} + \frac{41 - 10d}{30} + c + d$$

$$30d = 5d - 9 + 41 - 10d$$

$$+ 30c + 30d$$

$$-2 = 30c + 25d$$

$$c = \frac{-2 - 25d}{30}$$

**14 a**  $AX + C = X'$

$$A^{-1} = \begin{bmatrix} \frac{k}{\det(A)} & 0 \\ 0 & \frac{-4}{\det(A)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{-4} & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{-4} & 0 \\ 0 & \frac{1}{k} \end{bmatrix} \begin{bmatrix} x' - 3 \\ y' - 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x' - 3}{-4} \\ \frac{y' - 2}{k} \end{bmatrix}$$

**b**  $y = \frac{1}{x}$   
 $\frac{y' - 2}{k} = \frac{1}{\left(\frac{x' - 3}{-4}\right)}$  (from (a))

$$y' - 2 = \frac{-4k}{x' - 3}$$

$$y' = \frac{-4k}{x' - 3} + 2$$

**c**  $0 = \frac{-4k}{0 - 3} + 2$   
 $\frac{4}{3}k = -2$   
 $k = \frac{-6}{4} = \frac{-3}{2}$

**15 a**

$$AX + C = X'$$

$$X = A^{-1}(X' - C)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \left( \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} a \\ -2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \det(A) = -4 \times 2 = -8$$

$$A^{-1} = \begin{bmatrix} \frac{2}{\det(A)} & 0 \\ 0 & \frac{-4}{\det(A)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{-8} & 0 \\ 0 & \frac{-4}{-8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{-4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{-4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x' - a \\ y' + 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x' - a}{-4} \\ \frac{y' + 2}{2} \end{bmatrix}$$

**b**

$$y = 2^x$$

$$\frac{y' + 2}{2} = 2^{\left(\frac{x' - a}{4}\right)}$$

$$y' + 2 = 2^{\left(\frac{x' - a}{4} + 1\right)}$$

$$y' = 2^{\left(\frac{x' + 4 - a}{4}\right)} - 2 = 2 \times 2^{\frac{(x' - a)}{4}} - 2$$

**c**  $0 = 2^{\left(\frac{0 + 4 - a}{4}\right) - 2}$

$$2 = 2^{\left(\frac{4 - a}{4}\right)}$$

$$\frac{4 - a}{4} = 1$$

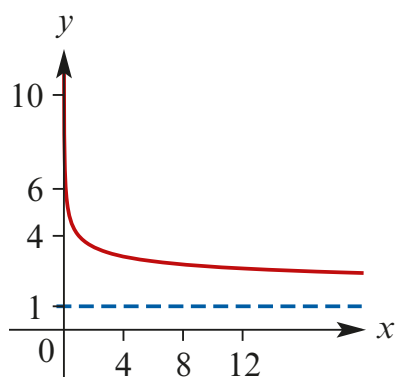
$$4 - a = 4$$

$$-a = 0$$

$$a = 0$$

## Solutions to Technology-free questions

1 a

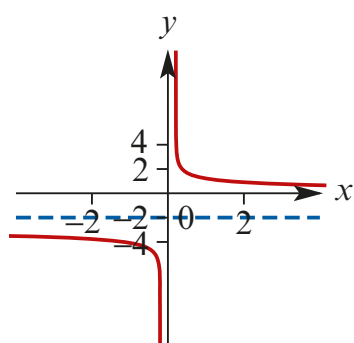


Domain =  $\mathbb{R}^+$

Range =  $(1, \infty)$

Neither

b



Domain =  $\mathbb{R} \setminus \{0\}$

Range =  $\mathbb{R} \setminus \{-2\}$

Neither

2 a  $243^{\frac{2}{5}} = 3^2 = 9$

b  $(-243)^{\frac{2}{5}} = (-3)^2 = 9$

c  $243^{\frac{3}{5}} = 3^3 = 27$

d  $(-243)^{\frac{2}{5}} = (-3)^2 = 9$

e  $(-27)^{\frac{5}{3}} = (-3)^5 = -243$

f  $(-125)^{\frac{4}{3}} = (-5)^4 = 625$

3 a i  $f \circ g(x) = 3 \cos(2x^2)$ ,  $g \circ f(x) = 9 \cos^2(2x)$

ii  $\text{dom}(f \circ g) = \mathbb{R}$ ,  $\text{ran}(f \circ g) = [-3, 3]$ ,  $\text{dom}(g \circ f) = \mathbb{R}$ ,  $\text{ran}(g \circ f) = [0, 9]$

- b i**  $f \circ g(x) = \log_e(3x^2)$ ,  $g \circ f(x) = (\log_e(3x))^2$
- ii**  $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$ ,  $\text{ran}(f \circ g) = \mathbb{R}$ ,  $\text{dom}(g \circ f) = \mathbb{R}^+$ ,  $\text{ran}(g \circ f) = [0, \infty)$
- c i**  $f \circ g(x) = \log_e(2 - x^2)$ ,  $g \circ f(x) = (\log_e(2 - x))^2$
- ii**  $\text{dom}(f \circ g) = (-\sqrt{2}, \sqrt{2})$ ,  $\text{ran}(f \circ g) = (-\infty, \log_e 2)$ ,  $\text{dom}(g \circ f) = (-\infty, 2)$ ,  $\text{ran}(g \circ f) = [0, \infty)$
- d i**  $f \circ g(x) = -\log_e(2x^2)$ ,  $g \circ f(x) = (\log_e(2x))^2$
- ii**  $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$ ,  $\text{ran}(f \circ g) = \mathbb{R}$ ,  $\text{dom}(g \circ f) = (0, \infty)$ ,  $\text{ran}(g \circ f) = [0, \infty)$

- 4 a**  $h(x) = f \circ g(x)$ ,  $g(x) = x^2$ ,  $f(x) = \cos x$  (*Note: answer not unique*)
- b**  $h(x) = f \circ g(x)$ ,  $g(x) = x^2 - x$ ,  $f(x) = x^n$  (*Note: answer not unique*)
- c**  $h(x) = f \circ g(x)$ ,  $g(x) = \sin x$ ,  $f(x) = \log_e x$  (*Note: answer not unique*)
- d**  $h(x) = f \circ g(x)$ ,  $g(x) = \sin(2x)$ ,  $f(x) = -2x^2$  (*Note: answer not unique*)
- e**  $h(x) = f \circ g(x)$ ,  $g(x) = x^2 - 3x$ ,  $f(x) = x^4 - 2x^2$  (*Note: answer not unique*)

**5 a i**  $(f + g)(x) = 2 \cos\left(\frac{\pi x}{2}\right) + e^{-x}$

**ii**  $(fg)(x) = 2e^{-x} \cos\left(\frac{\pi x}{2}\right)$

**b i**  $(f + g)(0) = 3$

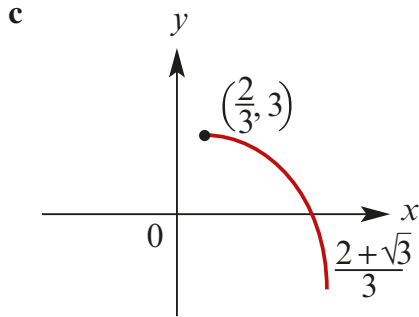
**ii**  $(fg)(0) = 2$

**6**  $f : [a, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = -(3x - 2)^2 + 3$

Turning point at  $\left(\frac{2}{3}, 3\right)$

**a**  $\frac{2}{3}$

**b**  $(-\infty, 3]$



**d**

$$f(f^{-1}(x)) = x$$

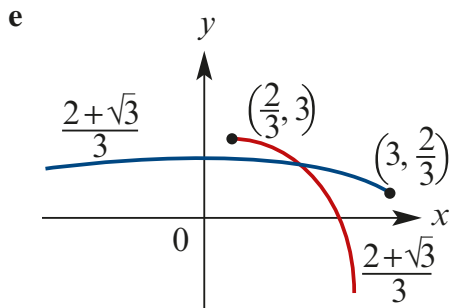
$$-3(f^{-1}(x) - 2)^2 + 3 = x$$

$$(f^{-1}(x) - 2)^2 = -\frac{x-3}{3}$$

$$f^{-1}(x) = 2 \pm \sqrt{-\frac{x-3}{3}}$$

Because of domain and range of  $f$

$$f^{-1}(x) = \frac{2 + \sqrt{3-x}}{3}, \text{ ran} = [\frac{2}{3}, \infty), \text{ dom} = (-\infty, 3]$$



**7**  $f(x) = c \log_e(x - a)$

**a**  $x = a$

**b**  $c \log_e(x - a) = 0 \Rightarrow x - a = e^0 \Rightarrow x = a + 1$   
Therefore coordinates of the  $x$ -axis intercept are  $(a + 1, 0)$

**c**  $c \log_e(x - a) = c \Rightarrow \log_e(x - a) = 1 \Rightarrow x = a + e^1$   
Therefore coordinates of the point where the curve crosses the line  $y = c$  is  $(e + a, c)$

**d**  $f(f^{-1}(x)) = x$

$$c \log_e(f^{-1}(x) - a) = x$$

$$\log_e(f^{-1}(x) - a) = \frac{x}{c}$$

$$f^{-1}(x) = e^{\frac{x}{c}} + a$$

**e**  $(a, \infty)$

**f**  $f^{-1}(1) = 2 \Rightarrow f(2) = 1$

$$f^{-1}(2) = 4 \Rightarrow f(4) = 2$$

$$c \log_e(2 - a) = 1 \dots (1)$$

$$c \log_e(4 - a) = 2 \dots (2)$$

Equation (2)  $\div$  Equation (1)

$$\frac{\log_e(4 - a)}{\log_e(2 - a)} = 2$$

$$\log_e(4 - a) = 2 \log_e(2 - a)$$

$$4 - a = (2 - a)^2$$

$$4 - a = 4 - 4a + a^2$$

$$a^2 - 3a = 0$$

$$a(a - 3) = 0$$

$$a = 0 \text{ or } a = 3$$

But  $a = 3$  does not satisfy our equations

$$\therefore c = \frac{1}{\log_e 2}, a = 0$$

**8** Consider,

$$af^{-1}(x) + b = x$$

$$\therefore f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$$

$$\text{If } f^{-1}(x) = 4x - 6$$

$$a = \frac{1}{4} \text{ and } -\frac{b}{a} = -6$$

$$\therefore a = \frac{1}{4} \text{ and } b = -\frac{3}{2}$$

**9 a**  $f^{-1}(x) = \left(\frac{x-1}{3}\right)^3$

**b**  $f^{-1}(x) = \left(\frac{x+2}{4}\right)^3$

**c**  $f^{-1}(x) = \frac{1}{3}\left((x-4)^{\frac{1}{3}} + 2\right)$

**d**  $f^{-1}(x) = \left(\frac{3-x}{2}\right)^{\frac{1}{3}}$

## Solutions to multiple-choice questions

1 B

$$h(x) = \frac{x^4 + 2}{x^2}$$

Split  $h(x)$  into two separate fractions:

$$h(x) = x^2 + \frac{2}{x^2}$$

$$\therefore f(x) = x^2, g(x) = \frac{2}{x^2}$$

2 E The graph of

$$f: R \rightarrow R, f(x) = \cos(x)$$

Is not a one to one function.

3 E

A:  $e^{x+y} = e^x \times e^y$ , so A is true.

B:  $\log_e xy = \log_e x + \log_e y$ , so B is true.

C:  $\log_e x^y = y \log_e x$ , so C is true.

D:  $f^{-1}(1) = \log_e 1 = 0$ , true for any  $x, y$ .

4 D

$$f(x) = \cos x$$

$$g(x) = 3x^2$$

$$g(f(x)) = 3(\cos x)^2$$

$$g\left(f\left(\frac{\pi}{3}\right)\right) = 3\left(\cos \frac{\pi}{3}\right)^2$$

$$g\left(f\left(\frac{\pi}{3}\right)\right) = 3\left(\frac{1}{2}\right)^2$$

$$g\left(f\left(\frac{\pi}{3}\right)\right) = \frac{3}{4}$$

5 E

$f: R \rightarrow R, f(x) = (x - 2)^2$  is not an even function as it is a parabola that has been translated 2 units right, so it is not symmetrical about the y-axis.

6 E

$$y = 2ax + \cos 2x$$

When  $x = \pi, y = 0$

$$0 = 2\pi a + \cos 2\pi$$

$$-1 = 2\pi a$$

$$a = -\frac{1}{2\pi}$$



**7 B**  $x > 5, g(x) = \log_e(x - 5)$   
 $2[g(x)] = g(f(x))$   
 $2[g(x)] = \log_e(x - 5)^2$   
 $\log_e(x - 5)^2 = \log_e(f(x) - 5)$   
 $(x - 5)^2 = f(x) - 5$   
 $x^2 - 10x + 25 = f(x) - 5$   
 $f(x) = x^2 - 10x + 30$

**8 C**

**9 D**

**10 C** Domain of  $(f + g) = (-\infty, 3) \cap [2, \infty) = [2, 3)$

**11 B**  $\frac{y' - 2}{-4} = \sin\left(3x' - \frac{\pi}{3}\right)$   
 $\therefore$  choose  $y' = -4y + 2$  and  $x' = \frac{1}{3}x + \frac{\pi}{9}$

**12 D**  $\frac{y + 3}{2} = \sin\left(2x - \frac{\pi}{4}\right)$  to  $y' = \sin x' \therefore$  choose  $y' = \frac{y + 3}{2} = \frac{1}{2}y + \frac{3}{2}$  and  $x' = 2x - \frac{\pi}{4}$

**13 E**

**14 B**

## Solutions to extended-response questions

**1 a** The range of  $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $f(x) = e^{-x}$  is  $(0, 1)$   
 The range of  $g: (-\infty, 0) \rightarrow \mathbb{R}$ ,  $g(x) = \frac{1}{x-1}$  is  $\mathbb{R}^-$

**b** domain of  $f^{-1} = \text{range of } f = (0, 1)$   
 domain of  $g^{-1} = \text{range of } g = \mathbb{R}^+$   
 To determine the rule for  $f^{-1}$  consider

$$x = e^{-y}$$

$$\log_e x = -y$$

$$-\log_e x = y$$

$$\text{Therefore } f^{-1}(x) = -\log_e x$$

To determine the rule for  $g^{-1}$  consider

$$x = \frac{1}{y-1}$$

Taking the reciprocal of both sides

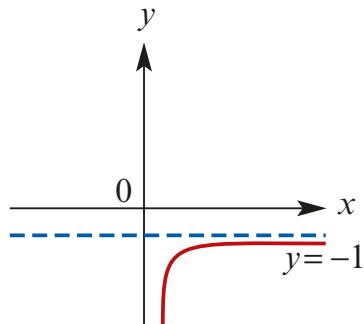
$$y-1 = \frac{1}{x}$$

$$\text{and } y = \frac{1}{x} + 1$$

$$g^{-1}(x) = \frac{1}{x} + 1$$

**c i**  $g \circ f(x)$  is defined as range of  $f \subseteq \text{domain of } g$   
 $g \circ f(x) = g(f(x)) = g(e^{-x}) = \frac{1}{e^{-x}-1} = \frac{e^x}{1-e^x}$

**ii**



$$g \circ f(x) = \frac{e^x}{1-e^x} = -1 + \frac{1}{1-e^x}$$

**d i**  $g \circ f(x) = \frac{e^x}{1-e^x}$  with domain  $= \mathbb{R}^+$  For the inverse consider

$$\frac{e^y}{1-e^y} = x$$

Solve for  $y$

$$x(1 - e^y) = e^y$$

$$x - xe^y = e^y$$

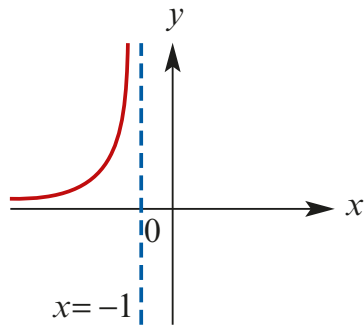
$$\text{Therefore } e^y(1 + x) = x$$

$$\text{and } e^y = \frac{x}{1 + x}$$

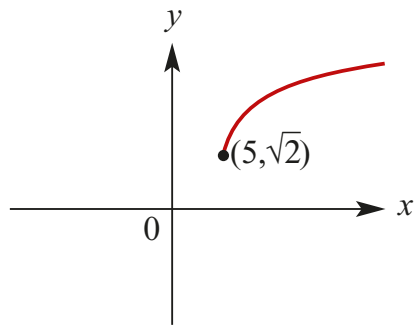
$$\text{Therefore } y = \log_e \left( \frac{x}{1 + x} \right)$$

$$(g \circ f)^{-1}(x) = \log_e \left( \frac{x}{1 + x} \right) \text{ and domain of } (g \circ f)^{-1} = \text{range of } g \circ f = (-\infty, -1)$$

ii



2 a i  $f: [5, \infty) \rightarrow \mathbb{R}, f(x) = \sqrt{x - 3}$



ii range of  $f = [\sqrt{2}, \infty)$

iii For the inverse rule consider  $x = \sqrt{y - 3}$   
Square both sides and make  $y$  the subject.

$$y = x^2 + 3$$

and  $f^{-1}(x) = x^2 + 3$ . The domain of the inverse function is  $[\sqrt{2}, \infty)$

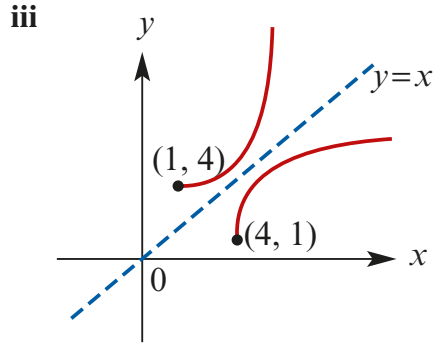
b i  $h: [4, \infty) \rightarrow \mathbb{R}, h(x) = \sqrt{x - p}$

The inverse function has domain  $[1, \infty)$ .

The function  $h$  is increasing and therefore  $\sqrt{4 - p} = 1$

Therefore  $p = 3$ .

ii Proceeding as above the rule is  $h^{-1}(x) = x^2 + 3$ .



3  $f: (0, \pi) \rightarrow \mathbb{R}, f(x) = \sin x$

$g: [1, \infty) \rightarrow \mathbb{R}, g(x) = \frac{1}{x}$

a range of  $f = (0, 1)$

b range of  $g = (0, 1]$

c  $f \circ g$  is defined as the range of  $g \subseteq$  domain of  $f$  as  $1 < \pi$

$$f \circ g(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \sin\left(\frac{1}{x}\right)$$

d  $g \circ f$  is not defined as the range of  $f \not\subseteq$  domain of  $g$

e Consider

$$x = \frac{1}{y} \text{ and solve for } y.$$

$$\text{Therefore } g^{-1}(x) = \frac{1}{x}$$

The domain of  $g^{-1} =$  range of  $g = (0, 1]$

The range of  $g^{-1} =$  domain of  $g = [1, \infty)$

f Range of  $f = (0, 1) \subseteq (0, 1] =$  domain of  $g^{-1}$ .

Therefore  $g^{-1}$  of is defined and

$$g^{-1} \circ f(x) = g^{-1}(f(x)) = g^{-1}(\sin x) = \frac{1}{\sin x}$$

The domain of  $g^{-1} \circ f =$  domain of  $f = (0, \pi)$

The range of  $g^{-1} \circ f = [1, \infty)$

4 a  $a = 2$

b  $c = 2 - k \log_e(2)$

c  $k = \frac{10}{\log_e\left(\frac{d+2}{2}\right)}$

d  $k = 10$

# Chapter 8 – Revision of Chapters 1–7

## Solutions to technology-free questions

1 a Domain =  $\mathbb{R} \setminus \{0\}$ ; Range =  $\mathbb{R} \setminus \{2\}$

b  $3x - 2 \geq 0 \Rightarrow x \geq \frac{2}{3}$

The endpoint is  $\left(\frac{2}{3}, 3\right)$

Domain =  $\left[\frac{2}{3}, \infty\right)$ ; Range =  $(-\infty, 3]$

c Domain =  $\mathbb{R} \setminus \{2\}$ ; Range =  $(3, \infty)$

d Domain =  $\mathbb{R} \setminus \{2\}$ ; Range =  $\mathbb{R} \setminus \{4\}$

e  $x - 2 \geq 0 \Rightarrow x \geq 0$

The endpoint is  $(2, -5)$

Domain =  $[2, \infty)$ ; Range =  $(-5, \infty]$

2  $\sqrt{f^{-1}(x) - 2} + 4 = x$

$\therefore \sqrt{f^{-1}(x) - 2} = x - 4$

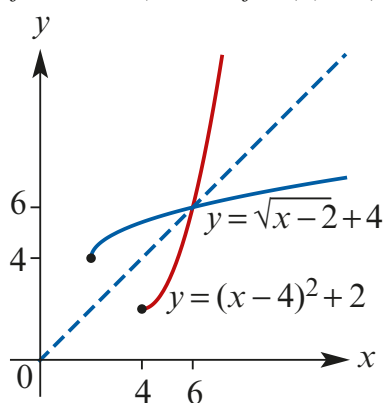
$\therefore f^{-1}(x) - 2 = (x - 4)^2$

$\therefore f^{-1}(x) = (x - 4)^2 + 2$

The domain of  $f^{-1}$  = range of  $f$  =  $[4, \infty)$

Hence,

$f^{-1}: [4, \infty) \rightarrow \mathbb{R}, f^{-1}(x) = (x - 4)^2 + 2$



3  $\frac{f^{-1}(x) - 2}{f^{-1}(x) + 1} = x$

$\therefore f^{-1}(x) - 2 = x(f^{-1}(x) + 1)$

$\therefore f^{-1}(x) - xf^{-1}(x) = x + 2$

$\therefore f^{-1}(x)(1 - x) = x + 2$

$\therefore f^{-1}(x) = \frac{x + 2}{1 - x}$

The domain of  $f^{-1}$  = range of  $f$  =  $\mathbb{R} \setminus \{1\}$

Hence,

$f^{-1}: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}, f^{-1}(x) = \frac{x + 2}{1 - x}$

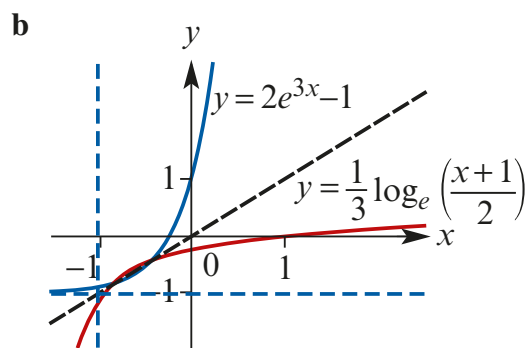
4 a  $2e^{f^{-1}(x)} - 1 = x$

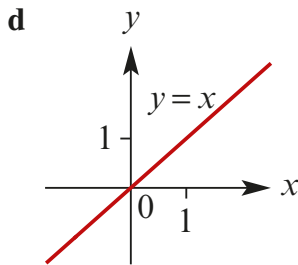
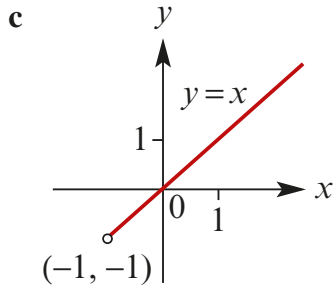
$\therefore e^{f^{-1}(x)} = \frac{x + 1}{2}$

$\therefore f^{-1}(x) = \log_e \left( \frac{x + 1}{2} \right)$

$f^{-1}(x) = \frac{1}{3} \log_e \left( \frac{x + 1}{2} \right)$

dom  $f^{-1} = (-1, \infty)$





**e**  $y = 2x$

**5**  $2 \log_{10} 5 + 3 \log_{10} 2 - \log_{10} 20$   
 $= \log_{10} 25 + \log_{10} 8 - \log_{10} 20$   
 $= \log_{10} \frac{200}{20}$   
 $= \log_{10} 10$   
 $= 1$

**6**  $3 \log_a x = 3 + \log_a 12$   
 $\log_a x^3 - \log_a 12 = 3$   
 $\log_a \left( \frac{x^3}{12} \right) = 3$   
 $\frac{x^3}{12} = a^3$   
 $x^3 = 12a^3$   
 $x = \sqrt[3]{12a^3}$

**7**  $2^{-x} = 2^9$   
 $x = -9$

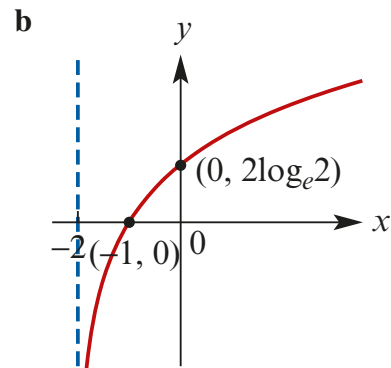
**8**  $4e^{2x} = 9$

$$e^{2x} = \frac{9}{4}$$

$$2x = \log_e \left( \frac{9}{4} \right)$$

$$x = \frac{1}{2} \log_e \left( \frac{9}{4} \right) = \log_e \left( \frac{3}{2} \right)$$

**9 a** When  $x = 0$ ,  $f(0) = 2 \log_e 2$   
 When  $f(x) = 0$   
 $2 \log_e (x + 2) = 0$   
 $\log_e (x + 2) = 0$   
 $x + 2 = e^0$   
 $x = -1$   
 $\therefore a = -1$  and  $b = 2 \log_e 2$



**10**  $2^{4x} - 5 \times 2^{2x} + 4 = 0$   
 Let  $a = 2^{2x}$   
 $a^2 - 5a + 4 = 0$   
 $(a - 4)(a - 1) = 0$   
 $a = 4$  or  $a = 1$   
 $\therefore 2^{2x} = 4$  or  $2^{2x} = 1$   
 $\therefore x = 0$  or  $x = 1$

**11**  $\sin \left( \frac{3x}{2} \right) = \frac{1}{2}$   
 $\frac{3x}{2} = -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

$$x = -\frac{7\pi}{9} \text{ or } x = \frac{\pi}{9} \text{ or } x = \frac{5\pi}{9}$$

**12 a** Range = [2, 8]; Period = 6

$$\begin{aligned} \mathbf{b} \quad \cos\left(2x + \frac{\pi}{6}\right) &= \frac{1}{2} \\ 2x + \frac{\pi}{6} &= \frac{\pi}{3}, \frac{5\pi}{3} \\ 2x + \frac{\pi}{6} &= \frac{2\pi}{6}, \frac{10\pi}{6} \\ 2x &= \frac{\pi}{6}, \frac{9\pi}{6} \\ x &= \frac{\pi}{12} \text{ or } x = \frac{3\pi}{4} \end{aligned}$$

**13** Consider the gradients of the two lines

$$\begin{aligned} \text{Gradient } \ell_1 &= -m \\ \text{and Gradient } \ell_2 &= -\frac{2}{m-1} \end{aligned}$$

If the gradients are equal

$$\begin{aligned} -m &= -\frac{2}{m-1} \\ m &= \frac{2}{m-1} \end{aligned}$$

$$m^2 - m - 20$$

$$(m-2)(m+1) = 0$$

$$m = 2 \text{ or } m = -1$$

**a** Therefore a unique solution when the lines are not parallel,  $m \in \mathbb{R} \setminus \{-1, 2\}$

**b** If  $m = 2$

$$2x + y = 2$$

$$2x + y = -4$$

The lines are parallel but do not coincide.

There is no solution.

$m = -1$  is checked in the next part.

**c** If  $m = -1$

$$-x + y = 2$$

$$2x - 2y = -4$$

The lines coincide and there are infinitely many solutions.

$$\mathbf{14} \quad y = \frac{a}{x^2} + b$$

$$\text{When } x = 1, y = -1$$

$$\text{When } x = -2, y = \frac{1}{2}$$

$$a + b = -1 \dots (1)$$

$$\frac{a}{4} + b = \frac{1}{2} \dots (2)$$

Equation (1) – Equation (2)

$$\frac{3a}{4} = -\frac{3}{2}$$

$$\therefore a = -2 \text{ and } b = 1$$

$$\mathbf{15} \quad \Delta = m^2 - 8$$

$$\mathbf{a} \quad \Delta = 0 \Rightarrow m = \pm 2\sqrt{2}$$

$$\mathbf{b} \quad \Delta > 0 \Rightarrow m > 2\sqrt{2} \text{ or } m < -2\sqrt{2}$$

$$\mathbf{c} \quad \Delta < 0 \Rightarrow -2\sqrt{2} < m < 2\sqrt{2}$$

$$\mathbf{16 a i} \quad \frac{a+3}{2} = 0, \therefore a = -3$$

$$\mathbf{ii} \quad \sqrt{(a-3)^2 + (-2-1)^2} = \sqrt{13}$$

$$a^2 - 6a + 9 + 9 = 13$$

$$a^2 - 6a + 5 = 0$$

$$(a-5)(a-1) = 0$$

$$a = 5 \text{ or } a = 1$$

$$\mathbf{iii} \quad \frac{3}{3-a} = \frac{1}{2}$$

$$6 = 3 - a$$

$$a = -3$$

**b** If  $a = -2$  the gradient of the line is  $\frac{3}{5}$

The equation of the line is

$$y - 1 = \frac{3}{5}(x - 3)$$

$$\text{or } 5y - 3x + 4 = 0$$

The angle the line makes with the positive direction of the  $x$ -axis is

$$\tan^{-1}\left(\frac{3}{5}\right).$$

**17 a** Odd

**b**  $f^{-1}(x) = \sqrt[3]{\frac{x}{2}}$

**c i** 2

**ii** -1

**iii**  $f^{-1}(x) = f(x)$

$$\sqrt[3]{\frac{x}{2}} = 2x^3$$

$$\frac{x}{2} = 8x^9$$

$$x - 16x^9 = 0$$

$$x(1 - 16x^8) = 0$$

$$x = 0 \text{ or } x = \left(\frac{1}{16}\right)^{\frac{1}{8}}$$

$$x = 0 \text{ or } x = 2^{-\frac{1}{2}} \text{ or } x = -2^{-\frac{1}{2}}$$

**18 a** 4

**b**  $\sqrt{5}$

**c**  $2 - 2a$

**d**  $\sqrt{2a - 5}$

**e**  $x = -8$

**f**  $x = \frac{103}{2}$

**g**  $x < 1$

**19 a i**  $f \circ g(x) = 4x^2 + 8x - 3$

**ii**  $g \circ f(x) = 16x^2 - 16x + 3$

**iii**  $g \circ f^{-1}(x) = \frac{1}{16}(x^2 + 14x + 33)$

**b** Dilation of factor  $\frac{1}{4}$  from the  $y$ -axis, then translation  $\frac{3}{4}$  units to the right

**c** Translation 1 unit to the left and 1 unit down

**20**  $x = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

**21**  $Ae^k = 4 \dots (1)$

$$Ae^{2k} = 10 \dots (2)$$

Equation (2)  $\div$  Equation (1)

$$e^k = \frac{5}{2}$$

$$k = \log_e\left(\frac{5}{2}\right)$$

$$A = \frac{8}{5} \text{ and } k = \log_e\left(\frac{5}{2}\right)$$

**22 a**  $2x^3 - 3x^2 - 11x + 6 \geq 0$

$$\Leftrightarrow (2x - 1)(x - 3)(x + 2) \geq 0$$

$$\Leftrightarrow -2 \leq x \leq \frac{1}{2} \text{ or } x \geq 3$$

**b**  $-x^3 + x^2 - 4x > 0$

$$\Leftrightarrow x(x^2 - x + 4) < 0$$

$$\Leftrightarrow x < 0$$



## Solutions to multiple-choice questions

1 **D** Domain =  $[-1, 3)$  since  $-1$  is included and  $3$  is excluded.

2 **A** For each value of  $x > 0$ , the rule  $x = 2y^2$ ,  $x \geq 0$  gives two values for  $y$ , so is not a function.

3 **B** Require  $2 - x > 0$ , i.e.  $x < 2$ . Implied domain =  $(-\infty, 2)$

$$\begin{aligned} 4 \text{ E } f\left(-\frac{1}{a}\right) &= \frac{-\frac{1}{a}}{-\frac{1}{a} - 1} \\ &= \frac{-1}{-1 - a} \\ &= \frac{1}{a + 1} \end{aligned}$$

$$\begin{aligned} 5 \text{ E } (f + g)\left(\frac{3\pi}{2}\right) &= f\left(\frac{3\pi}{2}\right) + g\left(\frac{3\pi}{2}\right) \\ &= \sin(3\pi) + 2 \sin\left(\frac{3\pi}{2}\right) \\ &= -2 \end{aligned}$$

$$\begin{aligned} 6 \text{ C } f(g(3)) &= f(18) \\ &= 56 \end{aligned}$$

$$\begin{aligned} 7 \text{ A } \text{dom } f &= [0, 6]; \text{ dom } g \\ &= (-\infty, 2] \\ \text{dom } (f + g) &= \text{dom } f \cap \text{dom } g \\ &= [0, 2] \end{aligned}$$

8 **A** For  $x \leq 0$ , the gradient is  $-2$  and the  $y$  intercept is  $(0, -2)$ ; the equation is  $y = -2x - 2$  for  $x \leq 0$ . For  $x > 0$ , the gradient is  $1$  and the  $y$  intercept is  $(0, -2)$ ; the equation is  $y = x - 2$  for  $x > 0$ .

$$\begin{aligned} 9 \text{ B } fg(x) &= (2x^2 + 1)(3x + 2) \\ &= 6x^3 + 4x^2 + 3x + 2 \end{aligned}$$

$$\begin{aligned} 10 \text{ C } \text{Require } 4 - x^2 &\geq 0, \text{ i.e.} \\ (2 - x)(2 + x) &\geq 0 \\ -2 \leq x &\leq 2 \\ \text{Implied domain} &= [-2, 2] \end{aligned}$$

11 **C** Reflect the graph of  $y = f(x)$  in the line  $y = x$ . Then the endpoint  $(4, -2)$  reflects to  $(-2, 4)$ . Only the third graph fits.

12 **B**

13 **A** For  $x < 2$ , the straight line has gradient  $1$  and the  $y$ -intercept is  $(0, -3)$ ; the equation is  $y = x - 3$  for  $x < 2$ . For  $x \geq 2$ , the curve has equation  $y = (x - 2)^2$ .

14 **E**  $f(2) = 0$ ,  $f(3) = 2$  so  $\text{dom } f^{-1} = \text{ran } f = [0, 2]$ . For  $f, y = 2x - 4$ . For  $f^{-1}$ , interchange  $x$  and  $y$  and solve for  $y$ .

$$x = 2y - 4$$

$$x + 4 = 2y$$

$$y = \frac{x + 4}{2}$$

Hence:

$$f^{-1} : [0, 2] \rightarrow \mathbb{R}, f^{-1}(x) = \frac{x + 4}{2}$$

15 **D** Require a subset of  $\mathbb{R}$  so that  $f$  is one-to-one. Either  $x \geq 0$  or  $x \leq 0$  would do. Only the fourth option fits.

16 **C** The graph of  $y = h(x)$  has its vertex at  $(1, 1)$ , so for an inverse function to

exist,  $x \geq 1$ . Hence  $a = 1$ .

- 17 E** For  $f, y = 3x - 2$   
For  $f^{-1}$ , interchange  $x$  and  $y$  and solve for  $y$ .

$$x = 3y - 2$$

$$x + 2 = 3y$$

$$y = \frac{1}{3}(x + 2)$$

**18 C**  $2x = \frac{3x}{2} - 4$

$$2x - \frac{3x}{2} = -4$$

$$\frac{x}{2} = -4$$

$$x = -8$$

- 19 B** The straight line has gradient 2 and  $y$  intercept  $(0, -2)$ .

$$\text{Hence } y = 2x - 2$$

**20 D**  $\frac{2(x-1)}{3} - \frac{x+4}{2} = \frac{5}{6}$

$$\frac{4(x-1) - 3(x+4)}{6} = \frac{5}{6}$$

$$4x - 4 - 3x - 12 = 5$$

$$x - 16 = 5$$

$$x = 21$$

**21 C**  $m = \frac{0-3}{4-(-2)} = -\frac{3}{6} = -\frac{1}{2}$

$$\text{Use } y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{1}{2}(x - 4)$$

$$y = -\frac{1}{2}x + 2$$

$$2y = -x + 4$$

$$2y + x = 4$$

**22**

**C**  $y = \frac{4}{5}x - 4 = \frac{4}{5}(x - 5)$

$x$  - axis intercept is  $(5, 0)$

$y$  - axis intercept is  $(0, -4)$ .

$$\text{area } OAB = \frac{1}{2}(OA)(OB)$$

$$= \frac{1}{2}(5)(4)$$

$$= 10 \text{ square units}$$

**23 D**  $2x - 3y = 12$  ①

$$3x - 2y = 13$$
 ②

② - ① gives:

$$x + y = 1$$

(Note: in this case, you do not need to solve for  $x$  and  $y$  explicitly, although it is not wrong to do so.)

**24 D**  $7x - 6y = 20$  ①

$$3x + 4y = 2$$
 ②

$3 \times \text{②} + 2 \times \text{①}$  gives :

$$9x + 14x = 6 + 40$$

$$23x = 46$$

$$x = 2$$

- 25 E** The graph is that of a hyperbola with asymptotes  $x = 1$  and  $y = -3$ .

The equation is of the form

$$y = \frac{a}{x-1} - 3$$

$$x = 0, y = -4: \quad -4 = \frac{a}{-1} - 3$$

$$-4 = -a - 3$$

$$a = 1$$

$$y = \frac{1}{x-1} - 3$$

(Check:  $x = \frac{4}{3}, y = 0$  as expected.)

- 26 A** As  $x \rightarrow \pm\infty, y = f(x) \rightarrow -2$

$y = -2$  is an asymptote So the range is  $R \setminus \{-2\}$

**27 D** Vertex at  $(2, 3)$  means  
 $y = a(x - 2)^2 + 3$ .  
 Only the fourth option in which  
 $a = 1$  fits.

**28 D** Require  $f(-x) = f(x)$  for any value of  $x$ .  
 If  $f(x) = -x^2$ , then  
 $f(-x) = -(-x)^2$   
 $= -x^2$   
 $= f(x)$

So  $f(x) = -x^2$  is an even function of  $x$ .  
 (A quick check reveals that none of the other functions is even.)

**29 A** The factor ' $x + 2$ ' indicates a translation of 2 units to the left.  
 So  $(x, y) \rightarrow (x - 2, y)$ .  
 The factor ' $3$ ' indicates a dilation of factor 3 from the  $x$ -axis.

**30 E** Require  $x - 2 \geq 0$ , i.e.  $x \geq 2$ .  
 Maximal domain =  $[2, \infty)$

**31 D** The graph has endpoint  $(3, 1)$  so its equation must be of the form  
 $y = a\sqrt{x - 3} + 1$   
 $x = 4, y = 0: 0 = a\sqrt{1} + 1$   
 $a = -1$   
 $y = -\sqrt{x - 3} + 1$

**32 E**  $\frac{3}{(x - 2)^2} > 0$  for any  $x \neq 2$ .  
 So  $\frac{3}{(x - 2)^2} + 4 > 4$  for any  $x \neq 2$ .  
 The range is  $(4, \infty)$ .

**33 A**  $3(1)^2 + k(1) + 1 = k + 4$   
 $= 0$   
 $k = -4$

**34 A**  $(x - 5)(x + 7) = 0$   
 $x^2 + 2x - 35 = 0$

**35 E** Let  $P(x) = x^3 - 5x^2 + x + k$ .  
 $P(x)$  is divisible by  $x + 1$ , so  
 $P(-1) = 0$ .  
 $(-1)^3 - 5(-1)^2 + (-1) + k = 0$   
 $-1 - 5 - 1 + k = 0$   
 $k = 7$

**36 C** The graph could be a cubic with minimum turning point at  $(-2, 0)$  and another  $x$ -intercept at  $(2, 0)$ .  
 Equation is  $y = a(x + 2)^2(x - 2)$   
 $x = 0, y > 0: = 8a > 0$   
 $a < 0$   
 Only the third option fits.

**37 D** The graph could be a cubic with a Stationary points of inflexion as  $(-1, 2)$ .  
 Equation is  $y = a(x + 1)^3 + 2$ .  
 Only the fourth option fits.  
 $y = -\frac{1}{2}(x + 1)^3 + 2$ , then when  $x = 0$   
 (Check: If  
 $y = -\frac{1}{2} + 2$   
 $= 1\frac{1}{2}$

which is consistent with the graph.)

**38 C**  $P(-1) = -1 + 2 + 5 - 6 = 0$   
 So  $(x + 1)$  is a factor.  
 Option B expanded has a constant term of  $+6$ .  
 Option C expanded has constant

term of  $-6$ .

Option D expanded has constant term of  $+6$

Only option C fits.

(Alternatively, divide the cubic by  $(x + 1)$  and factorise the resulting quadratic.)

- 39 E** For  $f.y = mx + 3$   
For  $f^{-1}$ , interchange  $x$   
and  $y$  and solve for  $y$ .

$$x = m y + 3$$

$$x - 3 = m y$$

$$y = \frac{1}{m}(x - 3)$$

$$= \frac{1}{m}x - \frac{3}{m}$$

$$\text{Hence } a = \frac{1}{m}, b = -\frac{3}{m}$$

- 40 B** Remainder is given by  $P(2)$   
 $P(2) = 2(2)^3 - 2(2)^2 + 3(2) + 1$   
 $= 16 - 8 + 6 + 1$   
 $= 15$

- 41 B** Let  $P(x) = x^3 + 2x^2 + ax - 4$   
Given  $P(-1) = 1$   
 $(-1)^3 + 2(-1)^2 + a(-1) - 4 = 1$   
 $-1 + 2 - a - 4 = 1$   
 $-a - 3 = 1$   
 $a = -4$

- 42 C** The graph could be a  
quartic with minimum tun-  
ing point at  $(-2, 0)$  and  $(2, 0)$ .  
Equation is  $y = a(x + 2)^2(x - 2)^2$   
 $= a((x + 2)(x - 2))^2$   
 $= a(x^2 - 4)^2$   
When  $x = 0, y = a \times (-4)^2 = 16a$

For the graph the  $y$  intercept is  
posture so  $a > 0$   
Only the third alternative fits.

- 43 D** As  $x \rightarrow \infty, f(x) \rightarrow 1$ ,  
Since  $e^{-x} > 0$  for all  $x, f(x) > 1$  for  
all  $x$ .  
Hence the range of  $f$  is  $(1, \infty)$  and  
this is the domain of  $f^{-1}$

- 44 B** For  $f, y = 2 \log_e x + 1$   
for  $f^{-1}$ , interchange  $x$  and  $y$  and  
solve for  $y$ .

$$x = 2 \log_e y + 1$$

$$x - 1 = \log_e y$$

$$\log_e y = \frac{1}{2}(x - 1)$$

$$y = e^{\frac{1}{2}(x-1)}$$

$$\text{So } f^{-1}(x) = e^{\frac{1}{2}(x-1)}$$

- 45 B**  $\log_e(-1 + 2) = \log_e 1 = 0$ , so range  
of  $g = (0, \infty) \mathbb{R}^+$   
 $e^{-0} = 1$  and as  $x \rightarrow \infty, e^{-x} \rightarrow 0$   
Hence the range of the function with  
rule  $y = f(g(x))$  is  $(0, 1)$ .

- 46 E** For  $f, y = e^x - 1$   
For  $f^{-1}$ , interchange  $x$  and  $y$  and  
solve for  $y$ .

$$x = e^y - 1$$

$$x + 1 = e^y$$

$$y = \log_e(x + 1)$$

$$\text{So } f^{-1}(x) = \log_e(x + 1).$$

- 47 A**  $f(4) = \log_e(4 - 3) = \log_e 1 = 0$ , so  
 $f$  has range  $[0, \infty)$   
and this is the domain of the inverse.

- 48 C** For  $f, y = e^{x-1}$

For  $f^{-1}$ , interchange  $x$  and  $y$  and solve for  $y$ .

$$x = e^{y-1}$$

$$\log_e x = y - 1$$

$$y = 1 + \log_e x$$

$$\text{So } f^{-1}(x) = 1 + \log_e x$$

- 49 D** For  $f$ ,  $y = \log_e \frac{x}{2}$   
For  $f^{-1}$ , interchange  $x$  and  $y$  and solve for  $y$ .

$$x = \log_e \frac{y}{2}$$

$$e^x = \frac{y}{2}$$

$$y = 2e^x$$

$$\text{So } f^{-1}(x) = 2e^x.$$

- 50 B** Require  $3x - 2 > 0$ , i.e.

$$3x > 2$$

$$x > \frac{2}{3}$$

So  $f$  is defined for  $x \in \left(\frac{2}{3}, \infty\right)$ .

- 51 C** The Graph of  $f$  has asymptote  $x = -2$ .  
Reflecting it in the line  $y = x$  means its inverse has asymptote  $y = -2$ .  
Only the third option fits.

- 52 C** Method 1

$$\begin{aligned} \log_2 8x &= \log_2 8 + \log_2 x \\ &= \log_2 2^3 + \log_2 x \\ &= 3 \log_2 2 + \log_2 x \\ &= \log_2 x + 3 \end{aligned}$$

$$\begin{aligned} \log_2 2x &= \log_2 2 + \log_2 x \\ &= \log_2 x + 1 \end{aligned}$$

So the equation becomes

$$\log_2 x + 3 + \log_2 x + 1 = 6$$

$$2 \log_2 x = 2$$

$$\log_2 x = 1$$

$$x = 2$$

Method 2

$$\log_2 8x - \log_2 2x = 6$$

$$\log_2(8x \times 2x) = 6$$

$$\log_2(16x^2) = 6$$

$$16x^2 = 2^6$$

$$= 64$$

$$x^2 = 4$$

$$x = \pm 2$$

But  $x > 0$ , so  $x = 2$ .

- 53 A**  $\log_{10} x = y(\log_{10} 3) + 1$   
 $= \log_{10} 3^y + \log_{10} 10$   
 $= \log_{10}(10(3^y))$   
 $x = 10(3^y)$

- 54 B** Graph has gradient  $-2$  and  $y$  intercept  $(0, 2)$ .  
Equation is  $\log_e N = -2t + 2$   
 $N = e^{2-2t}$

- 55 A** As  $x \rightarrow -\infty$ ,  $y \rightarrow 1$ , so the rule must involve  $e^x$ , and  $e^{-x}$ .  
When  $x = 0$ ,  $y = 0$ .  
Only the first option fits both of these.

- 56 B**  $x = -2$  is a vertical asymptote and the domain is  $(-2, \infty)$ , so only the second and fourth options are possible.  
The graph through  $(0, 0)$ .  
B:  $\log_e \frac{1}{2}(0 + 2) = \log_e 1 = 0$   
E:  $\frac{1}{2} \log_e(0 + 2) = \frac{1}{2} \log_e(2) \neq 0$

So the second option fits.

- 57 D** Period =  $\frac{5\pi}{12} - \left(-\frac{\pi}{4}\right) = \frac{8\pi}{12} = \frac{2\pi}{3}$   
 Range =  $[-4, 0]$  so amplitude = 2  
 and there is a vertical translation of 2 units down.

these rule out options A and B.

When  $\theta = \frac{\pi}{4}$ ,  $y = 0$ .

In the third and fifth options, when  $\theta = -\frac{\pi}{4}$ ,  $y = -4$ ; in the fourth option:

$$y = 2 \cos 3\left(-\frac{\pi}{4} + \frac{\pi}{4}\right) - 2$$

$$2 \cos 0 - 2$$

$$= 2 - 2$$

$$= 0$$

So the fourth option fits.

- 58 D** The minimum value of  $f$  is  
 $2 - 3 = -1$   
 The maximum value of  $f$  is  $2 + 3 = 5$ .  
 The range of  $f$  is  $[-1, 5]$

- 59 D**  $2 \sin \theta + \sqrt{3} = 0$

$$2 \sin \theta = -\sqrt{3}$$

$$\sin \theta = -\frac{\sqrt{3}}{2}$$

$\theta$  is in the third or fourth quadrants.

$$\theta = \pi + \frac{\pi}{3} \text{ or } 2\pi - \frac{\pi}{3}$$

$$\theta = \frac{4\pi}{3} \text{ or } \frac{5\pi}{3}$$

- 60 A** When  $x = \frac{\pi}{2}$ ,  $y = 0$ . Only options A and C satisfy this.  
 When  $\frac{\pi}{6} < x < \frac{7\pi}{6}$ ,  $y > 0$ .  
 This is true for option A but false for option C.

- 61 A** Amplitude 3, period =  $\frac{2\pi}{2} = \pi$

- 62 D** The minimum value of  $f$  is  $-3$ .  
 The maximum value of  $f$  is 3.  
 The range of  $f$  is  $[-3, 3]$

- 63 B**  $P(x) = 0 \Rightarrow x - 2a = 0$  or  
 $x + a = 0$  or  $x^2 a = 0$ .

So  $x = 2a$  or  $x = -a$  or  $x^2 = -a$ .

But  $a > 0$  so  $x^2 = -a$  has no solutions.

The equation has 2 decimal real solutions.

- 64 D** The gradient of the given straight line is  $-2$ .  
 For perpendicular lines,  $m_1 m_2 = -1$ .  
 So  $-2m_2 = -1$ , giving  $m_2 = \frac{1}{2}$ .

- 65 B** Since  $x$  and  $y$  are interchanged for the inverse, there must be an asymptote with equation  $x = 6$  for the inverse function.  
 So the inverse has vertical asymptote with equation  $x = 6$ .

- 66 E** Period =  $\frac{2\pi}{6}$

- 67 C**  $f(18) = 32 = 2^5$ ,  $f(34) = 64 = 2^6$ .

$$g(2^5) = \log_2 2^5 = 5$$

$$g(2^6) = \log_2 2^6 = 6$$

The range of  $g \circ f$  is  $[5, 6]$

- 68 C** Interchange  $x$  &  $y$ :  $x = y^2 - 4y + 5$

Solve for  $y$ :  $x = (y - 2)^2 + 1$

$$(y - 2)^2 = x - 1$$

$$y = 2 \pm \sqrt{x - 1}$$

- 69 B**

- 70 C**

## Solutions to extended-response questions

1 a The graph is of the form

$$y = ax^2 + b$$

The vertex is at  $(0, 9)$ .

Therefore  $b = 9$

The width of the arch is 20m.

Therefore the  $x$ -axis intercepts are at  $(10, 0)$  and  $(-10, 0)$

When  $x = 10$ ,  $y = 0$ .

Hence  $b = 9$

$$\text{and } 0 = a \times 100 + 9$$

$$\therefore a = \frac{-9}{100} = -0.09$$

b The equation of the curve is  $y = \frac{-9}{100}x^2 + 9$

$$\text{When } x = -7 \quad y = \frac{-9}{100} \times 49 + 9 = 4.59$$

The man is 1.8 m high.

$\therefore E$  is  $(4.56 - 1.8) \text{ m} = 2.79 \text{ m}$  above the man's head.

c  $OH$  is 6.3 m

$\therefore$  Consider  $y = 6.3$

$$6.3 = \frac{-9}{100}x^2 + 9$$

$$\frac{-2.7}{-9} \times 100 = x^2$$

$$\therefore 30 = x^2$$

$$\therefore x = \pm \sqrt{30}$$

The length of the bar is  $2\sqrt{30} \text{ m} \approx 10.95 \text{ m}$ .

2 a Let  $P(x) = 2x^3 + ax^2 - 72x - 18$

By the Remainder Theorem

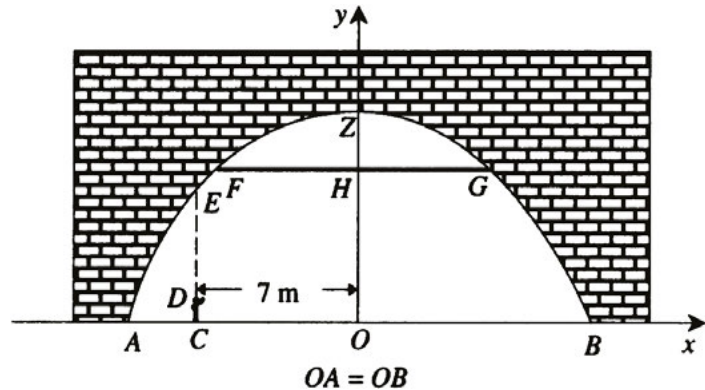
$$P(-5) = 17$$

$$\text{i.e. } 2 \times (-5)^3 + a(-5)^2 - (72 \times -5) - 18 = 17$$

$$250 + 25a + 360 - 18 = 17$$

$$\therefore 25a = -75$$

$$a = -3$$



**b**  $2x^3 = x^2 + 5x + 2$

Let  $P(x) = 2x^3 - x^2 - 5x - 2$

$$P(-1) = -2 - 1 + 5 - 2 = 0$$

$\therefore$  By the Factor Theorem  $x + 1$  is a factor.

Dividing  $P(x)$  by  $x + 1$

$$P(x) = (x + 1)(2x^2 - 3x - 2)$$

For  $P(x) = 0$

$$x = -1 \text{ or } 2x^2 - 3x - 2 = 0$$

and  $2x^2 - 3x - 2 = 0$

implies  $(2x + 1)(x - 2) = 0$

$$\therefore x = -\frac{1}{2} \text{ or } x = 2$$

i.e.  $x = -\frac{1}{2}$ ,  $x = 2$  and  $x = -1$  are solutions to the equation  $2x^3 = x^2 + 5x + 2$

**c**  $x^2 - 5x + 7$  leaves the same remainder when divided by  $x - b$  or  $x - c$

By the Remainder Theorem

$$b^2 - 5b + 7 = c^2 - 5c + 7$$

$$\Leftrightarrow b^2 - c^2 = 5(b - c)$$

$$\Leftrightarrow (b - c)(b + c) = 5(b - c)$$

$$\Leftrightarrow b + c = 5 \text{ as } b \neq c$$

$$\therefore b = 5 - c$$

and if  $4bc = 21$

$$4(5 - c)c = 21$$

$$20c - 4c^2 - 21 = 0$$

$$\therefore 4c^2 - 20c + 21 = 0$$

$$(2c - 3)(2c - 7) = 0$$

which implies

$$c = \frac{3}{2} \text{ or } c = \frac{7}{2}$$

If  $c = \frac{3}{2}$ ,  $4 \times b \times \frac{3}{2} = 21$

$$\therefore b = \frac{7}{2}$$

If  $c = \frac{7}{2}$ ,  $b = \frac{3}{2}$

As  $b > c$  the required values are  $b = \frac{7}{2}$ ,  $c = \frac{3}{2}$

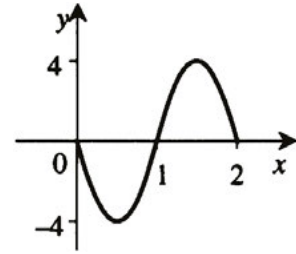


**3 a**  $x = -4 \sin \pi t$

**b i** When  $t = 0$ ,  $x = -4 \sin 0 = 0$

**ii** When  $t = \frac{1}{2}$ ,  $x = -4 \sin \frac{\pi}{2} = -4$

**iii** When  $t = 1$ ,  $x = -4 \sin \pi = 0$



**c** When  $x = 2$

$$2 = -4 \sin \pi t$$

$$\therefore -\frac{1}{2} = \sin(\pi t)$$

i.e.  $\frac{7\pi}{6} = \pi t$

$$\therefore t = \frac{7}{6}$$

**d** Period =  $\frac{2\pi}{n} = \frac{2\pi}{\pi} = 2$  seconds

**4 a**  $y = -1.25 \cos(2\pi t) + 1.25$

**i** When  $t = 0$

$$\begin{aligned} y &= -1.25 \cos(0) + 1.25 \\ &= 0 \end{aligned}$$

**ii** When  $t = \frac{1}{2}$

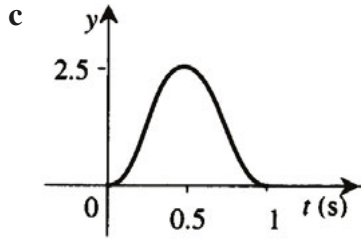
$$\begin{aligned} y &= -1.25 \cos(\pi) + 1.25 \\ &= 1.25 + 1.25 \\ &= 2.5 \end{aligned}$$

**iii** When  $t = 1$

$$\begin{aligned} y &= -1.25 \cos 2\pi + 1.25 \\ &= 0 \end{aligned}$$

**b** Period =  $\frac{2\pi}{n} = \frac{2\pi}{2\pi} = 1$

One revolution of the rope takes 1 second.



d  $2 = -1.25 \cos(2\pi t) + 1.25$

$$\frac{0.75}{-1.25} = \cos(2\pi t)$$

$$-0.6 = \cos(2\pi t)$$

$$\therefore 2\pi t = \cos^{-1}(-0.6) \quad (\text{only first solution required})$$

$$t = \frac{1}{2\pi} \cos^{-1}(-0.6)$$

$$\approx 0.3524$$

It is 2 metres above the ground after 0.35 seconds.

5  $P(t) = 150 \times 10^6 e^{kt}$

a From section 5.8, chapter 5 of *EMM Units 3 & 4*,  $k = 0.0296$  (i.e. 2.96% as a decimal).

b  $P(0) = 150 \times 10^6 e^0$

$\therefore$  Population on 1st Jan 1950 is  $150 \times 10^6$

c  $P(50) = 150 \times 10^6 \times e^{0.0296 \times 50}$

$$= 150 \times 10^6 \times 4.3929$$

$$= 658941852.1$$

$$\approx 6.589418521 \times 10^8$$

Population is approximately  $6.589 \times 10^8$  on January 1st 2000.

When  $P = 300 \times 10^6$

d  $300 \times 10^6 = 150 \times 10^6 e^{0.0296t}$

$$\therefore 2 = e^{0.0296t}$$

Taking logarithms of both sides of the equation

$$\frac{1}{0.0296} \log_e 2 = t$$

$$\therefore t \approx 23.417 \text{ years}$$

The population is  $300 \times 10^6$  after 23.417 years.

**6 a**  $T = Ae^{-kt} + 15$ , there  $0 \leq t \leq 10$

When  $t = 0$ ,  $T = 95$

$$95 = A + 15$$

$$\therefore A = 80$$

When  $t = 2$ ,  $T = 55$

$$55 = 80e^{-2k} + 15$$

$$\therefore 0.5 = e^{-2k}$$

Taking logarithms both sides

$$-\frac{1}{2} \log_e 5 = k$$

$$\therefore k = \log_e \left(2^{\frac{1}{2}}\right)$$

$$k \approx 0.3466$$

**b** At midnight  $t = 0$

$$\therefore T = 80e^{(-\log_e(2^{\frac{1}{2}})) \times 10}$$

$$= 80e^{\log_e 2^{-5}} + 15$$

$$= 80 \times \frac{1}{32} + 15$$

$$= 17.5$$

The temperature is  $17.5^\circ\text{C}$  at midnight.

**c** Graph is decreasing

When  $T = 24^\circ$

$$24 = 80e^{-\log_e(2^{\frac{1}{2}})t} + 15$$

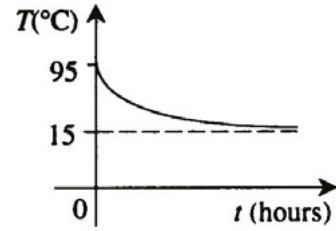
$$\frac{9}{80} = e^{-\log_e(2^{\frac{1}{2}})t}$$

$$\log_e\left(\frac{9}{80}\right) = -\log_e(2^{\frac{1}{2}})t$$

$$\frac{\log_e\left(\frac{9}{80}\right)}{-\log_e(2^{\frac{1}{2}})} = t$$

$$\therefore t = 6.304$$

This is 6 hours 18 minutes and 14 seconds after 2:00 pm, i.e. 8:18:14 pm.  
 Jenny first recorded a temperature less than  $24^\circ$  at 9:00 p.m. (Note: temperature is recorded on the hour)



$$\mathbf{d} \quad T = 80e^{-\log_e(2^{\frac{1}{2}})t} + 15$$

**7 a** If  $V = 25$  and  $\alpha = 45^\circ$

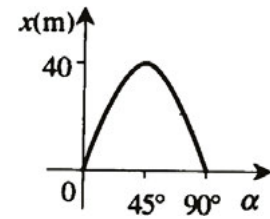
$$\begin{aligned} x &= \frac{25^2 \sin 90}{10} \\ &= \frac{625}{10} \\ &= 62.5 \end{aligned}$$

The distance the ball is kicked is 62.5 m.

**b** For  $V = 20$

$$\begin{aligned} x &= \frac{400 \sin 2\alpha}{10} \\ &= 40 \sin 2\alpha \end{aligned}$$

Period =  $180^\circ$ ; amplitude = 40



**c** If  $x = 30$  and  $V = 20$

$$30 = \frac{400 \sin 2\alpha}{10}$$

$$\frac{3}{4} = \sin 2\alpha$$

$$\therefore 2\alpha = \sin^{-1}\left(\frac{3}{4}\right) \text{ or } 180^\circ - \sin^{-1}\left(\frac{3}{4}\right)$$

$$\alpha = \frac{1}{2} \sin^{-1}\left(\frac{3}{4}\right) \text{ or } 90^\circ - \frac{1}{2} \sin^{-1}\left(\frac{3}{4}\right)$$

$$\approx 24.3^\circ \text{ or } 65.7^\circ$$

The angle projection is  $24.3^\circ$  or  $65.7^\circ$

$$\begin{aligned} \mathbf{8 a} \quad \text{Area} &= 0.02 \left(0.92^{\frac{1}{10}}\right)^x \\ &= 0.02(0.92)^{\frac{x}{10}} \end{aligned}$$

**b** When  $x = \frac{5}{3}$

$$\begin{aligned}\text{Area} &= (0.02)\left((0.92)^{\frac{5}{30}}\right) \\ &= (0.02)\left((0.92)^{\frac{1}{6}}\right) \\ &= 0.0197\end{aligned}$$

Area is  $0.0197 \text{ mm}^2$  when  $x = \frac{5}{3}$

**c** load = strength  $\times$  cross-sectional area

$$\begin{aligned}&= (0.92)^{10-3x} \times (0.02) \times (0.92)^{\frac{x}{10}} \\ &= (0.92)^{10-3x+\frac{x}{10}} \times 0.02 \\ &= (0.92)^{\frac{100-29x}{10}} \times 0.02 = 0.02(0.92)^{10-2.9x}\end{aligned}$$

**d** If load =  $0.02 \times (0.92)^{2.5}$

$$0.02 \times (0.92)^{2.5} = (0.92)^{\frac{100-29x}{10}} \times 0.02$$

$$(0.92)^{2.5} = (0.92)^{\frac{100-29x}{10}}$$

$$\therefore \frac{100-29x}{10} = 2.5$$

$$\text{i.e. } 100 - 29x = 25$$

$$\therefore 75 = 29x$$

$$\text{and } x \approx 2.59$$

Therefore the cable cannot exceed 2.59 m in length.

**9 a** The period of the function

$$= \frac{2\pi}{b} = 2\pi \div \frac{\pi}{6} = 12$$

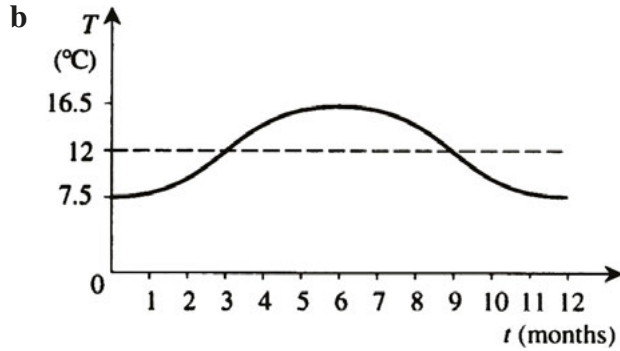
**i** Therefore length of  $OR$  is 12 units

**ii**  $Q$  is at the minimum value

$$\therefore OQ = h - k$$

$R$  is at the maximum value

$$\therefore OR = h + k$$



**c** From a  $h + k = 16.5$  and  $h - k = 7.5$

Consider as simultaneous equations

$$h + k = 16.5 \quad \text{①}$$

$$h - k = 7.5 \quad \text{②}$$

Add ① and ②

$$2h = 24$$

$$h = 12$$

and from ①  $k = 4.5$

**10 a** For Carriage A

$$\text{Stop 1 Illumination} = 0.83I$$

$$\text{Stop 2 Illumination} = (0.83)^2 I$$

$$\text{Stop } n \text{ Illumination} = (0.83)^n I$$

For Carriage B

$$\text{Stop 1 Illumination} = 0.89 \times 0.66I$$

$$\text{Stop 2 Illumination} = (0.89)^2 \times 0.66I$$

$$\text{Stop } n \text{ Illumination} = (0.89)^n \times 0.66I$$

**b** Illuminations equal implies

$$(0.83)^n I = (0.89)^n \times 0.66I$$

$$\therefore \left(\frac{0.83}{0.89}\right)^n = 0.66$$

Taking logarithms of both sides

$$n \log_e \left(\frac{0.83}{0.89}\right) = \log_e(0.66)$$

$$n = \frac{\log_e(0.66)}{\log_e\left(\frac{0.83}{0.89}\right)}$$

$$\approx 5.95$$

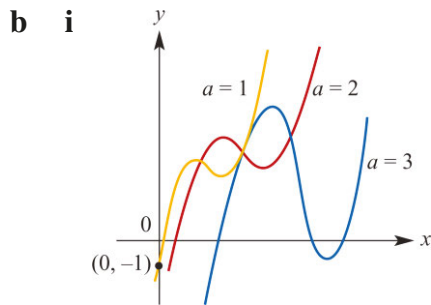
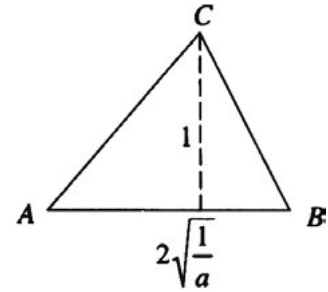
The illumination is approximately equal after the sixth stop.

**11 a i**  $y = 1 - a(x - 3)^2$

When  $y = 0$

$$\begin{aligned}
 1 - a(x - 3)^2 &= 0 \\
 \therefore 1 &= a(x - 3)^2 \\
 \therefore (x - 3)^2 &= \frac{1}{a} \\
 \therefore x &= 3 \pm \sqrt{\frac{1}{a}} \\
 &\left(3 + \sqrt{\frac{1}{a}}, 0\right) \text{ and } \left(3 - \sqrt{\frac{1}{a}}, 0\right)
 \end{aligned}$$

**ii**  $AB$  has length  $2\sqrt{\frac{1}{a}}$   
 $C$  has coordinates  $(3, 1)$   
 Therefore the area  $= \frac{1}{2} \times 2\sqrt{\frac{1}{a}} \times 1$   
 $= \frac{1}{\sqrt{a}}$  square units



**ii**  $-\frac{4}{27}a^3 + a = 0$   
 $\therefore a\left(-\frac{4}{27}a^2 + 1\right) = 0$   
 $\therefore a = 0$  or  $a = \pm\sqrt{\frac{27}{4}}$   
 $= \pm\frac{3\sqrt{3}}{2}$   
 But  $a > 0$ . Therefore  $a = \frac{3\sqrt{3}}{2}$

**iii**  $-\frac{4}{27}a^3 + a < 0$   
 $\Leftrightarrow -\frac{4}{27}a^2 + 1 < 0$  (as  $a > 0$ )  
 $\Leftrightarrow a^2 > \frac{27}{4}$   
 $\Leftrightarrow a > \frac{3\sqrt{3}}{2}$  (as  $a > 0$ )

iv  $-\frac{4}{27}a^3 + a = -1$   
 $-4a^3 + 27a + 27 = 0$

Using a CAS calculator yields  $a = 3$  is a solution.

Consider

$$(a - 3)(-4a^2 - 12a - 9) = 0$$

i.e.  $(a - 3)(4a^2 + 12a + 9) = 0$

and  $4a^2 + 12a + 9 > 0$  for all  $a$

$\therefore a = 3$  is the only solution.

v  $-\frac{4}{27}a^3 + a = 1$   
 $-4a^3 + 27a - 27 = 0$

Using a graphics calculator yields  $a = \frac{3}{2}$  is a solution.

$$\therefore -4a^3 + 27a - 27 = (2a - 3)(-2a^2 - 3a + 9)$$

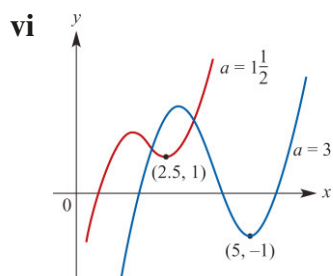
and  $-2a^2 - 3a + 9 = 0$

implies  $2a^2 + 3a - 9 = 0$

$$\therefore (2a - 3)(a + 3) = 0$$

$$\therefore a = \frac{3}{2} \text{ or } a = -3$$

$a = \frac{3}{2}$  is the only solution.



### Graphic calculator techniques for 12b

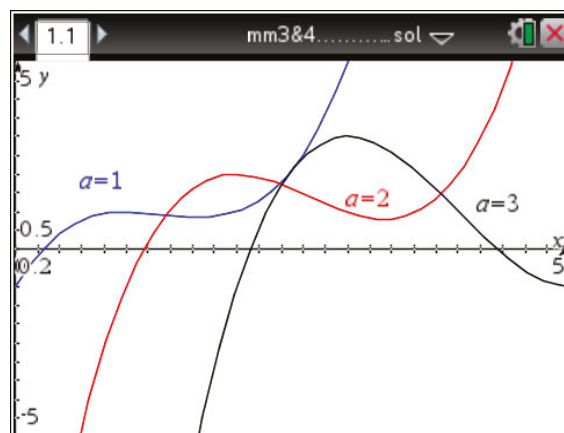
b i In a Graphs page enter

$$f1(x) = (x - 1)^2(x - 2) + 1,$$

$$f2(x) = (x - 2)^2(x - 4) + 2 \text{ and}$$

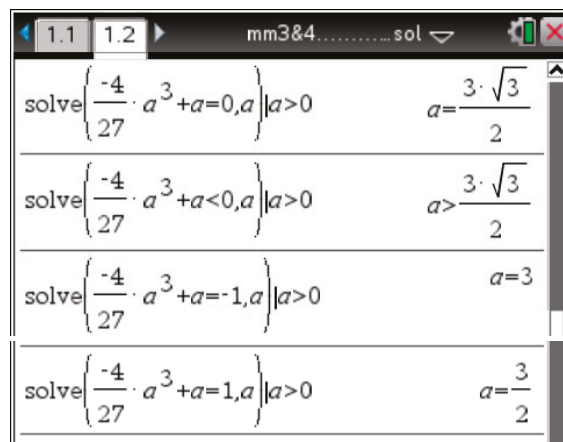
$$df3(x) = (x - 3)^2(x - 6) + 3$$

Set an appropriate window to show key points.

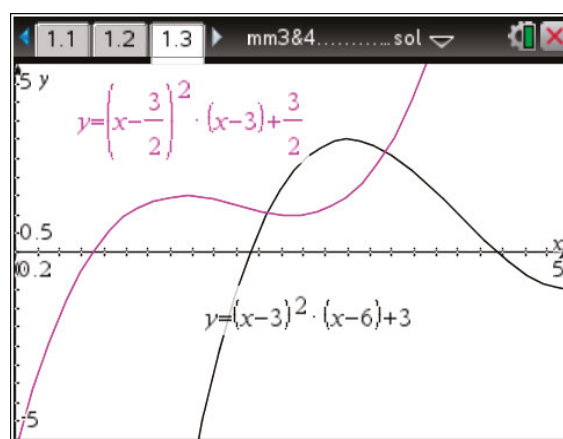




**b ii – v** In a **Calculator** page, use the **Solve** command. Note the domain restrictions.



**b vi** Plot in a **Graphs** page.



**c i**  $\left(a, -\frac{4}{27}a^3 + a\right)$

**ii**  $PS = a - \left(-\frac{4}{27}a^3 + a\right)$

$$= \frac{4}{27}a^3$$

$$SQ = \frac{5a}{3} - a$$

$$= \frac{2a}{3}$$

**iii**  $\text{Area} = \frac{1}{2} \times SQ \times PS$

$$= \frac{1}{2} \times \frac{2a}{3} \times \frac{4}{27}a^3$$

$$= \frac{4a^4}{81}$$

$$\text{iv } \frac{4a^4}{81} = 4$$

$$\therefore a^4 = 81$$

$$\therefore a = 3 \quad (\text{since } a > 0)$$

$$\text{v } \frac{4a^4}{81} = 1500$$

$$\therefore a^4 = \frac{81 \times 1500}{4}$$

$$\therefore a^4 = 81 \times 375$$

$$a = 3\sqrt[4]{375} \quad (\text{since } a > 0)$$

**12 a**  $D = at^2 + bt + c$

When  $t = 0, D = 1.8$

Therefore  $c = 1.8$

When  $t = 1, D = 1.6$

Therefore

$$1.6 = a + b + 1.8 \text{ and rearranging gives,}$$

$$-0.2 = a + b \quad \textcircled{1}$$

When  $t = 3, D = 1.5$

Therefore

$$1.5 = 9a + 3b + 1.8 \text{ and rearranging gives,}$$

$$-0.3 = 9a + 3b$$

Dividing both sides of the equation by 3 gives

$$-0.1 = 3a + b \quad \textcircled{2}$$

Subtract  $\textcircled{1}$  from  $\textcircled{2}$

$$0.1 = 2a$$

Therefore  $a = 0.05$ . Substituting in (1) gives that  $b = 0.25$

$$D = 0.05t^2 - 0.25t + 1.8$$

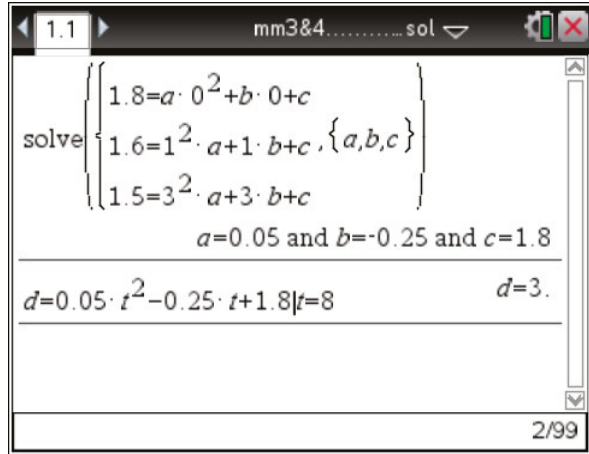
**b** When  $t = 8, D = 0.05 \times 64 - 0.25 \times 8 + 1.8 = 3$

The deficit is 3 000 000 Ningteak dollars

**Graphic calculator techniques for question**

In a **Calculator** page use:  
**b>Algebra>Solve System of Equations>Solve System of Equations** and enter as shown opposite.

Substitute  $t = 8$  into equation.



**13**  $R = at^2 + bt + c$

When  $t = 0, R = 7.5$

Therefore  $c = 7.5$

When  $t = 4, R = 9$

Therefore  $9 = 16a + 4b + 7.5$

and  $1.5 = 16a + 4b$

Dividing both sides by 4 gives

$$\frac{3}{8} = 4a + b \quad \text{①}$$

When  $t = 6, R = 8$

$$8 = 36a + 6b + 7.5$$

$$0.5 = 36a + 6b$$

Divide both sides by 6

$$\frac{1}{12} = 6a + b \quad \text{②}$$

Subtract ② from ①

$$\frac{3}{8} - \frac{1}{12} = -2a$$

Therefore  $a = -\frac{7}{48}$  and substituting in ① gives  $b = \frac{23}{24}$

Thus  $R = -\frac{7}{48}t^2 + \frac{23}{24}t + \frac{15}{2}$

When  $t = 8, R = \frac{35}{6}$ . The rate is  $\frac{35}{6}$  mm/h at 12:00 noon.

The rainfall is greatest when  $t = -\frac{-b}{2a} = -\frac{23}{24} \div -\frac{7}{24} = \frac{23}{7}$

The rainfall was heaviest at 7: 17 am.

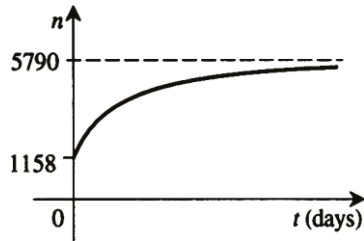
$$14 \quad n = \frac{c}{1 + ae^{-bt}} \quad t \geq 0$$

$$n = \frac{5790}{1 + 4e^{-0.03t}} \quad \text{for } c = 5790, a = 4 \text{ and } b = 0.03$$

a i  $n = 5790$  is the horizontal asymptote

ii when  $t = 0, n = \frac{5790}{5} = 1158$

iii



iv 
$$4000 = \frac{5790}{1 + 4e^{-0.03t}}$$

$$\therefore 1 + 4e^{-0.03t} = \frac{579}{400}$$

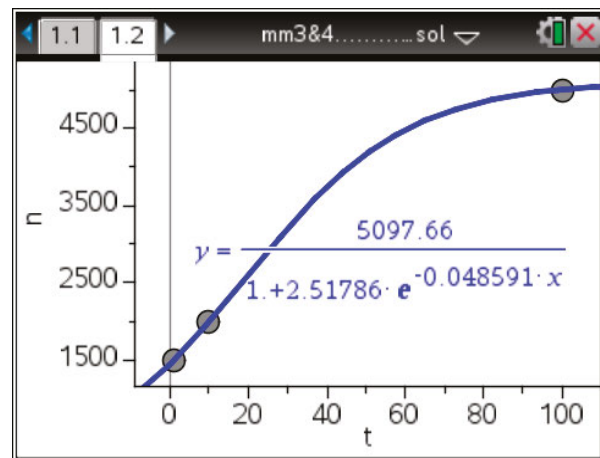
$$\therefore 4e^{-0.03t} = \frac{179}{400}$$

$$\therefore e^{-0.03t} = \frac{179}{1600}$$

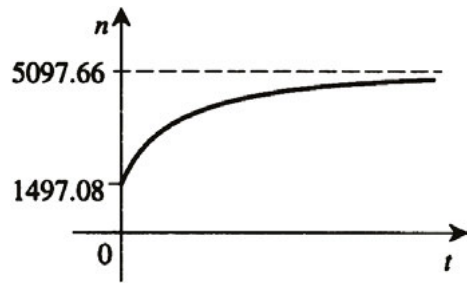
$$\therefore t = -\frac{100}{3} \log_e \left( \frac{179}{1600} \right)$$

$$= \frac{100}{3} \log_e \left( \frac{1600}{179} \right)$$

- b i Enter the data in a **Lists & Spreadsheet** page. Plot the data in a **Data & Statistics** page. Determine the logistic regression using **b>Analyze>Regression>Show Logistic (d=0)**. The result is as shown.



ii



# Chapter 9 – Differentiation of polynomials, power functions and rational functions

## Solutions to Exercise 9A

- 1**  $f(x) = -x^2 + 2x + 1$   
 $f(-1) = -(-1)^2 + 2(-1) + 1 = -2$   
 $f(4) = -(4)^2 + 2 \times 4 + 1 = -7$   
 Average rate of change  $= \frac{f(4) - f(-1)}{4 - (-1)}$   
 $= \frac{-7 - (-2)}{5}$   
 $= -1$
- 2**  $f(x) = 6 - x^3$   
 $f(-1) = 6 + 1 = 7$   
 $f(1) = 6 - 1 = 5$   
 Average rate of change  $= \frac{f(1) - f(-1)}{1 - (-1)}$   
 $= \frac{-2}{2}$   
 $= -1$
- 3**  $f(x) = x^2 + 5x$
- a** Gradient  
 $= \frac{(2+h)^2 + 5(2+h) - 14}{2+h-2}$   
 $= \frac{4 + 4h + h^2 + 10 + 5h - 14}{h}$   
 $= \frac{9h + h^2}{h}$   
 $= 9 + h$
- b**  $\lim_{h \rightarrow 0} 9 + h = 9$
- 4 a**  $\lim_{h \rightarrow 0} \frac{4x^2h^2 + xh + h}{h}$   
 $= \lim_{h \rightarrow 0} (4x^2h + x + 1)$   
 $= x + 1$
- b**  $\lim_{h \rightarrow 0} \frac{2x^3h - 2xh^2 + h}{h}$   
 $= \lim_{h \rightarrow 0} (2x^3 - 2xh + 1)$   
 $= 2x^3 + 1$
- c**  $\lim_{h \rightarrow 0} (40 - 50h)$   
 $= 40$
- d**  $\lim_{h \rightarrow 0} 5h$   
 $= 0$
- e**  $\lim_{h \rightarrow 0} 5$   
 $= 5$
- f**  $\lim_{h \rightarrow 0} \frac{30h^2x^2 + 20h^2x + h}{h}$   
 $= \lim_{h \rightarrow 0} (30hx^2 + 20hx + 1)$   
 $= 1$
- g**  $\lim_{h \rightarrow 0} \frac{3h^2x^3 + 2hx + h}{h}$   
 $= \lim_{h \rightarrow 0} (3hx^3 + 2x + 1)$   
 $= 2x + 1$
- h**  $\lim_{h \rightarrow 0} 3x$   
 $= 3x$
- i**  $\lim_{h \rightarrow 0} \frac{3x^3h - 5x^2h^2 + hx}{h}$   
 $= \lim_{h \rightarrow 0} (3x^3 + 5x^2h + x)$   
 $= 3x^3 + x$
- j**  $\lim_{h \rightarrow 0} (6x - 7h)$   
 $= 6x$

$$5 \quad y = x^3 - x$$

$$\begin{aligned} \mathbf{a} \quad \text{grad} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{(1+h)^3 - (1+h) - 0}{(1+h) - 1} \\ &= \frac{1 + 3h + 3h^2 + h^3 - 1 - h}{h} \\ &= \frac{h^3 + 3h^2 + 2h}{h} \\ &= h^2 + 3h + 2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{grad} &= \lim_{h \rightarrow 0} \text{grad}(PQ) \\ &= \lim_{h \rightarrow 0} (h^2 + 3h + 2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad f(x) &= x^2 - 2 \\ \frac{f(x+h) - f(x)}{h} &= \frac{((x+h)^2 - 2) - (x^2 - 2)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= 2x + h \\ f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} (2x + h) \\ f'(x) &= 2x \end{aligned}$$

7

$$\begin{aligned} y &= x^2 + 2x + 5 \\ \text{grad}(PQ) &= \frac{\text{rise}}{\text{run}} \\ &= \frac{((2+h)^2 + 2(a+h) + 5) - ((2)^2 + 2(2) + 5)}{(2+h) - (2)} \\ &= \frac{4 + 4h + h^2 + 4 + 2h + 5 - 4 - 4 - 5}{h} \\ &= \frac{h^2 + 6h}{h} \\ &= h + 6 \\ \text{grad}(P) &= \lim_{h \rightarrow 0} (\text{grad}(PQ)) \\ &= \lim_{h \rightarrow 0} (h + 6) \\ &= 6 \end{aligned}$$

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad f(x) &= 5x^2 \\ f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5(x+h)^2 - 5x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h} \\ &= \lim_{h \rightarrow 0} (10x + 5h) \\ &= 10x \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f(x) &= 3x + 2 \\ f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3(x+h) + 2 - 3x - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h}{h} \\ &= \lim_{h \rightarrow 0} 3 \\ &= 3 \end{aligned}$$

**c**  $f(x) = 5$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5 - 5}{h} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0 \end{aligned}$$

**d**

$$f(x) = 3x^2 + 4x + 3$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 4(x+h) + 3 - 3x^2 - 4x - 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 - 3x^2 + 6xh + 3h^2 + 4x - 4x + 4h + 3 - 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2 + 4h}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h + 4) \\ &= 6x + 4 \end{aligned}$$

**e**

$$f(x) = 5x^3 - 5$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5(x+h)^3 - 5 - 5x^3 + 5}{h} \\ &= \lim_{h \rightarrow 0} \frac{5x^3 + 15x^2h + 15xh^2 + 5h^3 - 5x^3}{h} \\ &= \lim_{h \rightarrow 0} (15x^2 + 15xh + 5h^2) \\ &= 15x^2 \end{aligned}$$

**f**

$$f(x) = 5x^2 - 6x$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5(x+h)^2 + 6(x+h) - 5x^2 + 6x}{h} \\ &= \lim_{h \rightarrow 0} \frac{5x^2 + 10xh + 5h^2 - 6x - 6h - 5x^2 + 6x}{h} \\ &= \lim_{h \rightarrow 0} (10x + 5h - 6) \\ &= 10x - 6 \end{aligned}$$



## Solutions to Exercise 9B

**1 a**  $f(x) = x^5$

$$f'(x) = 5x^4$$

**b**  $f(x) = 4x^7$

$$\begin{aligned} f'(x) &= 7 \times 4x^6 \\ &= 28x^6 \end{aligned}$$

**c**  $f(x) = 6x$

$$f'(x) = 6$$

**d**  $f(x) = 5x^2 - 4x + 3$

$$\begin{aligned} f'(x) &= 2 \times 5x - 4 \\ &= 10x - 4 \end{aligned}$$

**e**  $f(x) = 4x^3 + 6x^2 + 2x - 4$

$$\begin{aligned} f'(x) &= 3 \times 4x^2 + 2 \times 6x + 2 \\ &= 12x^2 + 12x + 2 \end{aligned}$$

**f**  $f(x) = 5x^4 + 3x^3$

$$\begin{aligned} f'(x) &= 4 \times 5x^3 + 3 \times 3x^2 \\ &= 20x^3 + 9x^2 \end{aligned}$$

**g**  $f(x) = -2x^2 + 4x + 6$

$$f'(x) = -4x + 4$$

**h**  $f(x) = 6x^3 - 2x^2 + 4x - 6$

$$f'(x) = 18x^2 - 4x + 4$$

**2 a**  $f(x) = 2x^3 - 5x^2 + 1$

$$f'(x) = 6x^2 - 10x$$

$$f'(1) = -4$$

**b**  $f(x) = -2x^3 - x^2 - 1$

$$f'(x) = -6x^2 - 2x$$

$$f'(1) = -8$$

**c**  $f(x) = x^4 - 2x^3 + 1$

$$f'(x) = 4x^3 - 6x^2$$

$$f'(1) = -2$$

**d**  $f(x) = x^5 - 3x^3 + 2$

$$f'(x) = 5x^4 - 9x^2$$

$$f'(1) = -4$$

**3 a**  $f(x) = 2x^3 - 5x^2 + 2$

$$f'(x) = 6x^2 - 10x$$

$$f'(1) = -4$$

**b**  $f(x) = -2x^3 - 3x^2 + 2$

$$f'(x) = -6x^2 - 6x$$

$$f'(1) = -12$$

**4 a**  $\frac{dy}{dt} = 3t^2$

**b**  $\frac{dx}{dt} = 3t^2 - 2t$

**c**  $\frac{dz}{dx} = x^3 + 9x^2$

**5 a**  $y = -2x$

$$\frac{dy}{dx} = -2$$

**b**  $y = 7$

$$\frac{dy}{dx} = 0$$

$$\text{c } y = 5x^3 - 3x^2 + 2x + 1$$

$$\frac{dy}{dx} = 15x^2 - 6x + 2$$

$$\text{d } y = \frac{2}{5}x^3 - \frac{8}{5}x + \frac{12}{5}$$

$$\frac{dy}{dx} = \frac{6}{5}x^2 - \frac{8}{5}$$

$$\text{e } y = (2x + 1)(x - 3)$$

$$= 2x^2 - 5x - 3$$

$$\frac{dy}{dx} = 4x - 5$$

$$\text{f } y = 3x(2x - 4)$$

$$= 6x^2 - 12x$$

$$\frac{dy}{dx} = 12x - 12$$

$$\text{g } y = \frac{10x^7 + 2x^2}{x^2}$$

$$= 10x^5 + 2$$

$$\frac{dy}{dx} = 50x^4$$

$$\text{h } y = \frac{9x^4 + 3x^2}{x}$$

$$= 9x^3 + 3x$$

$$\frac{dy}{dx} = 27x^2 + 3$$

$$\text{6 a } \frac{d}{dx}(2x^2 - 5x^3) = 4x - 15x^2$$

$$\text{b } \frac{d}{dz}(-2z^2 - 6z) = -4z - 6$$

$$\text{c } \frac{d}{dz}(6z^3 - 4z^2 + 3) = 18z^2 - 8z$$

$$\text{d } \frac{d}{dx}(-2x - 5x^3) = -2 - 15x^2$$

$$\text{e } \frac{d}{dz}(-2z^2 - 6z + 7) = -4z - 6$$

$$\text{f } \frac{d}{dz}(-z^3 - 4z^2 + 3) = -3z^2 - 8z$$

$$\text{7 a } y = 2x^2 - 4x + 1, \frac{dy}{dx} = -6$$

$$\frac{dy}{dx} = 4x - 4$$

$$-6 = 4x - 4$$

$$4x = -2$$

$$x = \frac{-1}{2}$$

$$y = \frac{1}{2} + 2 + 1 = \frac{7}{2}$$

$$\text{co-ords} = \left(\frac{-1}{2}, \frac{7}{2}\right)$$

$$\text{b } y = 4x^3, \frac{dy}{dx} = 48$$

$$\frac{dy}{dx} = 12x^2$$

$$48 = 12x^2$$

$$x^2 = 4$$

$$x = \pm 2$$

$$y = \pm 32$$

$$\text{co-ords} = (-2, -32) \text{ and } (2, 32)$$

$$\text{c } y = x(5 - x), \frac{dy}{dx} = 1$$

$$y = 5x - x^2$$

$$\frac{dy}{dx} = 5 - 2x$$

$$1 = 5 - 2x$$

$$-2x = -4$$

$$x = 2$$

$$y = 2(3) = 6$$

$$\text{co-ords} = (2, 6)$$

$$\mathbf{d} \quad y = x^3 - 3x^2, \quad \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 3x^2 - 6x$$

$$0 = 3x^2 - 6x$$

$$0 = 3x(x - 2)$$

$$x = 0, 2$$

$$y = 0 - 0 = 0, \quad y = 8 - 12 = -4$$

co-ords = (0, 0) and (2, -4)

$$\mathbf{8} \quad \mathbf{a} \quad \tan 45^\circ = 1$$

$$\therefore \text{gradient} = 1$$

$$\frac{dy}{dx} = 4x - 3 \quad \text{Therefore}$$

$$\text{When gradient} = 1$$

$$4x - 3 = 1$$

$$\therefore x = 1$$

$$f(1) = 7$$

the tangent line at the point (1, 7) makes an angle of  $\tan 45^\circ$  with the positive direction of the  $x$ -axis.

$$\mathbf{b} \quad \text{Gradient} = 2$$

$$\frac{dy}{dx} = 4x - 3$$

$$\text{When gradient} = 2$$

$$4x - 3 = 2$$

$$\therefore x = \frac{5}{4}$$

$$f\left(\frac{5}{4}\right) = \frac{59}{8}$$

Therefore the tangent line at the point  $\left(\frac{5}{4}, \frac{59}{8}\right)$  is parallel to the line  $y = 2x + 8$

$$\mathbf{9} \quad \frac{dy}{dx} = 2x - 1$$

$$\mathbf{a} \quad 2x - 1 = 1$$

$$x = 1$$

$$\mathbf{b} \quad 2x - 1 = -1$$

$$x = 0$$

$$\mathbf{c} \quad 2x - 1 = \sqrt{3}$$

$$x = \frac{1}{2}(1 + \sqrt{3})$$

$$= \frac{1 + \sqrt{3}}{2}$$

$$\mathbf{10} \quad \mathbf{a} \quad y = x^2 + 3x, \quad (1, 4)$$

Let  $\theta$  be the angle between the tangent line and the  $x$ -axis.

$$\frac{dy}{dx} = 2x + 3$$

$$\text{When } x = 1, \quad \frac{dy}{dx} = 5$$

$$\therefore \tan \theta = 5$$

$$\therefore \theta \approx 78.69^\circ$$

$$\mathbf{b} \quad y = -x^2 + 2x, \quad (1, 1) \quad \text{Let } \theta \text{ be the angle between the tangent line and the } x\text{-axis.}$$

$$\frac{dy}{dx} = -2x + 2$$

$$\text{When } x = 1, \quad \frac{dy}{dx} = 0$$

$$\therefore \tan \theta = 0$$

$$\therefore \theta = 0^\circ$$

$$\mathbf{c} \quad y = x^3 + x, \quad (0, 0)$$

Let  $\theta$  be the angle between the tangent line and the  $x$ -axis.

$$\frac{dy}{dx} = 3x^2 + 1$$

$$\text{When } x = 0, \quad \frac{dy}{dx} = 1$$

$$\therefore \tan \theta = 1$$

$$\therefore \theta = 45^\circ$$

- d**  $y = -x^3 - x$ ,  $(0, 0)$   
Let  $\theta$  be the angle between the tangent line and the  $x$ -axis.

$$\frac{dy}{dx} = -3x^2 - 1$$

When  $x = 0$ ,  $\frac{dy}{dx} = -1$

$$\therefore \tan \theta = -1$$

$$\therefore \theta = 135^\circ$$

- e**  $y = x^4 - x^2$ ,  $(1, 0)$   
Let  $\theta$  be the angle between the tangent line and the  $x$ -axis.

$$\frac{dy}{dx} = 4x^3 - 2x$$

When  $x = 1$ ,  $\frac{dy}{dx} = 2$

$$\therefore \tan \theta = 2$$

$$\therefore \theta \approx 63.43^\circ$$

- f**  $y = x^4 - x^2$ ,  $(-1, 0)$   
Let  $\theta$  be the angle between the tangent line and the  $x$ -axis.

$$\frac{dy}{dx} = 4x^3 - 2x$$

When  $x = -1$ ,  $\frac{dy}{dx} = -2$

$$\therefore \tan \theta = -2$$

$$\therefore \theta \approx 116.57^\circ$$

**11 a**  $y = (2x - 1)^2$   
 $= 4x^2 - 4x + 1$

$$\frac{dy}{dx} = 8x - 4$$

**b**  $y = \frac{x^3 + 2x^2}{x}$   
 $= x^2 + 2x$

$$\frac{dy}{dx} = 2x + 2$$

**c**  $y = 2x^3 - 6x^2 + 18x$

$$\frac{dy}{dx} = 6x^2 - 12x + 18$$

$$= 6(x^2 - 2x + 3)$$

$$b^2 - 4ac = 6(4 - 12) < 0$$

$\therefore \frac{dy}{dx}$  does not intersect the  $x$ -axis

and since  $x = 0$  gives  $\frac{dy}{dx} = 3$ ,  $\frac{dy}{dx} > 0$

for all  $x$  (as opposed to  $\frac{dy}{dx} < 0$  for all  $x$ )

**d**  $y = \frac{x^3}{3} - x^2 + x$

$$\frac{dy}{dx} = x^2 - 2x + 1$$

$$= (x - 1)^2$$

$$\therefore \frac{dy}{dx} \geq 0,$$

since any number squared is non-negative

**12 a**  $y = x^2 + 2x + 1$ ,  $x = 3$

$$y = 3^2 + 2(3) + 1$$

$$= 9 + 6 + 1$$

$$y = 16$$

$$\frac{dy}{dx} = 2x + 2$$

$$\frac{dy}{dx} = 2(3) + 2$$

$$= 8$$

**b**  $y = x^2 - x - 1$ ,  $x = 0$

$$y = -1$$

$$\frac{dy}{dx} = 2x - 1$$

$$\frac{dy}{dx} = -1$$

**c**  $y = 2x^2 - 4x, x = -1$

$$y = 2(-1)^2 - 4(-1)$$

$$= 2 + 4$$

$$y = 6$$

$$\frac{dy}{dx} = 4x - 4$$

$$\frac{dy}{dx} = 4(-1) - 4$$

$$= -8$$

**d**  $y = (2x + 1)(3x - 1)(x + 2), x = 4$

$$y = 6x^3 + 13x^2 + x - 2$$

$$= 6(4)^3 + 13(4)^2 + (4) - 2$$

$$y = 6 \times 64 + 13 \times 16 + 4 - 2$$

$$y = 384 + 208 + 2$$

$$y = 594$$

$$y = 6x^3 + 13x^2 + x - 2$$

$$\frac{dy}{dx} = 18x^2 + 26x + 1$$

$$x = 4, \frac{dy}{dx} = 18 \times 16 + 26 \times 4 + 1$$

$$= 393$$

**e**  $y = (2x + 5)(3 - 5x)(x + 1), x = +1$

$$y = -10x^3 - 25x^2 + 6x^2 - 10x + 6x$$

$$- 25x + 15x + 15$$

$$y = -10x^3 - 29x^2 - 4x + 15$$

$$x = +1, y = -10 - 29 - 4 + 15$$

$$y = -28$$

$$\frac{dy}{dx} = -30x^2 - 58x - 4$$

$$x = +1, \frac{dy}{dx} = -30 - 58 - 4$$

$$= -92$$

**f**  $y = (2x - 5)^2, x = 2\frac{1}{2}$

$$x = 2\frac{1}{2}, y = (5 - 5)^2$$

$$y = 0$$

$$y = 4x^2 - 20x + 25$$

$$\frac{dy}{dx} = 8x - 20$$

$$x = 2\frac{1}{2}, \frac{dy}{dx} = 4 \times 5 - 20$$

$$= 0$$

**13**  $f(x) = 3(x - 1)^2$

**a**  $0 = 3(x - 1)^2$

$$x = 1$$

**b**  $f'(x) = 3(x^2 - 2x + 1)$

$$f'(x) = 3(2x - 2)$$

$$= 6(x - 1)$$

$$0 = 6(x - 1)$$

$$x = 1$$

**c**  $0 < 6(x - 1)$

$$x - 1 > 0$$

$$x > 1; \text{i.e. } (1, \infty)$$

**d**  $0 > 6(x - 1)$

$$x - 1 < 0$$

$$x < 1; \text{i.e. } (-\infty, 1)$$

**e**  $10 = 6(x - 1)$

$$x - 1 = \frac{5}{3}$$

$$x = \frac{8}{3}$$

$$\begin{aligned} \mathbf{f} \quad 27 &= 3(x-1)^2 \\ 9 &= (x-1)^2 \\ x-1 &= \pm 3 \\ x &= -2, 4 \end{aligned}$$

$$\mathbf{14 a} \quad x < -1, x > 1 \\ \text{i.e. } x \in \mathbb{R} \setminus [-1, 1]$$

$$\mathbf{b} \quad -1 < x < 1 \\ \text{i.e. } x \in (-1, 1)$$

$$\mathbf{c} \quad x = -1, 1$$

$$\mathbf{15 a} \quad -1 < x < 0.5, x > 2 \\ \text{i.e. } x \in \left(-1, \frac{1}{2}\right) \cup (2, \infty)$$

$$\mathbf{b} \quad x < -1, \frac{1}{2} < x < 2 \\ \text{i.e. } x \in (-\infty, -1) \cup \left(\frac{1}{2}, 2\right)$$

$$\mathbf{c} \quad x = -1, \frac{1}{2}, 2$$

$$\mathbf{16 a} \quad x > -1, x \neq 2 \\ \text{i.e. } x \in \left(-\frac{1}{4}, 2\right) \cup (2, \infty)$$

$$\mathbf{b} \quad x < \frac{-1}{4} \\ \text{i.e. } x \in \left(-\infty, \frac{-1}{4}\right)$$

$$\mathbf{c} \quad x = \frac{-1}{4}, 2$$

$$\mathbf{17} \quad y = x^2 - 4x - 8$$

$$\mathbf{a} \quad \frac{dy}{dx} = 2x - 4$$

$$\frac{dy}{dx} = 0,$$

$$0 = 2x - 4$$

$$2x = 4$$

$$x = 2$$

$$y = 4 - 8 - 8 = -12$$

$$\text{co-ords} = (2, -12)$$

$$\mathbf{b} \quad \frac{dy}{dx} = 2$$

$$2 = 2x - 4$$

$$2x = 6$$

$$x = 3$$

$$y = -11$$

$$\text{co-ords} = (3, -11)$$

$$\mathbf{c} \quad 3x + 2y = 8$$

$$\Rightarrow y = \frac{8}{2} - \frac{3}{2}x$$

$$= 4 - \frac{3}{2}x$$

$$\frac{dy}{dx} = \frac{-3}{2}$$

$$\frac{-3}{2} = 2x - 4$$

$$x = \frac{5}{4}$$

$$y = -\frac{183}{16}$$

$$\text{co-ords} = \left(\frac{5}{4}, -\frac{183}{16}\right)$$

$$\mathbf{18 a} \quad f'(x) = x^2 > 0 \text{ for all } x \neq 0.$$

Therefore strictly increasing for  $\mathbb{R} \setminus \{0\}$ .

Also  $f(0) = 0$  and  $f(b) > 0$  for all  $b > 0$  and  $f(b) < 0$  for all  $b < 0$ .

Therefore strictly increasing for all

$x \in \mathbb{R}$ .

**b**  $f'(x) = -x^2 < 0$  for all  $x \neq 0$ .

Therefore strictly decreasing for  $\mathbb{R} \setminus \{0\}$ .

Also  $f(0) = 0$  and  $f(b) < 0$  for all  $b > 0$  and  $f(b) > 0$  for all  $b < 0$ .

Therefore strictly decreasing for all  $x \in \mathbb{R}$ .

**19 a** Assume  $x > y$  and  $x \geq 0$  and  $y \geq 0$ .  
Then

$$x > y$$

$$\Leftrightarrow x - y > 0$$

$$\Leftrightarrow (x - y)(x + y) > 0$$

$$\Leftrightarrow x^2 - y^2 > 0$$

$$\Leftrightarrow x^2 > y^2$$

**b** Assume  $x > y$  and  $x \leq 0$  and  $y \leq 0$ .  
Then

$$x > y$$

$$\Leftrightarrow x - y > 0$$

$$\Leftrightarrow (x - y)(x + y) < 0$$

$$\Leftrightarrow x^2 - y^2 < 0$$

$$\Leftrightarrow x^2 < y^2$$

**20**  $f'(x) = 2x - 1$

$$2x - 1 > 0 \Leftrightarrow x > \frac{1}{2}$$

$\therefore$  strictly increasing for  $x > \frac{1}{2}$  We also

know that  $f(x) > f(\frac{1}{2})$  for all  $x \in \mathbb{R} \setminus \{\frac{1}{2}\}$

$\therefore$  strictly increasing for  $[\frac{1}{2}, \infty)$

If  $x < \frac{1}{2}$  then  $f(x) > \frac{1}{2}$

Hence  $[\frac{1}{2}, \infty)$  is the largest interval for which  $f$  is strictly increasing.

**21** Note that answers in text are given for strictly increasing

**a**  $(\infty, -1]$

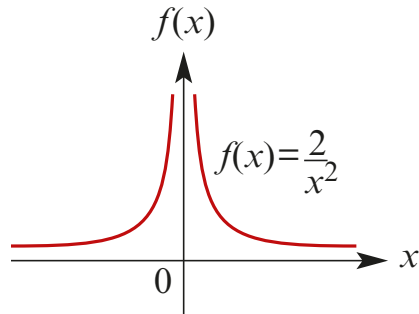
**b**  $[2, \infty)$

**c**  $[-\infty, 0]$

**d**  $[\frac{3}{2}, \infty)$

## Solutions to Exercise 9C

1 a



$$\begin{aligned} \text{b } \text{grad}(PQ) &= \frac{\text{rise}}{\text{run}} \\ &= \frac{f(1+h) - f(1)}{(1+h) - 1} \\ &= \frac{\frac{2}{(1+h)^2} - \frac{2}{1^2}}{h} \\ &= \frac{1}{h} \left( \frac{2}{(1+h)^2} - 2 \frac{(1+h)^2}{(1+h)^2} \right) \\ &= \frac{1}{h} \left( \frac{2 - 2(1+2h+h^2)}{1+2h+h^2} \right) \\ &= \frac{1}{h} \left( \frac{-4h - 2h^2}{1+2h+h^2} \right) \end{aligned}$$

$$\text{grad}(PQ) = \frac{-4 - 2h}{1 + 2h + h^2}$$

$$\begin{aligned} \text{c } \text{grad}(P) &= \lim_{h \rightarrow 0} \text{grad}(PQ) \\ &= \lim_{h \rightarrow 0} \frac{-4 - 2h}{1 + 2h + h^2} \\ &= \frac{-4}{1} \end{aligned}$$

$$\text{grad}(P) = -4$$

$$\begin{aligned} \text{2 a } \frac{f(x+h) - f(x)}{h} &= \left( \frac{1}{x+h-3} - \frac{1}{x-3} \right) \times \frac{1}{h} \\ &= \left( \frac{x-3 - (x+h-3)}{(x+h-3)(x-3)} \right) \times \frac{1}{h} \\ &= \left( \frac{-h}{(x+h-3)(x-3)} \right) \times \frac{1}{h} \\ &= \left( \frac{-1}{(x+h-3)(x-3)} \right) \end{aligned}$$

Hence

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{-1}{(x+h-3)(x-3)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x-3)(x-3)} \\ &= -\frac{1}{(x-3)^2} \end{aligned}$$

$$\begin{aligned} \text{b } \frac{f(x+h) - f(x)}{h} &= \left( \frac{1}{x+h+2} - \frac{1}{x+2} \right) \times \frac{1}{h} \\ &= \left( \frac{x+2 - (x+h+2)}{(x+h+2)(x+2)} \right) \times \frac{1}{h} \\ &= \left( \frac{-h}{(x+h+2)(x+2)} \right) \times \frac{1}{h} \\ &= \left( \frac{-1}{(x+h+2)(x+2)} \right) \end{aligned}$$

Hence

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{-1}{(x+h+2)(x+2)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+2)(x+2)} \\ &= -\frac{1}{(x+2)^2} \end{aligned}$$



**3**

$$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ &= \frac{(x+h)^{-4} - x^{-4}}{h} \\ &= \left( \frac{1}{(x+h)^4} - \frac{1}{x^4} \right) \times \frac{1}{h} \\ &= \left( \frac{x^4 - (x+h)^4}{x^4(x+h)^4} \right) \times \frac{1}{h} \\ &= \left( \frac{x^4 - (x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4)}{x^4(x+h)^4} \right) \times \frac{1}{h} \\ &= \left( \frac{-(4x^3h + 6x^2h^2 + 4xh^3 + h^4)}{x^4(x+h)^4} \right) \times \frac{1}{h} \\ &= \frac{-(4x^3 + 6x^2h + 4xh^2 + h^3)}{x^4(x+h)^4} \end{aligned}$$

Hence

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{-(4x^3 + 6x^2h + 4xh^2 + h^3)}{x^4(x+h)^4} \right) \\ &= -\frac{4x^3}{x^8} \\ &= -\frac{4}{x^5} \end{aligned}$$

**4 a**  $y = 3x^{-2} + 5x^{-1} + 6$

$$\frac{dy}{dx} = -6x^{-3} - 5x^{-2}$$

**b**  $y = 5x^{-3} + 6x^2$

$$\frac{dy}{dx} = -15x^{-4} + 12x$$

**c**  $f(x) = -5x^{-3} + 4x^{-2} + 1$

$$f'(x) = 15x^{-4} - 8x^{-3}$$

**d**  $f(x) = 6x^{-3} + 3x^{-2}$

$$f'(x) = -18x^{-4} - 6x^{-3}$$

**e**  $y = 4 + 2x^{-1}$

$$\frac{dy}{dx} = -2x^{-2}$$

**5 a**  $y = 2 - 4z^{-1}$

$$\frac{dy}{dz} = 4z^{-2}$$

**b**  $y = 6z^{-3} + z^{-2}$

$$\frac{dy}{dz} = -18z^{-4} - 2z^{-3}$$

**c**  $y = 16 - z^{-3}$

$$\frac{dy}{dz} = 3z^{-4}$$

**d**  $f(z) = 4z^{-1} + z - z^2$

$$f'(z) = -4z^{-2} + 1 - 2z$$

**e**  $f(z) = 6z^{-2} - 2z^{-3}$

$$f'(z) = -12z^{-3} + 6z^{-4}$$

**f**  $f(x) = 6x^{-1} - 3x^2$

$$f'(x) = -6x^{-2} - 6x$$

**6 a**  $y = x^{-2} + x^3$

$$\frac{dy}{dx} = -2x^{-3} + 3x^2$$

$$x = 2,$$

$$\frac{dy}{dx} = \frac{-2}{8} + 3 \times 4$$

$$= \frac{-1}{4} + 12$$

$$\frac{dy}{dx} = \frac{47}{4} = 11\frac{3}{4}$$

$$\mathbf{b} \quad y = x^{-2} - x^{-1}$$

$$\frac{dy}{dx} = -2x^{-3} + x^{-2}$$

$$x = 4,$$

$$\frac{dy}{dx} = \frac{-2}{64} + \frac{1}{16}$$

$$= \frac{-1}{32} + \frac{1}{16}$$

$$\frac{dy}{dx} = \frac{1}{32}$$

$$\mathbf{c} \quad y = x^{-2} - x^{-1}$$

$$\frac{dy}{dx} = -2x^{-3} + x^{-2}$$

$$x = 1,$$

$$\frac{dy}{dx} = -2 + 1$$

$$\frac{dy}{dx} = -1$$

$$\mathbf{d} \quad y = 1 + x^3 - x^{-2}$$

$$\frac{dy}{dx} = 3x^2 + 2x^{-3}$$

$$x = 1,$$

$$\frac{dy}{dx} = 3 + 2$$

$$\frac{dy}{dx} = 5$$

$$\mathbf{7} \quad f'(x) = 10x^{-4} > 0 \text{ for all } x \neq 0$$

$$\mathbf{8} \quad y = \frac{x^2 - 1}{x} = x - \frac{1}{x} = x - x^{-1}$$

$$\frac{dy}{dx} = 1 + x^{-2}$$

$$\frac{dy}{dx} = 5$$

$$5 = 1 + \frac{1}{x^2}$$

$$\frac{1}{x^2} = 4$$

$$x^2 = \frac{1}{4}$$

$$x = \pm \frac{1}{2}$$

$$\mathbf{9} \quad y = ax^2 + bx^{-1}$$

$$x = 2, y = -2$$

$$1 - 2 = 4a + \frac{b}{2}$$

$$\frac{dy}{dx} = 2ax - bx^{-2}$$

$$x = 2, \frac{dy}{dx} = -5$$

$$2 - 5 = 4a - \frac{b}{4}$$

$$1 - 2 \Rightarrow 3 = \frac{3b}{4}$$

$$b = 4$$

$$\text{Sub in 1} \Rightarrow -2 = 4a + 2$$

$$4a = -4$$

$$a = -1$$

$$y = -x^2 + \frac{4}{x}$$

$$10 \quad y = 2x^{-1} - 4x^{-2}$$

$$y = 0,$$

$$0 = 2x^{-1} - 4x^{-2}$$

$$0 = 2x - 4$$

$$x = 2$$

$$\frac{dy}{dx} = -2x^{-2} + 8x^{-3}$$

$$x = 2,$$

$$\frac{dy}{dx} = \frac{-2}{4} + \frac{8}{8}$$

$$= \frac{-1}{2} + 1$$

$$= \frac{1}{2}$$

$$11 \quad y = \frac{9}{x} + bx^2$$

$$= ax^{-1} + bx^2$$

$$x = 3, y = 6$$

$$16 = \frac{9}{3} + 9b$$

$$\frac{dy}{dx} = ax^{-2} + 2bx$$

$$x = 3, \frac{dy}{dx} = 7$$

$$27 = \frac{-a}{9} + 6b$$

$$31 \Rightarrow 18 = a + 27b \quad b = 1$$

$$92 \Rightarrow 63 = -a + 54b$$

$$31 + 92 \Rightarrow 81 = 81b$$

$$\text{Sub in 2} \Rightarrow 7 = \frac{-a}{9} + 6$$

$$\frac{-a}{9} = 1$$

$$a = -9$$

$$y = \frac{-9}{x} + x^2$$

$$12 \quad y = \frac{5}{3}x + kx^2 - \frac{8}{9}x^3$$

$$\frac{dy}{dx} = \frac{5}{3} + 2kx - \frac{8}{3}x^2$$

$$\text{at } x = \frac{-1}{2}$$

$$\frac{dy}{dx} = \frac{5}{3} - k - \frac{2}{3}$$

$$\frac{dy}{dx} = 1 - k$$

$$\text{at } x = 1,$$

$$\frac{dy}{dx} = \frac{5}{3} + 2k - \frac{8}{3}$$

$$\frac{dy}{dx} = 2k - 1$$

$$2k - 1 = \frac{-1}{1 - k} \text{ (perpendicular)}$$

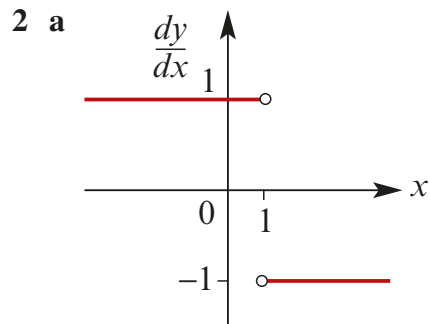
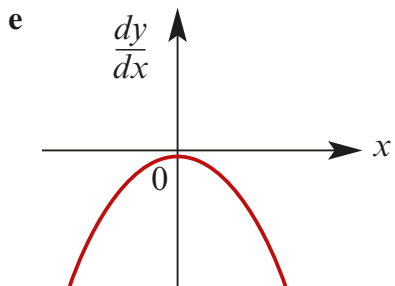
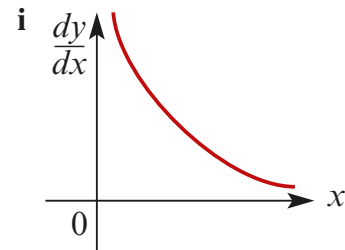
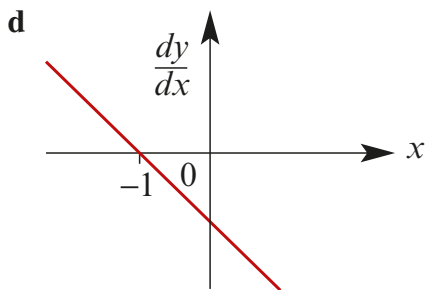
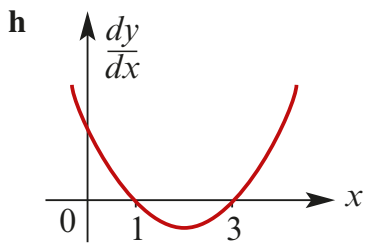
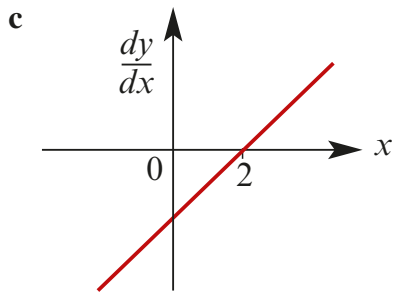
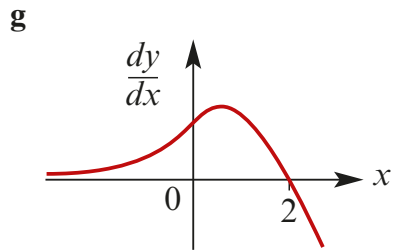
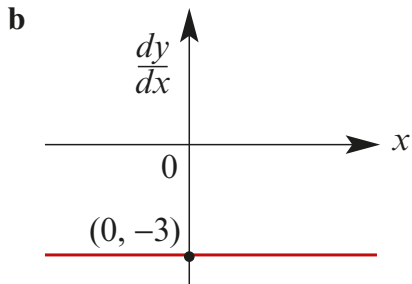
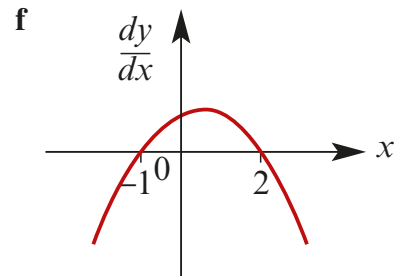
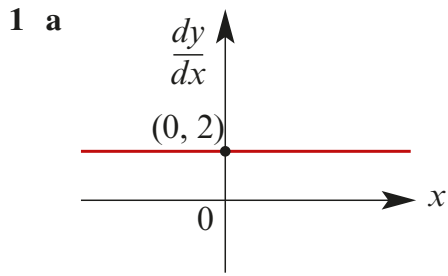
$$(2k - 1)(k - 1) = 1$$

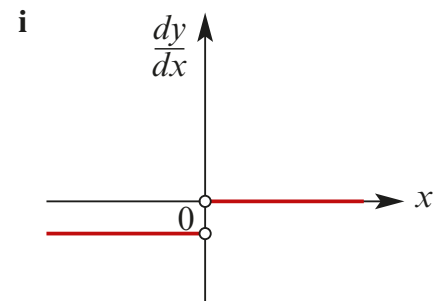
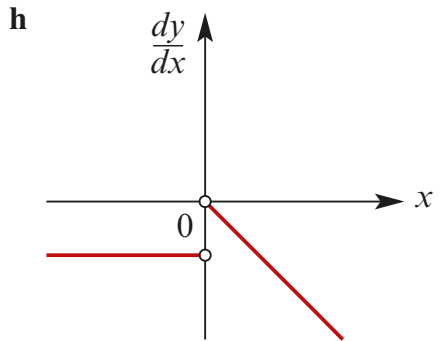
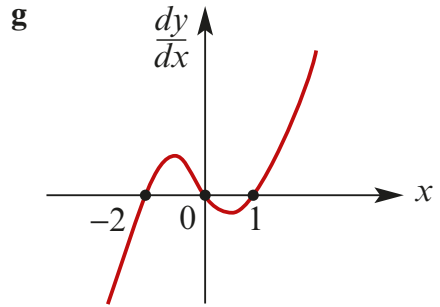
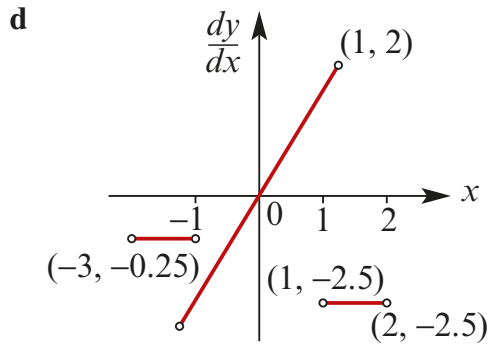
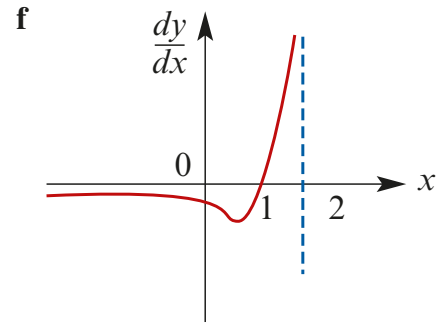
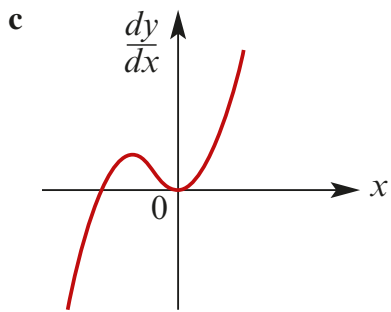
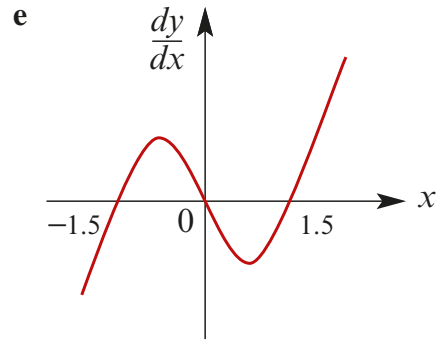
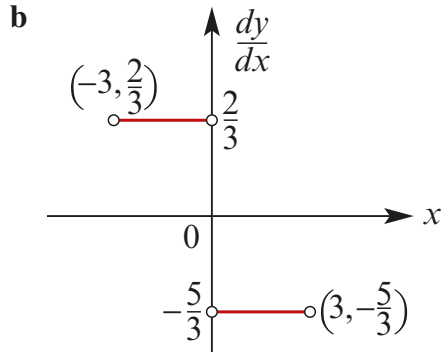
$$2k^2 - 3k + 1 = 1$$

$$2k^2 - 3k = 0$$

$$k(2k - 3) = 0 \Rightarrow k = 0, \frac{3}{2}$$

## Solutions to Exercise 9D





3 a D

b F

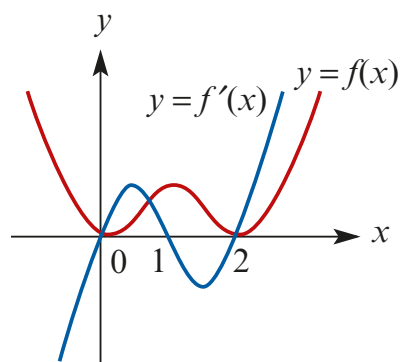
c B

d C

e A

f E

4 a



b i 0

ii 0

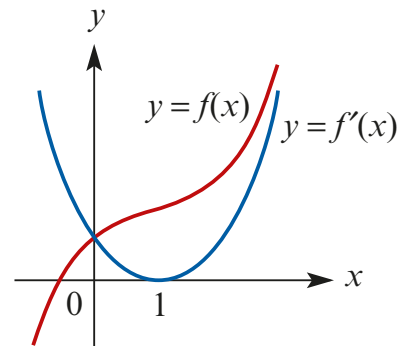
iii 0

iv 96

c i 1

ii 0.423

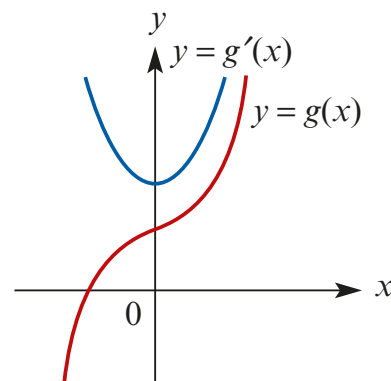
5



Gradient is 0 at  $(1, \frac{4}{3})$

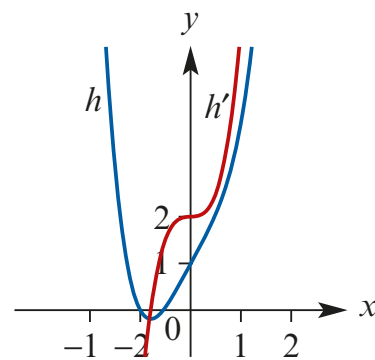
Gradient is positive for  $R \setminus \{1\}$

6



Gradient is always positive, minimum gradient where  $x=0$

7 a



b i  $x = -1.495$  or  $x = 0.798$

ii  $x = 0.630$

## Solutions to Exercise 9E

**1 a**  $y = (x^2 + 1)^4$

Let  $u = x^2 + 1$ ,  $y = u^4$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 4u^3 \times 2x \\ &= 4(x^2 + 1)^3 \times 2x \\ &= 8x(x^2 + 1)^3\end{aligned}$$

**b**  $y = (2x^2 - 3)^5$

Let  $u = 2x^2 - 3$ ,  $y = u^5$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 4x \times 5u^4 \\ &= 4x \times 5(2x^2 - 3)^4 \\ &= 20x(2x^2 - 3)^4\end{aligned}$$

**c**  $y = (6x + 1)^4$

Let  $u = 6x + 1$ ,  $y = u^4$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 6 \times 4u^3 \\ &= 24(6x + 1)^3\end{aligned}$$

**d**  $y = (ax + b)^n$

Let  $u = ax + b$ ,  $y = u^n$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= a \times nu^{(n-1)} \\ &= an(ax + b)^{n-1}\end{aligned}$$

**e**  $y = (ax^2 + b)^n$

Let  $u = ax^2 + b$ ,  $y = u^n$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2ax \times nu^{(n-1)} \\ &= 2anx(ax^2 + b)^{n-1}\end{aligned}$$

**f**  $y = (1 - x^2)^{-3}$

Let  $u = 1 - x^2$ ,  $y = u^{-3}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -2x - 3u^{-4} \\ &= 6x(1 - x^2)^{-4}\end{aligned}$$

**g**  $y = (x^2 - x^{-2})^{-3}$

Let  $u = x^2 - x^{-2}$ ,  $y = u^{-3}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= (2x + 2x^{-3}) \times -3u^{-4} \\ &= -6(x + x^{-3})(x^2 - x^{-2})^{-4}\end{aligned}$$

**h**  $y = (1 - x)^{-1}$

Let  $u = 1 - x$ ,  $y = u^{-1}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -1 \times -u^{-2} \\ &= (1 - x)^{-2}\end{aligned}$$

$$\begin{aligned}
 2 \text{ a} \quad y &= (x^2 + 2x + 1)^3 \\
 &= ((x + 1)^2)^3 \\
 &= (x + 1)^6
 \end{aligned}$$

$$\text{Let } u = x + 1, y = u^6$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= 1 \times 6u^5 \\
 &= 6(x + 1)^5
 \end{aligned}$$

**b**

$$y = (x^3 + 2x^2 + x)^4$$

$$\text{Let } u = x^3 + 2x^2 + x, y = u^4$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= (3x^2 + 4x + 1) \times 4u^3 \\
 &= 4(3x + 1)(x + 1)(x^3 + 2x^2 + x)^3 \\
 &= 4(3x + 1)(x + 1)(x(x + 1)^2)^3 \\
 &= 4x^3(3x + 1)(x + 1)(x + 1)^6 \\
 &= 4x^3(3x + 1)(x + 1)^7
 \end{aligned}$$

$$2 \text{ c} \quad y = (6x^3 + 2x^{-1})^4$$

$$\text{Let } u = 6x^3 + 2x^{-1}, y = u^4$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= (18x^2 - 2x^{-2}) \times 4u^3 \\
 &= 8(9x^2 - x^{-2})(6x^3 + 2x^{-1})^3
 \end{aligned}$$

$$\begin{aligned}
 2 \text{ d} \quad y &= (x^2 + 2x + 1)^{-2} \\
 &= ((x + 1)^2)^{-2} \\
 &= (x + 1)^{-4}
 \end{aligned}$$

$$\text{Let } u = x + 1, y = u^{-4}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= 1 \times -4u^{-5} \\
 &= -4(x + 1)^{-5}
 \end{aligned}$$

$$3 \text{ Let } y = \frac{16}{3x^3 + x} = 16(3x^3 + x)^{-1}$$

$$\text{Let } u = 3x^3 + x$$

$$\text{Then } y = 16u^{-1}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= -16u^{-2} \times (9x^2 + 1) \\
 &= \frac{-16(9x^2 + 1)}{(3x^3 + x)^2}
 \end{aligned}$$

$$\text{When } x = 1$$

$$\frac{dy}{dx} = -10$$

$$4 \text{ Let } y = \frac{1}{x^2 + 1} = (x^2 + 1)^{-1}$$

$$\text{Let } u = x^2 + 1$$

$$\text{Then } y = u^{-1}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= -u^{-2} \times (2x) \\
 &= \frac{-2x}{(x^2 + 1)^2}
 \end{aligned}$$

$$\text{When } x = 1$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

$$\text{When } x = -1$$

$$\frac{dy}{dx} = \frac{1}{2}$$

$$5 \text{ } F(x) = f(g(x))$$

$$F'(x) = g'(x)f'(g(x))$$

$$= 2x\sqrt{3g(x) + 4}$$

$$= 2x\sqrt{3x^2 + 1}$$



**6 a** Let  $h(x) = [f(x)]^n$

Let  $g(x) = x^n$

then  $h(x) = g(f(x))$

$$\begin{aligned}h'(x) &= g'(f(x)) \times f'(x) \\ &= n(f(x))^{n-1} \times f'(x)\end{aligned}$$

**b** Let  $h(x) = (f(x))^{-1}$

Let  $g(x) = x^{-1}$

then  $h(x) = g(f(x))$

$$\begin{aligned}h'(x) &= g'(f(x)) \times f'(x) \\ &= -(f(x))^{-2} \times f'(x)\end{aligned}$$

## Solutions to Exercise 9F

$$\begin{aligned}
 \mathbf{1} \quad & \frac{f(x+h) - f(x)}{h} \\
 &= (2\sqrt{x+h} - 2\sqrt{x}) \times \frac{1}{h} \\
 &= (2\sqrt{x+h} - 2\sqrt{x}) \times \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \times \frac{1}{h} \\
 &= \frac{2(x+h-x)}{\sqrt{x+h} + \sqrt{x}} \times \frac{1}{h} \\
 &= \frac{2}{\sqrt{x+h} + \sqrt{x}}
 \end{aligned}$$

Hence

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{2}{\sqrt{x+h} + \sqrt{x}} \\
 &= \frac{1}{\sqrt{x}}
 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad \frac{d(x^{\frac{1}{5}})}{dx} = \frac{1}{5}x^{-\frac{4}{5}}$$

$$\mathbf{b} \quad \frac{d(x^{\frac{5}{2}})}{dx} = \frac{5}{2}x^{\frac{3}{2}}$$

$$\mathbf{c} \quad \frac{d(x^{\frac{5}{2}} - x^{\frac{3}{2}})}{dx} = \frac{5}{2}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$

$$\begin{aligned}
 \mathbf{d} \quad \frac{d(3x^{\frac{1}{2}} - 4x^{\frac{5}{3}})}{dx} &= \frac{3}{2}x^{-\frac{1}{2}} - 4 \times \frac{5}{3}x^{\frac{2}{3}} = \\
 & \frac{3}{2}x^{-\frac{1}{2}} - \frac{20}{3}x^{\frac{2}{3}}
 \end{aligned}$$

$$\mathbf{e} \quad \frac{d(x^{-\frac{6}{7}})}{dx} = \frac{-6}{7}x^{-\frac{13}{7}}$$

$$\mathbf{f} \quad \frac{d(x^{\frac{1}{4}} + 4x^{\frac{1}{2}})}{dx} = \frac{-1}{4}x^{-\frac{5}{4}} + 2x^{-\frac{1}{2}}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & f(x) = x^{\frac{1}{3}} \\
 & f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \\
 & f'(27) = \frac{1}{3} \times \frac{1}{(27^{\frac{1}{3}})^2} \\
 & = \frac{1}{3} \times \frac{1}{3^2} = \frac{1}{27}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & f(x) = x^{\frac{1}{3}} \\
 & f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \\
 & f'(-8) = \frac{1}{3} \times (-8)^{-\frac{2}{3}} \\
 & = \frac{1}{3}(-2)^{-2} \\
 & = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & f(x) = x^{\frac{2}{3}} \\
 & f'(x) = \frac{2}{3}x^{-\frac{1}{3}} \\
 & f'(27) = \frac{2}{3} \times (27)^{-\frac{1}{3}} \\
 & = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & f(x) = x^{\frac{5}{4}} \\
 & f'(x) = \frac{5}{4}x^{\frac{1}{4}} \\
 & f'(16) = \frac{5}{4} \times (16)^{\frac{1}{4}} \\
 & = \frac{5}{4} \times 2 = \frac{5}{2}
 \end{aligned}$$

$$4 \text{ a } \frac{d}{dx}(\sqrt{2x+1})$$

$$= 2 \times \frac{1}{2\sqrt{2x+1}}$$

$$= \frac{1}{\sqrt{2x+1}}$$

$$b \frac{d}{dx}(\sqrt{4-3x})$$

$$= -3 \times \frac{1}{2\sqrt{4-3x}}$$

$$= \frac{-3}{2\sqrt{4-3x}}$$

$$c \frac{d}{dx}(\sqrt{x^2+2})$$

$$= 2x \times \frac{1}{2\sqrt{x^2+2}}$$

$$= \frac{x}{\sqrt{x^2+2}}$$

$$d \frac{d}{dx}(4-3x)^{\frac{1}{3}}$$

$$= -3 \times \frac{1}{3(4-3x)^{\frac{2}{3}}}$$

$$= -(4-3x)^{-2/3}$$

$$e \frac{d}{dx}\left(\frac{x^2+2}{\sqrt{x}}\right)$$

$$= \frac{d}{dx}\left(x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}\right)$$

$$= \frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{3}{2}}$$

$$f \frac{d}{dx}(3\sqrt{x}(x^2+2x))$$

$$= \frac{d}{dx}(3x^{\frac{5}{2}} + 6x^{\frac{3}{2}})$$

$$= \frac{15}{2}x^{\frac{3}{2}} + \frac{18}{2}x^{\frac{1}{2}}$$

$$= \frac{15}{2}x^{\frac{3}{2}} + 9\sqrt{x}$$

$$5 \text{ a } \text{Let } u = x^2 \pm a^2$$

$$LHS = \frac{d}{dx}(\sqrt{x^2 \pm a^2}) = \frac{d}{dx}(u^{\frac{1}{2}})$$

$$= \frac{d}{dx}(u^{\frac{1}{2}}) \times \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \times 2x$$

$$= \frac{x}{\sqrt{x^2 \pm a^2}} = RHS \quad QED$$

$$b \text{ Let } u = a^2 - x^2$$

$$LHS = \frac{d}{dx}(\sqrt{a^2 - x^2}) = \frac{d}{dx}(\sqrt{u})$$

$$= \frac{d}{dx}(\sqrt{u}) \times \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \times -2x$$

$$= \frac{-x}{\sqrt{a^2 - x^2}}$$

$$6 \quad y = (x + \sqrt{x^2 + 1})^2$$

$$\text{Let } u = x + \sqrt{x^2 + 1}, \quad y = u^2$$

$$\begin{aligned} \text{LHS} &= \frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du} \\ &= \left( \frac{d(x)}{dx} + \frac{d(\sqrt{x^2 + 1})}{dx} \right) \times \frac{dy}{du} \end{aligned}$$

$$\text{Let } w = x^2 + 1$$

$$\begin{aligned} \frac{dy}{dx} &= \left( \frac{d(x)}{dx} + \frac{d(\sqrt{w})}{dw} \times \frac{dw}{dx} \right) \times \frac{dy}{du} \\ &= \left( 1 + \frac{1}{2\sqrt{w}} \times 2x \right) \times 2u \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \left( 1 + \frac{x}{\sqrt{x^2 + 1}} \right) \times 2(x + \sqrt{x^2 + 1}) \\ &= \left( \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right) \times 2(x + \sqrt{x^2 + 1}) \\ &= \frac{2(x + \sqrt{x^2 + 1})^2}{\sqrt{x^2 + 1}} \\ &= \frac{2y}{\sqrt{x^2 + 1}} = \text{RHS QED} \end{aligned}$$

$$7 \quad \text{a} \quad \text{Let } u = x^2 + 2$$

$$\begin{aligned} \frac{d(\sqrt{x^2 + 2})}{dx} &= \frac{d(\sqrt{u})}{du} \times \frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}} \times 2x \\ &= \frac{x}{\sqrt{x^2 + 2}} \end{aligned}$$

$$\text{b} \quad \text{Let } u = x^3 - 5x$$

$$\begin{aligned} \frac{d\left((x^3 - 5x)^{\frac{1}{3}}\right)}{dx} &= \frac{d\left(u^{\frac{1}{3}}\right)}{du} \times \frac{du}{dx} \\ &= \frac{1}{3}u^{-\frac{2}{3}} \times (3x^2 - 5) \\ &= \frac{1}{3}(3x^2 - 5)(x^3 - 5x)^{-\frac{2}{3}} \\ &= \frac{3x^2 - 5}{3\sqrt[3]{(x^3 - 5x)^2}} \end{aligned}$$

$$\text{c} \quad \text{Let } u = x^2 + 2x$$

$$\begin{aligned} \frac{d\left((x^2 + 2x)^{\frac{1}{5}}\right)}{dx} &= \frac{d\left(u^{\frac{1}{5}}\right)}{du} \times \frac{du}{dx} \\ &= \frac{1}{5}u^{-\frac{4}{5}} \times 2x + 2 \\ &= \frac{2x + 2}{5(x^2 + 2x)^{\frac{4}{5}}} \end{aligned}$$

## Solutions to Exercise 9G

1 a  $f(x) = e^{5x}$

$$f'(x) = 5e^{5x}$$

b  $f(x) = 7e^{-3x}$

$$f'(x) = -21e^{-3x}$$

c  $f(x) = 3e^{-4x} + e^x - x^2$

$$f'(x) = -12e^{-4x} + e^x - 2x$$

d  $f(x) = e^x - 1 + e^{-x}$

$$f'(x) = e^x - e^{-x}$$

e  $f(x) = \frac{4e^{2x} - 2e^x + 1}{2e^{2x}}$

$$= -e^{-x} + \frac{1}{2}e^{-2x}$$

$$f'(x) = e^{-x} - e^{-2x}$$

$$= e^{-2x}(e^x - 1)$$

f  $f(x) = e^{2x} + e^4 + e^{-2x}$

$$f'(x) = 2e^{2x} - 2e^{-2x}$$

2 a  $-6x^2e^{-2x^3}$

b  $2xe^{x^2} + 3$

c  $(2x - 4)e^{x^2 - 4x} + 3$

d  $(2x - 2)e^{x^2 - 2x + 3} - 1$

e  $-\frac{1}{x^2}e^{\frac{1}{x}}$

f  $\frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$

3 Let  $y = e^{\frac{x}{2}} + 4x$

Then  $\frac{dy}{dx} = \frac{1}{2}e^{\frac{x}{2}} + 4$

a When  $x = 0, y = \frac{9}{2}$

b When  $x = 1, \frac{1}{2}e^{\frac{1}{2}} + 4$

4 Let  $y = e^{x^2 + 3x} + 2x$

Then  $\frac{dy}{dx} = (2x + 3)e^{x^2 + 3x} + 2$

a When  $x = 0, y = 5$

b When  $x = 1, 5e^4 + 2$

5 a  $2f'(x)e^{2f(x)}$

b  $2e^{2x}f'(e^{2x})$

6 a  $y = (e^{2x} - 1)^4$

Let  $u = e^{2x} - 1, y = u^4$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= (2e^{2x}) \times 4u^3 \\ &= 8e^{2x}(e^{2x} - 1)^3 \end{aligned}$$

b  $y = e^{\sqrt{x}}$

Let  $u = \sqrt{x}, y = e^u$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times \frac{1}{2\sqrt{x}} \\ &= e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}e^{\sqrt{x}} \end{aligned}$$

**c**  $y = (e^x - 1)^{\frac{1}{2}}$

Let  $u = e^x - 1, y = u^{\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{2}u^{-\frac{1}{2}} \times e^x \\ &= \frac{1}{2}e^x(e^x - 1)^{-\frac{1}{2}}\end{aligned}$$

**d**  $y = e^{x^{\frac{2}{3}}}$

Let  $u = x^{\frac{2}{3}}, y = e^u$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times \frac{2}{3}x^{-\frac{1}{3}} \\ &= \frac{2}{3}x^{-\frac{1}{3}}e^{x^{\frac{2}{3}}}\end{aligned}$$

**e**  $(2x - 3)e^{(x-1)(x-2)}$

**f**  $e^{e^x+x}$

## Solutions to Exercise 9H

1 a  $\frac{dy}{dx} = \frac{2}{x}$

b  $\frac{dy}{dx} = \frac{4}{2x} = \frac{2}{x}$

c  $\frac{dy}{dx} = 2x + \frac{3}{x}$

d  $\frac{dy}{dx} = \frac{3}{x} - \frac{1}{x^2} = \frac{3x-1}{x^2}$

e  $\frac{dy}{dx} = \frac{3}{x} + 1 = \frac{3+x}{x}$

f  $\frac{dy}{dx} = \frac{1}{x+1}$

g  $\frac{dy}{dx} = \frac{2}{2x+4} = \frac{1}{x+2}$

h  $\frac{dy}{dx} = \frac{3}{3x-1}$

i  $\frac{dy}{dx} = \frac{6}{6x-1}$

2 a  $\frac{dy}{dx} = \frac{3}{x}$

b  $\frac{dy}{dx} = \frac{3(\log_e x)^2}{x}$

c  $\frac{dy}{dx} = \frac{2x+1}{x^2+x-1}$

d  $\frac{dy}{dx} = \frac{3x^2+2x}{x^3+x^2}$

e  $\frac{dy}{dx} = \frac{4}{2x+3}$

f  $\frac{dy}{dx} = \frac{4}{2x-3}$

3 a  $f(x) = \log_e(x^2+1)$

$$f'(x) = 2x \times \frac{1}{x^2+1}$$

$$= \frac{2x}{x^2+1}$$

b  $f(x) = \log_e(e^x)$

$$f'(x) = e^x \times \frac{1}{e^x}$$

$$= 1$$

4 a  $y = \log_e x$

$$x = e,$$

$$y = \ln e = 1$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$x = e,$$

$$\frac{dy}{dx} = \frac{1}{e} = e^{-1}$$

b  $y = \ln(x^2+1)$

$$x = e,$$

$$y = \ln(e^2+1)$$

$$\frac{dy}{dx} = 2x \times \frac{1}{x^2+1}$$

$$= \frac{2x}{x^2+1}$$

$$x = e,$$

$$\frac{dy}{dx} = \frac{2e}{e^2+1}$$

**c**  $y = \ln(-x)$

$$x = -e,$$

$$y = \log_e e = 1$$

$$\frac{dy}{dx} = \frac{-1}{-x} = \frac{1}{x}$$

$$x = -e,$$

$$\frac{dy}{dx} = \frac{-1}{e} = -e^{-1}$$

**d**  $y = x + \log_e x$

$$x = 1,$$

$$y = 1 + \log_e 1 = 1$$

$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

$$x = 1,$$

$$\frac{dy}{dx} = 1 + 1 = 2$$

**e**  $y = \log_e(x^2 - 2x + 2)$

$$x = 1,$$

$$y = \log_e 1 = 0$$

$$\frac{dy}{dx} = (2x - 2) \frac{1}{x^2 - 2x + 2}$$

$$x = 1,$$

$$\frac{dy}{dx} = 0$$

**f**  $y = \log_e(2x - 1)$

$$x = \frac{3}{2},$$

$$y = \log_e 2$$

$$\frac{dy}{dx} = \frac{2}{2x - 1}$$

$$x = \frac{3}{2},$$

$$\frac{dy}{dx} = \frac{2}{2} = 1$$

**5**  $f(x) = \ln(\sqrt{x^2 + 1})$

$$\begin{aligned} f'(x) &= 2x \times \frac{1}{2\sqrt{x^2 + 1}} \times \frac{1}{\sqrt{x^2 + 1}} \\ &= \frac{x}{x^2 + 1} \end{aligned}$$

alternatively,

$$\begin{aligned} f(x) &= \ln\left((x^2 + 1)^{\frac{1}{2}}\right) \\ &= \frac{1}{2} \ln(x^2 + 1) \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{1}{2} \times 2x \times \frac{1}{x^2 + 1} \\ &= \frac{x}{x^2 + 1} \end{aligned}$$

$$f'(1) = \frac{1}{1 + 1} = \frac{1}{2}$$

**6**  $\frac{d}{dx}(\ln(x^2 + x + 1))$

$$\begin{aligned} &= (2x + 1) \times \frac{1}{x^2 + x + 1} \\ &= \frac{2x + 1}{x^2 + x + 1} \end{aligned}$$

**7**  $f(x) = \ln(x^2 + 1)$

$$\begin{aligned} f'(x) &= 2x \times \frac{1}{x^2 + 1} \\ &= \frac{2x}{x^2 + 1} \end{aligned}$$

$$f'(3) = \frac{6}{9 + 1} = \frac{3}{5}$$

**8**  $\frac{d}{dx}(\ln(f(x))) = f'(x) \times \frac{1}{f(x)}$   
$$= \frac{f'(x)}{f(x)}$$



$$x = 0,$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(0)}{f(0)}$$

$$= \frac{4}{2}$$

$$= 2$$

## Solutions to Exercise 9I

1 a  $5 \cos 5x$

b  $-5 \sin 5x$

c  $5 \sec^2 5x$

d  $\cos x \times 2 \sin x = \sin 2x$

e  $3 \sec^2(3x + 1)$

f  $-2x \sin(x^2 + 1)$

g  $2 \sin\left(x - \frac{\pi}{4}\right) \cos\left(x - \frac{\pi}{4}\right)$

h  $-2 \cos\left(x - \frac{\pi}{3}\right) \sin\left(x - \frac{\pi}{3}\right)$

i  $6 \sin^2\left(2x + \frac{\pi}{6}\right) \cos\left(2x + \frac{\pi}{6}\right)$

j  $6 \cos\left(2x + \frac{\pi}{4}\right) \sin^2\left(2x + \frac{\pi}{4}\right)$

2 a  $y = \sin 2x$

$$\frac{dy}{dx} = 2 \cos 2x$$

$$x = \frac{\pi}{8},$$

$$y = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\frac{dy}{dx} = 2 \cos \frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

b  $y = \sin 3x$

$$\frac{dy}{dx} = 3 \cos 3x$$

$$x = \frac{\pi}{6},$$

$$y = \sin \frac{\pi}{2} = 1$$

$$\frac{dy}{dx} = 3 \cos \frac{\pi}{2} = 0$$

c  $y = 1 + \sin 3x$

$$\frac{dy}{dx} = 3 \cos 3x$$

$$x = \frac{\pi}{6},$$

$$y = 1 + \sin \frac{\pi}{2} = 1 + 1 = 2$$

$$\frac{dy}{dx} = 3 \cos \frac{\pi}{2} = 0$$

d  $y = \cos^2 2x$

$$\frac{dy}{dx} = -2 \sin 2x \times 2 \cos 2x$$

$$= -4 \sin 2x \cos 2x$$

$$= -2 \sin 4x$$

$$x = \frac{\pi}{4},$$

$$y = \cos^2 \frac{\pi}{2} = 0$$

$$\frac{dy}{dx} = -2 \sin \pi = 0$$

e  $y = \sin^2 2x$

$$\frac{dy}{dx} = -2 \cos 2x \times 2 \sin 2x$$

$$= 4 \cos 2x \sin 2x$$

$$= 2 \sin 4x$$

$$x = \frac{\pi}{4},$$

$$y = \sin^2 \frac{\pi}{2} = 1$$

$$\frac{dy}{dx} = 2 \sin \pi = 0$$

**f**  $y = \tan 2x$

$$\frac{dy}{dx} = 2 \sec^2 2x$$

$$x = \frac{\pi}{8},$$

$$y = \tan \frac{\pi}{4} = 1$$

$$\frac{dy}{dx} = 2 \sec^2 \frac{\pi}{4}$$

$$= 2 \times (\sqrt{2})^2$$

$$= 4$$

**3 a**  $f(x) = 5 \cos x - 2 \sin 3x$

$$f'(x) = -5 \sin x - 6 \cos 3x$$

**b**  $f(x) = \cos x + \sin x$

$$f'(x) = -\sin x + \cos x$$

$$= \cos x - \sin x$$

**c**  $f(x) = \sin x + \tan x$

$$f'(x) = \cos x + \sec^2 x$$

**d**  $f(x) = \tan^2 x$

$$f'(x) = \sec^2 x \times 2 \tan x$$

$$= 2 \tan x \sec^2 x$$

**4 a**  $y = 2 \cos\left(\frac{\pi x}{180}\right)$

$$\frac{dy}{dx} = \frac{-2\pi}{180} \sin\left(\frac{\pi x}{180}\right)$$

$$= \frac{-\pi}{90} \sin(x^\circ)$$

**b**  $y = 3 \sin\left(\frac{\pi x}{180}\right)$

$$\frac{dy}{dx} = \frac{3\pi}{180} \cos\left(\frac{\pi x}{180}\right)$$

$$= \frac{\pi}{60} \cos(x^\circ)$$

**c**  $y = \tan\left(\frac{3\pi x}{180}\right)$

$$y = \tan\left(\frac{\pi x}{60}\right)$$

$$\frac{dy}{dx} = \frac{\pi}{60} \sec^2\left(\frac{\pi x}{60}\right)$$

$$= \frac{\pi}{60} \sec^2(3x^\circ)$$

**5 a**  $y = -\ln(\cos x)$

$$\frac{dy}{dx} = -\sin x \times -1 \times \frac{1}{\cos x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

**b**  $y = -\log_e(\tan x)$

$$\frac{dy}{dx} = -\sec^2 x \times \frac{1}{\tan x}$$

$$= -\frac{1}{\cos x \sin x}$$

**6 a**  $2 \cos x e^{2 \sin x}$

**b**  $-2 \sin(2x) e^{\cos 2x}$

## Solutions to Exercise 9J

**1 a**  $y = (2x^2 + 6)(2x^3 + 1)$

$$\begin{aligned}\frac{dy}{dx} &= (2x^2 + 6)\frac{d}{dx}(2x^3 + 1) \\ &\quad + (2x^3 + 1)\frac{d}{dx}(2x^2 + 6) \\ &= (2x^2 + 6)(6x^2) + (2x^3 + 1)(4x) \\ &= 12x^4 + 36x^2 + 8x^4 + 4x \\ &= 20x^4 + 36x^2 + 4x\end{aligned}$$

**b**  $y = 3x^{\frac{1}{2}}(2x + 1)$

$$\begin{aligned}\frac{dy}{dx} &= 3x^{\frac{1}{2}}\frac{d}{dx}(2x + 1) \\ &\quad + 3(2x + 1)\frac{d}{dx}x^{\frac{1}{2}} \\ &= 3x^{\frac{1}{2}} \times 2 + 3(2x + 1) \times \frac{1}{2x^{\frac{1}{2}}} \\ &= 6x^{\frac{1}{2}} + 3x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}} \\ &= 9x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}\end{aligned}$$

**c**  $y = 3x(2x - 1)^3$

$$\begin{aligned}\frac{dy}{dx} &= 3x\frac{d}{dx}((2x - 1)^3) \\ &\quad + 3(2x - 1)^3\frac{d}{dx}(x)\end{aligned}$$

Let  $u = 2x - 1$

$$\begin{aligned}\frac{dy}{dx} &= 3x\frac{du}{dx} \times \frac{d(u)^3}{dx} + 3(2x - 1)^3 \\ &= 3x(2 \times 3u^2) + 3(2x - 1)^3 \\ &= 18x(2x - 1)^2 + 3(2x - 1)^3 \\ &= 3(2x - 1)^2(6x + (2x - 1)) \\ &= 3(2x - 1)^2(8x - 1)\end{aligned}$$

**d**

$$\begin{aligned}y &= 4x^2(2x^2 + 1)^2 \\ \frac{dy}{dx} &= 4x^2\frac{d}{dx}(2x^2 + 1)^2 + 4(2x^2 + 1)^2\frac{dy}{dx}(x^2) \\ &= 4x^2\left((2x^2 + 1)\frac{d}{dx}(2x^2 + 1)\right. \\ &\quad \left.+ (2x^2 + 1)\frac{d}{dx}(2x^2 + 1)\right) \\ &\quad + 4(2x^2 + 1)^2 \times 2x \\ &= 8x^2(2x^2 + 1)(4x) + 8x(2x^2 + 1)^2 \\ &= 32x^3(2x^2 + 1) + 8x(2x^2 + 1)^2 \\ &= 8x(2x^2 + 1)(4x^2 + 2x^2 + 1) \\ &= 8x(2x^2 + 1)(6x^2 + 1)\end{aligned}$$

**e**

$$\begin{aligned}y &= (3x + 1)^{\frac{3}{2}}(2x + 4) \\ \frac{dy}{dx} &= (3x + 1)^{\frac{3}{2}}\frac{d}{dx}(2x + 4) \\ &\quad + (2x + 4)\frac{d}{dx}(3x + 1)^{\frac{3}{2}}\end{aligned}$$

Let  $u = 3x + 1$

$$\begin{aligned}\frac{dy}{dx} &= (3x + 1)^{\frac{3}{2}}(2) + (2x + 4)\left(\frac{d(u^{\frac{3}{2}})}{dx} \times \frac{du}{dx}\right) \\ &= 2(3x + 1)^{\frac{3}{2}} + (2x + 4)\left(\frac{3}{2}u^{\frac{1}{2}} \times 3\right) \\ &= 2(3x + 1)^{\frac{3}{2}} + \frac{9}{2}(2x + 4)(3x + 1)^{\frac{1}{2}} \\ &= 2(3x + 1)^{\frac{3}{2}} + 9(x + 2)(3x + 1)^{\frac{1}{2}} \\ &= (2(3x + 1) + 9(x + 2))(3x + 1)^{\frac{1}{2}} \\ &= (6x + 2 + 9x + 18)(3x + 1)^{\frac{1}{2}} \\ &= (15x + 20)(3x + 1)^{\frac{1}{2}} \\ &= 5(3x + 4)(3x + 1)^{\frac{1}{2}}\end{aligned}$$

**f**

$$y = (x^2 + 1)(2x - 4)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = (x^2 + 1) \frac{d}{dx}(2x - 4)^{\frac{1}{2}}$$

$$+ (2x - 4)^{\frac{1}{2}} \frac{d}{dx}(x^2 + 1)$$

$$\text{Let } u = 2x - 4$$

$$\frac{dy}{dx} = (x^2 + 1) \left( \frac{d(u^{\frac{1}{2}})}{dx} \times \frac{du}{dx} \right)$$

$$+ (2x - 4)^{\frac{1}{2}} + 2x$$

$$= (x^2 + 1) \left( \frac{1}{2u^{\frac{1}{2}}} \times 2 \right) + 2x(2x - 4)^{\frac{1}{2}}$$

$$= (x^2 + 1)(2x - 4)^{-\frac{1}{2}} + 2x(2x - 4)^{\frac{1}{2}}$$

$$= \frac{(x^2 + 1) + 2x(2x - 4)}{\sqrt{2x - 4}}$$

$$= \frac{(x^2 + 1) + 4x^2 - 8x}{\sqrt{2x - 4}}$$

$$= \frac{5x^2 - 8x + 1}{\sqrt{2x - 4}}$$

**g**

$$y = x^3(3x^2 + 2x + 1)^{-1}$$

$$\frac{dy}{dx} = x^3 \frac{d}{dx}(3x^2 + 2x + 1)^{-1}$$

$$+ (3x^2 + 2x + 1)^{-1} \frac{d}{dx}x^3$$

$$\text{Let } u = 3x^2 + 2x + 1$$

$$\frac{dy}{dx} = x^3 \left( \frac{d(u^{-1})}{dx} \times \frac{du}{dx} \right) + (3x^2 + 2x + 1)^{-1} + 3x^2$$

$$= x^3(-u^{-2} \times (6x + 2)) + 3x^2(3x^2 + 2x + 1)^{-1}$$

$$= -x^3(6x + 2)(3x^2 + 2x + 1)^{-2}$$

$$+ 3x^2(3x^2 + 2x + 1)^{-1}$$

$$= \frac{-x^3(6x + 2) + 3x^2(3x^2 + 2x + 1)}{(3x^2 + 2x + 1)^2}$$

$$= \frac{-6x^4 - 2x^3 + 9x^4 + 6x^3 + 3x^2}{(3x^2 + 2x + 1)^2}$$

$$= \frac{3x^4 + 4x^3 + 3x^2}{(3x^2 + 2x + 1)^2}$$

$$= \frac{x^2(3x^2 + 4x + 3)}{(3x^2 + 2x + 1)^2}$$

**h**

$$y = x^4(2x^2 - 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = x^4 \frac{d}{dx}(2x^2 - 1)^{\frac{1}{2}} + (2x^2 - 1)^{\frac{1}{2}} \frac{d}{dx}x^4$$

$$\text{Let } u = 2x^2 - 1$$

$$\frac{dy}{dx} = x^4 \left( \frac{d(u^{\frac{1}{2}})}{dx} \times \frac{du}{dx} \right) + (2x^2 - 1)^{\frac{1}{2}} + 4x^3$$

$$= x^4 \left( \frac{1}{2}u^{-\frac{1}{2}} \times 4x \right) + 4x^3(2x^2 - 1)^{\frac{1}{2}}$$

$$= 2x^5(2x^2 - 1)^{-\frac{1}{2}} + 4x^3(2x^2 - 1)^{\frac{1}{2}}$$

$$= (2x^5 + 4x^3(2x^2 - 1))(2x^2 - 1)^{-\frac{1}{2}}$$

$$= (2x^5 + 8x^5 - 4x^3)(2x^2 - 1)^{-\frac{1}{2}}$$

$$= (10x^5 - 4x^3)(2x^2 - 1)^{-\frac{1}{2}}$$

$$= 2x^3(5x^2 - 2)(2x^2 - 1)^{-\frac{1}{2}}$$

**i**

$$y = x^2(x^2 + 2x)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(x^2 + 2x)^{\frac{1}{3}} + (x^2 + 2x)^{\frac{1}{3}} \frac{d}{dx}(x^2)$$

$$\text{Let } u = x^2 + 2x$$

$$\begin{aligned} \frac{dy}{dx} &= x^2 \left( \frac{d(u^{\frac{1}{3}})}{dx} \times \frac{du}{dx} \right) + (x^2 + 2x)^{\frac{1}{3}} \times 2x \\ &= x^2 \left( \frac{1}{3} u^{-\frac{2}{3}} \times (2x + 2) \right) + 2x(x^2 + 2x)^{\frac{1}{3}} \\ &= \frac{2}{3} x^2(x + 1)(x^2 + 2x)^{-\frac{2}{3}} + 2x(x^2 + 2x)^{\frac{1}{3}} \\ &= \left( \frac{2}{3} x^3 + \frac{2}{3} x^2 + 2x(x^2 + 2x) \right) (x^2 + 2x)^{-\frac{2}{3}} \\ &= \left( \frac{2}{3} x^3 + \frac{2}{3} x^2 + 2x^3 + 4x^2 \right) (x^2 + 2x)^{-\frac{2}{3}} \\ &= x^2 \left( \frac{8}{3} x + \frac{14}{3} \right) (x^2 + 2x)^{-\frac{2}{3}} \\ &= \frac{2}{3} x^2(4x + 7)(x^2 + 2x)^{-\frac{2}{3}} \end{aligned}$$

**j**  $\frac{4(5x^2 - 4)^2(5x^2 + 2)}{x^3}$

**k**  $\frac{3(x^6 - 16)}{x^4}$

**l**  $\frac{2x^3(9x^2 - 8)}{5(x(x^2 - 1))^{4/5}}$

**2 a**  $f(x) = e^x(x^2 + 1)$

$$\begin{aligned} f'(x) &= e^x + 2x + (x^2 + 1) \times e^x \\ &= e^x(x^2 + 2x + 1) \\ &= ex(x + 1)^2 \end{aligned}$$

**b**

$$\begin{aligned} f(x) &= e^{2x}(x^3 + 3x + 1) \\ f'(x) &= e^{2x}(3x^2 + 3) + (x^3 + 3x + 1) \times 2e^{2x} \\ &= e^{2x}(3x^2 + 3 + 2x^3 + 6x + 2) \\ &= e^{2x}(2x^3 + 3x^2 + 6x + 5) \end{aligned}$$

**c**

$$\begin{aligned}
 f(x) &= e^{4x+1}(x+1)^2 \\
 f'(x) &= e^{4x+1} \times 2(x+1) + (x+1)^2 \times 4e^{4x+1} \\
 &= e^{4x+1}(4(x+1)^2 + 2(x+1)) \\
 &= e^{4x+1}(4x^2 + 8x + 4 + 2x + 2) \\
 &= e^{4x+1}(4x^2 + 10x + 6) \\
 &= e^{4x+1}(2x+2)(2x+3)
 \end{aligned}$$

**d**

$$\begin{aligned}
 f(x) &= e^{-4x}(x+1)^{\frac{1}{2}} \\
 f'(x) &= e^{-4x} \times \frac{1}{2}(x+1)^{-\frac{1}{2}} + (x+1)^{\frac{1}{2}} \times -4e^{-4x} \\
 &= e^{-4x} \left( \frac{1}{2}(x+1)^{-\frac{1}{2}} - 4(x+1)^{\frac{1}{2}} \right) \\
 &= e^{-4x}(x+1)^{-\frac{1}{2}} \left( \frac{1}{2} - 4(x+1) \right) \\
 &= e^{-4x}(x+1)^{-\frac{1}{2}} \left( -4x - \frac{7}{2} \right) \\
 &= \frac{-8x-7}{2e^{4x}\sqrt{x+1}}
 \end{aligned}$$

**3 a**  $f'(x) = \ln x \times 1 + x + \frac{1}{x}$   
 $= \ln x + 1$

**b**  $f'(x) = \ln x \times 4x + 2x^2 \times \frac{1}{x}$   
 $= 2x(1 + 2 \ln x)$

**c**  $f'(x) = e^x \times \frac{1}{x} + \ln x \times e^x$   
 $= e^x \left( \frac{1}{x} + \ln x \right)$

**d**  $f'(x) = \ln(-x) \times 1 + x + \frac{-1}{-x}$   
 $= \ln(-x) + 1$

**4 a**  $f'(x) = 4x^3 e^{-2x} - 2x^4 e^{-2x}$   
 $= 2x^3 e^{-2x}(2-x)$

**b**  $f'(x) = 2e^{2x+3}$

**c** Let  $y = (e^{2x} + x)^{\frac{3}{2}}$   
 Let  $u = e^{2x} + x$   
 Then  $y = u^{\frac{3}{2}}$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= \frac{3}{2} u^{\frac{1}{2}} \times (2e^{2x} + 1) \\
 &= \frac{3}{2} (e^{2x} + x)^{\frac{1}{2}} \times (2e^{2x} + 1)
 \end{aligned}$$

**d** Let  $y = \frac{1}{x} e^x$   
 $\frac{dy}{dx} = -\frac{1}{x^2} \times e^x + \frac{1}{x} \times e^x$   
 $= \frac{e^x(x-1)}{x^2}$

**e** Let  $y = e^{\frac{1}{2}x^2}$   
 $\frac{dy}{dx} = x e^{\frac{1}{2}x^2}$

**f** Let  $y = (x^2 + 2x + 2)e^{-x}$   
 $\frac{dy}{dx} = (2x+2)e^{-x} - (x^2 + 2x + 2)e^{-x}$   
 $= e^{-x}(2x+2-x^2-2x-2)$   
 $= -x^2 e^{-x}$

**5 a**  $\frac{d}{dx}(e^x f(x)) = e^x f(x) + e^x f'(x)$   
 $= e^x(f(x) + f'(x))$

**b**  $\frac{d}{dx}\left(\frac{e^x}{f(x)}\right) = \frac{e^x f(x) - e^x f'(x)}{(f(x))^2}$

$$\mathbf{c} \quad \frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$$

$$\mathbf{d} \quad \frac{d}{dx}(e^{x(f(x))^2}) = e^{x(f(x))^2} + 2e^x f(x)f'(x)$$

$$\mathbf{6 a} \quad \frac{d}{dx}(x^3 \cos x) = 3x^2 + (-\sin x)x^3 \\ = x^2(3 \cos x - x \sin x)$$

$$\mathbf{b} \quad 2x \cos x - (1 + x^2) \sin x$$

$$\mathbf{c} \quad \frac{d}{dx}(e^{-x} \sin x) \\ = e^{-x} \sin x + e^{-x} \cos x \\ = e^{-x}(\cos x - \sin x)$$

$$\mathbf{d} \quad 6 \cos x - 6x \sin x$$

$$\mathbf{e} \quad 3 \cos(3x) \cos(4x) - 4 \sin(4x) \sin(3x)$$

$$\mathbf{f} \quad 2 \sin(2x) + 2 \tan(2x) \sec(2x)$$

$$\mathbf{g} \quad 12 \sin x + 12x \cos x$$

$$\mathbf{h} \quad \frac{d}{dx}(x^2 e^{\sin x}) \\ = 2x e^{\sin x} + x^2 \cos x e^{\sin x} \\ = x e^{\sin x}(2 + x \cos x)$$

$$\mathbf{i} \quad \frac{d}{dx}(x^2 \cos^2 x) \\ = 2x \cos^2 x - 2 \sin x \cos x \times x^2 \\ = 2x \cos^2 x - x^2 \sin 2x$$

$$\mathbf{j} \quad \frac{d}{dx}(e^x \tan x) \\ = e^x \tan x + e^x \sec^2 x \\ = e^x(\tan x + \sec^2 x)$$

$$\mathbf{7 a} \quad f(x) = e^x \sin x \\ f'(x) = e^x \sin x + e^x \cos x \\ = e^x(\sin x + \cos x) \\ f'(\pi) = e^\pi(\sin \pi + \cos \pi) \\ = -e^\pi$$

$$\mathbf{b} \quad f(x) = \cos^2 2x \\ f'(x) = -2 \sin 2x \times 2 \cos 2x \\ = -4(\sin 2x \cos 2x) \\ = -2 \sin 4x \\ f'(\pi) = -2 \sin 4\pi \\ = -2 \sin 0 \\ = 0$$

$$\mathbf{8} \quad \text{Let } y = \frac{d}{dx}(f(x) \log_e x) \\ = f'(x) \log_e x + \frac{1}{x} f(x) \\ \text{When } x = 1 \\ y = 4 \log_e 1 + 2 = 2$$



## Solutions to Exercise 9K

$$\begin{aligned}
 \mathbf{1 \ a} \quad y &= \frac{x}{x+4} \\
 \frac{dy}{dx} &= \frac{(x+4)\frac{dx}{dx} - x\frac{d(x+4)}{dx}}{(x+4)^2} \\
 &= \frac{(x+4) - x}{(x+4)^2} \\
 &= \frac{4}{(x+4)^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= \frac{x^2 - 1}{x^2 + 1} \\
 \frac{dy}{dx} &= \frac{(x^2 + 1)\frac{d(x^2 - 1)}{dx} - (x^2 - 1)\frac{d(x^2 + 1)}{dx}}{(x^2 + 1)^2} \\
 &= \frac{(x^2 + 1) \times 2x + (1 - x^2) \times 2x}{(x^2 + 1)^2} \\
 &= \frac{4x}{(x^2 + 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= \frac{x^{\frac{1}{2}}}{1+x} \\
 \frac{dy}{dx} &= \frac{(1+x)\frac{d(x^{\frac{1}{2}})}{dx} - x^{\frac{1}{2}}\frac{d(1+x)}{dx}}{(1+x)^2} \\
 &= \frac{(1+x)\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{(1+x)^2} \\
 &= \frac{\frac{1}{2}(1+x) - x}{x^{\frac{1}{2}}(1+x)^2} \\
 &= \frac{\frac{1}{2} - \frac{1}{2}x}{x^{\frac{1}{2}}(1+x)^2} \\
 &= \frac{x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{2(1+x)^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad y &= \frac{(x+2)^3}{x^2 + 1} \\
 \frac{dy}{dx} &= \frac{(x^2 + 1)\frac{d(x+2)^3}{dx} - (x+2)^3\frac{d(x^2 + 1)}{dx}}{(x^2 + 1)^2} \\
 &= \frac{(x^2 + 1) \times 3(x+2)^2 - (x+2)^3 \times 2x}{(x^2 + 1)^2} \\
 &= \frac{(3(x^2 + 1) - 2x(x+2))(x+2)^2}{(x^2 + 1)^2} \\
 &= \frac{(3x^2 + 3 - 2x^2 - 4x)(x+2)^2}{(x^2 + 1)^2} \\
 &= \frac{(x^2 - 4x + 3)(x+2)^2}{(x^2 + 1)^2} \\
 &= \frac{(x-3)(x-1)(x+2)^2}{(x^2 + 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad y &= \frac{x-1}{x^2 + 2} \\
 \frac{dy}{dx} &= \frac{(x^2 + 2)\frac{d(x-1)}{dx} - (x-1)\frac{d(x^2 + 2)}{dx}}{(x^2 + 2)^2} \\
 &= \frac{(x^2 + 2) - (x-1) \times 2x}{(x^2 + 2)^2} \\
 &= \frac{x^2 + 2 - 2x^2 + 2x}{(x^2 + 2)^2} \\
 &= \frac{-x^2 + 2x + 2}{(x^2 + 2)^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad y &= \frac{x^2 + 1}{x^2 - 1} \\
 \frac{dy}{dx} &= \frac{(x^2 - 1)\frac{d(x^2 + 1)}{dx} - (x^2 + 1)\frac{d(x^2 - 1)}{dx}}{(x^2 - 1)^2} \\
 &= \frac{(x^2 - 1) \times 2x - (x^2 + 1) \times 2x}{(x^2 - 1)^2} \\
 &= \frac{-4x}{(x^2 - 1)^2}
 \end{aligned}$$

**g**

$$\begin{aligned}
 y &= \frac{3x^2 + 2x + 1}{x^2 + x + 1} \\
 (x^2 + x + 1) \frac{d}{dx}(3x^2 + 2x + 1) \\
 \frac{dy}{dx} &= \frac{-(3x^2 + 2x + 1) \frac{d}{dx}(x^2 + x + 1)}{(x^2 + x + 1)^2} \\
 &= \frac{(x^2 + x + 1)(6x + 2)}{-(3x^2 + 2x + 1) \times (2x + 1)} \\
 &= \frac{6x^3 + 8x^2 + 8x + 2 - 6x^3 - 7x^2 - 4x - 1}{(x^2 + x + 1)^2} \\
 &= \frac{x^2 + 4x + 1}{(x^2 + x + 1)^2}
 \end{aligned}$$

**h**

$$\begin{aligned}
 y &= \frac{2x + 1}{2x^3 + 2x} \\
 (2x^3 + 2x) \frac{d}{dx}(2x + 1) \\
 \frac{dy}{dx} &= \frac{-(2x + 1) \frac{d}{dx}(2x^3 + 2x)}{(2x^3 + 2x)^2} \\
 &= \frac{(2x^3 + 2x) \times 2 - (2x + 1)(6x^2 + 2)}{(2x^3 + 2x)^2} \\
 &= \frac{4x^3 + 4x - 12x^3 - 6x^2 - 4x - 2}{(2x^3 + 2x)^2} \\
 &= \frac{-8x^3 - 6x^2 - 2}{(2x^3 + 2x)^2} \\
 &= \frac{-(4x^3 + 3x^2 + 1)}{2(x^3 + x)^2}
 \end{aligned}$$

**2 a**

$$\begin{aligned}
 y &= (2x + 1)^4 x^2 \\
 x = 1, y &= (2(1) + 1)^4 \times (1)^2 \\
 y &= 3^4 \\
 y &= 81 \\
 \frac{dy}{dx} &= x^2 \frac{d((2x + 1)^4)}{dx} + (2x + 1)^4 \frac{d(x^2)}{dx} \\
 &= x^2 \times 2 \times 4(2x + 1)^3 + (2x + 1)^4 \times 2x \\
 &= 8x^2(2x + 1)^3 + 2x(2x + 1)^4 \\
 &= (8x^2 + 4x^2 + 2x)(2x + 1)^3 \\
 &= (8x^2 + 2x(2x + 1))(2x + 1)^3 \\
 &= 2x(6x + 1)(2x + 1)^3 \\
 x &= 1, \\
 \frac{dy}{dx} &= 2(1)(6(1) + 1)(2(1) + 1)^3 \\
 &= 2(7)(3)^3 \\
 &= 14 \times 27 \\
 \frac{dy}{dx} &= 378
 \end{aligned}$$

**b**  $y = x^2(x+1)^{\frac{1}{2}}$   
 $x = 0,$   
 $y = (0)^2(0+1)^{\frac{1}{2}}$   
 $y = 0$

$$\frac{dy}{dx} = x^2 \frac{d\left((x+1)^{\frac{1}{2}}\right)}{dx} + (x+1)^{\frac{1}{2}} \frac{d(x^2)}{dx}$$

$$= x^2 \times \frac{1}{2\sqrt{x+1}} + \sqrt{x+1} \times 2x$$

$$= \frac{x^2 + 2x(x+1) \times 2x}{2\sqrt{x+1}}$$

$$= \frac{x^2 + 4x^2 + 4x}{2\sqrt{x+1}}$$

$$= \frac{5x^2 + 4x}{2\sqrt{x+1}}$$
 $x = 0,$ 

$$\frac{dy}{dx} = \frac{5(0)^2 + 4(0)}{2\sqrt{0+1}}$$

$$\frac{dy}{dx} = 0$$

**c**  $y = x^2(2x+1)^{\frac{1}{2}}$   
 $x = 0,$   
 $y = 0$

$$\frac{dy}{dx} = x^2 \frac{d(2x+1)^{\frac{1}{2}}}{dx} + (2x+1) \frac{dx^2}{dx}$$

$$= x^2 \frac{d(2x+1)^{\frac{1}{2}}}{dx} + 2x(2x+1)$$
 $x = 0,$ 

$$\frac{dy}{dx} = 0 + 0$$

$$\frac{dy}{dx} = 0$$

**d**  $y = \frac{x}{x^2+1}$   
 $x = 1,$   
 $y = \frac{1}{1+1}$   
 $y = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{(x^2+1) \frac{dx}{dx} - x \frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{x^2+1 - x \times 2x}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$
 $x = 1,$ 

$$\frac{dy}{dx} = \frac{1-1}{(1+1)^2}$$

$$\frac{dy}{dx} = 0$$

**e**  $y = \frac{2x+1}{x^2+1}$   
 $x = 1,$   
 $y = \frac{2+1}{1+1}$   
 $y = \frac{3}{2}$

$$\frac{dy}{dx} = \frac{(x^2+1) \frac{d(2x+1)}{dx} - (2x+1) \frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{(x^2+1) \times 2 - (2x+1) \times 2x}{(x^2+1)^2}$$

$$= \frac{2x^2 + 2 - 4x^2 - 2x}{(x^2+1)^2}$$

$$= \frac{2(-x^2 - x + 1)}{(x^2+1)^2}$$
 $x = 1, \frac{dy}{dx} = \frac{2(-1 - 1 + 1)}{(2)^2} = -\frac{1}{2}$

**3 a**

$$\begin{aligned}f(x) &= (x+1)(x^2+1)^{\frac{1}{2}} \\f'(x) &= (x+1)\frac{d(x^2+1)^{\frac{1}{2}}}{dx} \\&\quad + (x^2+1)^{\frac{1}{2}}\frac{d(x+1)}{dx} \\&= (x+1)\left(\frac{d(x^2+1)^{\frac{1}{2}}}{d(x^2+1)} \times \frac{d(x^2+1)}{dx}\right) \\&\quad + (x^2+1)^{\frac{1}{2}} \\&= (x+1)\left(\frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x\right) + (x^2+1)^{\frac{1}{2}} \\&= (x^2+1)^{-\frac{1}{2}}(x^2+x+x^2+1) \\&= (x^2+1)^{-\frac{1}{2}}(2x^2+x+1)\end{aligned}$$

**b**

$$\begin{aligned}f(x) &= (x^2+1)(x^3+1)^{\frac{1}{2}} \\f'(x) &= (x^2+1)\frac{d(x^3+1)^{\frac{1}{2}}}{dx} \\&\quad + (x^3+1)^{\frac{1}{2}}\frac{d(x^2+1)}{dx} \\&= (x^2+1)\left(\frac{d(x^3+1)^{\frac{1}{2}}}{d(x^3+1)} \times \frac{d(x^3+1)}{dx}\right) \\&\quad + (x^3+1)^{\frac{1}{2}} \times 2x \\&= (x^2+1)\left(\frac{1}{2}(x^3+1)^{-\frac{1}{2}} \times 3x^2\right) + 2x(x^3+1)^{\frac{1}{2}} \\&= \frac{3}{2}x^2(x^2+1)(x^3+1)^{-\frac{1}{2}} + 2x(x^3+1)^{\frac{1}{2}} \\&= (x^3+1)^{-\frac{1}{2}}\left(\frac{3}{2}x^4 + \frac{3}{2}x^2 + 2x^4 + 2x\right) \\&= (x^3+1)^{-\frac{1}{2}}\left(7x^4 + \frac{3}{2}x^2 + 2x\right) \\&= x\left(\frac{7}{2}x^3 + \frac{3}{2}x + 2\right)(x^3+1)^{-\frac{1}{2}}\end{aligned}$$

**c**

$$\begin{aligned}f(x) &= \frac{2x+1}{x+3} \\f'(x) &= \frac{(x+3)\frac{d(2x+1)}{dx} - (2x+1)\frac{d(x+3)}{dx}}{(x+3)^2} \\&= \frac{(x+3) \times 2 - (2x+1)}{(x+3)^2} \\&= \frac{2x+6-2x-1}{(x+3)^2} \\&= \frac{5}{(x+3)^2}\end{aligned}$$

**4 a**

$$\begin{aligned}f'(x) &= \frac{e^x(e^{3x}+3) - 3e^{3x}e^x}{(e^{3x}+3)^2} \\&= \frac{3e^x - 2e^{4x}}{(e^{3x}+3)^2}\end{aligned}$$

**b**

$$\begin{aligned}f'(x) &= \frac{-\sin x(x+1) - \cos x}{(x+1)^2} \\&= -\frac{\sin x(x+1) + \cos x}{(x+1)^2}\end{aligned}$$

**c**

$$\begin{aligned}f'(x) &= \frac{\frac{1}{x} \times (x+1) - \log_e x}{(x+1)^2} \\&= \frac{(x+1) - x \log_e x}{x(x+1)^2}\end{aligned}$$

**5 a**

$$\begin{aligned}f'(x) &= \frac{\frac{1}{x} \times x - \log_e x}{(x^2)} \\&= \frac{1 - \log_e x}{x^2}\end{aligned}$$

**b**

$$\begin{aligned}f'(x) &= \frac{\frac{1}{x}(x^2+1) - 2x \log_e x}{(x^2+1)^2} \\&= \frac{x^2+1 - 2x^2 \log_e x}{x(x^2+1)^2}\end{aligned}$$

6 a

$$\begin{aligned}
 f(x) &= \frac{e^{3x}}{e^{3x} + 3} \\
 f'(x) &= \frac{(e^{3x} + 3) \frac{d}{dx} e^{3x} - e^{3x} \frac{d}{dx} (e^{3x} + 3)}{(e^{3x} + 3)^2} \\
 &= \frac{3e^{3x}(e^{3x} + 3) - 3e^{3x}(e^{3x})}{(e^{3x} + 3)^2} \\
 &= \frac{9e^{3x}}{(e^{3x} + 3)^2}
 \end{aligned}$$

b

$$\begin{aligned}
 f(x) &= \frac{e^x + 1}{e^x - 1} \\
 f'(x) &= \frac{(e^x - 1) \frac{d}{dx} (e^x + 1) - (e^x + 1) \frac{d}{dx} (e^x - 1)}{(e^x - 1)^2} \\
 &= \frac{(e^x - 1)e^x + (-e^x - 1)(e^x)}{(e^x - 1)^2} \\
 &= \frac{-2e^x}{(e^x - 1)^2}
 \end{aligned}$$

c

$$\begin{aligned}
 f(x) &= \frac{e^{2x} + 2}{e^{2x} - 2} \\
 f'(x) &= \frac{(e^x - 2) \frac{d}{dx} (e^{2x} + 2) - (e^{2x} + 2) \frac{d}{dx} (e^{2x} - 2)}{(e^{2x} - 2)^2} \\
 &= \frac{(e^{2x} - 2)2e^{2x} - (e^{2x} + 2)2e^{2x}}{(e^{2x} - 2)^2} \\
 &= \frac{-8e^{2x}}{(e^{2x} - 2)^2}
 \end{aligned}$$

7 a

$$\begin{aligned}
 f(x) &= \frac{2x}{\cos x} \\
 f'(x) &= \frac{\cos x \times 2 - 2x(-\sin x)}{\cos^2 x} \\
 &= \frac{2 \cos x + 2x \sin x}{\cos^2 x} \\
 f'(\pi) &= \frac{2 \cos(\pi) + 2\pi \sin(\pi)}{(\cos(\pi))^2} \\
 &= \frac{-2}{1} \\
 &= -2
 \end{aligned}$$

b

$$\begin{aligned}
 f(x) &= \frac{3x^2 + 1}{\cos x} \\
 f'(x) &= \frac{\cos x(6x) - (3x^2 + 1)(-\sin x)}{\cos^2 x} \\
 &= \frac{6x \cos x + (3x^2 + 1) \sin x}{\cos^2 x} \\
 f'(\pi) &= \frac{6\pi \cos(\pi) + (3\pi^2 + 1) \sin(\pi)}{(\cos(\pi))^2} \\
 &= \frac{-6\pi}{1} \\
 &= -6\pi
 \end{aligned}$$

c

$$\begin{aligned}
 f(x) &= \frac{e^x}{\cos x} \\
 f'(x) &= \frac{\cos x e^x + \sin x e^x}{\cos^2 x} \\
 f'(\pi) &= \frac{(\cos \pi + \sin \pi)e^\pi}{\cos^2 \pi} \\
 &= -e^\pi
 \end{aligned}$$

d

$$\begin{aligned}
 f(x) &= \frac{\sin x}{x} \\
 f'(x) &= \frac{x \cos x - \sin x}{x^2} \\
 f'(\pi) &= \frac{\pi \cos \pi - \sin \pi}{\pi^2} \\
 &= \frac{-\pi}{\pi^2} \\
 &= \frac{-1}{\pi}
 \end{aligned}$$

## Solutions to Exercise 9L

1 a  $\lim_{x \rightarrow 2} (17) = 17$

b  $\lim_{x \rightarrow 6} (x - 3) = 6 - 3 = 3$

c  $\lim_{x \rightarrow \frac{1}{2}} (2x - 5) = 1 - 5 = -4$

d  $\lim_{t \rightarrow -3} \left( \frac{t+2}{t-5} \right) = \frac{-3+2}{-3-5} = \frac{-1}{-8} = \frac{1}{8}$

e  $\lim_{t \rightarrow 2} \left( \frac{t^2 + 2t + 1}{t + 1} \right)$   
 $= \lim_{t \rightarrow 2} \left( \frac{(t+1)^2}{t+1} \right) = \lim_{t \rightarrow 2} (t+1) = 3$

f  $\lim_{x \rightarrow 0} \left( \frac{(x+2)^2 - 4}{x} \right)$   
 $= \lim_{x \rightarrow 0} \left( \frac{x^2 + 4x + 4 - 4}{x} \right)$   
 $= \lim_{x \rightarrow 0} \left( \frac{x^2 + 4x}{x} \right) = \lim_{x \rightarrow 0} (x + 4) = 4$

g  $\lim_{t \rightarrow 1} \left( \frac{t^2 - 1}{t - 1} \right)$   
 $= \lim_{t \rightarrow 1} \left( \frac{(t+1)(t-1)}{t-1} \right)$   
 $= \lim_{t \rightarrow 1} (t+1) = 2$

h  $\lim_{x \rightarrow 9} (\sqrt{x+3}) = \sqrt{9+3}$   
 $= \sqrt{12} = 2\sqrt{3}$

i  $\lim_{x \rightarrow 0} \left( \frac{x^2 - 2x}{x} \right)$   
 $= \lim_{x \rightarrow 0} (x - 2) = -2$

j  $\lim_{x \rightarrow 2} \left( \frac{x^3 - 8}{x - 2} \right)$   
 $= \lim_{x \rightarrow 2} \left( \frac{(x-2)(x^2 + 2x + 4)}{(x-2)} \right)$   
 $= \lim_{x \rightarrow 2} (x^2 + 2x + 4) = 12$

k  $\lim_{x \rightarrow 2} \left( \frac{3x^2 - x - 10}{x^2 + 5x - 14} \right)$   
 $= \lim_{x \rightarrow 2} \left( \frac{(3x+5)(x-2)}{(x+7)(x-2)} \right)$   
 $= \lim_{x \rightarrow 2} \frac{3x+5}{x+7} = \frac{11}{9}$

l  $\lim_{x \rightarrow 1} \left( \frac{x^2 - 3x + 2}{x^2 - 6x + 5} \right)$   
 $= \lim_{x \rightarrow 1} \left( \frac{(x-1)(x-2)}{(x-1)(x-4)} \right)$   
 $= \lim_{x \rightarrow 1} \frac{x-2}{x-4} = \frac{1}{4}$

2 a  $x = 3$ , since  $f(3) \neq \lim_{x \rightarrow 3} (f(x))$ ,  $x = 4$ ,  
 since  $\lim_{x \rightarrow 4^+} (f(x)) \neq \lim_{x \rightarrow 4^-} (f(x))$

b  $x = 7$ , since  $\lim_{x \rightarrow 7^+} (f(x)) \neq \lim_{x \rightarrow 7^-} (f(x))$

3 a value to test:  $x = 0$

$$\lim_{x \rightarrow 0^-} (f(x)) = \lim_{x \rightarrow 0} (f(-2x + 2)) = 2$$

$$\lim_{x \rightarrow 0^+} (f(x)) = \lim_{x \rightarrow 0} (3x) = 0 \neq$$

$$\lim_{x \rightarrow 0^-} (f(x))$$

$\therefore$  there is a discontinuity at  $x = 0$

b value to test:  $x = 1$

$$\lim_{x \rightarrow 1^-} (f(x)) = \lim_{x \rightarrow 1} (f(-2x + 1)) =$$

$$-2(1) + 1 = -1$$

$$\lim_{x \rightarrow 1^+} (f(x)) = \lim_{x \rightarrow 1} (x^2 + 2) = 1^2 + 2$$

$$= 3 \neq \lim_{x \rightarrow 1^-} (f(x))$$

$\therefore$  there is a discontinuity at  $x = 1$

**c** value to test:  $x = -1, 0$

$$\lim_{x \rightarrow 1^-} (f(x)) = \lim_{x \rightarrow -1} (-x) = -(-1) = 1$$

$$\lim_{x \rightarrow -1^+} (f(x)) = \lim_{x \rightarrow -1} (x^2) = (-1)^2 = 1$$

$$= \lim_{x \rightarrow -1^-} (f(x))$$

$$f(-1) = -(-1) = 1 = \lim_{x \rightarrow -1} (f(x))$$

$\therefore f(x)$  is continuous at  $x = -1$

$$\lim_{x \rightarrow 0} -(f(x)) = \lim_{x \rightarrow 0} (x^2) = (0)^2 = 0$$

$$\lim_{x \rightarrow 0^+} (f(x)) = \lim_{x \rightarrow 0} (-3x + 1) = -3(0) + 1$$

$$= 1 \neq \lim_{x \rightarrow 0^-} (f(x))$$

$\therefore$  there is one discontinuity at  $x = 0$

**4 a** value to test:  $x = 1, 7$

$$\lim_{x \rightarrow 1^-} (f(x)) = \lim_{x \rightarrow -1} (2) = 2$$

$$\lim_{x \rightarrow -1^+} (f(x))$$

$$= \lim_{x \rightarrow -1} ((x - 4)^2 - 9)$$

$$= (1 - 4)^2 - 9$$

$$= 0 \neq \lim_{x \rightarrow -1^-} (f(x))$$

$\therefore$  there is a discontinuity at  $x = 1$

$$\lim_{x \rightarrow 7^-} (f(x)) = (\lim_{x \rightarrow 7} (x - 4)^2 - 9)$$

$$= (7 - 4)^2 - 9 = 0$$

$$\lim_{x \rightarrow 7^-} (f(x)) = \lim_{x \rightarrow 7} (x - 7) = 7 - 7 =$$

$$\lim_{x \rightarrow 7^-} (f(x))$$

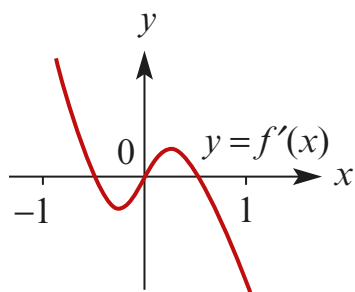
$$f(7) = 7 - 7 = 0 = \lim_{x \rightarrow 7} f(x)$$

$f(x)$  is continuous at  $x = 7$

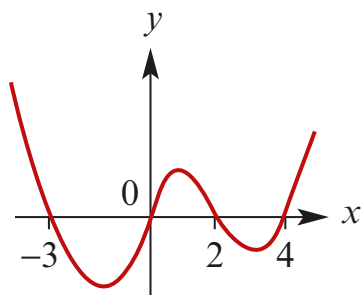
$\therefore f(x)$  is continuous for all  $x \in \mathbb{R} \setminus \{1\}$

## Solutions to Exercise 9M

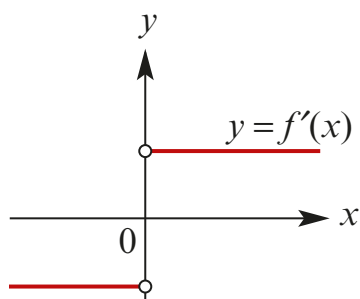
1 a



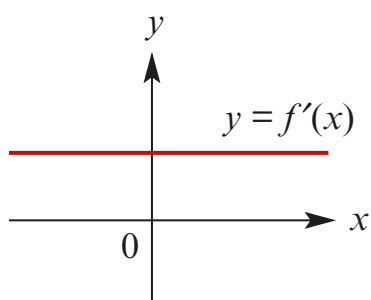
b



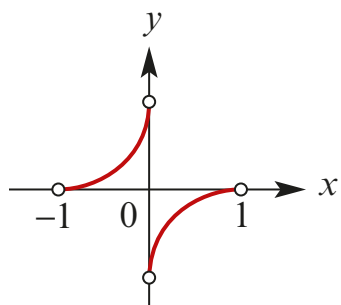
c



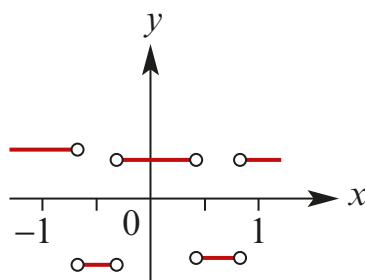
d



e



f



$$2 \quad x > 0, \quad f'(x) = \frac{d}{dx}(-x^2 + 3)$$

$$= -2x + 3$$

$$x < 0, \quad f'(x) = \frac{d}{dx}(3x + 1)$$

$$= 3$$

test  $x = 0$

$$\lim_{x \rightarrow 0^-} (f(x)) = \lim_{x \rightarrow 0} (3x + 1) = 1$$

$$\lim_{x \rightarrow 0^+} (f(x)) = \lim_{x \rightarrow 0} (-x^2 + 3x + 1) = 1 =$$

$$\lim_{x \rightarrow 0^-} (f(x))$$

$$f(0) = -(0)^2 + 3(0) + 1 = 1 = \lim_{x \rightarrow 0} (f(x))$$

$\therefore f(x)$  is continuous at  $x = 0$

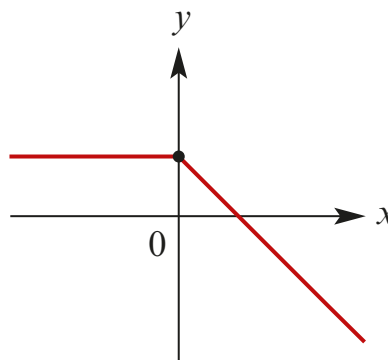
$$\lim_{x \rightarrow 0^-} (f'(x)) = \lim_{x \rightarrow 0} (3) = 3$$

$$\lim_{x \rightarrow 0^+} (f'(x)) = \lim_{x \rightarrow 0} (-2x + 3) = 3 =$$

$$\lim_{x \rightarrow 0^-} (f'(x))$$

$f(x)$  is differentiable at  $x = 0$

$$f'(x) = \begin{cases} -2x + 3 & \text{if } x \geq 0 \\ 3 & \text{if } x < 0 \end{cases}$$



3



$$x > 1, f'(x) = \frac{d}{dx}(x^2 + 2x + 1)$$

$$= 2x + 2$$

$$x < 1, f'(x) = \frac{d}{dx}(-2x + 3)$$

$$= -2$$

test  $x = 1$

$$\lim_{x \rightarrow 1^-} (f(x)) = \lim_{x \rightarrow 1} (-2x + 3) = -2 + 3 = 1$$

$$\lim_{x \rightarrow 1^+} (f(x)) = \lim_{x \rightarrow 1} (x^2 + 2x + 1) = 1 + 2 + 1$$

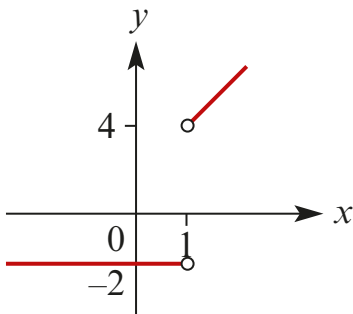
$$= 4 \neq \lim_{x \rightarrow 1^-} (f(x))$$

$\therefore f(x)$  is discontinuous &

$\therefore$  not differentiable at  $x = 1$

$\therefore f'(x)$  is defined for  $x \in \mathbb{R} \setminus \{1\}$

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x > 1 \\ -2 & \text{if } x < 1 \end{cases}$$



**4**  $x > -1, f'(x) = \frac{d}{dx}(-x^2 - 2x + 1)$

$$= -2x - 2$$

$$x < -1, f'(x) = \frac{d}{dx}(-2x + 3)$$

$$= -2$$

test  $x = -1$

$$\lim_{x \rightarrow -1^-} (f(x)) = \lim_{x \rightarrow -1} (-2x + 3) = 2 + 3 = 5$$

$$\lim_{x \rightarrow -1^+} (f(x)) = \lim_{x \rightarrow -1} (-x^2 - 2x + 1) = -1 + 2 + 1$$

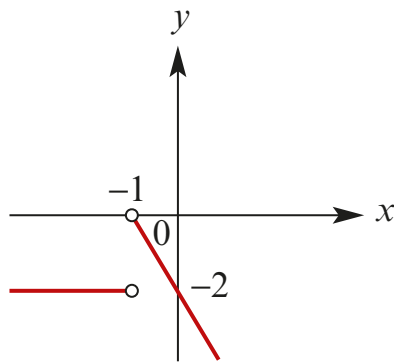
$$= 2 \neq \lim_{x \rightarrow -1^-} (f(x))$$

$\therefore f(x)$  is not continuous &

$\therefore$  not differentiable at  $x = -1$

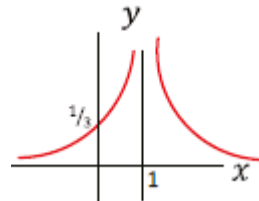
$\therefore f'(x)$  is defined for  $x \in \mathbb{R} \setminus \{-1\}$

$$f'(x) = \begin{cases} -2x - 2 & \text{if } x > -1 \\ -2 & \text{if } x < -1 \end{cases}$$



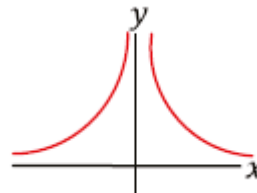
**5 a**  $f'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$

$f'(x)$  is defined for  $x \in \mathbb{R} \setminus \{1\}$  (since  $x = 1$  gives  $f'(x) = \frac{1}{0}$ )

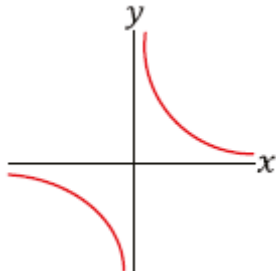


**b**  $f'(x) = \frac{1}{5}x^{-\frac{4}{5}}$

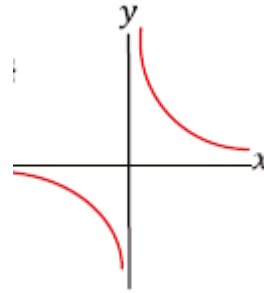
$f'(x)$  is defined for  $x \in \mathbb{R} \setminus \{0\}$  (since  $x = 0$  gives  $f'(x) = \frac{1}{0}$ )



**c**  $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$   
 $f'(x)$  is defined for  $x \in \mathbb{R} \setminus \{0\}$



**d**  $f'(x) = \frac{2}{5}(x+2)^{-\frac{3}{5}}$   
 $f'(x)$  is defined for  $x \in \mathbb{R} \setminus \{-2\}$



## Solutions to Technology-free questions

$$\begin{aligned} 1 \text{ a Average rate of change} &= \frac{26 - 10}{2} \\ &= 8 \end{aligned}$$

$$b \quad \frac{dy}{dx} = 2x$$

$$\text{When } x = -4, \frac{dy}{dx} = -8$$

$$2 \text{ a } y = x + \sqrt{1 - x^2} = x + (1 - x^2)^{\frac{1}{2}}$$

$$\begin{aligned} \frac{dy}{dx} &= 1 + \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} \times (-2x) \\ &= 1 - \frac{x}{\sqrt{1 - x^2}} \end{aligned}$$

$$b \quad y = \frac{4x + 1}{x^2 + 3}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^2 + 3)(4) - (4x + 1)(2x)}{(x^2 + 3)^2} \\ &= \frac{12 - 2x - 4x^2}{(x^2 + 3)^2} \end{aligned}$$

$$c \quad y = \sqrt{1 + 3x} = (1 + 3x)^{\frac{1}{2}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(1 + 3x)^{-\frac{1}{2}} \times 3 \\ &= \frac{3}{2\sqrt{1 + 3x}} \end{aligned}$$

$$d \quad y = \frac{2 + \sqrt{x}}{x} = 2x^{-1} + x^{-\frac{1}{2}}$$

$$\begin{aligned} \frac{dy}{dx} &= -2x^{-2} - \frac{1}{2}x^{-\frac{3}{2}} \\ &= -\frac{2}{x^2} - \frac{1}{2x^{\frac{3}{2}}} \end{aligned}$$

e

$$\begin{aligned} y &= (x - 9)\sqrt{x - 3} = (x - 9)(x - 3)^{\frac{1}{2}} \\ \frac{dy}{dx} &= (1)(x - 3)^{\frac{1}{2}} + (x - 9) \times \frac{1}{2}(x - 3)^{-\frac{1}{2}} \\ &= \frac{2(x - 3) + (x - 9)}{2\sqrt{x - 3}} \\ &= \frac{3x - 15}{2\sqrt{x - 3}} \end{aligned}$$

f

$$\begin{aligned} y &= x\sqrt{1 + x^2} = x(1 + x^2)^{\frac{1}{2}} \\ \frac{dy}{dx} &= (1)(1 + x^2)^{\frac{1}{2}} + x \times \frac{1}{2}(1 + x^2)^{-\frac{1}{2}}(2x) \\ &= \frac{1 + 2x^2}{\sqrt{1 + x^2}} \end{aligned}$$

g

$$\begin{aligned} y &= \frac{x^2 - 1}{x^2 + 1} \\ \frac{dy}{dx} &= \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} \\ &= \frac{4x}{(x^2 + 1)^2} \end{aligned}$$

h

$$\begin{aligned} y &= \frac{x}{x^2 + 1} \\ \frac{dy}{dx} &= \frac{(x^2 + 1)(1) - (x)(2x)}{(x^2 + 1)^2} \\ &= \frac{1 - x^2}{(x^2 + 1)^2} \end{aligned}$$

i

$$\begin{aligned} y &= (2 + 5x^2)^{\frac{1}{3}} \\ \frac{dy}{dx} &= \frac{1}{3}(2 + 5x^2)^{-\frac{2}{3}} \times 10x \\ &= \frac{10x}{3}(2 + 5x^2)^{-\frac{2}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad y &= \frac{2x+1}{x^2+2} \\ \frac{dy}{dx} &= \frac{(x^2+2)(2) - (2x+1)(2x)}{(x^2+2)^2} \\ &= \frac{4-2x-2x^2}{(x^2+2)^2} \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad y &= (3x^2+2)^{\frac{2}{3}} \\ \frac{dy}{dx} &= \frac{2}{3}(3x+2)^{-\frac{1}{3}} \times 6x \\ &= 4x(3x^2+2)^{-\frac{1}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{3 a} \quad y &= 3x^2 - 4 \\ \frac{dy}{dx} &= 6x \\ &= -6 \text{ (at } x = -1) \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= \frac{x-1}{x^2+1} \\ \frac{dy}{dx} &= \frac{(x^2+1)(1) - (x-1)(2x)}{(x^2+1)^2} \\ &= \frac{1+2x-x^2}{(x^2+1)^2} \\ &= 1 \text{ (at } x = 0) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= (x-2)^5 \\ \frac{dy}{dx} &= 5(x-2)^4 \\ &= 5 \text{ (at } x = 1) \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad y &= (2x+2)^{\frac{2}{3}} \\ \frac{dy}{dx} &= \frac{1}{3}(2x+2)^{-\frac{1}{3}} \times 2 \\ &= \frac{2}{3}(2x+2)^{-\frac{2}{3}} \\ &= \frac{2}{3}(8)^{-\frac{2}{3}} \text{ (at } x = 3) \\ &= \frac{2}{3}(2^3)^{-\frac{2}{3}} \\ &= \frac{2}{3} \times 2^{-2} \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \mathbf{4 a} \quad y &= \log_e(x+2) \\ \frac{dy}{dx} &= \frac{1}{x+2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= \sin(3x+2) \\ \frac{dy}{dx} &= 3 \cos(3x+2) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= \cos\left(\frac{x}{2}\right) \\ \frac{dy}{dx} &= -\frac{1}{2} \sin\left(\frac{x}{2}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad y &= e^{x^2-2x} \\ \frac{dy}{dx} &= (2x-2)e^{x^2-2x} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad y &= \log_e(3-x) \\ \frac{dy}{dx} &= -\frac{1}{3-x} = \frac{1}{x-3} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad y &= \sin(2\pi x) \\ \frac{dy}{dx} &= 2\pi \cos(2\pi x) \end{aligned}$$

**g**  $y = \sin^2(3x + 1)$   
 $\frac{dy}{dx} = 2 \sin(3x + 1) \times 3 \cos(3x + 1)$   
 $= 6 \sin(3x + 1) \cos(3x + 1)$   
 $= 3 \sin(6x + 2)$   
as  $\sin(2a) = 2 \sin(a) \cos(a)$

**h**  $y = \sqrt{\log_e x} = (\log_e x)^{\frac{1}{2}}$   
 $\frac{dy}{dx} = \frac{1}{2}(\log_e x)^{\frac{1}{2}} \times \frac{1}{x} = \frac{1}{2x\sqrt{\log_e x}}$

**i**  $y = \frac{2 \log_e 2x}{x} = 2x^{-1} \log_e 2x$   
 $\frac{dy}{dx} = -2x^{-2} \log_e 2x + 2x^{-1} \times \frac{2}{2x}$   
 $= -\frac{2 \log_e 2x}{x^2} + \frac{2}{x^2} = \frac{2 - 2 \log_e 2x}{x^2}$

**j**  $y = x^2 \sin(2\pi x)$   
 $\frac{dy}{dx} = 2x \sin(2\pi x) + 2\pi x^2 \cos(2\pi x)$

**5 a**  $y = e^x \sin 2x$   
 $\frac{dy}{dx} = e^x \sin 2x + 2e^x \cos 2x$

**b**  $y = 2x^2 \log_e x$   
 $\frac{dy}{dx} = 4x \log_e x + 2x^2 \times \frac{1}{x}$   
 $= 4x \log_e x + 2x$

**c**  $y = \frac{\log_e x}{x^3} = x^{-3} \log_e x$   
 $\frac{dy}{dx} = -3x^{-4} \log_e x + x^{-3} \times \frac{1}{x}$   
 $= \frac{1 - 3 \log_e x}{x^4}$

**d**  $y = \sin 2x \cos 3x$   
 $\frac{dy}{dx} = (2 \cos 2x) \cos 3x + \sin 2x(-3 \sin 3x)$   
 $= 2 \cos 2x \cos 3x - 3 \sin 2x \sin 3x$

**e**  $y = \frac{\sin 2x}{\cos 2x} = \tan 2x$   
 $\frac{dy}{dx} = 2 \sec^2 2x$   
(Alternatively, use the quotient rule.)

**f**  $y = \cos^3(3x + 2)$   
 $\frac{dy}{dx} = 3 \cos^2(3x + 2) \times -3 \sin(3x + 2)$   
 $= -9 \cos^2(3x + 2) \sin(3x + 2)$

**g**  $y = x^2 \sin^2(3x)$   
 $\frac{dy}{dx} = 2x \sin^2(3x)$   
 $+ x^2(2 \sin(3x) \times 3 \cos(3x))$   
 $= 2x \sin^2(3x) + 6x^2 \sin(3x) \cos(3x)$   
 $= 2x \sin^2(3x) + 3x^2 \sin(6x)$   
as  $\sin(2a) = 2 \sin(a) \cos(a)$

**6 a**  $y = e^{2x} + 1$   
 $\frac{dy}{dx} = 2e^{2x}$   
 $= 2e^2$  (at  $x = 1$ )

**b**  $y = e^{x^2+1}$   
 $\frac{dy}{dx} = 2xe^{x^2+1}$   
 $= 0$  (at  $x = 0$ )

**c**  $y = 5e^{3x} + x^2$   
 $\frac{dy}{dx} = 15e^{3x} + 2x$   
 $= 15e^3 + 2$  (at  $x = 1$ )

**d**  $y = 5 - e^{-x}$   
 $\frac{dy}{dx} = e^{-x}$   
 $= 1$  (at  $x = 0$ )

**7 a**  $y = e^{ax}$   
 $\frac{dy}{dx} = ae^{ax}$

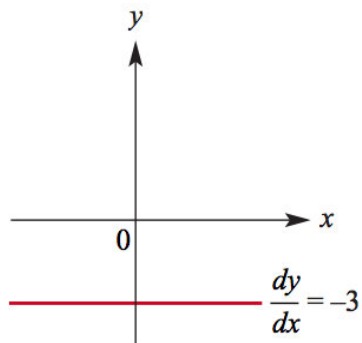
**b**  $y = e^{ax+b}$   
 $\frac{dy}{dx} = ae^{ax+b}$

**c**  $y = e^{a-bx}$   
 $\frac{dy}{dx} = -be^{a-bx}$

**d**  $y = be^{ax} - ae^{bx}$   
 $\frac{dy}{dx} = abe^{ax} - abe^{bx}$   
 $= ab(e^{ax} - e^{bx})$

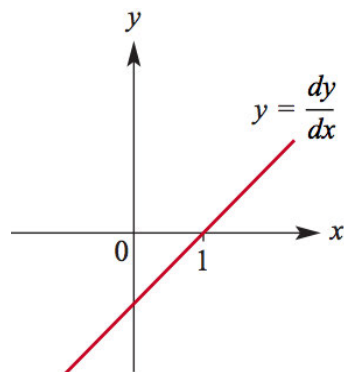
**e**  $y = \frac{e^{ax}}{e^{bx}} = e^{ax-bx} = e^{(a-b)x}$   
 $\frac{dy}{dx} = (a-b)e^{(a-b)x}$

**8 a**  $y = 3 - 3x$  so  $\frac{dy}{dx} = -3$

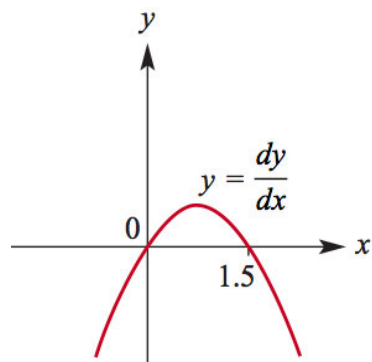


**b** Graph looks parabolic, so derivative graph will be linear. Also, there is a turning point where  $x = 1$ , so the

derivative function will be zero at  $x = 1$ .



**c** Graph looks cubic, so derivative graph will be quadratic. Also, there are turning points where  $x = 0$  and  $x = 1.5$ , so derivative function will be zero at  $x = 0, 1.5$ . Finally, the gradient goes from negative to positive to negative, so the gradient graph will be an inverted parabola.



**9**  $y = \left(4x + \frac{9}{x}\right)^2$   
 $\frac{dy}{dx} = 2\left(4x + \frac{9}{x}\right)\left(4 - \frac{9}{x^2}\right)$   
 $= \frac{2(4x^2 + 9)(4x^2 - 9)}{x^3}$

Then  $\frac{dy}{dx} = 0$  provided  $4x^2 - 9 = 0$   
 (since  $4x^2 + 9 > 0$  for all values of  $x$ ).  
 Hence  $x = \pm \frac{3}{2}$ .

$$\begin{aligned}
 \mathbf{10\ a} \quad y &= \frac{2x-3}{x^2+4} \\
 \frac{dy}{dx} &= \frac{(x^2+4)(2) - (2x-3)(2x)}{(x^2+4)^2} \\
 &= \frac{2x^2+8-4x^2+6x}{(x^2+4)^2} \\
 &= \frac{8+6x-2x^2}{(x^2+4)^2}
 \end{aligned}$$

**b** Note that  $x^2 + 4 > 0$  for all values of  $x$ . So only check the numerators.

$$y > 0 \text{ provided } 2x - 3 > 0, \text{ i.e. } x > \frac{3}{2}.$$

$$\frac{dy}{dx} = 0 \text{ provided } 8 + 6x - 2x^2 > 0,$$

which is equivalent to  $4 + 3x - x^2 > 0$ .

$4 + 3x - x^2 = (4 - x)(1 + x) > 0$   
 provided  $-1 < x < 4$  (since the corresponding quadratic graph is an inverted parabola with  $x$ -axis intercepts of  $-1$  and  $4$ ).

So  $y$  and  $\frac{dy}{dx}$  are both positive  
 provided  $x \in \left(\frac{3}{2}, \infty\right) \cap (-1, 4)$ , i.e.  
 $\left(\frac{3}{2}, 4\right)$ .

$$\begin{aligned}
 \mathbf{11\ a} \quad y &= xf(x) \\
 \frac{dy}{dx} &= (x)(f'(x)) + (1)(f(x)) \\
 &= xf'(x) + f(x)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= \frac{1}{f(x)} \\
 \frac{dy}{dx} &= \frac{-f'(x)}{[f(x)]^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= \frac{x}{f(x)} \\
 \frac{dy}{dx} &= \frac{f(x) + xf'(x)}{[f(x)]^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad y &= \frac{x^2}{[f(x)]^2} \\
 \frac{dy}{dx} &= \frac{[f(x)]^2(2x) - (x^2)(2f(x)f'(x))}{[f(x)]^4} \\
 &= \frac{[f(x)](2xf(x) - 2x^2f'(x))}{[f(x)]^4} \\
 &= \frac{2xf(x) - 2x^2f'(x)}{[f(x)]^3}
 \end{aligned}$$

$$\mathbf{12\ a} \quad f \circ g(x) = 2 \cos^3 x - 1$$

$$\mathbf{b} \quad g \circ f(x) = \cos(2x^3 - 1)$$

$$\mathbf{c} \quad g' \circ f(x) = -\sin(2x^3 - 1)$$

$$\mathbf{d} \quad (g \circ f)'(x) = -(6x^2) \sin(2x^3 - 1)$$

$$\mathbf{e} \quad \frac{3}{2}$$

$$\mathbf{f} \quad -\frac{3\sqrt{3}}{4}$$

## Solutions to multiple-choice questions

1 A

$$\begin{aligned} \text{Average rate of change} &= \frac{e + 1 - (1)}{1} \\ &= e \end{aligned}$$

2 B  $f(x) = \frac{4x^4 - 12x^2}{3x}$

$$f(x) = \frac{4x^3}{3} - 4x$$

$$\therefore f'(x) = 4x^2 - 4$$

3 C  $f : R \setminus \{7\} \rightarrow R, f(x) = 5 + \frac{5}{(7-x)^2}$

$$f(x) = 5 + 5(7-x)^{-2}$$

$$f'(x) = 10(7-x)^{-3}$$

$$f'(x) = \frac{10}{(7-x)^3}$$

$$f'(x) > 0$$

$$\therefore (7-x)^3 > 0$$

$$x < 7$$

4 A  $y = f(g(x))$

$$g(x) = 2x^4$$

$$\therefore y = f(2x^4)$$

$$\frac{dy}{dx} = 8x^3 f'(2x^4)$$

5 A  $f(x) = x^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$

As  $3x^{\frac{2}{3}} \neq 0$ , the gradient is undefined at this point.

6 B  $y = \frac{k}{2(x^3 + 1)}$

$$y' = \frac{-3kx^2}{2(x^3 + 1)^2}$$

$$\therefore 1 = \frac{-3kx^2}{2(x^3 + 1)^2}$$

$$1 = \frac{-3k}{8}$$

$$k = \frac{-8}{3}$$

7 C The gradient is positive when:  
 $x < -3$  or  $x > 2$

8 D  $f(x) = 4x(2 - 3x)$

$$f(x) = 8x - 12x^2$$

$$f'(x) = 8 - 24x$$

$$f'(x) < 0$$

$$8 - 24x < 0$$

$$x > \frac{8}{24}$$

$$x > \frac{1}{3}$$

9 E  $y = (3 - 2f(x))^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2} \times -2f'(x) \times (3 - 2f(x))^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{3 - 2f(x)}}$$



**10 A**  $y = (x + 3)(x - 2)$

$$y = x^2 + x - 6$$

$$\frac{dy}{dx} = 2x + 1$$

When  $\frac{dy}{dx} = -7$

$$-8 = 2x$$

$$x = -4$$

When  $x = -4$

$$y = (-4)^2 - 4 - 6$$

$$y = 6$$

Coordinates =  $(-4, 6)$

**11 B**  $y = ax^2 - bx$

$$\frac{dy}{dx} = 2ax - b$$

When  $\frac{dy}{dx} = 0, x = 2$

$$0 = 4a - b$$

$$4a = b$$

Sub into:  $y = ax^2 - bx$

$$y = ax^2 - 4ax$$

$$y = ax(x - 4)$$

Using null factor theorem:

$$x = 0, x = 4$$

**12 C** Derivative of  $e^{-2ax} \cos(ax)$

Using Product rule

$$u = e^{-2ax}$$

$$u' = -2ae^{-2ax}$$

$$v = \cos(ax)$$

$$v' = -a \sin(ax)$$

$$u'v + v'u$$

$$\therefore -2ae^{-2ax} \cos(ax) - ae^{-2ax} \sin(ax)$$

**13 B**  $f(x) = \frac{\cos(x)}{x - a}$

Using quotient rule

$$u = \cos(x)$$

$$u' = -\sin(x)$$

$$v = x - a$$

$$v' = 1$$

$$\frac{u'v - v'u}{v^2}$$

$$f'(x) = \frac{-\sin(x)(x - a) - \cos(x)}{(x - a)^2}$$

$$f'(x) = -\frac{\sin(x)}{(x - a)} - \frac{\cos(x)}{(x - a)^2}$$

## Solutions to extended-response questions

**1**  $f(1) = 6$ ,  $g(1) = -1$ ,  $g(6) = 7$  and  $f(-1) = 8$   
 $f'(1) = 6$ ,  $g'(1) = -2$ ,  $f'(-1) = 2$  and  $g'(6) = -1$

**a i**  $f \circ g'(1) = g'(1)f'(g(1)) = -2 \times f'(-1) = -2 \times 2 = -4$

**ii**  $(g \circ f)'(1) = f'(1)g'(f(1)) = 6 \times g'(6) = 6 \times (-1) = -6$

**iii**  $(fg)'(1) = f'(1)g(1) + g'(1)f(1) = 6 \times (-1) + (-2) \times 6 = -18$

**iv**  $(gf)'(1) = f'(1)g(1) + g'(1)f(1) = 6 \times (-1) + (-2) \times 6 = -18$

**v**  $\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - g'(1)f(1)}{[g(1)]^2} = \frac{6 \times (-1) - (-2 \times 6)}{[-1]^2} = 6$

**vi**  $\left(\frac{g}{f}\right)'(1) = \frac{g'(1)f(1) - f'(1)g(1)}{[f(1)]^2} = \frac{-2 \times 6 - (6 \times (-1))}{[6]^2} = -\frac{1}{6}$

**b** For  $f(x) = ax^3 + bx^2 + cx + d$ ,  $f(1) = 6$  and  $f(-1) = 8$

Therefore

$$a + b + c + d = 6 \quad 1$$

$$-a + b - c + d = 8 \quad 2$$

Also  $f'(x) = 3ax^2 + 2bx + c$ ,  $f'(1) = 6$  and  $f'(-1) = 2$

Therefore

$$3a + 2b + c = 6 \quad 3$$

$$3a - 2b + c = 2 \quad 4$$

Subtract 4 from 3 to give  $4b = 4$  and  $b = 1$

Add 1 and 2

$$2b + 2d = 14 \text{ and as } b = 1, d = 6$$

From 1

$$a + c = -1 \quad 5$$

and from 4

$$3a + c = 4 \quad 6$$

Subtract 5 from 6

$$2a = 5 \text{ and therefore } a = \frac{5}{2} \text{ and } c = -\frac{7}{2}$$

**2**  $f'(x) = 0$  for  $x = 1$  and  $x = 5$

$$f'(x) > 0 \text{ for } x > 5 \text{ and } x < 1$$

$$f'(x) < 0 \text{ for } 1 < x < 5$$

$$f(1) = 6 \text{ and } f(5) = 1$$

- a** The graph of  $y = f(x + 2)$  is obtained from the graph of  $y = f(x)$  by a translation of 2 units in the negative direction of the  $x$ -axis.
- i** Therefore  $\frac{dy}{dx} = 0$  for  $x = -1$  and  $x = 3$
  - ii**  $\frac{dy}{dx} > 0$  for  $x > 3$  and  $x < -1$
- b** The graph of  $y = f(x - 2)$  is obtained from the graph of  $y = f(x)$  by a translation of 2 units in the positive direction of the  $x$ -axis.
- i** Therefore  $\frac{dy}{dx} = 0$  for  $x = 3$  and  $x = 7$ .
  - ii** The coordinates at which the gradient is zero are  $(3, 6)$  and  $(7, 1)$
- c** The graph of  $y = f(2x)$  is obtained from the graph of  $y = f(x)$  by a dilation of factor  $\frac{1}{2}$  from the  $y$ -axis.
- i** Therefore  $\frac{dy}{dx} = 0$  for  $x = \frac{1}{2}$  and  $\frac{5}{2}$
  - ii** The coordinates at which the gradient is zero are  $(\frac{1}{2}, 6)$  and  $(\frac{5}{2}, 1)$
- d** The graph of  $y = f(\frac{x}{2})$  is obtained from the graph of  $y = f(x)$  by a dilation of factor 2 from the  $y$ -axis.
- i** Therefore  $\frac{dy}{dx} = 0$  for  $x = 2$  and  $x = 10$
  - ii** The coordinates at which the gradient is zero are  $(2, 6)$  and  $(10, 1)$
- e** The graph of  $y = 3f(\frac{x}{2})$  is obtained from the graph of  $y = f(x)$  by a dilation of factor 2 from the  $y$ -axis and factor 3 from the  $x$ -axis.
- i** Therefore  $\frac{dy}{dx} = 0$  for  $x = 2$  and  $x = 10$
  - ii** The coordinates at which the gradient is zero are  $(2, 18)$  and  $(10, 3)$
- 3**  $f(x) = (x - \alpha)^n(x - \beta)^m$  where  $m$  and  $n$  are positive integers with  $m > n$  and  $\beta > \alpha$
- a**  $f(x) = 0$  implies  $x = \alpha$  or  $x = \beta$
  - b** Using the product rule

$$\begin{aligned}
 f'(x) &= n(x - \alpha)^{n-1}(x - \beta)^m + m(x - \alpha)^n(x - \beta)^{m-1} \\
 &= (x - \alpha)^{n-1}(x - \beta)^{m-1}[n(x - \beta) + m(x - \alpha)] \\
 &= (x - \alpha)^{n-1}(x - \beta)^{m-1}[x(n + m) - (n\beta + m\alpha)]
 \end{aligned}$$

**c**  $f'(x) = 0$  implies  $x = \alpha$  or  $x = \beta$  or  $x = \frac{n\beta + m\alpha}{n + m}$

**d i** If  $m$  and  $n$  are odd then  $m - 1$  and  $n - 1$  are even.

Therefore  $(x - \alpha)^{n-1}(x - \beta)^{m-1} \geq 0$  for all  $x$

and  $f'(x) > 0$  for  $x > \frac{n\beta + m\alpha}{n + m}$  and  $x \neq \beta$

**ii** If  $m$  is odd then  $m - 1$  is even and  $(x - \beta)^{m-1} \geq 0$  for all  $x$

Therefore  $f'(x) > 0$  if and only if  $(x - \alpha)^{n-1}[x(n + m) - (n\beta + m\alpha)] > 0$

If  $n$  is even then  $(x - \alpha)^{n-1} > 0$  if and only if  $x - \alpha > 0$ .

Together gives

$(x - \alpha)^{n-1}[x(n + m) - (n\beta + m\alpha)] > 0$  is equivalent to both factors positive or both factors negative.

**If both are positive:**

$$x > \alpha \text{ and } x > \frac{n\beta + m\alpha}{n + m}$$

$$\text{and as } \beta > \alpha, \frac{n\beta + m\alpha}{n + m} > \alpha \text{ and thus } x > \frac{n\beta + m\alpha}{n + m}$$

**If both are negative**

$$x < \alpha \text{ and } x < \frac{n\beta + m\alpha}{n + m} \text{ and hence } x < \alpha$$

**4**  $f(x) = \frac{x^n}{1 + x^n}$  where  $n$  is an even integer.

**a**  $1 - \frac{1}{x^n + 1} = \frac{x^n + 1 - 1}{x^n + 1} = \frac{x^n}{1 + x^n}$

**b**  $f(x) = \frac{nx^{n-1}}{(x^n + 1)^2}$

**c**  $0 < \frac{1}{x^n + 1} \leq 1$  as  $n$  is even. Therefore  $-1 \leq -\frac{1}{x^n + 1} < 0$  and  $0 \leq 1 - \frac{1}{x^n + 1} < 1$

**d**  $f'(x) = 0$  implies  $\frac{nx^{n-1}}{(x^n + 1)^2} = 0$  implies  $x = 0$

**e**  $f'(x) > 0$  for  $\frac{nx^{n-1}}{(x^n + 1)^2} > 0$  which implies  $x > 0$

# Chapter 10 – Applications of differentiation

## Solutions to Exercise 10A

1  $y = x^2 - 1$

$$\frac{dy}{dx} = 2x$$

$$x = 2,$$

$$\frac{dy}{dx} = 4$$

tangent:  $y = 4x + c$

$$x = 2, y = 3$$

$$3 = 8 + c$$

$$c = -5$$

$$y = 4x - 5$$

2  $y = x^2 + 3x - 1$

$$x = 0, y = -1$$

$$\frac{dy}{dx} = 2x + 3$$

$$x = 0,$$

$$\frac{dy}{dx} = 3$$

normal:

$$\text{grad} = -\frac{-1}{4}$$

$$y = \frac{-1}{3}x + c$$

$$x = 0, y = -1$$

$$-1 = c$$

$$y = \frac{-x}{3} - 1$$

3  $y = x^2 - 5x + 6$

$$= (x - 3)(x - 2)$$

$$y = 0, x = 2, 3$$

$$\frac{dy}{dx} = 2x - 5$$

When  $x = 2,$

$$\frac{dy}{dx} = 4 - 5$$

$$= -1$$

Gradient of normal = 1

$$y = x + c$$

$$= x + c$$

$$x = 2, y = 0$$

$$0 = 2 + c$$

$$c = -2$$

$$y = x - 2$$

When  $x = 3,$

$$\frac{dy}{dx} = 6 - 5$$

$$= 1$$

Gradient of normal = -1

$$y = -x + c$$

$$= -x + c$$

$$x = 3, y = 0$$

$$0 = -3 + c$$

$$c = 3$$

$$y = 3 - x$$

$$\begin{aligned}
 4 \quad y &= (2x + 1)^9 \\
 \frac{dy}{dx} &= 2 \times 9(2x + 1)^8 \\
 &= 18(2x + 1)^8 \\
 x &= 0, \\
 \frac{dy}{dx} &= 18(1)^8 \\
 &= 18 \\
 \text{tangent:} \\
 y &= 18x + c \\
 x = 0, y &= 1 \\
 1 &= c \\
 y &= 18x + 1 \\
 \text{normal:} \\
 y &= \frac{-1}{18}x + c \\
 x = 0, y &= 1 \\
 1 &= c \\
 y &= \frac{-1}{18}x + 1
 \end{aligned}$$

$$\begin{aligned}
 5 \quad y &= x^2 - 5 \\
 \frac{dy}{dx} &= 2x \\
 \frac{dy}{dx} &= 3 \\
 3 &= 2x \\
 x &= \frac{3}{2} \\
 x &= \frac{3}{2}, \\
 y &= \left(\frac{3}{2}\right)^2 - 5 \\
 &= \frac{9}{4} - 5 \\
 y &= \frac{-11}{4} \\
 \text{co-ords} &= \left(\frac{3}{2}, \frac{-11}{4}\right) \\
 y &= 3x + c \\
 x = \frac{3}{2}, y &= \frac{-11}{4} \\
 \frac{-11}{4} &= 3 \times \frac{3}{2} + c \\
 c &= \frac{-11}{4} - \frac{9}{2} \\
 &= \frac{-11}{4} - \frac{18}{4} \\
 c &= \frac{-29}{4}
 \end{aligned}$$

$$\begin{aligned}
 6 \text{ a} \quad y &= x^2 - 2 \\
 x = 1, y &= 1 - 2 = -1 \\
 \frac{dy}{dx} &= 2x \\
 x = 1, \frac{dy}{dx} &= 2 \\
 \text{i tangent:}
 \end{aligned}$$

$$y = 2x + c$$

$$x = 1, y = -1$$

$$-1 = 2 + c$$

$$c = -3$$

$$y = 2x - 3$$

**ii** normal:

$$y = \frac{-1}{2}x + c$$

$$x = 1, y = -1$$

$$-1 = \frac{-1}{2} + c$$

$$c = \frac{-1}{2}$$

$$y = \frac{-1}{2}x - \frac{1}{2}$$

**b**  $y = x^2 - 3x - 1$

$$x = 0, y = -1$$

$$\frac{dy}{dx} = 2x - 3$$

$$x = 0, \frac{dy}{dx} = -3$$

**i** tangent:

$$y = -3x + c$$

$$x = 0, y = -1$$

$$-1 = c$$

$$y = -3x - 1$$

**ii** normal:

$$y = \frac{-1}{-3}x + c$$

$$= \frac{1}{3}x + c$$

$$x = 0, y = -1$$

$$-1 = c$$

$$y = \frac{1}{3}x - 1$$

**c**  $y = \frac{1}{x}$

$$x = -1, y = -1$$

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

$$x = -1, \frac{dy}{dx} = -1$$

**i** tangent:

$$y = -x + c$$

$$x = -1, y = -1$$

$$-1 = 1 + c$$

$$c = -2$$

$$y = -x - 2$$

**ii** normal:

$$y = \frac{-1}{-1}x + c$$

$$= x + c$$

$$x = -1, y = -1$$

$$-1 = -1 + c$$

$$c = 0$$

$$y = x$$

$$\begin{aligned} \mathbf{d} \quad y &= (x-2)(x^2+1) \\ &= x^3 - 2x^2 + x - 2 \\ x &= -1, \\ y &= -1 - 2 - 1 - 2 \\ &= -6 \end{aligned}$$

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

$$x = -1,$$

$$\begin{aligned} \frac{dy}{dx} &= 3 + 4 + 1 \\ &= 8 \end{aligned}$$

**i** tangent:

$$y = 8x + c$$

$$x = -1, y = -6$$

$$-6 = -8 + c$$

$$c = 2$$

$$y = 8x + 2$$

**ii** normal:

$$y = \frac{-1}{8}x + c$$

$$x = -1$$

$$y = -6$$

$$-6 = \frac{1}{8} + c$$

$$c = -6\frac{1}{8} = -\frac{49}{8}$$

$$y = \frac{-1}{8}x - \frac{49}{8}$$

$$\begin{aligned} \mathbf{e} \quad y &= \sqrt{3x+1} \\ x &= 0, \quad y = \sqrt{1} = 1 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 3 \times \frac{1}{2\sqrt{3x+1}} \\ &= \frac{3}{2\sqrt{3x+1}} \end{aligned}$$

$$x = 0,$$

$$\frac{dy}{dx} = \frac{3}{2\sqrt{1}}$$

$$\frac{dy}{dx} = \frac{3}{2}$$

**i** tangent:

$$y = \frac{3}{2}x + c$$

$$x = 0, y = 1$$

$$1 = c$$

$$y = \frac{3}{2}x + 1$$

**ii** normal:

$$y = \frac{-2}{3}x + c$$

$$x = 0, y = 1$$

$$1 = c$$

$$y = \frac{-2}{3}x + 1$$

$$\begin{aligned} \mathbf{f} \quad y &= \sqrt{x} \\ x &= 1, y = 1 \end{aligned}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$x = 1,$$

$$\frac{dy}{dx} = \frac{1}{2}$$



**i** tangent:

$$y = \frac{1}{2}x + c$$

$$x = 1, y = 1$$

$$1 = \frac{1}{2} + c$$

$$c = \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

**ii** normal:

$$y = -2x + c$$

$$x = 1, y = 1$$

$$1 = -2 + c$$

$$c = 3$$

$$y = -2x + 3$$

**g**  $y = x^{\frac{2}{3}} + 1$

$$x = 1, y = 2$$

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$$

$$x = 1,$$

$$\frac{dy}{dx} = \frac{2}{3}$$

**i** tangent:

$$y = \frac{2}{3}x + c$$

$$x = 1, y = 2$$

$$2 = \frac{2}{3} + c$$

$$c = \frac{4}{3}$$

$$y = \frac{2x + 4}{3}$$

**ii** normal:

$$y = \frac{-3}{2}x + c$$

$$x = 1, y = 2$$

$$2 = \frac{-3}{2} + c$$

$$c = \frac{7}{2}$$

$$y = \frac{7 - 3x}{2}$$

**h**  $y = x^3 - 8x$

$$x = 2,$$

$$y = 8 - 16$$

$$= -8$$

$$\frac{dy}{dx} = 3x^2 - 8$$

$$x = 2,$$

$$\frac{dy}{dx} = 12 - 8$$

$$= 4$$

**i** tangent:

$$y = 4x + c$$

$$x = 2, y = -8$$

$$-8 = 8 + c$$

$$c = -16$$

$$y = 4x - 16$$

**ii** normal:

$$y = \frac{-1}{4}x + c$$
$$x = 2, y = -8$$
$$-8 = \frac{-1}{2} + c$$
$$c = -7\frac{1}{2}$$
$$= \frac{-15}{2}$$
$$y = \frac{-x}{4} - \frac{15}{2}$$

**i**  $y = x^3 - 3x^2 + 2$

$$x = 2,$$
$$y = 8 - 3 \times 4 + 2$$
$$y = -2$$

$$\frac{dy}{dx} = 3x^2 - 6x$$

$$x = 2,$$

$$\frac{dy}{dx} = 3 \times 4 - 6 \times 2$$

$$= 0$$

**i** tangent:

$$y = c$$
$$x = 2, y = -2$$
$$c = -2,$$
$$y = -2$$

**ii** normal:

$$x = 2$$

**j**  $y = 2x^3 + x^2 - 4x + 1$

$$x = 1,$$

$$y = 2 + 1 - 4 + 1$$

$$y = 0$$

$$\frac{dy}{dx} = 6x^2 + 2x - 4$$

$$x = 1,$$

$$\frac{dy}{dx} = 6 + 2 - 4$$

$$= 4$$

**i** tangent:

$$y = 4x + c$$

$$x = 1, y = 0$$

$$0 = 4 + c$$

$$c = -4$$

$$y = 4x - 4$$

**ii** normal:

$$y = \frac{-1}{4}x + c$$

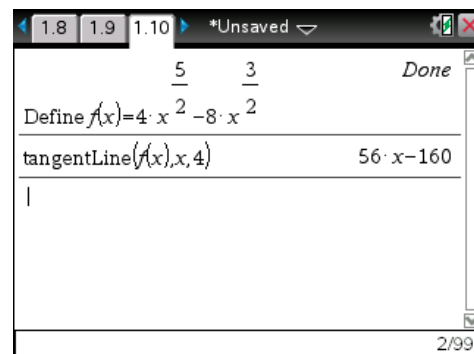
$$x = 1, y = 0$$

$$0 = \frac{-1}{4} + c$$

$$c = \frac{1}{4}$$

$$y = \frac{-1}{4}x + \frac{1}{4}$$

**7**  $y = 56x - 160$



$$\begin{aligned}
 \mathbf{8\ a} \quad y &= \frac{x^2 - 1}{x^2 + 1} \\
 x = 0, y &= \frac{-1}{1} = -1 \\
 y &= \frac{x^2 + 1 - 2}{x^2 + 1} \\
 y &= 1 - \frac{2}{x^2 + 1} \\
 y &= 1 - 2(x^2 + 1)^{-1} \\
 \frac{dy}{dx} &= 2x \times -1 \times -2(x^2 + 1)^{-2} \\
 &= \frac{4x}{(x^2 + 1)^2}
 \end{aligned}$$

$$x = 0, \frac{dy}{dx} = 0$$

tangent:

$$y = 0 \times x + c$$

$$x = 0, y = -1$$

$$-1 = c$$

$$y = -1$$

$$\begin{aligned}
 \mathbf{b} \quad y &= \sqrt{3x^2 + 1} \\
 x &= 1, \\
 y &= \sqrt{3 + 1} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dx} &= 6x \times \frac{1}{2\sqrt{3x^2 + 1}} \\
 &= \frac{3x}{\sqrt{3x^2 + 1}}
 \end{aligned}$$

$$x = 1,$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{3}{\sqrt{3 + 1}} \\
 &= \frac{3}{2}
 \end{aligned}$$

tangent:

$$y = \frac{3}{2}x + c$$

$$x = 1, y = 2$$

$$2 = \frac{3}{2} + c$$

$$c = \frac{1}{2}$$

$$y = \frac{3}{2}x + \frac{1}{2}$$

$$\mathbf{c} \quad y = \frac{1}{2x - 1}$$

$$x = 0,$$

$$y = \frac{1}{-1}$$

$$y = -1$$

$$\frac{dy}{dx} = 2 \times -1 \times \frac{1}{(2x - 1)^2}$$

$$= \frac{-2}{(2x - 1)^2}$$

$$x = 0,$$

$$\frac{dy}{dx} = \frac{-2}{(-1)^2}$$

$$= -2$$

tangent:

$$y = -2x + c$$

$$x = 0, y = -1$$

$$c = -1$$

$$y = -2x - 1$$

**d**  $y = \frac{1}{(2x-1)^2}$   
 $x = 1,$   
 $y = \frac{1}{(1)^2}$   
 $y = 1$   
 $\frac{dy}{dx} = -2 \times 2 \times \frac{1}{(2x-1)^3}$   
 $= \frac{-4}{(2x-1)^3}$   
 $x = 1,$   
 $\frac{dy}{dx} = \frac{-4}{(1)^3}$   
 $= -4$   
tangent:  
 $y = -4x + c$   
 $x = 1, y = 1$   
 $1 = -4 + c$   
 $c = 5$   
 $y = -4x + 5$

**9 a**  $y = \sin 2x$   
 $\frac{dy}{dx} = 2 \cos 2x$   
 $x = 0,$   
 $y = \sin 0 = 0$   
 $\frac{dy}{dx} = 2 \cos 0 = 2$   
 $y = 2x + c$   
 $x = 0, y = 0$   
 $c = 0$   
 $y = 2x$

**b**  $y = \cos 2x$   
 $\frac{dy}{dx} = -2 \sin 2x$   
 $x = \frac{\pi}{2},$   
 $y = \cos \pi = -1$   
 $\frac{dy}{dx} = -\sin \pi = 0$   
 $y = -1$

**c**  $y = \tan x$   
 $\frac{dy}{dx} = \sec^2 x$   
 $x = \frac{\pi}{4},$   
 $y = \tan \frac{\pi}{4} = 1$   
 $\frac{dy}{dx} = \sec^2 \frac{\pi}{4} = \sqrt{2}^2 = 2$   
 $y = 2x + c$   
 $x = \frac{\pi}{4}, y = 1$   
 $1 = \frac{\pi}{2} + c$   
 $c = 1 - \frac{\pi}{2}$   
 $y = 2x + 1 - \frac{\pi}{2}$

**d**  $y = \tan 2x$   
 $\frac{dy}{dx} = 2 \sec^2 2x$   
 $x = 0,$   
 $y = \tan 0 = 0$   
 $\frac{dy}{dx} = 2 \sec^2 0 = 2$   
 $y = 2x + c$   
 $x = 0, y = 0$   
 $0 = c$   
 $y = 2x$

**e**  $y = \sin x + x \sin 2x$

$$\frac{dy}{dx} = \cos x + \sin 2x + 2x \cos 2x$$

$$x = 0,$$

$$y = \sin 0 + 0 \sin 0 = 0$$

$$\frac{dy}{dx} = \cos 0 + 2 \sin 0 + 0 = 1$$

$$y = x + c$$

$$x = 0, \quad y = 0$$

$$c = 0$$

$$y = x$$

**f**  $y = x - \tan x$

$$\frac{dy}{dx} = 1 - \sec^2 x$$

$$x = \frac{\pi}{4},$$

$$y = \frac{\pi}{4} - \tan \frac{\pi}{4} = \frac{\pi}{4} - 1$$

$$\frac{dy}{dx} = 1 - \sec^2 \frac{\pi}{4} = 1 - 2 = -1$$

$$y = -x + c$$

$$x = \frac{\pi}{4}, \quad y = \frac{\pi}{4} - 1$$

$$\frac{\pi}{4} - 1 = \frac{-\pi}{4} + c$$

$$c = \frac{\pi}{2} - 1$$

$$y = -x + \frac{\pi}{2} - 1$$

**10 a**  $f(x) = e^x + e^{-x}$

$$f'(x) = e^x - e^{-x}$$

$$f'(0) = 1 - 1 = 0$$

$$y = c$$

$$f(0) = 1 + 1 = 2$$

$$2 = c$$

$$y = 2$$

**b**  $f(x) = \frac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{e^x + e^{-x}}{2}$$

$$f'(0) = \frac{1 + 1}{2} = 1$$

$$y = x + c$$

$$f(0) = \frac{1 - 1}{2} = 0$$

$$0 = c$$

$$y = x$$

**c**  $f(x) = x^2 e^{2x}$

$$f'(x) = 2xe^{2x} + 2x^2 e^{2x}$$

$$= 2xe^{2x}(x^2 + x)$$

$$f'(1) = 2e^2(1 + 1)$$

$$= 4e^2$$

$$y = 4e^2 x + c$$

$$f(1) = 1 \times e^2 = e^2$$

$$e^2 = 4e^2 + c$$

$$c = -3e^2$$

$$y = 4e^2 x - 3e^2$$

**d**  $f(x) = e^{\sqrt{x}}$

$$f'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$$

$$f'(1) = \frac{e}{2}$$

$$y = \frac{e}{2}x + c$$

$$f(1) = e^1 = e$$

$$e = \frac{e}{2} + c$$

$$c = \frac{e}{2}$$

$$y = \frac{e}{2}(x + 1)$$

**e**  $f(x) = xe^{x^2}$   
 $f'(x) = e^{x^2} + 2x^2e^{x^2}$   
 $= e^{x^2}(2x^2 + 1)$   
 $f'(1) = e^1(2 + 1)$   
 $= 3e$   
 $y = 3ex + c$   
 $f'(1) = 1 \times e^1 = e$   
 $e = 3e + c$   
 $c = -2e$   
 $y = 3ex - 2e$

**f**  $f(x) = x^2e^{-x}$   
 $f'(x) = 2xe^{-x} - x^2e^{-x}$   
 $= e^{-x}(2x - x^2)$   
 $f'(2) = e^{-2}(4 - 4) = 0$   
 $y = c$   
 $f(2) = 2^2e^{-2} = \frac{4}{e^2}$   
 $\frac{4}{e^2} = c$   
 $y = \frac{4}{e^2}$

**11 a**  $f(x) = \ln x$   
 $f'(x) = \frac{1}{x}$   
 $f'(1) = 1$   
 $y = x + c$   
 $(1, 0) \Rightarrow 0 = 1 + c$   
 $c = -1$   
 $y = x - 1$   
 For the normal the gradient is  $-1$   
 The equation of the normal is  
 $y = x + 1$

**b**  $f(x) = \ln(2x)$   
 $f'(x) = \frac{2}{2x} = \frac{1}{x}$   
 $f'\left(\frac{1}{2}\right) = 2$   
 $y = 2x + c$   
 $\left(\frac{1}{2}, 0\right) \Rightarrow 0 = 1 + c$   
 $c = -1$   
 $y = 2x - 1$

**c**  $f(x) = \ln(kx)$   
 $f'(x) = \frac{k}{kx} = \frac{1}{x}$   
 $f\left(\frac{1}{k}\right) = k$   
 $y = kx + c$   
 $\left(\frac{1}{k}, 0\right) \Rightarrow 0 = 1 + c$   
 $c = -1$   
 $y = kx - 1$

**12 a**  $y = x^{\frac{1}{5}}$   
 $\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}}$

When  $x = 0, y = 0, \frac{dy}{dx}$  not defined.

Therefore equation of tangent

$$x = 0$$

**b**  $y = x^{\frac{3}{5}}$   
 $\frac{dy}{dx} = \frac{3}{5}x^{-\frac{2}{5}}$

When  $x = 0, y = 0, \frac{dy}{dx}$  not defined.

Therefore equation of tangent

$$x = 0$$

**c**  $y = (x - 4)^{\frac{1}{3}}$   
 $y = 0 \Rightarrow x = 4$   
 $\frac{dy}{dx} = \frac{1}{3}(x - 4)^{-\frac{2}{3}}$   
 $x = 4,$   
 $\frac{dy}{dx}$  is undefined  
 $\therefore$  tangent is  $x = 4$

**d**  $y = (x + 5)^{\frac{2}{3}}$   
 $y = 0 \Rightarrow x = -5$   
 $\frac{dy}{dx} = \frac{2}{3}(x + 5)^{-\frac{1}{3}}$   
 $x = -5,$   
 $\frac{dy}{dx}$  is undefined  
 $\therefore$  tangent is  $x = -5$

**e**  $y = (2x + 1)^{\frac{1}{3}}$   
 $y = 0 \Rightarrow x = -\frac{1}{2}$   
 $\frac{dy}{dx} = \frac{2}{3}(2x + 1)^{-\frac{2}{3}}$   
 $x = -\frac{1}{2},$   
 $\frac{dy}{dx}$  is undefined  
 $\therefore$  tangent is  $x = -\frac{1}{2}$

**f**  $y = (x + 5)^{\frac{4}{5}}$   
 $y = 0 \Rightarrow x = -5$   
 $\frac{dy}{dx} = \frac{4}{5}(x + 5)^{-\frac{1}{5}}$   
 $x = -5,$   
 $\frac{dy}{dx}$  is undefined  
 $\therefore$  tangent is  $x = -5$

**13**  $y = \tan 2x$   
 $\frac{dy}{dx} = 2 \sec^2 2x$   
 $x = \frac{\pi}{8},$   
 $y = \tan \frac{\pi}{4} = 1$   
 $\frac{dy}{dx} = 2 \sec^2 \frac{\pi}{4} = 4$   
 $y = 4x + c$   
 $x = \frac{\pi}{8}, y = 1$   
 $1 = \frac{\pi}{2} + c$   
 $c = 1 - \frac{\pi}{2}$   
 $y = 4x + 1 - \frac{\pi}{2}$   
 $x = 0,$   
 $y = 1 - \frac{\pi}{2}$   
 $A = \left(0, 1 - \frac{\pi}{2}\right)$   
 $OA = \frac{\pi}{2} - 1$

**14**  $y = 2e^x$   
 $\frac{dy}{dx} = 2e^x$   
 $\therefore \frac{dy}{dx} = 2e^a$  when  $x = a$   
 Gradient of the line segment joining  
 $(a, 2e^a)$  and the origin is  $\frac{2e^a}{a}$   
 $\therefore \frac{2e^a}{a} = 2e^a$   
 $\therefore a = 1$

$$15 \quad y = \log_e x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{a} \text{ when } x = a$$

Gradient of the line segment joining  
( $a, \log_e a$ ) and the origin is  $\frac{\log_e a}{a}$

$$\therefore \frac{\log_e a}{a} = \frac{1}{a}$$

$$\therefore \log_e a = 1 \therefore a = e$$

$$16 \quad y = x^2 + 2x$$

$$\frac{dy}{dx} = 2x + 2$$

$$\therefore \frac{dy}{dx} = 2a + 2 \text{ when } x = a$$

Gradient of the line segment joining  
( $a, a^2 + 2a$ ) and the origin is  $\frac{a^2 + 2a}{a}$

$$\therefore \frac{a^2 + 2a}{a} = 2a + 2$$

$$\therefore a^2 + 2a = 2a^2 + 2a$$

$$\therefore a = 0$$

$$17 \quad y = x^3 + x$$

$$\frac{dy}{dx} = 3x^2 + 1$$

$$\therefore \frac{dy}{dx} = 3a^2 + 1 \text{ when } x = a$$

Gradient of the line segment joining  
( $a, a^3 + a$ ) and the point (1, 1) is

$$\frac{a^3 + a - 1}{a - 1}$$

$$\therefore 3a^2 + 1 = \frac{a^3 + a - 1}{a - 1}$$

$$\therefore (3a^2 + 1)(a - 1) = a^3 + a - 1$$

$$\therefore 3a^3 + a - 3a^2 - 1 = a^3 + a - 1$$

$$\therefore 2a^3 - 3a^2 = 0$$

$$\therefore a^2(2a - 3) = 0$$

$$\therefore a = 0 \text{ or } a = \frac{3}{2}$$



## Solutions to Exercise 10B

1 a

$$\begin{aligned} \text{Average rate of change} &= \frac{f(3) - f(2)}{3 - 2} \\ &= \frac{45 - 24}{1} \\ &= 21 \end{aligned}$$

b Average rate of change

$$\begin{aligned} &= \frac{f(2+h) - f(2)}{2+h-2} \\ &= \frac{3(2+h)^2 + 6(2+h) - 24}{h} \\ &= \frac{3(4+4h+h^2) + 6(2+h) - 24}{h} \\ &= \frac{18h + 3h^2}{h} \\ &= 18 + 3h \end{aligned}$$

c  $f'(x) = 6x + 6$   
 $f'(2) = 18$

2 a  $\frac{dV}{dt}$

b  $\frac{dS}{dr}$

c  $\frac{dV}{dx}$

d  $\frac{dA}{dt}$

e  $\frac{dV}{dh}$

3 a  $I = \frac{4}{(t+1)^2}$

$$\frac{dI}{dt} = \frac{-8}{(t+1)^3}$$

$$t = 10,$$

$$\frac{dI}{dt} = \frac{-8}{11^3}$$

$$= \frac{-8}{1331}$$

i.e.  $I$  wanes by  $\approx 0.006$  units/day

4  $V(t) = 1000(90-t)^3$

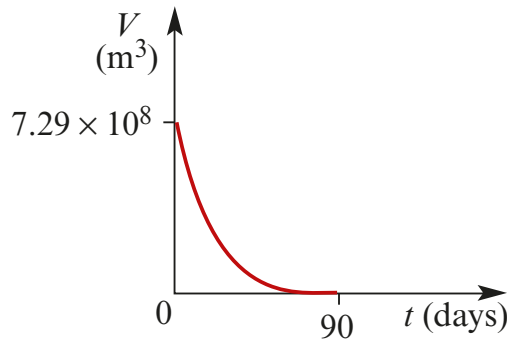
a  $V'(t) = -3000(90-t)^2$   
 it empties at  $3000(90-t)^2 \text{ m}^3/\text{day}$

b  $V(t) = 0,$   
 $1000(90-t)^3 = 0$   
 $t = 90$  days

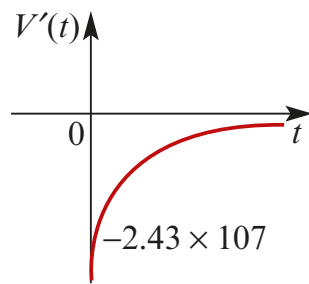
c  $V(0) = 1000(90)^3$   
 $= 729\,000\,000 \text{ m}^3$

d  $V'(t) = -300\,000$   
 $-300\,000 = -3000(90-t)^2$   
 $(90-t)^2 = 100$   
 $90-t = \pm 10$   
 $t = 90 \pm 10$   
 since  $t \in [0, 90]$   
 $t = 80$ th day

e



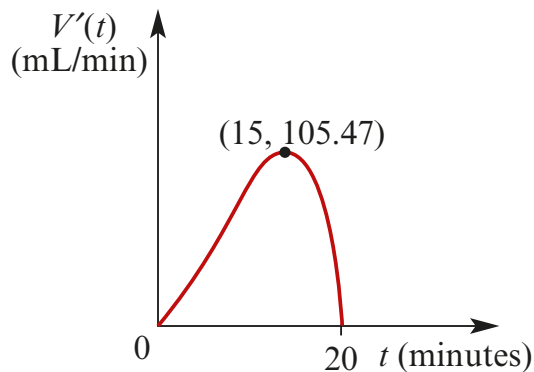
f



5  $V(t) = \frac{1}{160} \left( 5t^4 - \frac{t^5}{5} \right), 0 \leq t \leq 20$

a  $V'(t) = \frac{1}{160} (20t^3 - t^4)$  ml/min

b



c  $\frac{d}{dt} = \frac{1}{160} (60t^2 - 4t^3)$   
 $= \frac{1}{40} (15t^2 - t^3)$

$\frac{d}{dt} = 0,$

$15t^2 - t^3 = 0$

$t^2(15 - t) = 0$

$t = 0, 15$

max flow occurs at

$t = 15$

(using graph to determine max. or min. status)

6 a  $t \approx 100, 250, 500$

(read off graph-turning points)

b draw tangent at  $t = 200,$

use  $\frac{\text{rise}}{\text{run}}$  to find gradient

$\frac{dV}{dt} \approx 430\,000 \text{ m}^3/\text{day}$

(be careful *re*: vertical scale)

c  $t = 100, V \approx 4 \times 10^7$

$t = 250, V \approx 8 \times 10^7$

$\frac{\text{rise}}{\text{run}} \approx \frac{4 \times 10^7}{150} \approx 270\,000 \text{ m}^3/\text{day}$

d  $100 < t < 250$  or  $t > 500$

7 a

$$P = P_0 e^{-kt}$$

When  $t = 0, P = 30$

$$\therefore 30 = P_0 e^0$$

$$\therefore P_0 = 30$$

When  $t = 8, P = 10$

$$\therefore 10 = 30e^{-8k}$$

$$\therefore \frac{1}{3} = e^{-8k}$$

$$\therefore \log_e\left(\frac{1}{3}\right) = -8k$$

and  $k = -\frac{1}{8} \log_e\left(\frac{1}{3}\right)$   
 $= \frac{1}{8} \log_e(3) \approx 0.1373$

b When  $P = 8, 8 = 30 e^{\left(-\frac{1}{8} \log_e(3)\right)t}$

$$\therefore \frac{4}{15} = e^{\left(\log_e(3)\frac{1}{8}\right)t}$$

$$\therefore \frac{4}{15} = 3^{-\frac{t}{8}}$$

$$\therefore \frac{15}{4} = 3^{\frac{t}{8}}$$

$$\therefore t = \frac{8 \log_e\left(\frac{15}{4}\right)}{\log_e(3)} \approx 9.625$$

The pressure would be 8 units after approximately 9.625 hours.

c i  $\frac{dP}{dt} = -30ke^{-kt}$

where  $k = \frac{1}{8} \log_e 3$

When  $t = 0, \frac{dP}{dt} = -30ke^0$   
 $= -30k$

$$= -\frac{30}{8} \log_e 3$$

$$= -\frac{15}{4} \log_e 3$$

The rate of loss is

$$\frac{15}{4} \log_e 3 \approx 4.120 \text{ units per hour when } t = 0.$$

ii When  $t = 8, \frac{dP}{dt} = -30ke^{-8k}$

$$= -\frac{15}{4} \times \log_e 3 \times e^{-\log_e 3}$$

$$= -\frac{15}{4} \times \log_e 3 \times \frac{1}{3}$$

$$= -\frac{5}{4} \log_e 3$$

This rate of loss is

$$\frac{5}{4} \log_e 3 \approx 1.373 \text{ units per hour when } t = 8.$$

8 a

$$\frac{dT}{dt} = -\frac{45}{2} e^{-0.3t}$$

Also,  $e^{-0.3t} = \frac{1}{75}(T - 15)$

$$\therefore \frac{dT}{dt} = -\frac{45}{150}(T - 15)$$

$$= -0.3(T - 15)$$

b i When  $T = 90, t = 0$

$$\therefore \frac{dT}{dt} = -\frac{45}{2}$$

ii When  $T = 60$

$$\frac{dT}{dt} = -0.3(60 - 15) = -13.5$$

iii When  $T = 30$

$$\frac{dT}{dt} = -0.3(30 - 15) = -4.5$$

9

$$y = 3x + 2 \cos x$$

$$\frac{dy}{dx} = 3 - 2 \sin x$$

$$-1 \leq \sin 2x \leq 1$$

$$-2 \leq -2 \sin 2x \leq 2$$

$$-1 \leq 3 - 2 \sin 2x \leq 5 \therefore \frac{dy}{dx} > 0 \quad QED$$

10  $V(t) = 3 + 2 \sin \frac{t}{4}$

a  $V(10) = 3 + 2 \sin\left(\frac{5}{2}\right)$   
 $\approx 4.197$

b  $V'(t) = \frac{1}{2} \cos \frac{t}{4}$   
 $V'(10) = \frac{1}{2} \cos\left(\frac{5}{2}\right)$   
 $\approx -0.4$

11  $x = 2t^3 - 9t^2 + 12t$

a  $V = \frac{dx}{dt} = 6t^2 - 18t + 12$

b  $V = 0$   
 $\Rightarrow t^2 - 3t + 2 = 0$   
 $(t - 2)(t - 1) = 0$   
 $t = 1, 2$   
 $t = 1,$   
 $x = 2 - 9 + 12$   
 $t = 1, x = 5 \text{ cm}$   
 $t = 2, x = 16 - 36 + 24 = 4 \text{ cm}$

c  $a = \frac{dV}{dt} = 12t - 18$

$$t = 1, a = -6 \text{ cm/s}^2$$

$$t = 2, a = 6 \text{ cm/s}^2$$

d  $a = 0$

$$12t = 18$$

$$t = \frac{3}{2}$$

$$V = 6 \times \frac{9}{4} - 18 \times \frac{3}{2} + 12$$

$$= \frac{27}{2} - 27 + 12$$

$$= \frac{24}{2} - \frac{27}{2}$$

$$V = \frac{-3}{2} \text{ cm/s}$$

12  $x = 8 + 2t - t^2$

a  $t = 0,$   
 $x = 8 \text{ cm}$

b  $V = 2 - 2t$   
 $t = 0,$   
 $v = 2 \text{ cm/s}$

c  $V = 0$   
 $t = 1 \text{ s}$   
 $x = 8 + 2 - 1 = 9 \text{ cm}$

d  $a = -2 \text{ cm/s}^2$

13  $x = \sqrt{2t^2 + 2}$

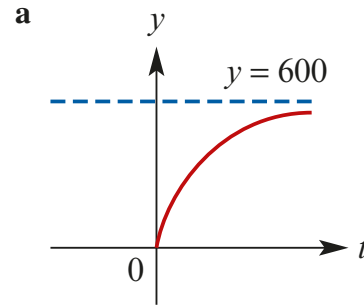
$$\begin{aligned} \mathbf{a} \quad V &= 4t \times \frac{1}{\sqrt{2t^2 + 2}} \\ &= \frac{\sqrt{2}t}{\sqrt{2t^2 + 2}} \\ V &= \frac{\sqrt{2}t}{\sqrt{t^2 + 1}} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad a &= \frac{\sqrt{t^2 + 1} \frac{d}{dt}(\sqrt{2}t) - \sqrt{2}t \frac{d}{dt}(\sqrt{t^2 + 1})}{t^2 + 1} \\ a &= \frac{\sqrt{2t^2 + 2} - \sqrt{2}t \times \frac{2t}{2\sqrt{t^2 + 1}}}{t^2 + 1} \\ &= \frac{\sqrt{2}(t^2 + 1) - \sqrt{2}t^2}{(t^2 + 1)^{\frac{3}{2}}} \\ &= \frac{\sqrt{2}}{(t^2 + 1)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad t &= 1 \\ V &= \frac{\sqrt{2}}{\sqrt{2}} = 1 \text{ cm/s} \\ a &= \frac{\sqrt{2}}{(2)^{\frac{3}{2}}} = \frac{1}{2} \text{ cm/s}^2 \end{aligned}$$

$$\begin{aligned} \mathbf{14} \quad x &= 0.4e^t \\ v &= 0.4e^t \\ t = 0, \quad v &= 0.4 \text{ m/s} \\ t = 1, \quad v &= 0.4e \approx 1.087 \text{ m/s} \\ t = 2, \quad v &= 0.4e^2 \approx 2.956 \text{ m/s} \end{aligned}$$

$$\mathbf{15} \quad y = 600(1 - e^{-0.5t})$$



$$\begin{aligned} \mathbf{b} \quad \frac{dy}{dx} &= 600(0.5e^{-0.5t}) \\ &= 300e^{-0.5t} \\ t &= 10, \\ \frac{dy}{dx} &= 300e^{-5} \approx 2.02 \end{aligned}$$

$$\begin{aligned} \mathbf{16} \quad \mathbf{a} \quad y &= e^{-2x} \\ -2x &= \ln y \\ x &= \frac{-1}{2} \ln y \\ \frac{dy}{dx} &= \frac{-1}{2y} \\ \frac{dy}{dx} &= -2y \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= Ae^{kx} \\ \frac{dy}{dx} &= Ake^{kx} \\ &= k(Ae^{kx}) \\ &= ky \end{aligned}$$

$$\mathbf{17} \quad m = 2e^{-0.2t}$$

$$\begin{aligned} \mathbf{a} \quad t &= 12, \\ m &= 2e^{-2.4} \\ &\approx 0.18 \text{ kg} \end{aligned}$$

**b**  $t = 0,$

$$m = 2$$

$$m - 1,$$

$$1 = 2e^{-0.2t}$$

$$-0.2t = \ln \frac{1}{2}$$

$$0.2t = \ln 2$$

$$t = 5 \ln 2 \approx 3.47 \text{ hours}$$

**c** **i**  $e^{-0.2t} = \frac{1}{4}$

$$0.2t = \ln 4$$

$$t = 10 \ln 2 \approx 6.93 \text{ hours}$$

**ii**  $e^{-0.2t} = \frac{1}{8}$

$$0.2t = \ln 8$$

$$t = 15 \ln 2 \approx 10.4 \text{ hours}$$

**d**  $\frac{dm}{dt} = -\frac{2}{5}e^{-\frac{t}{5}}$   
 $= -\frac{1}{5}(2e^{-\frac{t}{5}})$   
 $= -\frac{1}{5} \text{ m/hr}$

Rate of decay =  $\frac{1}{5} \text{ m}$

## Solutions to Exercise 10C

**1 a**  $f(x) = x^3 - 12x$

$$f'(x) = 3x^2 - 12$$

$$f'(x) = 0$$

$$3x^2 - 12 = 0$$

$$x^2 - 4 = 0$$

$$x = \pm 2$$

$$f(\pm 2) = \pm 8 \mp 24$$

$$= \mp 16$$

$$\text{co-ords} = (-2, 16), (2, -16)$$

**b**  $g(x) = 2x^2 - 4x$

$$g'(x) = 4x - 4$$

$$g'(x) = 0,$$

$$4x - 4 = 0$$

$$x = 1,$$

$$g(1) = 2 - 4$$

$$= -2$$

$$\text{co-ords} = (1, -2)$$

**c**  $h(x) = 5x^4 - 4x^5$

$$h'(x) = 20x^3 - 20x^4$$

$$h'(x) = 0,$$

$$20x^3 - 20x^4 = 0$$

$$x^3(1 - x) = 0$$

$$x = 0, 1$$

$$h(0) = 0,$$

$$h(1) = 1$$

$$\text{co-ords} = (0, 0), (1, 1)$$

**d**  $f(t) = 8t + 5t^2 - t^3, t > 0$

$$f'(t) = 8 + 10t - 3t^2$$

$$f'(t) = 0,$$

$$3t^2 + 10t + 8 = 0$$

$$t = \frac{10 \pm \sqrt{100 + 16}}{6}$$

$$= \frac{10 \pm 14}{6}$$

$$= \frac{-2}{3}, 4$$

$$t > 0, \therefore t = 4$$

$$f(4) = 32 + 80 - 64 = 48$$

$$\text{co-ords} = (4, 48)$$

**e**  $g(z) = 8z^2 - 3z^4$

$$g'(z) = 16z - 12z^3$$

$$g'(z) = 0,$$

$$16z - 12z^3 = 0$$

$$(3z^2 - 4)z = 0$$

$$z = 0, \frac{\pm 2}{\sqrt{3}}$$

$$g(0) = 0,$$

$$g\left(\frac{\pm 2}{\sqrt{3}}\right) = 8 \times \frac{4}{3} - 3 \times \frac{16}{9}$$

$$= \frac{32}{3} - \frac{16}{3}$$

$$= \frac{16}{3}$$

$$\text{co-ords} = \left(\frac{\pm 2}{\sqrt{3}}, \frac{16}{3}\right), (0, 0)$$

**f**  $f(x) = 5 - 2x + 3x^2$

$$f'(x) = -2 + 6x$$

$$f'(x) = 0,$$

$$6x - 2$$

$$x = \frac{1}{3}$$

$$f\left(\frac{1}{3}\right) = 5 - \frac{2}{3} + \frac{1}{3}$$

$$= 4\frac{2}{3} = \frac{14}{3}$$

$$\text{co-ords} = \left(\frac{1}{3}, \frac{14}{3}\right)$$

**g**  $h(x) = x^3 - 4x^2 - 3x + 20,$

$$x > 0$$

$$h'(x) = 3x^2 - 8x - 3$$

$$h'(x) = 0$$

$$3x^2 - 8x - 3 = 0$$

$$x = \frac{8 \pm \sqrt{64 + 36}}{6}$$

$$= \frac{-2}{6}, \frac{18}{6}$$

$$x > 0, \therefore x = \frac{18}{6} = 3$$

$$h(3) = 27 - 36 - 9 + 20$$

$$= 2$$

$$\text{co-ords} = (3, 2)$$

**h**

$$f(x) = 3x^4 - 16x^3 + 24x^2 - 10$$

$$f'(x) = 12x^3 - 48x^2 + 48x$$

$$f'(x) = 0,$$

$$x(x^2 - 4x + 4) = 0$$

$$x(x - 2)^2 = 0$$

$$x = 0, 2$$

$$f(0) = -10$$

$$f(2) = 3 \times 16 - 16 \times 8 + 24 \times 4 - 10$$

$$= 48 - 128 + 96 - 10$$

$$= -80 + 86$$

$$= 6$$

$$\text{co-ords} = (0, -10), (2, 6)$$

**2 a**  $f(x) = e^{2x} - 2x$

$$f'(x) = 2e^{2x} - 2$$

$$f'(x) = 0 \Rightarrow e^{2x} = 1$$

$$\Rightarrow x = 0$$

Coordinates of stationary point: (0,1)

**b**  $f(x) = x \log_e(3x)$

$$f'(x) = \log_e(3x) + 1$$

$$f'(x) = 0 \Rightarrow \log_e(3x) = -1$$

$$\Rightarrow x = \frac{1}{3e}$$

Coordinates of stationary point:  $\left(\frac{1}{3e}, -\frac{1}{3e}\right)$

**c**  $f(x) = \cos(2x), x \in [-\pi, \pi]$

$$f'(x) = -2 \sin(2x)$$

$$f'(x) = 0 \Rightarrow \sin(2x) = 0$$

$$\Rightarrow 2x = -2\pi, -\pi, 0, \pi, 2\pi$$

$$\Rightarrow x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$$

Coordinates of stationary point :

$$(-\pi, 1), \left(-\frac{\pi}{2}, -1\right), (0, 1), \left(\frac{\pi}{2}, -1\right), (\pi, 1)$$

**d**  $f(x) = xe^x$

$$f'(x) = e^x + xe^x = e^x(1 + x)$$

$$f'(x) = 0 \Rightarrow x = -1$$

Coordinates of stationary point:  $\left(-1, -\frac{1}{e}\right)$



e

$$f(x) = x^2 e^x$$

$$f'(x) = x^2 e^x + 2xe^x = xe^x(2 + x)$$

$$f'(x) = 0 \Rightarrow x = -2, 0$$

Coordinates of stationary point:

$$\left(-2, \frac{4}{e^2}\right), (0, 0)$$

f

$$f(x) = 2x \log_e(x)$$

$$f'(x) = 2 \log_e(3x) + 2$$

$$f'(x) = 0 \Rightarrow \log_e(x) = -1$$

$$\Rightarrow x = \frac{1}{e}$$

Coordinates of stationary point:  $\left(\frac{1}{e}, -\frac{2}{e}\right)$

3 a  $f(x) = x^2 - ax + 9$

$$f'(x) = 2x - a$$

$$f'(3) = 0,$$

$$6 - a = 0$$

$$a = 6$$

b  $h(x) = x^3 - bx^2 - 9x + 7$

$$h'(x) = 3x^2 - 2bx - 9$$

$$h'(-1) = 0,$$

$$3 + 2b - 9 = 0$$

$$2b = 6$$

$$b = 3$$

4

$$y = x^3 + bx^2 + cx + d$$

$$\frac{dy}{dx} = 3x^2 + 2bx + c$$

$$\text{When } x = 0, y = 3$$

$$\therefore d = 3$$

$$\text{When } x = 1, y = 3$$

$$\therefore 1 + b + c + 3 = 3$$

$$\therefore b + c = -1 \dots (1)$$

$$\text{When } x = 1, \frac{dy}{dx} = 0$$

$$\therefore 2b + c = -3 \dots (2)$$

Subtract (1) from (2)

$$b = -2$$

$$\therefore c = 1$$

5  $y = ax^2 + bx + c$

$$x = 1, y = -3$$

$$(1) \quad -3 = a + b + c$$

$$\frac{dy}{dx} = 2ax + b$$

$$x = 2, \frac{dy}{dx} = 4$$

$$(2) \quad 4 = 4a + b$$

$$x = 1, \frac{dy}{dx} = 0$$

$$(3) \quad 0 = 2a + b$$

$$(2) - (3) \Rightarrow 4 = 2a$$

$$a = 2$$

$$\text{sub in (3)} \Rightarrow b + 4 = 0$$

$$b = -4$$

$$\text{sub in (1)} \Rightarrow -3 = 2 - 4 + c$$

$$c = -1$$

$$y = 2x^2 - 4x - 1$$

**6**

$$y = ax^3 + bx^2 + cx + d$$

$$x = 0, y = 7\frac{1}{2}$$

$$d = \frac{15}{2}$$

$$x = 3, y = -6$$

$$-6 = 27a + 9b + 3c + \frac{15}{2}$$

$$-\frac{27}{2} = 27a + 9b + 3c$$

$$-\frac{9}{2} = 9a + 3b + c \dots (1)$$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$x = 0, \frac{dy}{dx} = -3$$

$$-3 = c$$

$$\text{sub in (1)} \Rightarrow -\frac{9}{2} = 9a + 3b - 3$$

$$-\frac{3}{2} = 9a + 3b \dots (2)$$

$$x = 3, \frac{dy}{dx} = 0$$

$$0 = 27a + 6b - 3$$

$$9a + 2b = 1 \dots (3)$$

$$(2) - (3) \Rightarrow b = \frac{-5}{2}$$

$$\text{sub in (3)} \Rightarrow 9a - 5 = 1$$

$$9a = 6$$

$$a = \frac{2}{3}$$

$$y = \frac{2x^3}{3} - \frac{5x^2}{2} - 3x + \frac{15}{2}$$

**7**  $y = ax + b(2x - 1)^{-1}$ **a**  $x = 2, y = 7$ 

$$1 \quad 7 = 2a + \frac{b}{3}$$

$$\frac{dy}{dx} = a - \frac{2b}{(2x - 1)^2}$$

$$x = 2, \frac{dy}{dx} = 0$$

$$0 = a - \frac{2b}{9}$$

$$2 \quad a = \frac{2b}{9}$$

$$\text{sub in 1} \Rightarrow 7 = \frac{4b}{9} + \frac{b}{3}$$

$$7 = \frac{7b}{9}$$

$$b = 9$$

$$\text{sub in 2} \Rightarrow a = 2$$

**b**

$$y = 2x + \frac{9}{(2x - 1)}$$

$$\frac{dy}{dx} = 2 - \frac{18}{(2x - 1)^2}$$

$$\frac{dy}{dx} = 0$$

$$\frac{18}{(2x - 1)^2} = 2$$

$$(2x - 1)^2 = 9$$

$$2x - 1 = \pm 3$$

$$2x = 1 \pm 3$$

$$x = -1, 2$$

$$x = -1,$$

$$y = -2 + \frac{9}{-3}$$

$$= -5$$

$$\text{co-ords} = (-1, -5)$$

**8**

$$y = (2x - 1)^n(x + 2)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(2x - 1)^n \times (x + 2)(2x - 1)^n \\ &\quad \times \frac{d}{dx}(x + 2) \\ &= 2n(2x - 1)^{n-1} \times (x + 2) \\ &\quad + (2x - 1)^n \\ &= (2x - 1)^{n-1}(2n(x + 2) + (2x - 1)) \\ &= (2x - 1)^{n-1}(2nx + 4n + 2x - 1) \\ &= (2x - 1)^{n-1}((2n + 2)x + (4n - 1)) \end{aligned}$$

$$\frac{dy}{dx} = 0,$$

$$0 = (2x - 1)^{n-1}((2n + 2)x + (4n - 1))$$

$$2x - 1 = 0 \text{ or } (2n + 2)x + (4n - 1) = 0$$

$$x = \frac{1}{2} \text{ or } x = \frac{1 - 4n}{2n + 2}$$

$$9 \quad y = (x^2 - 1)^n$$

$$\begin{aligned} \frac{dy}{dx} &= 2x \times n(x^2 - 1)^{n-1} \\ &= 2nx(x^2 - 1)^{n-1} \\ &= 2nx((x + 1)(x - 1))^{n-1} \\ \frac{dy}{dx} &= 0, \\ 2nx((x + 1)(x - 1))^{n-1} &= 0 \\ x &= 0, -1, 1 \end{aligned}$$

10

$$y = \frac{x}{x^2 + 1}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^2 + 1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2} \\ &= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \\ &= \frac{1 - x^2}{(x^2 + 1)^2} \end{aligned}$$

$$\frac{dy}{dx} = 0,$$

$$1 - x^2 = 0$$

$$x = \pm 1$$

$$y = \frac{\pm 1}{2}$$

$$\text{co-ords} = \left(1, \frac{1}{2}\right), \left(-1, \frac{-1}{2}\right)$$

## Solutions to Exercise 10D

**1 a**  $0 = 4x^2$

$x = 0$

	0	
+	0	+
/	—	/

inflexion

**b**  $0 = (x - 2)(x + 5)$

$x = -5, 2$

	-5		2	
+	0	-	0	+
/	—	\	—	/

max.                  min.

**c**  $0 = (x + 1)(2x - 1)$

$x = -1, \frac{1}{2}$

	-1		$\frac{1}{2}$	
+	0	-	0	+
/	—	\	—	/

max.                  min.

**d**  $0 = -x^2 + x + 12$

$0 = -(x^2 - x - 12)$

$0 = -(x - 4)(x + 3)$

$x = -3, 4$

	-3		4	
-	0	+	0	-
\	—	/	—	\

min.                  max.

**e**  $0 = x^2 - x - 12$

$0 = -(x - 4)(x + 3)$

$x = -3, 4$

	-3		4	
+	0	-	0	+
/	—	\	—	/

max.                  min.

**f**  $0 = 5x^4 - 27x^3$

$0 = x^3(5x - 27)$

$x = 0, \frac{27}{5}$

	0		$\frac{27}{5}$	
+	0	-	0	+
/	—	\	—	/

max.                  min.

**g**  $0 = (x - 1)(x - 3)$

$x = 1, 3$

	1		3	
+	0	-	0	+
/	—	\	—	/

max.                  min.

**h**  $0 = -(x - 1)(x - 3)$

$x = 1, 3$

	1		3	
-	0	+	0	-
\	—	/	—	\

min.                  max.

**2 a**  $y = x^3 - 12x$

$$\frac{dy}{dx} = 3x^2 - 12$$

$$\frac{dy}{dx} = 0$$

$$x^2 - 4 = 0$$

$$x = -2, +2$$

$$x = -2.5,$$

$$\frac{dy}{dx} > 0 \quad /$$

$$x = -1.5,$$

$$\frac{dy}{dx} < 0 \quad \backslash$$

$\therefore x = -2$  is a max

$$x = 1.5,$$

$$\frac{dy}{dx} < 0 \quad \backslash$$

$$x = 2.5,$$

$$\frac{dy}{dx} > 0 \quad /$$

$\therefore x = 2$  is a min

**b**  $y = 3x^2 - x^3$

$$\frac{dy}{dx} = 6x - 3x^2$$

$$\frac{dy}{dx} = 0,$$

$$2x - x^2 = 0$$

$$x(x - 2) = 0$$

$$x = 0, 2$$

$$x = -0.5,$$

$$\frac{dy}{dx} < 0 \quad \backslash$$

$$x = 0,$$

$$\frac{dy}{dx} = 0 \quad \text{—}$$

$$x = 1,$$

$$\frac{dy}{dx} > 0, \quad /$$

$$x = 2,$$

$$\frac{dy}{dx} = 0$$

$$x = 2.5,$$

$$\frac{dy}{dx} < 0 \quad \backslash$$

$\therefore x = 0$  is a min.

$\therefore x = 2$  is a max.

**c**  $y = x^3 - 5x^2 + 3x$

$$\frac{dy}{dx} = 3x^2 - 10x + 3$$

$$\frac{dy}{dx} = 0,$$

$$(3x - 1)(x - 3) = 0$$

$$x = \frac{1}{3}, 3$$

$$x = 0,$$

$$\frac{dy}{dx} > 0 \quad /$$

$$x = \frac{1}{3},$$

$$\frac{dy}{dx} = 0 \quad \text{—}$$

$$x = 1,$$

$$\frac{dy}{dx} < 0 \quad \diagdown$$

$$x = 3,$$

$$\frac{dy}{dx} = 0 \quad \text{—}$$

$$x = 4,$$

$$\frac{dy}{dx} > 0 \quad \diagup$$

$\therefore x = \frac{1}{3}$  is a max.

$x = 3$  is a min.

**d**

$$y = 3 - x^3$$

$$\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = 0,$$

$$x = 0$$

$$x = -1,$$

$$\frac{dy}{dx} < 0 \quad \diagdown$$

$$x = 0,$$

$$\frac{dy}{dx} = 0 \quad \text{—}$$

$$x = 1,$$

$$\frac{dy}{dx} < 0 \quad \diagdown$$

$x = 0$  is a stationary point of inflection

**e**

$$y = 3x^4 + 16x^3 + 22x^2 + 3$$

$$\frac{dy}{dx} = 12x^3 + 48x^2 + 48x$$

$$\frac{dy}{dx} = 0,$$

$$x(x^2 + 4x + 4) = 0$$

$$x(x + 2)^2 = 0$$

$$x = -2, 0$$

$$x = -3,$$

$$\frac{dy}{dx} = -27 \times 12$$

$$+ 48 \times 9 - 48 \times 3 < 0 \quad \diagdown$$

$$x = -2,$$

$$\frac{dy}{dx} = 0 \quad \text{—}$$

$$x = -1,$$

$$\frac{dy}{dx} = -12 + 48 - 48 < 0 \quad \diagdown$$

$$x = 0,$$

$$\frac{dy}{dx} = 0 \quad \text{—}$$

$$x = 1,$$

$$\frac{dy}{dx} = 12 + 48 + 48 > 0 \quad \diagup$$

$\therefore x = -2$  is a stationary point of inflection

$x = 0$  is a min.

**f**

$$y = x^3 - x$$

$$\frac{dy}{dx} = 3x^2 - 1$$

$$\frac{dy}{dx} = 0,$$

$$x^2 = \frac{1}{3}$$

$$x = \frac{\pm 1}{\sqrt{3}}$$

$$x = -1,$$

$$\frac{dy}{dx} > 0 \quad \nearrow$$

$$x = \frac{-1}{\sqrt{3}},$$

$$\frac{dy}{dx} = 0 \quad \text{—}$$

$$x = 0,$$

$$\frac{dy}{dx} < 0 \quad \searrow$$

$$x = \frac{+1}{\sqrt{3}},$$

$$\frac{dy}{dx} = 0 \quad \text{—}$$

$$x = 1,$$

$$\frac{dy}{dx} > 0 \quad \nearrow$$

$$\therefore x = \frac{-1}{\sqrt{3}} \text{ is a max}$$

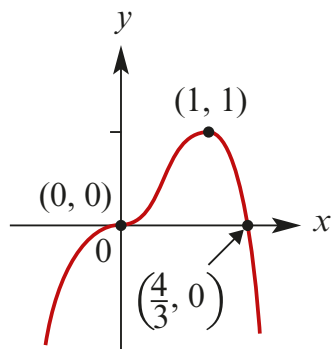
$$x = \frac{1}{\sqrt{3}} \text{ is a min}$$

**3 a i**  $y = 0$

$$4x^3 - 3x^4 = 0$$

$$x^3(4 - 3x) = 0$$

$$x = 0, \frac{4}{3}$$



**ii**

$$\frac{dy}{dx} = 12x^2 - 12x^3$$

$$\frac{dy}{dx} = 0,$$

$$12x^2 - 12x^3 = 0$$

$$x^2(1 - x) = 0$$

$$x = 0, 1$$

$$x = 0, y = 0$$

$(0, 0)$  is a stationary point of inflection

$$x = 1, y = 4 - 3 = 1$$

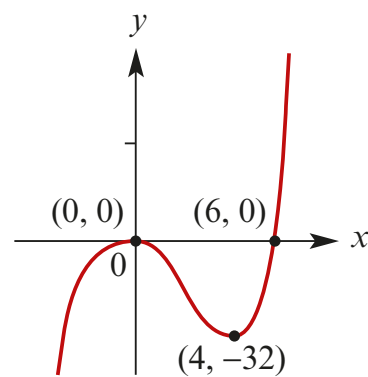
$(1, 1)$  is a maximum turning point

**b i**  $y = x^3 - 6x^2$

$$y = 0$$

$$x^2(x - 6) = 0$$

$$x = 0, 6$$



**ii**  $\frac{dy}{dx} = 3x^2 - 12x$

$$\frac{dy}{dx} = 0$$

$$3x(x - 4) = 0$$

$$x = 0, 4$$

$$x = 0, y = 0$$

(0, 0) is a maximum turning point

$$x = 4,$$

$$y = 64 - 96$$

$$= -32$$

(4, -32) is a minimum turning point

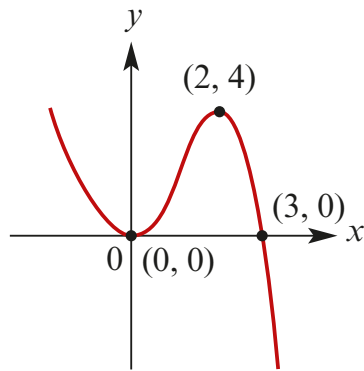
**c i**

$$y = 3x^2 - x^3$$

$$y = 0$$

$$x^2(3 - x) = 0$$

$$x = 0, 3$$



**ii**

$$\frac{dy}{dx} = 6x - 3x^2$$

$$\frac{dy}{dx} = 0$$

$$3x(2 - x) = 0$$

$$x = 0, 2$$

$$x = 0, y = 0$$

(0,0) is a minimum turning point

$$x = 2,$$

$$y = 3 \times 4 - 8$$

$$= 4$$

(2, 4) is a maximum turning point

**d i**

$$y = x^3 + 6x^2 + 9x + 4$$

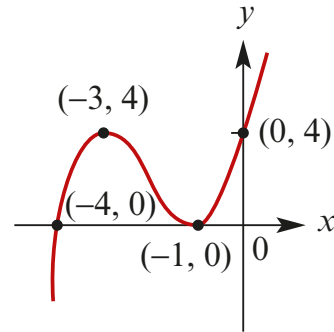
$$y = 0$$

$$x^3 + 6x^2 + 9x + 4 = 0$$

$$(x + 4)(x + 1)^2 = 0$$

$$x - \text{ints} : x = -4, -1$$

$$y - \text{int} : y = 4$$



**ii**

$$\frac{dy}{dx} = 3x^2 + 12x + 9$$

$$\frac{dy}{dx} = 0$$

$$3x^2 + 12x + 9 = 0$$

$$x^2 + 4x + 3 = 0$$

$$(x + 3)(x + 1) = 0$$

$$x = -3, -1$$

$$x = -3,$$

$$y = -27 + 6 \times 9 - 9 \times 3 + 4$$

$$y = -27 + 54 - 27 + 4$$

$$y = 4$$

$$(-3, 4)$$

is a maximum turning point

$$x = -1,$$

$$y = 0$$

$$(-1, 0)$$

is a minimum turning point



**e i**  $y = (x^2 - 1)^5$

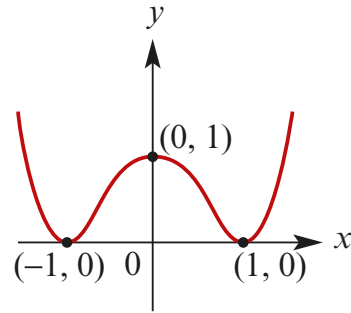
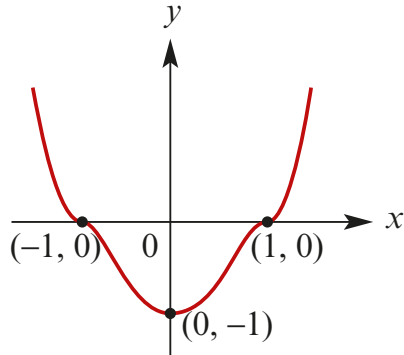
$$y = 0$$

$$x^2 - 1 = 0$$

$$x\text{-ints: } x = \pm 1$$

$$x = 0, y = -1$$

$$y\text{-int: } y = -1$$



**ii**  $\frac{dy}{dx} = 8x(x^2 - 1)^3$

$$\frac{dy}{dx} = 0,$$

$$x(x^2 - 1)^3 = 0$$

$$x = 0, \pm 1$$

$$(-1, 0), (1, 0)$$

are minimum turning points

$(0, 1)$  are maximum turning points

**ii**  $\frac{dy}{dx} = 2x \times 5(x^2 - 1)^4$

$$= 10x(x^2 - 1)^4$$

$$\frac{dy}{dx} = 0$$

$$10x(x^2 - 1)^4 = 0$$

$$x = 0, \pm 1$$

$$x = 0,$$

$$y = (-1)^5 = -1$$

$$(0, -1)$$

is a minimum turning point

$$x = \pm 1,$$

$$y = 0$$

$$(\pm 1, 0)$$

are stationary point of inflection

**f i**  $y = (x^2 - 1)^4$

$$x^2 = 1$$

$$x\text{-ints: } x = \pm 1$$

$$x = 0, y = 1$$

$$y\text{-int: } y = 1$$

**4 a**  $y = 2x^3 + 3x^2 - 12x + 7$

$$\frac{dy}{dx} = 6x^2 + 6x - 12$$

$$\frac{dy}{dx} = 0,$$

$$0 = 6x^2 + 6x - 12$$

$$x^2 + x - 2 = 0$$

$$(x - 1)(x + 2) = 0$$

$x = -2, 1$  both are turning points

$$x = -3,$$

$$\frac{dy}{dx} = 6(9 - 3 - 2) \\ > 0$$

$$x = 0,$$

$$\frac{dy}{dx} = -12 < 0$$

$\therefore x = -2$  is a max

$$x = -2,$$

$$y = -2 \times 8 + 3 \times 4 + 12 \times 2 + 7$$

$$y = -16 + 12 + 24 + 7$$

$$= 27$$

$(-2, 27)$  is a max

$$x = 2$$

$$\frac{dy}{dx} = 6(4 + 2 - 2)$$

$$> 0$$

$\therefore x = 1$  is a min

$$x = 1,$$

$$y = 2 + 3 - 12 + 7$$

$$x = 1, y = 0$$

$(1, 0)$  is a *min*

**b** see above,  $(1, 0)$

is a point on the curve

**c**  $y = 0$

$$2x^3 + 3x^2 - 12x + 7 = 0 \text{ (from a),}$$

we know  $(1, 0)$  is a turning point.

$\therefore (x - 1)^2$  is a factor,

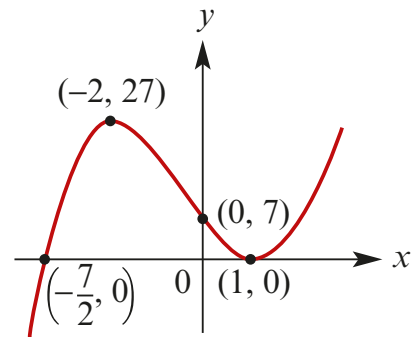
$$\therefore y = (x - 1)^2(2x + 7)$$

$$(x - 1)^2(2x + 7) = 0$$

$$x = \text{ints} : x = 1, \frac{-7}{2}$$

$$x = 0, y = 7, y\text{-int} : y = 7$$

**d**



**5 a**  $P(x) = x^3 + ax^2 + b$

$$P'(x) = 3x^2 + 2ax$$

$$= x(3x + 2a)$$

$$P'(0) = 0$$

$$\therefore x = 0$$

is a stationary point for all values of  $a$  and  $b$

**b**

$$P'(-2) = 0$$

$$3(-2)^2 + 2a(-2) = 0$$

$$12 - 4a = 0$$

$$a = 3$$

$$P(-2) = 6$$

$$(-2)^3 + 3(-2)^2 + b = 6$$

$$-8 + 12 + b = 6$$

$$4 + b = 6$$

$$b = 2$$

$$P(x) = x^3 + 3x^2 + 6$$

$$P'(x) = 3x^2 + 6x$$

$$P'(-1) = 3 - 6$$

$$< 0 \quad \searrow$$

$$P'(1) = 3 + 6$$

$$> 0 \quad \nearrow$$

$x = 0$  is a min.

Local minimum at  $(0, 2)$

$$P'(-3) = 27 - 18 > 0 \quad \nearrow$$

$x = -2$  is a max.

Local maximum at  $(-2, 6)$

**6 a**

$$f(x) = (2x - 1)^5(2x - 4)^4$$

$$f(0) = (-1)^5(-4)^4$$

$$y\text{-intercept} = -256$$

$$\text{co-ords}(0, -256)$$

$$f(x) = 0,$$

$$0 = (2x - 1)^5(2x - 4)^4$$

$$x = \frac{1}{2}, 2$$

$$x\text{-intercepts} = \frac{1}{2}, 2$$

$$\text{co-ords} \left( \frac{1}{2}, 0 \right), (2, 0)$$

**b**

$$f'(x) = (2x - 1)^5$$

$$\left( \frac{d}{dx}((2x - 4)^4) + (2x - 4)^4 \frac{d}{dx}((2x - 1)^5) \right)$$

$$= (2x - 1)^5 \times 2 \times 4(2x - 4)^3$$

$$+ (2x - 4)^4 \times 2 \times 5(2x - 1)^4$$

$$= (2x - 4)^3(2x - 1)^4((2x - 1) \times 8$$

$$+ (2x - 4) \times 10)$$

$$= (2x - 4)^3(2x - 1)^4(16x - 8 + 20x - 40)$$

$$= (2x - 4)^3(2x - 1)^4(36x - 48)$$

$$= 12(3x - 4)(2x - 4)^3(2x - 1)^4$$

$$f'(x) = 0,$$

$$12(3x - 4)(2x - 4)^3(2x - 1)^4 = 0$$

$$x = \frac{4}{3} \text{ (turning point)}$$

$$\text{or } x = 2 \text{ (turning point)}$$

$$\text{or } x = \frac{1}{2} \text{ (stationary point of inflection)}$$

$$f'(1) = 12(3-4)(2-4)^3(2-1)^4$$

$$> 0$$

$$f'(1.5) = 12(4.5-4)(3-4)^3(3-1)^4$$

$$< 0$$

$$f'(3) = 12(9-4)(6-4)^3(6-1)^4$$

$$> 0$$

$$x = \frac{4}{3} \text{ is a max.}$$

$$x = 2 \text{ is a min.}$$

$$f\left(\frac{4}{3}\right) = \left(\frac{8}{3} - 1\right)^5 \left(\frac{8}{3} - 4\right)^4 \left(\frac{5}{3}\right)^5 \left(\frac{-4}{3}\right)^4$$

$$\approx 40.6$$

$$\left(\frac{4}{3}, 40.6\right) \text{ is a max.}$$

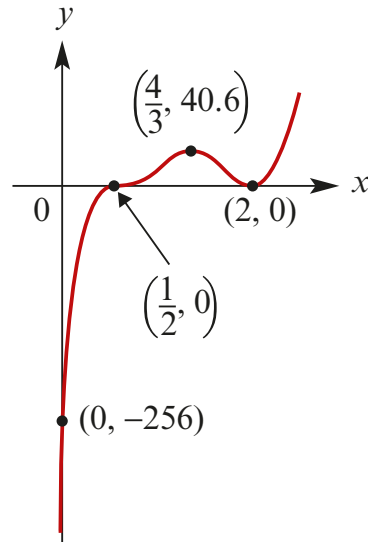
$$f(2) = (4-1)^5(4-4)^4$$

$$= 0$$

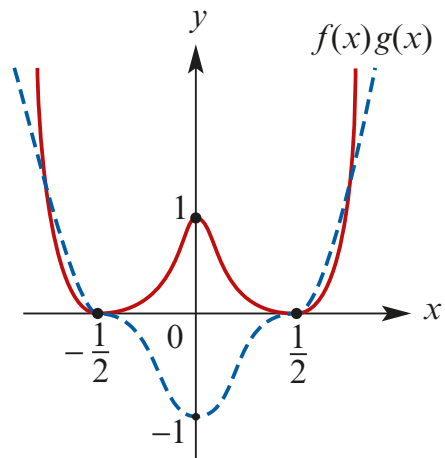
$$(2, 0) \text{ is a min.}$$

$$f\left(\frac{1}{2}\right) = 0$$

$$\left(\frac{1}{2}, 0\right) \text{ is a stationary point of inflection}$$



7 a



**b i**  $(4x^2 - 1)^6 > (4x^2 - 1)^5$   
 $4x^2 - 1 > 1$  if  $(4x^2 - 1)^5 > 0$   
 $4x^2 > 2$  if  $(4x^2 - 1)^5 > 0$   
 $x^2 > \frac{1}{2}$  if  $(4x^2 - 1)^5 > 0$   
 $|x| > \frac{1}{\sqrt{2}}$  if  $(4x^2 - 1)^5 > 0$   
 $x^2 < \frac{1}{4}$  if  $(4x^2 - 1)^5 < 0$   
 $|x| < \frac{1}{2}$  if  $(4x^2 - 1)^5 < 0$   
 $\therefore |x| > \frac{1}{\sqrt{2}}$  or  $|x| < \frac{1}{2}$

**ii**  
 $f'(x) = 8x \times 6(4x^2 - 1)^5$   
 $g'(x) = 8x \times 6(4x^2 - 1)^4$   
 $f'(x) > g'(x)$   
 $8x \times 6(4x^2 - 1)^5 > 8x \times 6(4x^2 - 1)^4$   
 $6x(4x^2 - 1) > 5x$   
if  $x > 0$ ,  
 $6(4x^2 - 1) > 5$   
 $(4x^2 - 1) > \frac{5}{6}$   
 $4x^2 > \frac{11}{6}$   
 $x^2 > \frac{11}{24}$   
 $x > \sqrt{\frac{11}{24}}$   
 $x > \frac{\sqrt{66}}{12}$

if  $x < 0$ ,  
 $6(4x^2 - 1) < 5$   
 $(4x^2 - 1) < \frac{5}{6}$   
 $4x^2 < \frac{11}{6}$   
 $x^2 < \frac{11}{24}$   
 $x^2 < \frac{66}{144}$   
 $x > \frac{-\sqrt{66}}{12}, x < 0$   
 $x \neq \pm \frac{1}{2}$   
 $\therefore \frac{-\sqrt{66}}{12} < x < \frac{-1}{2}$ ,

or  $\frac{-1}{2} < x < 0$ ,  
or  $x > \frac{\sqrt{66}}{12}$

**8 a**

$y = x^3 + x^2 - 8x - 12$   
 $x = 0, y = -12$   
y-intercept = (0, -12)  
 $y = 0$   
 $x^3 + x^2 - 8x - 12 = 0$   
try  $x = 3$  (a factor of -12)  
 $27 + 9 - 24 - 12$   
 $= 0$   
 $\therefore (x - 3)$  is a factor  
 $(x - 3)(x^2 + 4x + 4) = 0$   
 $(x - 3)(x + 2)^2 = 0$   
 $x = 3, -2$

$$x\text{-intercepts} = (-2, 0), (3, 0)$$

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 + 2x - 8 \\ &= (3x - 4)(x + 2)\end{aligned}$$

$$\frac{dy}{dx} = 0,$$

$$(3x - 4)(x + 2) = 0$$

$$x = -2, \frac{4}{3}$$

$$x = -2, y = 0$$

$$x = \frac{4}{3}$$

$$\begin{aligned}y &= \left(\frac{4}{3}\right)^3 + \left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) - 12 \\ &= \frac{64}{27} + \frac{16}{9} - \frac{32}{3} - 12 \\ &= \frac{64 + 48 - 288 - 324}{27} \\ &= \frac{-500}{27} \\ &= -18\frac{14}{27}\end{aligned}$$

stationary points are  $(-2, 0)$  max

and  $\left(\frac{4}{3}, \frac{-500}{27}\right)$  min

$$\begin{aligned}\mathbf{b} \quad y &= 4x - 18x^2 + 48x - 290 \\ &= 2(2x^3 - 9x^2 + 24x - 145)\end{aligned}$$

$$x = 0, y = -290$$

$$y\text{-intercept} = (0, -290)$$

$$y = 0$$

$$2x^3 - 9x^2 + 24x - 145 = 0$$

using CAS calculator

$$x = 5$$

$$y = 2(x - 5)(2x^2 + x + 29)$$

$$2x^2 + x + 29 = 0,$$

$$x = \frac{-1 \pm \sqrt{1 - 232}}{4}$$

no real solutions

$$y = 0, x = 5$$

$$x\text{-intercept} = (5, 0)$$

$$\begin{aligned}\frac{dy}{dx} &= 12x^2 - 36x + 48 \\ &= 12(x^2 - 3x + 4)\end{aligned}$$

$$\frac{dy}{dx} = 0,$$

$$x^2 - 3x + 4 = 0$$

$$x = \frac{3 \pm \sqrt{9 - 16}}{2}$$

no real solutions

$\therefore y$  has no stationary points

$$\mathbf{9 \ a} \quad f(x) = 3x^4 + 4x^3$$

$$f(x) = 12x^3 + 12x^2$$

$$f'(x) = 0,$$

$$12x^2(x + 1) = 0$$

$$x = -1, 0$$

$$f'(0) = 0$$

$(0, 0)$ , stationary point of inflection

$f(-1) = 3 - 4$   
 $= -1$   
 $(-1, -1)$  min., since  
 $f(x)$  is shaped and  $(0, 0)$   
 is a stationary point of inflection



**b**  $f(x) = x^4 + 2x^3 - 1$   
 $f'(x) = 4x^3 + 6x^2$   
 $f'(x) = 0,$   
 $2x^2(2x + 3) = 0$   
 $x = 0$  (stationary point of inflection)  
 $x = \frac{-3}{2}$  (turning point)  
 $f(0) = -1$   
 $(0, -1)$

is a stationary point of inflection  
 $f'(-2) = 4 \times -8 + 6 \times 4$

$< 0$

$$f'(-1) = -4 + 6$$

$> 0$

$$f'\left(-\frac{3}{2}\right) = \frac{81}{16} + \frac{-27}{4} - 1$$

$$= \frac{-43}{16}$$

$$\left(-\frac{3}{2}, \frac{-43}{16}\right) = (-1.5, -2.6875) \text{ is a min.}$$

**c**  $f(x) = 3x^3 - 3x^2 + 12x + 9$   
 $= 3(x^3 - x^2 + 4x + 3)$

$$f'(x) = 3(3x^2 - 2x + 4)$$

$$f'(x) = 0,$$

$$3x^2 - 2x + 4 = 0$$

$$x = \frac{2 \pm \sqrt{4 - 48}}{6}$$

no real solutions

$\therefore f(x)$  has no stationary points

**10**  $f(x) = \frac{1}{8}(x-1)^3(8-3x) + 1$

**a**  $f(0) = \frac{1}{8}(-1)^3(8) + 1$   
 $= 1 - 1$   
 $= 0$  QED

$f(3) = \frac{1}{8}(2)^3(-1) + 1$   
 $= -1 + 1$   
 $= 0$  QED

**b**  $f'(x) = \frac{1}{8}(x-1)^3 \frac{d}{dx}(8-3x)$   
 $+ \frac{1}{8}(8-3x) \frac{d}{dx}(x-1)^3$   
 $= \frac{-3}{8}(x-1)^3 \frac{3}{8}(8-3x)(x-1)^2$   
 $= \frac{3}{8}(x-1)^2((8-3x) - (x-1))$   
 $= \frac{3}{8}(x-1)^2(9-4x)$  QED

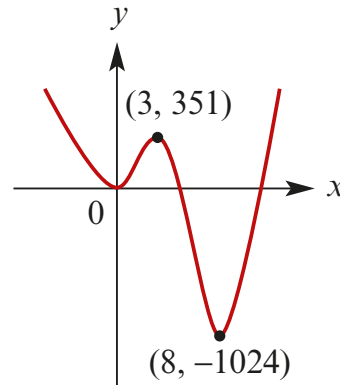
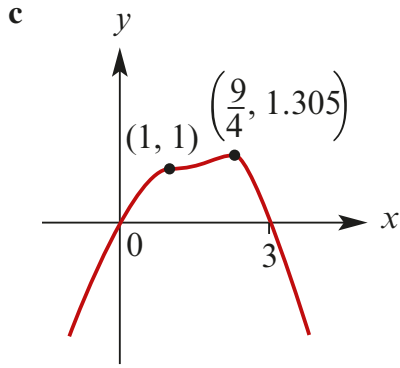
want  $x$  such that  $f'(x) \geq 0$

$$\frac{3}{8}(x-1)^2(9-4x) \geq 0$$

since  $(x-1)^2 \geq 0,$

$$9 - 4x \geq 0$$

$$x \leq \frac{9}{4}$$



**11**  $y = 3x^4 - 44x^3 + 144x^2$

$$\begin{aligned} \frac{dy}{dx} &= 12x^3 - 132x^2 + 288x \\ &= 12x(x^2 - 11x + 24) \\ &= 12x(x - 8)(x - 3) \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 0, \\ x &= 0, 3, 8 \end{aligned}$$

$$x = 0,$$

$$y = 0$$

$(0, 0)$  is a minimum turning point

$$x = 3,$$

$$\begin{aligned} y &= 3^5 - 44 \times 27 + 144 \times 9 \\ &= 243 - 1188 + 1296 \end{aligned}$$

$$= 351$$

$(3, 351)$  is a maximum turning point

$$x = 8,$$

$$\begin{aligned} y &= 3 \times 8^4 - 44 \times 8^3 + 144 \times 64 \\ &= 12288 - 22528 + 9216 \end{aligned}$$

$$= -1024$$

$(8, -1024)$  is a minimum turning point

**12 a**

$x = -1$  (stationary point of inflection)

$x = 1$  (*min*)

$x = 5$  (*max*)

**b**  $x = 0$  (*max*)

$x = 2$  (*min*)

**c**  $x = -4$  (*min*)

$x = 0$  (*max*)

**d**  $x = -3$  (*min*)

$x = 2$  (stationary point of inflection)

**13 a**  $y = x^4 - 16x^2$

$$\begin{aligned} \frac{dy}{dx} &= 4x^3 - 32x \\ &= 4x(x^2 - 8) \end{aligned}$$

$$\frac{dy}{dx} = 0,$$

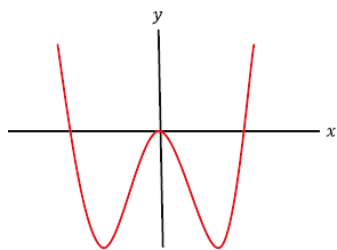
$$x = 0, \pm 2\sqrt{2}$$

$$x = 0, x = \pm 2\sqrt{2},$$

$$y = 0 \quad y = -64$$

Since the x-intercepts are  $\pm 4, 0$  we can sketch the graph.





hence  $(0,0)$  is a maximum  
 $(\pm 2\sqrt{2}, -64)$  are minimums

**b**  $y = x^{2m} - 16x^{2m-2}$

$$\begin{aligned} \frac{dy}{dx} &= (x^{2m} - 16x^{2m-2}) \\ &= 2mx^{2m-1} - 16(2m-2)x^{2m-3} \\ &= 2x^{2m-3}(mx^2 - 16(m-1)) \end{aligned}$$

$$\frac{dy}{dx} = 0$$

$$x^{2m-3}(mx^2 - 16(m-1)) = 0$$

$$x = 0, mx^2 - 16(m-1) = 0$$

$$x^2 = \frac{16(m-1)}{m}$$

$$x = \pm \sqrt{\frac{16(m-1)}{m}}$$

$$= \pm 4\sqrt{\frac{(m-1)}{m}}$$

If  $x = 0$  then  $y = 0$ .

$$\text{If } x = 4\sqrt{\frac{(m-1)}{m}}$$

$$y =$$

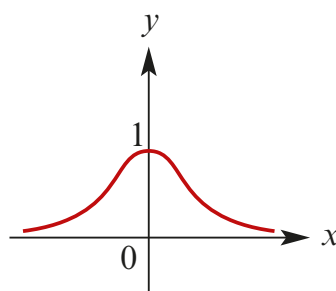
$$\begin{aligned} & \left(4\sqrt{\frac{(m-1)}{m}}\right)^{2m} - 16\left(4\sqrt{\frac{(m-1)}{m}}\right)^{2m-2} \\ &= 16m\left(\frac{(m-1)}{m}\right)^m - 16 \times 16^{m-1}\left(\frac{(m-1)}{m}\right)^{m-1} \\ &= 16m\left(\frac{(m-1)}{m}\right)^m - \left(\frac{(m-1)}{m}\right)^{m-1} \\ &= 16^m\left(\frac{(m-1)}{m}\right)^{m-1}\left(1 - \frac{(m-1)}{m}\right) \\ &= 16^m\left(\frac{(m-1)}{m}\right)^{m-1}\frac{1}{m} \end{aligned}$$

stationary points are :

$(0,0)_{min}$

$$\left(\pm 4\sqrt{\frac{(m-1)}{m}}, \frac{16^m(m-1)^{m-1}}{m^m}\right)_{max}$$

**14**



**15**  $f(x) = x^2 e^x$  in set

$$f'(x) = 2xe^x + x^2 e^x$$

$$= e^x(x^2 + 2x)$$

$$f'(x) < 0,$$

$$e^x(x^2 + 2x) < 0$$

$$x^2 + 2x < 0$$

$$x(x+2) < 0$$

$$x < 0 \text{ \& } x > -2$$

$$\therefore -2 < x < 0$$

notation,  $\{x: -2 < x < 0\}$

in set notation,  $\{x: -2 < x < 0\}$

16

$$f(x) = 100e^{-x^2+2x-5}$$

$$f'(x) = 100(-2x + 2)e^{-x^2+2x-5}$$

$$f'(x) > 0,$$

$$100(-2x + 2)e^{-x^2+2x-5} > 0$$

$$-2x + 2 > 0$$

$$x < 1$$

hence maximum  $f(x)$  occurs at  $x = 1$ .

$$f'(1) = 100e^{-1+2-5}$$

$$= 100e^{-4}$$

$$\cong 1.83$$

17  $f(x) = e^x - 1 - x$

a  $f'(x) = e^x - 1$

$$f'(x) = 0,$$

$$e^x = 1$$

$$x = 0$$

$$f(0) = e^0 - 1 - 0$$

$$\min f(x) = 0$$

b  $\min f(x) = 0$

$$\therefore f(x) \geq 0$$

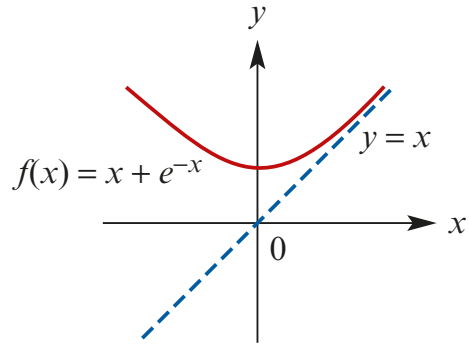
$$e^x - 1 - x \geq 0$$

$$e^x \geq 1 + x \quad QED$$

18 a (0, 1) min

b  $y = x$

c



19  $y = e^x(px^2 + qx + r)$

$$\frac{dy}{dx} = e^x(px^2 + qx + r) + e^x(2px + q)$$

$$= e^x(px^2(q + 2p)x + (r + q))$$

$$x = 0, y = 9$$

$$9 = e^0(0 + 0 + r)$$

$$9 = r$$

$$y = e^x(px^2 + qx + 9)$$

$$\frac{dy}{dx} = e^x(px^2(q + 2p)x + (q + 9))$$

$$x = 1, y = 0$$

$$0 = e^1(p + q + 2p + q + 9)$$

$$13p + 2q + 9 = 0$$

$$x = 3, y = 0$$

$$0 = e^3(9p + 3(q + 2p) + (q + 9))$$

$$9p + 3q + 6p + q + 9 = 0$$

$$215p + 4q + 9 = 0$$

$$2 - 21 \Rightarrow 9p - 9 = 0$$

$$p = 1$$

$$\text{sub in 1} \Rightarrow 3 + 2q + 9 = 0$$

$$2q = -12$$

$$q = -6$$

$$y = e^x(x^2 - 6x + 9)$$

**20 a**  $y = e^{4x^2-8x}$

$$\frac{dy}{dx} = (8x - 8)e^{4x^2-8x}$$

**b**  $\frac{dy}{dx} = 0$

$$8(x - 1)e^{4x^2-8x} = 0$$

$$x - 1 = 0$$

$$x = 1$$

$$x < 1, f'(x) < 0$$

$$x > 1, f'(x) > 0$$

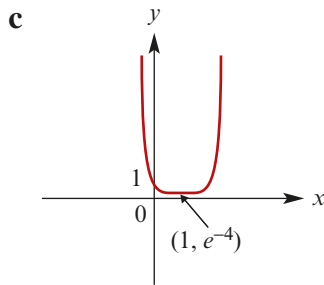
(since  $e^{4x^2-8x} > 0$ )

$\therefore x = 1$  is a min.

$$x = 1,$$

$$y = e^{4-8} = e^{-4}$$

$\therefore (1, e^{-4})$  is a min.



**d**  $x = 2,$

$$y = e^{16-16} = 1$$

$$\frac{dy}{dx} = 8(x - 1)e^{16-16}$$

$$= 8$$

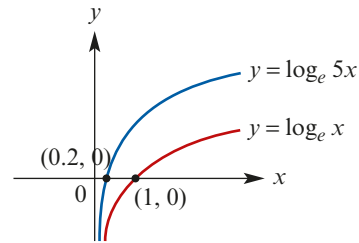
$$x = 2, y = 1$$

$$1 = -\frac{1}{4} + c$$

$$c = \frac{5}{4}$$

$$y = -\frac{1}{8}x + \frac{5}{4}$$

**21**



tangents are parallel for any given value of  $x$

**22**  $f(x) = x^2 \ln x$

**a**  $f'(x) = 2x \ln x + \frac{x^2}{x}$   
 $= x(2 \ln x + 1)$

**b**  $f(x) = 0,$   
 $x^2 \ln x = 0$

$$x^2 = 0, \quad \ln x = 0$$

$$x = 0, \quad x = 1$$

$$x = 0, 1 \quad \text{but } x > 0, \quad \therefore x = 1$$

**c**  $f'(x) = 0$

$$x(2 \ln x + 1) = 0$$

$$x = 0, 2 \ln x + 1 = 0$$

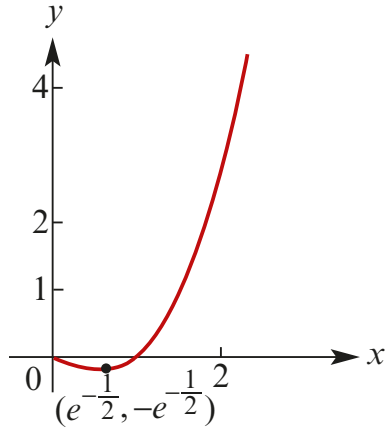
$$\ln x = \frac{-1}{2}$$

$$x = \frac{1}{\sqrt{e}}$$

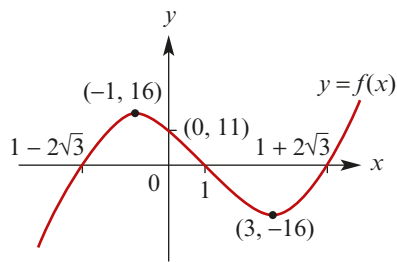
$$x = 0, \frac{1}{\sqrt{e}} \text{ but } x > 0,$$

$$\therefore x = \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$$

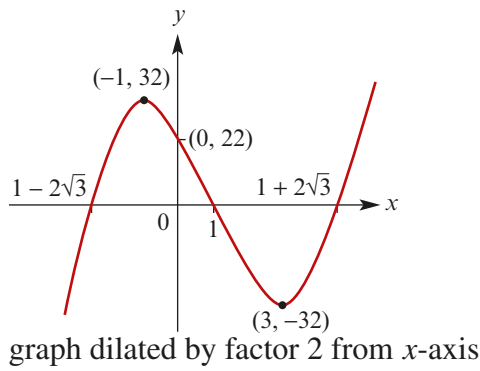
**d**  $x = e^{-\frac{1}{2}}, y = \left(e^{-\frac{1}{2}}\right)^2 \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2}e^{-1}$



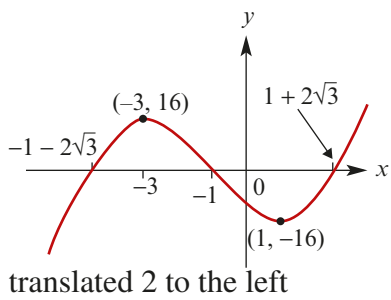
23 a



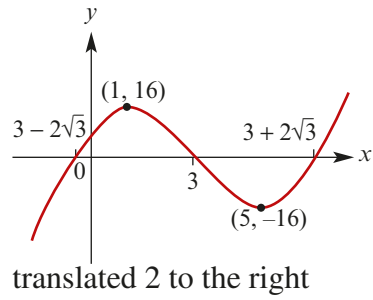
b



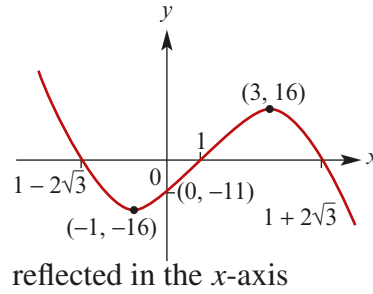
c



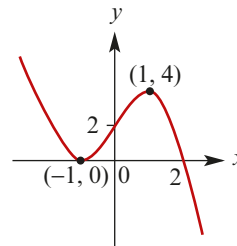
d



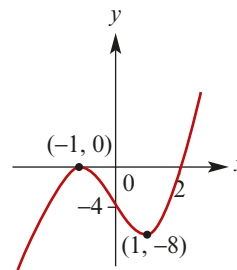
e



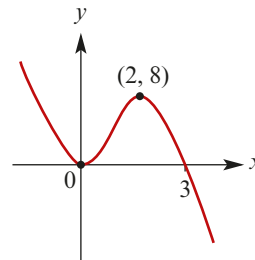
24 a

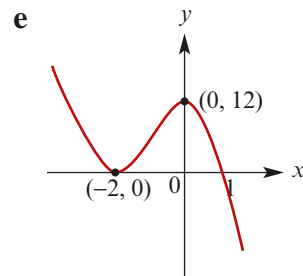
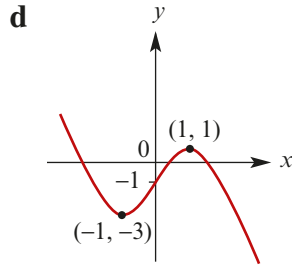


b



c





**25 a**  $A' = (a + l, 0)$

$B' = (b + l, 0)$

**b**  $P' = (h + l, kp)$

**26 a**

$$f(x) = 2 \cos x - 2 \cos^2 x + 1$$

$$f'(x) = -2 \sin x + 4 \sin x \cos x$$

$$f'(x) = 0 \Rightarrow 2 \sin x (2 \cos x - 1) = 0$$

$$\Rightarrow \sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$\Rightarrow x = 0, \pi, 2\pi \text{ or}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$(0, 1), (\pi, -3), (2\pi, 1)$  are min.

$(\frac{\pi}{3}, \frac{3}{2}), (\frac{5\pi}{3}, \frac{3}{2})$  are max.

**b**

$$f(x) = 2 \cos x + 2 \sin x \cos x$$

$$f'(x) = -2 \sin x + 2 \cos x \cos x - 2 \sin x \sin x$$

$$= -2 \sin x + 2(1 - 2 \sin^2 x)$$

$$= 2(-2 \sin^2 x - \sin x + 1)$$

$$f'(x) = 0,$$

$$\sin x = \frac{1 \pm \sqrt{1 + 8}}{-4}$$

$$\sin x = \frac{-1 \pm 3}{4}$$

$$\sin x = \frac{1}{2}, -1$$

$$x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{5\pi}{6}$$

$$f\left(\frac{\pi}{6}\right) = 2 \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)$$

$$= \sqrt{3} + \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2}$$

$$f\left(\frac{3\pi}{2}\right) = 2 \cos\left(\frac{3\pi}{2}\right) + \sin 3\pi$$

$$= 0$$

$$f\left(\frac{5\pi}{6}\right) = 2 \cos\left(\frac{5\pi}{6}\right) + \sin\left(\frac{10\pi}{6}\right)$$

$$= -\sqrt{3} - \frac{\sqrt{3}}{2}$$

$$= \frac{-3\sqrt{3}}{2}$$

$$\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right) \text{ max.}$$

$\left(\frac{3\pi}{2}, 0\right)$  stationary point of inflection

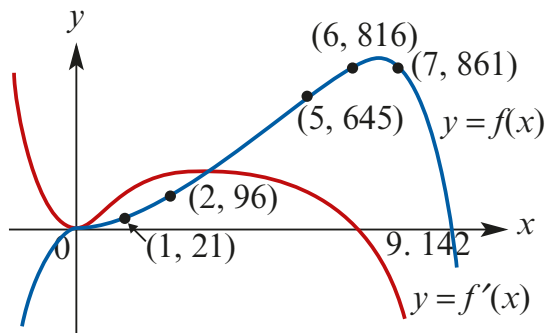
$$\left(\frac{5\pi}{6}, \frac{-3\sqrt{3}}{2}\right) \text{ min.}$$

**c** Max  $x = \frac{\pi}{2}, \frac{3\pi}{2}$ ; Min  $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

**d** Max  $x = \frac{\pi}{3}$ ; Infl  $x = \pi$ ; Min  $x = \frac{5\pi}{3}$

27 a and

b



Using a CAS calculator:

$$y = -x^4 + 8x^3 + 10x^2 + 4x$$

loc max at  $(6.761, 867.07)$

no stationary point of inflexion, since at

$$x = 0, \frac{dy}{dx} = 4.$$

c  $-960$

d Use the 'solve' command of a CAS calculator, giving:

$$x = 4.317 \text{ or } x = 8.404$$

## Solutions to Exercise 10E

- 1  $f : [-3, 3] \rightarrow \mathbb{R}, f(x) = 2 - 8x^2$   
Local maximum at  $(0, 2)$   
 $f(-3) = 2 - 8(-3)^2 = 2 - 72 = -70$   
 $f(3) = 2 - 8(3)^2 = 2 - 72 = -70$   
Therefore absolute maximum of  $f$  is 2  
and absolute minimum is  $-70$

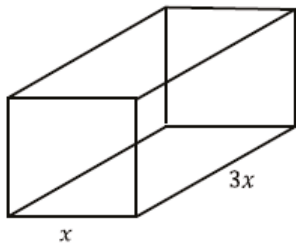
- 2  $f : [-3, 2] \rightarrow \mathbb{R},$   
 $f(x) = x^3 + 2x + 3$   
 $f'(x) = 3x^2 + 2$   
 $f'(x)$  has no real solution  
 $\therefore f(x)$  has no stationary points  
 $\therefore f(-3)$  is absolute minimum  
 $f(2)$  is absolute maximum  
 $f(-3) = -27 - 6 + 3$   
*abs. min.* =  $-30$   
 $f(2) = 8 + 4 + 3$   
*abs. max.* = 15

- 3  $f : \left[-\frac{3}{2}, \frac{5}{2}\right] \rightarrow \mathbb{R},$   
 $f(x) = 2x^3 - 6x^2$   
 $f'(x) = 6x^2 - 12x$   
 $f'(x) = 0,$   
 $x(x - 2) = 0$   
 $x = 0, 2$   
 $f(0) = 0$   
 $f(2) = 16 - 24$   
 $= -8$   
 $f\left(-\frac{3}{2}\right) = \frac{-27}{4} - \frac{27}{2}$   
 $= \frac{-81}{4} = -20.25$   
 $f\left(\frac{5}{2}\right) = \frac{125}{4} - \frac{75}{2}$   
 $= \frac{-25}{4}$   
absolute *min* =  $\frac{-81}{4}$   
absolute *max* = 0

4  $f : [-2, 6] \rightarrow R, f(x) = 2x^4 - 8x^2$   
 $f'(x) = 8x^3 - 16x$   
 $f'(x) = 0,$   
 $8x(x^2 - 2) = 0$   
 $x = \pm\sqrt{2}, 0$   
 $f(\pm\sqrt{2}) = 8 - 16$   
 $= -8$   
 $f(0) = 0$   
 $f(-2) = 32 - 32 = 0$   
 $f(6) = 2 \times 6^4 - 8 \times 6^2$   
 $= 2592 - 288$   
 $= 2304$

absolute *min* = -8  
 absolute *max* = 2304

5



$$4x + 4(3x) + 4y = 20$$

$$4x + y = 5$$

$$y = 5 - 4x$$

a  $V = x(3x)y$   
 $= 3x^2(5 - 4x)$   
 $V = 15x^2 - 12x^3$

*QED*

b  $\frac{dV}{dx} = 30x - 36x^2$

c  $\frac{dV}{dx} = 0,$   
 $30x - 36x^2 = 0$   
 $x(5 - 6x) = 0$   
 $x = 0, \frac{5}{6}$   
 since  $x = 0$  gives  
 $V = 0,$  it is not the *max*,  
 $\therefore x = \frac{5}{6}$  is the *max*  
 co-ords =  $(\frac{5}{6}, \frac{125}{36})$

d there are no turning points, so test the end points,  
 $x = 0,$   
 $V = 0$   
 $x = 0.8,$   
 $V = 15 \times \frac{16}{25} - 12 \times \frac{64}{125}$   
 $= \frac{1200 - 768}{125}$   
 absolute *max* :  $V = \frac{432}{125} = 3.456 \text{ cm}^3$   
 when  $x = 0.8$

e turning point at  $x = \frac{5}{6},$   
 test the endpoints,  $\frac{5}{6}$   
 $x = 0,$   
 $V = 0,$   
 $x = 1,$   
 $V = 15 - 12$   
 $= 3$   
 $x = \frac{5}{6},$   
 $V = 15 \times \frac{16}{25} - 12 \times \frac{125}{216}$   
 $= \frac{375 - 250}{36}$



absolute  $max : V = \frac{125}{36} = 3.472 \text{ cm}^3$

when  $x = \frac{5}{6}$

6  $x + y = 30, z = xy$

a  $x \in [2, 5],$

$y = 30 - x$

$y \in [25, 28], \text{ i.e. } 25 \leq y \leq 28$

b  $z = x(30 - x)$

$z = 30x - x^2$

$\frac{dz}{dx} = 30 - 2x$

$\frac{dz}{dx} = 0, x = 15$

this is outside the domain  $x \in [2, 5]$

$\therefore$  values to test are

$x = 2, 5$

$x = 2,$

$z = 60 - 4 = 56,$

$x = 5,$

$z = 150 - 25 = 125$

absolute minimum = 56

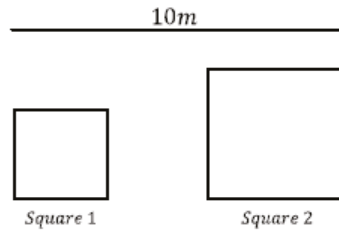
absolute maximum = 125

7 a  $\frac{1}{(x-4)^2} - \frac{1}{(x-1)^2}$

b  $\left(\frac{5}{2}, \frac{4}{3}\right)$

c Absolute max =  $\frac{4}{3}$ ; Absolute min =  $\frac{3}{2}$

8



a square 1 has perimeter

$x$ , i.e. side  $\left(\frac{x}{4}\right)$

$A_1 = \frac{x^2}{16}$

square 2 has perimeter

$(10 - x)$ , i.e. side  $\left(\frac{10 - x}{4}\right)$

$A_2 = \frac{(10 - x)^2}{16}$

$A = A_1 + A_2$

$= \frac{x^2 + (10 - x)^2}{16}$

$= \frac{x^2 + 100 - 20x + x^2}{16}$

$= \frac{2x^2 - 20x + 100}{16}$

$= \frac{1}{8}(x^2 - 10x + 50) \text{ QED}$

b  $\frac{dA}{dx} = \frac{1}{8}(2x - 10)$

$= \frac{1}{4}(x - 5)$

c  $\frac{dA}{dx} = 0,$

$x = 5$

d  $A = \frac{1}{8}(x^2 - 10x + 50)$

$x \in [0, 1] A(0) = \frac{25}{4}$  and  $A(1) = \frac{41}{8}$

The maximum is  $\frac{25}{4} \text{ m}^2$  but only one square is formed

9  $g : [2.1, 8] \rightarrow R, g(x) = x + \frac{1}{x-2}$

$$g'(x) = 1 - \frac{1}{(x-2)^2}$$

$$g'(x) = 0,$$

$$\frac{1}{(x-2)^2} = 1$$

$$(x-2)^2 = 1$$

$$x-2 = \pm 1$$

$$x = 2 \pm 1$$

$$x \in [2.1, 8]$$

$$x = 3,$$

values to test :

$$f(2.1), f(3), f(8)$$

$$\begin{aligned} f(2.1) &= 2.1 + \frac{1}{0.1} \\ &= 12.1 \end{aligned}$$

$$\begin{aligned} f(3) &= 3 + \frac{1}{1} \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(8) &= 8 + \frac{1}{6} \\ &= 8\frac{1}{6} \end{aligned}$$

absolute minimum = 4

absolute maximum = 12.1

10  $f : [0, 3] \rightarrow R, f(x) = \frac{1}{x+1} + \frac{1}{4-x}$

$$f(x) = \frac{1}{x+1} - \frac{1}{x-4}$$

a 
$$\begin{aligned} f'(x) &= \frac{-1}{(x+1)^2} - \frac{-1}{(x-4)^2} \\ &= \frac{1}{(x-4)^2} - \frac{1}{(x+1)^2} \end{aligned}$$

b  $f'(x) = 0,$

$$\frac{1}{(x-4)^2} = \frac{1}{(x+1)^2}$$

$$(x-4)^2 = (x+1)^2$$

$$(x-4) = \pm(x+1)$$

$$\text{if } x-4 = x+1$$

$$-4 = -1$$

does not work

$$\therefore x-4 = -(x+1) = -x-1$$

$$2x-3 = 0$$

$$x = \frac{3}{2}$$

$$\begin{aligned} f\left(\frac{3}{2}\right) &= \frac{1}{\left(\frac{5}{2}\right)} + \frac{1}{\left(\frac{5}{2}\right)} \\ &= \frac{4}{5} \end{aligned}$$

$$\text{co-ords} = \left(\frac{3}{2}, \frac{4}{5}\right)$$

c values to test:

$$f(0), f(3), f\left(\frac{3}{2}\right)$$

$$f\left(\frac{3}{2}\right) = \frac{4}{5}$$

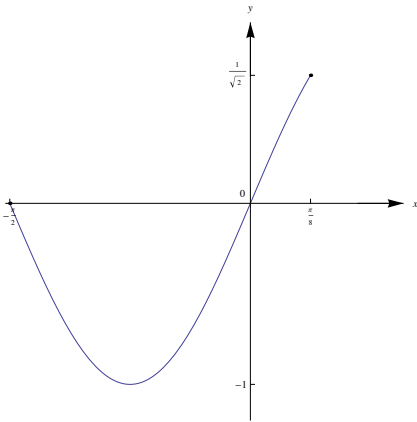
$$f(0) = \frac{1}{1} + \frac{1}{4} = \frac{5}{4}$$

$$f(3) = \frac{1}{4} + \frac{1}{1} = \frac{5}{4}$$

$$\text{absolute minimum} = \frac{4}{5}$$

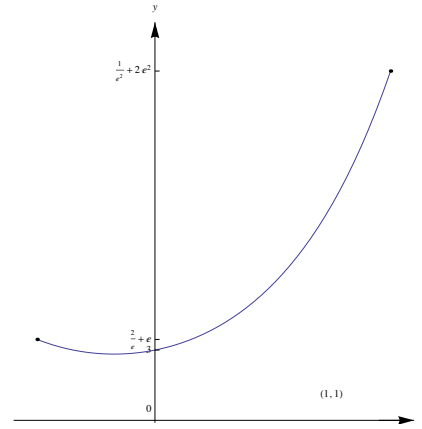
$$\text{absolute maximum} = \frac{5}{4}$$

11 Absolute max =  $\frac{\sqrt{2}}{2}$ ; Absolute min =  $-1$

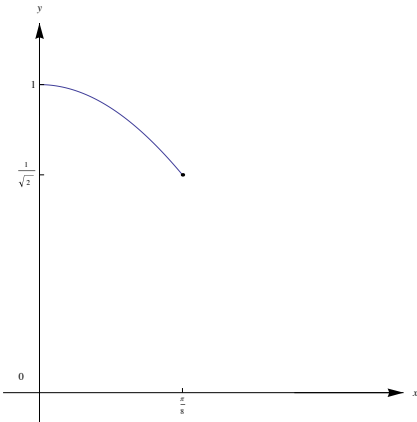


absolute maximum = 2  
absolute minimum =  $-2$

14 Absolute max =  $\frac{1}{e^2} + 2e^2$ ;  
Absolute min =  $2\sqrt{2}$



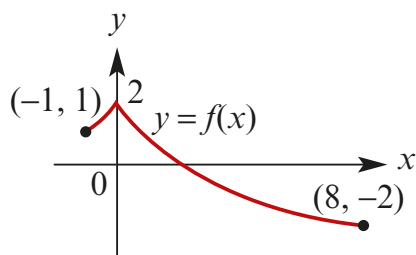
12 Absolute max = 1; Absolute min =  $\frac{\sqrt{2}}{2}$



15  $f(x) = 2e^{(x-1)^2}$   
 $f(-2) = 2e^9$  and  $f(2) = 2e$   
 $f'(x) = 4(x-1)e^{(x-1)^2}$   
 $f'(x) = 0$  implies  $x = 1$   
 $f(1) = 2$   
Absolute max =  $2e^9$ ; Absolute min = 2

16 Absolute max =  $-\log_e 10$ ;  
Absolute min =  $-\frac{10}{e}$

13



## Solutions to Exercise 10F

1 Let  $x$  m be the width of the rectangle

Let  $y$  m be the length of the rectangle

$$2x + 2y = 100 \Rightarrow x + y = 50$$

$$\text{Area, } A = xy = x(50 - x) = 50x - x^2$$

$$\frac{dA}{dx} = 50 - 2x$$

$$\frac{dA}{dx} = 0 \Rightarrow x = 25$$

$$\therefore \text{maximum area} = 25 \times 25 = 625 \text{ m}^2.$$

2  $x = y = 4; x, y > 0;$

$x^3 + y^2$  is a min.

$$\text{let } z = x^3 + y^2$$

$$y = 4 - x$$

$$z = x^3 + (4 - x)^2$$

$$= x^3 + 16 - 8x + x^2$$

$$= x^3 + x^2 - 8x + 16$$

$$\frac{dz}{dx} = 3x^2 + 2x - 8$$

$$\frac{dz}{dx} = 0,$$

$$3x^2 + 2x - 8 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 96}}{6}$$

$$x = \frac{-2 \pm 10}{6}$$

$$x = -2, \frac{4}{3}$$

but  $x > 0,$

$$\therefore x = \frac{4}{3}$$

$$y = 4 - x = \frac{8}{3}$$

3  $x + y = 100 \quad P = xy$

$$y = 100 - x$$

$$P = x(100 - x)$$

$$= 100x - x^2$$

$$\frac{dP}{dx} = 100 - 2x$$

$$\frac{dP}{dx} = 0,$$

$$x = 50$$

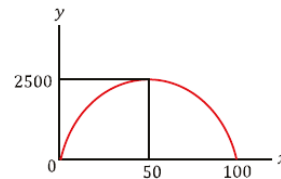
this gives max P

$$x = 50, y = 100 - 50 = 50 = x$$

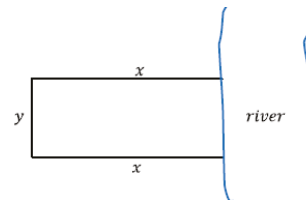
*QED*

$$P = 50^2$$

$$P = 2500$$



4



$$y + 2x = 4$$

$$y = 4 - 2x$$

$$A = xy$$

$$= x(4 - 2x)$$

$$= 4x - 2x^2$$

$$\frac{dA}{dx} = 4 - 4x$$

$$\frac{dA}{dx} = 0,$$

$$x = 1 \text{ km}$$

the farmer should make one side 2 km long and the other two sides 1 km long, using 2 km of river.

5  $p, q > 0$

$$p^3q = 9$$

$$\Rightarrow q = \frac{9}{p^3}$$

$$z = 16p + 3q$$

$$z = 16p + \frac{27}{p^3}$$

$$\frac{dz}{dx} = 16 - \frac{81}{p^4}$$

$$\frac{dz}{dx} = 0,$$

$$\frac{81}{p^4} = 16$$

$$\frac{3}{p} = 2 \text{ since } p > 0$$

$$p = \frac{3}{2}$$

$$q = \frac{9}{p^3} = \frac{9}{\left(\frac{27}{8}\right)} = \frac{8}{3}$$

6  $SA = 150$ , base has side  $x(x >, \text{not } \geq 0)$

a  $SA = 2x^2 + 4xh$

$$150 = 2x^2 + 4xh$$

$$h = \frac{75 - x^2}{2x} \text{ QED}$$

b  $V = x^2h$

$$= x^2 \left( \frac{75 - x^2}{2x} \right)$$

$$V = \frac{75x - x^3}{2}$$

c  $\frac{dv}{dx} = \frac{75}{2} - \frac{3}{2}x^2$

$$\frac{dv}{dx} = 0,$$

$$x^2 = \frac{75}{2} \times \frac{2}{3}$$

$$= 25$$

$$x = 5 \text{ cm}$$

$$V = \frac{75 \times 5 - 5^3}{2} = \frac{375 - 125}{2} = 125 \text{ cm}^3$$

7  $P = 100n - 0.4n^2 - 160$

a i  $\frac{dP}{dn} = 100 - 0.8n$

$$\frac{dP}{dn} = 0 \text{ implies } 100 = 0.8n$$

$$\therefore n = 125$$

A maximum occurs when  $n = 125$  as  $P$  is a quadratic with negative coefficient of  $n^2$ .

ii When  $n = 125$ ,  $P = 100 \times 125 - 0.4 \times 125^2 - 160 = 6090$

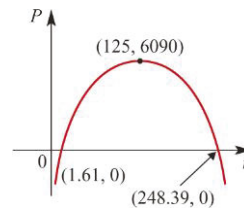
Maximum daily profit is \$ 6090.

b When  $P = 0$ ,

$$n = \frac{100 \pm \sqrt{100^2 - 4 \times 0.4 \times 160}}{-0.8}$$

$$\therefore n \approx 1.6, 248.4$$

In this problem a continuous model for a discrete situation has been used.



c  $P > 0$  implies  $2 \leq n \leq 248$  (Note:  $n$  can only take integer values)

d Let  $\$P$  be the profit per article

$$\begin{aligned} \therefore p &= \frac{P}{n} \left( = \frac{\text{total profit}}{\text{no. of articles}} \right) \\ &= \frac{100n - 0.4n^2 - 160}{n} \\ &= 100 - 0.4n - \frac{160}{n} \end{aligned}$$

In order to find the maximum profit per article consider the derivative of  $p$  with respect to  $n$ .

$$\frac{dp}{dn} = -0.4 + \frac{160}{n^2}$$

$$\frac{dp}{dn} = 0 \text{ implies } 0.4 = \frac{160}{n^2}$$

$$\therefore n^2 = \frac{160}{0.4}$$

$$\text{i.e. } n^2 = 400$$

$$\therefore n = 20$$

The gradient chart indicates maximum:

	< 20	20	> 20
sign $f'(x)$	+ve	0	-ve
shape	/	-	\

i.e. selling 20 articles maximises the profit per article.

8

$$S(x) = -x^3 + 3x^2 + 360x + 5000, \text{ values to test:}$$

$$x \in [6, 20]$$

$$S'(x) = -3x^2 + 6x + 360$$

$$S'(x) = 0,$$

$$x^2 - 2x - 120 = 0$$

$$(x + 10)(x - 12) = 0$$

$$x = -10, 12$$

$$\text{but } x \in [6, 20]$$

$$x = 12$$

values to test :

$$S(6), S(12), S(20)$$

$$S(6) = -216 + 108 + 2160 + 5000$$

$$= 2052 + 5000 = 7052$$

$$S(12) = -1728 + 432 + 4320 + 5000$$

$$= 3024 + 5000 = 8024$$

$$S(20) = -8000 + 1200 + 7200 + 5000$$

$$= 400 + 5000 = 5400$$

absolute maximum = 12°C

$$S(12) = 8024 \text{ salmon}$$

$$9 \quad M(x) = \frac{-1}{30}(x^3 - 14x^2 + 32x - 50),$$

$$0 \leq x \leq 10$$

$$M'(x) = \frac{-1}{30}(3x^2 - 28x + 32)$$

$$M'(x) = 0,$$

$$3x^2 - 28x + 32 = 0$$

$$x = \frac{28 \pm \sqrt{784 - 384}}{6}$$

$$x = \frac{28 \pm 20}{6}$$

$$x = \frac{4}{3}, 8$$

$$x = 0, x = \frac{4}{3}, x = 8, x \rightarrow \infty$$

$$M(0) = \frac{50}{30} = \frac{5}{3}$$

$$M\left(\frac{4}{3}\right) = \frac{1}{30} \left( 50 - 30 \times \frac{4}{3} + 14 \times \frac{16}{9} - \frac{64}{27} \right)$$

$$= \frac{1}{30} \left( \frac{1}{27} (1350 - 1152 + 672 - 64) \right)$$

$$= \frac{806}{810}$$

$$M\left(\frac{4}{3}\right) = \frac{403}{405}$$

$$M(8) = \frac{1}{30} (50 - 32 \times 8 + 14 \times 64 - 512)$$

$$= \frac{178}{30}$$

$$= \frac{89}{15}$$

Maximum M occurs at  $x = 8$  mm

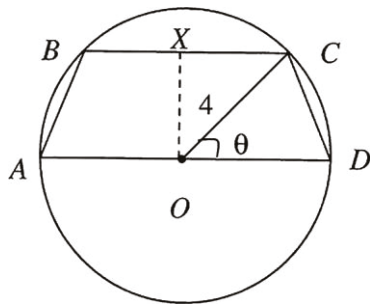
Minimum when  $x = \frac{4}{3}$

**10 a** Let X be the midpoint of BC.

Angle  $XCO = \theta$

Therefore  $XC = 4 \cos \theta$

$BC = 8 \cos \theta$



**b**

$$\text{Area} = \frac{1}{2} \times (8 + 8 \cos \theta) \times 4 \sin \theta$$

$$= 16 \sin \theta (1 + \cos \theta)$$

$$A = 16 \sin \theta (1 + \cos \theta)$$

$$\frac{dA}{d\theta} = 16[\cos \theta (1 + \cos \theta) - \sin^2 \theta]$$

$$= 16[\cos^2 \theta - \sin^2 \theta + \cos \theta]$$

$$= 16[\cos^2 \theta - (1 - \cos^2 \theta) + \cos \theta]$$

$$= 16[2 \cos^2 \theta + \cos \theta - 1]$$

$$\frac{dA}{d\theta} = 0 \text{ implies}$$

$$(2 \cos \theta - 1)(\cos \theta + 1) = 0$$

$$\cos \theta = \frac{1}{2} \text{ or } \cos \theta = -1$$

For the figure to exist  $\cos \theta = \frac{1}{2}$

which implies  $\theta = \frac{\pi}{3}$

Therefore maximum area

$$= 16 \sin \frac{\pi}{3} \left( 1 + \cos \frac{\pi}{3} \right)$$

$$= 16 \times \frac{\sqrt{3}}{2} \times \frac{3}{2} = 12\sqrt{3} \text{ square units}$$

**11** distance =  $\sqrt{(x-3)^2 + y^2}$

$$= \sqrt{x^2 - 6x + 9 + x^4}$$

want minimum distance

$$\frac{d}{dx}(x^4 + x^2 - 6x + 9) = 0$$

$$4x^3 + 2x - 6 = 0$$

$$2x^3 + x - 3 = 0$$

try  $x = 1$

$$2 + 1 - 3 = 0 \quad \checkmark$$

$$(x - 1)(2x^2 + 2x + 3) = 0$$

$$x - 1 = 0, 2x^2 + 2x + 3 = 0$$

$$x = 1 \quad x = \frac{-2 \pm \sqrt{4 - 24}}{4}$$

no solution

$x = 1$  is the only solution

$$y = 1$$

$\therefore (1, 1)$  is the point on  $y = x^2$

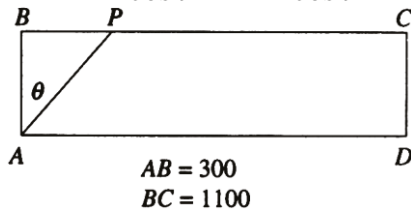
closest to  $(3, 0)$

**12 a**  $\frac{AB}{AP} = \cos \theta$

$$\therefore AP = \frac{300}{\cos \theta}$$

$\therefore$  time taken to run from

$$A \text{ to } P = \frac{300}{\cos \theta} \times \frac{1}{4} = \frac{75}{\cos \theta}$$



**b**

$$PC = BC - BA \tan \theta$$

$$= 1100 - 300 \tan \theta$$

$\therefore$  the time taken to run from  $P$  to  $C$

$$= \frac{1100 - 300 \tan \theta}{5}$$

$$= 220 - 60 \tan \theta$$

**c** Let  $T$  denote the total time

then  $T =$  time to run from  $A$  to

$P +$  time taken to run from  $P$  to  $C$

$$= \frac{75}{\cos \theta} + 220 - 60 \tan \theta$$

$$= \frac{75}{\cos \theta} + 220 - 60 \frac{\sin \theta}{\cos \theta}$$

$$= \frac{75 - 60 \sin \theta}{\cos \theta} + 220$$

**d** The quotient rule gives

$$\frac{dT}{d\theta} = \frac{\cos \theta(-60 \cos \theta) + \sin \theta(75 - 60 \sin \theta)}{\cos^2 \theta}$$

$$\text{(Note: } \frac{d}{d\theta}(220) = 0)$$

$$= \frac{-60 \cos^2 \theta + 75 \sin \theta - 60 \sin^2 \theta}{\cos^2 \theta}$$

$$= \frac{-60[\cos^2 \theta + \sin^2 \theta] + 75 \sin \theta}{\cos^2 \theta}$$

$$= \frac{75 \sin \theta - 60}{\cos^2 \theta}$$

**e**

$$\frac{dT}{d\theta} = 0$$

$$\text{implies } \frac{75 \sin \theta - 60}{\cos^2 \theta} = 0$$

$$\therefore \sin \theta = \frac{60}{75} = \frac{4}{5}$$

$$\therefore \theta = \sin^{-1}\left(\frac{4}{5}\right)$$

(Only the acute angle solution needs to be considered).

$$\theta \approx 53.13^\circ$$

In order to confirm a minimum consider the following

When  $\theta = 60^\circ$ ,

$$\frac{dT}{d\theta} = \frac{75 \sin 60^\circ - 60}{\cos^2 \theta} > 0$$

When  $\theta = 50^\circ$ ,

$$\frac{dT}{d\theta} = \frac{75 \sin 50^\circ - 60}{\cos^2 \theta} < 0$$

$\therefore$  a minimum occurs when

$$\theta = \sin^{-1}\left(\frac{4}{5}\right)$$

**f**

$$\text{When } \sin \theta = \frac{4}{5}, T = \frac{75 - 60 \times \frac{4}{5} + 220}{\frac{3}{5}}$$

$$= 45 + 220$$

$$= 265$$

$\therefore$  minimum time taken is

265 seconds.

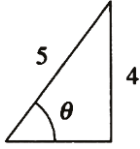


$$\text{If } \sin \theta = \frac{4}{5}, \tan \theta = \frac{4}{3}$$

$$\therefore BP = BA \tan \theta$$

$$= 300 \times \frac{4}{3}$$

$$= 400$$



$P$  is 400 metres from  $B$  for a minimum time.

13

$$N(t) = 50te^{-0.1t}$$

$$N'(t) = 50e^{-0.1t} - 5te^{-0.1t}$$

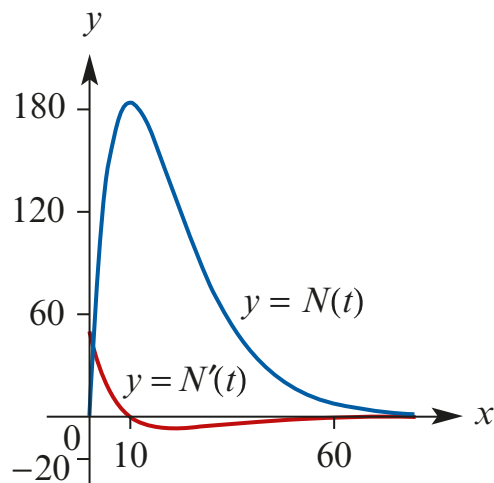
$$= 5e^{-0.1t}(10 - t)$$

$$N'(t) = 0 \Rightarrow t = 10$$

Therefore maximum population when  $t = 10$  The volume of water is 1000 mL

$$N(10) = 500e^{-1}$$

14 a



b Maximum rate of increase

$$= N'(0) = 50$$

$$N''(t) = -5e^{-0.1x} - 0.5e^{-0.1x}(10 - x)$$

$$N''(t) = 0 \Rightarrow t = 20$$

$$N'(20) = -\frac{50}{e^2}$$

Maximum rate of decrease

$$= N'(20) = \frac{50}{e^2}$$

$$15 \quad V(t) = \frac{3}{4} \left( 10t^2 - \frac{t^3}{3} \right) \quad 0 \leq t \leq 20$$

a i  $V(0) = 0$  The volume of water is 0 mL when  $t = 0$

$$\begin{aligned} \text{ii } V(20) &= \frac{3}{4} \left( 10 \times 20^2 - \frac{20^3}{3} \right) \\ &= \frac{3 \times 400}{4} \left( 10 - \frac{20}{3} \right) \\ &= 3 \times 100 \left( \frac{30 - 20}{3} \right) \\ &= 1000 \end{aligned}$$

The volume of water is 1000 mL when  $t = 20$

$$\begin{aligned} \text{b } V'(t) &= \frac{3}{4} \left( 20t - \frac{3t^2}{3} \right) \\ &= \frac{3}{4} (20t - t^2) \end{aligned}$$

c

Domain of  $V(t) = [0, 20]$

$$V(0) = 0 \text{ and } V(20) = 1000$$

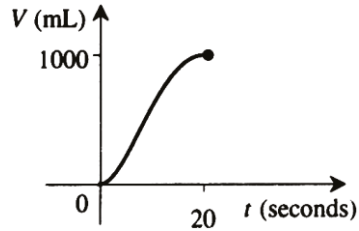
$$V'(t) = 0 \text{ implies } 20t - t^2 = 0$$

$$\therefore t(20 - t) = 0$$

$$\therefore t = 0 \text{ or } t = 20$$

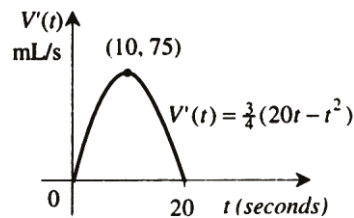
A gradient chart

$t$	$<$	$0$	$<<$	$20$	$>$
sign of $V'(t)$	$-ve$	$0$	$+ve$	$0$	$-ve$
shape	$\backslash$	$-$	$/$	$-$	$\backslash$



reveals a local minimum at  $(0,0)$   
and a local minimum at  $(20,1000)$

- d** The graph of  $V'(t)$  against  $t$  is a parabola with  $t$  intercepts 0 and 20.

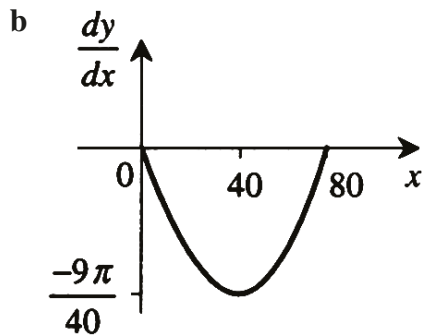


The maximum occurs when  $t = 10$   
and

$$V'(10) = \frac{3}{4}(200 - 100) = 75$$

- e** The flow is greatest after 10 seconds  
and the flow is 75 mL/s.

**16 a**  $\frac{dy}{dx} = -\frac{18\pi}{80} \sin\left(\frac{\pi x}{80}\right) \quad x \in [0, 80]$   
 $= -\frac{9\pi}{40} \sin\left(\frac{\pi x}{80}\right)$



For the graph of  $\frac{dy}{dx}$  against  $x$

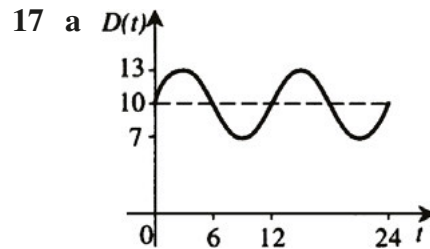
$$\text{amplitude} = \frac{18\pi}{80} = \frac{9\pi}{40}$$

$$\text{period} = 2\pi \div \frac{\pi}{80} = 2\pi \times \frac{80}{\pi} = 160$$

$$\text{When } x = 0, \frac{dy}{dx} = 0$$

$$\text{When } x = 80, \frac{dy}{dx} = \frac{18\pi}{80} \sin\left(\frac{\pi \times 80}{80}\right) = \frac{18\pi}{80} \sin \pi = 0$$

- c** Maximum gradient magnitude occurs  
where  $\sin\left(\frac{\pi x}{80}\right) = \pm 1$   
This occurs when  $x = 40$  for  
 $0 \leq x \leq 80$



The depth of the harbour at time  $t$  is  
given by

$$D(t) = 10 + 3 \sin\left(\frac{\pi t}{6}\right) \quad 0 \leq t \leq 24$$

amplitude = 3

$$\text{period} = 2\pi \div \frac{\pi}{6} = 2\pi \times \frac{6}{\pi} = 12$$

centre  $D = 10$

$$\text{range} = [10 - 3, 10 + 3] = [7, 13]$$

**b**  $D(t) \geq 8.5$

$$\Leftrightarrow 10 + 3 \sin\left(\frac{\pi t}{6}\right) \geq 8.5$$

which is equivalent to

$$3 \sin\left(\frac{\pi t}{6}\right) \geq -1.5$$

$$\sin\left(\frac{\pi t}{6}\right) \geq -\frac{1}{2}$$

Consider

$$\sin\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$$

then

$$\frac{\pi t}{6} = \frac{7\pi}{6}$$

$$\text{or } \frac{11\pi}{6} \quad \text{or } \frac{19\pi}{6} \quad \text{or } \frac{23\pi}{6} \quad \text{or } \dots$$

$$t = 7 \text{ or } 11 \text{ or } 19 \text{ or } 23 \text{ or } \dots$$

From the graph and considering the domain  $[0, 24]$

$$\{t : D(t) \geq 8.5\} = [0, 7] \cup [11, 19] \cup [23, 24]$$

**c** The rate of change of depth is given by the derivative function

$$D'(t) = \frac{3\pi}{6} \cos\left(\frac{\pi t}{6}\right) = \frac{\pi}{2} \cos\left(\frac{\pi t}{6}\right)$$

**i**  $D'(t) = \frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) = 0$

The rate at which the depth is changing when  $t = 3$  is 0 metres/hour.

**ii**  $D'(6) = \frac{\pi}{2} \cos\left(\frac{6\pi}{6}\right) = \frac{\pi}{2} \cos(\pi) =$

$$-\frac{\pi}{2}$$

The rate at which the depth

is changing when  $t = 3$  is  $-\frac{\pi}{2}$  metres/hours.

(This means that the depth is decreasing at a rate of  $\frac{\pi}{2}$  metres/hour).

**iii**  $D'(12) = \frac{\pi}{2} \cos(2\pi) = \frac{\pi}{2}$

The depth is increasing at a rate of  $\frac{\pi}{2}$  metres/hours

**d** The function which describes the rate is

$$D'(t) = \frac{\pi}{2} \cos\left(\frac{\pi t}{6}\right)$$

**i**  $D'(t)$  has a maximum when

$$\cos\left(\frac{\pi t}{6}\right) = 1$$

$$\therefore \frac{\pi t}{6} = 0 \text{ or } 2\pi \text{ or } 4\pi \text{ or } \dots$$

$$t = 0 \text{ or } 12 \text{ or } 24 \text{ or } \dots$$

For the required domain the depth is increasing most rapidly when  $t = 0$  or  $t = 12$  or  $t = 24$

**ii** The depth is decreasing most rapidly when

$$\cos\left(\frac{\pi t}{6}\right) = -1$$

$$\therefore \text{when } \frac{\pi t}{6} = \pi \text{ or } 3\pi \text{ or } 5\pi \text{ or } \dots$$

$$\therefore t = 6 \text{ or } 18 \text{ or } 30 \text{ or } \dots$$

For the required domain the depth is decreasing most rapidly when  $t = 6$  or  $18$

## Solutions to Exercise 10G

1  $f(x) = (x - 1)^2(x - b)$ ,  $b > 1$

**a**  $f(x) = (x^2 - 2x + 1)(x - b)$   
 $= x^3 - (2 + b)x^2 + (1 + 2b)x - b$   
 $f'(x) = 3x^2 - 2(2 + b)x + 1 + 2b$   
 $= (x - 1)(3x - 2b - 1)$

**b**  $f'(x) = 0$ ,  
 $(x - 1)(3x - (1 + 2b)) = 0$   
 $x = 1, x = \frac{1 + 2b}{3}$   
 $f(1) = 1 - 2 - b + 1 + 2b - b = 0$

$$f\left(\frac{1 + 2b}{3}\right) = \left(\frac{2b - 2}{3}\right)^2 \left(\frac{2b + 1 - 3b}{3}\right)$$

$$= \frac{4}{9}(b - 1)^2 \left(\frac{1 - b}{3}\right)$$

$$= \frac{-4}{27}(b - 1)^3$$

co-ords =  $(1, 0)$  &  $\left(\frac{1 + 2b}{3}, \frac{-4(b - 1)^3}{27}\right)$

**c**  $\frac{2b + 1}{3} > 1$  as  $b > 1$  so the other stationary point is a local minimum; hence the point  $(1, 0)$  is always a local maximum.

**d**  $\frac{1 + 2b}{3} = 4$   
 $1 + 2b = 12$   
 $b = \frac{11}{2}$

2  $y = x^4 - 4x^2$

**a**  $\frac{dy}{dx} = 4x^3 - 8x$   
 $\frac{dy}{dx} = 0$ ,  
 $4x(x^2 - 2) = 0$

$$x = 0, \pm\sqrt{2}$$

$$x = 0, y = 0$$

$$(0, 0)$$

$$x = \pm\sqrt{2}$$

$$y = 4 - 4(2)$$

$$= -4$$

$$(\pm\sqrt{2}, -4)$$

**b**  $(x, y) \rightarrow (x + a, y + b)$

$$(0, 0) \rightarrow (a, b)$$

$$(\pm\sqrt{2}, -4) \rightarrow (a \pm \sqrt{2}, b - 4)$$

3 **a**

$$f(x) = ax^3 + bx^2 + cx$$

$$f(1) = 10 \Rightarrow a + b + c = 10 \dots (1)$$

$$f'(x) = 3ax^2 + 2bx + c$$

$$f'(1) = 0 \Rightarrow 3a + 2b + c = 0 \dots (2)$$

Multiply (1) by 2 and subtract from (2)

$$a - c = -20 \Rightarrow a = c - 20$$

Substitute for  $a$  in (1)

$$c - 20 + b + c = 10$$

$$\therefore b = 30 - 2c$$

**b**

$$f'(3) = 0 \Rightarrow 27a + 6b + c = 0$$

$$\therefore 27(c - 20) + 6(30 - 2c) + c = 0$$

$$27c - 540 + 180 - 12c + c = 0$$

$$16c - 360 = 0$$

$$c = \frac{360}{16} = \frac{45}{2}$$

**4**  $f : [0, \infty] \rightarrow R, f(x) = x^2 - ax^3$   
 $a > 0$

**a**  $f'(x) = 2x - 3ax^2$

$$= x(2 - 3ax)$$

$$f'(x) < 0,$$

$$x(2 - 3ax) < 0$$

$$\text{since } x \geq 0, 2 - 3ax < 0$$

$$\Rightarrow x > \frac{2}{3a}$$

$f(x)$  is decreasing when  $x > \frac{2}{3a}$

$$f'(x) > 0,$$

$$x(2 - 3ax) > 0$$

$$\text{since } x \geq 0, 2 - 3ax > 0$$

$$\Rightarrow x < \frac{2}{3a}$$

$$0 < x < \frac{2}{3a}$$

$f(x)$  is increasing when  $0 < x < \frac{2}{3a}$

**b**  $f'\left(\frac{1}{a}\right) = \frac{1}{a}\left(2 - 3\frac{a}{a}\right)$

$$= \frac{-1}{a}$$

$$y = \frac{-x}{a} + c$$

$$x = \frac{1}{a}, y = 0$$

$$0 = \frac{-1}{a^2} + c$$

$$c = \frac{1}{a^2}$$

$$y = \frac{-x}{a} + \frac{1}{a^2}$$

**c**  $y = ax + c$

$$x = \frac{1}{a}, y = 0$$

$$0 = 1 + c$$

$$c = -1$$

$$y = ax - 1$$

**d**  $\text{max. at } x = \frac{2}{3a}$   
 $f\left(\frac{2}{3a}\right) = \frac{4}{9a^2} - \frac{a \times 8}{27a^3} = \frac{1}{a^2}\left(\frac{4}{9} - \frac{8}{27}\right)$

$$= \frac{4}{27a^2}$$

$$\text{range} = \left(-\infty, \frac{4}{27a^2}\right]$$

**5 a i**  $y = (x - 3)^2$

$$\frac{dy}{dx} = 2(x - 3)$$

$$= 2x - 6$$

$$x = a,$$

$$\frac{dy}{dx} = 2a - 6$$

**ii**  $m = 2a - 6$

**b**  $x = a$

$$\begin{aligned} y &= (a - 3)^2 \\ &= a^2 - 6a + 9 \\ &\Rightarrow (a, a^2 - 6a + 9) \end{aligned}$$

**c**  $y = mx + c$

$$\begin{aligned} &= (2a - 6)x + c \\ x &= a, y = (a - 3)^2 \\ (a - 3)^2 &= 2a(a - 3) + c \\ c &= (a - 3)(a - 3 - 2a) \\ c &= (a - 3)(-a - 3) \\ y &= (a - 3)(2x - a - 3) \\ &= 2(a - 3)x - a^2 + 9 \end{aligned}$$

**d**  $y = 0,$

$$2x - a - 3 = 0$$

$$\begin{aligned} 2x &= a + 3 \\ x &= \frac{a + 3}{2} \end{aligned}$$

**6 a**  $f(x) = x^4$

$$\begin{aligned} \Rightarrow f(x + h) &= (x + h)^4 \\ f(1 + h) &= 16 \\ (1 + h)^4 &= 16 \\ 1 + h &= \pm 2 \\ h &= -1 \pm 2 \\ h &= -3, 1 \end{aligned}$$

**b**  $f(x) = x^3$

$$\begin{aligned} \Rightarrow f(ax) &= (ax)^3 \\ f(a) &= 8 \\ a^3 &= 8 \\ a &= 2 \end{aligned}$$

**c**

$$\begin{aligned} y &= ax^4 - bx^3 \\ \frac{dy}{dx} &= 4ax^3 - 3bx^2 \end{aligned}$$

$$x = 1,$$

$$\frac{dy}{dx} = 0$$

$$4a = 3b \dots (1)$$

$$x = 1, y = 16$$

$$16 = a - b$$

$$a = 16 + b \dots (2)$$

$$\text{sub in (1)} \Rightarrow 4(16 + b) = 3b$$

$$64 + 4b = 3b$$

$$b = -64$$

$$\text{sub in (2)} \Rightarrow a = -48$$

**7 a**

$$\begin{aligned} f(x) &= (x - a)^2(x - 1) \\ &= (x^2 - 2ax + a^2)(x - 1) \\ &= x^3 - (2a + 1)x^2 + (a^2 + 2a)x - a^2 \end{aligned}$$

$$f'(x) = 3x^2 - (4a + 2)x + (a^2 + 2a)$$

$$f'(x) = 0,$$

$$3x^2 - (4a + 2)x + (a^2 + 2a) = 0$$

$$x = \frac{4a + 2 \pm \sqrt{4(2a + 1)^2 - 4(3a^2 + 6a)}}{6}$$

$$= \frac{2a + 1 \pm \sqrt{4a^2 + 4a + 1 - 3a^2 - 6a}}{3}$$

$$= \frac{2a + 1 \pm \sqrt{a^2 - 2a + 1}}{3}$$

$$= \frac{(2a + 1) \pm (a - 1)}{3}$$

$$x = a, \frac{a + 2}{3}$$

$$f(a) = 0,$$

$$\begin{aligned}
 f\left(\frac{a+2}{3}\right) &= \left(\frac{a+2}{3} - \frac{3a}{a}\right)^2 \left(\frac{a+2}{3} - \frac{3}{3}\right) \\
 &= \left(\frac{2-2a}{3}\right)^2 \left(\frac{a-1}{3}\right) \\
 &= \frac{4}{9}(1-a)^2 \frac{-1}{3}(1-a) \\
 &= \frac{-4(1-a)^3}{27}
 \end{aligned}$$

$$\text{co-ords} = (a, 0), \left(\frac{a+2}{3}, \frac{-4(1-a)^3}{27}\right)$$

**b** Since  $a > 1$ ,  $\frac{-4(1-a)^3}{27} > 0$ ,  
Hence  $(a, 0)$  is a local minimum and  
 $\left(\frac{a+2}{3}, \frac{-4(1-a)^3}{27}\right)$  is a local  
maximum.

**c i**  $f'(1) = 3 - 4a - 2 + a^2 + 2a$   
 $= a^2 - 2a + 1$   
 $= (a-1)^2$   
 $y = (a-1)^2x + c$   
 $f'(1) = 0$ ,  
 $0 = (a-1)^2 + c$   
 $c = -(a-1)^2$   
 $y = (a-1)^2(x-1)$

**ii**  $f'(a) = 0$   
 $y = c$   
 $f(a) = 0 = c$   
 $y = 0$

**iii**  $f'\left(\frac{a+1}{2}\right)$   
 $= \frac{3(a+1)^2}{4} - (2a+1)(a+1) + (a^2+2a)$   
 $= \frac{3a^2+6a+3-8a^2-12a-8+8a^2+8a}{4}$   
 $= \frac{-a^2+2a-1}{4}$   
 $= \frac{-(a-1)^2}{4}$

$$\begin{aligned}
 f\left(\frac{a+1}{2}\right) &= \left(\frac{a+1}{2} - a\right)^2 \left(\frac{a+1}{2} - 1\right) \\
 &= \frac{1}{4}(1-a)^2 \frac{1}{2}(a-1) \\
 &= \frac{1}{8}(a-1)^3
 \end{aligned}$$

$$y = \frac{-(a-1)^2}{4}x + c$$

$$x = \frac{a+1}{2}, y = \frac{1}{8}(a-1)^3$$

$$\begin{aligned}\frac{1}{8}(a-1)^3 &= \frac{-1}{4}(a-1)^2\left(\frac{a+1}{2}\right) + c \\ c &= \frac{1}{4}(a-1)^2\left(\frac{a-1}{2} + \frac{a+1}{2}\right) \\ &= \frac{1}{4}(a-1)^2 \\ y &= \frac{1}{4}(a-1)^2(-x+a) \\ &= \frac{-1}{4}(a-1)^2(x-a)\end{aligned}$$

$$\begin{aligned}\mathbf{8 a} \quad f'(x) &= (x-1)^2 \frac{d}{dx}(x-b)^2 + (x-b)^2 \frac{d}{dx}(x-1)^2 \\ &= 2(x-1)(x-b)((x-1) + (x-b)) \\ &= 2(x-1)(x-b)(2x-b-1)\end{aligned}$$

$$\mathbf{b} \quad f'(x) = 0,$$

$$x = 1, b, \frac{b+1}{2}$$

$$f(1) = 0, f(b) = 0,$$

$$\begin{aligned}f\left(\frac{b+1}{2}\right) &= \left(\frac{b-1}{2}\right)^2 \left(\frac{1-b}{2}\right)^2 \\ &= \left(\frac{b-1}{2}\right)^4\end{aligned}$$

$$\text{co-ords} = (1, 0) (b, 0) \left(\frac{b+1}{2}, \frac{(b-1)^4}{2}\right)$$

$$\mathbf{c} \quad \frac{b+1}{2} = 2$$

$$b = 1 = 4$$

$$b = 3$$



$$9 \quad a = \frac{1}{486}, \quad b = 0, \quad c = \frac{-1}{161}, \quad d = \frac{1459}{243}$$

$$10 \quad f(x) = ax^4 + bx^3 + cx^2 + dx$$

$$\mathbf{a} \quad f'(x) = 4ax^3 + 3bx^2 + 2cx + d$$

$$f(1) = 1$$

$$1 = a + b + c + d \dots (1)$$

$$f'(1) = 0$$

$$0 = 4a + 3b + 2c + d \dots (2)$$

$$f'(-1) = 4$$

$$4 = a - b + c - d \dots (3)$$

$$(1) - (3) \Rightarrow -3 = 2b + 2d$$

$$b = \frac{-3}{2} - d$$

$$(2) - (1) - (3) \Rightarrow -5 = 2a + 3b + d$$

$$-5 = 2a - \frac{9}{2} - 3d + d$$

$$-\frac{1}{2} = 2a - 2d$$

$$a = d - \frac{1}{4}$$

$$\text{sub in (1)} \Rightarrow 1 = \left(d - \frac{1}{4}\right) + \left(\frac{-3}{2} - d\right)c + d$$

$$1 = -\frac{7}{4} + c + d$$

$$c = \frac{11}{4} - d$$

$$\mathbf{b} \quad f'(4) = 0$$

$$0 = 4a(64) + 3b(16) + 2c(4) + d$$

$$0 = 256a + 48b + 8c + d$$

$$0 = 256\left(\frac{4d-1}{4}\right) + 48\left(\frac{-3-2d}{2}\right)$$

$$+ 8\left(\frac{11-4d}{4}\right) + d$$

$$0 = 256d - 64 - 72 - 48d + 22 - 8d + d$$

$$0 = 201d - 114$$

$$d = \frac{114}{201}$$

$$d = \frac{38}{67}$$

## Solutions to Technology-free questions

**1 a**  $y = x^3 - 8x^2 + 15x$

$$\frac{dy}{dx} = 3x^2 - 16x + 15$$

$$= -1 \text{ (at } x = 4\text{)}$$

For the tangent:

$$y + 4 = -1(x - 4)$$

$$y = -x$$

**b** Tangent meets curve again when

$$x^3 - 8x^2 + 15x = -x$$

$$x^3 - 8x^2 + 16x = 0$$

$$x(x^2 - 8x + 16) = 0$$

$$x(x - 4)^2 = 0$$

Thus  $x = 0$  and then  $y = 0$  ( $x = 4$  corresponds to the given point).

The tangent meets the curve again at the point  $(0, 0)$ .

**2** At  $x = a, y = 3a^2$

$$y = 3x^2$$

$$\frac{dy}{dx} = 6x$$

$$= 6a \text{ (at } x = a\text{)}$$

For the tangent:

$$y - 3a^2 = 6a(x - a)$$

$$y = 6ax - 3a^2$$

$x = 0 : y = -3a^2$ , so the tangent meets the  $y$  axis where  $y = -3a^2$ .

**3**  $y = x^3 - 7x^2 + 14x - 8$

$$\frac{dy}{dx} = 3x^2 - 14x + 14$$

$$= 3 \text{ (at } x = 1\text{)}$$

For the tangent,  $x = 1$  gives  $y = 0$ , so:

$$y - 0 = 3(x - 1)$$

$$y = 3x - 3$$

A parallel tangent has gradient 3, so:

$$\frac{dy}{dx} = 3$$

$$3x^2 - 14x + 14 = 3$$

$$3x^2 - 14x + 11 = 0$$

$$(3x - 11)(x - 1) = 0$$

$$x = 1, \frac{11}{3}$$

The  $x$  coordinate of a second point with the same gradient is  $x = \frac{11}{3}$ .

**4 a** Average rate is given by

$$\frac{A(3) - A(2)}{3 - 2} = \frac{9\pi - 4\pi}{1} = 5\pi$$

**b**  $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

$$= 6\pi \text{ (at } r = 3\text{)}$$

Instantaneous rate is  $6\pi$

**5 a**  $f(x) = 4x^3 - 3x^4$

$$f'(x) = 12x^2 - 12x^3$$

$$= 12x^2(1 - x)$$

$$= 0 \text{ if } x = 0, 1$$

$$x = 0, y = 0; x = 1, y = 1$$

The stationary points have coordinates  $(0, 0)$  and  $(1, 1)$ .

$$x < 0, f'(x) > 0; f'(0) = 0;$$

$0 < x < 1, f'(x) > 0$ ; so  $(0, 0)$  is a stationary point of inflexion.

$$0 < x < 1, f'(x) > 0; f'(1) = 0;$$

$x > 1, f'(x) < 0$ ; so  $(1, 1)$  is a maximum.

- b**  $g(x) = x^3 - 3x - 2$   
 $g'(x) = 3x^2 - 3$   
 $= 3(x+1)(x-1)$   
 $= 0$  if  $x = -1, 1$   
 $x = -1, y = 0; x = 1, y = -4$   
The stationary points have coordinates  $(-1, 0)$  and  $(1, -4)$ .  
 $x < -1, f'(x) > 0; f'(-1) = 0;$   
 $-1 < x < 1, f'(x) < 0;$  so  $(-1, 0)$  is a maximum.  
 $-1 < x < 1, f'(x) < 0; f'(1) = 0;$   
 $x > 1, f'(x) > 0;$  so  $(1, -4)$  is a minimum.

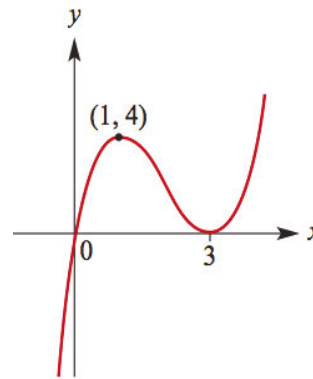
- c**  $h(x) = x^3 - 9x + 1$   
 $g'(x) = 3x^2 - 9$   
 $= 3(x^2 - 3)$   
 $= 3(x + \sqrt{3})(x - \sqrt{3})$   
 $= 0$  if  $x = -\sqrt{3}, \sqrt{3}$   
 $x = -\sqrt{3}, y = 6\sqrt{3} + 1$   
 $x = \sqrt{3}, y = -6\sqrt{3} + 1$   
The stationary points have coordinates  $(-\sqrt{3}, 6\sqrt{3} + 1)$  and  $(\sqrt{3}, -6\sqrt{3} + 1)$ .  
 $x < -\sqrt{3}, f'(x) > 0; f'(-\sqrt{3}) = 0;$   
 $-\sqrt{3} < x < \sqrt{3}, f'(x) < 0;$   
so there is a maximum at  $(-\sqrt{3}, 6\sqrt{3} + 1)$ .  
 $-\sqrt{3} < x < \sqrt{3}, f'(x) < 0;$   
 $f'(\sqrt{3}) = 0;$   
 $x > \sqrt{3}, f'(x) > 0;$   
so there is a minimum at  $(\sqrt{3}, -6\sqrt{3} + 1)$ .

- 6**  $y = x^3 - 6x^2 + 9x = x(x-3)^2$   
 $\frac{dy}{dx} = 3x^2 - 12x + 9$   
 $= 3(x^2 - 4x + 3)$   
 $= 3(x-1)(x-3)$   
 $= 0$  if  $x = 1, 3$   
 $x = 1, y = 4; x = 3, y = 0$   
The stationary points have coordinates  $(1, 4)$  and  $(3, 0)$ . Also it is evident from the factorised form that  $(3, 0)$  is a stationary point of inflexion. Then  $(1, 4)$  must be a maximum.

For the intercepts:

$$y = 0, x = 0, 3$$

The graph is shown below.



- 7**  $\frac{dy}{dx} = (x-1)^2(x-2)$   
 $= 0$  if  $x = 1, 2$   
There are stationary points where  $x = 1$  and  $x = 2$ .  
 $x < 1, \frac{dy}{dx} < 0; \frac{dy}{dx} = 0$  at  $x = 1;$   
 $1 < x < 2, \frac{dy}{dx} < 0;$   
so there is a stationary point of inflexion at  $x = 1$ .  
 $1 < x < 2, \frac{dy}{dx} < 0; \frac{dy}{dx} = 0$  at  $x = 2;$   
 $x > 2, \frac{dy}{dx} > 0;$  so there is a minimum at  $x = 2$ .

$$8 \quad y = x^3 - 3x^2 - 9x + 11$$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

$$= -9 \text{ (at } x = 2)$$

$$\text{Also when } x = 2, y = -11$$

For the tangent:

$$y + 11 = -9(x - 2)$$

$$y = -9x + 7$$

$$9 \quad f(x) = 3 + 6x^2 - 2x^3$$

$$f'(x) = 12x - 6x^2$$

$$= 6x(2 - x)$$

$$= 0 \text{ if } x = 0, 2$$

The graph of the gradient function is an inverted parabola with  $x$  intercepts 0 and 2. So the gradient function is positive if  $0 < x < 2$ , i.e. the graph of  $y = f(x)$  has a positive gradient in the interval  $(0, 2)$ .

10

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$$

$$y = x^3 + x^2 + x - 2 \Rightarrow \frac{dy}{dx} = 3x^2 + 2x + 1$$

So the graphs have the same gradient

when  $3x^2 = 3x^2 + 2x + 1$ , i.e. when

$$2x + 1 = 0, \text{ or } x = -\frac{1}{2}.$$

$$11 \quad f(x) = (x - 1)^{\frac{4}{5}}$$

a The function is differentiable for  $R \setminus \{1\}$ .  $f'(x) = \frac{4}{5}(x - 1)^{-\frac{1}{5}}$

b  $f'(0) = \frac{4}{5}$ ,  $f'(2) = -\frac{4}{5}$   
For the tangent at  $(2, 1)$ :

$$y - 1 = \frac{4}{5}(x - 2)$$

$$y = \frac{4}{5}x - \frac{3}{5}$$

For the tangent at  $(0, 1)$ :

$$y - 1 = -\frac{4}{5}(x - 0)$$

$$y = -\frac{4}{5}x + 1$$

$$c \quad \frac{4}{5}x - \frac{3}{5} = -\frac{4}{5}x + 1$$

$$\frac{8}{5}x = \frac{8}{5}$$

$$x = 1$$

When  $x = 1$ ,  $y = \frac{1}{5}$ , so  $(1, \frac{1}{5})$  is the point of intersection of the tangents.

12 For a sphere of radius  $r$  and volume  $V$ ,

$$V = \frac{4}{3}\pi r^3.$$

$$a \quad \frac{dV}{dr} = 4\pi r^2$$

$$= 64\pi \text{ if } r = 4$$

The rate of increase of volume with respect to the change in radius is  $64\pi \text{ cm}^3/\text{cm}$  when the radius is 4 cm.

$$b \quad \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$= 4\pi r^2 \times 1$$

$$= 4\pi r^2$$

$$= 64\pi \text{ if } r = 4$$

The rate of increase of volume with respect to time is  $64\pi \text{ cm}^3/\text{s}$  when the radius is 4 cm.

(An alternative is to use the initial conditions to express  $r$  in terms of  $t$ , i.e.  $r = 1 + t$ , and then  $V$  in terms of  $t$ ; then differentiate to get the result directly.)

**13**  $x = 0.25e^t$

$$v = \frac{dx}{dt} = 0.25e^t$$

At  $t = 0, 1, 2, 4$ , the velocity  $v$  in m/s is respectively  $0.25, 0.25e, 0.25e^2, 0.25e^4$ .

**14**  $\theta = \frac{1}{4}e^{100t}$

**a**  $\frac{d\theta}{dt} = 25e^{100t} \text{ }^\circ\text{C/s}$

**b**  $\frac{d\theta}{dt} = 25e^5 \left( \text{at } t = \frac{1}{20} \right)$

So the rate of increase is  $25e^5 \text{ }^\circ\text{C/s}$ .

**15**  $y = e^x$

$$\frac{dy}{dx} = e^x$$

$$= e \text{ (at } x = 1) \quad y - e = e(x - 1)$$

$$y - e = ex - e$$

$$y = ex$$

**16**  $D = 50e^{kt}$

**a**  $\frac{dD}{dt} = 50ke^{kt}$

$$= k \times 50e^{kt} = kD$$

Thus  $\frac{dD}{dt} = cD$ , where  $c = k$ .

**b**  $\frac{dD}{dt} = kD = 0.2 \times 100 = 20 \text{ cm/year}$

**17**  $y = e^{3x} + e^{-3x}$

$$\frac{dy}{dx} = 3e^{3x} - 3e^{-3x}$$

$$= 0 \text{ if}$$

$$3e^{3x} = 3e^{-3x}$$

$$e^{6x} = 1$$

$$x = 0$$

When  $x = 0, y = 2$ .

Since  $y \rightarrow \infty$  when  $x \rightarrow \pm\infty$ , it is evident that  $y = 2$  is a minimum.

**18 a**  $y = \log_e x$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$= \frac{1}{e} \text{ (at } x = e)$$

$$y - 1 = \frac{1}{e}(x - e)$$

$$y - 1 = \frac{1}{e}x - 1$$

$$y = \frac{1}{e}x$$

**b**  $y = 2 \sin\left(\frac{x}{2}\right)$

$$\frac{dy}{dx} = \cos\left(\frac{x}{2}\right)$$

$$= \frac{1}{\sqrt{2}} \left( \text{at } x = \frac{\pi}{2} \right)$$

$$y - \sqrt{2} = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{2}\right)$$

$$y = \frac{1}{\sqrt{2}}x - \frac{\pi}{2\sqrt{2}} + \sqrt{2}$$

**c**  $y = \cos x$

$$\frac{dy}{dx} = -\sin x$$

$$= 1 \left( \text{at } x = \frac{3\pi}{2} \right)$$

$$y = (1) \left( x - \frac{3\pi}{2} \right)$$

$$y = x - \frac{3\pi}{2}$$

**d**  $y = \log_e(x^2)$

$$\frac{dy}{dx} = \frac{2}{x}$$

$$= -\frac{2}{\sqrt{e}} \left( \text{at } x = -\sqrt{e} \right)$$

$$y - 1 = -\frac{2}{\sqrt{e}}(x + \sqrt{e})$$

$$y = -\frac{2}{\sqrt{e}}x - 1$$

## Solutions to multiple-choice questions

1 A  $y = x^2 - x - 5$

$$\frac{dy}{dx} = 2x - 1$$

Gradient of tangent equation = 4

$$\therefore \frac{dy}{dx} = 4$$

$$\therefore 4 = 2x - 1$$

$$x = \frac{5}{2}$$

$$y = \left(\frac{5}{2}\right)^2 - \frac{5}{2} - 5$$

$$y = -\frac{5}{4}$$

Sub  $x$  and  $y$  values into  $y = 4x + c$

$$-\frac{5}{4} = 10 + c$$

$$c = -\frac{45}{4}$$

2 E Tangent of  $y = x^4$  at  $x = 1$

$$\frac{dy}{dx} = 4x^3$$

$$\frac{dy}{dx} = 4$$

Equation of tangent:

$$y = 4x + c$$

At  $x = 1, y = 1$

$$1 = 4 + c$$

$$c = -3$$

$$\therefore y = 4x - 3$$

3 E Since the gradient changes from negative to positive at point  $a$ , this is a local minimum.

Since the gradient remains the same at, before and after point  $b$ , this is a stationary point of inflection.

4 E The graph of the second function is obtained from the graph of the first function by this sequence of transformations:

- (1) a reflection in the  $x$ -axis
- (2) a dilation of factor 2 from the  $x$ -axis
- (3) a dilation of factor 2 from  $y$ -axis
- (4) a translation of  $k$  units vertically up

The point  $(0, 0)$  transforms to  $(0, k)$  and is now a maximum due to the reflection (the dilation leave no effect).

The maximum point  $a, f(a)$  of the original graph transforms as follows:

- (1)  $(a, -f(a))$ ; local minimum
- (2)  $(a, -2f(a))$ ; local minimum
- (3)  $(2a, -2f(a))$ ; local minimum
- (4)  $(2a, -2f(a) + k)$  local minimum

5 B  $f(x) = x^3 - x^2 - 1$

$$f'(x) = 3x^2 - 2x$$

Stationary points occur when

$$f'(x) = 0$$

$$3x^2 - 2x = 0$$

$$x(3x - 2) = 0$$

Using the null factor theorem:

$$x = 0 \text{ and } x = \frac{2}{3}$$

6 C As it is a local minimum the gradient of the tangent is 0. Therefore it is a horizontal line which goes through the point  $(2, 4)$

$$y = mx + c$$

$$m = 0$$

$$\therefore y = 4$$

**7 B**  $V = -10x(2x^2 - 6)$   
 $V = -20x^3 + 60x$   
 $\frac{dV}{dx} = -60x^2 + 60$   
 $0 = -60x^2 + 60$   
 $60x^2 = 60$   
 $x = \pm 1$   
 When  $x = -1, V = -80$   
 $V \neq -80$   
 $\therefore$  Maximum volume occurs when  
 $x = 1$

**8 B**  $y = x^2$   
 $\frac{dy}{dx} = 2x$   
 At  $x = a$ , gradient of tangent =  $2a$   
 Normal at  $x = a : -\frac{1}{2a}$   
 $\therefore y = -\frac{1}{2a}x + c$   
 At  $x = a, y = a^2$   
 $\therefore a^2 = -\frac{1}{2a}a + c$   
 $\therefore c = a^2 + \frac{1}{2}$   
 Equation of normal:  
 $y = -\frac{1}{2a}x + \frac{1}{2} + a^2$

**9 D**  $f : R \rightarrow R, f(x) = e^x - ex$   
 $f'(x) = e^x - e$   
 $f'(x) = 0$   
 $e^x - e = 0$   
 $e^x = e$   
 $x = 1$   
 Turning point occurs at  $x = 1$   
 Sub into  $f(x)$  to find  $y$  coordinate:  
 $y = e - e$   
 $y = 0$   
 $\therefore (1, 0)$

**10 E**  $y = e^{ax}$   
 Tangent at point  $\left(\frac{1}{a}, e\right)$   
 $\frac{dy}{dx} = ae^{ax}$   
 At  $x = \frac{1}{a}$   
 $\frac{dy}{dx} = ae^{\frac{a}{a}}$   
 $\frac{dy}{dx} = ae$   
 Equation of tangent:  $y = aex + c$   
 Sub in point  $\left(\frac{1}{a}, e\right)$   
 $e = ae\frac{1}{a} + c$   
 $e = e + c$   
 $c = 0$   
 $\therefore$  equation of tangent:  
 $y = aex$

**11 A**  $N = 4000e^{0.2t}$   
 $\frac{dN}{dt} = 800e^{0.2t}$   
 When  $t = 3$   
 $\frac{dN}{dt} = 800e^{0.6} \approx 1458$

**12 E**  $y = x^2 \cos(5x)$   
 Tangent at  $x = \pi$   
 Using product rule to find derivative:  
 $\frac{dy}{dx} = 2x \cos(5x) - 5x^2 \sin(5x)$   
 Gradient of tangent:  
 $\frac{dy}{dx} = 2\pi \cos(5\pi) - 5\pi^2 \sin(5\pi)$   
 $\frac{dy}{dx} = -2\pi$

**13 B**  $y = e^{-x} - 1$   
 Point where equation crosses the  
 y-axis:  
 $x = 0, y = 0$  coordinate:  $(0, 0)$



$$\frac{dy}{dx} = -e^{-x}$$

Gradient of tangent at  $x = 0$ :

$$\frac{dy}{dx} = -1$$

Equation of tangent:

$$y = -x + c$$

Sub in point  $(0, 0)$ :

$$c = 0$$

$\therefore$  Equation of tangent:

$$y = -x$$

**14 C**

$$f(x) = e^{ax} - \frac{ax}{e}$$

$$f'(x) = ae^{ax} - \frac{a}{e}$$

$$= 0 \text{ if}$$

$$ae^{ax} = \frac{a}{e}$$

$$e^{ax} = \frac{1}{e} = e^{-1}$$

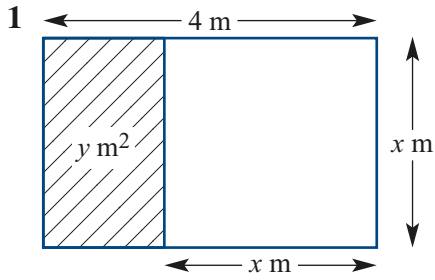
$$ax = -1$$

$$x = -\frac{1}{a}$$

$$f\left(-\frac{1}{a}\right) = e^{-1} - \frac{-1}{e} = \frac{2}{e}$$

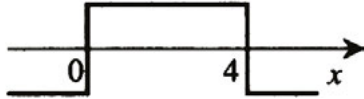
The coordinates of the turning point are  $\left(-\frac{1}{a}, \frac{2}{e}\right)$ .

## Solutions to extended-response questions



a Shaded area =  $4x - x^2$   
 i.e.  $y = 4x - x^2$

b As  $y > 0$ ,  $4x - x^2 > 0$   
 i.e.  $x(4 - x) > 0$



$\therefore y > 0$  for  $0 < x < 4$

The possible values of  $x$  are  $0 < x < 4$ .

c  $\frac{dy}{dx} = 4 - 2x$

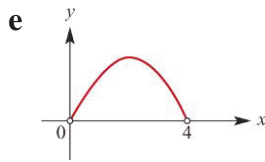
$\frac{dy}{dx} = 0$  implies  $x = 2$

Note:  $y = 4x - x^2$  is a quadratic with negative coefficient of  $x^2$ .

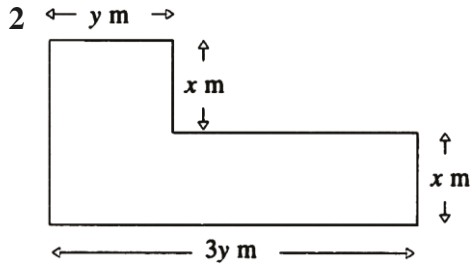
When  $x = 2$ ,  $y = 8 - 4 = 4$ .

The maximum value of  $y$  is 4 and this occurs when  $x = 2$ .

d  $y = 4x - x^2$  is a quadratic with negative coefficient of  $x^2$   
 or gradient to the left of  $x = 2$  is positive and to the right negative.



f From the graph the possible values for  $y$  are  $0 < y < 4$ .



a  $A = xy + 3xy = 4xy$

b Perimeter = 48

$$\therefore 48 = 6y + 4x$$

$$y = \frac{48 - 4x}{6}$$

$$= 8 - \frac{2}{3}x$$

c  $A = 4xy$

$$= 4x\left(8 - \frac{2}{3}x\right)$$

$$= 32x - \frac{8x^2}{3}$$

d  $\frac{dA}{dx} = 32 - \frac{16x}{3}$

$\frac{dA}{dx} = 0$  implies  $x = \frac{96}{16} = 6$ . Maximum as quadratic with negative coefficient of  $x^2$

When  $x = 6$ ,  $y = 8 - \frac{2}{3} \times 6 = 4$

e When  $x = 6$

$$A = 32 \times 6 - \frac{8}{3} \times 36$$

$$= 96$$

The maximum area is  $96 \text{ m}^2$ .

3 a Cost is  $(12 + 0.008x)$  dollars per kilometre plus \$14.40 per hour for the driver, where  $x$  is the speed of the truck in km/h

i Cost per kilometre for truck travelling at 40 km/h

$$= (12 + 0.008 \times 40) + 14.40 \times \frac{1}{40}$$

$$= 12.68$$

i.e. the cost per kilometre is \$12.68.

ii Cost per kilometre for truck travelling at 64 km/h

$$= (12 + 0.008 \times 64) + \frac{1}{64} \times 14.40$$

$$= 12.737$$

i.e. the cost per kilometre is \$12.74.

b Let  $C$  be the cost per kilometre.

$$C = (12 + 0.008x) + \frac{14.40}{x}$$

$$= 12 + 0.008x + \frac{14.40}{x}$$

c To sketch the graph we first differentiate to determine turning points.

$$\text{For } C = 12 + 0.008x + \frac{14.40}{x}$$

$$\frac{dC}{dx} = 0.008 - \frac{14.40}{x^2}$$

and stationary points occur for  $\frac{dC}{dx} = 0$ .

This implies

$$0.008x^2 = 14.40$$

$$x^2 = 1800$$

$$x = 30$$

$$\approx 42.426$$

A sign chart is used to determine the nature of the stationary point.

	$< 30\sqrt{2}$	$30\sqrt{2}$	$> 30\sqrt{2}$
sign $f'(x)$	-ve	0	+ve
shape	\	-	/

$\therefore$  A minimum occurs where  $x = 30\sqrt{2}$ .

When  $x = 30\sqrt{2}$

$$C = 12 + 0.008 \times 30\sqrt{2} + \frac{14.40}{30\sqrt{2}}$$

$$= 12 + 0.24\sqrt{2} + \frac{0.48}{\sqrt{2}}$$

$$= 12 + 0.24\sqrt{2} + 0.24\sqrt{2}$$

$$= 12 + 0.48\sqrt{2}$$

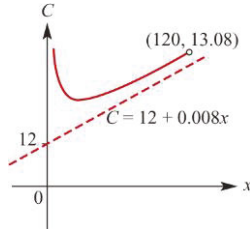
$$\approx 12.679$$

$\therefore$  minimum at  $(30\sqrt{2}, 12 + 0.48\sqrt{2})$

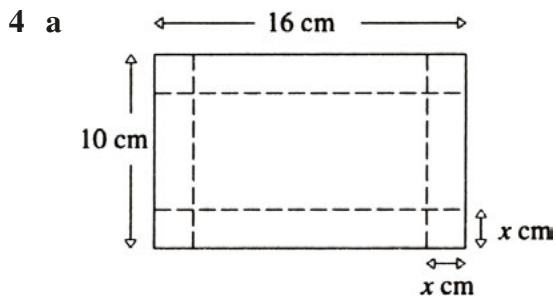
When  $x = 120$

$$\begin{aligned}
 C &= 12 + 0.008 \times 120 + \frac{14.40}{120} \\
 &= 12 + 0.96 + 0.12 \\
 &= 13.08
 \end{aligned}$$

It is also observed that as  $x \rightarrow 0$ ,  $C \rightarrow \infty$  and that as  $x$  large, the graph gets close to that of  $c = 12 + 0.08x$ .



**d** From the above, the truck should be driven at  $30\sqrt{2} \approx 42.43$  km/hr.



$$\begin{aligned}
 \therefore \text{length of box} &= (16 - 2x) \text{ cm} \\
 \text{width of box} &= (10 - 2x) \text{ cm} \\
 \text{height of box} &= x \text{ cm} \\
 \text{Volume} &= \text{length} \times \text{width} \times \text{height} \\
 &= (16 - 2x)(10 - 2x)x \\
 &= 4(8 - x)(5 - x)x \\
 &= 4(40 - 13x + x^2)x \\
 &= 4(x^3 - 13x^2 + 40x) \text{ cm}^3
 \end{aligned}$$

**b** All dimensions are positive.  
 $10 - 2x > 0$  and  $16 - 2x > 0$  and  $x > 0$   
 $\therefore x < 5$  and  $x < 8$  and  $x > 0$   
 $\therefore 0 < x < 5$

c Let  $V = 4(x^3 - 13x^2 + 40x)$

$$\frac{dV}{dx} = 4(3x^2 - 26x + 40)$$

$$\frac{dV}{dx} = 0 \text{ implies } 4(3x^2 - 26x + 40) = 0$$

$$\therefore 3x^2 - 26x + 40 = 0$$

$$\therefore (3x - 20)(x - 2) = 0$$

$$\therefore x = \frac{20}{3} \text{ or } x = 2$$

but  $0 < x < 5 \therefore x = 2$

d A gradient chart reveals there is a maximum when  $x = 2$ :

	$< 2$	$2$	$> 2$
sign $\frac{dV}{dx}$	+ve	0	-ve
shape	/	-	\

When  $x = 2$ ,

$$16 - 2x = 12$$

$$10 - 2x = 6$$

$\therefore$  The dimensions of the box for maximum volume are:

2 cm, 6 cm, 12 cm

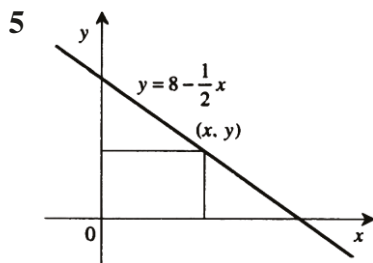
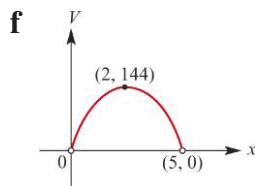
e Maximum when  $x = 2$

$$\therefore V_{\max} = 4(5 - 2)(8 - 2)2$$

$$= 4 \times 3 \times 6 \times 2$$

$$= 144$$

The maximum volume is  $144 \text{ cm}^3$



Area of rectangle = length  $\times$  width

Let  $A$  denote the area.

Let  $x$  denote the width.

Let  $y$  denote the length.

$$A = xy$$

$$= x\left(8 - \frac{x}{2}\right)$$

$$= 8x - \frac{x^2}{2}$$

Consider the derivative of  $A$  with respect to  $x$ .

$$\frac{dA}{dx} = 8 - x$$

$$\frac{dA}{dx} = 0 \text{ implies } x = 8$$

As  $A$  is a quadratic function with negative coefficient of  $x^2$ , a maximum occurs where  $x = 8$ .

$$\text{When } x = 8, A = 8 \times 8 - \frac{8^2}{2} = 32$$

$\therefore$  Maximum area = 32 square units.

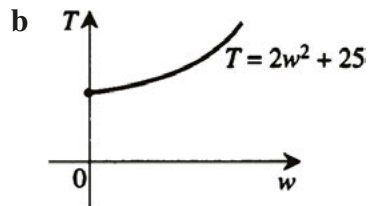
**6 a**  $T = k + 2w^2$

When  $w = 5, T = 75$

$$\therefore 75 = k + 50$$

i.e.  $k = 25$

So:  $T = 2w^2 + 25$



**c** Average time in seconds per kg =  $\frac{T}{w} = \frac{25}{w} + 2w$

**d i** Let  $A$  be the average time.

$$A = \frac{25}{w} + 2w$$

Minimum occurs when  $\frac{dA}{dw} = 0$ .

$$\frac{dA}{dw} = -25w^{-2} + 2 = 0$$

which implies  $w^2 = \frac{25}{2}$

$$\text{i.e. } w = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

A gradient chart confirms minimum:

$w$	$< \frac{5}{\sqrt{2}}$	$\frac{5}{\sqrt{2}}$	$> \frac{5}{\sqrt{2}}$
sign of $\frac{dA}{dw}$	-ve	0	+ve
shape	/	-	\

$\therefore \frac{5\sqrt{2}}{2} \text{ kg} \approx 3.54 \text{ kg}$  yields the minimum average machinery time.

**ii** When  $w = \frac{5\sqrt{2}}{2}$

$$A = \frac{2 \times 5\sqrt{2}}{2} + \frac{25}{\left(\frac{5\sqrt{2}}{2}\right)}$$

$$= 5\sqrt{2} + 5\sqrt{2}$$

$$= 10\sqrt{2}$$

$\therefore$  minimum average machine time is  $10\sqrt{2} \approx 14.14$  seconds.

**7 a** Area of bottom =  $x^2 + x^2 = 2x^2$

$$\text{Area of top} = x^2$$

$$\text{Area of sides} = xh + xh + xh + xh = 4xh$$

$$\therefore \text{total area} = 4xh + 3x^2$$

$$\text{i.e. } C = 4xh + 3x^2$$



**b** Volume  $V = x^2h$

For Volume =  $12 \text{ m}^3$

$$12 = x^2h$$

i.e.  $h = \frac{12}{x^2}$

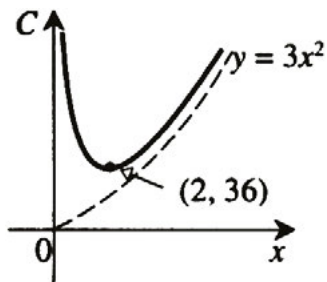
$$\begin{aligned} \text{and } C &= 4x\left(\frac{12}{x^2}\right) + 3x^2 \\ &= \frac{48}{x} + 3x^2 \end{aligned}$$

**c** It is preferable to complete **d** before sketching the graph.

**d i**  $\frac{dC}{dx} = -\frac{48}{x^2} + 6x$   
 $\frac{dC}{dx} = 0$  implies  $-\frac{48}{x^2} + 6x = 0$

which implies  $6x = \frac{48}{x^2}$

$\therefore x^3 = 8$  and  $x = 2$



The gradient chart is as shown:

$x$	$< 2$	$2$	$> 2$
$\text{sign } \frac{dC}{dx}$	-ve	0	+ve
shape	\	-	/

$\therefore$  a minimum when  $x = 2$  When  $x = 2$ , the dimensions are  $2 \text{ m}, 2 \text{ m}, 3 \text{ m}$  ( $h = \frac{12}{2^2}$ )

**ii** When  $x = 2$

$$C = 12 + \frac{48}{2} = 12 + 24 = 36$$

$\therefore$  The minimum area is  $36 \text{ m}^2$

8 Let the base have dimension  $x$  m by  $x$  m and  $h$  m be the height of the tank.

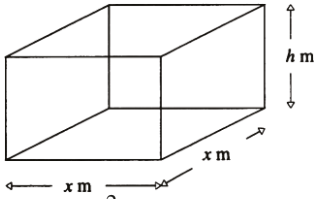
The volume of a cuboid = length  $\times$  width  $\times$  height

$$= x \times x \times h$$

$$= x^2 h$$

For this tank volume =  $500 \text{ m}^3$

$$\therefore x^2 h = 500$$



Let  $A \text{ m}^2$  be the area of sheet metal required.

$$A = x^2 + 4xh$$

(Note: The tank is open.)

From equation (1)

$$h = \frac{500}{x^2}$$

$$\therefore A = x^2 + 4x \left( \frac{500}{x^2} \right)$$

$$= x^2 + \frac{2000}{x}$$

Differentiating to find a minimum:

$$\frac{dA}{dx} = 2x - \frac{2000}{x^2}$$

$$\therefore \frac{dA}{dx} = 0 \text{ implies } x^3 = 1000$$

$$\therefore x = 10$$

The gradient chart shows a minimum occurs when  $x = 10$ .

$x$	$<$	10	$>$
sign of $\frac{dA}{dx}$	-ve	0	+ve
shape	\	-	/

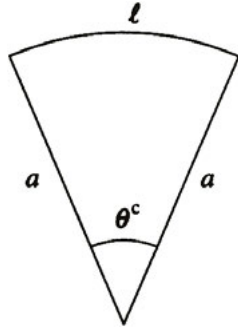
When  $x = 10, h = 5$

Therefore the dimensions necessary for a minimum surface area are  $10 \text{ m} \times 10 \text{ m} \times 5 \text{ m}$

9 a The area of a sector  $A = \frac{1}{2} r^2 \theta$

In this case  $r = a$

$$\therefore A = \frac{1}{2} a^2 \theta$$



**b** The length of the wire =  $a + a + \ell$

$$= 2a + \ell$$

where  $\ell = a\theta$

Therefore as the wire is 1 m = 100 cm in length

$$100 = 2a + a\theta$$

$$\therefore 100 = a(\theta + 2)$$

$$\text{i.e. } a = \frac{100}{\theta + 2}$$

$$\therefore A = \frac{1}{2} \left( \frac{100}{\theta + 2} \right)^2 \theta$$

**c** Differentiating to find maximum

$$A = \frac{10^4}{2} \left( \frac{1}{\theta + 2} \right)^2 \theta$$

Using the product rule

$$\frac{dA}{d\theta} = 5000 \left[ \frac{1}{(\theta + 2)^2} - \frac{2\theta}{(\theta + 2)^3} \right]$$

$$\frac{dA}{d\theta} = 0 \text{ implies } \frac{1}{(\theta + 2)^2} = \frac{2\theta}{(\theta + 2)^3}$$

$$\therefore (\theta + 2)^3 - 2\theta(\theta + 2)^2 = 0$$

$$\therefore (\theta + 2)^2[\theta + 2 - 2\theta] = 0$$

$$\therefore \theta = 2 \text{ or } \theta = -2$$

but  $\theta > 0 \therefore \theta = 2$

The gradient chart will show a maximum i.e.  $A$  is maximum when  $\theta = 2$

**d** When  $\theta = 2, A = \frac{1}{2} \left( \frac{100}{2 + 2} \right)^2 \times 2$

$$= \frac{1}{2} \times 25^2 \times 2$$

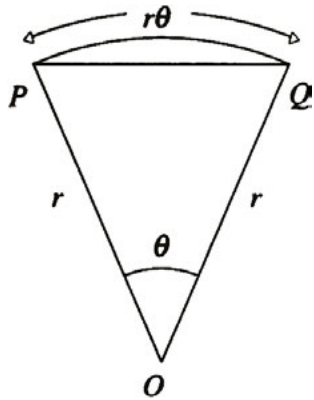
$$= 625$$

The maximum area is  $625 \text{ cm}^2$

**10 a**  $L = 2r + r\theta$

$$\therefore \theta = \frac{1}{r}(L - 2r)$$

$$\begin{aligned} \text{Area of sector} &= \frac{1}{2}r^2\theta \\ &= \frac{1}{2}r^2\left(\frac{1}{r}(L - 2r)\right) \\ &= \frac{1}{2}r(L - 2r) \\ &= \frac{1}{2}rL - r^2 \end{aligned}$$



**b i** The area of the sector  $A = \frac{1}{2}rL - r^2$

$$\therefore \frac{dA}{dr} = \frac{1}{2}L - 2r$$

and  $\frac{dA}{dr} = 0$  implies  $\frac{1}{2}L - 2r = 0$ , so  $r = \frac{L}{4}$ .

**ii** Substituting  $r = \frac{L}{4}$  in 1 gives

$$\theta = \frac{1}{\frac{L}{4}}\left(L - 2 \times \frac{L}{4}\right)$$

$$= \frac{4}{L}\left(L - \frac{L}{2}\right)$$

$$= \frac{4}{L} \times \frac{L}{2} = 2$$

**iii** A stationary point occurs when  $r = \frac{L}{4}$ .

If  $r < \frac{L}{4}$ ,  $\frac{dA}{dr} > 0$

$r > \frac{L}{4}$ ,  $\frac{dA}{dr} < 0$  (gradients considered locally)

So the stationary point is a maximum.

c Area of sector =  $\frac{1}{2}r^2\theta$

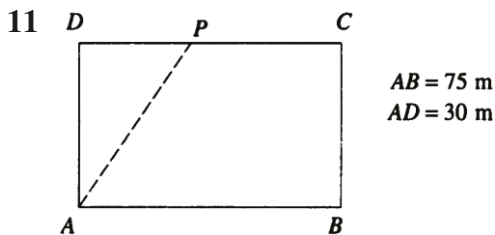
When  $\theta = 2, r = \frac{L}{4}$

$$\begin{aligned}\therefore \text{Area of sector} &= \frac{1}{2} \times \frac{L^2}{16} \times 2 \\ &= \frac{L^2}{16}\end{aligned}$$

$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2}r^2 \sin 2 \\ &= \frac{1}{2} \times \frac{L^2}{16} \sin 2 \\ &= \frac{L^2 \sin 2}{32}\end{aligned}$$

$$\begin{aligned}\frac{\text{Area of triangle}}{\text{Area of sector}} &= \frac{L^2 \sin 2}{32} \div \frac{L^2}{16} \\ &= \frac{L^2 \sin 2}{32} \times \frac{16}{L^2} \\ &= \frac{\sin 2}{2} = 0.4546 \dots\end{aligned}$$

$\therefore$  Area of triangle  $\approx 45.5\%$  area of sector



a Let  $T$  be the total time in seconds and  $DP = x$  (m) (Note: the position of  $P$  varies)

$T =$  time to swim to  $AP +$  time to run  $PC +$  time to get out

$$\text{time to swim } AP = \frac{AP}{\text{speed of swimming}} = \frac{\sqrt{900 + x^2}}{1} \text{ (Pythagoras' Theorem)}$$

$$\text{time to run } PC = \frac{75 - x}{1\frac{2}{3}} = \frac{3}{5}(75 - x)$$

time to get out = 2

$$T = \sqrt{900 + x^2} + \frac{3}{5}(75 - x) + 2$$

$$\mathbf{b} \quad \frac{d}{dx}(\sqrt{900+x^2}) = 2x \times \frac{1}{2}(900+x^2)^{-\frac{1}{2}} \text{ (Chain rule)}$$

$$\therefore \frac{dT}{dx} = x(x^2+900)^{-\frac{1}{2}} - \frac{3}{5}$$

**c i** Minimum occurs when  $\frac{dT}{dx} = 0$

$$\frac{dT}{dx} = 0 \text{ implies } x(x^2+900)^{-\frac{1}{2}} = \frac{3}{5}$$

$$\therefore \frac{x}{x(x^2+900)^{\frac{1}{2}}} = \frac{3}{5}$$

$$\text{and } 5x = 9(x^2+900)^{\frac{1}{2}}$$

Squaring both sides yields

$$25x^2 = 9x^2 + 8100$$

$$\therefore 6x^2 = 8100$$

$$\therefore x^2 = \frac{8100}{6}$$

$$\text{and } x = \frac{90}{4} \text{ (Note: } x \geq 0 \text{ and so positive root is chosen)}$$

$$= 22\frac{1}{2}$$

A gradient chart reveals a local minimum when  $x = 22\frac{1}{2}$

**ii** The minimum time occurs when  $x = \frac{90}{4}$

$$\text{When } x = \frac{90}{4}, T = \sqrt{900 + \left(\frac{90}{4}\right)^2} + \frac{3}{5}\left(75 - \frac{90}{4}\right) + 2$$

$$= \sqrt{\frac{22500}{16}} + 3\left(15 - \frac{18}{4}\right) + 2$$

$$= \frac{150}{4} + 3 \times \frac{42}{4} + 2$$

$$= 71$$

The minimum time is 71 seconds.

**d** If the boy runs from  $A$  to  $D$  and then from  $D$  to  $C$

$$\text{time} = \frac{30}{1\frac{2}{3}} + \frac{75}{1\frac{2}{3}}$$

$$= \frac{30}{\frac{5}{3}} + \frac{75}{\frac{5}{3}}$$

$$= \frac{3}{5} \times \frac{105}{1} = 63$$

It takes 63 seconds to run from  $A$  to  $D$  and then from  $D$  to  $C$ .

**12 a** For  $y = e^x$ ,  $\frac{dy}{dx} = e^x$

When  $x = 1$ ,  $y = e$  and  $\frac{dy}{dx} = e$

Therefore the equation of the tangent is given by

$$y - e = e(x - 1)$$

i.e.  $y - e = ex - e$

$\therefore y = ex$  is the equation of the tangent.

**b** For  $y = e^{2x}$ ,  $\frac{dy}{dx} = 2e^{2x}$

when  $x = \frac{1}{2}$ ,  $y = e + \frac{dy}{dx} = 2e$

The equation of the tangent at  $(\frac{1}{2}, e)$  is given by

$$y - e = 2e\left(x - \frac{1}{2}\right)$$

i.e.  $y - e = 2ex - e$

$\therefore y = 2ex$  is the equation of the tangent at  $(\frac{1}{2}, e)$

**c** For  $y = e^{kx}$ ,  $\frac{dy}{dx} = ke^{kx}$

when  $x = \frac{1}{k}$ ,  $y = e$  and  $\frac{dy}{dx} = ke$

The equation of the tangent is

$$y - e = ke\left(x - \frac{1}{k}\right)$$

i.e.  $y = kex$

**d** Consider the equation of the tangent at the point  $(a, e^{ka})$

which passes through the origin for the curve with equation  $y = e^{kx}$

$$\frac{dy}{dx} = ke^{kx} \text{ and at } (a, e^{ka}), \frac{dy}{dx} = ke^{ka}$$

$\therefore$  The equation of the tangent is

$$y - 0 = ke^{ka}(x - 0)$$

i.e.  $y = ke^{ka}x$

Also the gradient of the tangent can be determined as the gradient of a straight line joining the point  $(a, e^{ka})$  and  $(0, 0)$

$$\text{Gradient} = \frac{e^{ka} - 0}{a - 0} = \frac{e^{ka}}{a}$$

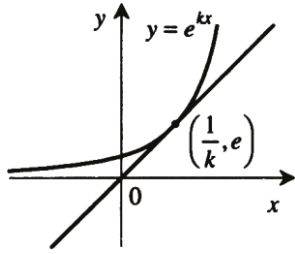
$$\therefore ke^{ka} = \frac{e^{ka}}{a}$$

$$\therefore a = \frac{1}{k}$$

and  $e^{ka} = e$

$\therefore$  Equation of tangent is  $y = kex$

e



Solving  $e^{kx} = x$  is equivalent to solving the pair of equations  
 $y = e^{kx}$   
 $y = x$   
 simultaneously.

i There is a single solution to the equation  $e^{kx} = x$  if  $y = x$  is a tangent to the curve  $y = e^{kx}$ .

From (d) this occurs only if  $ke = 1$  i.e. if  $k = \frac{1}{e}$  for  $k > 0$ .

There is always a unique real root for  $k \leq 0$  (check the graph of  $y = e^{kx}$  for  $k \leq 0$ )

ii For no real roots, there are no solutions to the pair of equations

$$y = e^{kx}$$

and  $y = x$

For  $k = \frac{1}{e}$ ,  $y = x$  is a tangent.

For  $k > \frac{1}{e}$ , the curve  $y = e^{kx}$  does not meet the line  $y = x$ .

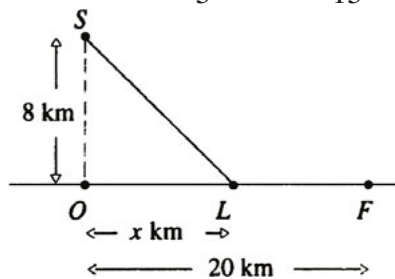
13 a Distance  $SL = \sqrt{64 + x^2}$  (Pythagoras' Theorem)

Time taken = time taken for  $SL$  + time taken for  $LF$

$$= \frac{SL}{\text{rowing speed}} + \frac{LF}{\text{running speed}}$$

$$= \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$

$$\therefore T(x) = \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$



b Differentiating to find minimum



$$T'(x) = \frac{x}{5}(64 + x^2)^{-\frac{1}{2}} - \frac{1}{15}$$

$$T'(x) = 0$$

$$\text{implies } \frac{x}{5(64 + x^2)^{\frac{1}{2}}} = \frac{1}{15}$$

$$\therefore 15x = 5(64 + x^2)^{\frac{1}{2}}$$

Squaring both sides yields

$$225x^2 = 25(64 + x^2)$$

$$\text{i.e. } 200x^2 = 25 \times 64$$

$$\text{and } x = \frac{5 \times 8}{10 \sqrt{2}} \text{ (Note: positive root is chosen as } x \geq 0)$$

$$= \frac{4 \sqrt{2}}{2} = 2 \sqrt{2}$$

A gradient chart reveals a minimum.

When  $x = 2 \sqrt{2}$

$$\begin{aligned} T &= \frac{\sqrt{64 + 8}}{5} + \frac{20 - 2 \sqrt{2}}{15} \\ &= \frac{6 \sqrt{2}}{5} + \frac{4}{3} - \frac{2 \sqrt{2}}{15} \\ &= \frac{18 \sqrt{2} - 2 \sqrt{2} + 20}{15} \\ &= \frac{16 \sqrt{2} + 20}{15} \\ &\approx 2.84 \end{aligned}$$

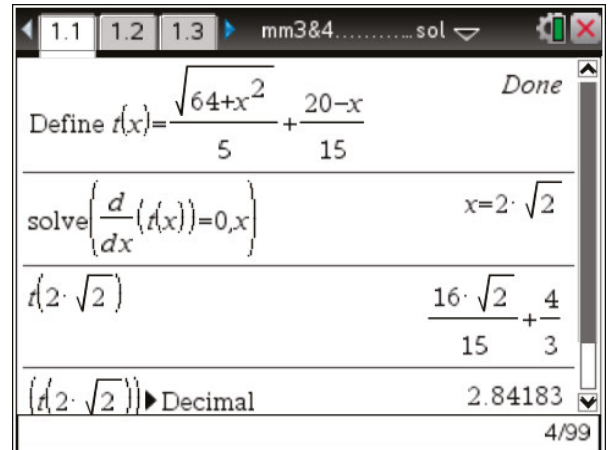
The minimum time is  $\frac{16 \sqrt{2} + 20}{15}$  hours  $\approx 2.84$  hours  $\approx 2$  hours 50 minutes 31 seconds

### Graphic calculator techniques for question 13

In a **Calculator** page, define the function.

To find the  $x$ -value where the minimum occurs, solve the derivative equalling zero.

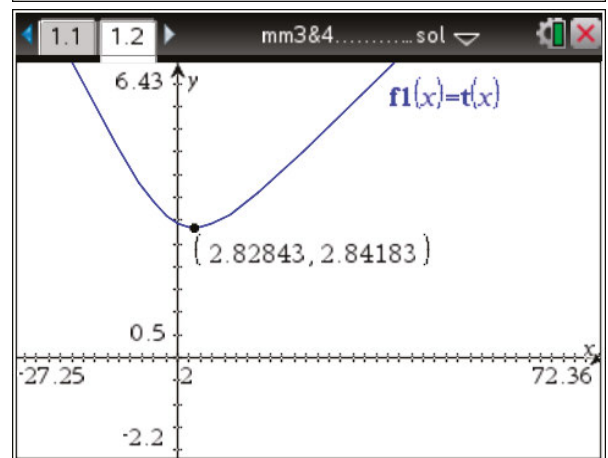
Hence minimum occurs at  $(2\sqrt{2}, \frac{16\sqrt{2}}{15} + \frac{4}{3})$



In a **Graphs** page enter the function  $t(x)$ .

Find the minimum

using **b>Analyze Graph>Minimum**.



A further investigation can be made by considering

$$T(x) = \frac{\sqrt{64+x^2}}{A} + \frac{20-x}{B}$$

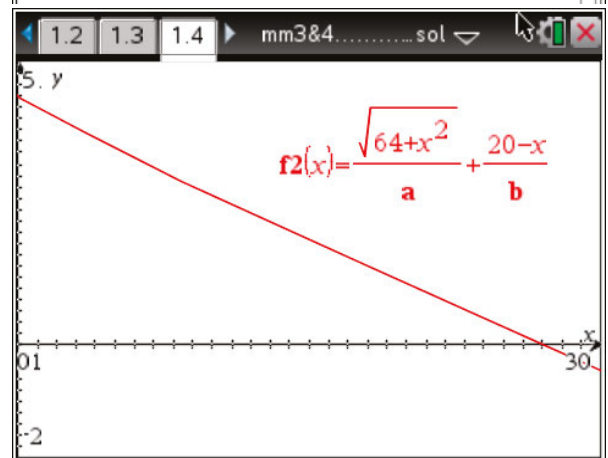
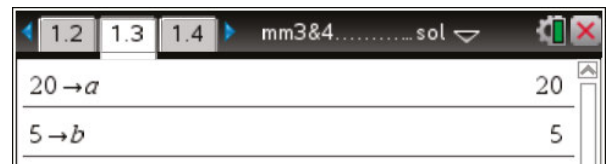
where  $A$  is the rowing speed and  $B$  is the running speed.

In the problem, store  $A = 20$  and  $B = 5$  in a

**Calculator** page

Enter the formula in the **Function Entry Line** as shown by the graph label.

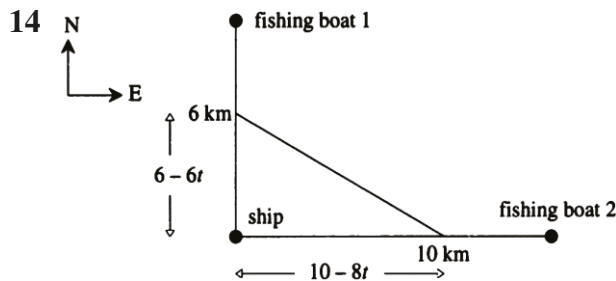
The result is as expected. It is best to row straight to F.



Investigate the minimum value, holding  $A = 5$  and varying  $B$ . The use of **fMin**(,  $b$ ) > **Calculus** > **Function Minimum**, is a good way of doing this. The screen shows a possible investigation. The first value of **fMin**( was obtained with  $A = 5$ ,  $B = 15$ . i.e.  $f_1(x)$ . The following values are obtained by altering the  $B$  value ( $A$  is constant) in function  $f_2(x)$ .

Note: **approx**( is used to give decimal answers for easier comparison. Other options to give decimal answers can also be used.

Function	Minimum Value (x)
$\text{approx}\{f\text{Min}\{f_1(x), x, 0, 20\}\}$	$x = 2.82843$
$5 \rightarrow a$	5
$20 \rightarrow b$	20
$\text{approx}\{f\text{Min}\{f_2(x), x, 0, 20\}\}$	$x = 2.06559$
$10 \rightarrow b$	10
$\text{approx}\{f\text{Min}\{f_2(x), x, 0, 20\}\}$	$x = 4.6188$



Position of fishing boat 1 after  $t$  hours =  $(6 - 6t)$  km North

Position of fishing boat 2 after  $t$  hours  $(10 - 8t)$  km East

Distance apart after  $t$  hours

$$= \sqrt{(6 - 6t)^2 + (10 - 8t)^2} \text{ (Pythagoras' Theorem)}$$

Let  $D$  km be the distance apart after  $t$  hours

$$D = \sqrt{(6 - 6t)^2 + (10 - 8t)^2}$$

$$\text{and } D^2 = (6 - 6t)^2 + (10 - 8t)^2$$

The minimum value of  $D$  will occur for the same value of  $t$  as the minimum of  $D^2$ .

$$\frac{d(D^2)}{dt} = -12(6 - 6t) - 16(10 - 8t)$$

$$= -72 + 72t - 160 + 128t$$

$$= 200t - 232$$

$$\frac{d(D^2)}{dt} = 0 \text{ implies}$$

$$t = \frac{232}{200} = 1.16$$

This is a local minimum as  $D^2$  vs  $t$  is a parabola with positive coefficient of  $t^2$ .

The boats are closest 1.16 hours after noon, i.e. after 1 hour 9 minutes and 36 seconds.

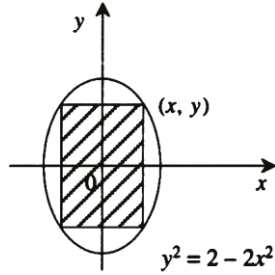
When  $t = 1.16$

$$D = \sqrt{(6 - 6 \times 1.16)^2 + (10 - 8 \times 1.16)^2}$$

$$= 1.2$$

The least distance between the two fishing boats is 1.2 km.

15 a



Area of rectangle,  $A = 2x \times 2y$

$$= 4x \sqrt{2 - 2x^2}$$

$$= 4x(2 - 2x^2)^{\frac{1}{2}}$$

b  $2 - 2x^2 > 0$  for the relation to be defined

$$\therefore 1 > x^2$$

and  $-1 < x < 1$

But  $0 < x < 1$  (as  $x$  is the half-width of the beam)

$\therefore$  allowable values are  $x \in (0, 1)$

c Using the product rule and chain rule

$$A = 4x(2 - 2x^2)^{\frac{1}{2}}$$

$$\frac{dA}{dx} = 4(2 - 2x^2)^{\frac{1}{2}} - 2x(2 - 2x^2)^{\frac{1}{2}} \times 4x$$

$$= 4(2 - x^2)^{\frac{1}{2}} - \frac{8x^2}{(2 - 2x^2)^{\frac{1}{2}}}$$

$$= \frac{4(2 - x^2) - 8x^2}{(2 - 2x^2)^{\frac{1}{2}}}$$

$$= \frac{8 - 8x^2 - 8x^2}{(2 - 2x^2)^{\frac{1}{2}}} = \frac{8 - 16x^2}{(2 - 2x^2)^{\frac{1}{2}}}$$

Maximum will occur when  $\frac{dA}{dx} = 0$

$$\text{When } \frac{dA}{dx} = 0$$

$$8 = 16x^2$$

$$x^2 = \frac{1}{2}$$

$$\therefore x = \pm \frac{1}{\sqrt{2}}$$

$$\text{but } x \in (0, 1) \therefore x = \frac{1}{\sqrt{2}}$$

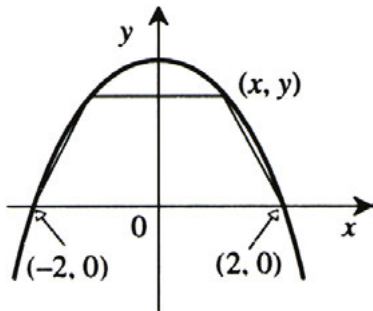
A gradient chart shows local maximum when  $x = \frac{1}{\sqrt{2}}$ . When  $x = \frac{1}{\sqrt{2}}$ ,  $y = \pm 1$

d When  $x = \frac{1}{\sqrt{2}}$

$$\begin{aligned} A &= \frac{4}{\sqrt{2}} \times \left( 2 - 2 \times \left( \frac{1}{\sqrt{2}} \right)^2 \right)^{\frac{1}{2}} \\ &= \frac{4}{\sqrt{2}} \times (2 - 1)^{\frac{1}{2}} \\ &= 2\sqrt{2} \end{aligned}$$

$\therefore$  The maximum cross-sectional area of the beam is  $2\sqrt{2}$  square units.

16



a Area of a trapezoid =  $\frac{h}{2}(a + b)$   
 where  $h$  is the height of the trapezoid and  $a$  and  $b$  are the lengths of the opposite parallel sides.

$$\therefore \text{Area of the trapezoid} = \frac{y}{2}(4 + 2x)$$

$$\text{But } y = 4 - x^2$$

$$\begin{aligned} \therefore \text{Area, } A &= \frac{(4 - x^2)}{2}(4 + 2x) \\ &= \frac{1}{2}(4 - x^2)(2x + 4) \end{aligned}$$

b Using the product rule

$$\begin{aligned} \frac{dA}{dx} &= \frac{1}{2}[-2x(2x + 4) + 2(4 - x^2)] \\ &= \frac{1}{2}[-4x^2 - 8x + 8 - 2x^2] \\ &= \frac{1}{2}[-6x^2 - 8x + 8] = -3x^2 - 4x + 4 \end{aligned}$$

$$\frac{dA}{dx} = 0 \text{ implies } 3x^2 + 4x - 4 = 0$$

$$\therefore (3x - 2)(x + 2) = 0$$

$$\therefore x = \frac{2}{3} \text{ or } x = -2$$

$$\frac{dA}{dx} = -(3x - 2)(x + 2)$$

$$\text{When } x > \frac{2}{3}, \frac{dA}{dx} < 0 \text{ (locally)}$$

$$\text{When } x < \frac{2}{3}, \frac{dA}{dx} > 0$$

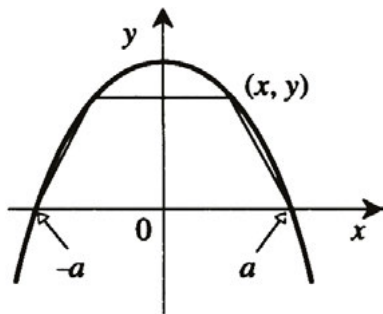
$$\therefore \text{ local maximum when } x = \frac{2}{3}$$

$$\therefore \text{ The trapezoid has its greatest area when } x = \frac{2}{3}$$

**c i**  $A = \frac{1}{2} \times y(2a + 2x)$

$$= y(a + x)$$

$$= (a^2 - x^2)(a + x)$$



**ii** Using the product rule

$$\frac{dA}{dx} = a^2 - x^2 + (-2x)(a + x)$$

$$= a^2 - x^2 - 2xa - 2x^2$$

$$= a^2 - 2xa - 3x^2$$

$$= (a + x)(a - 3x)$$

iii  $\frac{dA}{dx} = 0$  implies  $x = \frac{a}{3}$  or  $x = -a$

when  $x > \frac{a}{3}$ ,  $\frac{dA}{dx} < 0$  (locally)

When  $x < \frac{a}{3}$ ,  $\frac{dA}{dx} > 0$

$\therefore$  maximum when  $x = \frac{a}{3}$

17  $N(t) = 24te^{-0.2t}$

$$N'(t) = 24e^{-\frac{t}{5}} - \frac{24}{5}te^{-\frac{t}{5}}$$

$$= e^{-\frac{t}{5}}\left(24 - \frac{24t}{5}\right)$$

$N'(t) = 0,$

$$24 - \frac{24t}{5} = 0$$

$$\frac{t}{5} = 1$$

$$t = 5$$

$$N(5) = 120e^{-1}$$

$$= \frac{120}{e} = 44 \text{ bacteria}$$

(round because it is a discrete quantity not a continuous one)

18 a  $y = -t^3 + bt^2 + ct$

When  $t = 1, y = 10$

When  $t = 2, y = 24$

$$\therefore 10 = -1 + b + c$$

and  $24 = -8 + 4b + 2c$

$$\therefore 11 = b + c \quad 1$$

and  $32 = 4b + 2c \quad 2$

Subtract  $2 \times 1$  from 2

$$10 = 2b$$

$$\therefore b = 5 \text{ and from 1, } c = 6$$

$$\therefore y = -t^3 + 5t^2 + 6t$$

b  $y = -t^3 + 5t^2 + 6t$

**i**  $y$  is the rate of increase.

$\therefore$  to determine when area covered by the plant is a maximum consider  $y = 0$

$$\text{i.e. } -t^3 + 5t^2 + 6t = 0$$

$$-t(t^2 - 5t - 6) = 0$$

$$\therefore t = 0 \text{ or } (t - 6)(t + 1) = 0$$

$$\therefore t = 6 \text{ or } t = -1 \text{ or } t \text{ (Note: } t \geq 0 \text{ and when } t = 0, y = 0)$$

$$y = -t(t - 6)(t + 1)$$

When  $t > 6, y < 0$  (locally)

When  $t < 6, y > 0$

$\therefore$  local maximum then  $t = 6$

The area is a maximum 6 weeks after planting.

**ii** The rate of increase  $y = -t^3 + 5t^2 + 6t$

To determine maximum rate consider

$$\frac{dy}{dt} = -3t^2 + 10t + 6$$

$$\frac{dy}{dt} = 0 \text{ implies } -3t^2 + 10t + 6 = 0$$

The quadratic formula gives

$$t = \frac{-10 \pm \sqrt{100 + 72}}{-6}$$

$$= \frac{-10 \pm \sqrt{172}}{-6}$$

$$= \frac{-10 \pm 2\sqrt{43}}{-6}$$

$$= \frac{10 \mp 2\sqrt{43}}{6} = \frac{-5 \mp \sqrt{43}}{3} \approx 3.852 \text{ or } -0.519$$

$t \geq 0$  in this example and a gradient chart reveals that a maximum rate of increase occurs when  $t = 3.852$ .

i.e. The rate of increase is a maximum after 3.852 weeks.

**c** This question requires antidifferentiation at year 11 MM 1 & 2 standard.

$$y = -t^3 + 5t^2 + 6t$$

$$\therefore \text{Area} = -\frac{t^4}{4} + \frac{5t^3}{3} + \frac{6t^2}{2} + c$$

$$\text{When } t = 0, \text{ area} = 100 \text{ cm}^2 \therefore c = 100$$

$$\therefore \text{Area} = \frac{-t^4}{4} + \frac{5t^3}{3} + 3t^2 + 100$$



When  $t = 4$

$$\begin{aligned}\text{Area} &= \frac{-4^4}{4} + \frac{5 \times 4^3}{3} + 3 \times 16 + 100 \\ &= -4^3 + \frac{5}{3} \times 4^3 + 3 \times 16 + 100 \\ &= \frac{2 \times 4^3}{3} + 3 \times 16 + 100 \\ &= \frac{2 \times 64}{3} + 48 + 100 \\ &= 42\frac{2}{3} + 48 + 100 \\ &= 190\frac{2}{3}\end{aligned}$$

The plant will cover  $190\frac{2}{3}$  cm<sup>2</sup> after 4 weeks

**d** After 6 weeks the rate becomes negative which implies the plant begins to recede.

Area = 244 cm<sup>2</sup> after 6 weeks

7 = 218 cm<sup>2</sup> after 7 weeks

= 121 cm<sup>2</sup> after 8 weeks

The area becomes “negative” between 8 and 9 weeks. The model is not valid after this. Once the area begins to decrease the model is questionable.

**19**  $f(x) = x^3 - 3x^2 + 6x - 10$

**a**  $f'(x) = 3x^2 - 6x + 6$

$f'(x) = 3$  implies  $3x^2 - 6x + 6 = 3$

$\therefore x^2 - 2x + 2 = 1$

$\therefore x^2 - 2x + 1 = 0$

$\therefore (x - 1)^2 = 0$

$x = 1$

and  $f(1) = 1 - 3 + 6 - 10 = -6$

The coordinates of the point where  $f'(x) = 3$  are  $(1, -6)$

**b**  $f(x) = 3x^2 - 6x + 6$

$= 3[x^2 - 2x + 2]$

$= 3[x^2 - 2x + 1 + 1]$

$= 3[(x - 1)^2 + 1]$

$= 3(x - 1)^2 + 3$

**c**  $(x - 1)^2 > 0$  for all  $x \in \mathbb{R} \setminus \{1\}$

$$\therefore f(x) > 3 \text{ for all } x \in \mathbb{R} \setminus \{1\}$$

**20 a**  $y = ax^3 + bx^2 + cx + d$  passes through the  $x$ -axis at  $(1, 0)$

$$\therefore 0 = a + b + c + d \dots\dots (1)$$

$$\text{Gradient} = 0 \text{ when } x = 1 \text{ and } x = \frac{1}{3}$$

$$\text{gradient function, } \frac{dy}{dx} = 3ax^2 + 2abx + c$$

$$\therefore 0 = 3a + 2b + c \dots\dots (2)$$

$$\text{and } 0 = \frac{a}{3} + \frac{2}{3}b + c \dots\dots (3)$$

Finally it passes through the point  $\left(\frac{1}{3}, \frac{4}{27}\right)$

$$\therefore \frac{4}{27} = \frac{a}{27} + \frac{b}{9} + \frac{c}{3} + d \dots\dots (4)$$

Subtract 3 and 2

$$0 = \frac{8a}{3} + \frac{4b}{3}$$

$$\therefore 0 = 2a + b$$

$$\text{i.e. } b = -2a \dots (5)$$

Substitute in (1) for  $b$

$$0 = a - 2a + c + d$$

$$\text{i.e. } 0 = -a + c + d \dots (6)$$

Substitute in (4) for  $b$

$$\frac{4}{27} = \frac{a}{27} - \frac{2a}{9} + \frac{c}{3} + d$$

$$\frac{4}{27} = -\frac{5a}{27} + \frac{c}{3} + d \dots (7)$$

Subtract (7) from (6)

$$-\frac{4}{27} = -\frac{22a}{27} + \frac{2c}{3}$$

$$\text{i.e. } -4 = -22a + 18c$$

$$\text{and } -2 = -11a + 9c$$

$$\therefore c = \frac{-2 + 11a}{9}$$

Substitute in (2) for  $b$  and  $c$

$$0 = 3a - 4a + 2 - \frac{2 + 11a}{9}$$

$$0 = \frac{9a + -2 + 11a}{9}$$

$$\therefore a = 1$$

$$\text{and } b = -2a = -2$$

$$\text{and } c = -\frac{2 + 11}{9} = 1$$

From (1)

$$0 = a + b + c + d$$

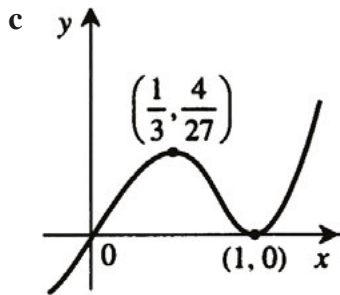
$$0 = 1 - 2 + 1 + d$$

$$\therefore d = 0$$

$$\text{i.e. } a = 1, b = -2, c = 1, d = 0$$

$$\begin{aligned} \mathbf{b} \quad \frac{dy}{dx} &= 3ax^2 + 2bx + c \\ &= 3x^2 - 4x + 1 \\ &= (3x - 1)(x - 1) \end{aligned}$$

$$\frac{dy}{dx} < 0 \text{ for } \frac{1}{3} < x < 1$$



$$\begin{aligned} y &= x^3 - 2x^2 + x \\ &= x(x - 1)^2 \end{aligned}$$

$$\frac{dy}{dx} = 0 \text{ when } x = \frac{1}{3}$$

$$\text{and } x = 1$$

$$\text{when } x = \frac{1}{3}, y = \frac{4}{27}$$

$$\text{and } \frac{dy}{dx} < 0 \text{ for } x \in \left(\frac{1}{3}, 1\right)$$

$$\mathbf{21} \quad V = \frac{\pi}{3}((y + 630)^3 - 630^3)$$

**a** When  $y = 40$

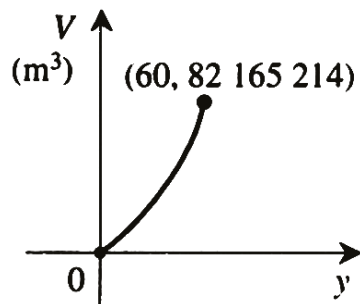
$$\begin{aligned} V &= \frac{\pi}{3}((40 + 630)^3 - 630^3) \\ &= \frac{\pi}{3}(670^3 - 630^3) \\ &= \frac{\pi}{3}(50\,716\,000) \\ &\approx 53\,109\,671.0 \end{aligned}$$

Volume of water in reservoir = 53 109 671.0 m<sup>3</sup>

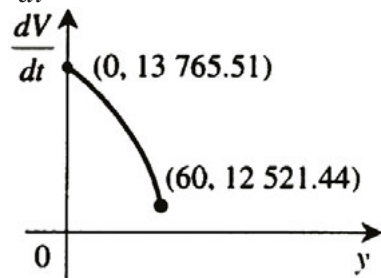
$$\begin{aligned} \mathbf{b} \quad \frac{dV}{dy} &= \frac{\pi}{3}(3(y + 630)^2) \\ &= \pi(y + 630)^2 \end{aligned}$$

c  $\frac{dV}{dy} > 0$  for all  $v \in R$ , gives that the function is increasing and the gradient increases as  $y$  increases.

$$\begin{aligned} \text{d } V &= \frac{\pi}{3}((y + 630)^3 - 630^3) \\ &= \frac{\pi}{3}((690)^3 + (630)^3) \text{ when } y = 60 \\ &= \frac{\pi}{3}(78\,462\,000) \\ &= 82\,165\,214 \text{ m}^3 \end{aligned}$$



$$\text{e } \frac{dV}{dt} = 20\,000 - 0.005\pi(y + 630)^2$$



The graph of  $\frac{dV}{dt}$  against  $y$  is a parabola.

It is the graph of  $z = -x^2$  transformed by a dilation of  $0.005\pi$  from the  $x$ -axis followed by a translation of 630 units “to the left” and 20 000 units “up”.

The domain is  $0 \leq y \leq 60$ .

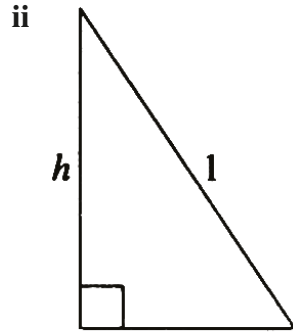
$$\text{When } y = 60, \frac{dV}{dt} = 12521.44$$

$$\text{When } y = 0, \frac{dV}{dt} = 13765.51$$

**22 a i** The circumference of the base of the cone is equal to the length of the sector formed. Hence  $2\pi r = (2\pi - \theta)$  (radius of circle is one)

$$\therefore 2\pi r = 2\pi - \theta$$

$$r = \frac{2\pi - \theta}{2\pi}$$



$$h^2 + r^2 = 1$$

$$\therefore h^2 = 1 - r^2$$

From (i),  $r = \frac{2\pi - \theta}{2\pi}$

$$\therefore h^2 = 1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2$$

$$\text{and } h = \sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$$

**iii**  $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi \left(\frac{2\pi - \theta}{2\pi}\right)^2 \sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$$

**b** When  $\theta = \frac{\pi}{2}$

$$V = \frac{\pi}{3} \left(\frac{2\pi - \frac{\pi}{4}}{2\pi}\right)^2 \sqrt{\frac{4\pi^2 - (4\pi^2 - 4\pi\theta + \theta^2)}{4\pi^2}}$$

$$= \frac{\pi}{3} \left(\frac{7}{8}\right)^2 \sqrt{\frac{4\pi \times \frac{\pi}{4} - \left(\frac{\pi}{4}\right)^2}{4\pi^2}}$$

$$= \frac{49\pi}{192} \sqrt{\frac{1 - \frac{1}{16}}{4}}$$

$$= \frac{49\pi}{384} \sqrt{\frac{15}{16}}$$

$$= \frac{49\pi}{1536} \sqrt{15}$$

$$\begin{aligned} \text{c } 0.3 &= \frac{\pi}{3} \left( \frac{2\pi - \theta}{2\pi} \right)^2 \sqrt{\frac{4\pi\theta - \theta^2}{4\pi^2}} \text{ Solving using a CAS calculator } \theta = 0.3281 \\ &= \frac{(2\pi - \theta)^2}{24\pi^2} \sqrt{4\pi\theta - \theta^2} \end{aligned}$$

**d** (Note:  $0 < \theta < \pi$ )

**i** maximum occurs at  $\theta \approx 1.153$

**ii** maximum volume is  $V \approx 0.403 \text{ cm}^2$

$$\begin{aligned} \text{e } V &= \frac{(2\pi - \theta)^2}{24\pi^2} (4\pi\theta - \theta^2)^{\frac{1}{2}} \\ \frac{dV}{d\theta} &= \frac{1}{24\pi^2} \left[ -2(2\pi - \theta)(4\pi\theta - \theta^2)^{\frac{1}{2}} + \frac{1}{2}(2\pi - \theta)^2(4\pi - 2\theta)(4\pi\theta - \theta^2)^{-\frac{1}{2}} \right] \\ &= \frac{(2\pi - \theta)}{24\pi^2} \left[ \frac{-2(4\pi\theta - \theta^2) + (2\pi - \theta)^2}{(4\pi\theta - \theta^2)^{\frac{1}{2}}} \right] \end{aligned}$$

$$\begin{aligned} \frac{dV}{d\theta} = 0 &\text{ implies } \theta = 2\pi \\ \text{or } -8\pi\theta + 2\theta^2 + 4\pi^2 - 4\pi\theta + \theta^2 &= 0 \end{aligned}$$

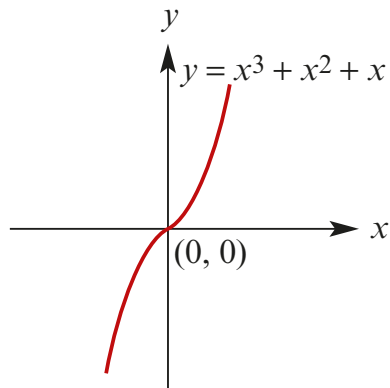
$$\begin{aligned} \text{i.e. } 3\theta^2 - 12\pi\theta + 4\pi^2 &= 0 \\ \therefore \theta &= \frac{12\pi \pm \sqrt{144\pi^2 - 48\pi^2}}{6} \end{aligned}$$

$$\theta = \frac{12\pi \pm 4\sqrt{6\pi^2}}{6}$$

$$\therefore \theta = \frac{6\pi - 2\pi\sqrt{6}}{3} \approx 1.153$$

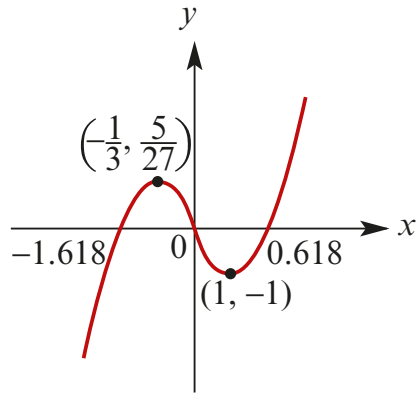
Maximum volume is  $0.403 \text{ cm}^2$

**23 a i**

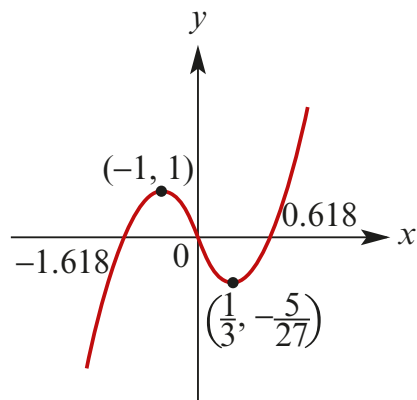


No stationary points

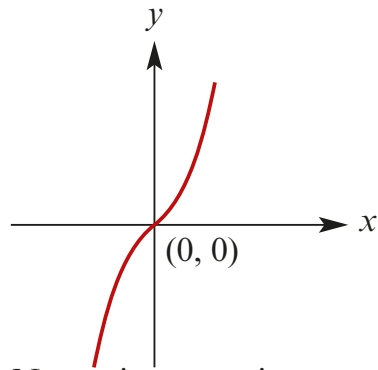
ii



iii



iv



No stationary point

**b i**  $f'(x) = 3x^2 + 2ax + b$

**ii**  $f'(x) = 0$  implies  $3x^2 + 2ax + b = 0$

$$\begin{aligned}\therefore x &= \frac{-2a \pm \sqrt{4a^2 - 4 \times 3 \times b}}{6} \\ &= \frac{-2a \pm \sqrt{4a^2 - 12b}}{6} \\ &= \frac{-a \pm \sqrt{a^2 - 3b}}{3}\end{aligned}$$

**c i** If  $a^2 - 3b = 0$ , the cubic has one stationary point given by  $x = \frac{-9}{3} = -3$ .





24 Let  $y = \frac{\log_e x}{x}$

$$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \log_e x}{x^2}$$

$$= \frac{1 - \log_e x}{x^2}$$

$$\frac{dy}{dx} = 0 \text{ implies } \frac{1 - \log_e x}{x^2} = 0$$

$$\therefore x = e$$

$$\frac{dy}{dx} < 0 \text{ for } 1 - \log_e x < 0 \Leftrightarrow \log_e x > 1 \Leftrightarrow x > e$$

$$\frac{dy}{dx} > 0 \text{ for } 1 - \log_e x > 0 \Leftrightarrow \log_e x < 1 \Leftrightarrow x < e$$

$\therefore$  a maximum for  $x = e$

$$\text{When } x = e, y = \frac{\log_e e}{e} = \frac{1}{e}$$

i.e. The ratio of the logarithm of a number to the number is a maximum when  $x = e$ .

25 a i  $f(x) = 6x^4 - x^3 + ax^2 - 6x + 8$

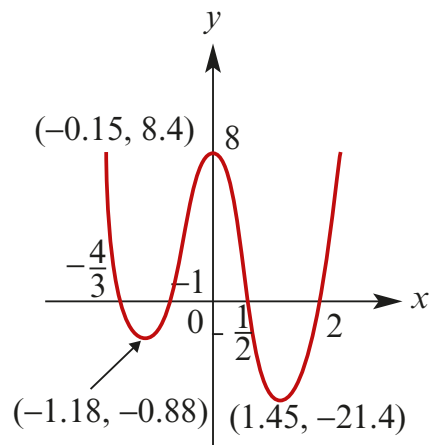
If  $x + 1$  is a factor  $f(-1) = 0$

i.e.  $f(-1) = 6 + 1 + a + 6 + 8 = 0$

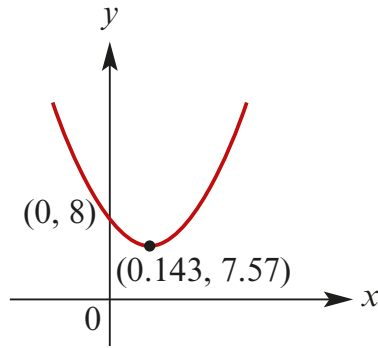
$$\therefore a + 21 = 0$$

$$a = -21$$

ii



**b i**



**ii** minimum = 7.57 when  $x = 0.143$

**iii**  $g'(x) = 24x^3 - 3x^2 + 42x - 6$

**iv**  $g'(x) = 0$

$$24x^3 - 3x^2 + 42x - 6 = 0$$

$$x = 0.1427$$

**v**  $g'(0) = -6; g'(10) = 24\ 114$

**vi**  $\frac{d}{dx}(g(x))$  can be written as  $g''(x)$ , meaning the derivative of derivative.  
 $g''(x) = 72x^2 - 6x + 42$

**vii**  $g''(x) = 0$  implies

$$12x^2 - x + 7 = 0$$

$$\text{But } \Delta = 1 - 4 \times 12 \times 7 < 0$$

$\therefore$  no stationary points

Hence the graph of

$y = g'(x)$  has positive gradient for all  $x$ . There is only one solution of  $g'(x) = 0$ .

**26 a**  $f(x) = (x - a)^2(x - b)^2a > 0 b > 0$

$$f'(x) = 2(x - a)(x - b)^2 + 2(x - b)(x - a)^2$$

$$= 2(x - a)(x - b)(x - b + x - a)$$

$$= 2(x - a)(x - b)(2x - (b + a))$$

**b i**  $f'(x) = 0$  implies  $x = a$  or  $x = b$  or  $x = \frac{b + a}{2}$

**ii**  $x = a$  or  $x = b$

**c** Stationary points

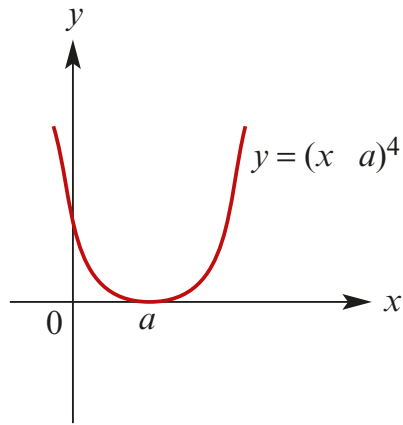
$$(a, 0) (b, 0)$$

When  $x = \frac{a+b}{2}$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \left(\frac{a+b}{2} - a\right)^2 \left(\frac{a+b}{2} - b\right)^2 \\ &= \left(\frac{a+b-2a}{2}\right)^2 \left(\frac{a+b-2b}{2}\right)^2 \\ &= \left(\frac{b-a}{2}\right)^2 \left(\frac{a-b}{2}\right)^2 \\ &= \frac{(a-b)^4}{16} \end{aligned}$$

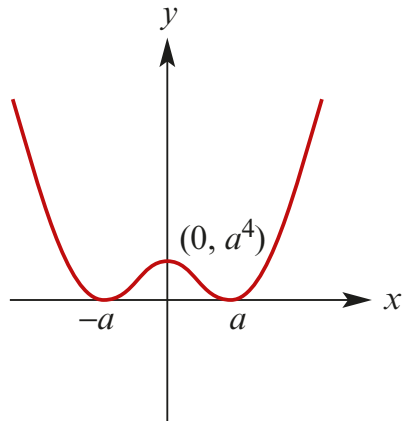
$\therefore$  coordinates  $\left(\frac{a+b}{2}, \frac{(a-b)^4}{16}\right)$

**d i**



**ii** If  $a = -b$  coordinates are  $(a, 0)$   $(-a, 0)$   $(0, a^4)$

**iii**



**27 a**  $f(x) = (x - a)^3(x - b)$

$$\begin{aligned} f'(x) &= 3(x - a)^2(x - b) + (x - a)^3 \\ &= (x - a)^2[3(x - b) + (x - a)] \\ &= (x - a)^2[4x - (3b + a)] \end{aligned}$$

**b i**  $f'(x) = 0$  implies  $x = a$  or  $x = \frac{3b+a}{4}$

**ii**  $f(x) = 0$  implies  $x = a$  or  $x = b$

**c**  $(a, 0)$  is a stationary point of inflection as  $f'(a+h)$  and  $f'(a-h)$  have the same sign where  $h$  is a small number.

If  $x = \frac{3b+a}{4}$

$$\begin{aligned} f(x) &= \left(\frac{3b+a}{4} - a\right)^3 \left(\frac{3b+a}{4} - b\right) \\ &= \left(\frac{3b+a-4a}{4}\right)^3 \left(\frac{3b+a-4b}{4}\right) \\ &= \left(\frac{3b-3a}{4}\right)^3 \left(\frac{a-b}{4}\right) \\ &= -\frac{27}{256}(b-a)^4 \end{aligned}$$

If  $x > \frac{3b+a}{4}$  then  $f'(x) > 0$

If  $x < \frac{3b+a}{4}$  then  $f'(x) < 0$

$\therefore$  local minimum at  $\left(\frac{3b+a}{4}, -\frac{27}{256}(b-a)^4\right)$

**d** Calculator

**e** If  $a = -b$

stationary points are  $(a, 0)$  and  $\left(-\frac{a}{2}, \frac{27a^4}{16}\right)$

**f i** If a local minimum at  $x = 0$ ,  $\frac{3b+a}{4} = 0$ , i.e.  $a = -3b$  or  $b = -\frac{a}{3}$ .

**g** If there is a turning point for  $x = \frac{a+b}{2}$

then  $\frac{a+b}{2} = \frac{3b+a}{4}$

$\therefore 2a + 2b = 3b + a$

$\therefore 0 = b - a$

$\therefore b = a$

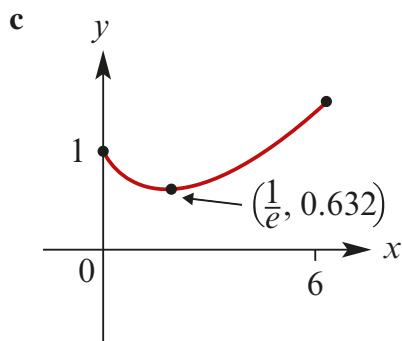
If  $b = a$   $f(x) = (x-a)^4$

**28**  $f: (0, 6] \rightarrow R, f(x) = x \log_e x + 1$

**a**  $f'(x) = \log_e x + x \times \frac{1}{x} = \log_e x + 1$  (product rule)

- b**  $f'(x) = 0$  implies  $\log_e x = -1$   
 $\therefore x = e^{-1}$   
 When  $x > e^{-1}$ ,  $\log_e x + 1 > 0$   
 When  $x < e^{-1}$ ,  $\log_e x + 1 < 0$   
 $\therefore$  a minimum when  $x = e^{-1} \approx 0.37$ ,  
 i.e. during the fourth month of its life.

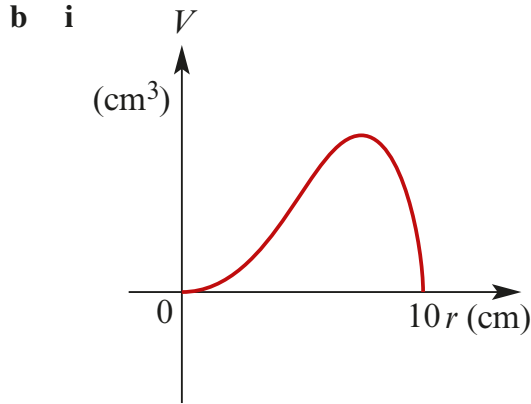
$$\begin{aligned} \text{When } x = e^{-1}, f(x) &= \frac{1}{e} \log_e e^{-1} + 1 \\ &= -\frac{1}{e} + 1 \\ &\approx 0.632 \end{aligned}$$



- d** The mouse's ability to memorise is a maximum after 6 years.

**29 a i**  $y^2 + r^2 = 100$   
 $\therefore y = \sqrt{100 - r^2}$   
 $\therefore$  height  $= 2y = 2\sqrt{100 - r^2}$

**ii**  $V = \pi r^2 h$   
 $= \pi r^2 (2\sqrt{100 - r^2})$   
 $= 2\pi r^2 \sqrt{100 - r^2}$



**ii** Maximum volume is  $2418.4 \text{ cm}^3$   
 This occurs when  $r = 8.165$  and  $h = 11.55$

**iii** Use the 'solve' command of a CAS calculator.  
 $r = 6.456$  or  $r = 9.297$

**c i**

$$V = 2\pi r^2(100 - r^2)^{\frac{1}{2}}$$

$$\frac{dV}{dr} = 4\pi r(100 - r^2)^{\frac{1}{2}} - \frac{2r}{2}(100 - r^2)^{-\frac{1}{2}} \times 2\pi r^2$$

$$= 2\pi r \left[ 2(100 - r^2)^{\frac{1}{2}} - \frac{r^2}{(100 - r^2)^{\frac{1}{2}}} \right]$$

$$= 2\pi r \frac{[2(100 - r^2) - r^2]}{(100 - r^2)^{\frac{1}{2}}}$$

$$= 2\pi r \frac{[200 - 3r^2]}{(100 - r^2)^{\frac{1}{2}}}$$

**ii** If  $\frac{dV}{dr} = 0$ ,  $200 - 3r^2 = 0$

$$\therefore 3r^2 = 200$$

$$\therefore r^2 = \frac{200}{3}$$

$$r = \sqrt{\frac{200}{3}} = \sqrt{\frac{600}{3}}$$

$$= \frac{10\sqrt{6}}{3}$$

$\therefore$  maximum volume is given by

$$\begin{aligned}
 V_{\max} &= 2\pi \times \frac{200}{3} \left(100 - \frac{200}{3}\right)^{\frac{1}{2}} \\
 &= \frac{400\pi}{3} \left(\frac{100}{3}\right)^{\frac{1}{2}} = \frac{4000\pi}{3} \times \frac{\sqrt{3}}{3} \\
 &= \frac{4000\sqrt{3}\pi}{9}
 \end{aligned}$$

**d i** Calculator

**ii**  $\frac{dV}{dr} > 0$  for  $r \in \left(0, \frac{20\sqrt{6}}{6}\right)$

**iii**  $\frac{dV}{dr}$  is increasing for  $r \in (0, 5.21)$

**30 a** Surface area =  $\pi r^2 + 2\pi r h + 2\pi r^2$

$$\therefore 3\pi r^2 + 2\pi r h = 100\pi$$

$$\therefore 3r^2 + 2rh = 100$$

$$\therefore h = \frac{100 - 3r^2}{2r}$$

**b**  $V = \pi r^2 h + \frac{2}{3}\pi r^3$

$$\begin{aligned}
 &= \pi r^2 \left(\frac{100 - 3r^2}{2r}\right) + \frac{2}{3}\pi r^3 \\
 &= \pi r \left(\frac{100 - 3r^2}{2}\right) + \frac{2}{3}\pi r^3 \\
 &= \frac{\pi r}{6}(300 - 9r^2 + 4r^2) \\
 &= \frac{\pi r}{6}(300 - 5r^2)
 \end{aligned}$$

**c** defined for  $r > 0$  and  $300 - 5r^2 > 0$

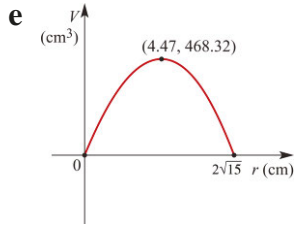
i.e.  $r^2 < 60$

$$r < 2\sqrt{15}$$

**d**  $V = \frac{\pi}{6}(300r - 5r^3)$

$$\frac{dV}{dr} = \frac{\pi}{6}(300 - 15r^2)$$





**31 a i**  $30x^2y = 3000$

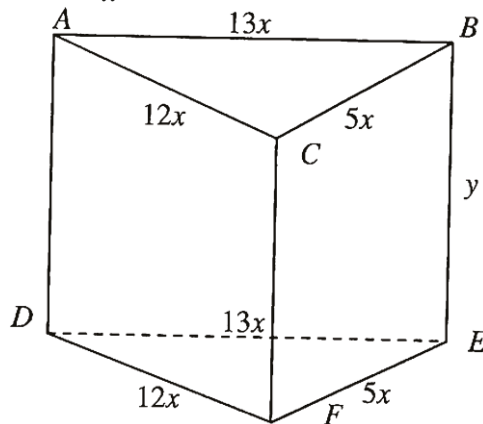
$$y = \frac{100}{x^2}$$

**ii**  $S = 12xy + 5xy + 13xy + 60x^2$

$$= 30xy + 60x^2$$

$$= 30x \frac{100}{x^2} + 60x^2$$

$$= \frac{3000}{x} + 60x^2$$



**b i**  $\frac{dS}{dx} = -\frac{3000}{x^2} + 120x$

**ii**  $\frac{dS}{dx} = 0$  implies  $3000 = 120x^3$

Therefore  $x^3 = 250$

and hence  $x = 5^{\frac{2}{3}}$

When  $x = 5^{\frac{2}{3}}$ ,  $S \approx 1539 \text{ cm}^2$

**c**  $\frac{dS}{dt} = \frac{dS}{dx} \frac{dx}{dt} = \left(-\frac{3000}{x^2} + 120x\right) \times 0.5$

When  $x = 10$ ,  $\frac{dS}{dt} = \left(-\frac{3000}{10^2} + 1200\right) \times 0.5$

$$= 585 \text{ cm}^2/\text{s}$$

$$32 \text{ a } f(x) = \frac{100\,000}{1 + 100e^{-3x}}$$

Using the Chain rule

$$\begin{aligned} f'(x) &= -\frac{100\,000}{(1 + 100e^{-0.3x})^2} \times -30e^{-0.3x} \\ &= \frac{3\,000\,000e^{-0.3x}}{(1 + 100e^{-0.3x})^2} \end{aligned}$$

$$\text{b i When } x = 0, f'(0) = \frac{3\,000\,000}{(1 + 100)^2} = 294.08$$

The rate of growth is 294 kangaroos per year when  $x = 0$

$$\text{ii When } x = 4, f'(4) = \frac{3\,000\,000e^{-1.2}}{(1 + 100e^{-1.2})^2} = 933.0498$$

The rate of growth is 933 kangaroos per year when  $x = 4$

33 a  $f$  is defined for

$$6 - 0.2x > 0$$

$$\Leftrightarrow 6 > 0.2x$$

$$\Leftrightarrow \frac{6}{0.2} > x$$

$$\Leftrightarrow 30 > x$$

$$\therefore a = 30$$

$$\text{b } f(0) = 8 \log_e 6$$

When  $f(x) = 0$

$$8 \log_e(6 - 0.2x) = 0$$

which implies

$$6 - 0.2x = 1$$

$$5 = \frac{1}{5}x$$

$$25 = x$$

$\therefore (25, 0)$  and  $(0, 8 \log_e 6)$  are the coordinates of the axes intercepts

$$\text{c } f(x) = 8 \log_e(6 - 0.2x)$$

$$f'(x) = \frac{-8}{5(6 - 0.2x)}$$

when  $x = 20$

$$\begin{aligned} f'(20) &= \frac{-8}{5(6 - 4)} \\ &= \frac{-4}{5} = -0.8 \end{aligned}$$

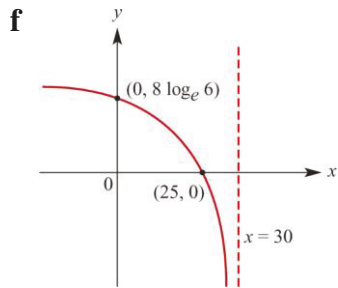
d Consider  $x = 8 \log_e(6 - 0.2y)$

$$e^{\frac{x}{8}} = 6 - \frac{y}{5}$$

$$\therefore y = 5\left(6 - e^{\frac{x}{8}}\right)$$

$$\therefore f^{-1}(x) = 5\left(6 - e^{\frac{x}{8}}\right)$$

e The domain of  $f^{-1}$  is  $R$



34 a Calculator

b  $g'(x) = \cos(x)e^{\sin x}$

$$g'(x) = 0 \text{ implies } \cos x = 0 \text{ as } e^{\sin x} \neq 0$$

$$\therefore \text{the stationary points occur at } x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{2}$$

The coordinates of the stationary points are

$$\left(\frac{\pi}{2}, e\right) \text{ and } \left(\frac{3\pi}{2}, \frac{1}{e}\right)$$

c range =  $\left[\frac{1}{e}, e\right]$

d period =  $2\pi$  as  $g(x + 2\pi) = g(x)$

35 a  $y = e^x$

$$\frac{dy}{dx} = e^x$$

When  $x = 0$ ,  $\frac{dy}{dx} = 1$

Therefore equation of tangent is  $y = x + 1$

b Identical transformations applied to the curve and the tangent retain the relationship, i.e. the image of the tangent is tangent to the image of the curve.

c Consider the curve with equation  $y = a f(bx)$

Then the gradient at  $\left(\frac{x_1}{b}\right)$  is given by  $\frac{dy}{dx} = ab f'(bx) = ab f'(x_1)$

But the gradient of  $y = f(x)$  at  $x_1$  is  $f'(x_1) = m$

$\therefore$  gradient of  $y = a f(bx)$  is  $abm$

$\therefore$  equation of the tangent at  $\left(\frac{x_1}{b}, y_1 a\right)$

is  $y - y_1 a = abm\left(x - \frac{x_1}{b}\right)$

$\therefore y = bam\left(x - \frac{x_1}{b}\right) + y_1 a$   
 $= bamx - amx_1 + y_1 a$

But  $y_1 = mx_1 + c$  and

$\therefore y = a(bmx - mx_1 + y_1)$

$y = a(bmx - mx_1 + mx_1 + c)$

$\therefore y = a(bmx + c)$

**36 a i** When  $t = 0$ ,  $x = \frac{60}{5e^0 - 3} = \frac{60}{2} = 30$

When  $t = 0$ , there are 30 g not dissolved.

**ii** When  $t = 5$ ,  $x = \frac{60}{5e^{5\lambda} - 3}$  where  $\lambda = \frac{1}{2} \log_e \frac{6}{5}$

$$= \frac{60}{5e^{\frac{5}{2} \log\left(\frac{6}{5}\right)} - 3}$$

$$= \frac{60}{5 \times \left(\frac{6}{5}\right)^{\frac{5}{2}} - 3}$$

$$\approx 12.2769$$

When  $t = 5$  there are 12.28 g not dissolved.

**b**  $\frac{dx}{dt} = 5\lambda e^{\lambda t} \times -\frac{60}{(5e^{\lambda t} - 3)^2}$  (Chain rule)

$$= -\frac{300\lambda e^{\lambda t}}{(5e^{\lambda t} - 3)^2}$$

**c i**

$$x = \frac{60}{5e^{\lambda t} - 3}$$

$$\therefore 5xe^{\lambda t} - 3x = 60$$

$$\therefore e^{\lambda t} = \frac{3x + 60}{5x}$$

$$\therefore \frac{dx}{dt} = -\frac{300\lambda e^{\lambda t}}{(5e^{\lambda t} - 3)^2}$$

$$= -300\lambda \left( \frac{3x + 60}{5x} \right) \div \left( \frac{5(3x + 60)}{5x} - 3 \right)^2$$

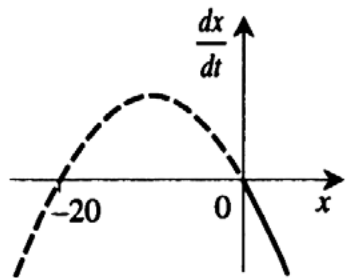
$$= -60\lambda \left( \frac{3x + 60}{x} \right) \div \left( \frac{3x + 60 - 3x}{x} \right)^2$$

$$= -60\lambda \left( \frac{3x + 60}{x} \right) \times \frac{x^2}{3600}$$

$$= -\lambda(x + 20) \times \frac{x}{20}$$

$$= -\frac{\lambda x^2}{20} - \lambda x$$

**ii**



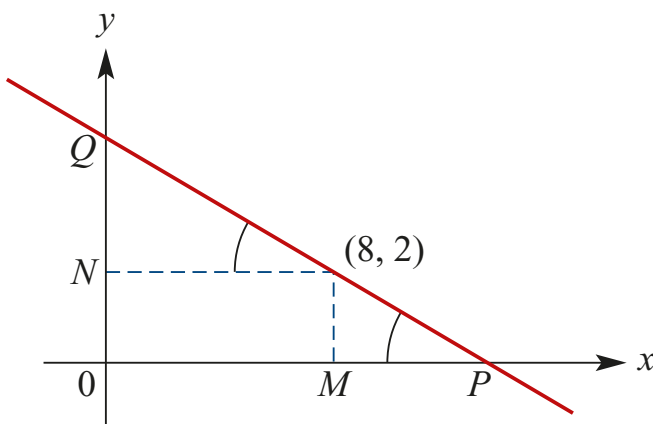
$$\frac{dx}{dt} = -\lambda \left( \frac{x^2}{20} + x \right)$$

$$= -\lambda x \left( \frac{x}{20} + 1 \right)$$

$x$ -axis intercepts  $x = 0$  and  $x = -20$  domain =  $[0, \infty)$

**iii** Rate of dissolving increases where  $x$  is the amount of material not dissolved.

**37**



**a** Let  $y = \frac{1}{\tan \theta}$   
Let  $u = \tan \theta$

$$\begin{aligned}
\text{Then } y &= \frac{1}{u} \text{ and } \frac{dy}{d\theta} = \frac{dy}{du} \times \frac{du}{d\theta} \\
&= -\frac{1}{u^2} \times \sec^2 \theta \\
&= -\frac{1}{\tan^2 \theta} \times \sec^2 \theta \\
&= -\frac{\cos^2 \theta}{\sin^2 \theta} \times \sec^2 \theta \\
&= -\frac{1}{\sin^2 \theta} \\
&= -\operatorname{cosec}^2 \theta
\end{aligned}$$

**b**  $\frac{2}{MP} = \tan \theta$

$$\therefore MP = \frac{2}{\tan \theta}$$

**c**  $\frac{NQ}{8} = \tan \theta$

$$\therefore NQ = 8 \tan \theta$$

**d**  $OP + OQ = OM + MP + ON + NQ$

$$= 8 + \frac{2}{\tan \theta} + 2 + 8 \tan \theta$$

$$= 10 + 8 \tan \theta + \frac{2}{\tan \theta}$$

**e** Let  $x = OP + OQ$

$$\text{i.e. } x = 10 + 8 \tan \theta + \frac{2}{\tan \theta}$$

$$\frac{dx}{d\theta} = -2 \operatorname{cosec}^2 \theta + 8 \sec^2 \theta$$

**f** minimum occurs when  $\frac{dx}{d\theta} = 0$

$$-2[\operatorname{cosec}^2 \theta + 8 \sec^2 \theta] = 0$$

$$\therefore -\frac{2}{\sin^2 \theta} + \frac{8}{\cos^2 \theta} = 0$$

$$\therefore \frac{2}{\sin^2 \theta} = \frac{8}{\cos^2 \theta}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} = \frac{2}{8} = \frac{1}{4}$$

$$\therefore \tan^2 \theta = \frac{1}{4}$$

$$\text{and } \tan \theta = \pm \frac{1}{2}$$

$$\text{We know } 0 < \theta < \frac{\pi}{2}$$

$$\therefore \tan \theta = \frac{1}{2}$$

$$\therefore \theta > 26.6^\circ$$

If  $\theta > 26.6^\circ$ ,  $\frac{dx}{d\theta} < 0$  (locally)

If  $\theta < 26.6^\circ$ ,  $\frac{dx}{d\theta} > 0$

$\therefore$  minimum when  $\theta = 26.6^\circ$

$$\text{If } \tan \theta = \frac{1}{2}$$

$$x = \frac{2}{\tan \theta} + 8 \tan \theta + 10$$

$$= \frac{2}{\frac{1}{2}} + 8 \times \frac{1}{2} + 10$$

$$= 4 + 4 + 10 = 18$$

The minimum value of  $x$  is 18 units.

**38** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = e^x - e^{-x}$

**a**  $f'(x) = e^x + e^{-x}$

**b**  $f(x) = 0$  implies  $e^x - e^{-x} = 0$

i.e.  $e^x - \frac{1}{e^x} = 0$

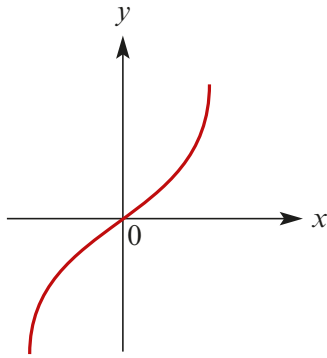
$$\therefore e^{2x} - 1 = 0$$

$$\therefore 2x = 0$$

which implies  $x = 0$

**c**  $f'(x) = e^x + e^{-x}$  and both  $e^x$  and  $e^{-x}$  are positive, so  $f'(x) > 0$  for all  $x$ .

d



The result that  $f'(x) > 0$  for all  $x$  is used

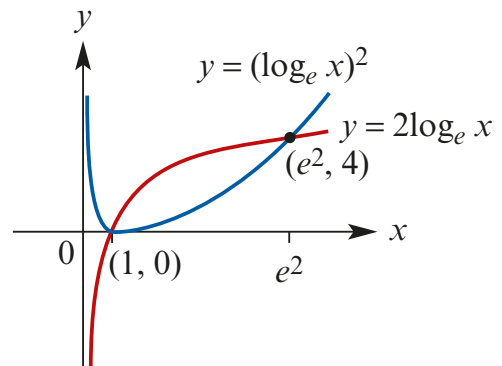
- 39 a  $(\log_e x)^2 = 2 \log_e x$   
 is equivalent to  $(\log_e x)^2 - 2 \log_e x = 0$   
 i.e.  $\log_e x [\log_e x - 2] = 0$   
 which implies  $\log_e x = 0$  or  $\log_e x = 2$   
 $\therefore x = 1$  or  $x = e^2$

b

For $y = 2 \log_e x$ $\frac{dy}{dx} = \frac{2}{x}$	For $y = (\log_e x)^2$ Let $u = \log_e x$ Then $y = u^2$ and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $= 2u \cdot \frac{1}{x}$ $= \frac{2 \log_e x}{x}$
---	---

The gradient of  $y = 2 \log_e x$  at  $x = 1$  is 2  
 The gradient of  $y = (\log_e x)^2$  at  $x = 1$  is 0  
 This information is now used to sketch the graphs.

- c Note:  $y = (\log_e x)^2 \geq 0$  for all  $x$  and  
 $(\log_e x)^2 \rightarrow \infty$  as  $x \rightarrow 0$
- d  $\therefore \left\{ x: 2 \log_e x > (\log_e x)^2 \right\} = \left\{ x: 1 < x < e^2 \right\}$





40 A cross-section of the solids is as shown.

**a**  $h = VA + AE$

$$= a + a \cos \theta$$

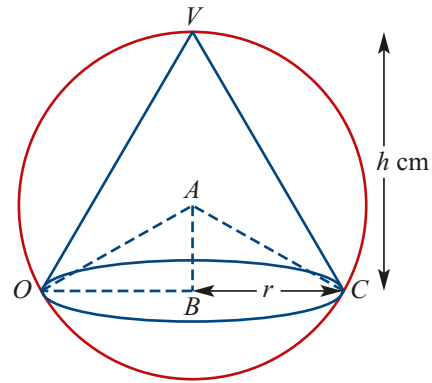
where  $a$  is the radius of the sphere

**b**  $r = a \sin \theta$

**c**  $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(a^2 \sin^2 \theta)(a + a \cos \theta)$$

$$= \frac{1}{3}\pi a^3 \sin^2 \theta(1 + \cos \theta)$$



**d** Using the product rule

$$\frac{dV}{d\theta} = \frac{1}{3}\pi a^3 [\sin^2 \theta \times -\sin \theta + 2 \sin \theta \cos \theta(1 + \cos \theta)]$$

$$= \frac{1}{3}\pi a^3 [-\sin^3 \theta + 2 \sin \theta \cos \theta(1 + \cos \theta)]$$

$$\frac{dV}{d\theta} = 0 \text{ implies } \frac{1}{3}\pi a^3 [-\sin^3 \theta + 2 \sin \theta \cos \theta(1 + \cos \theta)] = 0$$

$$\therefore \sin^3 \theta = 2 \sin \theta \cos \theta(1 + \cos \theta)$$

For  $\sin \theta \neq 0$

$$\sin^2 \theta = 2 \cos \theta + 2 \cos^2 \theta$$

Using  $\sin^2 \theta = 1 - \cos^2 \theta$

$$1 - \cos^2 \theta = 2 \cos \theta + 2 \cos^2 \theta$$

which implies  $3 \cos^2 \theta + 2 \cos \theta - 1 = 0$

This is a quadratic equation in  $\cos \theta$ . It factorises to give the following:

$$(3 \cos \theta - 1)(\cos \theta + 1) = 0$$

$$\therefore \cos \theta = \frac{1}{3} \text{ or } \cos \theta = -1$$

A gradient chart confirms a maximum volume occurs when

$$\cos \theta = \frac{1}{3} \left( \text{when } \theta = \cos^{-1}\left(\frac{1}{3}\right) \right), \text{ i.e. } \theta \approx 70.53^\circ$$

$$\mathbf{e} \quad V = \frac{1}{3}\pi a^3 \sin^2 \theta (1 + \cos \theta)$$

Using  $\sin^2 \theta = 1 - \cos^2 \theta$

$$V = \frac{1}{3}\pi a^3 (1 - \cos^2 \theta)(1 + \cos \theta)$$

When  $\cos \theta = \frac{1}{3}$

$$\begin{aligned} V &= \frac{1}{3}\pi a^3 \left(1 - \frac{1}{9}\right) \left(1 + \frac{1}{3}\right) \\ &= \frac{1}{3}\pi a^3 \times \frac{8}{9} \times \frac{4}{3} \\ &= \frac{32\pi a^3}{81} \end{aligned}$$

The maximum volume is  $\frac{32\pi a^3}{81} \text{ cm}^3$

$$\mathbf{41 a} \quad y = \frac{Ae^{bt}}{1 + Ae^{bt}}$$

Dividing through by  $1 + Ae^{bt}$  gives

$$y = 1 - \frac{1}{1 + Ae^{bt}}$$

and as  $Ae^{bt} > 0$  for all  $t$ ,  $\frac{1}{1 + Ae^{bt}} < 1$

Hence  $0 < y < 1$

**b** By using the quotient rule

$$\begin{aligned} \frac{dy}{dt} &= \frac{(1 + Ae^{bt})bAe^{bt} - bA^2e^{2bt}}{(1 + Ae^{bt})^2} \\ &= \frac{bAe^{bt}}{(1 + Ae^{bt})^2} \end{aligned}$$

$$\mathbf{c} \quad \text{As} \quad y = \frac{Ae^{bt}}{1 + Ae^{bt}}$$

$$y(1 + Ae^{bt}) = Ae^{bt}$$

$$\therefore y + yAe^{bt} = Ae^{bt}$$

and  $y = Ae^{bt}(1 - y)$

$$\therefore Ae^{bt} = \frac{y}{1 - y}$$

**d i** From the result of **b**

$$\frac{dy}{dt} = \frac{bAe^{bt}}{(1 + Ae^{bt})^2}$$

Substituting  $Ae^{bt} = \frac{y}{1-y}$

$$\begin{aligned} \frac{dy}{dx} &= b\left(\frac{y}{1-y}\right) \div \left(1 + \frac{y}{1-y}\right)^2 \\ &= b\left(\frac{y}{1-y}\right) \times (1-y)^2 \\ &= by(1-y) \end{aligned}$$

ii  $\frac{dy}{dx} = by(1-y)$

is a quadratic expression in  $y$  with negative coefficient of  $y^2$  ( $b$  is a positive constant)

$\therefore$  a maximum occurs when  $y = 0.5$ .

e From c  $Ae^{bt} = \frac{y}{1-y}$

$\therefore$  when  $A = 0.01$ ,  $b = 0.7$  and  $y = 0.5$

$$0.01e^{0.7t} = 1$$

$$\therefore e^{0.7t} = 100$$

$$\therefore 0.7t = \log_e 100$$

$$\therefore t = \frac{10}{7} \log_e 100 \approx 6.578$$

$\therefore$  The bacteria are increasing at the fastest rate when  $t = 7$  (to the nearest hour).

42 Let  $f(x) = \frac{e^x}{x}$

a  $f(x) = e^x \cdot x^{-1}$

The product rule gives

$$f'(x) = e^x x^{-1} - e^x x^{-2}$$

$$\begin{aligned} &= \frac{e^x}{x} - \frac{e^x}{x^2} \\ &= \frac{xe^x - e^x}{x^2} \end{aligned}$$

b If  $f'(x) = 0$

$$\frac{xe^x - e^x}{x^2} = 0$$

which implies  $e^x(x-1) = 0$

and as  $e^x \neq 0$  for all  $x \in \mathbb{R}^+$ ,  $x = 1$

c There is a stationary point when  $x = 1$  and  $f(1) = e$ .

Therefore there is a stationary point at  $(1, e)$ .

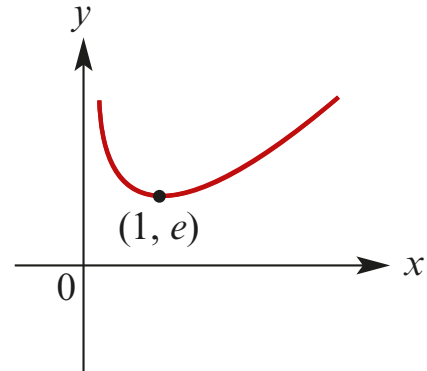
If  $x > 1$   $f'(x) = \frac{e^x(x-1)}{x^2} > 0$  (Note: domain of  $f'$  is  $R^+$ )

If  $x < 1$   $f'(x) = \frac{e^x(x-1)}{x^2} < 0$

$\therefore$  there is a minimum at  $(1, e)$

$$\begin{aligned} \text{d i } \frac{f'(x)}{f(x)} &= \frac{xe^x - e^x}{x^2} \times \frac{x}{e^x} \\ &= \frac{x-1}{x} \end{aligned}$$

$$\begin{aligned} \text{ii } \lim_{x \rightarrow \infty} \frac{f'(x)}{f(x)} &= \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right) = 1 \\ \therefore f'(x) &\approx f(x) \text{ when } x \text{ is very large} \end{aligned}$$



$$\text{e } n = \frac{ae^{kt}}{t}$$

$$\text{When } t = 65, n = \frac{ae^{65k}}{65}$$

$$\text{When } t = 30, n = \frac{ae^{30k}}{30}$$

The population of birds is the same for the years 1930 and 1965.

$$\therefore \frac{ae^{65k}}{65} = \frac{ae^{30k}}{30}$$

$$\therefore e^{35k} = \frac{65}{30} = \frac{13}{6}$$

$$\therefore 35k = \log_e\left(\frac{13}{6}\right)$$

$$\therefore k = \frac{1}{35} \log_e\left(\frac{13}{6}\right) \approx 0.0221$$

f Minimum occurs when  $\frac{dn}{dt} = 0$

Using the quotient rule

$$\frac{dn}{dt} = \frac{a(kte^{kt} - e^{kt})}{t^2}$$

$$= \frac{ae^{kt}(kt - 1)}{t^2}$$

$$\frac{dn}{dt} = 0 \text{ implies } t = \frac{1}{k}$$

This is a local minimum

$$\text{and } t = \frac{1}{\frac{1}{35} \log_e\left(\frac{13}{6}\right)} = \frac{35}{\log_e\left(\frac{13}{6}\right)} \approx 45.27$$

The minimum population occurred in 1945.

**43 a** When  $t = 0$ ,  $N = 1000$

When  $t = 5$ ,  $N = 10\,000$

As  $N = Ae^{kt}$

$$1000 = Ae^0$$

which implies  $A = 1000$

$$\text{Also } 10\,000 = 1000e^{5k}$$

which implies

$$e^{5k} = 10$$

$$\therefore k = \frac{1}{5} \log_e 10$$

$$\text{i.e. } A = 1000 \text{ and } k = \frac{1}{5} \log_e 10 \approx 0.46$$

**b**  $\frac{dN}{dt} = kAe^{kt}$ , where  $A = 1000$  and  $k = \frac{1}{5} \log_e 10$

**c**  $\frac{dN}{dt} = kN$  as  $N = Ae^{kt}$

**d i** When  $t = 4$

$$\frac{dN}{dt} = \frac{1}{5} \log_e 10 \times 1000e^{\frac{4}{5} \log_e 10}$$

$$= 200 \log_e 10 \times 10^{\frac{4}{5}}$$

i.e. the rate of growth when  $t = 4$  is 2905.7 bacteria/hour.

**ii** When  $t = 50$

$$\frac{dN}{dt} = 200(\log_e 10)e^{\frac{50}{5} \log_e 10}$$

$$= 200 \log_e 10 \times 10^{10}$$

$$= 2 \log_e 10 \times 10^{12}$$

$$\approx 4.61 \times 10^{12}$$

i.e. the rate of growth when  $t = 50$  is  $4.61 \times 10^{12}$  bacteria/hour.

**44 a** For the populations to be equal

$$2 \times 10^4 e^{0.03t} = 10^4 e^{0.05t}$$

which implies

$$e^{0.02t} = 2$$

$$\therefore t = \frac{1}{0.02} \log_e 2 \approx 34.657$$

The populations will be equal after 34.66 years.

**b**  $\frac{dA}{dt} = 600e^{0.03t}$  and  $\frac{dB}{dt} = 500e^{0.05t}$

Rates are equal implies

$$\frac{dA}{dt} = \frac{dB}{dt}$$

$$\therefore 600e^{0.03t} = 500e^{0.05t}$$

$$\frac{6}{5} = e^{0.02t}$$

which implies

$$t = \frac{1}{0.02} \log_e\left(\frac{6}{5}\right)$$

$$= 50 \log_e\left(\frac{6}{5}\right) \approx 9.116$$

Rates are equal after 9.12 years.

**45**  $h = 0.5 + 0.2 \sin(3\pi t) \geq 0$

where  $h$  metres is the height of the particular time  $t$  seconds.

**a** The greatest height above the floor is reached at values of  $t$  such that  $\sin(3\pi t) = 1$   
i.e. the greatest height is 0.7 metres

and this occurs when

$$3\pi t = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \dots$$

$$t = \frac{1}{6} \text{ or } \frac{5}{6} \text{ or } \dots$$

The question required the first time and this is given by  $t = \frac{1}{6}$

**b** Period of oscillation is  $2\pi \div 3\pi = \frac{2}{3}$  seconds.

**c**  $\frac{dh}{dt} = 0.6\pi \cos(3\pi t)$

**i** When  $t = \frac{1}{3}$ ,  $\frac{dh}{dt} = 0.6\pi \cos(\pi) = -0.6\pi$

The speed is  $0.6\pi$  metres/second moving downwards.

**ii** When  $t = \frac{2}{3}$ ,  $\frac{dh}{dt} = 0.6\pi \cos(2\pi) = 0.6\pi$

The speed is  $0.6\pi$  metres/second moving upwards.

**iii** When  $t = \frac{1}{6}$ ,  $\frac{dh}{dt} = 0.6\pi \cos\left(\frac{\pi}{2}\right) = 0$

The speed is 0 metres/second.

46  $T(t) = p + q \cos(\pi r t)$  where  $p, q$  and  $r$  are constants

a From the graph:

i The period is 12  $\therefore \frac{2\pi}{\pi r} = 12$   
 which implies  $r = \frac{1}{6}$

ii The amplitude is  $\frac{(20 - 4)}{2} = 8$  which implies  $q = 8$   
 The centre is  $T = \frac{20 + 4}{2} = 12$  which implies  $p = 12$

b  $T'(t) = -\pi r q \sin(\pi r t)$

$$\begin{aligned} \therefore T'(3) &= -\frac{4\pi}{3} \sin\left(\frac{3\pi}{6}\right) \\ &= -\frac{4\pi}{3} \sin\left(\frac{\pi}{2}\right) = -\frac{4\pi}{3} \end{aligned}$$

The hours of night are decreasing at a rate of  $\frac{4\pi}{3}$  hours/month when  $t = 3$

$$T'(9) = -\frac{4\pi}{3} \sin\left(\frac{3\pi}{2}\right) = \frac{4\pi}{3}$$

The hours of night are increasing at a rate of  $\frac{4\pi}{3}$  hours/month when  $t = 9$

c Average rate of change from  $t = 0$  to  $t = 6$

$$= \frac{T(6) - T(0)}{6 - 0}$$

when  $T(t) = 12 + 8 \cos\left(\frac{\pi t}{6}\right)$

$$\begin{aligned} \therefore \text{Average rate of change} &= \frac{12 + 8 \cos(\pi) - (12 + 8 \cos(0))}{6} \\ &= \frac{12 - 8 - 12 - 8}{6} \\ &= -\frac{16}{6} = -\frac{8}{3} \end{aligned}$$

i.e. the average rate of change for time interval  $[0, 6]$  is  $-\frac{8}{3}$  hours/month.

d  $T'(t) = -\frac{4\pi}{3} \sin\left(\frac{\pi t}{6}\right)$

Rate of change of hours is maximum (in the sense of maximum increasing rate)

when  $\sin\left(\frac{\pi t}{6}\right) = -1$

This occurs when

$$\frac{\pi t}{6} = \frac{3\pi}{2} \text{ or } \frac{7\pi}{2} \text{ or } \dots$$

i.e.  $t = 9$  or  $21$  or  $\dots$

The rate of change of hours of night is a maximum after 9 months.

47 a Area  $A = \text{length} \times \text{width}$

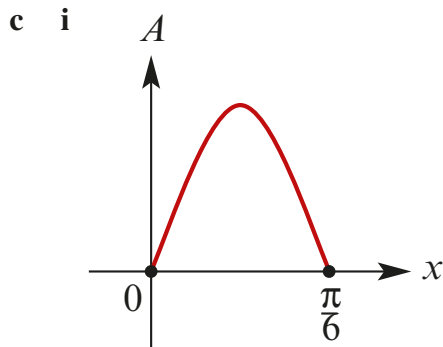
$$= x \times 2 \cos(3x)$$

$$= 2x \cos(3x)$$

b i  $\frac{dA}{dx} = 2 \cos(3x) - 6x \sin(3x)$

ii When  $x = 0$ ,  $\frac{dA}{dx} = 2$

When  $x = \frac{\pi}{6}$ ,  $\frac{dA}{dx} = -6 \times \frac{\pi}{6} \sin \frac{\pi}{2}$   
 $= -\pi$



ii Either use the 'Intersect' feature of a CAS calculator of the graph screen or use the 'solve' command at the calculator screen to solve the equation  $2x \cos 3x = 0.2$ .

$$x = 0.105 \text{ or } x = 0.449$$

iii maximum area is 0.374

Use the 'max' feature of a CAS calculator of the graph screen or use the 'flex' command at the calculator screen with the instruction  $0 < x < \frac{\pi}{6}$ . when  $x = 0.287$

d i  $\frac{dA}{dx} = 2 \cos(3x) - 6x \sin(3x)$   
 $\frac{dA}{dx} = 0$  implies  $\tan(3x) = \frac{1}{3x}$

ii The co-ordinates of the points of intersection are (0.287, 1.16), founded as with c ii.

48 a i  $N'(t) = -1 + \frac{1}{10}e^{\frac{t}{20}}$



**ii**  $N'(t) = 0$  implies  $10 = e^{\frac{t}{20}}$

$\therefore t = 20 \log_e 10 \approx 46.05$

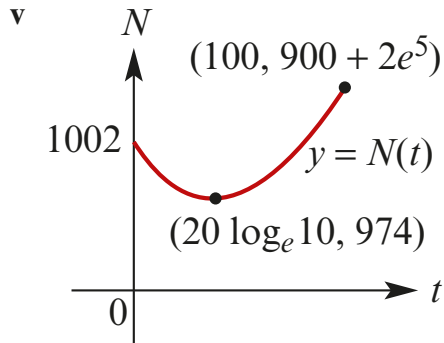
When  $t = 20 \log_e 10$

$$\begin{aligned} N(t) &= 1000 - 20 \log_e 10 + 2e^{\log_e 10} \\ &= 1000 - 20 \log_e 10 + 20 \\ &= 1020 - 20 \log_e 10 \\ &\approx 973.95 \end{aligned}$$

Minimum population is 974

**iii**  $N(0) = 1000 + 2 = 1002$

**iv**  $N(100) = 1000 - 100 + 2e^5$   
 $= 900 + 2e^5$   
 $\approx 1196.826$



**b**  $N_2(t) = 1000 - t^{\frac{1}{2}} + 2e^{\frac{t}{20}}$

**i**  $N_2(0) = 1000 - 0 + 2$   
 $= 1002$

**ii**  $N_2(100) = 1000 - 10 + 2e^{\frac{1}{2}}$   
 $= 990 + 2e^{\frac{1}{2}}$

**iii**  $N_2'(t) = -\frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{2}t^{-\frac{1}{2}} \times \frac{1}{20} \times 2e^{\frac{t}{20}}$   
 $= \frac{1}{2}t^{-\frac{1}{2}} \left( -1 + \frac{1}{10}e^{\frac{t}{20}} \right)$

$$N_2'(t) = 0 \text{ implies } e^{\frac{t}{20}} = 10$$

$$\therefore \frac{t}{20} = \log_e 10$$

$$\therefore t = 20 \log_e 10$$

$$\therefore t = (20 \log_e 10)^2$$

When  $t = (20 \log_e 10)^2$

$$\begin{aligned} N_2(t) &= 1000 - 20 \log_e 10 + 2e^{\log_e 10} \\ &= 1000 - 20 \log_e 10 + 20 \end{aligned}$$

Minimum population is 974

**c**  $N_3(t) = 1000 - t^{\frac{3}{2}} + 2e^{\frac{t}{20}}$

**i** Using a CAS calculator with the 'min' feature at the graph screen, the minimum population is

297 when  $t = 100.24$

**d i**  $N_3'(t) = -\frac{3}{2}t^{\frac{1}{2}} + \frac{1}{10}e^{\frac{t}{20}}$

**ii**  $N_3'(t) = 0$

$$\frac{3}{2}t^{\frac{1}{2}} = \frac{1}{10}e^{\frac{t}{20}}$$

$$15t^{\frac{1}{2}} = e^{\frac{t}{20}}$$

$$t = 20 \log_e(15 \sqrt{t})$$

**49 a**  $y = (2x^2 - 5x)e^{ax}$

$(3, 10): 10 = 3e^{3a}$

$$e^{3a} = \frac{10}{3}$$

$$a = \frac{1}{3} \log e\left(\frac{10}{3}\right)$$

**b i**  $y = 0: 2x^2 - 5x = 0$  (since  $e^{ax} > 0$ )

$$x(2x - 5) = 0$$

$$x = 0, \frac{5}{2}$$

**ii**  $\frac{dy}{dx} = (4x - 5)e^{ax} + (2x^2 - 5x) \times ae^{ax}$

$$= (2ax^2(4 - 5a)x - 5)e^{ax}$$

$$= 0 \text{ if}$$

$$2ax^2 + (4 - 5a)x - 5 = 0$$

$$x = \frac{-4 + 5a \pm \sqrt{16 - 40a + 25a^2 + 40a}}{4a}$$

$$= \frac{-4 + 5a \pm \sqrt{25a^2 + 16}}{4a}$$

# Chapter 11 – Integration

## Solutions to Exercise 11A

$$\begin{aligned}
 \mathbf{1 \ a} \quad A &\approx \frac{1}{2} \times \frac{1}{2} (2.5)^2 + \frac{1}{2} \times \frac{1}{2} (3)^2 \\
 &= \frac{1}{4} \times \frac{25}{4} + \frac{1}{4} \times 9 \\
 &= \frac{25}{16} + \frac{9}{4} \\
 &= \frac{25}{16} + \frac{36}{16} \\
 A &\approx \frac{61}{16} \approx 3.81 \text{ square units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad A &\approx \frac{\pi}{4} \times \cos 0 + \frac{\pi}{4} \times \cos \frac{\pi}{4} \\
 &= \frac{\pi}{4} + \frac{\pi}{4} \times \frac{1}{\sqrt{2}} \\
 A &\approx \frac{(1 + \sqrt{2}\pi)\pi}{4\sqrt{2}} \approx 1.34 \text{ square units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad A &\approx 1 \times \frac{1}{2} (2)^3 + 1 \times \frac{1}{2} (3)^3 \\
 &= 4 + \frac{27}{2} \\
 A &\approx \frac{35}{2} \approx 17.5 \text{ square units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2 \ a} \quad A &\approx 1 \times (f(1) + f(2) + f(3) + f(4)) \\
 &= 5 + 3.5 + 2.5 + 2.2 \\
 A &\approx 13.2 \text{ square units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad A &\approx 1 \times (f(2) + f(3) + f(4) + f(4)) \\
 &= 3.5 + 2.5 + 2.2 + 2 \\
 A &\approx 10.2 \text{ square units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3 \ a} \quad y &= x(4 - x) \\
 &\text{using left end estimate}
 \end{aligned}$$

$$\begin{aligned}
 A &\approx 1.0 \times ((0)(4 - 0) + (1)(4 - 1) \\
 &\quad + (2)(4 - 2) + (3)(4 - 3)) \\
 &= 0 + 3 + 4 + 3
 \end{aligned}$$

$$\begin{aligned}
 A &\approx 10 \text{ square units} \\
 &\text{using right end estimate} \\
 A &\approx 1.0 \times ((1)(4 - 1) + (2)(4 - 2) \\
 &\quad + (3)(4 - 3) + (4)(4 - 4)) \\
 &= 3 + 4 + 3 + 0
 \end{aligned}$$

$$A \approx 10 \text{ square units}$$

$$\begin{aligned}
 \mathbf{b} \quad &\text{using the CAS calculator} \\
 &10.64 \text{ square units}
 \end{aligned}$$

$$\mathbf{4 \ a} \quad y = \frac{1}{1 + x^2}$$

$$\begin{aligned}
 A &\approx 0.25 \left( \frac{1}{1 + (0.25)^2} + \frac{1}{1 + (0.5)^2} \right. \\
 &\quad \left. + \frac{1}{1 + (0.75)^2} + \frac{1}{1 + (1)^2} \right) \\
 &= \frac{1}{4} \left( \frac{1}{\left(\frac{17}{16}\right)} + \frac{1}{\left(\frac{5}{4}\right)} + \frac{1}{\left(\frac{25}{16}\right)} + \frac{1}{2} \right) \\
 &= \frac{1}{4} \left( \frac{16}{17} + \frac{4}{5} + \frac{16}{25} + \frac{1}{2} \right) \\
 &= \frac{14}{17} + \frac{1}{5} + \frac{4}{25} + \frac{1}{8} \\
 A &\approx \frac{2449}{3400} \approx 0.72 \text{ square units}
 \end{aligned}$$

$$\mathbf{b} \quad \frac{\pi}{4} = A \approx 0.72 \text{ square units}$$

$$\pi \approx 2.88$$

*this approximation could be improved by adding more sub-intervals to the area calculation*

**5 a**  $A \approx 1.0 \times (3 + 3.5 + 3.7 + 3.8 + 3.9$   
 $+ 3.9 + 4.0 + 4.0 + 3.7 + 3.3)$

$A \approx 36.8$  square units

**b**  $A \approx 1.0 \times (3.5 + 3.7 + 3.8 + 3.9 + 3.9$   
 $+ 4.0 + 4.0 + 3.7 + 3.3 + 2.9)$

$A \approx 36.7$  square units

**6 a**

$y = 2^x$

$A \approx 0.5 (2^{0.5} + 2^1 + 2^{1.5} + 2^2 + 2^{2.5} + 2^3)$

$= \frac{1}{\sqrt{2}} + 1 + \sqrt{2} + 2 + 2\sqrt{2} + 4$

$A \approx 7 + \frac{7\sqrt{2}}{2} \approx 11.9$  square units

**7 a**  $A \approx 1.0 \times (4 + 5.6 + 7 + 8 + 8.2$   
 $+ 8.1 + 7.6)$

$A \approx 48.5$  square units

*note: since the values are read off the graph, the value of A is approximate*

**b** this area represents the distance travelled between  $t = 2$  and  $t = 9$

**8 a** The definite integral represents the triangular region shown.

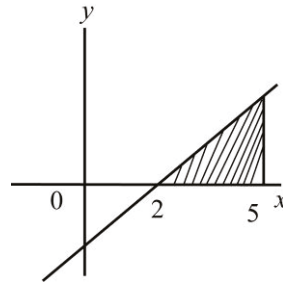
The triangle has base  $5 - 2 = 3$  units.

When  $x = 5$ ,  $y = 5 - 2 = 3$ , so the triangle has height 3 units Area

$= \frac{1}{2} \times 3 \times 3$

$= \frac{9}{2}$  square units

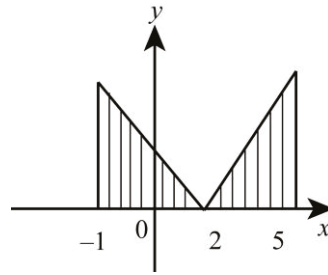
Hence  $\int_2^5 (x - 2) dx = \frac{9}{2}$ .



**b** The definite integral represents two equal triangular regions shown.

Formal, area  $= 2 \times \frac{9}{2} = 9$  square units

Hence  $\int_{-1}^5 |x - 2| dx = 9$ .



**c** The definite integral represents the trapezium region shown.

The distance between the parallel sides is 1 unit.

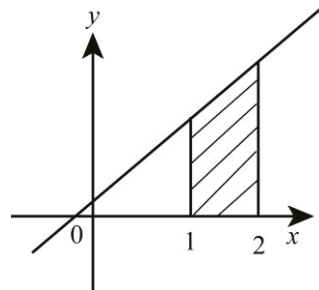
When  $x = 1$ ,  $y = 2 + 1 = 3$  units.

When  $x = 2$ ,  $y = 4 + 1 = 5$  units.

Area  $= \frac{1}{2} \times 1 \times (3 + 5)$

$= 4$  square units

Hence  $\int_1^2 (2x + 1) dx = 4$



## Solutions to Exercise 11B

$$1 \text{ a } \int \frac{1}{2}x^3 dx$$

$$= \frac{1}{2} \times \frac{1}{4}x^4 + c$$

$$= \frac{1}{8}x^4 + c$$

$$b \int 5x^3 - 2x dx$$

$$= 5 \times \frac{x^4}{4} - 2 \times \frac{x^2}{2} + c$$

$$= \frac{5}{4}x^4 - x^2 + c$$

$$c \int \frac{4}{5}x^3 - 3x^2 dx$$

$$= \frac{4}{5} \times \frac{x^4}{4} - 3 \frac{x^3}{3} + c$$

$$= \frac{1}{5}x^4 - x^3 + c$$

$$d \int 6z - 3z^2 - z + 2 dz$$

$$= \int 3z^2 + 5z + 2 dz$$

$$= -3 \frac{z^3}{3} + 5 \frac{z^2}{2} + 2z + c$$

$$= -z^3 + \frac{5}{2}z^2 + 2z + c$$

$$2 \text{ a } \frac{dy}{dx} = x^{-3}$$

$$y = -\frac{1}{2}x^{-2} + c = -\frac{1}{2x^2} + c$$

$$b \frac{dy}{dx} = 4\sqrt[3]{x} = 4x^{\frac{1}{3}}$$

$$y = 4 \times \frac{3}{4}x^{\frac{4}{3}} + c = 3x^{\frac{4}{3}} + c$$

$$c \frac{dy}{dx} = x^{\frac{1}{4}} + x^{-\frac{3}{5}}$$

$$y = \frac{4}{5}x^{\frac{5}{4}} + \frac{5}{2}x^{\frac{2}{5}} + c$$

$$3 \text{ a } \int 3x^{-2} dx$$

$$= 3 \frac{x^{-1}}{-1} + c$$

$$= \frac{-3}{x} + c$$

$$b \int 2x^{-4} + 6x dx$$

$$= 2 \frac{x^{-3}}{-3} + 6 \frac{x^2}{2} + c$$

$$= \frac{-2}{3}x^{-3} + 3x^2 + c$$

$$c \int 2x^{-2} + 6x^{-3} dx$$

$$= -2x^{-1} - 3x^{-2} + c$$

$$d \int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}} dx$$

$$= 3 \frac{x^{\frac{4}{3}}}{\frac{4}{3}} - 5 \frac{x^{\frac{9}{4}}}{\frac{9}{4}} + c$$

$$= \frac{9}{4}x^{\frac{4}{3}} - \frac{20}{9}x^{\frac{9}{4}} + c$$

$$e \int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}} dx$$

$$= \frac{12}{7}x^{\frac{7}{4}} - \frac{14}{3}x^{\frac{3}{2}} + c$$

$$f \int 4x^{\frac{3}{5}} + 12x^{\frac{5}{3}} dx$$

$$= \frac{5}{2}x^{\frac{8}{5}} + \frac{9}{2}x^{\frac{8}{3}} + c$$

$$\begin{aligned}
 \mathbf{4\ a} \quad \frac{dy}{dx} &= 2x - 3 \\
 y &= \frac{2x^2}{2} - 3x + c \\
 &= x^2 - 3x + c \\
 x = 1, y &= 1 \\
 1 &= 1 - 3 + c \\
 c &= 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{dy}{dx} &= x^3 \\
 y &= \frac{x^4}{4} + c \\
 x = 0, y &= 6 \\
 6 &= c \\
 y &= \frac{x^4}{4} + 6
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= \frac{x^{\frac{3}{2}}}{\frac{2}{2}} + \frac{x^2}{2} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 + c \\
 x = 4, y &= 6 \\
 6 &= \frac{2}{3} \times 8 + \frac{1}{2} \times 16 + c \\
 6 &= \frac{16}{3} + 8 + c \\
 c &= -2 - \frac{16}{3} = \frac{-22}{3} \\
 y &= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 - \frac{22}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5\ a} \quad \int \sqrt{x}(2+x) dx &= \int 2x^{\frac{1}{2}} + x^{\frac{3}{2}} dx \\
 &= \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int \frac{3z^4 + 2z}{z^3} dz &= \int 3z + 2z^{-2} dz \\
 &= \frac{3z^2}{2} - 2z^{-1} + c \\
 &= \frac{3z^2}{2} - \frac{2}{z} + c \\
 &= \frac{3z^3 - 4}{2z} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int \frac{5x^3 + 2x^2}{x} dx &= \int 5x^2 + 2x dx \\
 &= \frac{5x^3}{3} + x^2 + c \\
 &= \frac{5x^3 + 3x^2}{3} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \int \sqrt{x}(2x + x^2) dx &= \int 2x^{\frac{3}{2}} + x^{\frac{5}{2}} dx \\
 &= \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad \int x^2(2 + 3x^2) dx &= \int 2x^2 + 3x^4 dx \\
 &= \frac{2x^3}{3} + \frac{3x^5}{5} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad \int \sqrt[3]{x}(x + x^4) dx &= \int x^{\frac{4}{3}} + x^{\frac{13}{3}} dx \\
 &= \frac{3}{7}x^{\frac{7}{3}} + \frac{3}{16}x^{\frac{16}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad f'(x) &= 3x^2 - x^{-2} \\
 f(x) &= 3\frac{x^3}{3} - \frac{x^{-1}}{-1} + c
 \end{aligned}$$

$$= x^3 + \frac{1}{x} + c$$

$$f(2) = 0$$

$$0 = 8 + \frac{1}{2} + c$$

$$c = \frac{-17}{2}$$

$$f(x) = x^3 + \frac{1}{x} - \frac{17}{2}$$

$$7 \quad \frac{ds}{dt} = 3t - \frac{8}{t^2} = 3t - 8t^{-2}$$

$$s = 3\frac{t^2}{2} - \frac{8t^{-1}}{-1} + c$$

$$= \frac{3}{2}t^2 + \frac{8}{t} + c$$

$$t = 1, s = \frac{3}{2}$$

$$\frac{3}{2} = \frac{3}{2} + 8 + c$$

$$c = -8$$

$$s = \frac{3}{2}t^2 + \frac{8}{t} - 8$$

$$8 \quad \mathbf{a} \quad f'(x) = 16x + k$$

$$f'(2) = 0$$

$$0 = 32 + k$$

$$k = -32$$

$$f'(x) = 16x - 32$$

$$\mathbf{b} \quad f(x) = 16\frac{x^2}{2} - 32x + c$$

$$= 8x^2 - 32x + c$$

$$f(2) = 1$$

$$1 = 32 - 64 + c$$

$$c = 33$$

$$f(x) = 8x^2 - 32x + 33$$

$$f(7) = 8 \times 49 - 32 \times 7 + 33 \\ = 201$$



## Solutions to Exercise 11C

$$\begin{aligned} \mathbf{1 a} \quad & \int (2x - 1)^2 dx \\ &= \frac{1}{2 \times 3} (2x - 1)^3 + c \\ &= \frac{1}{6} (2x - 1)^3 + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int (2 - t)^3 dt \\ &= \frac{1}{-1 \times 4} (2 - t)^4 + c \\ &= -\frac{1}{4} (2 - t)^4 + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int (5x - 2)^3 dx \\ &= \frac{1}{5 \times 4} (5x - 2)^4 + c \\ &= \frac{1}{20} (5x - 2)^4 + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int (4x - 6)^{-2} dx \\ &= -\frac{1}{4} (4x - 6)^{-1} + c \\ &= -\frac{1}{4(4x - 6)} + c \\ &= \frac{1}{24 - 16x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \int (6 - 4x)^{-3} dx \\ &= \frac{1}{8} (6 - 4x)^{-2} + c \\ &= \frac{1}{8(6 - 4x)^2} + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \int (4x + 3)^{-3} dx \\ &= -\frac{1}{8} (4x + 3)^{-2} + c \\ &= -\frac{1}{8(4x + 3)^2} + c \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \int (3x + 6)^{\frac{1}{2}} dx \\ &= \frac{1}{3 \times \frac{3}{2}} (3x + 6)^{\frac{3}{2}} + c \\ &= \frac{2}{9} (3x + 6)^{\frac{3}{2}} + c \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \int (3x + 6)^{-\frac{1}{2}} dx \\ &= \frac{1}{3 \times \frac{1}{2}} (3x + 6)^{\frac{1}{2}} + c \\ &= \frac{2}{3} (3x + 6)^{\frac{1}{2}} + c \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \int (2x - 4)^{\frac{7}{2}} dx \\ &= \frac{1}{2 \times \frac{9}{2}} (2x - 4)^{\frac{9}{2}} + c \\ &= \frac{1}{9} (2x - 4)^{\frac{9}{2}} + c \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & \int (3x + 11)^{\frac{4}{3}} dx \\ &= \frac{1}{3 \times \frac{7}{3}} (3x + 11)^{\frac{7}{3}} + c \\ &= \frac{1}{7} (3x + 11)^{\frac{7}{3}} + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad & \int (2 - 3x)^{\frac{1}{2}} dx \\
 &= \frac{1}{-3 \times \frac{3}{2}} (2 - 3x)^{\frac{3}{2}} + c \\
 &= \frac{-2}{9} (2 - 3x)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad & \int (5 - 2x)^4 dx \\
 &= \frac{1}{-2 \times 5} (5 - 2x)^5 + c \\
 &= \frac{-1}{10} (5 - 2x)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2 a} \quad & \int \frac{1}{2} x^{-1} dx \\
 &= \frac{1}{2} \log_e x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{1}{3x + 2} dx \\
 &= \frac{1}{3} \log_e(3x + 2) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \frac{4}{1 + 4x} dx \\
 &= \log_e(1 + 4x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{5}{3x - 2} dx \\
 &= \frac{5}{3} \log_e(3x - 2) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int \frac{3}{1 - 4x} dx \\
 &= -\frac{3}{4} \log_e(1 - 4x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int \frac{3}{2 - \frac{x}{2}} dx \\
 &= -6 \log_e\left(\frac{4 - x}{2}\right) + c \\
 &= -6 \log_e(x - 4) + c_2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3 a} \quad & \int \frac{5}{x} dx \\
 &= 5 \log_e |x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{3}{x - 4} dx \\
 &= 3 \log_e |x - 4| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \frac{10}{2x + 1} dx \\
 &= \frac{10}{2} \log_e |2x + 1| + c \\
 &= 5 \log_e |2x + 1| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{6}{5 - 2x} dx \\
 &= \frac{6}{-2} \log_e |5 - 2x| + c \\
 &= -3 \log_e |2x - 5| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int 6(1 - 2x)^{-1} dx \\
 &= -3 \log_e |1 - 2x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int (4 - 3x)^{-1} dx \\
 &= \frac{1}{-3} \ln |4 - 3x| + c \\
 &= \frac{-1}{3} \ln |3x - 4| + c
 \end{aligned}$$

$$\mathbf{4 a} \quad 3x + \log_e |x| + c$$

$$\mathbf{b} \quad x + \log_e |x| + c$$

$$\mathbf{c} \quad -\frac{1}{x + 1} + c$$

$$\mathbf{d} \quad 2x + \frac{x^2}{2} + \log_e |x| + c$$

$$\mathbf{e} \quad -\frac{3}{2(x - 1)^2} + c$$

$$\mathbf{f} \quad -2x + \log_e |x| + c$$

$$\mathbf{5} \quad \mathbf{a} \quad \frac{dy}{dx} = \frac{1}{2x} \quad x > 0$$

$$y = \frac{1}{2} \log_e x + c$$

$$x = e^2, y = 2$$

$$2 = \frac{1}{2} \log_e e^2 + c$$

$$2 = \frac{1}{2} \times 2 + c$$

$$c = 1$$

$$y = \frac{1}{2} \log_e x + 1, x > 0$$

$$\mathbf{b} \quad \frac{dy}{dx} = \frac{2}{5-2x}$$

$$y = \frac{2}{-2} \log_e |5-2x| + c$$

$$y = -\log_e |2x-5| + c$$

$$x = 2, y = 10$$

$$10 = -\log_e 1 + c$$

$$c = 10$$

$$y = -\log_e 2x - 5 + 10, x < \frac{5}{2}$$

$$\mathbf{6} \quad f'(x) = \frac{10}{x-5}$$

$$f(x) = 10 \log_e |x-5| + c$$

$$f(5+e) = 10$$

$$10 = 10 \log_e e + c$$

$$c = 0$$

$$f(x) = 10 \log_e x - 5, x > 5$$

$$\mathbf{7} \quad \mathbf{a} \quad \int \frac{x}{x+1} dx$$

$$= \int 1 - \frac{1}{x+1} dx$$

$$= x - \log_e |x+1| + c$$

$$\mathbf{b} \quad \int \frac{1-2x}{x+1} dx$$

$$= \int -2 + \frac{3}{x+1} dx$$

$$= -2x + 3 \log_e |x+1| + c$$

$$\mathbf{c} \quad \int \frac{2x+1}{x+1} dx$$

$$= \int 2 - \frac{1}{x+1} dx$$

$$= 2x - \log_e |x+1| + c$$

$$\mathbf{8} \quad \frac{dy}{dx} = \frac{3}{x-2}$$

$$y = 3 \log_e |x-2| + c$$

$$x = 0, y = 10$$

$$10 = 3 \log_e |-2| + c$$

$$c = 10 - 3 \log_e 2$$

$$y = 3 \log_e |x-2| + 10 - 3 \log_e 2$$

$$y = 3 \log_e \left( \frac{|x-2|}{2} \right) + 10$$

You can complete it without using the absolute value function.

$$\frac{dy}{dx} = \frac{3}{x-2}$$

$$= -\frac{3}{2-x}$$

$$y = 3 \log_e(2-x) + c$$

$$x = 0, y = 10$$

$$10 = 3 \log_e 2 + c$$

$$c = 10 - 3 \log_e 2$$

$$y = 3 \log_e(2-x) + 10 - 3 \log_e 2$$

$$y = 3 \log_e\left(\frac{2-x}{2}\right) + 10$$

$$9 \quad \frac{dy}{dx} = \frac{5}{2-4x}$$

$$y = \frac{5}{-4} \log_e |2-4x| + c$$

$$y = \frac{-5}{4} \log_e |4x-2| + c$$

$$x = -2, y = 10$$

$$10 = \frac{-5}{4} \log_e |-8-4| + c$$

$$10 = \frac{-5}{4} \log_e 10 + c$$

$$c = 10 + \frac{5}{4} \log_e 10$$

$$y = \frac{5}{4} \log_e 10 - \frac{5}{4} \log_e |4x-2| + 10$$

$$y = \frac{5}{4} \log_e \left| \frac{10}{4x-2} \right| + 10$$

$$y = \frac{5}{4} \log_e \left| \frac{5}{2x-1} \right| + 10$$

To satisfy the conditions you can write the rule as  $y = \frac{5}{4} \log_e \frac{5}{1-2x} + 10$

$$10 \quad \frac{dy}{dx} = \frac{5}{2-4x}$$

$$y = -\frac{5}{4} \log_e |2x-1| + c$$

$$x = 1, y = 10$$

$$10 = -\frac{5}{4} \log_e |1| + c$$

$$c = 10$$

$$y = -\frac{5}{4} \log_e |2x-1| + 10$$

$$y = \frac{5}{4} \log_e \left| \frac{1}{2x-1} \right| + 10$$

To satisfy the conditions you can write the rule as  $y = \frac{5}{4} \log_e \frac{1}{2x-1} + 10$

## Solutions to Exercise 11D

1 a  $\frac{1}{6}e^{6x} + c$

b  $\frac{1}{2}e^{2x} + \frac{3}{2}x^2 + c$

c  $-\frac{1}{3}e^{-3x} + x^2 + c$

d  $-\frac{1}{2}e^{-2x} + \frac{1}{2}e^{2x} + c$

2 a  $\int e^{2x} - e^{\frac{x}{2}} dx$   
 $= \frac{1}{2}e^{2x} - 2e^{\frac{x}{2}} + c$

b  $\int e^x + e^{-x} dx$   
 $= e^x - e^{-x} + c$

c  $\int 2e^{3x} - e^{-x} dx$   
 $= \frac{2}{3}e^{3x} + e^{-x} + c$

d  $\int 5e^{\frac{x}{3}} - 3e^{\frac{x}{5}} dx$   
 $= 15e^{\frac{x}{3}} - 15e^{\frac{x}{5}} + c$

e  $\frac{9}{2}e^{\frac{2x}{3}} - \frac{15}{7}e^{\frac{7x}{5}} + c$

f  $\frac{15}{4}e^{\frac{4x}{3}} - \frac{9}{2}e^{\frac{2x}{3}} + c$

3 a  $\frac{dy}{dx} = e^{2x} - x$

$$y = \frac{1}{2}e^{2x} - \frac{x^2}{2} + c$$

$$x = 0, y = 5$$

$$5 = \frac{1}{2} - 0 + c$$

$$c = \frac{9}{2}$$

$$y = \frac{1}{2}(e^{2x} - x^2 + 9)$$

b  $\frac{dy}{dx} = 3e^{-x} - e^x$

$$y = -3e^{-x} - e^x + c$$

$$x = 0, y = 4$$

$$4 = -3 - 1 + c$$

$$c = 8$$

$$y = -3e^{-x} - e^x + 8$$

$$4 \quad \frac{dy}{dx} = ae^{-x} + 1$$

$$x = 0, \frac{dy}{dx} = 3$$

$$3 = a + 1$$

$$a = 2$$

$$\frac{dy}{dx} = 2e^{-x} + 1$$

$$y = -2e^{-x} + x + c$$

$$x = 0, y = 5$$

$$5 = -2 + 0 + c$$

$$c = 7$$

$$y = -2e^{-x} + x + 7$$

$$x = 2,$$

$$y = -2e^{-2} - 2 + 7$$

$$y = 9 - \frac{2}{e^2}$$

$$5 \quad \frac{dy}{dx} = e^{kx}$$

$$a \quad x = 1,$$

$$\frac{dy}{dx} = e^k$$

**Tangent**

$$y = e^k x + c$$

$$x = 0, y = 0$$

$$0 = c$$

$$y = e^k x$$

$$x = 1, y = e^2$$

$$e^2 = e^k$$

$$k = 2$$

$$\frac{dy}{dx} = e^{2x}$$

$$b \quad y = \frac{1}{2}e^{2x} + c$$

$$x = 1, y = e^2$$

$$e^2 = \frac{1}{2}e^2 + c$$

$$c = \frac{1}{2}e^2$$

$$y = \frac{1}{2}e^{2x} + \frac{1}{2}e^2$$

$$6 \quad \frac{dy}{dx} = -e^{kx}$$

$$a \quad x = 1,$$

$$\frac{dy}{dx} = -e^k$$

**Tangent**

$$y = -e^k x + c$$

$$x = 0, y = 0$$

$$0 = c$$

$$y = -e^k x$$

$$x = 1, y = -e^3$$

$$-e^3 = -e^k$$

$$k = 3$$

$$\frac{dy}{dx} = -e^{3x}$$

$$b \quad y = -\frac{1}{3}e^{3x} + c$$

$$x = 1, y = -e^3$$

$$-e^3 = -\frac{1}{3}e^3 + c$$

$$c = -\frac{2}{3}e^3$$

$$y = -\frac{1}{3}e^{3x} - \frac{2}{3}e^3$$

## Solutions to Exercise 11E

$$\begin{aligned} \mathbf{1 a} \quad & \int_1^2 x^2 dx \\ &= \left[ \frac{x^3}{3} \right]_1^2 \\ &= \frac{2^3}{3} - \frac{1^3}{3} \\ &= \frac{7}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int_{-1}^3 x^3 dx \\ &= \left[ \frac{x^4}{4} \right]_{-1}^3 \\ &= \frac{81}{4} - \frac{1}{4} \\ &= 20 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int_0^1 x^3 - x dx \\ &= \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_0^1 \\ &= \left( \frac{1}{4} - \frac{1}{2} \right) - (0 - 0) \\ &= -\frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int_{-1}^2 (x+1)^2 dx \\ &= \left[ \frac{1}{3}(x+1)^3 \right]_{-1}^2 \\ &= \frac{1}{3}(3)^3 - \frac{1}{3}(0)^3 \\ &= 9 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \int_1^2 \frac{1}{x^2} dx \\ &= \left[ \frac{-1}{x} \right]_1^2 \\ &= \frac{-1}{2} - \frac{-1}{1} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \int_1^4 x^{\frac{1}{2}} + 2x^2 dx \\ &= \left[ \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}x^3 \right]_1^4 \\ &= \frac{2}{3} \left( 4^{\frac{3}{2}} + 4^3 \right) - \frac{2}{3} \left( 1^{\frac{3}{2}} + 1^3 \right) \\ &= \frac{2}{3}(8 + 64) - \frac{4}{3} \\ &= \frac{140}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \int_0^2 x^3 + 2x^2 + x + 2 dx \\ &= \left[ \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + 2x \right]_0^2 \\ &= \frac{16}{4} + \frac{2}{3} \times 8 + \frac{4}{2} + 2 \times 2 - 0 \\ &= 4 + \frac{16}{3} + 2 + 4 \\ &= \frac{46}{3} = 15\frac{1}{3} \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int_1^4 2x^{\frac{3}{2}} + 5x^3 dx \\
 & = \left[ \frac{4}{5}x^{\frac{5}{2}} + \frac{5}{4}x^4 \right]_1^4 \\
 & = \left( \frac{4}{5} \times 32 + \frac{5}{4} \times 256 \right) - \left( \frac{4}{5} + \frac{5}{4} \right) \\
 & = \frac{128}{5} + 320 - \frac{4}{5} - \frac{5}{4} \\
 & = \frac{6871}{20} = 343\frac{11}{20}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_0^1 (3 - 2x)^{-2} dx \\
 & = \left[ \frac{1}{2}(3 - 2x)^{-1} \right]_0^1 \\
 & = \frac{1}{2}(1)^{-1} - \frac{1}{2}(3)^{-1} \\
 & = \frac{1}{2} - \frac{1}{6} \\
 & = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2 a} \quad & \int_0^1 (2x + 1)^3 dx \\
 & = \left[ \frac{1}{8}(2x + 1)^4 \right]_0^1 \\
 & = \frac{1}{8}(3)^4 - \frac{1}{8}(1)^4 \\
 & = \frac{81}{8} - \frac{1}{8} \\
 & = 10
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int_0^2 (3 + 2x)^{-3} dx \\
 & = \left[ \frac{1}{-4}(3 + 2x)^{-2} \right]_0^2 \\
 & = \frac{-1}{4}(7)^{-2} + \frac{1}{4}(3)^{-2} \\
 & = \frac{1}{36} - \frac{1}{196} \\
 & = \frac{10}{441}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int_0^2 (4x + 1)^{\frac{-1}{2}} dx \\
 & = \left[ \frac{2}{4}(4x + 1)^{\frac{1}{2}} \right]_0^2 \\
 & = \frac{1}{2}(9)^{\frac{1}{2}} - \frac{1}{2}(1)^{\frac{1}{2}} \\
 & = \frac{3}{2} - \frac{1}{2} \\
 & = 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int_{-1}^1 (4x + 1)^3 dx \\
 & = \left[ \frac{1}{16}(4x + 1)^4 \right]_{-1}^1 \\
 & = \frac{1}{16}(5)^4 - \frac{1}{16}(-3)^4 \\
 & = \frac{625}{16} - \frac{81}{16} \\
 & = \frac{544}{16} \\
 & = 34
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int_1^2 (1 - 2x)^2 dx \\
 & = \left[ \frac{1}{-6}(1 - 2x)^3 \right]_1^2 \\
 & = \frac{-1}{6}(-3)^3 + \frac{1}{6}(-1)^3 \\
 & = \frac{27}{6} - \frac{1}{6} \\
 & = \frac{13}{3}
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{g} \quad & \int_0^1 (2-x)^{\frac{1}{2}} dx \\
 & = \left[ \frac{2}{-3} (2-x)^{\frac{3}{2}} \right]_0^1 \\
 & = \frac{-2}{3} (1)^{\frac{3}{2}} + \frac{2}{3} (2)^{\frac{3}{2}} \\
 & = \frac{2}{3} (2^{\frac{3}{2}} - 1) \\
 & \approx 1.22
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int_3^4 (2x-4)^{\frac{-1}{2}} dx \\
 & = \left[ \frac{2}{2} (2x-4)^{\frac{1}{2}} \right]_3^4 \\
 & = \sqrt{4} - \sqrt{2} \\
 & = 2 - \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int_0^1 (3+2x)^{-2} dx \\
 & = \left[ \frac{1}{-2} (3+2x)^{-1} \right]_0^1 \\
 & = \frac{-1}{2} (5)^{-1} + \frac{1}{2} (3)^{-1} \\
 & = \frac{1}{6} - \frac{1}{10} \\
 & = \frac{1}{15}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3 a} \quad & \int_0^1 e^{2x} dx \\
 & = \left[ \frac{1}{2} e^{2x} \right]_0^1 \\
 & = \frac{1}{2} e^2 - \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int_0^1 e^{-2x} + 1 dx \\
 & = \left[ \frac{-1}{2} e^{-2x} + x \right]_0^1 \\
 & = \left( \frac{-1}{2} e^{-2} + 1 \right) - \left( \frac{-1}{2} \right) \\
 & = \frac{3}{2} - \frac{1}{2e^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int_0^1 2e^{\frac{x}{3}} + 2 dx \\
 & = \left[ 6e^{\frac{x}{3}} + 2x \right]_0^1 \\
 & = (6e^{\frac{1}{3}} + 2) - (6) \\
 & = 6e^{\frac{1}{3}} - 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_{-2}^2 \frac{e^x + e^{-x}}{2} dx \\
 & = \left[ \frac{e^x - e^{-x}}{2} \right]_{-2}^2 \\
 & = \frac{e^2 - e^{-2}}{2} - \frac{e^{-2} - e^2}{2} \\
 & = e^2 - e^{-2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4 a} \quad & \int_0^4 h(x) dx = 5 \\
 & \int_0^4 2h(x) dx \\
 & = 2 \int_0^4 h(x) dx = 10
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int_0^4 (h(x) + 3) dx \\
 & = \int_0^4 h(x) dx + \int_0^4 3 dx \\
 & = 5 + 12 \\
 & = 17
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int_4^0 h(x) dx \\
 &= - \int_0^4 h(x) dx \\
 &= -5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_0^4 (h(x) + 1) dx \\
 &= \int_0^4 h(x) dx + \int_0^4 dx \\
 &= 5 + 4 \\
 &= 9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int_0^4 (h(x) - x) dx \\
 &= \int_0^4 h(x) dx - \int_0^4 x dx \\
 &= 5 - \left[ \frac{x^2}{2} \right]_0^4 \\
 &= 5 - 8 \\
 &= -3
 \end{aligned}$$

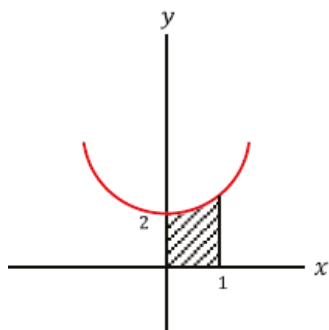
$$\begin{aligned}
 \mathbf{5 a} \quad & \int_0^4 \frac{1}{x-6} dx \\
 &= - \int_0^4 \frac{1}{6-x} dx \\
 &= \left[ \log_e(6-x) \right]_0^4 \\
 &= (\log_e(2) - \log_e(6)) \\
 &= \log_e\left(\frac{1}{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int_2^4 \frac{1}{2x-3} dx \\
 &= \left[ \frac{1}{2} \log_e(2x-3) \right]_2^4 \\
 &= \frac{1}{2} \log_e 5 - \frac{1}{2} \log_e 1 \\
 &= \frac{1}{2} \log_e 5
 \end{aligned}$$

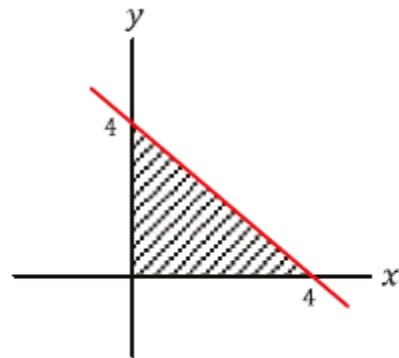
$$\begin{aligned}
 \mathbf{c} \quad & \int_5^6 \frac{3}{2x+7} dx \\
 &= \left[ \frac{3}{2} \log_e(2x+7) \right]_5^6 \\
 &= \frac{3}{2} \log_e 19 - \frac{3}{2} \log_e 17 \\
 &= \frac{3}{2} \log_e\left(\frac{19}{17}\right)
 \end{aligned}$$

## Solutions to Exercise 11F

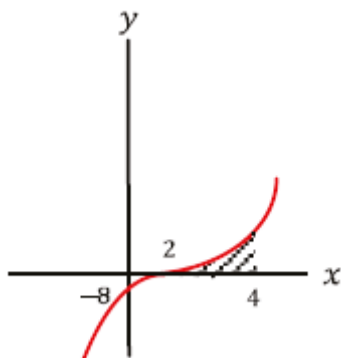
$$\begin{aligned}
 \mathbf{1\ a} \quad A &= \int_0^1 y \, dx \\
 &= \int_0^1 (3x^2 + 2) \, dx \\
 &= \left[ x^3 + 2x \right]_0^1 \\
 &= (1 + 2) - (0 + 0) \\
 &= 3
 \end{aligned}$$



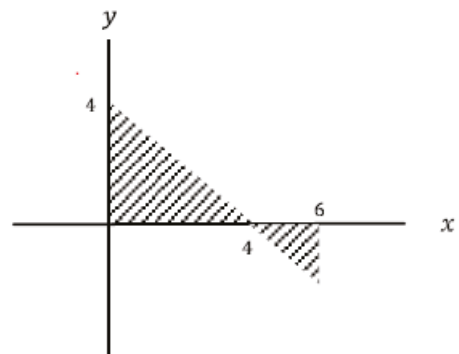
$$\begin{aligned}
 \mathbf{c\ i} \quad A &= \int_0^4 y \, dx \\
 &= \int_0^4 (4 - x) \, dx \\
 &= \left[ 4x - \frac{x^2}{2} \right]_0^4 \\
 &= (16 - 8) - 0 \\
 &= 8
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{b} \quad A &= \int_2^4 y \, dx \\
 &= \int_2^4 (x^3 - 8) \, dx \\
 &= \left[ \frac{x^4}{4} - 8x \right]_2^4 \\
 &= (64 - 32) - (4 - 16) \\
 &= 32 + 12 \\
 &= 44
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{ii} \quad A &= \int_0^4 y \, dx - \int_4^6 y \, dx \\
 &= 8 - \left[ 4x - \frac{x^2}{2} \right]_4^6 \quad (\text{from (i)}) \\
 &= 8 - ((24 - 18) - (16 - 8)) \\
 &= 8 - (6 - 8) \\
 &= 10
 \end{aligned}$$



$$\begin{aligned} \mathbf{2 a} \quad A &= x^2 - 2x \\ &= x(x - 2) \end{aligned}$$

$$y = 0, \quad x = 0, 2$$

$$\begin{aligned} A &= - \int_0^2 y \, dx \\ &= - \int_0^2 x^2 - 2x \, dx \\ &= - \left[ \frac{x^3}{3} - x^2 \right]_0^2 \\ &= - \left( \frac{8}{3} - 4 \right) \\ &= \frac{4}{3} \end{aligned}$$

$$\mathbf{b} \quad y = (4 - x)(3 - x)$$

$$y = 0, \quad x = 3, 4$$

$$\begin{aligned} A &= - \int_3^4 y \, dx \\ A &= - \int_3^4 (x - 4)(x - 3) \, dx \\ A &= - \int_3^4 -x^2 - 7x + 12 \, dx \\ &= - \left[ \frac{x^3}{3} - \frac{7x^2}{2} + 12x \right]_3^4 \\ &= \left( \left( \frac{64}{3} - \frac{7 \times 16}{2} + 48 \right) \right. \\ &\quad \left. - \left( \frac{27}{3} - \frac{7 \times 9}{2} + 36 \right) \right) \\ &= - \left( \left( \frac{40}{3} \right) - \left( \frac{27}{2} \right) \right) \\ &= \frac{1}{6} \end{aligned}$$

$$\mathbf{c} \quad y = (x + 2)(x + 7)$$

$$y = 0, \quad x = -2, 7$$

$$\begin{aligned} A &= \int_{-2}^7 y \, dx \\ &= \int_{-2}^7 -x^2 + 5x + 14 \, dx \\ &= \left[ -\frac{x^3}{3} + \frac{5x^2}{2} + 14x \right]_{-2}^7 \\ &= \left( \frac{-343}{3} + \frac{5 \times 49}{2} + 98 \right) \\ &\quad - \left( \frac{8}{3} + \frac{20}{2} - 28 \right) \\ &= \left( \frac{637}{6} \right) + \left( \frac{46}{3} \right) \\ &= 121.5 \end{aligned}$$

$$\mathbf{d} \quad y = x^2 - 5x + 6$$

$$= (x - 2)(x - 3)$$

$$y = 0, \quad x = 2, 3$$

$$\begin{aligned} A &= - \int_2^3 y \, dx \\ &= - \int_2^3 x^2 - 5x + 6 \, dx \\ &= - \left[ \frac{x^3}{3} - \frac{5x^2}{2} + 6x \right]_2^3 \\ &= - \left( \left( \frac{27}{3} - \frac{45}{2} + 18 \right) - \left( \frac{8}{3} - \frac{20}{2} + 12 \right) \right) \\ &= \left( \frac{14}{3} \right) - \left( \frac{9}{2} \right) \\ &= \frac{1}{6} \end{aligned}$$

$$\mathbf{e} \quad y = 3 - x^2$$

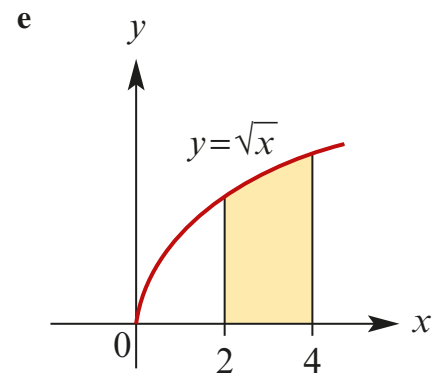
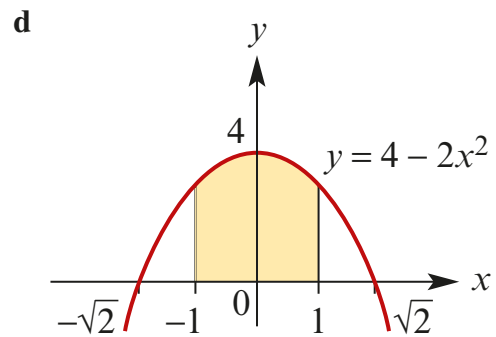
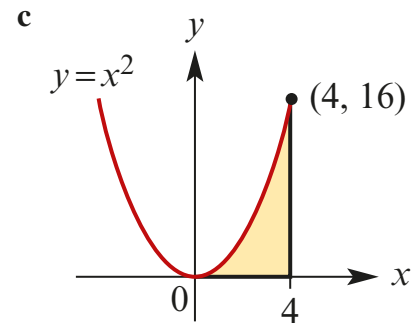
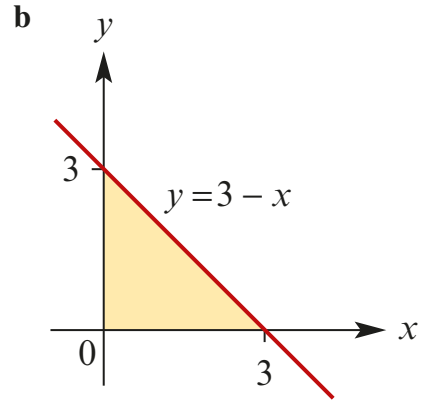
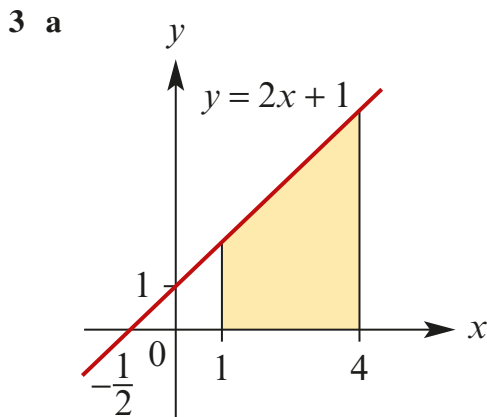
$$= (\sqrt{3} + x)(\sqrt{3} - x)$$

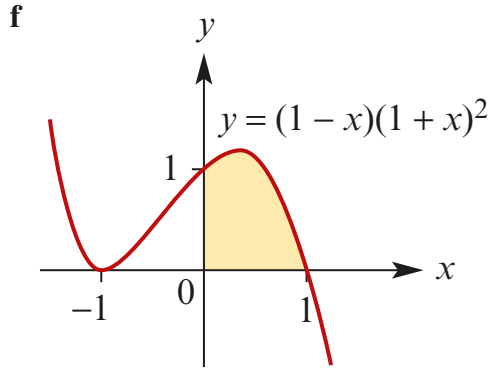
$$y = 0, \quad x = \pm \sqrt{3}$$

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} y \, dx$$

$$\begin{aligned}
&= \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 dx \\
&= \left[ 3x - \frac{x^3}{3} \right]_{-\sqrt{3}}^{\sqrt{3}} \\
&= \left( 3\sqrt{3} - \frac{3\sqrt{3}}{3} \right) - \left( -3\sqrt{3} + \frac{3\sqrt{3}}{3} \right) \\
&= 2\sqrt{3} + 2\sqrt{3} \\
&= 4\sqrt{3}
\end{aligned}$$

**f**  $y = x^3 - 6x^2$   
 $= x^2(x - 6)$   
 $y = 0, x = 0, 6$   
 $A = - \int_0^6 y dx$   
 $= - \int_0^6 x^3 - 6x^2 dx$   
 $= \int_0^6 6x^2 - x^3 dx$   
 $= \left[ 2x^3 - \frac{x^4}{4} \right]_0^6$   
 $= 2 \times (216) - \frac{1296}{4}$   
 $= 432 - 324$   
 $= 108$





**4**  $y = 3x + 2x^{-2}$

$y = 0,$

$3x = \frac{-2}{x^2}$

$3x^3 = -2$

$x^3 = \frac{-2}{3}$

$x = \left(\frac{-2}{3}\right)^{\frac{1}{3}}$

which is not in the region under consideration

$\therefore A = \int_2^5 3x + 2x^{-2} dx$

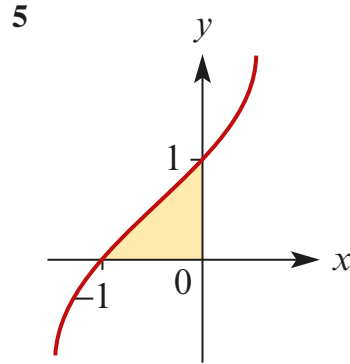
$= \left[ \frac{3}{2}x^2 - 2x^{-1} \right]_2^5$

$= \left( \frac{3}{2} \times 25 - \frac{2}{5} \right) - \left( \frac{3}{2} \times 4 - \frac{2}{2} \right)$

$= \frac{75}{2} - \frac{2}{5} - 6 + 1$

$= \frac{375 - 4 - 50}{10}$

$= \frac{321}{10}$  square units



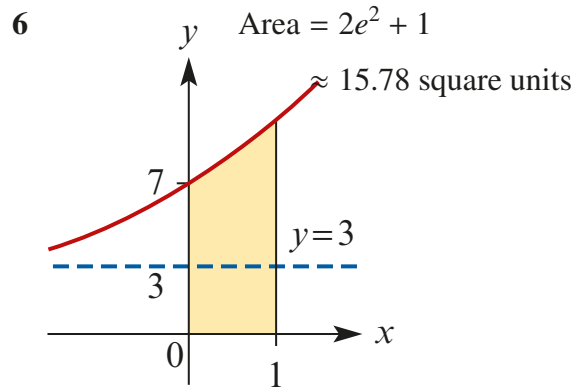
$A = \int_{-1}^0 f(x) dx$  (from graph)

$= \int_{-1}^0 1 + x^3 dx$

$= \left[ x + \frac{x^4}{4} \right]_{-1}^0$

$= 0 - \left( -1 + \frac{1}{4} \right)$

$= \frac{3}{4}$  square units



$A = \int_0^1 f(x) dx$  (from graph)

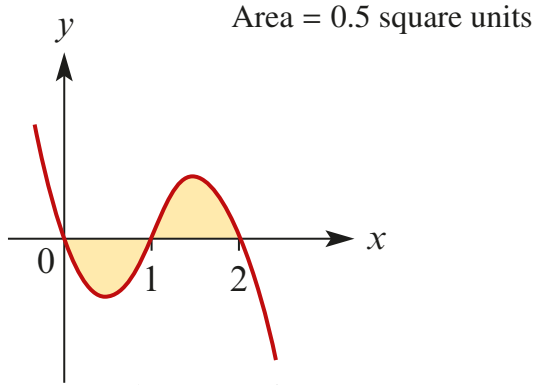
$= \int_0^1 4e^{2x} + 3 dx$

$= \left[ 2e^{2x} + 3x \right]_0^1$

$= (2e^2 + 3) - (2 + 0)$

$= 2e^2 + 1 \approx 15.78$  square units

7



$$\begin{aligned}
 A &= -\int_0^1 y \, dx + \int_1^2 y \, dx \quad (\text{from graph}) \\
 &= -\int_0^1 x(2-x)(x-1) \, dx \\
 &\quad + \int_1^2 x(2-x)(x-1) \, dx \\
 &= -\int_0^1 -x^3 + 3x^2 - 2x \, dx \\
 &\quad + \int_1^2 -x^3 + 3x^2 - 2x \, dx \\
 &= \left[ \frac{-x^4}{4} + \frac{3x^3}{3} - \frac{2x^2}{2} \right]_0^1 \\
 &\quad + \left[ \frac{-x^4}{4} + \frac{3x^3}{3} - \frac{2x^2}{2} \right]_1^2 \\
 &= \left[ \frac{-x^4}{4} + x^3 - x^2 \right]_0^1 + \left[ \frac{-x^4}{4} + x^3 - x^2 \right]_1^2 \\
 &= 0 - \left( \frac{-1}{4} + 1 - 1 \right) + \left( \frac{-16}{4} + 8 - 4 \right) \\
 &\quad - \left( \frac{-1}{4} + 1 - 1 \right) \\
 &= \frac{1}{2} \text{ square units}
 \end{aligned}$$

**8 a**

$$\begin{aligned}
 &\int_{-1}^4 x(3-x) \, dx \\
 &= \int_{-1}^4 3x - x^2 \, dx \\
 &= \left[ \frac{3x^2}{2} - \frac{x^3}{3} \right]_{-1}^4 \\
 &= \left( \frac{3 \cdot 16}{2} - \frac{64}{3} \right) - \left( \frac{3}{2} + \frac{1}{3} \right) \\
 &= 24 - \frac{64}{3} - \frac{3}{2} - \frac{1}{3} \\
 &= \frac{45}{2} - \frac{65}{3} \\
 &= \frac{5}{6} \text{ square units}
 \end{aligned}$$

**b** assuming the graph shown is

$$y = x(3-x),$$

$$\begin{aligned}
 A &= -\int_{-1}^0 x(3-x) \, dx + \int_0^3 x(3-x) \, dx \\
 &\quad - \int_3^4 x(3-x) \, dx \\
 &= \left[ \frac{3}{2}x^2 - \frac{x^3}{3} \right]_{-1}^0 - \left[ \frac{3}{2}x^2 - \frac{x^3}{3} \right]_{-1}^0 - \left[ \frac{3}{2}x^2 - \frac{x^3}{3} \right]_3^4 \\
 &\text{from (a)} \\
 &= \left( \frac{3 \times 9}{2} - \frac{27}{3} \right) - (0) - 0 + \left( \frac{3}{2} + \frac{1}{3} \right) \\
 &\quad - \left( \frac{3}{2} \times 16 - \frac{64}{3} \right) + \left( \frac{3 \times 9}{2} - \frac{27}{3} \right) \\
 &= \frac{49}{6}
 \end{aligned}$$

**9 a**  $y^2 = 9(1-x)$

A:  $x = 0,$

$y^2 = 9(1-0)$

$= 9$

$y = \pm 3$

but A is above the x-axis

so  $A = (0, 3)$

$B: y = 0$

$0 = 9(1 - x)$

$1 - x = 0$

$x = 1$

$B = (1, 0)$

**b**  $A = \int_0^3 \left(1 - \frac{y^2}{9}\right) dy$

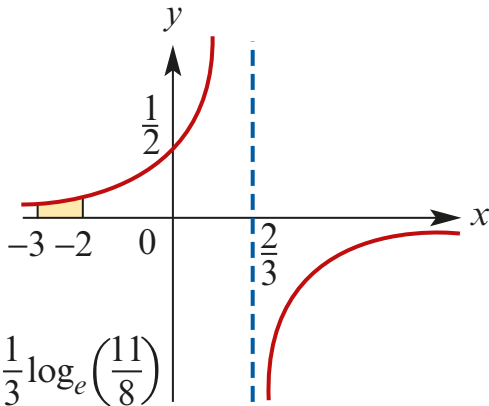
(0 to A, where 0 is the origin)

$$= \left[ y - \frac{y^3}{27} \right]_0^3$$

$$= \left( 3 - \frac{27}{27} \right) - 0$$

$$= 2 \text{ square units}$$

**10**



$$\frac{1}{3} \log_e \left( \frac{11}{8} \right)$$

$$A = - \int_{-3}^{-2} y dx \quad (\text{from graph})$$

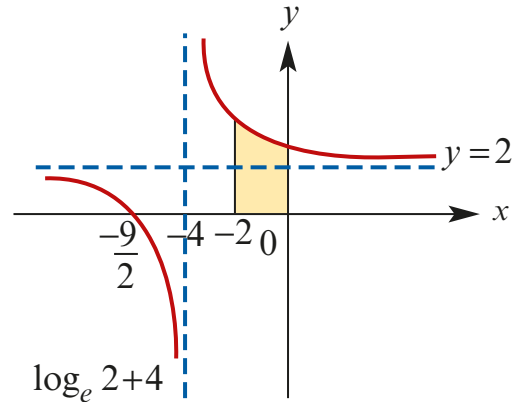
$$= - \int_{-3}^{-2} \frac{1}{2 - 3x} dx$$

$$= \left[ \frac{-1}{3} \log_e |2 - 3x| \right]_{-3}^{-2}$$

$$= \frac{-1}{3} \log_e |8| + \frac{1}{3} \log_e |11|$$

$$= \frac{1}{3} \log_e \left( \frac{11}{8} \right)$$

**11**



$$A = - \int_{-2}^0 y dx \quad (\text{from graph})$$

$$= \int_{-2}^0 2 + \frac{1}{x+4} dx$$

$$= \left[ 2x + \log_e |x+4| \right]_{-2}^0$$

$$= (0 + \log_e 4) - (-4 + \log_e 2)$$

$$= 2 \log_e 2 + 4 - \log_e 2$$

$$= \log_e 2 + 4$$

**12 a** RHS =  $e^{x(\ln a)}$

$$= e^{(\ln a^x)}$$

$$= a^x$$

$$= \text{LHS} \quad \text{QED}$$



$$\begin{aligned}
 \mathbf{b} \quad & \frac{d}{dx}(a^x) \\
 &= \frac{d}{dx}(e^{x(\ln a)}) \\
 &= \ln a e^{x \ln a} \\
 &= a^x \ln a \\
 & \int a^x dx \\
 &= \int e^{x \ln a} dx \\
 &= \frac{1}{\ln a} e^{x \ln a} + c \\
 &= \frac{a^x}{\ln a} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int_0^b a^x dx \\
 &= \left[ \frac{a^x}{\ln a} \right]_0^b \text{ from (b)} \\
 &= \frac{a^b}{\ln a} - \frac{1}{\ln a} \\
 &= \frac{1}{\ln a} (a^b - 1) \quad \text{QED}
 \end{aligned}$$

## Solutions to Exercise 11G

$$\begin{aligned} 1 \text{ a } \int \cos 3x \, dx \\ = \frac{1}{3} \sin 3x \end{aligned}$$

$$\begin{aligned} \text{b } \int \sin \frac{1}{2}x \, dx \\ = -2 \cos \frac{1}{2}x \end{aligned}$$

$$\begin{aligned} \text{c } \int 3 \cos 3x \, dx \\ = \sin 3x \end{aligned}$$

$$\begin{aligned} \text{d } \int 2 \sin \frac{1}{2}x \, dx \\ = -4 \cos \frac{1}{2}x \end{aligned}$$

$$\begin{aligned} \text{e } \int \sin \left(2x - \frac{\pi}{3}\right) \, dx \\ = \frac{-1}{2} \cos \left(2x - \frac{\pi}{3}\right) \end{aligned}$$

$$\begin{aligned} \text{f } \int \cos 3x + \sin 2x \, dx \\ = \frac{1}{3} \sin 3x - \frac{1}{2} \cos 2x \end{aligned}$$

$$\begin{aligned} \text{g } \int \cos 4x - \sin 4x \, dx \\ = \frac{1}{4} \sin 4x + \frac{1}{4} \cos 4x \end{aligned}$$

$$\begin{aligned} \text{h } \int \frac{-1}{2} \sin 2x + \cos 3x \, dx \\ = \frac{1}{4} \cos 2x + \frac{1}{3} \sin 3x \end{aligned}$$

$$\begin{aligned} \text{i } \int \frac{-1}{2} \cos \left(2x + \frac{\pi}{3}\right) \, dx \\ = \frac{-1}{4} \sin \left(2x + \frac{\pi}{3}\right) \end{aligned}$$

$$\begin{aligned} \text{j } \int \sin \pi x \, dx \\ = \frac{-1}{\pi} \cos \pi x \end{aligned}$$

$$\begin{aligned} 2 \text{ a } \int_0^{\frac{\pi}{4}} \sin x \, dx \\ = [-\cos x]_0^{\frac{\pi}{4}} \\ = \frac{-1}{\sqrt{2}} + 1 \\ = 1 - \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{b } \int_0^{\frac{\pi}{4}} \cos 2x \, dx \\ = \left[\frac{1}{2} \sin 2x\right]_0^{\frac{\pi}{4}} \\ = \frac{1}{2}(1) - 0 \\ = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{c } \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos \theta \, d\theta \\ = [\sin \theta]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \\ = \frac{1}{\sqrt{2}} - -1 \\ = 1 + \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{d } \int_0^{\frac{\pi}{2}} \sin \theta + \cos \theta \, d\theta \\ = [-\cos \theta + \sin \theta]_0^{\frac{\pi}{2}} \\ = (0 + 1) - (-1 + 0) \\ = 2 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta \\
 &= \left[ -\frac{1}{2} \cos 2\theta \right]_0^{\frac{\pi}{2}} \\
 &= \frac{1}{2}(-1) + \frac{1}{2}(1) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int_0^{\frac{\pi}{3}} \cos 3\theta + \sin 3\theta \, d\theta \\
 &= \left[ \frac{1}{3} \sin 3\theta - \frac{1}{3} \cos 3\theta \right]_0^{\frac{\pi}{3}} \\
 &= \left( 0 - \frac{1}{3}(-1) \right) - \left( 0 - \frac{1}{3}(1) \right) \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int_0^{\frac{\pi}{3}} \cos 3\theta + \sin\left(\theta - \frac{\pi}{3}\right) d\theta \\
 &= \left[ \frac{1}{3} \sin 3\theta - \cos\left(\theta - \frac{\pi}{3}\right) \right]_0^{\frac{\pi}{3}} \\
 &= \left( 0 - 1 \right) - \left( 0 - \frac{1}{2} \right) \\
 &= -\frac{1}{2}
 \end{aligned}$$

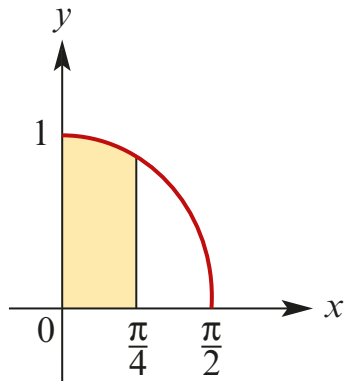
$$\begin{aligned}
 \mathbf{h} \quad & \int_0^{\pi} \sin \frac{x}{4} + \cos \frac{x}{4} \, dx \\
 &= \left[ -4 \cos \frac{x}{4} + 4 \sin \frac{x}{4} \right]_0^{\pi} \\
 &= \left( -4\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{2}}\right) \right) - (-4(1) + 0) \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int_0^{\frac{\pi}{4}} \sin\left(2x - \frac{\pi}{3}\right) \, dx \\
 &= \left[ -\frac{1}{2} \cos\left(2x - \frac{\pi}{3}\right) \right]_0^{\frac{\pi}{4}} \\
 &= -\frac{1}{2} \cos\left(\frac{\pi}{6}\right) + \frac{1}{2} \cos\left(-\frac{\pi}{3}\right) \\
 &= -\frac{\sqrt{3}}{4} + \frac{1}{4} \\
 &= \frac{1 - \sqrt{3}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{j} \quad & \int_0^{\pi} \cos 2x - \sin \frac{x}{2} \, dx \\
 &= \left[ \frac{1}{2} \sin 2x + 2 \cos \frac{x}{2} \right]_0^{\pi} \\
 &= \left( \frac{1}{2} \sin 2\pi + 2 \cos \frac{\pi}{2} \right) - \left( \frac{1}{2} \sin 0 + 2 \cos 0 \right) \\
 &= -2
 \end{aligned}$$

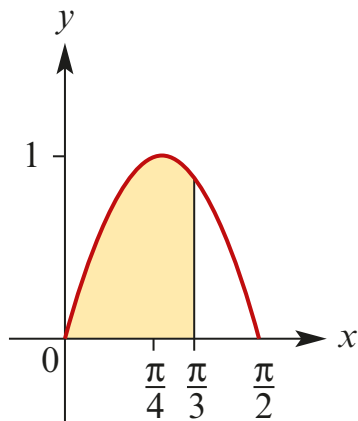
$$\begin{aligned}
 \mathbf{3} \quad A &= \int_0^{\frac{\pi}{2}} y \, dx \\
 &= \int_0^{\frac{\pi}{2}} \sin \frac{1}{2}x \, dx \\
 &= \left[ -2 \cos \frac{1}{2}x \right]_0^{\frac{\pi}{2}} \\
 &= -2 \cos \frac{\pi}{4} + 2 \cos 0 \\
 &= 2 - \sqrt{2}
 \end{aligned}$$

4 a



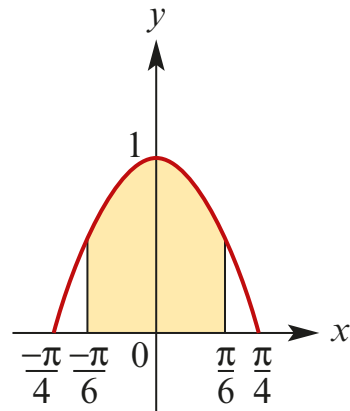
$$\begin{aligned} & \int_0^{\pi/4} \cos x \, dx \\ &= [\sin x]_0^{\pi/4} \\ &= \frac{1}{\sqrt{2}} - 0 \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

b



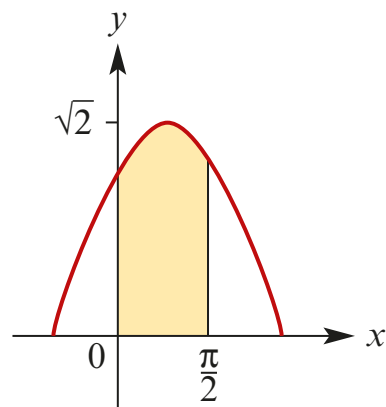
$$\begin{aligned} & \int_0^{\pi/3} \sin 2x \, dx \\ &= \left[ -\frac{1}{2} \cos 2x \right]_0^{\pi/3} \\ &= \frac{-1}{2} \left( \frac{-1}{2} \right) + \frac{-1}{2} (1) \\ &= \frac{3}{4} \end{aligned}$$

c

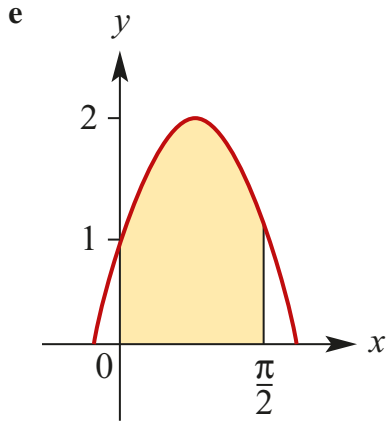


$$\begin{aligned} & \int_{-\pi/6}^{\pi/6} \cos 2x \, dx \\ &= \left[ \frac{1}{2} \sin 2x \right]_{-\pi/6}^{\pi/6} \\ &= \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) - \frac{1}{2} \left( \frac{-\sqrt{3}}{2} \right) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

d



$$\begin{aligned} & \int_0^{\pi/2} \cos \theta + \sin \theta \, d\theta \\ &= [\sin \theta - \cos \theta]_0^{\pi/2} \\ &= (1 - 0) - (0 - 1) \\ &= 2 \end{aligned}$$



$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \sin 2\theta + 1 \, d\theta \\ &= \left[ \frac{-1}{2} \cos 2\theta + \theta \right]_0^{\frac{\pi}{2}} \\ &= \left( \frac{-1}{2}(-1) + \frac{\pi}{2} \right) - \left( \frac{-1}{2}(-1) + 0 \right) \\ &= 1 + \frac{\pi}{2} \end{aligned}$$

**f**

$$\begin{aligned} & \int_{-\frac{x}{4}}^{\frac{x}{4}} \cos 2\theta \, d\theta \\ &= \left[ \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= \frac{1}{2}(1) - \frac{1}{2}(-1) \\ &= 1 \end{aligned}$$

**5 a**

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \sin\left(2x + \frac{\pi}{4}\right) dx \\ &= \left[ \frac{-1}{2} \cos\left(2x + \frac{\pi}{4}\right) \right]_0^{\frac{\pi}{2}} \\ &= \frac{-1}{2} \cos\left(\frac{5\pi}{4}\right) + \frac{1}{2} \cos\left(\frac{\pi}{4}\right) \\ &= \frac{-1}{2} \left( \frac{1}{\sqrt{2}} \right) + \frac{1}{2} \left( \frac{1}{\sqrt{2}} \right) \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

**b**

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{6}\right) dx \\ &= \left[ \frac{1}{3} \sin\left(3x + \frac{\pi}{6}\right) \right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{3} \sin\left(\frac{7\pi}{6}\right) - \frac{1}{3} \sin\left(\frac{\pi}{6}\right) \\ &= \frac{1}{3} \left( \frac{-1}{2} \right) - \frac{1}{3} \left( \frac{1}{2} \right) \\ &= \frac{-1}{3} \end{aligned}$$

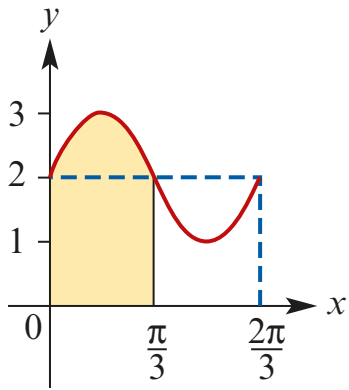
**c**

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{3}\right) dx \\ &= \left[ \frac{1}{3} \sin\left(3x + \frac{\pi}{3}\right) \right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{3} \sin\left(\frac{4\pi}{3}\right) - \frac{1}{3} \sin\left(\frac{\pi}{3}\right) \\ &= \frac{1}{3} \left( \frac{-\sqrt{3}}{2} \right) - \frac{1}{3} \left( \frac{\sqrt{3}}{2} \right) \\ &= \frac{-1}{\sqrt{3}} \end{aligned}$$

**d**

$$\begin{aligned} & \int_0^{\frac{\pi}{4}} \cos(3\pi - x) \, dx \\ &= \left[ -\sin(3\pi - x) \right]_0^{\frac{\pi}{4}} \\ &= \left[ -\sin(x - 3\pi) \right]_0^{\frac{\pi}{4}} \\ &= \sin\left(-2\pi - \frac{3\pi}{4}\right) - \sin(-2\pi - \pi) \\ &= \sin\left(-\frac{3\pi}{4}\right) - \sin(-\pi) \\ &= \frac{-1}{\sqrt{2}} - 0 \\ &= \frac{-1}{\sqrt{2}} \end{aligned}$$

6



$$\begin{aligned}
 A &= \int_0^{\frac{\pi}{3}} 2 + \sin 3x \, dx \\
 &= \left[ 2x - \frac{1}{3} \cos 3x \right]_0^{\frac{\pi}{3}} \\
 &= \left( \frac{2x}{3} - \frac{1}{3}(-1) \right) - \left( 0 - \frac{1}{3}(1) \right) \\
 &= \frac{2}{3}(\pi + 1)
 \end{aligned}$$

## Solutions to Exercise 11H

$$\begin{aligned}
 \mathbf{1\ a} \quad & \int_1^4 \sqrt{x} \, dx \\
 &= \left[ \frac{2}{3} x^{\frac{3}{2}} \right]_1^4 \\
 &= \frac{16}{3} - \frac{2}{3} \\
 &= \frac{14}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int_{-1}^1 (1+x)^2 \, dx \\
 &= \left[ \frac{1}{3} (1+x)^3 \right]_{-1}^1 \\
 &= \frac{1}{3} (2)^3 - \frac{1}{3} (0)^3 \\
 &= \frac{8}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int_0^8 x^{\frac{1}{3}} \, dx \\
 &= \left[ \frac{3}{4} x^{\frac{4}{3}} \right]_0^8 \\
 &= \frac{3}{4} \times 16 - 0 \\
 &= 12
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_0^{\frac{\pi}{3}} \cos 2x - \sin \frac{1}{2}x \, dx \\
 &= \left[ \frac{1}{2} \sin 2x + 2 \cos \frac{1}{2}x \right]_0^{\frac{\pi}{3}} \\
 &= \frac{1}{2} \sin \frac{2\pi}{3} + 2 \cos \frac{\pi}{6} - \frac{1}{2} \sin 0 + 2 \cos 0 \\
 &= \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) + 2 \left( \frac{\sqrt{3}}{2} \right) - 0 - 2(1) \\
 &= \frac{5\sqrt{3}}{4} - 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int_1^2 e^{2x} + \frac{4}{x} \, dx \\
 &= \left[ \frac{1}{2} e^{2x} + 4 \log_e |x| \right]_1^2 \\
 &= \frac{1}{2} e^4 + 4 \log_e 2 - \frac{1}{2} e^2 - 4 \log_e 1 \\
 &= \frac{1}{2} e^4 - \frac{1}{2} e^2 + 4 \log_e 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int_0^{\frac{\pi}{2}} \sin 2x + \cos 3x \, dx \\
 &= \left[ -\frac{1}{2} \cos 2x + \frac{1}{3} \sin 3x \right]_0^{\frac{\pi}{2}} \\
 &= \left( -\frac{1}{2}(-1) + \frac{1}{3}(-1) \right) - \left( -\frac{1}{2}(1) + \frac{1}{3}(0) \right) \\
 &= \frac{1}{2} - \frac{1}{3} + \frac{1}{2} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int_0^{\pi} \sin \frac{x}{4} + \cos \frac{x}{4} \, dx \\
 &= \left[ -4 \cos \frac{x}{4} + 4 \sin \frac{x}{4} \right]_0^{\pi} \\
 &= \left( -4 \left( \frac{1}{\sqrt{2}} \right) + 4 \left( \frac{1}{\sqrt{2}} \right) \right) - (-4(1) + 4(0)) \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int_0^{\frac{\pi}{2}} 5x + \sin 2x \, dx \\
 &= \left[ \frac{5x^2}{2} - \frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{2}} \\
 &= \left( \frac{5\left(\frac{\pi}{2}\right)^2}{2} - \frac{1}{2}(-1) \right) - \left( 0 - \frac{1}{2}(1) \right) \\
 &= \frac{5\pi^2}{8} + \frac{1}{2} + \frac{1}{2} \\
 &= 1 + \frac{5\pi^2}{8}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int_1^4 \left( 2 + \frac{1}{x} \right)^2 dx \\
 &= \int_1^4 \left( 4 + \frac{4}{x} + \frac{1}{x^2} \right) dx \\
 &= \left[ 4x + 4 \log_e |x| - \frac{1}{x} \right]_1^4 \\
 &= \left( 16 + 4 \log_e 4 - \frac{1}{4} \right) - \left( 4 + 4 \log_e 1 - 1 \right) \\
 &= 16 + 8 \log_e 2 - \frac{1}{4} - 4 + 1 \\
 &= 12\frac{3}{4} + 8 \log_e 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{j} \quad & \int_0^1 x^2 - x^3 \, dx \\
 &= \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 \\
 &= \frac{1}{3} - \frac{1}{4} \\
 &= \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad A &= \int_0^{\frac{\pi}{3}} \sin x \, dx \\
 &= \left[ -\cos x \right]_0^{\frac{\pi}{3}} \\
 &= -\cos \frac{\pi}{3} + \cos 0 \\
 &= -\frac{1}{2} + 1 \\
 &= \frac{1}{2} \text{ square units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right) \\
 &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
 &= \frac{1}{\cos^2 x}
 \end{aligned}$$

$$\begin{aligned}
 \text{Therefore,} \quad & \int \frac{1}{\cos^2 x} \, dx \\
 &= \frac{\sin x}{\cos x} + c \\
 &= \tan x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{d}{dx} \left( \frac{\cos 2x}{\sin 2x} \right) \\
 &= \frac{2 \sin 2x \frac{d}{dx}(\cos 2x) - 2 \cos 2x \frac{d}{dx}(\sin 2x)}{\sin^2(2x)} \\
 &= \frac{2(-\sin^2 2x - \cos^2 2x)}{\sin^2 2x} \\
 &= -\frac{2}{\sin^2 2x} \\
 \text{Therefore,} \quad & \int \frac{1}{\sin^2 2x} \, dx \\
 &= -\frac{\cos 2x}{2 \sin 2x} + c
 \end{aligned}$$



$$\begin{aligned}
 \text{c } \frac{d}{dx}(\log_e(3x^2 + 7)) &= 6x \times \frac{1}{3x^2 + 7} \\
 &= \frac{6x}{3x^2 + 7} \\
 &= \frac{x}{\frac{3x^2 + 7}{6}} \\
 &= \frac{1}{6} \int \frac{6x}{3x^2 + 7} dx \\
 &= \frac{1}{6} \log_e [3x^2 + 7]_0^2 \\
 &= \frac{1}{6} \log_e \left( \frac{19}{7} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \frac{d}{dx}(x \sin x) &= x \cos x + \sin x \\
 \text{Therefore,} \\
 \int x \cos x + \sin x dx &= x \sin x + c \\
 \int_0^{\frac{\pi}{4}} x \cos x + \sin x dx &= \left[ x \sin x \right]_0^{\frac{\pi}{4}} \\
 \int_0^{\frac{\pi}{4}} x \cos x dx &= \left[ \cos x + x \sin x \right]_0^{\frac{\pi}{4}} \\
 &= \frac{\pi \sqrt{2}}{8} + \frac{\sqrt{2}}{2} - 1
 \end{aligned}$$

**4 a**  $1 + \log_e(2x), -x + \log_e(2x)$

**b**  $x + 2x \log_e(2x), \frac{1}{2}x^2 \log_e(2x) - \frac{x^2}{4}$

$$\begin{aligned}
 \text{c } \frac{d}{dx}(x + \sqrt{1+x^2}) &= 1 + \frac{2x}{2\sqrt{1+x^2}} \\
 &= 1 + \frac{x}{\sqrt{1+x^2}} \\
 \frac{d}{dx}(\log_e(x + \sqrt{1+x^2})) &= \left(1 + \frac{x}{\sqrt{1+x^2}}\right) \times \frac{1}{x + \sqrt{1+x^2}} \\
 &= \left(\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}\right) \times \frac{1}{x + \sqrt{1+x^2}} \\
 &= \frac{1}{\sqrt{1+x^2}} \\
 \int_0^1 \frac{1}{\sqrt{1+x^2}} dx &= \left[ \log_e(x + \sqrt{1+x^2}) \right]_0^1 \\
 &= \log_e(1 + \sqrt{1+1}) - \log_e(0 + \sqrt{1+0}) \\
 &= \log_e(1 + \sqrt{2}) - \log_e 1 \\
 &= \log_e(1 + \sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 \text{5 } \frac{d}{dx}(e^{\sqrt{x}}) &= \frac{e^{\sqrt{x}}}{2\sqrt{x}}
 \end{aligned}$$

Therefore,

$$\int_1^2 \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx = \left[ e^{\sqrt{x}} \right]_1^2 = 2e^{\sqrt{2}} - 2e$$

**6**  $6 \sin^2(2x) \cos(2x), \frac{1}{6}$

**7** using the CAS calculator's 'integral' command:

**a** 139.68

**b** 18.50

**c** -0.66

**d** -23.76

**e** 2.06

**f** 0.43

**8 a**  $LHS = \frac{2x+3}{x-1}$   
 $= \frac{2x-2+2+3}{x-1}$   
 $= \frac{2(x-1)}{x-1} + \frac{5}{x-1}$   
 $= 2 + \frac{5}{x-1}$   
 $= RHS \text{ QED}$

**b**  $\int_2^4 \frac{2x+3}{x-1} dx$   
 $= \int_2^4 2 + \frac{5}{x-1} dx$   
 $= \left[ 2x + 5 \log_e |x-1| \right]_2^4$   
 $= (8 + 5 \log_e 3) - (4 + 5 \log_e 1)$   
 $= 4 + 5 \log_e 3$

**9 a**  $LHS = \frac{5x-4}{x-2}$   
 $= \frac{5x-4-6+6}{x-2}$   
 $= \frac{5(x-2)}{x-2} + \frac{6}{x-2}$   
 $= 5 + \frac{6}{x-2}$   
 $= RHS \text{ QED}$

**b**  $\int_3^4 \frac{5x-4}{x-2} dx$   
 $= \int_3^4 5 + \frac{6}{x-2} dx$   
 $= \left[ 5x + 6 \log_e |x-2| \right]_3^4$   
 $= (20 + 6 \log_e 2) - (15 + 6 \log_e 1)$   
 $= 5 + 6 \log_e 2$

**10 a**  $y = \left(1 - \frac{1}{2}x\right)^8$   
 $\frac{dy}{dx} = \frac{-1}{2} \times 8 \left(1 - \frac{1}{2}x\right)^7$   
 $= -4 \left(1 - \frac{1}{2}x\right)^7$   
 $\int \left(1 - \frac{1}{2}x\right)^7 dx$   
 $= \frac{-1}{4} \int -4 \left(1 - \frac{1}{2}x\right)^7 dx$   
 $= \frac{-1}{4} \left(1 - \frac{1}{2}x\right)^8 + c$

**b**  $y = \log_e |\cos x|$   
 $\frac{dy}{dx} = -\sin x \times \frac{1}{\cos x}$   
 $= -\tan x$   
 $\int_0^{\frac{\pi}{3}} \tan x dx$   
 $= \int_0^{\frac{\pi}{3}} -\tan x dx$   
 $= -\left[ \log_e |\cos x| \right]_0^{\frac{\pi}{3}}$   
 $= -\log_e \left| \cos \frac{\pi}{3} \right| + \log_e |\cos 0|$   
 $= -\log_e \frac{1}{2} + \log_e 1$   
 $= \log_e 2$

$$\begin{aligned}
 \mathbf{11} \quad f'(x) &= \sin\left(\frac{1}{2}x\right) \\
 f(x) &= -2 \cos\left(\frac{1}{2}x\right) + c \\
 f\left(\frac{4\pi}{3}\right) &= 2, \\
 2 &= -2 \cos\left(\frac{2\pi}{3}\right) + c \\
 2 &= -2\left(\frac{-1}{2}\right) + c \\
 2 &= 1 + c \\
 c &= 1 \\
 f(x) &= -2 \cos\left(\frac{1}{2}x\right) + 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{12} \quad \mathbf{a} \quad f'(x) &= \cos 2x \\
 f(x) &= \frac{1}{2} \sin 2x + c \\
 f(\pi) &= 1, \\
 1 &= \frac{1}{2} \sin 2\pi + c \\
 c &= 1 \\
 f(x) &= \frac{1}{2} \sin 2x + 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f'(x) &= \frac{3}{x} \\
 f(x) &= 3 \log_e |x| + c \\
 f(1) &= 6, \\
 6 &= 3 \log_e 1 + c \\
 c &= 6 \\
 f(x) &= 3 \log_e |x| + 6
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad f'(x) &= e^{\frac{x}{2}} \\
 f(x) &= 2e^{\frac{x}{2}} + c \\
 f(0) &= 1, \\
 1 &= 2 + c \\
 c &= -1 \\
 f(x) &= 2e^{\frac{x}{2}} - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{13} \quad \frac{d}{dx}(x \sin 3x) &= \sin 3x + 3x \cos 3x \\
 \int_0^{\frac{\pi}{6}} x \cos 3x \, dx & \\
 &= \frac{1}{3} \int_0^{\frac{\pi}{6}} 3x \cos 3x + \sin 3x - \sin 3x \, dx \\
 &= \frac{1}{3} \int_0^{\frac{\pi}{6}} 3x \cos 3x + \sin 3x - \frac{1}{3} \int_0^{\frac{\pi}{6}} \sin 3x \, dx \\
 &= \frac{1}{3} \left[ x \sin 3x \right]_0^{\frac{\pi}{6}} - \frac{1}{3} \left[ \frac{-1}{3} \cos 3x \right]_0^{\frac{\pi}{6}} \\
 &= \frac{1}{3} \left( \frac{\pi}{6} \sin \frac{\pi}{2} - 0 \right) - \frac{1}{3} \left( \frac{-1}{3} \cos \frac{\pi}{2} + \frac{1}{3} \cos 0 \right) \\
 &= \frac{\pi}{18} - \frac{1}{9}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{14} \quad y &= a + b \sin\left(\frac{\pi x}{2}\right) \\
 (0, 1) & \\
 \Rightarrow 1 &= a + b \sin 0 \\
 1 &= a \\
 (3, 3) & \\
 \Rightarrow 3 &= 1 + b \sin\left(\frac{3\pi}{2}\right) \\
 3 &= 1 - b \\
 b &= -2
 \end{aligned}$$

$$y = 1 - 2 \sin\left(\frac{\pi x}{2}\right)$$

$$x = 0, y = 1$$

$$x = 1, y = 1 - 2 \sin \frac{\pi}{2}$$

$$= -1$$

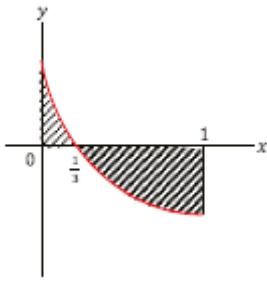
$\therefore$  there is an  $x$ -intercept

$$0 = 1 - 2 \sin\left(\frac{\pi}{2}x\right)$$

$$\sin \frac{\pi}{2}x = \frac{1}{2}$$

$$\frac{\pi}{2}x = \frac{\pi}{6}$$

$$x = \frac{1}{3}$$



$$\therefore A = \int_0^{\frac{1}{3}} y dx - \int_{\frac{1}{3}}^1 y dx$$

$$= \int_0^{\frac{1}{3}} 1 - 2 \sin \frac{\pi x}{2} dx$$

$$- \int_{\frac{1}{3}}^1 1 - 2 \sin \frac{\pi x}{2} dx$$

$$= \left[ x + \frac{4}{\pi} \cos \frac{\pi x}{2} \right]_0^{\frac{1}{3}} - \left[ x + \frac{4}{\pi} \cos \frac{\pi x}{2} \right]_{\frac{1}{3}}^1$$

$$= \left( \frac{1}{3} + \frac{4}{\pi} \cos \frac{\pi}{6} \right) - \left( 0 + \frac{4}{\pi} \cos 0 \right)$$

$$- \left( 1 + \frac{4}{\pi} \cos \frac{\pi}{2} \right) + \left( \frac{1}{3} + \frac{4}{\pi} \cos \frac{\pi}{6} \right)$$

$$= \frac{-1}{3} + \frac{4\sqrt{3}}{\pi} - \frac{4}{\pi} \approx 0.5987 \text{ square units}$$

**a** 1.450 square units

**b** 1.716 square units

**16** using the CAS calculator's 'integral' command:

0.1345

**17**  $f'(x) = x + \sin 2x$

$$f(x) = \frac{x^2}{2} - \frac{1}{2} \cos 2x + c$$

$$f(0) = 1,$$

$$1 = 0 - \frac{1}{2} \cos 0 + c$$

$$c = \frac{3}{2}$$

$$f(x) = \frac{x^2 - \cos 2x + 3}{2}$$

**18 a**  $\int f(x) dx$

$$= \int g'(x) dx$$

$$= g(x) + c$$

$$= (x^2 + 1)^3 + c$$

**b**  $\int h(x) dx$

$$= \int k'(x) dx$$

$$= k(x) + c$$

$$= \sin x^2 + c$$

**15** using the CAS calculator's 'integral' command:

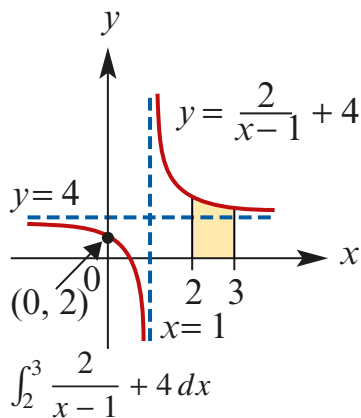
$$\begin{aligned}
 \text{c } \int f(x) + h(x) dx & \\
 &= \int g'(x) + k'(x) dx \\
 &= g(x) + k(x) + c \\
 &= (x^2 + 1)^3 + \sin x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \int -f(x) dx & \\
 &= - \int g'(x) dx \\
 &= -g(x) + c \\
 &= -(x^2 + 1)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \int f(x) - 4 dx & \\
 &= \int g'(x) dx - \int 4 dx \\
 &= g(x) - 4x + c \\
 &= (x^2 + 1)^3 - 4x + c
 \end{aligned}$$

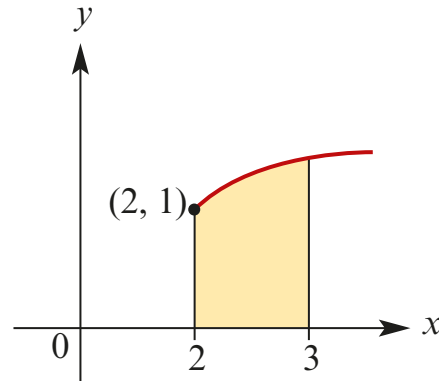
$$\begin{aligned}
 \text{f } \int 3h(x) dx & \\
 &= 3 \int k'(x) dx \\
 &= 3k(x) + c \\
 &= 3 \sin x^2 + c
 \end{aligned}$$

19



$$\begin{aligned}
 &= [2 \log_e |x - 1| + 4x]_2^3 \\
 &= (2 \log_e 2 + 12) - (2 \log_e 1 + 8) \\
 &= 4 + 2 \log_e 2
 \end{aligned}$$

20



$$\begin{aligned}
 &\int_2^3 \sqrt{2x-4} + 1 dx \\
 &= \left[ \frac{2}{3 \times 2} (2x-4)^{\frac{3}{2}} + x \right]_2^3 \\
 &= \left( \frac{1}{3} (2)^{\frac{3}{2}} + 3 \right) - \left( \frac{1}{3} (0)^{\frac{3}{2}} + 2 \right) \\
 &= 1 + \frac{1}{3} (2)^{\frac{3}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{21 a } \int_3^4 \sqrt{x-2} dx & \\
 &= \left[ \frac{2}{3} (x-2)^{\frac{3}{2}} \right]_3^4 \\
 &= \frac{2}{3} (2)^{\frac{3}{2}} - \frac{2}{3} (1)^{\frac{3}{2}} \\
 &= \frac{2}{3} (2\sqrt{2} - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_0^2 \sqrt{2-x} dx & \\
 &= \left[ \frac{-2}{3} (2-x)^{\frac{3}{2}} \right]_0^2 \\
 &= \frac{-2}{3} (0)^{\frac{3}{2}} + \frac{2}{3} (2)^{\frac{3}{2}} \\
 &= \frac{2\sqrt{2}}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int_0^1 \frac{1}{3x+1} dx \\
 & = \left[ \frac{1}{3} \log_e |3x+1| \right]_0^1 \\
 & = \frac{1}{3} \log_e 4 + \frac{1}{3} \log_e 1 \\
 & = \frac{2}{3} \log_e 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_1^2 \frac{1}{2x-1} + 3 dx \\
 & = \left[ \frac{1}{2} \log_e |2x-1| + 3x \right]_1^2 \\
 & = \left( \frac{1}{2} \log_e |3| + 6 \right) - \left( \frac{1}{2} \log_e 1 + 3 \right) \\
 & = \frac{1}{2} \log_e 3 + 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int_{2.5}^3 \sqrt{2x-5} - 6 dx \\
 & = \left[ \frac{2}{3} \times \frac{1}{2} \times (2x-5)^{\frac{3}{2}} - 6x \right]_{2.5}^3 \\
 & = \left( \frac{1}{3} (1)^{\frac{3}{2}} - 18 \right) - \left( \frac{1}{3} (0)^{\frac{3}{2}} - 15 \right) \\
 & = \frac{1}{3} - 3 \\
 & = \frac{-8}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int_3^4 \frac{1}{\sqrt{x-2}} dx \\
 & = \left[ 2(x-2)^{\frac{1}{2}} \right]_3^4 \\
 & = 2(2)^{\frac{1}{2}} - 2(1)^{\frac{1}{2}} \\
 & = 2\sqrt{2} - 2
 \end{aligned}$$

## Solutions to Exercise 11I

1  $y_1 = 12 - x - x^2, y_2 = x + 4$

$$12 - x - x^2 = x + 4$$

$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

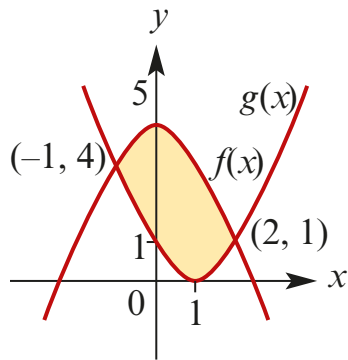
$$x = 2, -4$$

to test which graph is higher in this interval:

$$x = 0, y_1 = 12, y_2 = 4$$

$$\begin{aligned} A &= \int_{-4}^2 y_1 - y_2 \, dx \\ &= \int_{-4}^2 (12 - x - x^2) - (x + 4) \, dx \\ &= \int_{-4}^2 8 - 2x - x^2 \, dx \\ &= \left[ 8x - x^2 - \frac{x^3}{3} \right]_{-4}^2 \\ &= \left( 16 - 4 - \frac{8}{3} \right) - \left( -32 - 16 + \frac{64}{3} \right) \\ &= 36 \text{ units}^2 \end{aligned}$$

2



$$f(x) = 5 - x^2, g(x) = (x - 1)^2$$

$$5 - x^2 = (x - 1)^2$$

$$5 - x^2 = x^2 - 2x + 1$$

$$2x^2 - 2x - 4 = 0$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x - 1) = 0$$

$$x = -1, 2$$

to test which graph is higher in this interval:

$$f(0) = 5, g(0) = (-1)^2 = 1$$

$$\begin{aligned} \therefore A &= \int_{-1}^2 f(x) - g(x) \, dx \\ &= \int_{-1}^2 5 - x^2 - (x^2 - 2x + 1) \, dx \\ &= \int_{-1}^2 4 + 2x - 2x^2 \, dx \\ &= \left[ 4x + x^2 - \frac{2x^3}{3} \right]_{-1}^2 \\ &= \left( 8 + 4 - \frac{16}{3} \right) - \left( -4 + 1 + \frac{2}{3} \right) \\ &= 16 - 6 - 1 \\ &= 9 \text{ units}^2 \end{aligned}$$

3 a  $y_1 = x + 3, y_2 = 12 + x - x^2$

$$x + 3 = 12 + x - x^2$$

$$x^2 - 9 = 0$$

$$(x + 3)(x - 3) = 0$$

$$x = -3, 3$$

to test which graph is higher in this interval:

$$x = 0, y_1 = 3, y_2 = 12$$

$$\begin{aligned} A &= \int_{-3}^3 y_2 - y_1 \, dx \\ &= \int_{-3}^3 (12 + x - x^2) - (x + 3) \, dx \\ &= \int_{-3}^3 9 - x^2 \, dx \\ &= \left[ 9x - \frac{x^3}{3} \right]_{-3}^3 \\ &= \left( 27 - \frac{27}{3} \right) - \left( -27 + \frac{27}{3} \right) \\ &= 54 - \frac{54}{3} \\ &= 54 - 18 \\ &= 36 \text{ units}^2 \end{aligned}$$

**b**  $y_1 = 3x + 5, y_2 = x^2 + 1$

$$3x + 5 = x^2 + 1$$

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

$$x = -1, 4$$

*to test which graph is higher in this interval:*

$$x = 0, y_1 = 5, y_2 = 1$$

$$\begin{aligned} A &= \int_{-1}^4 y_1 - y_2 \, dx \\ &= \int_{-1}^4 (3x + 5) - (1 + x^2) \, dx \\ &= \int_{-1}^4 4 + 3x - x^2 \, dx \\ &= \left[ 4x + \frac{3}{2}x^2 - \frac{x^3}{3} \right]_{-1}^4 \\ &= \left( 16 - 24 - \frac{64}{3} \right) - \left( -4 + \frac{3}{2} + \frac{1}{3} \right) \\ &= \frac{125}{6} \text{ units}^2 \end{aligned}$$

**c**  $y_1 = 3 - x^2, y_2 = 2x^2$

$$3 - x^2 = 2x^2$$

$$x^2 = 1$$

$$x = \pm 1$$

*to test which graph is higher in this interval:*

$$x = 0, y_1 = 3, y_2 = 0$$

$$\begin{aligned} A &= \int_{-1}^1 y_1 - y_2 \, dx \\ &= \int_{-1}^1 (3 - x^2) - (2x^2) \, dx \\ &= \int_{-1}^1 3 - 3x^2 \, dx \\ &= [3x - x^3]_{-1}^1 \\ &= (3 - 1) - (-3 + 1) \\ &= 4 \text{ units}^2 \end{aligned}$$

**d**  $y_1 = x^2, y_2 = 3x$

$$x^2 = 3x$$

$$x^2 - 3x = 0$$

$$x(x - 3) = 0$$

$$x = 0, 3$$

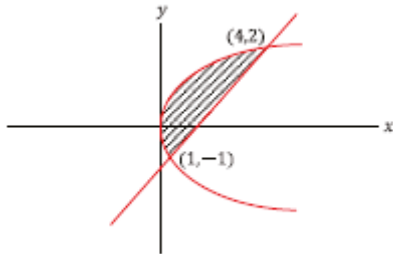
*to test which graph is higher in this interval:*

$$x = 1, y_1 = 1, y_2 = 3$$

$$\begin{aligned} A &= \int_0^3 y_2 - y_1 \, dx \\ &= \int_0^3 3x - x^2 \, dx \\ &= \left[ \frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_0^3 \\ &= \left( \frac{27}{2} - 9 \right) - (0 - 0) \\ &= \frac{9}{2} \text{ units}^2 \end{aligned}$$



**e**  $y_1^2 = x, x - y_2 = 2$   
 $y_1 = \pm \sqrt{x}, y_2 = x - 2$   
 $\pm \sqrt{x} = x - 2$   
 $x = x^2 - 4x + 4$   
 $x^2 - 5x + 4 = 0$   
 $(x - 4)(x - 1) = 0$   
 $x = 1, 4$



$$A = \int_0^1 \sqrt{x} - \sqrt{x} dx + \int_1^4 \sqrt{x} - (x - 2) dx$$

$$= \left[ \frac{4}{3} x^{\frac{3}{2}} \right]_0^1 + \left[ \frac{2}{3} x^{\frac{3}{2}} - \frac{x^2}{2} + 2x \right]_1^4$$

$$= \frac{4}{3} + \left( \frac{16}{3} - 8 + 8 \right) - \left( \frac{2}{3} - \frac{1}{2} + 2 \right)$$

$$= \frac{20}{3} - \frac{2}{3} - \frac{3}{2}$$

$$= 6 - \frac{3}{2}$$

$$= \frac{9}{2} \text{ units}^2$$

**4 a**  $P = \int_{-1}^0 e - e^{-x} dx + \int_0^1 e - e^x dx$

$$= 2 \int_0^1 e - e^x dx$$

$$= 2[ex - e^x]_0^1$$

$$= 2(e - e) - 2(0 - 1)$$

$$= 2 \text{ units}^2$$

**b**  $Q = \int_0^1 e^x - e^{-x} dx$

$$= \left[ e^x + e^{-x} \right]_0^1$$

$$= \left( e + \frac{1}{e} \right) - (1 + 1)$$

$$= e + \frac{1}{e} - 2 \approx 1.086 \text{ units}^2$$

**5**

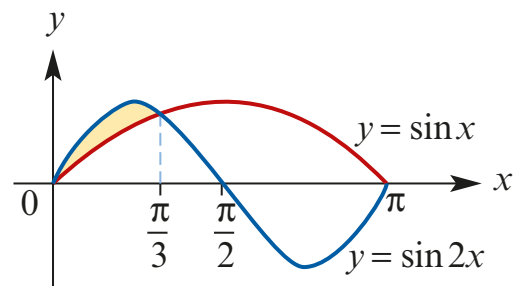
$$A = \int_0^{\frac{7\pi}{6}} (\sin x) - \left( \frac{-1}{2} \right) dx$$

$$= \left[ -\cos x + \frac{1}{2}x \right]_0^{\frac{7\pi}{6}}$$

$$= \left( -\left( \frac{-\sqrt{3}}{2} \right) + \frac{1}{2} \times \frac{7\pi}{6} \right) - \left( -(1) + \frac{1}{2}(0) \right)$$

$$= \frac{\sqrt{3}}{2} + \frac{7\pi}{12} + 1 \approx 3.699 \text{ units}^2$$

**6**



$$A = \int_0^{\frac{\pi}{3}} \sin 2x - \sin x dx \text{ (from the graph)}$$

$$= \left[ \frac{-1}{2} \cos 2x + \cos x \right]_0^{\frac{\pi}{3}}$$

$$= \left( \frac{-1}{2} \cos \frac{2\pi}{3} + \cos \frac{\pi}{3} \right)$$

$$- \left( \frac{-1}{2} \cos 0 + \cos 0 \right)$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{2}$$

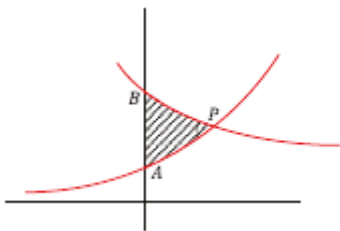
$$= \frac{1}{4} \text{ units}^2$$

7

$$\begin{aligned}
 A &= \int_0^{\frac{\pi}{6}} \cos x - \sin 2x \, dx \\
 &\quad + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x - \cos x \, dx \\
 &= \left[ \sin x + \frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{6}} \\
 &\quad + \left[ -\frac{1}{2} \cos 2x - \cos x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
 &= \left( \sin \frac{\pi}{6} + \frac{1}{2} \cos \frac{\pi}{3} \right) - \left( \sin 0 + \frac{1}{2} \cos 0 \right) \\
 &\quad + \left( -\frac{1}{2} \cos \pi - \sin \frac{\pi}{2} \right) - \left( -\frac{1}{2} \cos \frac{\pi}{3} - \sin \frac{\pi}{6} \right) \\
 &= \frac{1}{2} + \frac{1}{4} - \frac{1}{2} + \frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{4} \\
 &= \frac{1}{2} \text{ units}^2
 \end{aligned}$$

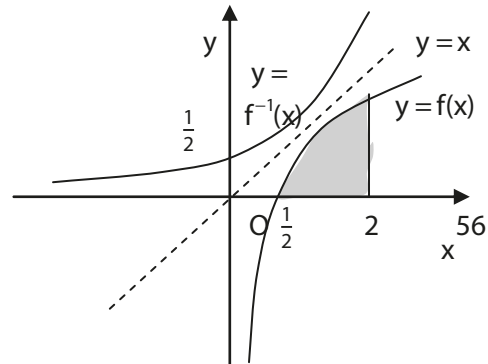
8

$$\begin{aligned}
 e^x &= 2 + 3e^{-x} \\
 e^{2x} - 2e^x - 3 &= 0 \\
 (e^x - 3)(e^x + 1) &= 0 \\
 e^x &= -1, 3 \\
 \text{Since } e^x > 0, e^x &= 3 \\
 x &= \log_e 3 \\
 y &= e^x = 3 \\
 P &= (\log_e 3, 3)
 \end{aligned}$$



$$\begin{aligned}
 \text{Area} &= \int_0^{\log_e 3} 2 + 3e^{-x} - e^x \, dx \\
 &= \left[ 2x - 3e^{-x} - e^x \right]_0^{\log_e 3} \\
 &= (2 \log_e 3 - 3e^{\log_e \frac{1}{3}} - e^{\log_e 3}) - (0 - 3 - 1) \\
 &= 4 + 2 \log_e 3 - 1 - 3 \\
 &= 2 \log_e 3 \\
 &\approx 2.197 \text{ units}^2
 \end{aligned}$$

9 a  $f : R^+ \rightarrow R, f(x) = \log_e(2x)$   
 Consider  $x = \log_e(2y)$ . Solving for  $y$  gives  $y = \frac{1}{2}e^x$   
 $f^{-1}(x) = \frac{1}{2}e^x$  and the domain of  $f^{-1} = R$



$$\begin{aligned}
 \text{b } b \int_0^{\log_e 4} f^{-1}(x) \, dx \\
 &= \int_0^{\log_e 4} \frac{1}{2} e^x \, dx \\
 &= \left[ \frac{1}{2} e^x \right]_0^{\log_e 4} = \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c By symmetry } \int_{\frac{1}{2}}^2 f(x) \, dx \\
 &= 4 \log_e(2) - \frac{3}{2}
 \end{aligned}$$

## Solutions to Exercise 11J

$$1 \text{ a } \text{av} = \frac{1}{b-a} \int_a^b f(x) dx$$

$$\text{av} = \frac{1}{2-0} \int_0^2 x(2-x) dx$$

$$= \frac{1}{2} \int_0^2 (2x - x^2) dx$$

$$= \frac{1}{2} \left[ x^2 - \frac{1}{3}x^3 \right]_0^2$$

$$= \frac{1}{2} \left( 4 - \frac{8}{3} \right) = \frac{2}{3}$$

$$b \text{ av} = \frac{1}{\pi-0} \int_0^\pi \sin(x) dx$$

$$= \frac{1}{\pi} \left[ -\cos(x) \right]_0^\pi$$

$$= \frac{1}{\pi} ((-(-1)) - (-1)) = \frac{2}{\pi}$$

$$c \text{ av} = \frac{1}{\frac{\pi}{2}-0} \int_0^{\frac{\pi}{2}} \sin(x) dx$$

$$= \frac{2}{\pi} \left[ -\cos(x) \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi} ((-0) - (-1)) = \frac{2}{\pi}$$

$$d \text{ av} = \frac{1}{\frac{2\pi}{n}-0} \int_0^{\frac{2\pi}{n}} \sin(nx) dx$$

$$= \frac{n}{2\pi} \left[ -\frac{1}{n} \cos(nx) \right]_0^{\frac{2\pi}{n}}$$

$$\frac{1}{2\pi} ((-1) - (-1)) = 0$$

$$= \frac{1}{2\pi} ((-1) - (-1)) = 0$$

$$e \text{ av} = \frac{1}{2-(-2)} \int_{-2}^2 (e^x + e^{-x}) dx$$

$$= \frac{1}{4} \left[ e^x - e^{-x} \right]_{-2}^2$$

$$= \frac{1}{4} ((e^2 - e^{-2}) - (e^{-2} - e^2))$$

$$= \frac{1}{2} (e^2 - e^{-2})$$

$$2 \text{ av temp} = \frac{1}{10-0} \int_0^{10} 50e^{-\frac{t}{2}} dt$$

$$= \frac{1}{10} \left[ \frac{50}{(-\frac{1}{2})} e^{-\frac{t}{2}} \right]_0^{10}$$

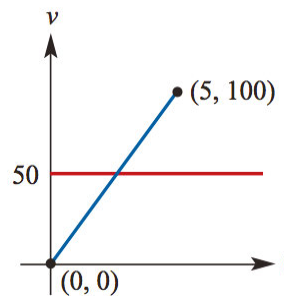
$$= -10(e^{-5} - e^{-0})$$

$$= 10(1 - e^{-5}) \approx 9.93^\circ\text{C}$$

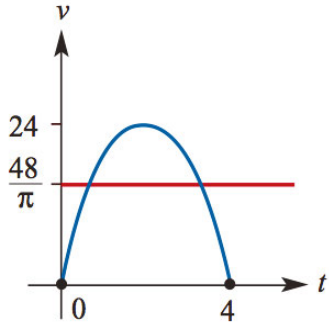
$$3 \text{ a av speed} = \frac{1}{5-0} \int_0^5 20t dt$$

$$= \frac{1}{5} \left[ 10t^2 \right]_0^5$$

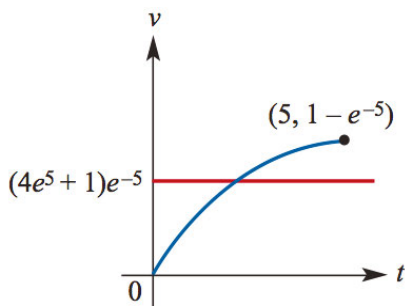
$$= 50 \text{ m/s}$$



$$\begin{aligned}
 \text{b av speed} &= \frac{1}{4-0} \int_0^4 24 \sin\left(\frac{1}{4}\pi t\right) dt \\
 &= \frac{1}{4} \left[ -\frac{24 \times 4}{\pi} \cos\left(\frac{1}{4}\pi t\right) \right]_0^4 \\
 &= -\frac{24}{\pi}(-1-1) \\
 &= \frac{48}{\pi} \text{ m/s}
 \end{aligned}$$



$$\begin{aligned}
 \text{c av speed} &= \frac{1}{5-0} \int_0^5 5(1-e^{-t}) dt \\
 &= \frac{1}{5} \left[ 5(t+e^{-t}) \right]_0^5 \\
 &= \frac{1}{5}(5(5+e^{-5})-5e^{-0}) \\
 &= 4+e^{-5} \text{ m/s}
 \end{aligned}$$



$$\begin{aligned}
 \text{4 av velocity} &= \frac{1}{3-0} \int_0^3 9.8t dt \\
 &= \frac{1}{3} \left[ 4.9t^2 \right]_0^3 \\
 &= 14.7 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 \text{5 mean value} &= \frac{1}{a-0} \int_0^a x(a-x) dx \\
 &= \frac{1}{a} \int_0^a (ax-x^2) dx \\
 &= \frac{1}{a} \left[ \frac{1}{2}ax^2 - \frac{1}{3}x^3 \right]_0^a \\
 &= \frac{1}{a} \left( \frac{1}{2}a^3 - \frac{1}{3}a^3 \right) \\
 &= \frac{a^2}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a } pv^{0.9} &= 300 \Rightarrow p = 300v^{-0.9} \\
 \text{av pressure} &= \frac{1}{1-\frac{1}{2}} \int_{\frac{1}{2}}^1 300v^{-0.9} dv \\
 &= 2 \left[ \frac{300}{0.1} v^{0.1} \right]_{\frac{1}{2}}^1 \\
 &= 6000 \left( 1 - \left( \frac{1}{2} \right)^{0.1} \right) \\
 &= 3000(2-2^{0.9}) \\
 &\approx 401.8 \text{ N/m}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{b } v &= 3t+1, \text{ so:} \\
 t=0, v &= 1, t=1, v=4 \\
 \text{av pressure} &= \frac{1}{4-1} \int_1^4 300v^{-0.9} dv \\
 &= \frac{1}{3} \left[ \frac{300}{0.1} v^{0.1} \right]_1^4 \\
 &= 1000((4)^{0.1}-1) \\
 &\approx 148.7 \text{ N/m}^2
 \end{aligned}$$

$$\text{7 } v = \frac{dx}{dt} \text{ and } v = 2t-3$$

$$\mathbf{a} \quad x = \int (2t - 3) dt$$

$$= t^2 - 3t + c$$

$$t = 0, x = 0 \text{ so } c = 0$$

$$x = t^2 - 3t$$

**b**  $t = 3, x = 0$ , so the body is at  $O$ .

$$\mathbf{c} \quad \text{av velocity} = \frac{x(3) - x(0)}{3 - 0} = 0$$

(Note: this is exactly the same as av velocity =  $\frac{1}{3-0} \int_0^3 (2t - 3) dt$ ; but the integration is already done in part a.)

**d**  $v = 0$  when  $2t - 3 = 0$ , i.e. when  $t = 1.5$ .

When  $t = 1.5, x = -2.25$ , so the body goes from  $x = 0$  to  $x = -2.25$  in the first 1.5 s and then returns to  $x = 0$  in the next 1.5 s. It travels 4.5 m in the first 3 s.

$$\mathbf{e} \quad \text{av speed} = \frac{4.5}{3} = 1.5 \text{ m/s}$$

$$\mathbf{8 a} \quad v = 2t^2 - 8t + 6, \quad v = \frac{dx}{dt}, \quad a = \frac{dv}{dt}$$

$$x = \int (2t^2 - 8t + 6) dt$$

$$= \frac{2}{3}t^3 - 4t^2 + 6t + c$$

$$t = 0, x = 4$$

$$\text{so } c = 4$$

$$x = \frac{2}{3}t^3 - 4t^2 + 6t + 4$$

$$a = \frac{d}{dt}(2t^2 - 8t + 6)$$

$$= 4t - 8$$

**b**  $v = 0$  when  $2t^2 - 8t + 6 = 0$ , i.e.

$$t^2 - 4t + 3 = 0$$

$$(t - 1)(t - 3) = 0$$

$$t = 1, 3$$

$$x(1) = \frac{20}{3} \text{ m}, \quad x(3) = 4 \text{ m}$$

$$\mathbf{c} \quad a(1) = -4 \text{ m/s}^2, \quad a(3) = 4 \text{ m/s}^2$$

$$\mathbf{9} \quad a = \frac{dv}{dt} \text{ so } \frac{dv}{dt} = 8$$

$$v = 8t + c$$

$$v = \frac{dx}{dt} \text{ so } \frac{dx}{dt} = 8t + c$$

$$x = 4t^2 + ct + d$$

$$t = 1, x = 0 \text{ so } 4 + c + d = 0, \text{ i.e.}$$

$$c + d = -4 \quad \textcircled{1}$$

$$t = 3, x = 30 \text{ so } 36 + 3c + d = 30, \text{ i.e.}$$

$$3c + d = -6 \quad \textcircled{2}$$

$$\textcircled{2} - \textcircled{1} \text{ gives } 2c = -2, \text{ so } c = -1$$

Substitution into (1) gives  $d = -3$ , so

$$x = 4t^2 - t - 3$$

$$t = 0, x = -3$$

Initial displacement is  $-3$  m.

$$\mathbf{10} \quad a = \frac{dv}{dt} \text{ so } \frac{dv}{dt} = 2t - 3$$

$$v = t^2 - 3t + c$$

$$t = 0, v = 3 \text{ so } c = 3$$

$$v = t^2 - 3t + 3$$

$$v = \frac{dx}{dt} \text{ so } \frac{dx}{dt} = t^2 - 3t + 3$$

$$x = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 3t + d$$

$$t = 0, x = 2 \text{ so } d = 2$$

$$x = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 3t + 2$$

$$v(10) = 73 \text{ m/s}, \quad x(10) = \frac{646}{3} \text{ m}$$

$$11 \quad a = \frac{dv}{dt} \text{ so } \frac{dv}{dt} = -10$$

$$\mathbf{a} \quad v = -10t + c; t = 0, v = 25, \text{ so} \\ c = 25 \quad v = -10t + 25$$

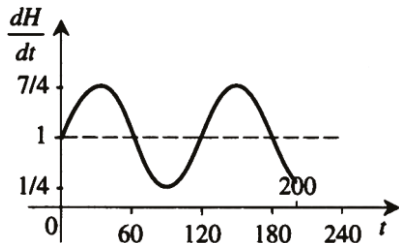
$$\mathbf{b} \quad x = -5t^2 + 25t + d; t = 0, x = 0, \text{ so} \\ d = 0 \quad x = -5t^2 + 25t$$

$\mathbf{c}$  max height when  $v = 0$ , i.e.  $t = 2.5$  s

$$\mathbf{d} \quad x(2.5) = \frac{125}{4} \text{ m}$$

$\mathbf{e}$   $x = 0$  when  $-5t^2 + 25t = 0$ , giving  $t = 0$  or  $t = 5$ ; so returns after 5 s.

$$12 \quad \mathbf{a} \quad \frac{dH}{dt} = 1 + \frac{3}{4} \sin\left(\frac{\pi t}{60}\right) \text{ for } t \in [0, 200]$$



$$\mathbf{b} \quad \frac{dH}{dt} > 1.375 \Leftrightarrow 1 + 4\frac{3}{4} \sin\left(\frac{\pi t}{60}\right) > \frac{11}{8} \\ \Leftrightarrow \frac{3}{4} \sin\left(\frac{\pi t}{60}\right) > \frac{3}{8} \\ \Leftrightarrow \sin\left(\frac{\pi t}{60}\right) > \frac{1}{2}$$

$$\text{Consider the equation } \sin\left(\frac{\pi t}{60}\right) = \frac{1}{2}$$

This is equivalent to

$$\frac{\pi t}{60} = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \dots \\ \text{i.e. } t = 10 \text{ or } 50 \text{ or } 130 \text{ or } 170 \text{ or } \dots$$

For the required domain and by observation from graph  $\frac{dH}{dt} > 1.375$  for  $t \in (10, 50) \cup (130, 170)$

$\mathbf{c}$  The rate of heat loss is greatest when  $\sin\left(\frac{\pi t}{60}\right) = 1$   
This occurs when

$$\frac{\pi t}{60} = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \dots \\ \Rightarrow t = 30 \text{ or } 150 \text{ or } 270 \text{ or } \dots$$

$\therefore$  rate of heat loss is greatest when  $t = 30$  or  $150$  for  $t$  in  $[0, 200]$

$$\mathbf{d} \quad \mathbf{i} \quad \text{The total heat loss for } t \in [0, 120] = \int_0^{120} 1 + \frac{3}{4} \sin\left(\frac{\pi t}{60}\right) dt \\ = \left[ t - \frac{3}{4} \times \frac{60}{\pi} \cos\left(\frac{\pi t}{60}\right) \right]_0^{120} \\ = 120 - \frac{45}{\pi} \cos 2\pi - \left( 0 - \frac{45}{\pi} \cos 0 \right) \\ = 120 - \frac{45}{\pi} + \frac{45}{\pi} \\ = 120 \\ \therefore 120 \text{ kilojoules lost over the 120 days.}$$

$$\mathbf{ii} \quad \text{Total heat lost for } t \in [0, 200] = \int_0^{200} 1 + \frac{3}{4} \sin\left(\frac{\pi t}{60}\right) dt \\ = \left[ t - \frac{45}{\pi} \cos\left(\frac{\pi t}{60}\right) \right]_0^{200} \\ = 200 - \frac{45}{\pi} \cos\left(\frac{200\pi}{60}\right) - \left( 0 - \frac{45}{\pi} \right) \\ = 200 - \frac{45}{\pi} \cos\left(\frac{10\pi}{3}\right) + \frac{45}{\pi} \\ = 200 - \frac{45}{\pi} \times -\frac{1}{2} + \frac{45}{\pi} \\ = 200 + \frac{45}{2\pi} + \frac{45}{\pi} \\ = 200 + \frac{135}{2\pi} \\ \approx 221.48 \text{ kilojoules}$$

$$13 \quad \frac{dV}{dt} = 1000 - 30t^2 + 2t^3 \quad 0 \leq t \leq 15$$

$\mathbf{a}$  When  $t = 0$ ,  $\frac{dV}{dt} = 1000$   
The rate of flow is 1000 million

litres/hour =  $10^9$  litres/hour.  
 When  $t = 2$ ,  
 $= 1000 - 30 \times 4 + 2 \times 2^3 =$   
 $1000 - 120 + 16$   
 $= 896$  million litres/hour =  $8.96 \times 10^8$   
 litres/hour

**b i** To find stationary points, let

$$R = \frac{dV}{dt} = 1000 - 30t^2 + 2t^3$$

Stationary points occur when

$$\frac{dR}{dt} = 0$$

$$\frac{dR}{dt} = -60t + 6t^2$$

$$= -6t(10 - t)$$

$$\frac{dR}{dt} = 0 \text{ implies } t = 0 \text{ or } t = 10$$

A gradient chart for  $\frac{dR}{dt}$  is as shown:

$t$	$< 0$	$0$	$<<$	$10$	$> 10$
sign of $\frac{dR}{dt}$	+ve	0	-ve	0	+ve
shape	/	-	\	-	/

$\therefore$  a local maximum at  $(0, 1000)$   
 and a local minimum when  
 $t = 10$ .

when  $t = 10$

$$R = \frac{dV}{dt} = 1000 - 30 \times 10^2 + 2 \times 10^3$$

$$= 1000 - 3000 + 2000$$

$$= 0$$

$\therefore$  local minimum at  $(10, 0)$

$$\text{When } t = 15, \frac{dV}{dt} =$$

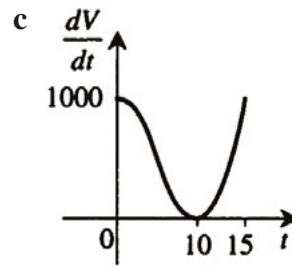
$$1000 - 30 \times 15^2 + 2 \times 15^3$$

$$= 1000 - 30 \times 225 + 2 \times 3375$$

$$= 1000$$

$\therefore$  The maximum flow occurs  
 when  $t = 0$  and  $t = 15$

**ii** The maximum flow is 1000  
 million litres/hour.



**d i** Area beneath the graph between  
 $t = 0$  and  $t = 10$

$$= \int_0^{10} 1000 - 30t^2 + 2t^3 dt$$

$$= \left[ 1000t - \frac{30 \times t^3}{3} + \frac{2t^4}{4} \right]_0^{10}$$

$$= 5000$$

**ii** 5000 million litres flowed out in  
 the first 10 hours.

**14 a**  $R: [0, \infty) \rightarrow \mathbb{R}$ ,  $R(t) = 10 \log_e(t + 1)$

When  $t = 5$ ,  $R(5) = 10 \log_e(6) \approx$   
 17.918

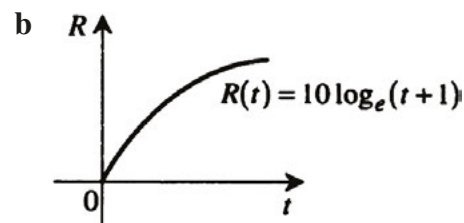
When  $t = 5$ , the rate of growth is  
 $\approx 17.918$  penguins per year.

When  $t = 10$ ,  $R(10) = 10 \log_e(10) \approx$   
 23.978

When  $t = 10$ , the rate of growth is  
 23.978 penguins per year.

When  $t = 100$ ,  $R(100) =$   
 $10 \log_e(100) \approx 46.151$

When  $t = 100$ , the rate of growth is  
 46.151 penguins per year.



c For the inverse function consider

$$t = 10 \log_e(y + 1)$$

$$\therefore \frac{t}{10} = \log_e(y + 1)$$

$$\therefore e^{\frac{t}{10}} = y + 1$$

$$\therefore y = e^{\frac{t}{10}} - 1$$

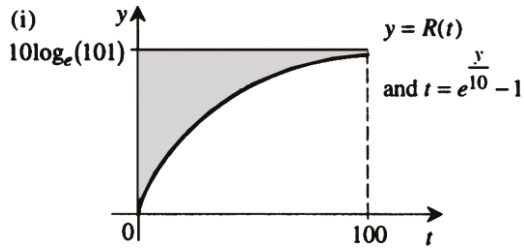
and the inverse function is

$$R^{-1}(t) = e^{\frac{t}{10}} - 1$$

The domain of  $R^{-1}$  = range of

$$R = R^+ \cup \{0\}$$

d i



$\therefore$  required area = area of rectangle – area

$$= 100 \times 10 \log_e(101) - \int_0^{10 \log_e(101)} (e^{\frac{y}{10}} - 1) dy$$

$$= 1000 \log_e(101) - \left[ 10e^{\frac{y}{10}} - y \right]_0^{10 \log_e 101}$$

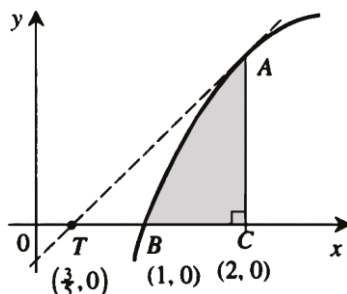
$$= 1000 \log_e(101) - \left[ 10e^{\log_e 101} - 10 \log_e(101) - (10e^0 - 0) \right]$$

$$= 1000 \log_e(101) - [1010 - 10 \log_e(101) - 10]$$

$$= 1000 \log_e(101) - 1000$$

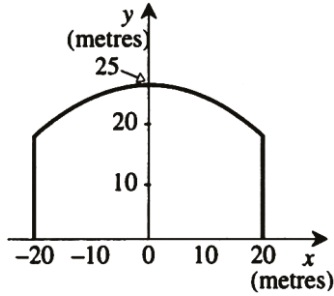
$$\approx 3661.27$$

ii The penguin population has increased by 3661 penguins over 100 years.





15



Area of cross section

$$= \int_{-20}^{20} (25 - 0.02x^2) dx$$

The symmetry of  $f$  gives that the area of cross section

$$= 2 \int_0^{20} (25 - 0.02x^2) dx$$

$$= 2 \left[ 25x - \frac{x^3}{150} \right]_0^{20}$$

$$= 2 \left( 25 \times 20 - \frac{20^3}{150} \right)$$

$$= 2 \left( 500 - \frac{8000}{150} \right)$$

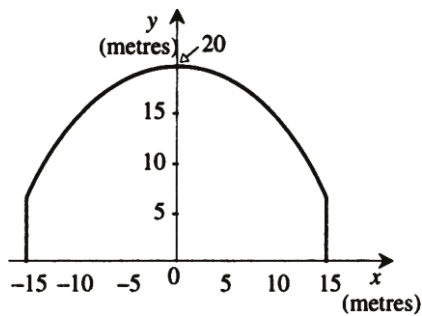
$$= 893 \frac{1}{3}$$

The volume of the hall = area of cross section  $\times$  length

$$= 893 \frac{1}{3} \times 80$$

$$= 71\#466 \frac{2}{3} \text{ m}^3$$

16 a



Area of cross section

$$= \int_{-15}^{15} (20 - 0.06x^2) dx$$

The symmetry of  $f$  gives that the area of cross section

$$= 2 \int_0^{15} (20 - 0.06x^2) dx$$

$$= 2 \left[ 20x - \frac{x^3}{50} \right]_0^{15}$$

$$= 2 \left( 20 \times 15 - \frac{15^3}{50} \right)$$

$$= 465$$

The area of the cross section is

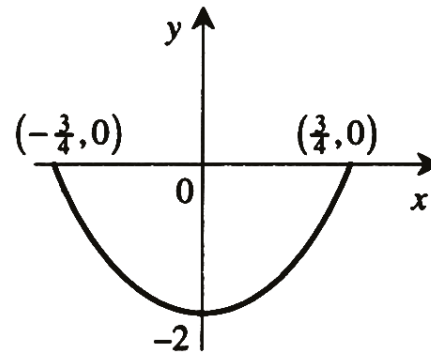
$$465 \text{ m}^2$$

b The volume of the hangar = area of cross section  $\times$  length

$$= 465 \times 100$$

$$= 46\ 500 \text{ m}^3$$

17



The parabola is of the form  $y = ax^2 + b$

$$\text{When } x = 0, y = -2$$

$$\text{When } y = 0, x = \pm \frac{3}{4}$$

$$0 = a \left( \frac{3}{4} \right)^2 - 2$$

$$\text{and } \frac{2 \times 16}{9} = a$$

$$\text{i.e. } a = \frac{32}{9}$$

$\therefore$  The equation of the parabola is

$$y = \frac{32}{9}x^2 - 2$$

The total volume of the trough = area of cross section  $\times$  length To determine the

area of cross section

$$\text{consider } 2 \int_0^{\frac{3}{4}} \left( \frac{32x^2}{9} - 2 \right) dx$$

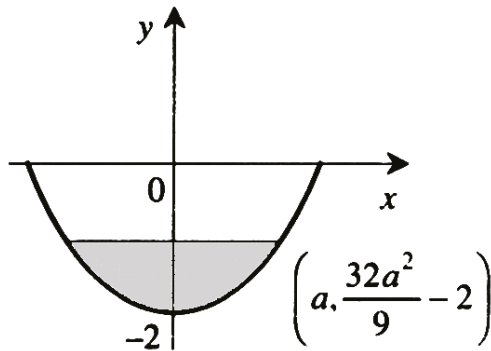
$$\begin{aligned}
 &= 2 \left[ \frac{32x^3}{27} - 2x \right]_0^{\frac{3}{2}} \\
 &= 2 \left( \frac{32}{27} \times \frac{27}{64} - 2 \times \frac{3}{4} \right) \\
 &= 2 \left( \frac{1}{2} - 1 \frac{1}{2} \right) \\
 &= -2
 \end{aligned}$$

∴ The cross sectional area is  $2 \text{ m}^2$   
 The total volume =  $2 \times l = 2l \text{ m}^3$ , where  $l$  is the length of the trough.

When the trough is half full the volume is  $l \text{ m}^3$  and the cross sectional area is

$$\begin{aligned}
 1 \text{ m}^3 \text{ The shaded area} &= 1 \text{ m}^2 \\
 \text{Shaded area} &= 2 \int_0^a \left( \frac{32}{9} a^2 - 2 \right) - \left( \frac{32}{9} a^2 - 2 \right) dx \\
 &= 2 \left[ \frac{32}{9} a^2 x - 2x - \frac{32}{27} x^3 + 2x \right]_0^a \\
 &= 2 \left[ \frac{32a^3}{9} - \frac{32a^3}{27} \right] \\
 &= 2 \times \left[ \frac{64a^3}{27} \right] \\
 &= \frac{128a^3}{27}
 \end{aligned}$$

$$\therefore \frac{128a^3}{27} = 1$$



$$\text{which implies } a^3 = \frac{27}{128}$$

$$\therefore a^3 = \frac{3^3}{2^3}$$

$$\therefore a = \frac{3}{2}$$

$$\begin{aligned}
 \text{When } x = \frac{3}{2} \quad y &= \frac{32}{9} \times \left( \frac{3}{2} \right)^2 - 2 \\
 &= \frac{32}{9} \times \frac{9}{4} - 2 \\
 &= \frac{2^5}{2^2} - 2 \\
 &= 2^{\frac{5}{2}} - 2 \\
 &= 2^{\frac{1}{2}} - 2
 \end{aligned}$$

$$\therefore \text{the depth} = 2 - (2 - 2^{\frac{1}{2}})$$

$$= 2^{\frac{1}{2}} \text{ metres} \approx 1.26 \text{ metres}$$

The depth of the water is 1.26 metres when it is half full.

**18 a**  $y = 3 - 3 \cos\left(\frac{x}{3}\right)$  for  $x \in [-3\pi, 3\pi]$ . The maximum value of the function is 6 and hence the maximum height of 6 metres

**b** The area =  $2 \int_0^{3\pi} \left( 3 - 3 \cos\left(\frac{x}{3}\right) \right) dx$   
 $= 2[3x - 9 \sin\left(\frac{x}{3}\right)]_0^{3\pi} = 18\pi$   
 The area is  $18\pi \text{ m}^2$

**c i**  $\frac{dy}{dx} = \sin\left(\frac{x}{3}\right)$ . When  $x = a$ ,  $y = 3 - 3 \cos\left(\frac{a}{3}\right)$  and  $\frac{dy}{dx} = \sin\left(\frac{a}{3}\right)$ . Therefore the equation of the normal is  $y - \left( 3 - 3 \cos\left(\frac{a}{3}\right) \right) = -\frac{1}{\sin\left(\frac{a}{3}\right)}(x - a)$

**ii** If it passes through (9,0),  $0 - 3 + 3 \cos\left(\frac{a}{3}\right) = -\frac{1}{\sin\left(\frac{a}{3}\right)}(9 - a)$   
 Solving numerically gives  $a = 5.409$

$$**19 a** \frac{dV}{dt} = 3 \left[ \cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2 \right]$$

$$**i** \text{ When } t = 0, \frac{dV}{dt} = 3[1 + 0 + 2] = 9$$

$$**ii** \text{ When } t = 2, \frac{dV}{dt} = 3 \left[ -1 + \frac{1}{\sqrt{2}} + 2 \right] = 3 \frac{(\sqrt{2} + 2)}{2}$$

iii When  $t = 4$ ,  $\frac{dV}{dt} = 3[1 + 1 + 2]$   
 $= 12$

b From the graph maximum value is 12 and the minimum value is 0.834

c The volume through the pipe in the first 8

minutes

$$= 3 \int_0^8 \left[ \cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2 \right] dt$$

$$= 3 \left[ \frac{2}{\pi} \sin\left(\frac{\pi t}{2}\right) - \frac{8}{\pi} \cos\left(\frac{\pi t}{8}\right) + 2t \right]_0^8$$

$$= 3 \left( \frac{8}{\pi} + 16 - \left(-\frac{8}{\pi}\right) \right) = 48 \left( 1 + \frac{1}{\pi} \right)$$

$$= \frac{48(\pi + 1)}{\pi} \text{ litres}$$

## Solutions to Technology-free questions

$$1 \text{ a } \int_2^3 x^3 dx = \left[ \frac{1}{4}x^4 \right]_2^3 = \frac{65}{4}$$

**b** Since  $\sin x$  is an odd function, the integral is 0. (Alternatively work through the integral.)

$$\begin{aligned} \text{c } \int_a^{4a} (a^{\frac{1}{2}} - x^{\frac{1}{2}}) dx \\ &= \left[ a^{\frac{1}{2}}x - \frac{2}{3}x^{\frac{3}{2}} \right]_a^{4a} \\ &= \left( 4a^{\frac{3}{2}} - \frac{2}{3} \times 8a^{\frac{3}{2}} \right) - \left( a^{\frac{3}{2}} - \frac{2}{3}a^{\frac{3}{2}} \right) \\ &= -\frac{5a^{\frac{3}{2}}}{3} \end{aligned}$$

$$\begin{aligned} \text{d } \int_1^4 \frac{3}{\sqrt{x}} - 5\sqrt{x} - x^{-\frac{3}{2}} dx \\ &= \int_1^4 3x^{-\frac{1}{2}} - 5x^{\frac{1}{2}} - x^{-\frac{3}{2}} dx \\ &= \left[ 6x^{\frac{1}{2}} - \frac{10}{3}x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} \right]_1^4 \\ &= \left( 12 - \frac{80}{3} + 1 \right) - \left( 6 - \frac{10}{3} + 2 \right) \\ &= -\frac{55}{3} \end{aligned}$$

$$\begin{aligned} \text{e } \int_0^{\frac{\pi}{4}} \cos 2\theta d\theta &= \left[ \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}} \\ &= \left( \frac{1}{2} \sin \frac{\pi}{2} \right) - 0 = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{f } \int_1^e \frac{1}{x} dx &= [\log_e x]_1^e \\ &= \log_e e - \log_e 1 = 1 \end{aligned}$$

$$\begin{aligned} \text{g } \int_0^{\frac{\pi}{2}} \sin 2\left(\theta + \frac{\pi}{4}\right) d\theta \\ &= \left[ -\frac{1}{2} \cos 2\left(\theta + \frac{\pi}{4}\right) \right]_0^{\frac{\pi}{2}} \\ &= \left( -\frac{1}{2} \cos \frac{3\pi}{2} \right) - \left( -\frac{1}{2} \cos \frac{\pi}{2} \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{h } \int_0^{\pi} \sin 4\theta d\theta &= \left[ -\frac{1}{4} \cos 4\theta \right]_0^{\pi} \\ &= \left( -\frac{1}{4} \cos 4\pi \right) - \left( -\frac{1}{4} \cos 0 \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} 2 \int_{-1}^2 x + 2f(x) dx \\ &= \int_{-1}^2 x dx + 2 \int_{-1}^2 f(x) dx \\ &= \left[ \frac{x^2}{2} \right]_{-1}^2 + 2 \times 5 \\ &= \frac{23}{2} \end{aligned}$$

$$\begin{aligned} 3 \int_1^5 f(x) dx &= \int_0^5 f(x) dx - \int_0^1 f(x) dx \\ &= 1 - (-2) = 3 \end{aligned}$$

$$\begin{aligned} 4 \int_3^{-2} f(x) dx &= - \int_{-2}^3 f(x) dx \\ &= - \int_{-2}^1 f(x) dx - \int_1^3 f(x) dx \\ &= -2 - (-6) = 4 \end{aligned}$$

$$5 \int_0^2 (x+1)^7 dx = \left[ \frac{1}{8}(x+1)^8 \right]_0^2 = 820$$

$$6 \int_0^1 (3x+1)^3 dx = \left[ \frac{1}{3 \times 4}(3x+1)^4 \right]_0^1 \\ = \frac{85}{4}$$

7 If  $F(x)$  is an antiderivative of  $f(x)$ , then

$$\int_0^9 f(x) dx = [F(x)]_0^9 \\ = F(9) - F(0) = 5$$

Also by the chain rule

$$\frac{d}{dx}(F(3x)) = 3f(3x), \text{ so:}$$

$$\int_0^3 f(3x) dx = \frac{1}{3}[F(3x)]_0^3 \\ = \frac{1}{3}(F(9) - F(0)) \\ = \frac{5}{3}$$

8 If  $F(x)$  is an antiderivative of  $f(x)$ , then

$$\int_1^4 f(x) dx = [F(x)]_1^4 \\ = F(4) - F(1) = 5$$

Also by the chain rule

$$\frac{d}{dx}(F(3x+1)) = 3f(3x+1), \text{ so:}$$

$$\int_0^1 f(3x+1) dx = \frac{1}{3}[F(3x+1)]_0^1 \\ = \frac{1}{3}(F(4) - F(1)) \\ = \frac{5}{3}$$

9 The area of the shaded region

from  $x = a$  to  $x = b$  is given by

$$\int_a^b f(x) - g(x) dx.$$

The area of the shaded region

from  $x = b$  to  $x = c$  is given by

$$\int_b^c g(x) - f(x) dx.$$

The area of the shaded region

from  $x = c$  to  $x = d$  is given by

$$\int_c^d f(x) - g(x) dx.$$

The area of the shaded region is

$$\int_a^b f(x) - g(x) dx + \int_b^c g(x) - f(x) dx \\ + \int_c^d f(x) - g(x) dx$$

10 a The curves intersect where

$$2x + x^2 = 15, \text{ i.e. } x^2 + 2x - 15 = 0.$$

Hence  $(x-3)(x+5) = 0$ , so  $P$  has coordinates  $(3, 9)$ .

$Q$  has coordinates  $(7.5, 0)$ .

b The area of the shaded region is

$$\int_0^3 x^2 dx + \int_3^{7.5} 15 - 2x dx \\ = \left[ \frac{1}{3}x^3 \right]_0^3 + [15x - x^2]_3^{7.5} \\ = 9 + (112.5 - 56.25) - (45 - 9) \\ = 29.25$$

$$11 \text{ a area } A = \int_1^2 10x^{-2} dx$$

$$= [-10x^{-1}]_1^2 \\ = -5 - (-10) = 5$$

$$\text{b } \int_2^p 10x^{-2} dx = \int_p^5 10x^{-2} dx$$

$$[-10x^{-1}]_2^p = [-10x^{-1}]_p^5$$

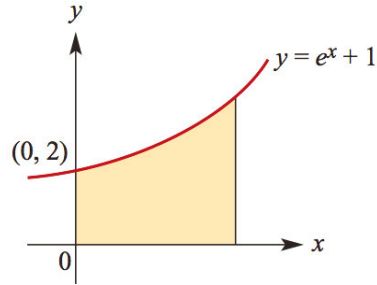
$$5 - \frac{10}{p} = -2 + \frac{10}{p}$$

$$\frac{20}{p} = 7$$

$$p = \frac{20}{7}$$

12 The area of the shaded region is

$$\begin{aligned} & \int_2^4 16x^{-2} - 0.5x + 1 \, dx \\ & + \int_4^5 0.5x - 1 - 16x^{-2} \, dx \\ & = [-16x^{-1} - 0.25x^2 + x]_2^4 \\ & \quad + [0.25x^2 - x + 16x^{-1}]_4^5 \\ & = (-4 - 4 + 4) - (-8 - 1 + 2) \\ & \quad + (6.25 - 5 + 3.2) - (4 - 4 + 4) \\ & = 3.45 \end{aligned}$$



b  $\int_0^2 e^x + 1 \, dx = [e^x + x]_0^2$   
 $= (e^2 + 2) - (1 + 0)$   
 $= e^2 + 1$

13 a When  $x = 0$ ,  $6y - y^2 = 0$ , so  $y = 0, 6$ .  
 Thus  $A$  has coordinates  $(0, 6)$ . For  
 point  $B$ , solve  $y = 6y - y^2$ ,  
 i.e.  $y^2 - 5y = 0$ , so  $y = 0$  or  $y = 5$ .  
 As  $y = x$ , then  $B$  has coordinates  $(5, 5)$ .

b area

$$\begin{aligned} P &= \int_0^5 y \, dy + \int_5^6 6y - y^2 \, dy \\ &= \left[ \frac{1}{2}y^2 \right]_0^5 + \left[ 3y^2 - \frac{1}{3}y^3 \right]_5^6 \\ &= \frac{25}{2} + (108 - 72) - \left( 75 - \frac{125}{3} \right) \\ &= \frac{91}{6} = 15\frac{1}{6} \end{aligned}$$

c Area bounded by the parabola and the  $y$  axis is given by

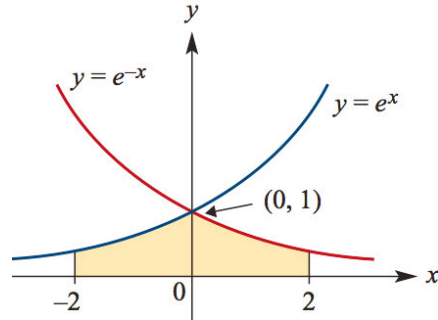
$$\begin{aligned} \int_0^6 6y - y^2 \, dy &= \left[ 3y^2 - \frac{1}{3}y^3 \right]_0^6 \\ &= 36 \end{aligned}$$

So area  $Q = 36 - \text{area}$

$$P = 20\frac{5}{6} = \frac{125}{6}.$$

14 a  $y$  intercept is  $(0, 2)$

15 a The graphs intersect at  $(0, 1)$ .



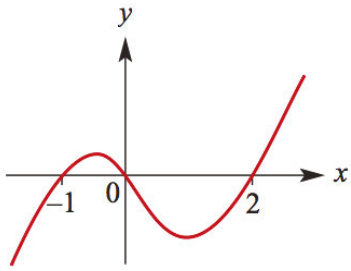
b  $\int_0^2 e^{-x} \, dx + \int_{-2}^0 e^x \, dx = 2 \int_{-2}^0 e^x \, dx$   
 $= 2[e^x]_{-2}^0$   
 $= 2 - 2e^{-2}$

16 a  $\int_0^1 e^x \, dx = [e^x]_0^1$   
 $= e - 1$

b area =  $2(e - 1)$

$$\begin{aligned}
 \text{17 area} &= \int_0^1 2e^{2x} + 3 dx \\
 &= [e^{2x} + 3x]_0^1 \\
 &= (e^2 + 3 + -(1 + 0)) \\
 &= e^2 + 2
 \end{aligned}$$

18 The intercepts are  $(-1, 0)$ ,  $(0, 0)$ ,  $(2, 0)$ .



$$\begin{aligned}
 \text{area} &= \int_{-1}^0 x(x-2)(x+1) dx \\
 &\quad - \int_0^2 x(x-2)(x+1) dx \\
 &= \int_{-1}^0 x^3 - x^2 - 2x dx \\
 &\quad - \int_0^2 x^3 - x^2 - 2x dx \\
 &= \left[ \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \right]_{-1}^0 \\
 &\quad - \left[ \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \right]_0^2 \\
 &= -\left(\frac{1}{4} + \frac{1}{3} - 1\right) - \left(4 - \frac{8}{3} - 4\right) \\
 &= \frac{37}{12} = 3\frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 \text{19 a} \int_0^2 e^{-x} + x dx &= \left[ -e^{-x} + \frac{1}{2}x^2 \right]_0^2 \\
 &= (-e^{-2} + 2) - (-1) \\
 &= 3 - e^{-2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \int_{-2}^{-1} x + \frac{1}{x-1} dx &= \left[ \frac{1}{2}x^2 + \log_e |x-1| \right]_{-2}^{-1} \\
 &= \left( \frac{1}{2} + \log_e |-2| \right) - \left( 2 + \log_e |-3| \right) \\
 &= \log_e \frac{2}{3} - \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \int_0^{\frac{\pi}{2}} \sin x + x dx &= \left[ -\cos x + \frac{1}{2}x^2 \right]_0^{\frac{\pi}{2}} \\
 &= \left( 0 + \frac{\pi^2}{8} \right) - (-1) \\
 &= \frac{\pi^2}{8} + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \int_{-4}^{-5} e^x + \frac{1}{2-2x} dx &= \int_{-4}^{-5} e^x + \frac{1}{2} \times \frac{1}{1-x} dx \\
 &= \left[ e^x - \frac{1}{2} \log_e |1-x| \right]_{-4}^{-5} \\
 &= \left( e^{-5} - \frac{1}{2} \log_e |6| \right) - \left( e^{-4} - \frac{1}{2} \log_e |5| \right) \\
 &= e^{-5} - e^{-4} + \frac{1}{2} \log_e \frac{5}{6}
 \end{aligned}$$

## Solutions to multiple-choice questions

$$\begin{aligned}
 1 \quad \mathbf{C} \quad & \int_0^2 3f(x) + 2 \, dx \\
 &= \int_0^2 3f(x) \, dx + \int_0^2 2 \, dx \\
 &= 3 \int_0^2 f(x) \, dx + [2x]_0^2 \\
 &= 3 \int_0^2 f(x) \, dx + 4
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \mathbf{C} \quad & \sqrt{(ax-b)^3} = (ax-b)^{\frac{3}{2}} \\
 & \int (ax-b)^{\frac{3}{2}} \, dx = \frac{1}{a \times \frac{5}{2}} (ax-b)^{\frac{5}{2}} \\
 &= \frac{2}{5a} (ax-b)^{\frac{5}{2}}
 \end{aligned}$$

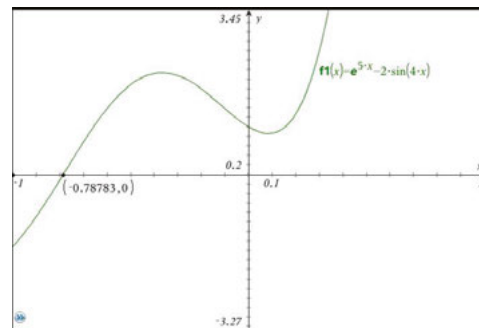
3 B The area of the shaded region from  $x = 0$  to  $x = 2$  is given by  $\int_0^2 f(x) - g(x) \, dx$ .  
The area of the shaded region from  $x = 2$  to  $x = 5$  is given by  $\int_2^5 g(x) - f(x) \, dx$ .  
The area of the shaded region is  $\int_0^2 f(x) - g(x) \, dx + \int_2^5 g(x) - f(x) \, dx$ .

$$\begin{aligned}
 4 \quad \mathbf{B} \quad & \int_a^b c \, dx = [cx]_a^b \\
 &= cb - ca
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \mathbf{A} \quad & \frac{dy}{dx} = \frac{ax}{2} + 1 \\
 & y = \frac{ax^2}{4} + x + c \\
 & x = 0, y = 1 \text{ so } c = 1 \\
 & y = \frac{ax^2}{4} + x + 1
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \mathbf{D} \quad & f'(x) = -6 \sin 3x \\
 & f(x) = 2 \cos 3x + c \\
 & f\left(\frac{2\pi}{3}\right) = 3 \\
 & c = 3 - 2 \cos 2\pi = 1 \\
 & f(x) = 2 \cos 3x + 1
 \end{aligned}$$

7 E Use a CAS calculator to plot the curve on the interval  $(-1, 1)$ . It shows one  $x$  intercept at  $x = -0.78783$ .



Hence the required area is given by

$$\begin{aligned}
 A &= \int_{-1}^{-0.78783} 2 \sin 4x - e^{5x} \, dx \\
 &+ \int_{-0.78783}^1 e^{5x} - 2 \sin 4x \, dx \\
 &= 30.02 \text{ to 2 dp}
 \end{aligned}$$

where the integrals have been evaluated using a CAS calculator.

$$\begin{aligned}
 8 \quad \mathbf{C} \quad & \frac{dy}{dx} = ae^{-x} + 2 \\
 & x = 0, \frac{dy}{dx} = 5, \text{ so } a + 2 = 5, \text{ i.e. } a = 3. \\
 & y = -3e^{-x} + 2x + c \\
 & x = 0, y = 1, \text{ so } -3 + c = 1, \text{ i.e. } c = 4. \\
 & y = -3e^{-x} + 2x + 4
 \end{aligned}$$



$$\text{If } x = 2, y = -3e^{-2} + 4 + 4 = -\frac{3}{e^2} + 8.$$

**9 C**  $R(t) = 5e^{-0.1t}$  litres/minute.

Since  $R(t)$  is the **rate** of flow, it is equal to  $\frac{dV}{dt}$  where  $V$  L is the volume of water at time  $t$ . Thus the outflow in the first 3 minutes is given by

$$\begin{aligned} \int_0^3 5e^{-0.1t} dt &= -\frac{5}{0.1} \left[ e^{-0.1t} \right]_0^3 \\ &= -50(e^{-0.3} - 1) \\ &= 12.959 \dots \end{aligned}$$

To the nearest litre, this is 13 litres.

**10 D** By symmetry, the shaded regions have equal area, so the total area is given by

$$\begin{aligned} 2 \int_{\pi-a}^{\pi} \sin x dx &= 2[-\cos x]_{\pi-a}^{\pi} \\ &= 2(-\cos \pi + \cos(\pi - a)) \\ &= 2(1 - \cos a) \end{aligned}$$

## Solutions to extended-response questions

**1 a** For  $y = x - \frac{1}{x^2} = x - x^{-2}$

$$\frac{dy}{dx} = 1 + 2x^{-3}$$

When  $x = 2$ ,  $\frac{dy}{dx} = 1 + \frac{2}{2^3} = 1\frac{1}{4}$

When  $x = 2$ ,  $y = 2 - \frac{1}{4} = \frac{7}{4}$

The equation of the tangent is

$$y - \frac{7}{4} = \frac{5}{4}(x - 2)$$

$$\therefore 4y - 7 = 5x - 10$$

$$\text{and } 4y - 5x = -3$$

**b** When  $y = 0$ ,  $x = \frac{3}{5}$  The coordinates are  $(\frac{3}{5}, 0)$

**c** When  $y = 0$ ,  $x - \frac{1}{x^2} = 0$

implies  $x^3 - 1 = 0$

i.e.  $x = 1$  The coordinates are  $(1, 0)$

**d** Required area = Area of triangle  $ATC$  – Area of shaded region

$$= \frac{1}{2} \left( 2 - \frac{3}{5} \right) \times \frac{7}{4} - \int_1^2 x - x^{-2} dx$$

$$= \frac{1}{2} \times \frac{7}{5} \times \frac{7}{4} - \left[ \frac{x^2}{2} + x^{-1} \right]_1^2$$

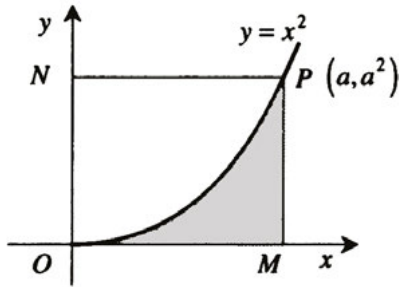
$$= \frac{49}{40} - \left( \left( 2 + \frac{1}{2} \right) - \left( \frac{1}{2} + 1 \right) \right)$$

$$= \frac{49}{40} - 1$$

$$= \frac{9}{40}$$

**e** The required ratio =  $\frac{9}{40} : \frac{49}{40} = 9 : 49$

2 a



Let  $M$ , have coordinates  $(a, 0)$

$$\begin{aligned} \text{Area of } OPM &= \int_0^a x^2 dx \\ &= \left[ \frac{x^3}{3} \right]_0^a \\ &= \frac{a^3}{3} \end{aligned}$$

The coordinates of  $P$  are  $(a, a^2)$

Area of  $OPN$  = area of rectangle  $OMPN$  – area of  $OPM$

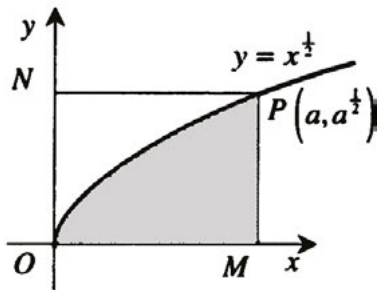


$$= a \times a^2 - \frac{a^3}{3}$$

$$= \frac{2a^3}{3}$$

$\therefore$  The ratio of the areas =  $\frac{2a^3}{3} : \frac{a^3}{3} = 2 : 1$

b



Let  $M$  have coordinates  $(a, 0)$ .

$$\begin{aligned} \text{Area shaded} &= \int_0^a x^{\frac{1}{2}} dx \\ &= \left[ \frac{2}{3} x^{\frac{3}{2}} \right]_0^a \\ &= \frac{2a^{\frac{3}{2}}}{3} \end{aligned}$$

Area of rectangle  $OMPN = a \times a^{\frac{1}{2}} = a^{\frac{3}{2}}$

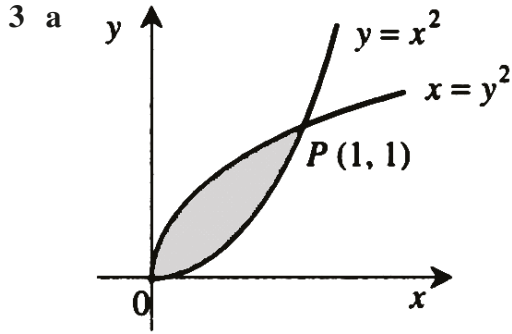
$\therefore$  shaded area =  $\frac{2}{3}$  of the area of rectangle  $OMPN$ .

c Let  $M$  have coordinates  $(a, 0)$ . Then the coordinates of  $P$  are  $(a, a^n)$  Area of region enclosed by  $PM$ , the  $x$ -axis and the curve

$$= \int_0^a x^n dx = \left[ \frac{x^{n+1}}{n+1} \right]_0^a = \frac{a^{n+1}}{n+1}$$

Area of rectangle  $OMPN = a \times a^n = a^{n+1}$

$\therefore$  Area of described region =  $\frac{1}{n+1}$  (area of rectangle)



The parabolas intersect at  $(1, 1)$

$$\begin{aligned} \therefore \text{the area} &= \int_0^1 x^{\frac{1}{2}} - x^2 dx \\ &= \left[ \frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

The area is  $\frac{1}{3}$  square units.

b For  $y = x^n$  and  $y^n = x$

$$x^{\frac{1}{n}} = x^n$$

which implies  $1 = x^{n - \frac{1}{n}}$

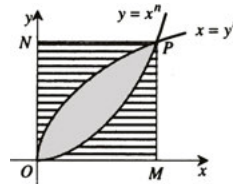
$$\text{i.e. } 1 = x^{\frac{n^2 - 1}{n}}$$

$$\therefore x = (1)^{\frac{n}{n^2 - 1}} = \pm 1$$

If  $n$  is even,  $x = 1$  is the only solution. All such pairs of curves intersect at  $(1, 1)$ .

c The coordinates of  $P$  are  $(1, 1)$

$$\begin{aligned} \text{Area} &= \int_0^1 x^{\frac{1}{n}} - x^n dx \\ &= \left[ \frac{x^{\frac{1}{n} + 1}}{\frac{1}{n} + 1} - \frac{x^{n+1}}{n+1} \right]_0^1 \\ &= \frac{1}{\frac{1}{n} + 1} - \frac{1}{n+1} \\ &= \frac{n}{n+1} - \frac{1}{n+1} \\ &= \frac{n-1}{n+1} \text{ square units} \end{aligned}$$



d Area with shading  $= 1 - \left( \frac{n-1}{n+1} \right)$

$$\begin{aligned} &= \frac{n+1 - n+1}{n+1} \\ &= \frac{2}{n+1} \text{ square units} \end{aligned}$$

e For  $n = 10$ , Area =  $\frac{10 - 1}{10 + 1} = \frac{9}{11}$  square units

For  $n = 100$ , Area =  $\frac{100 - 1}{100 + 1} = \frac{99}{101}$  square units

For  $n = 1000$ , Area =  $\frac{1000 - 1}{1000 + 1} = \frac{999}{1001}$  square units

f For  $\frac{n - 1}{n + 1} = 1 + \frac{2}{n - 1}$ , as  $n \rightarrow \infty$ ,  $\frac{n - 1}{n + 1} \rightarrow 1$

4 a  $\frac{d\theta}{dt} = e^{2.6t}$   
 $\therefore \theta = \frac{1}{2.6}e^{2.6t} + c$   
 $= \frac{5}{13}e^{2.6t} + c$

when  $t = 0$ ,  $\theta = 30$

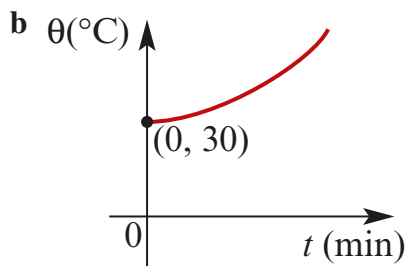
$\therefore c = 30 - \frac{5}{13} = \frac{385}{13}$

and  $\theta = \frac{5}{13}e^{2.6t} + \frac{385}{13}$

when  $t = 3$

$\theta = \frac{5}{13}e^{2.6 \times 3} + \frac{385}{13} = 968.3$

The temperature is  $968.3^\circ\text{C}$  after 3 minutes.



c When  $\theta = 500$

$500 = \frac{5}{13}e^{2.6t} + \frac{385}{13}$

$\therefore \frac{6115}{13} \times \frac{13}{5} = e^{2.6t}$

$1223 = e^{2.6t}$

$\therefore t = \frac{5}{13} \log_e(1223)$

$\approx 2.734$

The temperature is  $500^\circ$  after 2.734 minutes.

**d** The average rate of change for interval  $[1, 2]$

$$\begin{aligned} &= \frac{\theta(2) - \theta(1)}{2 - 1} \\ &= \frac{5}{13}e^{5.2} + \frac{385}{13} - \left( \frac{5}{13}e^{2.6} + \frac{385}{13} \right) \\ &= \frac{5}{13}(e^{5.2} - e^{2.6}) \\ &\approx 64.5 \end{aligned}$$

The average rate of change for the interval  $[1, 2]$  is  $64.5^\circ$  per minute.

**5**  $\frac{dx}{dt} = ve^{-t}$ , where  $v = 5 \times 10^4$  m/s

**a** When  $t = 0$ ,  $\frac{dx}{dt} = 5 \times 10^4$  m/s

**b**  $\frac{dx}{dt} = \frac{v}{e^t} = \frac{5 \times 10^4}{e^t}$   
as  $t \rightarrow \infty$ ,  $\frac{dx}{dt} \rightarrow 0$

**c** The distance travelled between  $t = 0$  and  $t = 20$

$$\begin{aligned} &= \int_0^{20} 5 \times 10^4 e^{-t} dt \\ &= \left[ -5 \times 10^4 e^{-t} \right]_0^{20} \\ &= -5 \times 10^4 \times e^{-20} + 5 \times 10^4 \\ &= 5 \times 10^4(1 - e^{-20}) \text{metres} \end{aligned}$$

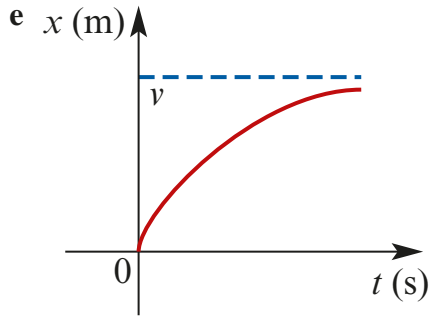
**d**  $\frac{dx}{dt} = ve^{-t}$   
 $\therefore x = -ve^{-t} + c$

When  $t = 0$ ,  $x = 0$

$$\therefore 0 = -v + c$$

i.e.  $c = v$

$$\begin{aligned} \therefore x &= v - ve^{-t} \\ &= v(1 - e^{-t}) \end{aligned}$$



6 a Let  $y = e^{-3x} \sin 2x$   
 then, using the produce rule,  

$$\frac{dy}{dx} = -3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x \quad \textcircled{1}$$

For  $y = e^{-3x} \cos 2x$

$$\frac{dy}{dx} = 3e^{-3x} \cos 2x - 2e^{-3x} \sin 2x \quad \textcircled{2}$$

b From  $\textcircled{1}$

$$\int (-3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x) dx = e^{-3x} \sin 2x + c_1$$

i.e.  $-3 \int e^{-3x} \sin 2x dx + 2 \int e^{-3x} \cos 2x dx = e^{-3x} \sin 2x + c_1 \quad \textcircled{3}$

From  $\textcircled{2}$

$$\int (-3e^{-3x} \cos 2x - 2e^{-3x} \sin 2x) dx = e^{-3x} \cos 2x + c_2$$

i.e.  $-3 \int e^{-3x} \cos 2x - 2 \int e^{-3x} \sin 2x dx = e^{-3x} \cos 2x + c_2 \quad \textcircled{4}$

c Let  $a = \int e^{-3x} \sin 2x dx$  and  $b = \int e^{-3x} \cos 2x dx$

Then the equations can be rewritten as

$$-3a + 2b = e^{-3x} \sin 2x + c_1$$

$$-3b - 2a = e^{-3x} \cos 2x + c_2$$

Multiply  $\textcircled{5}$  by 3 and  $\textcircled{6}$  by 2 and add:

$$-9a - 4a = 3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x + (3c_1 + 2c_2)$$

$$\therefore -13 \int e^{-3x} \sin 2x dx = 3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x + (3c_1 + 2c_2)$$

i.e.  $\int (e^{-3x} \sin 2x) dx = -\frac{1}{13} (3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x) + C$

where  $C = \frac{3c_1 + 3c_2}{-13}$

7 a

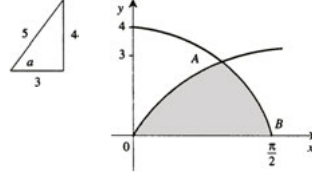
i  $3 \sin a = 4 \cos a$

$$\therefore \frac{3 \sin a}{\cos a} = 4$$

$$\tan a = \frac{4}{3}$$

ii Consider the triangle

$$\text{Then } \sin(a) = \frac{4}{5} \text{ and } \cos(a) = \frac{3}{5}$$



b Area of the shaded region =  $\int_0^a 3 \sin x dx + \int_a^{\frac{\pi}{4}} 4 \cos x dx$

$$= [-3 \cos x]_0^a + [4 \sin x]_a^{\frac{\pi}{4}}$$

$$= -3 \cos a - (-3) + 4 \sin \frac{\pi}{4} - 4 \sin a$$

$$= -3 \times \frac{3}{5} + 3 + 4 - 4 \times \frac{4}{5}$$

$$= -\frac{9}{5} + 7 - \frac{16}{5}$$

$$= -5 + 7$$

$$= 2$$

Area of the shaded region = 2 square units.

8 a  $y = x \log_e x$

Using the product rule gives

$$\frac{dy}{dx} = \log_e x + x \times \frac{1}{x}$$

$$= \log_e x + 1$$

$$\text{Also } \int (\log_e x + 1) dx = x \log_e x + c$$

$$\therefore \int \log_e x dx + x = x \log_e x + c$$

$$\text{and } \int_1^e \log_e x dx = [x \log_e x - x]_1^e$$

$$= e \log_e e - e - (e \log_e 1 - 1)$$

$$= e - e - 0 + 1$$

$$\therefore \int_1^e \log_e x dx = 1$$

b  $y = x(\log_e x)^n$

Using the product rule:

$$\frac{dy}{dx} = (\log_e x)^n + x \times \frac{1}{x} \times n(\log_e x)^{n-1}$$

$$= (\log_e x)^n + n(\log_e x)^{n-1}$$



**c**  $I_n = \int_1^e (\log_e x)^n dx$ , and  $I_{n-1} = \int_1^e (\log_e x)^{n-1} dx$

From  $\oplus$

$$\int (\log_e x)^n + n(\log_e x)^{n-1} dx = x(\log_e x)^n + c$$

$$\begin{aligned} \therefore I_n + nI_{n-1} &= \left[ x(\log_e x)^n \right]_1^e \\ &= e \end{aligned}$$

**d**  $I_3 = \int_1^3 (\log_e x)^3 dx$

From (c)  $I_3 = e - 3I_2$

$$= e - 3[e - 2I_1]$$

$$= e - 3e + 6I_1$$

$$= -2e + 6 \quad \text{by (a)}$$

**9** To find the point of intersection, consider  $x^2 = by$

and  $y^2 = ax$

as a simultaneous pair.

$$\therefore y = \frac{x^2}{b} \text{ and } \left(\frac{x^2}{b}\right)^2 = ax$$

which implies  $x^3 = ab^2$

and  $x = a^{\frac{1}{3}}b^{\frac{2}{3}}$  Substitute for  $x$  in

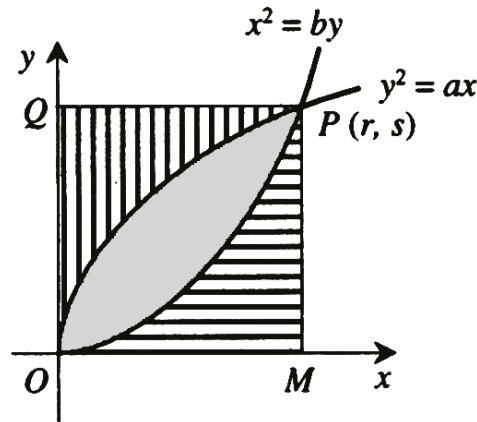
$$y^2 = ax$$

$$\therefore y^2 = a\left(a^{\frac{1}{3}}b^{\frac{2}{3}}\right)$$

$$= a^{\frac{4}{3}}b^{\frac{2}{3}}$$

$$\therefore y = a^{\frac{2}{3}}b^{\frac{1}{3}}$$

$$\therefore r = a^{\frac{1}{3}}b^{\frac{2}{3}} \text{ and } s = a^{\frac{2}{3}}b^{\frac{1}{3}}$$



The area of the region shaded horizontally

$$= \int_0^{a^{\frac{1}{3}}b^{\frac{2}{3}}} \frac{x^2}{b} dx$$

$$= \frac{1}{b} \left[ \frac{x^3}{3} \right]_0^{a^{\frac{1}{3}}b^{\frac{2}{3}}}$$

$$= \frac{1}{3b} \times a \times b^2$$

$$= \frac{ab}{3}$$

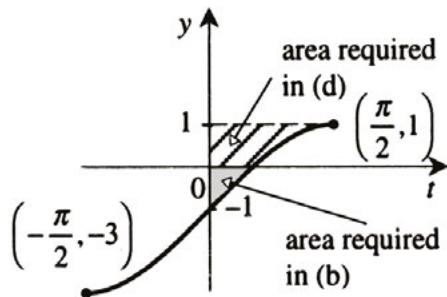
The area of the region shaded vertically

$$\begin{aligned}
&= \int_0^{a^{\frac{2}{3}}b^{\frac{1}{3}}} \frac{y^2}{a} dy \\
&= \left[ \frac{y^3}{3a} \right]_0^{a^{\frac{2}{3}}b^{\frac{1}{3}}} \\
&= \frac{ab}{3}
\end{aligned}$$

The area of rectangle  $OMPQ = a^{\frac{2}{3}}b^{\frac{1}{3}} \times a^{\frac{1}{3}}b^{\frac{2}{3}} = ab$

$\therefore$  All three regions have area  $\frac{ab}{3}$

10 a



b

$$\begin{aligned}
\int_0^{\frac{\pi}{6}} 2 \sin x - 1 dx &= [-2 \cos x - x]_0^{\frac{\pi}{6}} \\
&= -2 \cos \frac{\pi}{6} - \frac{\pi}{6} - (-2 \cos 0 - 0) \\
&= -2 \times \frac{\sqrt{3}}{2} - \frac{\pi}{6} + 2 \\
&= 2 - \sqrt{3} - \frac{\pi}{6}
\end{aligned}$$

c For the inverse of  $f(x) = 2 \sin x - 1$

consider  $x = 2 \sin y - 1$

$$\frac{(x+1)}{2} = \sin y$$

$$\text{and } y = \sin^{-1}\left(\frac{x+1}{2}\right)$$

$$\text{i.e. } f^{-1}(x) = \sin^{-1}\left(\frac{x+1}{2}\right)$$

The domain of  $f^{-1} = \text{range of } f = [-3, 1]$

$$\begin{aligned}
\mathbf{d} \quad \int_0^1 f^{-1}(x) dx &= \text{area of rectangle} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x) dx \\
&= \frac{\pi}{2} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2 \sin x - 1) dx \\
&= \frac{\pi}{2} - [-2 \cos x - x]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} + [2 \cos x + x]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} + \left[ 0 + \frac{\pi}{2} - \left( 2 \cos \frac{\pi}{6} + \frac{\pi}{6} \right) \right] \\
&= \frac{\pi}{2} + \frac{\pi}{2} - \sqrt{3} - \frac{\pi}{6} \\
&= \frac{5\pi}{6} - \sqrt{3}
\end{aligned}$$

The 'integral' command of a CAS could be used in this question.

**11 a** For  $y = e^{\frac{x}{10}}(10 - x)$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{10}e^{\frac{x}{10}}(10 - x) - e^{\frac{x}{10}} \\
&= e^{\frac{x}{10}}\left(1 - \frac{x}{10} - 1\right) \\
&= -\frac{x}{10}e^{\frac{x}{10}}
\end{aligned}$$

For  $y = \sqrt{100 - x^2} = (100 - x^2)^{\frac{1}{2}}$

$$\begin{aligned}
\frac{dy}{dx} &= -x(100 - x^2)^{-\frac{1}{2}} \\
&= \frac{-x}{(100 - x^2)^{\frac{1}{2}}}
\end{aligned}$$

**b** When  $x = 0$ ,  $\frac{dy}{dx} = 0$  for both functions

**c** When  $x = 10$ ,  $\frac{dy}{dx} = -e$

**d**  $\int_0^{10} e^{\frac{x}{10}}(10 - x) dx = 71.828 \dots$

$\int_0^{10} \sqrt{10^2 - x^2} dx = 78.5398 \dots$

$\therefore$  area between the curves = 6.7118 square units

**e** percentage error =  $\frac{6.7118}{25\pi} \times 100 = 8.55\%$

**f** Equation of the chord is  $y = 10 - x$

Area of the shaded region =  $25\pi - 50 = 28.54$  square units

$$\begin{aligned} \mathbf{g \ i} \quad \frac{d}{dx} \left( e^{\frac{x}{10}}(10 - x) \right) &= \frac{1}{10} e^{\frac{x}{10}}(10 - x) - e^{\frac{x}{10}} \\ \therefore \frac{1}{10} \int_0^{10} e^{\frac{x}{10}}(10 - x) dx &= \left[ e^{\frac{x}{10}}(10 - x) \right]_0^{10} + \int_0^{10} e^{\frac{x}{10}} dx \\ &= -(10) + \left[ 10e^{\frac{x}{10}} \right]_0^{10} \\ &= -10 + 10e - 10 \\ &= 10e - 20 \end{aligned}$$

$$\therefore \int_0^{10} e^{\frac{x}{10}}(10 - x) dx = 10(10e - 20)$$

**ii**  $\therefore$  exact area of shaded region =  $25\pi - 100e + 200$  square units

**12**  $R(t) = 10e^{-\left(\frac{t}{10}\right)} \sin\left(\frac{\pi t}{3}\right)$

**a i**  $R(0) = 0$

**ii**  $R(3) = 10e^{-\frac{3}{10}} \sin \pi = 0$

**b**  $R'(t) = -e^{-\frac{t}{10}} \sin\left(\frac{\pi t}{3}\right) + \frac{10\pi}{3} e^{-\frac{t}{10}} \cos\left(\frac{\pi t}{3}\right)$   
 $= e^{-\frac{t}{10}} \left[ \frac{10\pi}{3} \cos\left(\frac{\pi t}{3}\right) - \sin\left(\frac{\pi t}{3}\right) \right]$

**c i**  $R'(t) = 0$  implies  $\frac{10\pi}{3} \cos\left(\frac{\pi t}{3}\right) = \sin\left(\frac{\pi t}{3}\right)$  as  $e^{-\frac{t}{10}} \neq 0$

$$\therefore \tan\left(\frac{\pi t}{3}\right) = \frac{10\pi}{3}$$

$$\therefore \frac{\pi t}{3} = \tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } \pi + \tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 2\pi + \tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 3\pi + \tan^{-1}\left(\frac{10\pi}{3}\right)$$

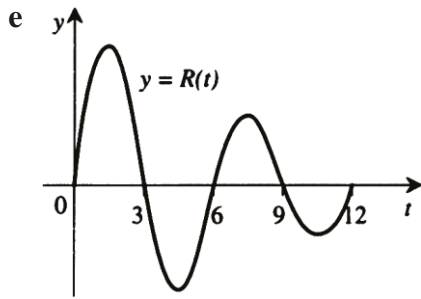
$$\therefore t = \frac{3}{\pi} \tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 3 + \frac{3}{\pi} \tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 6 + \frac{3}{\pi} \tan^{-1}\left(\frac{10\pi}{3}\right) \text{ or } 9 + \tan^{-1}\left(\frac{10\pi}{3}\right)$$

**ii** stationary points (1.409, 8.646) and (7.409, 4.745) loc max (4.409, -6.405) and (10.409, -3.515) loc min

**d**  $R(t) = 0$  implies  $\sin\left(\frac{\pi t}{3}\right) = 0$  as  $10 \therefore e^{-\frac{t}{10}} \neq 0$

$$\therefore \frac{\pi t}{3} = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } 4\pi$$

$t = 0$  or  $3$  or  $6$  or  $9$  or  $12$



- f i Use a CAS calculator to find areas  
 $\int_0^3 R(t) dt \approx 16.47337 \quad \therefore \quad 16.47$  litres flowed in
- ii  $\int_3^6 R(t) dt \approx -12.20377 \quad \therefore \quad 12.20$  litres flowed out
- iii Total amount of water in the device  
 $= 16.47337 \dots - 12.20377 \dots + 4$   
 $= 4.2695 + 4$   
 $= 8.2695$   
 There are approximately 8.27 litres in the device

- g  $\int_0^{30} R(t) dt \approx 8.9918 \dots$   
 $\therefore$  There are  $4 + 8.9918 = 12.992$  litres in the device after 30 minutes. (Use a CAS calculator with this problem.)

- 13 a If  $\cos 2x = 2 \cos^2 x - 1$  and  $\cos 2x = 1 - 2 \sin^2 x$

$$\begin{aligned} \frac{1 - \cos 2x}{1 + \cos 2x} &= \frac{1 - (1 - 2 \sin^2 x)}{1 + (2 \cos^2 x - 1)} \\ &= \frac{2 \sin^2 x}{2 \cos^2 x} \\ &= \tan^2 x \\ &= \sec^2 x - 1 \end{aligned}$$

b  $\int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{1 + \cos 2x} dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) dx$

$$\begin{aligned} &= \left[ \tan x - x \right]_0^{\frac{\pi}{4}} \\ &= \tan \frac{\pi}{4} - \frac{\pi}{4} = 1 - \frac{\pi}{4} \end{aligned}$$

# Chapter 12 – Revision of Chapters 9–11

## Solutions to Technology-free questions

1  $y = \frac{x^2 - 1}{x^4 - 1}$

a 
$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^4 - 1) \frac{d(x^2 - 1)}{dx} - (x^2 - 1) \frac{d(x^4 - 1)}{dx}}{(x^4 - 1)^2} \\ &= \frac{2x(x^4 - 1) - 4x^3(x^2 - 1)}{(x^4 - 1)^2} \\ &= \frac{2x^5 - 2x - 4x^5 + 4x^3}{(x^4 - 1)^2} \\ &= \frac{-2(x^5 - 2x^3 + x)}{(x^4 - 1)^2} \\ &= \frac{-2x(x^2 - 1)^2}{(x^4 - 1)^2} \end{aligned}$$

b  $\frac{dy}{dx} = 0,$   
 $0 = \frac{-2x(x^2 - 1)^2}{(x^4 - 1)^2}$

looking at the numerator

$$x = 0, x^2 = 1$$

$$x = -1, 0, 1$$

looking at the denominator

$$x^4 \neq 1$$

$$x \neq \pm 1$$

$$\therefore x = 0$$

in set notation,  $\{0\}$

2  $y = (3x^2 - 4x)^4$

Let  $u = 3x^2 - 4x$ .

Then  $y = u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 4u^3 \times (6x - 4)$$

$$= 8(3x - 2)(3x^2 - 4x)^3$$

3  $f(x) = x^2 \log_e(2x)$

$$\begin{aligned} f'(x) &= 2x \log_e(2x) + x^2 \times \frac{1}{x} \\ &= 2x \log_e(2x) + x \end{aligned}$$

4 a  $f(x) = e^{2x+1}$

$$f'(x) = 2e^{2x+1}$$

$$f'(b) = 2e^{2b+1}$$

The tangent is at the point  $(b, e^{2b+1})$

Gradient of line from the point to the

origin is  $\frac{e^{2b+1}}{b}$

$$\therefore \frac{e^{2b+1}}{b} = 2e^{2b+1}$$

$$\therefore b = \frac{1}{2}$$

b  $f(b) = e^{2b+1} + k$

$$f'(b) = 2e^{2b+1}$$

$$\therefore \frac{e^{2b+1} + k}{b} = 2e^{2b+1}$$

$$\therefore e^{2b+1} + k = 2be^{2b+1}$$

$$\therefore k = (2b - 1)e^{2b+1}$$

$$5 \quad y = x^{\frac{1}{3}} + c$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

When  $x = 8, y = a$

$$\therefore a = 2 + c \dots (1)$$

$$\text{When } x = 8, \frac{dy}{dx} = \frac{1}{12}$$

$$\therefore m = \frac{1}{12}$$

When  $x = 8, a = 8m - 8$

$$\therefore a = \frac{2}{3} - 8 = -\frac{22}{3}$$

Substitute in (1)

$$-\frac{22}{3} = 2 + c$$

$$c = -\frac{28}{3}$$

$$6 \quad \text{Average value} = \frac{1}{2} \int_0^2 \frac{1}{3x+1} dx$$

$$= \frac{1}{6} \left[ \log_e(3x+1) \right]_0^2$$

$$= \frac{1}{6} \log_e 7$$

$$7 \quad \mathbf{a} \quad \int \frac{3}{5x-2} dx = \frac{3}{5} \log_e(5x-2) + c$$

$$\mathbf{b} \quad \int \frac{3}{(5x-2)^2} dx = \frac{3}{10-25x}$$

$$8 \quad \mathbf{a} \quad g(x) = 3x^2 - 5f(x)$$

$$g'(x) = 6x - 5f'(x)$$

$$g'(3) = 6 \times 3 - 5f'(3x)$$

$$= 18 - 5 \times 5$$

$$g'(3) = -7$$

$$\mathbf{b} \quad g(x) = \frac{3x+1}{f(x)}$$

$$g'(x) = \frac{f(x) \frac{d(3x+1)}{dx} - f'(x)(3x+1)}{(f(x))^2}$$

$$= \frac{3f(x) - (3x+1)f'(x)}{(f(x))^2}$$

$$g'(3) = \frac{3f(3) - (9+1)f'(3)}{(f(3))^2}$$

$$= \frac{3 \times -2 - 10 \times 5}{(-2)^2}$$

$$= \frac{-6 - 50}{4}$$

$$g'(3) = -14$$

$$\mathbf{c} \quad g(x) = [f(x)]^2$$

$$g'(x) = 2f(x)f'(x)$$

$$\therefore g'(3) = 2 \times (-2) \times 5$$

$$= -20$$

$$9 \quad \mathbf{a} \quad g(x) = \sqrt{x}f(x)$$

$$g'(x) = \sqrt{x}f'(x) + \frac{f(x)}{2f\sqrt{x}}$$

$$g'(4) = \sqrt{4}f'(4) + \frac{f(4)}{2\sqrt{4}}$$

$$= 2 \times 2 + \frac{6}{4}$$

$$g'(4) = \frac{11}{2} = 5\frac{1}{2}$$

$$\begin{aligned} \mathbf{b} \quad g(x) &= \frac{f(x)}{x} \\ g'(x) &= \frac{xf'(x) - f(x)}{x^2} \\ g'(x) &= \frac{xf'(x) - f(x)}{x^2} \\ g'(4) &= \frac{4f'(4) - f(4)}{4^2} \\ &= \frac{4 \times 2 - 6}{16} \\ g'(4) &= \frac{1}{8} \end{aligned}$$

$$\begin{aligned} \mathbf{10} \quad f(x) &= f(g(x)) \\ f'(x) &= \sqrt{3x+4}, \quad g(x) = x^2 - 1 \\ \therefore g'(x) &= 2x \\ f'(x) &= f'(g(x)) \times g'(x) \\ &= \sqrt{3(x^2 - 1) + 4} \times 2x \\ &= \sqrt{3(x^2 - 3) + 4} \times 2x \\ f'(x) &= 2x \sqrt{3x^2 + 1} \end{aligned}$$

$$\begin{aligned} \mathbf{11} \quad f(x) &= 2x^2 - 3x + 5 \\ \mathbf{a} \quad f'(x) &= 4x - 3 \\ \mathbf{b} \quad f'(0) &= -3 \\ \mathbf{c} \quad f'(x) &= 1 \\ 4x - 3 &= 1 \\ 4x &= 4 \\ x &= 1 \\ &\text{in set notation}\{1\} \end{aligned}$$

$$\mathbf{12} \quad \frac{d}{dx} (\log_e 3f(x)) = \frac{f'(x)}{f(x)}$$

$$y = \sqrt{a-x} = (a-x)^{\frac{1}{2}}$$

Let  $u = a - x$ .

$$\text{Then } y = u^{\frac{1}{2}}$$

$$\begin{aligned} \mathbf{13} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ \frac{dy}{dx} &= \frac{1}{2} u^{-\frac{1}{2}} (-1) \\ &= -\frac{1}{2\sqrt{a-x}} \end{aligned}$$

$$\text{When } x = 1, \frac{dy}{dx} = -6$$

$$\therefore -\frac{1}{2\sqrt{a-1}} = -6$$

$$1 = 12\sqrt{a-1}$$

$$\frac{1}{144} = a - 1$$

$$a = \frac{145}{144}$$

$$\mathbf{14} \quad \text{Area of region A} = \int_0^1 -x^2 - x + 2 \, dx$$

$$\begin{aligned} &= \left[ -\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_0^1 \\ &= \frac{7}{6} \end{aligned}$$

$$\text{Area of region B} = -\int_1^m -x^2 - x + 2 \, dx$$

$$\begin{aligned} &= -\left[ -\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_1^m \\ &= \frac{m^3}{3} + \frac{m^2}{2} - 2m + \frac{7}{6} \end{aligned}$$



$$\text{Area A} = \text{Area B}$$

$$\frac{m^3}{3} + \frac{m^2}{2} - 2m = 0$$

$$2m^3 + 3m^2 - 12m = 0$$

$$m(2m^2 + 3m - 12) = 0$$

$$m = 0 \text{ or } m = \frac{-3 \pm \sqrt{9 + 96}}{4}$$

$$m = 0 \text{ or } m = \frac{-3 \pm \sqrt{105}}{4}$$

$$\text{But } m > 1, \therefore m = \frac{-3 + \sqrt{105}}{4}$$

$$f(x) = x^3 + 3x^2 - 4$$

$$f'(x) = 3x^2 + 6x$$

$$15 \text{ a } f'(x) = 0 \Rightarrow 3x(x + 2) = 0$$

$$\therefore x = 0 \text{ or } x = -2$$

$$f(0) = -4, f(-2) = 0$$

$$15 \text{ b } \int_{-2}^2 f(x) dx = \left[ \frac{x^4}{4} + x^3 - 4x \right]_{-2}^2 = 0$$

$$15 \text{ c } \int_0^2 f(x) dx = \left[ \frac{x^4}{4} + x^3 - 4x \right]_0^2 = 4$$

$$15 \text{ d } \text{Area} = - \int_0^1 f(x) dx + \int_1^2 f(x) dx \\ = \frac{19}{2}$$

$$16 \text{ } f(x) = \frac{1}{3x-1} = (3x-1)^{-1}$$

$$f'(x) = \frac{d(3x-1)}{dx} \times -(3x-1)^{-2}$$

$$= \frac{-3}{(3x-1)^2}$$

$$f'(2) = \frac{-3}{(6-1)^2}$$

$$f'(2) = \frac{-3}{25}$$

$$17 \text{ } y = 1 - x^2$$

$$\frac{dy}{dx} = -2x$$

$$LHS = x \frac{dy}{dx} + 2$$

$$= x \times -2x + 2$$

$$= 2 - 2x^2$$

$$= 2(1 - x^2)$$

$$= 2y = RHS \text{ QED}$$

$$18 \text{ } A = 4\pi r^2$$

$$\frac{dA}{dr} = 8\pi r$$

$$r = 3, \frac{dA}{dr} = 8\pi \times 3$$

$$= 24\pi$$

$$19 \text{ } y = 1.8x^2$$

$$\frac{dy}{dx} = 3.6x$$

$$\frac{dy}{dx} = 1, x = \frac{1}{3.6}$$

$$x = \frac{10}{36}$$

$$x = \frac{10}{36}$$

$$y = \frac{18}{10} \times \left(\frac{10}{36}\right)^2$$

$$= \frac{18 \times 10}{(36)^2}$$

$$= \frac{10}{2 \times 36}$$

$$= \frac{10}{72}$$

$$\text{co-ords} = \left(\frac{10}{36}, \frac{10}{72}\right)$$

$$\approx (0.28, 0.14)$$

$$20 \quad y = 3x^2 - 4x + 7$$

$$\frac{dy}{dx} = 6x - 4$$

$$\frac{dy}{dx} = 0,$$

$$6x - 4 = 0$$

$$x = \frac{2}{3}$$

$$21 \quad y = \frac{x^2 + 2}{x^2 - 2}$$

$$y = \frac{x^2 - 2 + 4}{x^2 - 2}$$

$$y = 1 + \frac{4}{x^2 - 2}$$

$$y = 1 + 4(x^2 - 2)^{-1}$$

$$\frac{dy}{dx} = 2x \times -4(x^2 - 2)^{-2}$$

$$= \frac{-8x}{(x^2 - 2)^2}$$

$$22 \quad z = 3y + 4,$$

$$y = 2x - 1$$

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

$$= 3 \times 2$$

$$= 6$$

$$23 \quad y = (5 - 7x)^9$$

$$\frac{dy}{dx} = -7 \times 9(5 - 7x)^8$$

$$= -63(5 - 7x)^8$$

$$24 \quad y = 3x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{3}{3} = x^{-\frac{2}{3}}$$

$$= x^{-\frac{2}{3}}$$

$$x = 27,$$

$$\frac{dy}{dx} = (27)^{-\frac{2}{3}}$$

$$= (3)^{-2}$$

$$= \frac{1}{9}$$

$$25 \quad y = \sqrt{5 + x^2}$$

$$\frac{dy}{dx} = 2x \times \frac{1}{2\sqrt{5 + x^2}}$$

$$= \frac{x}{\sqrt{5 + x^2}}$$

$$x = 2,$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{5 + 4}}$$

$$= \frac{2}{3}$$

$$26 \quad y = (x^2 + 3)(2 - 4x - 5x^2)$$

$$\frac{dy}{dx} = (x^2 + 3)(-4 - 10x)$$

$$+ (2 - 4x - 5x^2)(2x)$$

$$x = 1,$$

$$\frac{dy}{dx} = (1 + 3)(-4 - 10) + (2 - 4 - 5)(2)$$

$$= 4 \times -14 + (-7) \times 2$$

$$= 5 \times -14$$

$$= -70$$

$$\begin{aligned}
 27 \quad y &= \frac{x}{1+x^2} \\
 \frac{dy}{dx} &= \frac{(1+x^2)\frac{dx}{dx} - x\frac{d(1+x^2)}{2dx}}{(1+x^2)^2} \\
 &= \frac{(1+x^2) - x \times 2x}{(1+x^2)^2} \\
 &= \frac{1-x^2}{(1+x^2)^2} \\
 x &= 1, \\
 \frac{dy}{dx} &= \frac{1-1}{(1+1)^2} = 0
 \end{aligned}$$

$$\begin{aligned}
 28 \quad y &= \frac{2+x}{x^2+x+1} \\
 \frac{dy}{dx} &= \frac{x^2+x+1\frac{d(2+x)}{dx} - (2+x)\frac{d(x^2+x+1)}{dx}}{(x^2+x+1)^2} \\
 &= \frac{(x^2+x+1) - (2+x)(2x+1)}{(x^2+x+1)^2} \\
 &= \frac{x^2+x+1 - 2x^2 - 5x - 2}{(x^2+x+1)^2} \\
 &= \frac{-x^2 - 4x - 1}{(x^2+x+1)^2} \\
 x &= 0, \\
 \frac{dy}{dx} &= \frac{-1}{1} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 29 \quad f(x) &= \frac{1}{2x+1} \\
 \mathbf{a} \quad f'(x) &= \lim_{h \rightarrow 0} \left( \frac{f(x+h) - f(x)}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left( \frac{\frac{1}{2x+2h+1} - \frac{1}{2x+1}}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left( \frac{1}{h} \times \left( \frac{2x+1 - (2x+2h+1)}{(2x+1)(2x+2h+1)} \right) \right) \\
 &= \lim_{h \rightarrow 0} \left( \frac{1}{h} \times \left( \frac{-2h}{4x^2 + 4xh + 2x + 2x + 2h + 1} \right) \right) \\
 &= \lim_{h \rightarrow 0} \left( \frac{-2}{4x^2 + 4xh + 4x + 2h + 1} \right) \\
 &= \frac{-2}{4x^2 + 4x + 1} \\
 &= \frac{-2}{(2x+1)^2} \\
 \mathbf{b} \quad f'(0) &= \frac{-2}{(1)^2} \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 30 \quad f(x) &= x^3 + 3x^2 - 1 \\
 \mathbf{a} \quad f'(x) &= 3x^2 + 6x \\
 f'(x) &= 0, \\
 3x^2 + 6x &= 0 \\
 x(x+2) &= 0 \\
 x &= 0, -2 \\
 &\text{in set notation, } \{-2, 0\}
 \end{aligned}$$

**b**  $f'(x) > 0$   
 $3x^2 + 6x > 0$   
 $x(x + 2) > 0$   
 $x < -2, x > 0$   
in set notation,  $R \setminus [-2, 0]$

**c**  $f'(x) < 0$   
 $3x^2 + 6x < 0$   
 $x(x + 2) < 0$   
 $-2 < x < 0$   
in set notation,  $(-2, 0)$

**31**  $y = \frac{x}{1-x}$

**a**  $y = \frac{x-1+1}{1-x}$   
 $y = \frac{-(1-x)}{1-x} + \frac{1}{1-x}$   
 $= -1 - \frac{1}{1-x}$   
 $= -1 - (x-1)^{-1}$   
 $\frac{dy}{dx} = -1 - (x-1)^{-2}$   
 $= \frac{1}{(x-1)^2}$

**b**  $y = \frac{x}{1-x}$   
 $(1-x)y = x$   
 $y - yx = x$   
 $x(1+y) = y$   
 $x = \frac{y}{1+y}$   
 $x = \frac{y+1}{1+y} - \frac{1}{1+y}$   
 $x = 1 - \frac{1}{1+y}$   
 $= 1 - (y+1)^{-1}$   
 $\frac{dx}{dy} = -1 \times -(y+1)^{-2}$   
 $= (y+1)^{-2}$   
 $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = ((y+1)^{-2})^{-1}$   
 $\frac{dy}{dx} = (y+1)^2$

**32**  $y = (x^2 + 1)^{-\frac{3}{2}}$   
 $\frac{dy}{dx} = 2x \times \frac{-3}{2}(x^2 + 1)^{-\frac{5}{2}}$   
 $= -3x(x^2 + 1)^{-\frac{5}{2}}$

**33**  $y = x^4$   
 $\frac{dy}{dx} = 4x^3$   
LHS  $= x \times \frac{dy}{dx}$   
 $= x \times 4x^3$   
 $= 4x^4$   
 $= 4y$   
 $= \text{RHS QED}$

**34**  $f'(x) = 10x^4 > 0$  for all  $x \neq 0$   
 $f(b) = 2b^5 > f(0) = 0$  for all  $b > 0$   
 $f(b) = 2b^5 < f(0) = 0$  for all  $b < 0$

**35 a**  $\int_0^{\frac{\pi}{2}} 2 \sin\left(\frac{x}{2}\right) dx$   
 $= \left[ -4 \cos\left(\frac{x}{2}\right) \right]_0^{\frac{\pi}{2}}$   
 $= 4 - 2\sqrt{2}$

**b**  $\int_0^{\frac{3}{2}} e^{\frac{x}{2}} dx$   
 $= \left[ 2e^{\frac{x}{2}} \right]_0^{\frac{3}{2}}$   
 $= 2(e^{\frac{3}{4}} - 1)$

**c**  $\int_{\frac{1}{2}}^1 \frac{1}{2x} dx$   
 $= \left[ \frac{1}{2} \log_e(x) \right]_{\frac{1}{2}}^1$   
 $= \frac{1}{2} \log_e 2$

**d**  $\int_{-1}^{-\frac{1}{2}} \frac{1}{2x} dx$   
 $= - \int_{\frac{1}{2}}^1 \frac{1}{2x} dx$   
 $= - \left[ \frac{1}{2} \log_e(x) \right]_{\frac{1}{2}}^1$   
 $= -\frac{1}{2} \log_e 2$

**e**  $\int_3^4 \frac{1}{2(x-2)^2} dx$   
 $= \frac{1}{4}$

**f**  $\int_2^4 \frac{1}{(3x-2)^2} dx$   
 $= \frac{1}{20}$

**36**  $f'(x) = -6x^2 < 0$  for all  $x \neq 0$   
 $f(b) = -6b^3 < f(0) = 0$  for all  $b > 0$   
 $f(b) = -6b^3 > f(0) = 0$  for all  $b < 0$

**37 a**  $f(x) = e^{-mx+2} + 4x$   
 $f'(x) = -me^{-mx+2} + 4$   
 $f'(x) = 0 \Rightarrow e^{-mx+2} = \frac{4}{m}$   
 $\Rightarrow -mx + 2 = \log_e \frac{4}{m}$   
 $\Rightarrow x = \frac{1}{m} \left( 2 - \log_e \frac{4}{m} \right)$

**b**  $\frac{1}{m} \left( 2 - \log_e \frac{4}{m} \right) < 0$   
 $\Leftrightarrow 2 - \log_e \frac{4}{m} < 0$   
 $\Leftrightarrow \log_e \frac{4}{m} > 2$   
 $\Leftrightarrow \frac{4}{m} > e^2$   
 $\Leftrightarrow m < 4e^{-2}$

## Solutions to multiple-choice questions

$$\begin{aligned}
 \mathbf{1 \ E} \quad y &= \frac{x^4 + x}{x^2} \\
 &= x^2 + x^{-1} \\
 \frac{dy}{dx} &= 2x - x^{-2} \\
 &= 2x - \frac{1}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2 \ D} \quad f(x) &= x^5 + x^3 + x \\
 f'(x) &= 5x^4 + 3x^2 + 1 \\
 f'(1) &= 5 + 3 + 1 = 9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3 \ E} \quad y &= (4 - 9x^4)^{\frac{1}{2}} \\
 \frac{dy}{dx} &= \frac{1}{2}(4 - 9x^4)^{-\frac{1}{2}} \times (-36x^3) \\
 &= -18x^3(4 - 9x^4)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4 \ D} \quad y &= \sin 2x + 1 \\
 \frac{dy}{dx} &= 2 \cos 2x \\
 x = 0: \frac{dy}{dx} &= 2 \cos 0 = 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5 \ B} \quad \frac{dy}{dx} &= (e^{x^2+1}) = e^{x^2+1} \times (2x) \\
 &= 2xe^{x^2+1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6 \ C} \quad \frac{dy}{dx} &= \left( \frac{1}{1+x} \right) = \frac{1}{dx} (1+x)^{-1} \\
 &= -1(1+x)^{-2} \\
 &= -\frac{1}{(1+x)^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7 \ D} \quad \text{grad } PQ &= \frac{(2+h)^3 - 2^3}{(2+h) - 2} \\
 &= \frac{(2+h)^3 - 8}{h}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8 \ A} \quad \frac{[f(x+h) - f(x)]}{h} &= \frac{\frac{3}{x+h} - \frac{3}{x}}{h} \\
 &= \frac{3x - 3(x+h)}{hx(x+h)} \\
 &= \frac{-3h}{hx(x+h)} \\
 &= -\frac{3}{x(x+h)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9 \ D} \quad y &= ce^{2x} \\
 \frac{dy}{dx} &= 2ce^{2x} \\
 &= 2c \quad (\text{when } x = 0) \\
 &= 11 \\
 \Rightarrow c &= 5.5
 \end{aligned}$$

**10 A** The graph shows two local maximum and local minimum; there are no stationary point of inflexion. Here,  $f'(x) = 0$  at 3 points

**11 E**  $f(x) = 4 - e^{-2x}$   
 $f'(x) = 2e^{-2x}$   
 The graph is a decaying exponential with the  $x$ -axis as a horizontal asymptote.  
 Only the last graph fits.

**12 B**  $y = bx^2 - cx = x(bx - c)$   
 $y = 0$  if  $x = 0$  or  $x = \frac{c}{b}$ .

Since the graph crosses the x-axis at  $(4, 0)$ ,  $\frac{c}{b} = 4$ , i.e.  $c = 4b$  ①

$$\begin{aligned}\frac{dy}{dx} &= 2bx - c \\ &= 1 \text{ at } (4, 0)\end{aligned}$$

So  $8b - c = 1$ , i.e.  $c = 8b - 1$  ②

② - ① gives  $0 = 4b - 1$

$$\begin{aligned}b &= \frac{1}{4} \\ c &= 4 \times \frac{1}{4}\end{aligned}$$

**13 B** The gradient of the gives graph is zero at some negative value of  $x$  and again at  $x = 0$ .

The gradient goes from positive to negative to positive through these two stationary points.

Only the second graph fits.

**14 E**  $y = (3x^4 - 2)^4$

$$\begin{aligned}\frac{dy}{dx} &= 4(3x^4 - 2)^3 \times 12x^3 \\ &= 48x^3(3x^4 - 2)^3\end{aligned}$$

**15 D** Since the derivations are equal, the functions differ by at most of constant.

$$\begin{aligned}\text{So } g(x) &= f(x) + c \\ &= 3x^2 + 2 + c\end{aligned}$$

$$g(2) = 29$$

$$12 + 2 + c = 29$$

$$c = 15$$

$$\begin{aligned}g(x) &= 3x^2 + 2 + 15 \\ &= 3x^2 + 17\end{aligned}$$

(Alternatively,  $g(x) = 3x^2 + k$  for some constant  $k$ , so only options A, B and D are possible. Substitute  $x = 2$  in each to see which gives 29.)

**16 E**  $f'(x) = ke^{kx} - ke^{-kx}$

$$= ke^{-kx}(e^{2kx} - 1)$$

Case(1):  $k > 0$

Then  $f'(x) > 0$  provided  $e^{2kx} - 1 > 0$   
i.e.  $e^{2kx} > 1$

$$2kx < 0 \quad (\text{Since } e^0 = 1)$$

$$x > 0 \quad (\text{Since } k > 0)$$

Case(2):  $k < 0$

Then  $f'(x) > 0$  provided  $e^{2kx} - 1 < 0$   
i.e.  $e^{2kx} < 1$

$$2kx < 0$$

$$x > 0 \quad (\text{Since } k > 0)$$

In either case,  $f'(x) > 0$  for  $x > 0$

**17 B**  $y = 3x^2 + 2x - \frac{4}{x^2}$

$$= 3x^2 + 2x - 4x^{-2}$$

$$\frac{dy}{dx} = 6x + 2 + 8x^{-3}$$

$$= 6x + 2 + \frac{8}{x^3}$$

**18 C** Average rate =  $\frac{V(4) - V(2)}{4 - 2}$

$$= \frac{45 - 15}{2}$$

$$= 15 \text{ m}^3/\text{min}$$

**19**

**E** Gradient  $PQ = \frac{f(x+h) - f(x)}{(x+h) - (x)}$

$$\begin{aligned}&= \frac{[(x+h)^2 - 2(x+h) + 1] - [x^2 - 2x + 1]}{h} \\ &= \frac{x^2 + 2xh + h^2 - 2x - 2h + 1 - x^2 + 2x - 1}{h} \\ &= \frac{2xh + h^2 - 2h}{h} \\ &= 2x - 2 + 2h\end{aligned}$$

**20 D** The gradient of the gives graph is zero at a point in  $(-3, -1)$  and again at a point in  $(0, 2)$ . The gradient goes from position to negative to positive through these two points. Only the fourth graph fits.

**21 B** The gradient function is negative for  $x < -3$  and also for  $x > 2$ . It is zero or positive for  $-3 \leq x \leq 2$ , so the function is increasing for  $-3 \leq x \leq 2$ .

**22 E** The gradient of a tangent to  $y = f(x)$  at  $x = a$  is given by  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ . If  $a=2$ , this becomes  $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$

**23 B** 
$$\left(\frac{e^{2x} + e^{-2x}}{e^x}\right) = e^x + e^{-3x}$$
$$\frac{d}{dx}\left(\frac{e^{2x} + e^{-2x}}{e^x}\right) = \frac{d}{dx}(e^x + e^{-3x}) = e^x - 3e^{-3x}$$

**24 D** The graph of  $y = -x^2 + 4x + 3$  is an inverted parabola. The maximum value will be at the vertex.  $\frac{dy}{dx} = -2x + 4 = 0$  if  $x = 2$ . The  $y = -4 + 8 + 3 = 7$ , which is the required maximum value.

**25 B**  $y = 1 + e^{2x}$ 
$$\frac{dy}{dx} = 2e^{2x} = 2 \text{ when } x = 0$$
Equation of tangent is given by

$$y - 2 = 2(x - 0)$$
$$y = 2x + 2$$

**26 A**  $y = x^2 - x^3$ 
$$\frac{dy}{dx} = 2x - 3x^2 = x(2 - 3x) = 0 \text{ if } x = 0, \frac{2}{3}$$
There are stationary points where  $x = 0$  and  $x = \frac{2}{3}$

**27 D** 
$$\frac{d}{dx}\left(\frac{4x^2 + 6}{x}\right) = \frac{d}{dx}(4x + 6x^{-1}) = 4 - 6x^{-2} = \frac{4x^2 - 6}{x^2}$$

**28 D**  $f(x) = 4x^3 - 3x + 7 - 2x^{-1}$ 
$$f'(x) = 12x^2 - 3 + 2x^{-2}$$
$$f'(1) = 12 - 3 + 2 = 11$$

**29 A**  $f'(x) = x^2 + \frac{1}{x}$ 
$$f(x) = \frac{1}{3}x^3 + \log_e x + c$$
(Since the condition has  $x > 0$ )
$$f(1) = \frac{1}{3} + \log_e 1 + c = \frac{1}{3} + c = \frac{1}{3} \text{ if } c = 0$$
$$f(x) = \frac{1}{3}x^3 + \log_e x$$

**30 C** 
$$\int_2^3 f(x)dx = [F(x)]_2^3 = F(3) - F(2)$$



$$\begin{aligned}
 31 \text{ C } \text{ Area} &= \int_{\frac{\pi}{2}}^{\pi} \sin x \, dx - \int_{\pi}^{\frac{3\pi}{2}} \sin x \, dx \\
 &= \int_{\frac{\pi}{2}}^{\pi} \sin x \, dx + \int_{\frac{3\pi}{2}}^{\pi} \sin x \, dx
 \end{aligned}$$

32 D The Straight line crosses the x-axis at  $x = -1$

$$\text{Area} = \int_{-1}^2 x + 1 \, dx - \int_{-2}^{-1} x + 1 \, dx$$

33 B  $\frac{dy}{dx} = \frac{1}{x^2}$

$$= x^{-2}$$

$$y = -x^{-1} + c$$

$$= -\frac{1}{x} + c$$

$$y = 2 \text{ where } x = 1$$

$$2 = -1 + c$$

$$c = 3$$

$$y = -\frac{1}{x} + 3$$

34 A  $\int_0^{36} \frac{dx}{2x+9} = \left[ \frac{1}{2} \log_e(2x+9) \right]_0^{36}$

$$= \frac{1}{2} \log_e 81 - \frac{1}{2} \log_e 9$$

$$= \frac{1}{2} \log_e 9$$

$$= \frac{1}{2} \log_e 3^2$$

$$= \log_e 3$$

So  $k = 3$

35 E  $\text{Area} = -\int_{-3}^0 f(x) \, dx + \int_0^4 f(x) \, dx$   
Which is not the same as any a of the first four options.

36 D  $\int x^2 - \frac{1}{x^2} + \sin x \, dx$

$$= \int x^2 - x^{-2} + \sin x \, dx$$

$$= \frac{1}{3}x^3 + x^{-1} - \cos x + c$$

$$= \frac{1}{3}x^3 + \frac{1}{x} - \cos x + c$$

37 A  $\text{Area} = \int_0^2 \frac{1}{3-x} \, dx$

$$= [-\log_e(3-x)]_0^2$$

$$= -\log_e 1 + \log_e 3$$

$$= \log_e 3$$

38 A  $\int_a^b \sin 2x \, dx = \left[ -\frac{1}{2} \cos 2x \right]_a^b$

$$= -\frac{1}{2} \cos 2b + \frac{1}{2} \cos 2a$$

$$= \frac{1}{2}(\cos 2a - \cos 2b)$$

For A: Substituting

$$b = \frac{3\pi}{4} \text{ and } a = \frac{\pi}{4} \text{ give,}$$

$$\frac{1}{2} \left( \cos \frac{\pi}{2} - \cos \frac{3\pi}{2} \right) = \frac{1}{2}(0 - 0) = 0 \text{ as required.}$$

(Checking each other option shows that none of these gives zero.)

39 A Since the shaded region is below the x-axis, its area is given by  $\int_0^a -f(x) \, dx$ .

40 B  $\int x^2 - \frac{1}{x} \, dx = \frac{1}{3}x^3 - \log_e x$   
(taking  $x > 0$  and letting the constant of integration be 0 since an antiderivative is requested.)

**41 D**  $f'(x) = 3x^2 + 6x - 9$   
 $= 3(x^2 + 2x - 3)$   
 $= 3(x + 3)(x - 1)$   
 $> 0$  if  $x < -3$  or  $x > 1$   
 So the function is increasing if  
 $x < -3$  or  $x > 1$

**42 B**  $f(x) = \frac{\sin x}{x}$   
 $f'(x) = \frac{(x)(\cos x) - (\sin x)(1)}{x^2}$   
 $= \frac{x \cos x - \sin x}{x^2}$

**43 B**  $y = \log_e(\cos 2x)$   
 $\frac{dy}{dx} = \frac{1}{\cos 2x} x(-2 \sin 2x)$   
 $= \frac{2 \sin 2x}{\cos 2x}$

**44 C**  $f'(x) = \sin 2x$   
 $f(x) = -\frac{1}{2} \cos 2x + c$   
 $f(0) = 3$ , so  
 $3 = -\frac{1}{2} \cos 0 + c$   
 $= -\frac{1}{2} + c$   
 $c = 3\frac{1}{2}$   
 $f(x) = -\frac{1}{2} \cos 2x + 3\frac{1}{2}$

**45 D**  $y = 4e^{3x} - x$   
 $\frac{dy}{dx} = 12e^{3x} - 1$   
 $= 12 - 1 = 11$  at  $x = 0$   
 Equation of tangent at (0,4) is  
 $y - 4 = 11(x - 0)$   
 $y = 11x + 4$

**46 B**  $f'(x) = 3x^2 - 2x - 1$   
 $= (3x + 1)(x + 1)$   
 $= 0$  if  $x = -\frac{1}{3}, 1$   
 The gradient is positive if  $x < -\frac{1}{3}$

and negative if  $-\frac{1}{3} < x < 1$ .  
 So there is a local maximum at  
 $x = -\frac{1}{3}$ .

The gradient is negative if  
 $-\frac{1}{3} < x < 1$  and positive if  $x > 1$ .  
 So there is a local minimum at  
 $x = 1$ .  $f(1) = 1 - 1 - 1 + 2 = 1$  there  
 is a local minimum at (1, 1).

**47 D**  $\frac{d}{dx}\left(\frac{x-1}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)$   
 $= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$   
 $= \frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}}$   
 $= \frac{x+1}{2x\sqrt{x}}$

**48 C**  $\frac{d}{dx}(e^{\cos x}) = e^{\cos x} x(-\sin x)$   
 $= -\sin x e^{\cos x}$

**49 A** Area  $= \int_0^{\frac{\pi}{2}} \cos x \, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx$   
 $= [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$   
 $= (1 + 0) - (-1 - 1)$   
 $= 3$  square units

**50 C**  $x + y = 1$  so  $y = 1 - x$

$$p = x^2 + xy - y^2$$

$$= x^2 + x(1 - x) - (1 - x)^2$$

$$= x^2 + x - x^2 - 1 + 2x - x^2$$

$$= -x^2 + 3x - 1$$

$$\frac{dP}{dx} = -2x + 3$$

$$= 0 \text{ if } x = \frac{3}{2}$$

and this corresponds to a maximum since the graph of  $P$  against  $x$  is an inverted parabola.

**51 B**

$$y = e^{-\cos x}$$

$$\frac{dy}{dx} = e^{-\cos x} \times \sin x$$

$$= \sin x e^{-\cos x}$$

where  $x = \frac{\pi}{3}$ ,  $\frac{dy}{dx} = \sin \frac{\pi}{3} e^{-\cos \frac{\pi}{3}}$

$$= \frac{\sqrt{3}}{2} e^{-\frac{1}{2}}$$

and this is the gradient of the tangent at  $x = \frac{\pi}{3}$ .

Using  $m_1 m_2 = -1$

$$m_2 = -1 \div \left( \frac{\sqrt{3}}{2} e^{-\frac{1}{2}} \right)$$

$$= -\frac{2}{\sqrt{3}} e^{\frac{1}{2}}$$

$$= \frac{-2e^{\frac{1}{2}}}{\sqrt{3}}$$

**52 E**

$$\int_0^{\frac{\pi}{2}} (\cos x + \sin x) dx = [\sin x - \cos x]_0^{\frac{\pi}{2}}$$

$$= (1 - 0) - (0 - 1)$$

$$= 2$$

**53 C** Since  $(1, 3)$  is a maximum point on

the graph,  $f'(1) = 0$ .  
The equation of the tangent is  
 $y - 3 = 0(x - 1)$   
 $y = 3$

**54 C**  $y = 4 - x^2$

$$\frac{dy}{dx} = -2x$$

$$= -2 \text{ when } x = 1$$

Equation of the tangent is  
 $y - 3 = -2(x - 1)$   
 $y = -2x + 5$

**55 D**  $P = -x^2 + 6x + 4$

$$\frac{dP}{dx} = -2x + 6$$

$$= 0 \text{ if } x = 3$$

and this corresponds to a maximum since the graph of  $P$  against  $x$  is an inverted parabola.  
When  $x = 3$ ,  $P = -9 + 18 + 4$   
 $= 13$   
The maximum value is 13.

**56 B** Stationary points  
 $f'(x) = x^3 - x^2 - 1$  occur when  
 $f'(x) = 0$ .  
 $f'(x) = 3x^3 - 2x$   
 $= x(3x^2 - 2)$   
 $= 0 \text{ if } x = 0, \frac{2}{3}$

**57 D**  $f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$

$$= \lim_{h \rightarrow 0} h^2 + 6h + 12$$

$$= 12$$

**58 C**  $f(x) = \log_e 3x$   
 $f'(x) = \frac{1}{3x} \times 3$   
 $f'(1) = 1$

**59 B**  $y = x^2 e^x$   
 $\frac{dy}{dx} = x^2 e^x + 2x e^x$   
 $= x e^x (x + 2)$   
 $= 0$  if  $x = 0, 2$ .

Note that  $y \geq 0$  and when  $x = 0$ ,  
 $y = 0$ . So the minimum value is 0.

**60 E**  $f(x) = a \sin(3x)$   
 $f'(x) = 3a \cos(3x)$   
 $f'(\pi) = 3a \cos 3\pi$   
 $= -3a$   
 $= 2$  if  $a = -\frac{2}{3}$

**61 B**  $\int \frac{1}{(2x-5)^{\frac{5}{2}}} dx$   
 $= \int (2x-5)^{-\frac{5}{2}} dx$   
 $= \frac{1}{2 \times \left(-\frac{3}{2}\right)} (2x-5)^{-\frac{3}{2}}$   
 $= -\frac{1}{3} \times \frac{1}{(2x-5)^{-\frac{3}{2}}}$   
 $= \frac{1}{3(2x-5)^{-\frac{3}{2}}}$

Where the constant of integration is taken as 0 since an antiderivative is required.

**62 D** From the graph, there is a stationary point of inflexion at  $x = -3$  and a minimum stationary point at  $x = \frac{9}{4}$ . A quick check of each option shows that the fourth is not true, as there are two stationary point on the graph. (Checking the other options shows each is true.)

## Solutions to extended-response questions

1 a  $S = 50 + 30e^{-\frac{1}{3}t}$

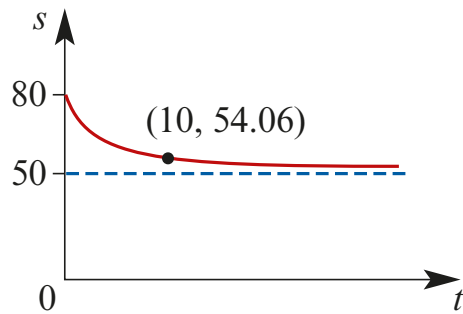
When  $t = 10$

$$s = 50 + 30e^{-2}$$

$$\approx 54.06$$

There are 54.06 g of salt in the mixture after 10 minutes.

b



c  $\frac{ds}{dt} = -6e^{\frac{1}{5}t}$

d Considering  $s = 50 + 30e^{-\frac{1}{5}t}$

Solve for  $e^{-\frac{1}{5}t}$

$$\therefore \frac{s-50}{30} = e^{-\frac{1}{5}t}$$

Substitute in  $\frac{ds}{dt} = -6e^{-\frac{1}{5}t}$

to yield  $\frac{ds}{dt} = -6\left(\frac{s-50}{30}\right)$

$$= \frac{1}{5}(50-s)$$

e When  $t = 0$

$$s = 50 + 30e^0$$

$$= 80$$

The volume of water is 100 litres.

$$\therefore \text{Concentration} = \frac{80}{100} = 0.8\text{g/litre}$$

f Concentration =  $\frac{s}{100}$

$$= \frac{50 + 30e^{-\frac{1}{5}t}}{100}$$

Concentration = 0.51 gram/litre

$$\text{implies } 0.51 = \frac{50 + 30e^{-\frac{1}{5}t}}{100}$$

$$\therefore 51 = 50 + 30e^{-\frac{1}{5}t}$$

$$\frac{1}{30} = e^{-\frac{1}{5}t}$$

and therefore

$$t = -5 \log_e \frac{1}{30}$$

$$= 5 \log_e 30$$

$$\approx 17.006$$

The concentration first reaches 0.51 g/litre after about 17 seconds.

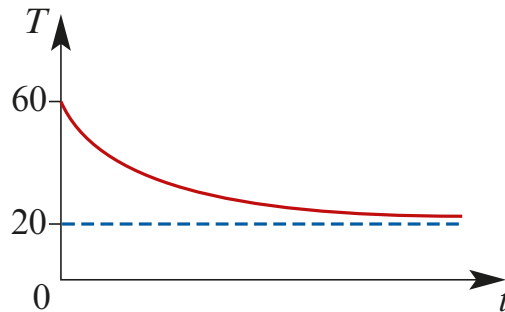
**2 a**  $T = 40e^{-0.36t} + 20, t \geq 0$

When  $t = 0$

$$T = 60$$

The initial temperature of the body was  $60^\circ \text{C}$ .

**b**



**c**  $\frac{dT}{dt} = -14.4e^{-0.36t}$

**d** Since  $T = 40e^{-0.36t} + 20$

$$e^{-0.36t} = \frac{T - 20}{40}$$

$$\text{Hence } \frac{dT}{dt} = -14.4 \left( \frac{T - 20}{40} \right) = -0.36(T - 20)$$

**3 a**  $f(t) = 1000e^{-0.5t}$

$$f(0) = 1000$$

Initially there were 1000 F-type spores.

50% of the initial number is 500.

Consider

$$500 = 1000e^{-0.5t}$$

$$0.5 = e^{-0.5t}$$

$$\therefore t = -2 \log_e(0.5)$$

$$= 2 \log_e 2$$

$$\approx 1.386$$

It takes about 1.386 minutes to kill half of the F-type spores.

**b**  $f(0) = 1000$  and  $g(1000) = 1200$

Initially there are 1000 F-type spores and 1200 G-type spores, so there are 2200 live spores of both types.

For  $t = 5$

$$f(5) = 1000e^{-0.5 \times 5} \quad \text{and} \quad g(5) = 1200e^{-0.7 \times 5}$$

$$= 1000e^{-2.5} \quad \quad \quad = 1200e^{-3.5}$$

$$\text{Percentage of spores still alive after 5 minutes} = \frac{f(5) + g(5)}{f(0) + g(0)} \times \frac{100}{1}$$

$$= \frac{1000e^{-2.5} + 1200e^{-3.5}}{2200} \times \frac{100}{1}$$

$$= \frac{1000e^{-2.5} + 1200e^{-3.5}}{22}$$

$$\approx 5.378$$

$\therefore$  Percentage of spores still alive after 5 minutes is 5.378. %

**c** Total no. of spores =  $1000e^{-0.5t} + 1200e^{-0.7t}$

i.e.  $T = 1000e^{-0.5t} + 1200e^{-0.7t}$  where  $T$  is the total number of spores

$$\frac{dT}{dt} = -500e^{-0.5t} - 840e^{-0.7t}$$

When  $t = 5$

$$\frac{dT}{dt} = -500e^{-0.5 \times 5} - 840e^{-0.7 \times 5}$$

$$= -500e^{-2.5} - 840e^{-3.5}$$

$$\approx -66.408$$

When  $t = 5$ , the rate at which the total number of spores is decreasing is 66.4 spores per minute.

**d** Live F-type spores = live G-type spores

When  $f(t) = g(t)$

i.e.  $1000e^{-0.5t} = 1200e^{-0.7t}$

which implies  $e^{0.2t} = \frac{1200}{1000}$

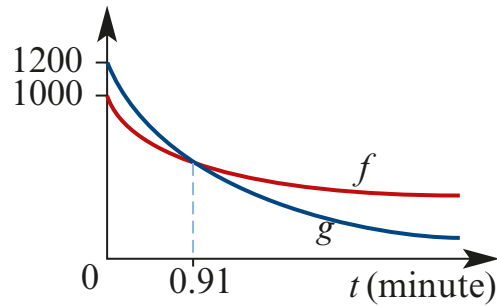
$$\text{and } t = \frac{1}{0.2} \log_e\left(\frac{6}{5}\right)$$

$$= 5 \log_e$$

$$\approx 0.9116$$

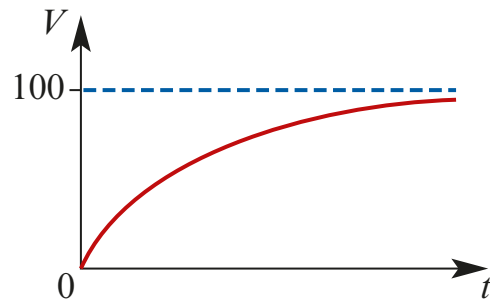
$\therefore$  the number of live F-type spores = the number of live G-type spores when  $t = 0.9116$ .

**e** number of spores



**4**  $V = 100(1 - e^{-0.2t})$

**a**



**b i** Acceleration  $= \frac{dV}{dt} = 100 \times 0.2e^{-0.2t}$   
 $= 20e^{-0.2t} \text{ m/s}^2$



**ii** From  $V = 100(1 - e^{-0.2t})$

$$\frac{V}{100} = 1 - e^{-0.2t}$$

$$\text{and } e^{-0.2t} = 1 - \frac{V}{100}$$

$$= \frac{100 - V}{100}$$

$$\therefore \frac{dV}{dt} = 20e^{-0.2t}$$

$$= 20\left(\frac{100 - V}{100}\right)$$

$$= \frac{1}{5}(100 - V)\text{m/s}^2$$

**c** when  $V = 80$

$$80 = 100(1 - e^{-0.2t})$$

$$0.8 = 1 - e^{-0.2t}$$

$$e^{-0.2t} = 0.2$$

$$\therefore -0.2t = \log_e(0.2)$$

$$\text{i.e. } t = -5 \log_e(0.2)$$

$$= 5 \log_e 5$$

When the velocity of the body is 80 m/s,  $t = 5 \log_e 5 \approx 8.05$  seconds.

**5**  $C = 0.05x^2 + 5x + 500$

The average cost  $A = \frac{C}{x}$

i.e.  $A = 0.05x + 5 + \frac{500}{x}$

$$\frac{dA}{dx} = 0.05 - \frac{500}{x^2}$$

$$\frac{dA}{dx} = 0 \text{ implies}$$

$$0.05 = \frac{500}{x^2}$$

i.e.  $x^2 = 10\,000$

$$x = 100$$

$$\frac{dA}{dx} > 0 \text{ for } x > 100$$

and  $\frac{dA}{dx} < 0$  for  $x < 100$

$\therefore$  a local minimum at  $x = 100$

i.e. 100 units per annum minimises the average cost per unit.

**6**  $T = T_0e^{-kt}$

$$\begin{aligned} \mathbf{a} \quad \frac{dT}{dt} &= -kT_0e^{-kt} \\ &= -kT \\ \therefore \frac{dT}{dt} &\text{ is proportional to } T \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{When } t = 0, T &= 100 - 30 = 70 \\ \text{i.e. } T_0 &= 70 \\ \text{When } t = 20, T &= 70 - 30 = 40 \\ \therefore 40 &= 70e^{-20k} \\ \frac{4}{7} &= e^{-20k} \\ \text{and } -20k &= \log_e\left(\frac{4}{7}\right) \\ k &= -\frac{1}{20} \log_e\left(\frac{4}{7}\right) \\ &= \frac{1}{20} \log_e\left(\frac{7}{4}\right) \\ &= 0.028 \text{ (correct to 3 decimal places)} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \frac{dT}{dt} &= -kT_0e^{-kt} \\ &= -\frac{70}{20} \log_e\left(\frac{7}{4}\right) e^{-\frac{30}{20} \log_e\left(\frac{7}{4}\right)} \\ &= -\frac{70}{20} \log_e\left(\frac{7}{4}\right) e^{\log_e\left(\frac{7}{4}\right)^{\frac{3}{2}}} \\ &= -\frac{70}{20} \times \left(\frac{4}{7}\right)^{\frac{3}{2}} \log_e\left(\frac{7}{4}\right) \\ &\approx -0.846 \end{aligned}$$

The temperature is decreasing at a rate of 0.846 degrees/minute.

$$\begin{aligned} \mathbf{7 a} \quad \mathbf{i} \quad p(t) &= 0.2 - 0.2e^{-\frac{t}{20}} + 0.1e^{-\frac{t}{10}} \geq 0 \\ \therefore p(10) &= 0.2 - 0.2e^{-\frac{1}{2}} + 0.1e^{-1} \\ &\approx 0.1155 \text{ (correct to four decimal places)} \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad \text{As } t &\rightarrow \infty \\ e^{-\frac{t}{20}} &\rightarrow 0 \text{ and } e^{-\frac{t}{10}} \rightarrow 0 \\ \therefore p(t) &\rightarrow 0.2 \\ \text{The proportion} &\text{ approaches } 0.2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad p'(t) &= 0.01e^{-\frac{t}{20}} - 0.01e^{-\frac{t}{10}} \\ \text{Let } N(t) &\text{ be the number of new cases per day} \\ N(t) &= kp'(t) \end{aligned}$$

$$= k\left(0.01e^{-\frac{t}{20}} - 0.01e^{-\frac{t}{10}}\right)$$

To find maximum, differentiate to find  $N'(t)$  and solve the equation  $N'(t) = 0$

$$N'(t) = \left(-0.0005e^{-\frac{t}{20}} + 0.001e^{-\frac{t}{10}}\right)k$$

$N'(t) = 0$  implies

$$0.0005 e^{-\frac{t}{20}} = 0.001e^{-\frac{t}{10}}$$

$$\therefore e^{\frac{t}{20}} = 2$$

$$\therefore t = 20 \log_e 2 \approx 13.86$$

$N'(t) < 0$  for  $t > 20 \log_e 2$  and  $N'(t) > 0$  for  $t < 20 \log_e 2$

which implies a local maximum at  $t = 20 \log_e 2$

The number of new cases per day is a maximum when  $t = 20 \log_e 2$

i.e. after 13.86 days.

**8** Let  $x$  be the rent per month from each apartment.

$$\begin{aligned} \text{The number of apartments occupied} &= 70 - 2 \frac{(x - 500)}{20} \\ &= \frac{700 - x + 500}{10} \\ &= \frac{1200 - x}{10} \end{aligned}$$

Let  $R$  be the total revenue

$$\text{then } R = \frac{x(1200 - x)}{10}$$

$$= \frac{1}{10}(1200x - x^2)$$

$$\frac{dR}{dx} = 0 \text{ implies } x = 600$$

this is a maximum as  $R$  is a quadratic function of  $x$  with negative coefficient of  $x^2$  i.e.

the price per apartment to maximise monthly revenue is \$600.

**9**  $V = \frac{5 \times 10^4}{(t + 1)^2}$

**a** When  $t = 0$ ,  $V = 5 \times 10^4$

i.e. the initial volume of liquid is  $5 \times 10^4 \text{ m}^3$

**b**  $V = (5 \times 10^4)(t + 1)^{-2}$

$$\therefore \frac{dV}{dt} = -10 \times 10^4 (t + 1)^{-3} \text{ (chain rule)}$$

$$= -\frac{10^5}{(t + 1)^3}$$

$$\text{When } t = 1, \frac{dV}{dt} = -\frac{10^5}{2^3} = -12\,500$$

i.e. the rate of change of the volume of liquid with respect to time is  $-12\,500 \text{ m}^3/\text{day}$ .

$$\text{c } V(4) = \frac{5 \times 10^4}{5^2} = \frac{10^4}{5} = 2000$$

$$V(1) = \frac{5 \times 10^4}{2^2} = 12\,500$$

$\therefore$  the average rate of change of  $V$  with respect to  $t$  for

$$\begin{aligned} \text{the interval } [1, 4] &= \frac{2000 - 12\,500}{4 - 1} \\ &= -3500 \end{aligned}$$

The average rate of change for the interval  $[1, 4]$  is  $-3500 \text{ m}^3/\text{day}$ .

$$\text{d } \frac{5 \times 10^4}{(t+1)^2} < 1$$

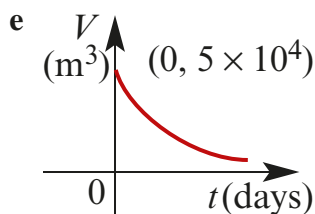
$$5 \times 10^4 < (t+1)^2$$

$$\therefore t+1 > \sqrt{5 \times 10^4}; t > 0$$

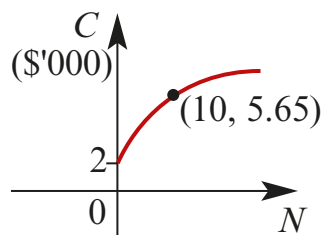
$$\therefore t > 100\sqrt{5} - 1$$

$$100\sqrt{5} - 1 \approx 222.61$$

$\therefore$  the amount of liquid in the pool is less than one cubic metre after  $100\sqrt{5} - 1 \approx 222.61$  days.



10 a



$$\begin{aligned} \text{b } \frac{dC}{dN} &= \frac{1}{4} \cdot 3N^2(N^3 + 16)^{-\frac{3}{4}} \text{ (chain rule)} \\ &= \frac{3N^2}{4(N^3 + 16)^{\frac{3}{4}}} \end{aligned}$$

c The rate of change of cost in \$ 1000 s with respect to the number of bottle tops produced.

**11** Profit = Selling Price – Cost Price

$$\text{Selling Price} = \frac{800}{p^2} \times p = \frac{800}{p}$$

$$\text{Cost Price} = \frac{800}{p^2} \times 2 = \frac{1600}{p^2}$$

$$\therefore \text{Profit} = \frac{800}{p} - \frac{1600}{p^2}$$

Let  $R$  denote profit

$$\text{then } R = \frac{800}{p} - \frac{1600}{p^2}$$

$$\frac{dR}{dp} = -\frac{800}{p^2} + \frac{3200}{p^3}$$

For maximum profit consider  $\frac{dR}{dp} = 0$

$$-\frac{800}{p^2} + \frac{3200}{p^3} = 0 \quad (p \neq 0)$$

$$-800p + 3200 = 0$$

$$p = 4$$

A sign diagram shows a local maximum

$p$	$<$	$4$	$>$
$\frac{dR}{dp}$	$+$	$0$	
sign	$/$	$-$	$\backslash$

$$\text{When } p = 4, R = \frac{800}{4} - \frac{1600}{4^2}$$

$$= 200 - 100$$

$$= 100$$

$\therefore$  The selling price is \$4 to maximise profit and the number of items sold is  $\frac{800}{16} = 50$

**12**  $y = (ax + b)^{-2}$

$$\text{When } x = 0, y = \frac{1}{4}$$

$$\frac{1}{4} = b^{-2}$$

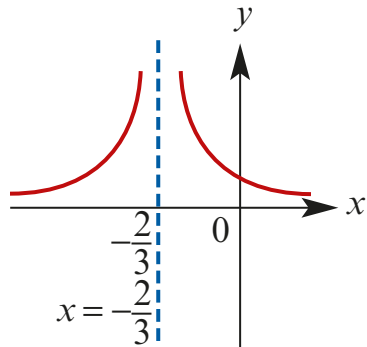
$$\therefore b^2 = 4$$

and  $b = \pm 2$

$$\frac{dy}{dx} = -2a(ax + b)^{-3}$$

$$\text{When } x = 0, \frac{dy}{dx} = -\frac{3}{4}$$

$$\therefore -\frac{3}{4} = -2a(b)^{-3}$$



Substituting  $b = 2$  gives

$$-\frac{3}{4} = -2a \times \frac{1}{8}$$

$$\therefore a = 3$$

For  $b = -2$

$$a = 3$$

$\therefore$  The possible pairs are  $(3, 2)$  and  $(-3, -2)$

For  $a = 3, b = 2$  (equivalently  $a = -3, b = -2$ )

$$y = \frac{1}{(3x + 2)^2}$$

**13 a** Cost per hour =  $160 + \frac{1}{100}V^3$  dollars

A journey of 1000 km at 10 km/hr takes  $\frac{1000}{10} = 100$  hours.

$$= (160 + \frac{1}{100} \times 10^3)100$$

$$\therefore \text{Cost of journey} = (160 + 10)100$$

$$= 17\,000$$

The cost of the journey = \$17 000

**b** Time for a journey of 1000 km at  $V$  km/hr =  $\frac{1000}{V}$  hours

$$\therefore C = \left(160 + \frac{1}{100}V^3\right)\frac{1000}{V}$$

$$= \frac{160\,000}{V} + 10V^2$$

**c** In order to sketch the graph it is necessary to investigate stationary points.

$$\frac{dC}{dV} = -\frac{160\,000}{V^2} + 20V$$

$$\frac{dC}{dV} = 0 \text{ implies } 20V = \frac{160\,000}{V^2}$$

$$\therefore V^3 = 8000$$

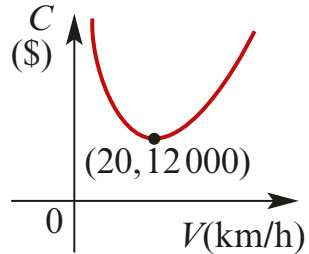
i.e.  $V = 20$

$$\text{When } V = 20, C = \frac{160\,000}{20} + 10 \times 20^2$$

$$= 12\,000$$

$$\text{When } V > 20, \frac{dC}{dV} > 0$$

$$\text{When } 0 < V < 20, \frac{dC}{dV} < 0$$



$\therefore$  there is a local minimum at  $(20, 12\,000)$

$$\text{For } C = \frac{160\,000}{V} + 10V^2$$

$$\text{as } V \rightarrow \infty, C \rightarrow 10V^2$$

$$\text{as } V \rightarrow 0, C \rightarrow \infty$$

$\therefore$  the graph is as shown here.

**d** From the above the most economical speed is 20 km/hr and the minimum cost is \$12 000.

**e** From the graph the minimum will occur when  $V = 16$

$$C = \left(160 + \frac{1}{100}V^3\right)\frac{1000}{V}$$

$$= \frac{160\,000}{V} + 10V^2$$

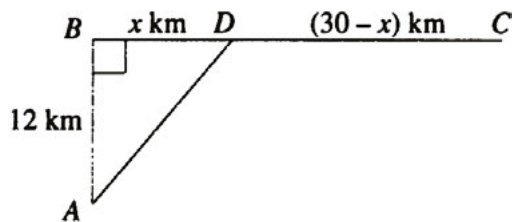
$$= \frac{160\,000}{16} + 10 \times 16^2$$

$$= 10\,000 + 2560$$

$$= 12\,560$$

$\therefore$  the minimum cost is \$12 560 when the maximum speed is 16 km/hr.

**14 a**



Let  $BD = x$ , where  $D$  is the point where the camper reaches the shore.

Then  $DC = 30 - x$

and by Pythagoras' Theorem

$$AD = \sqrt{x^2 + 12^2}$$

The camper rows at 5 km/hr. Therefore the time taken to row from

$$A \text{ to } D = \frac{\sqrt{x^2 + 144}}{5} \text{ hours.}$$

The camper walks at 8 km/hr. Therefore the time taken to walk from

$D$  to  $C = \frac{(30-x)}{8}$  hours.

The total time,  $T$  (hours), for the trip is given by

$$T = \frac{\sqrt{x^2 + 144}}{5} + \frac{30-x}{8} = \frac{(x^2 + 144)^{\frac{1}{2}}}{5} + \frac{30-x}{8}$$

To find the minimum time consider stationary point

$$\begin{aligned} \therefore \frac{dT}{dx} &= \frac{2x \times \frac{1}{2}(x^2 + 144)^{-\frac{1}{2}}}{5} - \frac{1}{8} \\ &= \frac{x}{5(x^2 + 144)^{\frac{1}{2}}} - \frac{1}{8} \\ \frac{dT}{dx} = 0 &\text{ implies } \frac{x}{5(x^2 + 144)^{\frac{1}{2}}} = \frac{1}{8} \end{aligned}$$

$$\therefore 8x = 5(x^2 + 144)^{\frac{1}{2}}$$

Squaring both sides

$$64x^2 = 25(x^2 + 144)$$

$$39x^2 = 25 \times 144$$

$$x^2 = \frac{25 \times 144}{39}$$

$$\therefore x = \frac{60}{\sqrt{39}} = \frac{60\sqrt{39}}{39} \approx 9.61$$

A gradient chart reveals a minimum

$x$	$<$	$\frac{60\sqrt{39}}{39}$	$>$
sign of $\frac{dT}{dx}$	$-$	$0$	$+$
shape	$\backslash$	$-$	$/$

$\therefore$  the camper should land 9.61 km from  $B$  to minimise the time of the journey.

**b** If  $C$  is 24 km from  $B$

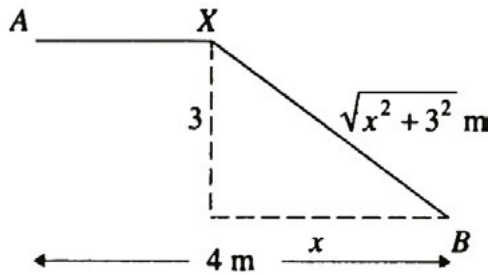
$$T = \frac{(x^2 + 144)^{\frac{1}{2}}}{5} + \frac{24-x}{8}$$

$$\frac{dT}{dx} = \frac{x}{5(x^2 + 144)^{\frac{1}{2}}} - \frac{1}{8}$$

$\therefore$  local minimum is the same as **a**.

i.e. the camper should still row to a point 9.61 km from  $B$ .





Let  $C$  be the cost of laying the pipe

Distance  $AX = 4 - x$

The cost of laying the pipe along  $AX = 10(4 - x)$  dollars

The cost of laying section  $XB = 25(x^2 + 9)^{\frac{1}{2}}$

$$\therefore C = 10(4 - x) + 25(x^2 + 9)^{\frac{1}{2}}$$

To find the minimum consider  $\frac{dC}{dx}$

$$\frac{dC}{dx} = -10 + 25x(x^2 + 9)^{-\frac{1}{2}}$$

$$= -10 + \frac{25x}{(x^2 + 9)^{\frac{1}{2}}}$$

$$\frac{dC}{dx} = 0 \text{ implies } 10 = \frac{25x}{(x^2 + 9)^{\frac{1}{2}}}$$

$$\therefore 10(x^2 + 9)^{\frac{1}{2}} = 25x$$

$$2(x^2 + 9)^{\frac{1}{2}} = 5x$$

Squaring both sides gives

$$4(x^2 + 9) = 25x^2$$

$$4x^2 + 36 = 25x^2$$

$$\therefore 36 = 21x^2$$

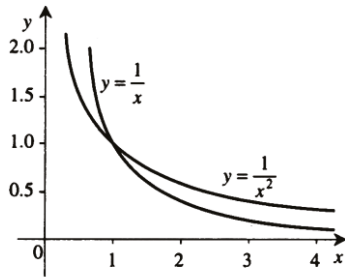
$$\therefore x = \sqrt{\frac{12}{7}}$$

$x$	$<$	$\sqrt{\frac{12}{7}}$	$>$
$\frac{dC}{dx}$	$-$	$0$	$+$
shape	$\backslash$	$-$	$/$

A minimum occurs when  $x = \sqrt{\frac{12}{7}}$

$\therefore$  Length of pipe on the surface should be  $\left(4 - \sqrt{\frac{12}{7}}\right) \approx 2.7$  metres in order to minimise costs.

16 a



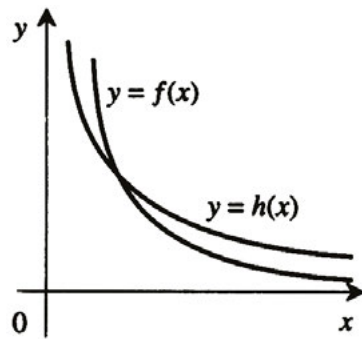
$$\begin{aligned}
 g(x) &> h(x) \\
 \Leftrightarrow \frac{1}{x} &> \frac{1}{x^2} \\
 \Leftrightarrow x &> 1 \text{ (Multiply both sides} \\
 &\text{by } x^2. \text{ Note } x > 0)
 \end{aligned}$$

$$\therefore \{x : g(x) > h(x)\} = \{x : x > 1\}$$

**b**

$$\begin{aligned}
 g'(x) &= -\frac{1}{x^2} \\
 h'(x) &= -\frac{2}{x^3} \\
 g'(x) &> h'(x) \\
 \Leftrightarrow -\frac{1}{x^2} &> -\frac{2}{x^3} \\
 \Leftrightarrow -x &> -2 \text{ (Multiply both sides by } x^3 : \text{ Note } x^3 > 0) \\
 \therefore x &< 2 \\
 \{x : g'(x) > h'(x)\} &= \{x : 0 < x < 2\}
 \end{aligned}$$

**c**



$$\begin{aligned}
 f(x) &= \frac{1}{x^3} \\
 h(x) &= \frac{1}{x^2} \\
 h(x) &> f(x) \\
 \Leftrightarrow \frac{1}{x^2} &> \frac{1}{x^3} \\
 \Leftrightarrow x &> 1 \text{ (Multiply both sides by} \\
 &x^3 : \\
 &\text{Note } x^3 > 0) \\
 \therefore \{x : h(x) > f(x)\} &= \{x : x > 1\}
 \end{aligned}$$

$$\begin{aligned}
 f'(x) &= -\frac{3}{x^4}, \quad h'(x) = -\frac{2}{x^3} \\
 h'(x) &> f'(x) \\
 \Leftrightarrow -\frac{2}{x^3} &> -\frac{3}{x^4} \\
 \text{(Multiply both sides by } x^4 : \text{ Note } x > 0) \\
 -2x &> -3 \\
 \therefore x &< \frac{3}{2} \\
 \therefore \{x : h'(x) > f'(x)\} &= \left\{x : 0 < x < \frac{3}{2}\right\}
 \end{aligned}$$

**d**

$$\begin{aligned}
 f_1(x) &= \frac{1}{x^n}, \quad f_2(x) = \frac{1}{x^{n+1}} \\
 f_1(x) &> f_2(x)
 \end{aligned}$$

$$\Leftrightarrow \frac{1}{x^n} > \frac{1}{x^{n+1}} \quad (\text{Multiply both sides by } x^{n+1}: \text{ Note } x^{n+1} > 0)$$

$$\Leftrightarrow x > 1$$

$$\therefore \{x: f_1(x) > f_2(x)\} = \{x: x > 1\}$$

$$f_1'(x) = -\frac{n}{x^{n+1}} \quad f_2'(x) = -\frac{(n+1)}{x^{n+2}}$$

$$f_1'(x) > f_2'(x)$$

$$\Leftrightarrow -\frac{n}{x^{n+1}} > -\frac{(n+1)}{x^{n+2}}$$

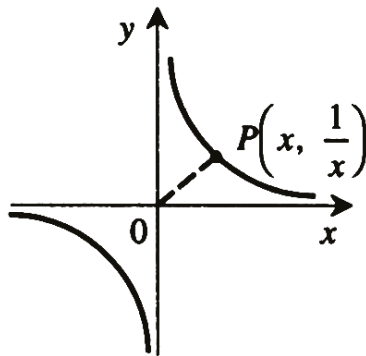
(Multiplying both sides by  $x^{n+2}$ : Note  $x^{n+2} > 0$ )

$$\therefore -nx > -(n+1)$$

$$\text{and } x < \frac{(n+1)}{n}$$

$$\therefore \{x: f_1'(x) > f_2'(x)\} = \left\{x: 0 < x < \frac{n+1}{n}\right\}$$

17 a



Let  $D = OP$   
 $D^2 = x^2 + \frac{1}{x^2}$

It is sufficient to minimise  $D^2$  to minimise  $D$

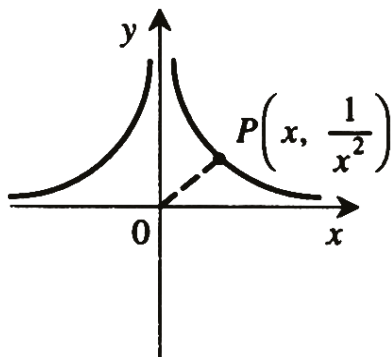
$$\therefore \frac{d(D^2)}{dx} = 2x - \frac{2}{x^3}$$

$$\frac{d(D^2)}{dx} = 0 \text{ implies } x^4 = 1$$

$$\therefore x = \pm 1$$

A sign diagram confirms a minimum at the points  $(1, 1)$  and  $(-1, -1)$

b



Let  $D = OP$   
then  $D^2 = x^2 + \frac{1}{x^4}$

$$\frac{d(D^2)}{dx} = 2x - \frac{4}{x^5}$$

$$\frac{d(D^2)}{dx} = 0 \text{ implies } x^6 = 2$$

$$\text{i.e. } x = \pm \sqrt[6]{2}$$

$$\text{When } x = \pm \sqrt[6]{2}, y = \frac{1}{\sqrt[3]{2}}$$

As before, a sign diagram reveals minimum distance for

$$P\left(\sqrt[6]{2}, \frac{1}{\sqrt[3]{2}}\right) \text{ and } P\left(-\sqrt[6]{2}, \frac{1}{\sqrt[3]{2}}\right)$$

c Let  $D = OP$   
then  $D^2 = x^2 + \frac{1}{x^{2n}}$   
 $\frac{d(D^2)}{dx} = 2x - \frac{2n}{x^{2n+1}}$   
 $\frac{d(D^2)}{dx} = 0$  implies  
 $2x = \frac{2n}{x^{2n+1}}$   
 $\therefore 2x^{2n+2} = 2n$   
and  $x = \pm n^{\frac{1}{2n+2}}$   
 $\therefore y = n^{-\frac{n}{2n+2}}$

18 a Let  $y$  m be the width of each window

$$\therefore 6xy = 36$$

$$y = \frac{6}{x}$$

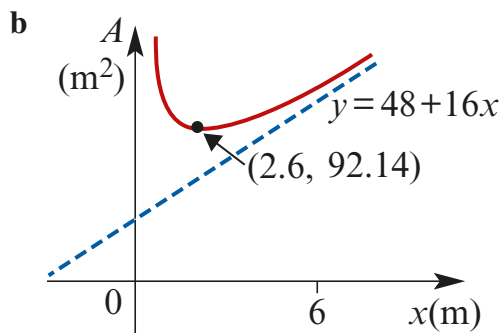
The width of the wall =  $8 + 3y$

The height of the wall =  $6 + 2x$

$$\therefore \text{Area of the brickwork } A = (8 + 3y)(6 + 2x) - 36$$

$$= \left(8 + \frac{18}{x}\right)(6 + 2x) - 36$$

$$= 48 + \frac{108}{x} + 16x$$



c In order to find the value of  $x$  which will give the minimum amount of brickwork,

consider:

$$\frac{dA}{dx} = -\frac{108}{x^2} + 16$$

$$\frac{dA}{dx} = 0 \text{ implies } x^2 = \frac{108}{16}$$

$$\therefore x = \pm \frac{3\sqrt{3}}{2}$$

But  $x > 0 \quad \therefore x = \frac{3\sqrt{3}}{2} \approx 2.62$

A sign diagram shows local minimum

$x$	$<$	$\frac{3\sqrt{3}}{2}$	$>$
$\frac{dA}{dx}$	$-$	$0$	$+$
shape	$\backslash$	$-$	$/$

When  $x = \frac{3\sqrt{3}}{2}$ ,  $y = 6 \div \frac{3\sqrt{3}}{2} = 6 \times \frac{2}{3\sqrt{3}}$   
 $= \frac{4\sqrt{3}}{3} \approx 2.3$

$\therefore$  The dimensions of each window are height  $\frac{3\sqrt{3}}{2}$  metres and width  $\frac{4\sqrt{3}}{3}$  metres;  
the minimum area of brickwork is  $48 + 48\sqrt{3} \approx 131.14$  metres.

**d**  $x \geq 1$  and  $y \geq 1$

implies  $x \geq 1$  and  $\frac{6}{x} \geq 1$

implies  $1 \leq x \leq 6$

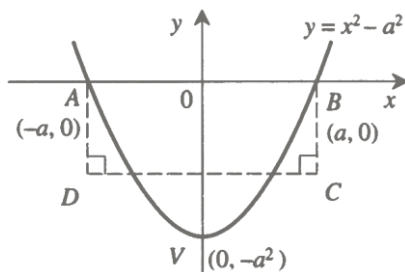
From the graph the maximum will occur at either  $x = 1$  or  $x = 6$

If  $x = 1$ ,  $A = 48 + 108 + 16 = 172$

If  $x = 6$ ,  $A = 48 + \frac{108}{6} + 16 \times 6 = 162$

$\therefore$  The maximum amount of brickwork which could be used is  $172 \text{ m}^2$

**19 a**



**b** Area  $= - \int_{-a}^a x^2 - a^2 dx$   
 $= -2 \int_0^a x^2 - a^2 dx$   
 $= -2 \left[ \frac{x^3}{3} - a^2 x \right]_0^a$   
 $= \frac{4a^3}{3}$  square units

See graph above.

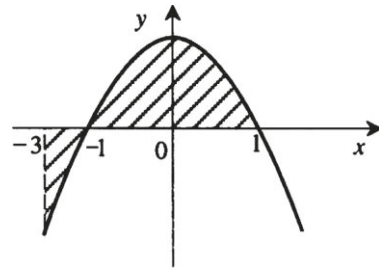
Length  $AB = 2a$

If the area of rectangle  $ABCD$  is  $\frac{4a^3}{3}$  square units

**b**  $BC = \frac{4a^3}{3} \div 2a$   
 $= \frac{4a^3}{3} \times \frac{1}{2a} = \frac{2a^2}{3}$  units

**c**  $\frac{\text{length of } BC}{\text{length of } OV} = \frac{2a^2}{3} \div a^2 = \frac{2}{3}$ , a ratio of 2:3.

$$\begin{aligned}
 \mathbf{20\ a} \quad \int_{-3}^1 (1 - t^2) dt &= \left[ t - \frac{t^3}{3} \right]_{-3}^1 \\
 &= 1 - \frac{1}{3} - \left( -3 - \frac{(-3)^3}{3} \right) \\
 &= \frac{2}{3} - (-3 + 9) \\
 &= \frac{2}{3} - (6) \\
 &= -5\frac{1}{3}
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{b} \quad \int_a^1 (1 - t^2) dt &= 0 \\
 \text{implies } \left[ t - \frac{t^3}{3} \right]_a^1 &= 0 \\
 \therefore 1 - \frac{1}{3} - \left( a - \frac{a^3}{3} \right) &= 0 \\
 \frac{2}{3} - a + \frac{a^3}{3} &= 0 \\
 \therefore a^3 - 3a + 2 &= 0
 \end{aligned}$$

**c** From  $\int_a^1 (1 - t^2) dt = 0$  is equivalent to  $a^3 - 3a + 2 = 0$

By the factor theorem  $(a - 1)$  is a factor  
 and  $a^3 - 3a + 2 = (a - 1)(a^2 + a - 2) = (a - 1)^2(a + 2)$   
 $\therefore \int_a^1 (1 - t^2) dt = 0$  for  $a = 1$  and  $a = -2$

21 a i When  $t = 0$ ,  $\frac{dV}{dt} = 10e^{-1} \times 5 = 50e^{-1}$  litres/minute

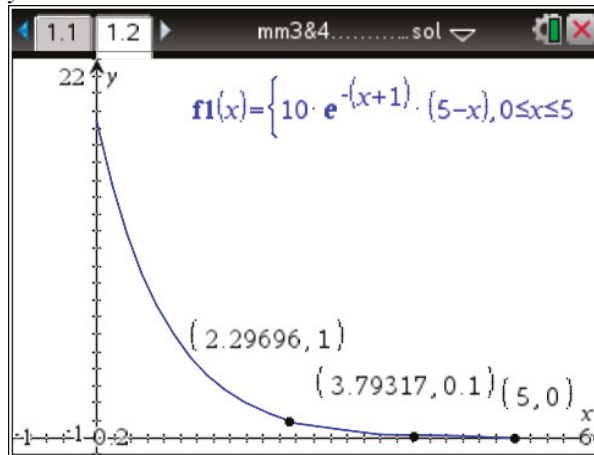
ii  $\frac{dV}{dt} = 0$  implies  $10e^{-(t+1)}(5-t) = 0$  and therefore  $t = 5$

iii The rate is 1 litre/minute when  $t = 2$  minutes and 18 seconds (to the nearest second)

iv The rate is less than 0.1 litres/minute for the first time when  $t = 3$  minutes and 48 seconds (to the nearest second)

$\frac{dv}{dt} = 10 \cdot e^{-(t+1)} \cdot (5-t)  _{t=0}$	$\frac{dv}{dt} = 50 \cdot e^{-1}$
$\left(\frac{dv}{dt} = 50 \cdot e^{-1}\right) \rightarrow \text{Decimal}$	$\frac{dv}{dt} = 18.394$
$\text{solve}\left(10 \cdot e^{-(t+1)} \cdot (5-t) = 0, t\right)$	$t = 5$
$\text{solve}\left(10 \cdot e^{-(t+1)} \cdot (5-t) = 1, t\right)$	$t = 2.29696$
$\text{solve}\left(10 \cdot e^{-(t+1)} \cdot (5-t) = 0.1, t\right)$	$t = 3.79317$

Solving graphically. Enter  $f1(x) = 10e^{-(x+1)}(5-x) | 0 \leq x \leq 5$  Use the **Point On** tool (b>Geometry>Points & Lines) and edit the y-coordinate.



**b** Use **Integral** from the **Calculus** menu. There are 14.74 litres of water in the tank after 5 minutes.

**c** The time (to the nearest second) that there is 10 litres in the tank is 53 seconds.

The screenshot shows a software window titled "mm3&4.....sol". It displays the following mathematical operations:

- The integral  $\int_0^5 (10 \cdot e^{-(x+1)} \cdot (5-x)) dx$  is evaluated to  $10 \cdot (4 \cdot e^5 + 1) \cdot e^{-6}$ , resulting in the decimal value 14.74.
- The equation  $\int_0^x (10 \cdot e^{-(x+1)} \cdot (5-x)) dx = 10$  is solved for  $x$ , yielding the solution  $x = 0.887292$ .



# Chapter 13 – Discrete random Variables and their probability distribution

## Solutions to Exercise 13A

1

$$\begin{aligned} 1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T &= \Pr(\text{Lit} \cup \text{Lan}) \\ &= \Pr(\text{Lit}) + \Pr(\text{Lan}) - \Pr(\text{Lan} \cap \text{Lit}) \\ &= 0.3 + 0.6 - 0.25 \\ &= 0.65 \end{aligned}$$

2 HH1, HH2, HH3, HH4, HH5, HH6,  
HT1, HT2, HT3, HT4, HT5, HT6,  
TH1, TH2, TH3, TH4, TH5, TH6,  
TT1, TT2, TT3, TT4, TT5, TT6

7 a  $0.05 + 0.02 - 0.003 = 0.067$

b  $0.05 - 0.003 = 0.047$

3 a  $\frac{4}{52} = \frac{1}{13}$

b  $\frac{3}{4}$

c  $\frac{16}{52} = \frac{4}{13}$

d  $\frac{8}{52} = \frac{2}{13}$

8  $1 - 0.75 - 0.12 - 0.08 = 0.05 = 5\%$

9 let  $\Pr(A)$  be the probability that an adult owns a car &  $\Pr(B)$  be the probability that an adult is employed

$$\Pr(A) = 0.7, \Pr(B) = 0.6$$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{0.6}{0.7} = \frac{6}{7}$$

4 a  $\frac{3}{6} = \frac{1}{2}$

b  $\frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}$

10 a  $\frac{17}{500}$

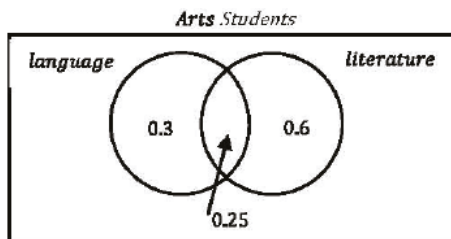
b  $\frac{18}{500} = \frac{9}{250}$

c  $\frac{30 + 45 + 33 + 39 + 17}{500}$   
 $= \frac{164}{500} = \frac{41}{125}$

d  $\frac{10 + 17 + 2 + 1 + 11}{500} = \frac{41}{500}$

5  $\Pr(S \cup L) = \Pr(S) + \Pr(L) - \Pr(S \cap L)$   
 $= 0.7 + 0.6 - 0.5$   
 $= 0.8$

6



11 a  $\Pr(\text{guns}) = \frac{130}{200} = \frac{13}{20}$

b  $\Pr(\text{guns} \cap \text{male}) = \frac{70}{200} = \frac{7}{20}$

- 12 a  $\Pr(\text{head}) \approx \frac{114}{200} = \frac{57}{100}$
- b  $\Pr(\text{ten}) \approx \frac{40}{380} = \frac{2}{19}$
- c  $\Pr(2 \text{ heads}) \approx \frac{54}{200} = \frac{27}{100}$
- d  $\Pr(3 \text{ sixes}) \approx \frac{2}{500} = \frac{1}{250}$

13  $\Pr(\text{White}) = \frac{\text{Area of white}}{\text{Total area}}$   
 $= \frac{30^2}{50^2}$   
 $= \frac{900}{2500}$   
 $= \frac{9}{25}$

- 14 a  $\Pr(\text{Green}) = \frac{\text{Area of green}}{\text{Total area}}$   
 $= \frac{\frac{1}{2}\pi r^2}{\pi r^2}$   
 $= \frac{1}{2}$
- b  $\Pr(\text{Yellow}) = \frac{\text{Area of yellow}}{\text{Total area}}$   
 $= \frac{\frac{1}{6}\pi r^2}{\pi r^2}$   
 $= \frac{1}{6}$
- c  $\Pr(\text{Not Yellow}) = 1 - \Pr(\text{Yellow})$   
 $= \frac{5}{6}$

15

	<i>C</i>	<i>C'</i>	
<i>T</i>	0.32	0.13	0.45
<i>T'</i>	0.33	0.22	0.55
	0.65	0.35	

- a  $\Pr(T \cap C') = 0.13$
- b  $\Pr(T \cap C) = 0.32$

16

	<i>S</i>	<i>S'</i>	
<i>D</i>	0.25	0.15	0.40
<i>D'</i>	0.42	0.18	0.60
	0.67	0.33	

- a  $\Pr(D) = 0.4$
- b  $\Pr(S) = 0.67$
- c  $\Pr(D' \cap S') = 0.18$

17

	<i>A</i>	<i>A'</i>	
<i>S</i>	0.53	0.12	0.65
<i>S'</i>	0.18	0.17	0.35
	0.71	0.29	

- a  $\Pr(S') = 0.35$
- b  $\Pr(A \cap S'S) = 0.18$
- c  $\Pr(A' \cap S) = 0.12$
- d  $\Pr(A' \cap S') = 0.17$

## Solutions to Exercise 13B

1 a  $\Pr(RR) = 0.25 \times 0.8 = 0.2$

b  $\Pr(R'R') = 0.75 \times 0.9 = 0.675$

c  $\Pr(R \text{ Sunday})$   
 $= \Pr(RR) + \Pr(R'R')$   
 $= 0.2 + 0.675 = 0.875$

2 a  $\Pr(B|A) = \frac{\Pr(B \cap A)}{\Pr(A)}$   
 $= \frac{1}{6}$

b  $\Pr(A|B) = \frac{\Pr(B \cap A)}{\Pr(B)}$   
 $= \frac{1}{3}$

3 a  $\Pr(A \cap B) = \Pr(B|A) \Pr(A)$   
 $= 0.1 \times 0.6 = 0.06$

b  $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1}{5}$

4  $\Pr(C|F) = \frac{\Pr(C \cap F)}{\Pr(F)} = \frac{0.3}{0.5} = \frac{3}{5}$

5 Let  $H$  be the event poor harvest.

Let  $D$  be the event disease.

$$\begin{aligned} \Pr(D|H) &= \frac{\Pr(D \cap H)}{\Pr(H)} \\ &= \frac{\Pr(H|D) \Pr(D)}{\Pr(H|D') \Pr(D') + \Pr(H|D) \Pr(D)} \\ &= \frac{0.8 \times 0.3}{0.8 \times 0.3 + 0.5 \times 0.7} \\ &= \frac{24}{59} \end{aligned}$$

**A second interpretation of the question which is entirely respectable.**

$\Pr(D) = 0.3$

let  $H$  be a poor harvest

$\Pr(H) = 0.5$

$\Pr(H|D) = 0.8$

$\Pr(H \cap D) = \Pr(H|D) \times \Pr(D)$   
 $= 0.8 \times 0.3$   
 $= 0.24$

$\Pr(D|H) = \frac{\Pr(H \cap D)}{\Pr(H)}$   
 $= \frac{0.24}{0.5} = 0.48$

6 a  $\frac{500}{1000} = \frac{1}{2}$

b  $\frac{385}{1000} = \frac{77}{200}$

c  $\frac{200}{385} = \frac{40}{77}$

d  $\frac{200}{500} = \frac{2}{5}$

7 a  $\Pr(S) = \frac{\text{total speed}}{\text{total}}$   
 $= \frac{130}{448} = \frac{65}{224}$

b  $\Pr(F) = \frac{\text{total fatal}}{\text{total}} = \frac{115}{448}$

c Look only at the Speed column:  
 $\Pr(F|S) = \frac{42}{130} = \frac{21}{65}$

d Look only at the Alcohol column:  
 $\Pr(F|A) = \frac{61}{246}$

8 a  $\Pr(J \cap S) = 0.8 \times 0.3 = 0.24$

b  $\Pr(J \cup S) = 0.8 + 0.3 - 0.24 = 0.86$

9  $\Pr(A) = 0.6, \Pr(B) = 0.5, \Pr(C) = 0.4$

a  $\Pr(A) \times \Pr(B) = 0.6 \times 0.5 = 0.3$

$A \cap B = \{1, 3, 5\} \quad \Pr(A \cap B) = 0.3$

$\Pr(A \cap B) \neq \Pr(A) \times \Pr(B)$

$\therefore A$  and  $B$  are not independent

b  $\Pr(A) \times \Pr(C) = 0.6 \times 0.4 = 0.24$

$A \cap C = \{2, 6\} \quad \Pr(A \cap C) = 0.2$

$\Pr(A \cap C) \neq \Pr(A) \times \Pr(C)$

$\therefore A$  and  $C$  are not independent

c  $\Pr(B) \times \Pr(C) = 0.5 \times 0.4 = 0.2$

$B \cap C = \{9\} \quad \Pr(B \cap C) = 0.1$

$\Pr(B \cap C) \neq \Pr(B) \times \Pr(C)$

$\therefore B$  and  $C$  are not independent

10  $\Pr(A) = 0.5, \Pr(B) = 0.4$

a 
$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \frac{\Pr(A) \times \Pr(B)}{\Pr(B)}$$

$$= \Pr(A)$$

$$= 0.5$$

b  $\Pr(A \cap B) = \Pr(A) \times \Pr(B) = 0.2$

c

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$= 0.5 + 0.4 - 0.2$$

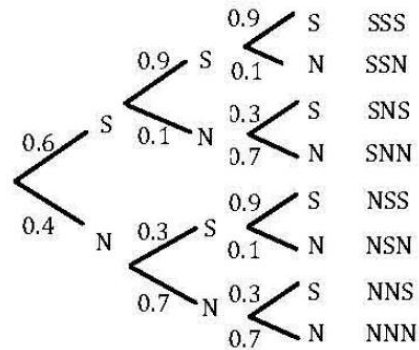
$$= 0.7$$

11  $0.3 \times 0.7 + 0.6 \times 0.3 = 1.3 \times 0.3 = 0.39$

12 
$$\frac{\Pr(HHH)}{1 - \Pr(TTT)} = \frac{\frac{1}{8}}{\frac{7}{8}} = \frac{1}{7}$$

13  $0.03 \times 0.95 + 0.97 \times 0.02 = 0.0479$

14  $S = stop \quad N = no stop$



a  $\Pr(SSS) = 0.6 \times 0.9 \times 0.9 = 0.486$

b  $\Pr(NSN) = 0.4 \times 0.3 \times 0.1 = 0.012$

c  $\Pr(SNN) = 0.6 \times 0.1 \times 0.7 = 0.042$

$\Pr(NSN) = 0.012$

$\Pr(NNS) = 0.4 \times 0.7 \times 0.3 = 0.084$

$\Pr(SNN) + \Pr(NSN) + \Pr(NNS) = 0.138$

15 a  $\frac{4}{6} \times \frac{3}{5} = \frac{2}{5}$

b  $\frac{2}{6} \times \frac{1}{5} = \frac{1}{15}$

c  $\frac{4}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{4}{5} = \frac{16}{30} = \frac{8}{15}$

16 a  $\frac{160}{400} = \frac{2}{5}$

b  $\frac{70}{400} = \frac{7}{40}$

c  $\frac{\frac{7}{40}}{\frac{2}{5}} = \frac{7}{16}$

$$\mathbf{d} \quad \frac{\frac{7}{40}}{\frac{150}{400}} = \frac{7}{15}$$

$$\mathbf{17 \ a} \quad \frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{2}{7} = \frac{5}{14}$$

$$\frac{\frac{3}{14}}{\frac{5}{14}} = \frac{3}{5}$$

$$\mathbf{18 \ a} \quad 0.3 \times 0.75 + 0.6 \times 0.8 + 0.1 \times 0.3$$

$$= 0.735$$

$$\mathbf{b} \quad \frac{0.6 \times 0.2}{1 - 0.735} = \frac{0.12}{0.265} = \frac{24}{53}$$

$$\mathbf{19} \quad \frac{3}{12} \times \frac{2}{11} \times \frac{1}{10} + \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$$

$$+ \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10}$$

$$= \frac{1}{220} + \frac{1}{55} + \frac{1}{22}$$

$$= \frac{1}{220} + \frac{4}{220} + \frac{10}{220}$$

$$= \frac{15}{220}$$

$$= \frac{3}{44}$$

## Solutions to Exercise 13C

1 a discrete

b continuous

c discrete

d discrete

2 a continuous

b discrete

c continuous

d discrete

3 a  $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

b  $X = 0, \{TTT\}$

$X = 1, \{HTT, THT, TTH\}$

$X = 2, \{HHT, HTH, THH\}$

$X = 3, \{HHH\}$

c  $\Pr(X \geq 2) = \frac{4}{8} = \frac{1}{2}$

4 a Yes, since  $p(x) \geq 0$  for all  $x$ , and  $\sum p(x) = 1$

b  $\Pr(X \leq 3) = 0.1 + 0.2 + 0.1 + 0.4 = 0.8$

5 a  $\Pr(X = 3) = \Pr(RRR) = 4/9 \times 4/9 \times 4/9 = 64/729$

$\Pr(X = 2) = \Pr(RRB) + \Pr(RBR) + \Pr(BRR) = 3 \times 4/9 \times 4/9 \times 5/9 = 240/729$

$\Pr(X = 1) = \Pr(RBB) + \Pr(BBR) + \Pr(BRB) = 3 \times 4/9 \times 5/9 \times 5/9 = 300/729$

$\Pr(X = 0) = \Pr(BBB) = 5/9 \times 5/9 \times 5/9 = 125/729$

b  $\Pr(X \geq 1) = 1 - \Pr(X = 0) = \frac{604}{729}$

c  $\Pr(X > 1) = 1 - \Pr(X = 0) - \Pr(X = 1) = \frac{304}{729}$

**6 a**  $\{(1, 1), (1, 2), (1, 3), \dots, (6, 4), (6, 5), (6, 6)\}$

		die 1					
		1	2	3	4	5	6
<b>b die 2</b>	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

**c i**  $\Pr(Y < 5) = \frac{6}{36} = \frac{1}{6}$

**ii**  $\Pr(Y = 3|Y < 5) = \frac{2}{6} = \frac{1}{3}$

**iii**  $\Pr(Y \leq 3|Y < 7) = \Pr(Y \leq 3) / \Pr(Y < 7) = (3/36) / (15/36) = 3/15 = 1/5$

**iv**  $\Pr(Y \geq 7|Y > 4) = \Pr(Y \geq 7) / \Pr(Y > 4) = (21/36) / (30/36) = 21/30 = 7/10$

**v**  $\Pr(Y = 7|Y > 4) = \Pr(Y = 7) / \Pr(Y > 4) = (6/36) / (30/36) = 6/30 = 1/5$

**vi**  $\Pr(Y = 7|Y < 8) = \Pr(Y = 7) / \Pr(Y < 8) = (6/36) / (21/36) = 6/21 = 2/7$

**7 a**

		die 1					
		1	2	3	4	5	6
<b>die 2</b>	1	1	1	1	1	1	1
	2	1	2	2	2	2	2
	3	1	2	3	3	3	3
	4	1	2	3	4	4	4
	5	1	2	3	4	5	5
	6	1	2	3	4	5	6

**b**  $Y = 1, 2, 3, 4, 5, 6$

**c**  $\Pr(Y = 1) = 0.1 + 0.1 - 0.1 \times 0.1$   
 $= 0.19$

**8 a**  $\Pr(X = 2) = \Pr(WWB) + \Pr(WBW)$   
 $+ \Pr(BWW)$

where  $B$  means ‘black ball drawn’ and  $W$  means ‘which ball drawn’.

$$\begin{aligned}\Pr(X = 2) &= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} + \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} \\ &\quad + \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5} \\ &= 3 \times \frac{12}{125} \\ &= \frac{36}{125} = 0.288\end{aligned}$$

**b**  $\Pr(X = 3) = \Pr(WWW)$

$$\begin{aligned}&= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \\ &= \frac{8}{125} = 0.064\end{aligned}$$

**c**  $\Pr(X \geq 2) = 0.288 + 0.064 = 0.352$ .

**d**  $\Pr(X = 3|X \geq 2) = \frac{\Pr(X = 3)}{\Pr(X \geq 2)}$

$$\begin{aligned}&= \frac{0.064}{0.352} = \frac{2}{11} \\ &\approx 0.182\end{aligned}$$

**9 a**  $\{(1, 1), (1, 2), (1, 3), \dots, (6, 4), (6, 5), (6, 6)\}$

**b**  $\Pr(A) = \frac{1}{6}$

$$\Pr(B) = \frac{1}{6}$$

$$\Pr(C) = \frac{15}{36}$$

(counting possibilities)

$$= \frac{5}{12}$$

$$\Pr(D) = \frac{6}{36}$$

(counting possibilities)

$$= \frac{1}{6}$$



$$\mathbf{c} \quad \Pr(A \cap B) = \frac{1}{36}$$

(counting possibilities)

$$\Pr(A|B) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

(counting possibilities)

$$\Pr(A \cap C) = \frac{3}{36}$$

$$= \frac{1}{12}$$

$$\Pr(A|C) = \frac{\frac{1}{12}}{\frac{1}{5}} = \frac{5}{12}$$

$$\Pr(A \cap D) = \frac{1}{36}$$

(counting possibilities)

$$\Pr(A|D) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

**d** A & B, A & D since

$$\Pr(A|B) = \Pr(A)$$

$$\Pr(A|D) = \Pr(A)$$

**10 a** Yes, since  $p(x) \geq 0$  for all  $x$ , and  $\sum p(x) = 1$

**b**  $\Pr(X \geq 2) = 0.2 + 0.3 = 0.5$

**11 a**, since the sum of the  $p(x)$  values  $> 1$ ; and **c** negative probabilities values is 0.

**12** Let  $x$  be the number of black balls in the sample.

$$\mathbf{a} \quad \Pr(X = 0) = \left(\frac{6}{10}\right)^3 = \left(\frac{27}{125}\right)$$

$$\Pr(X = 3) = \left(\frac{4}{10}\right)^3 = \frac{8}{125}$$

$$\begin{aligned} \Pr(X = 1) &= \frac{4}{10} \times \frac{6}{10} \times \frac{6}{10} + \frac{6}{10} \\ &\quad \times \frac{4}{10} \times \frac{6}{10} + \frac{6}{10} \times \frac{6}{10} \\ &= 3 \times \frac{18}{125} \\ &= \frac{54}{125} \end{aligned}$$

$$\begin{aligned} \Pr(X = 2) &= 1 - \frac{27}{125} - \frac{8}{125} - \frac{54}{125} \\ &= \frac{36}{125} \end{aligned}$$

$x$	0	1	2	3
$\Pr(X = x)$	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

$$\mathbf{b} \quad \Pr(X = 0) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6}$$

$$\Pr(X = 3) = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$$

$$\begin{aligned} \Pr(X = 1) &= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} + \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} \\ &\quad + \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} \\ &= 3 \times \frac{1}{6} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \Pr(X = 2) &= \frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} + \frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} \\ &\quad + \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} \\ &= 3 \times \frac{1}{10} = \frac{3}{10} \end{aligned}$$

$x$	0	1	2	3
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

$$\mathbf{13} \quad \Pr(X = 0) = 0.6^2 = 0.36$$

$$\Pr(X = 2) = 0.4^2 = 0.16$$

$$\Pr(X = 1) = 1 - 0.16 - 0.36 = 0.48$$

$x$	0	1	2
$\Pr(X = x)$	0.36	0.48	0.16

**14 a**

$x$	1	2	3	4	5
$\Pr(X = x)$	0.2	0.2	0.2	0.2	0.2

**b**  $\Pr(X \geq 3) = 0.2 \times 3 = 0.6$

**c**  $\Pr(X \leq 3 | X \geq 3) = \frac{0.2}{0.6} = \frac{1}{3}$

**15 a**  $\{(1, 1), (1, 2), (1, 3), \dots, (6, 4), (6, 5), (6, 6)\}$

**b**

		die 1					
		1	2	3	4	5	6
die 2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

$$\Pr(X = 2) = \frac{1}{36}$$

$$\Pr(X = 3) = \frac{2}{36} = \frac{1}{18}$$

$$\Pr(X = 4) = \frac{3}{36} = \frac{1}{12}$$

$$\Pr(X = 5) = \frac{4}{36} = \frac{1}{9}$$

$$\Pr(X = 6) = \frac{5}{36}$$

$$\Pr(X = 7) = \frac{6}{36} = \frac{1}{6}$$

$$\Pr(X = 8) = \frac{5}{36}$$

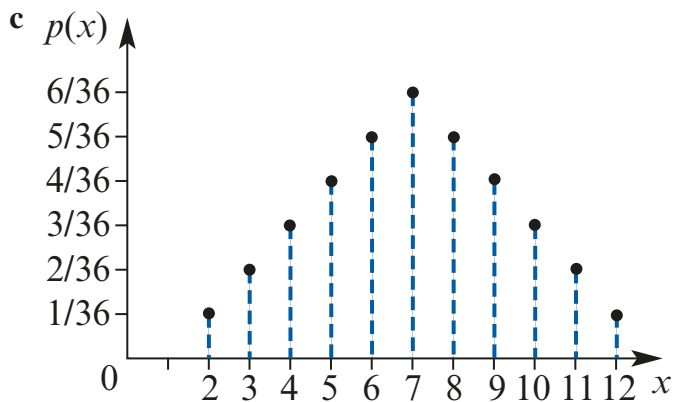
$$\Pr(X = 9) = \frac{4}{36} = \frac{1}{9}$$

$$\Pr(X = 10) = \frac{3}{36} = \frac{1}{12}$$

$$\Pr(X = 11) = \frac{2}{36} = \frac{1}{18}$$

$$\Pr(X = 12) = \frac{1}{36}$$

$x$	2	3	4	5	6	7	8	9	10	11	12
$\Pr(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



**d**

$$\Pr(X > 9) = \frac{4 + 3 + 2 + 1}{36} = \frac{10}{36}$$

$$= \frac{5}{18}$$

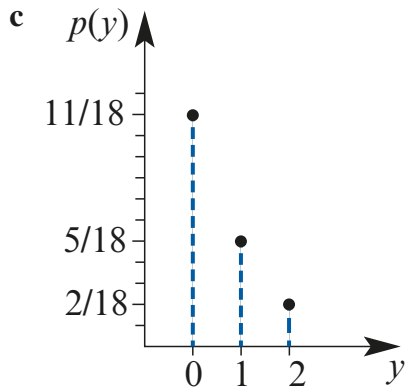
$$\mathbf{e} \Pr(X \leq 10 | X \geq 9) = \frac{\frac{7}{36}}{\frac{10}{36}} = \frac{7}{10}$$

**16 a**  $\{(1, 1), (1, 2), (1, 3), \dots, (6, 4), (6, 5), (6, 6)\}$

**b**

		dice 1					
		1	2	3	4	5	6
die 2	1	0	0	0	0	0	0
	2	0	0	1	0	1	0
	3	0	1	0	1	2	0
	4	0	0	1	0	1	2
	5	0	1	2	1	0	1
	6	0	0	0	2	1	0

$y$	0	1	2
$\Pr(Y = y)$	$\frac{22}{36}$	$\frac{10}{36}$	$\frac{4}{36}$



**17 a**

$$\Pr(X = 0) = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$$

$$\Pr(X = 2) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

$$\Pr(X = 1) = 1 - \frac{1}{3} - \frac{2}{15} = \frac{8}{15}$$

$x$	0	1	2
$\Pr(X = x)$	$\frac{1}{3}$	$\frac{8}{15}$	$\frac{2}{15}$

**b**  $\Pr(X \neq 1) = \frac{7}{15}$

- 18** centre circle =  $\pi(2)^2 = 4\pi$   
middle circle =  $\pi(10)^2 - \pi(2)^2 = 96\pi$   
outer circle =  $\pi(20)^2 - \pi(10)^2 = 300\pi$

**a**  $\Pr(X = 100) = \frac{4}{400} = \frac{1}{100}$   
 $\Pr(X = 20) = \frac{96}{400} = \frac{6}{25}$   
 $\Pr(X = 10) = \frac{300}{400} = \frac{3}{4}$

$x$	10	20	100
$\Pr(X = x)$	$\frac{3}{4}$	$\frac{6}{25}$	$\frac{1}{100}$

**b**  $\Pr(Y = 200) = \frac{1}{100} \times \frac{1}{100} = \frac{1}{10\,000}$   
 $\Pr(Y = 120) = \frac{1}{100} \times \frac{6}{25} \times 2 = \frac{3}{625}$   
 $\Pr(Y = 110) = \frac{3}{100} \times \frac{3}{4} \times 2 = \frac{3}{200}$   
 $\Pr(Y = 40) = \frac{6}{25} \times \frac{6}{25} = \frac{36}{625}$   
 $\Pr(Y = 30) = \frac{6}{25} \times \frac{3}{4} \times 2 = \frac{9}{25}$   
 $\Pr(Y = 20) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

$y$	20	30	40	110	120	200
$\Pr(Y = y)$	$\frac{9}{16}$	$\frac{9}{25}$	$\frac{36}{625}$	$\frac{3}{200}$	$\frac{3}{625}$	$\frac{1}{10\,000}$

**19 a**  $\Pr(X = 3) = \Pr(EEE) + \Pr(NNN)$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

**b**  $x = 4$ ,  
{NEEE, ENEE, EENE, ENNN, NENN, NNEN}

$$\Pr(X = 4) = 6 \times \frac{1}{16} = \frac{3}{8}$$

$$\begin{aligned}\mathbf{c} \quad \Pr(X = 5) &= 1 - \Pr(x \neq 5) \\ &= 1 - \frac{2}{8} - \frac{3}{8} \\ &= \frac{3}{8}\end{aligned}$$

## Solutions to Exercise 13D

1 There is:

30% chance of winning \$1 (\$2 prize less the \$1 cost to play)

10% chance of winning \$19 (\$20 prize less the \$1 cost to play)

60% chance of losing \$1 (the cost to play)

In 100 games the Expected win/loss =  $30 \times 1 + 10 \times 19 - 60 \times 1 = \$100$

2 Mean =  $1 \times 0.1 + 3 \times 0.3 + 5 \times 0.3$

$$+ 7 \times 0.3$$

$$= 0.1 + 0.9 + 1.5 + 2.1$$

$$= 4.6$$

Mean =  $0.25 \times -1 + 0.25 \times 0 + 0.25$

$$\times 1 + 0.25 \times 2$$

$$= 0.5$$

b Mean =  $0 \times 0.09 + 1 \times 0.22 + 2 \times 0.26$

$$+ 3 \times 0.21 + 4 \times 0.13 + 5 \times 0.06$$

$$+ 6 \times 0.02 + 7 \times 0.01$$

$$= 0.22 + 0.52 + 0.63 + 0.52 + 0.30$$

$$+ 0.12 + 0.07$$

$$= 2.38$$

c Mean =  $0.2 \times 0.08 + 0.3 \times 0.13$

$$+ 0.4 \times 0.09 + 0.5 \times 0.19$$

$$+ 0.6 \times 0.7 + 0.7 \times 0.03$$

$$+ 0.8 \times 0.10 + 0.9 \times 0.18$$

$$= 0.569$$

d Mean=7

e Mean=0



$$\begin{aligned}
3 \quad \mu &= \$10,000 \times 0.13 + \$5,000 \times 0.45 \\
&\quad + \$0 \times 0.25 - \$5,000 \times 0.15 \\
&= \$1,500 + \$2,250 - \$750 \\
&= \$3,000
\end{aligned}$$

4 assuming a payout (as opposed to a profit) of \$5 for a win,

$$\begin{aligned}
\mu &= \frac{5}{6} \times -\$1 + \frac{1}{6} \times \$4 \\
&= -\$ \frac{1}{6} = -\$0.17, \text{ i.e. a loss of } 17\text{c.}
\end{aligned}$$

$$\begin{aligned}
5 \quad \mu &= 0 \times 0.12 + 1 \times 0.36 + 2 \times 0.38 + 3 \times 0.14 \\
&= 0.36 + 0.76 + 0.42 \\
&= 1.54
\end{aligned}$$

$x$	1	2	3	4	5	7	8	9	10	11	12
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

$$\begin{aligned}
6 \quad \mu &= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} \\
&\quad + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 7 \times \frac{1}{36} \\
&\quad + 8 \times \frac{1}{36} + 9 \times \frac{1}{36} + 10 \times \frac{1}{36} \\
&\quad + 11 \times \frac{1}{36} + 12 \times \frac{1}{36} \\
&\quad 6 + 12 + 18 + 24 + 30 + 7 + 8 \\
&= \frac{+ 9 + 10 + 11 + 12}{36} \\
&= \frac{147}{36} \\
&= \frac{49}{12}
\end{aligned}$$

$$\begin{aligned}
7 \quad \mathbf{a} \quad E(X) &= 2 \times 0.01 + 3 \times 0.25 + 4 \times 0.40 \\
&\quad + 5 \times 0.30 + 6 \times 0.04 \\
&= 0.02 + 0.75 + 1.60 + 1.50 + 0.24 \\
&= 4.11
\end{aligned}$$

$$\begin{aligned}
 \text{b } E(X^3) &= 8 \times 0.01 + 27 \times 0.25 \\
 &\quad + 64 \times 0.40 + 125 \times 0.30 \\
 &\quad + 216 \times 0.04 \\
 &= 0.08 + 6.75 + 25.60 \\
 &\quad + 37.50 + 8.64 \\
 &= 78.57
 \end{aligned}$$

$$\begin{aligned}
 \text{c } E(5X - 4) &= 5E(X) - 4 = 5 \times 4.11 - 4 \\
 &= 16.55
 \end{aligned}$$

$$\begin{aligned}
 \text{d } E\left(\frac{1}{X}\right) &= \frac{1}{2} \times 0.01 + \frac{1}{3} \times 0.25 \\
 &\quad + \frac{1}{4} \times 0.40 + \frac{1}{5} \times 0.30 + \frac{1}{6} \times 0.04 \\
 &= 0.255
 \end{aligned}$$

$$\begin{aligned}
 \text{8 } E(X) &= \sum x \Pr(X = x) = 2.97 \\
 E(\text{commission}) &= 2.97 \times \$2000 = \$5940
 \end{aligned}$$

$$\begin{aligned}
 \text{9 a } p &= 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} \\
 p &= \frac{1}{16}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \mu &= 0 \times p + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} \\
 &\quad + 8 \times \frac{1}{16} \\
 &= 4 \times \frac{1}{2} \\
 \mu &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c } E(X^2) &= 0 \times p + 1 \times \frac{1}{2} + 4 \times \frac{1}{4} \\
 &\quad + 16 \times \frac{1}{8} + 64 \times \frac{1}{16} \\
 &= 0 + \frac{1}{2} + 1 + 2 + 4 \\
 &= \frac{15}{2} \\
 \sigma^2 &= E(X^2) - E(X)^2 \\
 &= \frac{15}{2} - 4 = \frac{7}{2}
 \end{aligned}$$

**10 a**  $k + 2k + 3k + 4k + 5k + 6k = 1$

$$21k = 1$$

$$k = \frac{1}{21}$$

**b**  $\mu = 1 \times k + 2 \times 2k + 3 \times 3k + 4 \times 4k$

$$+ 5 \times 5k + 6 \times 6k$$

$$= \frac{1 + 4 + 9 + 16 + 25 + 36}{21}$$

$$= \frac{91}{21}$$

$$= \frac{13}{3} \approx 4.33$$

**c**  $\sigma^2 = E(X^2) - \mu^2$

$$E(X^2) = 1 \times k + 4 \times 2k + 8 \times 3k$$

$$+ 16 \times 4k + 25 \times 5k + 36 \times 6k$$

$$= \frac{1 + 8 + 27 + 64 + 125 + 216}{21}$$

$$= \frac{441}{21}$$

$$= 21$$

$$\sigma^2 = 21 - \frac{169}{9}$$

$$= \frac{20}{9} \approx 2.22$$

**11**

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

$x$	1	2	3	4	6	8	9	12	16
<b>a</b> $\Pr(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

**b i**  $\Pr(X > 8) = \frac{1}{16} + \frac{2}{16} + \frac{1}{16}$

$$= \frac{4}{16} = \frac{1}{4}$$

$$\begin{aligned}
 \text{ii } E(X) &= 1 \times \frac{1}{16} + 2 \times \frac{2}{16} \\
 &\quad + 3 \times \frac{2}{16} + 4 \times \frac{3}{16} \\
 &\quad + 6 \times \frac{2}{16} + 8 \times \frac{2}{16} \\
 &\quad + 9 \times \frac{1}{16} + 12 \times \frac{2}{16} \\
 &\quad + 16 \times \frac{1}{16} \\
 &\quad 1 + 4 + 6 + 12 + 12 + 16 \\
 &\quad + 9 + 24 + 16 \\
 &= \frac{\quad}{16} \\
 &= \frac{100}{16} \\
 &= \frac{25}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } E(X)^2 &= 1 \times \frac{1}{16} + 4 \times \frac{2}{16} \\
 &\quad + 9 \times \frac{2}{16} + 16 \times \frac{3}{16} \\
 &\quad + 36 \times \frac{2}{16} + 64 \times \frac{2}{16} \\
 &\quad + 81 \times \frac{1}{16} + 144 \times \frac{2}{16} \\
 &\quad + 256 \times \frac{1}{16} \\
 &\quad 1 + 8 + 18 + 48 + 72 + 128 \\
 &\quad + 81 + 288 + 256 \\
 &= \frac{\quad}{16} \\
 &= \frac{900}{16} \\
 &= \frac{225}{4} \\
 \sigma^2 &= E(X^2) - \mu^2 = \frac{225}{4} - \frac{625}{16} \\
 &= \frac{275}{16}
 \end{aligned}$$

12

	H	T
1	1	2
2	2	4
3	3	6
4	4	8
5	5	10
6	6	12

$$\begin{aligned}
 \mathbf{a} \quad \mu &= 1 \times \frac{1}{12} + 2 \times \frac{2}{12} + 3 \times \frac{1}{12} \\
 &\quad + 4 \times \frac{2}{12} + 5 \times \frac{1}{12} + 6 \times \frac{2}{12} \\
 &\quad + 8 \times \frac{1}{12} + 10 \times \frac{1}{12} + 12 \times \frac{1}{12} \\
 &= \frac{1 + 4 + 3 + 8 + 5 + 12 + 8 + 10 + 12}{12} \\
 &= \frac{63}{12} = \frac{21}{4}
 \end{aligned}$$

$$\mathbf{b} \quad \Pr(X < \mu) = \frac{7}{12} \text{ (counting on the table)}$$

$$\begin{aligned}
 \mathbf{c} \quad E(X^2) &= 1 \times \frac{1}{12} + 4 \times \frac{2}{12} \\
 &\quad + 9 \times \frac{1}{12} + 16 \times \frac{2}{12} \\
 &\quad + 25 \times \frac{1}{12} + 36 \times \frac{2}{12} \\
 &\quad + 64 \times \frac{1}{12} + 100 \times \frac{1}{12} \\
 &\quad + 144 \times \frac{1}{12} \\
 &\quad 1 + 8 + 9 + 32 + 25 + 72 \\
 &\quad + 64 + 100 + 144 \\
 &= \frac{\quad}{12} \\
 &= \frac{455}{12} \\
 \sigma^2 &= E(X^2) - \mu^2 = \frac{455}{12} - \frac{441}{16} \\
 &= \frac{497}{48}
 \end{aligned}$$

$$\mathbf{13} \quad \mathbf{a} \quad \text{Var}(2X) = 2^2 \text{Var}(X) = 4 \times 16 = 64$$

$$\mathbf{b} \quad \text{Var}(X + 2) = 1^2 \text{Var}(X) = 16$$

$$\mathbf{c} \quad \text{Var}(1 - X) = (-1)^2 \text{Var}(X) = 16$$

$$\begin{aligned}
 \mathbf{d} \quad sd(3X) &= \sqrt{\text{Var}(3X)} = \sqrt{3^2 \text{Var}(X)} \\
 &= \sqrt{9 \times 16} \\
 &= 12
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{14} \quad \mathbf{a} \quad c &= 1 - 0.3 - 0.1 - 0.2 - 0.05 \\
 &= 0.35
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad E(X) &= 1 \times c + 2 \times 0.3 + 3 \times 0.1 \\
 &\quad + 4 \times 0.2 + 5 \times 0.05 \\
 &= 0.35 + 0.6 + 0.3 + 0.8 + 0.25 \\
 &= 2.3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad E(X^2) &= 1 \times c + 4 \times 0.3 + 9 \times 0.1 \\
 &\quad + 16 \times 0.2 + 25 \times 0.05 \\
 &= 0.35 + 1.2 + 0.9 + 3.2 + 1.25 \\
 &= 6.9
 \end{aligned}$$

$$\begin{aligned}
 \sigma^2 &= E(X^2) - (E(X))^2 \\
 &= 6.9 - (2.3)^2 \\
 &= 6.9 - 5.29 \\
 &= 1.61
 \end{aligned}$$

$$\sigma = \sqrt{\sigma^2} \approx 1.27$$

$$\begin{aligned}
 \mathbf{d} \quad \Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \\
 \Pr(2.3 - 2.5 \leq X \leq 2.3 + 2.5) \\
 \Pr(0 \leq X \leq 4.8) \\
 &= 0.35 + 0.3 + 0.1 + 0.2 \\
 &= 0.95
 \end{aligned}$$

$$\mathbf{15} \quad \mathbf{a} \quad k + 2k + 3k + 4k + 5k = 1$$

$$15k = 1$$

$$k = \frac{1}{15}$$

$$\begin{aligned}
 \mathbf{b} \quad \mu &= 1 \times \frac{1}{15} + 2 \times \frac{2}{15} + 3 \times \frac{3}{15} \\
 &\quad + 4 \times \frac{4}{15} + 5 \times \frac{5}{15} \\
 &= \frac{1 + 4 + 9 + 16 + 25}{15} \\
 &= \frac{55}{15}
 \end{aligned}$$

$$\mu = \frac{11}{3} \approx 3.667$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$\begin{aligned}
\text{c } E(X^2) &= 1 \frac{1}{15} + 4 \times \frac{2}{15} + 9 \times \frac{3}{15} \\
&\quad + 16 \times \frac{4}{15} + 25 \times \frac{5}{15} \\
&= \frac{1 + 8 + 27 + 64 + 125}{15} \\
&= \frac{225}{15} \\
&= 15 \\
\sigma^2 &= 15 - \frac{121}{9} \\
&= \frac{14}{9} \approx 1.556
\end{aligned}$$

$$\text{d } \sigma = \sqrt{\sigma^2} \approx 1.25$$

$$\begin{aligned}
&\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \\
&= \Pr(3.67 - 2.5 \leq X \leq 3.67 + 2.5) \\
&= \Pr(1.2 \leq X \leq 6.2) \\
&= \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15} \\
&= \frac{14}{15} \approx 0.933
\end{aligned}$$

$$\begin{aligned}
\text{16 a } E(X) &= \frac{1}{36} \times 2 + \frac{2}{36} \times 3 + \frac{3}{36} \times 4 \\
&\quad + \frac{4}{36} \times 5 + \frac{5}{36} \times 6 + \frac{6}{36} \times 7 \\
&\quad + \frac{5}{36} \times 8 + \frac{4}{36} \times 9 + \frac{3}{36} \times 10 \\
&\quad + \frac{2}{36} \times 11 + \frac{1}{36} \times 12 \\
&\quad 2 + 6 + 12 + 20 + 30 + 42 + 40 \\
&\quad + 36 + 30 + 22 + 12 \\
&= \frac{\quad}{36} \\
&= \frac{252}{36} \\
&= 7
\end{aligned}$$

alternatively, since we know the probability distribution is symmetrical, we also know that the mean is the central number, i.e.  $E(X) = 7$

$$\begin{aligned}
\mathbf{b} \quad E(X^2) &= \frac{1}{36} \times 4 + \frac{2}{36} \times 9 + \frac{3}{36} \times 16 \\
&\quad + \frac{4}{36} \times 25 + \frac{5}{36} \times 36 \\
&\quad + \frac{6}{36} \times 49 + \frac{5}{36} \times 64 \\
&\quad + \frac{4}{36} \times 81 + \frac{3}{36} \times 100 \\
&\quad + \frac{2}{36} \times 121 + \frac{1}{36} \times 144 \\
&\quad + 4 + 18 + 48 + 100 + 180 \\
&\quad + 294 + 320 + 324 + 300 \\
&\quad + 242 + 144 \\
&= \frac{\quad}{36} \\
&= \frac{1974}{36} \\
&= \frac{329}{6} \approx 54.833
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= E(X^2) - (E(X))^2 \\
&= \frac{329}{6} - 49 \\
&= \frac{35}{6} \approx 5.83
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad \sigma &= \sqrt{\sigma^2} \approx 2.4 \\
\Pr(\mu - 2\sigma &\leq X \leq \mu + 2\sigma) \\
&= \Pr(7 - 4.8 \leq X \leq 7 + 4.8) \\
&= \Pr(2.2 \leq X \leq 11.8) \\
&= 1 - \Pr(X = 2) - \Pr(X = 12) \\
&= 1 - \frac{1}{36} - \frac{1}{36} \\
&= \frac{17}{18} \approx 0.944
\end{aligned}$$

**17 a** *by symmetry*,  $E(X) = 3$

$$\begin{aligned}
\mathbf{b} \quad \text{Var}(X) &= E(X^2) - E(X)^2 \\
E(X^2) &= 0 \times 0.0156 + 1 \times 0.0937 \\
&\quad + 4 \times 0.2344 + 9 \times 0.3126 \\
&\quad + 16 \times 0.2344 + 5 \times 0.0937 \\
&\quad + 36 \times 0.0156 \\
&= 0.0937 + 0.9376 + 2.8134 \\
&\quad + 3.7504 + 2.3425 + 0.5616 \\
&= 10.4992 \approx 10.5
\end{aligned}$$



$$\begin{aligned}\text{Var}(X) &= 10.5 - 9 \\ &= 1.5\end{aligned}$$

$$\mathbf{c} \quad \sigma = \sqrt{\text{Var}(X)} \approx 1.2$$

$$\begin{aligned}\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \\ &= \Pr(3 - 2.4 \leq X \leq 3 + 2.4) \\ &= \Pr(0.6 \leq X \leq 5.4) \\ &= \Pr(1 \leq X \leq 5) \\ &= 0.0937 + 0.2344 + 0.3126 \\ &\quad + 0.2344 + 0.0937 \\ &= 0.9688\end{aligned}$$

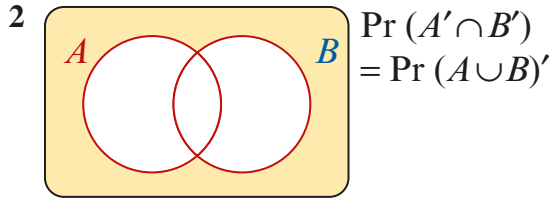
$$\mathbf{18} \quad \Pr(c_1 \leq X \leq c_2) \approx 0.95$$

$$c_1 = \mu - 2\sigma = 50 - 10 = 40$$

$$c_2 = \mu + 2\sigma = 50 + 10 = 60$$

## Solutions to Technology-free questions

- 1  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$   
 $0.7 = 0.5 + 0.2 - \Pr(A \cap B)$   
 $\Pr(A \cap B) = 0$   
 If  $A$  and  $B$  are mutually exclusive this is true.



$(A \cup B)'$  is the complement of  $A \cup B$ .  
 $\Pr(A \cup B) + \Pr[(A \cup B)'] = 1$   
 So  $\Pr(A \cup B) = 1 - \Pr[(A \cup B)']$   
 Therefore  $\Pr(A' \cap B') = \Pr[(A \cup B)']$   
 The intersection of the complements is the complement of the union, so the probability of the union of two sets is 1 – probability of the intersection of their complements.

3 a  $\Pr(BW \text{ or } WB) = \frac{5}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{5}{9}$   
 $= \frac{40}{81}$   
 $= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8}$

b  $\Pr(BW \text{ or } WB) = \frac{5}{9}$

- 4 Require  $\Pr(\text{coin } A | \text{'H\&T' tossed})$   
 $= \frac{\Pr(A \cap \text{'H\&T'})}{\Pr(\text{'H\&T'})}$   
 $\Pr(\text{selecting } A) = \Pr(\text{selecting } B) = \frac{1}{2}$   
 $\Pr(H|A) = 0.8, \Pr(T|A) = 0.2,$   
 $\Pr(H|B) = 0.4, \Pr(T|B) = 0.6$

$$\Pr(3 \cap \text{'H\&T'}) = \frac{1}{2} \times (0.8 \times 0.2 + 0.2 \times 0.8)$$

$$= 0.16$$

$$\Pr(\text{'H\&T'}) = \Pr(A \cap \text{'H\&T'}) + \Pr(B \cap \text{'H\&T'})$$

$$\Pr(A \cap \text{'K\&T'}) = \frac{1}{2} \times (0.4 \times 0.6 + 0.6 \times 0.4)$$

$$= 0.24$$

$$\text{So } \Pr(\text{'H\&T'}) = 0.16 + 0.24$$

$$= 0.40$$

$$\Pr(A | \text{'H\&T'}) = \frac{0.16}{0.40} = 0.4$$

5  $0.4p^2 + 0.1 + 0.1 + 1 - 0.6p = 1$   
 $0.4p^2 - 0.6p + 0.2 = 0$   
 $4p^2 - 6p + 2 = 0$   
 $2p^2 - 3p + 1 = 0$   
 $(2p - 1)(p - 1) = 0$   
 $\therefore p = \frac{1}{2} \text{ or } p = 1$

6 a  $k + 2k + 3k + 2k + k + k = 1$   
 $10k = 1$   
 $k = 0.1$

b  $E(x) = \sum xp(x)$   
 $= -k + 0 + 3k + 4k + 3k + 4k$   
 $= 13k$   
 $= 1.3$

$$\begin{aligned}
 \text{c } E(x^2) &= \sum x^2 p(x) \\
 &= k + 0 + 3k + 8k + 9k + 16k \\
 &= 37k = 3.7 \\
 \text{Var}(x) &= E(x^2) - [E(x)]^2 \\
 &= 3.7 - 1.69 = 2.01
 \end{aligned}$$

$$\begin{aligned}
 7 \text{ a } E(x) &= \sum xp(x) \\
 &= 2 \times \frac{1}{4} + 4 \times \frac{1}{4} \\
 &\quad + 16 \times \frac{1}{4} + 64 \times \frac{1}{4} \\
 &= 21 \frac{1}{2} (21.5)
 \end{aligned}$$

$$\begin{aligned}
 \text{b } E\left(\frac{1}{x}\right) &= \sum \frac{1}{x} p(x) \\
 &= \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} \\
 &\quad + \frac{1}{16} \times \frac{1}{4} + \frac{1}{64} \times \frac{1}{4} \\
 &= \frac{53}{256}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } E(x^2) &= \sum x^2 p(x) \\
 &= 4 \times \frac{1}{4} + 16 \times \frac{1}{4} + 256 \times \frac{1}{4} \\
 &\quad + 64^2 \times \frac{1}{4} \\
 &= 1 + 4 + 64 + 1024 \\
 &= 1093 \\
 \text{Var}(x) &= E(x^2) - [E(x)]^2 \\
 &= 1093 - (21.5)^2 \\
 &= 1093 - 462.25 \\
 &= 630.75 \\
 &= \frac{2523}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \text{From past c, } \text{Var}(x) &= \frac{2523}{4}. \\
 \text{But } 2523 &= 3 \times 841 \\
 &= 3 \times 29^2 \\
 \text{So } \text{Var}(x) &= \frac{3 \times 29^2}{2^2} \\
 \Rightarrow \text{sd}(x) &= \frac{29\sqrt{3}}{2}
 \end{aligned}$$

8 a Profit is  $\$(x - 2)$  if the cylinder is ok and  $-\$2$  if the cylinder is defective.

$$\frac{P}{\Pr(P = p)} \begin{vmatrix} x - 2 & -2 \\ \frac{4}{5} & \frac{1}{5} \end{vmatrix}$$

$$\begin{aligned}
 \text{b } E(P) &= \sum p \Pr(P = p) = \frac{4}{5}(x - 2) - \frac{2}{5} \\
 &= \frac{4}{5}x - 2.
 \end{aligned}$$

c To make a profit in the long term, require  $E(P) > 0$ , i.e.

$$\begin{aligned}
 \frac{4}{5}x - 2 &> 0 \\
 \frac{4}{5}x &> 2 \\
 x &> \frac{5}{2} = 2.5
 \end{aligned}$$

The manufacturer should sell the cylinders for more than  $\$2.50$ .

$$\begin{aligned}
 9 \text{ a } \Pr(' < 30' \cap ' > 1 \text{ acc}') &= \frac{470}{1000} \\
 &= 0.47
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \Pr(' < 30' \cap ' > 1 \text{ acc}') & \\
 &= \frac{\Pr(' < 30 \cap > 1 \text{ acc}')}{\Pr(' > 1 \text{ acc}')} \\
 &= \frac{0.47}{\left(\frac{470 + 230}{100}\right)} \\
 &= \frac{0.47}{0.70} \\
 &= \frac{47}{70}
 \end{aligned}$$

10

Let  $I$  = 'immunised',  $D$  = 'get disease'

$$\begin{aligned}
 \Pr(D) &= \Pr(D \cap I) + \Pr(D \cap I') \\
 &= \Pr(I) \Pr(D|I) + \Pr(I') \Pr(D|I') \\
 &= 0.7 \times 0.05 + 0.3 \times 0.6 \\
 &= 0.035 + 0.18 \\
 &= 0.215
 \end{aligned}$$

So 21.5% are expected to get the disease.

[NOTE: This is a probability way of saying: "5% of the 70% and 60% of the 30% get the disease, i.e. 3.5% + 18% = 21.5% get it]

$$11 \quad \Pr(A) = \frac{1}{2}, \Pr(B) = \frac{1}{4}, \Pr(A|B) = \frac{1}{6}$$

$$\begin{aligned}
 \text{a } \Pr(A \cap B) &= \Pr(A|B) \Pr(B) \\
 &= \frac{1}{6} \times \frac{1}{4} \\
 &= \frac{1}{24}
 \end{aligned}$$

b

$$\begin{aligned}
 \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\
 &= \frac{1}{2} + \frac{1}{4} - \frac{1}{24} \\
 &= \frac{17}{24}
 \end{aligned}$$

c

$$\Pr(A'|B) = \frac{\Pr(A' \cap B)}{\Pr(B)}$$

$$\text{But } \Pr(B) = \Pr(A \cap B) + \Pr(A' \cap B)$$

$$\text{So } \Pr(A' \cap B) = \Pr(B) - \Pr(A \cap B)$$

$$\begin{aligned}
 &= \frac{1}{4} - \frac{1}{24} \\
 &= \frac{5}{24} \\
 \text{So } \Pr(A'|B) &= \frac{\frac{5}{24}}{\frac{5}{24}} \\
 &= \frac{1}{4} \\
 &= \frac{5}{6}
 \end{aligned}$$

d

$$\Pr(A|B') = \frac{\Pr(A \cap B')}{\Pr(B')}$$

$$\text{But } \Pr(A') = \Pr(A \cap B) + \Pr(A \cap B')$$

$$\text{So } \Pr(A \cap B') = \Pr(A) - \Pr(A \cap B)$$

$$\begin{aligned}
 &= \frac{1}{2} - \frac{1}{24} \\
 &= \frac{11}{24} \\
 \Pr(B') &= 1 - \Pr(B) \\
 &= 1 - \frac{1}{4} \\
 &= \frac{3}{4} \\
 \text{So } \Pr(A|B') &= \frac{\frac{11}{24}}{\frac{3}{4}} \\
 &= \frac{11}{18}
 \end{aligned}$$

## Solutions to multiple-choice questions

$$\begin{aligned}
 \mathbf{1 \ A} \quad 2k + 3k + 0.1 + 3k + 2k &= 1 \\
 10k &= 0.9 \\
 k &= 0.09
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2 \ D} \quad \Pr(-3 \leq X < 0) &= \Pr(X = -3) \\
 &+ \Pr(X = -2) + \Pr(X = -1) \\
 &= 0.07 + 0.15 + 0.22 \\
 &= 0.44
 \end{aligned}$$

**3**

$$\begin{aligned}
 \mathbf{D} \quad E(X) &= \sum xp(x) = \Pr(X = x) \\
 &= 1 \times 0.46 + 2 \times 0.26 \\
 &+ 3 \times 0.14 + 4 \times 0.09 \\
 &+ 5 \times 0.07 \\
 &= 0.46 + 0.48 + 0.42 + 0.36 + 0.35 \\
 &= 2.07
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4 \ E} \quad \text{Var}(X) &= E(X^2) - [E(X)]^2 \\
 &= 1.69 - (1.20)^2 \\
 &= 1.69 - 1.44 \\
 &= 0.25 \\
 \text{sd}(X) &= \sqrt{0.25} \\
 &= 0.5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5 \ C} \quad E(Y) &= E(3X + 10) \\
 &= 3E(X) + 10 \\
 &= 3 \times 100 + 10 \\
 &= 310 \\
 \text{Var}(Y) &= \text{Var}(3X + 10) \\
 &= 9 \text{Var}(X) \\
 &= 9 \times 100 \\
 &= 900
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6 \ C} \quad E(x) &= \sum xp(x) \\
 &= -p + 0 + 1 - 3p \\
 &= 1 - 4p
 \end{aligned}$$

**7**

$$\begin{aligned}
 \mathbf{B/D} \quad a + b + 0.2 &= 1 \Rightarrow a + b = 0.8 \dots \textcircled{1} \\
 E(x) &= -2a + 0.4 = 0.2 \dots \textcircled{2} \\
 \text{From } \textcircled{2}, 2a &= 0.2 \text{ so } a = 0.1 \\
 \text{Substitute in } \textcircled{1}: &0.1 + b = 0.8 \text{ so} \\
 b &= 0.7
 \end{aligned}$$

## Solutions to extended-response questions

**1 a**

$$\Sigma \Pr(X = x) = 1$$

$$\therefore c + 2c + 2c + 3c + c^2 + 2c^2 + 7c^2 + c = 1$$

$$\therefore 10c^2 + 9c = 1$$

$$\therefore 10c^2 + 9c - 1 = 0$$

$$\therefore (10c - 1)(c + 1) = 0$$

$$\therefore c = 0.1 \text{ or } c = -1$$

but  $c > 0 \quad \therefore c = 0.1$

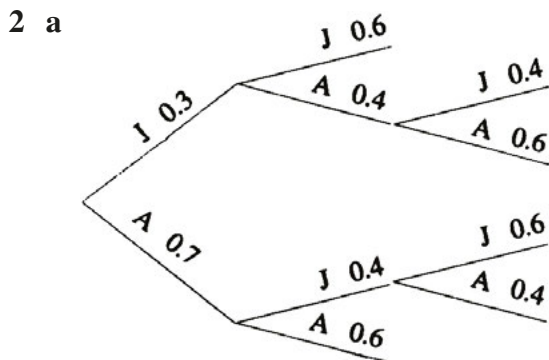
**b**  $\Pr(X \geq 5) = \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)$

$$= 10c^2 + c$$

$$= 10 \times (0.1)^2 + 0.1$$

$$= 0.2$$

**c** If  $\Pr(X \leq k) > 0.5$   
 then by considering cumulative probabilities  
 i.e.  $\Pr(X \leq 2) = 0.3$ ;  $\Pr(X \leq 3) = 0.5$   
 the minimum value of  $k$  is 4.



**b i** Probability of Janet winning

$$= 0.3 \times 0.6 + 0.3 \times 0.4 \times 0.4 + 0.7 \times 0.4 \times 0.6$$

$$= 0.396$$

**ii** Probability of Alan winning

$$= 1 - 0.396$$

$$= 0.604$$

**c i** Let  $X$  be the number of sets played until match is complete.

$$\begin{aligned}\Pr(X = 2) &= 0.3 \times 0.6 + 0.7 \times 0.6 \\ &= 0.6\end{aligned}$$

$$\therefore \Pr(X = 3) = 0.4$$

$t$	2	3
$\Pr(T = t)$	0.6	0.4

**ii**  $E(X) = 2 \times 0.6 + 3 \times 0.4 = 2.4$

**d**  $\Pr(\text{Alan wins} \mid \text{three sets}) = \frac{\Pr(\text{Alan wins in three sets})}{\Pr(\text{Three sets})}$

$$= \frac{0.3 \times 0.4 \times 0.6 + 0.7 \times 0.4 \times 0.4}{0.4}$$

$$= 0.46$$

**3** Let  $\$w$  be the amount a player pays to play.

\$5   \$5   \$5   \$10   \$10

Let  $X$  be the possible value From 2 cards

$$\therefore X = 10, 15, 20$$

A score of 10 is obtained if two \$5 cards are chosen

$$\therefore \Pr(X = 10) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \text{ (without replacement)}$$

A score of 15 is obtained with \$10 on the first and \$5 on the second or \$5 on the first and \$10 on the second.

$$\begin{aligned}\therefore \Pr(X = 15) &= \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{1}{2} \\ &= \frac{6}{20} + \frac{3}{10} \\ &= \frac{3}{5}\end{aligned}$$

A score of 20 is obtained with a \$10 on each card.

$$\begin{aligned}\Pr(X = 20) &= \frac{2}{5} \times \frac{1}{4} \\ &= \frac{1}{10}\end{aligned}$$

Let  $Y$  be the amount a player receives

$$Y = 10 - w \text{ or } 15 - w \text{ or } 20 - w$$

The probability distribution for  $Y$  is as shown

$y$	$10 - w$	$15 - w$	$20 - w$
$\Pr(Y = y)$	0.3	0.6	0.1

$$\begin{aligned} \therefore E(Y) &= 0.3(10 - w) + 0.6(15 - w) + 0.1(20 - w) \\ &= 3 - 0.3w + 9 - 0.6w + 2 - 0.1w \\ &= 14 - w \end{aligned}$$

If  $E(Y) = 0$ ,  $w = 14$

i.e. The player should pay \$14 to ensure that it is a fair game.

**4** Let  $F$  denote free from faults

Let  $N$  denote not free from faults (defective)

$$\Pr(F|A) = 0.95 \quad \Pr(F|B) = 0.98 \quad \Pr(F|C) = 0.99$$

$$\Pr(A) = 0.5 \quad \Pr(B) = 0.3 \quad \Pr(C) = 0.2$$

**a**  $\Pr(A) = 0.5$

**b**  $\Pr(N|A) = 0.05$

**c**  $\Pr(N) = \Pr(N|A)\Pr(A) + \Pr(N|B)\Pr(B) + \Pr(N|C)\Pr(C)$   
 $= 0.05 \times 0.5 + 0.02 \times 0.3 + 0.01 \times 0.2$   
 $= 0.033$

**d**  $\Pr(A|D) = \frac{\Pr(\text{produced by } A \text{ and defective})}{\Pr(\text{defective})}$   
 $= \frac{\Pr(N|A)\Pr(A)}{\Pr(N)}$   
 $= \frac{0.5 \times 0.5}{0.033}$   
 $= \frac{25}{33}$

**5**

$p$	0	1	2	3	4	5
$\Pr(P = p)$	0.39	0.27	0.16	0.12	0.04	0.02

**a i**  $E(P) = 0 \times 0.39 + 1 \times 0.27 + 2 \times 0.16 + 3 \times 0.12 + 4 \times 0.04 + 5 \times 0.02$   
 $= 0.27 + 0.32 + 0.36 + 0.16 + 0.1$   
 $= 1.21$

The mean number of passengers per car is 1.21



**ii**  $\text{Var}(P) = E(P^2) - [E(P)]^2$   
 $E(P^2) = 0^2 \times 0.39 + 1^2 \times 0.27 + 2^2 \times 0.16 + 3^2 \times 0.12 + 4^2 \times 0.04 + 5^2 \times 0.02$   
 $= 0.27 + 0.64 + 1.08 + 0.64 + 0.5 = 3.13$   
 $\text{Var}(P) = 3.13 - 1.4641$   
 $= 1.6659$   
 $\text{sd}(P) = \sqrt{1.6659} = 1.2907$  (correct to four decimal places)

**iii**  $\sigma = \text{sd}(P) = 1.2907$   
 $\mu - 2\sigma = -1.3714$   
 $\mu + 2\sigma = 3.7914$   
 $\Pr(\mu - 2\sigma \leq P \leq \mu + 2\sigma) = \Pr(-1.3714 \leq P \leq 3.7914)$   
 $= \Pr(P = 0) + \Pr(P = 1) + \Pr(P = 2) + \Pr(P = 3)$   
 $= 1 - [\Pr(P = 4) + \Pr(P = 5)]$   
 $= 0.94$

**b i** Let  $T$  be the cost per car in dollars.

$$\Pr(T = 1) = \Pr(P = 0) = 0.39$$

$$\Pr(T = 0.40) = \Pr(P = 1) = 0.27$$

$$\Pr(T = 0) = \Pr(P = 2) + \Pr(P = 3) + \Pr(P = 4) + \Pr(P = 5) = 0.34$$

$t$	1	0.40	0
$\Pr(T = t)$	0.39	0.27	0.34

**ii**  $E(T) = 1 \times 0.39 + 0.40 \times 0.27 + 0 \times 0.34$   
 $= 0.39 + 0.108$   
 $= 0.498$

$$\begin{aligned}
\text{iii } E(T^2) &= 1 \times 0.39 + 0.4^2 \times 0.27 \\
&= 0.39 + 0.0432 \\
&= 0.4332 \\
\text{Var}(T) &= E(T^2) - [E(T)]^2 \\
&= 0.4332 - 0.248004 \\
&= 0.1852 \\
\text{sd}(T) &= 0.4303 \\
\mu - 2\sigma &= 0.498 - 2 \times 0.4304 = -0.3628 \\
\mu + 2\sigma &= 0.498 + 2 \times 0.4304 = 1.3588 \\
\Pr(\mu - 2\sigma \leq T \leq \mu + 2\sigma) &= \Pr(-0.3628 \leq T \leq 1.3588) \\
&= \Pr(T = 0) + \Pr(T = 0.4) + \Pr(T = 1) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\text{6 a } E(Y) &= 0 \times 0.135 + 1 \times 0.271 + 2 \times 0.271 + 3 \times 0.180 + 4 \times 0.090 \\
&\quad + 5 \times 0.036 + 6 \times 0.012 + 7 \times 0.003 + 8 \times 0.002 \\
&= 2.002
\end{aligned}$$

The mean number of sales per week is 2.002.

$$\begin{aligned}
\text{b } E(Y^2) &= 6.002 \\
\text{Var}(Y) &= E(Y^2) - [E(Y)]^2 = 6.022 - 4.008004 \\
&= 2.013996 \approx 2.014 \\
\text{sd}(Y) &\approx 1.419
\end{aligned}$$

- c i** Let  $B$  be the bonus paid to each salesman.  
The possible values for  $B$  are 0, 100 and 200  
 $\Pr(B = 0) = \Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) = 0.677$   
 $\Pr(B = 100) = \Pr(Y = 3) + \Pr(Y = 4) = 0.27$   
 $\Pr(B = 200) = \Pr(Y \geq 4) = 0.053$   
The probability distribution is

$b$	0	100	200
$\Pr(B = b)$	0.677	0.27	0.053

$$\begin{aligned}
\text{ii } E(B) &= 0 \times 0.677 + 100 \times 0.27 + 200 \times 0.053 \\
&= 27 + 10.6 \\
&= 37.6
\end{aligned}$$

The mean bonus paid is \$37.60.

7 Let  $P$  denote the percentage profit

$p$	40	30	20	10	0	-10	-20
$\Pr(P = p)$	0.1	0.15	0.25	0.2	0.15	0.1	0.05

$$\begin{aligned} \mathbf{a} \quad E(P) &= 40 \times 0.1 + 30 \times 0.15 + 20 \times 0.25 + 10 \times 0.2 + 0 \times 0.15 - 10 \times 0.1 - 20 \times 0.05 \\ &= 13.5 \end{aligned}$$

The mean return is 13.5%

$$\begin{aligned} E(P^2) &= 1600 \times 0.1 + 100 \times 0.15 + 400 \times 0.25 + 100 \times 0.2 + 100 \times 0.1 + 400 \times 0.05 \\ &= 445 \end{aligned}$$

$$\therefore \text{Var}(P) = 445 - 182.25$$

$$= 262.75$$

$$\therefore \text{sd}(P) = \sqrt{262.75} \approx 16.2\%$$

$$\mathbf{b} \quad \Pr(13.5 - 2 \times 16.21 \leq P \leq 13.5 + 2 \times 16.21) = \Pr(-18.92 \leq P \leq 45.92)$$

$$= 1 - \Pr(P = -20)$$

$$= 1 - 0.05$$

$$= 0.95$$

$\mathbf{c}$  Return = Profit–Brokerage

$$\text{Percentage gain} = 0.6 \times \text{Return}$$

$$= 0.6(\text{Profit–Brokerage})$$

Let  $G$  be the percentage gain

$$\text{Then} \quad G = 0.6(P - 2)$$

$$\therefore E(G) = 0.6E(P) - 1.2$$

$$= 0.6 \times 13.5 - 1.2$$

$$= 6.9\%$$

$$\text{Var}(G) = (0.6)^2 \text{Var}(P)$$

$$= 0.36 \times 262.75$$

$$= 94.59$$

$$\text{sd}(G) \approx 9.726\%$$

8 Consider the case when the promoter takes out insurance.

If it rains:

$$\begin{aligned}
 (\$)\text{Profit} &= 250\,000 - 60\,000 + 20\,000 \\
 &= 210\,000
 \end{aligned}$$

(assuming the \$250 000 is paid by the insurance company and the \$20 000 profit is added.) If it does not rain:

$$\begin{aligned}
 (\$)\text{Profit} &= 25\,000 - 60\,000 \\
 &= 190\,000
 \end{aligned}$$

The probability distribution for this

$p$	190 000	210 000
$\Pr(P = p)$	0.67	0.33

$$E(P) = 196\,600$$

Then the promoter does not take the insurance, the probability distribution is as shown below:

$p$	250 000	20 000
$\Pr(P = p)$	0.67	0.33

$$\text{and } E(P) = 174\,100$$

$\therefore$  the promoter should buy the insurance.

**9** For the tossing of two dice the sums of the values may be recorded in a table as shown

die A						
die B	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The possible sums are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

$$\text{The probability of obtaining a sum of 7} = \frac{6}{36} = \frac{1}{6}$$

$$\text{The probability of obtaining a sum of 11 or 12} = \frac{3}{36} = \frac{1}{12}$$

$$\text{The probability of any other sum} = \frac{3}{4}$$

Let  $X$  be the amount obtained from game and let  $w$  be the amount obtained from obtaining a sum not equal to 7, 11 or 12

The probability distribution is as shown:

$x$	-10	11	$w$
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{3}{4}$

$$E(X) = -\frac{10}{6} + \frac{11}{12} + \frac{3w}{4}$$

$$\text{If } E(X) = 0$$

$$-\frac{20}{12} + \frac{11}{12} + \frac{9w}{12} = 0$$

$$\therefore \frac{9w}{12} = \frac{9}{12}$$

$$w = 1$$

There should be a payment of \$1.00 for a sum not equal to 7, 11 or 12.

**10 a i** Probability that the first prototype is successful is 0.65.

**ii** Probability of the first not successful, but the second successful

$$= 0.35 \times 0.65$$

$$= 0.2275$$

**iii** Probability of the first two not successful, but the third successful

$$= (0.35)^2 \times 0.65$$

$$= 0.079625$$

**iv** Probability that the project is abandoned

$$= (0.35)^3$$

$$= 0.042875$$

**b** The following cases have to be considered:

	Cost	Probability
A First is successful	\$7 million	0.65
B First is unsuccessful but second is successful	\$10.5 million	0.2275
C First two unsuccessful but third successful	\$12.25 million	0.079625
D The project is abandoned	\$12.25 million	0.042875

Let  $C$  be the cost of the project.

$C$	7	10.5	12.25
$\Pr(C = c)$	0.6	0.2275	0.1225

$$\therefore E(C) = 7 \times 0.65 + 10.5 \times 0.2275 + 12.25 \times 0.1225$$

$$= 8.439375$$

$\therefore$  the expected cost is \$8.439375 million

**c** Let  $P$  denote the profit

$P$	$20 - 7$	$20 - 10.5$	$20 - 12.25$	$-12.25$
$\Pr(P = p)$	0.65	0.2275	0.079625	0.042875

$$\therefore E(P) = 13 \times 0.65 + 9.5 \times 0.2275 + 7.75 \times 0.079625 - 12.25 \times 0.042875$$

$\therefore$  Expected profit is \$10.703125 million

**11** If the score is 5, 6, 7, 8, 9, 10, 11 or 12. Alfred pays \$ $x$  to Bertie. Therefore Alfred has  $100 - x$  dollars.

If the score is 2, 3 or 4 Alfred has  $100 + x + 8 = 108 + x$  dollars.

The tables gives the sum of the scores when the two die are tossed.

die 2						
die 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let  $Y$  be the score.

$$\Pr(Y \geq 5) = \frac{30}{36} = \frac{5}{6} \text{ and } \Pr(Y \leq 4) = \frac{1}{6}$$

**a** Let  $A$  be the amount of Alfred's cash

$A$	$100 - x$	$108 + x$
$\Pr(A = a)$	$\frac{5}{6}$	$\frac{1}{6}$

$$\begin{aligned} \therefore E(A) &= \frac{5}{6}(100 - x) + \frac{1}{6}(108 + x) \\ &= \frac{1}{6}(608 - 4x) \\ &= \frac{1}{3}(304 - 2x) \end{aligned}$$

**b** If the game is fair  $E(A) = 100$

$$\therefore \frac{1}{3}(304 - 2x) = 100$$

$$304 - 2x = 300$$

$$\therefore x = 2$$

**c**  $E(A^2) = 97^2 \times \frac{5}{6} + 111^2 \times \frac{1}{6}$  (given  $x = 3$ )

$$= 9894\frac{1}{3}$$

$$\therefore \text{Var}(A) = 2894\frac{1}{3} - \left(\frac{1}{3}[298]\right)^2$$

$$= 27\frac{2}{9}$$

**12** Let  $X$  be the values of the die

$$\Pr(X = 1) = \frac{x}{4} \quad \Pr(X = 2) = \frac{1}{4} \quad \Pr(X = 6) = \frac{1}{4}(1 - x)$$

$$\Pr(X = 3) = \Pr(X = 4) = \Pr(X = 5) = \frac{1}{6}$$

Table for total

2nd 1st	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

**a** Let  $Y$  be the total.

$$\Pr(Y = 7) = 2 \Pr(1 \text{ \& } 6) + 2 \Pr(3 \text{ \& } 4) + 2 \Pr(5 \text{ \& } 2)$$

$$\begin{aligned} &= 2 \times \frac{x}{4} \times \frac{1-x}{4} + 2 \times \frac{1}{6} \times \frac{1}{6} + 2 \times \frac{1}{6} \times \frac{1}{4} \\ &= \frac{x(1-x)}{8} + \frac{2}{36} + \frac{2}{24} \\ &= \frac{x(1-x)}{8} + \frac{1}{18} + \frac{1}{12} \\ &= \frac{9x(1-x) + 4 + 6}{72} \\ &= \frac{9x - 9x^2 + 10}{72} \end{aligned}$$

**b** Let  $P = \Pr(Y = 7)$

$$\frac{dP}{dx} = \frac{9 - 18x}{72}$$

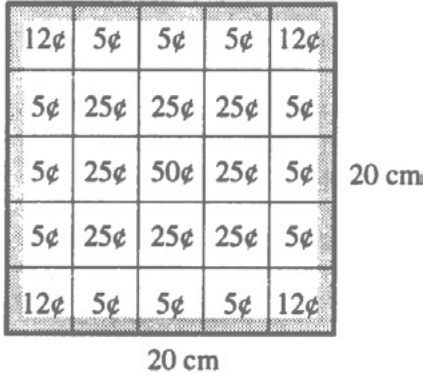
and  $\frac{dP}{dx} = 0$  implies  $x = \frac{1}{2}$

when  $x = \frac{1}{2}$ ,  $P = \frac{9 \times \frac{1}{2} - 9\left(\frac{1}{2}\right)^2 + 10}{72}$

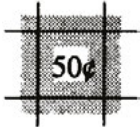
$$= \frac{\frac{9}{2} - \frac{9}{4} + 10}{72} = \frac{\frac{9}{4} + 10}{72} = \frac{49}{288}$$



13



Coin cannot land within 1 cm of ridge.  
 $\therefore$  Area in which “centre” of coin can fall is  $18^2 = 324 \text{ cm}^2$ .



In order to land in square and not overlap another square centre must fall at a distance of  $> 1$  from edge.

$\therefore$  Area in which “centre” of coin can fall is  $4 \text{ cm}^2$ .

a Let  $X$  be the prize money.

i  $\Pr(X = 50) = \frac{4}{324} = \frac{1}{81}$

ii  $\Pr(X = 25) = \frac{8}{81}$

iii  $\Pr(X = 12) = \frac{4}{81}$

iv  $\Pr(X = 5) = \frac{12}{81} = \frac{4}{27}$

v  $\Pr(X = 0) = \frac{56}{81}$

b Let  $P$  be the profit.

$P$	$C - 50$	$C - 25$	$C - 12$	$C - 5$	$C$
$\Pr(P = p)$	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{4}{81}$	$\frac{12}{81}$	$\frac{56}{81}$

$$E(P) = \frac{1}{81} [(C - 50) + 8(C - 25) + 4(C - 12) + 12(C - 5) + 56C] > 0$$

$$\Leftrightarrow 81C - 358 > 0$$

$$C > \frac{358}{81} \approx 4.4197$$

Five cents will yield a profit.

# Chapter 14 – The binomial distribution

## Solutions to Exercise 14A

1 a and b describe a Bernoulli sequence.

b  $\Pr(X = 2) = 0.2527$

2  $n = 7, p = 0.5$

$\Pr(X = 4) = \binom{7}{4}(0.5)^4(0.5)^3 = 0.2734$

7 a  $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 = 20 \times \frac{1}{216} \times \frac{125}{216}$   
 $\approx 0.0536$

3  $n = 4, p = 0.2$

a  $\Pr(X = 3) = \binom{4}{3}(0.2)^3(0.8)^1 = 0.0256$

$\binom{6}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2 + \binom{6}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right) + \binom{6}{6}\left(\frac{1}{6}\right)^6$

b  $\Pr(X = 4) = \binom{4}{4}(0.2)^4(0.8)^0 = 0.0016$

b  $\frac{375}{6^6} + \frac{30}{6^6} + \frac{1}{6^6} = \frac{406}{46656}$   
 $\approx 0.0087$

4  $n = 5, p = 0.4$

a  $\Pr(X = 0) = \binom{5}{0}(0.4)^0(0.6)^5 = 0.0778$

c  $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 + \binom{6}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2$   
 $+ \binom{6}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right) + \binom{6}{6}\left(\frac{1}{6}\right)^6$   
 $\approx 0.0623$

b  $\Pr(X = 3) = \binom{5}{3}(0.4)^3(0.6)^2 = 0.2304$

8  $n = 10, p = 0.1$

c  $\Pr(X = 5) = \binom{5}{5}(0.4)^5(0.6)^0 = 0.01024$

a  $\Pr(X = x) = \binom{10}{x}(0.1)^x(0.9)^{10-x}$   
 $x = 0, 1, 2, 3, \dots, 10$

5  $n = 3, p = 0.5$

a  $\Pr(X = x) = \binom{3}{x}(0.5)^x(0.5)^{3-x}$   
 $x = 0, 1, 2, 3$

b i  $\Pr(X = 0) = 0.3487$

ii  $\Pr(X \geq 1) = 1 - \Pr(X = 0)$   
 $= 0.6513$

b  $\Pr(X = 2) = 0.375$

9  $n = 11, p = 0.2$

6  $n = 6, p = 0.48$

a  $\Pr(X = x) = \binom{6}{x}(0.48)^x(0.52)^{6-x}$   
 $x = 0, 1, 2, 3$

a  $\Pr(X = x) = \binom{11}{x}(0.2)^x(0.8)^{11-x}$   
 $x = 0, 1, 2, 3, \dots, 11$

b i  $\Pr(X = 2) = 0.2953$

ii  $\Pr(X = 0) = 0.0859$

iii  $\Pr(X \geq 1) = 1 - \Pr(X = 0)$   
 $= 0.9141$

10  $n = 7, p = \frac{1}{5}$

a  $\Pr(X = x) = \binom{7}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{7-x}$   
 $x = 0, 1, 2, 3, \dots, 7$

b i  $\Pr(X = 7) = 0.000013$

ii  $\Pr(X = 0) = 0.2097$

iii  $\Pr(X = 2 \text{ or } X = 3) = 0.3899$

11  $1 - \binom{10}{0}(0.2)^0(0.8)^{10} - \binom{10}{1}(0.2)^1(0.8)^9$   
 $= 1 - \frac{4^{10}}{5^{10}} - \frac{10 \times 1 \times 4^9}{5^{10}}$   
 $\approx 0.624$

12  $n = 7, p = \frac{x}{100}$   
 $\left(\frac{x}{100}\right)^6$

ii  $\frac{6x^5(100 - x)}{100^6}$

iii  $\frac{x^6}{100^6} + \frac{6x^5(100 - x)}{100^6} + \frac{15x^4(100 - x)^2}{100^6}$

13  $1 - \binom{4}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^4$   
 $= 1 - \left(\frac{3}{4}\right)^4$   
 $\approx 0.6836$

14 *using the CAS calculator*

a 0.1156

b 0.7986

c 0.3170

15 *using the CAS calculator*  
 0.6791

16 *using the CAS calculator*

a 0.1123

b 0.5561

c 0.00001

d 0.00001

17  $\binom{6}{0}(0.4)^0(0.6)^6 + \binom{6}{1}(0.4)(0.6)^5$   
 $+ \binom{6}{2}(0.4)^2(0.6)^4$   
 $= \frac{3^6}{5^6} + \frac{6 \times 2 \times 3^5}{5^6} + \frac{15 \times 2^2 \times 3^4}{5^6}$   
 $= \frac{3^6 + 4 \times 3^6 + 5 \times 4 \times 3^5}{5^6}$   
 $= \frac{3 \times 3^5 + 4 \times 3^5}{5^5} = \frac{7 \times 3^5}{5^5} \approx 0.544$

18 a  $\left(\frac{1}{4}\right)^6 \approx 0.00024$

b using the CAS calculator  
 $\Pr(\geq 3 \text{ correct}) \approx 0.1694$

19 a  $\left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^3 = \frac{6^3}{5^6} \approx 0.0138$

b  $\left(\frac{6}{3}\right) \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^3 = 20 \times \frac{6^3}{5^6} \approx 0.2765$

c using the CAS calculator  
 $\Pr(\geq 3) \approx 0.8208$

d  $\Pr(\text{exactly } 3 | \geq 3) = \frac{\Pr(\text{exactly } 3)}{\Pr(\geq 3)} =$   
 $\frac{(b)}{(c)} = \frac{0.2765}{0.8202} \approx 0.3368$

20 a  $\left(\frac{4}{5}\right)^8 \approx 0.1678$

b using the CAS calculator  
 $\Pr(\geq 6 \text{ correct}) \approx 0.00123$

c  $\Pr(8 \text{ correct} | \geq 6 \text{ correct}) =$   
 $\frac{\Pr(8 \text{ correct})}{\Pr(\geq 6 \text{ correct})} = \frac{(0.2)^8}{(b)} =$

$$\frac{0.00000256}{0.00123} \approx 0.0021$$

21 a  $(0.15)^{10} \approx 0.000\ 000\ 006$

b  $1 - (0.85)^{10} \approx 1 - 0.1969 \approx 0.8031$

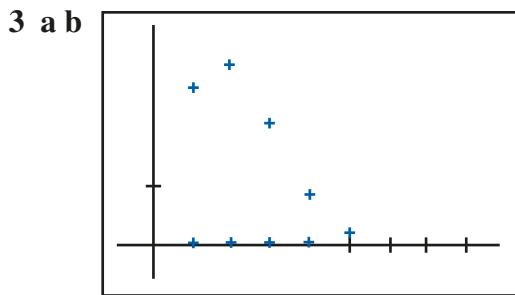
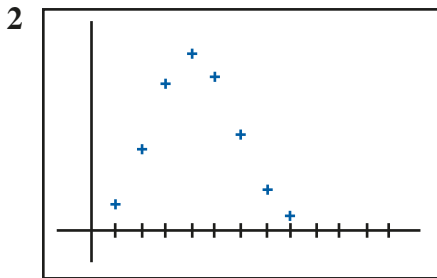
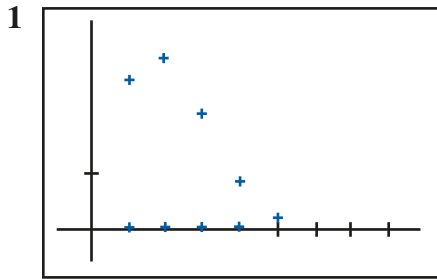
c  $\Pr(> | \text{goal} | \geq | \text{goal} |)$   
 $= \frac{\Pr(> | \text{goal} |)}{\Pr(\geq | \text{goal} |)}$   
 $= \frac{(b) - 10(0.15)(0.85)^9}{(b)}$   
 $= \frac{0.8031 - 0.3474}{0.8031}$   
 $\approx \frac{0.4557}{0.8031} \approx 0.5674$

22 a  $\left(\frac{4}{5}\right)^{20} \approx 0.0115$

b using the CAS calculator  
 $p = 0.2, n = 20,$   
 $\min = 10, \max = 20$   
 $\Pr(\geq 10 \text{ correct}) \approx 0.00259$

c  $\Pr(X \geq 12 | X \geq 10) = \frac{\Pr(X \geq 12)}{\Pr(X \geq 10)} \approx$   
 $0.0392$  (using the CAS calculator)

## Solutions to Exercise 14B



c the distribution in part b is a reflection of the distribution in part a in the line  $X = 5$

4 a  $\mu = np = 5$   
 $\sigma^2 = np(1 - p) = 5 \times (0.8) = 4$

b  $\mu = np = 6$   
 $\sigma^2 = np(1 - p) = 6 \times (0.4) = \frac{12}{5}$

c  $\mu = np = \frac{500}{3}$   
 $\sigma^2 = np(1 - p) = \frac{1000}{9}$

d  $\mu = np = 8$   
 $\sigma^2 = np(1 - p) = 8 \times \left(\frac{4}{5}\right) = \frac{32}{5}$

5 a  $\mu = np = 6 \times \frac{1}{6} = 1$

b  $\Pr(X > 1) = 1 - \Pr(X = 0) - \Pr(X = 1)$   
 $= 1 - \left(\frac{5}{6}\right)^6 - 6 \times \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5$   
 $= 1 - \frac{11}{6}\left(\frac{5}{6}\right)^5$   
 $\approx 1 - 0.7368$   
 $\approx 0.2632$

6  $\mu = np = 50 \times \frac{3}{4}$   
 $= 37.5$  people will survive on average

7  $\mu = np$   
 $\sigma^2 = np(1 - p) = \mu(1 - p)$   
 $\sigma^2 = 9, \mu = 12$   
 $9 = 12(1 - p)$

$$1 - p = \frac{3}{4}$$

$$p = \frac{1}{4}$$

$$12 = n \times \frac{1}{4}$$

$$n = 48$$

$$\Pr(X = 7) = \binom{48}{7} \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^{41}$$

$$\approx 0.0339$$

**8**

$$\mu = 30, \sigma^2 = 21$$

$$\sigma^2 = \mu(1 - p)$$

$$1 - p = \frac{21}{30}$$

$$p = \frac{9}{30} = \frac{3}{10}$$

$$\mu = np$$

$$30 = n \times \frac{3}{10}$$

$$n = 100$$

$$\begin{aligned} \Pr(X = 20) &= \binom{100}{20} \left(\frac{3}{10}\right)^{20} \left(\frac{7}{10}\right)^{80} \\ &\approx 0.0076 \end{aligned}$$

**9**

$$n = 20, p = 0.5$$

$$\mu = np = 10$$

$$\sigma^2 = np(1 - p) = 10 \times 0.5 = 5$$

$$\sigma = \sqrt{5} \approx 2.2$$

$$\mu \pm 2\sigma \approx 10 \pm 4.4$$

$$= 5.6, 14.4$$

$\therefore$  the probability of obtaining between 6 and 14 heads is about 0.95

**10**

$$n = 200, p = 0.6$$

$$\mu = np = 200 \times \frac{6}{10} = 120$$

$$\sigma^2 = \mu(1 - p) = 120 \times \frac{4}{10} = 48$$

$$\sigma = \sqrt{48} = 4\sqrt{3} \approx 6.9$$

$$\mu \pm 2\sigma \approx 120 \pm 13.8$$

$$= 106.2, 133.8$$

$\therefore$  the probability that between 107 and 133 students will have attended a government school is about 0.95

## Solutions to Exercise 14C

**1 a**  $n = 5, p = 0.2$

**i**  $\Pr(X = 0) = (0.8)^5 \approx 0.3277$

**ii**  $\Pr(X \geq 1) = 1 - \Pr(X = 0)$   
 $\approx 0.6723$

**b**  $\Pr(X \geq 1) > 0.95$

$$1 - \Pr(X = 0) > 0.95$$

$$\Pr(X = 0) < 0.05$$

$$(0.8)^n < 0.05$$

$$n \approx 13.43$$

$\therefore$  the smallest number of shots is 14

**c**  $\Pr(X \geq 1) > 0.95 \quad \therefore$

$$1 - \Pr(X = 0 - \Pr(X = 1)) > 0.95$$

$$\Pr(X = 0) + \Pr(X = 1) < 0.05$$

$$(0.8)^n + \binom{n}{1} 0.8^{n-1} \times 0.2 < 0.05$$

$$(0.8)^n + n0.8^{n-1} \times 0.2 < 0.05$$

$$n \approx 21.77$$

the smallest number of shots is 22

**2 a i**  $\Pr(X = 2) = \binom{10}{2} (0.1)^2 (0.9)^8$

$$\approx 0.1937$$

**ii**  $\Pr(X \geq 1) = 1 - \Pr(X = 0)$

$$= 1 - (0.9)^{10}$$

$$\approx 1 - 0.3487$$

$$\approx 0.6513$$

**b**  $\Pr(X \geq 1) > 0.7$

$$1 - \Pr(X = 0) > 0.7$$

$$\Pr(X = 0) < 0.3$$

$$(0.9)^n < 0.3$$

$$n \approx 11.43$$

$\therefore$  the smallest number of tickets is 12

**3**  $p = 0.6$

$$\Pr(X = 5) > 0.25$$

$$\binom{n}{5} (0.6)^5 (0.4)^{n-5} > 0.25$$

using CAS calculator, the minimum number of shots is 7

**4**  $p = 0.2$

$$\Pr(X = 3) > 0.1$$

$$\binom{n}{3} (0.2)^3 (0.8)^{n-3} > 0.1$$

using the CAS calculator, the minimum number of chocolates is 7

**5**  $p = 0.35$

$$\Pr(X \geq 2) > 0.9$$

$$1 - \Pr(X = 0) - \Pr(X = 1) > 0.9$$

$$(0.65)^n + n(0.35)(0.65)^{n-1} < 0.1$$

using the CAS calculator, the minimum number of games is 10

**6**

$$p = 0.07$$

*number of balls is 42*

$$\Pr(X > 1) > 0.8$$

$$1 - \Pr(X = 0) - \Pr(X = 1) > 0.8$$

$$(0.93)^n + n(0.07)(0.93)^{n-1} < 0.2$$

*using the CAS calculator, the minimum*

**7**  $p = 0.7$

$$\Pr(X \geq 50) > 0.99$$

*using the CAS calculator the minimum  
number of shots is 86*



## Solutions to Technology-free questions

**1**  $X$  is Bi  $\left(n = 4, p = \frac{1}{3}\right)$

**a**  $\Pr(X = 0) = q^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

**b**  $\Pr(X = 1) = \binom{4}{1} p^1 q^3 = 4 \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^3 = \frac{32}{81}$

**c**  $\Pr(X \leq 1) = \frac{16}{81} + \frac{32}{81} = \frac{48}{81} = \frac{16}{27}$

**d**  $\Pr(X \geq 1) = 1 - \Pr(X = 0)$   
 $= 1 - \frac{16}{81} = \frac{65}{81}$

**2**  $n = 3, p = 0.6, X$  is Bi  $(n, p)$

$$\begin{aligned} \Pr(X = 2) &= \binom{3}{2} (0.6)^2 (0.4) \\ &= 3 \times 0.36 \times 0.4 \\ &= 3 \times \frac{9}{25} \times \frac{2}{5} \\ &= \frac{54}{125} \end{aligned}$$

**3**  $n = 5, p = 0.1, X$  is Bi  $(n, p)$

$$\begin{aligned} \Pr(X \geq 1) &= 1 - \Pr(X = 0) \\ &= 1 - 0.9^5 \\ &= 1 - \frac{9^5}{10^5} \end{aligned}$$

Now  $9^2 = 81, 81^2 = 6561$  and so

$$9^5 = 9 \times 6561 = 59049$$

$$\begin{aligned} \text{Hence } \Pr(X \geq 1) &= 1 - 0.59049 \\ &= 0.40951 \end{aligned}$$

**4**  $n = 20, p = 0.1, X$  is Bi  $(n, p)$

**a**  $E(X) = np = 2$

**b**  $\text{sd}(X) = npq = 2 \times 0.9 = \frac{9}{5}$

$$\text{sd}(X) = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

**5**  $n = 4, X$  is Bi  $(n, p)$

**a**  $\Pr(\text{no successes}) = (1 - p)^4$

**b**  $\Pr(\text{one success}) = \binom{4}{1} p^1 (1 - p)^3$   
 $= 4p(1 - p)^3$

**c**  $\Pr(\text{at least one success})$

$$= 1 - \Pr(\text{no success})$$

$$= 1 - (1 - p)^4$$

**d**  $\Pr(\text{four successes}) p^4$

**e**

$$\Pr(\text{at least two successes})$$

$$= 1 - \Pr(\text{zero or one success})$$

$$= 1 - (1 - p)^4 - 4p(1 - p)^3 \text{ (from a/b)}$$

**6** We can assume the coin is unbound,

so  $\Pr(3 \text{ heads in } 10 \text{ hours})$

$$= \left(\frac{10}{3}\right) \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^7$$

$$= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10}$$

$$\text{so } n = \frac{10 \times 9 \times 8}{3 \times 2 \times 1}$$

$$= 10 \times 3 \times 4$$

$$= 120$$

$$\begin{aligned}
7 \quad n = 5, X \text{ is Bi } (n, p) \\
& \text{Pr(exactly one success given} \\
& \text{at least one success)} \\
&= \frac{\text{Pr('1 success 'n' } \geq 1 \text{ success ')}}{\text{Pr}(\geq 1 \text{ success)}} \\
&= \frac{\text{Pr('1 success')}}{1 - \text{Pr('0 successes')}} \\
&= \frac{\binom{5}{1} p^1 (1-p)^4}{1 - (1-p)^5} \\
&= \frac{5p(1-p)^4}{1 - (1-p)^5}
\end{aligned}$$

$$\begin{aligned}
8 \quad \text{In one throw of die (assuming unbiased),} \\
\text{Pr(even number uppermost)} = \frac{3}{6} = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
n = 5, p = \frac{1}{5}, X \text{ is Bi } (n, p) \\
\text{Pr}(X = 3) = \binom{5}{3} \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2 \\
= \frac{5 \times 4}{2 \times 1} \times \left(\frac{1}{5}\right)^5 \\
= \frac{5}{16}
\end{aligned}$$

$$\begin{aligned}
9 \quad n = 5, p = \frac{1}{5}, X \text{ is Bi } (n, p) \\
\text{Pr}(X = 3) = \binom{5}{3} \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2 \\
= 10 \times \frac{1}{125} \times \frac{16}{25} \\
= \frac{32}{625}
\end{aligned}$$

## Solutions to multiple-choice questions

**1 D**  $n = 5, p = 0.6, X$  is Bi  $(n, p)$

$$\begin{aligned}\Pr(X = 3) &= \binom{5}{3}(0.6)^3(0.4)^2 \\ &= \frac{5 \times 4}{2 \times 1}(0.6)^3(0.4)^2 \\ &= 10 \times (0.6)^3(0.4)^2\end{aligned}$$

**2 A**  $n = 5, p = 0.35, X$  is Bi  $(n, p)$

$$\begin{aligned}\Pr(\text{on time at least once}) \\ &= 1 - \Pr(\text{lets all 5 days}) \\ &= 1 - (0.65)^5\end{aligned}$$

**3 E**  $\Pr(\text{number} > 4) = \Pr(5 \text{ or } 6) = \frac{1}{3}$   
 $n = 4, p = \frac{1}{3}, X$  is Bi  $(n, p)$

$$\begin{aligned}\Pr(X = 2) &= \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 \\ &= \frac{4 \times 3}{2 \times 1} \times \frac{1}{9} \times \frac{4}{9} \\ &= \frac{8}{27}\end{aligned}$$

**4 B**  $n = 80, p = 0.4,$

$X$  is Bi  $(n, p)$

$$\begin{aligned}\Pr(X < 30) &= \Pr(0 \leq X \leq 29) \\ &= 0.2861\end{aligned}$$

using a CAS Calculator.

**5 A**  $E(X) = np = 18 \times \frac{1}{3} = 6$

$$\text{For } (X) = np(1 - p) = 6 \times \frac{2}{3} = 4$$

$$\text{So } \mu = 6, \sigma^2 = 4$$

**6 B** Since  $p = 0.7$ , the distribution has a long tail to the left. The greatest probability will be near the mean, which is  $10 \times 0.7 = 7$ . Hence the second graph is the best representation.

**7 C**

$$\mu = 10 \Rightarrow np = 10 \quad \dots \textcircled{1}$$

$$\sigma = 2\sigma^2 = 4$$

$$\Rightarrow 4pq = 4 \quad \dots \textcircled{2}$$

$$(2) \div (1) : q = \frac{4}{10} = 0.4$$

$$1 - p = 0.4$$

$$p = 0.6$$

[Note that if you try to find  $n$ , you get  $n = 16\frac{2}{3}$ . Taking  $n$  to be 17 and  $p$  to be 0.6 gives a mean of 10.2 and variance of 4.08, i.e. the mean and variance are 10 and 4 to the nearest integer.]

**8 C** Use  $(X)npq = np(1 - p) = 1.875n =$

$10 : p - p^2 = 0.1875$  Using a CAS *solve* command gives  $p = 0.25$  or  $p = 0.75$ . (Automatically, note that  $0.1875 = \frac{3}{16}$ , so the quadratic can

be expened as  $p^2 - p + \frac{3}{16} = 0 =$

$$\left(p - \frac{1}{4}\right)\left(p - \frac{3}{4}\right) = 0, \text{ so } p = \frac{1}{4} \text{ or}$$

$p = \frac{3}{4}$ . Since the coin is biased towards heads, the probability of a head is 0.75.

**9 E** Pr(Thomas wins at least one set)  
 $1 - \text{Pr}(\text{Thomas wins no set,}) =$   
 $1 - 0.76^n$   
 $1 - 0.76^n > 0.95$   
 $0.76^n < 0.05$   
 A CAS Calculator show that  
 $0.76^{10} = 0.065 \dots$  and  $0.76$   
 $n = 0.048 \dots$   
 So that fewest number of days is 11.  
 (Alternatively, taking  $\log_{10}$  of both  
 side) gives  $\log_{10} 0.76^n < \log_{10} 0.05$   
 $n \log_{10} 0.76 < \log_{10} 0.05$   

$$n > \frac{\log_{10} 0.05}{\log_{10} 0.76}$$
 (Since  $\log_{10} 0.76$  is negative)  
 so  $n > 10.91 \dots$  and hence the least  
 number of days is 11)

**10 B** Pr(Thomas wins at least one set)  
 $= 1 - \text{Pr}(\text{no wins or one win})$   
 $= 1 - 0.76^n - \binom{n}{1}(0.24)^1(0.76)^{4-1}$   
 $= 1 - 0.76^n - 0.24n(0.76)^{4-1}$   
 A CAS calculator shows that this  
 probability is 0.940... when  $n = 17$   
 and 0.952...  
 when  $n = 18$ .  
 So the fewest number of day is 18.  
 (Note: An efficient way to use a CAS  
 calculator is to first Define the func-  
 tion  $f(n) = 1 - 0.76^n - 0.264(0.76)^{4-1}$   
 It is then a simple matter to evaluate  
 $f(n)$  for various values of  $n$ .)

## Solutions to extended-response questions

- 1 a** For children without disability there is an equal chance of answering *A*, *B* or *C*.

Let  $X$  be the number of questions out of 10 which are answered *A* or *B*.  $X$  is a

binomial random variable with  $n = 10$  and  $p = \frac{2}{3}$

$$\Pr(X = 10) = \left(\frac{2}{3}\right)^{10} = 0.0173$$

The probability that the answers given by a child without either disability will be all *As* and *Bs* is 0.0173

- b**  $\Pr(\text{Answering } C \text{ five or more times})$

$$= \Pr(\text{Answering } A \text{ or } B \text{ 5 or less times})$$

$$= \Pr(X \leq 5)$$

$$= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$$

$$= \left(\frac{1}{3}\right)^{10} + \binom{10}{1}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^9 + \binom{10}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^8 + \binom{10}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^7$$

$$+ \binom{10}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^6 + \binom{10}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^5$$

$$= \left(\frac{1}{3}\right)^{10} + 10 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^9 + 45 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^8 + 120 \times \left(\frac{2}{3}\right)^3 \times \left(\frac{1}{3}\right)^7$$

$$+ 210 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^6 + 252 \times \left(\frac{2}{3}\right)^5 \times \left(\frac{1}{3}\right)^5$$

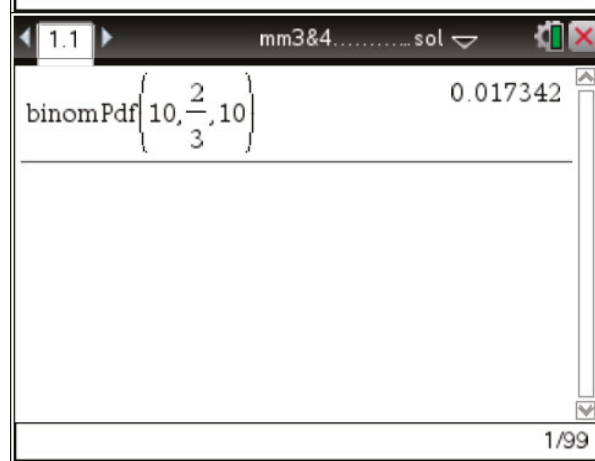
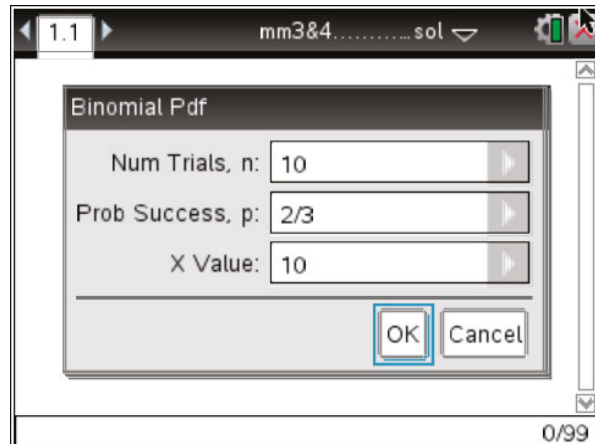
$$= \left(\frac{1}{3}\right)^{10} [1 + 20 + 180 + 960 + 3360 + 8064]$$

$$= \left(\frac{1}{3}\right)^{10} [12585]$$

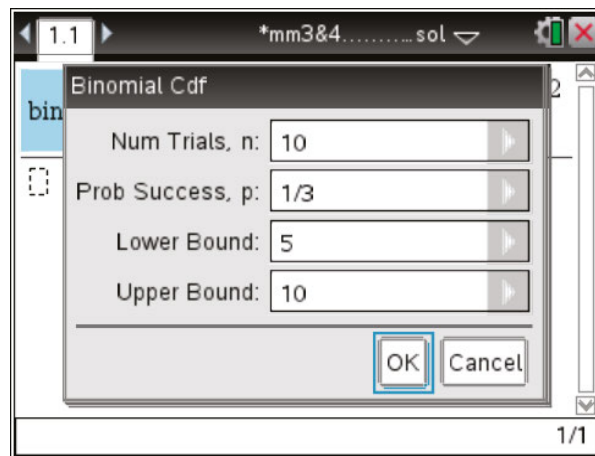
$$= 0.2131$$

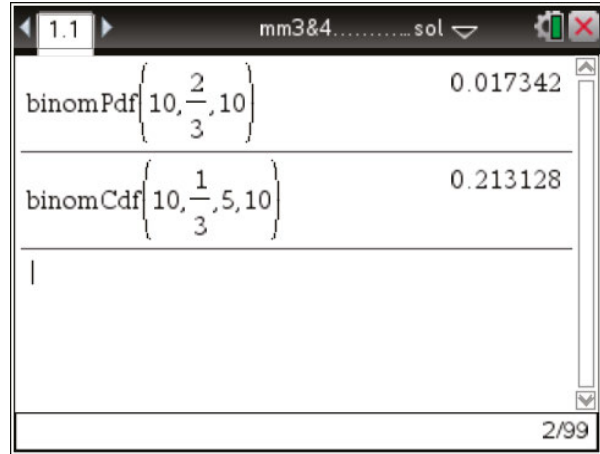
**Graphic calculator techniques for question 1**

- a In a **Calculator** page select **Binomial Pdf** from the **Probability > Distributions** menu and complete as shown.



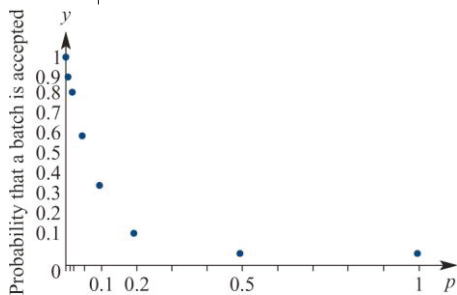
- b For the cumulative binomial select **Binomial Cdf** from the **Probability > Distributions** menu and complete the dialogue box as shown.





2 If the fraction of defective items is 0 the probability of acceptance is 1 If the fraction of defective items is 0.01 the probability of acceptance  $(0.99)^{10} = 0.9044$  The results are recorded in the table shown:

$p$	probability that a batch is accepted
0	1
0.01	0.904 4
0.02	0.817 1
0.05	0.598 7
0.1	0.348 7
0.2	0.107 4
0.5	0.000 98
1	0



3 a The probability of a defective = 0.04

Let  $X$  be the number of defectives in a sample of ten.

$$\Pr(X \geq 2) = 1 - [\Pr(X = 0) + \Pr(X = 1)]$$

$$= 1 - \left[ (0.96)^{10} + \binom{10}{1} (0.96)^9 (0.04) \right]$$

$$= 0.0582$$

**b** Mean number of defectives =  $np = 0.04 \times 10 = 0.4$

$$\begin{aligned}\text{sd of the number of defectives} &= \sqrt{np(1-p)} \\ &= \sqrt{10 \times 0.04 \times 0.96} \\ &= 0.6197\end{aligned}$$

$$\mu \pm 2\sigma = 0.4 \pm 2 \times 0.6197$$

$$0.4 + 2 \times 0.6197 = 1.6394$$

$$0.4 - 2 \times 0.6197 = -0.8394$$

**c** Yes, the claim of 4% defective must be questioned.

**4** Let  $X$  be the number of Pizzas delivered late.

$$\Pr(X \geq 12) = 1 - \Pr(X < 12)$$

Note:  $X$  is a binomial random variable with  $n = 67$  and  $p = 0.1$

A table obtained through a calculator

$x$	$\Pr(X = x)$	$\Pr(X \leq x)$
0	0.0009	0.009
1	0.0064	0.0073
2	0.0235	0.0307
3	0.0565	0.0872
4	0.1004	0.1876
5	0.1046	0.3282
6	0.1614	0.4896
7	0.1563	0.6459
8	0.1302	0.7761
9	0.0949	0.8710
10	0.0611	0.9321
11	0.0352	0.9673

$$\therefore \Pr(X \geq 12) = 1 - 0.9673$$

$$= 0.0327$$

**5 a i**  $p = \frac{1}{5}$ ,  $n = 6$  Let  $X$  be the number of defectives.

$$\begin{aligned}\Pr(X = 3) &= \binom{6}{3} \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^3 \\ &= 0.0819\end{aligned}$$



$$\begin{aligned}
 \text{ii } \Pr(X < 3) &= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \\
 &= (0.8)^6 + 6 \times (0.8)^5(0.2) + {}^6C_2(0.8)^4(0.2)^2 \\
 &= 0.9011
 \end{aligned}$$

**b i** Let  $X$  be the number of defectives.

$$\begin{aligned}
 \Pr(X = 2) &= \binom{6}{2} p^2 (1-p)^4 \\
 &= 15p^2(1-p)^4
 \end{aligned}$$

**ii** Let  $P = 15p^2(1-p)^4$

$$\begin{aligned}
 \frac{dP}{dp} &= 30p(1-p)^4 - 60p^2(1-p)^3 \\
 &= 30p[1-p]^3[1-3p]
 \end{aligned}$$

$$\frac{dP}{dp} = 0 \Rightarrow p = 1 \text{ or } p = \frac{1}{3} \text{ or } p = 0$$

Note when  $p = 1$  or  $p = 0$ ,  $P = 0$

$\therefore p = \frac{1}{3}$  gives a maximum probability.

$$\begin{aligned}
 \text{6 a Mean value} &= 1 \times \frac{53}{200} + 2 \times \frac{65}{200} + 3 \times \frac{45}{200} + 4 \times \frac{18}{200} + 5 \times \frac{2}{200} \\
 &= 2
 \end{aligned}$$

$$\text{b } np = 2, n = 6 \therefore p = \frac{1}{3}$$

**c**  $\therefore$  Probability distribution is as shown:

$x$	0	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{64}{729}$	$\frac{64}{243}$	$\frac{80}{243}$	$\frac{160}{729}$	$\frac{20}{243}$	$\frac{4}{243}$	$\frac{1}{729}$

Multiple the probabilities in the table to obtain the theoretical frequencies.

Theoretical frequencies are as follows:

$x$	0	1	2	3	4	5	6
Theoretical no. of occurrences	17.56	52.68	65.84	43.90	16.46	3.29	0.274

**7 a** Let  $X$  be the number of faulty articles in a sample of size 10.

Then  $X$  is Bi ( $n = 10$ ,  $p = 0.05$ )

$\Pr(\text{batch accepted after first sample}) = \Pr(X < 2)$

$$= \Pr(X = 0) \times \Pr(X = 1)$$

Using a CAS calculator given  $0.9138616 = 0.9139$  correct to 4 decimal places.

- b** Batch is rejected if 3 or more faulty articles or *if* there are exactly 2 faulty articles and then a second sample of size 10 contains any faulty articles.

$$\Pr(X \geq 3) = 0.0115036$$

$$\Pr(X = 2) = 0.0746348$$

$$\begin{aligned} \text{In a second sample, } \Pr(\geq 1 \text{ faulty articles}) &= 1 - \Pr(0 \text{ faulty articles}) \\ &= 1 - 0.95^{\circ} \\ &= 0.4012631 \end{aligned}$$

$$\begin{aligned} \Pr(\text{batch rejected}) &= 0.0115036 + 0.0766348 \times 0.4012631 \\ &= 0.0414517 \\ &= 0.04145 \text{ correct to 4 significant figures.} \end{aligned}$$

- c** Either 10 which are tested or, if 2 of the sample of 10 are faulty, a second 10 (giving a total of 20) are tested.

Let  $p' = \Pr(2 \text{ faulty article in first sample})$ , so

$$1 - p' = \Pr(0, 1, 3, \dots, 10 \text{ faulty articles in first sample}).$$

Then if  $y = \text{number of articles tested}$ , this gives:

$y$	10	20
$\Pr(y = q)$	$1 - p'$	$p'$

$$E(Y) = 10(1 - p') + 20p' = 10p' + 10$$

$$\text{From part b, } p' = 0.0746 \Rightarrow E(Y) = 10(0.0746) + 10 = 10.746$$

- 8 a** Let  $X$  be the number of people with a birthday in January.

$$\Pr(X = 2) = \binom{6}{2} \left(\frac{1}{12}\right)^2 \left(\frac{11}{12}\right)^4 = 0.0735$$

- b** Let  $Y$  be the number of people with a birthday in January.

$$\begin{aligned} \Pr(Y \geq 1) &= 1 - \Pr(Y = 0) = 1 - \left(\frac{11}{12}\right)^8 \\ &= 0.5015 \end{aligned}$$

- c** Let  $Z$  be the number of people with a birthday in January.

$$\Pr(Z \geq 1) = 1 - \Pr(Z = 0) = 1 - \left(\frac{11}{12}\right)^N$$

$$1 - \left(\frac{11}{12}\right)^N > 0.9$$

$$\Leftrightarrow \left(\frac{11}{12}\right)^N < 0.1$$

$$\Leftrightarrow N \log_e \left( \frac{11}{12} \right) < \log_e(0.1)$$

$$\Leftrightarrow N > \frac{\log_e(0.1)}{\log_e \left( \frac{11}{12} \right)}$$

$$\Leftrightarrow N > 26.46304$$

$\therefore$  Least value of  $N = 27$ .

### 9 For a two-engine plane

Let  $X$  be the number of engines which will fail.

The plane will successfully complete its journey if 0 or 1 engines fail.

$$\begin{aligned} \Pr(X = 0) + \Pr(X = 1) &= (1 - q)^2 + 2q(1 - q) \\ &= 1 - 2q + q^2 + 2q - 2q^2 \\ &= 1 - q^2 \end{aligned}$$

For a four-engine plane:

Let  $Y$  be the number of engines which will fail.

The plane will successfully complete its journey if 0, 1 or 2 engines fail.

$$\begin{aligned} \Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) &= (1 - q)^4 + 4q(1 - q)^3 + 6q^2(1 - q)^2 \\ &= (1 - q)^2[(1 - q)^2 + 4q(1 - q) + 6q^2] \\ &= (1 - q)^2[1 - 2q + q^2 + 4q - 4q^2 + 6q^2] \\ &= (1 - q)^2[1 + 2q + 3q^2] \end{aligned}$$

To find when a two-engine plane is to be preferred to a one-engine consider the inequality

$$\begin{aligned} 1 - q^2 &> (1 - q)^2(1 + 2q + 3q^2) \\ (1 - q)(1 + q) &> (1 - q)^2(1 + 2q + 3q^2) \\ \therefore (1 + q) &> (1 - q)(1 + 2q + 3q^2) \\ (1 + q) &> 1 + 2q + 3q^2 - q - 2q^2 - 3q^3 \\ 0 &> q^2 - 3q^3 \\ 0 &> q^2(1 - 3q) \\ \therefore \frac{1}{3} &\leq q \leq 1 \end{aligned}$$

A two-engine plane is to be preferred to a four-engine plane for  $\frac{1}{3} \leq q \leq 1$

# Chapter 15 – Continuous random variables and their probability distributions

## Solutions to Exercise 15A

$$1 \quad f(x) \begin{cases} \frac{24}{x^3} & 3 \leq x \leq 6 \\ 0 & x < 3 \text{ or } x > 6 \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_3^6 \frac{24}{x^3} dx \\ &= \left[ \frac{-12}{x^2} \right]_3^6 \\ &= \frac{-12}{6} + \frac{12}{3} \end{aligned}$$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$f(x) \geq 0$$

$\therefore$  is a probability density function

$$2 \quad \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_0^2 x^2 + kx + 1 dx = 1$$

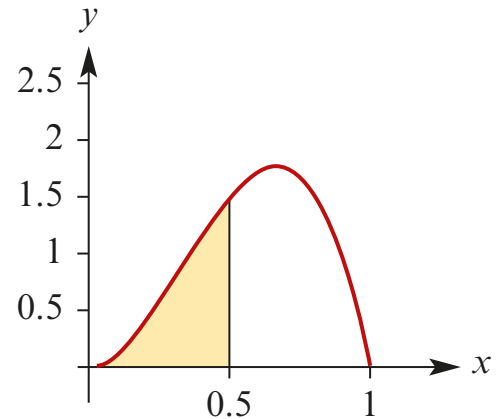
$$\left[ \frac{x^3}{3} + \frac{kx^2}{2} + x \right]_0^2 = 1$$

$$\frac{8}{3} + \frac{4k}{2} + 2 = 1$$

$$2k = \frac{-11}{3}$$

$$k = \frac{-11}{6}$$

3 a and c



b

$$\begin{aligned} \Pr(X < 0.5) &= \int_0^{0.5} 12x^2 - 12x^3 dx \\ &= \left[ 4x^3 - 3x^4 \right]_0^{0.5} \\ &= \frac{4}{8} - \frac{3}{16} \\ &= \frac{8}{16} - \frac{3}{16} \\ &= \frac{5}{16} \end{aligned}$$

$$4 \text{ a} \quad f(y) = \begin{cases} ke^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(y) dy = 1$$

$$\int_0^{\infty} ke^{-y} dy = 1$$

consider

$$\lim_{a \rightarrow \infty} \int_0^a ke^{-y} dy = 1$$

$$\lim_{a \rightarrow \infty} [-ke^{-y}]_0^a = 1$$

$$\lim_{a \rightarrow \infty} (-ke^{-a} + ke^0) = 1$$

$$-k \lim_{a \rightarrow \infty} (e^{-a}) + k = 1$$

$$k = 1$$

$$b \quad f(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

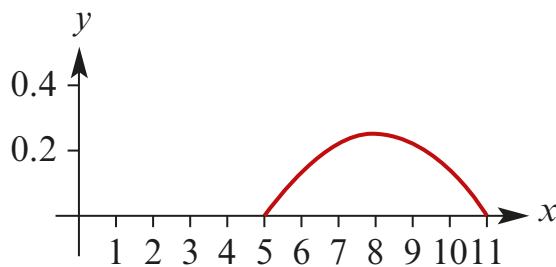
$$\Pr(Y \leq 2) = \int_0^2 e^{-y} dy$$

$$= [-e^{-y}]_0^2$$

$$= -e^{-2} + e^0$$

$$= 1 - \frac{1}{e^2} \approx 0.865$$

5 a



$$b \quad \Pr(T \leq 7) = \int_5^7 \frac{1}{36}(t-5)(11-t) dt$$

$$= \frac{-1}{36} \int_5^7 t^2 - 16t + 55 dt$$

$$= \frac{-1}{36} \left[ \frac{t^3}{3} - 8t^2 + 55t \right]_5^7$$

$$= \frac{1}{36} \left( \left( \frac{125}{3} - 200 + 275 \right) \right.$$

$$\left. - \left( \frac{343}{3} - 392 + 385 \right) \right)$$

$$= \frac{1}{36} \left( \frac{350}{3} - \frac{322}{3} \right)$$

$$= \frac{28}{108}$$

$$= \frac{7}{27} \approx 0.259$$

c

$$\Pr(T < 7 | T > 5.5) = \frac{\Pr(5.5 < T < 7)}{\Pr(T > 5.5)}$$

$$= \frac{\int_{5.5}^7 \frac{1}{36}(t-5)(11-t) dt}{\int_{5.5}^{11} \frac{1}{36}(t-5)(11-t) dt}$$

$$\approx 0.244$$

d

$$\Pr(T < 7 | T < 10) = \frac{\Pr(T < 7)}{\Pr(T < 10)}$$

$$= \frac{\int_5^7 \frac{1}{36}(t-5)(11-t) dt}{\int_5^{10} \frac{1}{36}(t-5)(11-t) dt}$$

$$\approx 0.28$$

6 a

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_7^{17} k \sin\left(\frac{1}{10}\pi(x-7)\right) dx = 1$$

$$\left[\frac{-10}{\pi}k \cos\left(\frac{\pi}{10}(x-7)\right)\right]_7^{17} = 1$$

$$\frac{-10}{\pi}k \cos(\pi) + \frac{10}{\pi}k \cos(0) = 1$$

$$\frac{10}{\pi}k + \frac{10}{\pi}k = 1$$

$$k = \frac{\pi}{20} \quad QED$$

b i

$$\Pr(X \geq 16) = \int_7^{17} \frac{\pi}{20} \sin\left(\frac{\pi}{10}(x-7)\right) dx$$

$$= \left[\frac{-1}{2} \cos\left(\frac{\pi}{10}(x-7)\right)\right]_7^{17}$$

$$= \frac{-1}{2} \cos(\pi) + \frac{1}{2} \cos\left(\frac{9\pi}{10}\right)$$

$$\approx 0.024$$

ii  $\Pr(12 \leq X \leq 13)$

$$\int_{12}^{13} \frac{\pi}{20} \sin\left(\frac{\pi}{10}(x-7)\right) dx$$

$$= \left[\frac{-1}{2} \cos\left(\frac{\pi}{10}(x-7)\right)\right]_{12}^{13}$$

$$= \frac{-1}{2} \cos\left(\frac{\pi}{10}\right) + \frac{1}{2} \cos\left(\frac{5\pi}{10}\right)$$

$$= \frac{-1}{2} \cos\left(\frac{3\pi}{5}\right)$$

$$\approx 0.155$$

7 a

$$\int_{-\infty}^{\infty} f(t) dt = 1$$

$$\int_0^{\infty} k e^{\left(\frac{-t}{200}\right)} dt = 1$$

Consider

$$\lim_{a \rightarrow \infty} \int_0^a k e^{\left(\frac{-t}{200}\right)} dt = 1$$

$$k \times \lim_{a \rightarrow \infty} \left[-200 e^{\left(\frac{-t}{200}\right)}\right]_0^a = 1$$

$$k \times \lim_{a \rightarrow \infty} \left(-200 e^{\left(\frac{-a}{200}\right)} + 200 e^0\right) = 1$$

$$200k \times \lim_{a \rightarrow \infty} \left(1 - e^{\left(\frac{-a}{200}\right)}\right) = 1$$

$$200k = 1$$

$$k = \frac{1}{200}$$

$$= 0.005$$

$$\text{b } \Pr(T \geq 1000) = \int_{1000}^{\infty} \frac{1}{200} e^{\left(\frac{-t}{200}\right)} dt$$

Consider

$$\lim_{a \rightarrow \infty} \int_{1000}^a \frac{1}{200} e^{\left(\frac{-t}{200}\right)} dt$$

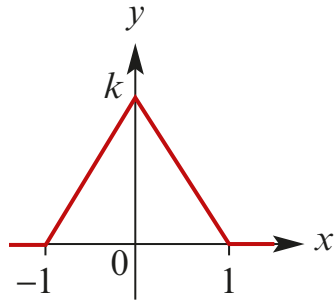
$$= \lim_{a \rightarrow \infty} \left[-e^{\left(\frac{-t}{200}\right)}\right]_{1000}^a$$

$$= \lim_{a \rightarrow \infty} \left(-e^{\left(\frac{-a}{200}\right)} + e^{-5}\right)$$

$$= \frac{1}{e^5}$$

$$\therefore \Pr(T \geq 1000) = \frac{1}{e^5} \approx 0.007$$

8 a



b 
$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_{-1}^0 k(1+x) dx + \int_0^1 k(1-x) dx = 1$$

$$\left[ x + \frac{x^2}{2} \right]_{-1}^0 + \left[ x - \frac{x^2}{2} \right]_0^1 = \frac{1}{k}$$

$$0 - \left( -1 + \frac{1}{2} \right) + \left( 1 - \frac{1}{2} \right) - 0 = \frac{1}{k}$$

$$\frac{1}{2} + \frac{1}{2} = \frac{1}{k}$$

$$k = 1$$

c 
$$\Pr\left(-\frac{1}{2} \leq X \leq \frac{1}{2}\right)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) dx$$

$$= \int_{-\frac{1}{2}}^0 1+x dx + \int_0^{\frac{1}{2}} 1-x dx$$

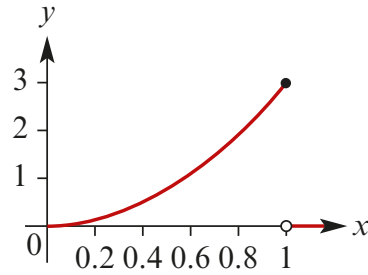
$$= \left[ x + \frac{x^2}{2} \right]_{-\frac{1}{2}}^0 + \left[ x - \frac{x^2}{2} \right]_0^{\frac{1}{2}}$$

$$= 0 - \left( -\frac{1}{2} + \frac{1}{8} \right) + \left( \frac{1}{2} - \frac{1}{8} \right) - 0$$

$$= \frac{3}{8} + \frac{3}{8}$$

$$= \frac{3}{4}$$

9 a



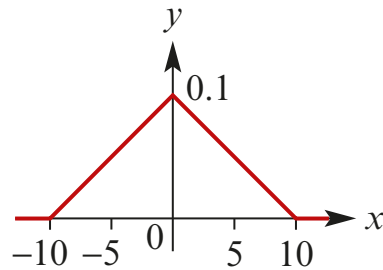
b 
$$\Pr(0.25 < X < 0.75) = \int_{\frac{1}{4}}^{\frac{3}{4}} 3x^2 dx$$

$$= \left[ x^3 \right]_{\frac{1}{4}}^{\frac{3}{4}}$$

$$= \frac{27}{64} - \frac{1}{64}$$

$$= \frac{13}{32} \approx 0.406$$

10 a





**b**  $\Pr(-1 \leq X < 1)$

$$\begin{aligned}
 &= \int_{-1}^1 f(x) dx \\
 &= \int_{-1}^0 \frac{1}{100}(10+x) dx \\
 &\quad + \int_0^1 \frac{1}{100}(10-x) dx \\
 &= \frac{1}{100} \left[ 10x + \frac{x^2}{2} \right]_{-1}^0 \\
 &\quad + \frac{1}{100} \left[ 10x - \frac{x^2}{2} \right]_0^1 \\
 &= \frac{1}{100} \left( 0 - \left( -10 + \frac{1}{2} \right) \right) \\
 &\quad + \left( 10 - \frac{1}{2} \right) - 0 \\
 &= \frac{19}{100} = 0.19
 \end{aligned}$$

**11 a**  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\int_{1000}^{\infty} \frac{k}{x^2} dx = 1$$

Consider

$$\lim_{a \rightarrow \infty} \int_{1000}^a \frac{k}{x^2} dx = 1$$

$$\lim_{a \rightarrow \infty} \left[ \frac{-k}{x} \right]_{1000}^a = 1$$

$$\lim_{a \rightarrow \infty} \left( \frac{-k}{a} + \frac{k}{1000} \right) = 1$$

$$\frac{k}{1000} = 1$$

$$k = 1000$$

**b**  $\Pr(X \geq 2000) = \int_{2000}^{\infty} \frac{1000}{x^2} dx$

Consider

$$\begin{aligned}
 \lim_{a \rightarrow \infty} \int_{2000}^a \frac{1000}{x^2} dx &= \lim_{a \rightarrow \infty} \left[ \frac{-1000}{x} \right]_{2000}^a \\
 &= \lim_{a \rightarrow \infty} \left( \frac{-1000}{a} + \frac{1000}{2000} \right) \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\therefore \Pr(X \geq 2000) = \frac{1}{2}$$

**12 a**  $\Pr(X \geq 1.5) = \int_{1.5}^{\infty} f(x) dx$

$$= \int_{1.5}^2 2 \left( 1 - \frac{1}{x^2} \right) dx$$

$$= \left[ 2 \left( x + \frac{1}{x} \right) \right]_{1.5}^2$$

$$= 2 \left( \left( 2 + \frac{1}{2} \right) - \left( \frac{3}{2} + \frac{2}{3} \right) \right)$$

$$= \frac{2}{3}$$

**b**  $\Pr(X \leq 1.8 | X \geq 1.5)$

$$= \frac{\Pr(1.5 \leq X \leq 1.8)}{\Pr(X \geq 1.5)}$$

$$= \frac{\int_{1.5}^{1.8} 2 \left( 1 - \frac{1}{x^2} \right) dx}{\frac{2}{3}}$$

$$= \frac{3}{2} \left[ 2 \left( x + \frac{1}{x} \right) \right]_{1.5}^{1.8}$$

$$= 3 \left( \left( \frac{9}{5} + \frac{5}{9} \right) - \left( \frac{3}{2} + \frac{2}{3} \right) \right)$$

$$= \frac{17}{30}$$

$$\begin{aligned} \mathbf{13\ a} \quad \Pr(X \geq 8) &= \int_8^{\infty} f(x) dx \\ &= \int_8^{\infty} \frac{1}{5} e^{\frac{-x}{5}} dx \end{aligned}$$

Consider

$$\begin{aligned} \lim_{a \rightarrow \infty} \int_8^a \frac{1}{5} e^{\frac{-x}{5}} dx &= \lim_{a \rightarrow \infty} \left[ -e^{\frac{-x}{5}} \right]_8^a \\ &= \lim_{a \rightarrow \infty} \left( -e^{\frac{-a}{5}} + e^{\frac{-8}{5}} \right) \\ &= e^{\frac{-8}{5}} \end{aligned}$$

$$\therefore \Pr(X \geq 8) = e^{\frac{-8}{5}} \approx 0.202$$

$$\begin{aligned} \mathbf{b} \quad \Pr(X \geq 12 | X \geq 8) &= \frac{\Pr(X \geq 12)}{\Pr(X \geq 8)} \\ &= \frac{\int_{12}^{\infty} \frac{1}{5} e^{\frac{-x}{5}} dx}{e^{\frac{-8}{5}}} \end{aligned}$$

Consider

$$\begin{aligned} \frac{\lim_{a \rightarrow \infty} \int_{12}^a \frac{1}{5} e^{\frac{-x}{5}} dx}{e^{\frac{-8}{5}}} &= \frac{\lim_{a \rightarrow \infty} \left[ -e^{\frac{-x}{5}} \right]_{12}^a}{e^{\frac{-8}{5}}} \\ &= e^{\frac{8}{5}} \lim_{a \rightarrow \infty} \left( -e^{\frac{-a}{5}} + e^{\frac{-12}{5}} \right) \\ &= e^{\frac{8}{5}} \times e^{\frac{-12}{5}} \\ &= e^{\frac{-4}{5}} \end{aligned}$$

$$\therefore \Pr(x \geq 12 | X \geq 8) = e^{\frac{-4}{5}} \approx 0.449$$

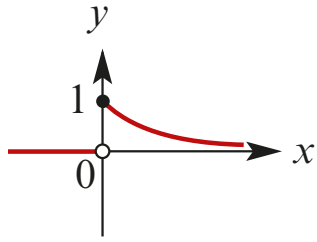
$$\mathbf{14\ a} \quad \Pr(X \leq 0.5 | = 0.45)$$

$$\begin{aligned} &\int_{-\infty}^{0.5} f(x) dx \\ &= \int_{-1}^0 0.2 dx + \int_1^{0.5} 0.2 + 1.2x dx \\ &= \frac{1}{5} + \left[ \frac{x}{5} + \frac{3x^2}{5} \right]_0^{\frac{1}{2}} \\ &= \frac{1}{5} + \left( \frac{1}{10} + \frac{3}{20} \right) \\ &= \frac{9}{20} = 0.45 \end{aligned}$$

$$\mathbf{b} \quad \Pr(X > 0.5 | X > 0.1)$$

$$\begin{aligned} &= \frac{\Pr(X > 0.5)}{\Pr(X > 0.1)} \\ &= \frac{1 - (a)}{\int_{0.1}^{\infty} f(x) dx} \\ &= \frac{0.55}{\int_{0.1}^1 \frac{1}{5} + \frac{6}{5}x dx} \\ &= \frac{11}{4 \int_{0.1}^1 1 + 6x dx} \\ &= \frac{11}{4 \left[ x + 3x^2 \right]_{0.1}^1} \\ &= \frac{11}{4 \left( (1 + 3) - \left( \frac{1}{10} + \frac{3}{100} \right) \right)} \\ &= \frac{11}{\frac{387}{25}} \\ &= \frac{275}{387} \approx 0.711 \end{aligned}$$

15 a



$$\begin{aligned}
 \text{b i } \Pr(X < 0.5) &= \int_0^{0.5} e^{-x} dx \\
 &= [-e^{-x}]_0^{0.5} \\
 &= -e^{-0.5} + e^0 \\
 &= 1 - \frac{1}{\sqrt{e}} \\
 &= 1 - e^{-\frac{1}{2}}
 \end{aligned}$$

ii  $\Pr(X \geq 1)$

$$\int_1^{\infty} e^{-x} dx$$

Consider

$$\begin{aligned}
 \lim_{a \rightarrow \infty} \int_1^a e^{-x} dx &= \lim_{a \rightarrow \infty} [-e^{-x}]_1^a \\
 &= \lim_{a \rightarrow \infty} (-e^{-a} + e^{-1}) \\
 &= \frac{1}{e}
 \end{aligned}$$

$$\therefore \Pr(X \geq 1) = \frac{1}{e} = e^{-1}$$

iii

$$\begin{aligned}
 \Pr(X \geq 1 | X > 0.5) &= \frac{\Pr(X \geq 1)}{\Pr(X > 0.5)} \\
 &= \frac{\frac{1}{e}}{\frac{1}{\sqrt{e}}} \\
 &= \frac{1}{\sqrt{e}} \\
 &= e^{-\frac{1}{2}}
 \end{aligned}$$

## Solutions to Exercise 15B

**1 a**  $f(x) = 2x, 0 < x < 1$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x) dx \\ &= \int_0^1 2x^2 dx \\ &= \left[ \frac{2}{3}x^3 \right]_0^1 \\ &= \frac{2}{3} \end{aligned}$$

**b**  $f(x) = \frac{1}{2\sqrt{x}}, 0 < x < 1$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x) dx \\ &= \int_0^1 \frac{\sqrt{x}}{2} dx \\ &= \left[ \frac{1}{3}x^{\frac{3}{2}} \right]_0^1 \\ &= \frac{1}{3} \end{aligned}$$

**c**  $f(x) = 6x - 6x^2, 0 < x < 1$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x) dx \\ &= \int_0^1 6x^2 - 6x^3 dx \\ &= \left[ 2x^3 - \frac{3}{2}x^4 \right]_0^1 \\ &= 2 - \frac{3}{2} = \frac{1}{2} \end{aligned}$$

**d**  $f(x) = \frac{1}{x^2}, x \geq 1$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x) dx \\ &= \int_0^1 \frac{1}{x} dx \end{aligned}$$

consider

$$\begin{aligned} \lim_{a \rightarrow \infty} \int_1^a \frac{1}{x} dx &= \lim_{a \rightarrow \infty} [\ln |x|]_1^a \\ &= \lim_{a \rightarrow \infty} (\ln a - \ln 1) \\ &= \lim_{a \rightarrow \infty} \ln a \end{aligned}$$

$\therefore E(X)$  does not exist

**2 a 1**

**b** 2.097

**c** 1.132

**d** 0.4444

**3 a**  $\mu = \int_{-\infty}^{\infty} xf(x) dx$

$$\begin{aligned} &= \int_0^1 2x^4 - x^2 + x dx \\ &= \left[ \frac{2}{5}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_0^1 \\ &= \frac{2}{5} - \frac{1}{3} + \frac{1}{2} \\ &= \frac{12 - 10 + 15}{30} \\ &= \frac{17}{30} \approx 0.567 \end{aligned}$$

**b**  $\Pr(X \leq \mu) = \int_{-\infty}^{\mu} f(x) dx$

$$\begin{aligned} &= \int_0^{\frac{17}{30}} 2x^3 - x + 1 dx \\ &= \left[ \frac{1}{2}x^4 - \frac{1}{2}x^2 + x \right]_0^{\frac{17}{30}} \\ &= \frac{1}{2} \left( \frac{17}{30} \right)^4 - \frac{1}{2} \left( \frac{17}{30} \right)^2 + \frac{17}{30} \\ &\approx 0.458 \end{aligned}$$

$$4 \quad f(x) = \frac{1}{2\pi} + \frac{1}{2\pi} \cos x, \quad -\pi \leq x \leq \pi$$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x) dx \\ &= \int_{-\pi}^{\pi} \frac{x}{2\pi} + \frac{x \cos x}{2\pi} dx \end{aligned}$$

using the CAS calculator = 0

Alternatively, notice that the function is symmetrical about the y-axis, so the mean must be 0.

$$5 \quad \int_{-\infty}^{\infty} f(y) dy = 1$$

$$\int_0^B Ay dy = 1$$

$$\left[ \frac{A}{2} y^2 \right]_0^B = 1$$

$$\frac{AB^2}{2} = 1$$

$$\textcircled{1} \quad AB^2 = 2$$

$$\mu = \int_{-\infty}^{\infty} yf(y) dy$$

$$2 = \int_0^B Ay^2 dy$$

$$2 = \left[ \frac{A}{3} y^3 \right]_0^B$$

$$\textcircled{2} \quad AB^3 = 6$$

$$\frac{\textcircled{2}}{\textcircled{1}} \Rightarrow B = 3$$

$$\text{Sub in } \textcircled{1} \Rightarrow A(3)^2 = 2$$

$$A = \frac{2}{9}$$

$$\begin{aligned} 6 \text{ a} \quad E\left(\frac{1}{X}\right) &= \int_{-\infty}^{\infty} \frac{1}{x} f(x) dx \\ &= \int_0^1 12x - 12x^2 dx \\ &= [6x^2 - 4x^3]_0^1 \\ &= 6 - 4 = 2 \end{aligned}$$

$$\begin{aligned} 6 \text{ b} \quad E(e^x) &= \int_{-\infty}^{\infty} e^x f(x) dx \\ &= \int_0^1 12x^2 e^x (1-x) dx \end{aligned}$$

using the CAS calculator = 1.858

$$7 \text{ a} \quad \Pr(X \leq 1)$$

$$\begin{aligned} \int_0^1 e^{-x} dx &= [-e^{-x}]_0^1 \\ &= -e^{-1} + e^{-0} \\ &= 1 - \frac{1}{e} \approx 0.632 \end{aligned}$$

$$\begin{aligned} 7 \text{ b} \quad \Pr(1 \leq X \leq 2) &= \int_1^2 e^{-x} dx \\ &= [-e^{-x}]_1^2 \\ &= -e^{-2} + e^{-1} \\ &= \frac{e-1}{e^2} \approx 0.233 \end{aligned}$$

$$7 \text{ c} \quad \int_0^m e^{-x} dx = \frac{1}{2}$$

$$[-e^{-x}]_0^m = \frac{1}{2}$$

$$-e^{-m} + e^0 = \frac{1}{2}$$

$$1 - e^{-m} = \frac{1}{2}$$

$$e^{-m} = \frac{1}{2}$$

$$e^m = 2$$

$$m = \ln 2 \approx 0.693$$

$$8 \text{ a } \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_0^1 k dx = 1$$

$$[kx]_0^1 = 1$$

$$k = 1$$

$$8 \text{ b } \int_0^m 1 dx = \frac{1}{2}$$

$$[m]_0^1 = \frac{1}{2}$$

$$m = \frac{1}{2}$$

$$9 \text{ } f(x) = 5(x-1)^4, 0 \leq x \leq 1$$

$$\int_0^m 5(x-1)^4 dx = \frac{1}{2}$$

$$[(x-1)^5]_0^m = \frac{1}{2}$$

$$(m-1)^5 - (-1)^5 = \frac{1}{2}$$

$$(m-1)^5 = -\frac{1}{2}$$

$$m-1 = \left(\frac{1}{2}\right)^{\frac{1}{5}}$$

$$m = 1 - \left(\frac{1}{2}\right)^{\frac{1}{5}} \approx 0.1294$$

$$10 \int_0^m \frac{1}{4} e^{\frac{-x}{4}} dx = \frac{1}{2}$$

$$\left[-e^{\frac{-x}{4}}\right]_0^m = \frac{1}{2}$$

$$-e^{\frac{-m}{4}} + e^0 = \frac{1}{2}$$

$$e^{\frac{-m}{4}} = \frac{1}{2}$$

$$e^{\frac{m}{4}} = 2$$

$$\frac{m}{4} = \ln 2$$

$$m = 4 \ln 2 \approx 2.773 \text{ minutes}$$

$$11 \text{ a } \mu = \int_{-\infty}^{\infty} xf(x) dx$$

$$= \int_0^1 x^2 dx + \int_1^2 2x - x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)$$

$$= \frac{1}{3} + 4 - \frac{10}{3}$$

$$= 1$$

$$11 \text{ b } \int_0^m f(x) dx = \frac{1}{2}$$

if  $m \leq 1$ ,

$$\int_0^m x dx = \frac{1}{2}$$

$$\left[\frac{x^2}{2}\right]_0^m = \frac{1}{2}$$

$$\frac{m^2}{2} = \frac{1}{2}$$

$$m^2 = 1$$

$$m = 1$$

**12**  $f(x) = 30x^4 - 30x^5, 0 < x < 1$

**a** 
$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} xf(x) dx \\ &= \int_0^1 30x^5 - 30x^6 dx \\ &= \left[ 5x^6 - \frac{30}{7}x^7 \right]_0^1 \\ &= 5 - \frac{30}{7} \\ \mu &= \frac{5}{7} \approx 0.714 \end{aligned}$$

**b** 
$$\begin{aligned} \int_{-\infty}^m f(x) dx &= \frac{1}{2} \\ \int_0^m 30x^4 - 30x^5 dx &= \frac{1}{2} \\ \left[ 6x^5 - 5x^6 \right]_0^m &= \frac{1}{2} \\ 6m^5 - 5m^6 &= \frac{1}{2} \end{aligned}$$
  
*using the CAS calculator*  
 $m \approx 0.736$   
 $\mu = 0.714 < m$  QED

**13**  $f(x) = \frac{\pi}{20} \sin\left(\frac{\pi}{10}(x-7)\right), 7 \leq x \leq 17$

$$\int_{-\infty}^m f(x) dx = \frac{1}{2}$$

$$\int_7^m \frac{\pi}{20} \sin\left(\frac{\pi}{10}(x-7)\right) dx = \frac{1}{2}$$

$$\left[ \frac{-1}{2} \cos\left(\frac{\pi}{10}(x-7)\right) \right]_7^m = \frac{1}{2}$$

$$-\cos\left(\frac{\pi}{10}(m-7)\right) + \cos 0 = 1$$

$$\cos\left(\frac{\pi}{10}(m-7)\right) = 0$$

$$\frac{\pi}{10}(m-7) = \frac{\pi}{2}$$

$$m-7 = 5$$

$$m = 12$$

**14 a** 
$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} xf(x) dx \\ &= \int_{-1}^0 \frac{x}{5} dx + \int_0^1 \frac{x}{5} + \frac{6x^2}{5} dx \\ &= \left[ \frac{x^2}{10} \right]_{-1}^0 + \left[ \frac{x^2}{10} + \frac{2x^3}{5} \right]_0^1 \\ &= 0 - \frac{1}{10} + \frac{1}{10} + \frac{2}{5} - 0 \\ &= \frac{2}{5} \end{aligned}$$

**b** 
$$\int_{-\infty}^m f(x) dx = \frac{1}{2}$$

if  $m \leq 0$ ,

$$\int_{-1}^m \frac{1}{5} dx = \frac{1}{2}$$

$$\left[ \frac{x}{5} \right]_{-1}^m = \frac{1}{2}$$

$$\frac{m}{5} + \frac{1}{5} = \frac{1}{2}$$

$m > 0$

$$\begin{aligned} \therefore \int_{-1}^0 \frac{1}{5} dx + \int_0^m \left( \frac{1}{5} + \frac{6}{5}x \right) dx &= \frac{1}{2} \\ \frac{1}{5} + \left[ \frac{x}{5} + \frac{3x^2}{5} \right]_0^m &= \frac{1}{2} \\ \frac{m}{5} + \frac{3m^2}{5} - \frac{3}{10} &= 0 \\ 3m^2 + m - \frac{3}{2} &= 0 \\ m &= \frac{-1 \pm \sqrt{1+18}}{6} \end{aligned}$$

since  $m > 0$ ,

$$m = \frac{-1 + \sqrt{19}}{6}$$

**15 a**  $\frac{d}{dx}(kxe^{-kx}) = ke^{-kx} - k^2xe^{-kx}$

$$\begin{aligned} \int kxe^{-kx} dx &= \frac{-1}{k} \int -k^2xe^{-kx} \\ &\quad + ke^{-kx} - ke^{-kx} dx \\ &= \frac{-1}{k} \int ke^{-kx} \\ &\quad - k^2xe^{-kx} dx \\ &\quad + \int e^{-kx} dx \\ &= \frac{-1}{k}(kxe^{-kx}) + \frac{-1}{k}e^{-kx} \\ &= -xe^{-kx} - \frac{-1}{k}e^{-kx} \\ &= -\frac{(kx+1)}{k}e^{-kx} \end{aligned}$$

**b**  $\mu = \int_{-\infty}^{\infty} xf(x) dx$

$$= \int_0^{\infty} \frac{x}{\lambda} e^{-\frac{x}{\lambda}} dx$$

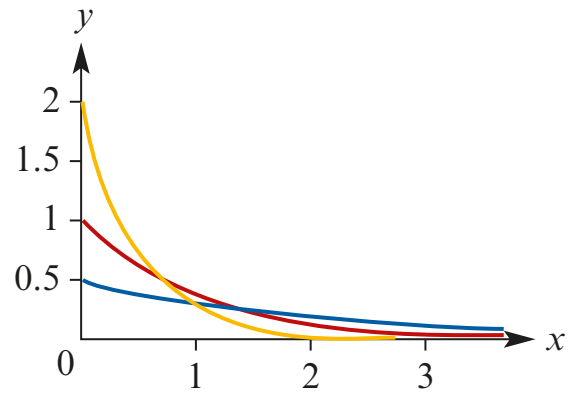


Consider

$$\begin{aligned} & \lim_{a \rightarrow \infty} \int_1^a x \left(\frac{1}{\lambda}\right)^x e^{-\left(\frac{1}{\lambda}\right)^x} dx \\ a &= \lim_{a \rightarrow \infty} \left[ x e^{-\frac{1}{\lambda}x} - \lambda e^{-\frac{1}{\lambda}x} \right]_0^a \\ &= \lim_{a \rightarrow \infty} \left( -e^{-\frac{a}{\lambda}}(a + \lambda) + e^0(0 + \lambda) \right) \\ &= \lambda + \lim_{a \rightarrow \infty} \left( -(a + \lambda) - e^{-\frac{a}{\lambda}} \right) \end{aligned}$$

$$\mu = \lambda$$

**c**



**d**  $y = e^{-x}$  is dilated by factor  $\frac{1}{\lambda}$  from the  $x$ -axis and by factor  $\lambda$  from the  $y$ -axis

## Solutions to Exercise 15C

$$\begin{aligned}
 \mathbf{1} \quad E(X) &= \int_{-\infty}^{\infty} xf(x) dx \\
 &= \int_0^1 2x^2 dx \\
 &= \left[ \frac{2}{3}x^3 \right]_0^1 \\
 &= \frac{2}{3}
 \end{aligned}$$

$$(E(X))^2 = \frac{4}{9}$$

$$\begin{aligned}
 E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
 &= \int_0^1 2x^3 dx \\
 &= \left[ \frac{1}{2}x^4 \right]_0^1 \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\sigma^2 = E(X^2) - (E(X))^2$$

$$= \frac{9}{18} - \frac{8}{18} = \frac{1}{18}$$

$$\sigma = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$$

**2** (384, 416)

$$\mathbf{3} \quad \mathbf{a} \quad \int_{-\infty}^a xf(x) dx = \frac{1}{4}$$

$$\int_0^a 3x^2 dx = \frac{1}{4}$$

$$[x^3]_0^a = \frac{1}{4}$$

$$a^3 = \frac{1}{4}$$

$$a = \left(\frac{1}{4}\right)^{\frac{1}{3}} \approx 0.630$$

$$\mathbf{b} \quad \int_{-\infty}^b f(x) dx = \frac{3}{4}$$

$$\int_0^b 3x^2 dx = \frac{3}{4}$$

$$[x^3]_0^b = \frac{3}{4}$$

$$b^3 = \frac{3}{4}$$

$$b = \left(\frac{3}{4}\right)^{\frac{1}{3}} \approx 0.909$$

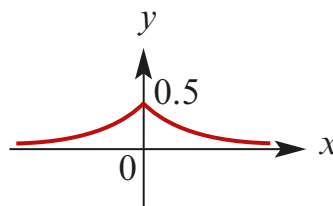
**c**

the interquartile range =  $b - a$

$$= \left(\frac{3}{4}\right)^{\frac{1}{3}} - \left(\frac{1}{4}\right)^{\frac{1}{3}}$$

$$\approx 0.279$$

**4 a**



$$\mathbf{b} \quad \int_{-\infty}^a f(x) dx = \frac{1}{4}$$

$$\int_{-\infty}^a 0.5e^x dx = \frac{1}{4}, \text{ since } a < 0.$$

consider

$$\lim_{k \rightarrow -\infty} \int_k^a \frac{1}{2} e^x dx = \frac{1}{4}$$

$$\lim_{k \rightarrow -\infty} [e^x]_k^a = \frac{1}{2}$$

$$\lim_{k \rightarrow -\infty} (e^a - e^k) = \frac{1}{2}$$

$$e^a = \frac{1}{2}$$

$$a = \ln \frac{1}{2}$$

$$a = -\ln 2$$

$$\int_b^\infty \frac{1}{2} e^{-x} dx = \frac{1}{4}, \text{ since } b > 0$$

$$\int_b^\infty e^{-x} dx = \frac{1}{2}$$

consider

$$\lim_{h \rightarrow \infty} \int_b^h e^{-x} dx = \frac{1}{2}$$

$$\lim_{h \rightarrow \infty} [-e^{-x}]_b^h = \frac{1}{2}$$

$$\lim_{h \rightarrow \infty} (-e^{-h} + e^{-b}) = \frac{1}{2}$$

$$e^{-b} = \frac{1}{2}$$

$$-b = \ln \frac{1}{2}$$

$$b = \ln 2$$

the interquartile range =  $b - a$

$$= \ln 2 - (-\ln 2)$$

$$= 2 \ln 2 \approx 1.386$$

$$5 \text{ a } \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_1^9 \frac{k}{x} dx = 1$$

$$[k \ln x]_1^9 = 1$$

$$k \ln 9 - k \ln 1 = 1$$

$$k \ln 9 = 1$$

$$k = \frac{1}{\ln 9}$$

$$b \quad \mu = \int_{-\infty}^{\infty} xf(x) dx$$

$$\int_1^9 \frac{1}{\ln 9} dx = \frac{8}{\ln 9} = \frac{4}{\ln 3} \approx 3.641$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_1^9 \frac{x}{\ln 9} dx$$

$$= \left[ \frac{x^2}{4 \ln 3} \right]_1^9$$

$$= \frac{81}{4 \ln 3} - \frac{1}{4 \ln 3}$$

$$= \frac{20}{\ln 3}$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$= \frac{20}{\ln 3} - \frac{16}{(\ln 3)^2}$$

$$= \frac{20 \ln 3 - 16}{(\ln 3)^2} \approx 4.948$$

$$\begin{aligned}
\mathbf{6 a} \quad \int_{-\infty}^a f(x) dx &= \frac{1}{4} \\
\int_0^a 2 - 2x dx &= \frac{1}{4} \\
[2x - x^2]_0^a &= \frac{1}{4} \\
2a - a^2 &= \frac{1}{4} \\
a^2 - 2a + \frac{1}{4} &= 0 \\
a &= \frac{2 \pm \sqrt{4-1}}{2} \\
a &= 1 - \frac{\sqrt{3}}{2}, \\
&\text{since } 0 \leq a \leq 1
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^b f(x) dx &= \frac{3}{4} \\
\int_0^b 2 - 2x dx &= \frac{3}{4} \\
[2x - x^2]_0^b &= \frac{3}{4} \\
2b - b^2 &= \frac{3}{4} \\
b^2 - 2b + \frac{3}{4} &= 0 \\
b &= \frac{2 \pm \sqrt{4-3}}{2} \\
b &= 1 \pm \frac{1}{2} \\
b &= \frac{1}{2}, \text{ since } 0 \leq b \leq 1 \\
&= b - a \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2} - 1 \\
&= \frac{\sqrt{3}}{2} - \frac{1}{2} \\
&\approx 0.366
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad \mu &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_0^1 2x - 2x^2 dx \\
&= \left[ x^2 - \frac{2}{3}x^3 \right]_0^1 \\
&= 1 - \frac{2}{3} \\
&= \frac{1}{3} \\
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
&= \int_0^1 2x^2 - 2x^3 dx \\
&= \left[ \frac{2}{3}x^3 - \frac{1}{2}x^4 \right]_0^1 \\
&= \frac{2}{3} - \frac{1}{2} \\
&= \frac{1}{6} \\
\sigma^2 &= E(X^2) - \mu^2 \\
&= \frac{1}{6} - \frac{1}{9} \\
&= \frac{1}{18}
\end{aligned}$$

$$\begin{aligned}
\mathbf{7} \quad \int_{-\infty}^a f(x) dx &= \frac{1}{4} \\
\int_0^a 2xe^{-x^2} dx &= \frac{1}{4} \\
\left[ e^{-x^2} \right]_0^a &= \frac{1}{4} \\
-e^{-a^2} + e^0 &= \frac{1}{4}
\end{aligned}$$

$$e^{-a^2} = \frac{3}{4}$$

$$-a^2 = \ln \frac{3}{4}$$

$$a = +\sqrt{\ln \frac{4}{3}} \approx 0.5364,$$

*since*  $a > 0$

$$\int_{-\infty}^b f(x) dx = \frac{3}{4}$$

$$\int_0^b 2xe^{-x^2} dx = \frac{3}{4}$$

$$\left[ e^{-x^2} \right]_0^b = \frac{3}{4}$$

$$e^{-b^2} + e^0 = \frac{3}{4}$$

$$e^{-b^2} = \frac{1}{4}$$

$$-b^2 = \ln \frac{1}{4}$$

$$b = +\sqrt{\ln 4} \approx 1.1774,$$

*since*  $b > 0$

*the interquartile range* =  $b - a$

$$\approx 0.641$$

**8 a**  $\int_{-\infty}^a f(x) dx = \frac{1}{4}$

$$\int_0^a \frac{x}{2} dx = \frac{1}{4}$$

$$\int_0^a 2x dx = 1$$

$$\left[ x^2 \right]_0^a = 1$$

$$a^2 = 1$$

$a = 1$ , *since*  $0 \leq a \leq 2$

$$\int_{-\infty}^b f(x) dx = \frac{3}{4}$$

$$\int_0^b \frac{x}{2} dx = \frac{3}{4}$$

$$\int_0^b 2x dx = 3$$

$$\left[ x^2 \right]_0^b = 3$$

$$b^2 = 3$$

$b = \sqrt{3}$ , *since*  $0 \leq b \leq 2$

*the interquartile range* =  $b - a$

$$= \sqrt{3} - 1$$

$$\approx 0.732$$

**b**  $\mu = \int_{-\infty}^{\infty} x f(x) dx$

$$= \int_0^2 \frac{x^2}{2} dx$$

$$= \left[ \frac{x^3}{6} \right]_0^2$$

$$= \frac{8}{6} = \frac{4}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^2 \frac{x^3}{3} dx$$

$$= \left[ \frac{x^4}{8} \right]_0^2$$

$$= \frac{16}{8} = 2$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$= 2 - \frac{16}{9} = \frac{2}{9}$$

$$\begin{aligned}
 \mathbf{9 a} \quad & \int_{-\infty}^{\infty} f(x) dx = 1 \\
 & \int_0^{10} kx(100 - x^2) dx = 1 \\
 & k \int_0^{10} 100x - x^3 dx = 1 \\
 & k \left[ 50x^2 - \frac{x^4}{4} \right]_0^{10} = 1 \\
 & k(5000 - 2500) = 1 \\
 & k = \frac{1}{2500} \\
 & = 0.0004
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \mu &= \int_{-\infty}^{\infty} x f(x) dx \\
 &= \int_0^{10} \frac{100x^2 - x^4}{2500} dx \\
 &= \frac{1}{2500} \left[ \frac{100}{3}x^3 - \frac{x^5}{5} \right]_0^{10} \\
 &= \frac{1}{2500} \left( \frac{100000}{3} - \frac{100000}{5} \right) \\
 &= \frac{1}{25} \left( \frac{2000}{15} \right) \\
 &= \frac{80}{15} \\
 &= \frac{16}{3}
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
 &= \int_0^{10} \frac{1}{2500} (100x^3 - x^5) dx \\
 &= \frac{1}{2500} \left[ 25x^4 - \frac{1}{6}x^6 \right]_0^{10} \\
 &= \frac{1}{2500} \left( 250000 - \frac{1000000}{6} \right) \\
 &= \frac{1}{25} \left( \frac{15000 - 10000}{6} \right) \\
 &= \frac{5000}{150} \\
 &= \frac{100}{3} \\
 \sigma^2 &= E(X^2) - \mu^2 \\
 &= \frac{100}{3} - \frac{256}{9} \\
 &= \frac{44}{9} \\
 \sigma &= \frac{2\sqrt{11}}{3} \approx 2.21
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10 a} \quad & \int_{-\infty}^{\infty} f(x) dx = 1 \\
 & k \int_{-a}^a (a^2 - x^2) dx = 1 \\
 & k \left[ a^2x - \frac{x^3}{3} \right]_{-a}^a = 1 \\
 & k \left( \left( a^3 - \frac{a^3}{3} \right) - \left( -a^3 + \frac{a^3}{3} \right) \right) = 1 \\
 & \frac{4}{3}a^3k = 1 \\
 & k = \frac{3}{4a^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \mu &= \int_{-\infty}^{\infty} x f(x) dx \\
 &= \int_{-a}^a \frac{3x}{4a} - \frac{3x^3}{4a^3} dx \\
 &= \left[ \frac{3x^2}{8a} - \frac{3x^4}{16a^3} \right]_{-a}^a \\
 &= \left( \frac{3a^2}{8a} - \frac{3a^4}{16a^3} \right) - \left( \frac{3a^2}{8a} - \frac{3a^4}{16a^3} \right) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
 &= \int_{-a}^a \frac{3x^2}{4a} - \frac{3x^4}{4a^3} dx \\
 &= \left[ \frac{x^3}{4a} - \frac{3x^5}{20a^3} \right]_{-a}^a \\
 &= 2 \left( \frac{a^3}{4a} - \frac{3a^5}{20a^3} \right) \\
 &= 2 \left( \frac{a^2}{4} - \frac{3a^2}{20} \right) \\
 &= \frac{5a^2}{10} - \frac{3a^2}{10} \\
 &= \frac{a^2}{5} \\
 \sigma^2 &= E(X^2) - \mu^2 \\
 &= \frac{a^2}{5} - 0 \\
 &= \frac{a^2}{5}
 \end{aligned}$$

but  $\sigma = 2$

$$\therefore \sigma^2 = 4$$

$$\frac{a^2}{5} = 4$$

$$a^2 = 20$$

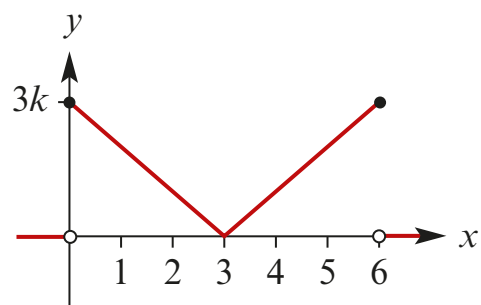
$$a = \pm \sqrt{20}$$

$$s^a = \pm 2\sqrt{5}$$

note: the way the question is stated implies that  $a$  is positive,

$$\therefore a = 2\sqrt{5}$$

**11 a**



**b**

$$\begin{aligned}
 \int_{-\infty}^{\infty} f(x) dx &= 1 \\
 \int_0^3 k(3-x) dx + \int_3^6 k(3-x) dx &= 1 \\
 k \left( \left[ 3x - \frac{x^2}{2} \right]_0^3 + \left[ \frac{x^2}{2} - 3x \right]_3^6 \right) &= 1 \\
 k \left( \left( \left( 9 - \frac{9}{2} \right) - 0 \right) + \left( \frac{36}{2} - 18 \right) - \left( \frac{9}{2} - 9 \right) \right) &= 1 \\
 k \times 9 &= 1 \\
 k &= \frac{1}{9}
 \end{aligned}$$

c

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} xf(x) dx \\ &= \int_0^3 \frac{x}{9}(3-x) dx + \int_3^6 \frac{x}{9}(x-3) dx \\ &= \int_0^3 \frac{x}{3} - \frac{x^2}{9} dx + \int_3^6 \frac{x^2}{9} - \frac{x}{3} dx \\ &= \left[ \frac{x^2}{6} - \frac{x^3}{27} \right]_0^3 + \left[ \frac{x^3}{27} - \frac{x^2}{6} \right]_3^6 \\ &= \left( \left( \frac{9}{6} - \frac{27}{27} \right) - 0 \right) + \left( \frac{216}{27} - \frac{36}{6} \right) \\ &\quad - \left( \frac{27}{27} - \frac{9}{6} \right) \\ &= \frac{1}{2} + 2 + \frac{1}{2} \\ &= 3 \quad \text{QED}\end{aligned}$$

$$\begin{aligned}E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_0^3 \frac{x^2}{3} - \frac{x^3}{9} dx \\ &\quad + \int_3^6 \frac{x^3}{9} - \frac{x^2}{3} dx \\ &= \left[ \frac{x^3}{9} - \frac{x^4}{36} \right]_0^3 + \left[ \frac{x^4}{36} - \frac{x^3}{9} \right]_3^6 \\ &= \left( \left( \frac{27}{9} - \frac{81}{36} \right) - 0 \right) \\ &\quad + \left( \frac{6^4}{36} - \frac{216}{9} \right) - \left( \frac{81}{36} - \frac{27}{9} \right) \\ &= 3 - \frac{9}{4} + 36 - 24 - \frac{9}{4} + 3 \\ &= 18 - \frac{9}{2} \\ &= \frac{27}{2}\end{aligned}$$

$$\begin{aligned}\sigma^2 &= E(X^2) - \mu^2 \\ &= \frac{27}{2} - 9 \\ &= \frac{9}{2} = 4.5\end{aligned}$$



## Solutions to Exercise 15D

**1**  $E(X) = 4$

$$C = 300X + 100$$

$$E(C) = E(300X + 100)$$

$$= 300E(X) + 100$$

$$= 1300$$

**2 a**  $E(X) = \int_{-\infty}^{\infty} xf(x) dx$

$$= \int_0^1 \frac{3x^3}{2} + x^2 dx$$

$$= \left[ \frac{3x^4}{8} + \frac{x^3}{3} \right]_0^1$$

$$= \frac{3}{8} + \frac{1}{3} = \frac{17}{24}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^1 \frac{3x^4}{2} + x^3 dx$$

$$= \left[ \frac{3x^5}{10} + \frac{x^4}{4} \right]_0^1$$

$$= \frac{3}{10} + \frac{1}{4} = \frac{11}{20}$$

$$\text{Var}(X) = E(X^2) - E(X)^2$$

$$= \frac{11}{20} - \frac{17^2}{24^2}$$

$$\approx 0.048$$

**b**  $E(V) = E(100 - 1.5X)$

$$= 100 - 1.5E(X)$$

$$= 100 - \frac{3}{2} \times \frac{17}{24}$$

$$= \frac{4800}{48} - \frac{51}{48}$$

$$= \frac{1583}{16} = \$98.94$$

$$\text{Var}(V) = \text{Var}(100 - 1.5X)$$

$$= 2.25\text{Var}(X)$$

$$\approx 0.1086$$

$$\text{sd}(V) = \sqrt{\text{Var}(V)} = \$0.33$$

**3 a**  $E(3X) = \int_{-\infty}^{\infty} 3xf(x) dx$

$$= \int_{-1}^1 \frac{9x^3}{2} dx$$

$$= \left[ \frac{9x^4}{8} \right]_{-1}^1$$

$$= \frac{9}{8} - \frac{9}{8} = 0$$

$$E(9X^2) = \int_{-\infty}^{\infty} 9x^2 f(x) dx$$

$$= \int_{-1}^1 \frac{27x^4}{2} dx$$

$$= \left[ \frac{27x^5}{10} \right]_{-1}^1$$

$$= \frac{27}{10} + \frac{27}{10} = \frac{27}{5} = 5.4$$

$$\text{Var}(3X) = E(9X^2) - E(3X)^2$$

$$= \frac{27}{5}$$

$$\mathbf{b} \quad E(3 - X) = 3 - E(X)$$

$$= 3 - \frac{1}{3}E(3X)$$

$$= 3 - 0 = 3$$

$$\text{Var}(3 - X) = \text{Var}(X)$$

$$= \frac{1}{9}\text{Var}(3X)$$

$$= \frac{3}{5} = 0.6$$

$$\mathbf{c} \quad 1, \quad 5.4$$

**d** Let  $g(x)$  be the function required

we know  $g(x) = ax^2$  and since

$$-1 \leq x \leq 1, \quad -3 \leq 3x \leq 3$$

$$\therefore g(x) = \begin{cases} ax^2 & -3 \leq 3x \leq 3 \\ 0 & x < -3 \text{ or } x > 3 \end{cases}$$

$$\int_{-\infty}^{\infty} g(x) dx = 1$$

$$\int_{-3}^3 ax^2 dx = 1$$

$$\left[ a \frac{x^3}{3} \right]_{-3}^3 = 1$$

$$9a - (-9a) = 1$$

$$a = \frac{1}{18}$$

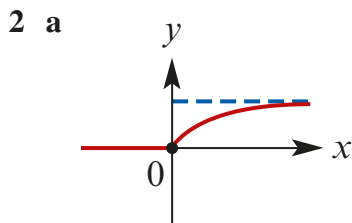
$$\therefore g(x) = \begin{cases} \frac{x^2}{18} & -3 \leq x \leq 3 \\ 0 & x < -3 \text{ or } x > 3 \end{cases}$$

$$\mathbf{e} \quad h(x) = \begin{cases} \frac{(x-1)^2}{18} & -2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

## Solutions to Exercise 15E

$$\begin{aligned}
 \mathbf{1\ a} \quad F(x) &= \int_0^x f(t) dt \\
 &= \int_0^x \frac{1}{5} dt \\
 &= \left[ \frac{t}{5} \right]_0^x \\
 F(x) &= \begin{cases} \frac{x}{5} & \text{if } 0 < x \leq 5 \\ 0 & \text{if } x \leq 0 \\ 0 & \text{if } x \geq 5 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \Pr(X \leq 3) &= F(3) \\
 &= \frac{3}{5}
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{b} \quad \Pr(X \geq 2) &= 1 - \Pr(X < 2) \\
 &= 1 - F(2) \\
 &= 1 - (1 - e^{-4}) \\
 &= e^{-4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \Pr(X \geq 2 | X < 3) &= \frac{\Pr(X \geq 2)}{\Pr(X < 3)} \\
 &= \frac{e^{-4}}{F(3)} \\
 &= \frac{e^{-4}}{1 - e^{-9}} \approx 0.0183
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3\ a} \quad F(6) &= 1 \\
 k(6)^2 &= 1 \\
 k &= \frac{1}{36}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \Pr\left(\frac{1}{2} \leq X \leq 1\right) &= \Pr(X \leq 1) - \Pr\left(X < \frac{1}{2}\right) \\
 &= F(1) - F\left(\frac{1}{2}\right) \\
 &= \frac{1}{36}(1) - \frac{1}{36}\left(\frac{1}{4}\right) \\
 &= \frac{1}{48}
 \end{aligned}$$

## Solutions to Technology-free questions

$$1 \text{ a} \quad k \int_1^{\sqrt{2}} x \, dx = 1$$

$$k \left[ \frac{1}{2} x^2 \right]_1^{\sqrt{2}} = 1$$

$$k \left( \frac{1}{2} \times 2 - \frac{1}{2} \times 1 \right) = 1$$

$$\frac{1}{2} k = 1$$

$$k = 2$$

$$b \text{ Pr}(1 < X < 1.1) = \int_1^{1.1} 2x \, dx$$

$$= \left[ x^2 \right]_1^{1.1}$$

$$= 1.21 - 1$$

$$= 0.21$$

$$c \text{ Pr}(1 < X < 1.2) = \int_1^{1.2} 2x \, dx$$

$$= \left[ x^2 \right]_1^{1.2}$$

$$= 1.44 - 1$$

$$= 0.44$$

$$2 \quad \int_0^1 (a + bx^2) \, dx = 1$$

$$\left[ ax + \frac{1}{3} bx^3 \right]_0^1 = 1$$

$$a + \frac{1}{3} b = 1 \quad \textcircled{1}$$

$$E(X) = \frac{2}{3}, \text{ so:}$$

$$\int_0^1 x(a + bx^2) \, dx = \frac{2}{3}$$

$$\int_0^1 x(ax + bx^3) \, dx = \frac{2}{3}$$

$$\left[ \frac{1}{2} ax^2 + \frac{1}{4} bx^4 \right]_0^1 = \frac{2}{3}$$

$$\frac{1}{2} a + \frac{1}{4} b = \frac{2}{3} \quad \textcircled{2}$$

$$2 \times (2) - (1): \left( \frac{1}{2} - \frac{1}{3} \right) b = \frac{4}{3} - 1$$

$$\frac{1}{6} b = \frac{1}{3}$$

$$b = 2$$

$$\text{Substituting in (1) gives } a = \frac{1}{3}.$$

- 3 The graph of  $\frac{1}{2} \sin x$ ,  $0 \leq x \leq \pi$ , has  $x$ -intercepts at  $(0, 0)$  and  $(\pi, 0)$  and a maximum at  $\left(\frac{\pi}{2}, \frac{1}{2}\right)$ .

Also, the graph is symmetrical about the line  $x = \frac{\pi}{2}$ , so the area under the curve from 0 to  $\frac{\pi}{2}$  is  $\frac{1}{2}$ .

(Alternatively, solve  $\int_0^{\pi} \frac{1}{2} \sin x \, dx = \frac{1}{2}$ .)

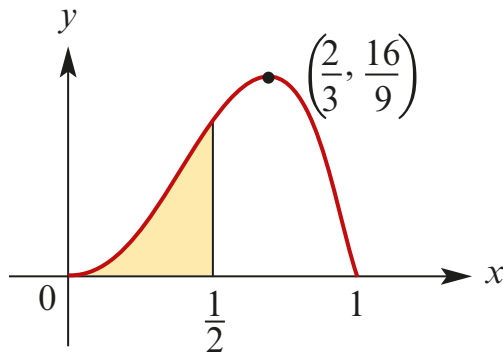
(Note that the symmetry also implies that the mean is  $\frac{\pi}{2}$ .)

$$4 \text{ a} \quad \text{Pr}(1 < x < 3) = \frac{1}{4}(3 - 1) - \frac{1}{4}(1 - 1) \\ = \frac{1}{2}$$

$$\begin{aligned}
 \text{b } \Pr(X > 2 | 1 < X < 3) &= \frac{\Pr(X > 2 \cap 1 < X < 3)}{\Pr(1 < X < 3)} \\
 &= \frac{\Pr(2 < X < 3)}{\Pr(1 < X < 3)} \\
 &= \frac{\Pr(X < 3) - \Pr(X < 2)}{\left(\frac{1}{2}\right)} \\
 &= \frac{\frac{1}{4}(3-1) - \frac{1}{4}(2-1)}{\frac{1}{2}} \\
 &= \frac{1}{2}(2-1) \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \Pr(X > 4 | X > 2) &= \frac{\Pr(X > 4 \cap X > 2)}{\Pr(X > 2)} \\
 &= \frac{\Pr(X > 4)}{\Pr(X > 2)} \\
 &= \frac{\frac{1}{4}}{\frac{3}{4}} \\
 &= \frac{1}{3}
 \end{aligned}$$

5 a



$$\begin{aligned}
 \text{b } P(X < 0.5) &= \int_0^{0.5} (12x^2 - 12x^3) dx \\
 &= \left[ 4x^3 - 3x^4 \right]_0^{0.5} \\
 &= 4 \times \frac{1}{8} - 3 \times \frac{1}{16} \\
 &= \frac{5}{16}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a } k \int_0^1 (x^2 - x^3) dx &= 1 \\
 k \left[ \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 &= 1 \\
 k \left( \frac{1}{3} - \frac{1}{4} \right) &= 1 \\
 \frac{1}{12}k &= 1
 \end{aligned}$$

$$k = 12$$

(Note that this agrees with the function given in Qn. 5 above.)

$$\begin{aligned}
 \text{b } f(x) &= 12x^2 - 12x^3 \\
 f'(x) &= 24x - 36x^2 \\
 &= 12x(2 - 3x)
 \end{aligned}$$

$$= 0 \text{ if } x = 0 \text{ or } x = \frac{2}{3}$$

Since the graph of  $y = f(x)$  is that of a negative cubic with  $x$  intercepts  $(0, 0)$  and  $(1, 0)$ ,  $x = \frac{2}{3}$  must correspond to a local maximum. So the mode is  $\frac{2}{3}$ .

$$\begin{aligned}\Pr\left(X < \frac{2}{3}\right) &= \int_0^{\frac{2}{3}} (12x^2 - 12x^3) dx \\ &= \left[4x^3 - 3x^4\right]_0^{\frac{2}{3}} \\ &= 4 \times \frac{8}{27} - 3 \times \frac{16}{81} \\ &= \frac{16}{27}\end{aligned}$$

$$\mathbf{c} \quad \Pr\left(X < \frac{1}{3} \mid X < \frac{2}{3}\right) = \frac{\Pr\left(X < \frac{1}{3}\right)}{\Pr\left(X < \frac{2}{3}\right)}$$

$$\begin{aligned}\Pr\left(X < \frac{1}{3}\right) &= \left[4x^3 - 3x^4\right]_0^{\frac{1}{3}} \\ &= \frac{1}{9}, \text{ so}\end{aligned}$$

$$\Pr\left(X < \frac{1}{3} \mid X < \frac{2}{3}\right) = \frac{\frac{1}{9}}{\frac{16}{27}} = \frac{3}{16}$$

$$\begin{aligned}\mathbf{7} \quad \mathbf{a} \quad \Pr(X < 0.2) &= \int_0^{0.2} 3x^2 dx \\ &= \left[x^3\right]_0^{0.2} \\ &= 0.008\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \Pr(X < 0.2 \mid X < 0.3) &= \frac{\int_0^{0.2} 3x^2 dx}{\int_0^{0.3} 3x^2 dx} \\ &= \frac{0.008}{0.027} \\ &= \frac{8}{27}\end{aligned}$$

$$\begin{aligned}\mathbf{8} \quad \Pr(X < m) &= \int_0^m \frac{\pi}{4} \cos\left(\frac{\pi x}{4}\right) dx \\ &= \left[\sin\left(\frac{\pi x}{4}\right)\right]_0^m \\ &= \sin\left(\frac{\pi m}{4}\right) \\ \sin\left(\frac{\pi m}{4}\right) &= 0.5 \\ \frac{\pi m}{4} &= \frac{\pi}{6} \\ m &= \frac{2}{3}\end{aligned}$$

$$\begin{aligned}\mathbf{9} \quad \mathbf{a} \quad E(X) &= \int_0^4 \frac{x(x+2)}{16} dx \\ &= \frac{1}{16} \int_0^4 x^2 + 2x dx \\ &= \frac{1}{16} \left[\frac{x^3}{3} + x^2\right]_0^4 \\ &= \frac{7}{3}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \int_0^a \frac{(x+2)}{16} dx &= \frac{5}{32} \\ \int_0^a (x+2) dx &= \frac{5}{2} \\ \left[\frac{x^2}{2} + 2x\right]_0^a &= \frac{5}{2} \\ \frac{a^2}{2} + 2a &= \frac{5}{2} \\ a^2 + 4a - 5 &= 0 \\ (a+5)(a-1) &= 0 \\ a &= -5 \text{ or } a = 1 \\ \therefore a &= 1\end{aligned}$$

$$10 \text{ a } \int_{-1}^1 c(1-x^2) dx = c \left[ x - \frac{x^3}{3} \right]_{-1}^1$$

$$= \frac{4c}{3}$$

For PDF  $\frac{4c}{3} = 1$

$$\therefore c = \frac{3}{4}$$

**b** 0

$$11 \int_0^1 n(1-x)^{n-1} dx = \left[ \frac{-n(1-x)^n}{n} \right]_0^1$$

$$= 1$$

$$12 \text{ a } \int_0^m \frac{1}{x} dx = \left[ \log_e(x) \right]_1^m$$

$$= \log_e m$$

For median

$$\log_e m = \frac{1}{2}$$

$$m = e^{\frac{1}{2}}$$

For the interquartile range:

$$13 \text{ b } \log_e m = \frac{1}{4}$$

$$m = e^{\frac{1}{4}}$$

$$13 \text{ c } \log_e m = \frac{3}{4}$$

$$m = e^{\frac{3}{4}}$$

$$\text{IQR} = e^{\frac{3}{4}} - e^{\frac{1}{4}} \approx 0.833$$

**13** For a continuous variable  $X$ ,  
 $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.95$

Here,  $\mu = 330$  and  $\sigma = 5$ ,

$$\mu + 2\sigma = 330 + 10 = 340$$

$$\mu - 2\sigma = 330 - 10 = 320$$

so (320, 340) is the required  
 (approximate) interval for 95%  
 of cans.

**14** Since the variance is 4, the standard deviation is  $\sqrt{4} = 2$ .

$$\mu + 2\sigma = 250 + 4 = 254$$

$$\mu - 2\sigma = 250 - 4 = 246$$

so  $(2 \times 6, 254)$  is an (approximate) 95%  
 interval.

## Solutions to multiple-choice questions

**1 B** The second graph partly lies below the  $x$ -axis. Since  $f(x) \geq 0$  for all  $x$ , this could not requirement a probability density function.

**2 D** An antiderivative of  $4x$  is  $2x^2$ .

If the domain is of the form  $0 < x < a$ , then

$$\int_0^a 4x \, dx = \left[ 2x^2 \right]_0^a$$

$$= 2a^2$$

$$= 1$$

$$\Rightarrow a = \frac{1}{\sqrt{2}} \text{ (since } a > 0\text{),}$$

so option **D** fits.

(Note that the above shows options  $A \rightarrow C$  are not possible.)

For option **E**:  $\int_{\frac{1}{\sqrt{2}}}^{\frac{2}{\sqrt{2}}} 4x \, dx$

$$= \left[ 2x^2 \right]_{\frac{1}{\sqrt{2}}}^{\frac{2}{\sqrt{2}}}$$

$$= 2\left(\frac{4}{2}\right) - 2\left(\frac{1}{2}\right)$$

$$= 3$$

$$\begin{aligned} \mathbf{3 D} \quad \int_0^L \frac{1}{2} \sin x \, dx &= \left[ -\frac{1}{2} \cos x \right]_0^k \\ &= -\frac{1}{2} \cos k + \frac{1}{2} \end{aligned}$$

$$= 1$$

$$\text{if } \cos k = -1$$

$$k = \pi$$

$$\begin{aligned} \mathbf{4 A} \quad \Pr(X \leq 1.3) &= \int_1^{1.3} \frac{3}{4}(x^2 - 1) \, dx \\ &= \left[ \frac{1}{4}x^3 - \frac{3}{4}x \right]_1^{1.3} \\ &\approx 0.0743 \end{aligned}$$

$$\begin{aligned} \mathbf{5 E} \quad E(X) &= \int_1^2 x \times \frac{3}{4}(x^2 - 1) \, dx \\ &= \int_1^2 \left( \frac{3}{4}x^3 - \frac{3}{4}x \right) \, dx \\ &= \left[ \frac{3}{16}x^4 - \frac{3}{8}x^2 \right]_1^2 \\ &= \frac{27}{16} \end{aligned}$$

$$\begin{aligned} \mathbf{6 B} \quad E(x^2) &= \int_1^2 x^2 \times \frac{3}{4}(x^2 - 1) \, dx \\ &= \int_1^2 \left( \frac{3}{4}x^4 - \frac{3}{4}x^2 \right) \, dx \\ &= \left[ \frac{3}{20}x^5 - \frac{1}{4}x^3 \right]_1^2 \\ &= \frac{29}{10} \end{aligned}$$

$$\text{var}(X) = E(X)^2 - [E(X)]^2$$

$$= \frac{29}{10} - \left( \frac{27}{16} \right)^2$$

$$= \frac{67}{1280}$$



$$7 \text{ C } \int_0^m \frac{x^3}{4} dx = \frac{1}{2}$$

$$\left[ \frac{x^4}{16} \right]_0^m = \frac{1}{2}$$

$$\frac{m^4}{16} = \frac{1}{2}$$

$$m^4 = 8$$

$$m = \sqrt[4]{8}$$

$$\approx 1.6818$$

- 8 E The graph of  $y = f(x)$  is that of a positive cubic which touches the  $x$ -axis at  $(2, 0)$ . The end points are  $(1, 0)$  and  $(3, 3)$ . Click for a local maximum turning point.

$$\begin{aligned} f(x) &= \frac{3}{2}(x-1)(x-2)^2 \\ &= \frac{3}{2}(x^3 - 5x^2 + 8x - 6) \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{3}{2}(3x^2 - 10x + 8) \\ &= \frac{3}{2}(x-2)(3x-4) \\ &= 0 \text{ if } x = 2 \text{ or } x = \frac{4}{3}. \end{aligned}$$

Then  $x = \frac{4}{3}$  must correspond to a local maximum turning point.

$$\begin{aligned} f\left(\frac{4}{3}\right) &= \frac{3}{2}\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)^2 \\ &= \frac{2}{9} \end{aligned}$$

But  $f(3) = 3$ , so the mode is 3

(NOT  $\frac{4}{3}$ !)

9

$$\begin{aligned} \text{C } E(X) &= \int_0^{20} x \times \frac{x}{40\,000}(400 - x^2) dx \\ &= \frac{32}{3} \end{aligned}$$

(using a CAS calculator)

Then the expected consultations time for three patients is  $3 \times \frac{32}{3} = 32$  min.

- 10 A Let  $s$  be the minimum score for an 'A'.

Then  $\Pr(X \geq S) = 0.10$  or equivalently  $\Pr(X < S) = 0.909$ .

$$\text{Hence } \int_0^s \frac{\pi}{100} \sin\left(\frac{\pi x}{50}\right) dx = 0.90$$

$$\left[ -\frac{1}{2} \cos\left(\frac{\pi x}{50}\right) \right]_0^s = 0.90$$

$$-\frac{1}{2} \cos\left(\frac{\pi s}{50}\right) + \frac{1}{2} = 0.90$$

$$\cos\left(\frac{\pi s}{50}\right) = -0.80$$

$$\frac{\pi s}{50} = \cos^{-1}(-0.80)$$

$$s = \frac{50}{\pi} \cos^{-1}(-0.80)$$

$$\approx 39.8$$

so the minimum score required is closest to 40.

## Solutions to extended-response questions

$$1 \quad f(x) = \begin{cases} \frac{a}{100} \left(1 - \frac{x}{100}\right) & \text{if } 100 \leq x \leq 1000 \\ 0 & \text{otherwise} \end{cases}$$

$$a \quad \int_{100}^{1000} f(x) dx = \left[ \frac{a}{100} \left(x - \frac{x^2}{200}\right) \right]_{100}^{1000} = -\frac{81a}{2}$$

For  $f$  to be a probability density function  $-\frac{81a}{2} = 1$  and hence  $a = -\frac{2}{81}$

$$b \quad E(x) = \int_{100}^{1000} xf(x) dx = \left[ -\frac{2}{8100} \left(\frac{x^2}{2} - \frac{x^3}{300}\right) \right]_{100}^{1000} = 700 \text{ hours}$$

c The cumulative probability density function

$$= \int_{100}^x f(t) dt = \frac{1}{810000} (x^2 - 200x + 10000)$$

Median = 736.4 hours

$$2 \quad F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x - x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

a  $\Pr(X > 0.5) = 1 - \Pr(X \leq 0.5)$

$$= 1 - F(0.5)$$

$$= \frac{1}{4}$$

b  $\Pr(X < a) = 0.8$ . This can be written as

$$F(a) = 0.8 \text{ which implies,}$$

$$2a - a^2 = 0.8$$

Solving the quadratic for  $a$  gives  $a = \frac{5 - \sqrt{5}}{5}$  or  $a = \frac{5 + \sqrt{5}}{5}$

But  $0 < a < 1$ . Therefore  $a = \frac{5 - \sqrt{5}}{5}$ .

c The corresponding probability density function is  $f(x) = 2 - 2x$

$$\begin{aligned}
E(X) &= \int_0^1 xf(x) dx \\
&= \int_0^1 2x - 2x^2 dx \\
&= \left[ x^2 - \frac{2x^3}{3} \right]_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
E(\sqrt{X}) &= \int_0^1 \sqrt{x}f(x) dx \\
&= \int_0^1 2x^{\frac{1}{2}} - 2x^{\frac{3}{2}} dx \\
&= \left[ \frac{4}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} \right]_0^1 \\
&= \frac{8}{15}
\end{aligned}$$

$$\mathbf{3 \ a} \quad f(x) = \begin{cases} \frac{\pi}{20} \cos\left(\frac{\pi}{10}(x-6)\right) & \text{if } 1 \leq x \leq 11 \\ 0 & \text{if } x < 1 \text{ or } x > 11 \end{cases}$$

For the median:

Let  $q_1$  be the first quartile and  $q_3$  be the third quartile. Then

$$\int_1^{q_1} f(x) dx = 0.25 \text{ and } \int_1^{q_3} f(x) dx = 0.75$$

$$\text{From the earlier calculation, } \int_1^{q_1} f(x) dx = \frac{1}{2} \left( \sin\left(\frac{\pi}{10}(q_1 - 6)\right) + 1 \right).$$

$$\text{For } q_1: \frac{1}{2} \left( \sin\left(\frac{\pi}{10}(q_1 - 6)\right) + 1 \right) = 0.25$$

$$\sin\left(\frac{\pi}{10}(q_1 - 6)\right) = -0.5$$

$$\frac{\pi}{10}(q_1 - 6) = -\frac{\pi}{6}$$

$$q_1 - 6 = \frac{-5}{3}$$

$$q_1 = \frac{13}{3}$$

$$\text{For } q_3 : \frac{1}{2} \left( \sin \left( \frac{\pi}{10} (q_3 - 6) \right) + 1 \right) = 0.75$$

$$\sin \left( \frac{\pi}{10} (q_3 - 6) \right) = 0.5$$

$$\frac{\pi}{10} (q_3 - 6) = \frac{\pi}{6}$$

$$q_3 - 6 = \frac{5}{3}$$

$$q_3 = \frac{23}{3}$$

$$\text{Interquartile range} = q_3 - q_1$$

$$= \frac{23}{3} - \frac{13}{3} = \frac{10}{3}$$

- b** The graph of  $y = f(x)$  is symmetrical about the line  $x = 6$ , so the mean is 6,  
Alternatively:

$$E(X) = \int_1^{11} x f(x) dx = \frac{\pi}{20} \int_1^{11} x \cos \left( \frac{\pi}{10} (x - 6) \right) dx$$

Since the integrand is not a 'standard' function for integration, use a CAS calculator for its evaluation. This gives  $E(X) = 6$ .

For the variance, first find  $E(X^2)$ .

$$E(X^2) = \int_1^{11} x^2 f(x) dx = \frac{\pi}{20} \int_1^{11} x^2 \cos \left( \frac{\pi}{10} (x - 6) \right) dx$$

Using the integration command of a CAS calculator gives  $E(X^2) \approx 40.7358$  Hence  
 $\text{var}(X) = E(X^2) - [E(X)]^2$

$$\approx 40.7358 - 36$$

$$= 4.736 \text{ correct to 3 decimal places.}$$

$$\int_1^a f(x) dx = 0.5$$

$$\frac{\pi}{20} \int_1^a \cos \left( \frac{\pi}{10} (x - 6) \right) dx = 0.5$$

$$\frac{\pi}{20} \times \frac{10}{\pi} \left[ \sin \left( \frac{\pi}{10} (x - 6) \right) dx \right]_1^a = 0.5$$

$$\sin \left( \frac{\pi}{10} (a - 6) \right) - \sin \left( -\frac{\pi}{2} \right) = 1$$

$$\sin \left( \frac{\pi}{10} (a - 6) \right) = 0$$

$$a = 6$$

4 a  $\Pr(4 \leq Y \leq 5)$

$$\begin{aligned} \int_4^5 f(y) dy \text{ (not } f(x) dx) &= \frac{2}{25} \int_4^5 y - 1 dy \\ &= \frac{2}{25} \left[ \frac{1}{2}y^2 - y \right]_4^5 \\ &= \frac{7}{25} \end{aligned}$$

b  $E(Y) = \int_1^6 yf(y) dy$

$$\begin{aligned} &= \frac{2}{25} \int_1^6 y(y-1) dy \\ &= \frac{2}{25} \left[ \frac{1}{3}y^3 - \frac{1}{2}y^2 \right]_1^6 \\ &= \frac{2}{25} \times \left( 54 + \frac{1}{6} \right) = \frac{13}{3} \end{aligned}$$

Also  $E(X) = 8 \times 0.6 = 4.80$

The expected money received =  $\$ \left( 4.80 + 4 \times \frac{13}{3} \right) = \$22.13$

5  $E(X - c)^2 = \int_2^4 (x^2 - 2cx + c^2)f(x) dx$

$$\begin{aligned} &= \frac{1}{2} \int_2^4 x^2 - 2cx + c^2(x-2) dx \\ &= \frac{1}{2} \int_2^4 (x^3 - (2c+2)^2 + (c^2 4c)x - 2c^2) dx \\ &= \frac{1}{3}(3c^2 - 20c + 34) \end{aligned}$$

If  $E(X - c)^2 = \frac{2}{3}$

implies  $3c^2 - 20c + 34 = 2$

$$3c^2 - 20c + 32 = 0$$

$$(3c - 8)(c - 4) = 0$$

Therefore  $c = \frac{8}{3}$  or  $c = 4$

6  $f(x) = \begin{cases} k(x-1)(3-x) & \text{if } 1 \leq x \leq 3 \\ 0 & \text{if } x < 1 \text{ or } x > 3 \end{cases}$

$$\mathbf{a} \quad \int_1^3 f(x) dx = 1$$

$$\text{Therefore } k \int_1^3 -x^2 + 4x - 3 dx = 1$$

$$\text{Hence } k = \frac{3}{4}$$

$$\begin{aligned} \mathbf{b} \quad E(X) &= \frac{3}{4} \int_1^3 xf(x) dx \\ &= \frac{3}{4} \int_1^3 x(-x^2 + 4x - 3) dx \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{3}{4} \int_1^3 xf(x) dx - 4 \\ &= \frac{21}{5} - 4 \\ &= \frac{1}{5} = 0.2 \end{aligned}$$

$$\mathbf{c} \quad \Pr(X > 2.5) = \frac{3}{4} \int_{2.5}^3 -x^2 + 4x - 3 dx = \frac{5}{32}$$

(Note: the integral, in this question have been evaluated using the ‘Integral’ command of a CAS calculator.)

$$\mathbf{7} \quad f(x) = \begin{cases} \frac{k}{12(x-1)^3} & \text{if } 0 \leq x \leq 4 \\ 0 & \text{if } x < 0 \text{ or } x > 4 \end{cases}$$

$$\begin{aligned} \mathbf{a} \quad \int_0^4 f(x) dx = 1 &\Rightarrow \frac{k}{12} \int_0^4 \frac{1}{(x+1)^3} dx = 1 \\ &\frac{k}{12} \left[ -\frac{1}{2(x+1)^2} \right]_0^4 = 1 \\ &\frac{k}{12} \left( -\frac{1}{50} + \frac{1}{2} \right) = 1 \\ &\frac{k}{25} = 1 \\ &k = 25 \end{aligned}$$

$$\begin{aligned}
 \text{b } E(X + 1) &= \frac{25}{12} \int_0^4 \frac{1}{(x + 1)^2} dy \\
 &= \frac{25}{12} \left[ -\frac{1}{(x + 1)} \right]_0^4 \\
 &= \frac{25}{12} \left( -\frac{1}{5} + 1 \right) \\
 &= \frac{5}{3}
 \end{aligned}$$

$$E(X + 1) = E(X) + 1$$

$$E(X) = E(X + 1) - 1$$

$$= \frac{5}{3} - 1 = \frac{2}{3}$$

$$\text{d } P(X \leq c) = c$$

$$\frac{25}{12} \int_0^c \frac{1}{(x + 1)^3} dx = c$$

$$-\frac{25}{24} \left[ \frac{1}{(x + 1)^2} \right]_0^c = c$$

$$-\frac{25}{24} \left( \frac{1}{(c + 1)^2} - 1 \right) = c$$

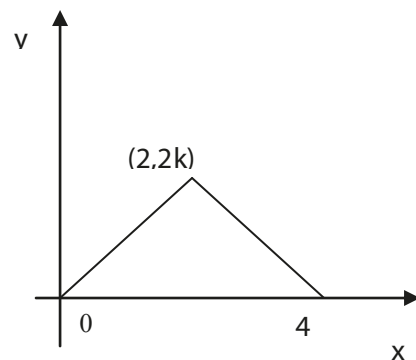
Using the 'solve' command of a CAS calculator gives  $c = \frac{-13}{6}$  or  $c = 0$  or  $c = \frac{2}{3}$ .

But  $c > 0$ , so  $c = \frac{2}{3}$ .

$$\text{8 } f(x) = \begin{cases} kx & \text{if } 0 \leq x < 2 \\ k(4 - x) & \text{if } 2 \leq x < 4 \\ 0 & \text{if } x < 0 \text{ or } x > 4 \end{cases}$$

$$\begin{aligned}
 \text{a } \text{The area of the triangle} &= \frac{1}{2} \times 4 \times 2k \\
 &= 4k
 \end{aligned}$$

$$\text{Therefore } k = \frac{1}{4}$$



**b** Since the graph of  $y = f(x)$  is symmetrical about  $x = 2$ ,  $E(X) = 2$ .

$$\begin{aligned}\text{var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{1}{4} \int_0^2 x^3 dx + \frac{1}{4} \int_2^4 x^2(4-x) dx - 4 \\ &= \frac{14}{3} - 4 \\ &= \frac{2}{3}\end{aligned}$$

**c**  $\Pr(|X - \mu| < 1) = \Pr(|X - 2| < 1)$

$$\begin{aligned}&= \Pr(1 < X < 3) \\ &= \frac{1}{4} \int_1^2 x dx + \frac{1}{4} \int_2^3 (4-x) dx \\ &= \frac{3}{4}\end{aligned}$$

**d**  $\Pr(X > a) = 0.6$  or equivalently  $\Pr(X \leq a) = 0.4$ .

Since the graph of  $y = f(x)$  is symmetrical about  $x = 2$ ,  $\Pr(X \leq 2) = 0.5$ .

Hence  $0 < a < 2$ .

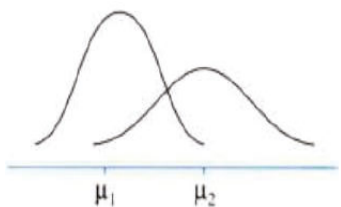
$$\begin{aligned}\int_0^a \frac{1}{4} x dx &= 0.4 = \frac{4}{10} \\ \left[ \frac{1}{8} x^2 \right]_0^a &= \frac{2}{5} \\ \frac{1}{8} a^2 &= \frac{2}{5} \\ a^2 &= \frac{16}{5} \\ a &= \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5} \quad (a > 0).\end{aligned}$$



# Chapter 16 – The normal distribution

## Solutions to Exercise 16A

1



2 (c) appears to be the only normally distributed curve

3 a using CAS calculator, integral = 1

$$\begin{aligned} \text{b i } E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{-\infty}^{\infty} \frac{x}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} dx \end{aligned}$$

ii using CAS calculator, integral=2

$$\begin{aligned} \text{c i } E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\ &= \int_{-\infty}^{\infty} \frac{x^2}{3\sqrt{2\pi}} \\ &\quad \times e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} dx \end{aligned}$$

ii using CAS calculator, integral = 13

$$\begin{aligned} \text{iii } \sigma &= \sqrt{\sigma^2} \\ &= \sqrt{E(X^2) - [E(X)]^2} \\ &= \sqrt{13 - 4} \\ &= 3 \end{aligned}$$

4 a using CAS calculator, integral = 1

$$\begin{aligned} \text{b i } E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{-\infty}^{\infty} \frac{x}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} dx \end{aligned}$$

ii using CAS calculator, integral = -4

$$\begin{aligned} \text{c i } E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\ &= \int_{-\infty}^{\infty} \frac{x^2}{5\sqrt{2\pi}} \\ &\quad \times e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} dx \end{aligned}$$

ii  $E(X^2) = 41$

$$\begin{aligned} \text{iii } \sigma &= \sqrt{\sigma^2} \\ &= \sqrt{E(X^2) - [E(X)]^2} \\ &= \sqrt{41 - 16} \\ &= 5 \end{aligned}$$

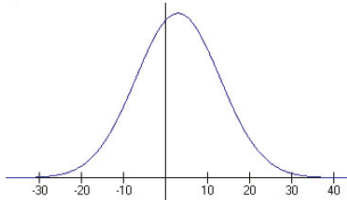
$$5 \quad f(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{10}\right)^2}$$

a  $\mu = 3$

$\sigma = 10$

$\left(\text{read off}\left(\frac{x-3}{10}\right)\right)$   
section of the equation)

**b**



$$6 \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+3}{1}\right)^2}$$

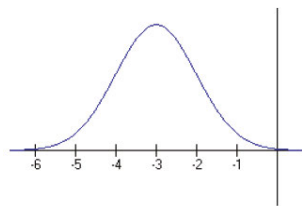
**a**  $\mu = -3,$

$\sigma = 1$

*(read off  $\left(\frac{x+3}{1}\right)$*

*section of the equation)*

**b**



$$7 \quad f(x) = \frac{1}{9\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+0}{3}\right)^2}$$

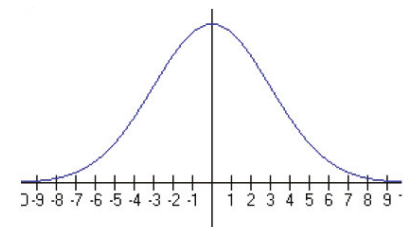
**a**  $\mu = 0$

$\sigma = 3$

*(read off  $\left(\frac{x+0}{3}\right)$*

*section of the equation)*

**b**



**8 a** translation + 3 along the x-axis

$(\mu = 3)$

dilation factor 2 from the y-axis

$(\sigma = 2)$

dilation factor  $\frac{1}{2}$  from the x-axis

**b** translation + 3 along the x-axis

$(\mu = 3)$

dilation factor  $\frac{1}{2}$  from the y-axis

$(\sigma = \frac{1}{2})$

dilation factor 2 from the x-axis

**c** translation -3 along the x-axis

$(\mu = -3)$

dilation factor 2 from the y-axis

$(\sigma = 2)$

dilation factor  $\frac{1}{2}$  from the x-axis

**9 a** translation -3 along the x-axis

$(\mu = 3)$

dilation factor 2 from the x-axis

dilation factor  $\frac{1}{2}$  from the y-axis

$(\sigma = 2)$

**b** translation -3 along the x-axis

$(\mu = 3)$

dilation factor  $\frac{1}{2}$  from the x-axis

dilation factor 2 from the y-axis

$(\sigma = \frac{1}{2})$

**c** translation +3 along the x-axis

$(\mu = -3)$

dilation factor 2 from the x-axis

dilation factor  $\frac{1}{2}$  from the y-axis

$(\sigma = 2)$

## Solutions to Exercise 16B

1 a 16%

b 16 %

c 2.5 %

d 2.5 %

2 a  $\mu \approx 135$

$3\sigma \approx 15$

$\sigma \approx 5$

b  $\mu \approx 10$

$3\sigma \approx 4$

$\sigma \approx \frac{4}{3}$

3 a  $\approx 68\%$

b  $\approx \frac{100\% - 68\%}{2} = 16\%$

c  $\approx \frac{100\% - 99.7\%}{2} = 0.15\%$

4  $\mu - 2\sigma$  and  $\mu + 2\sigma$

$27.3 - 6.2$  and  $27.3 + 6.2$

answer:

$21.1$  and  $33.5$

5 one; 95; 99.7; three

6  $\approx \frac{1 - 0.95}{2} = 0.025$ , i.e. 2.5%

7 a  $\approx \frac{1 - 0.68}{2} = 0.16$ , i.e. 16%

b  $\approx \frac{1 - 0.68}{2} = 0.16$  i.e. 16%

8 a  $\approx 68\%$

b  $\approx \frac{100\% - 68\%}{2} = 16\%$

c  $\approx \frac{100\% - 95\%}{2} = 2.5\%$

9 a  $\approx 95\%$

b  $\approx \frac{100\% - 68\%}{2} = 16\%$

c = 50%, since the mean = the median for normal distributions

d  $\approx 99.7\%$

10 a  $\frac{160 - 160}{8} = 0$

b  $\frac{150 - 160}{8} = -1.25$

c  $\frac{172 - 160}{8} = 1.5$

11 a  $\frac{256 - 270}{10} = -1.4$

b  $\frac{281 - 270}{10} = 1.1$

c  $\frac{305 - 270}{10} = 3.5$

12 Michael has a score of  $\frac{85 - 78}{5} = 1.4$  standard deviations

Cheryl has a score of  $\frac{27 - 18}{6} = 1.5$  standard deviations

$\therefore$  Cheryl performed better

- 13** Biology score is  $\frac{77 - 68.5}{4.9} \approx 1.73$   
 standard deviations  
 History score is  $\frac{79 - 75.3}{4.1} \approx 0.90$   
 standard deviations  
 $\therefore$  the student did better in Biology

Sue:

French:  $\frac{15 - 15}{5} = 0$   
 English:  $\frac{42 - 35}{10} = 0.7$   
 Mathematics:  $\frac{19 - 20}{5} = -0.2$

**14 a**

Mary:

French:  $\frac{19 - 15}{4} = 1$   
 English:  $\frac{42 - 35}{8} = 0.875$   
 Mathematics:  $\frac{20 - 20}{5} = 0$

Steve:

French:  $\frac{21 - 23}{4} = -0.5$   
 English:  $\frac{39 - 42}{3} = -1$   
 Mathematics:  $\frac{23 - 18}{4} = 1.25$

**b i** Mary

**ii** Mary

**iii** Steve

**c** if all the subjects are weighted equally, Mary is the best student, since her total standardised mark is higher

## Solutions to Exercise 16C

**1 a** 0.9772

**b** 0.9938

**c** 0.9938

**d** 0.9943

**e** 0.0228

**f** 0.0668

**g** 0.3669

**h** 0.1562

**2 a** 0.9772

**b** 0.6915

**c** 0.9938

**d** 0.9003

**e** 0.0228

**f** 0.0099

**g** 0.0359

**h** 0.1711

**3 a** 0.6826

**b** 0.9544

**c** 0.9974

These results are vary close to the  
'68%–95%–99.7%' rule.

**4 a** 0.0214

**b** 0.9270

**c** 0.0441

**d** 0.1311

**5**  $c = 1.2816$

**6**  $c = 0.6745$

**7**  $c = 1.96$

**8**  $-1.6449$

**9**  $-0.8416$

**10**  $-1.2816$

**11**  $-1.9600$

**12 a** 0.9522

**b** 0.7977

**c** 0.0478

**d** 0.1547

**13 a** 0.9452

**b** 0.2119

**c** 0.9452

**d** 0.1571

**14 a** 9.2897

**b** 8.5631

**15 a**  $c = 10$

**b**  $k = 15.88$

**16 a**  $a = 0.994$

**b**  $b = 1.96$

**c**  $c = 2.968$

**17 a** 0.7161

**b** 0.0965

**c**  $\Pr(x < 26 \mid 25 < x < 27)$   
 $= \frac{\Pr('x < 26' \cap '15 < x < 27')}{\Pr(25 < x < 27)}$   
 $= \frac{\Pr(25 < x < 26)}{0.096 \dots}$   
 $= 0.5204$

**d**  $c = 33.5143$

**e**  $k = 13.02913$

**f**  $c_1 = 8.28; c_2 = 35.72$   
(assumed symmetrical about the mean)

**18 a** 0.9772

**b**  $\Pr(x < 11 \mid x < 13)$   
 $= \frac{\Pr('x < 11 \mid ' \cap 'x < 13')}{\Pr(x < 13)}$   
 $= \frac{\Pr(x < 11)}{\Pr(x < 13)}$   
 $= \frac{0.9772}{0.9999}$   
 $= 0.9772$

**c** 10.822

**d** 9.5792

**e**  $c_2 = 10.98; c_1 = 9.02$   
(assumed symmetrical about the mean)

## Solutions to Exercise 16D

1 a i 0.2525

ii 0.0478

iii  $\Pr(\text{IQ} > 130 \mid \text{IQ} > 110)$

$$= \frac{\Pr(\text{IQ} > 130)}{\Pr(\text{IQ} > 110)}$$

$$= \frac{0.0227 \dots}{0.2524 \dots}$$

$$= 0.0901$$

b 124.7

2 a i 0.7340

ii 0.8944

iii  $\Pr(> 170 \mid \text{between } 168 \text{ \& } 174)$

$$= \frac{\Pr(\text{between } 170 \text{ \& } 174)}{\Pr(\text{between } 168 \text{ \& } 174)}$$

$$= \frac{0.0655 \dots}{0.1185 \dots}$$

$$= 0.5531$$

b 170.25 cm

c 153.267 cm

3 a i 0.0766

ii 0.9998

iii 0.1531

b 57.3

4 a 10.56%

b 78.51%

5 mean = 1.55 kg; sd = 0.194 kg

6 a 36.9%

b 69

7 a 0.0228

b 0.0005

c If  $Y$  is the number with heights exceeding 190 cm, then  $Y$  is Binomial with  $n = 10$ ,  $P = 0.02275 \dots$

$$\Pr(Y \geq 2) = \Pr(2 \leq Y \leq 10)$$

$$= 0.0206$$

using a CAS calculator's 'bimom CAS' function.

8 a  $\Pr(X < 295) = 0.05$

$$\Pr(Z < \frac{295 - 300}{\sigma}) = 0.05$$

$$\frac{5}{\sigma} = -1.6449$$

$$\sigma = 3.04 \text{ grams}$$

b  $\Pr(X < 340) = 0.02$

$$\Pr(Z < \frac{340 - \mu}{5}) = 0.02$$

$$\frac{340 - \mu}{5} = -2.0537$$

$$\mu = 350.27 \text{ grams}$$

9 1004 ml

10 a small 0.1587

medium 0.7745

large 0.0668

**b** Expected cost

$$= 100 \times (\$2.80 \times 0.1587 + 3.50 \times 0.7745 + 5.00 \times 0.0688)$$

$$= \$348.92$$

**11 a i** 0.1169

**ii** 17.7

**b** 0.0284

**12 a** 0.0228, 0.1587

**b** Let  $x$  be the amount of chemical in a type A call, so  $x$  is normal with mean 10 and sd 1. Let  $Y$  be the amount of chemical in a type 1 cell, so  $y$  is normal with mean 14 and sd 2

$$\Pr(x < c) = \Pr(y > c)$$

$$\Pr\left(\frac{x-10}{1} < \frac{c-10}{1}\right) = \Pr\left(\frac{y-14}{2} > \frac{c-14}{2}\right)$$

$$\text{i.e. } \Pr(z < c-10) = \Pr\left(z > \frac{c-14}{2}\right)$$

where  $z$  has a standard normal distribution. Since the graph of  $y = f(z)$  is symmetrical about the  $y$ -axis, the number  $c-10$  and  $\frac{c-14}{2}$  are equidistant from the origin.

$$\text{Hence } \frac{c-14}{2} = -(c-10)$$

$$c-14 = -2c+20$$

$$3c = 34$$

$$c = \frac{34}{3}$$



## Solutions to Exercise 16E

**1**  $n = 100, p = \frac{1}{6}$   
 $\mu = np = \frac{100}{6} \approx 16.667$   
 $\sigma = \sqrt{np(1-p)} \approx 3.727$   
 $\Pr(X > 10) = 0.9632$       calculator

**2**  $n = 300, p = 0.5$   
 $\mu = np = 150$   
 $\sigma = \sqrt{np(1-p)} \approx 8.660$   
 $\Pr(X > 156) = 0.2442$       calculator

**3**  $n = 100, p = 0.1$   
 $\mu = np = 10$   
 $\sigma = \sqrt{np(1-p)} = 3$   
**a**  $\Pr(X \geq 15) = 0.0478$       calculator

**b**  $\Pr(X \leq 15) = 0.2525$       calculator

**4**  $n = 400, p = 0.4$

$\mu = np = 16$   
 $\sigma = \sqrt{np(1-p)} \approx 6.898$

**a**  $\Pr(10 \leq X < 20) = 0.7834$   
calculator

**b**  $\Pr(X \geq 15) = 0.0108$       calculator

**5**  $n = 200, p = 0.4$   
 $\mu = np = 80$   
 $\sigma = \sqrt{np(1-p)} \approx 6.928$   
 $\Pr(X < 76) = 0.2819$       calculator

**6**  $n = 25, p = 0.25$   
 $\mu = np = 6.25$   
 $\sigma = \sqrt{np(1-p)} \approx 2.165$

**a**  $\Pr(X \geq 10) = 0.0416$   
calculator

**b**  $\Pr(12 \leq X \leq 14) = 0.0038$       calculator

## Solutions to Technology-free questions

$$\begin{aligned} 1 \text{ a } \Pr(Z > a) &= 1 - \Pr(Z \leq a) \\ &= 1 - p \end{aligned}$$

$$\begin{aligned} \text{b } \Pr(Z < -a) &= \Pr(Z > a) \\ &= 1 - p \end{aligned}$$

$$\begin{aligned} \text{c } \Pr(-a \leq Z \leq a) &= \Pr(Z \leq a) \\ &\quad - \Pr(Z < -a) \\ &= P - (1 - p) \\ &= 2p - 1 \end{aligned}$$

$$\begin{aligned} 2 \text{ a } \Pr(X < 3) &= \Pr\left(\frac{X - 4}{1} < \frac{3 - 4}{1}\right) \\ &= \Pr(Z < -1) \end{aligned}$$

$$\text{So } a = -1$$

$$\begin{aligned} \text{b } \Pr(X > 5) &= \Pr\left(\frac{X - 4}{1} > \frac{5 - 4}{1}\right) \\ &= \Pr(Z > 1) \end{aligned}$$

$$\text{So } b = 1$$

$$\begin{aligned} \text{c } \Pr(x > 4) &= \Pr(Z > 0) \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} 3 \text{ } (x, y) &\rightarrow \left(\frac{x - \mu}{\sigma}, \sigma y\right) \\ \text{Sine } \mu &= 8 \text{ and } \sigma = 3, \text{ then} \\ (x, y) &\rightarrow \left(\frac{x - 8}{3}, 3y\right) \end{aligned}$$

$$\begin{aligned} 4 \text{ a } \Pr(x < a | x < b) &= \frac{\Pr('x < a' \cap 'x < b')}{\Pr(x < b)} \\ &= \frac{\Pr(x < a)}{\Pr(x < b)} \\ &= \frac{q}{p} \end{aligned}$$

$$\begin{aligned} \text{b } \Pr(X < 2\mu - a) &= \Pr\left(\frac{x - \mu}{\sigma} < \frac{\mu - a}{\sigma}\right) \\ &= \Pr\left(Z < \frac{\mu - a}{\sigma}\right) \\ &= \Pr\left(Z > \frac{a - \mu}{\sigma}\right) \\ &= 1 - \Pr\left(Z < \frac{a - \mu}{\sigma}\right) \end{aligned}$$

$$\text{Also, } \Pr(X < a) = q$$

$$\Pr\left(\frac{x - \mu}{\sigma} < \frac{a - \mu}{\sigma}\right) = q$$

$$\Pr\left(Z < \frac{a - \mu}{\sigma}\right) = q$$

$$\text{Hence } \Pr(X < 2\mu - a)$$

$$= 1 - \Pr\left(Z < \frac{a - \mu}{\sigma}\right)$$

$$= 1 - q$$

$$\begin{aligned} \text{c } \Pr(X > b | X > a) &= \frac{\Pr('x > b' \cap 'x > a')}{\Pr(x > a)} \\ &= \frac{\Pr(x > b)}{\Pr(x > a)} \\ &= \frac{1 - \Pr(x < b)}{1 - \Pr(x < a)} \\ &= \frac{1 - p}{1 - q} \end{aligned}$$

$$\begin{aligned} 5 \text{ a } \Pr(X < 5) &= \Pr\left(\frac{x-4}{2} < \frac{5-4}{2}\right) \\ &= \Pr\left(Z < \frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} \text{b } \Pr(X < 3) &= \Pr\left(\frac{x-4}{2} < \frac{3-4}{2}\right) \\ &= \Pr\left(Z < -\frac{1}{2}\right) \end{aligned}$$

$$\text{c } \Pr(X > 5) = \Pr\left(Z > \frac{1}{2}\right)$$

$$\begin{aligned} \text{d } \Pr(3 < X < 5) &= \Pr\left(-\frac{1}{2} < Z < \frac{1}{2}\right) \\ &= \Pr\left(-\frac{1}{2} < Z < 1\right) \end{aligned}$$

$$\text{e } \Pr(3 < X < 6) = \Pr\left(-\frac{1}{2} < Z < 1\right)$$

$$\begin{aligned} 6 \text{ a } \Pr(X < 2.55) &= \Pr\left(Z < \frac{2.55 - 2.5}{0.05}\right) \\ &= \Pr(Z < 1) \\ &= 0.84 \end{aligned}$$

$$\text{b } \Pr(X < 2.5) = 0.5 \quad \text{since } \mu = 2.5$$

$$\begin{aligned} \text{c } \Pr(X < 2.45) &= \Pr\left(Z < \frac{2.45 - 2.5}{0.05}\right) \\ &= \Pr(Z < -1) \\ &= \Pr(Z > 1) \\ &= 1 - \Pr(Z < 1) \\ &= 0.16 \end{aligned}$$

$$\begin{aligned} \text{d } \Pr(2.45 < X < 2.55) &= \Pr(-1 < Z < 1) \\ &= \Pr(Z < 1) - \Pr(Z < -1) \\ &= 0.84 - 0.16 \\ &= 0.68 \end{aligned}$$

$$\begin{aligned} 7 \text{ a } \Pr(W > 505) &= \Pr\left(Z > \frac{505 - 500}{5}\right) \\ &= \Pr(Z > 1) \\ &= 1 - \Pr(Z < 1) \\ &= 1 - 0.84 \\ &= 0.16 \end{aligned}$$

$$\begin{aligned} \text{b } \Pr(500 < W < 505) &= \Pr(0 < Z < 1) \\ &= \Pr(Z < 1) \\ &= -\Pr(Z < 0) \\ &= 0.84 - 0.5 \\ &= 0.34 \end{aligned}$$

$$\begin{aligned} \text{c } \Pr(W > 505 | W > 500) &= \frac{\Pr(W > 505)}{\Pr(W > 500)} \\ &= \frac{0.16}{0.5} \\ &= 0.32 \end{aligned}$$

$$\begin{aligned} \text{d } \Pr(W > 510) &= \Pr\left(Z > \frac{510 - 500}{5}\right) \\ &= \Pr(Z > 2) \\ &= 1 - \Pr(Z < 2) \\ &= 1 - 0.98 \\ &= 0.02 \end{aligned}$$

$$\begin{aligned} 8 \text{ a } \Pr(X < 6.5) &= \Pr(Z < 0.5) \\ &= 0.69 \end{aligned}$$

$$\begin{aligned} \text{b } \Pr(6 < X < 6.5) &= \Pr(0 < Z < 0.5) \\ &= 0.69 - 0.5 \\ &= 0.19 \end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad & \Pr(6.5 < X < 7) \\
& = \Pr(0.5 < Z < 1) \\
& = \Pr(Z < 1) - \Pr(Z < 0.5) \\
& = 0.84 - 0.69 \\
& = 0.15
\end{aligned}$$

$$\begin{aligned}
\mathbf{d} \quad & \Pr(5 < X < 7) = \Pr(-1 < Z < 1) \\
& = \Pr(Z < 1) \\
& \quad - \Pr(Z < -1) \\
& = 0.84 - (1 - 0.84) \\
& = 0.84 - 0.16 \\
& = 0.68
\end{aligned}$$

**9** The standardised scores are as follows.

$$\text{Test A: } \frac{62 - 50}{11} = \frac{12}{11} = 1.0909 \dots$$

$$\text{Test B: } \frac{64 - 48}{17} = 1$$

$$\text{Test C: } \frac{73 - 63}{8} = \frac{10}{8} = 1.25$$

So the best test was test C and the worst test was test B.

$$\mathbf{10} \quad \mathbf{a} \quad \Pr(X > 10) = \Pr(Z > 0) = 0.5$$

$$\begin{aligned}
\mathbf{b} \quad & \Pr(X > 13) = \Pr(Z > b) \\
& \therefore \Pr(Z > \frac{13 - 10}{2}) = \Pr(Z < b) \\
& \therefore \Pr(Z > 1.5) = \Pr(Z < b) \\
& \therefore \Pr(Z < -1.5) = \Pr(Z < b) \\
& \therefore b = -1.5
\end{aligned}$$

## Solutions to multiple-choice questions

- 1 A** The graph is symmetrical about the line  $x = 4$ , so  $\mu = 4$ .  
Almost all of the distribution lies between  $-5$  and  $13$ , i.e. 18 unit, so  
 $6\sigma = 18$   
 $\sigma = 3$
- 2 C** Use the *normCdf* command of a CAS calculator with lower bound 1.45 and upper bound  $\infty$ . This gives 0.0735 correct to 4 decimal places.
- 3 B** Use the *invNom* command of a CAS calculator with *Area* set to 0.25. This gives  $-0.6745$  correct to 4 decimal places.
- 4 B**  $X$  has mean 12 and variance 9, so the standard deviation is 3.  
$$\Pr(X > 15) = \Pr\left(\frac{x - 12}{3} > \frac{15 - 12}{3}\right)$$
$$= \Pr(Z > 1)$$
- 5 E** Use the *normCdf* command of a CAS calculator with lower bound 110, upper bound  $\infty$ ,  $\mu = 102$  and  $\sigma = 3$ . This gives 0.00383, so the percentage is about 0.38%
- 6 E** 10 goals is 6 below the mean of 16, and this is 3 standard deviations below the mean. Similarly, 22 goals is 3 standard deviations above the mean.  
So from 10 to 22 corresponds to  $\mu \pm 3\sigma$ , and this corresponds to approximately 99.7%.
- 7 C**  $(x, y) \rightarrow \left(\frac{x - \mu}{\sigma}, \sigma y\right)$   
Here  $\mu = 6$  and  $\sigma = 3$ , so  
 $(x, y) \rightarrow \left(\frac{x - 6}{3}, 3y\right)$
- 8 D** The given information means that  $\Pr(X > k) = 0.20$   
where  $x$  is normally distributed with  $\mu = 100$  and  $\sigma = 14$ .  
This can be re-written in the form  $\Pr(X \leq k) = 0.80$ .  
Use the *invNorm* command of a CAS calculator with *Area* set to 0.80 and the values 100 and 14 for the mean and standard deviation respectively. This gives  $k = 111.8$ , correct to one decimal place.
- 9 A** Angie's standardised scores are follows.  
Mathematics:  $\frac{72 - 72}{5} = 0$   
Indonesian:  $\frac{57 - 59}{4} = -1$   
Politics:  $\frac{68 - 64}{4} = 1$   
So her best subject was Politics, followed by Mathematics and then Indonesian.
- 10 D** Choosing equal areas in each tail means  $\Pr(X < c_1) = 0.05$ , and  $\Pr(X < c_2) = 0.95$   
(so that  $\Pr(c_1 < x < c_2) = 0.90$ ).  
Use the *invNorm* command of a CAS calculator with *Area* and 0.95 respectively and the values 11.3 for the mean and 2.9 for the standard deviation. This gives  $c_1 = 6.53$  and  $c_2 = 16.07$ , correct to two decimal places.

**11 D**  $\mu = ?, \sigma = 0.005$   
 $\Pr(X > 1) \approx 0.999$   
 $\therefore \Pr\left(Z > \frac{1 - \mu}{0.005}\right) \approx 0.999$   
 $\therefore \frac{1 - \mu}{0.005} = 3.0902$   
 $\therefore \mu = 1.015$

**12 C**  $\mu = 272, \sigma = ?$   
 $\Pr(X < 260) \approx 0.091$   
 $\therefore \Pr\left(Z < \frac{260 - 272}{\sigma}\right) \approx 0.091$   
 $\therefore \frac{12}{\sigma} = -1.3346$   
 $\therefore \sigma \approx 8.99$

## Solutions to extended-response questions

1  $\mu = 50, \sigma = 10$

Let  $X$  be the score.

For the top 10% consider

$$\Pr(X > k_1) = 0.1$$

$$\therefore \Pr(X \leq k_1) = 0.9$$

$$\Pr\left(Z \leq \frac{k_1 - 50}{10}\right) = 0.9$$

$$\therefore \frac{k_1 - 50}{10} = 1.2816$$

$$\therefore k_1 = 10 \times 1.2816 + 50$$

$$= 12.816 + 50 = 62.816$$

$\therefore$  63 and above indicate high aptitude.

For the next 20% consider

$$\Pr(X > k_2) = 0.3$$

$$\therefore \Pr(X \leq k_2) = 0.7$$

$$\therefore \Pr\left(Z \leq \frac{k_2 - 50}{10}\right) = 0.7$$

$$\frac{k_2 - 50}{10} = 0.5244$$

$$\therefore k_2 = 50 + 5.244 = 55.244$$

$\therefore$  Scores from 56 to 62 indicate moderate aptitude.

For the middle 40% consider

$$\Pr(X > k_3) = 0.7$$

$$\therefore \Pr(X \leq k_3) = 0.3$$

$$\Pr\left(Z \leq \frac{k_3 - 50}{10}\right) = 0.3$$

$\therefore$  From the diagram

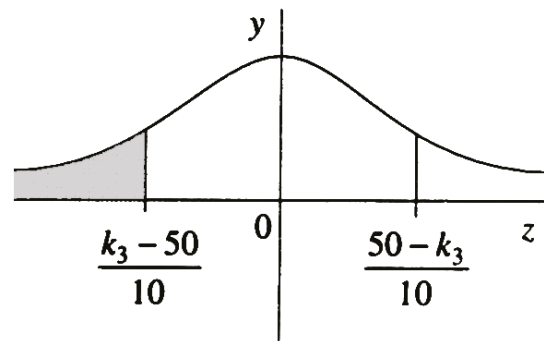
$$\Pr\left(Z \leq \frac{50 - k_3}{10}\right) = 0.7$$

$$\therefore \frac{50 - k_3}{10} = 0.5244$$

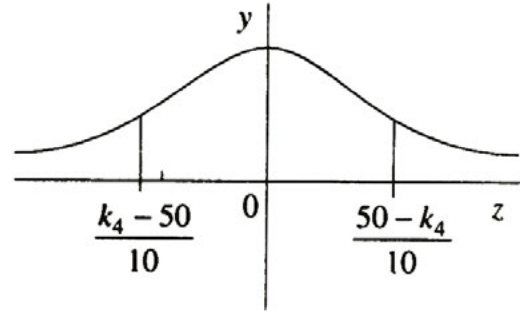
$$\therefore k_3 = 50 - 5.244 = 44.756$$

$\therefore$  Scores from 45 to 55 indicate average aptitude.

For the category of little aptitude consider



$$\begin{aligned} \Pr(X > k_4) &= 0.1 \\ \therefore \Pr\left(Z > \frac{k_4 - 50}{10}\right) &= 0.1 \\ \therefore \Pr\left(Z \leq \frac{50 - k_4}{10}\right) &= 0.9 \\ \frac{50 - k_4}{10} &= 1.2816 \\ 50 - k_4 &= 12.816 \\ \text{and } k_4 &= 50 - 12.816 \\ &= 37.184 \end{aligned}$$



Scores from 37 to 44 indicate little aptitude.  
 Scores less than 37 indicate no aptitude.  
 i.e. Scores 63 and above indicate high aptitude.  
 Scores from 56 to 62 indicate moderate aptitude.  
 Scores from 45 to 55 indicate average aptitude.  
 Scores from 37 to 44 indicate little aptitude.  
 Scores < 37 indicate no aptitude.

$$\begin{aligned} 2 \quad \Pr(\mu - k \leq X \leq \mu + k) &= 0.95 \\ \mu &= 10 \text{ and } \sigma = 2 \end{aligned}$$

$$\begin{aligned} \therefore \Pr(10 - k \leq X \leq 10 + k) &= 0.95 \\ \text{and transforming to the standard normal} \\ \Pr\left(\frac{10 - k - 10}{2} \leq Z \leq \frac{10 + k - 10}{2}\right) &= 0.95 \\ \text{i.e. } \Pr\left(-\frac{k}{2} \leq Z \leq \frac{k}{2}\right) &= 0.95 \end{aligned}$$

The graph of the standard normal with the region being considered is as shown:

$$\therefore \Pr\left(Z \leq \frac{k}{2}\right) - \Pr\left(Z \leq -\frac{k}{2}\right) = 0.95$$

$$\Pr\left(Z \leq \frac{k}{2}\right) - \left[1 - \Pr\left(Z \leq \frac{k}{2}\right)\right] = 0.95$$

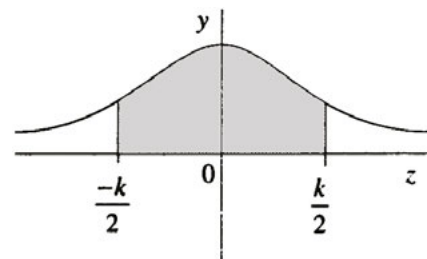
$$2\Pr\left(Z \leq \frac{k}{2}\right) - 1 = 0.95$$

$$\Pr\left(Z \leq \frac{k}{2}\right) = \frac{1.95}{2}$$

$$\text{i.e. } \Pr\left(Z \leq \frac{k}{2}\right) = 0.975$$

$$\therefore \frac{k}{2} = 1.96$$

$$\text{and } k = 3.92$$



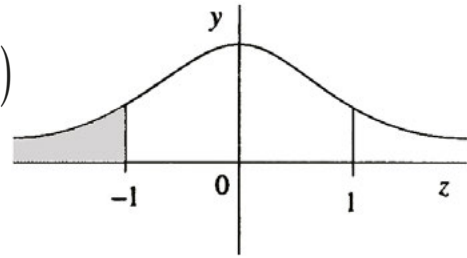


3  $\mu = 60\,000, \sigma = 5\,000$

a i Let  $X$  be the mileage for a tyre

$$\Pr(X \leq 55\,000) = \Pr\left(Z \leq \frac{55\,000 - 60\,000}{5\,000}\right)$$

where  $Z$  is the standard normal variable



$$\begin{aligned} &= \Pr\left(Z \leq -\frac{5\,000}{5\,000}\right) \\ &= \Pr(Z \leq -1) \\ &= 1 - 0.8413 \\ &= 0.1587 \end{aligned}$$

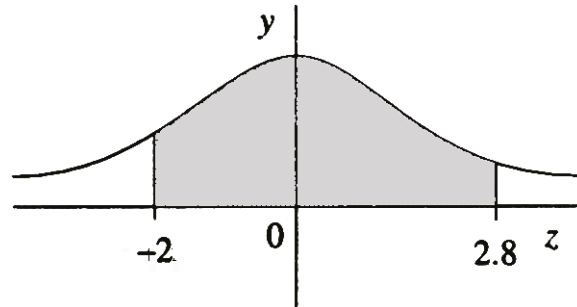
The proportion of the tyres which last less than 55 000 kilometres is 0.1587 or 15.87%

ii 
$$\Pr(50\,000 \leq X \leq 74\,000) = \Pr\left(\frac{50\,000 - 60\,000}{5\,000} \leq Z \leq \frac{74\,000 - 60\,000}{5\,000}\right)$$

$$= \Pr(-2 \leq Z \leq 2.8)$$

The required region is shown:

$$\begin{aligned} &\Pr(-2 \leq Z \leq 2.8) \\ &= \Pr(Z \leq 2.8) - \Pr(Z \leq -2) \\ &= \Pr(Z \leq 2.8) - [1 - \Pr(Z \leq 2)] \\ &= \Pr(Z \leq 2.8) + \Pr(Z \leq 2) - 1 \\ &= 0.99744 + 0.97725 - 1 \\ &= 0.9747 \end{aligned}$$

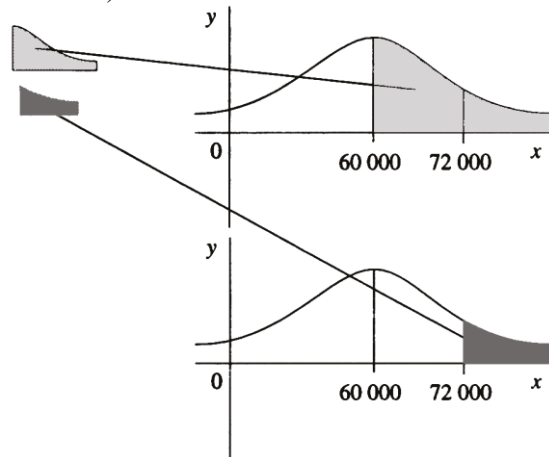


The proportion of tyres which last less than 74 000 kilometres but more than 50 000 is 0.9746 or 97.46%

$$\text{iii } \Pr(X \geq 72\,000 \mid X \geq 60\,000) = \frac{\Pr(X \geq 72\,000)}{\Pr(X \geq 60\,000)} \quad (\text{conditional probability})$$

The diagrams show that the required probability is given by Area divided by Area and transforming to the standard normal

$$\begin{aligned} \frac{\Pr(X \geq 72\,000)}{\Pr(X \geq 60\,000)} &= \frac{\Pr(Z \geq 2.4)}{\Pr(Z \geq 0)} \\ &= \frac{1 - \Pr(Z < 2.4)}{0.5} \\ &= \frac{1 - 0.9918}{0.5} \\ &= 0.0164 \end{aligned}$$

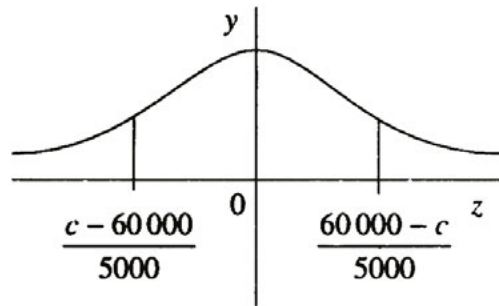


$$\text{b } \Pr(X \geq c) = 0.9$$

Transforming to the standard normal

$$\Pr\left(Z \geq \frac{c - 60\,000}{5000}\right) = 0.9$$

A graph of the standard normal curve helps:



$$\therefore \Pr\left(Z \geq \frac{c - 60\,000}{5000}\right) = \Pr\left(Z \leq \frac{60\,000 - c}{5000}\right)$$

$$\therefore \Pr\left(Z \leq \frac{60\,000 - c}{5000}\right) = 0.9$$

$$\therefore \frac{60\,000 - c}{5000} = 1.2816$$

$$\therefore 60\,000 - 5000 \times 1.2816 = c$$

$$\therefore c = 53\,592$$

The company's advertising manager can claim that 90% of their tyres last more than 53 592 kilometres.

$$\text{c } \Pr(X \geq 72\,000) = \Pr\left(Z \geq \frac{72\,000 - 60\,000}{5000}\right)$$

$$= \Pr\left(Z \geq \frac{12\,000}{5000}\right)$$

$$= \Pr(Z \geq 2.4)$$

$$= 1 - \Pr(Z < 2.4)$$

$$= 1 - 0.9918$$

$$= 0.0082$$

The probability of one tyre lasting more than 72 000 kilometres is 0.0082

The probability of 5 tyres lasting longer than 72 000 kilometres is  $(0.0082)^5 \approx 3.7 \times 10^{-11}$

### Graphic calculator techniques for question 3

- a i Choose **Normal Cdf** from the **Probability>Distributions** menu. Complete as shown.

Normal Cdf

Lower Bound:  $-\infty$

Upper Bound: 55000

$\mu$ : 60000

$\sigma$ : 5000

1.1 mm3&4..... sol

normCdf( $-\infty$ ,55000,60000,5000) 0.158655

1/99

- ii  $\Pr(X \geq c) = 0.9 \Leftrightarrow \Pr(X \leq c) = 0.1$   
Choose **Inverse Normal** from the **Probability>Distributions** menu. Complete as shown.

Inverse Normal

Area: 0.1

$\mu$ : 60000

$\sigma$ : 5000

1.1 mm3&4..... sol

normCdf( $-\infty$ ,55000,60000,5000) 0.158655

invNorm(0.1,60000,5000) 53592.2

2/99

4  $\mu = 15, \sigma = 0.75$

- a Let  $X$  be the number of litres/100 km used

$$\begin{aligned} \Pr(X \geq 18) &= \Pr\left(Z \geq \frac{18 - 15}{0.75}\right) \\ &= \Pr(Z \geq 4) \\ &= 0.0000317 = 3.17 \times 10^{-5} \end{aligned}$$

- b The manufacturer's claim is false.

c It is assumed, as in the text, that  $c_1$  and  $c_2$  are symmetric about the mean

$$\Pr(c_1 \leq X \leq c_2) = 0.95$$

$$\therefore \Pr\left(\frac{c_1 - 15}{0.75} \leq Z \leq \frac{c_2 - 15}{0.75}\right) = 0.95$$

$$\text{By symmetry } \frac{c_2 - 15}{0.75} = \frac{15 - c_1}{0.75}$$

$$\therefore \Pr\left(\frac{c_1 - 15}{0.75} \leq Z \leq \frac{15 - c_1}{0.75}\right) = 0.95$$

$$\therefore \Pr\left(Z \leq \frac{15 - c_1}{0.75}\right) - \left[\Pr\left(Z \leq \frac{15 - c_1}{0.75}\right) - 1\right] = 0.95$$

$$\text{i.e. } 2 \Pr\left(Z \leq \frac{15 - c_1}{0.75}\right) = 0.95 + 1$$

$$\Pr\left(Z \leq \frac{15 - c_1}{0.75}\right) = 0.975$$

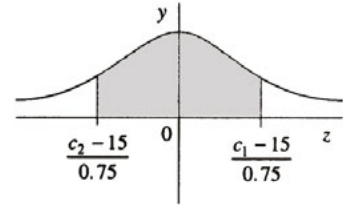
$$\therefore \frac{15 - c_1}{0.75} = 1.96$$

$$\therefore c_1 = 15 - 0.75 \times 1.96$$

$$= 13.53$$

$$\text{and by symmetry } c_2 = 15 + 0.75 \times 1.96$$

$$= 16.47$$



5 Let  $L$  be the useful life of a fluorescent tube

$$\mu = 600, \sigma = 4$$

$$\Pr(L \geq 605) = \Pr\left(Z \geq \frac{605 - 600}{4}\right)$$

$$= \Pr\left(Z \geq \frac{5}{4}\right)$$

$$= \Pr(Z \geq 1.25)$$

$$= 1 - \Pr(Z \leq 1.25)$$

$$= 1 - 0.8944$$

$$= 0.1056$$

The situation of ten tubes is described by a binomial distribution.

Let  $X$  be the number of tubes in a box which last longer than 605 hours.

$X$  is the random variable of a binomial distribution with  $n = 10$  and  $p = 0.1056$

$$\begin{aligned}
\Pr(X \geq 3) &= 1 - [\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)] \\
&= 1 - [0.3275 + 0.3868 + 0.2055] \\
&= 1 - 0.9198 \\
&= 0.0802
\end{aligned}$$

The probability of at least three tubes in a randomly selected box lasting longer than 605 hours is 0.0802

**6** Let  $L$  be the amount (mg) for a lethal dose

$$\mu = 110, \sigma = 20$$

Let  $D$  be the amount (mg) for a surgical anaesthesia

$$\mu = 50, \sigma = 10$$

Let  $c$  mg be the dose such that 90% of patients need less than this amount for surgical anaesthesia

$$\text{i.e. } \Pr(D \leq c) = 0.9$$

Transforming to the standard normal

$$\Pr\left(Z \leq \frac{c - 50}{10}\right) = 0.9$$

$$\frac{c - 50}{10} = 1.2816$$

$$\therefore c = 10 \times 1.2816 + 50$$

$$= 12.816 + 50$$

$$= 62.816$$

To find what percentage of patients would be killed by these amounts consider

$$\Pr(L \leq 62.816) = \Pr\left(Z \leq \frac{62.816 - 110}{20}\right)$$

$$= \Pr(Z \leq -2.3592)$$

$$= 1 - \Pr(Z \leq 2.3592)$$

$$= 1 - 0.9908$$

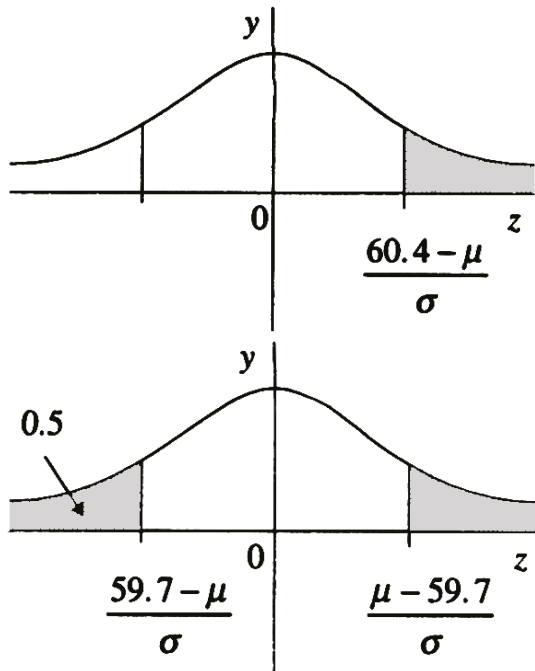
$$= 0.0092$$

i.e. 0.92% of patients would be killed by a dose of 62.816 mg or less.

7 a Let  $X$  be the length of the dimension

$$\Pr(X > 60.4) = 0.03$$

$$\Pr(X < 59.7) = 0.05$$



$$\Pr\left(Z < \frac{60.4 - \mu}{\sigma}\right) = 0.97 \quad \Pr\left(Z < \frac{\mu - 59.7}{\sigma}\right) = 0.95$$

$$\therefore \frac{60.4 - \mu}{\sigma} = 1.88079 \quad \text{and} \quad \frac{\mu - 59.7}{\sigma} = 1.64485$$

$$\therefore 60.4 - \mu = 1.88079\sigma \quad \text{①} \quad \text{and} \quad \mu - 59.7 = 1.64485\sigma \quad \text{②}$$

Add equations ① and ②

$$0.7 = 3.52564\sigma$$

$\therefore \sigma = 0.19854$ , i.e.  $\sigma = 0.2$ , correct to one decimal place.

Substitute in ①

$$60.4 - \mu = 1.88079\sigma$$

$$\therefore \mu = 60.4 - 1.88079\sigma$$

$$= 60.02658, \text{ i.e. } \mu = 60.0, \text{ correct to one decimal place.}$$

$$\begin{aligned}
\text{b } \Pr(X > 60.3) + \Pr(X < 59.6) &= \Pr\left(Z > \frac{60.3 - 60.02658}{0.19854}\right) \\
&\quad + \Pr\left(Z < \frac{59.6 - 60.02658}{0.19854}\right) \\
&= \Pr(Z > 1.37715) + \Pr(Z < -2.14858) \\
&= 1 - \Pr(Z < 1.37715) + 1 - \Pr(Z < 2.14858) \\
&= 2 - \Pr(Z < 1.37715) - \Pr(Z < 2.14858) \\
&= 2 - 0.915767 - 0.98416 \\
&= 0.1
\end{aligned}$$

These the percentage of rejects is 10%.

8 Let  $H$  denote the hardness of the metal  
 $\mu = 70$  and  $\sigma = 3$

$$\begin{aligned}
\text{a } \Pr(65 \leq H \leq 75) &= \Pr\left(\frac{65 - 70}{3} \leq Z \leq \frac{75 - 70}{3}\right) \\
&= \Pr\left(-\frac{5}{3} \leq Z \leq \frac{5}{3}\right) \\
&= \Pr\left(Z \leq \frac{5}{3}\right) - \Pr\left(Z \leq -\frac{5}{3}\right) \\
&= \Pr\left(Z \leq \frac{5}{3}\right) - \left[1 - \Pr\left(Z \leq \frac{5}{3}\right)\right] \\
&= 2\Pr\left(Z \leq \frac{5}{3}\right) - 1 \\
&= 0.9044
\end{aligned}$$

The probability that a randomly chosen specimen has acceptable hardness is 0.9044.

$$\text{b } \Pr(70 - c \leq H \leq 70 + c) = 0.95$$

$$\text{implies } \Pr\left(\frac{70 - c - 70}{3} \leq Z \leq \frac{70 + c - 70}{3}\right) = 0.95$$

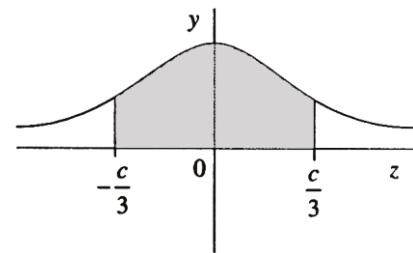
$$\Pr\left(-\frac{c}{3} \leq Z \leq \frac{c}{3}\right) = 0.95$$

$$\therefore 2\Pr\left(Z \leq \frac{c}{3}\right) - 1 = 0.95$$

$$\Pr\left(Z \leq \frac{c}{3}\right) = 0.975$$

$$\therefore \frac{c}{3} = 1.96$$

$$c = 5.88$$



c Let  $X$  be the number of acceptable specimens out of 10 randomly selected

specimens.

$X$  is a binomial random variable with  $n = 10$  and  $p = 0.9044$

$$E(X) = np = 9.044$$

The expected number of acceptable specimens is 9.044.

$$\begin{aligned} \text{d } \Pr(H < 73.84) &= \Pr\left(Z < \frac{73.84 - 70}{3}\right) \\ &= \Pr\left(Z < \frac{3.84}{3}\right) \\ &= \Pr(Z < 1.28) \\ &= 0.8997 \end{aligned}$$

Let  $X$  be the number of specimens out of the ten selected which have a hardness less than 73.84.

$$\begin{aligned} \Pr(X \leq 8) &= 1 - [\Pr(X = 9) + \Pr(X = 10)] \\ &= 1 - \binom{10}{9} (0.8997)^9 (0.1003) - (0.8997)^{10} \\ &= 0.2651 \text{ (to four decimal places)} \end{aligned}$$

e Let  $P$  be profit. The probability distribution for  $P$

$P$	20	-5
$\Pr(P = p)$	0.9044	0.0956

$$\begin{aligned} \therefore E(P) &= 20 \times 0.9044 - 5 \times 0.0956 \\ &= 17.61 \end{aligned}$$

The expected profit is \$17.61.

$$\begin{aligned} E(P^2) &= 400 \times 0.9044 + 25 \times 0.0956 \\ &= 364.15 \end{aligned}$$

$$\begin{aligned} \therefore \text{Var}(P) &= E(P^2) - [E(P)]^2 \\ &= 364.15 - 310.1121 \\ &= 54.04 \end{aligned}$$

9 Let  $\mu$  be the mean lifetime for a watch and  $\sigma$  the standard deviation.

a The mean error is 0

Let  $X$  be the error

$$\Pr(-5 \leq X \leq 5) = 0.94$$

$$\therefore \Pr\left(\frac{-5}{\sigma} \leq Z \leq \frac{5}{\sigma}\right) = 0.94$$

$$\therefore 2\Pr\left(Z \leq \frac{5}{\sigma}\right) - 1 = 0.94$$



$$\begin{aligned}\Pr\left(Z \leq \frac{5}{\sigma}\right) &= \frac{1.94}{2} \\ \frac{5}{\sigma} &= 1.8808 \\ \therefore \sigma &= \frac{5}{1.8808} \\ &= 2.658\end{aligned}$$

**b** Let  $Y$  be the number of watches rejected out of a batch of 10 watches.

This is a Binomial distribution with  $p = 0.06$  and  $n = 10$

$$\Pr(Y < 2) = \Pr(Y = 0) + \Pr(Y = 1)$$

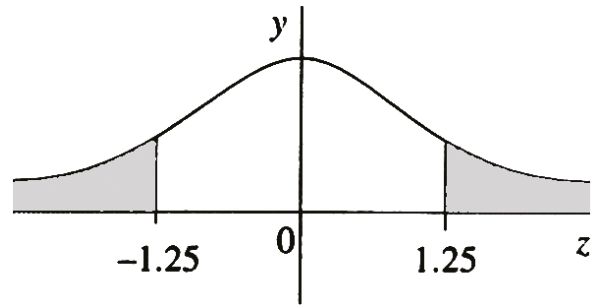
$$= (0.94)^{10} + \binom{10}{1}(0.06)(0.94)^9$$

$$= 0.5386 + 0.3438$$

$$= 0.882$$

**10 a** Let  $X$  be the number of litres in a standard bottle.

$$\begin{aligned}\Pr(X < 0.75) &= \Pr\left(Z < \frac{0.75 - 0.76}{0.008}\right) \\ &= \Pr(Z < -1.25) \\ &= 1 - \Pr(Z < 1.25) \\ &= 0.1056\end{aligned}$$



**b** Let  $N$  be the number of bottles out of ten which contain less than 0.75 litres. This is a binomial random variable with  $n = 10$  and  $p = 0.10565$ .

$$\therefore \Pr(N \geq 3) = 1 - [\Pr(N = 0) + \Pr(N = 1) + \Pr(N = 2)]$$

$$= 1 - 0.9197$$

$$= 0.0803$$

**c** Let  $Y$  be the number of litres in a large bottle.

$$\text{Define } W = 4X - 3Y$$

We require  $\Pr(W) \geq 0$

$$\text{i.e. } \Pr(4X - 3Y \geq 0)$$

$$\begin{aligned}\text{Note: } E(W) &= 4E(X) - 3E(Y) \\ &= 0.01\end{aligned}$$

$$\begin{aligned}\text{Var}(W) &= 16\text{Var}(X) + 9\text{Var}(Y) \\ &= 16 \times (0.008)^2 + 9 \times (0.009)^2 \\ &= 0.001753\end{aligned}$$

$$\therefore \text{sd}(W) = 0.04187$$

$$\begin{aligned}\therefore \Pr(W > 0) &= \Pr\left(Z > \frac{0.01}{0.04187}\right) \\ &= \Pr(Z > -0.23883) \\ &= \Pr(Z < 0.23883) \\ &= 0.5944\end{aligned}$$

# Chapter 17 – Sampling and estimation

## Solutions to Exercise 17A

- 1 No; sample will be biased towards the type of movie being shown.
- 2 a No; biased towards shoppers.  
b Randomly select a sample from telephone lists or an electoral roll.
- 3 No; only interested people will call, and they may call more than once.
- 4 a No; biased towards older, friendly or sick guinea pigs which may be easier to catch.  
b Number guinea pigs and then generate random numbers to select a sample.
- 5 No; a student from a large school has less chance of being selected than a student from a small school.
- 7 a Unemployed will be under represented.  
b Unemployed or employed may be under represented, depending on time of day.  
c Unemployed will be over represented.  
Use random sampling based on the whole population (e.g. electoral roll).
- 8 a Divide platform into a grid of  $1 \text{ m}^2$  squares. Select squares using a random number generator to give two digits, one a vertical reference and one a horizontal reference.  
b Yes, if crabs are fairly evenly distributed; otherwise, five squares may not be enough.
- 9 No; a parent's chance of selection depends on how many children they have at the school.
- 10 Not a random sample; only interested people will call, and they may call more than once.
- 11 People who go out in the evenings will not be included in the sample.
- 12 a All students at this school  
b  $p = 0.35$   
c  $\hat{p} = 0.42$
- 13 a 0.22  
b  $\hat{p}$

## Solutions to Exercise 17B

**1 a**  $p = \frac{5}{10} = \frac{1}{2}$

**b**  $0, \frac{1}{3}, \frac{2}{3}, 1$

**c**  $\Pr(\hat{P} = 0) = \frac{\binom{5}{0}\binom{5}{3}}{\binom{10}{3}} = \frac{1}{12}$

$\Pr(\hat{P} = \frac{1}{3}) = \frac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} = \frac{5}{12}$

$\Pr(\hat{P} = \frac{2}{3}) = \frac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} = \frac{5}{12}$

$\Pr(\hat{P} = 1) = \frac{\binom{5}{3}\binom{5}{0}}{\binom{10}{3}} = \frac{1}{12}$

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

**d**  $\Pr(\hat{P} > 0.5) = \frac{5}{12} + \frac{1}{12} = \frac{1}{2}$

**2 a**  $p = \frac{12}{20} = \frac{3}{5}$

**b** Values of  $\hat{P}$ :  $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

**c**

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$\Pr(\hat{P} = \hat{p})$	0.0036	0.0542	0.2384
$\hat{p}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	0.3973	0.2554	0.0511

**d**  $\Pr(\hat{P} > 0.7) = 0.2554 + 0.0511$   
 $= 0.3065$

$$\text{e } \Pr(\hat{P} < 0.7 | \hat{P} > 0) = \frac{\Pr(0 < \hat{P} < 0.7)}{\Pr(\hat{P} > 0)} = 0.6924$$

**3 a**  $p = 0.5$

**b** Values of  $\hat{P}$ :  $0, \frac{1}{2}, \frac{2}{3}, 1$

**c**

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.1	0.4	0.4	0.1

**d**  $\Pr(\hat{P} > 0.25) = 0.9$

**4 a**  $p = 0.4$

**b** Values of  $\hat{P}$ :  $0, \frac{1}{3}, \frac{2}{3}, 1$

**c**

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

**d**  $\Pr(\hat{P} > 0.5) = \frac{1}{3}$

**e**  $\Pr(\hat{P} < 0.5 | \hat{P} > 0)$   
 $= \frac{\Pr(0 < \hat{P} < 0.5)}{\Pr(\hat{P} > 0)} = \frac{4}{5}$

**5 a**  $p = 0.5$

**b** Values of  $\hat{P}$ :  $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

**c**

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

**d**  $\Pr(\hat{P} > 0.7) = \frac{5}{16}$

6 a Values of  $\hat{P}$  :  $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

b

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

c  $\Pr(\hat{P} < 0.4) = \frac{3}{16}$

d  $\Pr(\hat{P} > 0 | \hat{P} < 0.8)$   
 $= \frac{\Pr(0 < \hat{P} < 0.8)}{\Pr(\hat{P} < 0.8)} = \frac{25}{26}$

7 a Values of  $\hat{P}$  :  $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

b

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

c  $\Pr(\hat{P} > 0.5 | \hat{P} > 0)$   
 $= \frac{\Pr(\hat{P} > 0.5)}{\Pr(\hat{P} > 0)} = \frac{17}{369}$

8

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$$E(X) = 0 \times \frac{1}{16} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{8} + \frac{3}{4} \times \frac{1}{4} + 1 \times \frac{3}{16} = 0.5$$

$$E(X^2) = 0^2 \times \frac{1}{16} + \left(\frac{1}{4}\right)^2 \times \frac{1}{4} + \left(\frac{1}{2}\right)^2 \times \frac{3}{8} + \left(\frac{3}{4}\right)^2 \times \frac{1}{4} + (1)^2 \times \frac{1}{16} = \frac{5}{16}$$

$$\therefore \text{Var}(X) = \left(\frac{5}{16}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{16}$$

$$\therefore \text{sd}(x) = \frac{1}{4}$$

9

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

$$E(X) = 0 \times \frac{1}{32} + \frac{1}{5} \times \frac{5}{32} + \frac{2}{5} \times \frac{5}{16} + \frac{3}{5} \times \frac{5}{16} + \frac{4}{5} \times \frac{5}{32} + 1 \times \frac{1}{32} = 0.5$$

$$E(X^2) = 0^2 \times \frac{1}{32} + \left(\frac{1}{5}\right)^2 \times \frac{5}{32} + \left(\frac{2}{5}\right)^2 \times \frac{5}{16} + \left(\frac{3}{5}\right)^2 \times \frac{5}{16} + \left(\frac{4}{5}\right)^2 \times \frac{5}{32} + (1)^2 \times \frac{1}{32} = 0.340176$$

$$\therefore \text{Var}(X) = 0.340176 - \left(\frac{1}{2}\right)^2 = 0.050176$$

$$\therefore \text{sd}(X) = 0.224$$

$$\mu = 0.5, \sigma = 0.224$$

**10**

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

$$E(X) = 0 \times \frac{256}{625} + \frac{1}{4} \times \frac{256}{625} + \frac{3}{4} \times \frac{16}{625} + 1 \times \frac{1}{625} = 0.2$$

$$E(X^2) = 0^2 \times \frac{256}{625} + \left(\frac{1}{4}\right)^2 \times \frac{256}{625} + \left(\frac{3}{4}\right)^2 \times \frac{16}{625} + (1)^2 \times \frac{1}{625} = 0.08$$

$$\therefore \text{Var}(X) = 0.08 - (0.02)^2 = 0.04$$

$$\therefore \text{sd}(X) = 0.2$$

$$\mu = 0.2, \sigma = 0.2$$

**11**  $n = 30, p = 0.4, \mu = p = 0.3, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.3 \times 0.7}{30}} = 0.084$   
 $\mu = 0.3, \sigma = 0.084$

**12**  $n = 100, p = 0.4, \mu = p = 0.4, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.4 \times 0.6}{100}} = 0.049$   
 $\mu = 0.4, \sigma = 0.049$

**13**  $n = 100, p = 0.2, \mu = p = 0.2, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.2 \times 0.8}{100}} = 0.04$   $\mu = 0.2,$   
 $\sigma = 0.04$

**14 a**  $p = 0.65, n = 20$

$$\Pr(\hat{P} = 0.65) = \Pr(X = 13) = 0.1844$$

**b**  $\mu = 0.65, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.65 \times 0.35}{20}} = 0.1066$

$$\mu - \sigma = 0.543$$

$$\mu + \sigma = 0.757$$

$$\begin{aligned}\Pr(0.543 < \hat{P} < 0.757) &= \Pr(10.86 < X < 15.14) \\ &= \Pr(11 \leq X \leq 15) \\ &= 0.7600\end{aligned}$$

**c**  $\mu - 2\sigma = 0.4368$

$$\mu + 2\sigma = 0.8632$$

$$\begin{aligned}\Pr(0.4368 < \hat{P} < 0.8632) &= \Pr(8.74 < X < 17.26) \\ &= \Pr(9 \leq X \leq 17) \\ &= 0.9683\end{aligned}$$



## Solutions to Exercise 17C

**1**  $p = 0.5, n = 50$

$$\mu = 0.5, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.5 \times 0.5}{50}} = 0.0707$$

$$\Pr(\hat{P} < 0.46) \approx \Pr\left(Z \leq \frac{0.46 - 0.5}{0.0707}\right) = \Pr(Z \leq -0.5658)$$

The calculation can also be done directly with calculator:

$$\Pr(\hat{P} < 0.46) \approx 0.2858$$

**2**  $p = 0.12, n = 300$

$$\mu = 0.12, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.12 \times 0.88}{300}} = 0.018762$$

The calculation can also be done directly with calculator:

$$\Pr(\hat{P} > 0.1) \approx 0.8568$$

**3**  $p = 0.5, n = 25$

$$\mu = 0.5, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.5 \times 0.5}{25}} = 0.1$$

The calculation can also be done directly with calculator:

$$\Pr(\hat{P} > 0.6) \approx 0.1587$$

**4**  $p = 0.1, n = 200$

$$\mu = 0.1, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.1 \times 0.9}{200}} = 0.0212$$

The calculation can also be done directly with calculator:

$$\Pr(\hat{P} > 0.15) \approx 0.0092$$

**5**  $p = 0.3, n = 50$

$$\mu = 0.3, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.3 \times 0.7}{50}} = 0.0648$$

The calculation can also be done directly with calculator:

$$\Pr(\hat{P} < 0.2) \approx 0.0614$$

**6**  $p = 0.6, n = 100$

$$\mu = 0.6, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.6 \times 0.4}{100}} = 0.0490$$

**a**  $\Pr(\hat{P} < 0.8) \approx 1$

**b**  $\Pr(0.6 < \hat{P} < 0.8) \approx 0.5$

**c**  $\Pr(0.7 < \hat{P} < 0.8 | \hat{P} > 0.6) = \frac{\Pr(0.7 < \hat{P} < 0.8)}{\Pr(\hat{P} > 0.6)} \approx 0.0412$

**7**  $p = 0.5, n = 100$

$$\mu = 0.5, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.5 \times 0.5}{100}} = 0.05$$

The calculation can also be done directly with calculator:

$$\Pr(0.4 < \hat{P} < 0.6) \approx 0.9545$$

**8**  $p = 0.1, n = 1000$

$$\mu = 0.1, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.1 \times 0.9}{1000}} = 0.0095$$

**a**  $\Pr(0.08 < \hat{P} < 0.12) \approx 0.9650$

$$\mathbf{b} \quad \frac{\Pr(0.08 < \hat{P} < 0.12 | \hat{P} > 0.10)}{\Pr(\hat{P} < 0.12)} \approx 0.9650$$

$$\mathbf{9} \quad p = 0.52, n = 400$$

$$\mathbf{a} \quad \hat{p} = \frac{230}{400} = 0.575$$

$$\mathbf{b} \quad \mu = 0.52, \sigma = \sqrt{\frac{p(1-p)}{n}} = 0.0350$$

$$\Pr(\hat{P} \geq 0.575) \approx 0.0139$$

$$\mathbf{10} \quad p = 0.9, n = 250$$

$$\mathbf{a} \quad \hat{p} = \frac{212}{250} = 0.848$$

$$\mathbf{b} \quad \mu = 0.9, \sigma = \sqrt{\frac{p(1-p)}{n}} = 0.0190$$
$$\Pr(\hat{P} \leq 0.848) \approx 0.0031$$

$\mathbf{c}$  Yes, because the chance of the battery lasting only this short period of time is very small if the manufacturers claim is correct.

## Solutions to Exercise 17D

1 a 0.08

b (0.0268, 0.1332)

2 a 0.192

b (0.1432, 0.2408)

3 a 0.2

b (0.1216, 0.2784)

4 (0.28395, 0.3761)

5 a (0.4761, 0.5739)

b (0.5095, 0.5405)

c The second interval is narrower because the sample size is larger

6 a (0.8035, 0.8925)

b (0.8839, 0.8621)

c The second interval is narrower because the sample size is larger

7  $M = 0.02, \hat{p} = 0.8$

$$n = \left(\frac{1.96}{0.02}\right)^2 \times 0.8 \times 0.3 = 1536.64$$

Since  $n$  must be an integer larger than the calculated value to ensure the margin of error is no more than 0.02,  $n = 1537$

8  $M = 0.05, \hat{p} = 0.2$   $n =$

$$\left(\frac{1.96}{0.05}\right)^2 \times 0.2 \times 0.8 = 245.86$$

Since  $n$  must be an integer larger than the calculated value to ensure the margin of error is no more than 0.05,  $n = 246$

9  $p^* = 0.30$

$$M = 0.03$$

$$n = \left(\frac{1.96}{0.03}\right)^2 \times 0.3 \times 0.7 = 896.37$$

Since  $n$  must be an integer larger than the calculated value to ensure the margin of error is no more than 0.03,  $n = 897$

b  $M = 0.02, \left(\frac{1.96}{0.02}\right)^2 \times 0.3 \times 0.7 = 2017$

c Reducing margin of error by 1% requires the sample size to be more than doubled

10 a  $p^* = 0.3, M = 0.02$   $n =$

$$\left(\frac{1.96}{0.02}\right)^2 \times 0.3 \times 0.7 = 2016.94 \approx 2017$$

b  $p^* = 0.5, M = 0.02$

$$n = \left(\frac{1.96}{0.02}\right)^2 \times 0.5 \times 0.5 \approx 2401$$

c i  $p^* = 0.3, n = 2401$

$$M = 1.96 \sqrt{\frac{0.3 \times 0.7}{2401}} \approx 1.8$$

The margin of error is less than 2%

ii  $p^* = 0.5, n = 2017$

$$M = 1.96 \sqrt{\frac{0.5 \times 0.5}{2017}} \approx 2.2$$

The margin of error is greater than 2%

d 2401, as this ensures that  $M$  is 2% or less, whoever is correct

**11** 90%: (0.5194, 0.6801),  
95%: (0.5034, 0.6940),  
99%: (0.4738, 0.7262); Interval  
width increases as confidence level  
increases

**12** 90%: (0.5111, 0.5629),  
95%: (0.5061, 0.5679),  
99%: (0.4964, 0.5776); Interval  
width increases as confidence level  
increases

## Solutions to Technology-free questions

1 a All employees of the company

$$\therefore M = \frac{0.588}{\sqrt{n}}$$

b  $p = 0.35$

c Margin of error would decrease by a factor of  $\sqrt{2}$

c  $\hat{p} = 0.40$

2 a No; only people already interested in yoga

5 a  $40 \times 0.95 = 38$

b Use electoral roll

$$\text{b } \Pr(Y = 40) = \binom{40}{40}(0.95)^{40}(0.05)^0 = (0.95)^{40}$$

3 a  $\frac{k}{100}$

6 a  $50 \times 0.95 = 45$

$$\begin{aligned} \text{b } \hat{p} \pm 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= \frac{k}{100} \pm 1.96 \sqrt{\frac{\frac{k}{100}(1-\frac{k}{100})}{100}} \\ &= \frac{k}{100} \pm \frac{1.96 \sqrt{k(100-k)}}{1000} \end{aligned}$$

b

$$\begin{aligned} \Pr(Y \geq 49) &= \Pr(Y = 49) + \Pr(Y = 50) \\ &= \binom{50}{49}(0.1)^1(0.9)^{49} + \binom{50}{50}(0.1)^0(0.9)^{50} \\ &= 5(0.9)^{49} + (0.9)^{50} \\ &= 5.9(0.9)^{49} \end{aligned}$$

4 a  $\hat{p} = 0.9$

7 a  $\hat{p} = 0.60$

$$\begin{aligned} \text{b } M &= 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{100}} \\ &= 1.96 \times \sqrt{\frac{0.9 \times 0.1}{n}} \\ &= 1.96 \times \frac{0.3}{\sqrt{n}} \end{aligned}$$

b  $M = 0.10$

c Increase sample size

## Solutions to multiple-choice questions

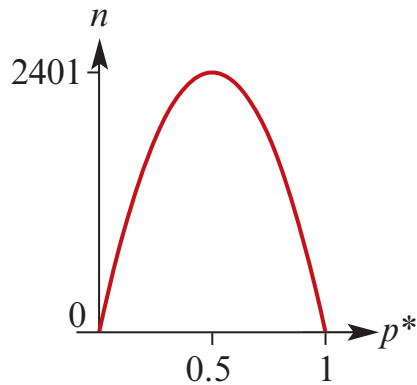
- 1 B** This class is a sample of the whole school population, so any statistics determined from this sample is called a sample statistic.
- 2 C** When the statistics is calculated from the whole population it is known as a population parameter.
- 3 D** All we can say about a 95% confidence interval is that 95% of such intervals will capture the true mean. Statement B is a common incorrect interpretation of a confidence interval.
- 4 E** 
$$M = 1.96 \sqrt{\frac{\hat{p}(1 - \hat{p})}{100}}$$

$$= 1.96 \times \sqrt{\frac{0.3 \times 0.7}{50}}$$

$$= 0.1270$$
- 5 C**  $\hat{p} = \frac{4}{50} = 0.08$   
95% CI = (0.005, 0.155) Use of calculator
- 6 E**  $\hat{p} = \frac{14}{88} = 0.08$   
95% CI = (0.083, 0.236) Use of calculator
- 7 B** As to be more confidence of capturing the true mean the interval will be wider.
- 8 E** I the centre of a confidence interval is a sample parameter not a population parameter  
II the bigger the margin of error the bigger the confidence interval  
III a point estimate is a single value estimate like  $\hat{p}$   
IV the sample proportion a point estimate
- 9 C** Since the width of the confidence interval is inversely proportional to the square root of the sample size, increasing the sample size by a factor of 4 decreases the width by a factor of 2.
- 10 E**  $M = 0.03$   $n = \left(\frac{1.96}{0.03}\right)^2 \times 0.3 \times 0.7 = 896.37 \approx 897$
- 11 A** See definitions
- 12 B** A sampling distribution is the distribution of a sample statistic, and as such shows how this statistic varies from sample to sample.
- 13 C**  $\hat{p} = 0.78, n = 100$   
95% CI = (0.6988, 0.8682)
- 14 D** The width of a confidence interval will decrease if the sample size is increased, or if the level of confidence is decreased.

## Solutions to extended-response questions

$$\begin{aligned}
 \mathbf{1 \ a} \quad n &= \left(\frac{1.96}{M}\right)^2 p^*(1 - p^*) \quad 0 \leq p^* \leq 1 \\
 &= \left(\frac{1.96}{0.02}\right)^2 p^*(1 - p^*) \\
 &= 9604^2 p^*(1 - p^*)
 \end{aligned}$$



**b** From the graph, the maximum occurs when  $p^* = 0.5$

**c** If they use the maximum samples size (2401) then they will ensure the margin of error stays within the desired range of  $\pm 2\%$

$$\mathbf{2} \quad p = 0.6, n = 100$$

$$\mu = \hat{p} = 0.6, \sigma = \sqrt{\frac{\hat{p}(1 - \hat{p})}{1n}} = 0.0490$$

$$\mathbf{a} \quad \Pr(\hat{P} > 0.65) = 0.1537$$

$$\mathbf{b} \quad \Pr(0.5 < \hat{P} < 0.65) = 0.8257$$

$$\mathbf{3 \ a} \quad \hat{p} = 0.57, n = 100$$

$$95\% \text{ CI} = (0.4730, 0.6670)$$

$$\mathbf{b \ i} \quad \Pr(Y = 5) = \binom{5}{5}(0.95)^5(0.05)^0 = 0.7738$$

$$\mathbf{ii} \quad \Pr(Y = 0) = \binom{5}{0}(0.95)^0(0.05)^5 = 0.0000003$$

$$\mathbf{iii} \quad \Pr(Y \leq 4) = 0.2262$$

$$\mathbf{iv} \quad 0.95 \times 5 = 4.75$$

$$\mathbf{c} \quad n = 500$$

$$X = 57 + 67 + 72 + 55 + 60 = 311$$

$$\hat{p} = \frac{311}{500} = 0.622$$
$$\text{CI} = (0.5795, 0.6645)$$

**4 a**  $p = \frac{500}{N}$

**b**  $\hat{p} = \frac{60}{400} = 0.15$

**c**  $\frac{500}{N} = 0.15 \quad N \approx \frac{500}{0.15} = 3333.33 \approx 3333$

**d** 95% CI for  $\hat{p}$

$$0.15 - 1.96 \sqrt{\frac{0.15 \times 0.85}{400}} < p < 0.15 + 1.96 \sqrt{\frac{0.15 \times 0.85}{400}}$$
$$0.15 - 1.96 \sqrt{\frac{0.1275}{400}} < p < 0.15 + 1.96 \sqrt{\frac{0.1275}{400}}$$

**e**  $0.1150 < \frac{500}{N} < 0.1850$

$$5.4056 < \frac{500}{N} < 8.6951$$
$$2703 < N < 4348$$



# Chapter 18 – Revision of chapters 13–17

## Solutions to Technology-free questions

1 a

$$\begin{aligned} \int_{\frac{3}{2}}^{\frac{5}{2}} k \cos(\pi x) dx &= \left[ \frac{k}{\pi} \sin \pi x \right]_{\frac{3}{2}}^{\frac{5}{2}} \\ &= \frac{k}{\pi} \left( \sin \frac{5\pi}{2} - \sin \frac{3\pi}{2} \right) \\ &= \frac{2k}{\pi} \end{aligned}$$

Since area = 1,  $\frac{2k}{\pi} = 1 \Rightarrow k = \frac{\pi}{2}$

b

$$\begin{aligned} \int_{\frac{3}{2}}^m \cos(\pi x) dx &= \left[ \frac{\pi}{2} \times \frac{1}{\pi} \sin(\pi x) \right]_{\frac{3}{2}}^m \\ &= \frac{1}{2} \left( \sin(m\pi) - \sin \frac{3\pi}{2} \right) \\ &= \frac{1}{2} \left( \sin(m\pi) + 1 \right) \end{aligned}$$

For the median;

$$\frac{1}{2} \left( \sin(m\pi) + 1 \right) = 0$$

$$m\pi = 0 \text{ or } m\pi = 2\pi$$

$$\therefore m = 2$$

$$\left( \text{since } \frac{3}{2} < m < \frac{5}{2} \right)$$

$$\begin{aligned} \text{c } \Pr\left(X < \frac{7}{4} \mid X < 2\right) &= \frac{\Pr\left(X < \frac{7}{4}\right)}{\Pr(X < 2)} \\ &= \frac{2 - \sqrt{2}}{2} \end{aligned}$$

$$\begin{aligned} \text{d } \Pr\left(X > \frac{9}{4} \mid X > \frac{7}{4}\right) &= \frac{\Pr\left(X > \frac{9}{4}\right)}{\Pr\left(X > \frac{7}{4}\right)} \\ &= 3 - 2\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{2 a } \Pr(X > 3 \mid X > 2) &= \frac{\Pr(X > 3)}{\Pr(X > 2)} \\ &= \frac{0.1}{0.5} \\ &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} \text{b } \Pr(X > 1 \mid X \leq 2) &= \frac{\Pr(2 \leq X \leq 3)}{\Pr(X \leq 2)} \\ &= \frac{0.4}{0.9} \\ &= \frac{4}{9} \end{aligned}$$

c  $E(X) = \sum x \Pr(X = x) = 1.7$

d  $E(X^2) = \sum x^2 \Pr(X = x) = 4.9$   
 $\text{Var}(X) = 4.9 - 1.7^2 = 2.01$

3 a

$$\begin{aligned} \int_0^6 kx(6-x) dx &= k \left[ 3x^2 - \frac{x^3}{3} \right]_0^6 \\ &= k(108 - 72) = 36k \\ 36k &= 1 \Rightarrow k = \frac{1}{36} \end{aligned}$$

$$\text{b } \frac{1}{36} \int_0^4 x(6-x) dx = \frac{20}{27}$$

$$\begin{aligned} \text{c } \int_0^m \frac{x}{36} (6-x) dx &= 0.5 \\ \frac{x}{36} \left[ 3x^2 - \frac{x^3}{3} \right]_0^m &= 0.5 \\ 3m^2 - \frac{1}{3}m^3 - 18 &= 0 \\ 9m^2 - m^3 - 54 &= 0 \\ -(m-3)(m^2 - 6m - 18) &= 0 \\ \therefore m &= 3 \end{aligned}$$

Other solutions are outside  $[0, 6]$

**d**  $\frac{1}{36} \int_0^6 x^2(6-x) dx = 3$  Symmetry  
can be used for this.

**e**

$$\begin{aligned} \Pr(X < 2 | X < 3) &= \frac{\Pr(X < 2)}{\Pr(X < 3)} \\ &= \frac{\frac{1}{36} \int_0^2 x(6-x) dx}{\frac{1}{36} \int_0^3 x(6-x) dx} \\ &= \frac{\int_0^2 x(6-x) dx}{\int_0^3 x(6-x) dx} \\ &= \frac{\frac{28}{3}}{\frac{14}{3}} \\ &= \frac{14}{27} \end{aligned}$$

**f**

$$\begin{aligned} \Pr(X > 2 | X < 4) &= \frac{\Pr(2 < X < 4)}{\Pr(X < 4)} \\ &= \frac{\frac{1}{36} \int_2^4 x(6-x) dx}{\frac{1}{36} \int_0^4 x(6-x) dx} \\ &= \frac{\int_2^4 x(6-x) dx}{\int_0^4 x(6-x) dx} \\ &= \frac{\frac{52}{3}}{\frac{80}{3}} \\ &= \frac{13}{20} \end{aligned}$$

**4 a**  $\Pr(RG) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

**b**  $\Pr(RG) + \Pr(GR) = \frac{3}{28} + \frac{3}{28} = \frac{3}{14}$

**c**  $\Pr(G_2 | R_1) + \Pr(B_2 | R_1) + \Pr(Y_2 | R_1) = \frac{2}{7} + \frac{2}{7} + \frac{1}{7} = \frac{5}{7}$

**d**  $\Pr(R'_1 \cap R'_2) = \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$

**e**  $\Pr(R_1 \cap R_2) + \Pr(B_1 \cap B_2) + \Pr(G_1 \cap G_2) = \frac{3}{14}$

**5**  $\Pr(A) = \frac{4}{7}, \Pr(B) = \frac{1}{3}, \Pr(A' \cap B) = ?$

**a**  $\Pr(A' \cap B) + \Pr(A \cap B) = \Pr(B)$   
 $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$   
 $\therefore$

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$$

$$\therefore \Pr(A \cap B) = \frac{4}{7} + \frac{1}{3} - \frac{5}{7} = \frac{4}{21}$$

Also

$$\Pr(A' \cap B) + \Pr(A \cap B) = \Pr(B)$$

$$\therefore \Pr(A' \cap B) = \frac{1}{3} - \frac{4}{21} = \frac{1}{7}$$

**b**  $\Pr(A' \cap B) = \Pr(B) = \frac{1}{3}$

**6 a**  $\Pr(A \cap B) = \Pr(B|A) \Pr(A)$

$$= \frac{1}{5} \times \frac{3}{4}$$

$$= \frac{3}{20}$$

**b**  $\Pr(B) = 1 - \Pr(A \cap B) - \Pr(A \cap B)$

$$\Pr(A' \cap B') = \Pr(B'|A') \Pr(A')$$

$$= \frac{4}{7} \times \frac{1}{4}$$

$$= \frac{1}{7}$$

$$\Pr(A \cap B') = \Pr(A) - \Pr(A \cap B)$$

$$= \frac{3}{4} - \frac{3}{20}$$

$$= \frac{3}{10}$$

$$\Pr(B) = 1 - \frac{3}{10} - \frac{1}{7}$$

$$= \frac{39}{70}$$

**c**  $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

$$= \frac{3}{20} \times \frac{70}{39}$$

$$= \frac{7}{26}$$

**7**  $n = 5, p = 0.45$

**a**  $\Pr(X = 2) = \binom{5}{2} \left(\frac{9}{20}\right)^2 \left(\frac{11}{20}\right)^3$   
 $(\approx 0.3369)$

**b**  $\Pr(\text{Janet and only one friend}) =$   
 $\frac{9}{20} \times \binom{4}{1} \left(\frac{9}{20}\right) \left(\frac{11}{20}\right)^3 (\approx 0.2995)$

**8 a**

$$\Sigma \Pr(X = x) = 1$$

$$\therefore a + 0.3 + 0.1 + 0.2 + b = 1$$

$$\therefore a + b = 0.4 \dots (1)$$

$$E(X) = \Sigma x \Pr(X = x) = 2.34$$

$$\therefore a + 0.6 + 0.3 + 0.8 + 5b = 2.34$$

$$\therefore a + 5b = 0.64 \dots (2)$$

$$(2) - (1)$$

$$4b = 0.24$$

$$b = 0.06$$

$$\text{From (1) } a = 0.34$$

**b**  $E(X^2) = 6.54$

$$\text{Var}(X) = 1.0644$$

**9 a**  $\Pr(\text{win}) = 0.7 \times 0.9 + 0.3 \times 0.4$

$$= 0.63 + 0.12$$

$$= 0.75$$

**b**

$$\Pr(\text{Fully fit} | \text{Did not win}) = \frac{\Pr(\text{Fully fit} \cap \text{Did not win})}{\Pr(\text{Did not win})}$$

$$= \frac{0.1 \times 0.7}{0.25}$$

$$= \frac{7}{25}$$

**10 a**

$$\int_a^{2a} (x-a)(2a-x) dx = \left[ -\frac{x^3}{3} + \frac{3ax^2}{2} - 2xa^2 \right]_a^{2a}$$

$$= \frac{a^3}{6}$$

Since the area = 1

$$\frac{a^3}{6} = 1$$

$$\therefore a^3 = 6$$

$$\begin{aligned}
 \mathbf{b} \quad E(X) &= \int_a^{2a} x(x-a)(2a-x) dx \\
 &= \left[ -\frac{x^4}{4} + \frac{3ax^3}{3} - (xa)^2 \right]_a^{2a} \\
 &= \frac{a^4}{4} \\
 &= \frac{6^4}{4}
 \end{aligned}$$

$$\mathbf{11} \quad \mu = 40, \sigma = 2$$

$$Pr(36 < X < 44) = q$$

$$Pr(X > 44) = \frac{1-q}{2}$$

$$\mathbf{12} \quad \int_a^0 2(1-x) dx = \frac{3}{4} \text{ since}$$

$$\left[ 2\left(x - \frac{x^2}{2}\right) \right]_0^a = \frac{3}{4}$$

$$2a - a^2 = \frac{3}{4}$$

$$8a - 4a^2 = 3$$

$$4a^2 - 8a + 3 = 0$$

$$(2a-1)(2a-3) = 0$$

$$a = \frac{1}{2} \text{ or } a = \frac{3}{2}$$

$$0 \leq x \leq 1, a = \frac{1}{2}$$

$$\mathbf{13} \quad n = 3, =?$$

$$\mathbf{a} \quad Pr(X = 0) = (1-p)^3$$

$$\mathbf{b} \quad Pr(X = 0) = p^3$$

$$(1-p)^3 = 8p^3$$

$$1-p = 2p$$

$$1 = 3p$$

$$p = \frac{1}{3}$$

$$\mathbf{14} \quad \mathbf{a} \quad \hat{p} = \frac{a+b}{2}$$

$$\begin{aligned}
 \mathbf{b} \quad M &= \text{half the width of the interval} \\
 &= \frac{b-a}{2}
 \end{aligned}$$

## Solutions to multiple-choice questions

$$\begin{aligned} 1 \quad \mathbf{E} \quad \Pr(\text{both green}) &= \frac{4}{16} \times \frac{3}{15} \\ &= \frac{1}{4} \times \frac{1}{5} \\ &= \frac{1}{20} \end{aligned}$$

$$\begin{aligned} 2 \quad \mathbf{D} \quad \Pr(\text{six correct}) &= \left(\frac{1}{2}\right)^6 \\ &= \frac{1}{64} \\ &= 0.0156 \end{aligned}$$

$$\begin{aligned} 3 \quad \mathbf{D} \quad 4c^2 + 5c^2 + 4c^2 + 3c^2 &= 1 \\ 16c^2 &= 1 \\ c^2 &= \frac{1}{16} \\ c &= \pm \frac{1}{4} \\ &= \pm 0.25 \end{aligned}$$

of the option a available, only option D fits.

4 **C** If  $X$  is the number of spins it takes to get a '3', then it could take 1 spin or 2 spins or . . . ; there is no theoretical upper limit. So the sample space is  $\{1, 2, 3, 4, \dots\}$ .

$$\begin{aligned} 5 \quad \mathbf{A} \quad E(X) &= \sum x \Pr(X = x) \\ &= 4 \times 0.3 + 6 \times 0.2 \\ &\quad + 7 \times 0.1 + 9 \times 0.4 \\ &= 1.2 + 1.2 + 0.7 + 3.6 \\ &= 6.7 \end{aligned}$$

$$\begin{aligned} 6 \quad \mathbf{B} \quad E(X^2) &= \sum x^2 \Pr(X = x) \\ &= 16 \times 0.3 + 36 \times 0.2 \\ &\quad + 49 \times 0.1 + 81 \times 0.4 \\ &= 4.8 + 7.2 + 4.9 + 32.4 \\ &= 49.3 \end{aligned}$$

$$\begin{aligned} \text{var}(X) &= E(X^2) - [E(X)]^2 \\ &= 49.3 - 6.7^2 \\ &= 4.41 \end{aligned}$$

7 **E** The values of  $X$  are 4, 6, 7, 9. Since  $Y = 2X - 1$ , the corresponding values of  $Y$  are 7, 11, 13, 17. The probabilities are unchanged, so the fifth option fits.

$$\begin{aligned} 8 \quad \mathbf{D} \quad \text{If } Z &= aX + b, \text{ then} \\ \text{var}(Z) &= a^2 \text{var}(x) \\ \text{Here, } a &= -1 \quad \text{and} \quad b = 4, \text{ so} \\ \text{var}(Z) &= (1)^2 \text{var}(X) \\ &= \text{var}(X) \\ &= 4.41 \end{aligned}$$

9 **E** The required probability is 0.46 (Alternatively use a tree diagram)

$$\begin{aligned} 10 \quad \mathbf{C} \quad \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 202 - 11^2 \\ &= 202 - 121 \\ &= 81 \\ \text{sd}(X) &= 9 \end{aligned}$$

11 **B** 95% of scores, assuming an approximate normal distribution, will lie in the interval  $(\mu - 2\sigma, \mu + 2\sigma)$ .

$$\begin{aligned}\mu - 2\sigma &= 50 - 20 = 30 \\ \mu + 2\sigma &= 50 + 20 = 70 \\ \text{So the required interval is } &(30, 70).\end{aligned}$$

**12 D** Pr(at last 2 heads)

$$\begin{aligned}&= {}^3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^3 \\ &= 3 \times \frac{1}{8} + \frac{1}{8} \\ &= \frac{4}{8} \\ &= \frac{1}{2}\end{aligned}$$

**13 C**  $E(X) = np$

$$\begin{aligned}&= 400 \times 0.1 \\ &= 40\end{aligned}$$

**14 B** If a die is rolled until a six is obtained the sample space is  $\{1, 2, 3, \dots\}$ . This can not be a binominal variable since there is no theoretical limit to the number of rolls.

**15 E**  $\text{var}(X) = np(1 - p)$

$$\begin{aligned}&= 900 \times 0.2 \times 0.8 \\ &= 9 \times 16 \\ \text{sd}(X) &= 3 \times 4 \\ &= 12\end{aligned}$$

**16 E**  $\text{var}(X) = np(1 - p)$

$$\begin{aligned}&= 42p(1 - p) \\ &= 9.4248\end{aligned}$$

$$p(1 - p) = 0.2244$$

Use the *solve* command of a CAS calculator, giving  $p = 0.34$  or  $p = 0.66$ . (Alternatively, solve the equation formula)

**17 A** If  $p$  is the probability of success, then  $\text{Pr}(5 \text{ successes}) = \left(\frac{7}{5}\right)p^5(1 - p)^2$ . But 5 successes is the same as 2 fails. So this represents the probability of exactly two failures.

**18 A**  $\text{Pr}(4 \text{ females}) = {}^{10}C_4(0.2)^4(0.8)^6$

$$\approx 0.0881$$

**19 D**  $\text{Pr}(\geq 1) = 1 - (0 \text{ at home})$

$$\begin{aligned}&= 1 - (0.4)^5 \\ &= 0.9898\end{aligned}$$

**20 D**  $\int_0^2 \left(kx^3 + \frac{3}{4}x\right) dx = 1$

$$\begin{aligned}\left[\frac{1}{4}kx^4 + \frac{3}{8}x^2\right]_0^2 &= 1 \\ 4k + \frac{3}{2} &= 1 \\ 4k &= -\frac{1}{2} \\ k &= -\frac{1}{8}\end{aligned}$$

$$\begin{aligned}
21 \quad \text{C} \quad \Pr(X \leq 2) &= \frac{1}{9} \int_0^2 (4x - x^2) dx \\
&= \frac{1}{9} \left[ 2x^2 - \frac{1}{3}x^3 \right]_0^2 \\
&= \frac{1}{9} \left( 8 - \frac{8}{3} \right) \\
&= \frac{16}{27} \\
&\approx 0.5926
\end{aligned}$$

$$22 \quad \text{E} \quad \frac{8}{3} \int_0^m (1-x) dx = \frac{1}{2} \text{ where } m \text{ is the median.}$$

$$\frac{8}{3} \left[ x - \frac{x^2}{2} \right]_0^m = \frac{1}{2}$$

$$m - \frac{1}{2}m^2 = \frac{3}{16}$$

Solving this quadratic with a CAS 'Solve' command (or by use of the quadratic formula) gives  $m \approx 0.209$  or  $m \approx 1.791$

But  $0 < m < \frac{1}{2}$ , so  $m \approx 0.209$

$$\begin{aligned}
23 \quad \text{B} \quad \text{E}(X) &= \int_1^2 2x \left( 1 - \frac{1}{x^2} \right) dx \\
&= \int_1^2 \left( 2x - \frac{2}{x} \right) dx \\
&= \left[ x^2 - 2 \log_e x \right]_1^2 \\
&= (4 - 2 \log_e 2) - (1 - 0) \\
&= 3 - 2 \log_e 2 \\
&\approx 1.614
\end{aligned}$$

$$\begin{aligned}
24 \quad \text{C} \quad \Pr(-1.0 < Z < 0) \\
&= \frac{1}{2} \Pr(-1.0 < Z < 1.0) \\
&\approx \frac{1}{2} \times 0.68 \\
&= 0.34
\end{aligned}$$

25 D From the definition of standard deviation, it is always positive for any distribution, including. (Checking the other options:  
A: a mean can be negative  
B: values for any normal distribution be any number in the interval  $(-\infty, \infty)$   
C: the area is *exactly* 1  
E: the standard deviation could be greater than the mean (it is for a standard normal distribution))

26 C  $\Pr(X > 2.6) \approx 0.1151$ , using the 'normCdf' command of a CAS calculator. (You do not need to standardise.)

27 E  $\Pr(X < -2) \approx 0.0228$ , using the 'normCdf' command of a CAS calculator. (In this case, you might note that:

$$\Pr(X < -2) = \Pr\left(Z < \frac{-2 - 2}{2}\right)$$

$$= \Pr(Z < -2)$$

$$\approx \frac{1}{2} \times 0.05 = 0.025$$

using the  $2\sigma$  limits. the only close option is the last option.)

28 A Since  $\sigma^2 = 0.4$ ,  $\sigma = \sqrt{0.4}$   
 $\Pr(X > -2.73) = 1$ , using the 'normCdf' command of a CAS calculator.

29 B Since  $\sigma^2 = 4$ ,  $\sigma = 2$ .  
 $\Pr(1 < X < 2.5) \approx 0.2902$ , using the 'normCdf' command of a CAS calculator. (If you mistakenly used 4 for  $\sigma$ , you would get 0.1484, which is not one of the option!)

**30 E** If  $X$  is the amount of cordial in a cup, then  $X$  is normal with  $\mu = 50$  and  $\sigma = 2$ .

$$\Pr(X > c) = 0.90, \text{ so}$$

$$\Pr(X < c) = 0.10, \text{ giving}$$

$c = 47.44$  mL using the 'invnorm' command of CAS calculator.

**31 C** If  $X$  cm is the length of a lock of cheese, then  $X$  is normal with  $\mu = 10$  and  $\sigma = \sqrt{0.5}$ .

$\Pr(X < c) = 0.95$ , giving  $c \approx 11.16$  cm using the 'invNorm' command of a CAS calculator.

**32 A**  $\Pr(\mu - k < x < \mu + k) = 0.7$

$$\Pr\left(-\frac{k}{\sigma} < \frac{X - \mu}{\sigma} < \frac{k}{\sigma}\right) = 0.7$$

$$\Pr\left(-\frac{k}{\sigma} < Z < \frac{k}{\sigma}\right) = 0.7$$

Thus an area of 0.3 remains in the two tail, or 0.15 in each tail.

$$\begin{aligned} \text{So } \Pr\left(Z < \frac{k}{\sigma}\right) &= 0.7 + 0.15 \\ &= 0.85 \end{aligned}$$

Using the 'invNorm' command of a CAS calculator shown that

$$\Pr(Z < 1.03643) = 0.85$$

$$\Rightarrow \frac{k}{\sigma} = 1.03643$$

$$\text{Now } \sigma^2 = 2.25, \text{ so } \sigma = 1.5.$$

Hence  $k = 1.555$ . to 3 decimal places.

(Note that the value of the mean  $\mu$  is not actually needed.)

**33 B** If  $X$  kg is the weight of a pocket, then  $X$  is normal  $\mu = 1$ . More than 0.05 kg underweight

means  $X < 0.95$  and 3% are underweight.

$$\Pr(X < 0.95) = 0.03$$

$$\Pr\left(Z < \frac{-0.05}{\sigma}\right) = 0.03$$

Using the 'invNorm' command of a CAS calculator shows that

$$\Pr(Z < -1.88079) = 0.03$$

$$\Rightarrow -\frac{0.05}{\sigma} = -1.88079$$

$$\begin{aligned} \sigma &= \frac{0.05}{1.88079} \\ &\approx 0.027 \end{aligned}$$

**34 B** The graphs have the same centre so  $\mu_1 = \mu_2$ .

The lower graph is more spread out than the upper graph so  $\sigma_1 > \sigma_2$ .

**35 B** The standard deviation is  $\sqrt{25} = 5$ . About 68% represent,  $\pm 1$  standard deviation from the mean of 173.

$$173 - 5 = 168$$

$$173 + 5 = 178$$

So the interval is (168, 178)

**36 B**  $n = 200, p = 0.38$

95% CI = (0.313, 0.447) (Calculator)

**37 B** Increasing the level of confidence means that the interval will be wider

**38 C** Only statement II is correct

**39 B** Since the width of the confidence interval is inversely proportional to the square root of the sample size, decreasing the sample size by a factor of 2 will increase the width of the interval by a factor of  $\sqrt{2}$



## Solutions to extended-response questions

- 1 In the following  $E$  denotes the event occurring,  $N$  the event not occurring. Three trials are considered first.

a i The outcomes to consider are

$$(E, E, E) \quad \Pr(E, E, E) = 0$$

$$(E, E, N) \quad \Pr(E, E, N) = 0$$

$$(E, N, E) \quad \Pr(E, N, E) = \frac{1}{2} \times 1 \times \frac{1}{2}$$

$$(N, E, E) \quad \Pr(N, E, E) = 0$$

$$(N, N, E)$$

$$(N, E, N) \quad \text{Note: Remember the event cannot occur in consecutive trials.}$$

$$(E, N, N)$$

$$(N, N, N)$$

$$\therefore \text{Probability of it occurring just twice} = \Pr\{(E, N, E)\} = \frac{1}{4}$$

ii Consider the following outcomes

$$(E, E, N, N) \quad \Pr(E, E, N, N) = 0$$

$$(E, N, E, N) \quad \Pr(E, N, E, N) = \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 = \frac{1}{4}$$

$$(E, N, N, E) \quad \Pr(E, N, N, E) = \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$(N, E, E, N) \quad \Pr(N, E, E, N) = 0$$

$$(N, E, N, E) \quad \Pr(N, E, N, E) = \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{8}$$

$$(N, N, E, E) \quad \Pr(N, N, E, E) = 0$$

$$\therefore \Pr(\text{the event occurs exactly twice}) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

b For 5 trials there are 10 possible outcomes to consider

$$(E, E, N, N, N) \quad \Pr(E, E, N, N, N) = 0$$

$$(E, N, E, N, N) \quad \Pr(E, N, E, N, N) = \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{8}$$

$$(E, N, N, E, N) \quad \Pr(E, N, N, E, N) = \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{8}$$

$$(E, N, N, N, E) \quad \Pr(E, N, N, N, E) = \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$$

$$(N, E, E, N, N) \quad \Pr(N, E, E, N, N) = 0$$

$$\begin{aligned}
(N, E, N, E, N) \quad \Pr(N, E, N, E, N) &= \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 = \frac{1}{8} \\
(N, E, N, N, E) \quad \Pr(N, E, N, N, E) &= \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16} \\
(N, N, E, E, N) \quad \Pr(N, N, E, E, N) &= 0 \\
(N, N, E, N, E) \quad \Pr(N, N, E, N, E) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{16} \\
(N, N, N, E, E) \quad \Pr(N, N, N, E, E) &= 0 \\
\therefore \Pr(\text{the event occurs exactly twice}) &= \frac{9}{16}
\end{aligned}$$

**2** Let  $X$  be the number of sixes obtained in 5 tosses of a die

$$\Pr(\text{an even number of sixes}) = \Pr(X = 0) + \Pr(X = 2) + \Pr(X = 4)$$

$$\begin{aligned}
&= \left(\frac{5}{6}\right)^5 + 10\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 + 10\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right) \\
&= \left(\frac{1}{6}\right)^5 [5^5 + 10 \cdot 5^3 + 10 \cdot 5] \\
&= \left(\frac{1}{6}\right)^5 [4425] \\
&= 0.5692
\end{aligned}$$

$$\Pr(\text{an odd number of sixes}) = 1 - \Pr(\text{an even number of sixes})$$

$$= 0.4309$$

Let  $Y$  be the amount won by Katia. The probability distribution is as shown

$y$	1	$-x$
$\Pr(Y = y)$	0.4309	0.5691

The game is fair if  $E(Y) = 0$

$$\text{i.e. if } 1 \times 0.4309 - 0.5691x = 0$$

$$\text{This implies } x = 0.7570$$

Therefore Mikki should receive 76 cents from Katia if there is an even number of sixes.

**3 a** Let  $x$  be the daily demand

Let  $s$  be the number of newspapers stocked

If the demand is less than the number stocked

$$P = 0.75x - 0.5s$$

If the demand is greater than the number stocked

$$P = 0.25s - 0.25(x - s) \quad \left( \text{Note: the newspaper seller is considered} \right)$$

$$= 0.5s - 0.25x \quad \left( \text{to lose money by not ordering enough.} \right)$$

$$\therefore P = \begin{cases} 0.75x - 0.5s & x \leq s \\ 0.5s - 0.25x & x > s \end{cases}$$

**b** Using the result of **a** a probability distribution for  $P$  is obtained with  $s = 26$

$p$	5	5.75	6.50	6.25	6	5.75	5.50
$\Pr(P = p)$	0.05	0.10	0.10	0.25	0.25	0.15	0.10

The computations are as follows

$$x = 24 \quad p = 0.75 \times 24 - 0.5 \times 26 = 5$$

$$x = 25 \quad p = 0.75 \times 25 - 0.5 \times 26 = 5.75$$

$$x = 26 \quad p = 0.75 \times 26 - 0.5 \times 26 = 6.5$$

$$x = 27 \quad p = 0.5 \times 26 - 0.25 \times 27 = 6.25$$

etc.

Reorganising the table

$p$	5	5.50	5.75	6	6.25	6.50
$\Pr(P = p)$	0.05	0.10	0.25	0.25	0.25	0.1

$$\therefore E(P) = 5 \times 0.05 + 5.50 \times 0.10 + 5.75 \times 0.25 + 6 \times 0.25 + 6.25 \times 0.25 + 6.25 \times 0.1$$

$$= 5.95$$

The expected profit is \$5.95.

$$\mathbf{c} \quad E(P) = \sum_{x=24}^s (0.75x - 0.5s)p(x) + \sum_{x=s+1}^{30} (0.5s - 0.25x)p(x)$$

**d** The newspaper seller should stock 27 (computation not shown).

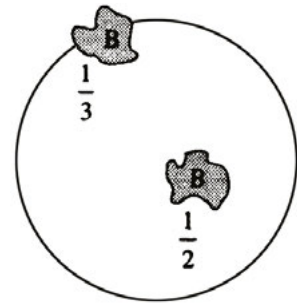
$$\begin{aligned}
4 \quad E(Z) &= E\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right) \\
&= E\left(\frac{X}{\sigma}\right) - \sigma \\
&= \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma} \quad (\text{Using } E(aX + b) = aE(X) + b) \\
&= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\text{Var}(Z) &= \text{Var}\left(\frac{X - \mu}{\sigma}\right) \\
&= \frac{1}{\sigma^2}\text{Var}(X) \quad (\text{Using } \text{Var}(aX + b) = a^2\text{Var}(X)) \\
&= \frac{\sigma^2}{\sigma^2} \\
&= 1
\end{aligned}$$

5 a i Probability bean bag lands outside =  $1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$

ii Probability of two consecutive throws landing outside the circle =  $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

iii Probability of first on the rim and second inside the circle =  $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$



b Let  $X$  be the score.

$X$	0	5	10
$\text{Pr}(X = x)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{2}{5}$

i With two shots to score a 20 requires two 10's.  $\therefore$  Probability of score 20 =  $\frac{4}{25}$

ii In order to score 10 the score could have resulted through 0 and 10 or 10 and 0 or 5 and 5.

$$\begin{aligned}
 \therefore \text{Probability of score 10} &= \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{2}{5} + \frac{1}{10} \times \frac{1}{10} \\
 &= \frac{1}{5} + \frac{1}{5} + \frac{1}{100} \\
 &= \frac{2}{5} + \frac{1}{100} \\
 &= \frac{41}{100}
 \end{aligned}$$

**c** For Jane to score a 10:

- It can be a ten from 2 shots (bean bag of Anne; outside).

$$\text{Probability of this} = \frac{41}{100} \times \frac{1}{6} = \frac{41}{600}$$

- It can be a ten from one throw (bean bag of Anne: rim)

$$\frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$$

$$\therefore \text{Probability of a ten} = \frac{41}{600} + \frac{2}{15} = \frac{121}{600}$$

**6 a**  $\mu = 400$ ,  $\sigma = 50$

Let  $X$  be the lifetime of a light globe

$$\begin{aligned}
 \Pr(X > 375) &= \Pr\left(Z > \frac{375 - 400}{50}\right) \\
 &= \Pr(Z > -0.5) \\
 &= \Pr(Z < 0.5) \\
 &= 0.6915
 \end{aligned}$$

**b** Let  $Y$  be the number of light globes which will last more than 375 hours when selected from a box of 10.

This is Binomial with  $n = 10$  and  $p = 0.6915$

$$\Pr(Y \geq 9) = \Pr(Y = 9) + \Pr(Y = 10)$$

$$\begin{aligned}
 &= \binom{10}{9} (0.6915)^9 (0.3085) + (0.6915)^{10} \\
 &= 0.1365
 \end{aligned}$$

The probability that at least 9 of the globes in a randomly selected box will last more than 375 hours is 0.1365.

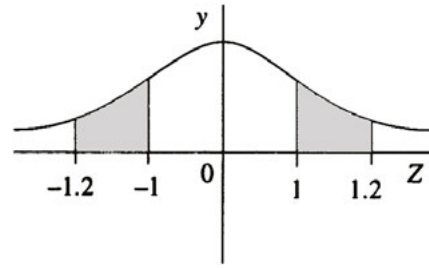
7  $\mu = 80\,000$ ,  $\sigma = 20\,000$

a Let  $X$  be the distance travelled annually

$$\Pr(56\,000 \leq X \leq 60\,000)$$

$$= \Pr\left(\frac{56\,000 - 80\,000}{20\,000} \leq Z \leq \frac{60\,000 - 80\,000}{20\,000}\right)$$

$$= \Pr(-1.2 \leq Z \leq -1)$$



From the graph it can be seen

$$\Pr(-1.2 \leq Z \leq -1)$$

$$= \Pr(1 \leq Z \leq 1.2)$$

$$= \Pr(Z \leq 1.2) - \Pr(Z \leq 1)$$

$$= 0.8849 - 0.8413$$

$$= 0.0436$$

The probability that a randomly selected taxi will travel between 50 000 km and 60 000 km is 0.0436.

b  $\Pr(\text{Below } 48\,000 \text{ or above } 96\,000)$

$$= \Pr(X \leq 48\,000) + \Pr(X \geq 96\,000)$$

$$= \Pr\left(Z \leq \frac{48\,000 - 80\,000}{20\,000}\right) + \Pr\left(Z \geq \frac{96\,000 - 80\,000}{20\,000}\right)$$

$$= \Pr(Z \leq -1.6) + \Pr(Z \geq 0.8)$$

$$= 1 - \Pr(Z \leq 1.6) + 1 - \Pr(Z \geq 0.8)$$

$$= 2 - \Pr(Z \leq 1.6) - \Pr(Z \leq 0.8)$$

$$= 2 - 0.9452 - 0.7881$$

$$= 0.2667$$

The percentage of taxis which travel below 48 000 km or have 96 000 km is 26.67%

c  $\Pr(48\,000 \leq X \leq 96\,000)$

$$= 1 - [\Pr(X \leq 48\,000) + \Pr(X \geq 96\,000)]$$

$$= 1 - 0.2667$$

$$= 0.7333$$

Let  $Y$  be the number of taxis out of the 250 which will travel between 48 000 and 96 000 km.

$Y$  is a Binomial random variable with  $n = 250$  and  $p = 0.7333$

$$\therefore E(Y) = np = 183.325$$

i.e. 183 taxis out of the 250 are expected to travel between 48 000 and 96 000 km.

**d** Let  $c$  be such that

$$\Pr(X \geq c) = 0.85$$

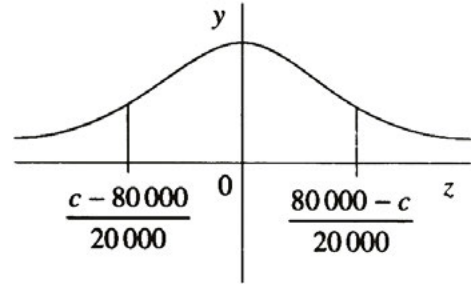
$$\text{Then } \Pr\left(Z \geq \frac{c - 80\,000}{20\,000}\right) = 0.85$$

From the graph

$$\Pr\left(Z \leq \frac{80\,000 - c}{20\,000}\right) = 0.85$$

$$\therefore \frac{80\,000 - c}{20\,000} = 1.03643$$

$$\begin{aligned} \therefore c &= 80\,000 - 20\,000 \times 1.03643 \\ &= 59\,271 \end{aligned}$$



85% of taxis travel at least 59 271 kilometres.

**8 a i** Let  $X$  be the weight of cereal in a box

$$9\mu = 505, \sigma = 5$$

$$\Pr(X \leq 500)$$

$$= \Pr\left(Z \leq \frac{500 - 505}{5}\right)$$

$$= \Pr(Z \leq -1)$$

$$= 1 - \Pr(Z \leq 1)$$

$$= 1 - 0.8413$$

$$= 0.1587$$

**ii**  $\mu = ?$ ,  $\sigma = 5$

$$\Pr(X \leq 500) = 0.1$$

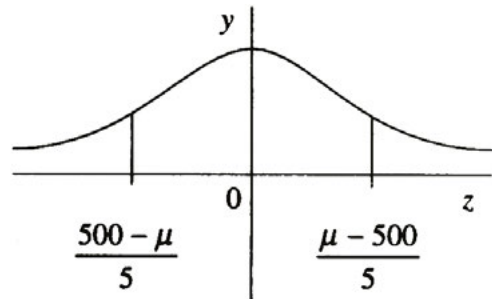
$$\text{implies } \Pr\left(Z \leq \frac{500 - \mu}{5}\right) = 0.1$$

From the graph

$$\Pr\left(Z \leq \frac{500 - \mu}{5}\right)$$

$$= \Pr\left(Z \geq \frac{\mu - 500}{5}\right)$$

$$= 1 - \Pr\left(Z \leq \frac{\mu - 500}{5}\right)$$



$$\begin{aligned} \therefore 1 - \Pr\left(Z \leq \frac{\mu - 500}{5}\right) &= 0.01 \\ 0.99 &= \Pr\left(Z \leq \frac{\mu - 500}{5}\right) \\ 2.3263 &= \frac{\mu - 500}{5} \\ \therefore \mu &= 5 \times 2.3263 + 500 \\ &= 511.63 \end{aligned}$$

**b** Let  $Y$  be the number of boxes under weight.  $Y$  is a Binomial random variable with  $n = 5$ ,  $p = 0.158655$

$$\begin{aligned} \Pr(Y > 1) &= 1 - (\Pr(Y = 0) + \Pr(Y = 1)) \\ &= 1 - (0.841345)^5 - \binom{5}{1} (0.158655)(0.841345)^4 \\ &= 0.1809 \end{aligned}$$

**9 a i** 
$$f(y) = \begin{cases} k(y - 8) & \text{if } 8 \leq y \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_8^{12} f(y) dy = \left[ k\left(\frac{1}{2}y^2 - 8y\right) \right]_8^{12} = 8k$$

For  $f$  to be a probability density function  $8k = 1$ .

Hence  $k = \frac{1}{8}$ .

**ii**  $X$  is normally distributed with mean = 10 and standard deviation = 2

$$\Pr(X < 11) = 0.6915$$

$$\begin{aligned} \Pr(Y < 11) &= \int_8^{11} f(y) dy \\ &= \left[ k\left(\frac{1}{2}y^2 - 8y\right) \right]_8^{11} \\ &= \frac{9}{16} \\ &= 0.5625 \end{aligned}$$



iii  $E(X) = 10$  and

$$\begin{aligned} E(Y) &= \int_8^{12} yf(y) dy \\ &= \frac{1}{8} \int_8^{12} y(y-8) dy \\ &= \frac{32}{3} \\ &\approx 10.67 \end{aligned}$$

Machine 1 is quicker or average.

b Probability of a widget being produced in less than 10 seconds

$$\begin{aligned} &= \Pr(\text{less than 10 s} \mid \text{machine 1})\Pr(\text{machine 1}) \\ &\quad + \Pr(\text{less than 10 s} \mid \text{machine 2})\Pr(\text{machine 2}) \\ &= 0.5 \times 0.6 + 0.25 \times 0.4 \\ &= 0.4 \end{aligned}$$

$\Pr(\text{machine 1} \mid \text{less than ten seconds to produce})$

$$\begin{aligned} &= \frac{\Pr(\text{Machine 1 and less than 10 stop roduce})}{\Pr(\text{less than 10 stop roduce})} \\ &= \frac{3}{4} \end{aligned}$$

10

$x$	1	2	3	4	5	6
$\Pr(X = x)$	$c$	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$	$\frac{c}{5}$	$\frac{c}{6}$

a Since we have a probability distribution

$$\begin{aligned} c + \frac{c}{2} + \frac{c}{3} + \frac{c}{4} + \frac{c}{5} + \frac{c}{6} &= 1 \\ \therefore c \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) &= 1 \\ \therefore \frac{49c}{20} &= 1 \\ \therefore c &= \frac{20}{49} \end{aligned}$$

b

$$\begin{aligned} E(X) &= c + 2 \times \frac{c}{2} + 3 \times \frac{c}{3} + 4 \times \frac{c}{4} + 5 \times \frac{c}{5} + 6 \times \frac{c}{6} \\ &= \frac{20}{49} (1 + 1 + 1 + 1 + 1 + 1) \\ &= \frac{120}{49} \end{aligned}$$

<b>c</b>	$x^2$	1	4	9	16	25	36
	$\Pr(X = x)$	$c$	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$	$\frac{c}{5}$	$\frac{c}{6}$

$$\begin{aligned} \therefore E(X^2) &= c + 2c + 3c + 4c + 5c + 6c \\ &= c(1 + 2 + 3 + 4 + 5 + 6) \\ &= 21c \\ &= \frac{21 \times 20}{49} \\ &= \frac{60}{7} \end{aligned}$$

$$\begin{aligned} \therefore \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{60}{7} - \left(\frac{120}{49}\right)^2 \\ &= \frac{6180}{2401} \end{aligned}$$

**11 a i** 
$$f(x) = \begin{cases} kx(100 - x^2) & \text{if } 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^{10} f(x)dx = \left[ k\left(50x^2 - \frac{x^4}{4}\right) \right]_0^{10} = 2500k$$

For  $f$  to be the function of a probability density  $k = \frac{1}{2500}$

**ii**  $E(X) = \int_0^{10} xf(x)dx = \frac{1}{2500} \int_0^{10} x^2(100 - x^2)dx = \frac{16}{3}$

**iii**  $\Pr(X > 3) = \int_3^{10} f(x)dx = \left[ k\left(50x^2 - \frac{x^4}{4}\right) \right]_3^{10} = 0.8281$

**iv**  $\Pr(X > 3 | X < 7) = \frac{\Pr(3 < X < 7)}{\Pr(X < 7)} = \frac{\int_3^7 f(x)dx}{\int_0^7 f(x)dx} = 0.7677$

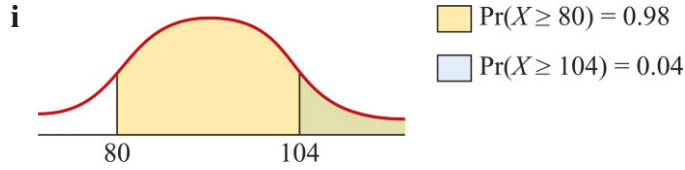
**b** This is a binomial distribution. Let  $W$  be the number of moviegoers who have to queue for more than 3 minutes

$$\Pr(W \geq 5) = ?$$

In this situation  $n = 10$  and  $p = 0.8281$

Using a calculator gives  $\Pr(W \geq 5) = 0.9971$

**12 a**  $T$  is the temperature at which the sensor fails.



**ii**  $\Pr(T \geq 80) = 0.98$

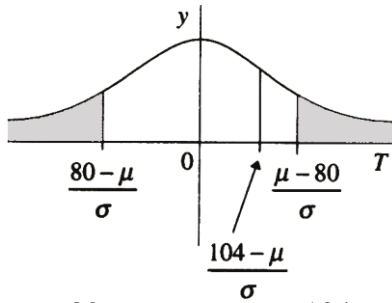
$\Pr(T \geq 104) = 0.04$

$\therefore \Pr\left(Z \geq \frac{80 - \mu}{\sigma}\right) = 0.98$

$\Pr\left(Z \geq \frac{104 - \mu}{\sigma}\right) = 0.04$

$\therefore \Pr\left(Z \leq \frac{\mu - 80}{\sigma}\right) = 0.98$

and  $\Pr\left(Z \leq \frac{104 - \mu}{\sigma}\right) = 0.96$



$\therefore \frac{\mu - 80}{\sigma} = 2.0537$  and  $\frac{104 - \mu}{\sigma} = 1.7507$

$\therefore \mu - 80 = 2.0537\sigma$  (1) and  $104 - \mu = 1.750\sigma$  (2)

Add (1) and (2)  $24 = (2.0537 + 1.7507)\sigma$

$6.3084 = \sigma$

$\therefore$  From (1)  $\mu = 80 + 2.0537 \times 6.3084 = 92.956$

**b i**  $\mu = 94.5$  and  $\sigma = 5.7$

$\Pr(T \leq 100) = \Pr\left(Z \leq \frac{100 - 94.5}{5.7}\right)$

$= \Pr\left(Z \leq \frac{5.5}{5.7}\right)$

$= \Pr\left(Z \leq \frac{55}{57}\right)$

$= 0.8327$

$\therefore \Pr(T \geq 100) = 1 - 0.8327 = 0.1673$

16.73% of sensors will operate in boiling water.

**ii**  $\Pr(T \geq c) = 0.99$

$$\therefore \Pr\left(Z \geq \frac{c - 94.5}{5.7}\right) = 0.99$$

$$\therefore \Pr\left(Z \leq \frac{c - 94.5}{5.7}\right) = 0.01$$

$$\therefore \frac{c - 94.5}{5.7} = -2.3263$$

$$\therefore c = 81.2398$$

They should quote 81°C as a temperature for the sensors to work.

**13 a i** Probability that the plane is early =  $1 - (0.5 + 0.3) = 0.2$

**ii** Probability that it does not arrive late =  $1 - 0.3 = 0.7$

**iii** Probability that the plane is on time on three consecutive days =  $(0.5)^3 = 0.125$

**iv** Late on Monday but one time all the remaining days =  $0.3 \times (0.5)^4 = \frac{3}{160}$

**b i** Let  $X$  be the number of times late

$$\Pr(X = 1) = {}^5C_1(0.3)(0.7)^4 = 0.36015$$

**ii** Let  $Y$  be the number of times early

$$\Pr(Y = 2) = {}^5C_2(0.2)^2(0.8)^3 = \frac{128}{625}$$

**c**  $\Pr(X > 2) = \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$

$$= {}^5C_3(0.3)^3(0.7)^2 + {}^5C_4(0.3)^4(0.7) + (0.3)^5$$

$$= 0.1323 + 0.02835 + 0.00243$$

$$= 0.16308$$

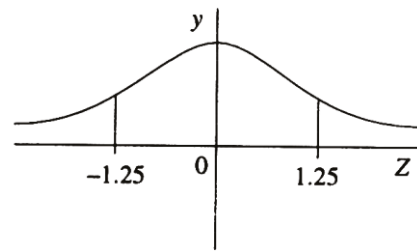
**14 a**  $\Pr(X \leq 985) = \Pr\left(Z \leq \frac{985 - 1000}{12}\right)$

$$= \Pr\left(Z \leq \frac{-15}{12}\right)$$

$$= \Pr(Z \leq -1.25)$$

$$= 1 - \Pr(Z \leq 1.25)$$

$$= 0.1056$$



**b**  $\Pr(X \leq 985) = 0.01$

$$\therefore \Pr\left(Z \leq \frac{985 - \mu}{12}\right) = 0.01$$

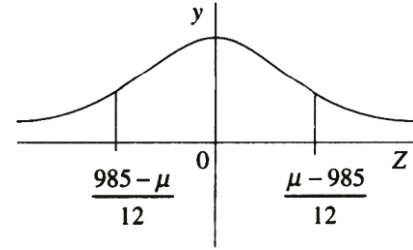
$$\therefore 1 - \Pr\left(Z \leq \frac{\mu - 985}{12}\right) = 0.01$$

$$\Pr\left(Z \leq \frac{\mu - 985}{12}\right) = 0.99$$

$$\therefore \frac{\mu - 985}{12} = 2.3263$$

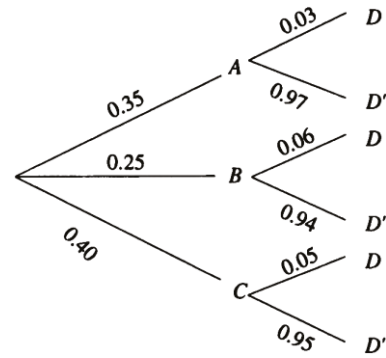
$$\therefore \mu = 12 \times 2.3263 + 985$$

$$\therefore \mu = 1012.92$$



The machine should be set at 1012.92

**15** In the tree diagram  $A$ ,  $B$  and  $C$  are the machines.  $D$  is defective.  $D'$  is not defective.



**a i**  $\Pr(A \cap D) = 0.35 \times 0.03 = 0.0105$

**ii**  $\Pr(D) = \Pr(A \cap D) + \Pr(B \cap D) + \Pr(C \cap D)$   
 $= 0.0105 + 0.25 \times 0.06 + 0.40 \times 0.05$   
 $= 0.0455$

**b**  $\Pr(C | D) = \frac{\Pr(C \cap D)}{\Pr(D)} = \frac{0.4 \times 0.05}{0.0455} = 0.4396$

**c**  $\Pr(A \cup B | D') = \frac{\Pr((A \cup B) \cap D')}{\Pr(D')} = \frac{\Pr(A \cap D') + \Pr(B \cap D')}{\Pr(D')} = \frac{0.5745}{0.9545} = \frac{1149}{1909}$

**16 a i**  $\mu = E(X) = \sum x \Pr(X = x)$   
 $= 4.25$

$$\begin{aligned}
 \text{ii} \quad \sigma &= \sqrt{\text{Var}(X)} \\
 \text{Var}(X) &= E(X^2) - [E(X)]^2 \\
 &= 1 \times 0.02 + 4 \times 0.03 + 9 \times 0.04 + 16 \times 0.45 + 25 \times 0.45 - (4.25)^2 \\
 &= 0.02 + 0.12 + 0.36 + 7.2 + 11.25 - (4.25)^2 \\
 &= 18.95 - 18.0625 \\
 &= 0.8875 \\
 \therefore \sigma &= \sqrt{0.8875} = 0.9421
 \end{aligned}$$

$$\begin{aligned}
 \text{iii} \quad \Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \\
 &= \Pr(2.366 \leq X \leq 6.134) \\
 &= 0.94
 \end{aligned}$$

$$\begin{aligned}
 \text{iv} \quad \Pr(X \geq 4) &= 0.45 + 0.45 \\
 &= 0.9
 \end{aligned}$$

**b i Binomial**

$$\begin{aligned}
 \text{ii} \quad \text{Expected number of working games in box} \\
 &= E(Y) \\
 &= 20 \times 0.9 \\
 &= 18
 \end{aligned}$$

$$\begin{aligned}
 \text{iii} \quad \text{Var}(Y) &= 20 \times 0.9 \times 0.1 \\
 &= 1.8 \\
 \sigma &= \sqrt{1.8} \\
 &= 1.342
 \end{aligned}$$

$$\begin{aligned}
 \text{iv} \quad \Pr(Y \geq 19) &= \Pr(Y = 19) + \Pr(Y = 20) \\
 &= 0.27017 \dots + 0.12158 \dots \\
 &= 0.3917 \text{ (correct to 4 decimal places)}
 \end{aligned}$$

$$\text{17 a i} \quad \Pr(\text{Black}) = \frac{n-3}{n}$$

$$\text{ii} \quad \Pr(\text{White}) = \frac{3}{n}$$

$$\begin{aligned}
 \mathbf{b} \quad \Pr(B_1|W_2) &= \frac{\Pr(B_1 \cap W_2)}{\Pr(W_2)} \\
 &= \frac{\frac{n-3}{n} \times \frac{n-3}{n+1}}{\frac{3}{n} \times \frac{n-2}{n+1} + \frac{n-3}{n} \times \frac{n-3}{n+1}} \\
 &= \frac{(n-3)^2}{n^2 - 3n + 3}
 \end{aligned}$$

**18 a**  $n = 1000, X = 100$  CI= (0.0814, 0.1186)

**b**  $m = 800, Y = 80$  CI= (0.0792, 0.1208)

**c** width female = 0.0372

width male = 0.0416

The confidence interval is narrower because the sample size for the females is larger.

**d** 900 of each sex

**e**

$$\begin{aligned}
 \frac{\hat{p}_1(1 - \hat{p}_1)}{n} &= \frac{\hat{p}_2(1 - \hat{p}_2)}{m} \\
 \frac{0.1 \times 0.9}{1000} &= \frac{\hat{p}_2(1 - \hat{p}_2)}{800} \\
 0.072 &= \hat{p}_2(1 - \hat{p}_2)
 \end{aligned}$$

$\hat{p}_2 = 0.078$  or  $0.922$

# Chapter 19 – Revision of Chapters 1–18

## Solutions to Technology-free questions

$$\begin{aligned}1 \quad f(g(x)) &= f(3x + 1) \\ &= (3x + 1)^2 + 6 \\ &= 9x^2 + 6x + 7\end{aligned}$$

2 Infinitely many solutions if the determined of the coefficients matrix is zero, i.e.

$$\begin{vmatrix} l_c & 3 \\ 4 & (l_c + 2) \end{vmatrix} = 0$$

$$l_c(l_c + 2) - 12 = 0$$

$$l_c^2 + 2l_c - 12 = 0$$

$$(l_c + 1)^2 - 13 = 0$$

$$l_c = -1 \pm \sqrt{13}$$

$$\begin{aligned}3 \quad \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2x \\ -3y \end{bmatrix}\end{aligned}$$

$$x' = 2x \text{ and } y' = -3y$$

$$x = \frac{1}{2}x' \text{ and } y = -\frac{1}{3}y'$$

$$y = \frac{1}{x} \text{ becomes } -\frac{1}{3}y' = \frac{1}{\frac{1}{2}x'}$$

i.e.  $y' = -\frac{6}{x'}$  or in terms of

$$x, y; \quad y = -\frac{6}{x}$$

Reflection in  $x$ -axis, dilation by factor 2 from  $y$ -axis, dilation by factor 3 from  $x$ -axis, OR (using the final rule) reflection in  $x$ -axis, then dilation by factor 6 from the  $x$ -(or  $y$ -) axis.

$$\begin{aligned}4 \quad \mathbf{a} \quad f'(x) &= 7(5x^3 - 3x)^6 \times (15x^2 - 3) \\ &= 21(5x^2 - 1)(5x^2 - 3x)^6 \\ &= 21x^6(5x^2 - 1)(5x^2 - 3)^6\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad f'(x) &= 2e^{4x} + 2x \times 4e^{4x} \\ &= 2e^{4x}(1 + 4x)\end{aligned}$$

$$\begin{aligned}f'(0) &= 2 \times 1 \times 1 \\ &= 2\end{aligned}$$

$$\begin{aligned}5 \quad \mathbf{a} \quad \frac{d}{dx} \left( x^2 \log_e(2x) \right) &= 2x \log_e(2x) + x^2 \times \frac{1}{x} \\ &= x(1 + 2 \log_e(2x))\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad f'(x) &= \frac{(2x + 1) \cos(x) - 2 \sin(x)}{(2x + 1)^2} \\ f'\left(\frac{\pi}{2}\right) &= \frac{\left(2 \times \frac{\pi}{2} + 1\right) \cos\left(\frac{\pi}{2}\right) - 1 \sin\left(\frac{\pi}{2}\right)}{\left(2 \times \frac{\pi}{2} + 1\right)^2} \\ &= -\frac{2}{(\pi + 1)^2}\end{aligned}$$

$$\begin{aligned}6 \quad \mathbf{a} \quad f'(x) &= e^{\sin(2x)} \times 2 \cos(2x) \\ &= 2 \cos(2x)e^{\sin(2x)}\end{aligned}$$



$$\begin{aligned} \mathbf{b} \quad f'(x) &= 3 \tan(2x) + 3x \times 2 \sec^2(2x) \\ &= 3 \tan(2x) + 6x \sec^2(2x) \end{aligned}$$

$$\begin{aligned} f'\left(\frac{\pi}{3}\right) &= 3 \tan\left(\frac{2\pi}{3}\right) + 2\pi \sec^2\left(\frac{2\pi}{3}\right) \\ &= -3\sqrt{3} + 2\pi \times \frac{1}{\cos^2\left(\frac{2\pi}{3}\right)} \\ &= -3\sqrt{3} + 2\pi \times 4 \\ &= 8\pi - 3\sqrt{3} \end{aligned}$$

$$7 \quad \sin(2x) - \cos(2x) = 0$$

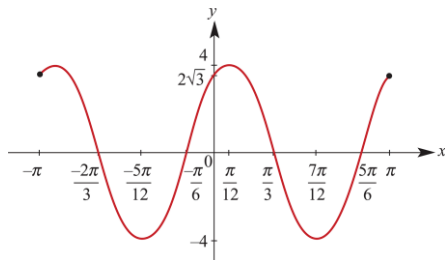
$$\sin(2x) = \cos(2x)$$

$$\tan(2x) = 1$$

$$\begin{aligned} 2x &= \frac{\pi}{4} + n\pi \\ &= \frac{\pi(4n+1)}{4} \\ x &= \frac{\pi(4n+1)}{8}, n \in \mathbb{Z} \end{aligned}$$

$$8 \quad \mathbf{a} \quad \text{Amplitude} = 4, \text{ period} = \frac{2\pi}{2} = \pi$$

**b**



$$\begin{aligned} 9 \quad y = f(x) &= 1 - \frac{4}{x-2} \\ x &\rightarrow \pm\infty, y \rightarrow 1; x \rightarrow 2, y \rightarrow \pm\infty \\ \text{S, the asymptotes have equations} \end{aligned}$$

$$x = 2 \text{ and } y = 1$$

$$x = 0 : y = 1 - \frac{4}{0-2}$$

$$= 1 - (-2)$$

$$= 3$$

$$y = 0 : \frac{4}{x-2} = 1$$

$$= x - 2 = 4$$

$$x = 6$$

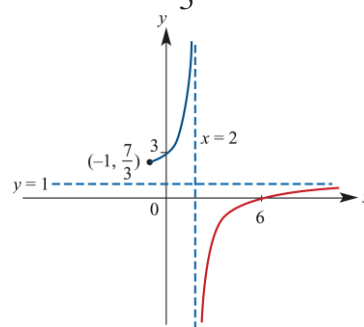
The intercepts are (6, 0) and (0, 3)

check the endpoint of the domain:

$$x = -1 : y = 1 - \frac{4}{-1-2}$$

$$= 1 + \frac{4}{3}$$

$$= \frac{7}{3}$$



$$10 \quad \mathbf{a} \quad y = 5e^{x-1} - 3$$

interchange  $x$  and  $y$  and solve for  $y$ :

$$x = 5e^{y-1} - 3$$

$$5e^{y-1} = x + 3$$

$$e^{y-1} = \log_e \frac{x+3}{5}$$

$$y - 1 = \log_e \left( \frac{x+3}{5} \right)$$

$$y = f^{-1}(x) = \log_e \left( \frac{x+3}{5} \right) + 1$$

**b** range of  $f = (-3, \infty) =$  domain of  $f^{-1}$

$$11 \quad \cos\left(\frac{5x}{2}\right) = \frac{1}{2}, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

General solution is given by

$$\frac{5x}{2} = \pm \frac{\pi}{3} + 2n\pi$$

$$x = \pm \frac{2\pi}{15} + \frac{4n\pi}{15}$$

$$n = 0: x = \pm \frac{2\pi}{15}$$

$$n = 1: x = \pm \frac{2\pi}{15} + \frac{4\pi}{5}$$

$$= \frac{2\pi}{3}, \frac{14\pi}{15} \text{ (outside interval)}$$

$$n = -1: x = \pm \frac{2\pi}{3} - \frac{4\pi}{5}$$

$$= -\frac{14\pi}{15}, -\frac{2\pi}{3} \text{ (outside interval)}$$

$$\text{Solutions are } x = -\frac{2\pi}{15}, \frac{2\pi}{15}$$

$$12 \quad \begin{aligned} g(u+v) &= 5(u+v)^2 \\ &= 5(u^2 + 2uv + v^2) \end{aligned}$$

$$\begin{aligned} g(u+v) &= 5(u-v)^2 \\ &= 5(u^2 - 2uv + v^2) \end{aligned}$$

$$\begin{aligned} g(u+v) + g(u-v) &= 10(u^2 + v^2) \\ &= 2(5u^2 + 5v^2) \\ &= 2(9(u) + g(v)) \end{aligned}$$

$$13 \quad \begin{aligned} \text{Average value} &= \frac{1}{4-0} \int_0^4 e^x dx \\ &= \frac{1}{4} \left[ e^x \right]_0^4 \\ &= \frac{1}{4} (e^4 - 1) \end{aligned}$$

$$14 \quad \text{a } x = 0, y = 6: \quad 6 = 0 + 0 + c$$

$$c = 6$$

$$x = -2, y = 0: 0 = -8a - 2b + 6$$

$$4a + b = 3 \quad \dots \textcircled{1}$$

$$\frac{dy}{dx} = 0 \text{ when } x = -1$$

$$\text{b } \frac{dy}{dx} = 3ax^2 + b$$

$$= 0 \text{ when } x = -1, \text{ so}$$

$$3a + b = 0 \quad \dots \textcircled{2}$$

$$\text{c } \textcircled{1} - \textcircled{2}: \quad a = 3$$

$$\begin{aligned} \text{Substitute into } \textcircled{2}: \quad b &= -3 \times 3 \\ &= -9 \end{aligned}$$

$$15 \quad \text{a } y = g(x) = 3 - e^{2x}$$

Interchange  $x$  and  $y$  and solve for  $y$ .

$$x = 3 - e^{2y}$$

$$e^{2y} = 3 - x$$

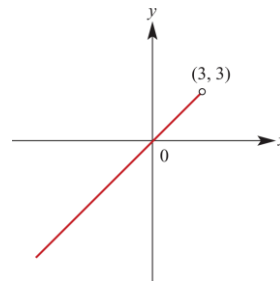
$$2y = \log_e(3 - x)$$

$$y = g^{-1}(x) = \frac{1}{2} \log_e(3 - x)$$

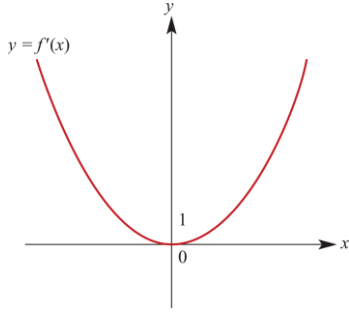
$$\text{domain of } g^{-1} = \text{range of } g = (-\infty, 3)$$

$$\text{b } y = g(g^{-1}(x))$$

$$= x, \text{ with domain } (-\infty, 3)$$



16 a The graph of  $y = f(x)$  is continuous and appears to be 'smooth' at  $(0, 1)$ , so the derivative exists at  $x = 0$  where the gradient appears to be zero. The gradient is positive for all other value of  $x$ . The graph of  $y = f'(x)$  is shown below.



**b**  $f'(x) = \begin{cases} -8x^3 & x \leq 0 \\ 8x^3 & \text{otherwise} \end{cases}$   
 (Note that  $f'(0) = 0$  as expected.)

**17**  $f(x) = \frac{1}{-3} \log_e(1 - 3x) + c$   
 $= -\frac{1}{3} \log_e(1 - 3x) + c$

**18**  $y = f(x) = \frac{3}{2x-1} + 3$   
 Interchange  $x$  and  $y$  and solve for  $y$   
 $x = \frac{3}{2y-1} + 3$   
 $\frac{3}{2y-1} = x - 3$   
 $\frac{2y-1}{3} = \frac{1}{x-3}$   
 $2y-1 = \frac{3}{x-3}$   
 $2y = \frac{3}{x-3} + 1$   
 $= \frac{3+x-3}{x-3}$   
 $= \frac{x}{x-3}$   
 $y = f^{-1}(x) = \frac{x}{2(x-3)}$

**19**  $\tan(2x) = -\sqrt{3}$

$$2x = \dots - \frac{\pi}{3} - \pi, \frac{\pi}{3}, -\frac{\pi}{3} + \pi, -\frac{\pi}{3} + 2\pi, \dots$$

$$2x = \dots - \frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \dots$$

$$x = \dots - \frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \dots$$

since  $x \in \left(\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ , the solution are  $x = -\frac{\pi}{6}, \frac{\pi}{3}$ .

**20**  $X$  is normal with mean 84 and standard deviation 6.

**a**  $\Pr(X > 84) = \Pr(Z > 0)$   
 $= 0.5$

**b**  $\Pr(78 < X < 90)$   
 $= \Pr\left(\frac{78-84}{6} < Z < \frac{90-84}{6}\right)$   
 $= \Pr(-1 < Z < 1)$   
 $= \Pr(Z < 1) - \Pr(Z > 1)$   
 $= \Pr(Z < 1) - \Pr(Z > 1)$   
 $= \Pr(Z < 1) - (1 - \Pr(Z < 1))$   
 $= 2 \Pr(Z < 1) - 1$   
 $= 2 \times 0.84 - 1$   
 $= 0.68$

**c**  $\Pr(X < 78 | X < 84)$   
 $= \frac{\Pr('X < 78' \cap 'X < 84')}{\Pr(X < 84')}$   
 $= \frac{\Pr(X < 78)}{\Pr(X < 84)}$   
 $\Pr(X < 78) = \Pr(Z < -1)$   
 $= 1 - \Pr(Z < 1)$   
 $= 1 - 0.84$   
 $= 0.16$

$$\Pr(X < 84) = 0.5$$

$$\begin{aligned}\Pr(X < 78 | X < 84) &= \frac{0.16}{0.5} \\ &= 0.32\end{aligned}$$

$$\begin{aligned}\mathbf{21\ a}\ \Pr(X < 3) &= \int_1^3 \frac{x}{24} dx \\ &= \left[ \frac{x^2}{48} \right]_1^3 \\ &= \frac{9-1}{48} \\ &= \frac{1}{6}\end{aligned}$$

$$\begin{aligned}\mathbf{b}\ \Pr(X \geq b) &= \int_b^7 \frac{x}{24} dx \\ &= \left[ \frac{x^2}{48} \right]_b^7 \\ &= \frac{49-b^2}{48} \\ &= \frac{3}{8}\end{aligned}$$

$$\text{if } 49 - b^2 = 18$$

$$b^2 = 31$$

$$b = \sqrt{31}, \text{ since } b \in [1, 7]$$

**22** The gradient of the tangent is  $\frac{1}{3}$

$$\begin{aligned}\text{Also, } \frac{dy}{dx} &= \frac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{1}{3}\end{aligned}$$

$$\text{if } x^{-\frac{2}{3}} = 1$$

$$x^{-2} = 1$$

$$x^2 = 1$$

$$x = \pm 1$$

$$x = -1, y = -1: y + 1 = \frac{1}{3}(x + 1)$$

$$\Rightarrow y = \frac{1}{3}x - \frac{2}{3}$$

$$x = -1, y = -1: y + 1 = \frac{1}{3}(x + 1)$$

$$\Rightarrow y = \frac{1}{3}x - \frac{2}{3}$$

$$\text{Hence } a = \pm \frac{2}{3}$$

$$\mathbf{23\ a}\ b = 16 - 4a^2$$

$$A = \text{area } XYZW$$

$$= 2ab$$

$$= 2a(16 - 4a^2)$$

$$= 32a - 8a^3$$

$$\mathbf{b}\ \frac{dA}{da} = 32 - 24a^2$$

$$= 0$$

$$\text{If } a^2 = \frac{32}{24}$$

$$= \frac{4}{3}$$

$$a = \pm \frac{2}{\sqrt{3}}$$

$$= \pm \frac{2\sqrt{3}}{3}$$

But  $a > 0$ , so  $a = \frac{2\sqrt{3}}{3}$ , and

$$A = \frac{128\sqrt{3}}{9}$$

(This clearly correspond, to a maximum since  $a \in [0, 2]$  and  $A = 0$  for  $a = 0$  or  $a = 2$ . Alternately check the sign of the derivative.)

$$\begin{aligned}
 24 \quad \int_{-1}^3 (-3x^2 + 2bx + 9) dx &= 32 \\
 \left[ -x^3 + bx^2 + 9x \right]_{-1}^3 &= 32 \\
 (-27 + 9b + 27) - (1 + b - 9) &= 32 \\
 8b + 8 &= 32 \\
 8b &= 24 \\
 b &= 3
 \end{aligned}$$

25 0.36

$$\begin{aligned}
 26 \quad \text{a} \quad \text{Mean of } X &= E(X) \\
 &= 0 \times 0.6 + 1 \times 0.2 + 2 \times 0.15 \\
 &\quad + 3 \times 0.0 \\
 &= 0.65 \\
 \text{the mean is } &\$0.65.
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \text{Pr(same amount)} &= \text{Pr}(0 \& 0 \text{ or } 1 \& 1 \\
 \text{or } 2 \& 2 \text{ or } 3 \& 3) \\
 &= 0.6^2 + 0.2^2 \\
 &\quad + 0.15^2 + 0.05^2 \\
 &= 0.36 + 0.04 \\
 &\quad + 0.0225 + 0.0025 \\
 &= 0.425
 \end{aligned}$$

$G \rightarrow G \rightarrow R \rightarrow R$  or

$G \rightarrow R \rightarrow G \rightarrow R$  or

$G \rightarrow R \rightarrow R \rightarrow G$

where  $G$  = goes to gym, and

$R$  = goes for run

Required probability =  $0.5 \times 0.5 \times 0.6$

$$+ 0.5 \times 0.4 \times 0.5$$

$$+ 0.5 \times 0.6 \times 0.4$$

$$= 0.15 + 0.10$$

$$+ 0.12$$

$$= 0.37$$

28 a Volume = area cross-section  $\times$  height

$$= \frac{1}{2}x^2h$$

$$= 2000$$

$$x^2h = 4000$$

$$h = \frac{4000}{x^2}$$

b The hypotenuse of the right-angled triangle cross-section has length  $\sqrt{2}x$ . The surface area is made up of three vertical rectangles and two equal triangular ends.

$$A = \sqrt{2}xh + xh + xh + 2 \times \frac{1}{2}x^2$$

$$= xh(2 + \sqrt{2}) + x^2$$

$$= x \times \frac{4000}{x^2} \times (2 + \sqrt{2}) + x^2$$

$$= \frac{4000\sqrt{2} + 8000}{x} + x^2$$

27 The possible sequences are:

$$\begin{aligned} \mathbf{c} \quad \frac{dA}{dx} &= -\frac{4000\sqrt{2} + 8000}{x^2} + 2x \\ &= 0 \end{aligned}$$

$$\text{if } 2x^3 = 4000\sqrt{2} + 8000$$

$$\begin{aligned} \text{i.e. } x^3 &= 2000\sqrt{2} + 4000 \\ &= 2000(2 + \sqrt{2}) \end{aligned}$$

- 29 a** No, these people may all be at the same restaurant because they have something in common eg may be members of a tennis club, and may have better reaction times than average
- b** Use a random selection method such

as the electoral role.

$$\mathbf{30 a} \quad \hat{p} = 0.53$$

$$\begin{aligned} \mathbf{b} \quad \hat{p} \pm 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ = 0.53 \pm 1.96 \sqrt{\frac{0.53 \times 0.47}{100}} \end{aligned}$$

$$\mathbf{31 a} \quad \hat{p} = 0.37$$

$$\begin{aligned} \mathbf{b} \quad M &= 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.96 \sqrt{\frac{0.37 \times 0.63}{n}} \end{aligned}$$

- c** Halving  $n$  will increase the margin of error by a factor of  $\sqrt{2}$

## Solutions to multiple-choice questions

1 B Write the equations in matrix form:

$$\begin{bmatrix} m & -2 \\ 6 & -(m+4) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These will be a unique solution if the coefficients matrix has a non-zero determinant, i.e.

$$\begin{vmatrix} m & -2 \\ 6 & -(m+4) \end{vmatrix} \neq 0$$

$$= m(m+4) - (-2)(6) \neq 0$$

$$-m^2 - 4m + 12 \neq 0$$

$$m^2 + 4m - 12 \neq 0$$

$$(m-2)(m+6) \neq 0$$

$$m \neq 2, -6$$

$$\text{So } m \in \mathbb{R} \setminus \{-6, 2\}$$

2 A Since  $\sin\left(\frac{\pi}{2}\right) = 1$ , then

$$2x = \frac{\pi}{2} + 2n\pi$$

$$x = n\pi + \frac{\pi}{4}$$

3 D  $f(x-y) = (x-y)^3$   
 $\neq x^3 - y^3 = f(x) - f(y)$

Checking the other options shows each one is true.

4 B The graph of  $f$  has a sharp point at  $x = -\frac{4}{5}$ , so  $f'\left(-\frac{4}{5}\right)$  is not defined.

Hence the graph of  $f'$  is discontinuous at  $x = -\frac{4}{5}$ .

Checking the other points shows that each one is true.

$$\begin{aligned} 5 \text{ C } k &= \int_{-6}^{-2} \frac{2}{x} dx \\ &= \left[ 2 \log_e |x| \right]_{-6}^{-2} \\ &= 2 \log_e 2 - 2 \log_e 6 \\ &= 2 \log_e \frac{x}{6} \\ &= \log_e \left( \frac{1^2}{3} \right) \\ &= \log_e \frac{1}{9} \\ e^k &= \frac{1}{9} \end{aligned}$$

$$\begin{aligned} 6 \text{ D } \text{Average value} &= \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{3 - (-1)} \\ &\quad \int_{-1}^3 \log_e(x+2) dx \\ &= \frac{5 \log_e 5 - 4}{4} \end{aligned}$$

using the integral command of a CAS calculator.

$$\begin{aligned} 7 \text{ A } \text{Average value} &= \frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} \sin(2x) dx \\ &= \frac{2}{\pi} \left[ -\frac{1}{2} \cos(2x) \right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{\pi} \left( -\frac{1}{2} \cos(\pi) \right) + \frac{1}{2} \cos(0) \\ &= \frac{2}{\pi} \end{aligned}$$

**8 D**  $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix}$  so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \end{bmatrix} \\ = \begin{bmatrix} x' - 5 \\ y' - 1 \end{bmatrix}$$

Multiply both sides by  $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$ , the

inverse of  $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ .

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' - 5 \\ y' - 1 \end{bmatrix} \\ = \begin{bmatrix} \frac{1}{3}(x' - 5) \\ y' - 1 \end{bmatrix}$$

Hence  $x = \frac{1}{3}(x' - 5)$  and  $y = y' - 1$

The image of  $y = x^2$  has equation

$$y^1 - 1 = \frac{1}{9}(x' - 5)^2$$

$$9y' = (x' - 5)^2 + 9$$

In terms of  $x$  and  $y$ :

**9 C**  $[f(x)]^3 = f(y)$

$$(e^{3x})^3 = e^{3y}$$

$$e^{9x} = e^{9y}$$

$$3y = 9x$$

$$y = 3x$$

**10 D**

$$\Pr(X > a) = 0.25$$

$$\int_a^{\frac{\pi}{2}} \sin(2x) dx = 0.25$$

$$\left[ -\frac{1}{2} \cos(2x) \right]_a^{\frac{\pi}{2}} = 0.25$$

$$-\frac{1}{2} \cos(\pi) + \frac{1}{2} \cos(2a) = 0.25 - (-1) + \cos(2a)$$

$$= 0.25 \cos(2a) = -0.5$$

$$2a = \frac{2\pi}{3}$$

$$a = \frac{\pi}{3}$$

$$\approx 1.05$$

**11 C**

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt}$$

$$= 4\pi r^2 \times 4$$

$$= 16\pi r^2$$

When  $r = 2$ :

$$\frac{dV}{dt} = 16\pi \times 4$$

$$= 64\pi \text{ cm}^3/\text{min}$$

**12 A**  $\int_0^{2k} (1 + 2e^{\frac{x}{k}}) dx = 1$

$$\left[ x + 2ke^{\frac{x}{k}} \right]_0^{2k} = 1$$

$$(2k + 2ke^2) - (2k) = 1$$

$$2ke^2 = 1$$

$$k = \frac{1}{2e^2}$$

$$= \frac{1}{2}e^{-2}$$



$$\begin{aligned}
 13 \text{ A } \Pr(X < 7.5) &= \Pr\left(Z < \frac{7.5 - 8}{0.25}\right) \\
 &= \Pr(Z < -2) \\
 &= \Pr(Z > 2)
 \end{aligned}$$

$$\begin{aligned}
 14 \text{ B } x^2 + 12x &= 2kx - 2 \\
 x^2 + (12 - 2k)x + 2 &= 0 \\
 \text{Quadratic has two solutions if} \\
 (12 - 2k)^2 - 4(1)(2) &> 0 \\
 4k^2 - 48k + 144 - 8 &> 0 \\
 k^2 - 12k + 34 &> 0 \\
 (k - 6)^2 - 2 &> 0 \\
 (k - 6 - \sqrt{2})(k - 6 + \sqrt{2}) &> 0 \\
 k < 6 - \sqrt{2} \text{ or } k > 6 + \sqrt{2}
 \end{aligned}$$

15

$$\begin{aligned}
 \text{D } e^{4x} - 7e^{2x} + 12 &= 0 \\
 (e^{2x} - 3)(e^{2x} - 4) &= 0 \\
 e^{2x} &= 3, 4 \\
 2x &= \log_e 3, \log_e 4 \\
 x &= \frac{1}{2} \log_e 3, \frac{1}{2} \log_e 4 \\
 &= \log_e \sqrt{3}, \log_e 2 \\
 \text{Solution set} &= \left\{ \log_e \sqrt{3}, \log_e 2 \right\}
 \end{aligned}$$

$$\begin{aligned}
 16 \text{ B } \text{Reflection in } x\text{-axis: } &-7x^{\frac{3}{2}} \\
 \text{Translated 3 units right: } &-7(x - 3)^{\frac{3}{2}} \\
 \text{Translated 4 units down:} \\
 -7(x - 3)^{\frac{3}{2}} - 4 \\
 \text{The equation of the new graph is} \\
 y = -7(x - 3)^{\frac{3}{2}} - 4
 \end{aligned}$$

$$\begin{aligned}
 17 \text{ C } E(x) &= \frac{1}{8} \int_0^4 x^2 dx \\
 &= \frac{1}{8} \left[ \frac{1}{3} x^3 \right]_0^4 \\
 &= \frac{1}{8} \left( \frac{64}{3} \right) \\
 &= \frac{8}{3}
 \end{aligned}$$

18 A Since  $f(2)$  does not exist, since  $\log_e 0$  is undefined, the graph of  $y = f(x) = 4 \log_e(x - 2)^4$  is symmetrical about the asymptote  $x = 2$ . For a one-one function, the domain must be restricted and for a domain of  $[a, \infty)$ , we must have  $a > 2$  of the available options, only the first fits.

$$\begin{aligned}
 19 \text{ A } e^{2x+4} - 3 &= e^{2(x+2)} - 3 \\
 &= f(2(x+2)) - 3
 \end{aligned}$$

So transform the graph of  $y = f(x)$  using this sequences:

- dilation of factor  $\frac{1}{2} = 0.5$  from the y-axis
- translations of 2 left and 3 down

The dilation can be represented by

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

The translation can be represented by

$$\begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- 20 E**  $f'(x) = g'(x)$ , so  
 $f(x) = g(x) + c$   
 Now  $f(1) = 2$  and  $g(x) = -xf(x)$ , so  
 So  $f(x) = g(x) + 4$

- 21 A** If  $f(x) = \frac{1}{x}$ , then  
 $f\left(\frac{xy}{2}\right) = \frac{2}{xy}$   
 $= 2 \times \frac{1}{x} \times \frac{1}{y}$   
 $= 2f(x)f(y) \quad (x, y \neq 0)$   
 Checking the other options shows none fit.

- 22 E**  $E(x) = 0 \times a + 1 \times b + 2 \times 0.6$   
 $= b + 1.2$   
 $= 1.6$  if  $b = 0.4$   
 Then  $a + 0.4 + 0.6 = 1 \rightarrow a = 0$

- 23 E** In matrix form:  
 $\begin{bmatrix} m-4 & 6 \\ 2 & m-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 2m-10 \end{bmatrix}$

For no solutions, the determinant of the coefficients matrix is zero, i.e.

$$(m-4)(m-3) - 12 = 0$$

$$m^2 - 7m = 0$$

$$m(m-7) = 0 \rightarrow m = 0 \text{ or } m = 7$$

$m = 0$ : Equations are  $-4x + by = 6$  and  $2x - 3y = 10$ , which have no solution.

$m = 7$ : Equations are  $3x + 6y = 6$  and  $2x + 4y = 4$ , which are both equal to  $x + 2y = 2$ .

- 24 D**  $n = 1000, \hat{p} = 0.52$   
 95% CI = (0.489, 0.551)

- 25 C** The candidate needs more than 50% of the vote to win. Based on the confidence interval they will get between 48.9% and 55.1% of the vote- they might win but its too close to tell.

## Solutions to extended-response questions

1 a i  $y = \frac{16x^3 + 4x^2 + 1}{2x^2}$

$$\frac{dy}{dx} = \frac{2x^2(48x^2 + 8x) - 4x(16x^3 + 4x^2 + 1)}{(2x^2)^2} \text{ (quotient rule)}$$

$$= \frac{96x^4 + 16x^3 - 64x^4 - 16x^3 - 4x}{4x^4}$$

$$= \frac{32x^4 - 4x}{4x^4} = \frac{8x^3 - 1}{x^3}$$

$$\frac{dy}{dx} = 0 \text{ implies } 8x^3 - 1 = 0$$

$$x^3 = \frac{1}{8}$$

$$\therefore x = \frac{1}{2}$$

$$\therefore \text{ Stationary point at } \left(\frac{1}{2}, 8\right)$$

ii  $y = 8x + 2 + \frac{1}{2}x^{-2}$  (achieved by dividing by  $2x^2$ )

Addition of coordinates gives the shape of the graph.

To establish minimum:

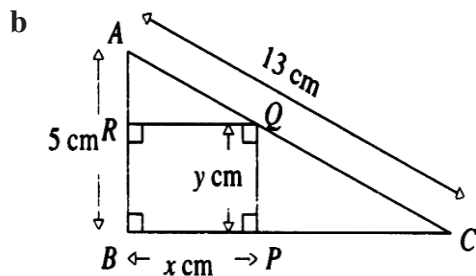
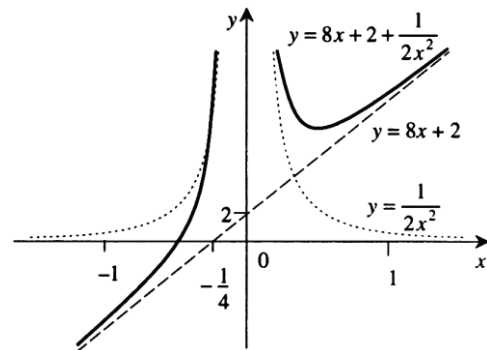
when  $x = 0.25$ ,  $y = 12$

when  $x = 0.75$ ,  $y = \frac{80}{9} = 8\frac{8}{9}$

or gradient

when  $x = 0.25$ ,  $\frac{dy}{dx} = -56$

when  $x = 0.75$ ,  $\frac{dy}{dx} = \frac{152}{27}$



i  $\triangle QPC \sim \triangle ABC$

and both are right angled triangles. By Pythagoras' Theorem

$$BC = \sqrt{13^2 - 5^2} = 12$$

and  $\frac{PC}{BC} = \frac{QP}{AB}$

$$\therefore \frac{12 - x}{12} = \frac{y}{5}$$

and  $y = \frac{60 - 5x}{12}$

ii Area of the rectangle  $A = xy = \frac{x(60 - 5x)}{12}$

iii The practical domain for  $A$  is  $0 \leq x \leq 12$

By the properties of parabolas for which the coefficient of  $x^2$  is negative, maximum point has coordinates  $(6, 15)$ .

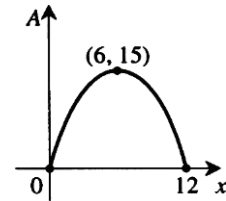
Alternately:  $A = 5x - \frac{5x^2}{12}$

$$\frac{dA}{dx} = 5 - \frac{5x}{6}$$

$$\frac{dA}{dx} = 0, \text{ implies } x = 6$$

when  $x = 6$ ,  $A = \frac{6(60 - 5 \times 6)}{12} = 15$

$\therefore$  maximum area is  $15 \text{ cm}^2$



2 a  $\begin{array}{c|ccc} x & 0 & 1 & 3 \\ \hline y & 6 & 0 & 0 \end{array}$

$$y = k(x - p)(x - q)$$

Since  $y = 6$  when  $x = 0$ ,  $6 = kpq$  ①

Also  $0 = k(x - p)(x - q)$

implies  $x = p$  or  $x = q$

hence  $p = 1$  and  $q = 3$  as  $p < q$

From equation ①  $k = 2$

b i For  $y = m(x - p)^2(x - q)$

As before  $p = 1$  and  $q = 3$

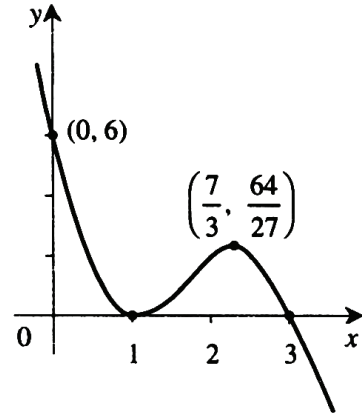
Now as then  $x = 2$ ,  $y = 2$

$$2 = m(2 - 1)^2(2 - 3)$$

$\therefore m = -2$  (Note: when  $x = 0$ ,  $y = 6$ )

$$\begin{aligned}
 \text{ii } y &= -2(x-1)^2(x-3) \\
 &= -2\left[(x^2-2x+1)(x-3)\right] \\
 &= -2[x^3-2x^2+x-3x^2+6x-3] \\
 &= -2[x^3-5x^2+7x-3] \\
 &= -2x^3+10x^2-14x+6
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } \frac{dy}{dx} &= -6x^2+20x-14 \\
 \frac{dy}{dx} = 0 &\text{ implies } -6x^2+20x-14=0 \\
 &\rightarrow -2(3x^2-10x+7)=0 \\
 &\rightarrow (3x-7)(x-1)=0 \\
 &\rightarrow x = \frac{7}{3} \text{ or } x = 1
 \end{aligned}$$



When  $x = 1, y = 0$ , When  $x = \frac{7}{3}, y = \frac{64}{27}$ .

There is a local min at  $(1,0)$  and a local max at  $(\frac{7}{3}, \frac{64}{27})$ .

Note:  $\frac{dy}{dx} = -\frac{11}{2}$  when  $x = \frac{1}{2}$

$\frac{dy}{dx} = 2$  when  $x = 2$

$\frac{dy}{dx} = -1.5$  when  $x = 2.5$

A gradient chart illustrates the nature of the stationary points

	$x < 1$	$1$	$1 < x < 2\frac{1}{3}$	$2\frac{1}{3}$	$x > 2\frac{1}{3}$
sign of $\frac{dy}{dx}$	-ve	0	+ve	0	-ve
shape	\	-	/	-	\

**3 a**  $y = ax - x^2$

When  $y = 0$

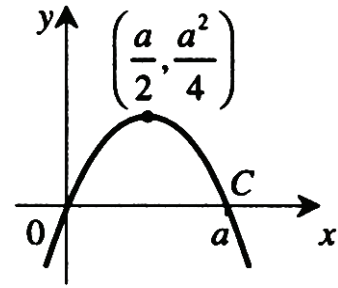
$x(a - x) = 0$

$\therefore x = 0, \text{ or } x = a$

By symmetry turning point occurs when

$$x = \frac{a}{2}$$

$$x = \frac{a}{2} \text{ When } y = \frac{a}{2} \left( a - \frac{a}{2} \right) = \frac{a^2}{4}$$



**b**  $\int_0^a ax - x^2 dx = \left[ \frac{ax^2}{2} - \frac{x^3}{3} \right]_0^a$

$$= \frac{a^3}{2} - \frac{a^3}{3}$$

$$= \frac{a^3}{6}$$

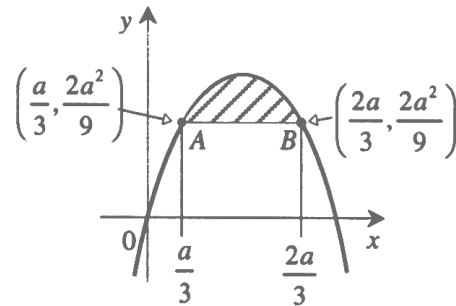
$\therefore$  The area is  $\frac{a^3}{6}$  square units.

**c i** When  $x = \frac{a}{3}$ ,  $y = a \times \frac{a}{3} - \left( \frac{a}{3} \right)^2$

$$= \frac{a^2}{3} - \frac{a^2}{9}$$

$$= \frac{2a^2}{9}$$

when  $x = \frac{2a}{3}$ ,  $y = \frac{2a^2}{9}$  by symmetry



$$\begin{aligned}
 \text{ii } \int_{\frac{1}{3}a}^{\frac{2}{3}a} ax - x^2 dx &= \left[ \frac{ax^2}{2} - \frac{x^3}{3} \right]_{\frac{1}{3}a}^{\frac{2}{3}a} \\
 &= \frac{a}{2} \times \frac{4}{9} a^2 - \frac{8a^3}{81} - \left( \frac{a}{2} \times \frac{1}{9} a^2 - \frac{1}{81} a^3 \right) \\
 &= \frac{2a^3}{9} - \frac{a^3}{18} - \frac{8a^3}{81} + \frac{1}{81} a^3 \\
 &= \frac{a^3}{81} \left[ 18 - \frac{9}{2} - 8 + 1 \right] \\
 &= \frac{13a^3}{162}
 \end{aligned}$$

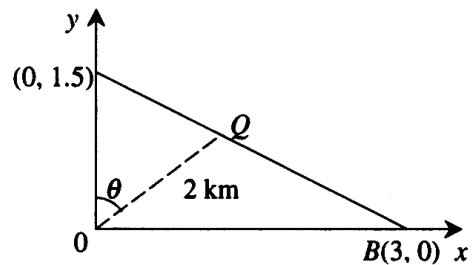
$$\begin{aligned}
 \text{Area of the rectangle} &= \frac{a}{3} \times \frac{2a^2}{9} \\
 &= \frac{12a^3}{162}
 \end{aligned}$$

$$\therefore \text{required area} = \frac{a^3}{162} \text{ square units}$$

Note: This area may also be found by evaluating  $\int_{\frac{1}{3}a}^{\frac{2}{3}a} ax - x^2 - \frac{2a^2}{9} dx$

4 a Equation of line

$$\begin{aligned}
 y &= \left( \frac{1.5 - 0}{0 - 3} \right) x + 1.5 \\
 &= -\frac{1}{2}x + \frac{3}{2}
 \end{aligned}$$



b i  $y = \sin \theta + 2 \cos \theta$

$$\therefore \frac{dy}{d\theta} = \cos \theta - 2 \sin \theta$$

ii  $\frac{dy}{d\theta} = 0$  implies  $\cos \theta = 2 \sin \theta$

$$\text{which implies } \tan \theta = \frac{1}{2} (\cos \theta \neq 0)$$

$$\therefore \theta = \tan^{-1} \left( \frac{1}{2} \right) \approx 26.57^\circ$$

iii  $y \approx 2.2361$  when  $\theta = \tan^{-1} \left( \frac{1}{2} \right)$

$\therefore (26.57, 2.2361)$  are the coordinates of the stationary point. The following shows the exact coordinates to be  $\left( \tan^{-1} \left( \frac{1}{2} \right), \sqrt{5} \right)$ .

iv A maximum occurs when  $\theta = \tan^{-1}\left(\frac{1}{2}\right)$

$$\begin{aligned} \text{Note: } & \sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right) + 2\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right) \\ &= \frac{1}{\sqrt{5}} + 2 \times \frac{2}{\sqrt{5}} \\ &= \frac{5}{\sqrt{5}} = \sqrt{5} \end{aligned}$$

$$\therefore r = \sqrt{5}$$

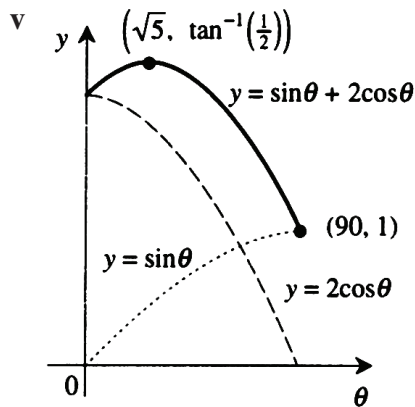
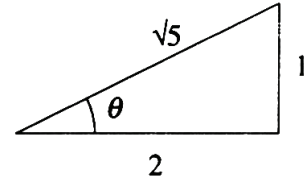
$$\therefore y = \sqrt{5} \sin(\theta + \alpha)$$

$$\text{when } \theta = 0, y = 2 \quad \therefore \sin \alpha = \frac{2}{\sqrt{5}}$$

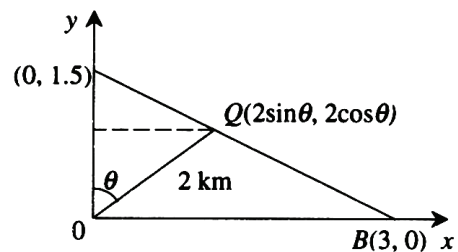
$$\therefore \alpha = 63.435^\circ \left[ \text{The smallest positive solution is chosen.} \right.$$

Any solution will work.  $\left. \vphantom{\alpha} \right]$

$$\therefore y = \sqrt{5} \sin(\theta + 63.435)$$



c i coordinates of  $Q = (2 \sin \theta, 2 \cos \theta)$



ii  $Q$  is on the line with equation

$$y = -\frac{1}{2}x + \frac{3}{2}$$



$$\therefore 2 \cos \theta = -\sin \theta + \frac{3}{2}$$

$$\therefore 2 \cos \theta + \sin \theta + \frac{3}{2}$$

i.e.  $4 \cos \theta + 2 \sin \theta = 3$

**iii**  $2 \cos \theta + \sin \theta = \frac{3}{2}$

From (b)(iv)

$$\sqrt{5} \sin\left(\theta + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \frac{3}{2}$$

$$\sin\left(\theta + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \frac{3}{2\sqrt{5}}$$

$$\therefore \theta + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) = \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right) \text{ or } 180 - \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right)$$

$$\therefore \theta = \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \text{ or } 180 - \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

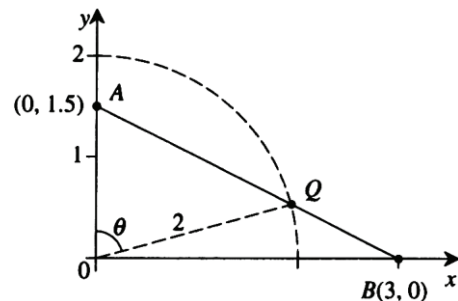
$$= -21.3045 \text{ or } 74.4346$$

for  $0^\circ < \theta \leq 90^\circ$ , required answer  $\theta = 74.4346^\circ$

**Alternative Method to find the point Q**

Q can be considered to be on a circle radius 2 km centre O.

The equation of this circle is  $x^2 + y^2 = 4$



$\therefore$  Solve simultaneously the equations

$$x^2 + y^2 = 4 \quad \text{①}$$

$$\text{and } y = -\frac{1}{2}x + \frac{3}{2} \quad \text{②}$$

Substitute from ② into ①

$$\therefore \left(-\frac{1}{2}x + \frac{3}{2}\right)^2 + x^2 = 4$$

$$\therefore 9 - 6x + x^2 + 4x^2 = 16$$

$$5x^2 - 6x - 7 = 0$$

$$\therefore x = \frac{6 \pm \sqrt{36 - 4 \times -7 \times 5}}{10}$$

$$= \frac{6 \pm \sqrt{176}}{10}$$

$$x = \frac{6 \pm 4\sqrt{11}}{10} = \frac{3 \pm 2\sqrt{11}}{5}$$

$$x \text{ must be positive } \therefore x = \frac{3 + 2\sqrt{11}}{5}$$

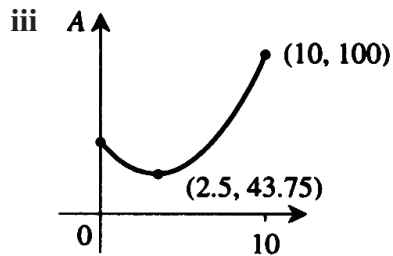
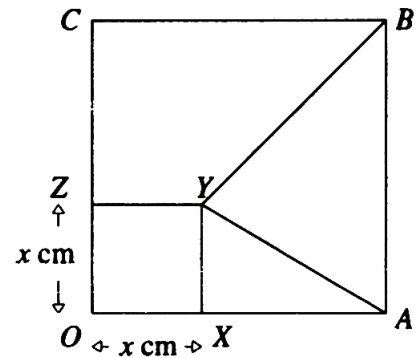
$$\text{i.e. } 2 \sin \theta = \frac{3 + 2\sqrt{11}}{5}$$

$$\text{i.e. } \sin \theta = \frac{3 + 2\sqrt{11}}{10}$$

$$\theta = 74.4346^\circ$$

- 5 a i** Area of  $OXYZ = x^2 \text{ cm}^2$   
 Area of  $ABY = \frac{1}{2} \times 10 \times (10 - x)$   
 $= 5(10 - x) \text{ cm}^2$   
 $\therefore$  total area  $A = x^2 + 50 - 5x$   
 $= x^2 - 5x + 50$

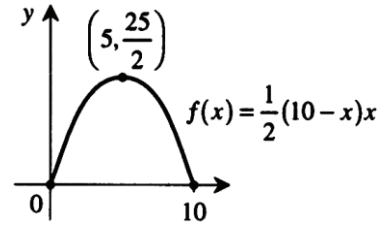
- ii** domain =  $(0, 10)$



- iv** minimum area =  $43.75 \text{ cm}^2$

- b i**  $f(x) = \frac{1}{2} \times \text{base} \times \text{height}$   
 $= \frac{1}{2} \times AX \times XY$   
 $= \frac{1}{2}(10 - x)x$  domain =  $(0, 10)$

- ii Maximum area of  $AYX = 12.5\text{cm}^2$   
 This occurs when  $x = 5$   
 When  $x = 5$   
 Area of square  $OXYZ = 25\text{cm}^2$   
 Area of triangle  $ABY = 25\text{cm}^2$   
 Area of trapezium  $CBYZ = 37.5\text{cm}^2$   
 $\therefore$  ratio of areas  $AYX : OXYZ : ABY : CBYZ$   
 $= 12.5 : 25 : 25 : 37.5$   
 $= 25 : 50 : 50 : 75$   
 $= 1 : 2 : 2 : 3$



- 6  $f(t) = 1000(t^2 - 10t + 44)e^{-\frac{t}{10}} \quad 0 \leq t \leq 35$   
 Using a CAS calculator it is interesting to graph the function for  $t \in [0, 35]$ .

a i  $f'(t) = 1000(2t - 10)e^{-\frac{t}{10}} - \frac{1}{10}\left(1000(t^2 - 10t + 44)\right)e^{-\frac{t}{10}}$   
 $= 100e^{-\frac{t}{10}}[20t - 100 - t^2 + 10t - 44]$   
 $= 100e^{-\frac{t}{10}}[30t - 144 - t^2]$   
 $= -100e^{-\frac{t}{10}}[t^2 - 30t + 144]$

ii  $f''(t) = -100\left[-\frac{1}{10}e^{-\frac{t}{10}}(t^2 - 30t + 144) + (2t - 30)e^{-\frac{t}{10}}\right]$   
 $= 10e^{-\frac{t}{10}}[t^2 - 30t + 144 - 20t + 300]$   
 $= 10e^{-\frac{t}{10}}[t^2 - 50t + 444]$

- b i Increasing if  $f'(t) > 0$   
 i.e.  $-100e^{-\frac{t}{10}}[t^2 - 30t + 144] > 0$   
 is equivalent to  $t^2 - 30t + 144 < 0$  as  $-100e^{-\frac{t}{10}} < 0$  for all  $t$   
 $\therefore (t - 24)(t - 6) < 0$   
 $\therefore t \in (6, 24)$   
 The number of unemployed was increasing for  $6 < t < 24$ .

- ii  $f''(t) < 0$   
 $10e^{-\frac{t}{10}}[t^2 - 50t + 444] < 0$   
 is equivalent to  $t^2 - 50t + 444 < 0$   
 First consider the equation

$$t^2 - 50t + 444 = 0$$

$$t = \frac{50 \pm \sqrt{50^2 - 4 \times 444}}{2}$$

$$= \frac{50 \pm \sqrt{724}}{2}$$

$$= 25 \pm \sqrt{181}$$

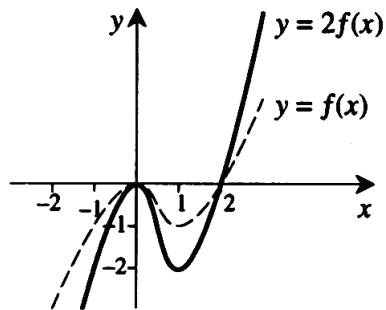
$$\therefore t^2 - 50t + 444 < 0 \text{ for } t \in (25 - \sqrt{181}, 25 + \sqrt{181})$$

However, the domain of the function is  $[0, 35]$ .

So  $f''(t) < 0$  for  $t \in (11.546, 35)$

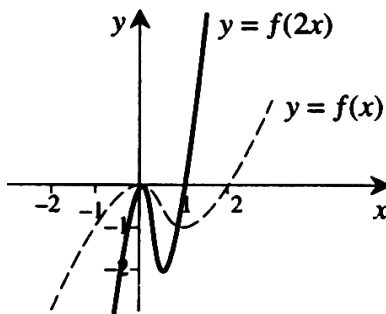
iii  $(6, 24) \cap (11.546, 38.454)$   
 $= (11.546, 24)$

7 a i



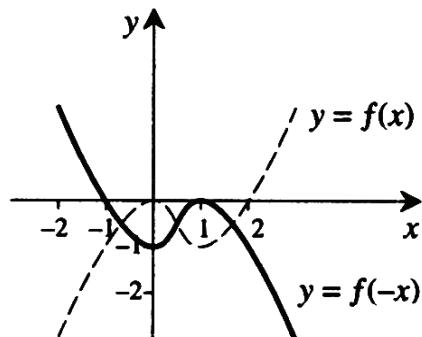
A dilation of factor 2 from the  $x$ -axis.

ii

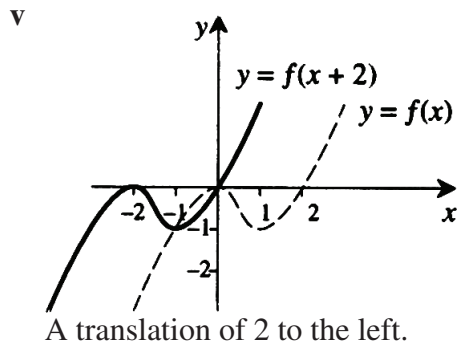
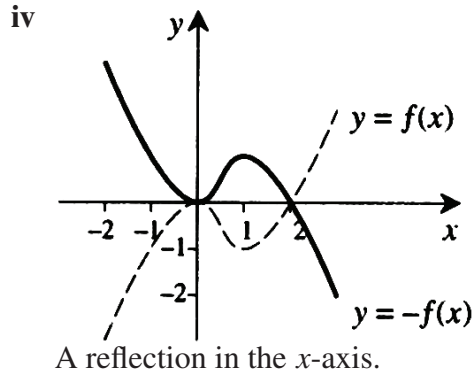


A dilation of factor  $\frac{1}{2}$  from the  $y$ -axis.

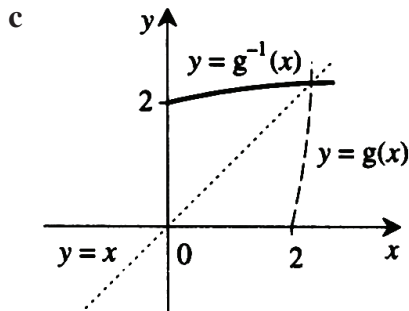
iii



A reflection in the  $y$ -axis.



b  $f$  does not have an inverse function as it is not one-to-one.



d i  $g(x) = x^2(x - 2)$  and  $g : (2, \infty) \rightarrow \mathbb{R}$   
 $g'(x) = 2x(x - 2) + x^2 = x(2x - 4 + x) = x(3x - 4)$   
 When  $x = 3$   $g'(x) = 15$

ii  $(g \circ g^{-1})'(x) = 1$   
 $\therefore g'(g^{-1}(x)) \left( (g^{-1})'(x) \right) = 1$  (by the chain rule)  
 $\therefore \left( g^{-1} \right)'(x) = \frac{1}{g'(g^{-1}(x))}$

$$\therefore (g^{-1})'(9) = \frac{1}{g'(g^{-1}(9))} = \frac{1}{g'(3)} = \frac{1}{15}$$

This can also be shown by the result  $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \frac{dy}{dx} \neq 0$

**8 a i**  $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

Actual value, correct to three decimal places = 0.995

**ii**  $\cos x = 0.98$

Consider the equation

$$1 - \frac{1}{2}x^2 = 0.98$$

$$\therefore 2 - x^2 = 1.96$$

$$0.04 = x^2$$

$$x = \pm 0.2$$

Actual value correct to three decimal places  $x = \pm 0.200 \leftarrow (\pm 0.200)$

**b i** Let  $f(x) = 1 - \frac{1}{2}x^2$

A reflection in the  $x$ -axis is given by

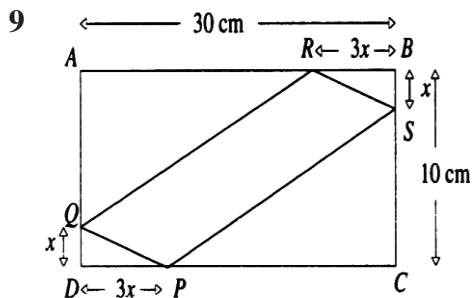
$$g(x) = -f(x) = \frac{1}{2}x^2 - 1$$

A translation of  $\pi$  units in the positive direction of the  $x$ -axis is given by

$$h(x) = g(x - \pi) = \frac{1}{2}(x - \pi)^2 - 1$$

**ii**  $h(3) = \frac{1}{2}(3 - \pi)^2 - 1 \approx -0.98998$

(Actual  $\cos(3) = -0.98999$  correct to five decimal places.)



**a** Area of a triangle  $RBS =$  area of triangle  $PDQ = \frac{3x^2}{2} \text{ cm}^2$

Area of a triangle  $CPS =$  area of triangle  $ARQ = \frac{1}{2} \times (30 - 3x)(10 - x)$

$$\begin{aligned} \therefore \text{Area of parallelogram} &= 300 - 3x^2 - 3(10 - x)^2 \\ &= [300 - 3x^2 - 3(100 - 20x + x^2)] \\ &= (60x - 6x^2) \text{ cm}^2 \end{aligned}$$

**b**  $0 < 3x < 30$  and  $0 < x < 10$

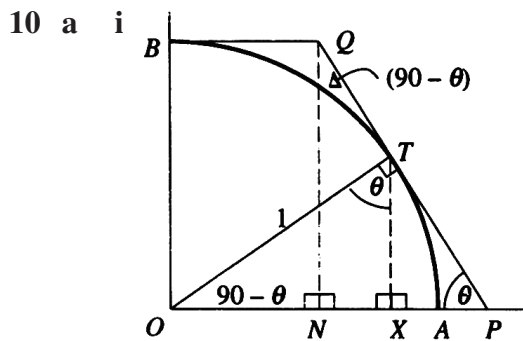
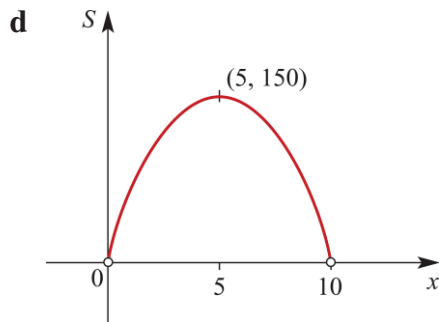
$$\therefore 0 < x < 10$$

**c**  $A = 60x - 6x^2$

and  $\frac{dA}{dx} = 60 - 12x$

$$\frac{dA}{dx} = 0 \text{ implies } x = 5$$

Since the expression is quadratic with negative coefficient of  $x^2$  a local maximum at  $(5, 150)$ .



From triangle  $OTP$

$$\frac{1}{OP} = \sin \theta$$

$$\therefore OP = \frac{1}{\sin \theta}$$

**ii**  $BQ = OP - NP$

$NP = TP$  as  $\triangle QNP$  is congruent to  $\triangle OTP$

and  $TP = \frac{1}{\tan \theta}$

$$\therefore BQ = \frac{1}{\sin \theta} - \frac{1}{\tan \theta}$$

$$\begin{aligned}
&= \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \\
&= \frac{1 - \cos \theta}{\sin \theta}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad \text{Area of the trapezium} &= \frac{1}{2} \left( \frac{1 - \cos \theta}{\sin \theta} + \frac{1}{\sin \theta} \right) \\
&= \frac{2 - \cos \theta}{2 \sin \theta}
\end{aligned}$$

$$\mathbf{c} \quad S = \frac{2 - \cos \theta}{2 \sin \theta}$$

$$\begin{aligned}
\frac{dS}{d\theta} &= \frac{\sin \theta \times 2 \sin \theta - 2 \cos \theta (2 - \cos \theta)}{(2 \sin \theta)^2} \\
&= \frac{2 \sin^2 \theta - 4 \cos \theta + 2 \cos^2 \theta}{(2 \sin \theta)^2} \\
&= \frac{2 - 4 \cos \theta}{4 \sin \theta}
\end{aligned}$$

$$\mathbf{d} \quad \frac{dS}{d\theta} = 0 \text{ implies, } \frac{2 - 4 \cos \theta}{2 - 4 \cos \theta} = 0 \quad 0 < \theta < \frac{\pi}{2}$$

$$\therefore \cos \theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$

$$\text{and } \frac{dS}{d\theta} < 0 \text{ when } \theta = \frac{\pi}{4}$$

$$\text{and } \frac{dS}{d\theta} > 0 \text{ when } \theta = \frac{\pi}{2}$$

$$\therefore \text{a minimum when } \theta = \frac{\pi}{3}$$

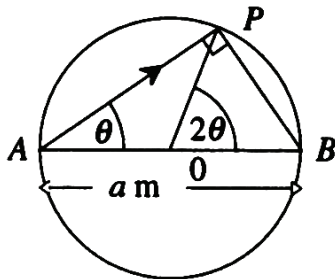


$$\text{When } \theta = \frac{\pi}{3} \quad S = \frac{2 - \frac{1}{2}}{2 \times \frac{\sqrt{3}}{2}} = \frac{3}{2} \times \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{2}$$

and  $AP = OP - 1$

$$\begin{aligned} &= \frac{1}{\sin(\frac{\pi}{3})} - 1 \\ &= \frac{1}{\frac{\sqrt{3}}{2}} - 1 \\ &= \frac{2}{\sqrt{3}} - 1 \\ &= \frac{2 - \sqrt{3}}{\sqrt{3}} \\ &= \frac{2\sqrt{3} - 3}{3} \end{aligned}$$

11



a i distance  $AP = a \cos \theta$

$$\text{distance } PB = \frac{a}{2} \times 2\theta = a\theta \text{ (for arc } PB)$$

$$\text{time for } AP = \frac{a \cos \theta}{\frac{1}{2}} = 2a \cos \theta$$

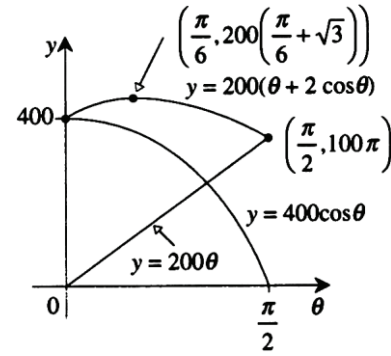
$$\text{time for } PB = \frac{a\theta}{1} = a\theta$$

$$\therefore \text{ total time, } T = a(\theta + 2 \cos \theta)$$

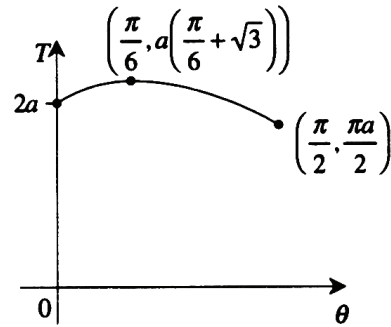
b  $\frac{dy}{d\theta} = 200(1 - 2 \sin \theta)$

$\frac{dy}{d\theta} = 0$  implies  $\sin \theta = \frac{1}{2}$

$\therefore$  maximum when  $\theta = \frac{\pi}{6}$



c The minimum value for  $T$  is  $\frac{\pi a}{2}$ . This is obtained by the dog running around outside of the lake.



12 a i  $f(x) = (x - 1)g(x)$  and  $f'(x) = (x - 1)h(x)$

$f'(x) = g(x) + (x - 1)g'(x)$  (product rule)

$\therefore g(x) + (x - 1)g'(x) = (x - 1)h(x)$

$\therefore g(x) = (x - 1)h(x) - (x - 1)g'(x)$

$= (x - 1)[h(x) - g'(x)]$

$\therefore (x - 1)$  is a factor of  $g(x)$

ii  $F(1) = 1 - k - 3 + 2k - k + 2 = 0$  where  $F(x) = x^3 - kx^2 - (3 - 2k)x - (k - 2)$

$F'(x) = 3x^2 - 2kx - (3 - 2k)$

$\therefore F'(1) = 3 - 2k - 3 + 2k = 0$

iii By the factor theorem  $x - 1$  is a factor of  $F(x)$  and  $F'(x)$ .

$\therefore F(x) = (x - 1)g(x)$  and  $F'(x) = (x - 1)h(x)$  where  $g(x)$  and  $h(x)$  are polynomials.

$\therefore x - 1$  is a factor of  $g(x)$

$\therefore F(x) = (x - 1)^2w(x)$  where  $w(x)$  is a linear polynomial)

$(x^2 - 2x + 1)(x - p) = x^3 - kx^2 - (3 - 2k)x - (k - 2)$

$\therefore -p = -(k - 2)$

i.e.  $p = k - 2$

$\therefore F(x) = (x - 1)^2(x - (k - 2))$

and  $F(x) = 0$  implies  $x = 1$  or  $x = k - 2$

**b i** For  $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

Given that  $(1, 1)$  is on the parabola and the gradient is the same as  $y = x^3$  at  $x = 1$  we have

$$a + b + c(1) = 1$$

$$2a + b = 3(2) \left( y = x^3, \frac{dy}{dx} = 3x^2 \text{ and when } x = 1, \frac{dy}{dx} = 3 \right)$$

$$\text{From } \textcircled{2} \quad b = 3 - 2a$$

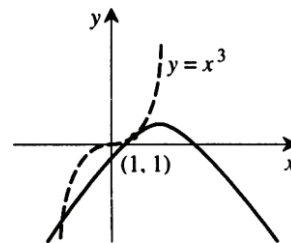
$$\text{From } \textcircled{1} \quad c = 1 - a - b$$

$$= 1 - a - (3 - 2a)$$

$$= 1 - a - 3 + 2a$$

$$= -2 + a$$

$$= a - 2$$



**ii**  $y = ax^2 + (3 - 2a)x + a - 2$

$$y = x^3$$

to find  $Q$  consider

$$ax^2 + (3 - 2a)x + a - 2 = x^3$$

$$\text{i.e. } x^3 - ax^2 + (2a - 3)x + (2 - a) = 0$$

Let  $F(x) = x^3 - ax^2 + (2a - 3)x + (2 - a)$  (the polynomial of  $a$ )

$$\therefore F(x) = (x - 1)^2(x - (a - 2))$$

$\therefore$  The parabola meets the curve  $y = x^3$  at the point  $((a - 2), (a - 2)^3)$  and  $h = a - 2$ .

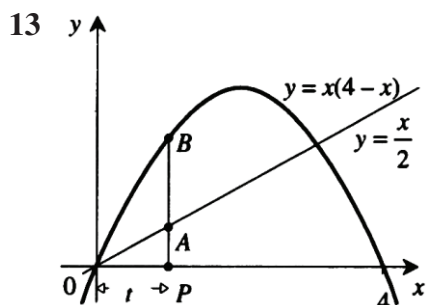
**iii** If  $a - 2 = -2$ ,  $a = 0$ ,  $b = 3$  and  $c = -2$

$Q$  is the point of intersection of  $y = x^3$  with the straight line  $y = 3x - 2$ .

Note:  $y = 3x - 2$  is the equation of the tangent to the curve  $y = x^3$  at the point with coordinates  $(1, 1)$

**iv** If  $a - 2 = -3$ ,  $a = -1$ ,  $b = 5$  and  $c = -3$

$Q$  is the point of intersection of  $y = -x^2 + 5x - 3$  and  $y = x^3$



a coordinates of  $A\left(t, \frac{t}{2}\right)$

coordinates of  $B(t, t(4-t))$

$$\text{Length of } AB = Z = t(4-t) - \frac{t}{2} = \frac{1}{2}(8t - 2t^2 - t) = \frac{1}{2}(7t - 2t^2)$$

b For the intercepts consider:

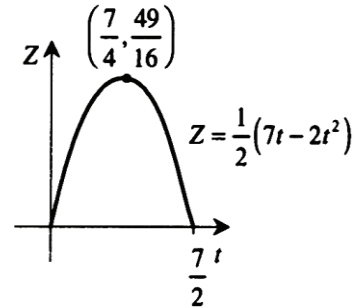
$$\frac{1}{2}(7t - 2t^2) = 0$$

$$\therefore t(7 - 2t) = 0$$

$$\therefore t = 0 \text{ or } t = \frac{7}{2}$$

Note:  $\left(\frac{7}{2}, \frac{7}{4}\right)$  is the point of intersection of

$$y = \frac{x}{2} \text{ and } y = x(4-x)$$



c The maximum value of  $Z = \frac{49}{16}$  and this occurs when  $t = \frac{7}{4}$ .

14 a Let  $X$  be the number of boys.

$X$  is the random variable of a Binomial distribution.

i  $\Pr(X = 2) = \binom{4}{2}(0.5)^2(0.5)^2 = \frac{3}{8}$

ii  $\Pr(X = 1 | X \geq 1) = ?$

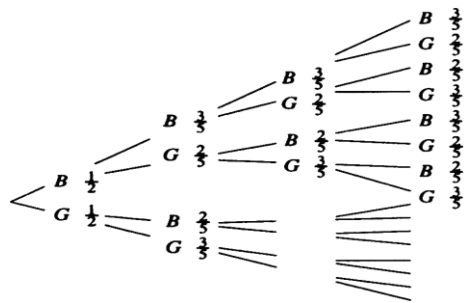
$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$

$$= 1 - (0.5)^4$$

$$= \frac{15}{16}$$

$$\Pr(X = 1 | X \geq 1) = \frac{\Pr(X = 1)}{\Pr(X \geq 1)} = \frac{\binom{4}{1}(0.5)^4}{\frac{15}{16}} = 4 \times \frac{1}{16} \times \frac{16}{15} = \frac{4}{15}$$

b Child 1 Child 2 Child 3 Child 4



i  $\Pr(\text{all boys}) = \frac{1}{2} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{250}$

$$\Pr(\text{all girls}) = \frac{1}{2} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{25} = \frac{27}{250}$$

$$\therefore \Pr(\text{same sex}) = \frac{27}{250} + \frac{27}{250} = \frac{27}{125}$$

$$\text{ii } \Pr(BGBG) = \frac{1}{2} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{250} = \frac{4}{125}$$

$$\Pr(GBGB) = \frac{1}{2} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{250} = \frac{4}{125}$$

$$\therefore \Pr(\text{no two consecutive children will be of the same sex}) = \frac{8}{125}$$

iii Two males and two females. The possible combinations are *BBGG GGBB*

*BGBG* Note: No. of ways of arranging =  $\frac{4!}{2!2!} = \frac{24}{4} = 6$

*GBGB*

*BGGB*

*GBBG*

$$\Pr(BBGG) = \frac{1}{2} \times \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{18}{250}$$

$$\Pr(GGBB) = \frac{18}{250}$$

$$\Pr(BGBG) = \frac{4}{125} \text{ (see part ii)}$$

$$\Pr(GBGB) = \frac{4}{125} \text{ (see part ii)}$$

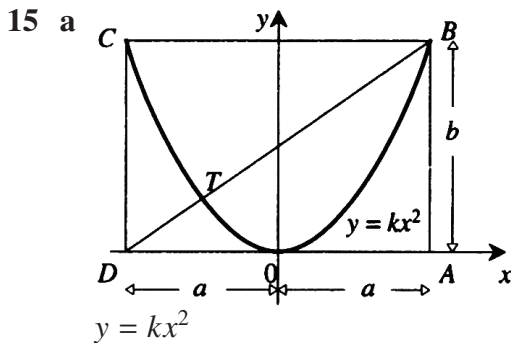
$$\Pr(BGGB) = \frac{1}{2} \times \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{12}{250}$$

$$\Pr(GBBG) = \frac{12}{250}$$

$$\therefore \Pr(\text{two males and two females}) = \frac{18 + 18 + 8 + 8 + 12 + 12}{250}$$

$$= \frac{76}{250}$$

$$= \frac{38}{125}$$



$$\therefore b = ka^2$$

$$\therefore k = \frac{b}{a^2}$$

**b i** Gradient of  $DB = \frac{b}{2a}$  and it passes through  $(-a, 0)$

$$\therefore y - 0 = \frac{b}{2a}(x + a)$$

$$\text{i.e. } y = \frac{b}{2a}x + \frac{b}{2}$$

**ii** crosses  $y = \frac{b}{a^2}x^2$

$$\text{where } \frac{b}{2a}x + \frac{b}{2} = \frac{b}{a^2}x^2$$

Multiply both sides by  $2a^2$

$$bax + ba^2 = 2bx^2$$

$$\text{i.e. } 2bx^2 - bax - ba^2 = 0$$

which implies

$$2x^2 - ax - a^2 = 0$$

$$\therefore (2x + a)(x - a) = 0$$

$$\therefore x = -\frac{a}{2} \text{ or } x = a$$

$$\therefore \text{ at } T \quad x = -\frac{a}{2}$$

$$\text{and } y = \frac{b}{a^2}\left(\frac{-a}{2}\right)^2 = \frac{b}{4}$$

$\therefore$  coordinates of  $T$  are  $\left(-\frac{a}{2}, \frac{b}{4}\right)$

**c** Area =  $\int_{-a}^a b - \frac{b}{a^2}x^2 dx$

$$= 2b \int_0^a 1 - \frac{x^2}{a^2} dx \quad (\text{by symmetry})$$

$$= 2b \left[ x - \frac{x^3}{3a^2} \right]_0^a$$

$$= 2b \left[ a - \frac{a^3}{3a^2} \right]$$

$$= 2b \left[ a - \frac{a}{3} \right]$$

$$= \frac{4}{3} ab$$

$$\begin{aligned}
\mathbf{d} \quad S_1 &= \int_{-\frac{a}{2}}^a \left( \frac{b}{2a}x + \frac{b}{2} - \left( \frac{b}{a^2}x^2 \right) \right) dx \\
&= b \int_{-\frac{a}{2}}^a \left( \frac{x}{2a} + \frac{1}{2} - \frac{x^2}{a^2} \right) dx \\
&= b \left[ \frac{x^2}{4a} + \frac{x}{2} - \frac{x^3}{3a^2} \right]_{-\frac{a}{2}}^a \\
&= b \left[ \left( \frac{a^2}{4a} + \frac{a}{2} - \frac{a^3}{3a^2} \right) - \left( \frac{a^2}{4} \times \frac{1}{4a} - \frac{a}{4} + \frac{a^3}{8} \times \frac{1}{3a^2} \right) \right] \\
&= b \left[ \frac{a}{4} + \frac{a}{2} - \frac{a}{3} - \left( \frac{a}{16} - \frac{a}{4} + \frac{a}{24} \right) \right] \\
&= \frac{ba}{48} [12 + 24 - 16 - (3 - 12 + 2)] \\
&= \frac{ba}{8} [20 + 7] \\
&= \frac{27ba}{48} = \frac{9ba}{16}
\end{aligned}$$

Now  $S_2 = \frac{4}{3}ab - \frac{9ba}{16}$  from **c**

$$= \frac{(64 - 27)ba}{48} = \frac{37ba}{48}$$

$\therefore$  ratio  $S_1 : S_2 = 27:37$

**16 a** Let  $X$  be the thickness of the washer

Let  $Y$  be the diameter of the hole

For  $X$ :  $\mu = 0.25, \sigma = 0.002$

For  $Y$ :  $\mu = 0.5, \sigma = 0.05$

**i**  $\Pr(X < 0.253)$

$$= \Pr\left(z < \frac{0.253 - 0.25}{0.002}\right)$$

$$= \Pr\left(Z < \frac{3}{2}\right)$$

$$= 0.9332$$

**ii**  $\Pr(X < 0.247)$

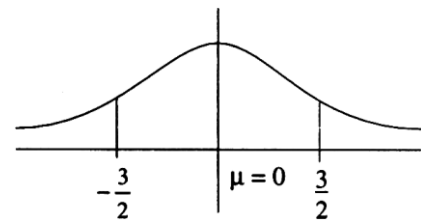
$$= \Pr\left(Z < \frac{0.247 - 0.25}{0.002}\right)$$

$$= \Pr\left(Z < -\frac{3}{2}\right)$$

$$= 1 - \Pr\left(Z < -\frac{3}{2}\right)$$

$$= 1 - 0.9332$$

$$= 0.0668$$



**iii**  $\Pr(Y > 0.56)$

$$= \Pr\left(Z > \frac{0.56 - 0.5}{0.05}\right)$$

$$= \Pr\left(Z > -\frac{6}{5}\right)$$

$$= \Pr(Z > 1.2)$$

$$= 1 - \Pr(Z < 1.2)$$

$$= 0.1151$$

**iv**  $\Pr(Y < 0.44)$

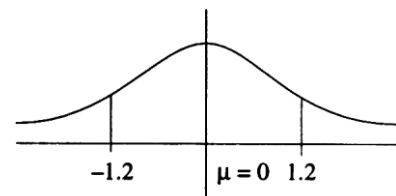
$$= \Pr\left(Z < \frac{0.44 - 0.5}{0.05}\right)$$

$$= \Pr\left(Z < -\frac{0.06}{0.05}\right)$$

$$= \Pr(Z < -1.2)$$

$$= \Pr(Z > 1.2)$$

$$= 0.1151$$





**b i** Let  $A$  be the event  $0.247 < X \leq 0.253$

Let  $B$  the event  $0.44 \leq Y \leq 0.56$

$$\Pr(A) = \Pr(X \leq 0.253) - \Pr(X \leq 0.247)$$

$$= 0.9332 - 0.0668$$

$$= 0.8664$$

$$\Pr(B) = \Pr(Y \leq 0.56) - \Pr(Y \leq 0.44)$$

$$= 0.8849 - 0.1151$$

$$= 0.7698$$

$\Pr(A \cap B) = \Pr(A) \Pr(B)$   $X$  and  $Y$  are independent and therefore

$$= 0.8664 \times 0.7698$$
  $A$  and  $B$  are independent events.

$$= 0.6670$$

$\therefore$  Probability of rejecting a washer is 0.333.

$\therefore$  33.3% of washers are rejected.

**ii**  $\Pr(A) = 0.8664$

$\therefore$  expected number of washers of acceptable thickness in a batch of 1000 is

866.4.

**iii**  $\Pr(A \cap B') = \Pr(A) \Pr(B')$

$$= 0.8664 \times (1 - 0.7698)$$

$$= 0.8664 \times 0.2302$$

$$= 0.1994$$

$\therefore$  Expected number with acceptable thickness but not acceptable diameter is

199.4.

**17** Let  $AC = x$

Then  $CE = 90 - x$

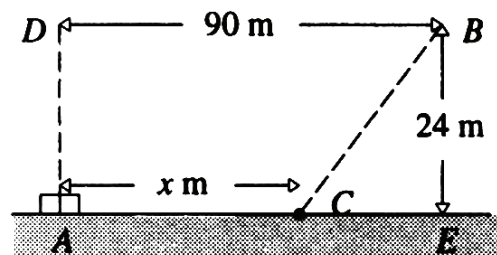
and  $CB = \sqrt{(90 - x)^2 + 24^2}$

$\therefore$  total cost  $C = 100(90 - x) + 200(90 - x)^2 + 576)^{\frac{1}{2}}$

$$\frac{dC}{dx} = -100 + 200 \left[ \frac{1}{2} \times ((90 - x)^2 + 576)^{-\frac{1}{2}} \times \right.$$

$$\left. -2(90 - x) \right]$$

$$\frac{dC}{dx} = 0 \text{ implies } 100 = \frac{200(x - 90)}{((90 - x)^2 + 576)^{\frac{1}{2}}}$$



which implies  $[(90 - x)^2 + 576]^{\frac{1}{2}} = 2(x - 90)$

$$\therefore [(90 - x)^2 + 576] = 4(x - 90)^2$$

$$\therefore 3(x - 90)^2 = 576$$

$$\therefore x - 90 = \pm \frac{24}{\sqrt{3}}$$

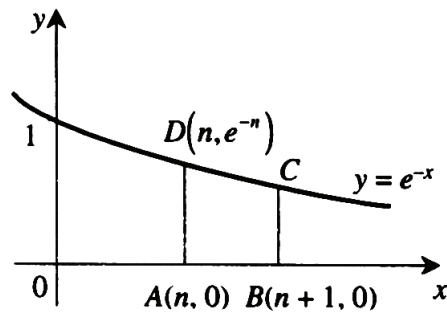
$$\therefore x = 90 \pm 8\sqrt{3}$$

as  $0 \leq x \leq 90$

$x = 90 - 8\sqrt{3}$ , [ $x = 90 - 8\sqrt{3}$ , as  $0 \leq x \leq 9$ .]

$\frac{dC}{dx} > 0$  when  $x > 90 - 8\sqrt{3}$  and  $\frac{dC}{dx} < 0$  when  $x < 90 - 8\sqrt{3}$

$\therefore$  a minimum when  $x = 90 - 8\sqrt{3} \approx 76.1436$  m



18 a i  $\frac{dy}{dx} = -e^{-x}$

When  $x = n$ ,  $\frac{dy}{dx} = -e^{-n}$

$\therefore$  equation of tangent is  $y - e^{-n} = -e^{-n}(x - n)$

$\therefore y = -e^{-n}x + e^{-n}n + e^{-n}$

ii When  $y = 0$

$\therefore e^{-n}x = e^{-n}n + e^{-n}$

$x = n + 1$  ( $e^{-n} \neq 0$ )

The line  $DB$  is a segment of the tangent at  $D$ .

b i 
$$\int_n^{n+1} e^{-x} dx = [-e^{-x}]_n^{n+1}$$

$$= -(e^{-(n+1)} - e^{-n})$$

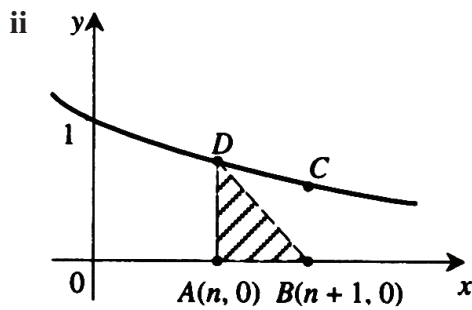
$$= -e^{-n}(e^{-1} - 1)$$

$\therefore$  area of region  $ABCD = \frac{1}{e^n} \left(1 - \frac{1}{e}\right)$

$\therefore$  The area under the curve  $y = e^{-x}$  between  $x = n$  and  $x = n + 1$  is  $\frac{1}{e^n} \left(1 - \frac{1}{e}\right)$ .

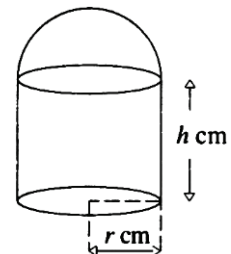
$$\begin{aligned} \therefore \text{The area of the second part} &= \frac{1}{e^n} - \frac{1}{e^{n+1}} - \frac{1}{2e^n} \\ &= \frac{1}{2e^n} - \frac{1}{e^{n+1}} \\ &= \frac{1}{e^n} \left( \frac{1}{2} - \frac{1}{e} \right) \end{aligned}$$

$$\begin{aligned} \therefore \text{The ratio of the two parts} &= \frac{1}{2e^n} : \frac{1}{e^n} \left( \frac{1}{2} - \frac{1}{e} \right) \\ &= \frac{1}{2} : \frac{1}{2} - \frac{1}{e} \\ &= e : e - 2 \end{aligned}$$



$$\begin{aligned} \text{The shaded area} &= \int_n^{n+1} -e^{-n}x + e^{-n}n + e^{-n} dx \\ &= \left[ -\frac{e^{-n}x^2}{2} + (e^{-n}n + e^{-n})x \right]_n^{n+1} \\ &= \left( -\frac{e^{-n}(n+1)^2}{2} + e^{-n}(n+1)(n+1) \right) - \left( -\frac{e^{-n}n^2}{2} + e^{-n}(n+1)n \right) \\ &= \frac{e^{-n}(n+1)^2}{2} - e^{-n} \left[ -\frac{n^2}{2} + n^2 + n \right] \\ &= \frac{e^{-n}}{2} [(n+1)^2 - (n^2 + 2n)] \\ &= \frac{e^{-n}}{2} [n^2 + 2n + 1 - n^2 - 2n] \\ &= \frac{e^{-n}}{2} \end{aligned}$$

19 a i Volume of cylinder =  $\pi r^2 h$   
 Volume of hemisphere =  $\frac{2}{3}\pi r^3$   
 $\therefore$  total volume,  $V = \frac{2}{3}\pi r^3 + \pi r^2 h$   
 $= \frac{\pi r^2}{3}(2r + 3h)$



**ii** Surface areas of capsule = surface area of hemisphere  
+ curved surface of cylinder + base  
 $= 2\pi r^2 + 2\pi rh + \pi r^2$   
 $= \pi r(3r + 2h)$

**b i** If  $V = \pi a^3$

$$\pi a^3 = \frac{\pi r^2}{3}(2r + 3h)$$

$$3\pi a^3 - 2\pi r^3 = 3\pi r^2 h$$

$$\therefore h = \frac{a^3}{r^2} - \frac{2}{3}r$$

$$= \frac{3a^3 - 2r^3}{3r^2}$$

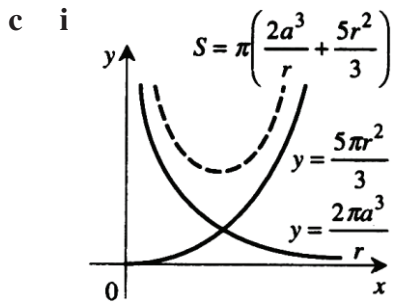
**ii**  $S = \pi r(2h + 3r)$

$$= \pi r \left( \frac{2 \times (3a^3 - 2r^3)}{3r^2} + 3r \right)$$

$$= \pi \left( \frac{2 \times (3a^3 - 2r^3)}{3r} + 3r^2 \right)$$

$$= \pi \left( \frac{2a^3}{r} - \frac{4r^2}{3} + 3r^2 \right)$$

$$= \pi \left( \frac{2a^3}{r} + \frac{5r^2}{3} \right)$$



**ii**  $S = \pi \left( \frac{2a^3}{r} + \frac{5r^2}{3} \right)$

$$\frac{dS}{dr} = \pi \left( -\frac{2a^3}{r^2} + \frac{10r}{3} \right)$$

$$\frac{dS}{dr} = 0 \text{ implies } \frac{2a^3}{r^2} = \frac{10r}{3}$$

$$\begin{aligned} \therefore r^3 &= 0.6a^3 \\ \therefore r &= (\sqrt[3]{0.6})a \\ s_{\min} &= \pi \left( \frac{2a^3}{\sqrt[3]{0.6}a} + \frac{5}{3} \left[ \sqrt[3]{0.6}a \right]^2 \right) \\ &= \pi a^2 \left( \frac{2}{\sqrt[3]{0.6}} + \frac{5}{3} (\sqrt[3]{0.6})^2 \right) \end{aligned}$$

**20 a** Let  $X$  be the cylinder diameter.

$$\Pr(3 - d < X < 3 + d) = 0.75$$

$$\therefore \Pr\left(\frac{-d}{0.002} < Z < \frac{d}{0.002}\right) = 0.75$$

$$\therefore 2\Pr\left(Z < \frac{d}{0.002}\right) - 1 = 0.75$$

$$\Pr\left(Z < \frac{d}{0.002}\right) = 0.875$$

$$\therefore \frac{d}{0.002} = 1.15$$

$$d = 0.0023$$

$$\mathbf{b} \quad \frac{q}{\Pr(Q = q)} \quad \left| \begin{array}{cc} s-1 & -1 \\ \frac{3}{4} & \frac{1}{4} \end{array} \right.$$

$$\mathbf{c} \quad E(Q) = \frac{3}{4}(s-1) - 1 \times \frac{1}{4} = \frac{3}{4}s - 1$$

$$E(Q^2) = (s-1)^2 \times \frac{3}{4} + 1 \times \frac{1}{4}$$

$$\begin{aligned} \text{Var}(Q) &= (s-1)^2 \times \frac{3}{4} + \frac{1}{4} - \left(\frac{3s-4}{4}\right)^2 \\ &= \frac{3}{4}(s^2 - 2s + 1) + \frac{1}{4} - \frac{1}{16}(9s^2 - 24s + 16) \end{aligned}$$

$$= \left(\frac{3}{4} - \frac{9}{16}\right)s^2 + \left(\frac{24}{16} - \frac{6}{4}\right)s + \frac{3}{4} + \frac{1}{4} - 1$$

$$= \left(\frac{3}{16}\right)s^2$$

$$\therefore \text{sd}(Q) = \frac{\sqrt{3}}{4}s$$

**21** Let  $X$  be the length of a worm.

$$\mu = 20 \text{ and } \sigma = 1.5.$$

$$\begin{aligned}
 \text{a } \Pr(X \geq 22) &= \Pr\left(Z \geq \frac{22 - 20}{1.5}\right) \\
 &= \Pr\left(Z \geq \frac{2}{1.5}\right) \\
 &= \Pr(Z \geq 1.3333) \\
 &= 1 - \Pr(Z \leq 1.3333) \\
 &= 0.09121
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \Pr(19.5 \leq X \leq 20.5) &= \Pr\left(\frac{19.5 - 20}{1.5} \leq Z \leq \frac{20.5 - 20}{1.5}\right) \\
 &= \Pr\left(-\frac{1}{3} \leq Z \leq \frac{1}{3}\right) \\
 &= \Pr(0.3333 \leq Z \leq 0.3333) \\
 &= 2 \Pr(Z \leq 0.3333) - 1 \\
 &= 2 \times 0.63056 - 1 \\
 &= 0.2611
 \end{aligned}$$

c Let  $Y$  be the number of worms out of five of 20 cm in length, So  $Y$  has a binomial distribution with  $n = 5$  and  $p = 0.2611$ .

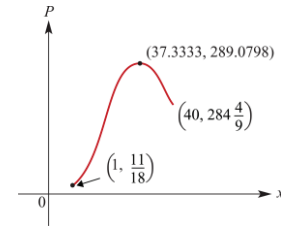
$$\begin{aligned}
 \Pr(Y = 2) &= \binom{5}{2} (0.2611)^2 (0.7389)^3 \\
 &= 10 \times (0.2611)^2 (0.7389)^3 \\
 &= 10 \times 0.0682 \times 0.4039 \\
 &= 0.275
 \end{aligned}$$

$$\begin{aligned}
 \text{22 a } P &= \frac{x^2}{90} (56 - x) \quad x \in [1, 40] \\
 &= \frac{1}{90} (56x^2 - x^3) \\
 \frac{dP}{dx} &= \frac{1}{90} (112x - 3x^2)
 \end{aligned}$$

b i

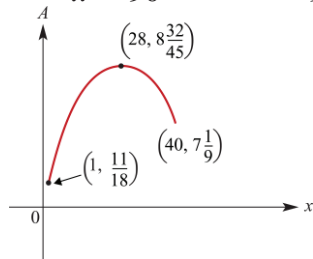
$$\begin{aligned}
 \text{ii } P(1) &= \frac{1}{90} \times (56 - 1) \quad P(40) = \frac{40^2}{90} \times [56 - 40] \\
 &= \frac{11}{18} \quad = 284 \frac{4}{9}
 \end{aligned}$$

$$\begin{aligned} \frac{dP}{dx} = 0 &\Rightarrow \frac{1}{90}(112x - 3x^2) = 0 \\ \therefore x(112 - 3x) &= 0 \\ \therefore x = 0 \text{ or } x &= 37\frac{1}{3} \end{aligned}$$



when  $x = 37\frac{1}{3}$   $P(x) = \frac{351232}{1215} \approx 289.0798$   
 The maximum value of  $P$  is 289.0798 tonnes

**c i**  $A = \frac{1}{x} \times \frac{x^2}{90}(56 - x) = \frac{x}{90}(56 - x)$

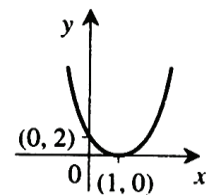


**ii** The maximum value of  $A$  is  $8\frac{32}{45}$  tonnes/man, when  $x = 28$ .

**23**  $f(x) = (k + 2)x^2 + (6k - 4)x + 2$

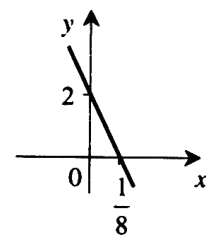
**a i** When  $k = 0$

$$\begin{aligned} f(x) &= 2x^2 - 4x + 2 \\ &= 2(x^2 - 2x + 1) \\ &= 2(x - 1)^2 \end{aligned}$$



**ii** When  $k = -2$

$$f(x) = -16x + 2$$



iii When  $k = -4$

$$\begin{aligned} f(x) &= -2x^2 - 28x + 2 \\ &= -2x[x^2 + 14x - 1] \\ &= -2[x^2 + 14x + 49] - 1 - 49 \\ &= -2[(x + 7)^2 - 50] \end{aligned}$$

$$f(0) = 2; \text{ when } f(x) = 0, (x + 7)^2 = 50$$

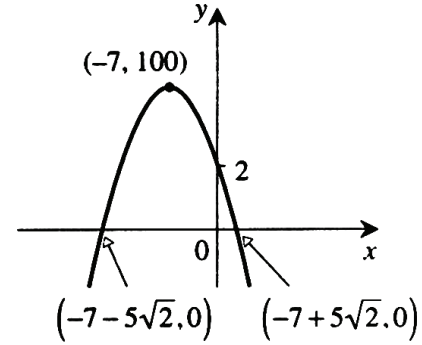
$$\therefore x = -7 \pm \sqrt{50}$$

$$= -7 \pm 5\sqrt{2}$$

$$f(x) = -2((x + 7)^2) + 100$$

$\therefore$  axes intercepts are  $(0, 2)$   $(-7 - 5\sqrt{2}, 0)$  and  $(-7 + 5\sqrt{2}, 0)$

Vertex is at  $(-7, 100)$



b  $f'(x) = 2(k + 2)x + (6k - 4)$

$$\begin{aligned} f'(x) = 0 \text{ implies} \\ x &= \frac{4 - 6k}{2(k + 2)} = \frac{2 - 3k}{k + 2} \end{aligned}$$

$$\begin{aligned} f\left(\frac{2 - 3k}{k + 2}\right) &= (k + 2) \times \left(\frac{2 - 3k}{k + 2}\right)^2 + (6k - 4)\frac{(2 - 3k)}{k + 2} + 2 \\ &= \frac{(2 - 3k)^2}{k + 2} + 2\frac{(3k - 2)(2 - 3k)}{k + 2} + 2 \\ &= \frac{(2 - 3k)^2 - 2(2 - 3k)^2 + 2(k + 2)}{k + 2} \\ &= \frac{-(2 - 3k)^2 + 2(k + 2)}{k + 2} \end{aligned}$$

**A check from previous results**

$$\text{When } k = 0, x = 1, f(1) = \frac{-4 + 4}{2} = 0$$

When  $k = -2$ ,  $f$  is undefined

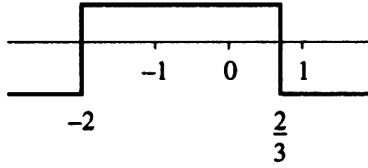
$$\text{When } k = -4, x = \frac{14}{-2} = -7$$

$$\begin{aligned} f(-7) &= -\frac{-(2 + 12)^2 + 2(-2)}{-2} \\ &= -\frac{-196 - 4}{-2} \\ &= 100 \end{aligned}$$

i If  $a > 0$ ,  $\frac{2 - 3k}{k + 2} > 0$

Multiply both sides of the inequality by  $(k + 2)^2$   $(2 - 3k)(k + 2) > 0$





A sign diagram reveals  $\{k : a > 0\} = \left\{k : -2 < k < \frac{2}{3}\right\}$

ii  $a = 0$  implies  $k = \frac{2}{3}$

$$\{k : a = 0\} = \left\{\frac{2}{3}\right\}$$

iii If  $b > 0$ ,  $\frac{-(2-3k)^2 + 2(k+2)}{k+2} > 0$

Multiply both sides of inequality by  $(k+2)^2$

$$(-(2-3k)^2 + 2(k+2))(k+2) > 0$$

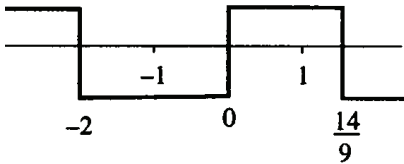
$$(-4 - 12k + 9k^2) + 2k + 4)(k+2) > 0$$

$$\therefore (-4 + 12k - 9k^2 + 2k + 4)(k+2) > 0$$

$$(14k - 9k^2)(k+2) > 0$$

$$k(14 - 9k)(k+2) > 0$$

Consider the sign diagram



$$\therefore \{k : b > 0\} = \left\{k : 0 < k < \frac{14}{9}\right\} \cup \{k : k < -2\}$$

iv  $\{k : b < 0\} = \{k : -2 < k < 0\} \cup \left\{k : k > \frac{14}{9}\right\}$

c  $f$  has a local maximum when  $k+2 < 0$

i.e. when  $k < -2$

d For  $f(x) = (k+2)x^2 + (6k-4)x + 2$

$$\Delta = (6k-4)^2 - 4(k+2)2$$

$$= 36k^2 - 48k + 16 - 8k - 16$$

$$= 36k^2 - 56k$$

i  $f(x)$  is a perfect square if  $\Delta = 0$

$$\text{i.e. } 36k^2 - 56k = 0$$

$$4k(9k - 14) = 0$$

$$k = 0 \text{ or } k = \frac{14}{9}$$

ii If there are no solutions  $\Delta < 0$

i.e.  $4k(9k - 14) < 0$

$$0 < k < \frac{14}{9}$$

24 a  $e^{2-2x} = 2e^{-x}$

$$\therefore e^2 = 2e^x$$

$$\therefore e^x = \frac{e^2}{2}$$

$$\therefore x = \log_e \left( \frac{e^2}{2} \right)$$

$$\therefore = \log_e (e^2) - \log_e 2$$

$$= 2 - \log_e 2$$

b i  $y = e^{2-2x} - 2e^{-x}$

$$\frac{dy}{dx} = -2e^{2-2x} + 2e^{-x}$$

ii  $\frac{dy}{dx} = 0$  implies  $e^{2-2x} = e^{-x}$

$$\therefore e^x = e^2$$

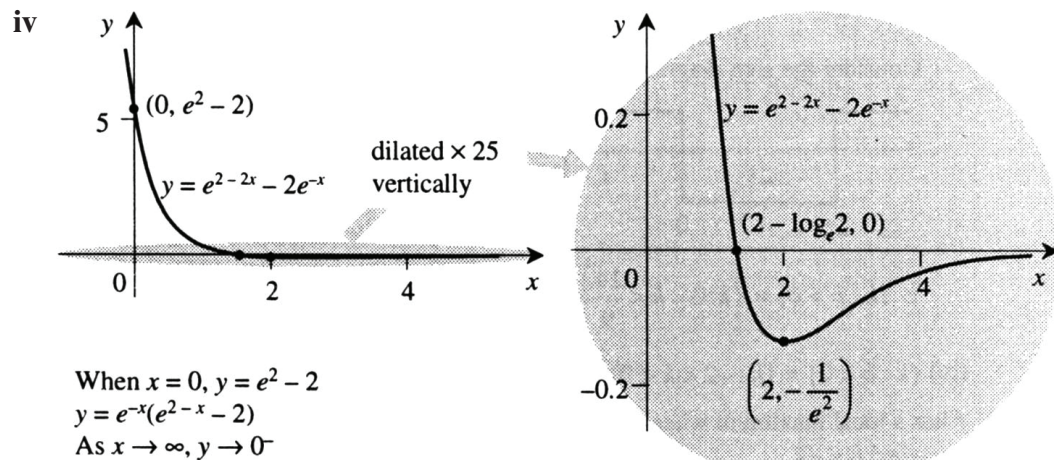
$$x = 2$$

iii When  $x = 2$ ,  $y = e^{2-4} - 2e^{-2}$

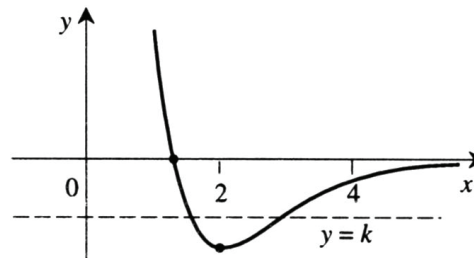
$$= e^{-2} - 2e^{-2}$$

$$= -\frac{1}{e^2}$$

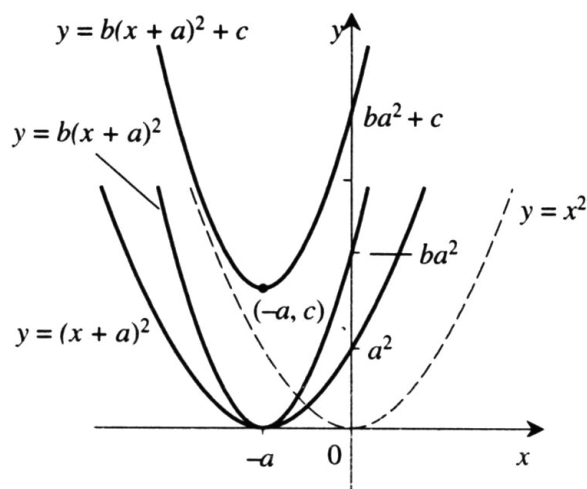
$\therefore$  coordinates of turning point  $(2, -\frac{1}{e^2})$



- c The equation  $e^{2-2x} - 2e^{-x} = k$  has two distinct positive solutions for  $k \in \left(-\frac{1}{e^2}, 0\right)$



25 a

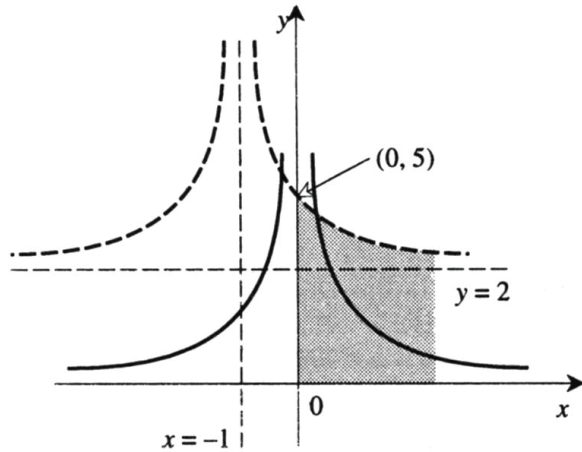


$$\begin{aligned} \text{b } \frac{3}{(x+1)^2} + 2 &= \frac{3}{x^2 + 2x + 1} + 2 \\ &= \frac{3 + 2x^2 + 4x + 2}{x^2 + 2x + 1} \\ &= \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}, x \neq -1 \end{aligned}$$

- c
- A dilation of factor 3 from the  $x$ -axis
  - A translation of 1 unit in the negative direction of the  $x$ -axis.
  - A translation of 2 units in the positive direction of the  $y$ -axis

$$\begin{aligned} \text{d } \int_0^1 \frac{2x^2 + 4x + 5}{x^2 + 2x + 1} dx &= \int_0^1 \frac{3}{(x+1)^2} + 2 dx \\ &= \int_0^1 3(x+1)^{-2} + 2 dx \\ &= [-3(x+1)^{-1} + 2x]_0^1 \\ &= -\frac{3}{2} + 2 - (-3) \\ &= \frac{7}{2} \end{aligned}$$

e



26 a i  $y = 50$

ii  $y - 25 = \frac{25 - 0}{50 - 25}(x - 50)$   
 $\therefore y - 25 = x - 50$   
 $\therefore y = x - 25$

b  $y = ax^2 + 4x + c$

$\therefore 50 = 25^2a + 100 + c$  ①

$25 = 50^2a + 200 + c$  ②

Subtract ② from ①

$25 = (25^2 - 50^2)a - 100$

$\frac{125}{25^2 - 50^2} = a$

$a = \frac{125}{75 \times -25} = \frac{-1}{15}$

Substitute in ①

$50 = 625 \times -\frac{1}{15} + 100 + c$

$\therefore c = -50 + 625 \times \frac{1}{15}$

$= -\frac{25}{3}$

$\therefore$  equation of parabola

$y = -\frac{1}{15}x^2 + 4x - \frac{25}{3}$

$= -\frac{1}{15}(x^2 - 60x + 125)$

c i area of rectangle  $OABE$

$= 25 \times 50$

$= 1250$  square units

$$\begin{aligned}
 \text{ii area of region } EBC &= \int_{25}^{50} -\frac{1}{15}(x^2 - 60x + 125) - (x - 25) dx \\
 &= -\frac{1}{15} \int_{25}^{50} x^2 - 45x - 250 dx \\
 &= -\frac{1}{15} \left[ \frac{x^3}{3} - \frac{45x^2}{2} - 250x \right]_{25}^{50} \\
 &= \frac{14375}{18}
 \end{aligned}$$

$$\text{iii total area} = \frac{36875}{18} \text{ square units}$$

$$27 \text{ a Area of rectangle } PQST = (4 \cos \theta + 4 \cos \theta) \times 2$$

$$= 16 \cos \theta$$

$$\text{Area of triangle } QRS = \frac{1}{2} \times 8 \cos \theta \times 4 \sin \theta$$

$$= 16 \cos \theta \sin \theta$$

$$\therefore \text{Area of metal plate} = 16(\cos \theta + \cos \theta \sin \theta), 0 < \theta < \frac{\pi}{2}$$

$$\text{b } \frac{dA}{d\theta} = 16[-\sin \theta + \sin \theta(-\sin \theta) + \cos \theta \cos \theta]$$

$$= 16[-\sin \theta + \cos^2 \theta - \sin^2 \theta]$$

$$= 16[-\sin \theta + (1 - \sin^2 \theta) - \sin^2 \theta]$$

$$= 16[1 - \sin \theta - 2 \sin^2 \theta]$$

$$\text{c } \frac{dA}{d\theta} = 0 \text{ implies } = 16[1 - a - 2a^2] = 0 \text{ (where } a = \sin \theta)$$

$$\Leftrightarrow 2a^2 + a - 1 = 0$$

$$\Leftrightarrow (2a - 1)(a + 1) = 0$$

$$\Leftrightarrow a = \frac{1}{2} \text{ or } a = -1$$

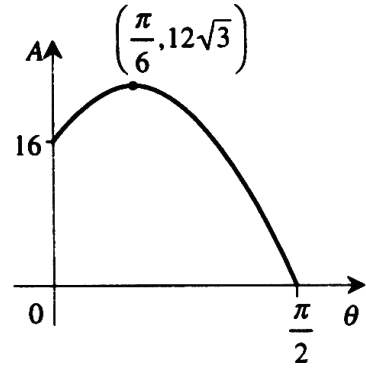
$$\therefore \sin \theta = \frac{1}{2} \text{ or } \sin \theta = -1$$

$$\theta = \frac{\pi}{6} \text{ since } 0 < \theta < \frac{\pi}{2}$$

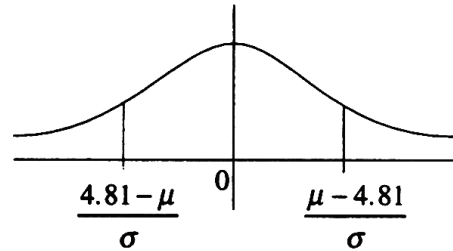
d  $A(0) = 16$

$$A\left(\frac{\pi}{2}\right) = 0$$

$$\begin{aligned} A\left(\frac{\pi}{6}\right) &= 16\left(\cos \frac{\pi}{6} + \cos \frac{\pi}{6} \sin \frac{\pi}{6}\right) \\ &= 16 \times \frac{\sqrt{3}}{2} \times \frac{3}{2} \\ &= 12\sqrt{3} \end{aligned}$$



- 28 a Let  $X$  be the length of the engine part.  
The engine part must be between 4.81 cm and 5.20 cm.  
 $\Pr(X < 4.81) = 0.008$   
 $\Pr(X > 5.20) = 0.03$



$$\Pr\left(Z < \frac{4.81 - \mu}{\sigma}\right) = 0.008$$

$$\therefore \Pr\left(Z < \frac{\mu - 4.81}{\sigma}\right) = 0.992$$

$$\Pr\left(Z > \frac{5.20 - \mu}{\sigma}\right) = 0.03$$

$$\therefore \Pr\left(Z < \frac{5.20 - \mu}{\sigma}\right) = 0.97$$

$\therefore$  we have the equations

$$\frac{\mu - 4.81}{\sigma} = 2.41 \text{ and } \frac{5.20 - \mu}{\sigma} = 1.881$$

$$\mu - 4.81 = 2.41\sigma \quad \textcircled{1} \text{ and } 5.20 - \mu = 1.881\sigma \quad \textcircled{2}$$

Add  $\textcircled{1}$  and  $\textcircled{2}$

$$5.20 - 4.81 = (2.41 + 1.881)\sigma$$

$$0.39 = 4.291\sigma$$

$$0.0909 = \sigma \text{ (correct to four decimal places)}$$

Substitute in  $\textcircled{1}$

$$\mu - 4.81 = 2.41 \times 0.0909$$

$$\mu = 5.0290$$

- b Let  $\$C$  be the cost to produce a part that meets the specifications. Then with probability 0.962, the cost is  $\$4$ ; with probability 0.03, the part is priced at a cost of  $\$(4+2) = \$6$ ; with probability 0.008, the part is rejected and the process begins again: so with probability  $0.008 \times 0.962 = 0.007697$ , a good part is made at a cost of

$\$(4+4) = \$8$ , and with probability  $0.008 \times 0.03 = 0.00024$ , a good part is made at a cost of  $\$(4 + 4 + 2) = \$10$ .

But with probability  $(0.008)^2 = 0.000064$ , its part is rejected and the process repeats ad infinitum.

METHOD 1

$C$	4	6	8	10	12	$> 12$
$\Pr(C = c)$	0.962	0.03	0.001696	0.00024	0.0000616	insignificant

$$E(C) = 4 \times 0.962 + 6 \times 0.03 + 8 \times 0.007696 + 10 \times 0.00024 + 12 \times 0.0000616 = 4.092707$$

The expected cost of producing 100 parts is \$409.27.

**29 a**  $\theta = 21$

$$T = 21 + Ae^{-kt}$$

When  $t = 0$ ,  $T = 100$

$$\therefore 100 = 21 + A$$

$$\therefore A = 79$$

$$\therefore T = 21 + 79e^{-kt}$$

When  $t = 10$ ,  $T = 84$

$$\therefore 84 = 21 + 79e^{-10k}$$

$$\frac{63}{79} = e^{-10k}$$

$$\therefore -10k = \log_e \frac{63}{79}$$

$$\therefore k = \frac{1}{10} \log_e \frac{79}{63} \approx 0.02$$

**b**  $70 = 21 + 79e^{-kt}$

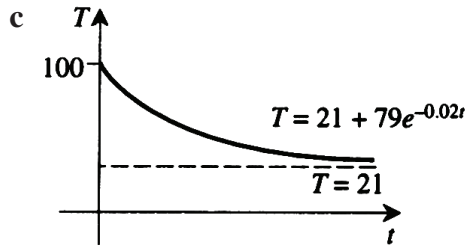
$$\therefore \frac{49}{79} = e^{-kt}$$

$$\therefore -kt = \log_e \frac{49}{79}$$

$$\therefore t = -\frac{1}{k} \log_e \frac{49}{79}$$

$$\text{As } \frac{1}{k} = \frac{10}{\log_e \frac{79}{63}}, t = \frac{10}{\log_e \left(\frac{79}{63}\right)} \times \log_e \left(\frac{79}{49}\right) \approx 21.1$$

The temperature of the kettle will be  $70^\circ\text{C}$  after 21.1 minutes i.e. at approximately 2.44 pm.



d When  $t = 0$ ,  $T = 100$

When  $t = 10$ ,  $T = 84$

$$\therefore \text{the average rate of change} = \frac{84 - 100}{10} \text{ } ^\circ\text{C min}$$

$$= \frac{-16}{10} \text{ } ^\circ\text{C/min}$$

$$= -1.6 \text{ } ^\circ\text{C/min}$$

e  $\frac{dT}{dt} = -kAe^{-kt}$

i When  $t = 6$

$$\frac{dT}{dt} = -k \times 79 \times e^{-6k}$$

$$\approx -2.0479 \text{ } ^\circ\text{C/min}$$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T - \theta)$$

ii  $= -k(60 - 21)$

$$= -k(39)$$

$$= -39k$$

$$= -0.8826 \text{ } ^\circ\text{C/min}$$



**30** Let  $X$  be the number of good components

**a** Probability of a batch being accepted

$$\begin{aligned}
 &= \Pr(X = 4) + \Pr(X = 5) = \binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5 \\
 &= 5 \times \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 \\
 &= 6 \times \left(\frac{1}{2}\right)^5 \\
 &= \frac{6}{32} \\
 &= \frac{3}{16} = 0.1875
 \end{aligned}$$

$$\begin{aligned}
 A(p) &= \Pr(X = 4) + \Pr(X = 5) \\
 &= \binom{5}{4} (1-p)^4 p + (1-p)^5
 \end{aligned}$$

**b**

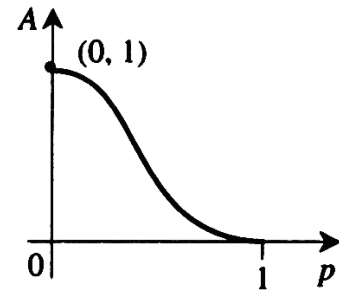
$$\begin{aligned}
 &= (1-p)^4 [5p + (1-p)] \\
 &= (1-p)^4 [4p + 1]
 \end{aligned}$$

$\therefore b = 4$

**c**

$$\begin{aligned}
 A'(p) &= -4(1-p)^3(1+4p) + 4(1-p)^4 \\
 &= (1-p)^3 [-4(1+4p) + 4(1-p)] \\
 &= (1-p)^3 [-4 - 16p + 4 - 4p] \\
 &= (1-p)^3 [-20p] \\
 &= -20p(1-p)^3
 \end{aligned}$$

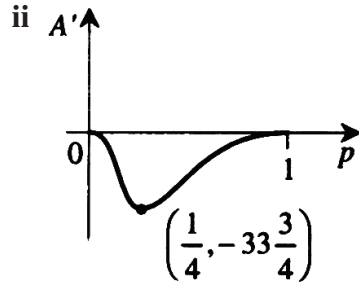
Note: no stationary point for  $p \in (0, 1)$



**d i**  $A(p) = 0.95$ : using the 'solve' command of a CAS calculate with  $0 < P < 1$  gives  $P \approx 0.076$

**ii**  $A(p) = 0.05$ : again using 'solve' gives  $P \approx 0.657$

**e i**  $A'(p) = -20p(1-p)^3$



$$\begin{aligned}
 A''(p) &= -3(1-p)^2(-20p) - 20(1-p)^3 \\
 &= -20[1-p]^2[-3p + (1-p)]
 \end{aligned}$$

iii

$$= -20(1-p)^2(1-4p)$$

$$A''(p) = 0 \text{ implies } p = 1 \text{ or } p = \frac{1}{4} \text{ so } A'(p) \text{ is a minimum in } p = \frac{1}{4}.$$

iv Most rapid rate of change of probabilities occurs when  $p = \frac{1}{4}$ .

31  $h(t) = (4.5 - 0.3t)^3$

a When  $t = 0$ ,  $h(0) = 4.5^3 = 91.125$  cm

b  $h(t) \geq 0$  and  $t \geq 0$

$$\therefore (4.5 - 0.3t)^3 \geq 0 \text{ and } t \geq 0$$

equivalently  $4.5 - 0.3t \geq 0$  and  $t \geq 0$

$$\therefore \frac{4.5}{0.3} \geq t \text{ and } t \geq 0$$

$$\therefore t \leq 15 \text{ and } t \geq 0$$

i.e.  $t \in [0, 15]$

c  $V = (0.8)^2(4.5 - 0.3t)^3$   
 $= 0.64(4.5 - 0.3t)^3$

d  $h$  is a 1 to 1 function

domain of  $h$  is  $[0, 15]$

range of  $h = [0, 91.125]$

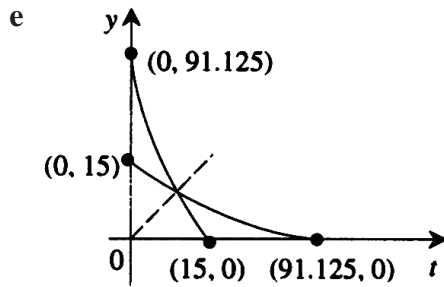
Consider  $x = (4.5 - 0.3y)^3$

$$x^{\frac{1}{3}} = 4.5 - 0.3y$$

$$\therefore 0.3y = 4.5 - x^{\frac{1}{3}}$$

$$\therefore y = 15 - \frac{10x^{\frac{1}{3}}}{3}$$

$\therefore$  inverse function is  $h^{-1}(t) = 15 - \frac{10t^{\frac{1}{3}}}{3}$   
 domain =  $[0, 91.125]$



32  $\mu = 3$  mm

Let  $X$  be the diameter

$$\Pr(X < 2.9) = 0.063$$

$$\Pr(X > 3.1) = 0.063$$

a  $\Pr\left(Z > \frac{3.1 - 3}{\sigma}\right) = 0.063$

$$\therefore \Pr\left(Z > \frac{0.1}{\sigma}\right) = 0.063$$

$$\therefore \Pr\left(Z \leq \frac{0.1}{\sigma}\right) = 0.937$$

$$\therefore \frac{0.1}{\sigma} = 1.53$$

$$\begin{aligned} \sigma &= \frac{0.1}{1.53} \\ &= 0.06536 \end{aligned}$$

b Let  $Y$  be the number of ball bearings accepted out of 8.

The probability of rejection =  $\Pr(X < 2.9) + \Pr(X > 3.1)$

$$= 0.063 \times 2$$

$$= 0.126$$

For the binomial distribution,  $p = 0.126$  and  $n = 8$

$$\Pr(Y \geq 1) = 1 - \Pr(Y = 0)$$

$$= 1 - (0.874)^8$$

i  $= 1 - 0.34047$

$$= 0.6595$$

$$\begin{aligned} \text{ii} \quad \Pr(Y = 2) &= \binom{8}{2} (0.126)^2 (0.874)^6 \\ &= 0.19814 \end{aligned}$$

$$\text{c i} \quad \mu = 3.05, \sigma = 0.06536$$

$$\begin{aligned} \Pr(X \leq 2.9) + \Pr(X \geq 3.1) &= \Pr\left(Z \leq \frac{2.9 - 3.05}{0.06536}\right) + 1 - \Pr\left(Z \leq \frac{3.1 - 3.05}{0.06536}\right) \\ &= \Pr(Z \leq -2.295) + 1 - \Pr(Z \leq 0.765) \\ &= 2 - \Pr(Z \leq 2.295) - \Pr(Z \leq 0.765) \\ &= 2 - 0.9891 - 0.7779 \\ &= 0.233 \end{aligned}$$

So 23.3% will now fill outside the given range.

$$\text{ii} \quad \Pr(3.05 - c \leq X \leq 3.05 + c) = 0.9$$

$$\therefore \Pr\left(\frac{-c}{0.06536} \leq Z \leq \frac{c}{0.06536}\right) = 0.9$$

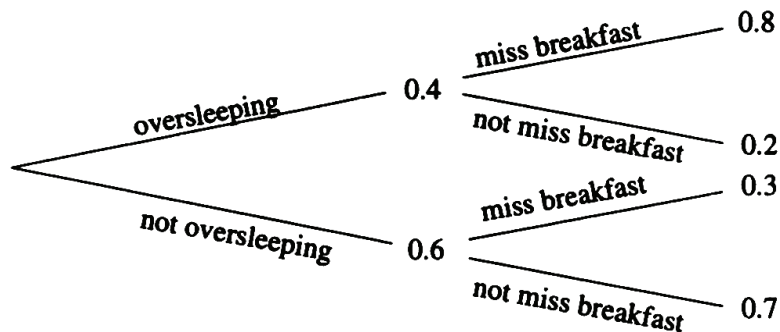
$$\therefore 2 \Pr\left(Z \leq \frac{c}{0.06536}\right) - 1 = 0.9$$

$$\therefore \Pr\left(Z \leq \frac{c}{0.06536}\right) = 0.95$$

$$\therefore \frac{c}{0.06536} = 1.6449$$

$$\therefore c = 0.1075$$

33



a From tree diagram

$$\text{i} \quad \Pr(\text{oversleeping} \cap \text{missing breakfast}) = 0.4 \times 0.8 = 0.32$$

$$\text{ii} \quad \Pr(\text{not oversleeping} \cap \text{missing breakfast}) = 0.6 \times 0.3 = 0.18$$

$$\text{iii} \quad \Pr(\text{oversleeping} \cap \text{missing breakfast}) + \Pr(\text{not oversleeping} \cap \text{missing breakfast}) = 0.32 + 0.18 = 0.5$$

$$\Pr(\text{overslept} \mid \text{missing breakfast}) = \frac{\Pr(\text{overslept} \cap \text{missing breakfast})}{\Pr(\text{missing breakfast})}$$

$$\begin{aligned} \mathbf{b} \quad &= \frac{0.32}{0.5} \\ &= 0.64 \end{aligned}$$

**c** This is a binomial distribution problem if it is assumed that a student's behaviour is independent of any other students behaviour.

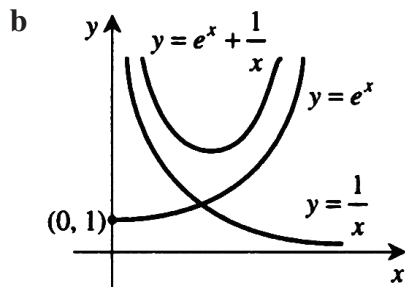
Let  $X$  be the number of students who miss breakfast

$$\mathbf{i} \quad \Pr(X = 2) = {}^{10}C_2(0.5)^2(0.5)^8 = 0.043955$$

$$\mathbf{ii} \quad \Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - (0.5)^{10} = 0.999$$

$$\begin{aligned} \mathbf{iii} \quad &\text{Probability of at least 8 not missing breakfast} \\ &= \Pr(X \leq 2) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \\ &= (0.5)^{10} + 10 \times (0.5)^{10} + {}^{10}C_2(0.5)^2(0.8)^8 \\ &= \frac{7}{128} \end{aligned}$$

**34 a**



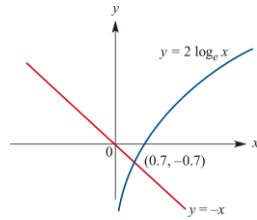
$$\begin{aligned} \mathbf{c} \quad &y = \frac{1}{x} + e^x \\ &\frac{dy}{dx} = -\frac{1}{x^2} + e^x \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \mathbf{i} \quad &\frac{dy}{dx} = 0 \Leftrightarrow -\frac{1}{x^2} + e^x = 0 \\ &\text{which implies } \frac{1}{x^2} = e^x \\ &\therefore x^2 = e^{-x} \\ &\therefore \log_e(x^2) = -x \\ &\text{i.e. } 2 \log_e x = -x \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad &\text{As } x > 0, 2 \log_e x = -x < 0 \\ &\therefore 2 \log_e x < 0 \end{aligned}$$

$\therefore x < 1$

$\therefore$  local minimum lies in the interval  $(0, 1)$



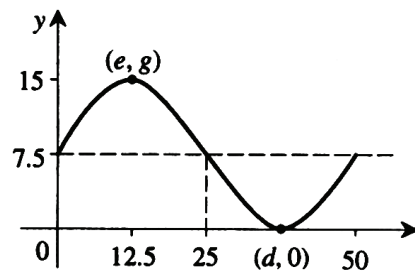
**iii**

**iv** local minimum occurs when  $x = 0.7$

$$\begin{aligned} \therefore y &= \frac{1}{0.7} + e^{0.7} \\ &= 3.4 \text{ correct to one decimal place} \\ \text{i.e. coordinates local minimum are } &(0.7, 3.4) \end{aligned}$$

**35 a i** From the diagram

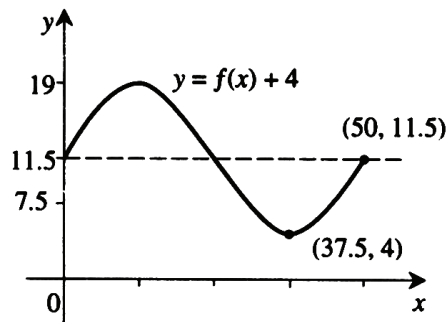
amplitude = 7.5  $\therefore b = 7.5$   
and centre is at  $y = 7.5 \therefore a = 7.5$



$$\text{period} = 2\pi \div \frac{2\pi}{50} = 50$$

$$\therefore m = 12.5, n = 15 \text{ and } d = 37.5$$

**ii**



**b**  $10 = 7.5 + 7.5 \sin \frac{(2\pi x)}{50}$

$$\therefore \frac{2.5}{7.5} = \sin\left(\frac{2\pi x}{50}\right)$$

$$\frac{1}{3} = \sin\left(\frac{2\pi x}{50}\right)$$

$$\text{Let } \theta = \frac{2\pi x}{50}$$

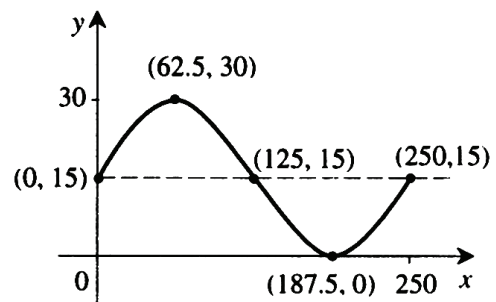
$$\therefore \theta = \sin^{-1}\left(\frac{1}{3}\right) \text{ or } \pi - \sin^{-1}\left(\frac{1}{3}\right)$$

$$\begin{aligned} \therefore x &= \frac{50}{2\pi} \sin^{-1}\left(\frac{1}{3}\right) \text{ or } \frac{50}{2\pi} \left(\pi - \sin^{-1}\left(\frac{1}{3}\right)\right) \\ &= 2.704 \text{ or } 22.296 \end{aligned}$$

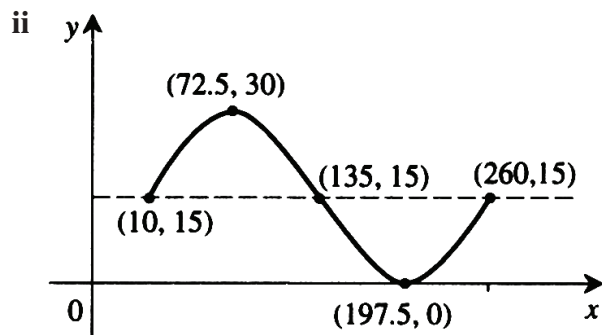
$$\begin{aligned} \text{c } g(x) &= 2f\left(\frac{x}{5}\right) = 2\left(7.5 + 7.5 \sin\left(\frac{2\pi}{50}\left(\frac{x}{5}\right)\right)\right) \\ &= 15 + 15 \sin\left(\frac{2\pi x}{250}\right) \\ &= 15 + 15 \sin\left(\frac{\pi x}{125}\right) \\ \therefore \text{amplitude} &= 15 \end{aligned}$$

$$\text{centre } y = 15$$

$$\text{period} = 2\pi \div \frac{\pi}{125} = 250$$



$$\begin{aligned} \text{d i } \text{the new function has rule } h(x) &= g(x - 10) \\ &= 15 + 15 \sin\left(\frac{\pi}{125}(x - 10)\right) \end{aligned}$$



$$36 \text{ a } f(x) = \begin{cases} 0 & \text{if } x < 20 \\ k(5 - 2x) & 2 < x \leq \frac{5}{2} \\ 0 & x > \frac{5}{2} \end{cases}$$

$$\int_2^5 f(x) dx = [k(5x - x^2)]_2^5 = \frac{k}{4}$$

For  $f$  to be a probability density function  $k = 4$ .

**b i**  $E(X) = \int_2^5 xf(x) dx = 4 \int_2^5 5x - 2x^2 dx = \frac{13}{6}$

**ii** Solve  $\int_0^a f(x) dx = 0.5$  for  $a$

$$4(5a - a^2 - (10 - 4)) = 0.5$$

$$8(-a^2 + 5a - 6) = 1$$

$$-8a^2 + 40a - 49 = 0$$

Therefore  $a = \frac{10 - \sqrt{2}}{4}$  as  $2 < a < \frac{5}{2}$

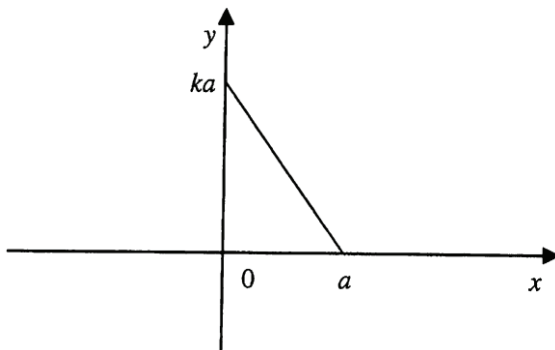
The median is  $\frac{10 - \sqrt{2}}{4}$

**iii**  $\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{1}{72}$

Therefore  $\sigma = \frac{\sqrt{2}}{12}$

**iv**  $\Pr(X < \mu - \sigma) = \Pr\left(x < \frac{13}{6} - \frac{\sqrt{2}}{12}\right)$   
 $= 0.1857$

37



**a**  $\int_0^a f(x) dx = 1$

Therefore  $\frac{1}{2}ka^2 = 1$

and  $k = \frac{2}{a^2}$



$$\begin{aligned} E(X) &= \int_0^a xf(x) dx \\ &= \frac{a^3k}{6} \\ &= \frac{a}{3} \end{aligned}$$

$$\begin{aligned} \text{b } \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \int_0^a x^2 f(x) dx - \frac{a^2}{9} \\ &= \frac{a^4k}{12} - \frac{a^2}{9} \\ &= \frac{a^2}{18} \end{aligned}$$

$$\text{c } \Pr(X > \mu + 2\sigma) = \Pr\left(x > \frac{a}{3} + \frac{2a}{3\sqrt{2}}\right) = \frac{6 - 4\sqrt{2}}{9}$$

$$\begin{aligned} \text{d } \text{Solve } \int_0^{1000} f(x) dx &= 0.5 \text{ for } a \\ a &= 1000(\sqrt{2} + 2) \end{aligned}$$

$$38 \quad y = \frac{x}{10} - \log_e(x + 3), \quad x > -3$$

$$\begin{aligned} \text{a } \frac{dy}{dx} &= \frac{1}{10} - \frac{1}{x+3} \\ \text{and } \frac{dy}{dx} &= 0 \text{ implies } x + 3 = 10. \text{ Hence } x = 7 \end{aligned}$$

$$\text{b } \frac{dy}{dx} = \frac{1}{10} - \frac{1}{x+3} > \frac{1}{10} \text{ for } x > -3$$

$$\text{c } \text{The coordinates of } M \text{ are } \left(7, \frac{7}{10} - \log_e(10)\right)$$

$$\text{Equation of line is } y - \left(\frac{7}{10} - \log_e(10)\right) = \frac{1}{10}(x - 7), \text{ i.e. } y = \frac{1}{10}x - \log_e 10$$

**d i** "The line in *c* has gradient  $\frac{1}{10}$  and hence intersects the *x*-axis at a point to the left of *P* (since the gradient of the curve  $\pi < \frac{1}{10}$ ).

For the line, when  $y = 0$ ,

$$x = 10 \log_e 10.$$

Hence the *x*-axis intercept at *P* is greater than  $10 \log_e 10$ ."

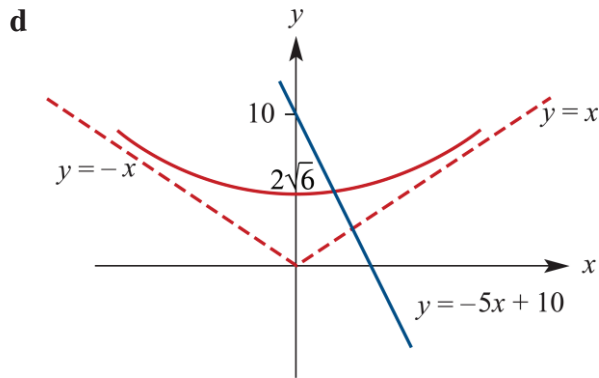
**ii** Using the 'solve' command of a CAS calculator shows that the intercept at *P* has *x* coordinate 36.852

39 a  $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 24}}$

b  $\frac{dy}{dx} = 0 \rightarrow x = 0$ ; then  $y = \sqrt{24} = 2\sqrt{6}$ .

So the coordinates of the local minimum are  $(0, 2\sqrt{6})$ .

c  $f(-x) = \sqrt{(-x)^2 + 24} = \sqrt{x^2 + 24} = f(x)$ , so the function is even.



e When  $x = 1$ ,  $\frac{dy}{dx} = \frac{1}{5}$ .

So the gradient of the normal at  $(1, 5)$  is  $-5$ .

Its equation is  $y - 5 = -5(x - 1)$

$y = -5x + 10$ :

f  $\frac{dy}{dt} = 10$  at the point  $(5, 7)$ :

$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$  (chain rule)

$10 = \frac{x}{\sqrt{x^2 + 24}} \frac{dx}{dt}$

$10 = \frac{5}{7} \frac{dx}{dt}$  at  $(5, 7)$

$\frac{dx}{dt} = 14$  units/second

g  $\frac{d}{dx} \left( 12 \log_e \left| \sqrt{x^2 + 24} + x \right| + \frac{\sqrt[3]{x^2 + 24}}{2} \right)$

$$= 12 \times \frac{\frac{x}{\sqrt{x^2 + 24}} + 1}{\sqrt{x^2 + 24} + x} + \frac{\sqrt{x^2 + 24}}{2} + \frac{x^2}{\sqrt{x^2 + 24}}$$

$$= \frac{12}{\sqrt{x^2 + 24}} + \frac{x^2 + 24}{2\sqrt{x^2 + 24}} + \frac{x^2}{2\sqrt{x^2 + 24}}$$

$$= \frac{x^2 + 24}{\sqrt{x^2 + 24}} = \sqrt{x^2 + 24} \text{ as required.}$$

**h** Area =  $\int_2^5 \sqrt{x^2 + 24} \, dx$

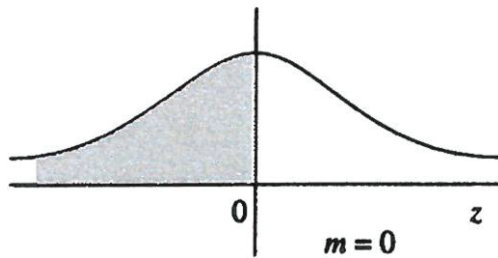
$$= \left[ 12 \log_e \left| \sqrt{x^2 + 24} + x \right| + \frac{x \sqrt{x^2 + 24}}{2} \right]_2^5$$

$$= \left( 12 \log_e \left( 12 + \frac{35}{2} \right) \right) - \left( 12 \log_e (2\sqrt{7} + 2) + 2\sqrt{7} \right)$$

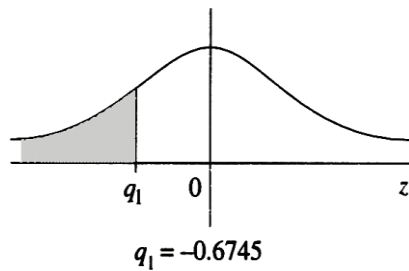
$$= 12 \log_e \left( \frac{6}{\sqrt{7} + 1} \right) - 2\sqrt{7} + \frac{35}{2}$$

$$= 12 \log_e (\sqrt{7} - 1) - 2\sqrt{7} + \frac{35}{2}$$

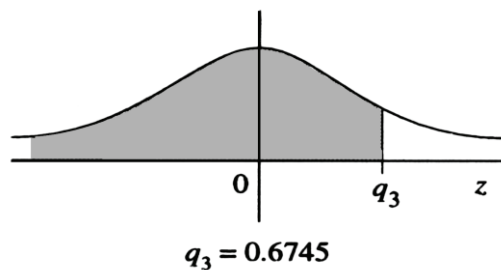
**40 a i**



**ii**



**iii**



**iv** interquartile range = 1.3490

**v**  $\Pr(q_1 - 1.5 \times IQR < Z \leq q_3 + 1.5 \times IQR)$

$$= \Pr(-0.6745 - 1.5 \times 1.3490 < Z < 0.6745 + 1.5 \times 1.3490)$$

$$= \Pr(-2.698 < Z < 2.698)$$

$$= 0.993 \text{ or } 99.3\%$$

vi 0.7%

b i  $\mu$

ii  $\mu - 0.6745\sigma$

iii  $\mu + 0.6745\sigma$

iv  $1.3490\sigma$

v 0.993 or 99.3%

vi 0.7%

41 a  $\int_0^1 f(x) dx = 1$   
 $\frac{k}{n+1} = 1$

Therefore  $k = n + 1$

b  $E(X) = \int_0^1 xf(x) dx = \frac{n+1}{n+2}$

c  $E(X^2) = \int_0^1 x^2 f(x) dx = \frac{n+1}{n+3}$   
 $\text{Var}(X) = \frac{n+1}{(n+2)^2(n+3)}$

d If  $m$  is the median, then

$$k \int_0^m x^n dx = \frac{1}{2}$$
$$k \left[ \frac{1}{n+1} x^{n+1} \right]_0^m = \frac{1}{2}$$

$$k \left( \frac{m^{n+1}}{n+1} \right) = \frac{1}{2}$$

$$m^{n+1} = \frac{n+1}{2k}$$
$$= \frac{1}{2} \text{ since } n = x +$$

$$m = n + 1 \sqrt{\frac{1}{2}}$$

e No turning pts  $\rightarrow$  mode = 1.

$$\begin{aligned}
 \text{42 a i Gradient } AB &= \frac{\frac{1}{b-1} - 1}{b-2} \\
 &= \frac{2-b}{(b-1)(b-2)} \\
 &= -\frac{1}{b-1} \\
 &= \frac{1}{1-b}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } g'(x) &= -\frac{1}{(x-1)^2} \\
 &= \frac{1}{1-b}
 \end{aligned}$$

$$\text{if } (x-1)^2 = b-1$$

$$x-1 = \sqrt{b-1} \text{ (positive square root since } x > 1)$$

$$x = 1 + \sqrt{b-1}$$

$$\begin{aligned}
 \text{b i } \int_2^{e+1} \frac{1}{x-1} dx &= [\log_e(x-1)]_2^{e+1} \\
 &= \log_e e - \log_e 1 \\
 &= 1 - 0 \\
 &= 1
 \end{aligned}$$

$$\text{ii } \int_c^{1+e} \frac{1}{x-1} dx = 8$$

$$[\log_e(x-1)]_c^{1+e} = 8$$

$$\log_e e - \log_e(c-1) = 8$$

$$1 - \log_e(c-1) = 8$$

$$\log_e(c-1) = -7$$

$$c-1 = e^{-7}$$

$$c = 1 + e^{-7}$$

$$\begin{aligned}
 \text{c i Area of trapezium} &= \frac{1}{2}(b-2)\left(1 + \frac{1}{b-1}\right) \\
 &= \frac{1}{2}(b-2)\left(\frac{b}{b-1}\right) \\
 &= \frac{b(b-2)}{2(b-1)}
 \end{aligned}$$

$$\text{ii} \quad \frac{b(b-2)}{2(b-1)} = 8$$

$$b^2 - 2b = 16b - 16$$

$$b^2 - 18b + 16 = 0$$

Solving by the formula or completing the square gives  $b = 9 \pm \sqrt{65}$  but  $b > 2$ . so  $b = 9 + \sqrt{65}$ .

$$\text{d} \quad \int_2^{mn+1} \frac{1}{x-1} dx + \int_2^{\frac{m}{n}+1} \frac{1}{x-1} dx = 2$$

Now the upper terminals must be greater than 1 since we can not integrate over the discontinuity at  $x = 1$ . Hence:

$$[\log_e(x-1)]_2^{mn+1} + [\log_e(x-1)]_2^{\frac{m}{n}+1} = 2$$

$$\left( \log_e(mn) + \log_e\left(\frac{m}{n}\right) \right) = 2 \quad (n \text{ positive so } m \text{ positive})$$

$$\log_e \left[ (mn) \times \left(\frac{m}{n}\right) \right] = 2$$

$$\log_e m^2 = 2$$

$$m^2 = e^2$$

$$m = e \quad (m > 0)$$

$$\begin{aligned} \text{43 a i Gradient } AB &= \frac{\frac{1}{b^2} - 1}{b - 1} \\ &= \frac{1 - b^2}{b^2(b - 1)} \\ &= \frac{(1 - b)(1 + b)}{b^2(b - 1)} \\ &= -\frac{b + 1}{b^2} \end{aligned}$$

$$\begin{aligned} f'(x) &= -\frac{2}{x^3} \\ &= -\frac{b + 1}{b^2} \end{aligned}$$

$$\text{ii} \quad \text{if } x^3 = -\frac{2b^2}{(b + 1)}$$

$$x = \left( \frac{2b^2}{b + 1} \right)^{\frac{1}{3}}$$

$$\text{b i Area of trapeziums} = \frac{1}{2}(b - 1) \left( \frac{1}{b^2} + 1 \right)$$

$$S(b) = \frac{(b^2 + 1)(b - 1)}{2b^2}$$

$$\text{ii } \frac{(b^2 + 1)(b - 1)}{2b^2} = \frac{10}{9}$$

$$9(b^3 - b^2 + b - 1) = 20b^2$$

$$9b^3 - 29b^2 + 9b - 9 = 0$$

Using the factor theorem or a CAS calculator shows that  $b - 3$  is a factor of the cubic, giving  $(b - 3)(9b^2 - 2b + 3) = 0$

The quadratic has no zeroes ( $B^2 - 4AC < 0$ ), so  $b = 3$  is the only solution.

$$\begin{aligned} \text{iii } \int_1^b f(x) dx &= \int_1^b \frac{1}{x^2} dx \\ &= \left[ -\frac{1}{x} \right]_1^b \\ &= \frac{-1}{b} + 1 \\ &= 1 - \frac{1}{b} \end{aligned}$$

$< 1$  since  $b > 1$  and so  $0 < \frac{1}{b} < 1$

$$\begin{aligned} D(b) &= S(b) - \int_1^b f(x) dx \\ &= \frac{(b^2 + 1)(b - 1)}{2b^2} - \left(1 - \frac{1}{b}\right) \text{ from } b \text{ i and } b \text{ iii} \\ &= \frac{(b^2 + 1)(b - 1)}{2b^2} - \left(\frac{b - 1}{b}\right) \quad [t] \\ &= \frac{b - 1}{2b^2}(b^2 + 1 - 2b) \\ &= \frac{b - 1}{2b^2}(b - 1)^2 \\ &= \frac{(b - 1)^3}{2b^2} \end{aligned}$$

To show that the function is strictly increasing for  $b > 1$ , it is sufficient to show that  $D'(b) > 0$  for  $b > 1$ .

$$\begin{aligned}
D'(b) &= \frac{(2b^2)(3(b-1)^2) - ((b-1)^3)(4b)}{4b^4} \\
&= \frac{3b(b-1)^2 - 2(b-1)^3}{2b^3} \\
&= \frac{(b-1)^2(3b - 2(b-1))}{2b^3} \\
&= \frac{(b-1)^2(b+2)}{2b^3} \\
&> 0 \text{ for all } b > 1
\end{aligned}$$

**44 a**  $f'(x) = x^m(-ne^{-nx+n}) + mx^{m-1}e^{-nx+n}$

$$= x^{m-1}e^{-nx+n}(-nx + m)$$

$$= 0$$

if  $x = 0$  or  $x = \frac{m}{n}$ .

So for the stationary point not at the origin,  $x = \frac{m}{n}$  and then

$$f\left(\frac{m}{n}\right) = \left(\frac{m}{n}\right)^m e^{-m+n}$$

The Point with coordinates  $\left(\frac{m}{n}, \left(\frac{m}{n}\right)^m e^{-m+n}\right)$  is a local maximum turning point (by reference to the given graph or by checking the sign of the first derivative which goes from positive to negative through  $x = \frac{m}{n}$ ).

**b** Find the equation of the tangent at a general point  $x = a$  on the curve.

$$x = a : f(a) = a^m e^{-an+n}$$

$$f'(a) = a^{m-1} e^{-an+n}(-an + m)$$

using  $y - y_1 = m(x - x_1)$ , the equation of the tangent is

$$y - a^m e^{-an+n} = a^{m-1} e^{-an+n}(-an + m)(x - a)$$

The tangent passes through the origin, so  $(0, 0)$  satisfies the equation.

$$-a^m e^{-an+n} = a^{m-1} e^{-an+n}(-an + m)(-a)$$

$$1 = (-an + m)(am + 0, e^{-an+n} + 0)$$

$$an = m - 1$$

$$a = \frac{m-1}{n}$$

substitute to find the y-coordinate:

$$\begin{aligned}
f(a) &= f\left(\frac{m-1}{n}\right) \\
&= \left(\frac{m-1}{n}\right)^m e^{n-m+1}
\end{aligned}$$



So the tangent at  $\left(\frac{m-1}{n}, \left(\frac{m-1}{n}\right)^m e^{n-m+1}\right)$  passes through the origin.  
 (Note: the tangent at  $(0, 0)$  also passes through the origin!)

**c i** Using CAS calculator, we find that

$$\int_0^{\infty} x^2 e^{-2x+2} dx = \frac{e^2}{4}$$

$$\text{So: } \frac{4}{e^2} \int_0^{\infty} x^2 e^{-2x+2} dx = 1 \text{ and } k = \frac{4}{e^2}$$

$$\Pr(X < 1) = \int_0^1 \frac{4}{e^2} x^2 e^{-2x+2} dx$$

$$\begin{aligned} \text{ii} \quad &= (e^2 - 5)e^{-2} \quad (\text{using a CAS calculator}) \\ &= 1 - 5e^{-2} \\ &= 1 - \frac{5}{e^2} \end{aligned}$$

**iii** The mode is the value for which  $f$  is a maximum. Use calculus to solve

$$f'(x) = 0.$$

$$\begin{aligned} f'(x) &= x^2(-2e^{-2x+2}) + 2xe^{-2x+2} dx \\ &= 2xe^{-2x+2}(-x + 1) \\ &= 0 \end{aligned}$$

$$\text{if } x = 1$$

So the mode is 1.

Alternatively, note that  $x^2 e^{-2x+2}$  is the function from part **a** with  $m = 2$  and  $n = 2$

From That question, the  $x$ -coordination of the stationary point is  $x = \frac{m}{n} = \frac{2}{2} = 1$  in this case.

$$\begin{aligned} \text{45 a i} \quad \int_0^{\infty} e^{-qx} dx &= \lim_{a \rightarrow \infty} \int_0^a e^{-qx} dx \\ &= \lim_{a \rightarrow \infty} \left[ -\frac{1}{q} e^{-qa} \right]_0^a \\ &= \lim_{a \rightarrow \infty} \left( -\frac{1}{q} e^{-qa} + \frac{1}{q} \right) \\ &= 0 + \frac{1}{q} \\ &= \frac{1}{q} \quad (\text{since } e^{-qa} \rightarrow 0 \text{ as } a \rightarrow \infty) \end{aligned}$$

$$\begin{aligned} \text{Hence } \int_0^{\infty} k e^{-qx} dx &= \frac{k}{q} \\ &= 1 \text{ if } k = q \end{aligned}$$

$$\begin{aligned} \text{ii } E(x) &= \int_0^{\infty} x \times qe^{-qx} dx \\ &= \frac{1}{q} \text{ (using a CAS calculator)} \end{aligned}$$

$$\begin{aligned} E(x^2) &= \int_0^{\infty} x^2 \times qe^{-qx} dx \\ &= \frac{2}{q^2} \text{ (using a CAS calculator)} \end{aligned}$$

$$\begin{aligned} \text{iii } \text{var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{2}{q^2} - \left(\frac{1}{q}\right)^2 \\ &= \frac{1}{q^2} \end{aligned}$$

**iv** If  $m = \frac{1}{2} \log_e(2)$ , then

$$\begin{aligned} \int_0^m qe^{-qx} dx &= [-e^{-qx}]_0^m \\ &= -e^{-qm} + 1 \\ &= -e^{-\log_e 2} + 1 \\ &= -e^{-\log_e \frac{1}{2}} + 1 \\ &= -\frac{1}{2} + 1 \\ &= \frac{1}{2} \end{aligned}$$

So  $m$  is the medium.

Alternatively, solve  $\int_0^m f(x) dx = \frac{1}{2}$  for  $m$ .

$$\text{b } \Pr\left(X > \frac{1}{q} \log_e(3) \mid X > \frac{1}{q} \log_e(2)\right) = \frac{\Pr\left(X > \frac{1}{q} \log_e(3)\right)}{\Pr\left(X > \frac{1}{q} \log_e(2)\right)}$$

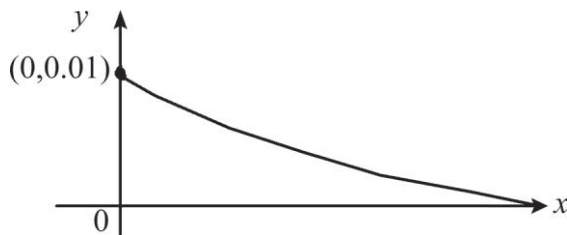
Since the median is  $\frac{1}{q} \log_e(2)$  from part **a iv**, the denominator is  $\frac{1}{2}$ .

$$\begin{aligned}
\Pr\left(X > \frac{1}{q} \log_e 3\right) &= 1 - \Pr\left(X \leq \frac{1}{q} \log_e(3)\right) \\
&= 1 - \int_0^{\frac{1}{q} \log_e(3)} q e^{-qx} dx \\
&= 1 - \left[-e^{-qx}\right]_0^{\frac{1}{q} \log_e(3)} \\
&= 1 + e^{\log_e 3} - 1 \\
&= e^{\log_e 3} \\
&= \frac{1}{3}
\end{aligned}$$

For the numerator:  $= \frac{1}{3}$   
 So

$$\begin{aligned}
\Pr\left(X > \frac{1}{q} \log_e(3) \mid X > \frac{1}{q} \log_e(2)\right) &= \frac{\frac{1}{3}}{\frac{1}{2}} \\
&= \frac{2}{3}
\end{aligned}$$

- c i** The graph of  $y = f(x) = 0.01e^{-0.01x}$ ,  $x \geq 0$ , is that of an exponential function with y-axis intercept  $(0, 0.01)$  and horizontal asymptote  $y = 0$  (the x-axis).



- ii**  $\Pr(X > 100) = 1 - \Pr(X \leq 100)$

$$\begin{aligned}
&= 1 - \int_0^{100} 0.01 e^{-0.01x} dx \\
&= 1 - \left[-e^{-0.01x}\right]_0^{100} \\
&= 1 + e^{-1} - 1 \\
&= e^{-1} \approx 0.37
\end{aligned}$$

- iii** From part **a iv**,  $m = \frac{1}{0.01} \log_e(2)$

$$= 100 \log_e(2) \approx 69.31$$

46 a 0.527

b (0.4961, 0.5580)

c For a 95% CI,  $M = 1.96 \times \sqrt{\frac{0.527 \times 0.473}{1000}} \approx 0.0309$  Half this width is 0.0155

Thus, we need to find a such that

$$a \times \sqrt{\frac{0.527 \times 0.473}{1000}} = 0.0155$$

$$a = 0.981$$

To find the level of confidence associated with  $a = 0.981$  we use the normal cdf function.

$$\text{Level of confidence} = \Pr(-0.981 < Z < 0.981) = 0.6734$$

Ie, 67.34% confidence interval

d Twice this width is 0.0618 Thus, we need to find a such that

$$a \times \sqrt{\frac{0.527 \times 0.473}{1000}} = 0.0618$$

$$a = 3.914$$

To find the level of confidence associated with  $a = 3.914$  we use the normal cdf function. Level of confidence =  $\Pr(-3.914 < Z < 3.914) = 0.9999$

ie, 99.99% confidence interval